REAL NUMBER SYSTEM

- **Set:** A Collection of Objects.

- Real No: System : 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.
  Uses base 10 system

↳ Natural Nos: 'N' : 1, 2, 3 .....
They are closed under addition & multiplication

↳ set of Integers 'Z'.
Positive, Negatives & 0.

↳ Rational Numbers :
Those which can be represented as
P/Q ; $Q \neq 0$

↳ Irrational Numbers :
Eg : $\sqrt{2}$, $\sqrt{3}$, π, e
Decimals are non-terminating. non repeating.

- **Geometric Representation :**

→ Canter Dedekind Axiom
For every real no: there is a point on the number line

Between any 2 rational nos: there exists infinitely many rational nos:.
∴, Everywhere dense set.

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▲ Everywhere dense set :-
In the neighbourhood of every rational number there are infinitely many irrational nos: ie: set of ~~nat~~ rational nos:

A set having additive, ~~multiplicative~~ commutative, × & distributive properties is called a FIELD.

+	1.	closure	$a+b \in R$
	2.	Commutative	$a+b = b+a$.
	3.	Associative	$(a+b)+c = a+(b+c)$.
	4.	Additive Identity	$a+0 = 0+a$.
×	5.	Closure	$ab \in R$
	6.	Commutative	$ab = ba$
	7.	Associative	$(ab)c = a(bc)$.
	8.	Identity	$a.1 = 1.a$.
+	9.	Inverse	$a+(-a) = 0$.
×	10.	Multiplicative	$a \cdot \frac{1}{a} = 1$.
	11.	Distributive	$a\cdot(b+c) = ab + ac$.

▲ POINT SET :
A set of points on the real axis is called a one-dimensional point set.

$a \leq x \leq b \rightarrow$ set of all x is called interval.

$[a, b]$ closed

(a, b) open

No: of elements in a set : Cardinal Number.
for countably infinite :- N_0

- COUNTABILITY :

A ~~said~~ set is said to be countable if its elements can be placed in a 1-1 correspondance with the set of Natural Nos.

$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{array}$$

Eg: stars in the sky.
countably finite or infinite

→ Uncountable Set:

Not every element can be accounted for.
Eg: (0, 1). In this interval, there are numerous decimal nos.
Eg: Spectrum of colours.

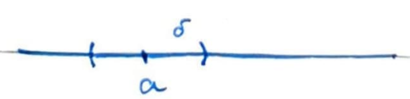
Cross product of set of real nos:-
$R \times R \rightarrow R^2$ (plane).
Euclidean space R^3.

→ Neighbourhood

Mathematically :- Neighbourhood $|x-a| < \delta$

It is an open set

$\boxed{\text{Open set}}$ is a set such that every point is an interior point.



Neighbourhood is a set of all pts x $\forall$ $|x-a|<\delta$.

$\boxed{\text{closed set}}$

Limit Point : A limit point (accumulation pt.) is a point 'l' such that every deleted δ neighbourhood contains point of the set.

→ Set containing all limit points when the limit point is at in the set it is a closed set.

Interior pt : contains at least one neighbourhood is contained within the set

Eg:

1. $|z|<1 \in \mathbb{R}^2$: open } circle
2. $|z|\leq 1 \in \mathbb{R}^2$: closed }

3. Set of integers : closed set.

$\because$ 0 1 2

$(-\infty,0)\cup(0,1)\cup(1,2)$ → open sets

4. $\frac{1}{n}$, $n=1,2,3$:
5. complex nos: closed.

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→ To prove the everywhere dense :-
All rational no:s are countable.
consider the interval [0, 1]

To prove that the set of real no:s lying in the interval (0,1) is uncountable

Assume the contrary, (it is countable). So we can list out all the elements in this interval.

0.973256....

0.80800 8000....

0.765 764 763....

Consider, 0.873256 or 0.984367
these aren't a part of the set.
Hence proved.

- If every point in a set is an interior point then the set is an open set.

- Every neighbourhood is an open set.

- A closed set is a compliment of an open set.

Eg: The ∞ set of integers is a closed set.

To prove $\{1, 2, 3, 4\}$ is a closed set.

Let $A = \{1, 2, 3, 4\}$

$\bar{A} = (-\infty, 1) \cup (1, 2) \cup (2, 3)$

These are all open sets.

$\therefore$ A is a closed set.

- Limit Point:

Let l be the limit point of a set E. Then every neighbourhood of 'l' contains a point of the set E other than l.

Finite sets do not have limit points.

→ Transendental no:s

are those no:s which cannot be expressed as a solution of a polynomial equation with integer co-efficients.

Eg: e, π.

- BOUNDS:

If for all no:s x of a set there exists a number M such that $x \leq M$

M is the upper bound of the set

$M \leq x \rightarrow$ lower bound of the set.

If both bounds exists, the set is bounded.

Greatest lower bound infimum

Lowest upper bound suprimum.

→ Least Upper Bound Prop :-

Let M be a least upper bound

ie: $\exists$ (there exists) one element of the set such that $x > M - \epsilon$

Let there be no such x.

$\therefore$ $M - \epsilon$ is the lowest upper bound which isn't true.

- Blzano - Weierstrase Th.

Every bounded infinite set has atleast one limit point.

→ SEQUENCES AND SERIES.

Eg: Seq. of natural no:s.

$1, 2, 3, 4, 5, \ldots$

If the sequence has infinitely many terms it is an infinite sequence.

$a_0, a_1, a_2, a_3, \ldots a_n$

$\{a_i\}_{i=1}^{\infty}$

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Examples.

- $1, 2, 3, 4 \ldots\ldots, n$ $(n = 1, \ldots \infty)$
- $2, 4, 6, 8 \ldots\ldots, 2n$ ~~(n = 2, 4, 6~~.
- $5, 10, 15, 20 \ldots\ldots 5n$
- $1, 1, 2, 3, 5, 8 \ldots\ldots$

Write the sequence : $a_n = \frac{n^2+n}{2}$.

$1, 3, 6, 10 \ldots\ldots$

1. Depiction through an explicit formula ↑

2. Depiction through
   a). Recursive formula
   Eg: $\left.\begin{array}{l} a_{n+1} = a_n + a_{n-1} \\ a_{n+2} = a_{n+1} + a_n \end{array}\right\}$ Fibonacci nos:

→ ARITHEMATIC PROGRESSION :-

Eg: $1, 2, 3, 4 \ldots\ldots, n$] $A = 1$, $B = 1$

→ GEOMETRIC PROGRESSION :-

$a, ar, ar^2, \ldots\ldots ar^{n-1}$

$2, 2, 2, 2 \ldots\ldots$
$a = 2$
$r = 1$

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Q: A batch of homemade bread is treated with yeast. It is observed that the yeast population grows for the first several hours at a rate which is proportional to the population at any given time

A: Let P_n be the yeast population after n hrs

$\therefore$, $P_{n+1} = \alpha P_n$ $[\alpha > 1]$.

$\therefore$, If $P_0 \rightarrow$ initial pop.

$P_1 = \alpha P_0$

$P_2 = \alpha P_1 = \alpha(\alpha) P_0 = \alpha^2 P_0$.

$\therefore$, the GP is :- $P_0, \alpha P_0, \alpha^2 P_0 \ldots\ldots \alpha^n P_0$

- A Sequence is a set of numbers u_1, u_2, u_3 in a definite order. And each term is formed by a definite rule.

→ Limit of a Sequence.

If for any ϵ we can find a +ve no: N (depends on ϵ) such that for $n > N$

$$|u_n - l| < \epsilon$$

for all such n,

$$\lim u_n = l.$$

Th. If $\lim\limits_{n\to\infty} a_n = A$ & $\lim\limits_{n\to\infty} b_n = B$

then, $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = A + B$.

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$\lim_{n\to\infty} (a_n . b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = A.B$.

If $U_n \leq M$ for $n = 1, 2, 3$
The sequence is bounded above
$U_n \geq m$ bounded below.
$m \rightarrow$ lower bound
$M \rightarrow$ upper bound.

$U_{n+1} \geq U_n$ monotonic increasing.
$U_{n+1} > U_n$ Strictly increasing.

Eg: • 1, 1.1, 1.11, 1.111 ....
Monotonic increasing.
bounded $\rightarrow$ 1, 1.2

• 1, -1, 1, -1 .....
Neither inc nor dec.
LB = -1 UB = 1

• -1, -1.5, -2, -2.5
Bounded above : -1
Monotonic decreasing

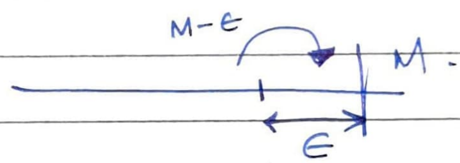
★ Every bounded monotonic sequence (inc. or dec.) seq. ~~is bounded~~ has a limit

— Convergent Seq: If the limit of the seq. exists & the seq. attains the limit, Every convergent sequence is bounded. Converse isn't necessarily true.

It is a convergent seq.

- If the lim of the seq. exists and the seq. does not attain the limit, you just say 'the limit exists'

A number M is called the least upper bound of the seq. $\{U_n\}$ $U_n \le M$ for all $n = 1, 2, 3 \ldots$
and at least one term $> M-\epsilon$ for $\epsilon > 0$



A number $\bar{l}$ is called the limit superior or upper limit of a sequence $\{U_n\}$ if infinitely many terms of the seq. are greater than $\bar{l} - \epsilon$ whereas only finitely many terms are greater than $\bar{l} + \epsilon$

$\{\ \}$ Form the set of sub-sequential limits

A number $\underline{l}$ is called the limit inferior of a seq. $\{U_n\}$ if infinitely many terms are less than $\underline{l} + \epsilon$ while only a finite no: of terms are less than $\underline{l} - \epsilon$

Upper bound → Lim superior
Lower bound → Lim inferior

Consider :-

$2, -2, 1, -1, 1, -1, 1, -1 \ldots$

lub ~~glb~~ :- 2 $[2 > 2-\epsilon]$

glb :- -1

→ lim superior :

$\boxed{1}$ $\because$, ∞ many terms are greater than $1-\epsilon$. [all 1s]

& finite terms greater than $1+\epsilon$. [only 2]

Eg:

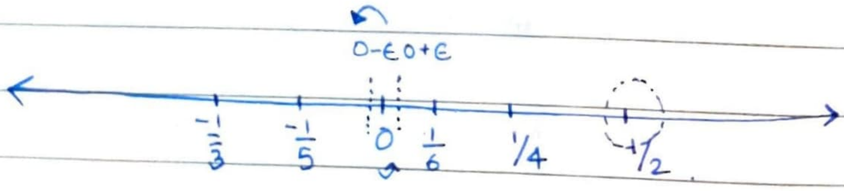
$1, -1, 1, -1, \ldots$

lub $= 1$.

Lim sup $= 1$!

Eg:

$\frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}, \ldots$



lub $= 1/2$.

glb $= -1/3$.

Lim Superior :- 0

Lim Inferior :- 0.

Neither monoton incr. nor decr.

$\frac{(-1)^n}{n}$

Eg: $2,\ 1.9,\ 1.8 \ldots\ldots\ 2 \cdot \frac{(n-1)}{10}$.

Bounded above . Monotonic increasing

- Cauchy's ~~&~~ Convergence Criterion :-

A seq. $\{a_n\}$ converges iff $\forall\ \epsilon > 0$ we can find a number (N ; large) $|u_n - u_m| < \epsilon$ for $n, m > N$ then the limit exist

$$\boxed{\text{TUTORIAL} - 1}$$

Q12. Assignment :

$$\left.\begin{aligned} A &: \{a_1, a_2, a_3 \ldots \infty\}. \\ B &= \{b_1, b_2, b_3 \ldots \infty\}. \end{aligned}\right] \text{Infinitely countable.}$$

$$A \cup B = \{ \underset{\substack{\updownarrow \\ 1}}{a_1}, \underset{\substack{\updownarrow \\ 2}}{b_1}, \underset{\substack{\updownarrow \\ 3}}{a_2}, \underset{\substack{\updownarrow \\ 4}}{b_2}, \underset{\substack{\updownarrow \\ 5}}{a_3}, \underset{\substack{\updownarrow \\ 6.}}{b_3} \ldots \infty \}$$

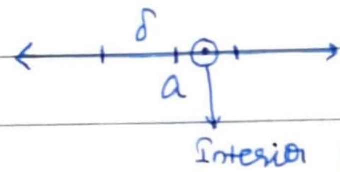
Q: Find the limit of the sequence :-

$$u_n = 3 + \frac{1}{n}$$

$$\lim_{n \to \infty} 3 + \frac{1}{n} = \lim_{n \to \infty} \frac{3n+1}{n}.$$

$\therefore\ = \frac{3}{1} = \underline{3}.$ [consider larger num/den

Neighbourhood.



$(a-\delta, a+\delta)$.

Limit point :-
Rational no:s are not closed set ∵
$\sqrt{2}$, $\sqrt{3}$ etc. are limit points but ~~a~~ not a part of the set.

Open set :- ~~a~~ all points are interior pts
Closed set :- contains all limit points.

16). for any int N there are only finitely many equations with

$$n + |a_0| + |a_1| + \dots + |a_n| = N.$$

Let A_N be the set of no:s satisfying this eq. for one N,

Squeeze Th:

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ there exists an N ∋ $a_n \le c_n \le b_n$ for all $n > N$
$\lim_{n\to\infty} c_n = L$.

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→ Every convergent sequence is bounded
Converse not necessarily true.
Eg :- $\{1, -1, 1, -1 \dots\}$.

→ Every bounded sequence has a limit.

PROBLEMS :

Q: Write out the first 5 terms of each of the following sequences.

a) $\left\{\frac{2n-1}{3n+2}\right\}$.

$a_1 = \left\{\frac{2.1-1}{3.1+2}\right\} = \frac{1}{5}$

$a_3 = \left\{\frac{2.3-1}{3.3+2}\right\} = \frac{5}{11}$

$a_2 = \left\{\frac{2.2-1}{3.2+2}\right\} = \frac{3}{8}$

$a_4 = \left\{\frac{2.4-1}{3.4+2}\right\} = \frac{7}{14}$

$a_5 = \left\{\frac{2.5-1}{3.5+2}\right\} = \frac{9}{17}$.

b) $a_n = \left(\frac{-2}{3}\right)^n$.

$= \frac{-2}{3}, \frac{4}{9}, \frac{-8}{27}, \frac{16}{81}$.

c) $a_n = \cos(n\pi/2)$.

$= 0, -1, 0, 1, 0$.

d) $a_{k+1} = \left(\frac{k+1}{2}\right) a_k$

$a_1 = 4$

$a_2 = \left(\frac{k+1}{2}\right) \cdot 4 = \cancel{8}\ 4$

$a_3 = \frac{2+1}{2} \cdot 4 = 6$

$a_4 = \frac{3+1}{2} \cdot \cancel{6} = 12$

$a_5 = \frac{4+1}{2} \cdot 12 = 30$

Q: Find the limit of this seq.

$a_n = 5 - \frac{1}{n^2}$

a) $\lim_{n \to \infty} \{a_n\} = 5$

b) $\lim_{n \to \infty} \frac{5n}{\sqrt{n^2+4}} = 5$

Q: Determine the convergence or divergence of the sequence.

① $\lim_{n \to \infty} [1 + (-1)^n]$

Lim doesn't exist

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(2) $\frac{\sqrt[3]{n}}{\sqrt[3]{n} + \cancel{1}1}$ Lim tends to 1.

Q: Determine the 5^{th} term of the seq.

1, 16, 81, 256.

$U_n = n^4$

$\times$, $U_n = 10n^3 - 35n^2 + 50n - 24$.

$\therefore$,

$U_5 = 625$.

Q: Prove that a convergent sequence is bounded.

$\lim_{n \to \infty} a_n = A$.

To show that $|a_n| < P$ for all n.

$|a_n| = |a_n + A - A| \leq |a_n - A| + |A|$

Also, $|a_n - A| < \epsilon$ for $n \geq N$.

ie: $|a_n| < \epsilon + |A|$ $\forall$ $n > N$

$|a_n| < P$ $\forall$ $n > N$.

The sequence is bounded.

$\therefore$, Every convergent seq. is bounded.

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Q: Prove that the seq. $u_n = \frac{2n-7}{3n+2}$ is

1. monotone increasing
2. Bounded above
3. Bounded below
4. Has a limit.

Sol: 1. $u_{n+1} \geq u_n$

$$u_{n+1} = \frac{2(n+1)-7}{3(n+1)+2} = \frac{2n-5}{3n+5}.$$

$\therefore$, $\frac{2n-5}{3n+5} \geq \frac{2n-7}{3n+2}$. Should be.

$$(2n-5)(3n+2) \geq (2n-7)(3n+5).$$

$$6n^2 + 4n - 15n - 10 \geq 6n^2 + 10n - 21n - 35.$$

$$-11n - 10 \geq -11n - 35.$$

$$-10 \geq -35.$$ ie: $-10 > -35$

This is correct.

$\therefore$, It is monotone increasing.

2. Claim :- The bound is 2

ie; $u_n \leq 2$.

ie: $\frac{2n-7}{3n+2} < 2$.

$$2n-7 < 6n+4.$$

$$4n > -11.$$

This is true.

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③. Claim :- Lower bound -2.

$$\frac{2n-7}{3n+2} \geq -2.$$

$$2n-7 \geq -6n-4.$$
$$8n \geq 3.$$
$$n \geq 3/8. \quad \text{True}.$$

④. $\lim_{n \to \infty} \{U_n\} = \boxed{\frac{2}{3}}$.

gives e

Q: Prove that the sequence with the n^{th} term $U_n = \left(1+\frac{1}{n}\right)^n$ is monotone increasing, bounded & limit exists.

- Let $x = \frac{1}{n}$. $\quad U_n = (1+x)^n$.

Binomial expansion.

$$1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots + \frac{n(n-1)\cdots n-(n-1)}{n!} x^n.$$

$$= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)\left(\frac{1}{n}\right)^2}{2!} + \frac{n(n-1)(n-2)\left(\frac{1}{n}\right)^3}{3!} + \cdots \frac{n(n-1)\cdots n-(n-1)\left(\frac{1}{n}\right)^n}{n!}$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(n-1)}{n}\right)$$

With each term in the sequence, positive terms are being added.

$\therefore$ this is a monotonic increasing sequence.

- $u_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!}$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}}.$$

$$< 2 + \frac{\frac{1}{2}}{1 - \frac{1}{2}} = < 2 + 1$$

$$< 3.$$

$\boxed{Q:}$ Prove that: $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

where $x \to \infty$ in any manner, not necessarily along the +ve integers

Sol. Let n = largest int $\leq x$ → real no:

$$n \leq x \leq n+1.$$

$$\frac{1}{n+1} \leq \frac{1}{x} < \frac{1}{n}$$

$$1 + \frac{1}{n+1} \leq 1 + \frac{1}{x} \leq 1 + \frac{1}{n}$$

$$\left(1 + \frac{1}{n+1}\right)^n \leq \left(1 + \frac{1}{x}\right)^x \leq \left(1 + \frac{1}{n}\right)^{n+1}.$$

$$\lim_{n \to \infty} \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{1 + \frac{1}{n+1}} \leq \left(1 + \frac{1}{x}\right)^x \leq \left(1 + \frac{1}{n}\right)^n \cdot \left(1 + \frac{1}{n}\right)$$

$$e \leq \left(1 + \frac{1}{x}\right)^x \leq e.$$

Sandwiched between 2 e

$\therefore \left(1+\frac{1}{x}\right)^x$ is equivalent to e.

when $x \to \infty$.

INFINITE SERIES:

$\{a_n\}$ is an infinite series

$$\sum_{n=1}^{n} a_n = a_1 + a_2 + \cdots\cdots a_n$$

is an infinite series.

Consider seq of partial sums $\{S_n\}$.

$S_1 = a_1$

$S_2 = a_1 + a_2$.

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + \cdots + a_n$

If $\{S_n\}$ converges, series is converged.

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = S.$$

Eg: $\sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)$.

finding sequence of partial terms.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 7/8.$$

$\vdots$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

$$= \frac{\cancel{\frac{1}{2}} \cdot \left(1 - \left(\frac{1}{2}\right)^n\right)}{\cancel{1 - \frac{1}{2}}}$$

$$= \frac{2^n - 1}{2^n} = \lim_{n \to \infty} \left(\frac{1 - \frac{1}{2^n}}{\frac{1}{2^n}} \right)$$

$$= \underline{\underline{1}}.$$

Eg: ~~$\sum_{n=1}^{\infty} \left(\frac{1}{2n}\right)$ convergent series~~

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

$$= 1 \cancel{-\frac{1}{2}} + \cancel{\frac{1}{2}} \cancel{-\frac{1}{3}} + \cancel{\frac{1}{3}} \cdots\cdots$$

$$= 1 - \frac{1}{n+1}.$$

$$\lim_{n \to \infty} = \underline{1}.$$

Eg: ~~Lim~~ $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}.$

$$\sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n-1)}.$$

split as partial fractions

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n+1}.$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \dots \right).$$

$$= 1.$$

GEOMETRIC SERIES :-

Convergence of a geometric series :-

$$\sum_{n=0}^{\infty} = a + ar + ar^2 + \dots + ar^{n-1}$$

$a \neq 0$ $\quad r \neq \pm 1$

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

$$rS_n = ar + ar^2 + \dots + ar^n.$$

Subtracting :-

$$S_n(1-r) = a - ar^n.$$

$$S_n = \frac{a(1-r^n)}{1-r}.$$

(i) $0 < |r| < 1.$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{a}{1-r} (1 - r^n) \right].$$

$$= \frac{a}{1-r}.$$

(ii) $\lim_{n\to\infty} S_n \to \infty$ $\qquad |r| > 1.$

limit is a finite no.

