MA 399 Intro to Quantum Information Theory

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Abstract

No idea what is happening in this class lol do this later

Contents

1   Intro to Vector Spaces

Let's jump right in. This section is an abbreviation of the introduction to linear algebra session.

A vector space is a group of objects (vectors) which may be added together and multiplied by compatible scalars from $\mathbb{R}$ or $\mathbb{C}$. In this class, we primarily care about vector spaces $\mathbb{C}^n$ from $\mathbb{C}$ and $\mathbb{R}^n$ from $\mathbb{R}$. Recall that $\mathbb{C}$ is the scalar field of complex numbers $a + bi$, where multiplication is defined by the rule $i^2 = -1$, and is equipped with:

(a) a complex conjugation operation – $\overline{a + bi} = (a + bi)^* = a - bi$ and

(b) a size function called the **modulus** – $|a + bi| = \sqrt{a^2 + b^2}$.

Note that the modulus is similar to magnitude.
To work with complex numbers, it's useful to have an understanding of the basic operations.

(a) To add/subtract complex numbers, add/subtract the corresponding real/imaginary parts. For example – $(a + bi) + (c + di) = (a + c) + (b + d)i$

(b) To multiply/divide complex numbers, multiply both parts of the complex number by the real number. For example – $(a + bi) * (c + di) = ac + adi + bcj - bd = (ac - bd) + (ad + bc)i$. This form will be useful for the duration of the class.

1.1 Representing Vectors in Complex Spaces

As mentioned, this class primarily works in complex number spaces. For this reason, having useful tools for representing vectors in these abstract spaces is useful. Vectors are represented in **bra-ket** notation, where *bra* represents a *row* vector, and *ket* represents a *column* vector.

Definition 1.1 If $|v\rangle \in \mathbb{C}^n$ is a *ket* vector which consists of n complex numbers,

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix} \text{ for } v_1, v_2, ..., v_n \in \mathbb{C} \tag{1.1}$$

◆

Definition 1.2 If $\langle v| \in \mathbb{C}^n$ is a *bra* vector which consists of n complex numbers,

$$\langle v| = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix} \text{ for } v_1, v_2, ..., v_n \in \mathbb{C} \tag{1.2}$$

◆

Note that the the integer n in definitions 1.1 and 1.2 is called the **dimension** of the vector space $\mathbb{C}^n$.

Example 1.3 $\mathbb{C}^2$ is a 2-dimensional vector space over $\mathbb{C}$.

$$\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ Note: $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ 'fills up' $\mathbb{C}^2$; IE it represents all values contained within $\mathbb{C}^2$

◆

1.2 Linear Combinations

A linear combination is a useful tool for this class which is the 'combination' or multiplication of each term in a set by constants, and then adding the result. The interesting property of linear combinations is that the result is still contained within the initial set.

Definition 1.4 A linear combination of $\{|v_1\rangle, ..., |v_n\rangle\} \subset \mathbb{C}^n$ is a single vector in the form $\lambda_1 |v_1\rangle + \lambda_2 |v_2\rangle + ... + \lambda_n |v_n\rangle$ for some $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{C}^n$. ◆

Remark 1.5 If $|w\rangle$ is a linear combination of $\{|v_1\rangle, ..., |v_n\rangle\} \subset \mathbb{C}^n$, we can say that it belongs to the **span** of the set of $\{|v_1\rangle, ..., |v_n\rangle\} \subset \mathbb{C}^n$. ◆

Example 1.6 Create a linear combination of $|v_1\rangle = \begin{bmatrix} i \\ 2 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} -1 \\ i+1 \end{bmatrix}$

$$(foil)$$
$$(3+2i) \begin{bmatrix} i \\ 2 \end{bmatrix} + (2+i) \begin{bmatrix} -1 \\ i+1 \end{bmatrix}$$
$$(add)$$
$$= \begin{bmatrix} 3i-2 \\ 6+4i \end{bmatrix} + \begin{bmatrix} -2-i \\ 3i+1 \end{bmatrix}$$
$$= \begin{bmatrix} 2i-4 \\ 7i+7 \end{bmatrix} = |w\rangle \text{ } spans \text{ } \{|v_1\rangle, |v_2\rangle\}$$

◆

1.3   Linear Independence

Another important concept is that of a set being *linearly independent*. Being linearly independent essentially means that every piece of 'information' given by a set of vectors adds some sort of new information/perspective on the problem.

Remark 1.7 Two vectors are **linearly independent** as long as they are not *parallel*.   ♦

A good way of looking at this is that a system that is not linearly independent has more than one element that is some offset of the same constant. These elements are not giving new perspective, because they give the same information.

Definition 1.8 A set of vectors $\{|v_1\rangle, \ldots, |v_v\rangle\}$ is linearly independent if no vector is a linear combination of any other vectors. Algebraically, $[if \lambda_1 |v_1\rangle + \lambda_2 |v_2\rangle + ... + \lambda_n |v_k\rangle = |0\rangle, then \lambda_1 = \lambda_2 = \lambda_k = 0]$ ♦

The only way that the zero vector ($|0\rangle$) can be possible is if the complex constants (λ) are all the same, *and* zero.

Theorem 1.9 *If a set of k vectors in $\mathbb{C}^n$ is linearly independent, then $k \leq n$. Equivalently, if you have a set of k vectors in $\mathbb{C}^n$ such that $k > n$, then the set is linearly independent.*

This theorem is important because it establishes the notion that a set can't contain more vectors than the space allows for. Said differently, the dimension of M is the number of vectors contained within that are linearly independent.

1.4   Basis

Theorem 1.10 *A **basis** of a subspace $M \subseteq \mathbb{C}^n$ is a set of vectors such that*

1. *S is linearly independent*

2. *Span(S) = M $\implies$ "S Spans M"*

It follows that the standard basis of $\mathbb{C}^N$ is
$\begin{bmatrix} 1 \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ ... \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ ... \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ ... \\ 1 \end{bmatrix}$

1.5   Inner Products

Recall that $|v\rangle$ is a column vector and $\langle v|$ is a row vector. Before talking about inner products, we must define operations that help the inner product process.

Remark 1.11 $(\mathbb{C}^n)^*$ is the "dual space of $\mathbb{C}^n$.   ♦

The dual space is the space of all row vectors. This is useful for the next definition, which is critical in completing inner products.

Definition 1.12 We equip $\mathbb{C}^n$ with an "involution" (*conjugate transpose*) operation:

$$\dagger : \mathbb{C}^n \longrightarrow (\mathbb{C}^n)^* \text{ given by } \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1^* & \dots & x_n^* \end{bmatrix} \text{ given a } |v\rangle \in \mathbb{C}^n, \langle v| = |v\rangle^\dagger \qquad \blacklozenge$$

A quick example of this operation is helpful to explain the significance of these definitions.

Example 1.13 Given $|v\rangle = \begin{bmatrix} 7 \\ 8i \\ \pi + 3i \\ 0 \end{bmatrix} \in \mathbb{C}^4$, find $|v\rangle^\dagger$

Take the complex conjugate of each entry
$\langle v| = \begin{bmatrix} 7 & -8i & \pi - 3i & 0 \end{bmatrix}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\blacklozenge$

Now, we can talk about inner products. Of primary importance for this section, the inner product can determine if two vectors are orthogonal, and operates the same as a dot product on $\mathbb{R}^n$.

Definition 1.14 Given $|v\rangle, |w\rangle \in \mathbb{C}^n$, the **inner product** of $|v\rangle$ and $|w\rangle$ is $\langle w|v\rangle = \langle w|^\dagger \cdot |v\rangle$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\blacklozenge$

Remark 1.15 The inner product on $\mathbb{R}^n$ is the usual dot product. $\qquad\qquad\qquad\blacklozenge$

The following is a simple example showing the inner product on simple vectors in $\mathbb{R}^3$.

Example 1.16 Given $|v\rangle = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, |w\rangle = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \in \mathbb{R}^3$, find $\langle w|v\rangle$

Perform a complex conjugate on $|w\rangle$ (In $\mathbb{R}^3$, this is the same as flipping $nx1$ to $1xn$)

$$\langle w|v\rangle = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = (5*1) + (-2*1) + (1*3) = 6 \qquad\qquad\blacklozenge$$

Now, the following example shows the inner product on two simple vectors in $\mathbb{C}^2$.

Example 1.17 Given $|v\rangle = \begin{bmatrix} i \\ 1 \end{bmatrix}, |w\rangle = \begin{bmatrix} i \\ -1 \end{bmatrix} \in \mathbb{C}^2$, find $\langle w \mid v\rangle$

$$\langle w \mid v\rangle = \begin{bmatrix} -i & -1 \end{bmatrix} \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} = 1 + (-1*1) = 0 \qquad\qquad\qquad\blacklozenge$$

Recall from calculus III that the dot product of two vectors can give us insight on orthogonality. In the case of example 1.17, the inner product of $\langle w|v\rangle$ was zero. This tells us that the two vectors are orthogonal.

