17.810/17.811 – Game Theory

Lecture 4: Extensive Form Games with Complete Information

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Where We Are/Where We Are Headed

- We have now learned the concept of Nash Equilibrium in both pure and mixed strategies
- We have focused on static games with complete information
- We now consider dynamic games, where players make multiple sequential moves
- We still consider complete information, meaning the players' payoff functions are common knowledge
- But, we will study games with:
  - Perfect information: at each move, all players know the history of play thus far
  - Imperfect information: at some move some player does not know the full history of the game

Reading

These slides will focus on the following readings:

- Dynamic Games of Complete and Perfect Information
  - Gibbons, 2.1A and 2.1B
- Dynamic Games of Complete and Imperfect Information
  - Gibbons, 2.2A, 2.2B, and 2.4

Adding Dynamic Aspects to Theory

- So far we have thought about games where players choose strategies once and for all at the beginning of an interaction.
- In these games, actions and strategies were the same.
    - In dynamic games they won't be.
- Sometimes we are interested in dynamics: how incentives change as players learn where they are in a game.

An Example: Colonial Control

- A country generates revenue from control of its colony's resources and from direct taxes on its people.
- Given a policy, residents of the colony decide whether to Rebel or Consent to the status quo.
- If the colony revolts, the country decides: grant independence or suppress the revolution.
- If the country suppresses, $\rightarrow$ war.

Colonial Control

- In the event of a war, the colony wins with probability p.
- At stake is control of the resources, which generate a payoff of \$4 to the side that controls them, as well as \$2 in taxes.
- Starting a revolution costs the colony \$1 if the country does not suppress.
- Suppression by the country (war) costs each side \$6.
- If colony does not revolt, country can continue to tax the colony's residents at \$2 or it can eliminate these taxes.
- If colony revolts and loses the war, country maintains resource *and* taxes.

The Game

The following table gives the game's payoffs (colony, country).

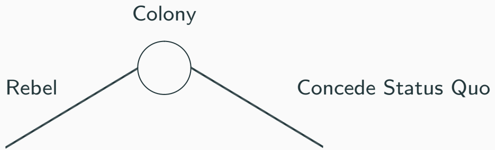
Revolution Game
Colony does not revolt and the country eliminates taxes, (0,4)
Colony does not revolt and the country continues to tax, (-2,6)
Colony revolts and the country grants independence, (3,0)
Colony revolts and the country suppresses, $(4p - 2(1-p) - 6,\ 6(1-p) - 6)$

Figure 1: Outcomes and payoffs.

The Game Form, Game Tree, or Extensive Form

The Game Form, Game Tree, or Extensive Form

The Game Form, Game Tree, or Extensive Form



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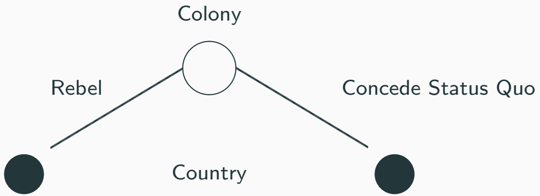

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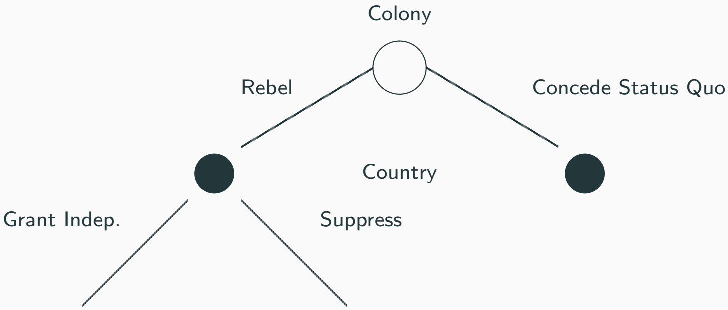

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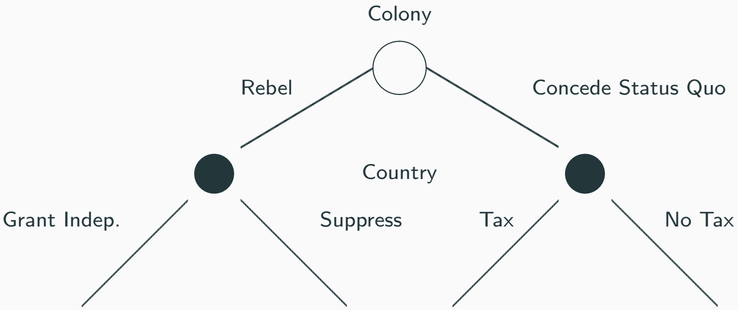

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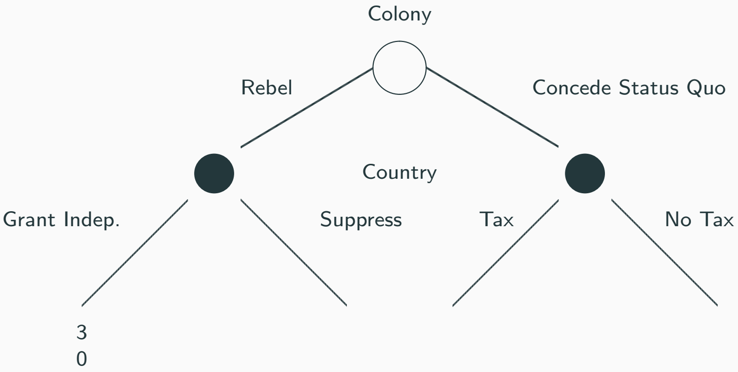

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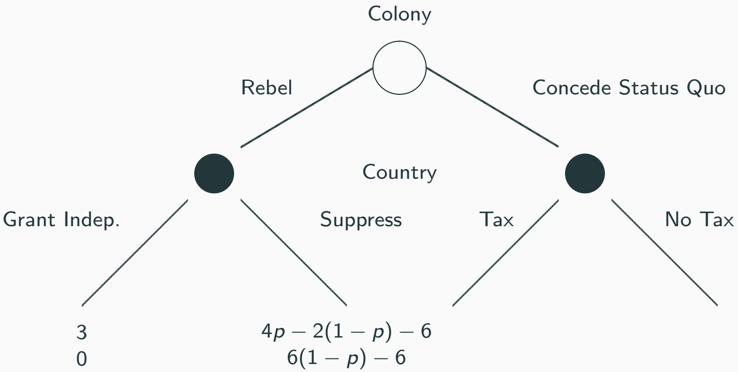

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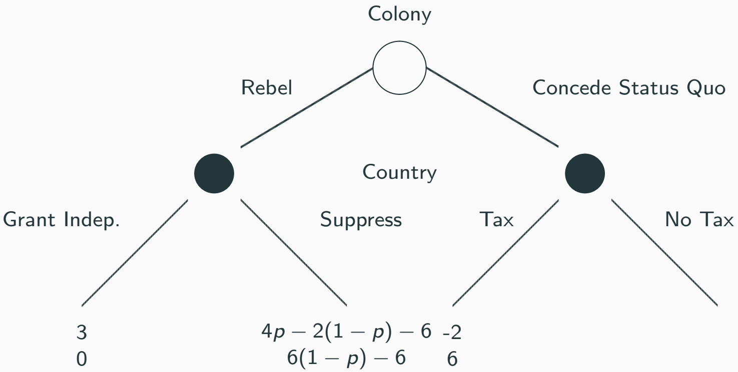

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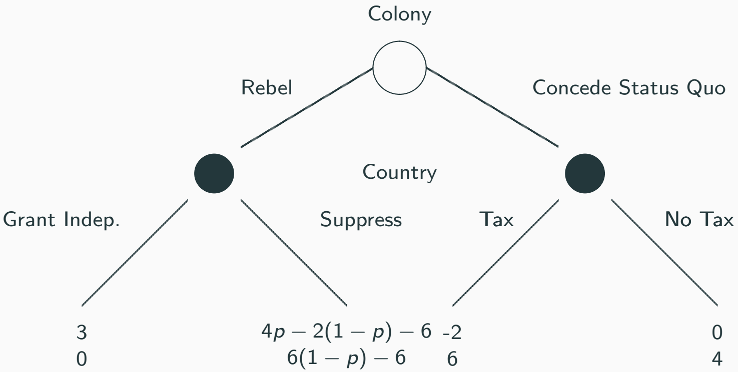


Extensive form

Formalizing the Extensive Form

1. A set of players, N
2. A set H of histories, where $h^k \in H$ is a history up to kth stage of the game.
   - We denote $H^T \subset H$ as the set of terminal histories, e.g., (`Rebel`, `Grant Independence`)
3. A mapping $p(h) : H - H^T \rightarrow N$ designates which players move at a given history h.
4. A set of actions $A(h)$ for each history for each of the $p(h)$ players.
5. Information sets that form a partition of the set of non-terminal histories.
   - A partition of a set X is a set of non-empty subsets of X such that every element $x \in X$ is in exactly one of these subsets.
6. Payoffs for each player at each terminal node.

Strategies in Dynamic Games

Strategies in dynamic games are complete, contingent plans.

Intuitive definition

Imagine you know there is a dynamic game to be played, but you can't show up to play. Instead you send your friend to play for you. The instructions you give them comprise a strategy if, for any circumstance your friend might face, they know exactly what they should do.

Formal definition

For an extensive form game Γ, a strategy profile for a player $i \in N$ is a mapping $s_i(h) : H_i \to A(h)$, where $s_i(h) = s_i(\hat{h})$ if h and $\hat{h}$ are in the same information set and H_i is the set of all histories for which $p(h) = i$. A strategy profile is a collection of mappings, one for each player.

From Extensive Form to Normal Form

We can use this definition of strategies to write down the normal form representation of an extensive form game.

