17.810/17.811 – Game Theory

Lecture 1: Rationality and Rational Choice

Asya Magazinnik

MIT

Course Overview

- A "graduate-level" introduction to formal theoretical analysis in political science
- After this course, you will:
  - understand several key theoretical concepts in political science, e.g., median voter theorem, probabilistic voting model, etc.
  - be able to critically read and understand formal theoretical studies in social science
  - be prepared to take more advanced courses
- The course covers:
  - Basic principles of preferences of rational political actors (understanding rational decisionmaking by political actors)
  - Static/Dynamic Games of Complete and Incomplete Information (understanding strategic interactions among rational actors)

Who Should Take This Course?

- Take this course if:
    - You are interested in politics and want to learn how formal mathematical tools are used to study politics.
    - You want to get an introduction to formal political science
    - You want to critically read/understand game theoretical models
    - You plan to utilize formal modeling in your research
    - You plan to take advanced game theory courses in social science
      (e.g., 14.121, 14.122, 14.126, 14.129)
- This course may not be for you if:
    - You just want a superficial introduction to game theory
    - You have a econ M.A. or taken a Ph.D.-level microeconomics sequence

Why Models?

Social scientists are interested in causal explanations for social phenomena.

- A non-social example: lightning $\rightarrow$ thunder
- We seek the causal mechanisms behind empirical regularities
- Social examples abound:
  - economic downturn $\rightarrow$ incumbent loses election
  - democracies are less likely than autocracies to go to war with each other?

Underlying many causal arguments about social outcomes are strategic interactions.

- Goal-oriented people, rationally pursuing their own goals, engaging in conflict and cooperation with each other

The Basic Tenets of Rational Choice Theory

1. The actor assumption:
    - Accounts of aggregate behavior are grounded in individual actions: votes require voters, vetoes require vetoers
2. The intentions assumption:
    - "The behavior of individual actors is to be understood in terms of their goals, opportunities, incentives, and constraints" (Cameron).
3. The aggregation assumption:
    - We can recover aggregate behavior from the intentional actions of individual agents.

"To a large extent, rational choice theory is nothing more than our everyday method for understanding the social world around us, elevated to a method of systematic research" (Cameron, *Veto Bargaining*).

What Rational Choice Theory Is Not

- History, behavioral economics, social psychology...
  - "[Vetoes] do not occur, at least in a rational choice account, because the zeitgeist was working itself out, or because social forces distinguishable from any human agency somehow compelled the political system to produce vetoes."
  - "we would not assume the president vetoed the bill in a fit of absentmindedness, or in a compulsive spasm triggered by a deeply repressed childhood trauma, or in the inexorable grip of a historical cycle far outside his merely conscious awareness."
- But let's be clear about what rationality demands:
  - We have no normative commitments about people's goals
  - Actors may be extremely constrained in what they can do
  - Actors may be operating with limited information
  - Outcomes may therefore be unintended, unanticipated, or suboptimal; the relationship between goals and choices is complex.

Game Theory versus Decision Theory

- Decision theory concerns how individuals make choices.
- Game theory analyzes how individuals make choices in strategic setting.
    - How do players make choices when outcomes depend on the choices of other players?

What political questions can we ask with models?

International relations assumes states are unitary rational actors seeking survival. Why does North Korea pursue nuclear weapons?

American politics debates whether rational politicians are office motivated or policy motivated. Why do politicians run for office? Why do voters vote even though one vote is not likely to change electoral outcomes?

Comparative politics considers how a rational dictator trades off the incentives to extract rents from being leader and the risk of revolt. Under what conditions can citizens solve their collective action problems and revolt against their leader?

Political economy considers how a rational consumer chooses labor and leisure facing various tax regimes. What is the optimal tax rate?

Elements of Game Theoretic Modeling

- What agents play the game (players)?
- What agents can do (strategies)?
- What is the structure of the problem (game form)?
- What are agents' preferences over outcomes (payoffs)?
- What do the agents know (information)?

Requirements

- *Willingness to work hard!*
- Required readings: Listed on syllabus for each topic
  - Take notes on readings, read slow. Skip no equation.
  - Try to do the reading before class; things will make much more sense.
- 8 Homework assignments (50% of final grade)
  - Posted on Wednesday and due at 1:00 the following Tuesday (before start of class)
    - Posted and submitted through Canvas. Please only submit problem sets through Canvas; emailed problem sets will not be accepted.
    - You are strongly encouraged to use LaTeX
  - Working in groups is encouraged, so long as you write up your work on your own and list your collaborators
  - Take a solo effort first
- Final problem set (40% of final grade)
- Participation (10% of final grade)
  - Can be earned through participation in lecture, weekly recitations, and on the online Piazza discussion board

Other Logistics

- Lectures: I strongly recommend but do not require synchronous attendance.
  - Lectures will be recorded and uploaded to Canvas within 24 hours of class (Canvas $\rightarrow$ Zoom $\rightarrow$ Cloud Recordings)
- Recitations: Fridays (TBD) with Sean
  - Attendance is strongly encouraged!
  - Please fill out our poll by the end of today (**February 16**) for timing
  - Please also let us know if you'd rather have recitations recorded or not (in survey)
- Piazza: Link
  - Available 24/7 for questions about problem sets, lectures, logistics...
  - Sean and I will monitor the site and answer questions within 24 hours on weekdays
  - But please help your peers by participating as well
  - We won't respond to emails unless they're of a personal nature

Books

All course readings are available online, under Canvas $\rightarrow$ Files $\rightarrow$ Readings

- Required textbook:
    - Robert Gibbons. *Game Theory for Applied Economists*. Princeton University Press.
- Optional books:
    - McCarty and Meirowitz. *Political Game Theory*
    - Torsten Persson and Guido Tabellini. *Political Economics: Explaining Economic Policy*. The MIT Press.
    - Martin J. Osborne. *An Introduction to Game Theory*. Oxford Univ.
    - Drew Fudenberg and Jean Tirole *Game Theory*. The MIT Press.
    - Mas-Colell, Whinston, and Green (MWG) *Microeconomic Theory*.