Definition 1.18 The **norm** (similar to magnitude) of $|v\rangle \in \mathbb{C}^n$ is $\||v\rangle\| := \sqrt{\langle w \mid v\rangle}$ $\qquad\blacklozenge$

Example 1.19 $|v\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2 \longrightarrow \| |v\rangle \| = \sqrt{1^2 + 1^2} = \sqrt{2}$ $\qquad\qquad\qquad\blacklozenge$

Remark 1.20 • $|v\rangle \in \mathbb{C}^n$ with $|||v\rangle|| = 1$ is called a unit vector

• Unit vectors in $\mathbb{C}^n$ represent a *quantum state*

• An important property of a norm is the *triangle inequality* $\longrightarrow |||v\rangle + |w\rangle|| \leq |||v\rangle|| + |||w\rangle||$ ◆

Definition 1.21 A set of non-zero vectors $S = \{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\} \leq \mathbb{C}^n$ is called an orthogonal set if $\langle v \mid w \rangle = 0$ if $i \neq j$ ◆

Theorem 1.22 *If $S = \{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\}$ is an orthogonal set of non-zero vectors in $\mathbb{C}^n$, then S is linearly independent.*

Proof. Let $c_1, \ldots, c_k \in \mathbb{C}$ such that $c_1 |v_1\rangle + \ldots + c_k |v_k\rangle$.
Goal: show that $c_1 = c_2 = c_k = 0 \longrightarrow$ linearly independent using S is orthogonal
Additionally, let
$\quad j \in 1, \ldots, k.$
Then,
$\quad \langle v_j| (c_1 |v_1\rangle + \ldots + c_k) = \langle v_j \mid 0 \rangle$
$\quad \longrightarrow c_1 \langle v_j \mid v_1 \rangle + \ldots + c_k \langle v_j \mid v_k \rangle$
$\quad c_j \langle v_j \mid v_j \rangle = (v_j$ *is the only component 'allowed' to have magnitude*$)$
Thus, $c_j = 0$ since $|v_j\rangle \neq |0\rangle$
Note: Since j was arbitrary, $c_1 = c_2 = \ldots = c_k = 0$. Hence, S is linearly independent. ∎

Remark 1.23 S is an orthonormal basis if:

• S is an orthonormal set if n unit vectors

• $\text{Span}(S) = \mathbb{C}^n$ ◆

1.6 Projections

Definition 1.24 Define $|f_i\rangle \langle f_i|$ to be the **projection** operator onto $\text{Span}(|f_i\rangle)$ ◆

Theorem 1.25 *Let $\beta = \{|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle\}$ be an orthonormal basis for $\mathbb{C}^n$. Then, any $|x\rangle = c_1 |f_1\rangle + \ldots + c_n |f_n\rangle = \langle f_1 \mid x \rangle |f_1\rangle + \ldots + \langle f_n \mid x \rangle |f_n\rangle.$*

It follows that $\sum_{j=1}^{n} |f_j\rangle \langle f_j| = \begin{bmatrix} 1 & & \\ & \cdots & \\ & & 1 \end{bmatrix}$

Example 1.26 Let $|f_1\rangle = \frac{-2}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, |x\rangle = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$
$|x_|\rangle = \langle f_1 \mid x \rangle |f_1\rangle = P_1(|x\rangle$
$P_1(|x\rangle) = |f_1\rangle \langle f_1 \mid x \rangle$ ◆

In the above example, $|x_|\rangle$ is the projection of x onto f_1 in the direction of f_1. Note that the projection operation is similar to *comp* from calculus III. Now, let's look at some properties of the projection operator.

Proposition 1.27 *Let* $\beta = \{|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle\}$ *be an orthonormal basis for* $\mathbb{C}^n$. *Then,* $\{P_1, P_2, P_n\}$ *is a set of* $n \times n$ *matrices such that*

1. $P_i(|v\rangle) \in Span\left(()\,|f_i\rangle\right)$

2. $|v\rangle - P_i(|v\rangle)$ *is orthogonal to* $|f_i\rangle$

3. $P_i^2 = P_i * P_i = P_i$

4. $P_i P_j = 0$ *when* $i \neq $ ʝ, *and* $P_i^\dagger = P_i$

5. $\sum_{i=1}^n P_i = I_n$ *IE the sum of* P's *gives us the identity matrix.*

Projection operators make it easy for us to find c_1, c_2, c_n for $|x\rangle = c_1\,|f_1\rangle + \ldots + c_k\,|f_k\rangle$ when $\{|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle\}$ is an orthonormal set. Side note, in quantum land, P_i's are how we measure the probability that $|x\rangle$ is actually in $|f_i\rangle$.

1.7 Gram-Schmidt

Let $\beta = \{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ be a basis for $\mathbb{C}^n$

- Linearly independent

- $Span\,(\beta) = \mathbb{C}^n$ (IE no redundancy)

Goal: Turn β into an orthonormal basis, $\{|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle\}$

1. First, make an orthogonal basis.
   $|f_{i+1}\rangle := |b_{i+1}\rangle - \left(\sum_{j=1} i\frac{\langle f_j|b_{i+1}\rangle}{\langle f_j|f_j\rangle}\,|f_j\rangle\right)$

2. Then, make an orthonormal basis by normalizing the resultant ($|f\rangle$) vectors.
   For each $|f\rangle$, $|f\rangle = \frac{1}{\||f\rangle\|}\,|f\rangle$

1.8 Exercise 1

Consider $S = |v_1\rangle, |v_2\rangle, |v_3\rangle$ where $|v_1\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} i \\ 1 \end{bmatrix}, |v_3\rangle = \begin{bmatrix} 0 \\ i \end{bmatrix}$

(a) Give a linear combination of the vectors in S.

$$\alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} i \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ i \end{bmatrix} = |w\rangle$$

Note that the next problem asks to determine if a vector is in span(S). To 'kill two birds with one stone', try to find a linear combination that satisfies the question. To do this, assign values for α, β, and γ. Note that $1 + i = i + i$ (the top row), and $-1 + 1 = 0$. This allows us to set α and β to 1 and operate on γ.

$$i(x + yi) = 200 - i$$
$$xi - y \implies x = -1 \implies \gamma = -1i - 200$$

Now plug in values. $1 * \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 * \begin{bmatrix} i \\ 1 \end{bmatrix} + (-200 - 1i) * \begin{bmatrix} 0 \\ i \end{bmatrix} = |w\rangle$

(b) Determine if $\begin{bmatrix} 1 + i \\ 200 - i \end{bmatrix}$ in Span(S)

Start with where we left off in the last part.

$$1 * \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 * \begin{bmatrix} i \\ 1 \end{bmatrix} + (-200 - 1i) * \begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} 1 + i \\ 200 - i \end{bmatrix}$$

$\begin{bmatrix} 1 + i \\ 200 - i \end{bmatrix}$ is in Span(S) because there existed values α, β, and γ such that $|w\rangle$ of S is a possible outcome of S.

(c) Describe Span(S) "geometrically"

Span(S) can be thought of as every possible vector ($\alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} i \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ i \end{bmatrix}$) contained within $\mathbb{C}^2$

1.9 Exercise 2

Find the condition under which the following two vectors are linearly independent:

$$|v_1\rangle = \begin{bmatrix} x \\ y \\ 3 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} 2 \\ x - y \\ 1 \end{bmatrix} \in \mathbb{R}^3$$

Multiply $|v_2\rangle$ by a scalar that makes $|v_1\rangle$ and $|v_2\rangle$ *not* linearly independent. This way, we can build a contradiction.

$$|v_1\rangle = \begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = 3 * |v_2\rangle = \begin{bmatrix} 2 \\ x - y \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 * (x - y) \\ 3 \end{bmatrix}$$

It can be seen that in this case, $\implies x = 6$ and $y = 4.5$.
Therefore, as long as $x \neq 6$ and $y \neq 4.5$, $|v_1\rangle$ and $|v_2\rangle$ are linearly independent.

1.10 Exercise 3

Show that the set formed by the following vectors is a basis for $\mathbb{C}^3$

$$|v_1\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, |v_3\rangle = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Recall that for the vectors to be a basis for $\mathbb{C}^3$, they must be linearly independent, and Span($|v_1\rangle, |v_2\rangle, |v_3\rangle$)= $\mathbb{C}^3$ (every vector must give additional information, IE no redundancy). To prove linear independence, we need to find where the constants (λ) are zero, by 1.4.

