MA 399 Intro to Quantum Information Theory

Hayden Roszell

January 31, 2022

Abstract

No idea what is happening in this class lol do this later

Contents

1 Intro to Linear Algebra

Let's jump right in. This section is an abbreviation of the introduction to linear algebra session.

A vector space is a group of objects (vectors) which may be added together and multiplied by compatible scalars from $\mathbb{R}$ or $\mathbb{C}$. In this class, we primarily care about vector spaces $\mathbb{C}^n$ from $\mathbb{C}$ and $\mathbb{R}^n$ from $\mathbb{R}$. Recall that $\mathbb{C}$ is the scalar field of complex numbers $a + bi$, where multiplication is defined by the rule $i^2 = -1$, and is equipped with:

(a) a complex conjugation operation – $\overline{a + bi} = (a + bi)^* = a - bi$ and

(b) a size function called the **modulus** – $|a+bi| = \sqrt{a^2+b^2}$.

Note that the modulus is similar to magnitude.
To work with complex numbers, it's useful to have an understanding of the basic operations.

(a) To add/subtract complex numbers, add/subtract the corresponding real/imaginary parts. For example – $(a+bi)+(c+di) = (a+c)+(b+d)i$

(b) To multiply/divide complex numbers, multiply both parts of the complex number by the real number. For example – $(a+bi)*(c+di) = ac+adi+bcj-bd = (ac-bd)+(ad+bc)i$. This form will be useful for the duration of the class.

1.1 Representing Vectors in Complex Spaces

As mentioned, this class primarily works in complex number spaces. For this reason, having useful tools for representing vectors in these abstract spaces is useful. Vectors are represented in **bra-ket** notation, where *bra* represents a *row* vector, and *ket* represents a *column* vector.

Definition 1.1 If $|v\rangle \in \mathbb{C}^n$ is a *ket* vector which consists of n complex numbers,

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} \text{ for } v_1, v_2, \dots, v_n \in \mathbb{C} \tag{1.1}$$

♦

Definition 1.2 If $\langle v| \in \mathbb{C}^n$ is a *bra* vector which consists of n complex numbers,

$$\langle v| = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \text{ for } v_1, v_2, \dots, v_n \in \mathbb{C} \tag{1.2}$$

♦

Note that the the integer n in definitions 1.1 and 1.2 is called the **dimension** of the vector space $\mathbb{C}^n$.

Example 1.3 $\mathbb{C}^2$ is a 2-dimensional vector space over $\mathbb{C}$.

$$\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \text{ Note: } \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \text{ 'fills up' } \mathbb{C}^2\text{; IE it represents all values contained within } \mathbb{C}^2$$

♦

1.2 Linear Combinations

A linear combination is a useful tool for this class which is the 'combination' or multiplication of each term in a set by constants, and then adding the result. The interesting property of linear combinations is that the result is still contained within the initial set.

Definition 1.4 A linear combination of $\{|v_1\rangle, ..., |v_n\rangle\} \subset \mathbb{C}^n$ is a single vector in the form $\lambda_1 |v_1\rangle + \lambda_2 |v_2\rangle + ... + \lambda_n |v_n\rangle$ for some $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{C}^n$. ♦

Remark 1.5 If $|w\rangle$ is a linear combination of $\{|v_1\rangle, ..., |v_n\rangle\} \subset \mathbb{C}^n$, we can say that it belongs to the **span** of the set of $\{|v_1\rangle, ..., |v_n\rangle\} \subset \mathbb{C}^n$. ♦

Example 1.6 Create a linear combination of $|v_1\rangle = \begin{bmatrix} i \\ 2 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} -1 \\ i+1 \end{bmatrix}$

$$\begin{aligned} &(foil) \\ &(3+2i)\begin{bmatrix} i \\ 2 \end{bmatrix} + (2+i)\begin{bmatrix} -1 \\ i+1 \end{bmatrix} \\ &(add) \\ &= \begin{bmatrix} 3i-2 \\ 6+4i \end{bmatrix} + \begin{bmatrix} -2-i \\ 3i+1 \end{bmatrix} \\ &= \begin{bmatrix} 2i-4 \\ 7i+7 \end{bmatrix} = |w\rangle \ spans\ \{|v_1\rangle, |v_2\rangle\} \end{aligned}$$ ♦

1.3 Linearly Independent

Another important concept is that of a set being *linearly independent*. Being linearly independent essentially means that every piece of 'information' given by a set of vectors adds some sort of new information/perspective on the problem.