Country/Colony	Rebel	Concede
Grant Indep. and Tax	$0, 3$	$6, -2$
Grant Indep. and Eliminate Tax	$0, 3$	$4, 0$
Suppress and Tax	$6(1-p)-6,\ 4p-2(1-p)-6$	$6, -2$
Suppress and Eliminate Tax	$6(1-p)-6,\ 4p-2(1-p)-6$	$4, 0$

Figure 2: Normal Form Representation of Colonial Control

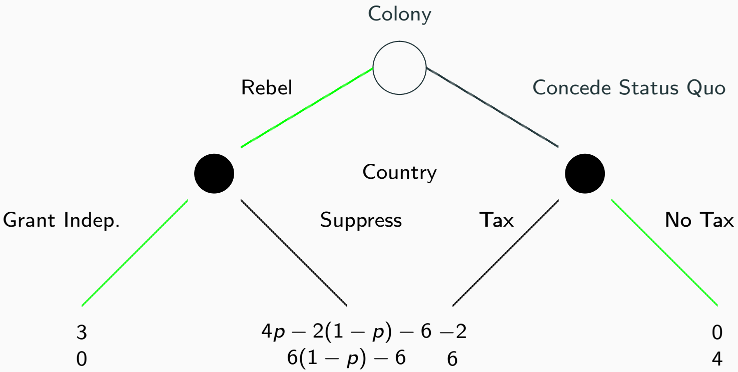
- Any extensive form game has a normal form representation.
- What are the Nash equilibria?

Nash Equilibria of Colonial Control Game

Country/Colony	Rebel	Concede
Grant Indep. and Tax	$0^*, 3^*$	$6^*, -2$
Grant Indep. and Eliminate Tax	$0^*, 3^*$	$4, 0$
Suppress and Tax	$6(1-p)-6,\ 4p-2(1-p)-6$	$6^*, -2^*$
Suppress and Eliminate Tax	$6(1-p)-6,\ 4p-2(1-p)-6$	$4, 0^*$

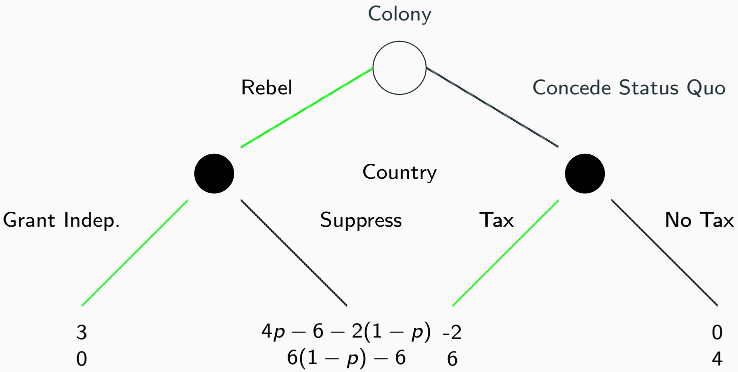
- Note that there are three NE.
- Two are (Rebel, Grant Indep. and Tax) and (Rebel, Grant Indep. and Eliminate Tax).
    - In both, the colony will rebel and the country will grant independence.
- However, the third equilibrium, (Concede, Suppress and Tax), has the threat of retaliation (suppression) leading the colony not to rebel.
    - Is something wrong with this last equilibrium?

Nash Equilibrium: Colonial Control



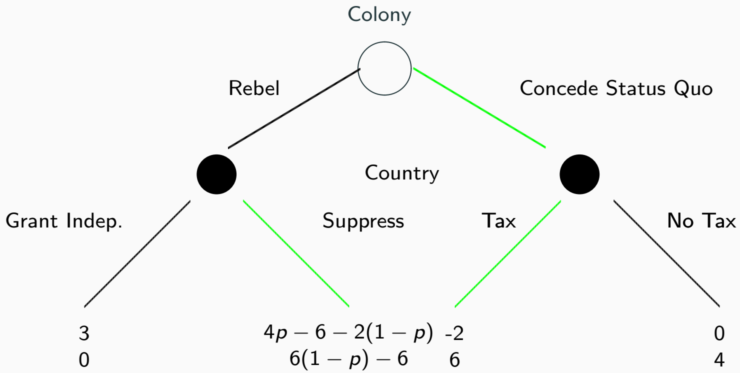
Nash equilibrium : (Rebel; Grant Indep., No tax)

Nash Equilibrium: Colonial Control



Nash equilibrium : (Rebel; Grant Indep., Tax)

Nash Equilibrium: Colonial Control



Nash equilibrium : (Concede; Suppress, Tax)

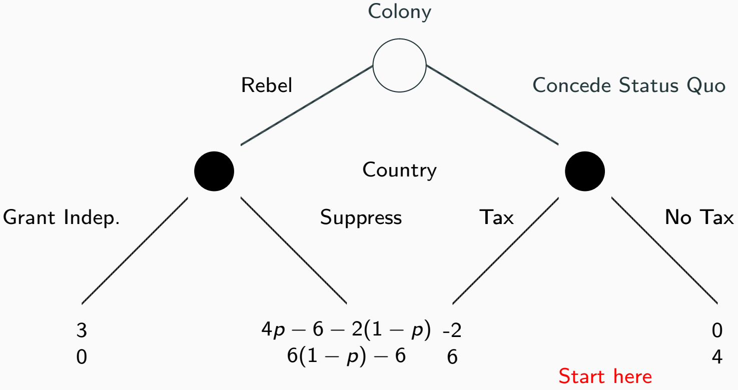
The equilibrium is not credible.

The Concede, Tax Equilibrium

- Note that the threat is not "credible." That is, if the players ever reached the world in which the colony rebelled, the country would strictly prefer to grant independence rather than suppress: 0 vs. $-6p$
- The reason that the country can make the threat in this equilibrium is because the threat is "off the equilibrium path."
- Can we refine our equilibrium concept to eliminate equilibria that rely on non-credible threats?

Credible Commitments and Backward Induction

Let's consider solving this game backward.



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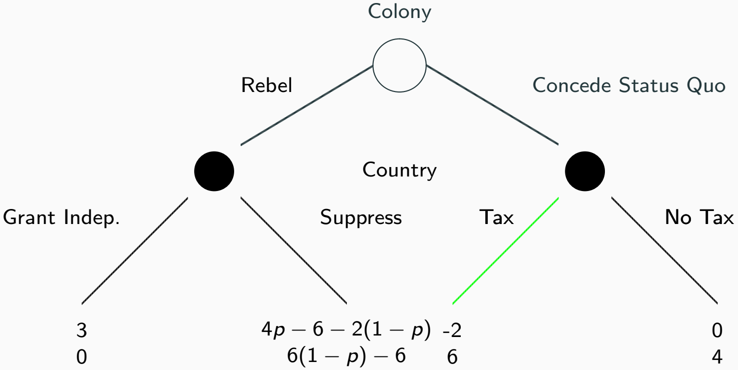

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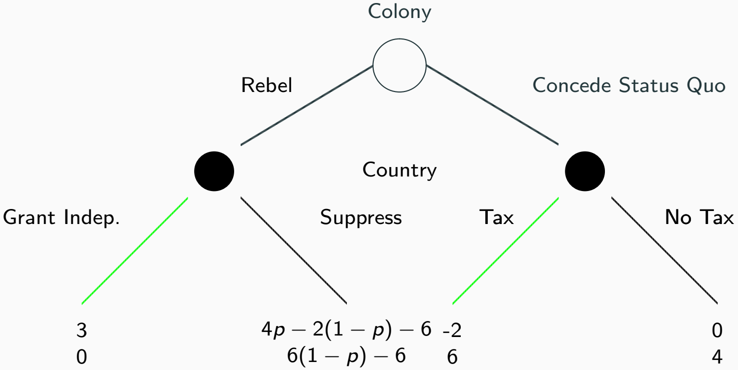

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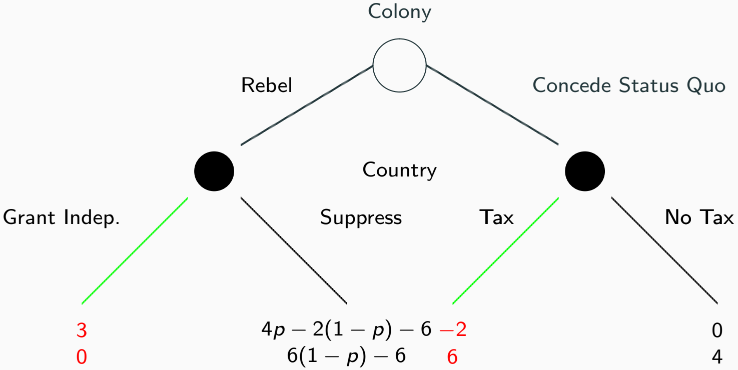

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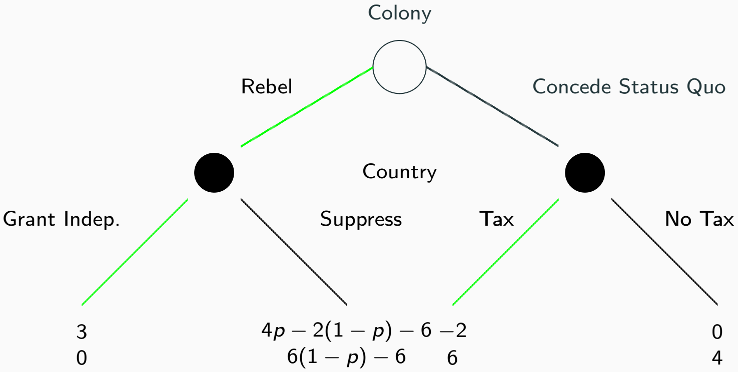

Credible Commitments and Backward Induction

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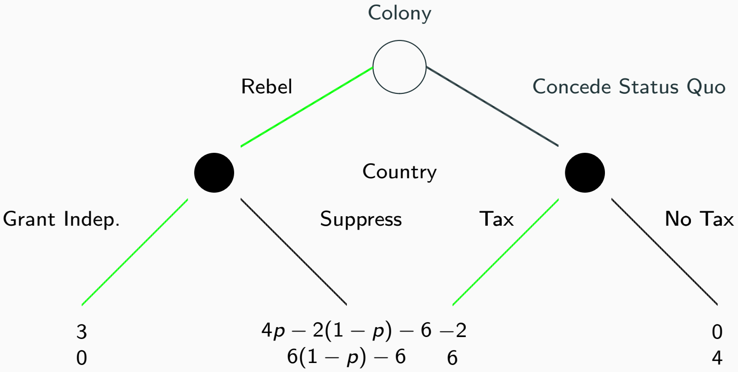


And we have found the credible Nash equilibrium.