Goal is to find Eigen values

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix} \longrightarrow (R_2 - R_3) \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix} \longrightarrow (R_3 - R_1) \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

Note that with linear algebra experience, this stage is enough to prove that we have linear independence. At the time of writing, I haven't taken this class so I'm going to continue.

$$(\text{multiply by constant}) \longrightarrow (-1R_2 - \tfrac{1}{2}R_3) \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

To prove that the vectors provided are a basis for $\mathbb{C}^3$, we determined that the vectors are linearly independent such that α, β, and γ are zero. Then, noting that we're operating in $\mathbb{C}^3$, the vectors must be a basis. *This is a bad way of explaining this, go to office hours*

1.11 Exercise 4

Let $|x\rangle = \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix}$, $|y\rangle = \begin{bmatrix} 2-i \\ 1 \\ 2+i \end{bmatrix}$, $|z\rangle = \frac{\sqrt{2+\sqrt{5}}}{2+\sqrt{5}}|x\rangle$.

1. Find $\| |x\rangle \|$.
   $$\| |x\rangle \| = \sqrt{1^2 + i^2 + (2+i)^2} = \sqrt{1 - 1 + (2+i)^2} = 2+i$$

2. Find $\langle x | y \rangle$

   $$\langle x | y \rangle = \langle x|^\dagger \cdot |y\rangle = \begin{bmatrix} 1 & -i & 2-i \end{bmatrix} \begin{bmatrix} 2-i \\ 1 \\ 2+i \end{bmatrix} = 2 - i - i + 5 = 7 - 2i$$

3. find $\| |z\rangle \|$

   $$\| |z\rangle \| = \frac{\sqrt{2+\sqrt{5}}}{2+\sqrt{5}} \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix} = (2+i) \cdot \frac{\sqrt{2+\sqrt{5}}}{2+\sqrt{5}}$$

1.12 Exercise 5

Prove that for all $|u\rangle, |v\rangle \in \mathbb{C}^n$, $\langle u | v \rangle = \langle v | u \rangle^*$
Proof

1.13 Exercise 6

Prove that the inner product on $\mathbb{C}^n$ is a sesquilinear form. That is, show that the inner product is linear in the "ket" component and conjugate linear in the "bra" component.
Proof Let $|a\rangle , ketb \in \mathbb{C}^n$. By definition of inner product, $\langle a \mid b \rangle = |a\rangle^\dagger \cdot |b\rangle$

1.14 Exercise 7

Let $\beta = \{|b_1\rangle , |b_2\rangle\}$, where $|b_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, |b_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(a) Show that β is an orthonormal basis for $\mathbb{C}^2$

   (i) Show that they're orthogonal

$$\langle w \mid v \rangle = \langle w|^\dagger \cdot |v\rangle$$
$$\frac{1}{\sqrt{2}}(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}((1 * 1) + (1 * -1)) = 0 \longleftarrow \text{Recall that if the inner product is zero, the}$$
$$\text{vectors are orthogonal}$$

   (ii) Determine if $|b_1\rangle$ and $|b_2\rangle$ are unit vectors

$$\frac{1}{\sqrt{2}}\sqrt{1^2 + 1^2} = \frac{\sqrt{2}}{\sqrt{2}} = 1 \longleftarrow \text{Unit vector}$$

Therefore, β is an orthonormal basis.

(b) Find the coordinates (or components) of $|x\rangle = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ relative to β. IE find the scalars $c_1, c_2 \in \mathbb{C}$ such that $c_1 |b_1\rangle + c_2 |b_2\rangle = |x\rangle$.

   (i) Multiply both sides by $\langle b_1|$

$$\langle b_1| (c_1 |b_1\rangle + c_2 |b_2\rangle) = \langle b_1 \mid x \rangle$$
$$c_1 \langle b_1 \mid b_1 \rangle + c_2 \langle b_1 \mid b_2 \rangle = c_1 * 1 + c_2 * 0 = c_1$$
Note that inner product of itself is parallel, and since β is an orthonormal basis,
$$\langle b_1 \mid b_2 \rangle = 1 \text{ (IE they're orthogonal).}$$
$$c_1 = \langle b_1 \mid x \rangle$$
$$c_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \longrightarrow = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$
$$\frac{-2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

   (ii) Multiply both sides by $\langle b_2|$

$$\langle b_2| (c_1 |b_1\rangle + c_2 |b_2\rangle) = \langle b_2 \mid x \rangle$$
$$c_1 \langle b_2 \mid b_1 \rangle + c_2 \langle b_2 \mid b_2 \rangle = c_1 * 0 + c_2 * 1 = c_2$$
$$c_2 = \langle b_2 \mid x \rangle$$
$$c_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \longrightarrow = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$
$$\frac{-2}{\sqrt{2}} + \frac{-1}{\sqrt{2}} = \frac{-3}{\sqrt{2}}$$

Therefore, $c_1 = \frac{-1}{\sqrt{2}}$ and $c_2 = \frac{-3}{\sqrt{2}}$ when $|x\rangle = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

1.15 Exercise 8

Given $S = \} |v_1\rangle = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, |v_3\rangle = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \{$, turn S into an orthonormal basis for

$\text{Span}(S) = \mathbb{R}^3$.

1. Find $|f_1\rangle$ $\||v_1\rangle\| = 3$

$$\longrightarrow |f_1\rangle = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

2. Find $|f_2\rangle$

$$|f_2\rangle = |v_2\rangle - \left(\frac{\langle v_1|v_2\rangle}{\langle v_1|v_1\rangle} |v_1\rangle \right)$$

$$\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} - \left[\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right] = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} - |0\rangle$$

$$|f_2\rangle = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

3. Find $|f_3\rangle$ $|f_3\rangle = |v_3\rangle - \left[\left(\frac{\langle v_1|v_3\rangle}{\langle v_1|v_1\rangle} |v_1\rangle \right) + \left(\frac{\langle v_2|v_3\rangle}{\langle v_2|v_2\rangle} |v_2\rangle \right) \right]$

$$= \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} - \left[\left(\frac{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}} \right) + \left(\frac{\begin{bmatrix} 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}} \right) \right]$$

$$= \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} + \begin{bmatrix} \frac{8}{3} \\ \frac{-4}{3} \\ \frac{8}{3} \end{bmatrix} \right) = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{-1}{3} \end{bmatrix}$$

$\|f_3\| = 1$ *already normalized*

Therefore, $|f_{final}\rangle = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{-1}{3} \end{bmatrix}$

1.16 Exercise 9

Use the Gram-Schmidt Process to produce an orthonormal basis β for $M = \text{Span}\left(\{|v_1\rangle, |v_2\rangle\} \right)$

where $|v_1\rangle = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} 3 \\ 1 \\ i \end{bmatrix}$.

Step 1:

$$|f_1\rangle = |b_1\rangle = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$$

Step 2:

$$|f_2\rangle = |b_2\rangle - \frac{\langle f_1|b_2\rangle}{\langle f_1|f_1\rangle}|f_1\rangle = \begin{bmatrix} 3 \\ 1 \\ i \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-i \\ -1+i \end{bmatrix}$$

Step 3:

$\||f_1\rangle\| = 1$, and $\||f_2\rangle\| = \sqrt{4 + (1-i)^2 + (-1+i)^2} = \sqrt{4-4i} = 2\sqrt{1-i}$

Therefore, $|f_1\rangle = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$ and $|f_2\rangle = \frac{1}{2\sqrt{1-i}}\begin{bmatrix} 2 \\ 1-i \\ -1+i \end{bmatrix}$.

2 Matrices and Linear Transformations

Understanding matrices and linear transformations will be critical when it comes time to talk about quantum mechanics. Specifically, this section will discuss matrix properties and operations, as well as special theorems that will help us to work with more complex systems.

2.1 Intro to Matrices

Definition 2.1 A map $T : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is a linear transformation if:

1. $T(|v\rangle + |u\rangle) = T(|v\rangle\,|\,|u\rangle)$

2. $T(C\,|\,V) = CT(|v\rangle)\forall\,|u\rangle\,,|v\rangle \in \mathbb{C}^n$ ◆

Note that every linear transformation arises from a matrix.

Definition 2.2 $M_{mn}(\mathbb{C})$ is a set of all $m \times n$ matrices with complex entries $M_N(\mathbb{C}) := M_{nm}(\mathbb{C})$ ◆

This is traditionally written as M_n or M_{mn}. M_{mn} is a vector space with usual scalar multiplication and matrix addition.

