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- COURSE TOPICS:

1). Introduction: Transition from classical to quantum.
2). Foundations of Quantum Th: States. Ensembles, Pure, Mixed states, Mult Qubit States, Tensor products, Unitary transformations, Spectral decomposition Th. Generalized measurement, projector measurement (POVM).

3) Quantum Entropy & Entaglement: Quantum Entropy, EPR paradox Schmidt decomposition.

4). Basic Quantum Information Processing Protocols: Teleportation, Superdense Coding, Entanglement Swapping.

5). Intro to Quantum computing: Pauli Gates, Hadamard Gates, Universal gates.

→ Books:
- Quantum Information & Computation: Cambridge University Press (Michael & ~~I~~)

- Quantum computer Science: Camb. Uni Press.

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- Information : Any meaningful data. Vaguely: some kind of quantity that reduces uncertainity in a system.
When 2 systems which have uncertainities about each others interact, uncertainity reduces over time.

→ How to measure uncertainity?
Uncertainity is a function which is dependent on P
$U = f(P)$. ST: $U \to 0$ when $p \to 0,1$ $U \to$ high o/w.
Shannon defined the uncertainity of a system as entropy.

Information is addititive. Hence the function chosen should be additive. It should also be positive.
$H(p) = -p\log p - (1-p)\log(1-p)$

→ Shannon's function was: $H(p) = -\sum p_i \log p_i$
$\because$ p_i goes from $0 \to 1$ $\log(p_i)$ will be negative
$\therefore$ shannon included a minus sign

To differentiate between 2 possibilities eg: ON & OFF, we require 1 bit ||ly for 4 poss → 2 bits, this is syntactic information.

→ Quantum mechanical effects - superposition, entanglement.

→ A computer utilizes/harnesses these quantum mechanics properties then it is called ~~a~~ quantum computation

- Study of quantum mechanics is very different from classical mechanics.

- $\exists$ 2 types of probability - classical & quantum (again!)
→ classical situation : A coin being tossed.
$1/2 : H$ $1/2 : T$

Quantum model: A coin is tossed in vaccum - each trial has the same conditions, prediction becomes possible ∴ not intrinsically random.

→ Quantum probability: Intrinsically random values, even when all external conditions are kept constant.

- Reality in classical sense exists ~~of~~ whether or not it is measured. But in quantum sense: it exists only if measured.

In the classical systems information is measured in 'bits' but in the Quantum system: qubits are used.

→ A ~~qu~~qubit :-

$$\alpha|0\rangle + \beta|1\rangle$$

where : $|\alpha|^2 + |\beta|^2 = 1$

In the microscopic world, photons, ē spin of photons/ ē can be written as above.

Eg: Searching for a marked atom in an unsorted list $O(n)$. But with quantum computation: quadratic speedup of $O(\sqrt{n})$ → Grover search algorithm.

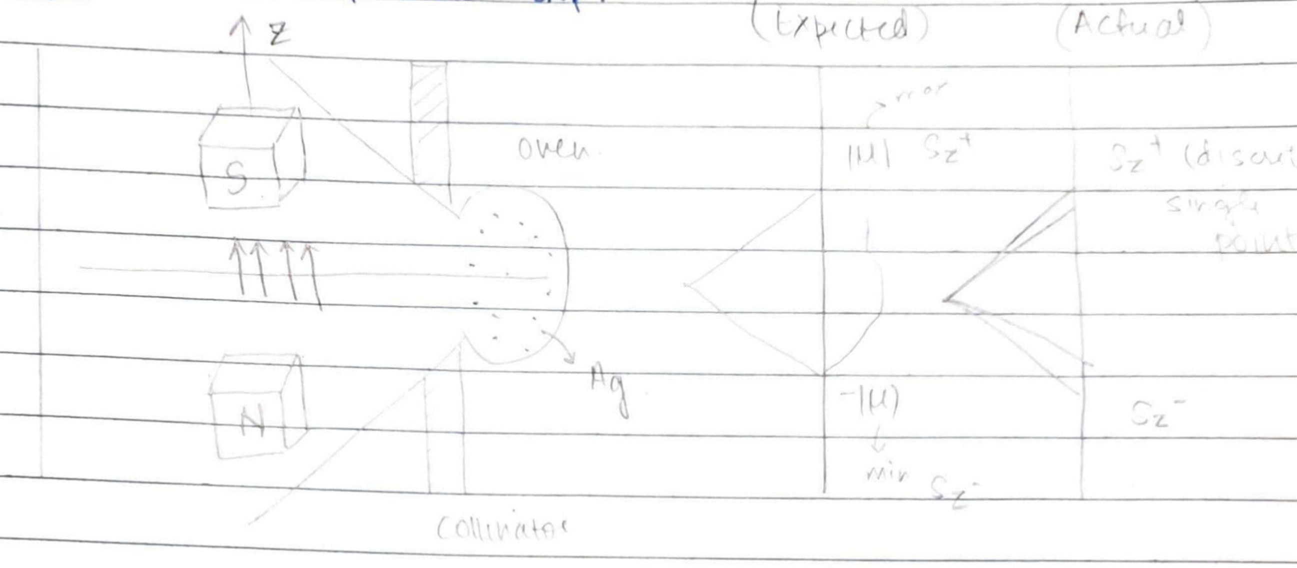
- EXPERIMENTS:

2 Experiments that help us visualize the existance of qubits as $\alpha|0\rangle + \beta|1\rangle$

a) Stern Garlach Exp.

→ Sequential

① Exp1: Stern Garlach Exp.



- Silver atoms being heated in the oven.
- $47\bar{e}$ 46 bound to the nucleus, one free.
- Placed in a magnetic field. in the Z direction

Magnetic moment is proportional to the spin $\mu \propto S_z$.

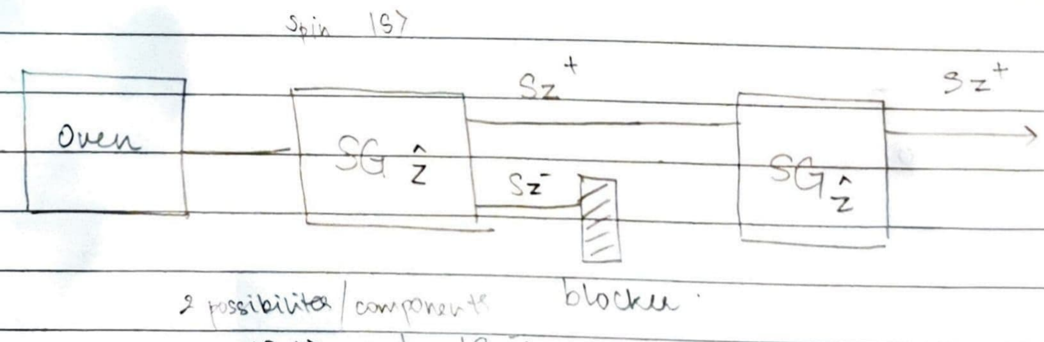
A gaussian distribution is expected for the silver atoms on the screen. Considering : spin is a classical model
However, a discretization of the values

$$S_z \begin{cases} S_z^+ & |0\rangle \\ S_z^- & |1\rangle . \end{cases}$$

Nature behaves in such a way that the spin ~~takes~~ discrete values.

Sequential Diagram :

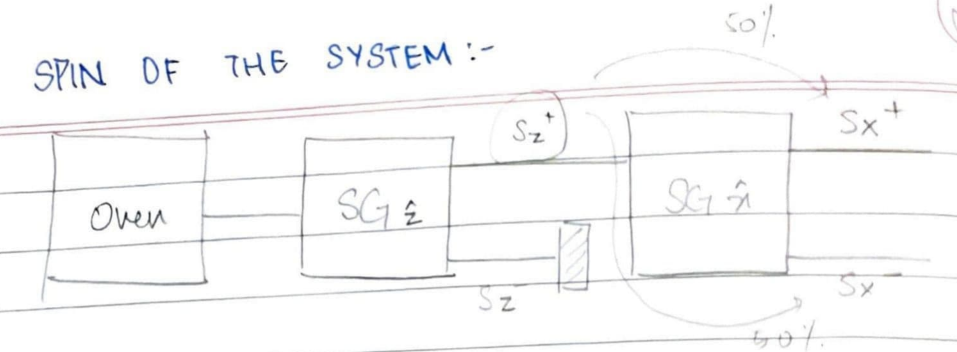
Fig1



2 possibilities / components
$|S_z^+\rangle$ and $|S_z^-\rangle$.

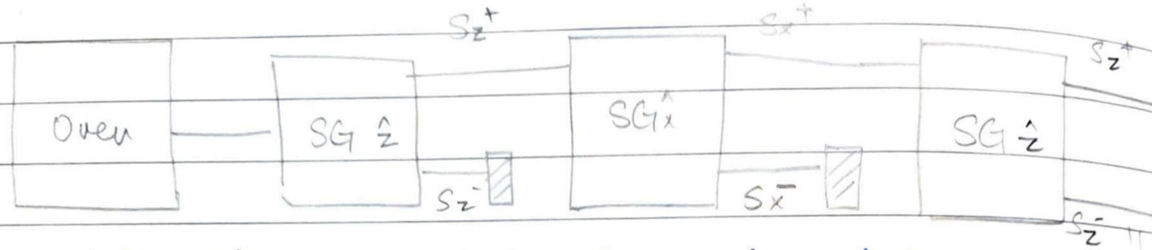
▲ SPIN OF THE SYSTEM :-

Fig 2:



Classical explanation 50% of S_z^+ are of S_x^+ property & remaining 50% are S_x^-

Fig 3:



Here information that S_z^- is blocked is lost.

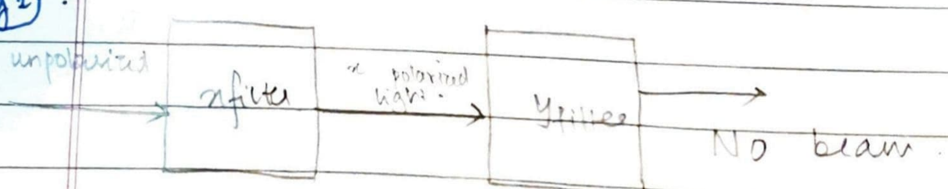
Existance of the system comes because of the measurement.

Here we observe we can represent $S_z^+ < \substack{S_x^+ \\ S_x^-}$ & $S_x^+ < \substack{S_z^+ \\ S_z^-}$

→ Similar situation happens even in the case of polarization. Here too, some kind of information is lost but reverting the right apparatus brings back that info.

▲ POLARIZATION OF PHOTON :

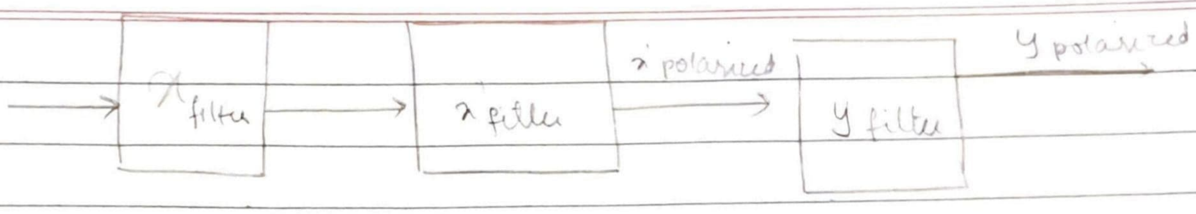
Fig 2.



Expectation : no light ∵ y filter ⊥ to x polarized.

depends on measurements in case of quantum particle

Fig 2.



x' is at an angle $45°$ to x filter.
The reality of the atoms gets changed at every stage.
Properties of microscopic particles depends on the choice of measurement.

→ Drawing an analogy b/w the 2 exp

Let: $S_z^{\pm}$ atoms $\leftrightarrow$ x, y polarized light.
$S_x^{\pm}$ atoms $\leftrightarrow$ x', y' polarized light
(x', y', $\perp$).

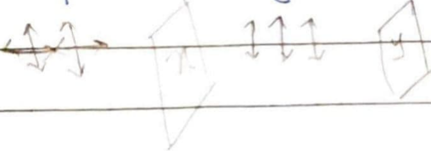
Now, $S_z^+ \rightarrow \{S_x^+, S_x^-\}$
ie: S_x^+, S_x^- will be some component of S_z^+

$\therefore$ $S_z^+ = \frac{1}{\sqrt{2}} S_x^+ + \frac{1}{\sqrt{2}} S_x^-$ such that norm of this vector $= 1$ ie: $|C_1|^2 + |C_2|^2 = 1$

ie:
$$|S_x^+\rangle = \frac{1}{\sqrt{2}}|S_z^+\rangle + \frac{1}{\sqrt{2}}|S_z^-\rangle$$
And:
$$|S_x^-\rangle = \frac{1}{\sqrt{2}}|S_z^+\rangle - \frac{1}{\sqrt{2}}|S_z^-\rangle.$$

This happens because x, y $\perp$ to each other, x', y' $\perp$ to each other

A linearly polarized light with a polarization vector in the x-direction which we call as x-polarized light has an electric field : direction of polarization



$$\mathcal{E} = \hat{n}\cos(kz - \omega t)$$

y-polarized : $\mathcal{E} = \cancel{\hat{x}}\ \hat{y}\cos(kz - \omega t)$.