Backward Induction

What is true about the backward induction exercise?

- Play is rational at every node.
- Backward induction in finite extensive form games of perfect information yields a strategy profile.
- This strategy profile is a pure strategy NE in the associated normal form game.

Perfect and Complete Information

- The games we have looked at so far have been of complete and perfect information.
- Games of complete and perfect information have the property that every player's information set is a singleton.
- Every decision node starts a "subgame."

Let's Talk More About Information Sets

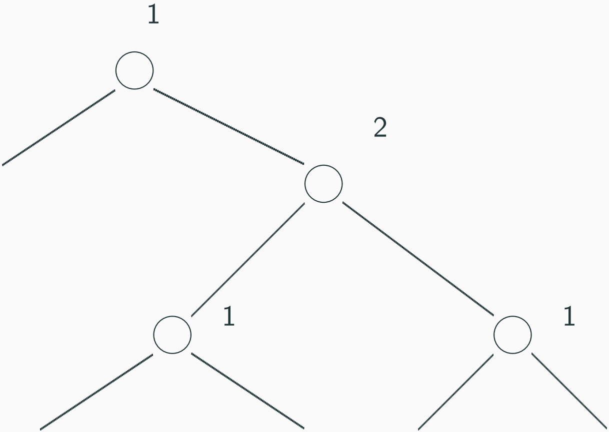
Definition

An information set for a player is a collection of decision nodes satisfying:

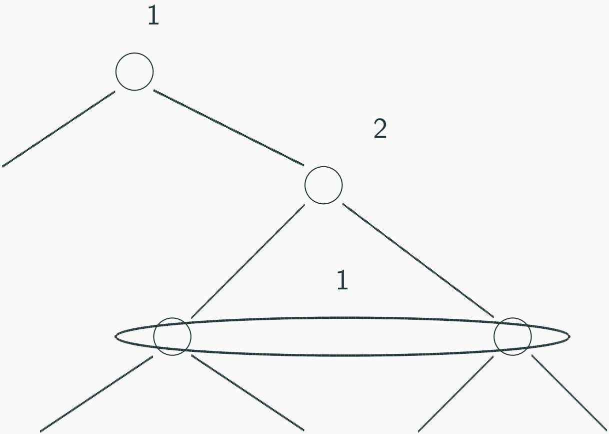
1. the player has the move at every node in the information set
2. when the play of the game reaches a node in the information set, the player with the move does not know which node in the set has (or has not) been reached.

This implies that the player must have the same set of feasible actions at each decision node in an information set, else they would be able to infer the node from the set of available actions.

Example: Perfect Information



Example: Imperfect Information



Subgames

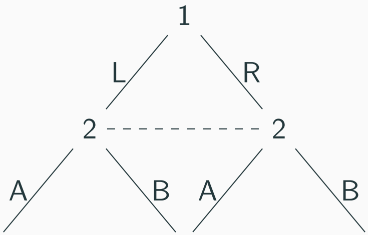
Notice that in an extensive form game, at any single nodes we could cut off what remains of the game tree and we would have a well defined extensive form game. We call this a subgame.

Definition

A subgame in an extensive-form game:

1. begins at a decision node n that is a singleton information set (but is not the game's first decision node)
2. includes all the decision and terminal nodes following n in the game tree
3. does not cut any information sets

Example: What Are the Subgames of this Game?



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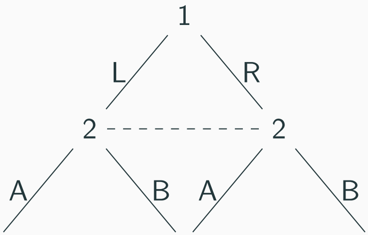
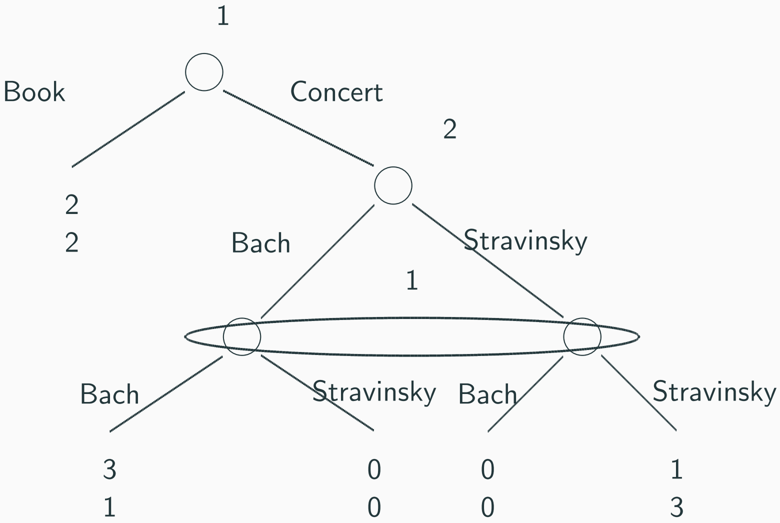


This game has no subgames.

What About This Game?



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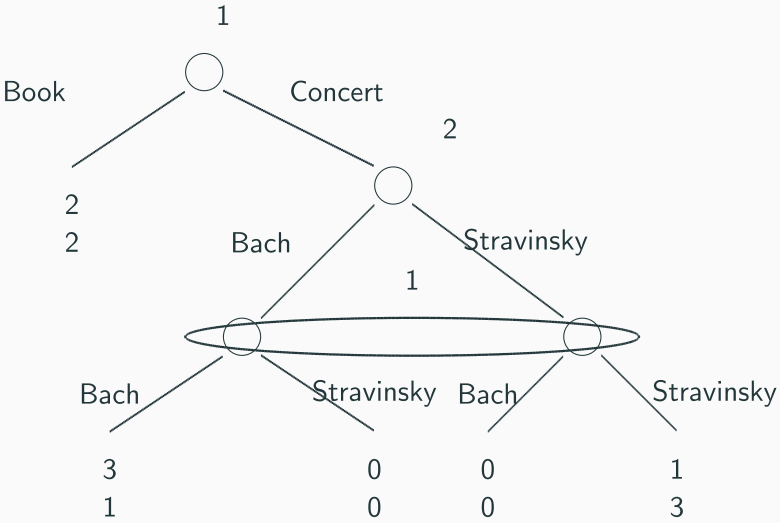


The subgame beginning at Player 2's node is the only subgame.

Our First Equilibrium Refinement

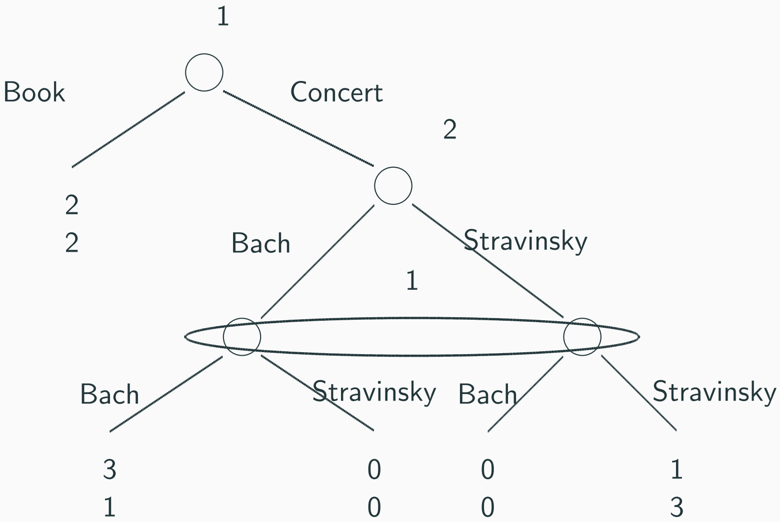
We can now refine the set of Nash equilibria by restricting our attention to those that satisfy the credibility condition of playing rationally in every subgame. We call such equilibria subgame perfect.

Definition

A Nash equilibrium is subgame perfect if the players' strategies constitute a Nash equilibrium in every subgame.

Subgame perfection is a generalized and useful extension of the idea of backward induction.

Modified Bach or Stravinsky Game



Nash Equilibria of the Modified Bach or Stravinsky Game

- What is the normal form?
- Player 1 has 4 strategies: 2 possible actions at each of her 2 information sets.
- Player 2 has two strategies.
- Let's write down the normal form representation of this game and find its Nash equilibria in pure strategies.