Reading

These slides will focus on the following readings:

- McCarty and Meirowitz, Chapter 1
    - Definition of Rationality
    - Rational Choice on a Finite Set
    - Rational Choice on a Non-finite Set
    - Utility Representations

- Mas-Colell, Whinston, and Green, Chapter 1
    - A Formal Model of Choice (Sections 1.C and 1.D)
    - (But read all of Chapter 1 for a concise second presentation of the McCarty & Meirowitz material.)

A Working Definition of Rationality

Today, we are going to learn important building blocks of formal theory. Let's start with the definition of "rationality."

In classical terms, rationality lies in

1. Preferences (we will study this in terms of "binary relation": $\succsim$, $\succ$, $\sim$)
2. Actions ($A = \{a_1, \cdots, a_k\}$)

Understanding rationality in actions is easy: do what is optimal given "constraints" where we will define constraints later.

Put simply:

Rationality in actions is the purposeful pursuit of goals.

The Theory of Choice: A Formal Foundation

How can we formally define rational preferences?

1. Confronted with any two options, x and y, a person can determine whether he prefers x to y, y to x, or neither.
   - Such preferences are complete.
2. Confronted with three options x, y and z, if a person prefers x to y and y to z, then she must prefer x to z.
   - Preferences satisfying this property are transitive.
   - Sounds reasonable? Well...
     - Individual Choice: apple $\succ$ banana & banana $\succ$ orange & orange $\succ$ apple?
     - Social Choice: What happens if we do pairwise majority voting? Person 1 (A $\succ$ B $\succ$ C), Person 2 (B $\succ$ C $\succ$ A), Person 3 (C $\succ$ A $\succ$ B) $\rightarrow$ You will see A $\succ$ B

The Theory of Choice: A Formal Foundation

How can we formally define rational preferences?

1. Confronted with any two options, x and y, a person can determine whether he prefers x to y, y to x, or neither.
    - Such preferences are complete.
2. Confronted with three options x, y and z, if a person prefers x to y and y to z, then she must prefer x to z.
    - Preferences satisfying this property are transitive.
    - Sounds reasonable? Well...
        - Individual Choice: apple $\succ$ banana & banana $\succ$ orange & orange $\succ$ apple?
        - Social Choice: What happens if we do pairwise majority voting? Person 1 (A $\succ$ B $\succ$ C), Person 2 (B $\succ$ C $\succ$ A), Person 3 (C $\succ$ A $\succ$ B) $\rightarrow$ You will see A $\succ$ B $\succ$ C $\succ$ A This is known as the Condorcet Paradox (group decision-making is very dificult!)

Finite Sets of Actions and Outcomes

- To start, we will have a finite set of actions $A = \{a_1, \ldots, a_k\}$.
- A leader involved in a international crisis might face the following set of alternatives:

$$A = \{\text{invade, negotiate, do nothing}\}$$

- A voter might choose among:

$$A = \{\text{vote Democrat, vote Republican, abstain}\}$$

Complete Information

We say there is complete information when actors are sufficiently knowledgeable that they can predict perfectly the consequences of each action.

- Denote outcomes by the set $X = \{x_1, \ldots, x_n\}$.
- E.g. $X = \{$win major concession and lose troops, win minor concession, status quo$\}$.
- Complete information in a choice setting implies each action $a \in A$ maps directly onto one and only one $x \in X$.
- Formally, there exists a function $f : A \to X$ that maps each action $a \in A$ into a specific outcome $x \in X$

Preferences

We define R as a weak preference relation that will help us build up our theory of choice.

- We define R (or $\succsim$) where the notation, $x_i R x_j$ (or $x_i \succsim x_j$) means that the outcome x_j is not preferred to x_i. We will say when we see $x_i R x_j$, x_i is "weakly" preferred to x_j.
- Formally, suppose we have a set X. $R \subseteq X \times X$ includes all the ordered pairs (x, y) such that if $(x, y) \in R$ then xRy.
- Note that R is similar to the binary relation $\geq$ that operates on the real numbers $\mathbb{R}$.

Strict Preference and Indifference

We can build further preference relations up from the weak preference relation R: strict preference (P or $\succ$) and indifference (I or $\sim$).

Definition

For any $x, y \in X$, xPy (x is strictly preferred to y) if and only if xRy and not yRx. Alternatively, xIy (x is indifferent to y) if and only if xRy and yRx.

An Example

Suppose we have a set $X = \{a, b, c\}$ such that $R \subseteq X \times X$, where the ordered pair (a, b) means a is weakly preferred to b. That is, $(a, b) \iff a \succsim b$

- Take $R = \{(a, a), (b, a), (b, b), (c, b), (b, c), (a, c)\}$.
- What is $P \subseteq X \times X$?
- What is $I \subseteq X \times X$?

