31-12-18

- **Scalar:** A scalar is a single number.

- **Vector:** An array of numbers

  Notation: $\mathbb{R}^n \rightarrow$ all entries are real no:s.

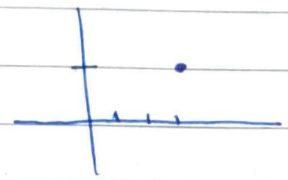
  $\mathbb{C}^n$ if all entries are comp. nos.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

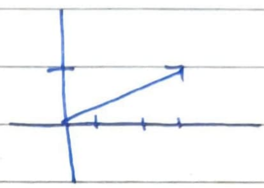
Vectors can be rep. as:-

$$v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

coloumn.

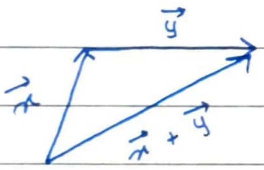


Point.

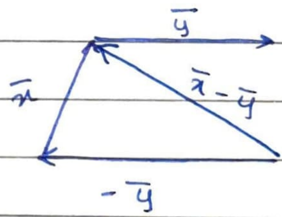


Arrow.

Addition:




Subtraction:




- <u>Field</u>: $(\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C})$

Let $\mathbb{F}$ denote the set of scalars which has the following properties:

(1) Addition is commutative

$$x + y = y + x \quad \forall \; x, y \in \mathbb{F}$$

(2) Addition is associative.

$$x + (y + z) = (x + y) + z \quad \forall \; x, y, z \in \mathbb{F}$$

(3) There exists unique el. $0$ in $\mathbb{F}$ such that

$$x + 0 = x \quad \forall \; x \in \mathbb{F}$$

(4). For every $x \in \mathbb{F}$, there exists a unique el $(-x)$ in $\mathbb{F}$ such that $x + (-x) = 0$.

(5). Multiplication is commutative

$$xy = yx \quad \forall \; x, y \in \mathbb{F}$$

(6). Multiplication is associative.

$$x(yz) = (xy)z \quad x, y, z \in \mathbb{F}$$

(7) There exists a unique el. 1 in $\mathbb{F}$ such that $1 \cdot x = x \quad \forall \; x \in \mathbb{F}$

(8). For every $x \in \mathbb{F}$ there exists a unique el. $\left(\frac{1}{x}\right)$ in $\mathbb{F}$ s.t. $x \cdot \frac{1}{x} = 1 \quad x \neq 0$.

- Vector Space :

A vector space consists of the following

a) A field $\mathbb{F}$ of scalars

b) A set $V$ of vectors

Vector space should satisfy following properties

$\boxed{a)}$ To every pair of vectors $\bar{x}, \bar{y}$ in $V$ there corresponds a vector $\bar{x} + \bar{y}$ (called sum of vector) in such a way that $(\bar{x} + \bar{y} \in V)$ closure

i) Addition is commutative :

$$\bar{x} + \bar{y} = \bar{y} + \bar{x} \quad ; \quad \bar{x}, \bar{y} \in V.$$

ii) Addition is associative :

$$\bar{x} + (\bar{y} + \bar{z}) = (\bar{x} + \bar{y}) + \bar{z})$$

iii) There exists a unique el. $\bar{0} \in V$ such that $\bar{0} + \bar{x} = \bar{x}$ (origin)

$$\alpha x + \beta x = \begin{bmatrix} (\alpha+\beta) x_1 \\ (\alpha+\beta) x_2 \\ \vdots \\ (\alpha+\beta) x_n \end{bmatrix} \rightarrow$$

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(iv) For every vector $\bar{x} \in V$, there exists a unique el. $(-\bar{x})$ in $V$ such that

$$\bar{x} + (-\bar{x}) = \bar{0}$$

$\boxed{b)}$ For every pair $\bar{x} \in V$ $\alpha \in \mathbb{F}$ there corresponds a vector $\alpha \bar{x}$ (called scalar multiplication) such that :-

i) Multiplication is associative.

$$\alpha(\beta \bar{x}) = (\alpha \beta) \bar{x}.$$

ii) $1 \bar{x} = \bar{x}$. [$1 \in \mathbb{F}$ which satisfy this property].

iii) $\alpha(\bar{x} + \bar{y}) = \alpha \bar{x} + \alpha \bar{y}$ $\forall$ $\bar{x}, \bar{y} \in V$ & $\alpha \in \mathbb{F}$.

iv). $(\alpha + \beta) \bar{x} = \alpha \bar{x} + \beta \bar{x}$ $\forall$ $\alpha, \beta \in \mathbb{F}$ & $\bar{x} \in V$.

★ $$\alpha x + \beta x = \begin{bmatrix} (\alpha+\beta) x_1 \\ \vdots \\ (\alpha+\beta) x_n \end{bmatrix}$$

For this to be well defined $\alpha + \beta = \beta + \alpha$
this implies that $\alpha, \beta \in \mathbb{F}$. [where $\mathbb{F}$ is the field].

Eg 1: $\mathbb{R}^n$ : vector space. (Real nos:).
Eg 2: $\mathbb{C}^n$ : (complex no:s).
Eg 3: $R^{m \times n}$.

Eg4: Space of polynomials.

$$P = \{ f(x) = c_0 + c_1 x + \dots \; c_n x^n \; ; \; c_0, c_1, c_2 \dots c_n \in \mathbb{F} \}$$

$$f(x) = c_0^1 + c_1^1 x + \dots + c_n^1 x^n$$
$$g(x) = c_0^2 + c_1^2 x + \dots + c_n^2 x^n$$

$$f(x) + g(x) = (c_0^1 + c_0^2) + (c_1^1 + c_1^2) x \dots \dots$$

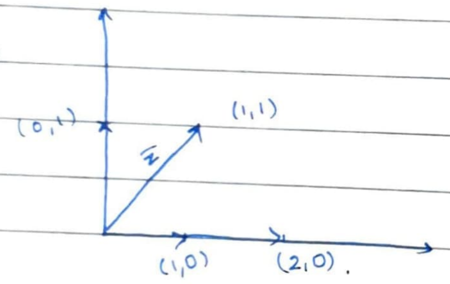
Hoffman & Kunze.
Halmos
Gilbert Strang.

Eg 5: The space of functions from a set to a field
$\mathcal{H} = \{ f(x) \mid f : S \to \mathbb{F} \}$.
where $S \neq \phi$.

→ $R^n$ & $R^{m \times n}$ are special cases of (5) as:-
$\mathbb{R}^n$ : $\bar{x}\ (x_1 \ldots\ldots x_n) \in \mathbb{R}^n$ can be thought of as $x_i : i \to \mathbb{R}$ $i \in \{1, 2 \ldots n\}$.

- Linear combination of vectors:
$\bar{z}$ is a linear combination of vectors $\bar{x}_1, \bar{x}_2, \ldots, x_n$ if there exists scalars $\alpha_1, \alpha_2, \ldots \alpha_n$ such that
$$\bar{z} = \Sigma \alpha_i \bar{x}_i$$

Eg:



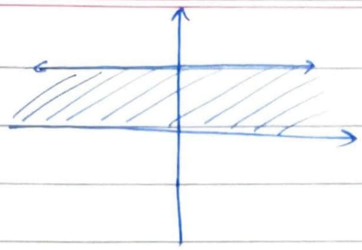

The vector $\bar{z}$ can be uniquely represented with the 2 vectors (1,0) & (0,1).
Further the linear combination of the vectors (1,0) & (2,0) can be used to represent any vector on the $x$-axis.

- <u>Subspace</u>:

Subspaces are subsets of vector spaces provided the subset is also a vector space.

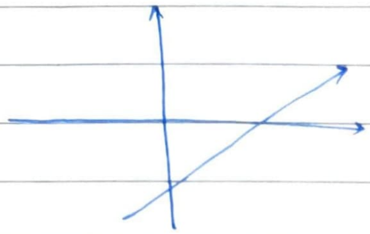
→ Let V be a vector space over field F. A non-empty subset W of V is a vector space over F with vector addition & scalar mult in V then we say W is subspace of V.

Eg:



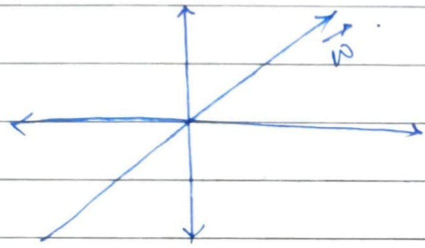
Here $R^2$ is a vector space. But the shaded area isn't a subspace since inverses of vectors don't exist. Also some vector scale out of the subspace.

Eg:



Not a subspace since $\vec{0}$ is not a part of the subset.

Eg:



Yes, a vector space $\because$ satisfies all properties.

→ Theorem (subspaces):
A subset $W$ of $V$ is subspace of $V$ iff $\forall\ \bar{x}, \bar{y} \in W$ & $c \in F$, the vector $c\bar{x} + \bar{y} \in W$.

Proof: Let A: $W$ is a subspace.
B: $c\bar{x} + \bar{y} \in W$. ; $\forall\ \bar{x}, \bar{y} \in W$

- $A \Rightarrow B$ true $\because$ $c\bar{x}$ is a vector ~~space~~ itself. $\therefore$ $c\bar{x} \in W$
Also, if $c\bar{x} = \bar{z}$ $\therefore$ $\bar{z} + \bar{y}$ also $\in W$.

- $B \Rightarrow A$

Let $c = 1$, $\bar{x} + \bar{y} \in W$ $\forall\ \bar{x}, \bar{y} \in V$.
Let $\bar{y} = \bar{x}$ & $c = -1$.
$\therefore$ $-\bar{x} + \bar{x} = \bar{0} \in W$. — (1).

- Let $\bar{y} = \bar{0}$ (From (1))
$\therefore$ $c\bar{x} \in W$.

- $\cancel{\alpha\vec{x}}$ $\vec{y} = \vec{0}$, $c = -1$.

$\Rightarrow$ W is a vectorspace.

$\boxed{Q:}$ $V = \mathbb{R}^n$

$W = \{(0, x_2, x_3 \dots x_n)\}$.

$= \{\bar{x} \in \mathbb{R} \mid x_1 = 0\}$.

Prove W is a subspace.

Using theorem :-

$\bar{x} \in W$ $(0, x_2, x_3 \dots x_n)$

$\bar{y} \in W$ $(0, y_2, y_3 \dots y_n)$.