		Bach	Stravinsky
Book	Bach	$2, \underline{2}$	$\underline{2}, \underline{2}$
Book	Stravinsky	$2, \underline{2}$	$\underline{2}, \underline{2}$
Concert	Bach	$\underline{3}, \underline{1}$	$0, 0$
Concert	Stravinsky	$0, 0$	$1, \underline{3}$

Credibility

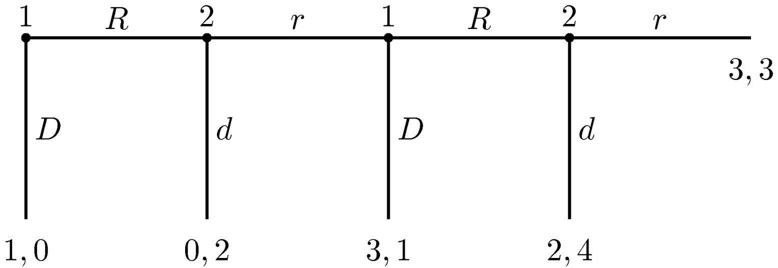
- Look closely at the (Book, Bach; Stravinsky) equilibrium.
- Why does 1 choose Book?
    - Because if she deviates to playing Concert she will get the payoff from a BOS game where the players don't coordinate $(0, 0)$.
- Is that a reasonable expectation?
    - No. We want players to always be playing rational strategies even in the parts of the game that are never reached.
    - This way, results of our analysis do not depend on threats to take actions that would not be carried out.
    - For example, if Player 1 wanted to challenge Player 2 by choosing to go to the Bach concert, would Player 2 really choose Stravinsky?

Some Facts about Subgame Perfect Nash Equilibria

- In every finite extensive form game, a subgame perfect equilibrium (SPE) exists (possibly in mixed strategies).
- An SPE is a Nash equilibrium by definition.
- In a finite game of perfect information, the SPE is the backward induction solution.

Criticism of Backward Induction and Subgame Perfection

What is the SPE of the following game?



- In the unique SPE, players defect at every node, precluding any mutually beneficial cooperative outcome
- Backward induction and subgame perfection may imply an unrealistic view of rationality and learning

Mixed Strategies in Extensive Forms

- In a normal form game, mixed strategies extend the players' possibilities by allowing them to choose a pure strategy randomly.

- In an extensive form game random choice can be executed two ways:
    1. The player can randomly choose a contingent plan for the whole game (a standard mixed strategy)
    2. At every decision node, the player can randomly choose one of her available actions (a behavioral strategy)

Mixed Strategies

- Let H_i be the set of player i's information sets, and let $A_i = \cup_{h_i \in H_i} A(h_i)$ be the set of actions available to player i.

- A pure strategy is a map $s_i : H_i \to A_i$ with $s_i(h_i) \in A_i(h_i)$ for all the $h_i \in H_i$, with $S_i = \times_{h_i \in H_i} A(h_i)$.

- A mixed strategy is then $\Delta(S_i)$.

Behavioral Strategies

- Let H_i be the set of player i's information sets, and let $A_i = \cup_{h_i \in H_i} A(h_i)$ be the set of actions available to player i.
- And let $\Delta(A_i(h_i))$ be a probability distribution on $A_i(h_i)$.
- A behavioral strategy b_i is an element of the Cartesian product $\times_{h_i \in H_i}$ that specifies a probability distribution over actions at each h_i, and the probability distributions at different information sets are independent.
- A **Nash Equilibrium in behavioral strategies** is a profile $b = (b_1, \ldots, b_I)$ such that no player can increase their expected payoff using a different behavioral strategy.

Equivalence

Theorem (Kuhn's Theorem)

In a game of perfect recall, mixed and behavioral strategies are equivalent. That is, every mixed strategy is equivalent to the unique behavioral strategy it generates and each behavioral strategy is equivalent to every mixed strategy that generates it.

Equivalence

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So we will talk of behavioral strategies as strategies and denote them as σ_i.

A Behavioral Strategy with No Equivalent Mixed Strategy

Consider the one player game where

- Two actions: exit, continue.
- At X the driver can exit and get to A or continue.
- At Y the driver can exit and get to B or continue.
- If at Y the driver continues, then he gets to C.
- Suppose that the driver cannot distinguish between intersections X and Y (cannot remember whether he has already gone through one of them)

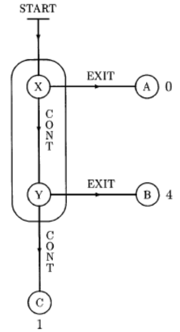
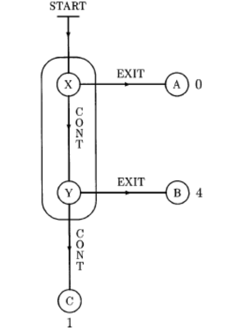


FIG. 1. The absent-minded driver problem.

A Behavioral Strategy with No Equivalent Mixed Strategy

Facts

- There are two pure strategies, exit or continue, resulting in A or C.
- Any mixed strategy is a probability distribution over pure strategies, so can only get A and C, but never B.
- By contrast, a behavioral strategy of $\frac{1}{2}$ `Exit`, $\frac{1}{2}$ `Continue`, where the agent chooses an action at each possible node produces a distribution $\frac{1}{2}A$, $\frac{1}{4}B$, $\frac{1}{4}C$.



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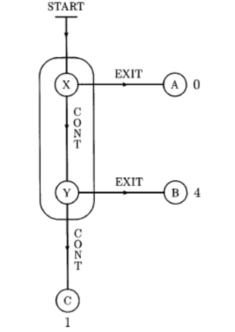
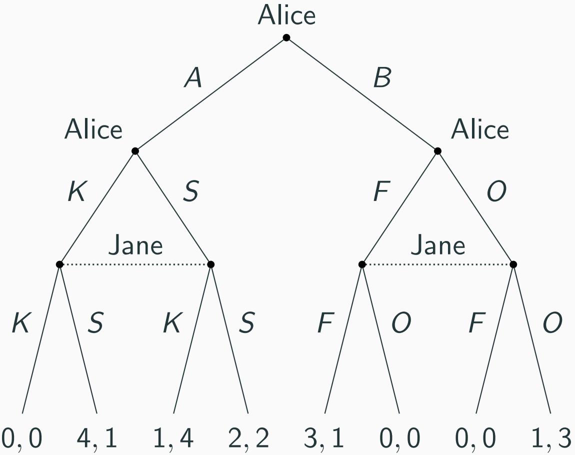


FIG. 1. The absent-minded driver problem.

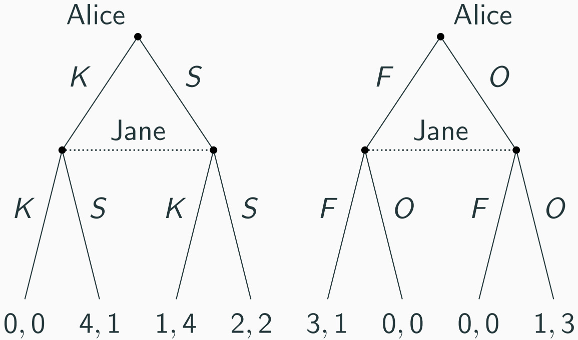
So there is no mixed strategy that produces a distribution the same as this behavioral strategy in this game of imperfect recall.

Backward Induction with Imperfect Information



Backward Induction with Imperfect Information

- Alice chooses first: if A, they play a game of chicken, if B they play a battle of the sexes (Bach or Stravinsky)
- What are the subgames?



Backward Induction with Imperfect Information

Start with the subgame after Alice chooses *A*.

		Jane	
		K	S
Alice	K	0,0	4*,1*
	S	1*,4*	2,2

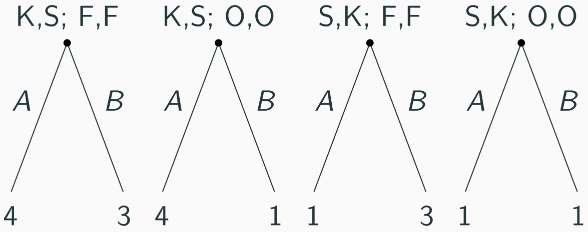
Backward Induction with Imperfect Information

Then go to the subgame after Alice chooses B.

		Jane	
		K	S
Alice	K	3*,1*	0,0
	S	0,0	1*,3*

Backward Induction with Imperfect Information

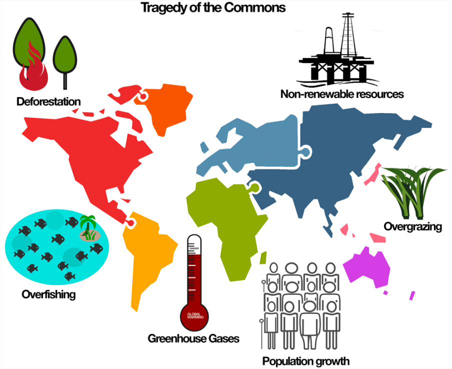
So we have four cases to consider:



SPNE:

1. ((A,K,F), (S,F))
2. ((A,K,O), (S,O))
3. ((B,S,F), (K,F))
4. ((A,S,O), (K,O)) and ((B,S,O), (K,O))

Tragedy of the Commons: Individuals' Own Utility Maximization Depletes Common Resources



The overutilization of a common resource is called the tragedy of the commons, and today we will talk about how this might come about by thinking about the SPNE of a "consumption game."

The Model

- There is a resource of size $y > 0$:
  - Think fuel in the ground, fish in the ocean.
- Two periods: today and tomorrow
- Two players: Player 1 and Player 2

Definition

Given a stage game G, let $G(T)$ denote the finitely repeated game in which G is played T times, with the outcomes of all preceding plays observed before the next play begins. The payoffs for $G(T)$ are simply the sum of the payoffs from the T stage games.

In the 1st period:

- 2 players consume amounts c_1 and c_2 and choose simultaneously
- In the case that $c_1 + c_2 > y$, then suppose that $c_1 = c_2 = y/2$.

The Common Resource Game

In the 1st period:

- 2 players consume amounts c_1 and c_2 and choose simultaneously
- In the case that $c_1 + c_2 > y$, then suppose that $c_1 = c_2 = y/2$.

In the 2nd period:

- The amount of resources is $y - (c_1 + c_2)$
- Players choose d_1 and d_2
- If $d_1 + d_2 > y - (c_1 + c_2)$ then $d_1 = d_2 = \frac{y-(c_1+c_2)}{2}$.