An Example

Suppose we have a set $X = \{a, b, c\}$ such that $R \subseteq X \times X$, where the ordered pair (a, b) means a is weakly preferred to b. That is, $(a, b) \iff a \succsim b$

- Take $R = \{(a, a), (b, a), (b, b), (c, b), (b, c), (a, c)\}$.
- What is $P \subseteq X \times X$?
- What is $I \subseteq X \times X$?

$P = \{(b, a), (a, c)\}$, $I = \{(a, a), (b, b), (c, b), (b, c)\}$

Rational Preferences

We now place further restrictions on R that guarantee that one can make a meaningful and well-defined "best" choice given one's preferences. This is our basic notion of rationality.

Formally, a best choice is in the maximal set.

Definition

Given a set X and weak preference relation R on X, the maximal set $M(R, X) \subset X$ is defined as follows:
$M(R, X) = \{x \in X : xRy \ \forall \ y \in X\}$

We seek conditions that guarantee that this maximal set has at least one element.

Rational Preferences: Completeness

The first thing that might derail us is if R is silent about how to compare two members of X. That is why we need completeness.

Definition

A binary relation is complete if, given a set of alternatives and a selection $x, y \in \{a, b, c, d, \ldots\}$, either x is at least as good as y or y is at least as good as x, or both.

Completeness means simply that the agent can compare any two (possibly non-unique) alternatives.

Definition

A binary relation R on X is

1. Transitive if for all $x, y, z \in X$ if xRy and yRz then xRz
2. Quasi-transitive if for all $x, y, z \in X$ if xPy and yPz then xPz
3. Acyclic if for all $\{x, y, z, ..., a, b\} \in X$ if xPy and yPz ... and aPb then xRb.

Note the subtle differences among these definitions.

Sugar

Transitivity may seem innocuous, but it is an assumption that might be violated even by very reasonable preferences.

An example that does not satisfy transitivity:

- Indifferent between 7g and 8g of sugar
- Indifferent between 8g and 9g of sugar
- Strictly prefer 9g to 7g of sugar

Can you formally show the violation of transitivity?

- Write these in terms of R: $7I8 \implies 7R8$ and $8R7$, $8I9 \implies 8R9$ and $9R8$, $9P7 \implies 9R7$ but not $7R9$
- Transitivity violation: $7R8$ and $8R9$ should imply $7R9$
- Quasi-transitivity and acyclicity: we are ok.

Beer

Another example:

- Suppose X is a set of 1000 different bottles of beer. Beer b_1 has had one drop of beer replaced with one drop of plain water, b_2 has had two drops replaced, and so on to b_{1000}.
- Unless one is a master brewer, $b_1 I b_2$ and $b_2 I b_3, \ldots$ and $b_{999} I b_{1000}$. Because xIy implies xRy (by the definition of I), then $b_{1000} R b_{999} \ldots R b_2 R b_1$. Transitivity implies $b_{1000} R b_1$.
- But clearly, $b_1 P b_{1000}$.

The assumption of acyclicity does not suffer from this problem:

- Acyclic if xPy and $yPz \ldots$ and aPb implies xRb for all $\{x, y, z, \ldots, a, b\} \in X$.

However, the stronger assumption of transitivity greatly simplifies many of the results that are coming, so we will stick with it.

An Example

Returning to our example of
$R = \{(a, a), (b, a), (b, b), (c, b), (b, c), (a, c)\}$

1. Is R complete?

An Example

Returning to our example of
$R = \{(a, a), (b, a), (b, b), (c, b), (b, c), (a, c)\}$

1. Is R complete?
   No, because we don't know about (c,c)
2. Is R transitive?

Returning to our example of
$R = \{(a,a),(b,a),(b,b),(c,b),(b,c),(a,c)\}$

1. Is R complete?
   No, because we don't know about (c,c)
2. Is R transitive?
   No, because (c,b), and (b,c) doesn't imply (c,c) although (b,a), and (a,c) does imply (b,c)

It all comes together: Weak Orderings

Putting together our definition of rationality:

Definition

Given a set X, a weak ordering (a.k.a, rational weak preference relation) is a binary relation that is complete and transitive.

We are now ready for our first big result:

Theorem

If X is a finite set and R is a weak ordering, then $M(R, X) \neq \emptyset$.

Proof.

By induction on the size of A.

Proof.

By induction on the size of A.

1. Obviously if A is a singleton, then by completeness its only element is maximal.

Proof.

By induction on the size of A.

1. Obviously if A is a singleton, then by completeness its only element is maximal.
2. For the induction step, let A be of cardinality $n+1$ and let $x \in A$. The set $A - x$ is of cardinality n and has a maximal element by assumption. Call that element y.

Proof.

By induction on the size of A.

1. Obviously if A is a singleton, then by completeness its only element is maximal.
2. For the induction step, let A be of cardinality $n+1$ and let $x \in A$. The set $A - x$ is of cardinality n and has a maximal element by assumption. Call that element y.
3. There are three possibilities: yRx, xRy, or both. (There are no other options by completeness.)
   - If yRx, then $y \in M(R, X)$
   - If xRy, then by transitivity, $xRz\ \forall z \in A - x$ and thus $x \in M(R, X)$

   Both could be true (A can have more than one maximal element). But either way, A has at least one maximal element.

