

# CS231a HW2

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\*Adapted from last year's slides



# Overview

Focus on

1. Problem 2 - Image Rectification
2. Problem 4 - Structure from Motion

Will briefly cover Problem 1 - Fundamental Matrix Estimation

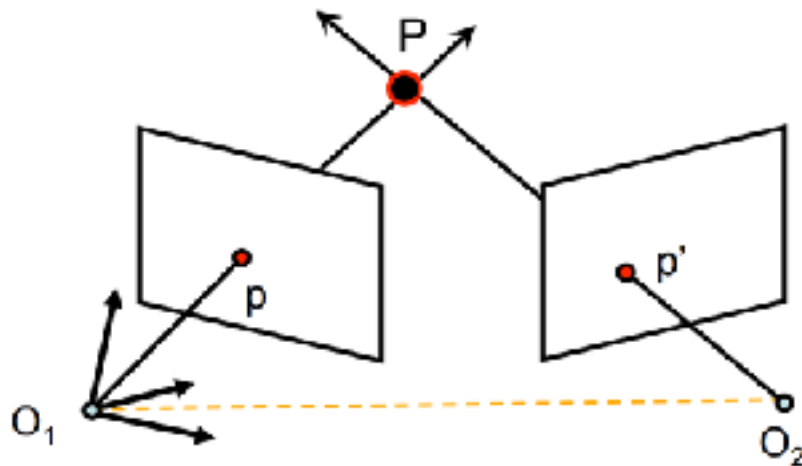
Will not cover Problem 3 - Factorization Method

(Enough details covered in class. Please visit our office hour if you need help)

# Problem 1- Fundamental Matrix Estimation

## Fundamental Matrix

- A matrix which maps the relationship of correspondences between stereo images



[Eq. 13]

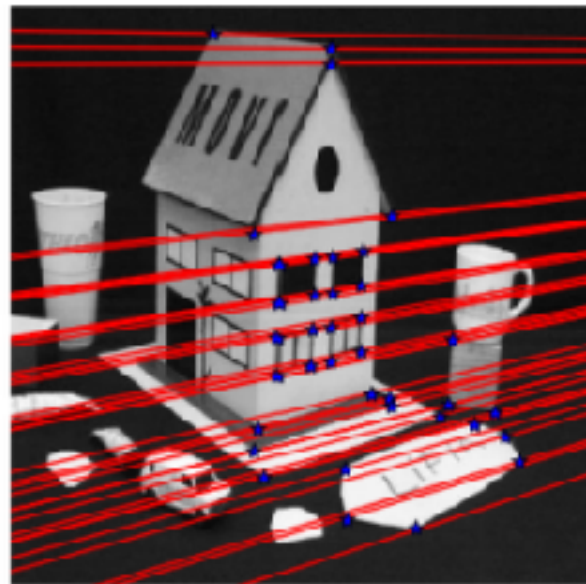
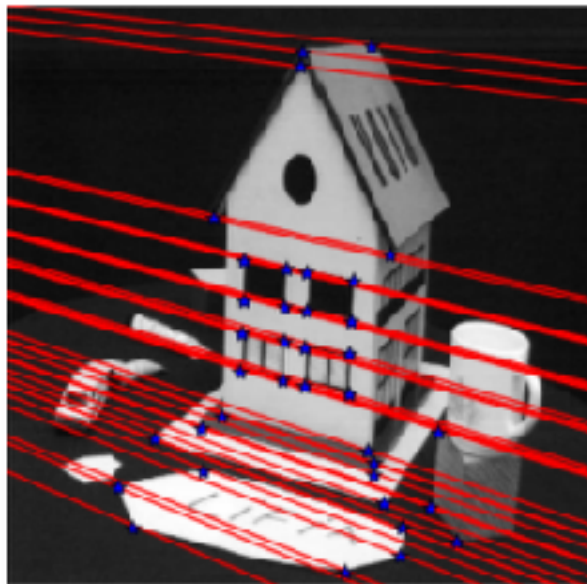
$$p^T F p' = 0$$

$$F = K^{-T} \cdot [T_x] \cdot R K'^{-1}$$

# Problem 1- Fundamental Matrix Estimation

Ex)

Image and  
correspondences  
given in the  
homework



# Problem 1- Fundamental Matrix Estimation

How to compute F?

- Eight point algorithm

$$\text{[Eq. 13]} \quad \mathbf{p}^T \mathbf{F} \mathbf{p}' = 0 \quad \Rightarrow \quad \mathbf{p} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad \mathbf{p}' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$(u, v, 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow (uu', uv', u, vu', vv', v, u', v', 1) \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

Let's take 8 corresponding points

[Eq. 14]

# Problem 1- Fundamental Matrix Estimation

How to compute F?

- Eight point algorithm

Problem?

- W is highly unbalanced  
(not well conditioned)

$$\mathbf{W}\mathbf{f} = 0$$

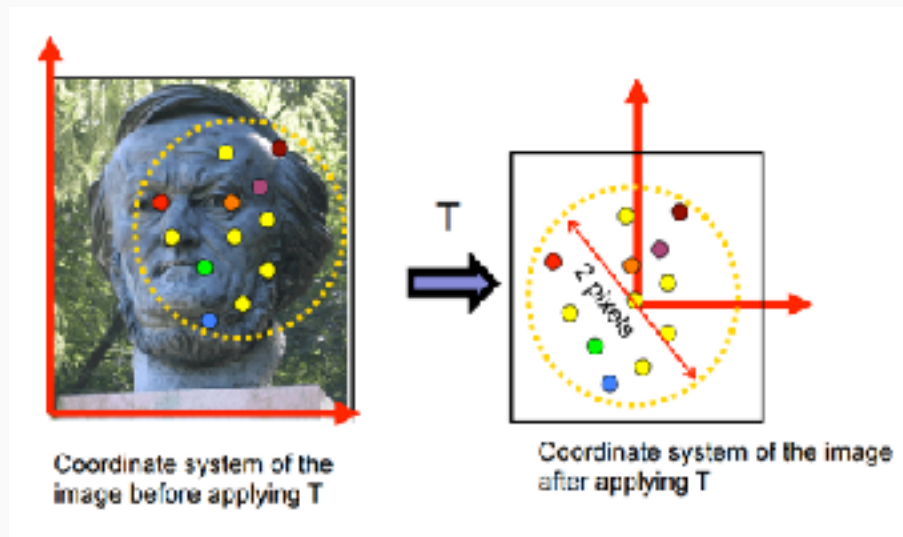
$$\begin{pmatrix} u_1u_1' & u_1v_1' & u_1 & v_1u_1' & v_1v_1' & v_1 & u_1' & v_1' & 1 \\ u_2u_2' & u_2v_2' & u_2 & v_2u_2' & v_2v_2' & v_2 & u_2' & v_2' & 1 \\ u_3u_3' & u_3v_3' & u_3 & v_3u_3' & v_3v_3' & v_3 & u_3' & v_3' & 1 \\ u_4u_4' & u_4v_4' & u_4 & v_4u_4' & v_4v_4' & v_4 & u_4' & v_4' & 1 \\ u_5u_5' & u_5v_5' & u_5 & v_5u_5' & v_5v_5' & v_5 & u_5' & v_5' & 1 \\ u_6u_6' & u_6v_6' & u_6 & v_6u_6' & v_6v_6' & v_6 & u_6' & v_6' & 1 \\ u_7u_7' & u_7v_7' & u_7 & v_7u_7' & v_7v_7' & v_7 & u_7' & v_7' & 1 \\ u_8u_8' & u_8v_8' & u_8 & v_8u_8' & v_8v_8' & v_8 & u_8' & v_8' & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