Utilities and Payoffs

- In both periods, utility is the log of the amount consumed
- $u_i = \ln(c_i) + \ln(d_i)$

Analysis

Lets look for some of the implications of a subgame perfect equilibrium.

- Start at the last stage, period 2, and suppose that c_1' and c_2' were consumed in the period 1.
- Then in the second period consuming everything is the result of any pair of equilibrium strategies. (why?)
- So, the subgame perfection requirement is that both players consume as much as possible in the second period.
- In fact, in every equilibrium each player's consumption is:

$$d_1 = d_2 = \frac{y - (c_1' + c_2')}{2}.$$

- Knowing this will happen in the second period we then want to ask, what are equilibrium levels of consumption in period 1.

SPNE

Now let's look for a mutual best response in the first period.

- For player 1, what is her best response to consumption $\bar{c}_2$?

$$u_1 = \ln(c_1) + \ln\left(\frac{y - (c_1 + \bar{c}_2)}{2}\right)$$

- We need to find a c_1 that maximizes this thing.

Calculating Best Responses

1. Player 1's Best Response

$$u_1 = \ln(c_1) + \ln\left(\frac{y - (c_1 + \bar{c}_2)}{2}\right)$$

$$\begin{aligned}
[FOC] \quad & du_1/dc_1 = 0 \\
\iff \quad & \frac{1}{c_1} + \left(\frac{2}{y - (c_1 + \bar{c}_2)}\right)\left(-\frac{1}{2}\right) = 0 \\
\iff \quad & \frac{1}{c_1} = \frac{1}{y - (c_1 + \bar{c}_2)} \\
\iff \quad & c_1 = y - (c_1 + \bar{c}_2) \\
\iff \quad & 2c_1 = y - \bar{c}_2 \\
\iff \quad & c_1 = \frac{y - \bar{c}_2}{2}
\end{aligned}$$

Calculating Best Responses

2. Player 2's Best Response: An exactly parallel argument gives a best response for player 2 of:

$$c_2 = \frac{y - \bar{c}_1}{2}$$

Calculating Equilibrium Values

Now given these best responses, we need equilibrium values

In equilibrium we need (c_1^*, c_2^*)

$$c_1^* = \frac{y - c_2^*}{2}, \qquad c_2^* = \frac{y - c_1^*}{2}$$

Plugging c_2^* into the first equation,

$$c_1^* = \frac{y}{2} - \frac{1}{2}\left(\frac{y - c_1^*}{2}\right)$$

Solving for c_1^* and recalling that $d_1 = d_2 = \frac{y-(c_1+c_2)}{2}$,

$$c_1^* = c_2^* = \frac{y}{3} \quad \text{and} \quad d_1^* = d_2^* = \frac{y}{6}$$

Equilibrium Welfare

What is the (social) utility of this equilibrium ?

$$\begin{aligned} u_i &= \ln(c_i) + \ln(d_i) \\ &= \ln(y/3) + \ln(y/6) \end{aligned}$$

Therefore, the social utility is,

$$u_1 + u_2 = 2\ln(y^2/18)$$

Can We Describe the Social Optimum?

- Let's think about a society where all we try to is maximize the welfare of consumption.
- In such a world we would want to maximize:

$$
\begin{aligned}
u_1 + u_2 &= \ln c_1 + \ln d_1 + \ln c_2 + \ln d_2 \\
&= \ln c_1 + \ln c_2 + 2 \ln \frac{y - (c_1 + c_2)}{2}
\end{aligned}
$$

Socially Optimal Extraction

At the maximum $c = c_1 = c_2$, so we really need to maximize:

$$2 \ln c + 2 \ln \left(\frac{y - (2c)}{2} \right)$$

Socially Optimal Extraction

At the maximum $c = c_1 = c_2$, so we really need to maximize:

$$2 \ln c + 2 \ln \left(\frac{y - (2c)}{2} \right)$$

Some calculus gives us:

$$\begin{aligned} U_c &= 2/c - 2/\left(\frac{y - (2c)}{2} \right) = 0 \\ 1/c &= \frac{2}{y - (2c)} \\ c &= \frac{y}{4} \end{aligned}$$

And this implies a social utility of:

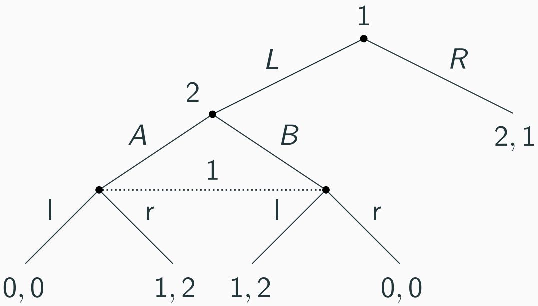
$$2 \ln(y^2/16).$$

Social Welfare

What is the implication?

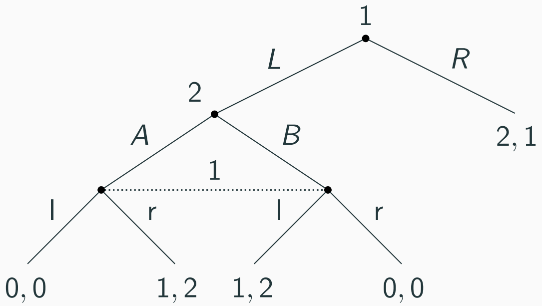
- Comparing social welfare to equilibrium extraction then comes down to seeing $2\ln(\frac{y^2}{16}) > 2\ln(\frac{y^2}{18})$.
- This is bad news for utilizing common resources and suggests that government intervention or at least the establishment of private property rights is important.
- For example, if a single player were allowed to extract the resource, then they would maximize their utility over the two periods and better utilize what is available.

A Little Review



Players:

A Little Review

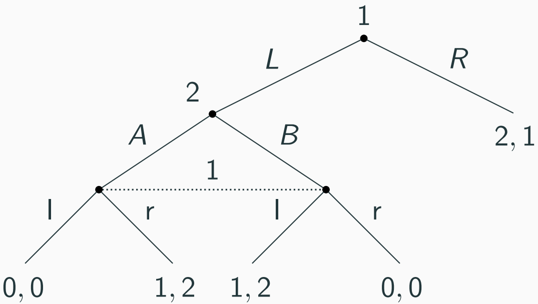


Players: 1, 2

Information Sets:

Player 1:

A Little Review

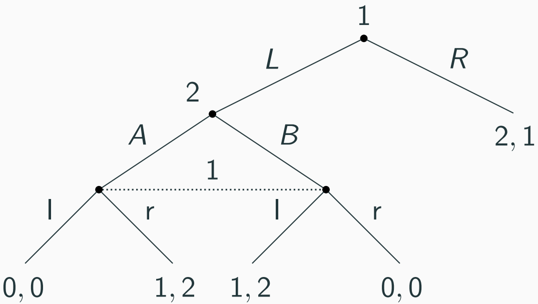


Players: 1, 2

Information Sets:

Player 1: $\{\emptyset\}$, $\{(L, A), (L, B)\}$

A Little Review



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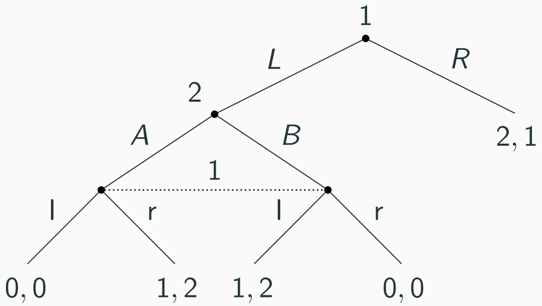
Player 2:

A Little Review



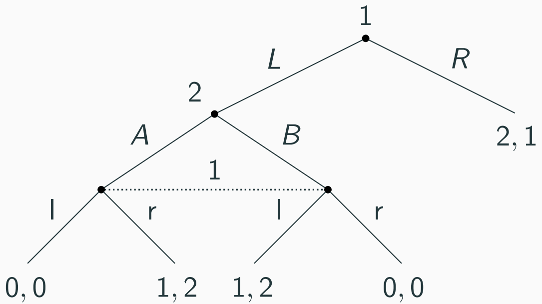
Players: 1, 2

Information Sets:

Player 1: $\{\emptyset\}$, $\{(L, A), (L, B)\}$

Player 2: $\{L\}$

A Little Review



Players: 1, 2

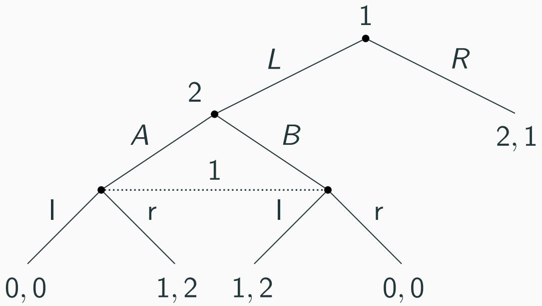
Information Sets:

Player 1: $\{\emptyset\}$, $\{(L, A), (L, B)\}$

Player 2: $\{L\}$

Subgames:

A Little Review



Players: 1, 2

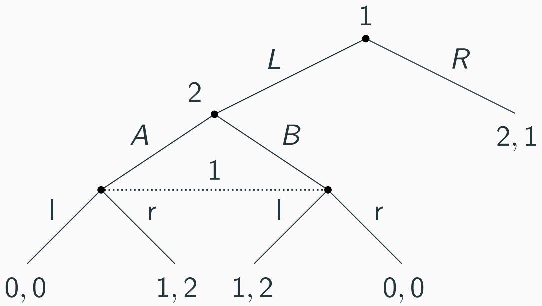
Information Sets:

Player 1: $\{\emptyset\}$, $\{(L, A), (L, B)\}$

Player 2: $\{L\}$

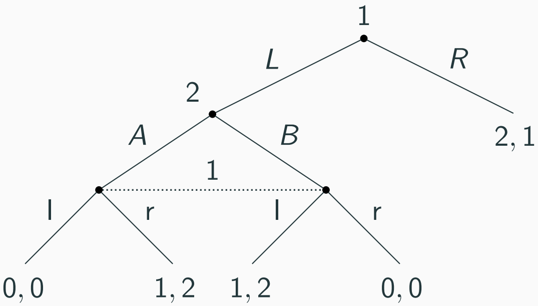
Subgames: The subgame beginning at Player 2's move.

