Newton's Divided Difference Polynomial Method of Interpolation

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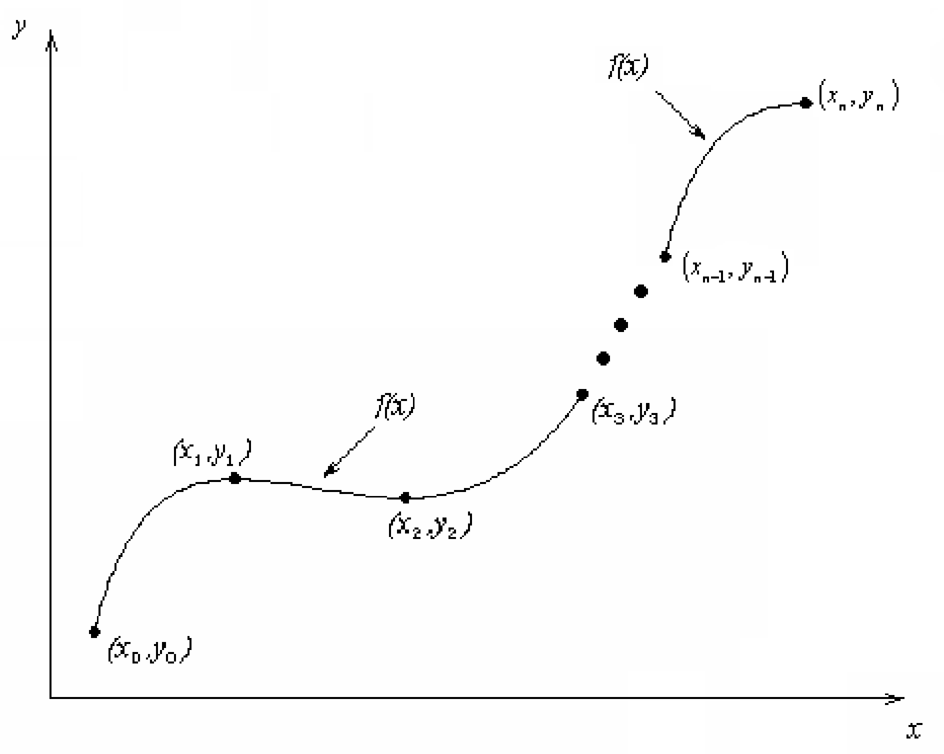
Textbook: TEXTBOOK: NUMERICAL METHODS WITH APPLICATIONS

Newton’s Divided Difference Method of Interpolation

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… ,(x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... ,(x_n,y_n), find the value of 'y' at a value of 'x' that is not given.

Interpolation is the process of estimating unknown values that fall between known values

Application: In engineering and science, one often has a function for a limited number of data points. Interpolation is required for finding the function's value at some intermediate points

Interpolants

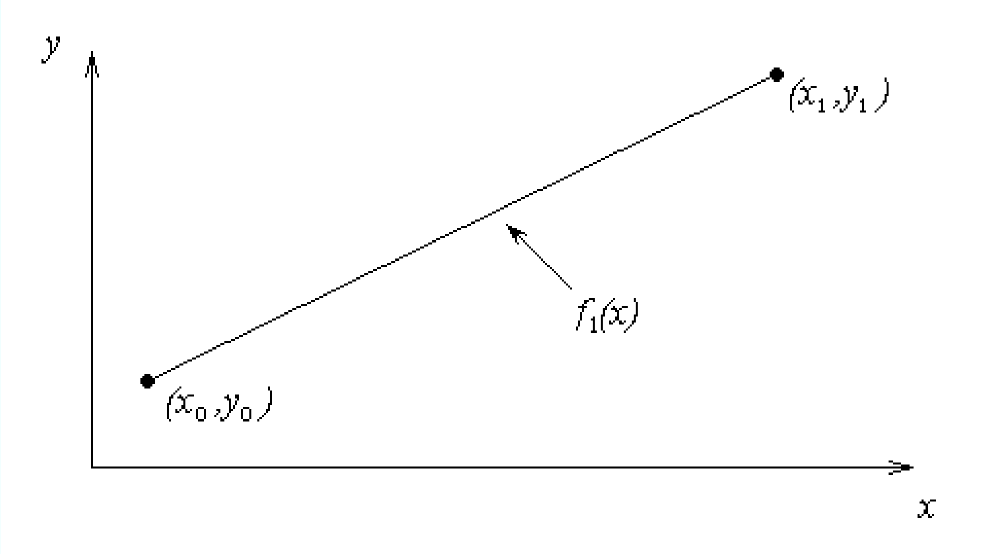
Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