Proof.

By induction on the size of A.

1. Obviously if A is a singleton, then by completeness its only element is maximal.
2. For the induction step, let A be of cardinality $n+1$ and let $x \in A$. The set $A - x$ is of cardinality n and has a maximal element by assumption. Call that element y.
3. There are three possibilities: yRx, xRy, or both. (There are no other options by completeness.)
   - If yRx, then $y \in M(R, X)$
   - If xRy, then by transitivity, $xRz \; \forall z \in A - x$ and thus $x \in M(R, X)$

   Both could be true (A can have more than one maximal element). But either way, A has at least one maximal element.
4. By the induction theorem, $M(R, X)$ is not empty.

□

Note

Note that we can derive alternative theories of choice without the completeness assumption and/or with weaker cousins of transitivity; it's just a bit harder. (See Austen-Smith and Banks 1999)

What if X is a set with the order of the continuum?

Some choice sets are not finite, and it is important to understand how rational preferences are defined in such an environment.

What could go wrong?

Example 1

Let $X = (0, 1)$ and let R on $\mathbb{R}^1$ be equivalent to $\geq$ so that xRy *iff* $x \geq y$. Then, the set $M(R, X)$ is empty.

Why? Because $(0, 1)$ has no maximal element. (We wouldn't have this problem with $[0, 1]$.)

This is a hint that we'll need X to be closed.

What if X is a set with the order of the continuum?

Some choice sets are not finite, and it is important to understand how rational preferences are defined in such an environment.

What could go wrong?

Example 2

Let $X = [0, 1]$ and define R on $\mathbb{R}^1$ as follows:

- xRy if $x, y \leq \frac{1}{2}$ and $x \geq y$
- xRy if $x, y > \frac{1}{2}$ and $x \leq y$
- xRy if $x > \frac{1}{2}$ and $y \leq \frac{1}{2}$

Then, the set $M(R, X)$ is empty.

Now, the set X is okay but the problem is with R (Prove to yourself that you cannot find a maximum in this set.) So we need to put some conditions on both X and R to move forward.

Interlude: Where we are

Building choice theory from the ground up:

- We've formally defined what it means to "prefer" one alternative over another, building up from the weak preference relation (R)
- We've formally defined what it means to make a "best choice" from a choice set (the maximal set $M(R, X)$)
- We've put conditions on R that assure that the actor can always make a best choice: transitivity and completeness
    - This is our working definition of rationality.
- We have proven that a best choice always exists under these conditions (formally, $M(R, X)$ is non-empty).

We are now trying to extend this logic to sets in continuous space:

- e.g. how much money or effort to optimally invest in something
- We need conditions on both X (closed, bounded) and R (complete, transitive, and lower continuous) to assure that $M(R, X)$ is non-empty.

Conditions on X: X is closed

Definition

An open ball of radius $\varepsilon > 0$ and center $x \in X$ is denoted
$B(x, \varepsilon) = \{y \in X : ||x - y|| < \varepsilon\}$
where $||x - y||$ is the Euclidean norm in an n-dimensional space:

$$||x - y|| = \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2}$$

Definition

A set $A \subset \mathbb{R}^n$ is open if for every $x \in A$ there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subset A$.

Definition

A set $A \subset \mathbb{R}^n$ is closed if its complement $B = \mathbb{R}^n \backslash A$ is an open set.

Conditions on X: X is bounded

Definition

A set $A \subset \mathbb{R}^n$ is bounded if there exists a finite number b such that for every $x \in A$ it is the case that $||x - \mathbf{0}|| < b$ where $\mathbf{0}$ is the vector $(0, ..., 0)$.

This is most intuitive in $\mathbb{R}^1$. Consider the set $(-\infty, 0] \cup [1, \infty)$:

This set is unbounded because there does not exist a b such that $x < b \; \forall \; x \in A$.

Putting it together: compact sets

Definition

A set $A \subset \mathbb{R}^n$ is compact if it is closed and bounded.

Conditions on R: Continuity

Having put some conditions on the choice set X, we will now put one further condition on the binary relation R.

First, we need a few more definitions:

Definition

Given a binary relation R on $\mathbb{R}^n$:

- The strict upper contour set of a point $x \in \mathbb{R}^n$ is $P(x) \equiv \{y \in \mathbb{R}^n : yPx\}$.
- The strict lower contour set of a point $x \in \mathbb{R}^n$ is $P^{-1}(x) \equiv \{y \in \mathbb{R}^n : xPy\}$.
- The level set of a point $x \in \mathbb{R}^n$ is $I(x) \equiv \{y \in \mathbb{R}^n : yRx \text{ and } xRy\}$.

Conditions on R: Continuity

We now define "continuity" for a binary relation:

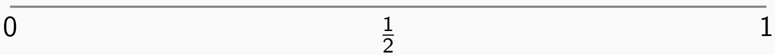
Definition

A binary relation R on $\mathbb{R}^n$ is:

- upper continuous if for all $x \in \mathbb{R}^n$, $P(x)$ is open
- lower continuous if for all $x \in \mathbb{R}^n$, $P^{-1}(x)$ is open
- continuous if it is both upper and lower continuous

Conditions on R: Continuity

Let's see why this rules out the bad behavior in Example 2 above.