$\therefore$ $c\bar{x} + \bar{y} = (0, cx_2 + y_2, \dots cx_n + y_n)$

$\therefore$, Proved that W is a vector space.

$\boxed{Q:}$ $W = \{\bar{x} \in \mathbb{R}^n \mid x_1 = 1 + x_2\}$.

$\bar{x} = (x_1, x_2 \dots x_n)$

$= (1 + x_2, x_2 \dots x_n)$

$\bar{y} = (y_1, y_2 \dots y_n)$

$= (1 + y_2, y_2, y_3 \dots y_n)$

$\therefore$ $c\bar{x} + \bar{y} = (c + cx_2 + 1 + y_2, cx_2 + y_2 \dots)$

But $\vec{0}$ will not belong $\because$

$c + cx_2 + 1 + y_2 \neq 0$

$\therefore$, NO, W is not a subspace. FAIL

$\boxed{Q:}$ $P = \{f(x) = c_0 + c_1 x_1 + \dots c_n x^n\}$ $c_0, c_1 \dots c_n \in F$

$H = \{f(x) \mid f: F \to F\}$.

Given P is a subspace of H.

Let $P_1, P_2 \in P$.

then $cP_1 + P_2 \in P$ $\therefore$, P is a ~~set~~ subspace

$\because$, $cP_1 \to \text{Poly}^n$, $P_2 \to \text{Poly}^n$.

Now, H is the set of all possible functions $F \to F$.

**Q:** Is $S = \{ A_{ji} = A_{ij} \}$ Symmetric matrices.
is $S$ a subspace of $R^{n \times n}$.

Let $A, B$ be two matrices $\in R^{n \times n}$.

$\therefore$ $(CA+B)_{ij}$

$CA_{ij} + B_{ij}$

$CA_{ji} + B_{ji}$ $\because$ symm.

$(CA+B)_{ji}$

$\therefore$ $CA+B \in S$.

$\therefore$ $S$ is a subspace of $R^{n \times n}$.

**Q:** $F = C$

$V = \{ A \in C^{n \times n} \}$.

(V is a vector space yes).

$S = \{ A \in V \mid A_{ij} = \overline{A_{ji}} \}$. Hermetian

Here $A_{ii} = \overline{A_{ii}}$

$\therefore$ Diagonal el. must be real nos:-

$\therefore$ for any matrix. (identity).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $c = i$, $iA = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$ this $\notin S$.

$\therefore$ NO, S isn't a subspace of V.

Eg: $W_1 = \{ \bar{x} \in R^n \mid A\bar{x} = \bar{0} \}$ $A \in R^{m \times n}$.

consider $\bar{x_1}$ & $\bar{x_2} \in W$.

$\therefore$ $A\bar{x_1} = \bar{0}$ $A\bar{x_2} = \bar{0}$

And $CA\bar{x_1} + A\bar{x_2}$

$= C.\bar{0} + \bar{0} = \bar{0}$

$\therefore$ $C\bar{x_1} + \bar{x_2} \in W$

$\therefore$ $W_1$ is a subspace.

null space of A.

$CX_1A + AX_2$
$C\bar{b} + \bar{b} = (C+1)\bar{b}$ $\quad A(CX_1 + X_2) = (C+1)\bar{b}$

eg: $W_2 = \{ \bar{x} \in \mathbb{R}^n \mid A\bar{x} = \bar{b} \}$
This is not a subspace.

- **INTERSECTION OF SUBSPACES :-**

Intersection of any no: of subspaces is a subspace.

Proof:

Let $\bar{x}, \bar{y} \in S_1 \cap S_2$.

$\therefore$

$\left.\begin{array}{l} \bar{x} \in S_1 \\ \bar{y} \in S_1 \end{array}\right] \rightarrow S_1$ is a subspace
$\therefore\ c\bar{x} + \bar{y} \in S_1$.

$\left.\begin{array}{l} \bar{x} \in S_2 \\ \bar{y} \in S_2 \end{array}\right] \rightarrow S_2$ is a subspace.
$c\bar{x} + \bar{y} \in S_2$.

$\therefore\ c\bar{x} + \bar{y} \in S_1 \cap S_2$.

eg: $S_1 = \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid y, z \in \mathbb{R}$

$S_2 = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \in \mathbb{R}^3 \mid y \in \mathbb{R}$.

$S_1 \cap S_2 \Rightarrow$ Y-axis ; which is a subspace.

- **Sum of Subsets :-**

If $S_1 \ldots S_k$ are subsets of a vector space $V$, the set of all sums.

★ $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \ldots + \bar{x}_k$ of vectors $(\bar{x}_i \in S_i)$ is called sum of subsets & is rep by.
$S_1 + S_2 + \ldots S_n$.

$\Rightarrow S_1 + S_2 + \ldots S_k = \{ \bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_k \mid \bar{x}_i \in S_i,\ i = 1, 2, 3 \ldots \}$

- UNION OF SUBSPACES:

Let $W_1 \dots W_k$ be subspaces of vector space $V$. then;

$W_1 + W_2 + \dots + W_k$ is also a subspace.

Let $\bar{X} = \bar{X_1} + \bar{X_2} + \dots + \bar{X_k}$
$\bar{Y} = \bar{Y_1} + \bar{Y_2} + \dots + \bar{Y_k}$ ] $\bar{X}, \bar{Y} \in W_1 + W_2 + \dots + W_k$.

where $\bar{x_i} \in W_i$, $\bar{Y_i} \in W_i$.

$\therefore$ Now $C\bar{X} + \bar{Y}$

$= (C\bar{X_1} + \bar{Y_1}) + (C\bar{X_2} + Y_2) + (C\bar{X_3} + \bar{Y_3}) + \dots + (C\bar{X_k} + \bar{Y_k})$

$= \bar{Z_1} + \bar{Z_2} + \bar{Z_3} + \dots + \bar{Z_k}$.

$\bar{Z_i} = C\bar{X_i} + \bar{Y_i}$ But $\bar{x_i}, \bar{y_i} \in W_i$
$\Rightarrow C\bar{x_i} + \bar{y_i} \in W_i \Rightarrow \bar{Z_i} \in \bar{W_i}$

$\therefore$

$\bar{Z_1} + \bar{Z_2} + \bar{Z_3} + \dots \bar{Z_k}$. $\in W$.

Proved.

→ SPANNING SET :

Eg: consider :-

$\bar{X_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. $S = \{\bar{X_1}, \bar{X_2}, \bar{X_3}\}$.

Q: Does $S$ form a spanning set of $\mathbb{R}^5$?

$\bar{X} = C_1\bar{X_1} + C_2\bar{X_2} + C_3\bar{X_3}$.

$= \begin{bmatrix} C_1 \\ 2C_1 \\ C_2 \\ 3C_1 + 4C_2 \\ C_3 \end{bmatrix}$

Eg: • For the x & y axes, the spanning set is the entire x-y plane.
• For the x-axis, spanning set is x-axis.

**Def.** ~~If a vector space V consists of~~ Vectors $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ form a spanning set of vector space V if for any vectors $\bar{x} \in V$

$\exists \; c_1, c_2, \ldots, c_n \in \mathbb{F}$ such that

$\bar{x}$ can be expressed as :-

$$\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_n\bar{x}_n$$

Eg: $\bar{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\bar{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\bar{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\bar{w}_4 = \begin{bmatrix} 1 \\ 0 \\ 10 \end{bmatrix}$ $\bar{w}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\bar{w}_6 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

If $\bar{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ $= 3\bar{w}_1 + 4\bar{w}_2 + 5\bar{w}_3$
$= 3\bar{w}_6 + \bar{w}_5 + 4\bar{w}_3$ NO UNIQUENESS!

Drawback: No compact representation of the spanning set

• **Linearly Independent Set :-**

A set of vectors ~~xs~~ $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ are called linearly independent vectors iff

$c_1 f(x) + c_2 f(x) + c_3 f_n(x)$

$$c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_n\bar{x}_n = \bar{0} \Rightarrow$$
$$c_1 = c_2 = \ldots = c_n = \bar{0}$$

⚠ Any set containing the $\bar{0}$ will not be linearly independent.

## • BASIS OF A VECTOR SPACE:

Let V be a vector space. A **Basis** for a vector space V is a linearly independent set of vectors in V which spans the space V.

**Q:** $W_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad W_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad W_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Verify $\bar{w}_1, \bar{w}_2, \bar{w}_3$ are linearly independent.

$c_1 W_1 + c_2 W_2 + c_3 W_3 = 0$

$\left.\begin{array}{l} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = c_3 = 0.$

★ Every vector $\bar{x} \in V$ will have a unique representation in terms of the basis vectors.

Proof: Assume a vector $\bar{x}$ not having a unique rep.

$\therefore \quad \bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \dots + c_n\bar{x}_n$

and: $\quad \bar{x} = d_1\bar{x}_1 + d_2\bar{x}_2 + \dots + d_n\bar{x}_n$

(−)

$\bar{0} = (c_1 - d_1)\bar{x}_1 + (c_2 - d_2)\bar{x}_2 + \dots$

But since ~~$\bar{x}_i$~~ $c_1 \dots c_n \neq d_1 \dots d_n$.

$\therefore$ $c_i - d_i$ is $\neq 0$ for at least one $c_i - d_i$

$\therefore$ Contradiction.

## • Properties of Basis:-

→ Minimal spanning set

→ Maximal linearly independent set.

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# TUTORIAL -1

ex:

(1) Given : F is the field
T.P $F^n$ : forms a vector space.
Let $\bar{x}$ , $\bar{y}$ $\in F^n$
then,
$$\bar{x}+\bar{y} = \bar{y}+\bar{x}$$
$$(\bar{x}+\bar{y})+\bar{z} = (\bar{x}+)+(\bar{y}+\bar{z})$$
$$\bar{x}+\bar{0} = \bar{x} \quad ; \quad \bar{x}+(-\bar{x}) = \bar{0}$$

∴, All prop. satisfied
⇒ $F^n$ is a vector space

(2) ∵ V is a vector space
Associativity of vec. follows
$$\Rightarrow (\alpha_1+\alpha_2)+(\alpha_3+\alpha_4) = [\alpha_2+(\alpha_3+\alpha_1)]+\alpha_4.$$

(3)
$$c_1\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}c_1+c_3\\c_2+c_3\\c_2+c_3-c_1\end{bmatrix}$$

(4) Q: Even/odd func. Question.