Remark 1.7 Two vectors are **linearly independent** as long as they are not *parallel*. ♦

A good way of looking at this is that a system that is not linearly independent has more than one element that is some offset of the same constant. These elements are not giving new perspective, because they give the same information.

Definition 1.8 A set of vectors $\{|v_1\rangle, \ldots, |v_v\rangle\}$ is linearly independent if no vector is a linear combination of any other vectors. Algebraically, $[if \lambda_1 |v_1\rangle + \lambda_2 |v_2\rangle + ... + \lambda_n |v_k\rangle = |0\rangle, then \lambda_1 = \lambda_2 = \lambda_k = 0]$ ♦

The only way that the zero vector ($|0\rangle$) can be possible is if the complex constants (λ) are all the same, *and* zero.

Theorem 1.9 *If a set of k vectors in $\mathbb{C}^n$ is linearly independent, then $k \leq n$. Equivalently, if you have a set of k vectors in $\mathbb{C}^n$ such that $k > n$, then the set is linearly independent.*

This theorem is important because it establishes the notion that a set can't contain more vectors than the space allows for. Said differently, the dimension of M is the number of vectors contained within that are linearly independent.

1.4 Basis

Theorem 1.10 *A **basis** of a subspace $M \subseteq \mathbb{C}^n$ is a set of vectors such that*

1. *S is linearly independent*
2. *Span(S) = M $\implies$ "S Spans M"*

It follows that the standard basis of $\mathbb{C}^N$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \dots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 1 \end{bmatrix}$

1.5 Exercise 1

Consider $S = |v_1\rangle, |v_2\rangle, |v_3\rangle$ where $|v_1\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} i \\ 1 \end{bmatrix}, |v_3\rangle = \begin{bmatrix} 0 \\ i \end{bmatrix}$

(a) Give a linear combination of the vectors in S.

$$\alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} i \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ i \end{bmatrix} = |w\rangle$$

Note that the next problem asks to determine if a vector is in span(S). To 'kill two birds with one stone', try to find a linear combination that satisfies the question. To do this, assign values for α, β, and γ. Note that $1 + i = i + i$ (the top row), and $-1 + 1 = 0$. This allows us to set α and β to 1 and operate on γ.

$$i(x + yi) = 200 - i$$
$$xi - y \implies x = -1 \implies \gamma = -1i - 200$$

Now plug in values. $1 * \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 * \begin{bmatrix} i \\ 1 \end{bmatrix} + (-200 - 1i) * \begin{bmatrix} 0 \\ i \end{bmatrix} = |w\rangle$

(b) Determine if $\begin{bmatrix} 1+i \\ 200-i \end{bmatrix}$ in Span(S)

Start with where we left off in the last part.

$$1 * \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 * \begin{bmatrix} i \\ 1 \end{bmatrix} + (-200 - 1i) * \begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} 1+i \\ 200-i \end{bmatrix}$$

$\begin{bmatrix} 1+i \\ 200-i \end{bmatrix}$ *is* in Span(S) because there existed values α, β, and γ such that $|w\rangle$ of S is a possible outcome of S.

(c) Describe Span(S) "geometrically"

Span(S) can be thought of as every possible vector ($\alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} i \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ i \end{bmatrix}$) contained within $\mathbb{C}^2$

1.6 Exercise 2

Find the condition under which the following two vectors are linearly independent:

$$|v_1\rangle = \begin{bmatrix} x \\ y \\ 3 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} 2 \\ x-y \\ 1 \end{bmatrix} \in \mathbb{R}^3$$

Multiply $|v_2\rangle$ by a scalar that makes $|v_1\rangle$ and $|v_2\rangle$ *not* linearly independent. This way, we can build a contradiction.

$$|v_1\rangle = \begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = 3 * |v_2\rangle = \begin{bmatrix} 2 \\ x-y \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3*(x-y) \\ 3 \end{bmatrix}$$

It can be seen that in this case, $\implies x = 6$ and $y = 4.5$.
Therefore, as long as $x \neq 6$ and $y \neq 4.5$, $|v_1\rangle$ and $|v_2\rangle$ are linearly independent.