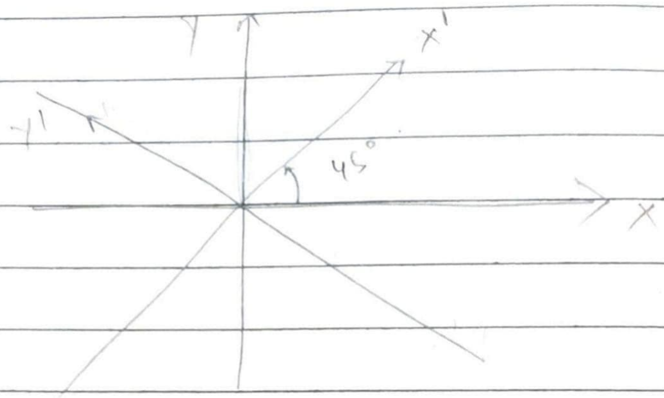
x is cross sectional ... Check
what is z ?

Now x' polarized light [45° with x axis] has 2 components - along x & y.

ie: : $\hat{x}' \cos(kz-\omega t) = \frac{1}{\sqrt{2}}\hat{x}\cos(kz-\omega t) + \frac{1}{\sqrt{2}}\hat{y}\cos(kz-\omega t)$

check

And : $\hat{y}' \cos(kz-\omega t) = \frac{-1}{\sqrt{2}}\hat{x}\cos(kz-\omega t) + \frac{1}{\sqrt{2}}\hat{y}\cos(kz-\omega t)$



To represent systems involving vectors we require a vector space.

Q: How will $|S_y^{\pm}\rangle$ be represented.

- **Vector Space:** (key Space).

- $|\rangle$: key notation.

- Any quantum property can be represented as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

do not correlate geometry in spin

$|S_x^+\rangle = |+\rangle = \frac{1}{\sqrt{2}}\{|0\rangle + |1\rangle\}$

$|S_x^-\rangle = |-\rangle = \frac{1}{\sqrt{2}}\{|0\rangle - |1\rangle\}$.

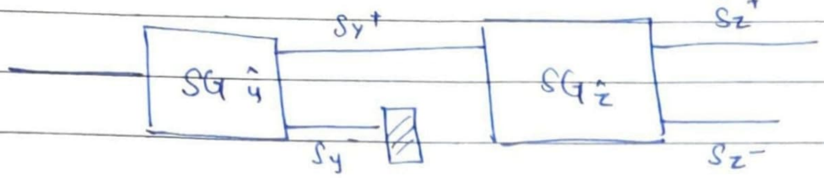
$|S_z^+\rangle = |0\rangle = \frac{1}{\sqrt{2}}\{|+\rangle + |-\rangle\}$

$|S_z^-\rangle = |1\rangle = \frac{1}{\sqrt{2}}\{|+\rangle - |-\rangle\}$

are $\{|0\rangle, |1\rangle\}$ $\{|+\rangle, |-\rangle\}$ linearly indep.

Check: circularly polarized light.

Now considering the case of $S_y^{\pm}$



→ S_y^+ atoms: ⟶ x'' right circularly polarized.

A right c. p. light is a lin comb. of polarized & $\hat{x}$ - $\hat{y}$ polarized light where the osc. of the electric field for the y-polarized component is 90° out of phase with the x-polarized component

$$x'' \cos(kz-\omega t) = \frac{1}{\sqrt{2}}\hat{x}\cos(kz-\omega t) + \frac{i}{\sqrt{2}}\hat{y}\cos(kz-\omega t+\pi/2)$$

$\sin(kz-\omega t)$

$$= \boxed{\frac{1}{\sqrt{2}}\hat{x}e^{i(kz-\omega t)} + \frac{i}{\sqrt{2}}\hat{y}e^{i(kz-\omega t)}}$$

$$\therefore, \quad |S_y^{\pm}\rangle = \frac{1}{\sqrt{2}}|S_z^+\rangle \pm \frac{i}{\sqrt{2}}|S_z^-\rangle .$$

Postulate 1: Associated to any isolated phy. sys. is a complex vector space with inner product known as the state space of the system.

→ The system is completely described by its state vector which is a unique vector in the sys. state space. The dim of the complex vector space is specified acc. to the nature of the phy. sys. under consideration.

[Dimension in case of the spin is 2].

∵ 2 vectors (linearly independent) are required to represent the spin.

Fermions / Bozones
qubits qutrits

Spin of the silver atoms:-

$$|\alpha\rangle = a|0\rangle + b|1\rangle \rightarrow \text{qubit.}$$
$$= a|S_z^+\rangle + b|S_z^-\rangle$$

Also
$$= c|S_x^+\rangle + d|S_x^-\rangle$$
$$= \cancel{c}e|S_y^+\rangle + f|S_y^-\rangle$$

→ Mathematically, the state of a quantum system is an element of a vector space (defined over a field of complex numbers).

$|\psi\rangle$: Ket space : mathematically entity rep the state.
↓
ket

Eg of ψ : spin, pos, momentum etc., polarization.
ψ is an umbrella that has properties describing the microparticle.
$|\psi\rangle$ is an element of the vector space ∴, all properties

$$|\psi\rangle = a|\alpha\rangle + b|\beta\rangle \quad ; \quad |\alpha|^2 + |\beta|^2 = 1$$

ie: Example: if $|v_1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \equiv |0\rangle$ $\quad |v_2\rangle = \begin{bmatrix}0\\1\end{bmatrix} \equiv |1\rangle$

Any vector $v = \begin{bmatrix}a_1\\a_2\end{bmatrix} \in \mathbb{C}^2$ $\quad$ a, b in $\mathbb{C}^2$

ie: Any arbitrary vector $|v\rangle$ can be rep. as a lin combination $|v\rangle = a_1|0\rangle + a_2|1\rangle$.

∴, these $|0\rangle$, $|1\rangle$ are spanning the vector space
ie: Any vector can be written as a lin comb. of these vectors.

ψ is the unit of information of a quantum system

Spin qubits is not exhaustive set of quantum mech. vector space

Further $c_1|0\rangle + c_2|1\rangle = 0$

$$\Rightarrow c_2\begin{bmatrix}0\\1\end{bmatrix} + c_1\begin{bmatrix}1\\0\end{bmatrix} = 0 \Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\Rightarrow c_1 = c_2 = 0$ $\therefore$ Lin Independent.

Now in terms of S_x^+ and S_x^- :-

$$S_x^+ = \frac{1}{\sqrt{2}}\{|S_z^+\rangle + |S_z^-\rangle\} \quad \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ 1/\sqrt{2}\end{bmatrix} = |+\rangle$$

And $S_x^- = |-\rangle = \begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}$

To prove lin. indep :-

$$c_1|+\rangle + c_2|-\rangle = 0$$

$$c_1\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix} + c_2\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix} = 0.$$

$$\begin{bmatrix}(c_1+c_2)\frac{1}{\sqrt{2}}\\ (c_1-c_2)\,1/\sqrt{2}\end{bmatrix} = 0 \quad c_1 = -c_2 \quad c_1 = c_2$$

$$\therefore \quad c_1 = -c_1 \quad 2c_1 = 0$$

$$\boxed{c_1 = 0 = c_2}$$

- Linear op. in a vector space is a function that takes a vector $|v\rangle$ to another vector $|v'\rangle$

  Logic gates by which these linear operators are realised in the laboratory.

- A linear operator b/w vector spaces V & W is defined to be a function $A: V \to W$ which is linear inputs.

$$A\left(\sum a_i v_i\right) = \sum a_i A(v_i)$$

Here W can be V itself.

Now $A(\ )_{n\times 1} = (\cdot)_{n\times 1}$ if going from $V \to V$.

Now if: $A: V \to W$.
V: dim n; W: dim m.

then: $A(\ )_{n\times 1} = (\ )_{m\times 1}$

then $\dim(A) = m \times n$

→ Every linear operator can be represented as a matrix
Consider matrix $[A]$

$$[A]_{m\times n}\left(\sum_{i=1}^{n} a_i |v_i\rangle\right)_{n\times 1}$$

$$= [A]_{m\times n}\left[a_1|v_1\rangle_{n\times 1} + a_2|v_2\rangle_{n\times 1} + \cdots + a_n|v_n\rangle_{n\times 1}\right]$$

Matrix multi is distributive.

$$= a_1 [A]_{m\times n}|v_1\rangle + a_2[A_{m\times n}]|v_2\rangle + \cdots$$

$$= \underline{\underline{\sum a_i [A]|v_i\rangle}} \Rightarrow \text{Linear operator.}$$

Reverse: Every linear op can be rep as a matrix. $A(\sum a_i v_i) = \sum a_i A v_i$

Suppose $A: \underset{(n)}{V} \to \underset{(m)}{W}$ be a linear operator

Pf: Suppose $\{|v_1\rangle, |v_2\rangle \ldots |v_n\rangle\}$ is a basis of V
& $\{|w_1\rangle, |w_2\rangle \ldots |w_m\rangle\}$ basis of W.

Now, $(A|v_1\rangle)$ is an element of of W

ie: Every basis vector in V, under the lin. op. can be represented as a linear combination of basis vector of W.

ie: $A|v_j\rangle = A_{1j}|w_1\rangle + A_{2j}|w_2\rangle + A_{3j}|w_3\rangle + \dots + A_{mj}|w_m\rangle$

$\therefore$, In general, all basis vectors in V:-

$$A|v_1\rangle = A_{11}|w_1\rangle + A_{12}|w_2\rangle + \dots + A_{1m}|w_m\rangle$$
$$A|v_2\rangle = A_{21}|w_1\rangle + A_{22}|w_2\rangle + \dots + A_{2m}|w_m\rangle$$
$$\vdots$$
$$A|v_n\rangle = A_{n1}|w_1\rangle + A_{n2}|w_2\rangle + \dots + A_{nm}|w_m\rangle.$$

ket vectors

$$\therefore, \quad A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & & & \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}$$

Q: Suppose $A : \underset{(2)}{V} \longrightarrow \underset{(2)}{V}$

such that $A|0\rangle = |1\rangle$
$A|1\rangle = |0\rangle$.

$$\therefore, \quad A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix}$$
$$\text{IIly}: \quad A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$

Bit flip cannot be for any arbitrary ψ

Q: Find $A : V^2 \longrightarrow V^2$ where
$A|0\rangle = |+\rangle$ $\quad A|1\rangle = |-\rangle$.

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} = 1/\sqrt{2}\begin{bmatrix}1\\0\end{bmatrix} + 1/\sqrt{2}\begin{bmatrix}0\\1\end{bmatrix}$$
$$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} = 1/\sqrt{2}\begin{bmatrix}1\\0\end{bmatrix} - 1/\sqrt{2}\begin{bmatrix}0\\1\end{bmatrix}.$$

$$\therefore, \quad A = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}.$$

→ Linear functional : is a map that goes from a vector to a scalar (Inner product is an eg. of linear func.)

ie: $A : V \times V \longrightarrow \mathbb{C}$ or $A : V \longrightarrow \mathbb{C}$

All these functionals together form a vector space called the dual space

Corresponding to every vector in the ~~dual~~ ket space $\exists$ a vector in the dual space.

ie: $|\alpha\rangle \Longleftrightarrow \langle\alpha|$

$|\alpha\rangle \xrightarrow{DC} \langle\alpha|$

& $|\alpha\rangle + |\beta\rangle \xrightarrow{DC} \langle\alpha| + \langle\beta|$

→ Certain linear operations are as follows.

$A'|\alpha\rangle = k|\alpha\rangle$

In such cases, $|\alpha\rangle$ is the eigen vector of the operator A' and k is the eigen value.

→ Whenever we have a linear functional we can represent it as

$\langle\alpha|\big(|\psi\rangle\big) = \langle\alpha|\psi\rangle. \in \mathbb{C}.$

- We now define inner product of bra and a ket

ie: $\langle\beta|\alpha\rangle = (\langle\beta|)\,|\alpha\rangle.$

i) It is linear in the second argument ie:

$\langle v | \left(\sum \lambda_i |w_i\rangle\right) = \sum \lambda_i \langle v_i|w_i\rangle.$

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NOTE: • $c_\alpha|\alpha\rangle + c_\beta|\beta\rangle \xleftrightarrow{DC.} c_\alpha^*\langle\alpha| + c_\beta^*\langle\beta|$

• If $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. $|\alpha|^2 + |\beta|^2 = 1$

then ~~$|\psi\rangle^*$~~ $\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$

$P(\langle\alpha|\alpha\rangle) = 1$

ii) $\langle v|w\rangle = \langle w|v\rangle^*$

iii) $\langle v|v\rangle > 0$ with equality if $|v\rangle = 0$.

Example $\mathbb{C}^n$.

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{n\times 1} \qquad |w\rangle = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

$$\langle v|w\rangle = \begin{bmatrix} v_1^* & v_2^* & \dots & v_n^* \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

$\langle\alpha|\alpha\rangle =$

Eg: Consider in $\mathbb{C}^2$: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Find $\langle 0|\psi\rangle$, $\langle 1|\psi\rangle$.