Q: What is the sum of this series

$$\sum_{n=0}^{\infty} \frac{3}{2^n}.$$

$$\therefore \; = \frac{3}{1-\frac{1}{2}} = 6.$$

$$\sum \left(\frac{3}{2}\right)^n \longrightarrow \text{diverges.}$$

- Recurring Decimal as Geometric series

Eg: $0.080808\ldots$

$= 0.\overline{08}$

$= \frac{8}{100} + \frac{8}{10^4} + \frac{8}{10^6} + \ldots$

$= \cancel{\sum \frac{8}{(10)^2}} \quad \sum_{n=1}^{\infty} (8)\cdot\left(\frac{1}{10^2}\right)^n.$

$= \; = \frac{\frac{8}{100}}{1-\frac{1}{100}} = \frac{8}{99}$

- Properties of Infinite series :-

Let: $\sum_{i=1}^{n} a_n = A$, $\sum_{i=1}^{n} b_n = B$

then $\sum_{i=1}^{n} ca_n = cA$

$\sum (a_n + b_n) = A + B.$

$\sum (a_n - b_n) = A - B.$

If $\lim_{n\to\infty} a_n = p$
then $p+p+p+\cdots$ will not lead to a finite L.

→ n^{th} term test for a convergent series :-

If $\sum_{i=1}^{\infty} a_n$ converges

cannot be a finite no!.

then $\lim_{n\to\infty} a_n = 0$ [converse not necessarily true].

If $a_1 + a_2 + a_3 + \cdots + a_n + \cdots = L$.
then $\Rightarrow \lim_{n\to\infty} a_n = 0$.

<u>Proof:</u>

$$\sum_{i=1}^{n} a_n = \lim_{n\to\infty} S_n = L$$

$$\lim_{n\to\infty} (a_1 + a_2 + \dots a_n).$$

Now, $S_n = S_{n-1} + a_n$.

$a_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}).$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}.$$

$L - L$

$$\lim_{n\to\infty} a_n = 0.$$

The logical equivalence $p \to q$.
Contrapositive :- $\neg q \to \neg p$.

$\neg Q \to \neg P$
Statement :- If $\lim_{n\to\infty} a_n \not\to 0$ then $\sum_{n=1}^{\infty} a_n$ doesn't converge

$P \to Q$ If a sequence converges, $\lim_{n\to\infty} a_n = 0$.

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Q: Use the n^{th} term test & verify if the series converges / diverges.

1). $\sum_{n=0}^{\infty} 2^n$.

The $a_n = 2^n$.

Here, $\lim_{n \to \infty} 2^n = \infty$

$\therefore$ Series diverges.

2). $\sum_{n=1}^{\infty} \frac{1}{n}$ given : $\lim_{n \to \infty} \frac{1}{n} = 0$.

Here, converse isn't true.

$\therefore$ No conclusion drawn.

→ **Test for Convergence of a Series :-**

$\boxed{\text{I)}}$ **The Integral Test :-**

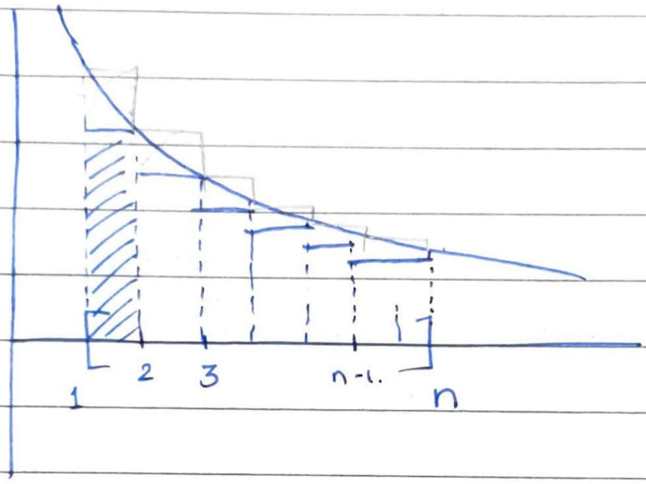
If f is positive, continuous and decreasing

and $a_n = f(n)$. then.

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx.$$

then either both converge or both diverge.

Proof: Let $f(x)$ be a positive, decreasing, continuous fn.
Consider interval $[1, n]$
Divide into $(n-1)$ no: of parts.



Area of rectangles (inscribed).

$$\sum_{i=2}^{n} f(i) = f(2) + f(3) + f(4) + \dots f(n).$$

Area circumscribed

$$= \sum_{i=1}^{n-1} f(1) + f(2) + \dots + f(n-1) = \sum_{i=1}^{n-1} f(i)$$

The exact area under $f(x)$ lies between the two areas.

$$\sum_{i=2}^{n} f(i) \le \int_{1}^{n} f(x)\,dx \le \sum_{i=1}^{n-1} f(i).$$

Again, ~~S~~ $f(n) = a_n$.

$$S_n = a_1 + a_2 + \dots + a_n.$$
$$= f(1) + f(2) + \dots + f(n).$$

Now, on eliminating a_1

$$S_n - f(1) \leq \int_1^n f(x)\,dx \leq S_{n-1}$$

Let $n \to \infty$

Assume that

$\int_1^\infty f(x)\,dx$ converges to L

$$S_n - f(1) \leq L$$
$$S_n \leq L + f(1).$$

$\{S_n\}$ is bounded, monotonic, it converges [By Balzano Weierstrase Th].

$\Rightarrow \Sigma a_n$ converges.

If integral

$\int_1^\infty f(x)\,dx$ diverges $\to \infty$

$$S_{n-1} \geq \int_1^\infty f(x)\,dx \longrightarrow \infty$$

$\Rightarrow \Sigma a_n$ diverges

$\boxed{Q:}$ Use Integral Test to test if the following series converges or diverges

1) $\sum_{n=1}^{\infty} \frac{2}{3n+5}$.

$$a_n = \frac{2}{3n+5} \qquad f(n) = a_n.$$

$$f(x) = \frac{2}{3x+5}.$$

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$f(1) = \frac{2}{8}$ $f(2) = \frac{2}{11}$.

f is continuous, positive & decreasing.

Consider $\int_1^\infty \frac{2}{3x+5}\,dx$.

$= \frac{2}{3} \log|3x+5|\Big]_1^\infty$.

$= \infty$ $\rightarrow$ Diverges

$\therefore$, The series diverges.

2). $\sum_{n=1}^\infty ne^{-n/2}$.

$a_n = ne^{-n/2}$. $f(n) = a_n$.

$f(x) = xe^{-x/2}$.

$f(1) = e^{-1/2}$ $f(2) = 2e^{-1}$

$f'(x) = x(e^{-x/2})\frac{-1}{2} + e^{-x/2}$.

$= -\frac{xe^{-x/2}}{2} + e^{-x/2}$.

$= \frac{(2-x)}{2e^{x/2}}$

At $x = 1$, $f'(x) > 0$

$x = 2$, $f'(x) = 0$.

$x = 3$, $f'(x) < 0$.

$\therefore$, f is inc from 1-2.

~~from~~ ignore $f(1)$ & $f(2)$.

$\therefore\ f(x) = xe^{-x/2}$

Since $f'(x) \geq 0$ for $n = 1, 2$ the func. is increasing upto 2.
So the integral test can be applied from $x = 3$ onwards where the function $f'(x) < 0$ or $f(x) \downarrow$

$$\int_3^\infty f(x) = \int_3^\infty xe^{-x/2}\,dx.$$

$$\left[\left(\frac{-1}{2}\right)x.e^{-x/2}\right]_3^\infty + \int_3^\infty 2e^{-x/2}\,dx$$

$$\left[\frac{-1}{1/2}x\,e^{-x/2}\right]_3^\infty - \cancel{+}\ \frac{1}{1/2}.e^{-x/2}\Big]_3^\infty.$$

$$\left[-2xe^{-x/2}\right]_3^\infty \mp \left[4e^{-x/2}\right]_3^\infty.$$

$$= 2\left[0 - \frac{3}{e^{3/2}}\right] - 4\left[0 - e^{-3/2}\right]$$

= a finite no:

$\therefore$ converges.

37. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+3}.$

$a_n = \dfrac{n}{n^2+3}.$ $\qquad f(n) = a_n$

$f(x) = \dfrac{x}{x^2+3}$

$$f'(x) = \frac{x^2+3 - x(2x)}{(x^2+3)^2}.$$

$$f'(x) = \frac{3-x^2}{(x^2+3)^2}.$$

$f'(1) = > 0$
$f'(2) = < 0$.

Since $f'(1) > 0$ ∴ increasing.
Exclude $f(1)$.

∴ $\int_2^\infty \frac{x}{x^2+3}\, dx.$ $\quad x^2 = t$, $2x\,dx = dt$, $x\,dx = \frac{1}{2} dt.$

$= \frac{1}{2}\int_4^\infty \frac{dt}{t+3}.$

$= \frac{1}{2}\left[\log|t+3|\right]_4^\infty.$

$= \infty \quad ∵ \log|\infty+3| \to \infty.$

∴ Diverges.

$\boxed{II}$) p-series Test :-

Convergence of the p-series

$$\sum_{n=1}^{p} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

① Converges if $p > 1$.
② Diverges if. $0 < p \leq 1$
As all three criteria of I.T. are met.

$$\frac{x^{1-p}}{(1-p)^2}\left[-1+(-1+p)\ln(x)\right]_2^{\infty}$$

Further $\int_1^{\infty} \frac{1}{x^p}\,dx$.

converges if $p > 1$.
diverges if $0 \leq p \leq 1$

Q: Convergence or divergence of

$$\sum_{n=2}^{\infty} \frac{\log n}{n^p}.$$

$$a_n = \frac{\log x}{x^p}. \qquad f(n) = a(n).$$

$$\therefore, \quad f(x) = \frac{\log x}{x^p}$$

f is decreasing, positive & cont.

Q: Show whether increasing / decreasing

$$a_n = \frac{2n}{3n+1}$$

$$a_{n+1} = \frac{2(n+1)}{3(n+1)+1} = \frac{2n+2}{3n+3+1}$$

$$= \frac{2n+2}{3n+4}.$$

To check $\quad a_{n+1} \geq a_n.$

$$\frac{2n+2}{3n+4} \geq \frac{2n}{3n+1}$$

$$(2n+2)(3n+1) \geq 2n(3n+4).$$

$$6n^2+8n+2 \geq 6n^2+8n$$

(0, 1).

$2 \geq 0$

True $\therefore$ Statement

$\therefore$ Yes Increasing.

Q: Check if the series conv. or div.

(1) $\sum_{n=1}^{\infty} \frac{n}{2n+3}$.

$a_n = \frac{n}{2n+3}$.

$\lim_{n \to \infty} \frac{n}{2n+3} = \lim_{n \to \infty} \frac{1}{2+3/n} = \frac{1}{2} \neq 0$

Therefore series diverges

(2) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$a_n = \frac{n}{\sqrt{n^2+1}}$ $\lim_{n \to \infty} \frac{n}{\sqrt{n^2+1}}$

$= \lim_{n \to \infty} \frac{1}{\sqrt{1+1/n^2}} = 1 \neq 0$.

$\therefore$ Diverges.

(3). $\sum_{n=1}^{\infty} (3)(0.85)^{n-1}$

$a_n = 3(0.85)^{n-1}$

$\lim_{n \to \infty} 3(0.85)^{n-1} = 0$

$r < 1$;

Converges.

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- P-series Test: (continuation)

eg1: $\sum_{i=1}^{\infty} \frac{3}{n^{5/3}}$

This converges as $p > 1$

eg2: $\sum_{i=1}^{\infty} \frac{1}{n^{2/3}}$ Diverges.

eg3: $\sum_{n=2}^{\infty} \frac{\log n}{n^p}$.

$f(x) = \frac{\log n}{n^p}$ here numerator grows slower than denominator

∴, Using IT:-

$$\int_2^{\infty} \frac{\log n}{n^p} dn = \int_2^{\infty} \frac{\log x}{x^p} dx.$$

$$= \log x . \left[\frac{x^{-p+1}}{-p+1} \right]_2^{\infty} - \int \frac{1}{x} \cdot \frac{x^{-p+1}}{(-p+1)} dx$$

$$= \log x \left[\frac{x^{-p+1}}{-p+1} \right] - \int \frac{x^{-p}}{-p+1} dx.$$

$$\log x \left[\frac{x^{-p+1}}{-p+1} \right]_2^{\infty} + \frac{1}{p-1} \left[\frac{x^{-p+1}}{-p+1} \right]$$

$$\left\{ \frac{x^{-p+1}}{(1-p)^2} \left[\log x (1-p) - 1 \right] \right\}_2^{\infty}$$

Now, as $n \to \infty$, $x \to \infty$.

ie:- $1-p$ increases & ∴, to have the function to be decreasing

$-p+1$ should be negative-

ie: $-p+1 < 0$

$\Rightarrow$ $\underline{\underline{p > 1}}$.

- **Comparison Test :-**

→ Direct Comparison test :-

Let $0 \leq \underbrace{a_n}_{\text{given}} \leq \underbrace{b_n}_{\text{unknown}}$, then if

$\sum_{n=1}^{\infty} b_n$ converges, then

$\sum_{n=1}^{\infty} a_n$ converges.

[A series greater than the one given converges $\therefore$ the given series converges].

Conversely :-

If $\sum_{i=1}^{\infty} a_n$ diverges.

then $\sum_{i=1}^{\infty} b_n$ diverges.

- **Balzano Weierstrass Th. [series] :-**

Every bounded sequence of real nos: has a convergent sub-sequence.

Proofs: Let

$$L = \sum_{n=1}^{\infty} b_n$$

Let s_n be the partial sums

ie: $S_n = a_1 + a_2 + a_3 + \dots + a_n$

∴ the sequence :- $S_1, S_2, S_3, \dots$ is bounded by b_ns.

∴ $S_1 = a_1$

$S_2 = a_1 + a_2$

⇒ This seq. converges

Let $\lim_{n \to \infty} S_n$ = finite.

ie: $\sum_{i=1}^{\infty} a_n$ is finite.

Similarly; By contrapositive equivalence; If a_n diverges, b_n diverges.

$\boxed{Q:}$ Verify if it converges

$$\sum_{n=1}^{\infty} \frac{1}{3n^2+2}.$$

Now, $\sum_{n=1}^{\infty} b_n \quad \frac{1}{3n^2}$ assume.

Term by term $\frac{1}{3n^2+2} < \frac{1}{3n^2}$

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Now, by the p series test $\frac{1}{3n^2}$ converges since $p > 1$.

$\therefore$ By comparison test $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ converges.

$\boxed{Q:}$ $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

consider $\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}-1}$ $\forall\ n \geq 2$.

By p-series test, $\frac{1}{\sqrt{n}}$ diverges as $p = 1/2 < 1$.

$\therefore$ By comparison test $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diveges

$\boxed{Q:}$ $\sum_{n=1}^{\infty} \frac{3^n}{4^n+5}$

Now, $\sum \frac{3^n}{4^n}$ $\cancel{\geq}$ $\geq \frac{3^n}{4^n+5}$

Here, $\left(\frac{3}{4}\right)^n$ is a GP with $r < 1$.

$\therefore$ it converges as it is a a geometric series with $r < 1$.

ie: $S = \frac{3/4}{1-3/4} = 3$.

$\therefore$ By comparison test, $\sum_{n=1}^{\infty} \frac{3^n}{4^n+5}$ converges.

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18.] $\sum \frac{1}{\sqrt{n^3}+1}$

Consider $\frac{1}{n^{3/2}} > \frac{1}{\sqrt{n^3}+1}$

By p-series test. $\frac{1}{n^{3/2}}$ ~~diverges~~ converges as $p > 1$.

$\therefore$ By comparison test

$\sum \frac{1}{\sqrt{n^3}+1}$ converges.

Q: $\sum \frac{1}{4(\sqrt[3]{n})-1}$

consider :- $\frac{1}{4\sqrt[3]{n}-1} > \frac{1}{4\sqrt[3]{n}}$

By p-series test. $\frac{1}{4}\sum \frac{1}{n^{1/3}}$ is a divergent series.

$\therefore$, $\frac{1}{4\sqrt[3]{n}-1}$ diverges.

Q: $\sum_{i=1}^{\infty} \frac{4^n}{3^n - 1}$

Consider $\sum \frac{4^n}{3^n}$ This is a divergent geometric series.

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& $\frac{4^n}{3^n-1} > \frac{4^n}{3^n}$.

$\therefore \sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$ diverges.

- Limit Comparison test :-

→ Suppose :- $a_n > 0$, $b_n > 0$ for all $n \geq N$

if $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ [a constant]

then both Σa_n & Σb_n either converge together or diverge.

$n > N$:-

$$\left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}.$$

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}.$$

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2}.$$

$$\frac{c b_n}{2} < a_n < \frac{3 c b_n}{2}.$$

By comparison test-

If $\Sigma \frac{3c}{2} b_n$ converges : Σa_n converges.

If $\Sigma \frac{\cancel{3} c b_n}{2}$ diverges ; Σa_n diverges.

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→ If $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = 0$

~~then~~ and Σb_n converges then Σa_n converges.

Proof: $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = 0$

$\exists$ an integer such than $n > N$

$$\left|\frac{a_n}{b_n} - 0\right| < 1.$$

$$-1 < \frac{a_n}{b_n} < 1$$

~~$b_n < a_n$~~ By comparision test, $\sum^{\infty} b_n$ ~~is~~ converges, $\sum^{\infty} a_n$ converges

→ If $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \infty$

If $b_n \to$ diverges

$\Rightarrow$ a_n diverges.

for some $n > N$.

$$\frac{a_n}{b_n} > 1.$$

$$a_n > b_n$$

If b_n diverges, a_n diverges

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$\boxed{Q:}$ $\sum_{n=1}^{\infty} \frac{2}{3^n - 5}$

$a_n = \frac{2}{3^n - 5}$ $b_n = \frac{\cancel{2}\ 1}{3^n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2 \cdot 3^n}{3^n - 5}$

$\lim_{n\to\infty} \frac{2}{1 - \frac{5}{3^n}} = \boxed{2}$

$\therefore$ Either both converge or both diverge.

ie: $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is the geometric series, $r < 1$ $\therefore$ this series converges.

$\Rightarrow \sum \frac{2}{3^n - 5}$ converges.

$\boxed{Q:}$ $\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2 - 4}}$

Consider $\frac{1}{n} = b_n$. $a_n = \frac{3}{\sqrt{n^2 - 4}}$.

$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3}{\sqrt{n^2-4}}}{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{3n}{\sqrt{n^2 - 4}} = \lim_{n\to\infty} \frac{3}{\sqrt{1 - 4/n^2}} = \underline{3}$

Either both converge or both diverges

$$\sum_{n=3}^{\infty} b_n = \frac{1}{n}$$ this series diverges $\because$, $p \leq 1$.

$$\therefore, \sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2-4}}$$ also diverges

$\boxed{Q:}$ $$\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$$ { Order net result is $\frac{1}{n}$

consider $\frac{1}{n} = b_n$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(5n-3)\,n}{n^2-2n+5}.$$

$$= \lim_{n\to\infty} \frac{5 - 3/n}{1 - \frac{2}{n} + \frac{5}{n^2}} = 5.$$

$\therefore$, Either both converge or both diverges

$\because$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\therefore$, $\sum \frac{5n-3}{n^2-2n+5}$ diverges

$\boxed{Q:}$ $$\sum \frac{1}{n(n^2+1)}.$$

consider $\frac{1}{n^3}$.

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$$\lim_{n \to \infty} \frac{n^3}{n^3 + n^2} = \frac{1}{1 + \frac{1}{n}} = 1.$$

$\therefore$ Converges.

$\therefore \sum_{n=1} \frac{1}{n^3}$ converges.

$\boxed{Q:}$ $\sum_{n=1}^{\infty} \frac{n}{(n+1) 2^{n-1}}$.

Consider $\frac{1}{2^{n-1}}$.

Converges. $\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$

$\boxed{Q:}$ $\frac{5}{n + \sqrt{n^2+4}}$.

Consider $\frac{1}{n}$.

$$\lim_{n \to \infty} \frac{5n}{n+\sqrt{n^2+4}} = \frac{5}{2}.$$

$\boxed{1/n}$ diverges. $\therefore \sum \frac{5}{n+\sqrt{n^2+4}}$ diverges.

$\boxed{Q:}$ $\sum_{n=2}^{\infty} \frac{1}{\ln n}$. Consider $1/n$.

$$\lim_{n \to \infty} \frac{n}{\ln n} = \lim_{n \to \infty} \frac{n}{} = \infty.$$

$\therefore$ cannot be applied.

MEAN VALUE THEOREM

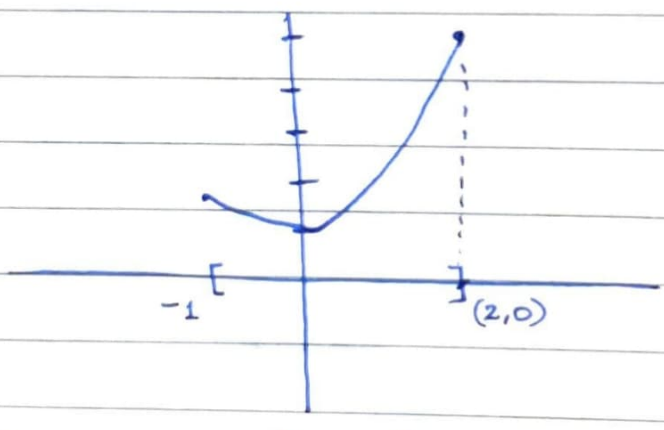
→ Extrema of a function.

Let f be defined on an interval I containing c

① $f(c)$ is the min. of f ($f(x)$ on I $\forall x \in I$)
if $f(c) \leq f(x)$ $\forall x \in I$.

② $f(c)$ is the max of f on I $\forall x \in I$
if $f(c) \geq f(x)$ $\forall x \in I$.

Graphically :-



Min on $[-1, 2]$ $= 1$

Max on $[-1, 2]$ $= 5$.

A continuous fn on closed intervals attains its bounds.

For a fn :-

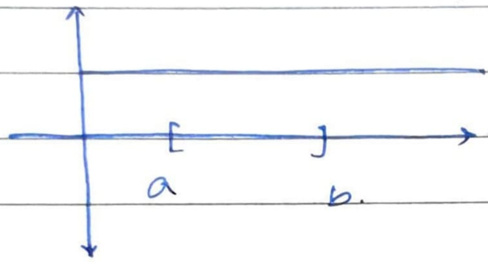
$$f(x) = \begin{cases} x^2+1 & x \neq 0. \\ 2 & x = 0 \end{cases}$$

Does not attain its bounds

→ Rolle's Theorem:

Let f be continuous on a closed $[a,b]$ interval & differentiable on (a,b). If $f(a) = f(b)$ then there exists at least one $c \in (a,b)$ such that $f'(c) = 0$.