Linear interpolation: Given $(x_0, y_0), (x_1, y_1),$ pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$



where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the Newton Divided Difference method for linear interpolation.

Table. Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

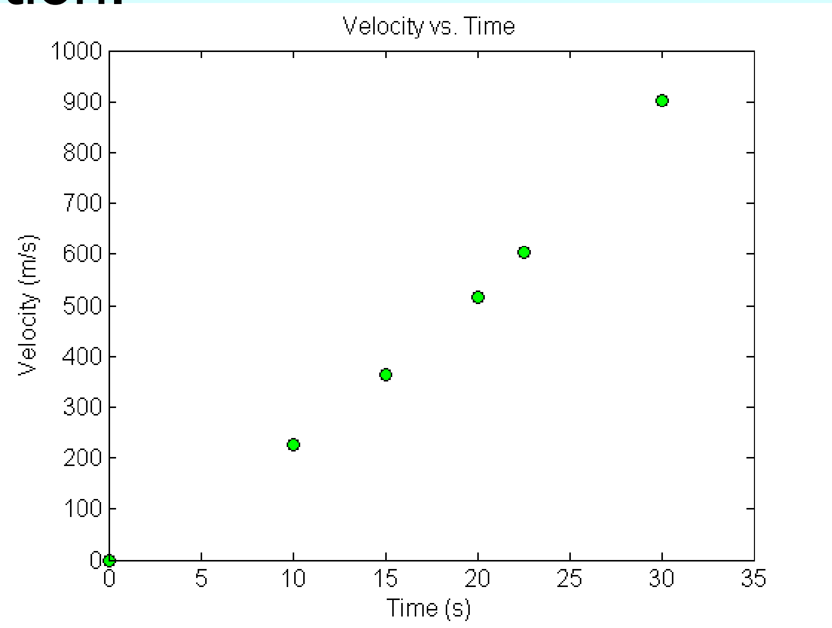