Ans: Now $\langle 0|\psi\rangle = \alpha\langle 0|0\rangle + \beta\langle 0|1\rangle$.

$= \alpha \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \alpha$. modular square of inner product will give probability

$||^{ly}$: for : $\langle 0|+\rangle$.

$= \langle 0| \left\{ \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right\}$.

$= \frac{1}{\sqrt{2}}\langle 0|0\rangle + \frac{1}{\sqrt{2}}\langle 0|1\rangle$.

$= \frac{1}{\sqrt{2}}$. in accordance with Stern Gerlach exp.

→ Inner Product space : Taking any 2 vectors we can obtain a scalar.

If $\langle \alpha | \beta \rangle = 0$

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Q: Show that the inner product is conjugate linear in the first argument.

Prove

ie: $\left(\sum \lambda_i^* \langle w_i | \right) | v \rangle = \sum \lambda_i^* \langle w_i | v_i \rangle$.

Two vectors α, β are said to be orthogonal if $\langle \alpha | \beta \rangle = 0$

Eg: $[1 \quad 0]$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .... orthogonal basis.

→ Norm of a vector can be obtained as:-

$|\alpha\rangle \longrightarrow |\bar{\alpha}| = \left(\frac{1}{\sqrt{\langle \alpha | \alpha \rangle}} \right) |\alpha\rangle$.

using the property that $\langle \alpha | \alpha \rangle = 1$.

→ A set of vectors $\{ \{ |i\rangle \}$ such that $\langle i | j \rangle = \delta_{ij}$ is an orthonormal basis.

$$\left\{ \begin{array}{ll} \delta_{ij} = 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{array} \right\}$$

- 'ket' vectors are the states of the system and not the observables.

Observables like momentum & spin components are to be represented by operators

Eigen operators are observables — 2 pictures in QM

All observables are operators but not all operators are observables. That is it may be that ∃ some mathematical operator that does not have any physical meaning, rather it only has mathematical notion.

- **Properties of Operators:** $X(|\alpha\rangle) = X|\alpha\rangle$.

→ 2 operators X, Y are said to be equal if $X|\alpha\rangle = Y|\alpha\rangle$ for any arbitrary $|\alpha\rangle$.

→ Null Operator :-
$X|\alpha\rangle = 0$ for any arbitrary $|\alpha\rangle$.

→ $X+Y = Y+X$. Commutative

→ $X+(Y+Z) = (X+Y)+Z$ Associative.

→ $X(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle) = c_\alpha X|\alpha\rangle + c_\beta X|\beta\rangle$.

→ ~~$(X|\alpha\rangle)$~~ $(\langle\alpha|)X = \langle\alpha|X$.

NOTE(2): But $\langle\alpha|X \overset{DC}{\not\longleftrightarrow} X|\alpha\rangle$.

NOTE(2) :-
$X|\alpha\rangle \overset{DC}{\longleftrightarrow} \langle\alpha|X^+$.
$((X|\alpha\rangle)^*)^T$
$=$

Since $X|\alpha\rangle$ ---- ket vector.

∴, DC: $(|\alpha\rangle)^* X^+ = \langle\alpha|X^+$

- $(X|\alpha\rangle)^+ = \langle\alpha|X^+$

where X^+ is a Hermetian matrix wherein it is the transpose of the complex conjugate.
Eigen values of X^+ is real values

Most operators in quantum mechanics ~~is~~ are hermetian operators.

→ $XY \neq YX$.

→ $X(YZ) = (XY)Z = XYZ$.

→ $X(Y|\alpha\rangle) = XY(|\alpha\rangle)$ .... ket vectors.

→ $(\langle\beta|X)Y = (\langle\beta|)XY$. .... bra vectors.

→ $(XY)^{+} = Y^{+}X^{+}$

Using earlier arguments of equality of 2 operators, this reduces to $(XY)^{+}|\alpha\rangle = Y^{+}X^{+}(|\alpha\rangle)$

- Types of operators:

$X|\alpha\rangle$: operator × ket vector → vector.

$\langle\alpha|X$: operator × bra vector → vector. dual space.

$\langle\alpha|\beta\rangle$: inner product

XY : Product of 2 operators.

→ Outer Product :

$$(|\beta\rangle\langle\alpha|)|\gamma\rangle \equiv \underbrace{|\beta\rangle\langle\alpha|\gamma\rangle}_{\text{associativity axiom}}.$$

Here we can consider an operator acting on a vector that results in a vector (RHS).

NOTE: $X = |\beta\rangle\langle\alpha|$

then $X^\dagger = (|\beta\rangle\langle\alpha|)^*$

$= (\langle\alpha|)^* (|\beta\rangle)^*$

$= |\alpha\rangle\langle\beta|$.

→ $\underbrace{(\langle\beta|)}_{\text{bra}}\underbrace{(X|\alpha\rangle)}_{\text{ket}} = \langle\beta|X|\alpha\rangle$.

Now,

$\langle\beta|X|\alpha\rangle = (\langle\alpha|X^\dagger|\beta\rangle)^*$ $\left\{ \because (ab)^* = b^* a^* \right\}$

$\langle\alpha|X|\beta\rangle = \langle\beta|X|\alpha\rangle$ if X is hermetian.

- **Theorem:** "The eigen values of a Hermitian operator A are real; the eigenkets corresponding to different eigenvalues are orthogonal" Hermetian : $(A^\dagger = A)$

$A|v_1\rangle = \lambda_1|v_1\rangle = A^\dagger|v_1\rangle$

$\langle v_1|A|v_1\rangle \neq \langle v_1|A^\dagger = \lambda_1^*\langle v_1|$

$\langle v_1|A^\dagger|v_1\rangle = \lambda_1^*\langle v_1|v_1\rangle$

$\lambda_1^* = \lambda_1$

$\lambda_1 - \lambda_1^* = 0$

Pf: Let A be a hermitian operator.

$A|a'\rangle = \lambda_1|a'\rangle$ — (1)

$A|a''\rangle = \lambda_2|a''\rangle$ — (2).

Taking conj. transpose of (2).

$\langle a''|A^\dagger = \lambda_2^*\langle a''|$ — (3)

Multiplying both sides of (3) by $|a'\rangle$, we get :

$\langle a''|A^\dagger|a'\rangle = \lambda_2^*\langle a''|a'\rangle$ — (4)

Mult (1) by $\langle a''|$: $\langle a''|A|a'\rangle = \lambda_1\langle a''|a'\rangle$. — (5).

From (4) & (5) $\because$ LHS are equal :-

$(\lambda_1 - \lambda_2^*)(\langle a''|a'\rangle) = 0$

If $a' = a''$ then $\langle a'' | a' \rangle = 1$.

$\therefore$, $\lambda_1 - \lambda_2^* = 0$

$\Rightarrow \lambda_1 = \lambda_2 \quad \because a' = a''$

then $\lambda_2^* = \lambda_1^*$

$\therefore$, $\lambda_1 - \lambda_1^* = 0$

$\Rightarrow \lambda_1$: eigenvalue is real.

Now, let us assume: that a', a'' are different then

$(a' - a''^*) \longrightarrow (a' - a'')$

then $(a' - a'') \langle a'' | a' \rangle = 0$

$\langle a'' | a' \rangle = 0$

$\Rightarrow |a''\rangle$, $|a'\rangle$ are orthogonal.

→ Let $|\alpha\rangle$ be spanned by the eigenkets of A.

$|\alpha\rangle = \sum_a C_a |a\rangle$. $\quad \because \langle a|\alpha\rangle = C_a \quad \because$ eigen vectors orthogonal

Now, $\langle a|\alpha\rangle = C_a \rightarrow$ How?

ie:

completeness

$$|\alpha\rangle = \sum_a \langle a|\alpha\rangle |a\rangle = \boxed{\sum_a \left(|a\rangle \langle a|\right)} |\alpha\rangle$$

$$\sum_a \left(|a\rangle\langle a|\right) = I$$

This is the Hilbert Space : final space we have reached

then :-

$$\sum |C_a|^2 = \sum |\langle a|\alpha\rangle|^2 = 1$$

- ## Gram Schmidt Procedure: Converting a basis orthonormal

Let $\{|w_1\rangle, |w_2\rangle, |w_3\rangle \dots |w_d\rangle\}$ is a set of basis vectors of a vector space V with inner product

Let $\{|v_1\rangle, |v_2\rangle \dots |v_d\rangle\}$ be the set of orthonormal vectors.

$$|v_2\rangle = \frac{|w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle}{\| |w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle \|}$$

Such that $\langle v_1|v_2\rangle = \underline{\langle v_1|w_2\rangle - \langle v_1|w_2\rangle \langle v_1|v_1\rangle}$; $\langle v_1|v_1\rangle = 1$

$= 0$

Likewise for the remaining vectors also this can be extended.

→ Now, consider $\{|a''\rangle\} \rightarrow$ N el. 0 to N

$\{|a'\rangle\} \rightarrow$ N el 0 to N.

$$X = I \times I = \sum_{a'}\sum_{a''} \underbrace{|a''\rangle\langle a''| \, X \, |a'\rangle\langle a'|}_{N^2 \text{ elements.}}$$

$$= \boxed{\sum_{a'}\sum_{a''} \langle a''|X|a'\rangle \, |a''\rangle\langle a'|}$$ ... Any op. can be written as sum of app. co-efficients times outer product.

$$X = \begin{bmatrix} \langle a^1|X|a^1\rangle & \langle a^1|X|a_2\rangle & \cdots & \langle a^1|X|a^N\rangle \\ \langle a^2|X a^1\rangle & \langle a^2|X|a^2\rangle & & \\ \vdots & & & \\ \langle a^N|X|a^1\rangle & & & \end{bmatrix}$$

Eg: $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{array}{c|cc} & |0\rangle & |1\rangle \\ \hline |0\rangle & 1 & 0 \\ |1\rangle & 0 & -1 \end{array}$

$\langle 0|\sigma_z|0\rangle = 1$ $\qquad$ $\langle \phi|\sigma_z|0\rangle = 0$

$\langle 0|\sigma_z|1\rangle = 0$ $\qquad$ $\langle 1|\sigma_z|1\rangle = -1$

σ_z = Sum of $\uparrow$ $= \underbrace{\langle 0|\sigma_z|0\rangle}_{1}\, |0\rangle\langle 0| + \cancel{\langle 1|\sigma_z|0\rangle}\, |1\rangle\langle 0| + \cancel{\langle 0|\sigma_z|1\rangle} + \underbrace{\langle 1|\sigma_z|1\rangle}_{-1}\, |1\rangle\langle 1|$

$$\boxed{\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|}$$

Now. $\sigma_z|0\rangle = |0\rangle$ $\quad S_z^+$ $\quad$ Observable in the z

$\sigma_z|1\rangle = -|1\rangle.$ $\quad S_z^-$ $\quad$ direction.

Likewise $\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$.

$$\boxed{\begin{aligned} \sigma_x &= |0\rangle\langle 1| + |1\rangle\langle 0|. \\ \sigma_y &= i|0\rangle\langle 1| - i|1\rangle\langle 0|. \end{aligned}}$$

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle$

Now for $\sigma_x|0\rangle = |1\rangle$ $\}$ bit flip.

$\sigma_x|1\rangle = |0\rangle$

$\therefore$ $\sigma_x|+\rangle = |+\rangle$ $\quad$ Observable in x direction

$\sigma_x|-\rangle = |-\rangle$ $\quad$ with eigen vectors $|+\rangle, |-\rangle$

& eigen values 1

→ Perform a change of basis for σ_z, σ_x

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|.$$
$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|.$$

$$\sigma_z = \cancel{\tfrac{1}{2}(|0\rangle + |1\rangle)(\langle 0| +} \ \tfrac{1}{2}\left(|+\rangle + |-\rangle\right)\left(\langle +| + \langle -|\right) -$$
$$\tfrac{1}{2}\left(|+\rangle - |-\rangle\right)\left(\langle +| - \langle -|\right)$$
$$= \tfrac{1}{2}\Big[\Big(\cancel{|+\rangle\langle +|} + \overset{2}{\cancel{|+\rangle\langle -|}} + |-\rangle\langle +| + \cancel{|-\rangle\langle -|}\Big) - \Big(\cancel{|+\rangle\langle +|} - |+\rangle\langle -| - |-\rangle\langle +| + \cancel{|-\rangle\langle -|}$$
$$= \boxed{|+\rangle\langle -| + |-\rangle\langle +|}$$

$$\sigma_x = \boxed{|+\rangle\langle +| - |-\rangle\langle -|}$$

$$\sigma_y = ? \quad i|0\rangle\langle 1| - i|1\rangle\langle 0|.$$

× Resuming after Quarantine ×

- Cauchy Shwatrtz Inequality:

It states that for any 2 vectors $|v\rangle$ & $|w\rangle$ we have the inequality

$$|\langle v|w\rangle|^2 \leq \langle v|v\rangle + \langle w|w\rangle.$$

Proof: $\left(\sum_i |i\rangle\langle i|\right) = I$ ... completeness − (1)

By Gram Schmidt procedure let us construct an orthonormal basis $\{|i\rangle\}$.
The first member of this orthonormal basis is given by : $\frac{|w\rangle}{\sqrt{\langle w|w\rangle}}$.