1.7 Exercise 3

Show that the set formed by the following vectors is a basis for $\mathbb{C}^3$

$$|v_1\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, |v_3\rangle = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Recall that for the vectors to be a basis for $\mathbb{C}^3$, they must be linearly independent, and Span($|v_1\rangle, |v_2\rangle, |v_3\rangle$)= $\mathbb{C}^3$ (every vector must give additional information, IE no redundancy). To prove linear independence, we need to find where the constants (λ) are zero, by 1.4.

Goal is to find Eigen values

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix} \longrightarrow (R_2 - R_3) \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix} \longrightarrow (R_3 - R_1) \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

Note that with linear algebra experience, this stage is enough to prove that we have linear independence. At the time of writing, I haven't taken this class so I'm going to continue.

$$(\textit{multiply by constant}) \longrightarrow (-1R_2 - \tfrac{1}{2}R_3) \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

To prove that the vectors provided are a basis for $\mathbb{C}^3$, we determined that the vectors are linearly independent such that α, β, and γ are zero. Then, noting that we're operating in $\mathbb{C}^3$, the vectors must be a basis. *This is a bad way of explaining this, go to office hours*

2 Inner Products

Recall that $|v\rangle$ is a column vector and $\langle v|$ is a row vector. Before talking about inner products, we must define operations that help the inner product process.

Remark 2.1 $(\mathbb{C}^n)^*$ is the "dual space of $\mathbb{C}^n$. ♦

The dual space is the space of all row vectors. This is useful for the next definition, which is critical in completing inner products.

Definition 2.2 We equip $\mathbb{C}^n$ with an "involution" (*conjugate transpose*) operation:

$$\dagger : \mathbb{C}^n \longrightarrow (\mathbb{C}^n)^* \text{ given by } \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} x_1^* & ... & x_n^* \end{bmatrix} \text{ given a } |v\rangle \in \mathbb{C}^n,\ \langle v| = |v\rangle^\dagger$$ ♦

A quick example of this operation is helpful to explain the significance of these definitions.

Example 2.3 Given $|v\rangle = \begin{bmatrix} 7 \\ 8i \\ \pi + 3i \\ 0 \end{bmatrix} \in \mathbb{C}^4$, find $|v\rangle^\dagger$
Take the complex conjugate of each entry
$\langle v| = \begin{bmatrix} 7 & -8i & \pi - 3i & 0 \end{bmatrix}$ ♦

Now, we can talk about inner products. Of primary importance for this section, the inner product can determine if two vectors are orthogonal, and operates the same as a dot product on $\mathbb{R}^n$.

Definition 2.4 Given $|v\rangle, |w\rangle \in \mathbb{C}^n$, the **inner product** of $|v\rangle$ and $|w\rangle$ is $\langle w|v\rangle = \langle w|^\dagger \cdot |v\rangle$ ♦

Remark 2.5 The inner product on $\mathbb{R}^n$ is the usual dot product. ♦

The following is a simple example showing the inner product on simple vectors in $\mathbb{R}^3$.

Example 2.6 Given $|v\rangle = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, |w\rangle = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \in \mathbb{R}^3$, find $\langle w|v\rangle$

Perform a complex conjugate on $|w\rangle$ (In $\mathbb{R}^3$, this is the same as flipping $nx1$ to $1xn$)

$$\langle w|v\rangle = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = (5 * 1) + (-2 * 1) + (1 * 3) = 6$$ ♦

Now, the following example shows the inner product on two simple vectors in $\mathbb{C}^2$.

Example 2.7 Given $|v\rangle = \begin{bmatrix} i \\ 1 \end{bmatrix}, |w\rangle = \begin{bmatrix} i \\ -1 \end{bmatrix} \in \mathbb{C}^2$, find $\langle w \mid v\rangle$

$$\langle w \mid v\rangle = \begin{bmatrix} -i & -1 \end{bmatrix} \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} = 1 + (-1 * 1) = 0$$ ♦

Recall from calculus III that the dot product of two vectors can give us insight on orthogonality. In the case of example 2.7, the inner product of $\langle w|v\rangle$ was zero. This tells us that the two vectors are orthogonal.