Figure. Velocity vs. time data for the rocket example

Linear Interpolation

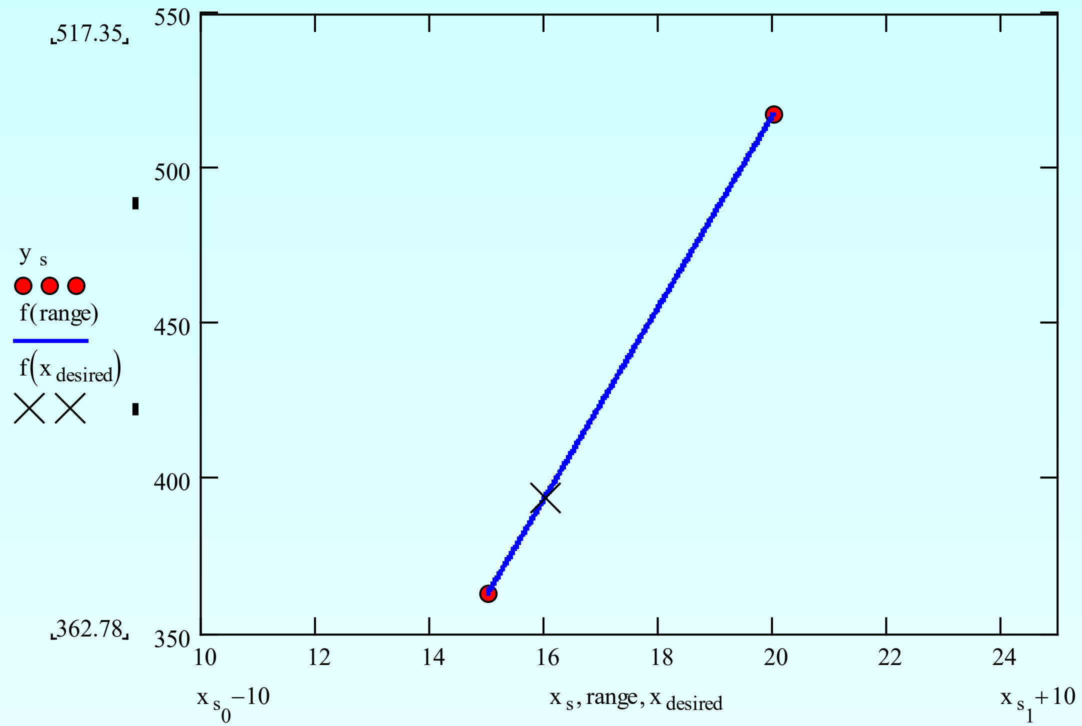
$$v(t) = b_0 + b_1(t - t_0)$$

$$t_0 = 15,\ v(t_0) = 362.78$$

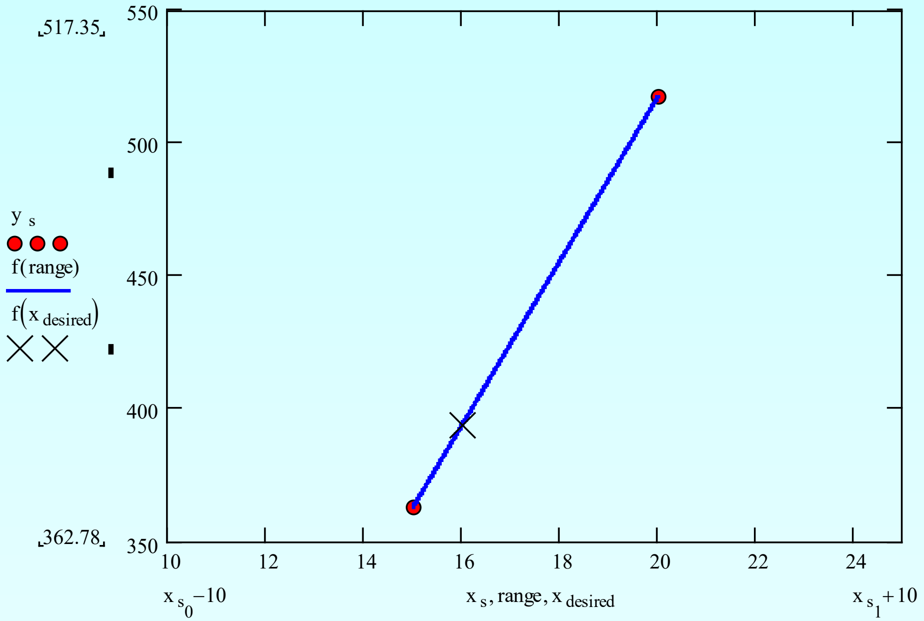
$$t_1 = 20,\ v(t_1) = 517.35$$

$$b_0 = v(t_0) = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = 30.914$$



Linear Interpolation (contd)