# Problem 1- Fundamental Matrix Estimation

Possible improvement?

Pre-condition our linear system to get more stable result

- origin = centroid of the image
- mean square distance of the image points from origin is  $\sim 2\text{px}$



# Problem 1- Fundamental Matrix Estimation

Final step

- Reduce rank(F) to 2

Find  $F$  that minimizes  $\|F - \hat{F}\|$  Frobenius norm (\*)  
Subject to  $\det(F)=0$

$$F = U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

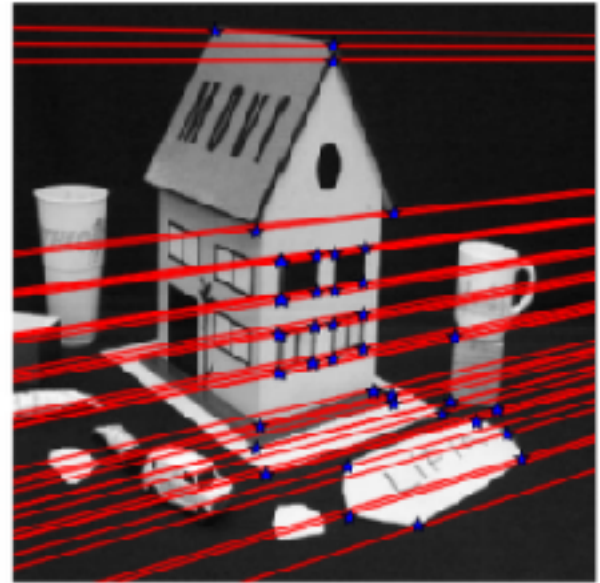
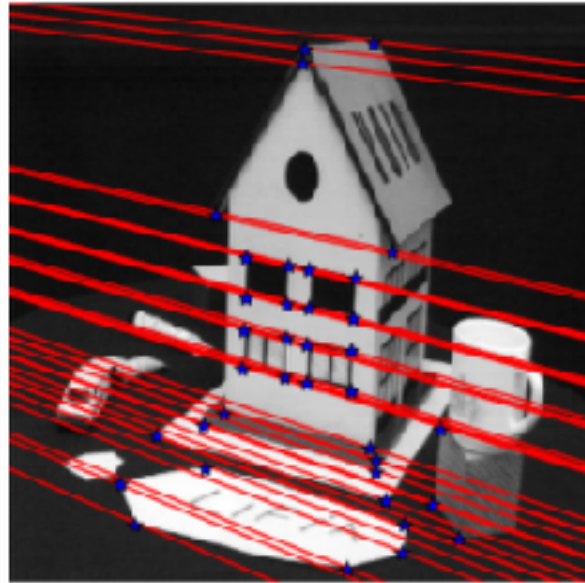
Where:

$$U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^T = SVD(\hat{F})$$



# Problem 1- Fundamental Matrix Estimation

Epipolar lines

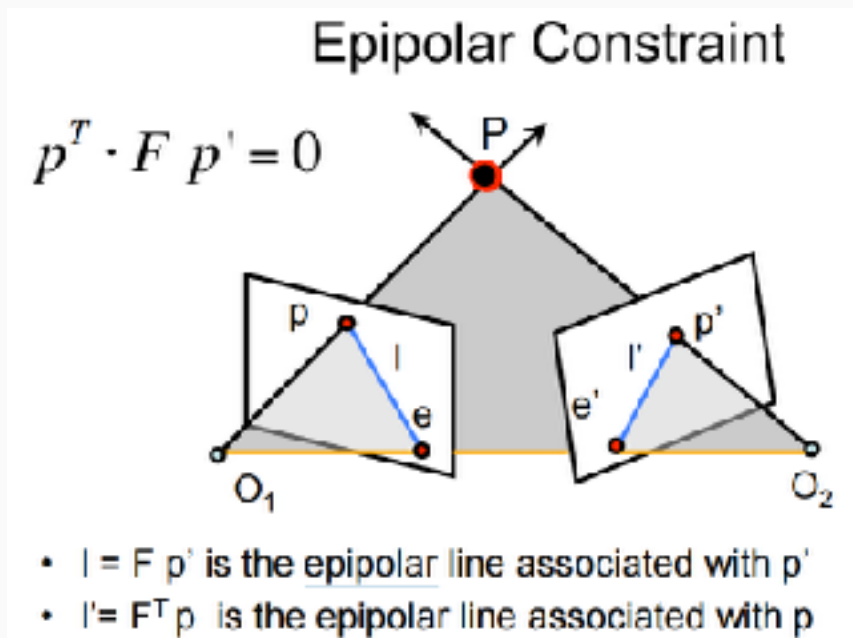


# Problem 1- Fundamental Matrix Estimation

Computing epipolar  
lines from  $F$

$$l = Fp'$$

$$l' = F^T p$$

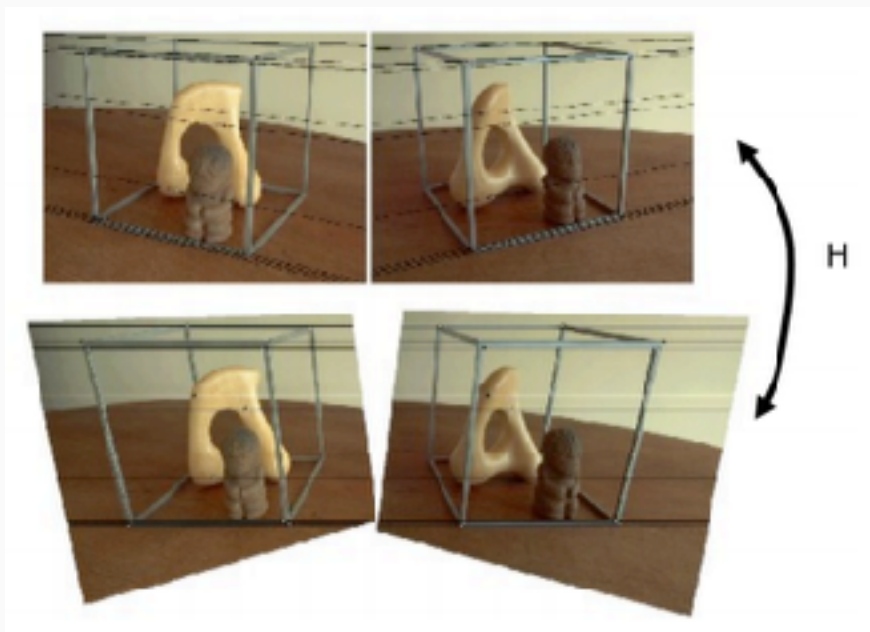


# Problem 2 - Image Rectification

Image Rectification

Making two images “parallel”

All correspondences lie on the same  $y$  axis



# Problem 2 - Image Rectification

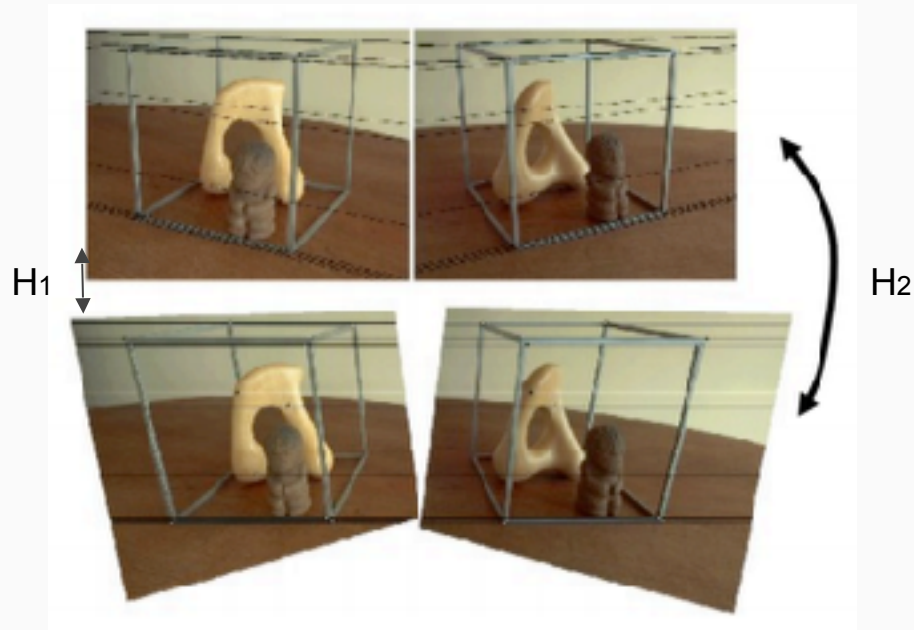
Application

Novel view synthesis  
(view morphing)  
based on rectified  
images



# Problem 2 - Image Rectification

Find a homography  $H_1$  and  $H_2$  such that corresponding points of image 1 and image 2 lie on the same  $y$  axis



# Problem 2 - Image Rectification

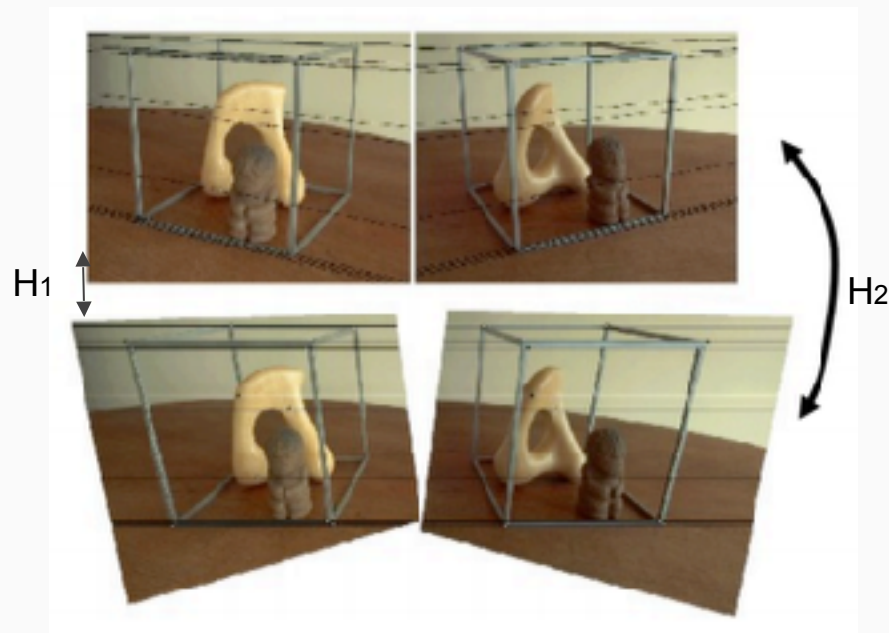
## Overview

- Find  $H_2$ : morph image 2 to make all epipolar lines parallel
  - Find epipole of image 2 ( $e_2$ )
  - Find homography  $H_2$  which brings  $e_2$  to a point at infinity
- Find  $H_1$ : minimize square distance from rectified image 2 to image 1

# Problem 2 - Image Rectification

Find a homography  $H_1$  and  $H_2$  such that corresponding points of image 1 and image 2 lie on the same y axis

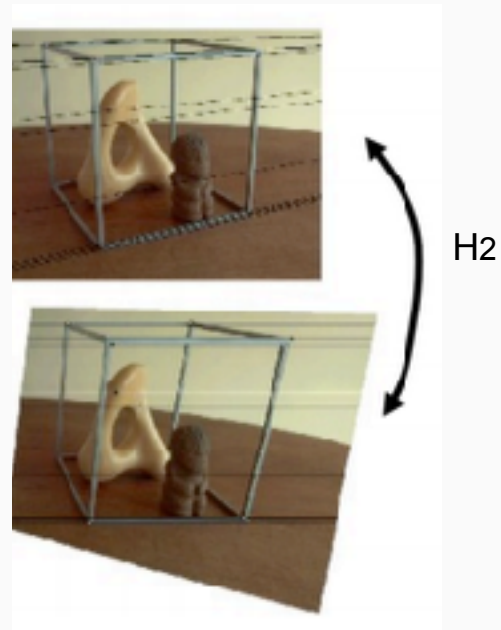
1. Find  $H_2$  such that epipolar lines are all horizontally aligned (parallel)
2. Find  $H_1$  which minimizes square distance between corresponding points of the image