Definition 2.8 The **norm** (similar to magnitude) of $|v\rangle \in \mathbb{C}^n$ is $\||v\rangle\| := \sqrt{\langle w \mid v\rangle}$ ♦

Example 2.9 $|v\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2 \longrightarrow ||\,|v\rangle\,|| = \sqrt{1^2 + 1^2} = \sqrt{2}$ ♦

Remark 2.10

- $|v\rangle \in \mathbb{C}^n$ with $||\,|v\rangle\,|| = 1$ is called a unit vector
- Unit vectors in $\mathbb{C}^n$ represent a *quantum state*
- An important property of a norm is the *triangle inequality* $\longrightarrow \||v\rangle + |w\rangle\| \leq \||v\rangle\| + \||w\rangle\|$ ♦

Definition 2.11 A set of non-zero vectors $S = \{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\} \leq \mathbb{C}^n$ is called an orthogonal set if $\langle v \mid w\rangle = 0$ if $i \neq j$ ♦

Theorem 2.12 *If $S = \{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\}$ is an orthogonal set of non-zero vectors in $\mathbb{C}^n$, then S is linearly independent.*

Proof. Let $c_1, ..., c_k \in \mathbb{C}$ such that $c_1 |v_1\rangle + ... + c_k |v_k\rangle$.
Goal: show that $c_1 = c_2 = c_k = 0 \longrightarrow$ linearly independent using S is orthogonal
Additionally, let
$j \in 1, ..., k$.
Then,
$\langle v_j| (c_1 |v_1\rangle + ... + c_k) = \langle v_j \mid 0\rangle$
$\longrightarrow c_1 \langle v_j \mid v_1\rangle + ... + c_k \langle v_j \mid v_k\rangle$
$c_j \langle v_j \mid v_j\rangle =$ (v_j *is the only component 'allowed' to have magnitude*)
Thus, $c_j = 0$ since $|v_j\rangle \neq |0\rangle$
Note: Since j was arbitrary, $c_1 = c_2 = ... = c_k = 0$. Hence, S is linearly independent. ■

Remark 2.13 S is an orthonormal basis if:

- S is an orthonormal set if n unit vectors
- $\text{Span}(S) = \mathbb{C}^n$ ♦

2.1 Projections

Definition 2.14 Define $|f_i\rangle \langle f_i|$ to be the **projection** operator onto $\text{Span}(|f_i\rangle)$ ♦

Theorem 2.15 *Let $\beta = \{|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle\}$ be an orthonormal basis for $\mathbb{C}^n$. Then, any $|x\rangle = c_1 |f_1\rangle + ... + c_n |f_n\rangle = \langle f_1 \mid x\rangle |f_1\rangle + ... + \langle f_n \mid x\rangle |f_n\rangle$.*

It follows that $\sum_{j=1}^{n} |f_j\rangle \langle f_j| = \begin{bmatrix} 1 & & \\ & ... & \\ & & 1 \end{bmatrix}$

Example 2.16 Let $|f_1\rangle = \frac{-2}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, |x\rangle = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
$|x_|\rangle = \langle f_1 \mid x\rangle |f_1\rangle = P_1(|x\rangle$
$P_1(|x\rangle) = |f_1\rangle \langle f_1 \mid x\rangle$ ♦

In the above example, $|x_|\rangle$ is the projection of x onto f_1 in the direction of f_1. Note that the projection operation is similar to *comp* from calculus III. Now, let's look at some properties of the projection operator.

Proposition 2.17 *Let $\beta = \{|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle\}$ be an orthonormal basis for $\mathbb{C}^n$. Then, $\{P_1, P_2, P_n\}$ is a set of $n \times n$ matrices such that*

1. $P_i(|v\rangle) \in Span\left(()\, |f_i\rangle\right.$
2. $|v\rangle - P_i(|v\rangle)$ *is orthogonal to* $|f_i\rangle$
3. $P_i^2 = P_i * P_i = P_i$
4. $P_i P_j = 0$ *when* $i \neq \jmath$, *and* $P_i^\dagger = P_i$
5. $\sum_{i=1}^{n} P_i = I_n$ *IE the sum of P's gives us the identity matrix.*

Projection operators make it easy for us to find c_1, c_2, c_n for $|x\rangle = c_1 |f_1\rangle + ... + c_k |f_k\rangle$ when $\{|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle\}$ is an orthonormal set. Side note, in quantum land, P_i's are how we measure the probability that $|x\rangle$ is actually in $|f_i\rangle$.