$$v(t) = b_0 + b_1(t - t_0)$$

$$= 362.78 + 30.914(t - 15),\ 15 \le t \le 20$$

At $t = 16$

$$v(16) = 362.78 + 30.914(16 - 15)$$

$$= 393.69 \text{ m/s}$$

Quadratic Interpolation

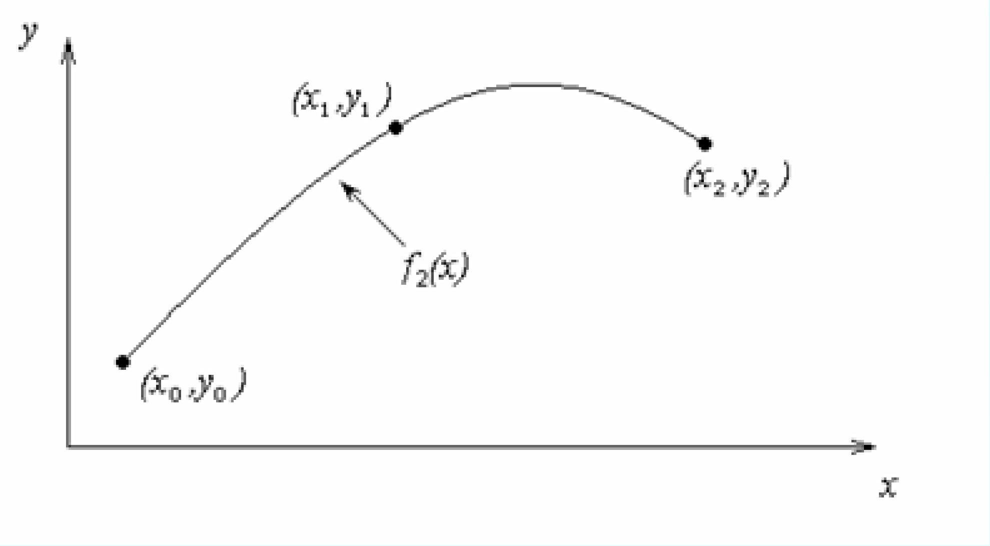
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the Newton Divided Difference method for quadratic interpolation.

Table. Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

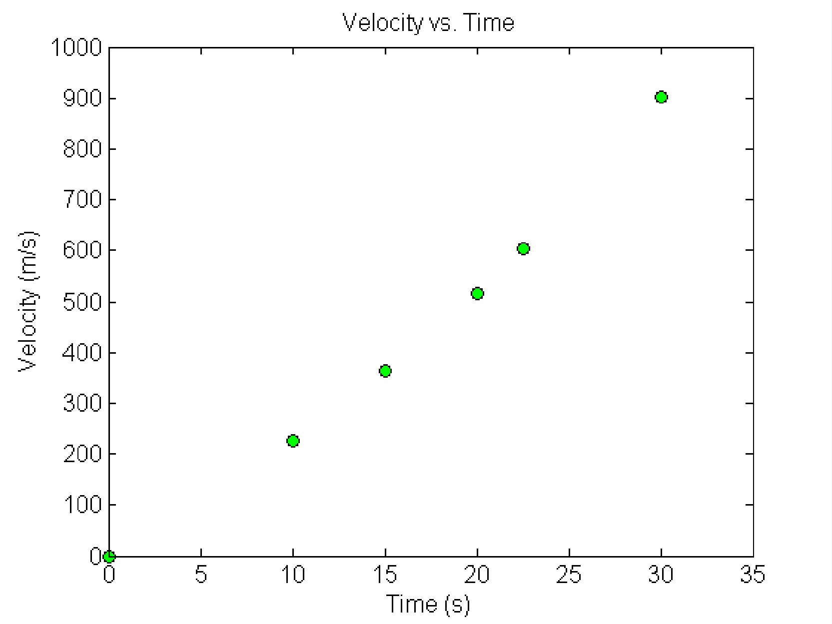
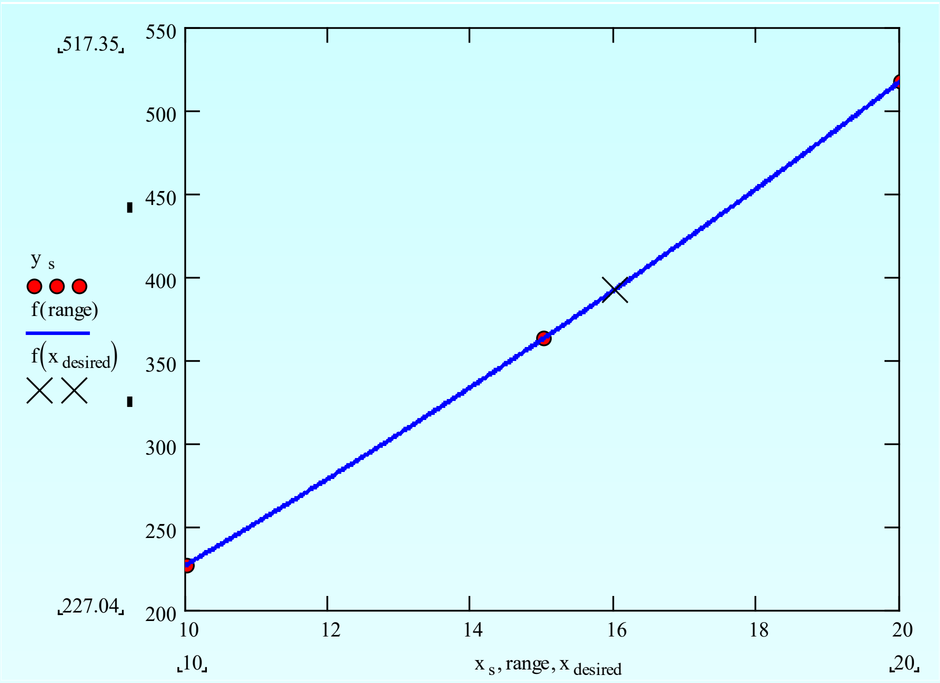