V: $\{f \mid f:\mathbb{R}\to\mathbb{R}\}$. U: $\{f \mid f$ is even$\}$ v: $\{f \mid f$ is odd$\}$

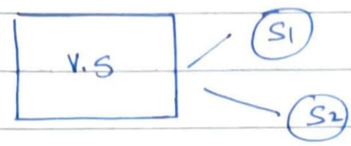
T.P. $V = U \oplus V$.

Case :-

1. f is even Take f from u , 0 from v.
2. f is odd Take f from v , 0 from 0
3. f is 0 Take 0
4. f is neither even nor odd.

$$\Rightarrow f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{Take from } u} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{Take from } v}$$

- **Direct Sum:**




if there are no common vectors except $\bar{0}$

$$\alpha = \alpha_{w_1} \oplus \alpha_{w_2} \rightarrow \text{Direct sum.}$$

Here, $W_1$ & $W_2$ are 2 subspaces s.t.

$$W_1 + W_2 = V$$
$$W_1 \cap W_2 = \{\phi\}.$$

$\boxed{Q:}$ $\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is linearly dependent iff $\bar{\alpha_r}$ $2 \leq r \leq k$ is linear combination of its preceding vectors.

$\rightarrow$ $\bar{\alpha_r} = c_1\bar{\alpha_1} + c_2\bar{\alpha_2} + \ldots + c_{r-1}\bar{\alpha_{r-1}}$

$\therefore$ $\bar{0} = c_1\bar{\alpha_1} + c_2\bar{\alpha_2} + \ldots + c_{r-1}\bar{\alpha_{r-1}} - \bar{\alpha_r}$

$\because$ Sum $= \bar{0}$

But not all co-efficients $= 0$

$\therefore$ Linearly dependent.

$\rightarrow$ If ~~it~~ linearly dependent :-

$\Rightarrow$ $\bar{0} = c_1\bar{\alpha_1} + c_2\bar{\alpha_2} + \ldots + c_k\bar{\alpha_k}$ ; at least

One of the co-efficients is $\neq 0$

Removing all co-efficients $= 0$ from right we can obtain;

$$\alpha_r = \frac{c_1\bar{\alpha_1}}{c_r} + \frac{c_2}{c_r}\bar{\alpha_2} + \frac{c_3}{c_r}\bar{\alpha_3} + \ldots \frac{c_{r-1}}{c_r}\bar{\alpha_r}$$

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# DIMENSION OF A VECTOR SPACE :-

If the size of the basis is finite then the vector space is finite dimension.

Eg: Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Let $\bar{a}_1 \dots \bar{a}_n \in \mathbb{R}^n$ be the columns of A. Then $\bar{a}_1 \dots \bar{a}_n$ form a basis of $\mathbb{R}^n$

Step 1: To show that $\bar{a}_1 \dots \bar{a}_n$ are LI,
$c_1\bar{a}_1 + c_2\bar{a}_2 + \dots c_n\bar{a}_n = \bar{0} \Rightarrow c_1 = c_2 = \dots = c_n = 0$

Essentially this means:-

$$A\bar{C} = \bar{0}$$

$$\because A = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \bar{a}_1 c_1 + c_2\bar{a}_2 \dots + c_n\bar{a}_n = 0.$$

$$\Rightarrow A\bar{C} = 0$$

$$\bar{C} = A^{-1}\bar{0} = \bar{0}$$

$$\Rightarrow c_1 = c_2 = \dots = c_n = 0.$$

Step 2: $\bar{a}_1, \bar{a}_2 \dots \bar{a}_n$ should span $\mathbb{R}^n$.
Let $Y \in \mathbb{R}^n$
then Y should be rep as a linear comb of $\bar{a}_1 \dots \bar{a}_n$ which means there exist $c_1, c_2 \dots c_n \in \mathbb{R}$ such that

$$Y = c_1\bar{a}_1 + c_2\bar{a}_2 + \dots + c_n\bar{a}_n$$

$$\therefore \bar{Y} = A\bar{C}$$

$$\bar{C} = A^{-1}\bar{Y} \rightarrow \text{unique combination}$$

Eg: of Infinite basis :
Let $P$ be the space of $poly^n$ functions over $\mathbb{F}$

$$f(x) = c_0 + c_1 x + \dots + c_n x^n \qquad x \in \mathbb{F}$$

Let $f_k(x) = x^k$ ; $k = 0, 1 \dots$

$\therefore$ The infinite set $\{f_0, f_1 \dots f_n\}$ is a basis for $\mathbb{P}$

$$\mathbb{P} = \{ f \mid f = c_0 + c_1 x + \dots + c_n x^n \text{ for some } n \}.$$

$f = c_0 f_0 + c_1 f_1 + \dots + c_n f_n \rightarrow$ spans the set $P$.

To show, the set $\{f_0, f_1, f_2 \dots f_k\}$ is linearly independent we show that every finite subset of it is linearly indep. without loss of generality we show

$\{f_0, f_1, f_2 \dots f_n\}$ is linearly indep $\forall n$.

$$c_0 f_0(x) + c_1 f_1(x) + \dots + c_k f_k(x) = 0$$
$$\Rightarrow c_0 = c_1 = c_2 = \dots = c_k = 0$$

ie: $c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n = 0$

$\Rightarrow c_0 = c_1 = c_2 = \dots = c_n = 0 \quad \forall x \in \mathbb{F}$

This means that if

$c_0, c_1, \dots c_n$ are not all $= 0$

then if $c_0 + c_1 x + c_2 x^2 + \dots = 0 \quad \forall x$

$\Rightarrow x$ will be the roots of the $poly^n$.

That will be countably infinite no: of complex/real roots.

$\therefore$, $c_0 = c_1 = c_2 = \dots c_n = 0 \rightarrow$ necessary for

$c_0 f_0 \quad c_0 + c_1 x + \dots + c_n x^n = 0$

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- **THEOREM:** every spanning set in a vector space can be reduced to a basis of the V.S.

Proof: Let $\{\bar{w}_1, \bar{w}_2, \dots \bar{w}_m\}$ is a spanning set for V

$B = \{\bar{w}_1\}$; $i = 2 \dots m$

if $\bar{w}_i \notin$ span $(B)$

$B = B \cup \{\bar{w}_i\}$ end $\ddot{\frown}$

- **THEOREM:** every linearly independent set of vectors in finite dimensional VS can be extendended to a basis of V.S

Let $\{\bar{\alpha}_1, \bar{\alpha}_2, \dots \bar{\alpha}_m\}$ be linearly indep. set then find a vector $\beta$ in V. If $\beta \notin$ span $(\bar{\alpha}_1, \dots \bar{\alpha}_m)$ then include $\beta$ in the set.

- **THEOREM:** ★ In a finite dim vector space, the length of every linearly independent set of vectors is <= the length of every spanning set of vectors.

Proof: Let $\{\bar{u}_1, \bar{u}_2 \dots \bar{u}_m\}$ is a lin. indep set in V & $\{w_1, \dots w_n\}$ is a spanning set of V we have to show that $m \le n$

Step 1: The set $\{\bar{w}_1, \bar{w}_2 \dots \bar{w}_n\}$ spans V then $\{\bar{u}_1, \bar{w}_1, \dots \bar{w}_n\}$ also spans V. But it will become linearly dependent. Thus we can remove one of the $\bar{w}_i$'s from the set $\{\bar{u}_1, \bar{w}_1, \dots w_n\}$.

and the remaining set still spans V. Let $W_i$ is removed.

$$B = \{ u_1, w_1 \dots w_{i-1}, W_{i+1}, \dots w_n \}.$$

Step j: we start with

$$B = \{ u_1, \cancel{w_1}, \dots u_{j-1}, w_1, \dots w_{i-1}, W_{i+1} \dots w_{i_{2}-1}, w_{i_2+1} \dots \dots w_n \}$$

If this process goes till step m then the resulting spanning set contains $\bar{u_1}, \bar{u_2} \dots \bar{u}_m$ & there are still w's left in the set

clearly $m \leq n$.

If at some step there is no $\bar{w}$ to remove then we reach a contradiction that $\{ w_1, \dots, w_n \}$ is a spanning set

- THEOREM:

Let V be a finite dimension vector space ~~of~~ over field $\mathbb{F}$. Let $\{ \bar{v}_1, \bar{v}_2, \dots \bar{v}_n \}$ be the ~~vector~~ spanning set for V. Let $v_i$ be such that it can be expressed as a linear combination of some $\bar{v} \in V$ & ~~for~~ and all other $\bar{v}_j = j \neq i$ then

$\{ \bar{v}, \bar{v}_1 \dots v_{i-1}, v_{i+1}, \dots \bar{v}_n \}$ is the spanning set for V.

Proof: Let $\bar{x}$ be any vector in V

then

$$\bar{x} = c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_i\bar{v}_i + \dots c_n v_n \quad \text{— (1)}.$$

Also,

$$v_i = d_1\bar{v}_1 + d_2 v + \dots d_{i-1} v_{i-1} + d_{i+1} V_{i+1} + \dots d \bar{v} \quad \text{— (2)}$$

② in ①:

$$\bar{x} = \sum_{j \neq i} c_j \bar{v}_j + c_i \bar{v}_i$$

$$= \sum_{j \neq i} c_j v_j + c_i \left[ \sum_{j \neq i} d_j \bar{v}_j + d \bar{v} \right]$$

$$= \sum_{j \neq i} (c_j + c_i d_j) v_j + c_i d \bar{v}$$

Thus $\bar{x}$ is expressed as a linear combination of all $v_j$ $j \neq i$ & $\bar{v}$. Since it can be done for any $\bar{x} \in V$ thus

$$V = \text{span}(v, v_1, v_2 \ldots v_{i-1}, v_{i+1}, \ldots v_n)$$

→ Theorem ★ Proof exp:

Let $\{\bar{v}_1, \bar{v}_2 \ldots \bar{v}_n\}$ be the spanning set
$\{\bar{w}_1, \bar{w}_2 \ldots w_m\}$ be the L.I set in V

To show $m \leq n$.

(By contradiction)

Assume $m > n$.