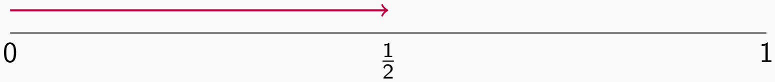


Let $X = [0, 1]$ and define R on $\mathbb{R}^1$ as follows:

- xRy if $x, y \leq \frac{1}{2}$ and $x \geq y$
- xRy if $x, y > \frac{1}{2}$ and $x \leq y$
- xRy if $x > \frac{1}{2}$ and $y \leq \frac{1}{2}$

Conditions on R: Continuity

Let's see why this rules out the bad behavior in Example 2 above.

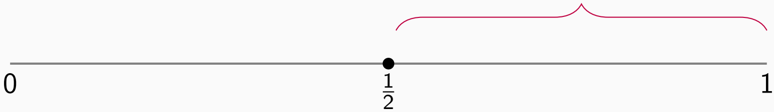


Let $X = [0, 1]$ and define R on $\mathbb{R}^1$ as follows:

- xRy if $x, y \leq \frac{1}{2}$ and $x \geq y$
- xRy if $x, y > \frac{1}{2}$ and $x \leq y$
- xRy if $x > \frac{1}{2}$ and $y \leq \frac{1}{2}$

Conditions on R: Continuity

Let's see why this rules out the bad behavior in Example 2 above.

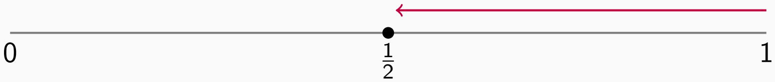


Let $X = [0, 1]$ and define R on $\mathbb{R}^1$ as follows:

- xRy if $x, y \leq \frac{1}{2}$ and $x \geq y$
- xRy if $x, y > \frac{1}{2}$ and $x \leq y$
- xRy if $x > \frac{1}{2}$ and $y \leq \frac{1}{2}$

Conditions on R: Continuity

Let's see why this rules out the bad behavior in Example 2 above.



Let $X = [0, 1]$ and define R on $\mathbb{R}^1$ as follows:

- xRy if $x, y \leq \frac{1}{2}$ and $x \geq y$
- xRy if $x, y > \frac{1}{2}$ and $x \leq y$
- xRy if $x > \frac{1}{2}$ and $y \leq \frac{1}{2}$

Conditions on R: Continuity

Let's see why this rules out the bad behavior in Example 2 above.

- When preferences are complete, any point y that is very close to x is either in $P(x)$, $P^{-1}(x)$, or $I(x)$.
- When preferences are continuous, $y \in P(x)$ or $y \in P^{-1}(x)$ implies that points sufficiently close to y will also be in the same set.
- We can state the violation in the example as a violation of lower continuity. The violation occurs just to the right of $\frac{1}{2}$:

$$P^{-1}\left(\frac{1}{2}+\varepsilon\right) = \left(-\infty, \frac{1}{2}\right] \cup \left(\frac{1}{2}+\varepsilon, 1\right]$$

  which is not an open set.

Putting it all together

We are now ready to state the sufficient conditions for a non-empty maximal set in continuous space.

Theorem

If $X \subset \mathbb{R}^n$ is non-empty and compact (closed and bounded), and R on $\mathbb{R}^n$ is complete, transitive, and lower continuous, then $M(R, X) \neq 0$.

Note: these conditions are sufficient but not necessary, meaning we can still have a nonempty maximal set if these conditions are violated. But, they guarantee a nonempty maximal set.

Generally, violations of compactness of X require stronger assumptions on R, while violations of continuity require more structure on X.

Uniqueness

It would be helpful to know if $M(R, X)$ has a unique element.

When the choice set is finite, we can typically guarantee uniqueness by assuming strict preference (no indifference).

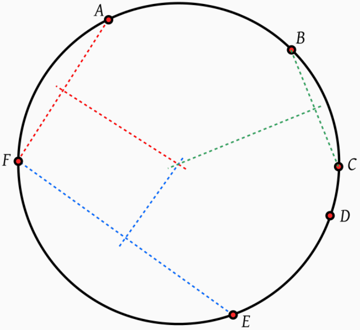
When the choice set is finite, we need convexity on both the set X and the binary relation R.

Convex Sets

Definition

$X \subset \mathbb{R}^n$ is convex if for any $x, y \in X$, the point $\lambda x + (1-\lambda)y$ is also in X for every $\lambda \in [0,1]$.

Not convex: a hand



Convex: the interior of a circle

Definition

Preference R on the convex set X is strictly convex if for any distinct points $x, y \in X$ if xRy then $[\lambda x + (1 - \lambda)y]Py$ for any $\lambda \in (0, 1)$.

Convex preferences have the property that if the agent prefers x to y, she also prefers convex combinations of x and y to y.

Strictly convex preferences go one step further: if the agent is only indifferent between x and y, she still prefers the convex combination of x and y to either of x or y.

- Diminishing returns, or a taste for variety

Uniqueness of an optimal choice

We are now ready to state the conditions for uniqueness of an optimal choice, and to prove that this is so.

Theorem

If X is convex and R on X is strictly convex, then if $M(R, X)$ is non-empty, it contains a single element.

Proof.

Suppose that X is convex, R is strictly convex, and two distinct choices, x and y, are both in $M(R, X)$.