Figure. Velocity vs. time data for the rocket example

Quadratic Interpolation (contd)



$$t_0 = 10,\ v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$

$$t_2 = 20,\ v(t_2) = 517.35$$

Quadratic Interpolation (contd)

$$b_0 = v(t_0)$$

$$= 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10}$$

$$= 27.148$$

$$b_2 = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{\dfrac{517.35 - 362.78}{20 - 15} - \dfrac{362.78 - 227.04}{15 - 10}}{20 - 10}$$

$$= \frac{30.914 - 27.148}{10}$$

$$= 0.37660$$

Quadratic Interpolation (contd)

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \qquad 10 \le t \le 20$$

At $t = 16,$

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) \; = 392.19 \text{ m/s}$$

The absolute relative approximate error $\left|\in_a\right|$ obtained between the results from the first order and second order polynomial is

$$\left|\in_a\right| = \left|\frac{392.19 - 393.69}{392.19}\right| \text{x}100$$

$$= 0.38502 \ \%$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n+1)$ data points, $\left(x_0, y_0\right), \left(x_1, y_1\right), \ldots\ldots, \left(x_{n-1}, y_{n-1}\right), \left(x_n, y_n\right)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

General form

The third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), is

$$
\begin{aligned}
f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \\
+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)
\end{aligned}
$$

	b_0			
x_0	$f(x_0)$	b_1		
		$f[x_1, x_0]$	b_2	
x_1	$f(x_1)$		$f[x_2, x_1, x_0]$	b_3
		$f[x_2, x_1]$		$f[x_3, x_2, x_1, x_0]$
x_2	$f(x_2)$		$f[x_3, x_2, x_1]$	
		$f[x_3, x_2]$		
x_3	$f(x_3)$			

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the Newton Divided Difference method for cubic interpolation.