Now,
$$\langle v|v\rangle \langle w|w\rangle = \langle v|I|v\rangle \langle w|w\rangle.$$

ie : $\sum_i \langle v|i\rangle \langle i|v\rangle \langle w|w\rangle$ — (2)
orthonormal basis

Now, we know that :- $\sum |i\rangle\langle i| \geq \frac{|w\rangle\langle w|}{\sqrt{\langle w|w\rangle}\sqrt{\langle w|w\rangle}}$.

$$\therefore (2) \geq \frac{\langle v|w\rangle \langle w|v\rangle \cancel{\langle w|w\rangle}}{\cancel{\langle w|w\rangle}}$$
$$= \langle v|w\rangle \langle w|v\rangle$$
$$= |\langle v|w\rangle|^2$$
Proved.

- **Eigen Values & Vectors :**

An eigen vector of a linear operator A on the vector space is a non-zero vector $|v\rangle$ S.T. :-
$$A|v\rangle = v|v\rangle$$
$|v\rangle$ → eigen vector
v → eigen value

→ Characteristic eqn :-
$$\det(A - \lambda I) = 0.$$

→ A diagonal representation for an operator A on a vector space is a representation
$$A = \sum_i \lambda_i |i\rangle\langle i| \quad \text{— (1)}$$

Any operator is diagonal in its diagonal basis where the vectors $\{|i\rangle\}$ form an orthonormal set of eigen vectors for A with corresponding eigen values λ_i

An operator is said to be diagonalisable if it has a diagonal representation.

We know that

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

And this is diagonalizable in $\{|0\rangle, |1\rangle$ basis$\}$.

$\boxed{Q:}$ Find eigenvectors, eigenvalues & diagonal representation of all pauli matrices.

$\sigma_x, \sigma_y, \sigma_z.$

($\sigma_0 = I$ diagonal in any basis)

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- ## Adjoint & Hermitian Operators:

Suppose A is any linear operator on the hilbert space V. It turns out that $\exists$ a linear op A^{+} on V such that for all vectors $|v\rangle, |w\rangle \in V$

$$(|v\rangle, A|w\rangle) = (A^{+}|v\rangle, |w\rangle)$$

inner product.

This linear op. is known as the adjoint or Hermetian conjugate of the op. A.

→ Easy to see that $(AB)^{+} = B^{+}A^{+}$.

Basically $|v\rangle^{+} = \langle v|$ ... (since conj. transpose)

$$\therefore (A|v\rangle)^{+} = \cancel{A^{+}} v^{+} A^{+}$$
$$= \underline{\langle v| A^{+}}.$$

$\boxed{Q}$ If $|w\rangle$ & $|v\rangle$ are any two vectors, show that $(|w\rangle\langle v|)^{+} = (|v\rangle\langle w|)$.

Proof: LHS: $(|w\rangle\langle v|)^{+} = (\langle v|)^{*+} (|w\rangle)^{+}$
$= |v\rangle\langle w|$. ... proved.

$\boxed{Q:}$ show that :-

a) $\left(\sum_i a_i A_i\right)^{+} = \sum_i a_i^{*} A_i^{+}$

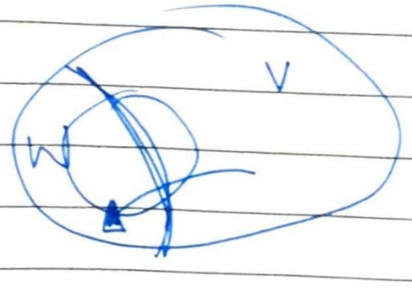
b) $(A^{+})^{+} = A$.

→ An operator whose adjoint is equal to the operator ie: $A^{\dagger} = A$
then the operator is known as Hermitian or self adjoint.

- (Projection Op) :-
Suppose W is a subspace (k-dim) of vector space V (d-dim).

Using gram schmidth orthonormalization it is possible to create an orthonormal basis $\{|1\rangle \dots |d\rangle\}$ for V such that $\{|1\rangle \dots |k\rangle\}$ is an orthonormal basis for W.

Projection operator is defined as:-
$P = \sum_{i=1}^{k} |i\rangle\langle i|$ is a projector on the subspace W.



It can be shown that $P^\dagger = P$.

$$\left[\begin{aligned} P &= |0\rangle\langle 0| + \cdots + |d\rangle\langle d| \\ P^\dagger &= (|0\rangle\langle 0|)^\dagger + \cdots (|d\rangle\langle d|)^\dagger \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| \cdots \end{aligned}\right]$$

$\boxed{Q:}$ Show that $P^2 = P$

$P = |0\rangle\langle 0| + \dots + |d\rangle\langle d|$

$P^2 = (|0\rangle\langle 0| \cdots |d\rangle\langle d|)(|0\rangle\langle 0| \cdots |d\rangle\langle d|)$.

Since orthonormal basis

$\langle i|j\rangle = \delta_{ij} = 0 \quad i \neq j$

$\qquad\qquad\qquad 1 \quad i = j$.

$\therefore$ this becomes:-

$|0\rangle\langle 0|0\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 1| + \cdots$

- A matrix is said to be a normal matrix if $A^\dagger A = A^\dagger A$

~ Hermetian matrix is normal

$\because \quad A^\dagger = A$

$AA^\dagger = A^2 = A^\dagger A$.

→ Matrix is said to be unitary if

$U^\dagger U = UU^\dagger = I$

Imp. prop: preserve inner products.

Eg: consider 2 vectors $|v\rangle$, $|w\rangle$

Apply U, $U|v\rangle$, $U|w\rangle$.

$\therefore$ ~~Eq~~ $\langle v|U^\dagger U|w\rangle = \langle v|I|w\rangle$

$= \langle v|w\rangle$ ... preserved

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- **Positive operator:**

$\langle v | A | v \rangle > 0$ & real for $|v\rangle \neq 0$

$\Rightarrow$ A is a positive definite op.

$\langle v | A | v \rangle \geq 0$.

$\downarrow$ positive op.

- **TENSOR PRODUCT:**

We are dealing with single particle ~~ts~~ till now but when we go beyond a single, particle how can we represent them in terms of ket notations!

$\underset{\dim(m)}{V}$ & $W \longrightarrow \dim(n)$

$\underset{V}{|v\rangle} \otimes |w\rangle \longrightarrow W$

can be $|v\rangle|w\rangle \longrightarrow |vw\rangle$

→ Properties:

i) For any scalar z & elements $|v\rangle$, $|w\rangle$ of V & W respectively ~~of~~ we have

$$z(|v\rangle \otimes |w\rangle) = (z|v\rangle) \otimes |w\rangle$$
$$= |v\rangle \otimes (z|w\rangle).$$

ii) For $|v_1\rangle, |v_2\rangle \in V$, $|w\rangle \in W$

$$(|v_1\rangle + |v_2\rangle) \otimes w = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle.$$

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(iii) For arbitary $|v\rangle \in V$, $|w_1\rangle, |w_2\rangle \in W$

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle)$$

$$= |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$$

Suppose $|v\rangle, |w\rangle$ are vectors in V & W s.t.

$$A \xrightarrow{L.O} V$$

$$B \xrightarrow{L.O} W$$

then operator

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle.$$

Generally :- $(A \otimes B)\left(\sum_i a_i |v_i\rangle \otimes |w_i\rangle\right)$

$$= \sum a_i\, A|v_i\rangle \otimes B|w_i\rangle).$$

$$C = \sum_i c_i\, A_i \otimes B_i$$

$$\sum c_i\, A_i \otimes B_i)\, |v\rangle \otimes |w\rangle = \sum c_i\, A_i|v\rangle \otimes B_i|w\rangle.$$

[screen shot]

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & & \\ \vdots & & \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\times 2 \\ 1\times 3 \\ 2\times 2 \\ 2\times 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \end{bmatrix}.$$

2/04/20

Postulates of QM :

1) Associated to any physical system (quantum system) (isolated) is a complex vector space with inner product known as the state space (ie: Hilbert space) of the system. The system is completely described by its state vector, which is unit vector in the system's state space

For any $|\psi\rangle \in H$ ... Hilbert space.
mathematical notation

State space: → complex, linear v. space
→ inner product
→ completeness condition

In reality they can be spin of silver atoms / polarization of the photons

Simplest QM system is a 2 level system which we otherwise refer to as Qubit

Eg: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

where $\{|0\rangle, |1\rangle\}$ forms an orthonormal basis

& $|\alpha|^2 + |\beta|^2 = 1.$

2) Evolution of a Quantum System :-

The evolution of a closed Quantum sys is described by unitary transformation.

ie: the state of the system at the time t_1 is $|\psi\rangle$ and at the time t_2 is $|\psi'\rangle$ then

$$|\psi'\rangle = U|\psi\rangle$$

(U depends only on the time t_1 & t_2.)

$t = t_1 \longrightarrow U \longrightarrow t = t_2$

2 pictures : $|\psi\rangle \rightsquigarrow |\psi'\rangle$

Also represented by schrödinger eqn.

A QM syst. starting to evolve.

$$\underbrace{|\psi\rangle \overset{U}{\rightsquigarrow} |\psi'\rangle}_{\text{evolution}} \left\{ \begin{matrix} |0\rangle \\ |1\rangle \end{matrix} \right\} \text{ Measurement}$$

with $|\alpha|^2$ prob collapse to $|0\rangle$

$|\beta^2|$ prob " " $|1\rangle$

collapse picture comes only when we're interacting with a quantum system with classical interface

For N experiments $N_1 \to |0\rangle$ & $N_2 |1\rangle$

then $\frac{N_1}{N} \to |\alpha^2|$ $\quad \frac{N_2}{N} = |\beta|^2$.

→ **Unitary operators**

$|0\rangle \to |1\rangle$

& $|1\rangle \to |0\rangle$ $\Big\}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ σ_x. $\qquad U^{+} = U^{-1}$

or $UU^{+} = I$

And $|0\rangle \to |0\rangle$

$|1\rangle \to -|1\rangle$ $\Big\}$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

→ $|0\rangle \longrightarrow \frac{1}{2}(|0\rangle + |1\rangle)$

$|1\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle$. $\Big\}$ $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\boxed{Q:}$ Verify this is unitary, $H^2 = 1$ ... verify.

Also verify $\sigma_x, \sigma_y, \sigma_z$ unitary

The time evolution of the state of a closed system is described by Schro

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$ H ... hamiltonian

3) Quantum Measurement (collapse part).

Quantum measurements are described by collection of measurement operators. These operators are acting on the state space of the system

The sys being measured.

$\{M_m\}$ ... m index represents measurement outcomes.

Given the measurement operators we verify what state it is in.

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evolution

$\sim\sim\sim$ $|\psi\rangle$ M_m ... state of the system being observed.

The probability of getting the measurement outcome m is given by :-

$$p(m) = \langle\psi| M_m^{+} M_m |\psi\rangle.$$

Since $|\psi\rangle \xrightarrow{M_m} M_m|\psi\rangle.$ $\Big\}$ $|\psi'\rangle$ $\therefore$ $\langle\psi'|\psi'\rangle^2$

$\langle\psi| \rightarrow \langle\psi| M_m^{+}$ $\Big\}$ $\langle\psi'|$ $= P(m)$.

And state of the system after measurement is given by

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^{+}M_m|\psi\rangle}}.$$

m: index telling you what are the outcomes.

Now, $\sum_m P(m) = 1$ ... total prob $= 1$.

Now, $\sum P(m) = \sum_m \langle\psi|M_m^{+} M_m|\psi\rangle.$

$= \cancel{\Sigma} \langle\psi| \sum_m M_m^{+} M_m|\psi\rangle.$

$= \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle = 1.$

Eg: $|\psi\rangle = a|0\rangle + b|1\rangle.$

$M_0 = |0\rangle\langle 0|$ $M_1 = |1\rangle\langle 1|$

$p(0) = \langle\psi|M_0^{+} M_0|\psi\rangle.$

Now, $\langle\psi|0\rangle\langle 0|\psi\rangle = |\langle\psi|0\rangle|^2 = |a|^2$ ... $P(0)$

Now,

$$\frac{M_0|\psi\rangle}{|a|} = \frac{a|0\rangle}{|a|}. \qquad \frac{M_1|\psi\rangle}{|b|} = \frac{b|1\rangle}{|b|}$$

$p(1) = |b|^2.$

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Q: 1 Let $|\psi\rangle = |0\rangle$ $M_0 = |0\rangle\langle 0|$

Find $P(0)$, $M_0|\psi\rangle$.

Since only 1 state exists

$P(0) = 1$

State measurement

$M_0|\psi\rangle = |0\rangle$.

Q 2: $|\psi\rangle = |1\rangle$; $M_1 = |1\rangle\langle 1|$

find $P(1)$, $M_1|\psi\rangle$

Since only 1 state $P(1) = 1$.

State measurement $M_1|\psi\rangle = |1\rangle$

Q 3: $|\psi\rangle = |0\rangle$, $M_1 = |1\rangle\langle 1|$

Find $P(1)$, $M_1|\psi\rangle$.