For some arbitrary $\lambda \in (0, 1)$ the point $[\lambda x + (1 - \lambda)y]$ is also in X, since X is convex.

This point, $\lambda x + (1 - \lambda y)$, is preferred to y since R is strictly convex.

But this contradicts the assumption that $y \in M(R, X)$. $\square$

What We Have Established

To summarize, we now know the following about rational choice on a continuum:

- When the choice set X is compact (closed and bounded) and a weak ordering (a complete and transitive weak preference relation R) is lower continuous, a “rational” choice exists (the set $M(R, X)$ is non-empty).
- Further, when X is convex and R is strictly convex, any optimal choice is unique.

From the Binary Relation to the Utility Function

- The model of choice used so far is based on the use of a binary relation, but in general these can be hard to work with.
- Numbers, on the other hand, are easier to work with.
- So if we associate a number with each outcome, then we can just use the $\geq$ operator to compare alternatives.
- We would like to represent preferences using a utility function such that:

$$
\begin{aligned}
u(x) \geq u(y) &\Rightarrow xRy \\
u(x) > u(y) &\Rightarrow xPy \\
u(x) = u(y) &\Rightarrow xIy.
\end{aligned}
$$

From Lists to Numbers: The "Utility Representation"

Sometimes a choice problem can be naturally represented by a numerical statement:

$$\text{I weakly prefer } x \text{ to } y \text{ if } u(x) \geq u(y).$$

Example: If the set X is types of politicians, the statement "I prefer politicians who produce lower taxes" can be expressed by $u(x)$, where x is a tax rate.

Even if a preference relation does not involve "numbers," we are interested in a numerical representation because it's easier to work with.

Utility Representations and Utility Functions

Definition

We say a function $u : X \rightarrow \mathbb{R}^1$ **represents** R if for all x and y in X, xRy iff $u(x) \geq u(y)$. Such a $u(x)$ is a **utility function**.

From here it can be easily shown that $u(x) > u(y)$ iff xPy, and that $u(x) = u(y)$ iff xIy.

Things to note about utility representations

1. There are many different functions that can represent the same preference relation
   - Take $u(x) = x$. The functions $log(u(x))$, $\alpha + \beta(u(x))$, $e^{u(x)}$ can be used to represent the same preference relation

2. Utility functions are ordinal, meaning they can only be used to rank alternatives
   - They cannot tell us how much one prefers something to something else: the value $u(x) - u(y)$ has no directly interpretable meaning
   - In general, comparing utilities across agents is not a meaningful exercise (the problem of interpersonal utility comparisons)

3. A preference relation can be represented by a utility function only if it is rational (complete and transitive)

Characterizing optimal choice using a utility function

Theorem

If the function $u(\cdot)$ is a utility representation of R on X, then $M(R, X) = arg\ max_{x \in X}\{u(x)\}$

$\arg\max_{x \in X}\{u(x)\}$ is the value of x that yields the highest value of $u(x)$, or, put simply, the maximizer.

Characterizing optimal choice using a utility function

Note: a good way to prove equality of the sets X and Y is to show that $X \subset Y$ and $Y \subset X$. We will take this strategy here.

Proof.

1. First let's show that $M(R, X) \subset \arg\max_{x \in X}\{u(x)\}$
   - Assume $u(\cdot)$ represents R on X and that some x' is in $M(R, X)$
   - This implies that $x' R y$ for all y in X
   - This implies that $u(x') \geq u(y)$ for all y in X $\implies x' \in \arg\max_{x \in X}\{u(x)\}$
2. Now let's show that $\arg\max_{x \in X}\{u(x)\} \subset M(R, X)$
   - Assume $u(\cdot)$ represents R on X and that some x' is in $\arg\max_{x \in X}\{u(x)\}$
   - This implies that $u(x') \geq u(y)$ for all y in X
   - This implies that $x' R y$ for all $y \in X \implies x' \in M(R, X)$

$\square$

Existence of a maximizer

For finite sets:

- If R is complete and transitive, we know that $M(R, X)$ is non-empty, and by the proof we just did we know a maximizer must therefore also exist.

For non-finite sets:

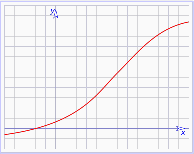
- We need our old condition of compactness on X, and, not surprisingly, continuity of the utility function

Continuity of functions

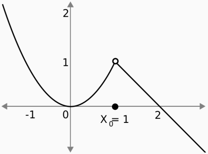
Definition

We say a function $f : X \to R^1$ is continuous if for every $x \in X$ the following is true: For every $\varepsilon > 0$ there exists some $\delta > 0$ such that if $||x - y|| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Informally, a continuous function is one you can draw without lifting your pencil. Substantively, a continuous utility function produces close utilities for close outcomes.



Continuous



Not continuous

Existence of a maximizer for non-finite sets

Theorem

If $X \subset \mathbb{R}^n$ is compact and $u : X \to \mathbb{R}^1$ is continuous, then a maximizer exists.