# Problem 2 - Image Rectification

Find a homography  $H_1$  and  $H_2$  such that corresponding points of image 1 and image 2 lie on the same  $y$  axis

1. Find  $H_2$  such that epipolar lines are all horizontally aligned (parallel)
2. Find  $H_1$  which minimizes square distance between corresponding points of the image

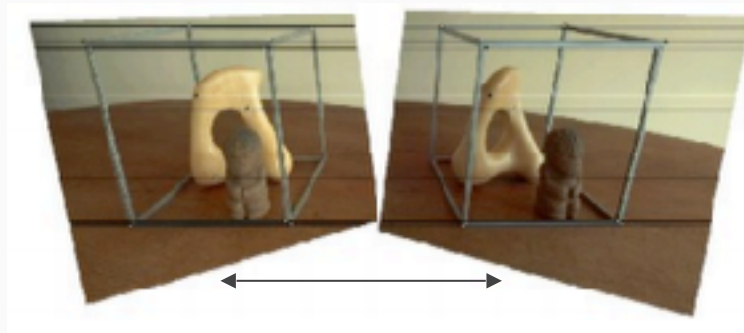




# Problem 2 - Image Rectification

Find a homography  $H_1$  and  $H_2$  such that corresponding points of image 1 and image 2 lie on the same y axis

1. Find  $H_2$  such that epipolar lines are all horizontally aligned (parallel)
2. **Find  $H_1$  which minimizes square distance between corresponding points of the image**



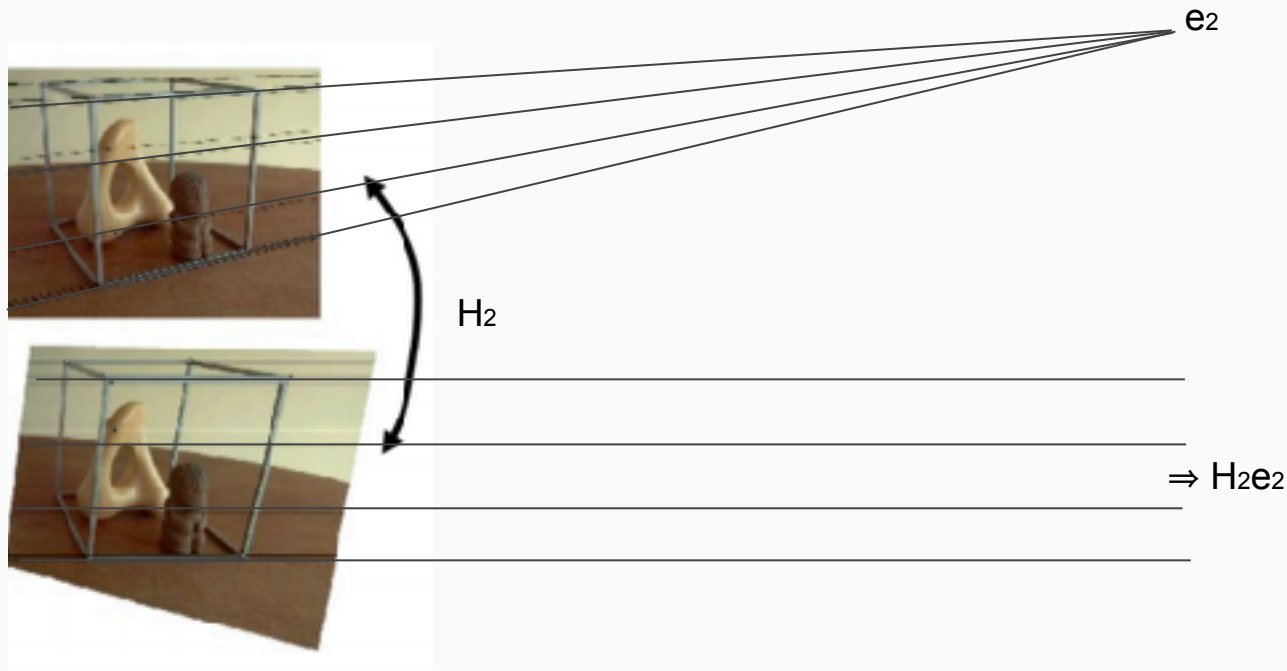
# Problem 2 - Image Rectification

## Overview

- **Find  $H_2$ : morph image 2 to make all epipolar lines parallel**
  - Find epipole of image 2 ( $e_2$ )
  - Find homography  $H_2$  which brings  $e_2$  to a point at infinity
- Find  $H_1$ : minimize square distance from rectified image 2 to image 1

# Problem 2 - Image Rectification

1. Find epipole of image 2,  $e_2$
2. Find homography  $H_2$  which brings  $e_2$  to a point at infinity



# Problem 2 - Image Rectification

## Overview

- Find  $H_2$ : morph image 2 to make all epipolar lines parallel
  - **Find epipole of image 2 ( $e_2$ )**
  - Find homography  $H_2$  which brings  $e_2$  to a point at infinity
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# Problem 2 - Image Rectification

Find epipole of image 2 ( $e_2$ )

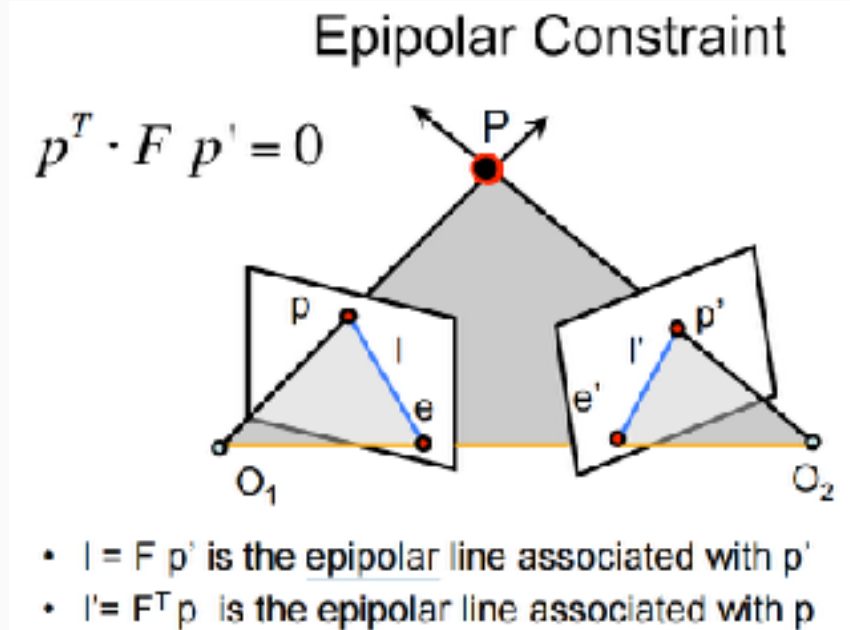
- Epipole is an intersection of epipolar lines
  - $\Leftrightarrow$  epipole lies on every epipolar lines
  - defining epipolar line  $\ell$  such that all points on the line are in set  $\{x | \ell^T x = 0\}$ , formulate a linear system of equations on the right
  - Solve using SVD