Here $P(1) = 0$

$M_1|\psi\rangle = |0\rangle$.

Q 4 $|\psi\rangle = |1\rangle$. $M_0 = |0\rangle\langle 0|$

Find $P(0)$, $M_0|\psi\rangle$.

Here $P(0) = 0$.

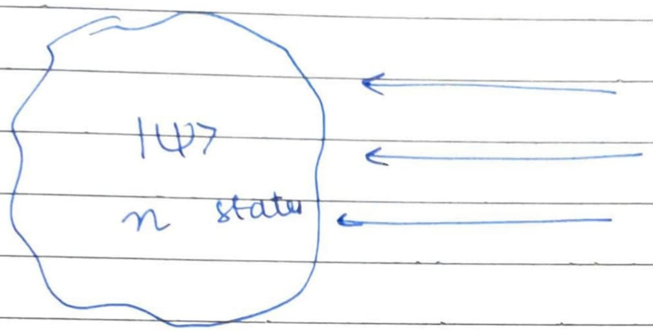
$M_0|\psi\rangle = |1\rangle$.

Distinguishing Quantum States:

Two classical obj. can be distinguished by size shape, weight, mass etc.

Consider 2 parties:-

Alice   Bob



If alice chooses any one state $|\psi_i\rangle$ $(1 \le i \le n)$ from her box, Bob has to guess this state ie: guess the index i.

Alice gives some state to Bob.
Bob now applies some measurement op.
ie: $M_i = |\psi_i\rangle\langle\psi_i|$. for each possible index.

Bob is completely unaware of the fact which state Alice has chosen.

For each state Bob will have different measurement operators: $|\psi_i\rangle$

$$\begin{cases} |\psi_1\rangle\langle\psi_1| \\ \vdots \\ |\psi_n\rangle\langle\psi_n| \end{cases}$$

Now, if $\sum M_i^{+} M_i \neq I$

$$M_0 = \sqrt{I - \sum_{i \neq 0} M_i^{\dagger} M_i}$$

$$= \sqrt{I - \sum_{i \neq 0} |\psi_i\rangle\langle\psi_i|}$$

For any state measurement

$$P(i) = \langle\psi_i| M_i |\psi_i\rangle = 1.$$

$$\Rightarrow \quad P(j) = \langle\psi_i| M_j |\psi_i\rangle = 0.$$

$|\psi_i\rangle$ is taken from an orthonormal basis.

$\therefore$ we can distinguish them when $\{|\psi_i\rangle\}$ forms an ON basis.

<u>Counter Eg</u>:

Alice:

$|0\rangle$

$|+\rangle = \frac{1}{\sqrt{2}}\{|0\rangle + |1\rangle\}$.

Bob:

$M_0 = |0\rangle\langle 0|$

$M_1 = |1\rangle\langle 1|$

$\sum M_i = I$ nec.

When alice chooses $|0\rangle$.

$$M_0|0\rangle = |0\rangle\langle 0|0\rangle = |0\rangle \longrightarrow |0\rangle.$$

$$P(0) = 1.$$

$$M_0|+\rangle = |+\rangle\langle +|0\rangle = \frac{1}{\sqrt{2}}|0\rangle \longrightarrow \text{with half probability } |0\rangle$$

$$P(0) = \frac{1}{2}.$$

Impossible for Bob to guess where the $|0\rangle$ has come from.

→ Indistinguishability of quant. states

→ Non orthogonal quant. states cannot be distinguished

Orthogonal $\Rightarrow$ cannot distinguish
Parallel $\Rightarrow$ full probability

- **Projective Measurement:**

A projective measurement is described by an Observable M, a hermitian op. on state space of the system ~~the~~ being observed.
The observable has a spectral decomposition

$$M = \sum m P_m .$$

projective measurement

[$P_m \rightarrow$ projector on the eigen space of M with ev: m]

Now, $M = \sum \lambda_i |\psi_i\rangle\langle\psi_i|$
$= \sum m |\psi_m\rangle\langle\psi_m|$ } Observable can be exp. as hermitian op.

For eg. $S_z = |0\rangle\langle 0| - |1\rangle\langle 1|$
$S_x = |+\rangle\langle +| - |-\rangle\langle -|$.

Now, $p(m) = \langle\psi| P_m |\psi\rangle$.
& $\dfrac{P_m |\psi\rangle}{\sqrt{p(m)}}$ .... output.

→ Follows completeness condition $\sum M_m^\dagger M_m = I$.
Eg : $|0\rangle\langle 0|0\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 1| = I$.

→ Also, $M_m M_{m'} = \delta_{mm'} M_m$ ⚠ only for Projective measurement

→ Average measurement :-

$$E(M) = \sum_m m\, p(m) = \sum_m m \langle\psi| P_m |\psi\rangle = \langle\psi| \textstyle\sum_m P_m |\psi\rangle$$

$$= \langle\psi| M |\psi\rangle = \langle M \rangle .$$

And. $[\Delta\langle M\rangle]^2 = \langle M^2\rangle - (\langle M\rangle)^2$.

- **Mixed States:**

Eg: No QM involved in chosing a studen from 2 categories with prob p, $(1-p)$

→ Consider QM case:-

$|\psi_1\rangle$ ... prob p_1.
$|\psi_2\rangle$ ... prob p_2.

S_z^+ $|0\rangle\langle 0|$	S_x^+ 50%
50%.	$\{|+\rangle\}$.

4). The state space of a composite system system (physical) is the tensor product of the states spaces of the composite physical system. Moreover if we have systems numbered 1 through n & the system i is prepared in the state $|\psi_i\rangle$ then the joint state of the total system is:

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \otimes \cdots \otimes |\psi_n\rangle.$$

basically if

$|\psi_1\rangle \in H_1$, $|\psi_2\rangle \in H_2$ .... $|\psi_n\rangle \in H_n$

then $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle \in H_1 \otimes H_2 \otimes \cdots \otimes H_n$

Eg: $|\psi_{AB}\rangle \in H_1 \otimes H_2$

$|0\rangle \otimes |1\rangle = \underset{H_2}{|0\rangle |1\rangle} \to H_2$ } tensor product of 2 states, one from H_1 & other from H_2.

- $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ | $|\psi_{AB}\rangle = \underset{H_A}{|x_A\rangle} \otimes \underset{H_B}{|x'_B\rangle}$

separable states.

Q: $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ .... can this be separated out as a product?

No! Cannot be separated. Such states are called entangled states.

$|\psi_{AB}\rangle_{AB} \in H_A \otimes H_B$ which cannot be written as a product of 2 states $|X\rangle_A$ from H_A & $|X'\rangle_B$ from H_B is called as entangled state.

→ there is no classical analogue of this state & also there is no counter part of the correlation given by these states in our classical world.

continuation: with a random dist

any particular state will be $p(S_z^+) + (1-p)(S_x^-)$] classical prob.

Uncertainity comes not because it is inherently uncertain but because we do not know which particle we are picking from the black box.

How can we write exp. for such states?

Consider a black box →

$|\psi_i\rangle = p_i$

$\vdots$

$i = 1, 2, \ldots n.$

The system is described by $\{p_i, |\psi_i\rangle\}$. $\Sigma p_i = 1$.

A physical system (quantum mechanical) $\longrightarrow$ rep. by the state $|\psi\rangle$ (ket vector) $\hookrightarrow H_A$.

$\rightarrow |\psi\rangle\langle\psi|$
$\hookrightarrow$ operator space $B(H_A)$.

This is done to make a distinction between pure states & mixed states.

- Pure state : $|\psi\rangle\langle\psi| = \rho$ ... only one kind of particle.
- Mixed states $\rightarrow \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
here you have selected a particle from an ensemble.

$\therefore$ $\sum p_i |\psi_i\rangle = \rho$
can also be a superposition and a pure states

- The ensemble of ~~pu~~ mixed states can be represented as $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Pos 1: Associated to any isolated physical system is a complex vector space with inner product known as the state space of the system. The system is completely described by the density operator, which is a positive operator with trace 1, acting on the state space of the system. If the quantum sys. ~~is~~ is in the state ρ_i with prob p_i then the density operator of the system is $\sum p_i \rho_i$.

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Thm: An operator ρ is the density operator associated to an ensemble $\{p_i, |\psi_i\rangle\}$ iff it satisfies the condition

i) (Trace condition) ρ has trace 1.

ii) (Positivity condition) : ρ is positive op.

Proof: $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ .... density op.

$$\text{trace}(\rho) = \sum_i p_i \, tr(|\psi_i\rangle\langle\psi_i|).$$

$$= \sum p_i \langle\psi_i|\psi_i\rangle\langle\psi_i|\psi_i\rangle \quad ? \text{ HOW}$$

$$= \underline{\sum p_i = 1}.$$

Let $|\psi\rangle$ be any arbitrary vector in the state space. Then :-

$$\langle\psi|\rho|\psi\rangle = \sum_i p_i \langle\psi|\psi_i\rangle\langle\psi_i|\psi\rangle.$$

$$= \sum p_i |\langle\psi|\psi_i\rangle|^2.$$

(c. prob ≥ 0) prob of $|\psi\rangle$ getting collapsed in $|\psi_i\rangle$. (q. prob ≥ 0)

→ Conversely: suppose ρ is a positive op.

By spectral decomp:- $\rho = \sum_j \lambda_j |j\rangle\langle j|$ $\left\{ \begin{array}{l} |j\rangle \to \text{orthogonal} \\ \lambda_j \to \text{ev. of } \rho. \end{array} \right.$

From trace condⁿ

$tr(\rho) = 1 \Rightarrow \sum \lambda_j = 1.$

$\Rightarrow (\lambda_j)$ nothing but classical prob.

∴ we can say, the sys. is in a state $|j\rangle$ with probability λ_j.

∴ The ~~[illegible]~~ ensemble is $\{\lambda_j, |j\rangle\}$.

Pos2 : The evolution of a closed system (quantum) is described by the unitary transformation.
That is, the state of a system ρ at a time t_1 is related to the state of the system ρ' at a time t_2 is described by the unitary op. U which depends on the time t_1 & t_2, &

$$\rho' = U \rho U^{\dagger}$$

Suppose $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$$\xrightarrow{U} \sum p_i U|\psi_i\rangle\langle\psi_i|U^{\dagger} = U\left(\sum p_i |\psi_i\rangle\langle\psi_i|\right)U^{\dagger}$$

$$= \underline{U\rho U^{\dagger}}$$

Pos3 : Quantum measurements are described by the collection $\{M_m\}$ of measurement op. These are the operators on the state space of the system being measured. The index m, refers to the measurement outcomes that may occur in the experiment.
If the state of the system is in ρ immediately before the measurement, then the prob that result m occurs is given by:

$$p(m) = tr(M_m^{\dagger} M_m \rho)$$

& the state of the sys after measurement will be given by

$$\frac{M_m \rho M_m^{\dagger}}{tr(M_m^{\dagger} M_m \rho)}$$

$\sum_m M_m^{\dagger} M_m = I$ satisfies completeness

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Suppose you have a state $|\psi_i\rangle$ & apply the op M_m

$\{p_i, |\psi_i\rangle\}$

ensemble.

$$p(m|i)$$
$$= \cancel{tr}\ \langle\psi_i| M_m^{+} M_m |\psi_i\rangle$$
$$= tr\left(M_m^{+} M_m |\psi_i\rangle\langle\psi_i|\right)$$

$$p(m) = \sum p(m|i)\, p_i$$
$$= \sum_i p_i\ tr\left(M_m^{+} M_m |\psi_i\rangle\langle\psi_i|\right)$$

$$= tr\left(M_m^{+} M_m \sum p_i |\psi_i\rangle\langle\psi_i|\right)$$
$$= tr\left(M_m^{+} M_m \rho\right)$$

Post measurement : $|\psi_i\rangle \xrightarrow{M_m} |\psi_i^{m}\rangle$

$$|\psi_i^{m}\rangle = \frac{M_m |\psi_i\rangle}{\sqrt{\langle\psi_i| M_m^{+} M_m |\psi_i\rangle}}$$

∴, After measurement, which yields result m ensemble of the system $\{|\psi_i^{m}\rangle, p(i|m)\}$.

Now, density matrix :- $\rho_m = \sum p(i|m)\, |\psi_i^{m}\rangle\langle\psi_i^{m}|$

$$= \sum p(i|m)\, \frac{M_m |\psi_i\rangle\langle\psi_i| M_m^{+}}{\langle\psi_i| M_m^{+} M_m |\psi_i\rangle}.$$

$$p(i|m) = \frac{p(m,i)}{P(m)}$$

$$= \frac{p(m|i)\, p_i}{P(\cancel{q}m)}$$

$$\therefore, \frac{p(i|m)}{p(m|i)} = \frac{p_i}{p_m} \Rightarrow \rho_m = \frac{\sum p_i\, M_m|\psi_i\rangle\langle\psi_i| M_m^{+}}{p_m}.$$

$$= \sum p_i\, \frac{M_m |\psi_i\rangle\langle\psi_i| M_m^{+}}{tr(M_m^{+} M_m \rho)}.$$

So, the state of the system after measurement is given by:

$$\rho = \sum_m p(m)\, \rho_m$$

$$= \sum_m \cancel{tr(M_m^{\dagger} M_m \rho)} \frac{M_m |\psi_i\rangle\langle\psi_i| M_m^{\dagger}}{\cancel{tr(M_m^{\dagger} M_m \rho)}}$$

$$= \sum_m M_m \rho M_m^{\dagger}.$$

Pos 4 The state space of a composite phy. system is the tensor product of the state spaces of the component phy. system. Moreover if we have systems numbered $1 \ldots n$ & the system i is prepared in the state ρ_i, then the joint state of the total system $= (\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n)$ ~~is given~~

Prob:

$$\rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$\rho_2 = \frac{3}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -|$$

Find $\rho_1 \otimes \rho_2$.