2.2 Gram-Schmidt

Let $\beta = \{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ be a basis for $\mathbb{C}^n$

- Linearly independent
- Span $(\beta) = \mathbb{C}^n$ (IE no redundancy)

Goal: Turn β into an orthonormal basis, $\{|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle\}$

1. First, make an orthogonal basis.
   $|f_{i+1}\rangle := |b_{i+1}\rangle - (\sum_{j=1} i \frac{\langle f_j | b_{i+1}\rangle}{\langle f_j | f_j\rangle} |f_j\rangle)$
2. Then, make an orthonormal basis by normalizing the resultant ($|f\rangle$) vectors.
   For each $|f\rangle$, $|f\rangle = \frac{1}{\||f\rangle\|} |f\rangle$

2.3 Exercise 7

Let $\beta = \{|b_1\rangle, |b_2\rangle\}$, where $|b_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, |b_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

(a) Show that β is an orthonormal basis for $\mathbb{C}^2$

(i) Show that they're orthogonal

$$\langle w \mid v\rangle = \langle w|^{\dagger} \cdot |v\rangle$$
$$\tfrac{1}{\sqrt{2}}(\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}$$
$$= \tfrac{1}{\sqrt{2}}((1*1)+(1*-1)) = 0 \longleftarrow \text{Recall that if the inner product is zero, the vectors are orthogonal}$$

(ii) Determine if $|b_1\rangle$ and $|b_2\rangle$ are unit vectors

$$\tfrac{1}{\sqrt{2}}\sqrt{1^2+1^2} = \tfrac{\sqrt{2}}{\sqrt{2}} = 1 \longleftarrow \text{Unit vector}$$

Therefore, β is an orthonormal basis.

(b) Find the coordinates (or components) of $|x\rangle = \begin{bmatrix}-2\\1\end{bmatrix}$ relative to β. IE find the scalars $c_1, c_2 \in \mathbb{C}$ such that $c_1|b_1\rangle + c_2|b_2\rangle = |x\rangle$.

(i) Multiply both sides by $\langle b_1|$

$$\langle b_1|(c_1|b_1\rangle + c_2|b_2\rangle) = \langle b_1 \mid x\rangle$$
$$c_1\langle b_1 \mid b_1\rangle + c_2\langle b_1 \mid b_2\rangle = c_1 * 1 + c_2 * 0 = c_1$$

Note that inner product of itself is parallel, and since β is an orthonormal basis, $\langle b_1 \mid b_2\rangle = 1$ (IE they're orthogonal).

$$c_1 = \langle b_1 \mid x\rangle$$
$$c_1 = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} \longrightarrow= \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}$$
$$\tfrac{-2}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} = \tfrac{-1}{\sqrt{2}}$$

(ii) Multiply both sides by $\langle b_2|$

$$\langle b_2|(c_1|b_1\rangle + c_2|b_2\rangle) = \langle b_2 \mid x\rangle$$
$$c_1\langle b_2 \mid b_1\rangle + c_2\langle b_2 \mid b_2\rangle = c_1 * 0 + c_2 * 1 = c_2$$
$$c_2 = \langle b_2 \mid x\rangle$$
$$c_2 = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} \longrightarrow= \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}$$
$$\tfrac{-2}{\sqrt{2}} + \tfrac{-1}{\sqrt{2}} = \tfrac{-3}{\sqrt{2}}$$

Therefore, $c_1 = \frac{-1}{\sqrt{2}}$ and $c_2 = \frac{-3}{\sqrt{2}}$ when $|x\rangle = \begin{bmatrix}-2\\1\end{bmatrix}$.

2.4 Exercise (Gram-Schmidt)

Given $S = \} \left|v_1\right\rangle = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \left|v_2\right\rangle = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \left|v_3\right\rangle = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \{$, turn S into an orthonormal basis for $\operatorname{Span}(S) = \mathbb{R}^3$.