Step 1: because $\{\bar{v}_1 \ldots \bar{v}_n\}$ is a spanning set

$$\bar{w}_1 = c_1 \bar{v}_1 + c_2 \bar{v}_2 + \ldots + c_n \bar{v}_n$$

at least one of $c_1 \ldots c_n$ is non-zero

∵ $\bar{w}_1 \in$ L.I set. We can reindex vectors $\bar{v}_1 \ldots \bar{v}_n$ such that $c_1 \neq 0$

Thus

$$\bar{v}_1 = \frac{\bar{w}_1}{c_1} - \frac{c_2}{c_1}\bar{v}_2 \ldots - \frac{c_n}{c_1} v_n.$$

$$V = \text{span}(w_1, v_2 \ldots v_n).$$

Step 2: Write

$\bar{w_2} = c_1\bar{w_1} + c_2\bar{v_2} + \dots + c_n\bar{v_n}$

$\therefore$ at least one of $c_2 - c_n$ must be non-zero since otherwise $w_2 = c_1\bar{w_1} \dots$ ~~cont~~ But $w_2 \in$ L.I set.

Reindexing vectors $\bar{v_2} \dots \bar{v_n}$ $c_2 \neq 0$ then.

$$V_2 = \frac{\bar{w_2}}{c_2} - \frac{c_1}{c_2}\bar{w_1} - \dots \frac{c_n}{c_2} V_n.$$

Swapping $v_2$ with $w_2$.

$$V = \text{span}(\bar{w_1}, \bar{w_2}, v_3 \dots V_n)$$

because $\bar{w_1} \dots w_n$ spanning set

$$w_{n+1} = d_1\bar{w_1} + d_2\bar{w_2} + \dots d_n\bar{w_n}$$

but $w \in$ L.I set.

$\therefore$ contradiction FAIL

- THEOREM:

Let $B$ & $B'$ are ~~for~~ 2 different bases for a finite dim V.S. Let size $(B) = m$ & size $(B') = n$ then $m = n$.

Step 1: Assume $B$ is spanning set & $B'$ is a L.I set.
then $n \leq m$. (th. ★)

Step 2: Assume $B'$ is spanning set & $B$ is a L.I set
then $m \leq n$ (th. ★).

$$\Rightarrow \boxed{m = n}$$

* If V is an n-dim. V.S. any subspace W of V is also finite dim. with dim W at most n

* Let $A \in \mathbb{R}^{n \times n}$ let row vectors of A are lin independent. Then A is invertible.

Pf: Let $\bar{v_1}, \dots \bar{v_n}$ be the row vectors of A.
Because $\bar{v_1} \dots \bar{v_n}$ are lin ind. any vector $w \in \mathbb{R}^{n \times n}$ can be expressed as lin. combination of them

$\bar{v_1} \dots \bar{v_n}$ forms bases of $\mathbb{R}^n$

Let $\bar{e_i}$ be n-dim vector where $i^{th}$ el is 1 rest are 0.

$$\bar{e_i} = \sum_{j=1}^{n} B_{ij} \bar{v_j} = B_{i1} \bar{v_1} + B_{i2} \bar{v_2} + \dots$$

$$I_{n \times n} = \begin{bmatrix} \bar{e_1} \\ \bar{e_2} \\ \vdots \\ \bar{e_n} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} B_{1j} \bar{v_j} \\ \vdots \\ \sum_{j=1}^{n} B_{nj} \bar{v_j} \end{bmatrix} = BA$$

$\therefore$ $I = BA$

$\therefore$ $B = A^{-1}$ .... $A^{-1}$ exists.

* **Theorem:**

If $U_1$ & $U_2$ are ~~as~~ subspaces of finite dimensional V.S then

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$$

Pf: Let $\{u_1 \dots u_p\}$ be the basis of $U_1 \cap U_2$.
$\therefore$ Since $U_1 \cap U_2 \subset U_1$
then $\{u_1 \dots u_p\}$ is a L.I set in $U_1$. We can extend it to a basis

Let $\{u_1 \dots u_p, v_1 \dots v_q\}$ basis for $U_1$
Sim. $\{u_1, \dots u_p, w_1 \dots w_r\}$ forms basis for $U_2$.

To show
$\dim(U_1+U_2) = p+q+p+r-p$.
$= p+q+r$

$= \{u_1 \dots u_p, v_1 \dots v_q, w_1 \dots w_r\}$

Span of $\{u_1 \dots u_p, v_1 \dots v_q, w_1 \dots w_r\}$ contains $U_1$ & $U_2$ & hence it also contains $U_1 + \cancel{U_2}\, U_2$
also we have to show these are lin. indep.

$a_1\bar{u}_1 \dots + a_p u_p + b_1 v_1 + \dots + b_q v_q + c_1 w_1 + \dots c_r w_r = 0$.

$$\underbrace{b_1 v_1 + \dots b_q v_q}_{\bar{c}} = -a_1\bar{u}_1 - \dots - a_p u_p - c_1\bar{w}_1 - \dots c_r \bar{w}_r \quad \text{—} (1).$$

$\Rightarrow$ c lies in $U_2$
$\because$ RHS basis for $U_2$.

But $\bar{c}$ lies in $U_1$.

$\Rightarrow b_1\bar{v}_1 + \dots b_q\bar{v}_q \in U_1 \cap U_2$
$\Rightarrow b_1\bar{v}_1 + \dots + b_q\bar{v}_q = d_1\bar{u}_1 + \dots + d_p\bar{u}_p$

$b_1\bar{v}_1 + \dots + b_q\bar{v}_q - d_1\bar{u}_1 - \dots d_p\bar{u}_p = 0$ [$\because$ u & v form basis for $U_1$]
$\Rightarrow b_1 = b_q \dots = d_1 = d_p = 0$ — (2)

Sub (2) in (1)
$\Rightarrow a_1 u_1 + \dots a_p u_p + c_1\bar{w}_1 + \dots \bar{c}_r w_r = 0$
$\Rightarrow a_1 = a_2 = \dots = c_1 = c_r = 0$ [basis for $U_1$].

# TUTORIAL 2:

- Basis :- LI set & spans the set.
  
  LI set $\downarrow$ maximal; spans $\downarrow$ minimum.

LI :- { set of vectors }.

if a vector $\bar{v} \in V$ is added

$\bar{v}$ = L.C of { }

∴ contradiction

Span :- { } if on removing one vector, still spans the set, then the vector removed = L.C of { }

- Cardinality of $S \leq$ card. of $V$.

→ Rings :-

Two operators + & ·

+ : abelian

· : semigroup ( closure & associative)

& $a \cdot (b+c) = a \cdot b + a \cdot c$

Q: Can 2 subrings have the same identity?

Let $R = \{(a,b)\}$ $a, b \in \mathbb{R}$.

$(a,b) + (c,d) =$ ~~a+~~ $(ac, bd)$

For R, identity is $(1,1)$.

# LINEAR TRANSFORMATIONS :

Definition : Let $V$ & $W$ be 2 v.s. A linear transformation from $V$ into $W$ is a function $T : V \to W$ such that :-

a). $T(\bar{a} + \bar{b}) = T(\bar{a}) + T(\bar{b}) \quad \forall \ \bar{a}, \bar{b} \in V$.

b). $T(c\bar{a}) = c \cdot T(\bar{a}) . \quad \forall \ \bar{a} \in V \ \& \ \forall \ c \in \mathbb{F}$

Eg: ① Identity map $I : V \to V$

$I(\bar{v}) = \bar{v} \qquad \forall \ v \in V$

② Differentiation transformation on polyⁿ.

$V = P(\mathbb{R}) = W$.

$D(P(x)) = P'(x)$.

$P(x) = c_0 + c_1 x + \dots + c_n x^n$

$P'(x) = c_1 + 2c_2 x + \dots + n c_n x^{n-1}$.

③ Zero Transformation

$0 : V \to W$

$0(\bar{v}) = \bar{0} \qquad \forall \ \bar{v} \in V$.

④ Let $\mathbb{R}$ be the real line. Let $V$ be the space of all functions from $\mathbb{R}$ to $\mathbb{R}$

$V = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$.

Define a transformation $T$ as

$$T(f)(x) = \int_0^x f(t)\, dt .$$

$T$ is a lin. transformation from $V$ to $V$

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(5). Let $A \in \mathbb{R}^{m\times n}$

Define $T(\bar{x}) = A\bar{x}$ $\quad \bar{x} \in \mathbb{R}^n$.

then $T$ is a transformation from $\mathbb{R}^n \to \mathbb{R}^m$.

(6). $P \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{n\times n}$.

If $P, Q$ are fixed.

Define $T(A) = PAQ$ $\quad A \in \mathbb{R}^{m\times n}$.

then $T$ is a L.T from $\mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$.

- Properties of Linear Transformation:

Claim $T: V \to W$ is a linear Transformation then,

$$T(c\bar{v}_1 + \bar{v}_2) = cT(\bar{v}_1) + T(\bar{v}_2)$$

Let $\bar{v}_1 = \bar{v}_2 = 0$.

$$T(\bar{0}) = c.T(\bar{0}) + T(\bar{0}).$$

$$T(\bar{0}) = \bar{0}$$

$\therefore$ For any linear transformation $T(\bar{0}) = \bar{0}$ !

To rotate any vector by $\theta$: multiply by $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

- THEOREM:

Let $V$ be a finite dim. vector space over the field $\mathbb{F}$. Let $(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n)$ be an ordered basis of $V$. Let $W$ be another vector space over the same field $\mathbb{F}$. Let $\bar{\beta}_1, \ldots, \bar{\beta}_n$ be any vectors in $W$. Then there is precisely one linear transformation from $V \to W$ ST $T(\bar{\alpha}_i) = \bar{\beta}_i$ $\quad i = 1, 2, 3, \ldots, n$.