Prob: If $|\psi\rangle = \frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}$.

Find $\rho = |\psi\rangle\langle\psi|$

Eg: $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$.

$$\left.\begin{aligned} |a\rangle &= \sqrt{\tfrac{3}{4}}\,|0\rangle + \sqrt{\tfrac{1}{4}}\,|1\rangle \\ |b\rangle &= \sqrt{\tfrac{3}{4}}\,|0\rangle \cancel{+} - \sqrt{\tfrac{1}{4}}\,|1\rangle. \end{aligned}\right\} \text{define}$$

$$\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|$$

Two diff ensembles can give rise to same density matrix.

$$\begin{matrix}\{\frac{1}{2}, |a\rangle\} \\ \{\frac{1}{2}, |b\rangle\}\end{matrix} \longrightarrow \rho \longleftarrow \begin{matrix}\{\frac{3}{4}, |0\rangle\} \\ \{\frac{1}{4}, |1\rangle\}\end{matrix}$$

same density matrix.

- ## Reduced Density Operator:

→ Given a state $|\psi\rangle$, ρ density matrix $= |\psi\rangle\langle\psi|$

→ Density matrix for a composite system
$\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n$ .... individual mat

Eg. of composite system:
$|\psi\rangle = |0\rangle|0\rangle$
or $|\psi\rangle = \underset{A}{|0\rangle} \dfrac{[|0\rangle + |1\rangle]_B}{\sqrt{2}}$. → state of systemB.
State of sys A

Clearly able to identify the system's state in A, B.

When the composite systems are separable it is easy to understand what is system A & B.

But: consider eg:-
$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\{|00\rangle_{AB} + |11\rangle_{AB}\}$.

→ ? (A) sys & ? (B) sys
No way that you can separate sys A & B.

- This is where reduced density op. comes into the picture.
It gives you the state of the system A & system B

Now.
$\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$ ... density op. of system AB.
$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\{|00\rangle + |11\rangle\}\{\langle 00| + \langle 11|\}$.

$tr(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle$ ↪ proof ?

★ # ie: $\rho_{AB} = \frac{1}{2}\{|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|\}$. ↳ ①

→ Suppose we have phy. sys. A, B whose state is described by the density operator ρ_{AB} then the reduced density operator for the system A is defined by

$\rho_A = tr_B(\rho_{AB})$. → tracing out the subsystem B out of A to get description of A.

(Partial trace) different from trace of a matrix. ⚠

continuation

→ Method to do partial trace : ?

where tr_B is a map of operators known as the partial trace over system B.

The partial trace is defined by

$$tr_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \; tr(|b_1\rangle\langle b_2|)$$

where $|a_1\rangle$ & $|a_2\rangle$ are any two vectors in the state space A & $|b_1\rangle$, $|b_2\rangle$ are any 2 vectors in state space B.

So, $tr(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle$

Now, if $\{|b_1\rangle, |b_2\rangle\}$ is an orthonormal basis

$$tr(|b_i\rangle\langle b_j|) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

• Revisiting the ★ eq:-

$$\rho_{AB} = \frac{1}{2}\{|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|\}$$

$$\therefore \quad \rho_A = tr_B(\rho_{AB})$$

$$= \frac{tr_B(|00\rangle\langle 00|) + tr_B(|11\rangle\langle 00|) + tr_B(|00\rangle\langle 11|) + tr_B(|11\rangle\langle 11|)}{2}.$$

Also: $$= \frac{tr_B(|00\rangle\langle 00|) + tr_B(|00\rangle\langle 11|) + tr_B(|11\rangle\langle 00|) + tr(|11\rangle\langle 11|)}{2}$$

consider this term.

why this?

$$tr_B(|00\rangle_{AB}\langle 00|_{AB}) = tr_B\{|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B\}$$

$$= |0\rangle\langle 0| \underbrace{\langle 0|0\rangle}_{=1}.$$

$$= |0\rangle\langle 0|$$

$$tr_B(|11\rangle\langle 00|) = tr_B\{|1\rangle\langle 0| \otimes |1\rangle\langle 0|\}$$

$$= |1\rangle\langle 0| \underbrace{\langle 0|1\rangle}_{\to 0}$$

$$= 0.$$

$||^{ly}$

$$tr_B(|00\rangle\langle 11|) = 0$$

$$tr_B(|11\rangle\langle 11|) = |1\rangle\langle 1|$$

$$\therefore \quad \rho_A = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}.$$

[which is also $= \rho_B$ in this particular case]

$\boxed{\text{Prob:}}$ Find the reduced density operator of ~~ρ_{AB}~~

$\rho_{AB} = \cancel{\frac{1}{4} I}$

$$\rho_{AB} = p\, I/4 + (1-p)\, |\Psi\rangle_{AB}\langle\Psi|.$$

where

$I_{4\times 4}$ is the identity matrix with order 4.

& $|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}$. $\quad \rho_A, \rho_B = ?$

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Ans: $\cancel{\rho_{AB}}$ $\rho_A = tr_B(\rho_{AB})$.

$= tr_B(p\,I/4) + tr_B\left(\frac{(1-p)}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)\right)$.

$= tr_B\left(p\frac{I}{4}\right) + \frac{1-p}{2}\left(\cancel{0 + 1 + 0 +} |0\rangle\langle 0| + |1\rangle\langle 1|\right)$

Check how to get this.

$= 4 \cdot \frac{p}{4} + \frac{(1-p)}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ is it ?

$= p + \frac{(1-p)}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

- ## Schmidt Decomposition:

Theorem: Suppose $|\psi\rangle$ is a pure state of a composite system AB then $\exists$ orthonormal states $|i\rangle_A$ for system A & orthonormal states $|i\rangle_B$ of the system B such that
$$|\psi\rangle = \sum_i \lambda_i |i\rangle_A |i\rangle_B .$$

where λ_i are non-negative real no:s satisfying $\sum \lambda_i^2 = 1$ ... known as schmidt's co-efficients.

Proof: Let us assume that the state space of A & B are of the same dimensions (in the resp. hilbert spaces, same no: of basis vetors).
Let $|j\rangle$ & $|k\rangle$ be any fixed orthonormal basis for systems A & B respectively. then $|\psi\rangle$ can be written as:
$$|\psi\rangle_{AB} = \sum_{jk} a_{jk} |j\rangle |k\rangle$$
↳ complex no:s of some matrix A

By singular value decomposition th. we can always write

$$A = UDV$$ (U, V are unitary matrices & D is diagonal matrix with non-neg. els.)

Then $|\psi\rangle$ becomes:-

$$|\psi\rangle = \sum_{ijk} \underbrace{u_{ji}\, d_{ii}\, v_{ik}}_{\text{how?}} |j\rangle|k\rangle.$$

Define $|i\rangle_A = \sum_j U_{ji} |j\rangle$ ... basis obtained from the

$|i\rangle_B = \sum_k v_{ik} |k\rangle.$

& $\lambda_i = d_{ii}$

$$\Rightarrow |\psi\rangle = \sum_i \lambda_i |i\rangle_A |i\rangle_B.$$

? CHECK! It is easy to check that $|i\rangle_A$ forms an orthonormal set from the unitarity of U & the orthonormality of $|j\rangle$. $|{}^{ly}$ $|i\rangle_B$ forms an orthonormal set.

- PURIFICATION:

Suppose we are given a state ρ_A ? of a quantum system A.

(For every system A). It is possible to introduce another system R & define a pure state $|AR\rangle$ for the join system such that ρ_A can be thought of as a subsystem of a joint composition AR

ie: $\rho_A = \mathrm{tr}_R \left(|AR\rangle\langle AR| \right).$

ie: Given a mixed state it is always possible to purify the state by projecting it into a larger hilbert space AR. [called the church of larger hilbert space? lmao].

- → Process is known as the purification of a quant state.
- → R : Reference system.

- To show that purification can be done for any state:
  → construct the system R & purification $|AR\rangle$ for the system ρ_A.

Suppose ρ_A has an orthonormal decomposition

$$\rho_A = \sum p_i |i_A\rangle\langle i_A|$$

To purify ρ_A we introduce a system R which has the same state space of system A (described by the same basis) with orthonormal basis $|i\rangle_R$.

Define a pure state :- $|AR\rangle = \sum \sqrt{p_i}\, |i_A\rangle|i_R\rangle$

We now calculate the density op. of the sys A corresponding to the state $|AR\rangle$. which is:-

$$tr_R\,(|AR\rangle\langle AR|).$$

$$= \left(\sum_{ij} \sqrt{p_i}\sqrt{p_j}\, |i_A\rangle\langle j|_A\right) tr\,(|i_R\rangle\langle j_R|)$$

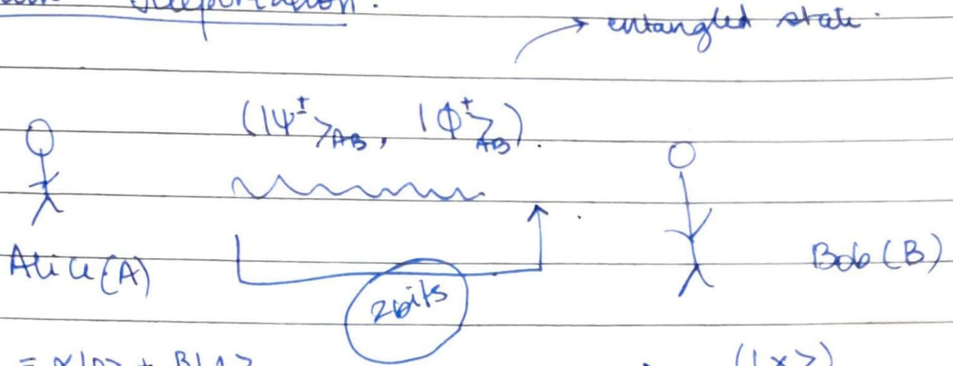
$$= \sum_{ij} \sqrt{p_ip_j}\, |i\rangle_A\langle j|_A\, \delta_{ij}$$

ie: when $i=j$ ⇒ $\sum_{ij} p_i\, |i\rangle_A\langle i| = \rho_A$.

Thus $|AR\rangle$ is the purification of ρ_A.

Quantum Teleportation:

Protocol.



$|x\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow (|x\rangle)$

(α, β unknown to Alice)

Teleportation does not mean the particle itself is being transmitted. It is the information that is teleported. Alice & Bob can separate arbitrary.

Consider the bell states:

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\{|01\rangle + |10\rangle\}$$

$$\& \quad |\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}.$$

These 4 states together called as Bell basis
$\{|\Psi^{+}\rangle, |\Psi^{-}\rangle, |\Phi^{+}\rangle, |\Phi^{-}\rangle\}$... O.N basis.
ie: $\langle\Psi^{\pm}|\Psi^{\pm}\rangle = 1.$

The entangled state can be any one of the bell states.

→ Write $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ in this basis.

$$|00\rangle = \sqrt{2}\{|\Phi^{+}\rangle - |\Phi^{-}\rangle\}.$$

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Teleportation:

Let us assume that Alice & Bob share an entangled state $|\phi^+\rangle_{AB}$

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\{ |00\rangle_{AB} + |11\rangle_{AB}$$

Alice Bob.

∴ the composite system

$$|\chi\rangle_A \otimes |\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle)(|00\rangle_{AB} + |11\rangle_{AB})$$

$$= \frac{1}{\sqrt{2}}\{ \alpha(|00\rangle)|0\rangle + \beta(|10\rangle)_{AA}|0\rangle_B + \alpha(|01\rangle_A)|1\rangle_B + \beta(|11\rangle)_B|1\rangle_B \}$$

After re-arranging :-

Alice

$$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \Big[|\phi^+\rangle_{AA} \otimes (\alpha|0\rangle + \beta|1\rangle)_B + |\phi^-\rangle_{AA} \otimes (\alpha|0\rangle - \beta|1\rangle)_B + |\psi^+\rangle_{AA} \otimes (\alpha|1\rangle + \beta|0\rangle)_B + |\psi^-\rangle_{AA} \otimes (\alpha|1\rangle - \beta|0\rangle)_B \Big].$$

If Alice carries out the measurements in the $\{|\psi^+\rangle, |\psi^-\rangle, |\phi^+\rangle, |\phi^-\rangle\}$ basis,

If Alice's measurement comes out to be $\cancel{|\phi^+\rangle}$

encode as		Alice	Bob	
	00	$\lvert\phi^+\rangle$	$\alpha\lvert 0\rangle + \beta\lvert 1\rangle$	→ The state gets t
	01	$\lvert\phi^-\rangle$	$\alpha\lvert 0\rangle - \beta\lvert 1\rangle$	Bob doesnt get the exact state
	10	$\lvert\psi^+\rangle$	$\alpha\lvert 1\rangle + \beta\lvert 0\rangle$.	
	11.	$\lvert\psi^-\rangle$	$\alpha\lvert 1\rangle - \beta\lvert 0\rangle$.	