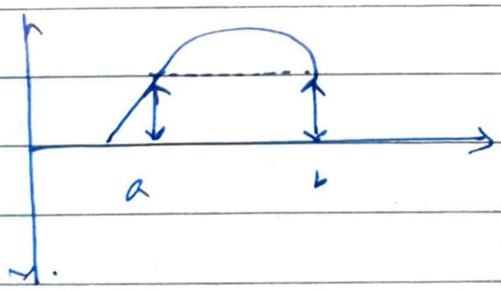
Case 1 If $f(x) = d \quad \forall \quad x \in [a,b]$.



$f'(x) = 0$
$\forall \; x \in (a,b)$.

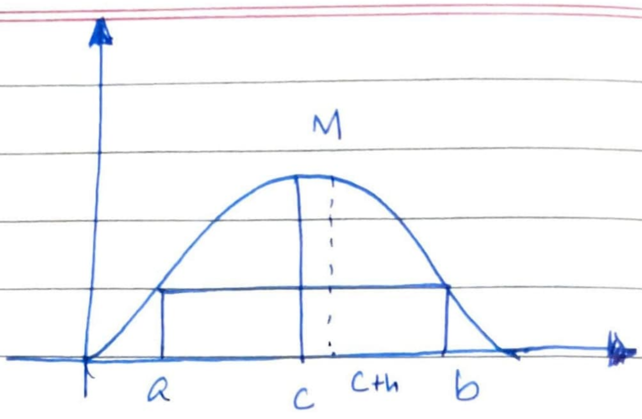
$\therefore$ Result is true.

Case 2: Let $f(x) > d$ for x in (a,b).



$f(x)$ is a continuous fn $\exists$ points in $[a,b]$ such that $f(x)$ attains its max/min.

If $f(x) = M$ ($m = f(x)$, both can be treated equally.



$$f(c+h) < f(c)$$

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0 \qquad [h \geq 0]$$

Let $h \leq 0$, $h = -h'$ $h' > 0$

$f(c-h') = f(c+h) \leq f(c)$ is a cont. function

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0 \qquad \left[\begin{matrix} Nr : -ve \\ Dr : -ve \end{matrix}\right]$$

By hypothesis $f(x)$ is differentiable at all points $c \in (a,b)$

we have that

Rt hand derivative < 0

Lt " " ≥ 0.

Both derivatives equal.

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = 0.$$

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Q: find the two intercepts of $f(x)$ & show that $f'(x) = 0$ ~~is~~ at some points between the intercepts.

$$f(x) = x^2 - 3x + 2.$$

Sol:

$$f(x) = 0.$$

$$x^2 - 3x + 2 = 0.$$

$$(x-2)(x-1) = 0$$

$$x = 2 \quad \text{or} \quad x = 1.$$

f is continuous, differentiable ~~$f'(x)$~~ in the interval $[1, 2]$

$$f(1) = f(2) = 0.$$

$$\therefore \quad f'(x) = 2x - 3.$$

For some c, $f'(c) = 2c - 3 = 0$

ie: $2c - 3 = 0$

$$c = 3/2. \in [1, 2].$$

$\boxed{Q:}$ Find all values of c in ~~$[-2, 2]$~~ $(-2, 2)$ such that $f'(c) = 0$

$$f(x) = x^4 - 2x^2.$$

f is continuous & differentiable on $[-2, 2]$ & $(-2, 2)$ resp.

$$f(-2) = 16 - 8 = 8 = f(2)$$

$$f'(x) = 4x^3 - 4x.$$

For $f'(x) = 0 \Rightarrow 4x^3 - 4x = 0.$

$$x(x^2 - 1) = 0 \qquad x(x-1)(x+1) = 0.$$

$$\therefore \quad x = 0, 1, -1. \in (-2, 2)$$

Values of c : $(0, 1, -1) \in I.$

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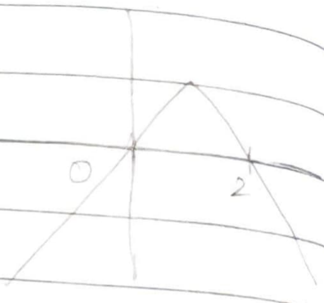
Q: Verify if Rolle's Th. applies or not to the following :-

$f(x) = 1 - |x-1|$ on $[0,2]$.

$f(0) = 0$

$f(2) = 0$.

This $f(x)$ is not differentiable at $x = 1$. & $1 \in [0,2]$



$\therefore$, Rolle's th. doesn't apply.

[The f. is not

Q: Find the x intercepts of the function $f(x) = x\sqrt{x+4}$ and if possible show that Rolle's th. is satisfied.

$f(x) = x\sqrt{x+4}$.

For intercepts :-

$f(x) = 0$ ie: $x\sqrt{x+4} = 0$.

$x^2(x+4) = 0$

$x = 0$ or $x = -4$

f is cont., diff.

$f'(x) = x \cdot \dfrac{1}{2\sqrt{x+4}} + \sqrt{x+4} = 0$

$= \dfrac{x + 2(x+4)}{2\sqrt{x+4}}$

$= \dfrac{3x+8}{2\sqrt{x+4}} = 0$.

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$3x + 8 = 0$

$x = \frac{-8}{3} \in [-4, 0]$.

$\therefore \; c = \frac{-8}{3} \quad \forall \; f'(c) = 0$.

$\boxed{Q:}$ Check if R.T. can be applied.

$f(x) = x^{2/3} - 1. \qquad [-8, 8]$.

$f(-2) = (-8)^{2/3} - 1 = \cancel{\sqrt[3]{4}-1}. \quad 4 - 1 = 3$.

$f(8) = (8)^{2/3} - 1 = 4 - 1 = 3$.

$f'(x) = \frac{2}{3} x^{-1/3} - 1$.

At $x = 0$, $f'(x) = \infty$.

$\therefore$, R.T. doesn't apply.

$\boxed{Q:}$ Verify

$f(x) = (x-1)(x-2)(x-3) \qquad [1, 3]$.

$f(1) = 0$, $f(3) = 0$.

$\therefore \; f(a) = f(b)$.

f is cont. & diff on $[1, 3]$.

$f(x) = (x^2 - 3x + 2)(x - 3)$

$= x^3 - 6x^2 + 11x - 6$.

$f'(x) = 3x^2 - 12x + 11$.

$f'(x) = 0 \Rightarrow 3x^2 - 12x + 11 = 0$.

$x = \frac{12 \pm \sqrt{144 - 132}}{6} = \frac{12 \pm \sqrt{12}}{6}$.

$= \frac{6 \pm \sqrt{3}}{3}$ both $\in [1, 3]$.

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Q: The ht. of a ball t seconds after it is thrown upward from a ht of 32 feet with an initial velocity of 48feet/s is:-

$$f(t) = -16t^2 + 48t + 32$$

Acc. to Rolles Th. verify that What must be the velocity at some time in the interval at which $f(1) = f(2)$. reaches a maximum

$f(1) = -16 + 48 + 32 = 64$

$f(2) = -64 + 96 + 32 = 64.$

$f'(t) = -32t + 48$

For vel. $= 0$, $f'(t) = 0.$

ie: $-32t + 48 = 0$

$32t = 48.$

$t = \frac{\cancel{48}\,3}{\cancel{32}\,2} = \boxed{1.5s}.$

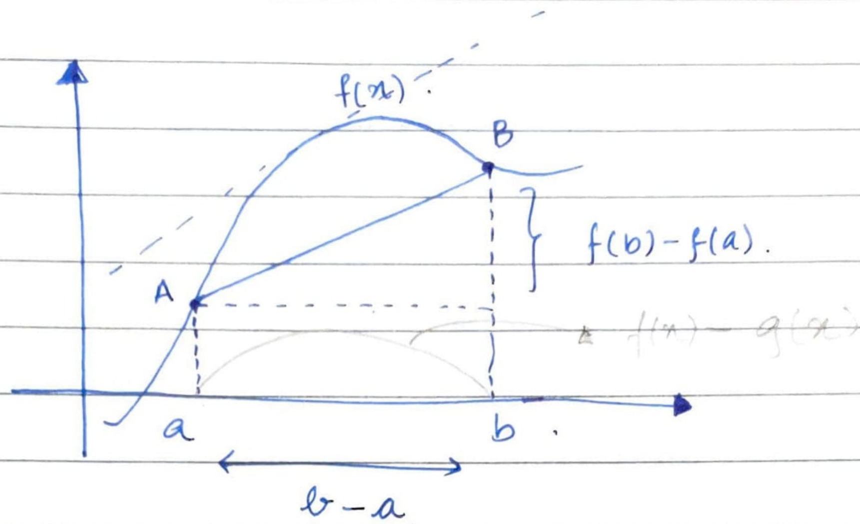
→ Mean Value Theorem:

Suppose $f(x)$ is continuous on $[a,b]$ and diff on (a,b) then $\exists$ a value c such that $f'(c) = \frac{f(b) - f(a)}{b - a}.$

→ Intermediate value Th.

If f is a cont. func. on $[a,b]$, K is any number then there exists

at least one no: c such that
$f(c) = k$.



eqn of the secant line (AB)

$$(y - f(a)) = \frac{f(b) - f(a)}{(b-a)} [x - a].$$

$$g(x) = y = \frac{f(b) - f(a)}{b - a} [x - a] + f(a).$$

Consider $H(x) = f(x) - g(x)$.

$H(a) = f(a) - g(a) = 0$.

$H(b) = f(b) - g(b) = 0$.

$H(x)$ is cont. on $[a, b]$ $\because$ $H(x) = f(x) - g(x)$

it is diff on (a, b) $\because$ $H(x) = f(x) - g(x)$.

$H(x)$ satisfies all conditions of Rolles th.

$\therefore$, $H'(c) = 0$ $[c \in (a, b)]$.

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{(b-a)}[x-a]$$

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{(b-a)} = 0 \text{ for some}$$

Rolles th.

ie: $f'(x) = \frac{f(b)-f(a)}{b-a}$

Physical interpretation :-

$\frac{f(b)-f(a)}{(b-a)}$ is the average change in f over the interval $[a, b]$.

$f'(c)$ is the instantaneous change.

The MVT. says that at some interior point the instantaneous change must equal the average change over the entire interval.

Eg: Suppose a car is accelerating from 0, it takes 8s to go to 352 feet

Sol. Here guaranteed that the speedometer of the car reads 44 feet p.s at some point during the journey.

Q: Determine if the MVT. can be applied to f on $[a, b]$. If so, find all values of c in (a, b) such that f'(c)

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(i) $f(x) = x^2.$ on $[-2, 1]$.

f is continuous, diff.

$\therefore$, MVT. applied.

& $f'(x) = 2x$

$f'(c) = \dfrac{f(b) - f(a)}{(b-a)}$.

$f(b) = f(1) = 1$

$f(a) = f(-2) = 4$

ie:

$2c = \dfrac{1-4}{1-\cancel{4}(-2)} = \dfrac{-3}{3} = \boxed{-1}$

$\boxed{c = -1/2.}$

Yes, $c \in [-2, 1]$. $\therefore$,

since $-2 < -1/2 < 1$ MVT is verified.

(ii) $f(x) = x^{2/3}$ $[0, 1]$.

Cont. & Diff. [not diff at $x = 0$. but diff. on $(0, 1)$]

$f'(x) = \dfrac{2}{3} x^{-1/3}$.

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$ $f(b) = 1$ $f(a) = 0$.

ie: $\dfrac{2c^{-1/3}}{3} = \dfrac{1-0}{1-0} = 1.$

$\dfrac{2}{3c^{1/3}} = 1$ $c^{1/3} = \dfrac{2}{3}.$

$\boxed{c = \dfrac{8}{27}}$

& $c \in [0, 1]$ $\therefore$, MVT is applicable.

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(iii) $f(x) = \sqrt{2-x}$ $[-7, 2]$.

continuous.

$f'(x) = \frac{-1}{2\sqrt{2-x}}$ Diff on $(-7, 2)$.

$f(a) = f(-7) = \sqrt{2+7} = \sqrt{9} = 3$

$f(b) = f(2) = \sqrt{2-2} = \sqrt{0} = 0$

$\therefore f'(c) = \frac{f(b)-f(a)}{b-a}$.

$$\frac{-1}{2\sqrt{2-c}} = \frac{0-3}{2+7}$$

$$\frac{\cancel{-}1}{2\sqrt{2-c}} = \frac{\cancel{-3}\ 1}{\cancel{9}\ 3}$$

$$9 = 4(2-c).$$

$$9 = 8 - 4c.$$

$$4c = -1 \qquad \boxed{c = -1/4} \in [-7, 2]$$

(iv) $f(x) = \frac{x}{1+x}$ $[-1/2, 2]$.

Find equ of the tangent line & secant line.

$f'(x) = \frac{(1+x) - x}{(1+x)^2} = \frac{1}{(1+x)^2}$.

$f(b) = \frac{2}{3}$ $f(a) = \frac{-1/2}{1-1/2} = -1$

$\therefore f'(c) = \frac{1}{(1+c)^2} = \frac{\frac{2}{3}+1}{2+\frac{1}{2}}$

$1+c = -\frac{\sqrt{3}}{\sqrt{2}}$ NEGLECT

$c = -1-\frac{\sqrt{3}}{2} < \frac{-1}{2}$

$$\frac{1}{(1+c)^2} = \frac{\frac{8}{3}}{\frac{5}{2}} = \frac{2}{3}.$$

$$(1+c)^2 = \frac{3}{2}$$

$$\boxed{c = \sqrt{\frac{3}{2}} - 1.} \Rightarrow f(c) = \frac{\sqrt{\frac{3}{2}}-1}{\frac{\sqrt{3}}{\sqrt{2}}} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}}.$$

Secant line through :-

$$(y-y_1) = m(x-x_1)$$

$$\left(y-\frac{2}{3}\right) = \left(\frac{2}{3}\right)(x-2).$$

$$3y-2 = 2x-4.$$

$$\underline{2x-3y-2=0}.$$

Tangent line :-

$$\left(y-\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}}\right) = \frac{2}{3}\left(x-\sqrt{\frac{3}{2}}+1\right).$$

$$\frac{\sqrt{3}y-\sqrt{3}+\sqrt{2}}{\cancel{\sqrt{3}}} = \frac{\cancel{2}\sqrt{2}}{\cancel{3}\sqrt{3}} \cdot \left(\frac{\sqrt{2}x-\sqrt{3}+\sqrt{2}}{\cancel{\sqrt{2}}}\right).$$

$$3y-3+\sqrt{6} = 2x-\sqrt{6}+2.$$

$$\underline{2x-3y+5\cancel{x}-2\sqrt{6}=0}$$

Q: The ht. of an object t-seconds after it is dropped from a ht. of 500M is $f(t) = 4.9t^2+500$.
Find the average velocity of the object during the first 3s.

$s(t) = -4.9t^2 + 500$ $\qquad [0, 3]$

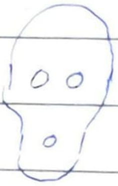
$s'(t) = -9.8t$

$s'(c) = \frac{s(b) - s(a)}{b - a}$

$s'(c) = \frac{4.9 \times \cancel{3^2}^{\,9} + \cancel{500} - \cancel{500}}{3}$

$\cancel{9.8}^{\,2}\, c = \frac{\cancel{49} \times \cancel{9}\, 3}{\cancel{3}}$

$\boxed{c = 1.5}$



$\therefore$ Av. velocity $= \frac{-4.9 \times 9}{3}$

$= -14.7$ m/s.

Use the MVT to verify that at some time during first 3 seconds of the fall, inst. vel = avg. vel.

Required $t = 1.5$ s $\in [0, 3]$.

Q: A company introduces a new product for which the no: of units sold is

$S(t) = 200\left(5 - \frac{9}{2+t}\right)$ $\quad$ t in months.

i) Find av. value of products sold in first year

ii) During which month does $S'(t)$ = avg

$S(t) = 200\left(5 - \frac{9}{2+t}\right)$

$S'(t) = \frac{1800}{(2+t)^2}$.

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$$s'(t) = \frac{f(b) - f(a)}{b - a}$$ [0, 12]

$$f(12) = 200\left(5 - \frac{9}{14}\right)$$

$$= 200 \times \left(\frac{70 - 9}{14}\right) = 200\left(\frac{61}{14}\right).$$

$$f(0) = 200\left(5 - \frac{9}{2}\right) = 200\left(\frac{1}{2}\right).$$

$$\therefore s'(t) = \frac{\frac{200 \times 61}{14} - \cancel{200}\,100}{12} = \frac{450}{7}.$$

$$S'(t) = \frac{450}{7}.$$

$$s'(t) = \frac{450}{7}$$

$$\Rightarrow \frac{\cancel{1800}\,4}{(2+t)^2} = \frac{\cancel{450}}{7}.$$

$$28 = (2+t)^2.$$

$$28 = 4 + t^2 + 4t$$

$$t^2 + 4t - 24 = 0.$$

$$2 + t = \pm 2\sqrt{7}.$$

$$t = \pm 2\sqrt{7} - 2$$

$$t = 2\sqrt{7} - 2.$$ ~~mon~~

$\therefore$ in the third month.

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$\boxed{Q:}$ Using MVT, prove the inequality:

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2} \quad a<b.$$

Choose $f(x) = \tan^{-1}(x)$

$$f'(x) = \frac{1}{1+x^2}.$$

$$f'(c) = \frac{1}{1+c^2} \qquad a<c<b.$$

By MVT:-

$$\cancel{\frac{1}{1+c}} \quad \frac{f(b)-f(a)}{b-a} = f'(c). = \frac{1}{1+c^2}.$$

Now,

$$a < c < b$$

$$a^2 < c^2 < b^2$$

$$1+a^2 < 1+c^2 < 1+b^2.$$

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+\cancel{b^2}\,a^2}$$

ie:

$$\frac{1}{1+b^2} < \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} < \frac{1}{1+a^2}$$

ie:

$$\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}$$

$\boxed{Q:}$ Show that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}.$$

Choose $a = 1$, $b = \frac{4}{3}$.

Sub:-

$$\frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{6}$$

$$\frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{1}{6}$$

$$\frac{b-a}{1+b^2} = \frac{\frac{4}{3}-1}{1+\frac{16}{9}} = \frac{3}{25}.$$

$$\frac{b-a}{1+a^2} = \frac{\frac{1}{3}}{1+1} = \frac{1}{6}.$$

Q: Show:-

$$\frac{h}{1+h^2} < \tan^{-1}(h) < h. \qquad 0 \le x \le h. \quad h > 0.$$

Choose: $f(x) = \tan^{-1}(x)$

$b = h \qquad a = 0.$

$$f'(x) = \frac{1}{1+x^2}$$

$\therefore$ By MVT:- $\frac{f(b)-f(a)}{b-a} = f'(c).$

$$= \frac{1}{1+c^2}.$$

$0 \le x \le h. \qquad 0 \le x \le b.$

$0 \le x^2 \le h^2.$

$1 \le 1+x^2 \le 1+h^2$

$$\frac{1}{1+h^2} \le \frac{1}{1+x^2} \le \frac{1}{1}.$$

$$\frac{1}{1+h^2} \le \frac{\tan^{-1}(h) - \tan^{-1}(0)}{h-0} \le 1.$$

$$\frac{h}{1+h^2} \le \tan^{-1}(h) \le h.$$

Q: Prove that $\frac{\sin x}{x}$ is decreasing by $(0, \pi/2)$.

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2}.$$

For this to be decreasing ;,

$$\frac{x\cos x - \sin x}{x^2} < 0.$$

$$x\cos x - \sin x < 0$$

$$x < \tan x \quad \text{on } (0, \pi/2).$$

or $\tan x - x > 0$

Choose $f(x) = \tan x - x$

$$f'(x) = \sec^2 x - 1 = \tan^2 x > 0$$

on $(0, \pi/2)$

$\therefore$ $(\tan x - x)$ increases on $(0, \pi/2)$

$\therefore$ $\frac{\sin x}{x}$ decreases.

$\boxed{Q:}$ Prove that :-

$$\underbrace{\frac{2x}{\pi} < \sin x}_{②} \quad \underbrace{\sin x < x}_{①} \quad \cdots \quad (0, \pi/2).$$

Let $f(x) = x - \sin x$

$$f'(x) = 1 - \cos x \quad [\text{on } (0, \pi/2)]$$

& $1 - \cos x > 0$

$f(0) = 0$ & $f(x)$ is increasing

$\therefore$ $f(x) > 0 \Rightarrow x - \sin x > 0.$

$x > \sin x.$ $\longrightarrow$ part ①.

T.S. $\frac{2x}{\pi} < \sin x.$

$$\frac{2}{\pi} < \frac{\sin x}{x}.$$

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

$h(x) = x\cos x - \sin x.$

$h'(x) = -x\sin x + \cos x - \cos x = -x\sin x.$

$< 0.$

$\because$ on $(0, \pi/2)$ $x > 0$, $\sin x > 0$.

$\therefore$, $h'(x) \leq 0.$

$f'(x) \leq 0.$

~~On~~ or $f(x)$ is decreasing

$\frac{\sin x}{x}$ starts at 1 & ends at $\frac{2}{\pi}$

$$\frac{\sin(\pi/2)}{(\pi/2)} = \frac{2}{\pi}.$$

$\therefore$, $\frac{\sin x}{x} > \frac{2}{\pi}.$

Q: Show that the following func. has exactly 1 zero in interval $[-2, -1]$.

$f(x) = x^4 + 3x + 1$

$f(-2) = 16 - 6 + 1 = 11 > 0.$

$f(-1) = 1 - 3 + 1 = -1 < 0$

$f'(x) = 4x^3 + 3.$

$-2 \leq x \leq -1.$

$-8 \leq x^3 \leq -1.$

$$-32 \le 4x^3 \le -4$$

$$-29 \le 4x^3 + 3 \le -1$$

$$\therefore \quad f'(x) < 0$$

$\therefore$ the func. is +ve at one end, -ve at the other & decreasing

$\therefore$ it crosses the x-axis only once & hence has only 1 root.