$$\begin{bmatrix} \ell_1^T \\ \vdots \\ \ell_n^T \end{bmatrix} e = 0$$

# Problem 2 - Image Rectification

Computing epipolar lines from  $F$

$$l = Fp'$$
$$l' = F^T p$$



# Problem 2 - Image Rectification

## Overview

- Find  $H_2$ : morph image 2 to make all epipolar lines parallel
  - Find epipole of image 2 ( $e_2$ )
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# Problem 2 - Image Rectification

Find homography  $H_2$  which brings  $e_2$  to a point at infinity

Intuition (from Hartley, Zisserman):

To keep image as realistic as possible after transformation, we want to keep  $H_2$  to act as a rigid transformation in the neighborhood of a given selected point  $x_0$  of the image

Center of the image is often a good choice for  $x_0$



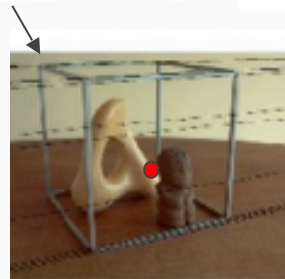
# Problem 2 - Image Rectification

Find homography  $H_2$  which brings  $e_2$  to a point at infinity

1. **Translate image coordinate so that the origin will be at the image center**
2. Rotate the image so that  $e_2$  will lie horizontal axis at some point  $(f, 0, 1)$
3. Bring  $e_2$  to infinity on  $(f, 0, 0)$
4. Translate image coordinate back to the original origin



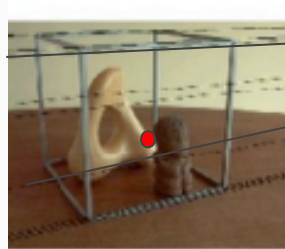
$$T = \begin{bmatrix} 1 & 0 & -\frac{\text{width}}{2} \\ 0 & 1 & -\frac{\text{height}}{2} \\ 0 & 0 & 1 \end{bmatrix}$$



# Problem 2 - Image Rectification

Find homography  $H_2$  which brings  $e_2$  to a point at infinity

1. Translate image coordinate so that the origin will be at the image center
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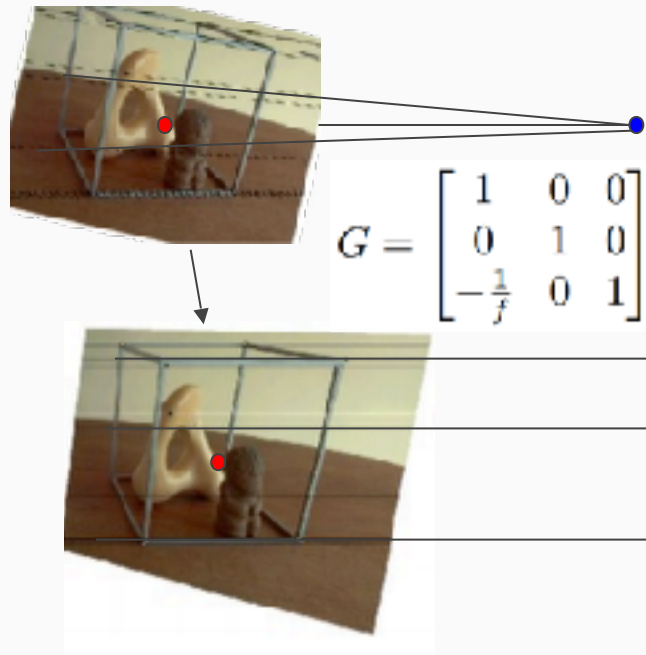
$$H = \begin{bmatrix} \alpha \frac{e_1}{\sqrt{e_1 + e_2}} & \alpha \frac{e_2}{\sqrt{e_1 + e_2}} & 0 \\ -\alpha \frac{e_2}{\sqrt{e_1 + e_2}} & \alpha \frac{e_1}{\sqrt{e_1 + e_2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Problem 2 - Image Rectification

Find homography  $H_2$  which brings  $e_2$  to a point at infinity

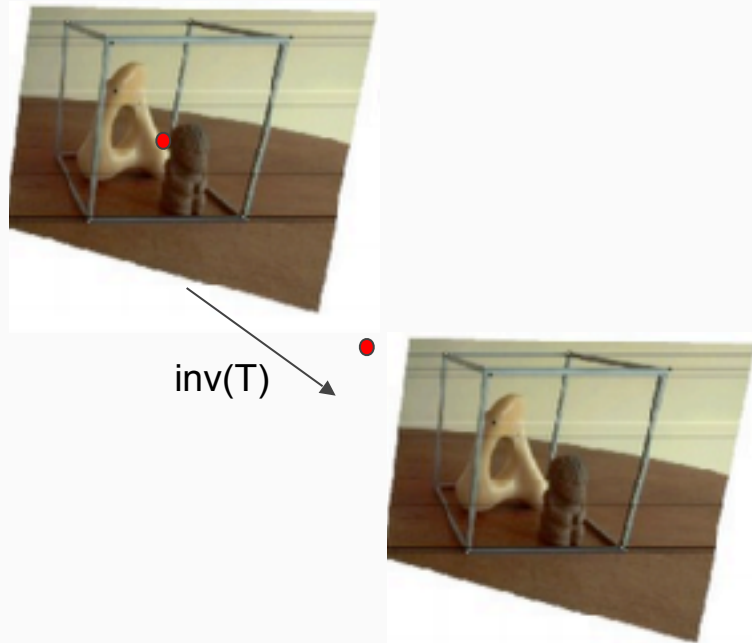
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# Problem 2 - Image Rectification

Find homography  $H_2$  which brings  $e_2$  to a point at infinity

1. Translate image coordinate so that the origin will be at the image center
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4. **Translate image coordinate back to the original origin**