A Little Review



Actions:

A Little Review



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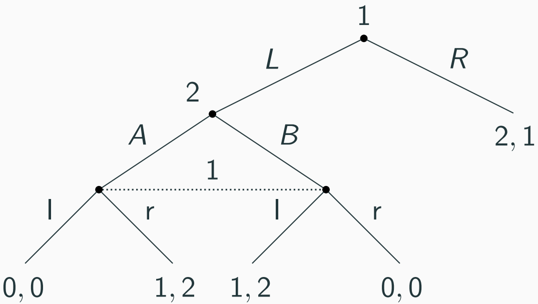
Player 1:

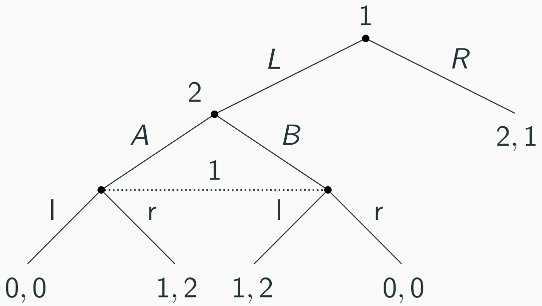
A Little Review



Actions:

Player 1: $A(\emptyset) = \{L, R\}$; $A(L, A) = \{l, r\}$; $A(L, B) = \{l, r\}$

A Little Review



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Player 2:

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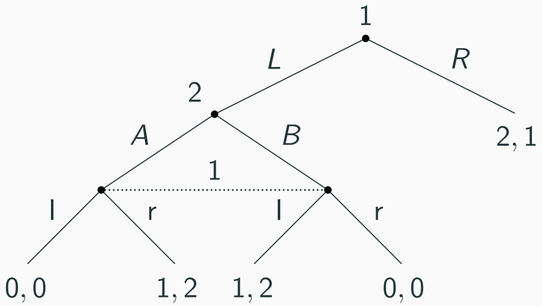
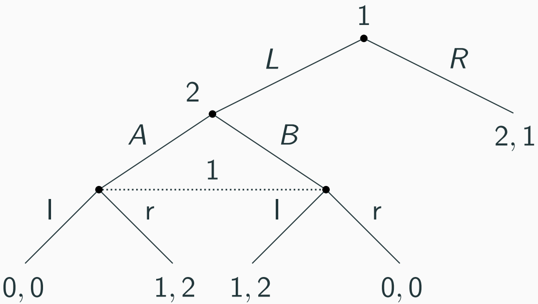


Actions:

Player 1: $A(\emptyset) = \{L, R\}$; $A(L, A) = \{l, r\}$; $A(L, B) = \{l, r\}$

Player 2: $A(L) = \{A, B\}$

A Little Review



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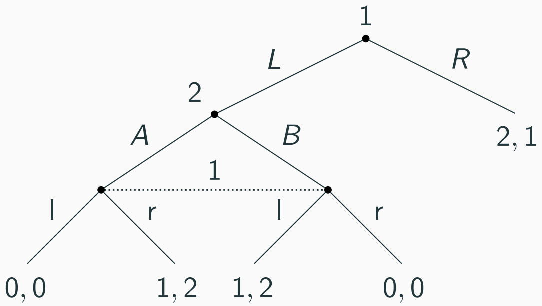
Strategies:

A Little Review



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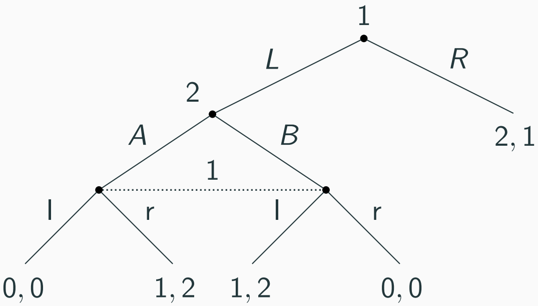
Player 1:

A Little Review



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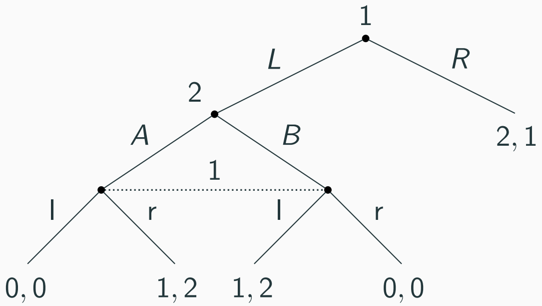
Player 2:

A Little Review



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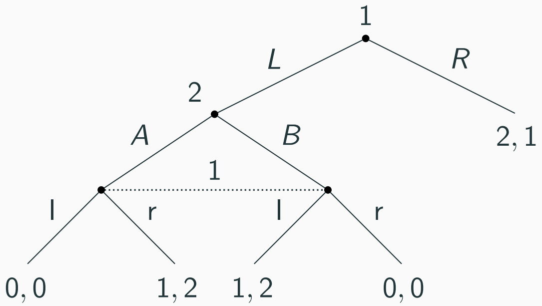
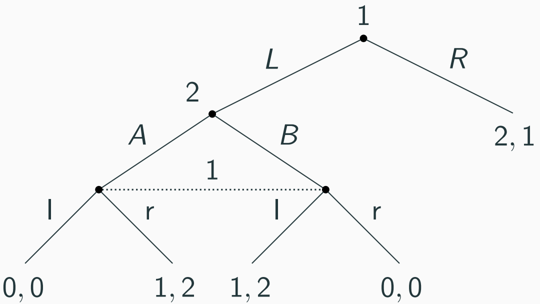
Review: Mixed and Behavioral Strategies



A mixed strategy is a probability distribution over pure strategies, e.g. for Player 1:

- Play (L, l) with probability $\frac{1}{3}$
- Play (L, r) with probability $\frac{1}{3}$
- Play (R, l) with probability $\frac{1}{6}$
- Play (R, r) with probability $\frac{1}{6}$

Review: Mixed and Behavioral Strategies



A behavioral strategy is a probability distribution over actions at every node, e.g. for Player 1:

- First play L with probability $\frac{1}{2}$ and R with probability $\frac{1}{2}$
- At second move (if we get there), play l with probability $\frac{1}{4}$ and r with probability $\frac{3}{4}$

Review: Mixed and Behavioral Strategies

For any mixed strategy of a player in a finite extensive game with perfect recall there is an outcome-equivalent behavioral strategy.

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Two (mixed or behavioral) strategies are outcome-equivalent if for every collection of pure strategies of the other players, the two strategies induce the same outcome.

Review: Mixed and Behavioral Strategies

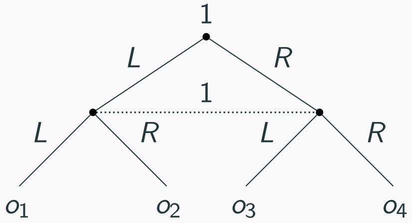
For any mixed strategy of a player in a finite extensive game with perfect recall there is an outcome-equivalent behavioral strategy.

Two (mixed or behavioral) strategies are outcome-equivalent if for every collection of pure strategies of the other players, the two strategies induce the same outcome.

Perfect recall simply means that every player knows whether or not they have already made a choice (and what that choice was) at every prior opportunity to move.

Example: A Game with Imperfect Recall

Can we find a mixed strategy for which there is no outcome-equivalent behavioral strategy for this game of imperfect recall?



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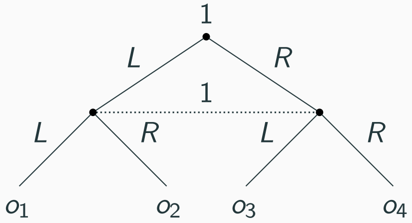

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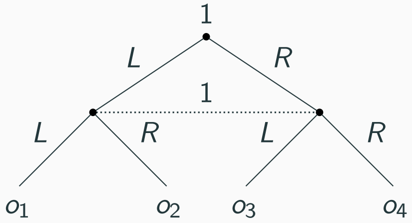

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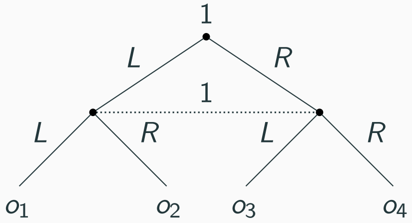

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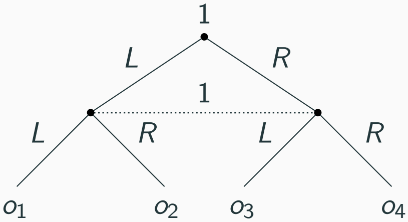

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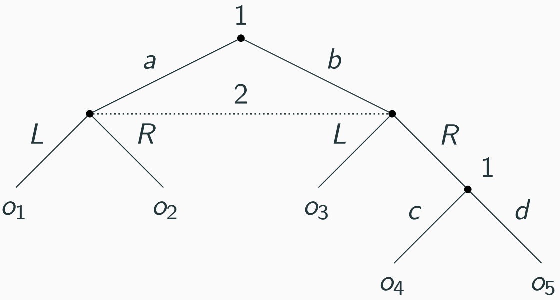
What is the induced distribution over outcomes? $\left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$

Can we achieve this with any behavioral strategy of the form $((p, 1-p), (q, 1-q))$? No.