- CAUCHY'S MVT:

Let $f(x)$ & $g(x)$ be continuous on the closed interval $[a,b]$, differentiable on (a,b) AND $g'(x) \neq 0$ for all x in (a,b), there exists a point c $a < c < b$.

such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

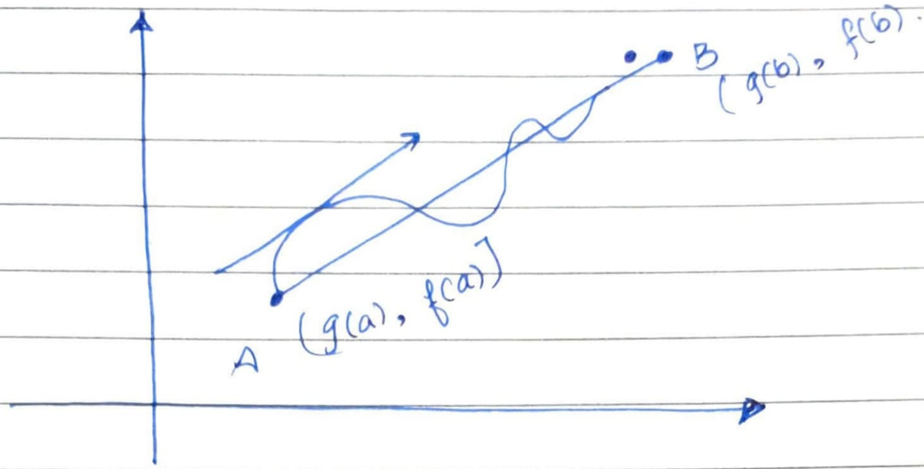
Let r be the curve described by the parametric equations:-

$$x = g(t)$$
$$y = f(t) \qquad t \in (a,b)$$

consider parametric form

PROOF.

~~Let γ be the curve described by the~~



$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(t)}{g'(t)}.$$

Slope of the secant $= \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

If $g(b) = g(a)$ $g(x)$ satisfies all conditions of Rolles th. &
$g'(\xi) = 0$ for some $a < \xi < b$

But the assumption is $g'(x) \neq 0$.
$\forall x \ (a, b)$.

Introduce an auxilliary function.
$F(x) = f(x) + \lambda g(x)$.
so that $F(a) = F(b)$.

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$$F(a) = f(a) + \lambda g(a)$$
$$F(b) = f(b) + \lambda g(b).$$

$$\lambda = \frac{-(f(b) - f(a))}{g(b) - g(a)}.$$

$$F(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g(x).$$

$F(x)$ is cont. on $[a,b]$

diff on (a,b)

& $F(a) = F(b)$.

$\therefore$, Rolle's th. can be applied

& $\exists$ c $a \leq c \leq b$

$F'(c) = 0$.

$$F^{*}(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g(x)$$

$$F'(c) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x) = 0$$

$$\therefore, \quad \frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(c)}.$$

$\boxed{Q:}$ Verify the validity of Cauchy's MVT for the functions.

- $f(x) = x^{*}$ $\qquad g(x) = x^2$ $\qquad [1,2]$.

$f(x), g(x)$ cont. on $[1,2]$

$f(x), g(x)$ diff on $(1,2)$

$g(x) \neq 0.$

$\therefore$ For some c.

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{16-1}{4-1}$$

$$\frac{f'(c)}{g'(c)} = \frac{15}{3} = 5. \quad = \frac{\cancel{4}^{2}\cancel{x}^{3}x^{2}}{\cancel{2}\cancel{x}} = \frac{5}{2}.$$

$$c = \sqrt{\frac{5}{2}}$$

- $f(x) = x^3$

$g(x) = \tan^{-1}(x).$ $[0, 1].$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a).} = \frac{1-0}{\pi/4 - 0}.$$

$$\frac{3x^2}{\cancel{1+}\frac{1}{(1+x^2)}} = \frac{4}{\pi}.$$

$$3x^2(1+x^2) = \frac{4}{\pi}.$$

$$3x^4 + 3x^2 = \frac{4}{\pi}$$

$$3x^4 + 3x^2 - \frac{4}{\pi} = 0.$$

$$x^2 = \frac{-3 \pm \sqrt{9 + \frac{48}{\pi}}}{6}.$$

$$x = \sqrt{\frac{-1 \pm \sqrt{1 + \frac{16}{3\pi}}}{2}}.$$

$$f(x) = \frac{x^3}{3} - 4x.$$

$$g(x) = x^2 \qquad [0,3).$$

f, g cont on $[0,3]$, diff on $(0,3)$.

$g(x) \neq 0$.

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

$$\frac{x^2 - 4}{2x} = \frac{-3 - 0}{9 - 0}$$

$$\frac{c^2 - 4}{2c} = \frac{-1}{3}.$$

$$3c^2 - 12 = -2c.$$

$$3c^2 + 2c - 12 = 0$$

$$c = \frac{-2 \pm \sqrt{4 + 144}}{6}$$

$$= \frac{\cancel{-2+} -1 \pm \sqrt{1 + 36}}{\cancel{6}\, 3}$$

Negelecting -ve, $c = \dfrac{-1 + \sqrt{\cancel{148}\, 37}}{3}.$

L'Hospitals Rule :-

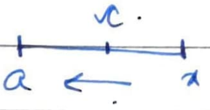
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad g(x) \neq 0 .$$

If $f(x) = g(x) = 0$
or indeterminant forms $\frac{0}{0}$, $\frac{\infty}{\infty}$.

Proof using Cauchy's MVT.

For $x \to a^+$

Suppose x lies to the right of a.



$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} .$$

To apply L' Hospitals rule we have.
$f(x) = g(x) = 0$ As $x \to a^+$, $c \to a^+$

$$\lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f(x)}{g(x)} .$$

This implies.

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \lim_{x \to a^+} \frac{f(x)}{g(x)} .$$

In a similar manner it can be shown for $x \to a^-$

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Q: $\lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}$.

$= \lim_{x \to \pi/2} \frac{\frac{1}{\cos x}}{1 + \frac{\sin x}{\cos x}} = \lim_{x \to \pi/2} \frac{1}{\cos x + \sin x} = 1.$

- Power series :- about $x = 0$.

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$$

about $x = a$.

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 x + \dots + C_n (x-a)^n.$$

For: $1 + x + x^2 + \dots + x^n = \frac{1}{1-x}$
converges to a finite no: if $|x| <$
The values of x here is collectively called radius of convergence.

- Taylor Series

Suppose $f(x) = \boxed{\sum_{n=0}^{\infty} a_n (x-a)^n}$

$f(x) = a_0 + a_1 (x-a) + \dots + a_n (x-a)^n \dots$
The derivatives of $f(x)$ exists

$f(a) = a_0$.
$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2$
$f''(x) = 2a_2 + 6a_3(x-a) \dots$

Here, $a_n = \dfrac{f^n(x)}{n!}$.

$f'(a) = a_1$

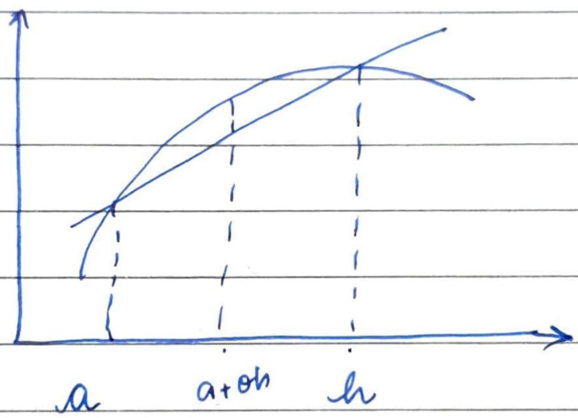
$f''(a) = \cancel{a_1+} \; 2a_2$

$f'''(a) = \cancel{3a} \; 3 \cdot 2 \cdot a_3$.

$\therefore f = n!\, a_n$

Taylor's series.

$$f(x) = a_0 + (x-a) f'(x) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(x) \cdot (x-a)^n}{n!} \dots \infty$$



$$f'(a+\theta h) = \frac{f(a+h) - f(a)}{\cancel{a}+h-\cancel{a}}$$

$$f'(a+h) = f(a) + h f(a+\theta h)$$

→ Mc Lauren's series is a Taylor series about $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n.$$

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Q: $f(x) = \frac{1}{x}$ at $x = 2$

Find Taylor's series

$f(x) = \frac{1}{x}$ $\quad f'(x) = \frac{-1}{x^2}$ $\quad f''(x) = \frac{2}{x^3}$.

$$f^{(n)}(x) = \frac{(-1)^n \cdot n!}{x^{n+1}} \quad \left\{ \frac{f^n(x)}{n!} = \frac{(-1)^n}{x^{n+1}} \right\}$$

$f(2) = \frac{1}{2}$

$f'(2) = \frac{-1}{4}$

$f''(2) = \frac{1}{2^3}$ $\quad f^n(x) = \frac{(-1)^n n!}{2^{n+1}}$

$$\therefore \frac{1}{x} = f(a) + \frac{f'(2) \cdot (x-2)}{1!} + \frac{f''(2)(x-2)^2}{2!}$$

$$+ \frac{f'''(3)(x-2)^3}{3!} + \dots \frac{f^n(x)(x-2)^n}{2^{n+1}}$$

Ans: ie: $\frac{1}{x} = \frac{1}{2} - \frac{1}{2^2}(x-2) + \frac{1}{2^3}(x-2)^2 \dots + \frac{(-1)^n (x-2)^n}{2^{n+1}}$

Here, $r = \left| \frac{x-2}{2} \right| < 1$. $\quad r = \frac{-(x-2)}{2}$.

ie: $\text{sum} = \frac{\frac{1}{2}}{1 + \frac{x-2}{2}} = \frac{1}{-2+x+2} = \boxed{\frac{1}{x}}$

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Q: Find Taylor's series for :-

$f(x) = e^x$ at $x = 0$.

$f'(x) = e^x$, $f''(x) = e^x$, $f^{(n)}(x) = e^x$

$f^n(0) = 1.$

$f(0) = 1.$ $f'(0) = 1 \ldots\ldots$

$\therefore e^x = f(0) + \frac{f'(0)(x)}{1!\cancel{2!}} + \frac{f''(0)(x)^2}{2!}$

$e^x = 1 + \frac{x}{\cancel{2!}1!} + \cancel{\xi}\frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{\cancel{4}n!}$

Q: $f(x) = \log(1+x)$ about $a = 0$.

$f'(x) = \frac{1}{1+x}$ $f'(0) = \cancel{0}.1.$

$f''(x) = \frac{-1}{(1+x)^2}$ $f''(0) = -1.$

$f'''(x) = \frac{2}{(1+x)^3}$ $f'''(0) = 2$

$\cancel{f^{(n)}(x) = \frac{(-1)}{(1+x)^n}}$

$f(x) = f(a) + f'(a)x + f''(a)\frac{x^2}{2!} + \ldots + \frac{f^n(a)\,x^n}{n!}$

$P_0(x) = 0.$

$P_1(x) = 0 + x.1 = x.$

$P_2(x) = 0 + 1x + \frac{x^2(-1)}{2} = x - \frac{x^2}{2}$

$P_3(x) = 0 + x - \frac{x^2}{2} + \frac{2(-1).x^3}{6} = x - \frac{x^2}{2} - \frac{x^3}{3}$

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Q: $f(x) = \sqrt{x}$ about $a = 4$.

Write the Taylor Polynomials P_0, P_1, P_2, P_3

$f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad f'(4) = \frac{1}{4}$.

$f''(x) = \frac{-1}{4x^{3/2}}$ $\qquad f''(4) = \frac{-1}{32}$.

$f'''(x) = \frac{3}{8x^{5/2}}$ $\qquad f'''(4) = \frac{3}{256}$

$P_0 = \sqrt{4} = 2$.

$P_1 = 2 + \frac{1}{4}(x-4)$

$P_2 = 2 + \frac{1}{4}(x-4) - \frac{1}{32}\frac{(x-4)^2}{2!}$

$P_3 = 2 + \frac{1}{4}(x-2) - \frac{1}{32}\frac{(x-4)^2}{2!} + \frac{3}{256}\frac{(x-4)^3}{3!}$.

Q: Find the Taylor Series :- at $x = 0$

$f(x) = xe^x$.

$f(0) = 0$.

$f'(x) = xe^x + e^x$.

$f''(x) = xe^x + 2e^x$.

$f'''(x) = xe^x + 3e^x$

$f^n(x) = xe^x + ne^x$

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$$f(x) = f(0) + \frac{f'(0)(x)}{1!} + \frac{f''(0)\cdot x^2}{2!} + \dots \frac{f^n(0)\,x^n}{n!}$$

$$f(x) = 0 + \frac{\cancel{2}\cdot x}{1} + \frac{2x^2}{2} + \frac{3x^3}{6} + \dots + \frac{\cancel{x}\; nx^n}{n!}$$

$$xe^x = x + x^2 + \frac{x^3}{3} + \dots + \frac{\cancel{x}\; x^n}{(n-1)!}$$

Q: $f(x) = \frac{2+x}{1-x}$ about $x=0$.

$f(0) = 2$.

$$f'(x) = \frac{(1-x) - (2+x)(-1)}{(1-x)^2} = \frac{1-x+2+x}{(1-x)^2} = \frac{3}{(1-x)^2}$$

$$f''(x) = \frac{3\cdot 2}{(1-x)^3} = \frac{3\times 2!}{(1-x)^3}. \qquad \frac{+2(3)}{(1-x)^2}$$

$$f'''(x) = \frac{3\cdot 3\cdot 2}{(1-x)^4} = \frac{3\cdot 3!}{(1-x)^4}$$

$$f^n(x) = \frac{3\cdot n!}{(1-x)^{n+1}}$$

$$\therefore f(x) = 2 + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)\cdot x^2}{2!} + \frac{f'''(0)\cdot x^3}{3!} + \dots + \frac{f^n(0)\cdot x^n}{n!}$$

$$= 2 + \frac{3x}{1!} + \frac{3\times 2!\,x^2}{2!} + \frac{3\cdot 3!\,x^3}{3!} + \dots \frac{3\cdot\cancel{n!}\,x^n}{\cancel{n!}}$$

$$= 2 + 3(x + x^2 + x^3 + \dots + x^n)$$

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Q: Find the Taylor Series ~~for~~ at $x = 2$,

$f(x) = x^3 - 2x + 4$

$f(2) = 2^3 - 2.2 + \cancel{4} = 8.$

$f'(x) = 3x^2 - 2.$

$f''(x) = 6x$

$f'''(x) = 6$

$f''''(x) = 0.$

$\therefore f(x) = 2 + \frac{(x-2).10}{1!} + \frac{(x-2)^2.12}{2!} + \frac{(x-3)^3.6}{3!}$

$f(x) = 2 + (x-2)10 + 6(x-2)^2 + (x-3)^3$

Q: $f(x) = 2x^3 + x^2 + 3x - 8$ at $a = 1$

$f(1) = 2 + 1 + 3 - 8 = -2.$

$f'(x) = 6x^2 + 2x + 3$ $f'(1) = 11$

$f''(x) = 12x + 2.$ $f''(1) = 14$

$f'''(x) = 12.$ $f'''(1) = 12$

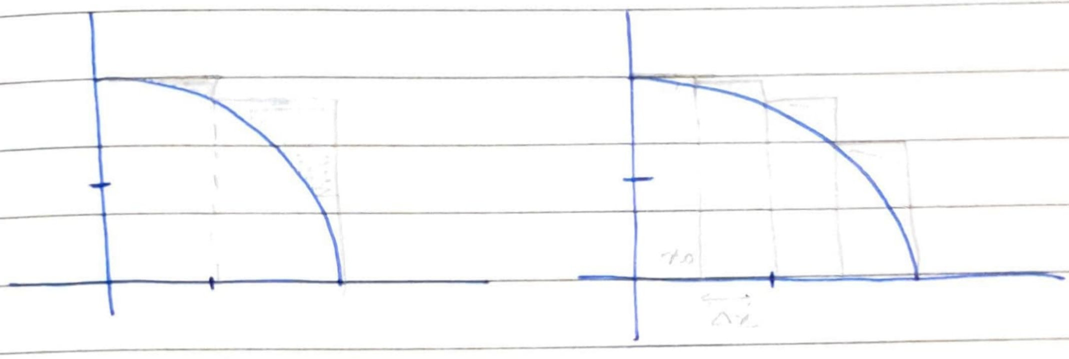
$f''''(x) = 0.$

$\therefore f(x) = -2 + \frac{(x-1).11}{1!} + \frac{(x-1)^2.14}{2!} + \frac{(x-1)^3.12}{3!}$

$= -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$

Area under the curve:

$y = 1-x^2$.



Upper & lower Sums

In the i^{th} interval $min = m_i$, $max = M_i$

$f(m_i)$ = Min value of $f(x)$

$f(M_i)$ = Max value of $f(x)$.

$\cancel{\Delta x \quad a + 0.\Delta x + \Delta x + 2\Delta x +}$

$a + 0\cdot\Delta x + a + \Delta x + 2\Delta x + a + a + 3\Delta x$
$+ \ldots + a + n\Delta x$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n} f(M_i)\Delta x.$$

Let f be continious & non-negative on $[a,b]$
the area bounded by $f(x)$ b/w a, b

$$\cancel{\lim_{n\to\infty}\sum} \quad \lim_{n\to\infty} \sum_{n=1}^{\infty} f(c_i)\Delta x \qquad \Delta x = \frac{b-a}{n}$$

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- Definition Of Riemann Sum

Let f be defined on at interval [a, b]
& 'Δ' be a partition given by

$$a = x_0 < x_1 < x_2 \dots < x_{n-1} < x_n = b$$

Δx_i is the width of the i^{th} interval

$a + n\Delta x$

$$\sum_{i=1}^{\infty} f(c_i) \cdot \Delta x_i \qquad x_{i-1} < c_i < x$$

→ Definite Integral :-
To define a definite integral :-

$$\sum_{\substack{i=1 \\ \|\Delta\| \to 0}}^{n} f(c_i) \Delta x_i = L \quad \text{(say)}$$

$$L - \sum_{i=1}^{n} f(c_i) \Delta x_i < \epsilon$$

If f is defined on [a, b]
&

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i \quad \text{exists.}$$

then f is integrable on [a, b].

$$\& \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_a^b f(x)$$

$F(x) = f'(x)$

Fundamental Th. of Calculus:-

If a function f is continuous on $[a,b]$
af F is the anti-derivative
(ie: $F'(x) = f(x)$.)
of f on $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Let Δ be a partition of $[a,b]$
$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

RHS: $F(b) - F(a)$

$$= F(x_n) - F(x_0)$$

$$= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \dots [F(x_1) - F(x_0)]$$

$$= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$$

By the mean value theorem :- to each interval.
There exists a number.

$$f(c_i) = F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

$x_{i-1} < c_i < x_i$

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$

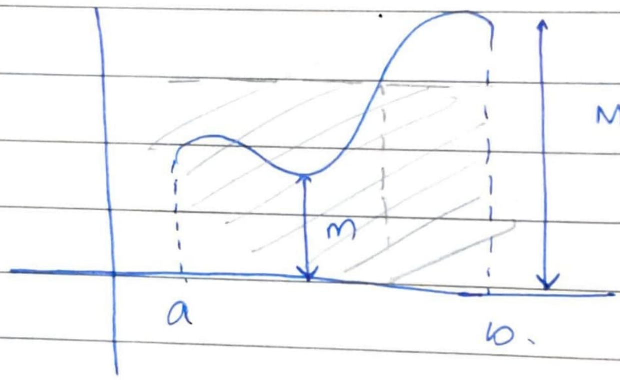
$$= \int_a^b f(x)\,dx \; : \; F(b) - F(a)$$

→ Mean value th. for integrals :-

If f is a continuous on $[a, b]$ then $\exists$ a c in $[a, b]$ such that.

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

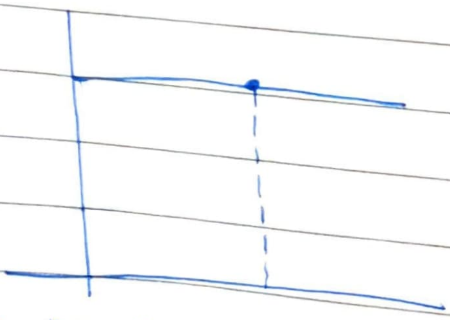
- **Mean Value Th for Integrals:**



If f is continuous on a closed interval $[a,b]$, there is a no: c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Case 1: If f is a constant func.



Then the result is true $\forall\, c \in [a,b]$

$\therefore$ Area beneath the curve

$$A = f(c)\cdot(b-a).$$

Case 2: Let

$f(m) \to$ min on $[a,b]$.

$f(M) \to$ max on $[a,b]$.