## Problem 2 - Image Rectification

Find homography  $H_2$  which brings  $e_2$  to a point at infinity

$$H_2 = T^{-1}GRT$$

# Problem 2 - Image Rectification

## Overview

- Find  $H_2$ : morph image 2 to make all epipolar lines parallel
  - Find epipole of image 2 ( $e_2$ )
  - Find homography  $H_2$  which brings  $e_2$  to a point at infinity
- **Find  $H_1$ : minimize square distance from rectified image 2 to image 1**

# Problem 2 - Image Rectification

Find  $H_1$ : minimize square distance from rectified image 2 to image 1

$$\arg \min_{H_1} \sum_i \|H_1 p_i - H_2 p'_i\|^2$$

Why this?  $H_1^{-T} \ell_i = H_2^{-T} \ell'_i$  Reduce matching lines to matching points

We will take a simpler approach for this problem

# Problem 2 - Image Rectification

Find  $H_1$ : minimize square distance from rectified image 2 to image 1

We can show that  $H_1 = H_A H_2 M$

where

$$H_A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This form of  $H_A$  allows a transformation of  $H_1$  where its epipole is at  $(1, 0, 0)$  - a page long proof of this in Chapter 11 of Hartley & Zisserman's textbook



# Problem 2 - Image Rectification

Find  $H_1$ : minimize square distance from rectified image 2 to image 1

We can also show that  $M = [e]_{\times} F + ev^T$

because 1)  $M = [e]_{\times} F$

- we know  $F$  up to scale
- any skew-symmetric matrix  $X$  (including  $[e]_{\times}$ ) is  $A = A^3$  up to scale

$$F = [e]_{\times} M = [e]_{\times} [e]_{\times} [e]_{\times} M = [e]_{\times} [e]_{\times} F$$

# Problem 2 - Image Rectification

Find  $H_1$ : minimize square distance from rectified image 2 to image 1

We can also show that  $M = [e]_{\times} F + ev^T$

because 2) if columns of  $M$  are added by any scalar multiple of  $e$ ,

up to scale,  $F = [e]_{\times} M$

Therefore,  $M = [e]_{\times} F + ev^T$  is more general case of defining  $M$

Where in practice, we use  $v = [1 \ 1 \ 1]$

# Problem 2 - Image Rectification

Find  $H_1$ : minimize square distance from rectified image 2 to image 1

Which reduces to

$$\arg \min_{H_A} \sum_i \|H_A \hat{p}_i - \hat{p}'_i\|^2$$

where  $\hat{p}_i = H_2 M p_i$   $\hat{p}'_i = H_2 p_i$

# Problem 2 - Image Rectification

Find  $H_1$ : minimize square distance from rectified image 2 to image 1

Which reduces down to solving least squares  $W[a_1, a_2, a_3]^T = b$

Where

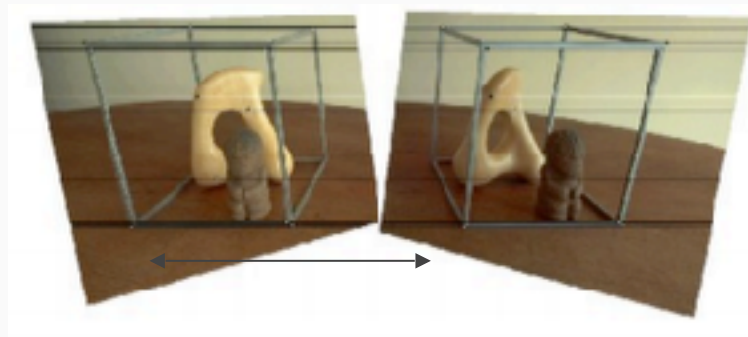
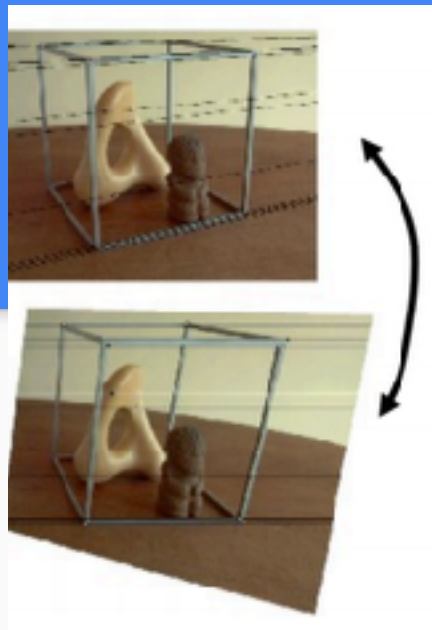
$$W = \begin{bmatrix} \hat{x}_1 & \hat{y}_1 & 1 \\ & \vdots & \\ \hat{x}_n & \hat{y}_n & 1 \end{bmatrix} \quad b = \begin{bmatrix} \hat{x}'_1 \\ \vdots \\ \hat{x}'_n \end{bmatrix}$$

and  $\hat{p}_i = H_2 M p_i$   $\hat{p}'_i = H_2 p_i$

# Problem 2 - Image Rectification

Find a homography  $H_1$  and  $H_2$  such that corresponding points of image 1 and image 2 lie on the same y axis

1. Find  $H_2$  such that epipolar lines are all horizontally aligned (parallel)
2. Find  $H_1$  which minimizes square distance between corresponding points of the image



# Problem 4: Structure from Motion

Structure from Motion (SfM)

Estimating 3D structure from  
2D images that may be  
coupled with local motions

Input: 2D images

Output: 3D structure  
(+ camera extrinsic)



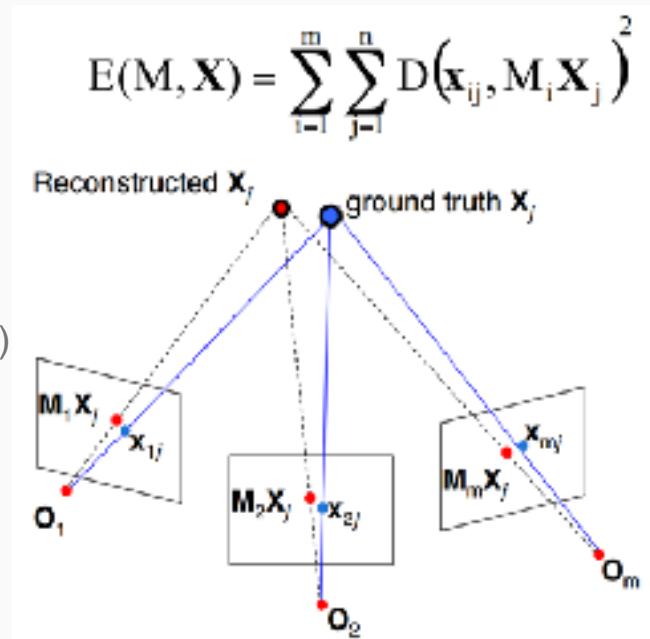
# Problem 4: Structure from Motion

In this homework, we explore two different approaches for SfM

- Factorization Method (problem 3) - Tomasi & Kanade algorithm
- **Bundle Adjustment (problem 4)**