Proof:

Let $\bar{\alpha} \in V$ then there exists a unique ~~to~~ tuple such that

$$\bar{\alpha} = x_1\bar{\alpha}_1 + \dots + x_n\bar{\alpha}_n \quad \text{—— ①}$$

Then we define $T(\bar{\alpha})$ as :-

$$T(\bar{\alpha}) = x_1\bar{\beta}_1 + x_2\bar{\beta}_2 + \dots + x_n\bar{\beta}_n \quad \text{—— ②}$$

Since $T$ is a linear transformation

$$T(\bar{\alpha}) = x_1 T(\bar{\alpha}_1) + \dots + x_n T(\bar{\alpha}_n) \quad \text{—— ③}$$

comparing ② & ③ :-

$$T(\alpha_i) = \beta_i \qquad i = 1 \dots n.$$

Let $U(\alpha_i) = \beta_i$

& $T(\alpha_i) = \beta_i$

where $U$ is another transformation.

then $U(\alpha_i) = T(\alpha_i)$.

eg: $\bar{\alpha}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\beta_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $\beta_2 = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$

then unique $T$ exists such that

$T\alpha_1 = \beta_1$ & $T\alpha_2 = \beta_2$.

ie :-

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T(-2\alpha_1 + \alpha_2)$$

$$= -2T\alpha_1 + (\alpha_2)$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}. \quad \text{unique}$$

- **Null space of a Linear Transformation:**

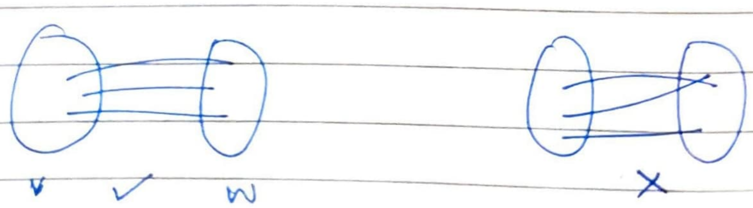
Let $T$ be a linear transformation
$T: V \rightarrow W$
then null space of $T$ is defined as:-

$$\text{null}(T) = \{\bar{v} \in V \mid T(\bar{v}) = \bar{0}\}.$$

(HW) Show that null space of $T$ is a subspace of $V$:-

Let $v_1, v_2 \in V \mid T(v_1) = T(v_2) = 0$
$\therefore$ $v_1, v_2 \in$ null space $(T)$.
Now, $cT(v_1) = 0$
$\therefore$ $cT(v_1) + T(v_2) = 0$.
$T(cv_1 + v_2) = 0$
$\Rightarrow$ $cv_1 + v_2 \in$ null. Hence proved.

— Injective:
Injective (one-one) linear transformation
A linear transformation
$T: V \rightarrow W$ is injective iff
$T(\bar{v}_1) = T(\bar{v}_2) \Rightarrow \bar{v}_1 = \bar{v}_2$




Claim: Let $T: V \rightarrow W$ be a linear trans. $T$ is injective iff $\text{null}(T) = \{\bar{0}\}$.

## Injective Linear Transformation:

If there is a one-to-one mapping.

$T: V \to W$ is an inj. linear transformation if

$$T(\bar{v}_1) = T(\bar{v}_2)$$
$$\Rightarrow \bar{v}_1 = \bar{v}_2$$

**Proposition:** Let $V$ & $W$ be vector spaces.
A linear transformation $T: V \to W$ is injective iff $null(T) = \{\bar{0}\}$.

Pf: Let $T$ be injective ... then,
Let $\bar{v} \in null(T)$. ie: $T(\bar{v}) = 0$.
But we know that $\bar{0} \in null(T)$.
ie: $T(\bar{0}) = 0$
But since $T$ is injective, we have $\bar{v} = \bar{0}$.

Now,
Let $null(T) = \{\bar{0}\}$.
TP: $T$ is an injective transformation.
Let $v_1, v_2 \in V$ such that

$$T(\bar{v}_1) = T(\bar{v}_2)$$
$$T(\bar{v}_1) - T(\bar{v}_2) = 0.$$
$$T(\bar{v}_1 - \bar{v}_2) = 0.$$

$\bar{v}_1 - \bar{v}_2 \in null(T)$ $\because$ it is mapped to 0.

$\therefore$ $\bar{v}_1 - \bar{v}_2 = 0$

$\Rightarrow \bar{v}_1 = \bar{v}_2 \Rightarrow$ injective.

- **Range of (T)**

$\{ \bar{y} \in W \mid T(\bar{v}) = \bar{y}$ for some $\bar{v} \in V \}$.
$\longrightarrow W$

HW Verify that range(T) is a subspace.

## Surjective transformation :

A lin. transformation $T: V \to W$ is called surjective if for every $\bar{\alpha} \in W$, there exists $\bar{v} \in V$ such that $T(\bar{v}) = \bar{\alpha}$.

Eg: 1. Let D be the differentiation transformation

$D: P(\mathbb{R}) \to P(\mathbb{R})$ defined as

$D(P) = P'$.

D is surjective map from $P(\mathbb{R})$ to $P(\mathbb{R})$.

Eg 2: D is a diff transformation from $P_m(\mathbb{R})$ to $P_m(\mathbb{R})$
This is not surjective.
Since the range space will not include polynomials of degree m. $\therefore$, those polyⁿ won't be mapped to.

Eg 3: Let D be a diff. transformation from $P_m(\mathbb{R})$ to $P_{m-1}(\mathbb{R})$
Here it is surjective.

Eg 4: $T(x) = x^2 P(x)$. from $P(\mathbb{R}) \to P(\mathbb{R})$
Not surjective $\because$ no constant or single degree polyⁿ.

★ Null space is a subset of vectors $\bar{\alpha}$ in V such that $T(\bar{\alpha}) = \bar{0}$

● **Theorem:**

Let V be a finite dimension vector space. Let $T: V \to W$ be a lin trans then range (T) is

Rank - Nullity theorem :

a finite dim subspace of $W$ &
$$\dim(V) = \dim \text{null}(T) + \dim \text{range}(T).$$

Pf : null (T) is a subspace.
Let $\{\bar{u}_1, \bar{u}_2 \dots u_m\}$ be the basis of null (T).
Because $\{u_1 \dots u_m\}$ is lin independent in null (T) it is also lin indep in V.
Thus we can extend $\{u_1 \dots u_m\}$ to a basis of V.

Let $\{u_1 \dots u_m, w_1 \dots w_n\}$ form a basis of V
$\therefore$ $\dim(V) = m + n$.
$\dim(\text{null}) = m$.

To complete this proof, we have to show that
$\dim(\text{range}) = n$.

Further :
let $\bar{\alpha} \in V$ thus
$$\bar{\alpha} = a_1\bar{u}_1 + a_2\bar{u}_2 + \dots + a_m\bar{u}_m + b_1\bar{w}_1 + b_2\bar{w}_2 + \dots + b_n\bar{w}_n$$
apply T on both sides.

$$T(\bar{\alpha}) = \underbrace{\bar{0} + \bar{0} + \dots \bar{0}}_{m} + b_1 T(w_1) + b_2 T(w_2) + \dots + b_m T(w_n).$$

[ Show $T(\alpha) = b_1 T(w_1) + b_2 T(w_2) \dots$ spans W ].

Because $\bar{\alpha} \in V$ is arbitrary, we can represent $T(\bar{\alpha})$ as a lin. comb of $T(w_1) + \dots T(w_n)$.

1 $\therefore$ $T(w_1) \dots T(w_n)$ spans the range (T).
Let $c_1 \dots c_n \in \mathbb{R}$.
& ~~$c_1w_1 + c_2w_2$~~
$$c_1 T(w_1) + c_2 T(w_2) + \dots = 0.$$
$$\Rightarrow T(c_1w_1 + c_2w_2 + \dots c_nw_n) = 0$$

$\therefore$, $c_1\bar{w}_1 + c_2\bar{w}_2 + \dots + c_n\bar{w}_n \in$ null $(T)$.

$\exists$ $d_1 \dots d_m$ such that :- [l.c of basis of null]

$c_1\bar{w}_1 + c_2\bar{w}_2 + \dots + c_n\bar{w}_n$

$= d_1\bar{u}_1 + d_2\bar{u}_2 + \dots d_m\bar{u}_m$. — ①

Now because $\{u_1 \dots u_m, w_1 \dots w_n\}$ forms the basis for $V$, $\Rightarrow$ linearly independent — ②

① is possible with ② only if
$c_1 \dots c_n = d_1 \dots d_m = 0$

Thus $T(w_1) \dots T(w_n)$ forms a basis for range$(T)$
Thus dim range$(T)$ $= n$.

× ——— ×

- **Corollary: 1**

If $V$ & $W$ are finite dim vector spaces such that
dim $V$ > dim $W$

Then no linear transformation from $V$ to $W$ is injective

dim $V$ = dim (null space) + ~~dim~~ dim Range $(T)$

Dim (null) = dim$(V)$ − dim Range $(T)$ — ①
but range $(T)$ subspace of $W$

$\therefore$, dim range $\leq$ dim $W$.

dim (null) $\geq$ dim $V$ − dim (range) $> 0$
$\Rightarrow$ $\exists$ $\bar{v} \in$ null $(T)$ s.t. $\bar{v} \neq 0$
$\Rightarrow$ $T$ is not injective.

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- Corollary 2:

Let $V$ & $W$ be finite dimension vector spaces
Let $\dim V < \dim W$
Then no linear transformation from $V$ to $W$ is surjective.

Pf: $\dim(\text{range } T) = \dim(V) - \dim(\text{null space})$.
But $\dim(\text{null}) \geq 0$.

$\Rightarrow \dim \text{range}(T) \leq \dim V < \dim W$.
contradiction
$\Rightarrow T$ is not surjective.

Eg: Let $A \in \mathbb{R}^{m \times n}$ ~~$m > n$~~ $n > m$
Define $T : \mathbb{R}^n \to \mathbb{R}^m$ as,
$T(\bar{x}) = A\bar{x}$.
$n = \dim(V)$ $m = \dim(W)$.

$A\bar{x} = \bar{0}$ .... since $n > m$
∴ not injective.
And therefore null space contains vectors other than $\bar{0}$
∴ $A\bar{x} = \bar{0}$ system has at least 1 non-trivial (not all zero) solution.

Eg: $n < m$.
$T(\bar{x}) = A\bar{x}$
So $T$ is not surjective.
$\Rightarrow \exists$ $Y$ in $W$ such that
$A\bar{x} = Y$

# TUTORIAL

Game of chess:

→ $V \in$ moves

Pawn moves forward only.
But no inverse exists. ∴, No, not a vector space

x———x

- DIRECT SUM :-

Let $W_1, W_2, \dots, W_k$ be subspaces of vector space $V$. Then $V$ is called direct sum ($V = W_1 \oplus W_2 \oplus \dots \oplus W_k$) if for every vector $\bar{\alpha} \in V$ there is a unique rep. as $\bar{\alpha} = \bar{u_1} + \bar{u_2} + \dots + \bar{u_k}$ such that $\bar{u_j} \in W_j$.