To distinguish b/w 4 states : 2 bits

But success probability = 1/4.

00 : whenever Alice measures $|\Phi^+\rangle$, she sends 00 ... classical bits to Bob.

→ Bob simply applies I.

01 : For 01 :- apply $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\sigma_z(\alpha|0\rangle - \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle$

10 For 10 :- apply $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Bitflip op.

$\sigma_x(\alpha|1\rangle + \beta|0\rangle) \rightarrow \alpha|0\rangle + \beta|1\rangle$.

11 $i\sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

It may apparently look like faster than light communication is happening which is not the case.

∵ Bob doesn't know his state until Alice sends the 2 classical qubits.

→ Bob's density matrix (reduced) before & after the measurement

↳ you'll find both of them to be same

ie: $\rho'_B = \rho_B$. (Try).

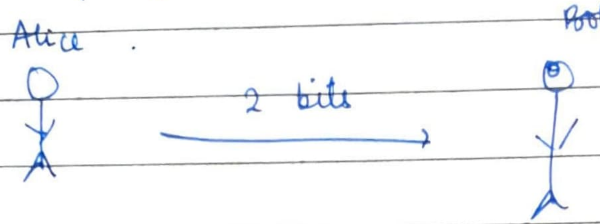
∴ By simply looking the the density matrix Bob is oblivious of the state he has.

• Superdense Coding :

Protocol:

It involves 2 parties (Alice & Bob) who are far apart from each other.

Target : transmit some classical info from Alice to Bob with the help of quantum resources.



Suppose Alice posses 2 bits of classical info but she is only allowed to send a single qubit to Bob. Is this possible ? Yes.

Generally : a qubit is an infinite extension of ~~qubits~~ bits

How : if $\exists$ an entangled state shared with Alice & Bob

Let the entangled shared state be :

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\{|00\rangle_{AB} + |11\rangle_{AB}\}.$$

Alice Bob

→ Note that $|\phi^+\rangle$ is a fixed state prepared by some third party & shared it b/w Alice & Bob by sending 1 qubit to each.

Alice : by sending here one qubit to Bob it is possible to comm. 2 classical bits.

$00 \xrightarrow{I\otimes I} (I\otimes I)|\phi^+\rangle \to |\phi^+\rangle = 1/\sqrt{2}\,(|00\rangle + |11\rangle)$.

$01 \xrightarrow{\sigma_z\otimes I} (\sigma_z\otimes I)|\phi^+\rangle \to |\phi^-\rangle = 1/\sqrt{2}\,(|00\rangle - |11\rangle)$.

$10 \xrightarrow{\sigma_x\otimes I} (\sigma_x\otimes I)|\phi^+\rangle \to |\psi^+\rangle = 1/\sqrt{2}\,(|10\rangle - |01\rangle)$

$11 \xrightarrow{i\sigma_y\otimes I} \to |\psi^-\rangle = 1/\sqrt{2}\,(|01\rangle - |10\rangle)$

Bob has: $|\phi^+\rangle_{BB}$ for [00]
$|\phi^-\rangle_{BB}$ for [01]
$|\psi^+\rangle_{BB}$ for [10]
$|\psi^-\rangle_{BB}$ for [11].

ON basis ⇒ each of these can be distinguished by applying measurement op.

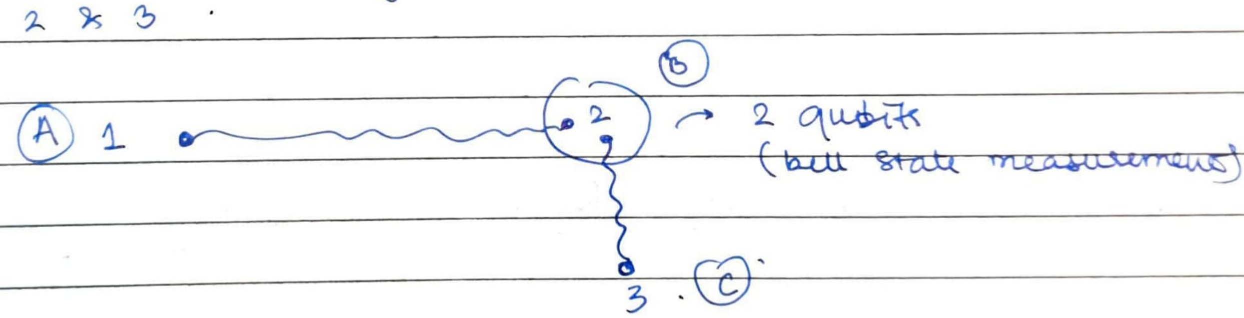
∴, Alice is successful in communicating 2 bits of classical info to Bob.

For that process to happen she needs to send one qubit to Bob.

- **Entangled Swapping:**

Protocol:

Suppose ∃ an entanglement b/w 1 & 2 and b/w 2 & 3.



By doing the bell state measurement, a new entanglement is created between 1 & 3.

Now,

$$\frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}_{AB} \otimes \frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}_{BC}.$$

$$= \frac{1}{2}\{|0\rangle_A(|00\rangle)_B|0\rangle_C + |1\rangle_A(|10\rangle_B)|0\rangle_C + |0\rangle_A(01)_B|1\rangle_C + |1\rangle_A(|11\rangle_B|1\rangle_C\}$$

Each of Bob's qubit can be rep in terms of bell basis.

$$= \frac{1}{2\sqrt{2}} \Big\{ |0\rangle \left[|\phi^+\rangle + |\phi^-\rangle \right] |0\rangle + |1\rangle \left[|\psi^+\rangle - |\psi^-\rangle \right] |0\rangle$$

$$+ |0\rangle \left[|\psi^+\rangle + |\psi^-\rangle \right] |1\rangle + |1\rangle \left[|\phi^+\rangle - |\phi^-\rangle \right] |1\rangle$$

$$= \frac{1}{2\sqrt{2}} \cdot \Big[|\phi^+\rangle_{BB} \left(|00\rangle + |11\rangle \right)_{AC} + |\phi^-\rangle \left(|00\rangle - |11\rangle \right)_{AC} + |\psi^+\rangle_{BB} \left(|10\rangle + |01\rangle \right)_{AC} + |\psi^-\rangle_{BB} \left(|10\rangle - |01\rangle \right)_{AC}$$

For each measured state of BB we have a definitive entangled state b/w A & C.

ie: $$= \frac{1}{2} \left[|\phi^+\rangle_{BB} \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right)_{AC} + |\phi^-\rangle_{BB} \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}} \right)_{AC} + |\psi^+\rangle_{BB} \left(\frac{|10\rangle + |01\rangle}{\sqrt{2}} \right)_{AC} + |\psi^-\rangle_{BB} \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}} \right)_{AC} \right]$$

x.———————x

Entropy & Quantum Entropy:

Information → present in the quantum system
How do I quantify information in general.

Consider egs:

1) Sun rises in east
2) There will be a phone call in 1/2 hour.
3) ~~Rs~~ Snowfall in hyderabad.

Which is the most random? (2nd one)

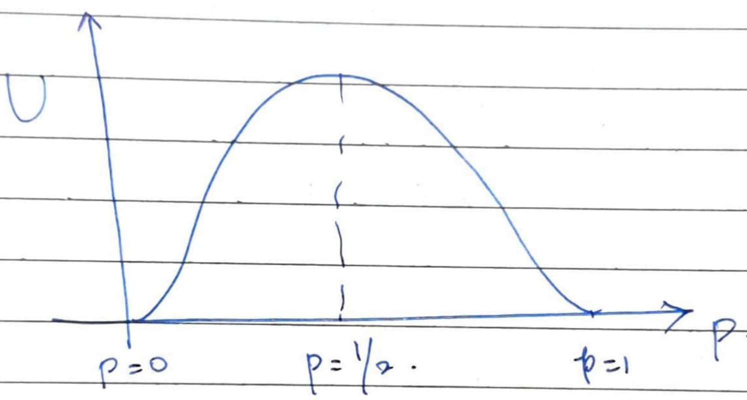
→ For statement (2) randomness is highest ⇒ uncertainity ↑.
How to quantify uncertainity in these sentences.

Intuitively: uncertainity is low when $p \to 1$ or $p \to 0$
uncertainity is max when $p \to 1/2$.

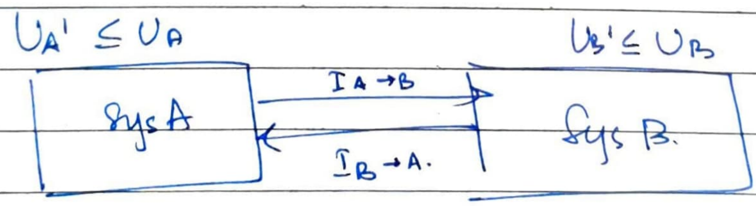
$\therefore$, $U = f(p)$.

$U \to 0$ $p \to 1$
$U \to 0$ $p \to 1$] $U \to$ max when $p \to 1/2$.

$\therefore$, Graph:



Measuring uncertainity in a sys ⇔ measuring info



Interaction b/w systems reduces uncertainity.
↓ in uncertainity ⇒ gain of information.

$\therefore$,

1. I is a function of probability.
2. $I(p) \to$ max: $p \to 1/2$ $I(p) \to 0$ $p \to 0, 1$.
3. $I(AB) = I(A) + I(B)$. (Additive property).

[Idea to give a mathematical formula of quantifying info]

(4) $I(A) \geq 0$ (cannot be negative).

Suppose there is a source emmitting symbols.

$X(x_i)$	$P(x_i)$.
x_1	p_1
x_2	p_2
$\vdots$	
x_n	p_n.

The info content of the ~~system~~ R.V X is given by the entropy $H(x) = H(p_1, p_2 .. p_n)$.

$H(x) = -\sum_x p_x \log p_x$. $\longrightarrow$ Shannon entropy.

$\because \log(AB) = \log(A) + \log(B)$ ... used in additivity of information.

For a binary case :-

$H(p) = -p\log p - (1-p)\log(1-p)$.

When $p=0$ $H(x) = 0\log 0$.

When $p \to 0$ $H(p) = 0\log 0$.

$\lim_{p\to 0} H(p) = \lim_{p\to 0} p\log p = 0$

→ probability does not depend on the nature of the sources X_1 & X_2.

- Now,

(1) $I(E)$ is the function (only of) of the probability of the event E. (where p is the p. of the event E)

$I(E) = I(p)$.

(2) I is a smooth function of probability.

(3) $I(pq) = I(p) + I(q)$ $p, q > 0$. where two events occur with prob p, q

$$\boxed{H_{bin}(p) = -p\log p - (1-p)\log(1-p)} \rightarrow \text{binary entropy.}$$

max when $p = 1/2$.

(4) $H(p)$ is a concave function of p

ie:- $H(q P_u + (1-q) P_A)$ $P_u \rightarrow$ prob dist

$\geq \underbrace{q H(P_u) + (1-q) H(P_A)}_{\text{average}}$ $P_A \rightarrow$ prob dist.

more info $\because$ there is an uncertainity of choosing the prob dist P_u or P_A.

- Relative entropy :

Suppose there are 2 prob dist

$p(x)$ $q(x) \rightarrow$ real prob. dist that you're looking over.

estimated in calculating $q(x)$.

$H(p(x) \| q(x))$ ... rel entropy gives you the error in estimating the P.D $q(x)$ by $p(x)$.

$$\boxed{H(p(x) \| q(x)) = \sum_x p(x) \log \frac{p(x)}{q(x)}}$$

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$= \sum_x p(x) \log p(x) - \sum p(x) \log q(x).$

$= -H(x) - \sum p(x) \log(q(x))$

→ R. Entropy can be thought about as an error in estimating the prob dist $q(x)$.

→ Can be thought about as the dist b/w prob dist $p(x)$ & $q(x)$.

→ Also a measure of correlation (b/w 2 prob dist)

If it has to be distance then it must satisfy:-

1. $D(p,q) \geq 0$
2. $D(p,q) = D(q,p)$.
3. $D(p,q) \leq D(p,r) + D(r,q)$.

If a measure satisfies all these three it is called a metric.

→ Our R. Entropy satisfies only condition ①.

Thm: Relative entropy is non-negative.