# Problem 4: Structure from Motion

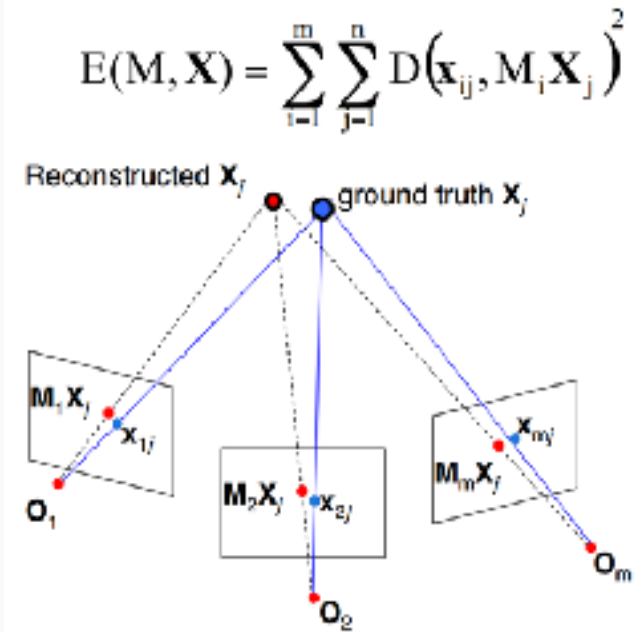
1. Compute essential matrix  $E$  from two views
2. Use  $E$  to make initial estimate of relative rotation  $R$  and translation  $T$
3. Estimate 3D location of the reconstruction given RT
4. Optimize (bundle adjustment)
  - Jointly optimize all relative camera motions ( $R$ 's and  $T$ 's)
  - Minimize total reprojection error with respect to all 3D point and camera parameters
5. Repeat 3 and 4 for pairs of frames





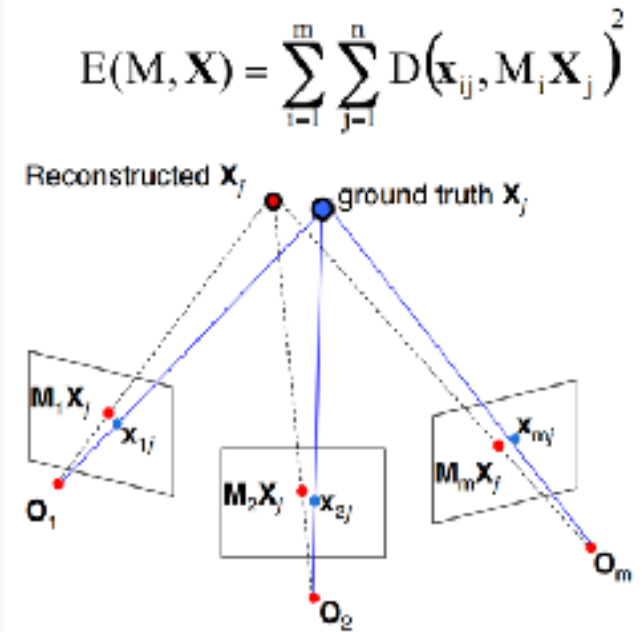
# Problem 4: Structure from Motion

1. Compute essential matrix  $E$  from two views
2. **Use  $E$  to make initial estimate of relative rotation  $R$  and translation  $T$**
3. **Estimate 3D location of the reconstruction given  $RT$**
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  - Jointly optimize all relative camera motions ( $R$ 's and  $T$ 's)
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# Problem 4: Structure from Motion

1. Compute essential matrix  $E$  from two views
2. Use  $E$  to make initial estimate of relative rotation  $R$  and translation  $T$
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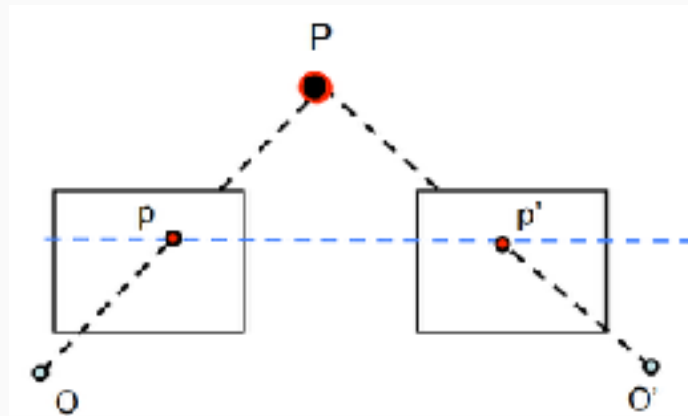


# Problem 4: Structure from Motion

Estimate 3D location of the reconstruction  
given 1. projective camera matrix 2. their image  
coordinates

Two different possible approaches:

1. Formulating a linear equation to solve
2. Nonlinear optimization to minimize reprojection error



# Problem 4: Structure from Motion

Estimate 3D location of the reconstruction  
given 1. projective camera matrix 2. their image coordinates

Formulating a linear equation to solve, using  $[p=MP]$

Solve for  $AP=0$  (using SVD) where:

$$A = \begin{bmatrix} p_{1,1}m^3 - m^1 \\ p_{1,2}m^3 - m^2 \\ \vdots \\ p_{n,1}m^3 - m^1 \\ p_{n,2}m^3 - m^2 \end{bmatrix}$$

$p_{ij}$ :  $(x, y)[j]$  coordinate of  $i$ th image  
 $m^{\{k\}}$ :  $k$ -th row of  $M$

# Problem 4: Structure from Motion

Estimate 3D location of the reconstruction  
given 1. projective camera matrix 2. their image coordinates

Nonlinear optimization to minimize reprojection error

Gauss-Newton algorithm

$$\hat{P} = \hat{P} - (J^T J)^{-1} J^T e$$

Begin from linear estimation for better initialization

How to define error and Jacobian?