[ $Z = W_1 + W_2 + \dots + W_k = \{ \bar{u_1} + \bar{u_2} + \dots + \bar{u_k} \} : u_j \in W_j$ ]
$Z$ is shown to be a subspace of $V$ but there can be multiple representations for a vector.

Eg: 1

$W_1 = \{ (x, y, 0) \in R^3, \ x, y \in \mathbb{R} \}$
$W_2 = \{ (0, 0, z) \in R^3, \ z \in \mathbb{R} \}$
$W_3 = \{ (0, y, y) \in \mathbb{R}^3, \ y \in R \}$.

$Z = W_1 + W_2 + W_3$.

$\bar{0} \in Z$

$\Rightarrow \bar{0} = \underset{\in W_1}{\bar{0}} + \underset{\in W_2}{\bar{0}} + \underset{\in W_3}{\bar{0}}$.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underset{\in W_1}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + \underset{\in W_2}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} + \underset{\in W_3}{\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}} = \bar{0}$$

Multiple Rep.

Eg 2: Let $W_i = \{(0, 0, \dots, x_i, 0, \dots 0) \in \mathbb{R}^n, x_i \in \mathbb{R}\}$.

$$\mathbb{R}^n = W_1 \oplus W_2 \oplus W_3 \oplus \dots \oplus W_n$$

Eg 3: $V$ = Spaces of all polynomials.

$$V = \underbrace{W_1}_{\text{even degree}} \oplus \underbrace{W_2}_{\text{Odd degree poly}^n}.$$

$x + \cancel{x^2} + x^3 + \cancel{x^4} + x^5$

## → Proposition:

Let $W_1, W_2 \dots W_n$ be subspaces of $V$ then $V = W_1 \oplus W_2 \oplus \dots \oplus W_n$ iff both the following conditions hold.

a) $V = W_1 + W_2 + \dots + W_n$.

b) The only way to write $\bar{0}$ as a sum $\bar{u}_1 + \bar{u}_2 + \dots + \bar{u}_n$ where each each $\bar{u}_i \in W_i$ by taking all $\bar{u}_i = \bar{0}$

Pf: Let $V = W_1 \oplus W_2 \oplus \dots \oplus W_n$

By definition of the direct sum WKT,

$$V = W_1 + W_2 + \dots + W_n$$

To show (b) holds:-

$$\bar{0} = \bar{u}_1 + \dots + \bar{u}_n$$

but $\bar{0} = \bar{0} + \bar{0} + \dots + \bar{0}$

because $V$ is direct sum of $W_i$'s $\bar{0}$ has a unique rep. as a sum of $\bar{u}_i \in W_i$. This makes each $\bar{u}_i = \bar{0}$.

....... Proving the reverse .......

Let $\bar{\alpha} \in V$ can be written in 2 diff ways as follows:-

$$\bar{\alpha} = \bar{u}_1 + \bar{u}_2 + \dots + \bar{u}_n \quad — ①$$

& . $$\bar{\alpha} = \bar{p}_1 + \bar{p}_2 + \dots + \bar{p}_n \quad — ②$$

① - ②

$$\bar{0} = (\bar{u}_1 - \bar{p}_1) + (\bar{u}_2 - \bar{p}_2) + \dots + (\bar{u}_n - \bar{p}_n)$$

$\bar{u}_1 - \bar{p}_1 \in W_1$

$\bar{u}_2 - \bar{p}_2 \in W_2$.

$\vdots$

$\bar{u}_n - \bar{p}_n \in W_n$

because (b) holds. $\bar{0}$ can be written as

$$\bar{0} = \bar{0} + \bar{0} + \dots + \bar{0}$$

Thus $\bar{u}_i - \bar{p}_i = 0$

$$\Rightarrow \bar{u}_i = \bar{p}_i = 0 \quad \forall i.$$

→ Proposition 2: HW

Suppose $W_1$ & $W_2$ are subspaces of $V$. then $V = W_1 \oplus W_2$ iff :-

a) $V = W_1 + W_2$.

b) $W_1 \cap W_2 = \{0\}$.

$V = W_1 \oplus W_2$ $\qquad$ $V = W_1 + W_2$

if $W_1 \cap W_2 \neq \{0\}$

$\exists\ \bar{d} \in W_1 \cap W_2 \quad d \neq 0$

$\bar{d} \in V \quad \bar{d} = \bar{0} + \bar{\alpha} \quad \bar{d} = \bar{d} + \bar{0}$

∴ No contradiction.

$W_1 \cap W_2 = \{0\}$ $\qquad$ Let $\bar{\alpha} = \bar{u}_1 + \bar{u}_2$

$\bar{0} = \bar{0} + \bar{0}$ $\qquad$ $\bar{\alpha} = \bar{v}_1 + \bar{v}_2$

∴ $\bar{0} = (\bar{u}_1 - \bar{v}_1) + (\bar{u}_2 - \bar{v}_2)$

$\Rightarrow \bar{u}_1 - \bar{v}_1 = \bar{v}_2 - \bar{u}_2 = k$

$k \in W_1 \cap W_2$

contradiction

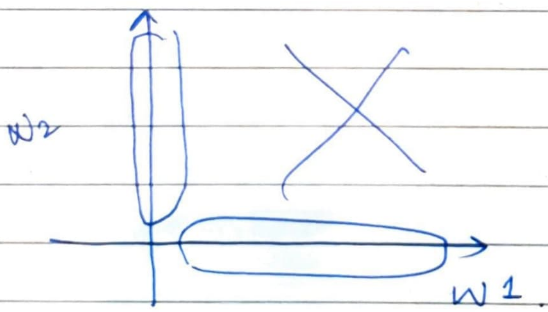
Prop 2 ~~is~~ works only when there are 2 subspaces. If there are more than 2 subspaces & we check pairwise intersection contains only $\bar{0}$ that will not be sufficient to show that V is direct sum of subspaces.

Eg: $W_1 = \{(x,y,0) \in \mathbb{R}^3, x, y \in \mathbb{R}\}$.
$W_2 = \{(0,0,z) \in \mathbb{R}^3, z \in \mathbb{R}\}$.
$W_3 = \{(0,y,y) \in \mathbb{R}^3, y \in \mathbb{R}\}$.

$\mathbb{R}^3 \neq W_1 \oplus W_2 \oplus W_3$ ∴ false. FAIL

→ Proposition 3 ::
Let $W_1$ & $W_2$ be subspaces of V. Then $W_1 \cup W_2$ is a subspace iff either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Eg:




$W_1 = \{(x,0) \in \mathbb{R}^2, x \in \mathbb{R}\}$.
$W_2 = \{(0,y) \in \mathbb{R}^2; y \in \mathbb{R}\}$.

Pf: Let $W_1 \subseteq W_2$ then $W_1 \cup W_2 = W_2$ which is a subspace.
Let $W_2 \subseteq W_1$ then $W_1 \cup W_2 = W_1$ which is a subspace.

Reverse Pf :-

$W_1 \not\subseteq W_2$ $\qquad$ $W_2 \not\subseteq W_1$.

Assume $W_1 \cup W_2$ is a subspace.

we will arrive at a contradiction.

$W_1 \not\subseteq W_2$ there exists :-

$\bar{\alpha}$ such that :- $\quad \bar{\alpha} \in W_1 \setminus W_2$

$W_2 \not\subseteq W_1$.

$\exists$ $\beta$. such that $\quad \bar{\beta} \in W_2 \setminus W_1$

But $\bar{\alpha} + \bar{\beta} \in W_1 \cup W_2$.

then either $\bar{\alpha} + \bar{\beta} \in W_1$ (OR) $\bar{\alpha} + \bar{\beta} \in W_2$.

a) $\bar{\alpha} + \bar{\beta} \in W_1$

$\bar{\alpha} \in W_1$.

$\therefore$, $\bar{\beta} = (\bar{\alpha} + \bar{\beta}) - (\bar{\alpha}) \in W_1$

$\bar{\beta} \in W_1$. Contradiction

b) Similarly the same for $W_2$

eg: Let $A \in \mathbb{R}^{m \times n}$ & $T: \mathbb{R}^n \to \mathbb{R}^m$ is such that

$T(\bar{x}) = A\bar{x}$. $\quad \bar{x} \in \mathbb{R}^n$.

$\therefore$, Range (T)

$= \{ \bar{y} \in \mathbb{R}^m \mid \bar{y} = A\bar{x} \}$.

$=$ space of all possible linear combinations of columns of A.

dim (Range(T)) = column rank

row-space (A) = Space spanned by rows of A.

If $m < n \rightarrow$ Rank $= m$.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{1m_1} & \cdot & \cdot & \cdot \\ a_{1m_2} & & & \\ a_{1m_3} & & & \\ \vdots & & & \end{bmatrix}}_{m \times n} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$
linear combination of columns.

- **Theorem:** Let $A \in \mathbb{R}^{m \times n}$. Then
column rank $(A)$ = ~~column~~ row rank $(A)$.

Pf: Let $r$ = row rank of $(A)$.
which means there are $r$ linearly independent rows of $A$ which span the rowspace of $A$.

Let $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_r \in \mathbb{R}^n$ be the linearly independent rows.
Then $A\bar{x}_1, A\bar{x}_2, \ldots A\bar{x}_r$ will be in the column space of $A$.
We will show that these vectors are L.I.

ie: $c_1 A\bar{x}_1 + c_2 A\bar{x}_2 + \cdots + c_r A\bar{x}_r = \bar{0}$.
$A(c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_r\bar{x}_r) = \bar{0}$.
$\therefore$, $c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_r\bar{x}_r \in$ null $(A)$.

But $c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_r\bar{x}_r \in$ rowspace $(A)$.

?? $\therefore$, $c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_r\bar{x}_r \in$ null $(A) \cap$ rowspace$(A)$
$= \{\bar{0}\}$

$\therefore$, $c_1\bar{x}_1 + \cdots + c_r\bar{x}_r = \bar{0}$
but $\bar{x}_1 \ldots \bar{x}_r$ are L.I. by assumption.
$\Rightarrow$ $c_1 = c_2 = c_3 \ldots c_r = 0$
$\Rightarrow$ $c_1(A\bar{x}_1) + c_2(A\bar{x}_2) + \cdots + c_r(A\bar{x}_r)$ are lin indep.
$r \leq$ col. rank $(A)$.
row rank $A \leq$ col. rank $(A)$.