Example 2.3 Given $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \in M_2$, $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, find $2\omega_x - i\omega_y$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$$ ◆

Note that $\sigma_x, \sigma_y, \sigma_z$ are called Pauli or 'spin' matrices.

I_n is called the identity matrix whose 0's are the standard basis for $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\} \in \mathbb{C}^n$.

IE: $I_N = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$ M_N has well-defined multiplication of matrices.

In general, if $A \in M_{mn}$, $B = m_{nk}$, $A = (a_{ij})_{j=1 j=1}^{mn}$, $B = (b_{rs})_{r=1,2=1}$. Additionally, the matrix $AB = \begin{bmatrix} A|b_1\rangle & \ldots & B|b_k\rangle \end{bmatrix} = (C_{pq})$ where C_{pq} is the dot product of the pth row of A against the qth column of B.

Here are some nice facts that will help us in further examples:

1. $AB = \begin{bmatrix} \langle a_1| B \\ \ldots \\ \langle a_m| B \end{bmatrix}$ where $\{|a_1\rangle, |a_2\rangle, \ldots, |a_n\rangle\}$ are the rows of A.

2. $AB = \sum_{i=1}^{n} |a_j\rangle \langle b_j|$ (IE "the columns of A against the rows of B")

It's useful to know some matrix operations to proceed. In general, $\begin{bmatrix} |a_1\rangle & \ldots & |a_n\rangle \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & \ldots & \\ 0 & & d_n \end{bmatrix} = \begin{bmatrix} d_1|a_1\rangle & \ldots & d_n|a_n\rangle \end{bmatrix}$. Here's an example:

Example 2.4 $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -3i \\ i & 0 \end{bmatrix}$ ◆

2.2 Eigenvalues and Eigenvectors

Definition 2.5 An **eigenvalue** for $A \in M_n$ is a complex number $\lambda \in \mathbb{C}^n$ such that there is a non zero vector $|x\rangle \in \mathbb{C}^n$ satisfying $A|x\rangle = \lambda|x\rangle$. ◆

Definition 2.6 A matrix $A \in M_n$ is diagonalizable if:

1. There is a diagonal matrix D and an invertable matrix P such that $A = PDP^{-1}$.

2. There exists a basis for $\mathbb{C}^n$ consisting of eigenvectors for A ◆

Example 2.7 Consider $A \begin{bmatrix} I_2 & 0 \\ 0 & \sigma_y \end{bmatrix}$

First, let's get the eigenvalues and normalized vectors. Note that eigenvalues are the *roots* of the characteristic equation $det(A - \lambda I) = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} - \begin{bmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & \lambda & \\ 0 & & & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 0 & 0 & 0 \\ 0 & 1 - \lambda & 0 & 0 \\ 0 & 0 & -\lambda & -i \\ 0 & 0 & i & \lambda \end{bmatrix} \longrightarrow det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 0 & 0 & 0 \\ 0 & 1 - \lambda & 0 & 0 \\ 0 & 0 & -\lambda & -i \\ 0 & 0 & i & \lambda \end{vmatrix}$$

$$= (1 - \lambda) \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & -\lambda & -i \\ 0 & i & \lambda \end{vmatrix} = (1 - \lambda) \left[1 - \lambda \begin{bmatrix} -\lambda & -i \\ i & \lambda \end{bmatrix} -0 \quad 0 \right]$$

$$= (1 - \lambda)(1 - \lambda) \left[\lambda^2 \quad -1 \right] = 0$$

$$\lambda = 1, 1, 1, -1$$

These are our eigenvalues. Now, we need to find our eigenvectors

Note: $A \, |e_1\rangle = |e_1\rangle$ and $B \, |e_2\rangle = |e_2\rangle$.

$$A - (1)I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -i \\ 0 & 0 & i & 1 \end{bmatrix} \xrightarrow{R_4 + iR_3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & i \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-1*R_1]{R_1 \leftrightarrow R_3} = \begin{bmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, our eigen vector is $|x\rangle = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$.

Our normalized eigen vectors are $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix} \right\}$ ♦

2.3   Matrix Classifications

Some, but not all matrices are diagonalizable; this property is highly desirable. Matrices are diagonalizable if and only if the eigenvalues form a basis.
put some shit here

1. Vectors must be self-adjoint (*hermition*) IE $A \in M_n$ is **hermition** if $A^\dagger = A$, $sigma + \dagger_x = \sigma_x$, $\sigma + \dagger_y = sigma_y$, $\sigma + \dagger_z = \sigma_z$

2. As it turns out, *all* hermition matricies are diagonalizable $\longleftrightarrow$ the eigenvectors for a basis (IE linearly independent and Span). Said differently, hermition matrices are only hermition matrices if their eigenvectors form an orthonormal basis (IE unit vector, norm, and Span). It's worth noting that the eigenvalues must be real for this to be the case.

3. A matrix in M_n is positive-semidefinite if for all $|x\rangle \in \mathbb{C}^n$, we have $\langle x| \, A \, |x\rangle \geq 0$

4. The following are equivalent:

   (a) $u \in M_n$ is unitary

   (b) $u^\dagger = u - 1$ (IE left *and* right inverse) $\longrightarrow u^{-1}u = uu^{-1}$

(c) $uu^\dagger = I_n$ and $u^\dagger u = I_n$ (unitaries are normal, but not necessarily hermition)

(d) The columns of u form an orthonormal basis for $\mathbb{C}^n$

(e) $\forall \left|x\right\rangle, \left|y\right\rangle \in \mathbb{C}^n$, $\langle u_x \mid u_y \rangle = \langle x \mid y \rangle$. Think of this as a rotation of the entire matrix, such that the angle between the vectors are preserved. Related to calculus III, this operation gives us the angle.

Now, let's go over some examples of these properties in action.

Example 2.8 Example of a matrix that is *not* Hermition:

$$N = \begin{bmatrix} 2 & -i \\ -i & 2 \end{bmatrix} \longrightarrow N^\dagger = \begin{bmatrix} 2 & i \\ i & 2 \end{bmatrix}$$

Note: the columns of N are not orthogonal, so N is not unitary. ◆

Example 2.9 Example of a unitary matrix that's not Hermition:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \longrightarrow u^{-1} = u^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

◆

Example 2.10 $\forall \left|x\right\rangle \in C^n$, $\langle x| A \left|x\right\rangle \geq 0$: $A = \left|e_2\right\rangle \langle e_2|$ where $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ This is positive semidefinite if:

proof Let $\left|x\right\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{C}^2$.

$\langle x| A \left|x\right\rangle = \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = 0 + b^* b = |b|^2 \geq 0$ ◆

2.4 Spectral Theorem for Normal Matrices

To perform decomposition of matrices, we can use the *spectral theorem for normal matrices*, and the fact that in general, $A \in M_n$ has a singular value decomposition. Note that both of these use eigenvectors of the given matrix. Note that $AB = BA$ doesn't always work for $A, B \in M_n$.

Theorem 2.11 *A matrix $N \in M_n$ is normal $\longleftrightarrow$ there is a unitary matrix $u = \begin{bmatrix} \left|u_1\right\rangle & \cdots & \left|u_n\right\rangle \end{bmatrix}$ and a diagonalizable matrix $D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda_n \end{bmatrix}$ such that $N = uDu^\dagger$*

Let's prove the forward case for this theorem. If you take $\left|u\right\rangle, ..., \left|u_n\right\rangle$ to be eigenvectors for N with respect to eigenvalues $\lambda_1, ..., \lambda_n$, then $u := [\left|u_1\right\rangle ... \left|u_n\right\rangle]$ is unitary and $D = \begin{bmatrix} \lambda_1 & & \\ & \cdots & \\ & & \lambda_n \end{bmatrix}$. Therefore, $N = uDu^{-1} = \sum_{j=1}^n \lambda_j \left|u_j\right\rangle \langle u_j|$. Note that this projection is P_j where $P_j \left|x\right\rangle$ is the projection of $\left|x\right\rangle$ onto $\left|u_j\right\rangle$. Now, let $\left|x\right\rangle = c_1 \left|u_1\right\rangle + ... + c_n \left|u_n\right\rangle$ for some $c_j \in \mathbb{C}^n$. Therefore,

$$N \ket{x} = \left(\sum_{j=1}^{n} \lambda_j \ket{u_j} \bra{u_j} \right) \left(\sum_{i=1}^{n} c_i \ket{u_i} \right)$$
$$= \sum_{j=1}^{n} \sum_{i=1}^{n} c_i \lambda_j \ket{u_j} \braket{u_j | u_i}$$
$$= \sum_{j=1}^{n} c_j \lambda_j \ket{u_j} = D \ket{x}$$
(fancy way of saying $Nu = uD$)

Let's do an example to illustrate this theorem.