Example: Finding an Outcome-Equivalent Mixed Strategy



Consider the behavioral strategies:

- Player 1: Play a with probability p and b with $1 - p$; play c with probability q and d with $1 - q$
- Player 2: Play L with probability $\frac{1}{2}$ and R with $\frac{1}{2}$

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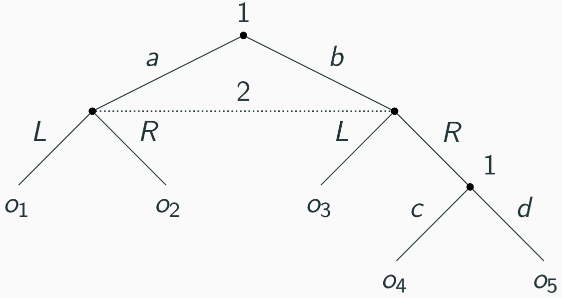

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First let's derive the outcome distribution.

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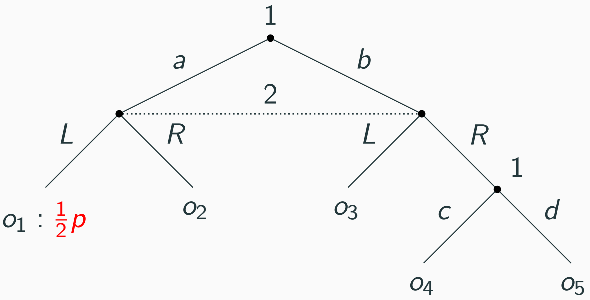

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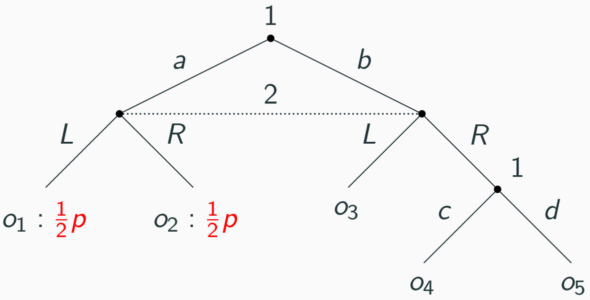

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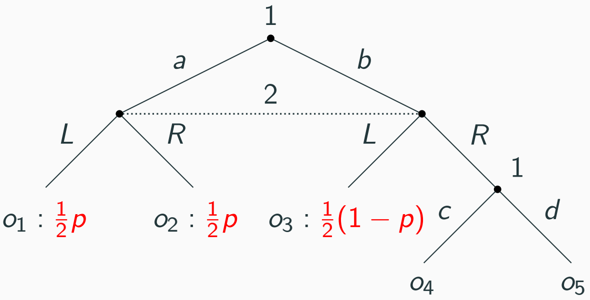

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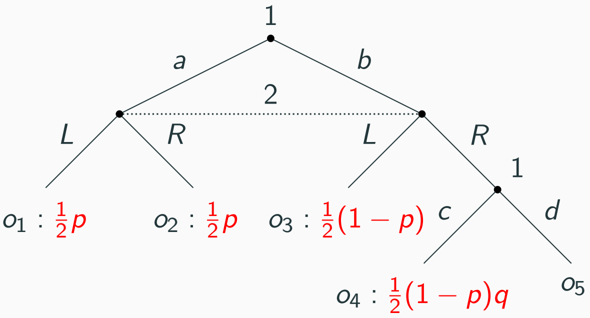

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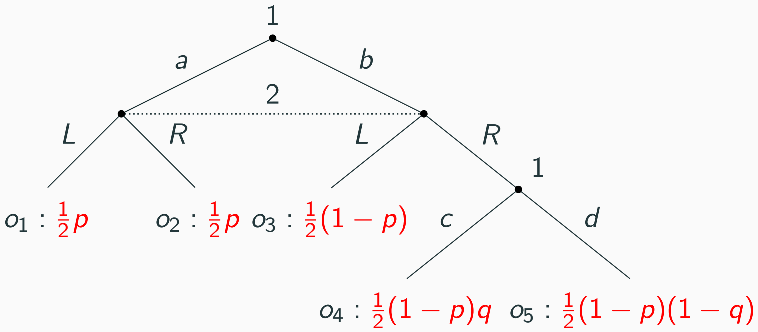
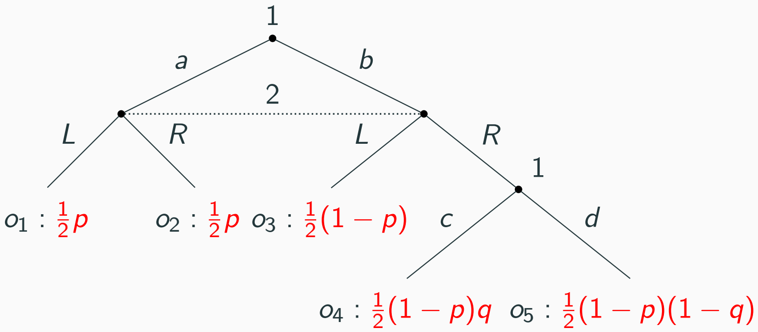


What are Player 1's pure strategies?

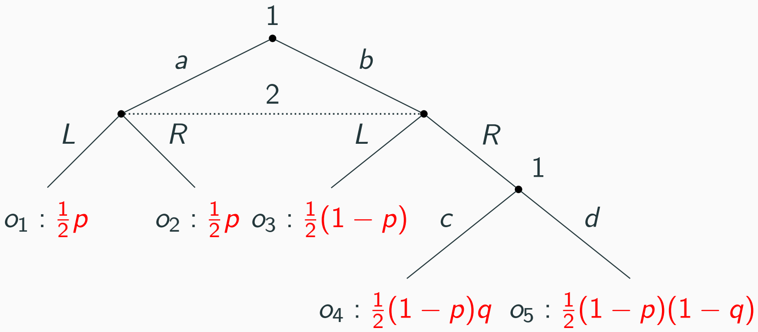
Example: Finding an Outcome-Equivalent Mixed Strategy



What are Player 1's pure strategies?

- (a, c) and (a, d):
- (b, c):
- (b, d):

Example: Finding an Outcome-Equivalent Mixed Strategy



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- (a, c) and (a, d): any two probabilities that sum to p
- (b, c):
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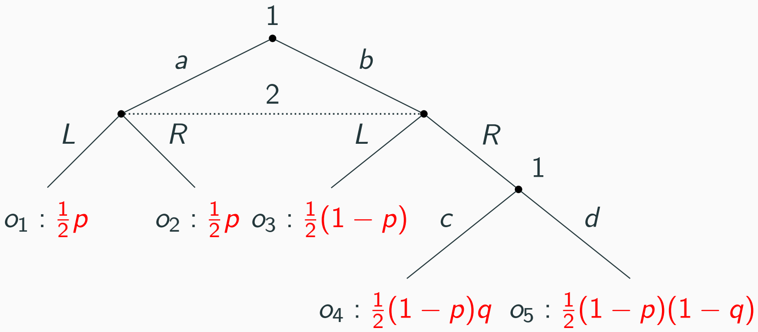
Example: Finding an Outcome-Equivalent Mixed Strategy



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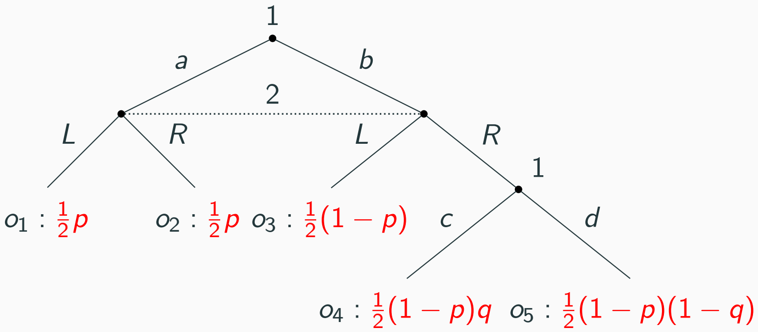
- (b, c): $(1-p)q$
- (b, d):

Example: Finding an Outcome-Equivalent Mixed Strategy



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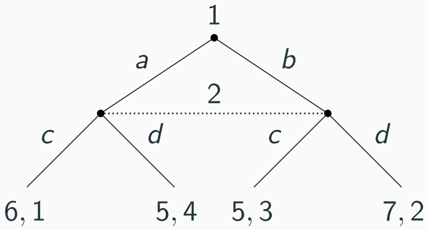
- (a, c) and (a, d): any two probabilities that sum to p
- (b, c): $(1-p)q$
- (b, d): $(1-p)(1-q)$

Example: Solving for Nash Equilibrium in Behavioral Strategies



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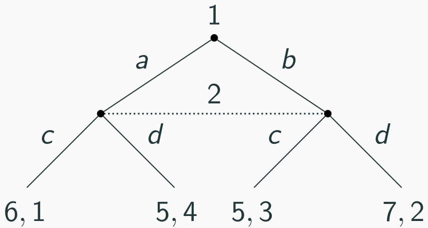

Write the normal form representation:

		Player 2	
		c	d
Player 1	a	6,1	5,4
	b	5,3	7,2

Example: Solving for Nash Equilibrium in Behavioral Strategies

There are no pure strategy Nash equilibria of this game.

		Player 2	
		c	d
Player 1	a	6*,1	5,4*
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There are no pure strategy Nash equilibria of this game.

		Player 2	
		c	d
Player 1	a	6*,1	5,4*
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What about mixed strategies?