Table. Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

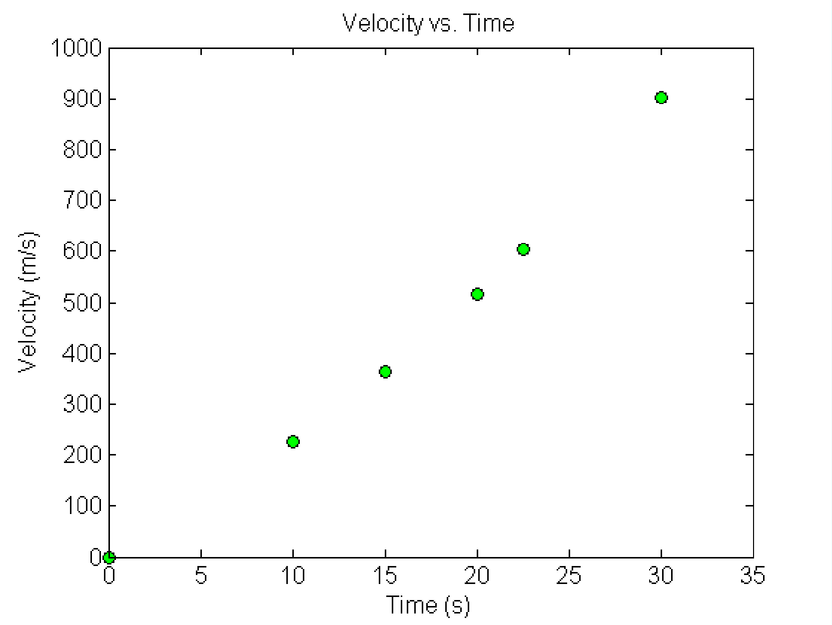


Figure. Velocity vs. time data for the rocket example

Example

The velocity profile is chosen as

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

we need to choose four data points that are closest to $t = 16$

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

$$t_2 = 20, \quad v(t_2) = 517.35$$

$$t_3 = 22.5, \quad v(t_3) = 602.97$$

The values of the constants are found as:

$$b_0 = 227.04; \ b_1 = 27.148; \ b_2 = 0.37660; \ b_3 = 5.4347 \times 10^{-3}$$

Example

$$
\begin{array}{llllll}
t_0 = 10 & 227.04 & & & \\
 & & 27.148 & & \\
t_1 = 15, & 362.78 & & 0.37660 & \\
 & & 30.914 & & 5.4347\times10^{-3} \\
t_2 = 20, & 517.35 & & 0.44453 & \\
 & & 34.248 & & \\
t_3 = 22.5, & 602.97 & & & \\
\end{array}
$$

$b_0 = 227.04$; $b_1 = 27.148$; $b_2 = 0.37660$; $b_3 = 5.4347\times10^{-3}$

Example

Hence

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$
$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15)$$
$$+ 5.4347 * 10^{-3}(t - 10)(t - 15)(t - 20)$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15)$$
$$+ 5.4347 * 10^{-3}(16 - 10)(16 - 15)(16 - 20)$$
$$= 392.06 \text{ m/s}$$

The absolute relative approximate error $|\in_a|$ obtained is

$$|\in_a| = \left|\frac{392.06 - 392.19}{392.06}\right| \text{x}100$$

$$= 0.033427 \ \%$$

Comparison Table

Order of Polynomial	1	2	3
v(t=16) m/s	393.69	392.19	392.06
Absolute Relative Approximate Error	----------	0.38502 %	0.033427 %

Distance from Velocity Profile

Find the distance covered by the rocket from t=11s to t=16s ?

$$v(t) = 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15) + 5.4347*10^{-3}(t-10)(t-15)(t-20) \qquad 10 \le t \le 22.5$$

$$= -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3 \qquad 10 \le t \le 22.5$$

So

$$s(16) - s(11) = \int_{11}^{16} v(t)\,dt$$

$$= \int_{11}^{16} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3)\,dt$$

$$= \left[-4.2541t + 21.265\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4} \right]_{11}^{16}$$

$$= 1605\ m$$

Acceleration from Velocity Profile

Find the acceleration of the rocket at t=16s given that

$$v(t) = -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}\left(-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3\right)$$

$$= 21.265 + 0.26408t + 0.016304t^2$$

$$a(16) = 21.265 + 0.26408(16) + 0.016304(16)^2$$

$$= 29.664\, m/s^2$$

THE END