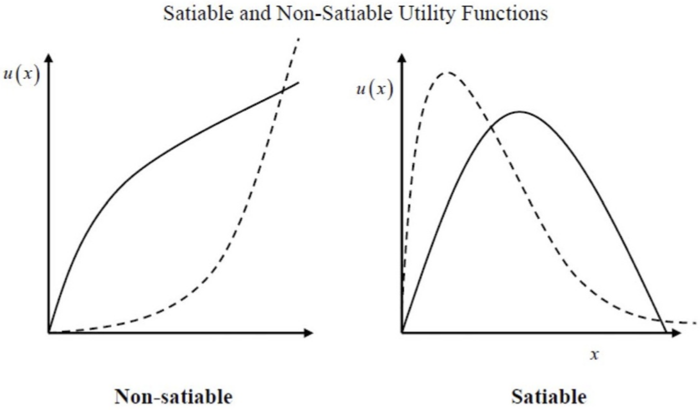
This result is known as the Weierstrass Theorem.

If we further assume differentiability, we can use the basic tools of calculus to characterize our optimal choices. (Stay tuned.)

Satiable vs. non-satiable preferences

- In most economic applications, outcomes are money (income, wealth, profits) or commodities (widgets, cookies, beer).
- In these cases it is sensible to assume that larger outcomes are always preferred to smaller ones: non-satiable preferences.
- In politics agents often have a most preferred outcome that is neither zero nor infinite: satiable preferences.
    - A voter might want enough taxation to fund a generous social safety net but not so much that she is left with no income
    - She might prefer restrictions on carbon emissions that don't go so far as to ban automobiles
- Formally, an agent has satiable preferences if $M(R, X)$ contains elements interior to the outcome space X, or when the maximizer of $u(x)$ is in the interior of X.
- Logic of diminishing returns vs. logic of trade-offs

Satiable vs. non-satiable preferences



The Spatial Model

- The most common application of satiable preferences is the spatial model that represents policy outcomes as points in $\mathbb{R}^d$
- It is generally assumed that voters have single-peaked and symmetric preferences
    - Single-peaked preferences mean that the agent has a single policy that maximizes her utility. We call this policy the agent's ideal point.
    - Symmetric preferences decline at the same rate in every direction moving away from the agent's ideal point.

One-dimensional models

If the policy space is one-dimensional, then single-peaked, symmetric preferences are represented by utility functions of the form:

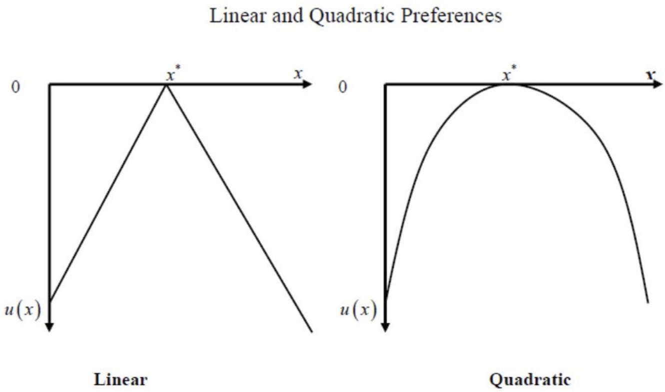
$$u_i(x) = h(-|x - z_i|)$$

where z_i is agent i's ideal point and h is an increasing function.

The two most popular examples are

- Linear utility: $u_i(x) = -|x - z_i|$.
- Quadratic utility: $u_i(x) = -(x - z_i)^2$.

Linear and quadratic utility functions



Multidimensional models

In outcome spaces with more than one dimension, distances are usually given by the Euclidian norm,

$$\|x - y\| = \sqrt{\sum_{j=1}^{n} (x^j - y^j)^2}$$

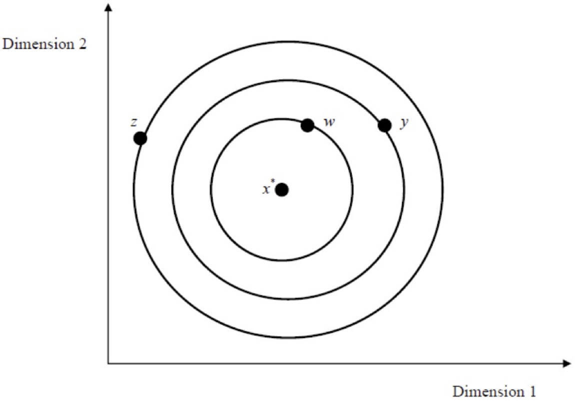
Thus a symmetric, single-peaked utility function takes the form:

$$u_i(x) = h(-\|x - z_i\|)$$

It is difficult to visualize functions over many dimensions, but for two dimensions:

Multidimensional models

Indifference Curves for Two-Dimensional Quadratic Preferences



$$u(x^*) > u(w) > u(y) > u(z)$$

Observable choice and preferences

- Preferences describe decisionmakers' tastes, but they are not observable
- Choices are observable, e.g., votes on bills, or what you had for lunch
- So far, we have taken preferences as our primitive feature
  - Note that all our assumptions about rationality were developed on this unobservable object
- But we can also start with actions and see what we can infer about preferences from there. This approach has some distinct advantages:
  - It leaves room for more general forms of individual behavior
  - It makes assumptions about directly observable objects
  - It bases individual decisionmaking not on a process of introspection but on an entirely behavioral foundation

A model of choice

To infer preferences from choices, we'll need a model of choice.

We'll formally represent choice behavior with a choice structure $(\mathcal{B}, C(\cdot))$, which consists of two ingredients:

1. A set of budget sets $\mathcal{B}$
2. A choice rule $C(\cdot)$

A model of choice

Definition

$\mathcal{B}$ is a family of subsets of X that represents all the choice problems that an individual might face.