Proof: Property: $-\log x \geq \frac{(1-x)}{\ln 2}$

$H(p(x) \| q(x)) = - \sum p(x) \log \frac{q(x)}{p(x)} \geq \frac{1}{\ln 2} \sum p(x) \left(1 - \frac{q(x)}{p(x)}\right)$

$= \left(\frac{1}{\ln 2} \sum_x p(x) - q(x)\right)$

$= \frac{1}{\ln 2} (1-1) = 0 \quad \therefore$ non-negative

$$\Rightarrow \quad H(p(x) \,||\, q(x)) \geq 0.$$

Now If $q(x) = \frac{1}{d}$ be a uniform distribution

$$H(p(x) \,||\, q(x)) = H(p(x) \,||\, 1/d)$$

$$= H(x) - \sum p_x \log\left(\frac{1}{d}\right)$$

$$= \log d \sum p_x - H(x).$$

$$\log d - H(x) \geq 0.$$

$$\therefore, \quad H(x) \leq \log d. \quad [\text{no. of bits required}]$$

If $d = 2 \ldots \quad H(x) \leq 1.$

- **Joint Entropy:**

The joint entropy of a pair of R.V. X & Y is given by:

$$H(x,y) = -\sum_{x,y} \underbrace{p(x,y)}_{\text{joint PD}} \log p(x,y).$$

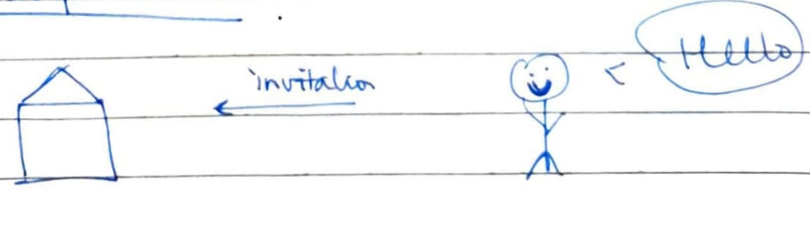
The joint entropy measures the total uncertainty about the pair (x,y).

- **Conditional Entropy:**

It is the remaining uncertainty about pair (x,y) given that now you know Y.

$$H(X|Y) = H(X,Y) - H(Y)$$

- **Mutual Information:**



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Uncertainity of arrival
If I get to know about a tyre puncture

$\therefore$ $H(X,Y) = \underset{\text{initial}}{H(X)} - \underset{\text{final}}{H(X|Y)}$.

How event X got affected by Y
Mutual info b/w X & Y.

$\therefore$ $= H(X) - (H(X,Y - H(Y))$
$= H(X) + H(Y) - H(X,Y)$.

This gives you the correllation b/w 2 R.Vs X & Y

Q: Can you express mutal info in terms of rel. Entropy
A: Yes!

$D(p(xy) \| p(x)p(y))$
$= \sum p(x,y) \log\left(\frac{p(x,y)}{p(x)q(x)}\right)$. derive.

$= H(p(x)) + H(p(y)) - H(p(x,y))$.
$= I(X:Y)$.

Distance measures the correlation b/w 2 vars X & Y

- Basic properties of Shannon Entropy:

① $H(X,Y) = H(Y,X)$; $H(X:Y) = H(Y:X)$

② $H(Y|X) \geq 0$. & thus $H(X:Y) \leq H(Y)$ with equality iff Y is a func. of X ie: $Y = f(x)$

③ $H(X) \leq H(X, Y)$ ... eq. when $Y = f(x)$.
$\because \uparrow$ 2 R.V.

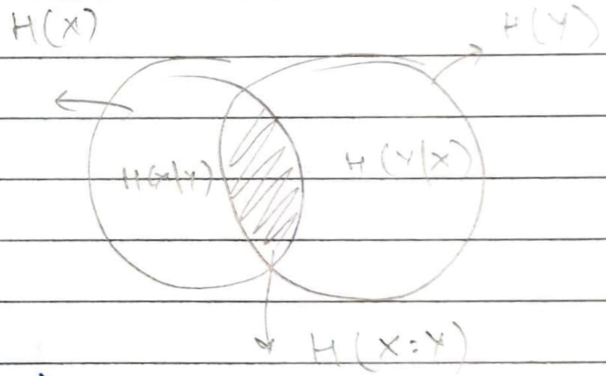
④ Subaditivity : $H(X,Y) \leq H(X) + H(Y)$
with eq. iff X, Y indep. R.V.

⑤ Conditioning reduces ~~of~~ the entropy
$H(Y|X) \leq H(Y)$ & thus
$H(X:Y) \geq 0$ with equality iff X, Y indep R.V.

⑥ $H(X|Y,Z) \leq H(X|Y)$.

- Venn diagramatic Representation of Shanon Entropy.

→ Chain Rule of conditional entropies :



Let $X_1, X_2 \ldots X_n$ & Y be any set of random variables. Then
$$H(X_1, X_2 \ldots X_n|Y) = \sum_{i=1}^{n} H(X_i|Y, X_1 \ldots X_{i-1}).$$

Proof: We prove this result for $n=2$ first.
$$H(X_1, X_2|Y) = H(X_1, X_2, Y) - H(Y)$$
↳ defn of conditional entropy.
$$= [H(X_1, X_2, Y) - H(X_1, Y) + H(X_1, Y) - H(Y)]$$
$$= H(X_2|Y; X_1) + H(X_1|Y). \quad \text{—— ①}$$

Let us assume the result for general n and show it holds for $n+1$.

$$H(\underbrace{X_1, X_2 \ldots X_{n+1}}_{1 \text{ line}} | Y) = H(X_2 \ldots X_{n+1} | Y, X_1) + H(X_1|Y)$$

$$= \sum_{i=2}^{n+1} H(X_i | Y, X_1 \ldots X_{i-1}) + H(X_1|Y)$$

$$= \sum_{i=1}^{n+1} H(X_i | Y, X_1 \ldots X_{i-1}).$$

- Von Neumann Entropy :-

$|\psi\rangle \rightarrow |\psi\rangle\langle\psi| = \rho_p$ (pure state).

$\{p_i, |\psi_i\rangle\} \longrightarrow \rho_m$. (mixed state).

↳ prob dist.

prob	Pure state.
p_1	$\lvert\psi_1\rangle$
p_2	$\lvert\psi_2\rangle$
p_3	$\lvert\psi_3\rangle$
$\vdots$	$\vdots$

$\sum p_i = 1$

how to measure this uncertainty

→ uncertainty induced in the quantum system because of classical mixing

The von-neumann entropy measures the uncertainty in the quantum system because of the classical mixing of $\underbrace{\text{pure states}}$.
↳

$p_1, p_2 \ldots p_n \quad | \quad |\psi_1\rangle, |\psi_2\rangle \ldots |\psi_n\rangle$

⇒ Why Shannon entropy will not suffice?

$H(p) = -\sum p_i \ln(p_i)$ → wrong to give this measure !

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Density matrix can be written as :-

$$\rho = p_1 |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2| \quad \text{——} \; (1)$$
$$= p_i' |\psi_i'\rangle\langle\psi_i'| + \ldots$$

By just looking at the density matrix it is impossible to tell from which combination it has come.

∴ For (1) the shannon entropies will be $H(p_i)$ in case 1 & $H(p_i'$ in case 2)

We need a way by which we can quantify the uncertainty

From the spectral decomposition

$$\rho = \sum \lambda_i |i\rangle\langle i| \qquad (\{|i\rangle \to \text{eigen basis}\})$$

∴ $\boxed{S(\rho) = -\sum \lambda_i \log \lambda_i}$ ↳ von neuman entropy.

$$S(\rho) = -\,\mathrm{tr}(\rho \log \rho)$$

$0 \log 0 = 0$ completely mixed operator in 'd' dimension space) → max entropy : $\log d$.

Probs: Calculate $S(\rho)$ for following ρ's.

i) $\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = |\psi\rangle\langle\psi| = |0\rangle\langle 0|$

ii) $\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right).$

iii) $\rho = \frac{1}{3}\,[$

i) $\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

pure states 1 eigen value will be 1 & rem will be 0

$\therefore \quad S(\rho) = \sum_{i=1}^{2} \lambda_i \log(\lambda_i)$.

$= 1\log 1 + 0\log 0 = 0$.

ii) $\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ ~~$A\lambda =$~~ $A\vec{v} = \lambda\vec{v}$ $\det(A - \lambda I) = 0$.

$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1/2-\lambda & 1/2 \\ 1/2 & 1/2-\lambda \end{bmatrix}$.

$\det = \left(\frac{1}{2}-\lambda\right)^2 - \frac{1}{4} = 0$.

$\lambda^2 + \cancel{\frac{1}{4}} - \lambda - \cancel{\frac{1}{4}} = 0$ $\lambda(\lambda-1) = 0$.

$\lambda = 0$ or $\lambda = 1$.

$\therefore \quad S(\rho) = 1\log 1 + 0\log 0$

This is diagonal matrix in the $|+\rangle$ $|-\rangle$ basis

All pure states will have $S(\rho) = 0$.

Mixed states. $S(\rho) \neq 0$

$tr(\rho^2) = 1$ pure states

$tr(\rho^2) < 1$ mixed states

- Quantum Relative Entropy:

(Analogous to classical)

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$S(\rho\|\sigma)$
- Error in estimating state σ by ρ
- dist b/w 2 states ρ & σ
- its a measure of correlation.

The quantum relative entropy of ρ to σ is defined by

$$S(\rho\|\sigma) = tr(\rho \log \rho) - trace(\rho \log \sigma).$$

[Similar to classical R.E, quantum R.E can be ∞]

→ (Klein's inequality) :-
The quantum relative entropy is non-negative.

$$\cancel{S(\rho)} \quad S(\rho\|\sigma) \geq 0$$

with equality holding iff $\sigma = \rho$.

Proof: Let $\rho = \sum_i p_i |i\rangle\langle i|$ and

$\sigma = \sum_j q_j |j\rangle\langle j|$ be orthonormal decomposition for ρ & σ

By R.E defn:-

$$S(\rho\|\sigma) = tr(\rho \log\rho) - tr(\rho \log \sigma).$$

$$= \sum p_i \log p_i - \underbrace{\sum_i \langle i|\rho \log \sigma|i\rangle}_{\text{How?}} \quad \text{--- (1)}$$

$$\rho = \sum_i p_i |i\rangle\langle i|$$

$$\langle i|\rho = p_i \langle i|i\rangle\langle i| + p_i \sum_{j\neq i} \langle i|j\rangle\langle i|$$

$$= p_i \langle i| \quad \text{--- (2)}$$

$$\langle i|\log\sigma|i\rangle = \langle i| \left(\sum_j \log q_j |j\rangle\langle j|\right)|i\rangle.$$

$$= \sum_j \log q_j \langle i|j\rangle\langle j|i\rangle = \sum_j \log q_j P_{ij}$$

where $\langle i|j\rangle\langle j|i\rangle \geq 0$.

$$S(\rho\|\sigma) = \sum p_i \log p_i - \sum p_i \langle i|\log\sigma|i\rangle$$

$$= \sum p_i \log p_i - \sum_i p_i \sum_j P_{ij} \log q_j$$

$$= \sum_i p_i \left(\log p_i - \sum_j p_{ij} \log q_j \right).$$

Now, $p_{ij} \geq 0$ $\quad \sum_i p_{ij} = 1, \quad \sum_j p_{ij} = 1.$

$\log()$ is a strictly concave function.

$$\sum p_{ij} \log q_j \leq \log r_i \quad \text{where} \quad r_i \equiv \sum_j p_{ij} q_j$$

With equality holds if $\exists$ a value of j for which $p_{ij} = 1$.

$$S(\rho\|\sigma) \geq \sum p_i \log p_i - \sum p_i \log r_i$$

$$= \sum p_i \log\left(\frac{p_i}{r_i}\right)$$

$$= D(p_i \| r_i) \geq 0.$$

$$\Rightarrow \boxed{S(\rho\|\sigma) \geq 0} \quad \text{Proved.}$$

→ Basic Properties of Quantum Von-Neuman Entropy

① The von-neuman entropy is non-negative.

② In a d-dimensional Hilbert space the entropy is at most $\log d$. iff the system is completely mixed state I/d

(3) Suppose AB is a composite pure system then

$$S(A) = S(B).$$

$$\rho_A = \rho_B = \sum \lambda_i |i\rangle\langle i|.$$

(4) Suppose p_i are the probabilities & the state ρ_i has a support on an orthonormal subspace then.

$$\boxed{S\left(\sum p_i \rho_i\right) = H(p_i) + \sum p_i S(\rho_i)}$$

(5) Joint Entropy Th :-

Suppose p_i are probabilities, $|i\rangle$ are orthogonal states for the system A & ρ_i are the density operator of the system B then we have

$$S\left(\sum p_i |i\rangle\langle i| \otimes \rho_i\right) = H(p_i) + \sum p_i S(\rho_i).$$

→ Quantum conditional entropy & Mutual Info :

$$\cancel{S(A|)} \quad S(A|B) = S(A,B) - S(B).$$

$$S(A:B) = S(A) + S(B) - S(A,B)$$
$$= S(A) - S(A|B)$$
$$= S(B) - S(B|A).$$

Consider a composite system in entangled state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \Big] \rightarrow \text{pure state.}$$

$$S(AB) = S(\psi_{AB}) = 0.$$

$$\rho_A = \rho_B = \frac{1}{2}\left\{|0\rangle\langle 0| + |1\rangle\langle 1|\right\}.$$

$$\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}.$$

$$S(A) = S(B) = 1.$$

$$S(A|B) = 0 - 1 = -1.$$

Negative entropy ⇒ entangled state