Such that: $f(m) \leq f(c) \leq f(M) \quad \forall x \in [a,b]$

Also, $\int_a^b f(m)dx \leq \int_a^b f(c)dx \leq \int_a^b f(M)\,dx.$

$\therefore$ $f(m)\int_a^b dx \leq \int_a^b f(c)dx \leq f(M)\int_a^b dx$

ie: $f(m)(b-a) \leq \int_a^b f(c)dx \leq f(M)(b-a).$

$\div$ by $(b-a)$:-

$$f(m) \leq \frac{\int_a^b f(c)\,dx}{(b-a)} \leq f(M).$$

Since f is continuous on $[a,b]$
By intermediate value th. [Extreme value]

$\therefore$ $\cancel{\int_a^b f(x)\,dx = (b-a).f(c)}.$

$\frac{1}{(b-a)}\int_a^b f(x)dx$ corresponds to $f(x)$ at some point say 'c' $\in [a,b]$

$\therefore$ $\int_a^b f(x)\,dx = (b-a)\,f(c)$

→ Find the value of c, guaranteed by the MVT, for integrals

1) $\int_0^2 x - 2\sqrt{x}\,dx \qquad [0, 2].$

$$= \frac{x^2}{2} - \frac{4}{3}x^{3/2}\Big]_0^2$$

$$\Rightarrow 2 - \frac{8\sqrt{2}}{3} = f(c) \times (b-a)$$

$$f(c) = \frac{2 - \frac{8\sqrt{2}}{3}}{2} = \frac{3 - 4\sqrt{2}}{3}$$

$$c - 2\sqrt{c} = \frac{3 - 4\sqrt{2}}{3}$$

$$\cancel{3c - 6\sqrt{c} = 3 - 4\sqrt{2}}$$

$$\cancel{6\sqrt{c} = 3c - 3 + 4\sqrt{2}}$$

$$(c - 2\sqrt{c} + 1) = \frac{3 - 4\sqrt{2}}{3} + 1$$

$$(\sqrt{c} - 1)^2 = \frac{6 - 4\sqrt{2}}{3}$$

$$\sqrt{c} - 1 = \pm\sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$\therefore \sqrt{c} = 1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$c = \left(1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}\right)^2$$

$c = 0.43$ or $c = 1.79$

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2). $f(x) = 2\sec^2 x \qquad [-\pi/4, \pi/4]$

$$\int_{-\pi/4}^{\pi/4} 2\sec^2 x \, dx$$

$$= \left[2\tan x\right]_{-\pi/4}^{\pi/4}$$

$$= 2\,[1-(-1)] = 2\,[2] = 4$$

By Mean value th:-

Now, $\int_a^b f(x)\,dx = (b-a)\, f(c) \qquad c \in [a,b]$

ie : $4 = \left[\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)\right] f(c).$

$$4 = \pi/2 \cdot [2\sec^2 c].$$

$$4 = \frac{\pi}{\cos^2 c}.$$

$$\cos^2 c = \frac{\pi}{4}. \qquad \cos c = \pm \frac{\sqrt{\pi}}{2}.$$

$$c = \cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right) \quad \text{or} \quad c = \cos^{-1}\left(\frac{-\sqrt{\pi}}{2}\right)$$

$$= \pi - \cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right).$$

$$\underline{c = 0.48^c.} \qquad \in [a,b].$$

3). $f(x) = \cancel{x^3}\ \frac{9}{x^3} \qquad [1, 3]$

$$\int_1^3 \frac{9}{x^3}\, dx.$$

$$= \left.\frac{9}{-2x^2}\right]_1^3 = \frac{-9}{2}\left[\cancel{\sqrt{3}/\frac{1}{\sqrt{3}} - 1}\right].$$

$$= \frac{-9}{2}\left[\frac{1}{9} - 1\right] \Rightarrow \frac{-1}{2} + \frac{9}{2} = 4.$$

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By MVT:-

$\int_a^b f(x)\,dx = (b-a)\,f(c)$.

$\cancel{4} = 2 \cdot \frac{9}{c^3}$.

$c^3 = 4.5$

$c = 1.63 \in [1,3]$

4>. $\int_{-\pi/3}^{\pi/3} \cos x \, dx$

$= [\sin x]_{-\pi/3}^{\pi/3} = \frac{\sqrt{3}}{2} - \left(\frac{-\sqrt{3}}{2}\right) = \sqrt{3}$.

By mean value th:-

$\int_a^b f(x)\,dx = (b-a)\,f(c)$.

i.e. $\sqrt{3} = \left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] \times \cos c$

$\sqrt{3}\,\cancel{\sqrt{3}} = \frac{2\pi}{3}\cos c$.

$\cos c = \frac{3\sqrt{3}}{2\pi}$.

$c = \cos^{-1}\left(\frac{3\sqrt{3}}{2\pi}\right)$.

$c = 0.59$.

And $[-\pi/3, \pi/3] = [-1.05, +1.05]$

$\therefore -1.05 < 0.59 < 1.05$

$\therefore c \in [a,b]$.

Average value of a function :-

If f is integrable on the closed interval $[a,b]$, then the average value of f on $[a,b]$, is :

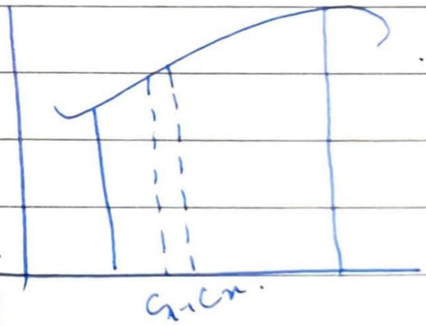
$$\frac{1}{(b-a)} \int_a^b f(x)\,dx$$

Divide the interval into n equal parts

$\Delta x = \frac{b-a}{n}$ length of each sub interval.

Suppose c_i is some point in the i^{th} interval

The average value of the function :-



$$a_n = \frac{1}{n}\left[f(c_1) + f(c_2) + \dots + f(c_n)\right]$$

$$a_n = \frac{1}{n} \cancel{\sum_{n=1}^{\infty}} \sum_{i=1}^{n} f(c_i) \times \frac{b-a}{(b-a)}$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(c_i) \times \frac{b-a}{n}$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(c_i) \cdot \Delta x$$

Taking limit :-

$$\frac{1}{b-a} \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

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$$\text{Average} = \frac{1}{b-a} \times \int_a^b f(x)\,dx.$$

Q: Find the average value of the function

1) $f(x) = 3x^2 - 2x$ on $[1, 4]$.

$$\int_1^4 3x^2 - 2x. = \left[x^3 - x^2\right]_1^4$$
$$= (64 - 16) - (1-1).$$
$$= 48.$$

$$\therefore \text{Av} = \frac{1}{b-a}\left[\int_a^b f(x)\,dx\right]$$

$$= \frac{1}{3} \times 48 = \underline{16.\text{ sq. units}}.$$

Q: Find :-

2) $f(x) = \cos x$ on $[0, \pi/2]$

$$\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \cos x\,dx = \left[\sin x\right]_0^{\pi/2}$$
$$= 1$$

$$\therefore \text{Av} = \frac{1}{\pi/2}[1] = \frac{2}{\pi}$$

Q: The force F of a hydraulic cylinder in a press is proportional to the square of $\sec x$. where x is the distance in m that the cylinder is extended in its

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cycle. The domain of f is $[0, \pi/3]$ &
$f(0) = 500$.

i) find F as a function of x.

$F \propto \sec^2 x$.
$F = K \sec^2 x$.

at $f(0)$:-
$500 = F(0) = K$.
$\therefore F(x) = 500\sec^2 x$

ii) Find the av. force expended by the press over the interval $[0, \pi/3]$.

Here: by av. value of func:-

$$\frac{1}{b-a} \int_a^b f(x)\, dx.$$

$$= \frac{1}{\frac{\pi}{3}} \int_0^{\pi/3} 500\sec^2 x\, dx.$$

$$= \frac{3}{\pi} \cdot 500 \left[\tan x\right]_0^{\pi/3}$$

$$= \frac{3}{\pi} \times 500 \times \sqrt{3}.$$

$$= \underline{\underline{\frac{1500\sqrt{3}}{\pi}}} \text{ N}.$$

$\boxed{Q:}$ The vol. V in litres of air in the lungs during a 5s, respiratory cycle is approximated by the model

$\cancel{V = 0.1729t + 0.1522t^2}$

$V = 0.1729t + 0.1522t^2 - 0.0374t^3$

where t is the time in s.

Approximate the average vol. of air in the lungs during 1 cycle.

$\int V dt = \int 0.1729t + 0.1522t^2 - 0.0374t^3 \, dt$

$= \left[0.8645t^2 + 0.5073t^3 - 0.00935t^4\right]_0^5$

$\therefore av = \frac{\int V dt}{5}.$

$= 0.8645 \times 5 + 0.5073 \times 25 - 0.00935 \times 125$

$=$ ______

- Fundamental Th. of Calculus (II)

If f is continuous on an open interval I containing a then $\forall x$ in the interval

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

$$F(x) = \int_a^x f(t)\,dt$$

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Eq: $F(x) = \int_0^x \cot t \, dt$

Also, $F(x)$ is the accumulating area under the curve.

Proof:

$$F(x) = \int_a^x f(t)\, dt.$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}.$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left\{ \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt \right\}.$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left\{ \int_a^{x+\Delta x} f(t)\,dt + \int_x^a f(t)\,dt \right\}.$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left\{ \int_x^{x+\Delta x} f(t)\,dt \right\}.$$

Using MVT. for integrals.

$$= \lim_{\Delta x \to 0} \cdot \frac{1}{\Delta x} \left[f(c) \{ \cancel{x} + \Delta x - \cancel{x} \right].$$

$$\left[\because \; c \in (x, \Delta x) \right]$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{1}{\cancel{\Delta x}} f(c) \cdot \cancel{\Delta x}$$

$$= f(x).$$

Hence. Proved

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→ Using 2nd Fundamental th. of Int. Calculus

1) Find $\frac{d}{dt}\left[\int_0^x \sqrt{t^2+1}\, dt\right]$

$\because$ $\sqrt{t^2+1}$ is continuous

$\therefore$ $= \sqrt{x^2+1}.$

2). $F(x) = \int_{\pi/2}^{x^3} \cos t\, dt$

Let $x^3 = u.$

$\therefore$

$$\frac{dF(x)}{dx} = \frac{dF(x)}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{d(u)}\left[\int_{\pi/2}^{u} \cos t\, dt\right] \cdot \frac{du}{dx}$$

$$= \cos u \cdot 3x^2$$

$$= \cos x^3 \cdot 3x^2$$

→ Find F as a function of x and evaluate it at $x = 2, 5, 8.$

$$F(x) = \int_2^x (t^3 + 2t - 2)\, dt$$

$$F(x) = \left[\frac{t^4}{4} + t^2 - 2t\right]_2^x$$

$$F(x) = \frac{x^4}{4} + x^2 - 2x - (4 + 4 - 4)$$

$$\frac{x^4}{4} + x^2 - 2x - 4$$

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$F(2) = 4 + \cancel{4} - \cancel{4} - 4$

$= 0.$

$F(5) = \frac{625}{4} + 25 - 10 - 4.$

$= \frac{625}{4} + 11.$

$F(8) = \frac{\cancel{64}\cdot 64}{4} + 64 - 16 - 4.$

$= 64.16 + 64 - 16 - 4$

→ $F(x) = \int_0^x \sin\theta \, d\theta.$ Eval. at $x = 2, 5, 8.$

$f(x) = \cancel{[\sin x]_0^x}$ $[-\cos\theta]_0^x.$

$= -\cos x - (-\cos 0)$

$= -\cos x + 1$

$= 1 - \cos x$

$F(2) = 1 - \cos 2.$

$F(5) = 1 - \cos 5.$

→ Integrate to find F as a function of x & demonstrate the second fundamental th.

$F(x) = \int_0^x t(t^2 + 1)\, dt.$

$F(x) = \left[\frac{t^4}{4} + \frac{t^2}{2}\right]_0^x.$

$$F(x) = \frac{x^4}{4} + \frac{x^2}{2}.$$

To demonstrate $\frac{d}{dx}\left[\int \frac{f(x)}{\cancel{F(x)}}\right] = F(x).$

ie: $\frac{d}{dx}\left(\frac{x^4}{4} + \frac{x^2}{2}\right) = x^3 + x$

$= x(x^2+1) = F(x)$

→ The total cost C of purchasing & maintaining a piece of equipment for x years is

$$C(x) = \cancel{500}\ 5050\left(25 + 3\int_0^x t^{1/4}\,dt\right).$$

$C(x) = ?$ $C(1) = ?$

$C(5) = ?$

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→ Consider a particle moving along the x-axis where :
$x(t)$ is the position of the particle at time t.

$x'(t) \longrightarrow$ velocity.

$\int_a^b \left| \frac{d(x)}{dt} \right| dt$ is the dist. travelled by the particle in the interval of time.

$\int_a^b |x'(t)|$

$x(t) = t^3 - 6t^2 + 9t - 2 \qquad 0 \le t \le 5$

Find the dist. the particle travels in 5 units of time.

$x'(t) = 3t^2 - 12t + 9$

$= 3(t^2 - 4t + 3)$.

$= 3(t-1)(t-3)$

$\therefore, \int_a^b |x'(t)|\, dt. = 3\int_a^b |t-1||t-3|\, dt.$

If $0 < x < 1$ $\qquad (t-1) < 0, \quad (t-3) < 0.$

$\therefore,$ Dist $= 3(t-1)(t-3)$.

If $1 < x < 3$. $\qquad (t-1) > 0 \qquad t-3 < 0.$

Dist $= -3(t-1)(t-3) = 3(1-t)(t-3)$

If $5 > x > 3$. $\qquad 3(t-1)(t-3)$.

$$x(t) = 3\left[\int_0^1 (t-1)(t-3)\,dt + \int_1^3 (1-t)(t-3) + \int_3^5 (t-1)(t-3)\,dt\right]$$

$$= 3\left[(t^2-4t+3)\Big|_0^1 + (4t-3-t^2)\Big|_1^3 + (t^2-4t+3)\Big|_3^5\right]$$

$$= 3\left[\cancel{0}\,(-3) + \cancel{\;\;} + 8 - (\right.$$

$$= 15$$

Q: Let f be a func. defined on $[a,b]$. What condition could you place on f to guarantee that, ~~min of f'~~

$$\min f' \le \frac{f(b)-f(a)}{b-a} \le \max f'$$

where $\min f'$ & $\max f'$ refer to min & max values of f' on $[a,b]$

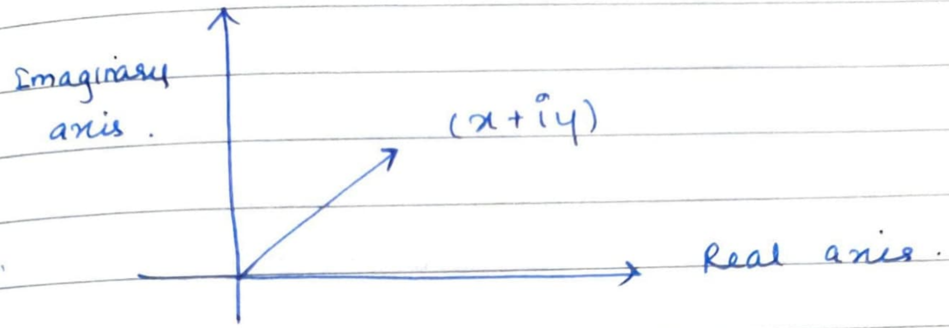
→ f' should be continuous on the interval in order for the function f' to ~~at~~ attain the minimum

11/10/18

COMPLEX NOs:

any complex no: can be represented as $Z = x + iy$

or $Z = (x, y)$,



→ Polar Form:-

$Z = x + iy.$

Let $x = r\cos\theta$ $\quad y = r\sin\theta$.

$\therefore \quad Z = r(\cos\theta + i\sin\theta)$

$r = \sqrt{x^2+y^2}$ modulus of z.

$\theta \rightarrow$ argument of z.

- $e^{i\theta} = \cos\theta + i\sin\theta$.
- $e^{in\theta} = \cos n\theta + i\sin n\theta$.

Euler's Formulae.

$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

[Lookup Riemann surfaces ; branches].

→ Principal argument:-

$-\pi < \text{Arg} \leq \pi$.

$\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$

$Z_1\overline{Z_1} = |Z_1|^2$.

circle $|z - a| = R \Rightarrow z = a + Re^{i\theta}$

$z_1 = r_1 e^{i\theta_1}$ $\quad z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Q: Represent in Polar form:- & plot on a graph.

1) $1+i$

$z = 1 + 1.i$ $\qquad x = 1,\ y = 1.$

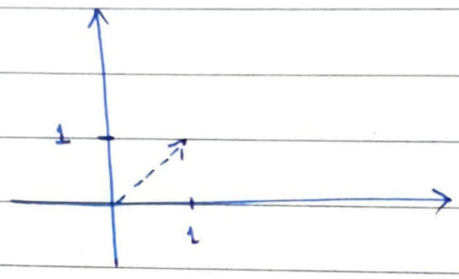
$r = \sqrt{1+1} = \sqrt{2}.$

$x = r\cos\theta.$ $\qquad y = r\sin\theta$

$\cos\theta = \frac{1}{\sqrt{2}}$ $\qquad \frac{y}{r} = \sin\theta = \frac{1}{\sqrt{2}}.$

$\theta = \pi/4$

$\therefore\ z = \sqrt{2}\,[\cos(\pi/4) + i\sin(\pi/4)]$



2). $-4 + 4i$

$z = -4 + 4i$ $\qquad x = -4,\ y = 4$

$r = \sqrt{16+16} = 4\sqrt{2}$

$\cos\theta = \frac{x}{r} = \frac{-4}{4\sqrt{2}} = \frac{-1}{\sqrt{2}}$ $\qquad \sin\theta = \frac{1}{\sqrt{2}}.$

$\therefore\ \theta = \pi - \pi/4 = \frac{3\pi}{4}.$

$\therefore\ z = 4\sqrt{2}\,[\cos(3\pi/4) + i\sin(3\pi/4)]$

→ Unit circle :-

$|z| = 1$

$\therefore\ \sqrt{x^2+y^2} = 1 \Rightarrow x^2+y^2 = 1.$

Centre $(0,0)$

Radius $= 1$

→ $|z-a| = \rho$

circle with centre a & radius ρ.

$|z-a| \leq \rho$ closed set [includes all pts on the rim also]

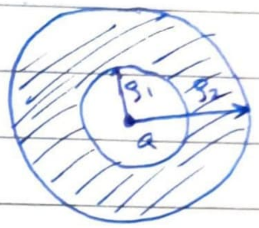
$|z-a| < \rho$ open set.

→ Annulus :

Any pt can be written as :-

$|z-a| \geq \rho_1$

$|z-a| \leq \rho_2$.

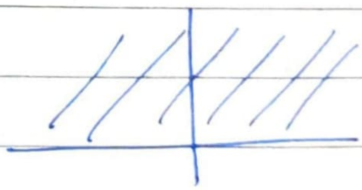


$\therefore$ $\underline{\rho_1 \leq |z-a| \leq \rho_2}$

→ <u>Half Plane :</u>

Upper :- $z = x+iy$ $y > 0$.

Similarly other planes can be defined.



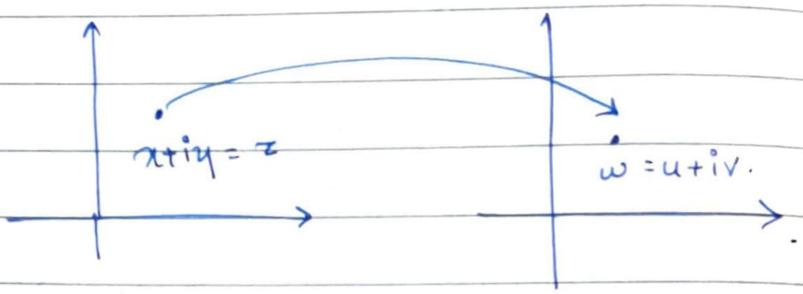
<u>COMPLEX FUNCTIONS</u> :

For real no: functions, every x of the domain is mapped to an $f(x)$.

However, for complex function, a 2-D point on a plane is mapped to another pt. on a 2D plane.

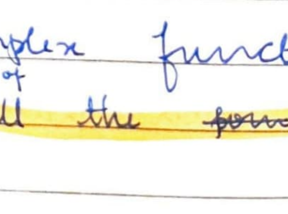
$$f(z) = w.$$

$$z \xrightarrow{f} w.$$



→ Limit of a function :-

$$\lim_{z \to z^0} (f(z)) = l.$$



There are multiple paths by which z can approach z₀.

The limit of a complex function will exist only if limit of all the ~~poss~~ paths is the same

→ Continuity

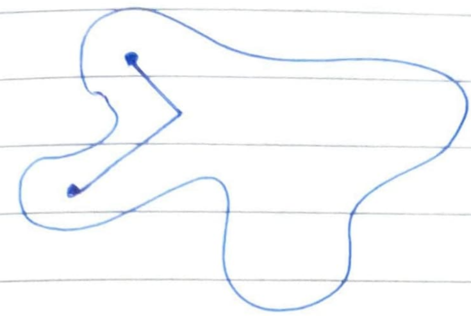
A function is continuous on a domain if it is continuous at each point.

- BOUNDED SET :-

A set S is bounded if $\exists$ a constant M, $|z| < M$ $\forall$ pts z in S.

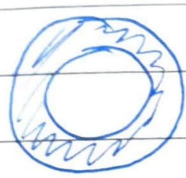
• CONNECTED SET :-

If any two pts of the set can be joined by a path of straight line segments

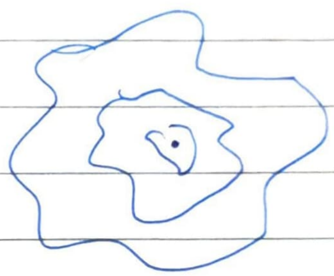


[path must lie in the set]

→



multiple connected region.



simply connected region.
can be reduced to a single point.

→ Region:

An open connected set is called an open region or domain.

• DERIVATIVE:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

eg: To check if derivative of $f(z) = \bar{z}$ exists

$z = x + iy$

then; $\Delta z = \Delta x + i\Delta y$

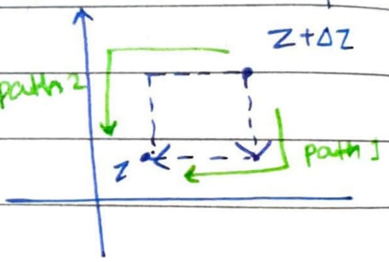
$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

$$= \lim_{\Delta z \to 0} \frac{(\overline{z+\Delta z}) - (\bar{z})}{\Delta z}.$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Here limits are applied to variables separately



case 1: $\Delta y \to 0$ first

then $\Delta x \to 0$

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = 1.$$

case 2: $\Delta x \to 0$ first.

$$\lim_{\Delta z \to 0} \quad -1.$$

$\because$ since the limits aren't equal,

$\therefore$ lim doesn't exist

$\Rightarrow$ Derivative doesn't exist

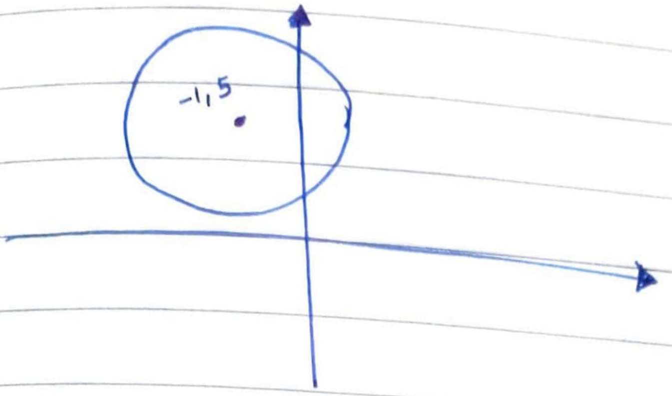
x ———— x

- Region:-

It is a domain together with some or none of its boundary points

Plot : $|z + 1 - 5i| \leq 3/2$.