Example 2.12 Consider $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Recall that a matrix $A \in M_n$ is

1. Normal if $A^\dagger A = AA^\dagger$, and

2. Hermitian if $A^\dagger = A$

Therefore, σ_x is Hermition because $\sigma_x^\dagger = \sigma_x$, so, $\sigma_x \sigma_x = \sigma_x \sigma_x^\dagger$ is normal. Thus, σ_x has a spectral decomposition. With this in mind, we're looking for $\sigma_x = uDu^\dagger$. First, let's find some eigenvalues.

$$\det(\sigma_x - \lambda I) = 0$$
$$\left| \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right| = 0$$
$$\lambda^2 - 1 = 0$$
$$\lambda \pm 1$$

Now, use the eigenvalues to find eigenvectors.

$$\text{First, for } \lambda = 1, \ \sigma_x \begin{bmatrix} a \\ b \end{bmatrix} = 1 \begin{bmatrix} a \\ b \end{bmatrix}.$$
$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}.$$
$$\text{Therefore, } a = b \text{ and } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \ket{u_1}$$
$$\text{Second, for } \lambda = -1, \ \sigma_x \begin{bmatrix} a \\ b \end{bmatrix} = -1 * \begin{bmatrix} a \\ b \end{bmatrix}$$
$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$
$$\text{Therefore, } a = -b \text{ and } \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \ket{u_2}$$

We now have $u = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Additionally, by theorem 2.11, $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Note that

$u^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Therefore,

$$\sigma_x = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$= 1 * \ket{u_1} \bra{u_1} + (-1) * \ket{u_2} \bra{u_2}$$
$$= 1 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + (-1) \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$\blacklozenge$

Let's take a break from this example to talk about an application of functional calculus for spectral projections.

Theorem 2.13 *Suppose $N \in M_n$. Write $N = uDu^\dagger = \sum_{i=1}^{n} \lambda_i |u_i\rangle \langle u_i|$ ($|u_i\rangle \langle u_i|$ are spectral projections) is an analytical function on $\mathbb{C}$ (IE a function f has a maclarin series $f(z) = \sum_{k=0}^{\inf} c_n z^k$). Then, $f(N) = \sum_{i=1}^{n} f(\lambda_i) |u_i\rangle \langle u_i|$.*

This exploits a useful property of projections, where the power of any projection is the same projection. Now, let's expand on example 2.12.

Example 2.14 Consider $f(z) = e^{i\alpha z}$, which is analytic on $\mathbb{C}$. Let's compute $f(\sigma_x) = \exp(i\alpha\sigma_x)$.

$$\exp(i\alpha\sigma_x) = e^{i\alpha(1)} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + e^{-i\alpha} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Recall that Eulier's Identity says $e^{i\alpha} = \cos\alpha + i\sin\alpha$ and $e^{-i\alpha} = \cos\alpha - i\sin\alpha$

$$\exp(i\alpha\sigma_x) = (\cos\alpha + i\sin\alpha)P_1 + (\cos\alpha - i\sin\alpha)P_2$$
$$= \cos\alpha(P_1 + P_2) + (i\sin\alpha)(P_1 - P_2). \text{ Note that } P_1 + P_2 = I.$$
$$= \cos(\alpha)I + (i\sin\alpha)(P_1 - P_2)$$
$$\text{Now, we can see that } \exp(i\alpha\sigma_x) = \begin{bmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{bmatrix}$$

$\blacklozenge$

2.5 Exercise 2.1

Let $A, B, C \in M_n$, and $c \in \mathbb{C}$. Show the following relationships:

(a) $(cA)^\dagger = c^* A^\dagger$
Conjugate transpose is distributive - $(cA)^\dagger = c^{*^T} A^\dagger$. Transpose of a scalar is the same scalar. Basically, transpose is a linear map from $m \times n$ to $n \times m$. So, $c^{*^T} = c^*$ and $(cA)^\dagger = c^* A^\dagger$

(b) $(A + B)^\dagger = A^\dagger + B^\dagger$

Let $A = \begin{bmatrix} \alpha_{11} & \cdots\cdots & \alpha_{1n} \\ \vdots & \ddots\ddots, & \vdots \\ \alpha_{m1} & \cdots\cdots & \alpha_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} \beta_{11} & \cdots\cdots & \beta_{1n} \\ \vdots & \ddots\cdots, & \vdots \\ \beta_{m1} & \cdots\cdots & \beta_{mn} \end{bmatrix}$ where $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$

$$(A + B)^\dagger = \left(\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix} + \begin{bmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{m1} & \cdots & \beta_{mn} \end{bmatrix} \right)^\dagger$$

$$= \left(\begin{bmatrix} \alpha_{11} + \beta_{11} \cdots & x_{1n} + \beta_{1n} \\ & \vdots \\ \alpha_{m1} + \beta_{m1} \cdots & \alpha_{mn} + \beta_{mn} \end{bmatrix} \right)^\dagger$$

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$$
= \begin{bmatrix} \alpha_{11}+ & \beta_{11} \cdot \ldots & \alpha_{m1} & +\beta_{m1} \\ \cdot & \ddots & & \cdot \\ \vdots & \cdot & & \vdots \\ \alpha_{1n} & +\beta_{1n} \cdots & \alpha_{mn} & +\beta_{mn} \end{bmatrix} = \begin{bmatrix} \alpha_{11} + \beta_{11} & \ldots & \alpha_{m1} + \beta_{m1} \\ \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot \\ \alpha_{1n} + \beta_{1n} & \cdots & \alpha_{mn} + \beta_{mn} \end{bmatrix}
$$

Where $\alpha_{mn} = \gamma_{mn} - \sum_{mn} i$
and $\beta_{mn} = \mu_{mn} - P_{mn}i$

$$
A^\dagger + B^\dagger = \begin{bmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & \ddots & ,\vdots \\ \alpha_{m1} & -\ldots- & \alpha_{mn} \end{bmatrix}^\dagger + \begin{bmatrix} \beta_{,}1 & i^{\ldots\beta} & \frac{\beta}{\vdots}n \\ \vdots & \ddots & \vdots \\ \beta_{m1} & -\ldots- & \beta_{mn} \end{bmatrix}^\dagger
$$

$$
= \begin{bmatrix} \alpha_{11} & \ldots & x_{m,} \\ \vdots & \ddots\ddots & \vdots \\ \alpha_{1n} & -\ldots & \alpha_{mn} \end{bmatrix} + \begin{bmatrix} \beta_{11} & \ldots & \beta_{m1} \\ \vdots & \ddots & \vdots \\ \beta_{1n} & \ldots & \beta_{mn} \end{bmatrix}
$$

Where $\alpha_{mn} = \gamma - \sum i$ and $\beta_{mn} = \mu_{mn} - \rho_{mn}$

$$
= \begin{bmatrix} \alpha_{11} + \beta_{1,} & \ldots & \alpha_{m1} + \beta_{m1} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} + \beta_{1n} & \ldots \alpha_{mn} & +\beta_{mn} \end{bmatrix}
$$

$$
\longrightarrow \begin{bmatrix} \alpha_{11} + \beta_{11} & \cdot\ldots & \alpha_{m1} + \beta_{m1} \\ \cdot & & \cdot \\ \vdots & \ddots & \vdots \\ \alpha_{1n} + \beta_{1n} & \ldots & \alpha_{mn} + \beta_{mn} \end{bmatrix} = \begin{bmatrix} \alpha_{11} + \beta_{11} & \ldots & \alpha_{m1} + \beta_{m1} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} + \beta_{1n} & \ldots & \alpha_{mn} + \beta_{mn} \end{bmatrix}
$$