- Every element of $\mathcal{B}$ is a budget set B which is a subset of X
- However, $\mathcal{B}$ need not include every possible subset of X

Example:

- $X = \{x, y, z\}$
- $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$

A model of choice

Definition

A choice rule $C(\cdot)$ is a correspondence that assigns every budget set $B \in \mathcal{B}$ a nonempty set of chosen elements, $C(B) \subset B$.

Intuitively, $C(\cdot)$ is the rule someone uses in deciding what they do for every set of possible alternatives they might face.

Example:

- $X = \{x, y, z\}$
- $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$
- $C_1(\{x, y\}) = \{x\}$ and $C_1(\{x, y, z\}) = \{x\}$
- $C_2(\{x, y\}) = \{x\}$ and $C_2(\{x, y, z\}) = \{x, y\}$

The Weak Axiom of Revealed Preference

As before, we seek a refinement on this choice structure to make sure an individual's choices are consistent.

Definition

The choice structure $(\mathcal{B}, C(\cdot))$ satisfies the weak axiom of revealed preference if the following property holds:
If for some $B \in \mathcal{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$.

Put simply: if x is *ever* chosen when y is available, then there can't be any budget sets where the choice rule picks y but not x.

The Weak Axiom of Revealed Preference

Example: Do the following choice structures satisfy the weak axiom of revealed preference?

Again, let $X = \{x, y, z\}$ and $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$.

1. $C_1(\{x, y\}) = \{x\}$ and $C_1(\{x, y, z\}) = \{x\}$
2. $C_2(\{x, y\}) = \{x\}$ and $C_2(\{x, y, z\}) = \{x, y\}$

Example: Do the following choice structures satisfy the weak axiom of revealed preference?

Again, let $X = \{x, y, z\}$ and $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$.

1. $C_1(\{x, y\}) = \{x\}$ and $C_1(\{x, y, z\}) = \{x\}$
2. $C_2(\{x, y\}) = \{x\}$ and $C_2(\{x, y, z\}) = \{x, y\}$

(1): yes, (2): no

The relationship between preference relations and choice rules

The weak axiom restricts choice behavior in a way that parallels the use of the rationality assumption for preference relations. This raises the question: what is the connection between the two approaches?

Specifically, we might ask two questions:

1. If a decisionmaker has a rational preference ordering R, do her decisions necessarily satisfy the weak axiom?
2. If a decisionmaker's behavior satisfies the weak axiom, is there necessarily a rational preference relation that is consistent with these choices?

The answers are (1) yes, and (2) maybe. Let's see.

Rational R implies choice structure satisfying the weak axiom

Suppose an individual has a rational preference relation R on X. Her preference-maximizing behavior is given by:

$$C^*(B, R) = \{x \in B : xRy \text{ for every } y \in B\}$$

Theorem

Suppose that R is a rational preference relation. Then the choice structure generated by R, $(\mathcal{B}, C^*(\cdot, R))$, satisfies the weak axiom.

Rational R implies choice structure satisfying the weak axiom

Proof.

Take some $B \in \mathcal{B}$ with $x, y \in B$. Suppose $x \in C^*(B, R)$. Then it must be the case that xRy.

Now take some $B' \in \mathcal{B}$ with $x, y \in B'$ and suppose $y \in C^*(B', R)$. Then yRz for all $z \in B'$.

Since we know that xRy, then by transitivity $xRz\ \forall\ z \in B'$. So $x \in C^*(B', R)$. This is precisely the conclusion that the weak axiom demands. $\square$

We've shown that if behavior is generated by rational preferences, then it satisfies the consistency requirements embodied in the weak axiom. What about going the other way?

Does choice structure satisfying weak axiom imply rational R?

We'll need to define a concept along the way:

Definition

Given a choice structure $(\mathcal{B}, C(\cdot))$, we say that the rational preference relation R rationalizes $C(\cdot)$ relative to $\mathcal{B}$ if:

$$C(B) = C^*(B, R) \ \forall \ B \in \mathcal{B}$$

Recalling that:

$$C^*(B, R) = \{x \in B : xRy \text{ for every } y \in B\}$$

Thus we derive preferences from behavior and not the other way around.

Does choice structure satisfying weak axiom imply rational R?

Example:

- $X = \{x, y, z\}$
- $\mathcal{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$
- $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$

First verify that this satisfies the weak axiom (it does).

Nevertheless, we cannot have a rational preference relation that rationalizes these choices:

- To rationalize $C(\{x, y\}) = \{x\}$ we need xPy
- To rationalize $C(\{y, z\}) = \{y\}$ we need yPz
- Transitivity demands xPz, which contradicts the choice behavior.

Does choice structure satisfying weak axiom imply rational R?

The more budget sets there are in $\mathcal{B}$, the more the weak axiom restricts choice behavior.

- Note that the fact that $\{x, y, z\}$ was not in $\mathcal{B}$ in the previous example was crucial!

We can rule out such violations with an additional restriction on $\mathcal{B}$.

Theorem

If $(\mathcal{B}, C(\cdot))$ is a choice structure such that:

1. *the weak axiom is satisfied*
2. *$\mathcal{B}$ includes all subsets of X of up to three elements*

then there is a rational preference relation that rationalizes $C(\cdot)$ relative to $\mathcal{B}$.