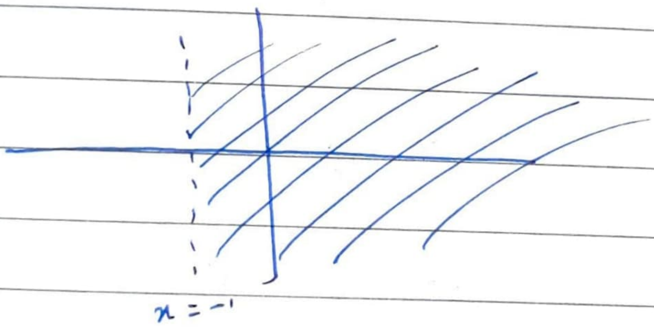
$|z - (5i - 1)| \leq 3/2$.



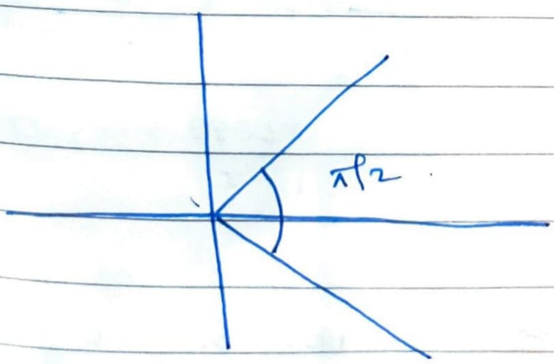
→ $\text{Re } z \geqslant -1$.

$z = x + iy$.

$\text{Re } z = x \geqslant -1$.



→ $|\text{Arg } z| \leq \pi/4$.



Q: Determine & give reasons whether $f(z)$ is continuous at $z = 0$ where $f(0) = 0$

$$f(z) = \frac{Re(z^2)}{|z|^2} \quad \text{if } z \neq 0.$$

Working with polar co-ordinates

$$z = r\cos\theta + ir\sin\theta.$$

$$z^2 = r^2\cos 2\theta + ir\sin 2\theta.$$

$$Re(z)^2 = r^2\cos 2\theta.$$

$$|z|^2 = r^2.$$

convert to polar!

$$\frac{Re(z)^2}{|z|^2} = \frac{\cancel{r^2}\cos 2\theta}{\cancel{r^2}} = \cos 2\theta = f(z)$$

clearly $\lim_{\theta \to 0} \cos 2\theta = 1.$

$\therefore$ This is not continuous.

Q: $f(z) = \dfrac{|Re(z)|}{1-|z|.}$ $\quad f(0) = 0.$

$$= \frac{x}{1-\sqrt{x^2+y^2}} = \frac{\cancel{x}\, r\cos\theta}{1-r}$$

As $r \to 0$, $\dfrac{r\cos\theta}{1-r} \longrightarrow 0$

$\therefore$ This func. is continuous.

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Q: $|z^2| \cdot Re\left(\frac{1}{z}\right)$.

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x}{r^2} - \frac{iy}{r^2}.$$

$$|z^2| \cdot Re\left(\frac{1}{z}\right) = \cancel{r^2} \cdot \frac{r\cos\theta}{\cancel{r^2}} = r\cos\theta \to 0.$$

as $r \to 0$.

$\therefore$ continuous.

→ Derivative :-

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

along every path.

~ Analyticity: A function is analytic in a domain (D) if f(z) is defined & differentiable at all points of D.

A function is analytic at a pt. $x = z_0$ in D if f(z) is analytic in a neighbourhood of z_0. / Holomorphic.

→ ~~Holomorphic~~

n^{th} root:

$$z = r(\cos\theta + i\sin\theta)$$

$$z^{1/n} = r^{1/n}\left(\cos\left(\frac{\phi + 2k\pi}{n}\right) + i\sin\left(\frac{\phi + 2k\pi}{n}\right)\right)$$

~~Rela~~ Necessary conditions for $f = u + iv$ to be analytic
$\Rightarrow$ CR eqns. satisfied.

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→ Cauchy Riemann Eqn

$w = f(z) = u(x,y) + iv(x,y)$ be defined and continuous in a neighbourhood of a pt. $z = x+iy$ & differentiable at z itself. Then ~~the~~ at that point the 1st order partial derivative of u & v exist & will satisfy the Cauchy Riemann eqn.

$$u_x = v_y \quad \left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right]$$

$$u_y = -v_x \quad \left[\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$$

Hence $f(z)$ is analytic in a dom D.

$f'(z)$ at z exists.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

$$\Delta z = \Delta x + i\Delta y.$$

$$z + \Delta z = (x+\Delta x) + i(y+\Delta y).$$

$$f(z) = u(x,y) + iv(x,y).$$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y).$$

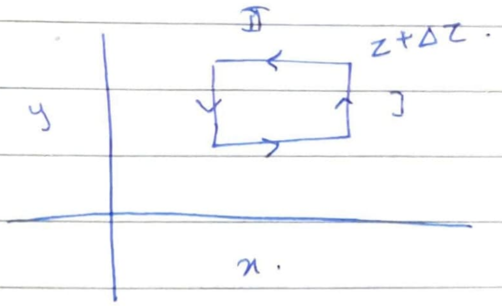
Subtracting: $-f(z)$.

$$= u\,\Delta x + i\,\Delta y.$$

Taking limits:-

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{u(x+\Delta x, y + \Delta y) - u(x, y) + i[v(x+\Delta x, y + \Delta y) - v(x, y)]}{\Delta x}.$$

$f'(z)$



Case I: $f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y) + i[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$.

Set $\Delta y = 0$.
sending $\Delta z = \Delta x + i\Delta y$ to 0
along x-axis
i.e. $\Delta x \to 0$

$$= \frac{\delta u}{dx} + \frac{i\delta v}{dx}.$$

Case II :- $f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y) + i[v(x+\Delta x, y) - v(x,y)]}{i\Delta y}$.

Set $\Delta x = 0$
sending $\Delta z \to 0$
along y-axis.
i.e. $\Delta y \to 0$

$$= \frac{\delta v}{\delta y} - i\frac{\delta u}{dy}.$$

But both of these should be equal as $f'(z)$ is diff ∴ should be same along both paths.

$$\therefore \frac{\delta u}{dx} + \frac{i\delta v}{\delta x} = \frac{\delta v}{\delta x} - \frac{i\delta u}{dy}$$

$$\underline{u_x = v_y} \qquad \underline{u_y = -v_x}.$$

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→ CONVERSE:

If a real valued continuous func. $u(x,y)$ & $v(x,y)$ of a real variables x & y & have continuous first partial derivative, & these satisfy the Cauchy Riemann eqn.

ie: $u_x = v_y$ $\quad u_y = -v_x$ in some domain then

$$f(z) = u(x,y) + iv(x,y) \text{ is analytic.}$$

$$\Delta z = \Delta x + i\Delta y.$$
$$\Delta w = f(z_0 + \Delta z) - f(z_0).$$
$$\Delta w = \Delta u + i\Delta v.$$

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0 y_0)$$
$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0).$$

Applying MVT: ~~to~~ to func. $u(x,y)$ & $v(x,y)$

to $$\frac{u(x + \Delta x, y_0) - u(x,y)}{\Delta x} = \frac{\partial u}{\partial x}(x_1, y_0)$$

at some pt. x_1,

where $x_0 < x_1 < x_0 + \Delta x$

Since the partial derivatives are continuous

$$\frac{\partial u}{\partial x}(x, y_0) - \frac{\partial u}{\partial x}(x_0, y_0) = \epsilon_1$$

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$$\frac{\delta u}{d x}(x_1, y_0) = \epsilon_1 + \frac{\delta u}{d x}(x_0, y_0) \quad (2)$$

Similarly

$$\frac{\delta u}{d y}(x_0, y_1) = \epsilon \cdot \frac{\delta u}{d y}(x_0, y_0)$$

$$y_0 < y_1 < y_0 + \Delta y$$

$$\Delta u = \frac{\delta u}{d y}(x_0 + \Delta x_0, y_1$$

$$\Delta u = u(\overset{x_0+\Delta x}{\cancel{x + x_0}}, y_0 + \Delta y) - u(x_0 + \Delta x, y_0) + \quad (1)$$
$$u(x_0 + \Delta x, y_0) - u(x_0, y_0).$$

By MVT :-

$$\frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x + \Delta x, y_0)}{\Delta y} = \frac{\delta u}{d x}(x_0 + \Delta x, y_1) \quad (2a)$$

where $\boxed{y_0 < y_1 < y_0 + \Delta y}$

??

$$\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = \frac{\delta u}{d x}(x_1, y_0)$$

$$\boxed{x_0 < x_1 < x_0 + \Delta x}$$

Similarly : $\frac{\delta u}{d y}(x_0, y_1) - \frac{\delta u}{d y}(x_0, y_0) = \epsilon_2.$

Using (2a) & (2b) in (1) :-

$$\frac{\delta u}{\delta y}(x_0 + \Delta x, y_1)\Delta y + \frac{\delta u}{d x}(x_1, y_0)\Delta x$$

$$\therefore \Delta u = \frac{\partial u}{\delta y}(x_0, y_0)\Delta y + \epsilon_2 \Delta y$$
$$+ \frac{\delta u}{d x}(x_0, y_0)\Delta x + \epsilon_1 \Delta x$$

$$\Delta v = v_x \Delta x + i\epsilon_3 \Delta x + v_y \Delta y + \epsilon_4 \Delta y$$

$$\Delta u = u_x \Delta x + i\epsilon_1 \Delta x + u_y \Delta y + \epsilon_2 \Delta y$$

Now, $f(z) = w$

$z = x + iy$ $\quad$ $w = u + iv$.

To find $\lim \frac{\Delta w}{\Delta z}$

$\therefore \Delta u + i\Delta v = \Delta w$

$$= u_x \Delta x + u_y \Delta y + i v_x \Delta x + i v_y \Delta y + (\epsilon_1 + i\epsilon_3)\Delta x + (\epsilon_2 + i\epsilon_4)\Delta y$$

$$= (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + (\epsilon_1 + \epsilon_3)\Delta x + (\epsilon_2 + \epsilon_4)\Delta y$$

$\therefore |\Delta x| \le |\Delta z|$

$\therefore \left|\frac{\Delta x}{\Delta z}\right| \le 1.$

Also: $(\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} \to 0$

& $(\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z} \to 0$

→ Using the Cauchy Riemann Eqn :-

$$u_x \Delta x - v_x \Delta y + i v_x \Delta x + i u_y \Delta y.$$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{u_x(\Delta x + i\Delta y)}{(\Delta x + i\Delta y)} + \frac{iv_x(\Delta x + i\Delta y)}{\Delta x + i\Delta y}$$

$$f'(z) = \frac{dw}{dz} = u_x + i v_x.$$

- Polar Co-ordinates :

$$z = x + iy$$
$$= re^{i\theta}$$

$x = r\cos\theta$

$y = r\sin\theta .$

$$\frac{\delta x}{\delta r} = \cos\theta \qquad \frac{\delta y}{\delta r} = \sin\theta .$$

$$\frac{\delta x}{\delta\theta} = -r\sin\theta \qquad \frac{\delta y}{\delta\theta} = r\cos\theta .$$

$$\frac{\delta u}{\delta\theta} = \frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta\theta} + \frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta\theta}.$$

$$= \frac{\delta u}{\delta x}\cdot(-r\sin\theta) + \frac{\delta u}{\delta y}(r\cos\theta).$$

$$\boxed{u_\theta = -r\,u_x\sin\theta + r\,u_y\cos\theta}$$

$$\frac{\delta u}{\delta r} = \frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta r} + \frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta r}.$$

$$= u_x(\cos\theta) + u_y\cdot\sin\theta .$$

$$\boxed{u_r = u_x\cos\theta + u_y\sin\theta .}$$

$$\frac{\delta v}{\delta\theta} = \frac{\delta v}{\delta x}\cdot\frac{\delta x}{\delta\theta} + \frac{\delta v}{\delta y}\cdot\frac{\delta y}{\delta\theta}$$

$$\boxed{v_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\boxed{v_r = v_x \cos\theta + v_y \sin\theta}$$

$$u_r = u_x\cos\theta + u_y \sin\theta \quad — \quad (1)$$

$$u_\theta = -r\, u_x \sin\theta + r u_y \cos\theta. \quad — \quad (2)$$

Using CR. eqns.

$$v_r = v_x \cos\theta + v_y \sin\theta.$$

$$v_r = -u_y \cos\theta + u_x \sin\theta. \quad — \quad (3)$$

$$v_\theta = -v_x\, r\sin\theta + v_y\, r\cos\theta.$$

$$= r u_y \sin\theta + r u_x r \cos\theta. \quad — \quad (4)$$

$\therefore$ From (1) & (4).

$$r u_r = v_\theta.$$

$$-r V_r = u_\theta.$$

Alternative :-

$$u + iv = f'(z) = f'(re^{i\theta})$$

$$u_r + iv_r = f'(re^{i\theta}) \cdot e^{i\theta}.$$

$$u_\theta + i v_\theta = f'(re^{i\theta}) \cdot ire^{i\theta}.$$

$$= ir(u_r + iv_r)$$

$$= -rV_r + ir u_r.$$

$\therefore$ $\boxed{u_\theta = -rV_r.}$ & $\boxed{v_\theta = r u_r}$

Q: Show that $f'(x)$ does not exist at any point.

1). $f(z) = \bar{z}$

$f(z) = x - iy$

$u_x = 1$ $\quad$ $v_y = -1$

Cauchy Riemann Eq. not satisfied.

$\therefore$ $f'(x)$ doesn't exist.

2). $f(z) = z - \bar{z}$.

$= (x+iy) - (x-iy)$.

$= x + iy - x + iy. = 2iy.$

$u_x = 0$ $\quad$ $u_y = 0$

$v_x = 0$ $\quad$ $v_y = 2.$ $\quad$] not satisfied.

Q: Verify that the following functions are entire analytic.

1). $f(z) = \underbrace{3x + y}_{u} + i\underbrace{(3y - x)}_{v}.$

$u_x = 3$ ... continuous.

$u_y = 1$ cont.

$v_x = -1$ cont.

$v_y = 3.$

$\underline{u_y = -v_x}$ $\quad$ $\underline{u_x = v_y}$

$e^z = e^x [\cos y + i \sin y]$

2). $f(z) = \sin x \cdot \cosh y + i \cos x \sinh y$.

$u = \sin x \cosh y$ $\qquad v = \cos x \sinh y$.

$u_x = \cosh y \cos x$ $\qquad v_x = \sinh y \cdot (-\sin x)$

$u_y = -\sinh y \sin x$ $\qquad v_y = \cosh y \cos x$.

∴ continuous function
& C.R. eqns. are verified.

→ Le Place Eqn:-

$u_{xx} + u_{yy} = 0$.

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

and any u which satisfies this eqn is called a Harmonic function.

$u_x = v_y$. $\qquad u_{xx} = v_{yx}$

$u_y = -v_x$. $\qquad u_{yy} = -v_{xy}$

If it's a continuous function, second derivatives will be the same for whatever order considered

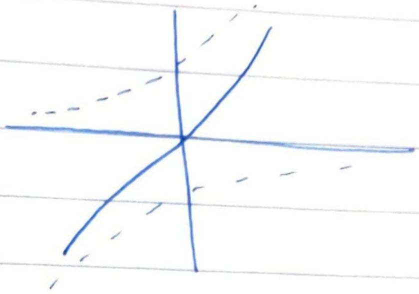
→ If 2 funcs. u & v satisfy the C.R. eqns then V is the conjugate harmonic func. of u.

$\cos z = \frac{1}{2}[e^{iz} + e^{-iz}]$

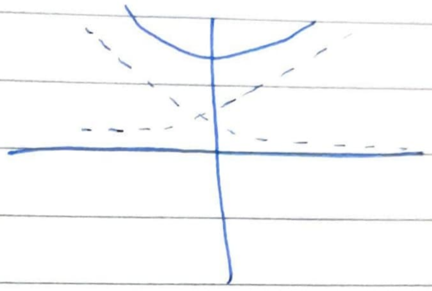
$\sin z = \frac{1}{2}[e^{iz} - e^{-iz}]$

Hyperbolic func:-

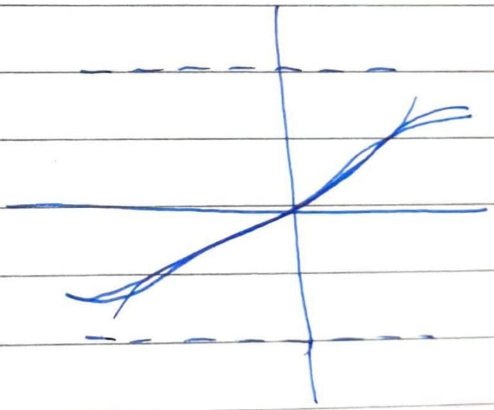
$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$



$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$



$\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$.



- Properties :

1. $\sinh x + \cosh x = e^{x}$.
2. $\cos^2 hx - \sin^2 hx = 1$.
3. $1 - \tan^2 hx = \sec^2 hx$.
4. $\sinh(2x) = 2\sinh(x) \cdot \cosh(x)$.

5. $\cosh(x+y) = \dfrac{e^{x+y} + e^{-x-y}}{2}$

$= \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

6. $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^{x}-e^{-x}}{2}\right) = \frac{e^{x}+e^{-x}}{2}.$

$= \cosh x$

7. $\frac{d}{dx}(\cosh x) = \sinh(x).$

8. $\frac{d}{dx}(\sec h(x)) = \cancel{\frac{d}{dx}\left(\frac{1}{\cosh(x)}\right)} \cdot \frac{d}{dx}\left(\frac{1}{\cosh x}\right)$

$= \cancel{\frac{-1}{[\cosh(x)]^2} \cdot \sinh(x)}. \quad -\frac{\sinh(x)}{[\cosh(x)]^2}.$

$= -\operatorname{sech}(x) \cdot \tanh(x).$

$\frac{d}{dx}(\tan h x) = \cancel{\frac{d}{dx}\left(\frac{e^{+x}-e^{-x}}{e^{x}+e^{-x}}\right)}$

$\cancel{= \frac{d}{dx}\left(\frac{e^{2x}-1}{e^{2x}+1}\right) =}$

$\cancel{= (e^{2x}+1)(2e^{2x}) - (e^{2x}-1)(2e^{2x})}$

$\frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$

$= \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{(\cosh x)^2}.$

$= \frac{\cos^2 hx - \sin^2 hx}{\cos^2 hx} = \boxed{\sec^2 hx}$

9. $\frac{d}{dx}(\operatorname{cosec} hx) = \frac{d}{dx}\left(\frac{1}{\sinh x}\right)$

$= \frac{-1}{\sin^2 hx} \cdot \cosh x = -\coth x \cdot \operatorname{cosech} x$

$\ln z = \ln |z| + i \,\text{Arg}\,(z)$ $\qquad \because e^{w} = z$.

$w = u + iv$

$e^{u} = r \qquad \theta = v$

→ To calculate lim in complex analysis we need to consider all possible paths of approaching the limit.

× ———— ×

Q: Show that $u(x,y)$ is harmonic in some domain where $u(x,y) = 2x(1-y)$.

(1) $u_x = 2(1-y) \qquad u_y = -2x$

$u_{xx} = 0 \qquad u_{yy} = 0$.

$u_{xx} + u_{yy} = 0$.

$\therefore u$ is harmonic.

$u = 2x(1-y) \qquad u_y = -2x$

$u_x = 2(1-y)$.

$u_x = v_y \Rightarrow \dfrac{\partial v}{\partial y} = \cancel{2}\ 2(1-y)$. — (1)

Integrating WRT y :-

$$V = 2y - y^2 + h(x) \quad \text{— (2)}$$

$u_y = -V_x = -2x \qquad V_x = 2x$. — (3)

Diff. (2) with resp. to x.

$\dfrac{\partial v}{\partial x} = h'(x) = 2x$ from (3).

Int. WRT x :-

$h(x) = x^2 + c$.

Sub in (2) :-

$V = 2y - y^2 + x^2 + c \Rightarrow \underline{x^2 - y^2 + 2y + c}$.

$$f(z) = 2x(1-y) + i(x^2-y^2+2y+c)$$

(2) $u(x,y) = 2x - x^3 + 3xy^2$.

$u_x = v_y \qquad u_y = -v_x.$

$u_x = 2-3x^2+3y^2 \qquad u_y = 6xy$

$u_{xx} = -6x \qquad u_{yy} = 6x$

$u_{xx} + u_{yy} = 0$

$\therefore$ u is harmonic.

$u_x = \frac{\partial v}{\partial y} = 2-3x^2+3y^2.$

Int WRT. y:-

$V = 2y - 3x^2y + y^3 + h(x).$

$\frac{\partial v}{\partial x} = -u_y = -6xy + h'(x) = -6xy$

$h'(x) = 0. \qquad h(x) = c$

$\therefore$ $v(x,y) = 2y - 3x^2y + y^3 + c$

(3) $u(x,y) = \sinh x \cdot \sin y.$

$u_x = \cosh x \sin y \qquad u_y = \sinh x \cos y.$

$u_{xx} = \sinh x \sin y \qquad u_{yy} = -\sinh x \sin y$

$u_{xx} + u_{yy} = 0$

$\Rightarrow$ u is harmonic

$u_x = v_y \qquad u_y = -v_x$.

ie:- $u_x = \frac{dv}{dy} = \cancel{\sinh x \cos y}$. $\cosh x \sin y$.

Int. WRT y:-

$V = -\cosh x . \cos y + h(x)$.