⚠ if columns of $A$ are lin. Indep. $\Leftrightarrow$ null$(A) = \{\bar{0}\}$

Rank: No. of linearly independent rows

Consider $A^T$

$A^T_{ij} = A_{ji}$

rowrank $(A^T) \leq$ col. rank $(A^T)$

col. rank $(A) \leq$ row rank $(A)$.

row rank $(A)$ = column rank $(A)$

- Co-ordinates :-

Let $V$ be a finite dimensional vector space

Let $B = \{ \bar{\alpha}_1, \bar{\alpha}_2, \ldots \bar{\alpha}_n$ be an ordered basis for $V$ Let $\bar{\alpha} \in V$.

$\bar{\alpha} = x_1 \bar{\alpha}_1 + x_2 \bar{\alpha}_2 + \cdots + x_n \bar{\alpha}_n$ then.

then $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is co-ordinate of $\bar{\alpha}$ with respect to $B$

Vector rep of $\bar{\alpha}$ WRT [B].

Eq: $\mathbb{R}^3$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

then $(3, 4, 5) \rightarrow 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$[\bar{\alpha}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

→ Theorem:

Let $V$ be an n-dimensional vector space over field $\mathbb{F}$. Let $B$ & $B'$ be 2 basis of $V$. Then there exists a unique, necessarily invertible $n \times n$ matrix $P$ with entries in $\mathbb{F}$ s.T:-

$B = \{ \alpha_1, \alpha_2, \ldots \alpha_n \}$ $B' = \{ \bar{\alpha}_1', \bar{\alpha}_2', \ldots \alpha_n' \}$

a). $[\bar{\alpha}]_B = P [\bar{\alpha}]_{B'}$

$[\bar{\alpha}]_{B'} = P^{-1} [\bar{\alpha}]_B$.

For every vector $\bar{\alpha} \in V$, columns of $P$ are

$\bar{P}_j = [\bar{\alpha}_j']_B$ $j = 1 \ldots n$

Pf: Because $B = \{\bar{\alpha}_1, \bar{\alpha}_2 \ldots \bar{\alpha}_n\}$ is a basis.

we can write uniquely

$$\bar{\alpha}_j' = \sum_{i=1}^{n} P_{ij}\,\bar{\alpha}_i \qquad j = 1, 2, \ldots, n.$$

$$\bar{\alpha}_j' = P_{1j}\bar{\alpha}_1 + P_{2j}\bar{\alpha}_2 + \cdots + P_{nj}\bar{\alpha}_n \quad \text{— (1)}.$$

Let $\bar{\alpha} \in V$, then

$$\bar{\alpha} = x_1\bar{\alpha}_1 + x_2\bar{\alpha}_2 + \cdots + x_n\bar{\alpha}_n$$

$$\Rightarrow [\bar{\alpha}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i\bar{\alpha}_i \quad \text{— (2)}.$$

$$\bar{\alpha} = x_1'\bar{\alpha}_1' + x_2'\bar{\alpha}_2' + \cdots + x_n'\bar{\alpha}_n'$$

$$[\bar{\alpha}]_{B'} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

$$\bar{\alpha} = \sum_{j=1}^{n} x_j'\bar{\alpha}_j'$$

from (1) :-

$$\bar{\alpha} = \sum_{j=1}^{n} x_j' \left(\sum_{i=1}^{n} P_{ij}\bar{\alpha}_i\right).$$

$$= \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{n} x_j' P_{ij}\right) \quad \text{— (3)}.$$

from (2) & (3) :-

$$x_i = \sum_{j=1}^{n} x_j' P_{ij}.$$

ie:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} x_j' P_{1j} \\ \vdots \\ \sum_{j=1}^{n} x_j' P_{nj} \end{bmatrix} = P \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

$$[\bar{\alpha}]_B = P[\bar{\alpha}]_{B'}.$$

$$[\alpha_j']_B = \begin{bmatrix} P_{1j} \\ P_{2j} \\ \vdots \\ P_{nj} \end{bmatrix} = \bar{P_j}$$

classmate

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## Proof continuation:

Let F be a finite dim vector space.

Let $\beta = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ is an ordered basis of V.

Let $\bar{\alpha} \in V$ then $\exists\ c_1, c_2 \dots c_n \in \mathbb{F}$

such that:- $\bar{\alpha} = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \dots + c_n\bar{\alpha}_n$

$$[\bar{\alpha}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$\boxed{Q:}$ If there are 2 ordered basis:-

$B = \{\bar{\alpha}_1, \bar{\alpha}_2, \dots \bar{\alpha}_n)\}$

$B' = \{\bar{\alpha}_1', \bar{\alpha}_2' \dots \bar{\alpha}_n'\}$.

If given $[\bar{\alpha}]_B$ can $[\bar{\alpha}]_{B'}$ be obtained?

Pf: Let $\bar{\alpha} \in V$.

$$[\bar{\alpha}]_B = \bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$$[\bar{\alpha}]_{B'} = \bar{X}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}.$$

which means that:-

$$\bar{\alpha} = x_1'\alpha_1' + x_2'\alpha_2' + \dots + x_n'\alpha_n'$$

$$= \sum_{j=1}^{n} x_j'\alpha_j'. \quad \text{---} (1)$$

but $\bar{\alpha}_j' = \sum_{i=1}^{n} P_{ij}\,\alpha_i$

$$\alpha_j' = P_{1j}\alpha_1 + P_{2j}\alpha_2 + \dots + P_{nj}\alpha_j. \quad \text{---} (2)$$

Sub (2) in (1) :-

$$\bar{\alpha} = \sum_{j=1}^{n} x_j' \sum_{i=1}^{n} P_{ij}\, \alpha_i$$

$$= \sum_{i=1}^{n} \left( \sum_{j=1}^{n} P_{ij}\, x_j' \right) \bar{\alpha_i} \qquad \text{— (3)}$$

But we know that :-

$$\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha_i}$$

$\therefore$

$$x_i = \sum_{j=1}^{n} P_{ij}\, x_j'$$

ie: $[\bar{\alpha}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} P_{1j}\, x_j' \\ \vdots \\ \sum_{j=1}^{n} P_{nj}\, x_j' \end{bmatrix}$

ie: $PX' = P[\alpha]_{B'}$.

Now, $\bar{P_j}$ is the $j^{th}$ col. of $P$

or $\bar{P_j} = \begin{bmatrix} P_{1j} \\ P_{2j} \\ \vdots \\ P_{nj} \end{bmatrix}$

From (2) : $P_j = [\alpha_j']_B$.

Well $\bar{X} = P\bar{X}'$

when $\bar{X}' = 0$ clearly

$\bar{X} = P\bar{0} = 0$

Let $\bar{X} = 0$ then $P\bar{X} = \bar{0}$

We have to show that $P\bar{X}' = 0$ has only 1 sol
$X' = \bar{0}$

(by contradiction...)

Let there exist some $\bar{x}' \neq 0$

s.t. $P\bar{x}' = 0$.

$$P_1 x_1' + P_2 x_2' + \dots + P_n x_n' = 0$$

$\Rightarrow$ col of $P$ linearly dependend.

$\exists$ some $k$ such that

$k = \{1 \dots n\}$.

$$P_k = -\sum_{i \neq k} \frac{x_i}{x_k} \bar{P_i}$$

$$\Rightarrow P_{ik} = -\sum_{j \neq k} \frac{x_j'}{x'_k} P_{ij} \qquad i = 1, 2 \dots n.$$

But we know that :-

$$\bar{\alpha_j}' = \sum_{j=1}^{n} P_{ij} \bar{\alpha_j}$$

$$= \sum_{i=1}^{n} \left( -\sum_{j \neq k} \frac{x_j'}{x_k'} P_{ij} \right) \bar{\alpha}$$

ie: $$\alpha_k' = -\sum_{j \neq k} \left( \sum_{i=1}^{n} P_{ij} \bar{\alpha_i} \right) \left( \frac{x_j'}{x_k'} \right)$$

$$= -\sum_{j \neq k} \frac{x_j'}{x_k'} \bar{\alpha_j}'$$

which contradicts that

$\bar{\alpha_1}', \bar{\alpha_2} \dots \bar{\alpha_n}'$ is a basis

that becomes $P\bar{x}' = \bar{0} \Rightarrow \bar{x}' = 0$

$P$ is inv.

Eg: (I) Let $\mathbb{F}$ be a field.

Let

$$\bar{\alpha} = (x_1, x_2, \dots x_n) \in \mathbb{F}^n$$

Let

$\mathcal{B}$ is a standard ordered basis of $\mathbb{F}^n$

$$B = \{ \bar{e_1}, \bar{e_2} \dots \bar{e_n} \}$$

where $e_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow$ ~~$e^{th}$~~ $i^{th}$.

$$[\alpha]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Eg (2): Let $\theta \in \mathbb{R}$ & $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

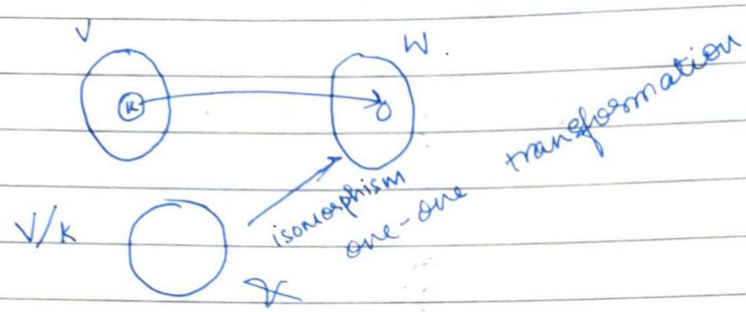
$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

TUTORIAL :-

Orthogonal subspaces :-
Every vector in S1 is orth. to every vect in S2

Kernel : Null space.

→




→ Linear functional :
Mapping vectors in V to scalars in $\mathbb{F}$

- THEOREM:

Let V & W be the vector spaces, over the field $\mathbb{F}$. Then the set of linear transformations from $V \to W$. $L(V, W)$, together with addition & & scalar multiplication is a ~~field~~ vector space over $\mathbb{F}$

Addition & scalar multiplication are defined as:

$$(T+V)(\bar{\alpha}) := T(\bar{\alpha}) + V(\bar{\alpha}).$$

$$cT(\bar{\alpha}) := T(c\bar{\alpha})$$

where $T, U \in L(V \to W)$.