Example: Solving for Nash Equilibrium in Behavioral Strategies

Player 2

		c (q)	d ($1-q$)
Player 1	a (p)	6*,1	5,4*
	b ($1-p$)	5,3*	7*,2

Example: Solving for Nash Equilibrium in Behavioral Strategies

		Player 2: c (q)	d ($1-q$)
Player 1	a (p)	6*,1	5,4*
	b ($1-p$)	5,3*	7*,2

$$EU_1(a) = 6q + 5(1-q) \qquad EU_2(c) = p + 3(1-p)$$

$$EU_1(b) = 5q + 7(1-q) \qquad EU_2(d) = 4p + 2(1-p)$$

Example: Solving for Nash Equilibrium in Behavioral Strategies

		Player 2	
		c (q)	d ($1-q$)
Player 1	a (p)	6^*,1	5,4^*
	b ($1-p$)	5,3^*	7^*,2

$$EU_1(a) = 6q + 5(1-q) \qquad EU_2(c) = p + 3(1-p)$$
$$EU_1(b) = 5q + 7(1-q) \qquad EU_2(d) = 4p + 2(1-p)$$

Player 1 is indifferent when $6q + 5(1-q) = 5q + 7(1-q) \rightarrow q = \frac{2}{3}$
Player 2 is indifferent when $p + 3(1-p) = 4p + 2(1-p) \rightarrow p = \frac{1}{4}$

Example: Solving for Nash Equilibrium in Behavioral Strategies

		Player 2	
		c (q)	d ($1-q$)
Player 1	a (p)	6*,1	5,4*
	b ($1-p$)	5,3*	7*,2

$$EU_1(a) = 6q + 5(1-q) \qquad EU_2(c) = p + 3(1-p)$$

$$EU_1(b) = 5q + 7(1-q) \qquad EU_2(d) = 4p + 2(1-p)$$

Player 1 is indifferent when $6q + 5(1-q) = 5q + 7(1-q) \rightarrow q = \frac{2}{3}$
Player 2 is indifferent when $p + 3(1-p) = 4p + 2(1-p) \rightarrow p = \frac{1}{4}$

Note that here, mixed and behavioral strategies are equivalent because each player takes only one turn.

Grossman and Helpman, “Protection for Sale”

How is trade policy constructed in a representative democracy?

A classic example of a political economy model, which combines a political (strategic) model with an economic model.

Consider a democracy with:

- Two-party electoral competition
- Voters who select representatives based on economic policy, other personal characteristics, and campaigns
- Interest groups who seek to influence trade policy in their favor through the use of campaign contributions

When asked why free trade is so often preached and so rarely practiced, most international economists blame “politics.” In representative democracies, governments shape trade policy in response not only to the concerns of the general electorate, but also to the pressures applied by special interests. Interest groups participate in the political process in order to influence policy outcomes. Politicians respond to the incentives they face, trading off the financial and other support that comes from heeding the interest groups’ demands against the alienation of voters that may result from the implementation of socially costly policies.

Grossman and Helpman, "Protection for Sale"

Players:

- Interest groups $i = 1, 2, ..., L$
- Voters $j = 1, 2, ..., N$
- Policy maker (government) A
- A challenger B

There is a vector of prices for goods on the international market $\mathbf{p}^* = p_1^*, p_2^*, ...$ and a vector of domestic prices $\mathbf{p} = p_1, p_2, ...$ that the government can set:

- A domestic price in excess of the world price implies an import tariff and an export subsidy
- A domestic price below the world price corresponds to an import subsidy and an export tariff

Grossman and Helpman, "Protection for Sale"

Interest groups (organized industries) and voters have different preferences over $\mathbf{p}$, but only interest groups are empowered to make campaign contributions.

- Motivation: must be organized to make an impact.

Actions/Sequence of Play:

1. All interest groups simultaneously propose a contribution schedule $C_i(\mathbf{p})$ that maps every feasible vector of domestic prices to a contribution amount
2. The incumbent and challenger propose binding electoral platforms $\mathbf{p}^A$ and $\mathbf{p}^B$, collect the associated campaign contributions, and compete in a majority-rule election
3. The winner of the election implements their platform

Utilities

The interest group chooses a contribution schedule to maximize:

$$U_i = W_i(\mathbf{p}) - C_i(\mathbf{p})$$

Where $W_i(\mathbf{p})$ is a function that returns group i's total welfare from a given set of domestic prices.

- Note: think of all interest groups and voters as participants in the economic marketplace. Thus we can nest an economic model inside W_i.

The policy maker chooses $\mathbf{p}$ to maximize:

$$aW(\mathbf{p}) + \sum_{i \in L} C_i(\mathbf{p})$$

Where $W(\mathbf{p})$ is total welfare and a is a weight. We will derive this from the subgame between voters and politicians in the second period!

The Election Subgame: Setup

Let's focus on the second period: the subgame between voters and politicians. The setup is as follows:

- There are two candidates who compete for office, A (incumbent) and B (challenger).
- A policy platform is a vector $\mathbf{p}^K, K = A, B$
    - Platforms are binding commitments
    - Simultaneous exchange of platforms for campaign contributions before voting takes place (with ample time to campaign)

The Election Subgame: Setup

There are two types of voters: informed and uninformed, in proportion σ and $1 - \sigma$, respectively.

Informed voters judge candidates on economic performance plus other factors. Their utility from candidate K winning the election is given by:

$$U_j(K) = v_j(\mathbf{p}^K) + \eta_j^K$$

Where v_j is their welfare from $\mathbf{p}^K$ and η_j^K represents all other factors (again, v_j condenses an underlying economic model).

Uninformed voters are "impressionable," or influenced by campaign contributions:

$$U_j(K) = C^K + \eta_j^K$$

Thus politicians need some combination of favorable policy and campaign contributions to get the most voters.

Politicians and Informed Voters

This is a probabilistic voting model: η_j^K is a random variable and politicians know its overall distribution but not its value for any particular voter.

Voter j selects the incumbent when:

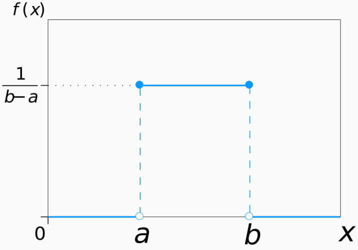
$$v_j(\mathbf{p}^A) + \eta_j^A > v_j(\mathbf{p}^B) + \eta_j^B$$

or otherwise,

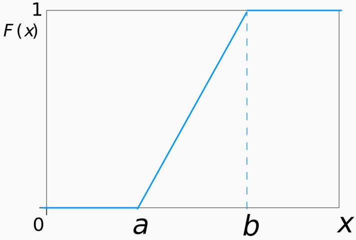
$$v_j(\mathbf{p}^A) - v_j(\mathbf{p}^B) > \eta_j^B - \eta_j^A \equiv \eta_j$$

Let η be distributed uniform on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Aside: The Uniform Distribution



Probability density function:
$Pr(X = x) = \frac{1}{b-a}$ for $x \in [a, b]$



Cumulative density function:
$Pr(X \leq x) = \frac{x-a}{b-a}$ for $x \in [a, b]$

Politicians and Informed Voters

Assume that politicians want to maximize expected vote share.

- In what ways does this make sense?
- In what ways does this not quite make sense?

For incumbent, that means maximizing:

$$Pr\left[\eta < v(\mathbf{p}^A) - v(\mathbf{p}^B)\right] = \frac{v(\mathbf{p}^A) - v(\mathbf{p}^B) - \left(-\frac{1}{2}\right)}{\frac{1}{2} - \left(-\frac{1}{2}\right)}$$

$$= \frac{1}{2} + v(\mathbf{p}^A) - v(\mathbf{p}^B)$$

Putting it Together: Politician's Utility Function

For uninformed voters, we derive a similar result with campaign contributions, such that the incumbent maximizes:

$$\frac{1}{2} + C^A - C^B$$

Weighting by population share of each group, we get utility:

$$\sigma \left(\frac{1}{2} + v(\mathbf{p}^A) - v(\mathbf{p}^B) \right) + (1 - \sigma) \left(\frac{1}{2} + C^A - C^B \right)$$

Focusing on what she can control, the incumbent maximizes:

$$\sigma v(\mathbf{p}^A) + (1 - \sigma) C^A$$

which corresponds to:

$$aW(\mathbf{p}) + \sum_{i \in L} C_i(\mathbf{p}) \quad \text{with } a = \frac{\sigma}{1 - \sigma}$$

Moving One Step Backward: Politician and Interest Groups

We can take the government currently in power, assume an exogenous challenger (that the incumbent can't control), and focus on the game between the politician and the interest groups, taking the politician's utility function as just derived. Thus we have:

Government choosing $\mathbf{p}$ to maximize:

$$aW(\mathbf{p}) + \sum_{i \in L} C_i(\mathbf{p}) \quad \text{with } a = \frac{\sigma}{1-\sigma}$$

Interest groups choosing contribution schedules to maximize:

$$W_i(\mathbf{p}) - C_i(\mathbf{p})$$

Note: The less informed citizens are, the more government is incentivized to pursue campaign contributions/satisfy organized interests

Insights of the Model

Solving even this simpler game requires knowing the functional forms of W (the economic model), the length of $\mathbf{p}$, and the number and membership of interest groups. It gets very complicated...

The game allows us to relate the level of political organization in a society to its equilibrium level of protectionism.

Equilibrium with one interest group:

- The lobby contributes an amount that is proportional to the excess burden that the equilibrium trade policies impose on society
- The factor of proportionality is the weight that the government attaches to aggregate gross welfare (relative to campaign contributions) in its own objective function
- Without competition, government ends up with the same utility as under a free trade regime; lobbying group captures all the surplus

Insights of the Model

Equilibrium with all voters represented by one interest group or another (densely organized society):

- When all voters are active in the process of buying influence, the rivalry among competing interests is most intense
- This rivalry actually results in a free trade outcome
- However, the lobbies make positive contributions and the government captures all of the surplus

The model's predictions seem to hold up to empirical testing (Goldberg and Maggi 1999).