Now, $u_y = -v_x$.

$V_x = -\sinh x \cos y + h'(x) = -\sinh x \cos y$.

$h'(x) = 0$.

$\therefore h(x) = c$.

$\therefore V(x,y) = -\cosh x \cos y + c$

4. Verify that

$u(x,y) = \frac{y}{x^2+y^2}$ is harmonic

& find the harmonic conj.

$u(x,y) = \frac{y}{x^2+y^2}$.

$u_x = \frac{-2yx}{(x^2+y^2)^2} \qquad u_y = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$u_{xx} =$

$\frac{(x^2+y^2)^2 . (-2y) + 2yx(2(x^2+y^2).2x)}{(x^2+y^2)^4}$

$\frac{(x^2+y^2)(-2yx^2-2y^3+8x^2y)}{(x^2+y^2)^4} = \frac{(6x^2y-2y^3)}{(x^2+y^2)^3}$.

$$U_y = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$U_{yy} = \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)(2(x^2+y^2)\cdot 2y)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\left[-2x^2y-2y^3 - 4yx^2+4y^3\right)}{(x^2+y^2)^4}$$

$$= \frac{2y^3-6x^2y}{(x^2+y^2)^3}$$

$\therefore$ $U_{xx} + U_{yy} = 0$. $\therefore$ Harmonic

Now $U_x = V_y$ $U_y = -V_x$.

$$\frac{\partial v}{\partial y} = U_x = \frac{-2xy}{(x^2+y^2)^2}.$$

Int WRT y :- $V(x,y) = \dfrac{-x\cdot(-1)}{(x^2+y^2)} = \dfrac{x}{x^2+y^2} +$

Now, $\dfrac{\partial v}{\partial x} = -U_y = \dfrac{y^2-x^2}{(x^2+y^2)^2}.$

ie:- $\dfrac{(x^2+y^2) - x(2x) + h'(x)}{\cancel{(x^2+y^2)^2}} \quad \dfrac{y^2-x^2}{\cancel{(x^2+y^2)^2}}$

$\therefore$ $h'(x) = 0$ $h(x) = c.$

$\Rightarrow$ $V(x,y) = \dfrac{x}{x^2+y^2} + c.$

Double integral

$$\iint_{R} f(x,y)\,(dx\,dy) \to dA \quad \text{(some area el.)}$$

R (region)

Eg: $$\int_{1}^{2}\int_{y=1}^{x} \left(2\frac{x^2}{y^2} + 2y\right) dy\Big)\, dx.$$

$$\int_{1}^{2} \left[-\frac{2x^2}{y} + y^2\right]_{1}^{x} dx.$$

$$\int_{1}^{2} \left[-2x + x^2 - (-2x^2 + 1)\right] dx.$$

$$\int_{1}^{2} \left[3x^2 - 2x - 1\right] dx$$

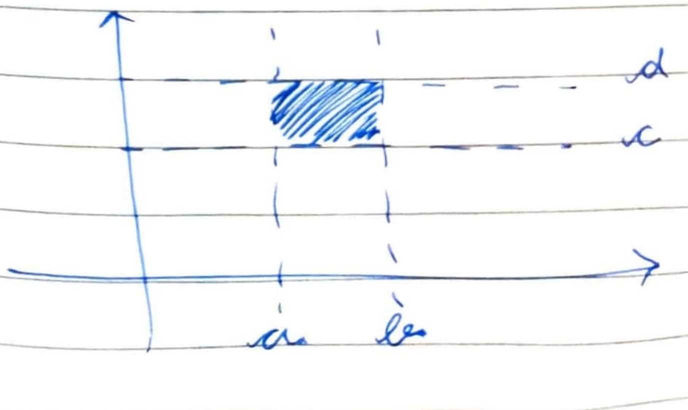
$$= \left[x^3 - x^2 - x\right]_{1}^{2}$$

$$= \left[8 - 4 - 2\right] - \left[\cancel{1} - \cancel{1} - 1\right]$$

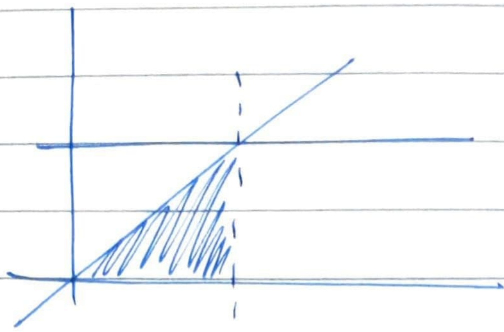
$$= \cancel{6 - 1 = 5} \quad 2 + 1 = 3$$

Region of int :-

eg: $$\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\, dx\, dy$$

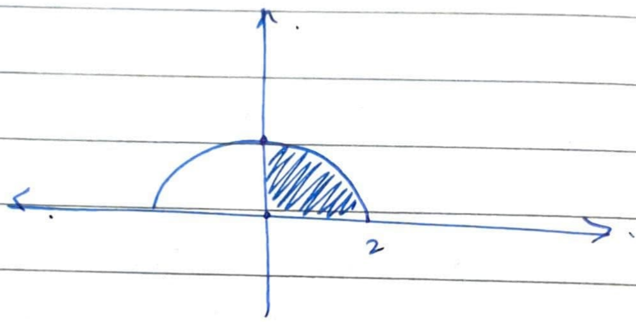


Eg: $$\int_{0}^{1} \int_{x=0}^{x=y} f(x,y)\, dy \cdot dx.$$

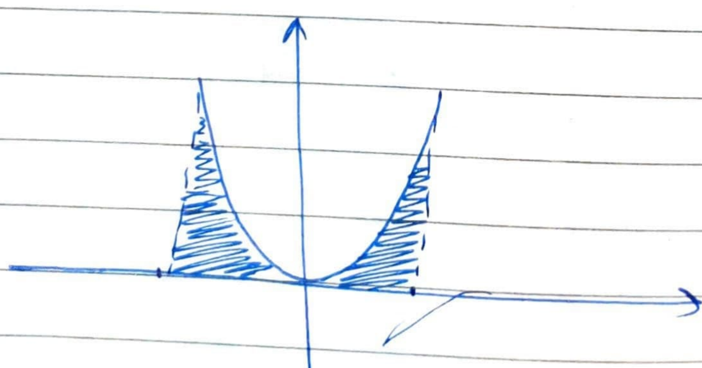


Shaded region: region of integration

Eg: $$\int_{0}^{2} \int_{y=0}^{y=\sqrt{4-x^2}} f(x,y)\, dy \cdot dx.$$



Eg: $$\int_{-1}^{1} \int_{y=x^2}^{y=1} f(x,y)\, dx\, dy.$$



... is continuous.

LINE INTEGRAL :

$\int_C f(z)\, dz$.

C: is the path of integration.

c is positive if in the anti-clockwise sense.

Let c be a smooth curve.

$a \leq t \leq b$

Partition $[a, b]$

$a = t_0, t_1 \ldots\ldots t_{n-1}, t_n = b$

$t_0 < t_1 < t_2 < \ldots\ldots < t_{n-1} < t_n$

z is taken as a parametric func. of t

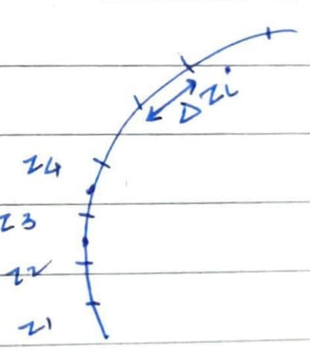
$z(t) = x(t) + iy(t)$.

Consider $z(t_0), z(t_1) \ldots\ldots z(t_n)$ pts. on curve

$z_0, z_1, \ldots z_n$.

consider a point on these subdivisions.

Let these pts :- ζ_m



Summing :-

$$S_n = \sum_{m=1}^{n} f(\zeta_m) . \Delta z_m . \qquad [\Delta z_m = z_m - z_{m-1}]$$

Consider the greatest of the partitions.

$|\Delta t_m| = |t_m - t_{m-1}| \longrightarrow 0$ as $n \to 0$.

$\therefore |\Delta z_m| = z_m - z_{m-1} \longrightarrow 0$

$\therefore$ Definition of line integral :-

$$\sum_{m=1}^{n} f(\zeta_m) \Delta z_m = \oint_C f(z)\, dz$$

$f(z) = z$ dt

- $\int_C k_1 f_1(z) + k_2 f_2(z) = \int_C k_1 f_1(z) + \int_C k_2 f_2(z)$

x ——— x

→ Milne Thomson Method:

$f(z)$ .... given $u(x, y)$

$f'(z) = u_x + i v_x$

$= u_x - i u_y.$

Now $x = \frac{z+\bar{z}}{2}$ $y = \frac{z-\bar{z}}{2i}$

$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

This is valid for all z ∴ valid even for $z = x$ ie: z is real.

∴ Replace $x \to z$ & $y \to 0$.

And integrate $f'(z)$ then.

Eg: $u(x, y) = 2x - x^3 + 3xy^2$.

$u_x = 2 - 3x^2 + 3y^2$

$u_y = 6xy$

$f'(z) = u_x - i u_y$

$f'(z) = (2 - 3x^2 + 3y^2) - i\,6xy.$

Replacing $x \to z$ $y = 0$.

$f'(z) = 2 - 3z^2$

$\int f'(z) = 2z - z^3 + C$

$f(z) = 2z - z^3 + C$

eg: $u(x,y) = \dfrac{y}{x^2+y^2}$.

$u_x = \dfrac{-y}{(x^2+y^2)^2}(2x) = \dfrac{-2xy}{(x^2+y^2)^2}$.

$u_y = \dfrac{(x^2+y^2) - y(2y)}{(x^2+y^2)} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$.

$f'(z) = \dfrac{-2xy}{(x^2+y^2)^2} - i\dfrac{(x^2-y^2)}{(x^2+y^2)^2}$.

$x = z \qquad y = 0$

$f'(z) = \dfrac{-iz^2}{z^4} = \dfrac{-i}{z^2}$.

$\therefore \quad f(z) = -i\int \dfrac{1}{z^2} = -i\dfrac{(-1)}{z} + c$

$f(z) = \dfrac{i}{z} + c$

x ——— x

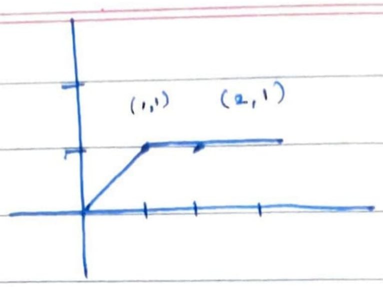
C

Q: Trace this polygonal line :-

$z = x + ix \qquad 0 \le x \le 1$

$= x + i \qquad 1 \le x \le 2$.

$(0,0) \to (1,1) \qquad (1,1)$ to $(2,1)$.



→ circle :

$|z| = r$ centre → origin

Radius $= r$

$z = re^{i\theta}$ $0 \leq \theta \leq 2\pi$.

$|z - z_0| = r$

ie: $z = z_0 + re^{i\theta}$ ($0 \leq \theta \leq 2\pi$)

[one int. around the path].

- **Evaluating line Integral:**

$$\int_C f(z)\, dz$$

Put $z(t) = x(t) + iy(t)$

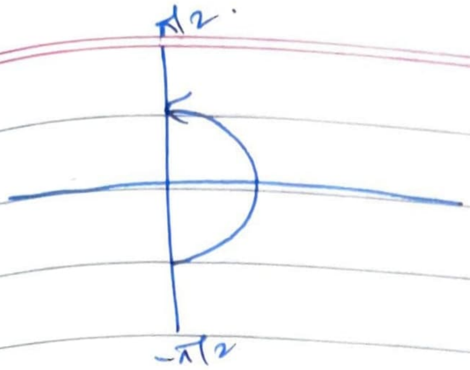
Eg: Eval: $\int_C \bar{z}\, dz$

where C is $z = 2e^{i\theta}$ $-\pi/2 \leq \theta \leq \pi/2$

$z = 2e^{i\theta}$.

$\bar{z} = 2e^{-i\theta}$

$dz = 2ie^{i\theta} \cdot d\theta$



contour
where
$-\pi/2 \le \theta \le \pi/2$.

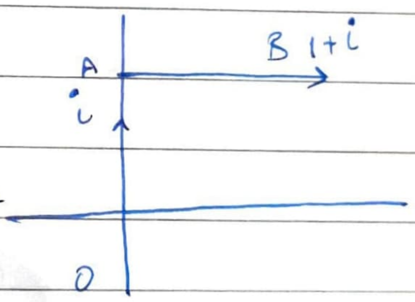
$$\therefore \int_{-\pi/2}^{\pi/2} \bar{z} \cdot dz = \int_{-\pi/2}^{\pi/2} 2e^{-i\theta} \cdot 2e^{i\theta} i \, d\theta.$$

$$= 4i \int_{-\pi/2}^{\pi/2} 1 \cdot d\theta$$

$$= 4i \left[\theta\right]_{-\pi/2}^{\pi/2} = \boxed{4\pi i}$$

eg: $f(z) = y - x - i3x^2$

$\int_{C_1} f(z)$



$$\int_{C_1} f(z) = \int_{OA} f(z)dz + \int_{AB} f(z)dz$$

On OA:-

$x = 0 \quad dx = 0$.

$y \to 0$ to 1.

$dz = dx + idy = idy$

On OA: $\int_{y=0}^{1} y \cdot idy = \frac{i}{2}\left[y^2\right]_0^1 = \frac{i}{2}$

On AB: $x \to 0$ to 1.

$y = 1$.

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$dz = dx + i dy$

$dy = 0.$

$dz = dx.$

On AB:-

$\therefore \int_0^1 (-x - i3x^2 + 1)\, dx.$

$= -\frac{1}{2}[x^2]_0^1 - i[x^3]_0^1 + [x]_0^1$

$= -\frac{1}{2} - i + 1$

$= \frac{1}{2} - i$

$\therefore$ Answer $= \frac{i}{2} + \frac{1}{2} - i = \boxed{\frac{1}{2} - \frac{i}{2}}$

If we integrate along $O \to B$

eqn of OB : $y = x$

$z = x + iy$

$= x + ix$

$dz = dx(1+i)$

$\int_{OB} f(z)\, dz = \int_{x=0}^{1} (\cancel{x} - \cancel{x} - i3x^2)(1+i)\, dx.$

$= -i(1+i) \int_0^1 3x^2\, dx.$

$= -i(1+i) \cdot [x^3]_0^1$

$= -i - i^2$

$= 1 - i$

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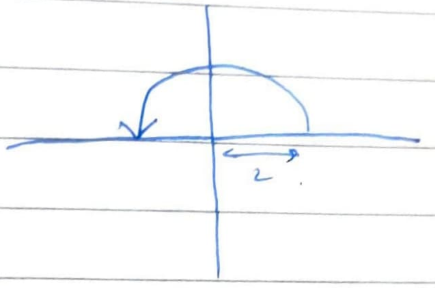
Q: $f(z) = \dfrac{z+2}{z}$

$C: \; z = 2e^{i\theta} \; (0 \le \theta \le \pi)$

$dz = 2e^{i\theta} \cdot i \, d\theta$



$\int_0^{\pi} f(z)\,dz$

$= \int_0^{\pi} \dfrac{2e^{i\theta}+2}{\cancel{2}e^{i\theta}} \cdot \cancel{2e^{i\theta}} \cdot i \, d\theta$

$= 2i \int_0^{\pi} (e^{i\theta} + 1) \, d\theta$

$= 2i \left[e^{i\theta} + \theta \right]_0^{\pi}$

$= 2i \left[\dfrac{e^{i\theta}}{i} + \theta \right]_0^{\pi}$

$\cos\pi + i\sin\pi$

$= 2i \left[\dfrac{e^{\pi i}}{i} + \pi - \dfrac{1}{i} \right]$

$= 2i \left[\dfrac{-1}{i} + \pi - \dfrac{1}{i} \right]$

$= 2i \left[\dfrac{-2}{i} + \pi \right]$

$= \underline{\underline{-4 + 2\pi i}}$

Q: $f(z) = \pi e^{\pi z}$

where C is the boundary of the square with vertices at points: in the counterclockwise

$z = 0$

$z = 1$

$z = 1+i$

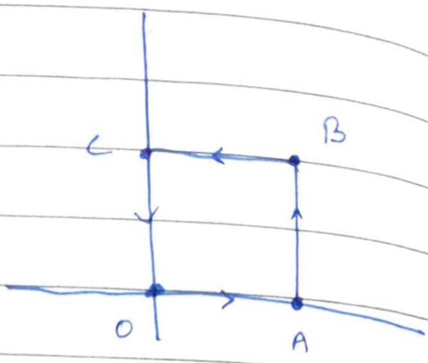
$z = i$

$z = 0$ $\qquad f(z) = \pi e^{\pi \bar{z}}$

$z = 1$ $\qquad f(z) = \pi e^{\pi(x - iy)}$.

$z = 1 + i$

$z = i$.



Along OA:-

$y = 0 \quad x \Rightarrow 0 \text{ to } 1$.

$dy = 0$

$f(z) = \pi e^{\pi(x)}$.

$dz = dx + i dy$

$= dx$.

$$\therefore \int_{OA} = \int_0^1 \pi e^{\pi x} \, . \, dx.$$

$$= \cancel{\pi} . \left[\frac{e^{\pi x}}{\cancel{\pi}} \right]_0^1 = \left[e^{\pi x} \right]_0^1$$

$$= e^{\pi} - e^{0} = \boxed{e^{\pi} - 1}$$

Along AB:-

$x = 1, \quad y \to 0, 1 \qquad dx = 0$

$f(z) = \pi e^{\pi(x - iy)}$.

$dz = dx + i dy = i dy$

$$\therefore \int_{AB} f(z) = \int_0^1 \pi e^{\pi(1 - iy)} \, i dy$$

$$= \pi i \int_0^1 e^{\pi - i\pi y} \, dy = \pi i \int_0^1 \frac{e^{\pi}}{e^{i\pi y}} dy$$

$$= \pi i e^{\pi} \int_0^1 e^{-i\pi y} dy$$

$$= \cancel{\pi} i e^{\pi} \left[\frac{e^{-i\pi y}}{-i\cancel{\pi}} \right]_0^1$$

$$e^{\pi} \left[e^{-i\pi} - 1 \right] = -2e^{\pi}$$

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Along BC $\quad y=1 \quad x \to (1,0)$

$dy = 0 \qquad dz = dx.$

$$\int_{BC} \pi e^{\pi(x-i)} = \int_1^0 \pi e^{\pi(x-i)}\, dx.$$

$$= \int_1^0 \frac{\pi [e^{\pi x}]}{e^{\pi i}}\, dx$$

$$= \frac{\pi}{e^{\pi i}} \int_1^0 e^{\pi x}\, dx$$

$$= \frac{\cancel{\pi}}{e^{\pi i}} \cdot \frac{[e^{\pi x}]_1^0}{\cancel{\pi}}$$

$$= \frac{1}{e^{\pi i}} [1 - e^{\pi}] = \boxed{e^{\pi} - 1.}$$

Along CO: $\quad x \to 0 \qquad y \to (1,0).$

$dz = i\,dy.$

$$\int_{CO} \pi e^{\pi(-iy)} = \int_1^0 \pi e^{-\pi i y}\, i\,dy$$

$$= \cancel{\pi i} \frac{[e^{-\pi i y}]_1^0}{\cancel{-\pi i}} = -1 [1 - e^{-\pi i}]$$

$$= -1 [1 - (-1)]$$

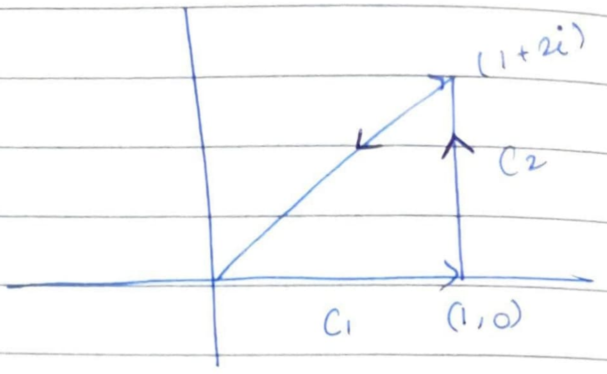
$$= -2$$

∴ Answer.

$$= \cancel{e^{\pi}} - 1 + (-2\cancel{e^{\pi}}) + \cancel{e^{\pi}} - 1 - 2.$$

$$= -4$$

Q: $f(z) = Re(z)$.



Parametric : (x_1, y_1) (x_2, y_2)

$x = (1-t)(x_1) + t x_2$

$y = (1-t)(y_1) + t(y_2)$ $0 \le t \le 1$.

C1 : $(x_1, y_1) = (0,0)$

$(x_2, y_2) = (1,0)$

$x = (1-t)0 + t = t$

$y = (1-t)(0) + 0 = 0$.

$z = x+iy$ $z = x$ $dz = dt$

C2 : $(x_1, y_1) = (1,0)$

$(x_2, y_2) = (1,2)$

$x = (1-t)(1) + 1(t) = 1$

$y = (1-t)(0) + 2t = 2t$.

$dz = 0 + i2dt = 2idt$.

C2: $f(z) = Re(z) = \int_0^1 2idt = 2i$

C1: $f(z) = Re(z) = \int_0^1 t\,dt = \frac{t^2}{2}\Big]_0^1 = \frac{1}{2}$

C_3 :-

$(x_1, y_1) \rightarrow (1, 2)$

$(x_2 y_2) \rightarrow (0, 0)$

$x = (1-t)(1) + 0 = 1-t$

$y = (1-t)2 + 0 = 2-2t$.

$dz = [-1-2i]\, dt$.

$\int_0^1 (1-t)(-1-2i)\, dt$

$= (-1-2i)\left[t - \frac{t^2}{2}\right]_0^1$

$= (-1-2i)(1-\frac{1}{2}) = \frac{-1-2i}{2} = \frac{-1}{2} - i$

$\therefore$ Answer $= 2i + \frac{1}{2} - \frac{1}{2} - i = i$

x———————x

Singularity ~~Computing~~ : Points where f^n not analytic.

$\oint (z-a)^n\, dz$

circle with centre a, radius R.

$|z-a| = R$.

$z = a + Re^{i\theta}$.