(c) $(AB)^\dagger = B^\dagger A^\dagger$

$$
A = \begin{bmatrix} \alpha_{11} & \ldots & \alpha_{m1} \\ \vdots & \ddots & \vdots \\ \alpha_{11} & \ldots & \alpha_{mn} \end{bmatrix} \quad B = \begin{bmatrix} \beta_{11} & \ldots & \beta_{m1} \\ \vdots & \ddots & \vdots \\ \beta_{11} & \ldots & \beta_{mn} \end{bmatrix}
$$

$$
(AB)^\dagger = \left(\begin{bmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \ldots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} \beta_{11} & \ldots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n_1} & \ldots & \beta_{nn} \end{bmatrix} \right)^\dagger
$$

$$
= \begin{bmatrix} (\alpha_{11}\beta_{11} + \ldots + \alpha_{1n}\beta_{n1}) & \ldots & (\alpha_{11}\beta_1 n^t \ldots + \alpha_n\beta_{nn}) \\ \vdots & \ddots & \vdots \\ (\alpha_n, \beta_{11} + \ldots + \alpha_{nn}\beta_{n1}) & \ldots & (\alpha_n, \beta_{n1} + \ldots + \alpha_{nn}\beta_{nn}) \end{bmatrix}^\dagger
$$

$$
= \begin{bmatrix} (\alpha_{11}\beta_{11} + \ldots + \alpha_{1n}\beta_{n1})\ldots & (\alpha_{n1}\beta_{11} + \ldots + \alpha_{nn}\beta_{n1}) \\ \vdots & \ddots & \vdots \\ (\alpha_{11}\beta_{11} + \ldots + \alpha_{1n}\beta_{nn}) & \cdots & (\alpha_n, \beta_{n1} + \ldots + \alpha_{nn}\beta_{nn}) \end{bmatrix}
$$

where $\alpha_{nn} = \alpha_{11} = \gamma_{nn} - \sum_{nn} i$ and $\beta_{nn} = \beta_{11} = M_{nn} - P_{nn}i$

$$
B^t A^t = \begin{bmatrix} \alpha_{11} & \ldots & \alpha_{in} \\ i & \ddots & \vdots \\ \alpha_{n1} & \ldots & \alpha_{nn} \end{bmatrix}^T \begin{bmatrix} \beta_{11} & \ldots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n_1} & \ldots & \beta_{nn} \end{bmatrix}^t == \begin{bmatrix} \alpha_{11} & \ldots & \alpha_{n1} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} & \ldots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} \beta_{11} & \ldots & \beta_{n1} \\ \vdots & \ddots & \vdots \\ \beta_{1n} & \ldots & \beta_{nn} \end{bmatrix}
$$

$$\begin{bmatrix} (\alpha_{11}\beta_{11} + \ldots + \alpha_{1n}\beta_{n1}) & \cdots & (x_n, \beta_{11} + \ldots + \alpha_{nn}\beta_{n1}) \\ \vdots & \ddots & \vdots \\ (\alpha_{11}\beta_1 n^t \ldots + A_{1n}B_{nn}) & \cdots & (\alpha_n, \beta_{n1} + \ldots + \alpha_{nn}\beta_{nn}) \end{bmatrix}$$

Where $\alpha_{nn} = \alpha_{11} = \gamma_{nn} \cdot \sum_{nn} i \,\& \, \beta_{nn} = \beta_{11} = \mu_{nn} - \rho_{nn}i$

$$\begin{bmatrix} (\alpha_{11}\beta_{11} + \ldots + \alpha_{1n}\beta_{n1}) & \cdots & (x_n, \beta_{11} + \ldots + \alpha_{nn}\beta_{n1}) \\ \vdots & \ddots & \vdots \\ (\alpha_{11}\beta_1 n^t \ldots + \alpha_{1n}\beta_{nn}) & \cdots & (\alpha_n, \beta_{n1} + \ldots + \alpha_{nn}\beta_{nn}) \end{bmatrix} =$$
$$\begin{bmatrix} (\alpha_{11}\beta_{11} + \ldots + \alpha_{1n}\beta_{n1}) & \cdots & (x_n, \beta_{11} + \ldots + \alpha_{nn}\beta_{n1}) \\ \vdots & \ddots & \vdots \\ (\alpha_{11}\beta_{1n}t_{\ldots} + A_n B_{nn}) & \cdots & (\alpha_n, \beta_{n1} + \ldots + \alpha_{nn}\beta_{nn}) \end{bmatrix}$$

2.6 Exercise 2.2

Let $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1+i \\ i-1 & 0 \end{bmatrix}$

(a) Find the eigenvalues and normalized eigenvectors for A

$$A - \lambda I = \begin{bmatrix} 0 & \dfrac{1+i}{\sqrt{2}} \\ \dfrac{1-i}{\sqrt{2}} & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & \dfrac{1+i}{\sqrt{2}} \\ \dfrac{1-i}{\sqrt{2}} & -\lambda \end{bmatrix}$$

$$\det |A - \lambda I| = \left| \begin{bmatrix} -\lambda & \dfrac{1+i}{\sqrt{2}} \\ \dfrac{1-i}{\sqrt{2}} & -\lambda \end{bmatrix} \right| = 0$$

$$\left(\lambda^2 - \frac{1}{2}(1+i)(1-i) \right) = 0$$

$$\lambda^2 - \frac{1}{2}(2) = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$$\lambda = 1 : A - 1I = \begin{bmatrix} -1 & \dfrac{1+i}{\sqrt{2}} \\ \dfrac{1-i}{\sqrt{2}} & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \longrightarrow |v_1\rangle = \begin{bmatrix} \dfrac{-1-i}{\sqrt{2}} \\ 1 \end{bmatrix}$$

$$\lambda = -1 : A + 1I = \begin{bmatrix} 1 & \dfrac{1+i}{\sqrt{2}} \\ \dfrac{1-i}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \longrightarrow |v_2\rangle = \begin{bmatrix} \dfrac{1+i}{\sqrt{2}} \\ 1 \end{bmatrix}$$

(b) Show that the eigenvectors are mutually orthogonal
To determine if the eigenvectors are mutually orthogonal, we can verify that the inner product is zero.

$$\langle v_1 \mid v_2 \rangle = \begin{bmatrix} \dfrac{-1+i}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1+i}{\sqrt{2}} \\ 1 \end{bmatrix} = \frac{-1+i}{\sqrt{2}} \cdot \frac{1+i}{\sqrt{2}} + 1 = 0 \longleftarrow \text{Therefore, the eigenvectors}$$

are mutually orthogonal.

(c) Show the projection operators associated to the mutually orthogonal eigenvectors satisfy the *completeness relation*; IE if $|u_1\rangle$, $|u_2\rangle$ are mutually orthogonal unit eigenvectors for A, then $I = \sum_{i=1}^{2} |u_i\rangle \langle u_i|$.

(d) Find a unitary matrix which diagonalizes A, IE find a unitary matrix U and a diagonal matrix D such that $A = UDU^\dagger$.

2.7 Exercise 2.3

2.8 Exercise 2.4

2.9 Exercise 2.5

2.10 Exercise 2.6

3 Framework of Quantum Mechanics

The axioms of quantum mechanics are choices made by humans to attempt to mathematically describe the behavior of particles. To dumb this down for my monkey brain, I'm going to describe these axioms with a story. To begin, let's suppose that we prepared a photon for experimentation in a laboratory. Realistically, despite 'preparing' this photon, IE understanding some of its initial conditions, we can never explicitly model it. Recall that $A \in M_n$ is:

1. normal if $A^\dagger A = AA^\dagger$

2. Hermitian if $A^\dagger = A$

3. unitary if $A^\dagger A = AA^\dagger = I$

3.1 Quantum Mechanics Axiom 1.1

A vector state $|x\rangle$ (IE $|x\rangle$ encodes all physical info about a photon) is a unit vector in a complex Hilbert state $(H = \mathbb{C}^n)$. In our story, we first want to figure out the photon's polarization. To figure this out, we're going to 'shoot' it at a vertically polarized filter. In classical mechanics, our particle should be polarized either vertically or horizontally. IE there are two possible outcomes:

1. If $|x\rangle$ goes through the vertically polarized filter, it was oriented vertically

2. If $|x\rangle$ is deflected by the vertically polarized filter, it was oriented horizontally

After preparing our $|x\rangle$ 100 times and repeating this same process, we're *appalled* to find that sometimes the particle was vertically oriented, but sometimes it was horizontally oriented. IE

1. 64 times it was oriented vertically ($|\alpha|^2 = 0.64$)

2. 36 times it was oriented horizontally ($|\alpha|^2 = 0.36$)

3.2   Quantum Mechanics Axiom 1.2

Linear combinations (aka superposition) of the physical states are allowed to act as $|x\rangle$. Let's consider $|v\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $|h\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We'll call this the polarization basis for $\mathbb{C}^2$. Therefore, $|x\rangle = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where α and β are probabilities and $|\alpha|^2 + |\beta|^2 = 1$. Since we've assigned representations to our states, we want some easy way to 'keep track' of the choices. Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, which behaves/represents the vertically polarized filter. This choice makes sense because $|v\rangle$ and $|h\rangle$ are eigenvectors for A, with eigenvalues 1 and 0.