# Problem 4: Structure from Motion

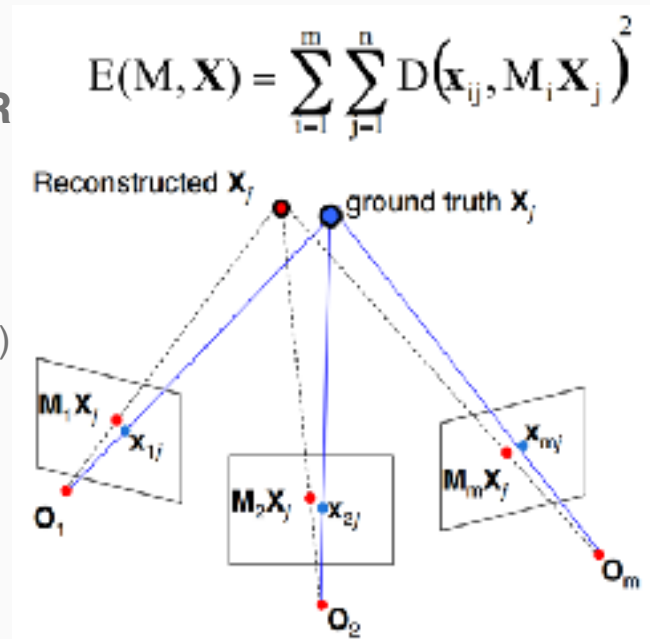
(reprojection) error: difference between the projected point (MiP) and ground-truth image coordinate  $p_i$

Jacobian:

$$J = \begin{bmatrix} \frac{\partial e_1}{\partial P_1} & \frac{\partial e_1}{\partial P_2} & \frac{\partial e_1}{\partial P_3} \\ \vdots & \vdots & \vdots \\ \frac{\partial e_m}{\partial P_1} & \frac{\partial e_m}{\partial P_2} & \frac{\partial e_m}{\partial P_3} \end{bmatrix}$$

# Problem 4: Structure from Motion

1. Compute essential matrix  $E$  from two views
2. **Use  $E$  to make initial estimate of relative rotation  $R$  and translation  $T$**
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  - Jointly optimize all relative camera motions ( $R$ 's and  $T$ 's)
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# Problem 4: Structure from Motion

Use  $E$  to make initial estimate of relative rotation  $R$  and translation  $T$

$$E = [T_x] \cdot R$$

1. To compute  $R$ : Given the singular value decomposition  $E = UDV^T$ , we can rewrite  $E = MQ$  where  $M = UZU^T$  and  $Q = UWV^T$  or  $UW^TV^T$ , where

$$Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that this factorization of  $E$  only guarantees that  $Q$  is orthogonal. To find a rotation, we simply compute  $R = (\det Q)Q$ .

2. To compute  $T$ : Given that  $E = U\Sigma V^T$ ,  $T$  is simply either  $u_3$  or  $-u_3$ , where  $u_3$  is the third column vector of  $U$ .

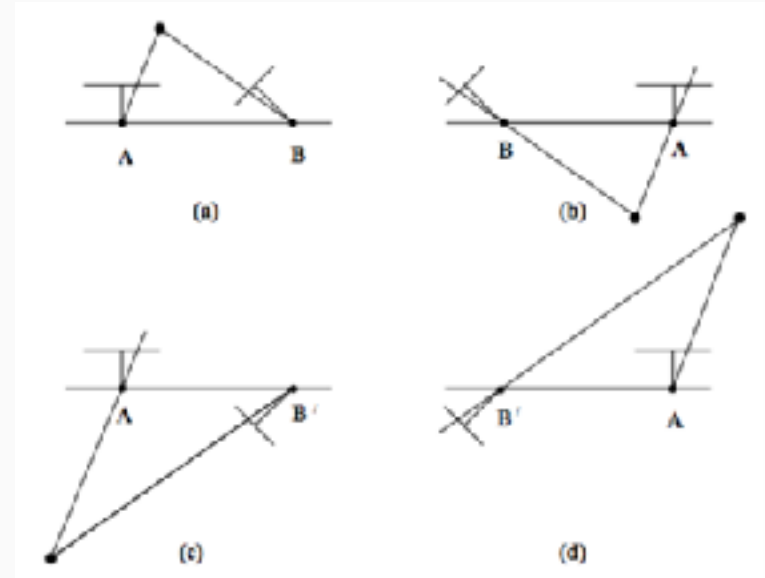


# Problem 4: Structure from Motion

Use E to make initial estimate of relative rotation  $R$  and translation  $T$

However, this gives four pairs of rotation and translation,  $(R_1, R_2) \times (T, -T)$

How do we find out which  $R$  and  $T$  is the correct one?

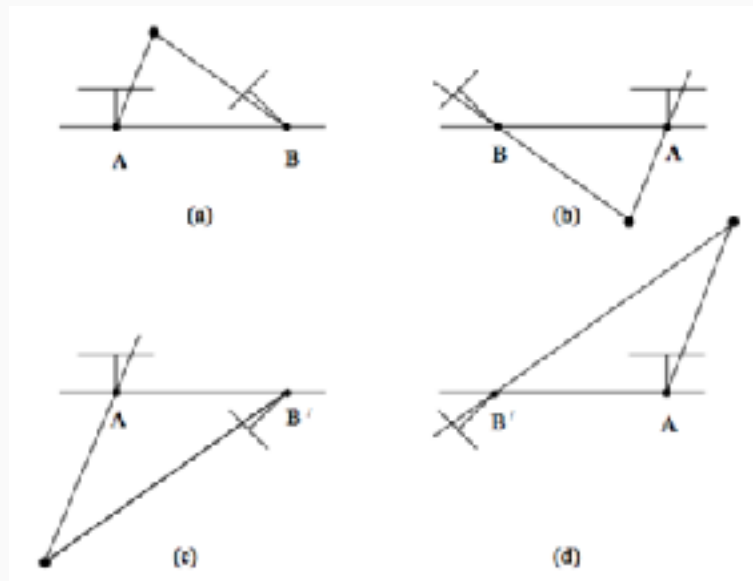


# Problem 4: Structure from Motion

There exists only **one** solution that will consistently produce 3D points which are both in front of camera

Compute 3D point's location in the RT frame!

- Find 3D location of the image points given RT frame
- Chose the one which has the most 3D points with positive depth (z-coordinate) with respect to both camera frame



# Problem 4: Structure from Motion

1. Compute fundamental matrix  $F$  from two views
2. **Use  $F$  to make initial estimate of relative rotation  $R$  and translation  $T$**
3. **Estimate 3D location of the reconstruction given RT**
4. Optimize (bundle adjustment)
  - Jointly optimize all relative camera motions ( $R$ 's and  $T$ 's)
  - Minimize total reprojection error with respect to all 3D point and camera parameters
5. Repeat 3 and 4 for pairs of frames