$dz = Rie^{i\theta}\, d\theta$.

for a full circle $0 \leq \theta \leq 2\pi$.

$\therefore \int_0^{2\pi} \underset{Re^{i\theta}}{\underbrace{(z-a)^n}}\, \underset{Rie^{i\theta}\, d\theta}{\underbrace{dz}}$

$$= \int_0^{2\pi} R^n e^{ni\theta} \cdot Rie^{i\theta}\, d\theta$$

$$= R^{n+1} \cdot i \int_0^{2\pi} e^{(n+1)i\theta}\, d\theta \cdot$$

$$= R^{n+1} \cancel{i} \cdot \left. \frac{e^{(n+1)i\theta}}{\cancel{i}(n+1)} \right]_0^{2\pi}$$

$$\frac{R^{n+1}}{n+1} \left[e^{2\pi i(n+1)} - 1 \right] = 0 \qquad n \neq 1$$

If: $n = -1$:,

$z = Re^{i\theta}$.

$\therefore \oint \frac{1}{(z-a)} \cdot dz \qquad 0 \leq \theta \leq 2\pi$

$$= \int_0^{2\pi} \frac{1}{\cancel{R}\cancel{e^{i\theta}}} \cdot \cancel{R}i\cancel{e^{i\theta}}\, d\theta$$

$$= 2\pi i$$

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- M – L Inequality :-

Bound for the contour integral.

$$\left| \int_C f(z)\, dz \right| \leq ML$$

L : length of the contour C.

$\therefore |f(z)| \leq M$ everywhere in C.

L ①

Now, $|S_n| = \left|\left[\sum_{m=1}^{n} f(\zeta_m)\right] \Delta z_m\right|$

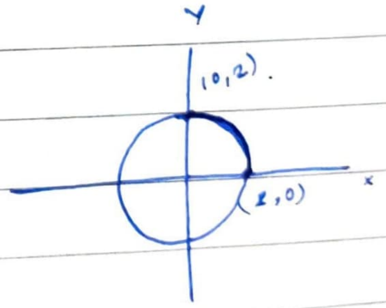
$$|S_n| \leq (M) \underbrace{\sum_{m=1}^{n} |\Delta z_m|}_{\text{length of contour}}$$

$$|S_n| \leq ML.$$

→ For every pt z, we obtain the value of z from the contour. Correspondingly we obtain $f(z)$.

Let C be the arc of circle $|z| = 2$, from $z = 2$ to $z = 2i$ that lies in the first quad. Find the bound of the integral.

$$\int_C \frac{z+4}{z^3+1} \, dz$$



$$L = \frac{1}{4} \cdot 2 \cdot \pi \cdot 2 = \pi$$

TIP: Look for a value ST, num < no: & den > no:

$$|z+4| \leq |z| + |4|$$
$$\leq 2 + |4| \leq 6.$$

$$|z^3 - 1| \geq ||z^3| - 1| \geq 7.$$

$$\therefore |f(z)| \leq \frac{6}{7}$$

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$$\therefore \int_C f(z)\, dz \le \frac{6\pi}{7}$$

Q: Let C_R be the semi-circular path

$$Z = Re^{i\theta} \qquad 0 \le \theta \le \pi \qquad R>1$$

Integrate

$$\int_C \frac{Z^{1/2}}{1+Z^2}\, dz$$

$$Z^{1/2} = e^{\log Z^{\frac{1}{2}}} = e^{\frac{1}{2}\log Z}$$

$$Z^{1/2} = \sqrt{R}\, e^{i\theta/2}$$

$$|Z| = R$$

$$|Z^{1/2}| = |\sqrt{R}\, e^{i\theta/2}|$$

$$= \sqrt{R}.$$

$$|1+z^2| \ge ||z^2|-1|$$

$$\ge R^2-1.$$

$$\therefore M = \frac{\sqrt{R}}{R^2-1}$$

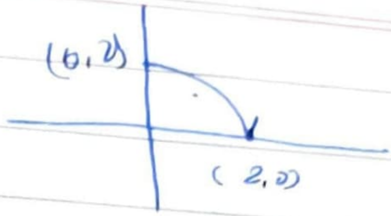
$$L = \pi R.$$

$$\therefore ML = \frac{R^{3/2}\pi}{R^2-1}$$

Q: Without evaluating the integral, show that

$$\left|\int_C \frac{dz}{z^2-1}\right| \le \pi/3$$

where contour :-



$|z| = 2$ $\quad 0 \leq \theta \leq \pi/2.$

$L = \frac{1}{4} \cdot 2\pi \cdot 2 = \pi.$

$|z^2 - 1| \geq ||z|^2 - 1|$
$\geq 4 - 1 \geq 3$

$\therefore$

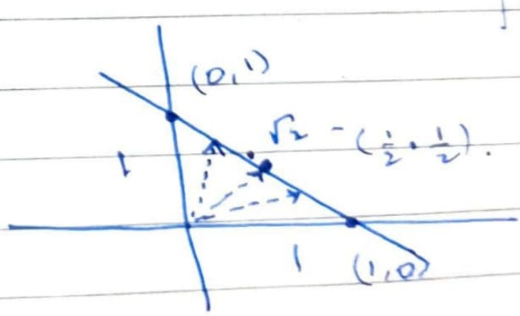
$M = 1/3 \qquad L = \pi$

$\therefore \left| \int_C \frac{dz}{z^2 - 1} \right| \leq \pi/3$

Q. Let C denote the segment from $z = i$ to $z = 1$. By observing that of all the pts. on that line seg., the mid pt. is the closest to the origin Estimate $\left| \int_C \frac{dz}{z^4} \right| \leq ML$.

$\sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$

$L = \sqrt{2}.$



Min. length
or $\min |z| = \frac{1}{\sqrt{2}}.$

$\therefore |z| \geq \frac{1}{\sqrt{2}}$ $\qquad \therefore \left| \frac{1}{z^4} \right| = \frac{1}{|z^4|} \leq 4.$

$\therefore \left| \int_C \frac{dz}{z^4} \right| \leq 4\sqrt{2}.$

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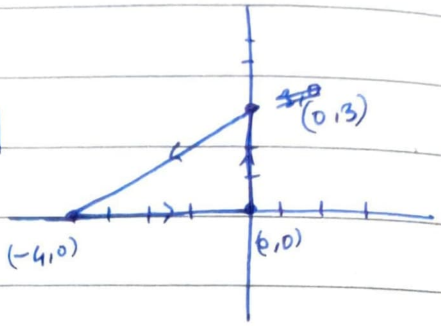
Q: Show that if c is the boundary of the triangle with vertices $(0, 3i$ & $-4)$ in the counter clockwise direction

$$\left| \int_C (e^z - \bar{z})\, dz \right| \leq 60 .$$



$|e^z - \bar{z}| \leq |e^z| + |\bar{z}|$

$= |e^x (\cos y + i \sin y)| + |\bar{z}|$

$= |e^x . 1| + |\bar{z}|$

$= |e^x| + \sqrt{x^2+y^2} .$

Now, $e^x \leq 1$ [since $x \leq 0$].

And $\sqrt{x^2+y^2} \leq 4$

$\therefore$ $e^x + \sqrt{x^2+y^2} \leq 5$

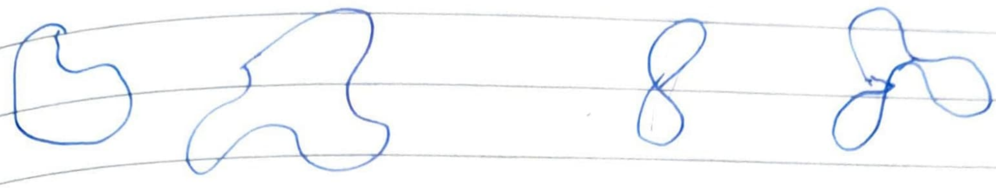
Now, $L = 3+4+5 = 12$.

$\therefore$ $ML = 5 \times 12 = 60$.

$\Rightarrow$ $\int_C |(e^x - \bar{z})\, dz| \leq 60$

Cauchy's Integral Theorem:

simple curves Not simple.



→ Simply connected path in a domain consists only of points belonging to the domain.

→ Multiply (doubly) connected domain are the paths with holes s.t. the path cannot be shrunk to a single pt.

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path c in D

$$\oint_C f(z)dz = 0$$

Eg:

$$\oint_C e^z dz = 0 \quad \because \quad e^z \text{ is analytic everywhere}$$

$||^{ly}$: $\oint_C \cos z = 0$

$$\oint_C (z-a)^n dz = 0 \quad n \neq -1$$

(converse not necessarily true).

- GREEN'S TH. :-

If

$$\oint P(x,y)\,dx + \oint Q(x,y)\,dy$$

& the partial ~~int~~ derivatives of P & Q are constant, then this integral is -

$$\iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \cdot dy.$$

Revisiting Cauchy's th.

→ $f(z)$ is analytic

$f(z)$ exists

→ $f'(z)$ is cont.

$$f'(z) = u_x + iv_x$$

& u_x, u_y, v_x, v_y → continuous.

$$\oint_C f(z)\,dz = \oint_C \left(u(x,y) + iv(x,y)\right)(dx + i\,dy)$$

$$= \cancel{\oint (u_x - v_y)\,dx}$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

$$= \iint_{C\ R}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy + i\int_R \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$= 0 + 0$$

$$= 0$$

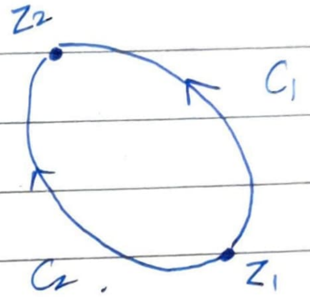
Independence of Path:-

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of the path

Pf: Take any 2 pts z_1, z_2 in D. Consider 2 different paths joining them

Now, $\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$.

$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.



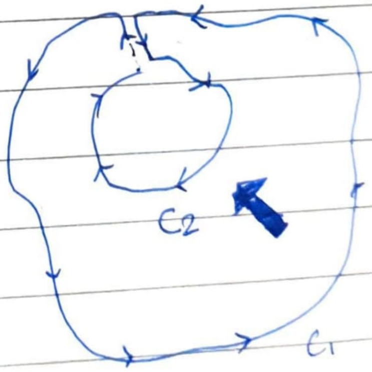
For a doubly connected domain.

Introduce a cut:-

Here:-

$\int_{C_1} f(z) + (-)\int_{C_2} f(z)\,dz = 0$.

$\therefore \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

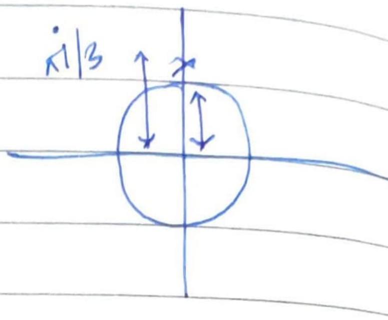


Here path C_1 is deformed to path C_2.

Similarly the same can be extended to multiply connected domains as well.

$\boxed{Q:}$ Integrate $f(z)$ in a counterclockwise dir. around the unit circle.
& $f(z) = \frac{1}{3z - \pi i}$ & indicate if C.T a

i) $f(z) = \frac{1}{3z - \pi i}$



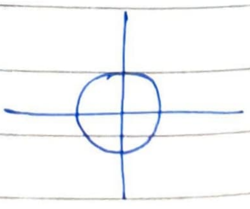
$\therefore \oint \frac{1}{3z - \pi i} \, dz$

$f(z)$ is analytic everywhere
$\because$, $f(z) = \infty$ when $z = \frac{\pi i}{3} > 1$.
Singularity lies outside.
$\therefore$, Yes C.T. applies
& $\oint \frac{1}{3z - \pi i} dz = 0$.

ii) $f(z) = \frac{1}{z}$

$\because$ singularity exists at $(0,0)$
C.T. does not apply. $\because$ f is not analytic at $z = 0$.

$\therefore$, $z = e^{i\theta}$.



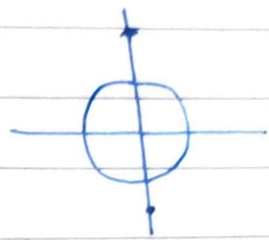
$dz = e^{i\theta} i \, d\theta$.

$\therefore$, $\int_0^{2\pi} \frac{1}{e^{-i\theta}} \cdot e^{i\theta} i \, d\theta$.

$= i \int_0^{2\pi} e^{2i\theta} d\theta = \frac{\cancel{i} \cdot [e^{2i\theta}]_0^{2\pi}}{2\cancel{i}} =$

$= \frac{1}{2}\left[(1-0)-(1-0)\right] = \underline{0}$

$f(z) = \sec\left(\frac{z}{2}\right)$



$f(z) = \frac{1}{0}$, when $z = \pi, 3\pi \dots$

The singularities ~~are~~ do not exist $\because \pi > 1$.

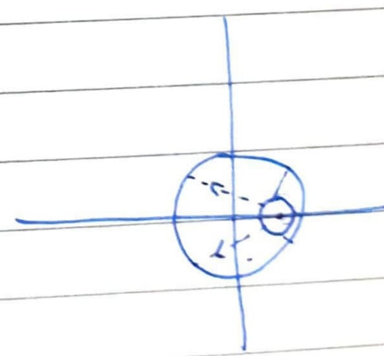
$\therefore$ C.T. applies

$\therefore \oint f(z)\,dz = \underline{0}$

iv) $f(z) = \frac{1}{4z-3}$.

$f(z) = \frac{1}{0}$ when $z = \frac{3}{4}$

Using path deformation



$$\oint (z-a)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1. \end{cases}$$

$\frac{1}{4}\oint \frac{1}{\left(z-\frac{3}{4}\right)} dz.$

$\left|z - \frac{3}{4}\right| = \rho.$

or $z = \frac{3}{4} + \rho e^{i\theta}$

$= \frac{1}{4} \times 2\pi i$

$= \underline{\frac{\pi i}{2}}$.

$\boxed{Q:}$ Can we conclude, using the result

$$\oint (z-a)^n \, dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = 1 \end{cases},$$

that $\displaystyle\int \frac{1}{z^2+4} \, dz$ is 0

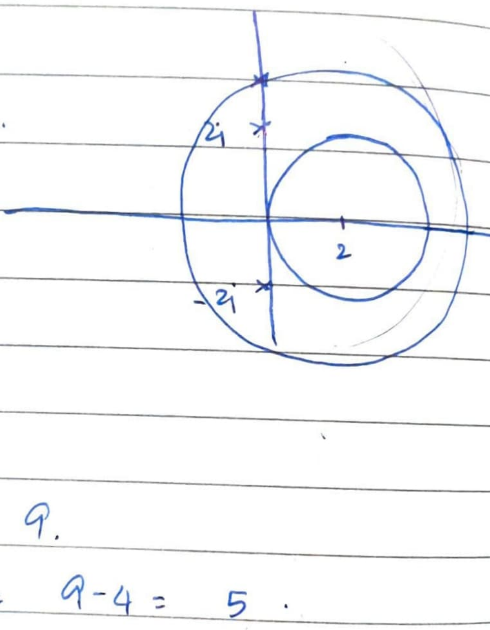
on $C_1: \; |z-2| = 2$

$C_2: \; |z-2| = 3$

Singularities : $z = \pm 2i$

For C_1: lie outside

$\therefore \displaystyle\int_{C_1} \frac{1}{z^2+4} \, dz = 0.$



For C_2 :-

$|z-2|$

$= |x-2+iy|$

ie: $(x-2)^2 + y^2 = 9.$

At $x = 0$, $y^2 = 9-4 = 5$.

ie: lies within the contour.

Put $z = 2+3e^{i\theta}$. $\quad dz = 3ie^{i\theta} d\theta$.

$\therefore \displaystyle\int_0^{2\pi} \frac{1}{4+9e^{2i\theta}+12e^{i\theta}+4} \, 3 \cdot ie^{i\theta} d\theta$

$e^{i\theta} = t$.
$e^{i\theta} d\theta =$
$\hookrightarrow \theta = 0,$
$\hookrightarrow \theta = 2\pi$

$= \displaystyle\int_1^1 \frac{3\,dt}{9t^2+12t+8}$

$= \underline{\underline{0}}$

Q: For what contours will Cauchy's th. apply for the following integrals

i) $\oint_c \frac{dz}{z} = 0$

Any contour that doesn't include (0,0).

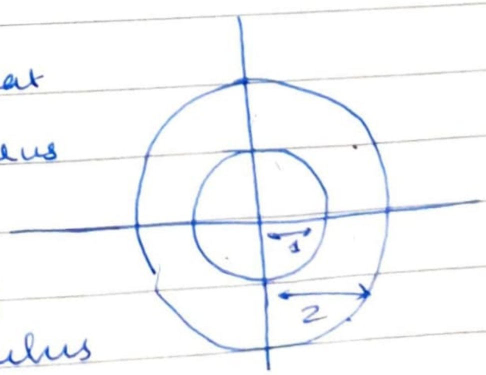
ii) $\oint \frac{\cos z}{z^6 - z^2} dz$. $z^6 - z^2 = z^2(z^4 - 1)$.
$z^2(z^2+1)(z+1)(z-1)$.

Any contour excluding origin & 4 roots of 1.

Q: If the integral of a fn. $f(z)$ over the unit circle
ie $\oint_{C_1} f(z) = 3$ $C \rightarrow$ unit circle

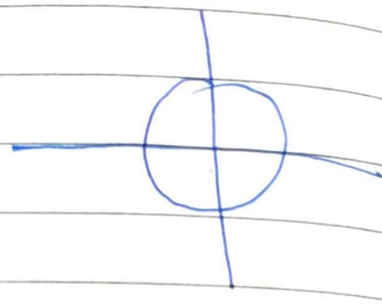
& over the circle $|z| = 2$
$\oint_{C_2} f(z) dz = 5$
can we conclude that $f(z)$ is analytic everywhere in the annulus $1 < |z| < 2$.

No, we cannot say that it is analytic in annulus
$\because$, the extra singularity for C_2 is contributed by a point in the annulus



Q: Integrate $f(z)$ counterclockwise around the unit circle & indicate if C.T applies

$$f(z) = \frac{1}{z^4 - 1.1}.$$



Q: Evaluate:

$\int_C (z^2+3z)\,dz$ along.

C_1: $|z| = 2$ (2,0) to (0,2).

C_2: St. line (2,0), (2, 2) & (2,2) to (0, 2).

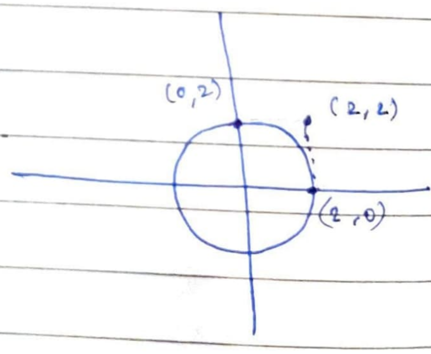
$|z| = 2$

No singularity

∴ Path is independent

$z = 2e^{i\theta}$ $dz = 2ie^{i\theta}\,d\theta$

∴ C_1:-



$$\int_0^{\pi/2} 2e^{i\theta} \cdot 2ie^{i\theta}\,d\theta$$

$$= 4i \int_0^{\pi/2} e^{2i\theta}\,d\theta$$

$$= \cancel{4i}^{\,2} \left[\frac{e^{2i\theta}}{\cancel{2i}}\right]_0^{\pi/2}$$

$$= 2\left[e^{2i\theta}\right]_0^{\pi/2}$$

$$= 2\left[(-1+0) - (1\right.$$

$= \int (2e^{2i\theta} + 3e^{i\theta} \cdot 2)\ 2ie^{i\theta}\, d\theta$

$= 4i \int e^{3i\theta}\, d\theta + 12i \int e^{2i\theta}\, d\theta$

$= \left[4i \frac{e^{3i\theta}}{3i} + \frac{\overset{6}{\cancel{12i}}\, e^{2i\theta}}{\cancel{2i}} \right]_0^{\pi/2}$

$= \frac{4}{3} \left[e^{3i\theta} \right]_0^{\pi/2} + 6 \left[e^{2i\theta} \right]_0^{\pi/2}$

$= \frac{4}{3} \left[0 + (-i) - (1+0) \right] + 6 \left[(-1) + 0 - (1) \right]$

$-\frac{4}{3} i \quad -\frac{4}{3} \quad - 12$

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$\boxed{Q:}$ $\oint_C \frac{2z^3+z^2+4}{z^4+4z^2} dz.$ $\qquad C: |z-2| = 4.$

$z = 2+4e^{i\theta}.$ $\quad dz = 4ie^{i\theta}.$

$= \oint \frac{2z^3+z^2+4}{z^2(z^2+4)} dz. = \oint \frac{(2z^3+z^2+4)}{z^2(z+2i)(z-2i)} dz$

$2z^3+z^2+4$

$= \frac{Az+B}{z^2} + \frac{C}{z+2i} + \frac{D}{z-2i}$

$=$

$2z^3+z^2+4 = (Az+B)(z+2i)(z-2i) + C(z^2)(z-2i) + D(z^2)(z+2i)$

$A=0, \quad B=1, \quad C=1, \quad D=1.$

$2 = A+C+D.$

$1 = -2Ai + B - 2Ci + 2Di$

$4 = 4B - 2Ci + 2Di.$

$1 = -2Ai + B + 4 - 4B$

$1 = -2Ai + 4 - 3B$

$3 = 3B + 2Ai.$

$\oint \frac{2z^3+z^2+4}{z^4+4z^2} = \int \frac{dz}{z^2} + \int \frac{dz}{z+2i} + \int \frac{dz}{z-2i}$

$(x-2)^2 + y^2 = 16$

$y^2 = 12.$

$y = 2\sqrt{6}.$

$= \int \frac{dz}{(z-0)^2} + \int \frac{dz}{z+2i} + \int dz$

$\Downarrow$

$\int_0^{2\pi} \frac{4ie^{i\theta} d\theta}{(2+4e^{i\theta})^2}.$

$\to 0 \quad 4$

$2\pi \Rightarrow 4.$

$\Rightarrow 4e^{i\theta} = t \qquad 4e^{i\theta} d\theta = \frac{dt}{i}$