1. Find $\left|f_1\right\rangle$

$$\begin{gathered} \|\left|v_1\right\rangle\| = 3 \\ \longrightarrow \left|f_1\right\rangle = \tfrac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \end{gathered}$$

2. Find $\left|f_2\right\rangle$

$$\begin{gathered} \left|f_2\right\rangle = \left|v_2\right\rangle - \left(\tfrac{\langle v_1|v_2\rangle}{\langle v_1|v_1\rangle} \left|v_1\right\rangle \right) \\ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} - \left[\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right] = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} - \left|0\right\rangle \\ \left|f_2\right\rangle = \tfrac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \end{gathered}$$

3. Find $\left|f_3\right\rangle$

$$\begin{gathered} \left|f_3\right\rangle = \left|v_3\right\rangle - \left[\left(\tfrac{\langle v_1|v_3\rangle}{\langle v_1|v_1\rangle} \left|v_1\right\rangle \right) + \left(\tfrac{\langle v_2|v_3\rangle}{\langle v_2|v_2\rangle} \left|v_2\right\rangle \right) \right] \\ = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} - \left[\left(\frac{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}} \right) + \left(\frac{\begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}} \right) \right] \\ = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} + \begin{bmatrix} \frac{8}{3} \\ \frac{-4}{3} \\ \frac{8}{3} \end{bmatrix} \right) = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{-1}{3} \end{bmatrix} \\ \|f_3\| = 1 \textit{ already normalized} \end{gathered}$$

Therefore, $\left|f_{final}\right\rangle = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{-1}{3} \end{bmatrix}$

3 Matrices and Linear Transformations

Lets goooo

Definition 3.1 A map $T : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is a linear transformation if:

1. $T(|v\rangle + |u\rangle = T(|v\rangle \mid |u\rangle)$
2. $T(C \mid V) = CT(|v\rangle) \forall |u\rangle, |v\rangle \in \mathbb{C}^n$ ♦

Note that every linear transformation arises from a matrix.

Definition 3.2 $M_{mn}(\mathbb{C})$ is a set of all $m \times n$ matrices with complex entries $M_N(\mathbb{C}) := M_{nm}(\mathbb{C})$ ♦

This is traditionally written as M_n or M_{mn}. M_{mn} is a vector space with usual scalar multiplication and matrix addition.

Example 3.3 Given $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \in M_2$, $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, find $2\omega_x - i\omega_y$

$$\begin{aligned} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \end{aligned}$$ ♦

Note that $\sigma_x, \sigma_y, \sigma_z$ are called Pauli or 'spin' matrices.
I_n is called the identity matrix whose 0's are the standard basis for $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\} \in \mathbb{C}^n$.

IE: $I_N = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}$ M_N has well-defined multiplication of matrices.

In general, if $A \in M_{mn}$, $B = m_{nk}$, $A = (a_{ij})^{mn}_{j=1j=1}$, $B = (b_{rs})_{r=1,2=1}$. Additionally, the matrix $AB = \begin{bmatrix} A\,|b_1\rangle & ... & B\,|b_k\rangle \end{bmatrix} = (C_{pq})$ where C_{pq} is the dot product of the pth row of A against the qth column of B.
Here are some nice facts that will help us in further examples:

1. $AB = \begin{bmatrix} \langle a_1|\, B \\ ... \\ \langle a_m|\, B \end{bmatrix}$ where $\{|a_1\rangle, |a_2\rangle, \ldots, |a_n\rangle\}$ are the rows of A.
2. $AB = \sum_{i=1}^{n} |a_j\rangle \langle b_j|$ (IE "the columns of A against the rows of B")

It's useful to know some matrix operations to proceed. In general, $\begin{bmatrix} |a_1\rangle & ... & |a_n\rangle \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & ... & \\ 0 & & d_n \end{bmatrix} = \begin{bmatrix} d_1\,|a_1\rangle & ... & d_n\,|a_n\rangle \end{bmatrix}$ Here's an example:

Example 3.4 $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -3i \\ i & 0 \end{bmatrix}$ ♦

Definition 3.5 An **eigenvalue** for $A \in M_n$ is a complex number $\lambda \in \mathbb{C}^n$ such that there is a non zero vector $|x\rangle \in \mathbb{C}^n$ satisfying $A\,|x\rangle = \lambda\,|x\rangle$. ♦

Said differently, the eigenvalue for A is somehow correlated to λ.