Example 3.1 Suppose $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. This might be a nice matrix to represent a horizontally polarized filter, because it has $|v\rangle$ and $|h\rangle$ as eigenvectors with eigenvalues 0 and 1. ♦

3.3   Quantum Mechanics Axiom 2

An observable of a state $|x\rangle$ corresponds to a Hermitian matrix A. IE $|x\rangle$ is in state $|u\rangle$; an eigenvector for A with probability $|\langle u \mid x \rangle|^2$. Note that we don't know the state of the particle *until* it hits the filter, at which point it's one state or the other. Remember that this is basically a Walmart version of this explanation, so this is more complex. When the particle is in the 'air,' it's in the state of *superposition*. Let's check this axiom:

$$\begin{aligned} |\langle v \mid u \rangle|^2 &= |\langle v|(\alpha|v\rangle \times \beta|h\rangle)|^2 \\ &= |\langle v \mid v \rangle + \beta\langle v \mid v \rangle|^2 \\ \longrightarrow |x\rangle &= \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \end{aligned}$$

3.4   Quantum Mechanics Axiom 3

The time dependence of a state is governed by the Schrödinger equation:

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H |\psi\rangle \tag{3.3}$$

where $\hbar$ is the reduced Planck's constant and H is a Hermitian matrix corresponding to the energy of the system, called "Hamiltonian." You'll note that this is just a differential equation at its core. The expectation value μ of the observable associated with A after

measurements with respect to many copies of $|\psi\rangle$ is the weighted average of the expected outcomes.

$$\langle A \rangle_\psi = \sum_{i=1}^{n} \lambda_i |\langle u_i \mid \psi \rangle|^2 \tag{3.4}$$

Note that λ_i's are the "locations" of the probabilistic outcomes.

$$|\psi\rangle = \sum_{i=1}^{n} c_i |u_1\rangle \tag{3.5}$$

When H is time-independent, the solution to the Schrödinger equation is

$$|\psi(t)\rangle = \exp(-it\frac{H}{\hbar})|\psi(0)\rangle \tag{3.6}$$

This is effectively saying $\frac{dy}{dt} = ky \longrightarrow y = e^{kt}$. IE when we take the derivative, we get an exponent.

Example 3.2 Consider a physical system with Hamiltonian $H = \frac{-\hbar}{2}\omega\sigma_x$ and suppose $|\psi(0)\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Recall that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the first column of σ_z, and the spin points in the z-direction at $t = 0$.

(a) Find the wave function $|\psi(t)\rangle$ when $t > 0$.
By the Schrödinger equation, $|\psi(t)\rangle = e^{\frac{-itH}{\hbar}} * |\psi(0)\rangle$.

$$|\psi(t)\rangle = \exp\left(i\left(\frac{-t}{\hbar}\right)\left(\frac{-\hbar}{2}\omega\sigma_x\right)\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \exp\left(i\left(\frac{-t}{2}\omega\right)\sigma_x\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ where } \alpha = \frac{-t}{2}\omega$$

$$= \left(\cos\left(\frac{t}{2}\omega\right)I + i\sin\left(\frac{t}{2}\omega\right)\sigma_x\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\left(\frac{t}{2}\omega\right) & i\sin\frac{t}{2} \\ i\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\omega\right) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\longrightarrow |\psi(t)\rangle = \begin{bmatrix} \cos\left(\frac{t\omega}{2}\right) \\ i\sin\left(\frac{t\omega}{2}\right) \end{bmatrix} \text{ This is our wave function; we can now use it to find proba-}$$
bilities, etc.

(b) Find the probability for the system to have outcome $+1$ upon measurement of ω_z. Key idea: The coefficients of $|\psi(t)\rangle$ in the orthonormal basis of eigenvectors associated to σ_z give the probabilities. *Use the projections!*
Recall that $N = uDu^\dagger = \sum_{j=1}^{n} \lambda_i |u_j\rangle\langle u_j|$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So, $P_1|\psi(t)\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\left(\frac{t\omega}{2}\right) \\ i\sin\left(\frac{t\omega}{2}\right) \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{t\omega}{2}\right) \\ 0 \end{bmatrix} \cos\left(\frac{t\omega}{2}\right)|u_1\rangle$. Now, recall that the probability of observing λ_i state is $|c_i|^2$. IE $|\psi(t)\rangle = \sum_{j=1}^{n} c_i |u_i\rangle$. Also note that $|u_i\rangle$ is an orthonormal basis of eigenvectors for an observable.
Therefore, the probability of seeing $+1$ upon measurement σ_z is $P(t) = |\cos\left(\frac{t\omega}{2}\right)|^2 = \cos^2\left(\frac{\omega t}{2}\right)$

(c) Find the probability of -1 upon measurement of σ_z

$$P_2\left|\psi(t)\right\rangle = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\left(\frac{t\omega}{2}\right) \\ i\sin\left(\frac{t\omega}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ i\sin\left(\frac{t\omega}{2}\right) \end{bmatrix} = i\sin\left(\frac{t\omega}{2}\right)$$

The probability of seeing -1 upon measurement σ_z is $P(t) = |i\sin\left(\frac{t\omega}{2}\right)|^2 = i\sin^2\left(\frac{\omega t}{2}\right)$

or $\left|\psi(t)\right\rangle = \langle u_1 \mid \psi(t)\rangle \left|u_1\right\rangle + \langle u_2 \mid \psi(t)\rangle \left|u_2\right\rangle$, so, $c_2 = \langle u_2 \mid \psi(t)\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\left(\frac{t\omega}{2}\right) \\ i\sin\left(\frac{t\omega}{2}\right) \end{bmatrix}$

(d) Find the expectation value under many measurements of σ_z.
Recall that $\langle A\rangle_\psi = \langle\psi| A |\psi\rangle$

$\langle\sigma_z\rangle_\psi = \langle\psi| \sigma_z |\psi\rangle = (+1)\left(\cos^2\left(\frac{\omega t}{2}\right)\right) + (-1)\sin^2\left(\frac{\omega t}{2}\right) = \left(\cos^2\left(\frac{\omega t}{2}\right)\right) - \sin^2\left(\frac{\omega t}{2}\right)$ ◆

3.5   Exercise 3.1

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(a) Explain why $\mathbb{C}^2$ must have an orthonormal basis of eigenvectors for A
Note that since $A^\dagger A = AA^\dagger$ (A is normal), AND $A^\dagger = A$ (A is Hermitian), we have an orthonormal basis. So, we can use the spectral theorem.

(b) Find eigenvalues for A, and its corresponding normalized eigenvectors.

$$\det\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \left|\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right|$$
$$= 4 - 2\lambda - 2\lambda + \lambda^2 - 1 = \lambda^2 - 4\lambda + 3$$
$$= \lambda^2 - 3\lambda - \lambda$$
$$\lambda(\lambda - 3) - \lambda - 3 = (\lambda - 3)(\lambda - 1) \longrightarrow \lambda = 3, 1$$

Now, let's find our eigenvectors. Start with $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$
Therefore, $\lambda = 3 : \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Next, find the eigenvector for $\lambda = 1$

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

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Therefore, $\lambda = 1 : \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(c) Find the spectral decomposition of A

$$|u_1\rangle \langle u_1| = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$|u_2\rangle \langle u_2| = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

So... $A = 3 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

3.6 Exercise 3.2

3.7 Exercise 3.3

3.8 Exercise 3.4

3.9 Exercise 3.5

4 Matrices and Linear Transformations

4.1 Tensor Product

Consider a vector space with inner product $H = H_1 \otimes H_2$ (where $\otimes$ is the tensor product) combining H_1 and H_2 in such a way that the elements of H_1 affect H_2, and vice versa. A general vector in H is a linear combination of vectors $\{|v_1\rangle \otimes |v_2\rangle \,|\, |v_1\rangle, |v_2\rangle \in H_2\}$ taking the tensor product of $A \in M_{mn}$ and $B \in M_{pq}$:

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & ... & A_{1n}B \\ A_{21}B & A_{22}B & ... & A_{2n}B \\ ... & ... & ... & ... \\ A_{m1}B & A_{m2}B & ... & A_{mn}B \end{bmatrix} \in M_{(mp)(nq)}$$

Example 4.1 $\sigma_x \otimes i\sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

$\sigma_y \otimes \sigma_x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

$\sigma_x \otimes \sigma_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ ♦

4.2 Nice facts of the Tensor Product

Here are some nice facts of the tensor product:

1. $(\lambda A) \otimes B = B \otimes (\lambda B)$ IE scaling A affects B and vice versa

2. In general, $A \otimes B \neq B \otimes A$ IE generally noncommutative

3. $(A \otimes B)(C \otimes D) = AC \otimes BD$ IE distributive

4. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$ IE distributive

5. $(A \otimes B) + (C \otimes D) \neq (A + C) \otimes (B + D)$

6. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ IE dagger can be distributed

7. If A and B are invertible, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

Example 4.2 $|1\rangle \otimes |1\rangle = |11\rangle$ (some funky notation) IE $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$|12\rangle = |1\rangle \otimes |2\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|21\rangle = |2\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|22\rangle = |2\rangle \otimes |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that $\{|11\rangle, |12\rangle, |21\rangle, |22\rangle\}$ forms a basis for $\mathbb{C}^4$, but these vectors live in $\mathbb{C}^2 \otimes \mathbb{C}^2$. IE $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ *(same shape)*. If $|v\rangle \in \mathbb{C}^m$ and $|w\rangle \in \mathbb{C}^n$, then $|vw\rangle = |v\rangle \otimes |w\rangle \in \mathbb{C}^m \oplus \mathbb{C}^n \cong \mathbb{C}^{mn}$.

Note that $\dim(\mathbb{C}^m \otimes \mathbb{C}^n) = \dim(\mathbb{C}^m) + \dim(\mathbb{C}^n) = m + n$ and $\dim(\mathbb{C}^m \otimes \mathbb{C}^n) = \dim(\mathbb{C}^m) \cdot \dim(\mathbb{C}^n) = m \cdot n$

Theorem 4.3 *If H_1 has orthonormal basis $\epsilon_1 = \{|e_{1,1}\rangle, |e_{1,2}\rangle, ..., |e_{1,m}\rangle\}$ and $\epsilon_2 = \{|e_{2,1}\rangle, |e_{2,2}\rangle, ..., |e_{2,n}\rangle\}$, then, $H = H_1 \otimes H_2$ has orthormal basis $\{|e_{1,i}\rangle \otimes |e_{2,j}\rangle \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$ written as $i \in [m], j \in [n]$*

4.3 Inner Product on H

Let $v_1, w_1 \in H_1$ and $v_2, w_2 \in H_2$. Then, $\langle |v_1\rangle \otimes |v_2\rangle \,|\, |w_1\rangle \otimes |w_2\rangle \rangle = \langle v_1 v_2 \,|\, w_1 w_2 \rangle = \langle v_1 \,|\, w_1 \rangle_{H_1} \cdot \langle v_2 \,|\, w_2 \rangle_{H_2}$. We need to check that ϵ is an orthonormal set. Let $\langle e_{1,i} e_{2,j} \,|\, e_{1,k} e_{2,j} \rangle = \langle e_{1,i} \,|\, e_{1,k} \rangle_{H_1} \cdot \langle e_{2,j} \,|\, e_{2,j} \rangle_{H_2}$.

4.4 Multipartite Physical Systems

Quantum systems comprised of multiple systems are called multipartite systems. This is to say $H = H_1 \otimes H_2 \otimes ... \otimes H_n$ is a multipartite system when $n = 2$, and $H = H_1 \otimes H_2$ is called bipartite. Note that a $|\psi\rangle$ state in $H = H_1 \otimes H_2$ is bipartite.

Definition 4.4 A vector $|\psi\rangle \in H = H_1 \otimes H_2$ is separable (IE elementary tensor) if $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ for some $|\psi_1\rangle \in H_1, |\psi_2\rangle \in H_2$. ◆

Recall that $H_1 \otimes H_2$ is a u.s. of linear combinations of vectors in the form $|\psi_1\rangle \otimes |\psi_2\rangle$. If $|\psi\rangle \in H_1 \otimes H_2$, $|\psi\rangle$ can be written as $\sum_{ij} c_{ij} |e_{1,i}\rangle \otimes |e_{2,j}\rangle$ where $|e_{1,i}\rangle$ is the basis vector for H_1 and $|e_{2,j}\rangle$ is a basis vector for H_2.

Example 4.5 Consider $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle \right)$
$\frac{1}{\sqrt{2}} \left(|11\rangle + |22\rangle \right) \in \mathbb{C}^2 \otimes \mathbb{C}^2$

$$= \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$
◆

How can we tell if a state $|\psi\rangle \in H_1 \otimes H_2$ is separable?

Definition 4.6 A vector that is **not separable** is called **entangled**. IE, there is a state that can't be expressed as a tensor of two vectors, and the state can't be analyzed because the system is in superposition. ◆

IE, if some $|\psi\rangle$ is separable, we should be able to write it as $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ where: $|\psi_1\rangle = c_1 |1\rangle + c_2 |2\rangle \in \mathbb{C}^2$ and $|\psi_2\rangle = d_1 |1\rangle + d_2 |2\rangle \in \mathbb{C}^2$. Note that the coefficient matrix for $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ is $C = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$. Note that to put this in terms of classical encryption, the *states* H_1 or H_2 are similar to single (large) primes p, q, and *separable states* is similar to RSA where $N = pq$. Additionally, *entangled* is similar to multiple of these systems together, IE $N_1 + N_2 + ... + N_k$.

Remark 4.7 In $\mathbb{C}^2$, a state is separable if and only if the coefficient matrix is rank 1. ◆

This is to say that a state $|\psi\rangle = c_1 1 |11\rangle + c_1 2 |11\rangle + c_2 1 |21\rangle + c_2 2 |22\rangle$ is separable if and only if the rows of $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ are scalar multiples of $|psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} c \\ d \end{bmatrix} \\ b \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix}$.

Definition 4.8 Let $A \in M_n$. The singular values for A are the square roots of the eigenvalues for $A^\dagger A \in M_n$ ◆

Let's do an example of the single value decomposition.

Example 4.9 Find the singular value decomposition (SVD) for $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ i & i \end{bmatrix}$.

Note: $A^\dagger A = \begin{bmatrix} 1 & 0 & -i \\ 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ i & i \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

1. First, let's compute eigenvalues and eigenvectors for $A^\dagger A$

   $\lambda = 0 \longrightarrow |x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

   $\lambda = 4 \longrightarrow |x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   Note: $A^\dagger A$ is positive semidefinite, so it must be normal. Therefore, it has a spectral decomposition.

   $$A^\dagger A = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^\dagger}_{V^\dagger}$$

2. Now, make sure our $V's$ are in decreasing order with respect to our eigenvalues. We'll also find our value Σ for the SVD.

   $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

   $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. Next let's make some more eigenvectors using $A |\lambda_1\rangle + A |\lambda_2\rangle$

   $A |\lambda_1\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ i & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$

   $A |\lambda_2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ i & i \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

   Use $|u_1\rangle = \sqrt{2} \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$ and $|u_2\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to create an orthonormal basis for $\mathbb{C}^3$. IE, find $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$. Our $|u_2\rangle$ doesn't do much for us in this case, so we'll try to separate $|u_1\rangle$ into 3 linearly independent vectors. Let's use $\{|u_1\rangle, |2\rangle, |3\rangle\}$. These are linearly independent both to each other and to $|u_1\rangle$. Now, we need to create an orthonormal basis using the Gram-Schmidt process.

   (a) Normalize $|u_1\rangle \longrightarrow |u_1\rangle = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{i}{\sqrt{2}} \end{bmatrix}$

(b) $|u_2\rangle = |2\rangle - \langle u_2 \mid 2\rangle \, |u_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (already orthogonal)

(c) $|u_3\rangle = |3\rangle - \langle u_1 \mid 3\rangle \, |u_1\rangle - \underbrace{\langle u_2 \mid 3\rangle \, |u_1\rangle}_{0} = |3\rangle - \left(\frac{-i}{\sqrt{2}} \cdot 1 \right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{i}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{i}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} =$

$\frac{1}{2} \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \longrightarrow |u_3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$

Ok now we have some wacky new vectors which form an orthonormal basis for $\mathbb{C}^3$.

4. Form $u = \begin{bmatrix} |u_1\rangle & |u_2\rangle & |u_3\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ By the way, u is unitary because its columns are orthonormal.

5. Finally, use $A = u\Sigma V^\dagger$ to find the SVD. $\blacklozenge$

OK so basically we found the factorization of the matrix A into the product of u, Σ, and $V^\dagger$. Now, we're going to use another tool to accomplish a similar task, but instead use the result to express a vector as the tensor product of two inner product spaces.

Theorem 4.10 *Let $H = H_1 \cdot H_2$ where $\dim H_1 = m < \inf$ and $\dim H_2 = n < \inf$. Then, every vector $|\psi\rangle \in H$ admits a Schmidt decomposition*

$$|\psi\rangle = \sum_{j=1}^{r} S_j \, |u_j\rangle \cdot |v_j\rangle$$

where $S_j > 0$ are the Schmidt coefficients satisfying $\sum_{j=1}^{r} S_j^2 = 1$, $\{|u_j\rangle\} \subseteq H_1$ and $\{|v_j\rangle\} \subseteq H_2$ are orthonormal and $r < \min\{m, n\}$

4.5 Exercise 4.1

4.6 Exercise 4.2

4.7 Exercise 4.3

4.8 Exercise 4.4