Definition 3.6 A matrix $A \in M_n$ is diagonalizable if:

1. There is a diagonal matrix D and an invertable matrix P such that $A = PDP^-1$.
2. There exists a basis for $\mathbb{C}^n$ consisting of eigenvectors for A ♦

Example 3.7 Consider $A \begin{bmatrix} I_2 & 0 \\ 0 & \sigma_y \end{bmatrix}$
First, let's get the eigenvalues and normalized vectors. Note that eigenvalues are the *roots* of the characteristic equation $det(A - \lambda I) = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} - \begin{bmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & \lambda & \\ 0 & & & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -i \\ 0 & 0 & i & \lambda \end{bmatrix} \longrightarrow det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -i \\ 0 & 0 & i & \lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -i \\ 0 & i & \lambda \end{vmatrix} = (1-\lambda) \begin{bmatrix} 1-\lambda & \begin{bmatrix} -\lambda & -i \\ i & \lambda \end{bmatrix} & -0 & 0 \end{bmatrix}$$

$$= (1-\lambda)(1-\lambda) \begin{bmatrix} \lambda^2 & -1 \end{bmatrix} = 0$$

$$\lambda = 1, 1, 1, -1$$

These are our eigenvalues. Now, we need to find our eigenvectors

Note: $A\,|e_1\rangle = |e_1\rangle$ and $B\,|e_2\rangle = |e_2\rangle$.

$$A - (1)I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -i \\ 0 & 0 & i & 1 \end{bmatrix} \xrightarrow{R_4 + iR_3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & i \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-1*R_1]{R_1 \leftrightarrow R_3} = \begin{bmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, our eigen vector is $|x\rangle = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$.

Our normalized eigen vectors are $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix} \right\}$ ♦

3.1 Subsection of something

Some, but not all matrices are diagonalizable; this property is highly desirable. Matrices are diagonalizable if and only if the eigenvalues form a basis.
put some shit here

1. Vectors must be self-adjoint (*hermition*) IE $A \in M_n$ is **hermition** if $A^\dagger = A$, $sigma + \dagger_x = \sigma_x$, $\sigma + \dagger_y = sigma_y$, $\sigma + \dagger_z = \sigma_z$

2. As it turns out, *all* hermition matricies are diagonalizable $\longleftrightarrow$ the eigenvectors for a basis (IE linearly independent and Span). Said differently, hermition matrices are only hermition matrices if their eigenvectors form an orthonormal basis (IE unit vector, norm, and Span). It's worth noting that the eigenvalues must be real for this to be the case.

3. A matrix in M_n is positive-semidefinite if for all $|x\rangle \in \mathbb{C}^n$, we have $\langle x| A |x\rangle \geq 0$

4. The following are equivalent:

   (a) $u \in M_n$ is unitary

   (b) $u^\dagger = u - 1$ (IE left *and* right inverse) $\longrightarrow u^{-1}u = uu^{-1}$

   (c) $uu^\dagger = I_n$ and $u^\dagger u = I_n$ (unitaries are normal, but not necessarily hermition)

   (d) The columns of u form an orthonormal basis for $\mathbb{C}^n$

   (e) $\forall |x\rangle, |y\rangle \in \mathbb{C}^n$, $\langle u_x \mid u_y\rangle = \langle x \mid y\rangle$. Think of this as a rotation of the entire matrix, such that the angle between the vectors are preserved. Related to calculus III, this operation gives us the angle.

Now, let's go over some examples of these properties in action.

Example 3.8 Example of a matrix that is *not* Hermition:

$$N = \begin{bmatrix} 2 & -i \\ -i & 2 \end{bmatrix} \longrightarrow N^\dagger = \begin{bmatrix} 2 & i \\ i & 2 \end{bmatrix}$$

Note: the columns of N are not orthogonal, so N is not unitary. ♦

Example 3.9 Example of a unitary matrix that's not Hermition:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \longrightarrow u^{-1} = u^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$ ♦

Example 3.10 $\forall |x\rangle \in C^n$, $\langle x| A |x\rangle \geq 0$: $A = |e_2\rangle \langle e_2|$ where $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ This is positive semidefinite if:
proof Let $|x\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{C}^2$.

$\langle x| A |x\rangle = \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = 0 + b^* b =\mid b \mid^2 \geq 0$ ♦

Theorem 3.11 *A matrix $N \in M_n$ is normal $\longleftrightarrow$ there is a unitary matrix $u = \begin{bmatrix} |u_1\rangle & ... & |u_n\rangle \end{bmatrix}$ and a diagonalizable matrix $D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda_n \end{bmatrix}$ such that $N = uDu^\dagger$*

To perform decomposition of matrices, we can use the *spectral theorem for normal matrices*, and the fact that in general, $A \in M_n$ has a singular value decomposition. Note that both of these use eigenvectors of the given matrix.