Pf: a) zero transformation belongs to $L(V \to W)$.

b) If $T \in L(V \to W)$ then $-T \in L(V \to W)$.

★ If rows/col. of a matrix are L.I the null space = $\{0\}$ !!

c). Let $T, U \in L(V,W)$ & $\bar{\alpha}, \bar{\beta} \in V$, $c \in F$ then

$(T+U)(c\bar{\alpha}+\bar{\beta})$.

To show $T+U \in L(V \to W)$.

$\Rightarrow (T+U)(c\bar{\alpha}+\bar{\beta})$

$= T(c\bar{\alpha}+\bar{\beta}) + U(c\bar{\alpha}+\bar{\beta})$.

$= T(c\bar{\alpha}) + T(\bar{\beta}) + U(c\bar{\alpha}) + U(\bar{\beta})$.

$c(T+U)(\bar{\alpha}) + (T+U)(\bar{\beta})$.

d). Let $T \in L(V \to W)$ & $c \in F$, then we have to show that $cT \in L(V \to W)$.

Let $d \in F$ ~~there~~, $\bar{\alpha}$, $\bar{\beta} \in V$. then

$= cT(d\bar{\alpha}+\bar{\beta})$

$= T[c(d\bar{\alpha}+\bar{\beta})]$.

$= T[cd\bar{\alpha} + c\bar{\beta}]$.

$= cd\{T(\bar{\alpha})\} + c\{T(\bar{\beta})\}$.

$d.c\{T(\bar{\alpha})\} + c\{T(\bar{\beta})\}$.

But $cT(\alpha) = (cT)(\bar{\alpha})$.

$\therefore$ $= d(cT)(\bar{\alpha}) + (cT)(\bar{\beta})$.

$\Rightarrow cT \in L(V \to W)$.

x———x

(HW) Prove remaining properties ~~of~~ $[-T \in L \to V \to W]$

NOTE: $L(V, W)$ is defined only when $V$, $W$ are individually vector spaces.

For $L: (V \to W)$

If $\dim(V) = m$, $\dim(W) = n$.

$\dim(L) = mn$.

→ THEOREM:-

Let $V$ be an ~~n~~ n-dimensional vector space over the field $\mathbb{F}$ & let $W$ be an $m$ dim vector space over ~~F~~. Then the space $L(V \to W)$ is finite dim & has dim $m \times n$.

Proof: Let $B = \{\bar{\alpha}_1, \bar{\alpha}_2 \dots \bar{\alpha}_n\}$ be the basis of $V$ & $B' = \{\bar{\beta}_1, \bar{\beta}_2 \dots \bar{\beta}_m\}$ be the basis of $W$

for each pair of integers $(p, q)$ with $p \in \{1 \dots m\}$ & $q = \{1 \dots n\}$ define a lin. trans as:- $E^{pq}$ as follows:-

$$E^{pq}(\bar{\alpha}_i) = \begin{cases} 0 & i \neq q. \\ \bar{\beta}_p & \text{if } i = q. \end{cases}$$

ie:

$E^{11}(\bar{\alpha}_1) = \bar{\beta}_1$

$E^{12}(\bar{\alpha}_2) = \bar{\beta}_1$

$\vdots$

$E^{1m}(\alpha_m)$

$E_{12}(\bar{\alpha}_2) = \bar{\beta}_1$

$E_{22}(\bar{\alpha}_2) = \bar{\beta}_2$

ie: $E^{11}(\bar{\alpha}_1) = \bar{\beta}_1$

$E^{21}(\bar{\alpha}_1) = \bar{\beta}_2$

Let $T \in L(V \to W)$

$T\bar{\alpha}_j \in W$ thus

$$T\bar{\alpha}_j = \sum_{p=1}^{m} A_{pj} \bar{\beta}_p \quad \text{—} (1)$$

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To show:-

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{pq}.$$

But:-

$$\sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} (E^{pq}) A(\bar{\alpha}_j)$$

$$= \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{pq} (\bar{\alpha}_j). \quad \text{--- } (1)$$

$$= \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} \delta_{qj} \bar{\beta}_p.$$

$$= \sum_{p=1}^{m} \bar{\beta}_p \sum_{q=1}^{n} A_{pq} \delta_{qj}$$

$$= \sum_{p=1}^{m} \bar{\beta}_p A_{pj}$$

$$\left[ \begin{array}{c} T(\alpha_j) = U(\alpha_j) \\ \forall j. \\ \text{if } T(\bar{\alpha}) = U(\bar{\alpha}) \\ \forall \bar{\alpha} \in V \end{array} \right] \Rightarrow T = U.$$

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}.$$

which means that

$\{ E^{pq} \quad p = 1 \dots m, \quad q = 1 \dots n \}$.

Spans ~~$L \to W$~~ $L(V \to W)$

we need to show that

$E^{pq}, \quad p = 1 \dots m \quad q = 1 \dots n$ are L.I.

$$\sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{pq} = 0_T$$

$$\sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{pq} (\bar{\alpha}_j) = 0(\bar{\alpha}_j)$$

From (1):-

$$\sum_{p=1}^{m} A_{pj} \underbrace{B_p}_{\text{Basis.}} = \bar{0}$$

$$\Rightarrow A_{pj} = \bar{0} \qquad \begin{array}{l} p = 1 \dots m \\ q = 1 \dots n \end{array}$$

$\Rightarrow E^{pq} \quad \begin{array}{l} p = 1 \dots m \\ q = 1 \dots n \end{array}$

are linearly independent.

- Matrix Representation of A lin. Transformation :-

Th: Let V be an $n$ dim space over $\mathbb{F}$ & W be $m$-dim vector space over $\mathbb{F}$. Let $B = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ is an ordered basis for V & $B' = \{\beta_1, \beta_2, \dots \beta_m\}$ be basis for ~~B'~~ W.

For any lin trans. T from V to W there is an $m \times n$ matrix A with entries in F s.T:-

$$[T.\bar{\alpha}]_{B'} = A[\bar{\alpha}]_B.$$

for every vect. $\bar{\alpha}$ in V.

Pf: $\bar{\alpha} \in V$

$$\bar{\alpha} = x_1\bar{\alpha}_1 + \dots + x_n\bar{\alpha}_n$$

$$= \sum_{j=1}^{n} x_j \bar{\alpha}_j$$

$$[\bar{\alpha}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$T\bar{\alpha} = T\left(\sum_{j=1}^{n} x_j \bar{\alpha}_j\right)$$

$$T\bar{\alpha} = \sum_{j=1}^{n} x_j T(\alpha_j) \longrightarrow (1)$$

But $T(\alpha_j) \in W$.

thus,

$$T\bar{\alpha}_j = \sum_{i=1}^{m} A_{ij} \bar{\beta}_i \qquad j = 1, 2, \dots n. \longrightarrow (2)$$

using (2) in (1):

$$T\bar{\alpha} = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij} \bar{\beta}_i$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} x_j\right) \bar{\beta}_i$$

$$[(T\bar{\alpha})]_{B'} = \begin{bmatrix} \sum_{j=1}^{n} A_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} A_{mj} x_j \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = A[\bar{\alpha}]$$

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* If T is a lin transformation from $V \to V$ it is called an operator.

Eg: sq. matrices : lin. operator.

Rect. matrices lin. transformation.

→ ~~Th~~. Let T be a lin. operator defined on V. Let $B = \{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \ldots \bar{\alpha}_n\}$ be an ordered basis for V. then $[T\bar{\alpha}]_B = [T]_B [\bar{\alpha}]_B \quad \forall \bar{\alpha} \in V.$

Eg 1: Let $T : \mathbb{F}^2 \to \mathbb{F}^2$ as follows:-

$T(x_1, x_2) = (x_1, 0)$.

Let $B = \{\bar{e}_1, \bar{e}_2\}$ be the basis for $\mathbb{F}^2$.

$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$T(1,0) = (1,0), \quad = 1.\bar{e}_1 + 0\bar{e}_2.$

$T(0,1) = (0,0) \quad = \bar{e}.0 + 0.\bar{e}_2.$

$\therefore [T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. ←

∴ → $[T]_B = \left[ \left[T\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_B \quad \left[T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]_B \right]$

CHECK · $\left[T\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1.\bar{e}_1 + 0.\bar{e}_2.$

$\left[T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.e_1 + 0e_2.$

Now, $B' = \{ \overset{\alpha_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \overset{\alpha_2}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \}$

$1e_1 + 0e_2$

$1e_1 + 1e_2$

$[T\alpha]_{B'} = [T]_{B'} . [\bar{\alpha}]_{B'}$

Eg 2: Let $V = P_3(\mathbb{R})$

$D : V \to V$ differentiation

Let $B = \{x^0, x^1, x^2, x^3\}$.

$D(x^0) = 0x^0 + 0x + 0x^2 + 0x^3$

$[D(x^0)] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $[D(x^2)] = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

$[D(x)] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $[D(x^3)] = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$

$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- Invertible Linear Transformation :-

$T: V \to W$ is called invertible if there exists a transformation $U_1 : W \to V$, $U_2 : W \to V$ such that

$TU_1: V \to V$ is an identity operator on $V$

& $TU_2: W \to W$ is an identity op. on $W$

If $T$ is invertible, then $U$ is unique &

$U = T^{-1}$.

$U_1 = U_1 I = U_1 T(U_2)$

$= (U_1 T) U_2$.

$= I U_2 = U_2$.

$\Rightarrow U_1 = U_2 = U$.

→ Th: A linear transformation $T: V \to W$ is invertible iff $T$ is injective & surjective

Pf: Let $T: V \to W$ be a lin transf. which is invertible.

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Let $\bar{\alpha}, \bar{\beta} \in V$ & $T\bar{\alpha} = T\bar{\beta}$.

But $\bar{\alpha} = T^{-1}(T\alpha))$. → invertible ∴ Identity map.

$= T^{-1}(T(\beta))$.

$= \bar{\beta}$. .

We have to show that T is also surjective.

Let $\bar{\beta} \in W$.

$\bar{\beta} = I\bar{\beta}$ (because T is inv).

$= TT^{-1}\beta$

$= T(T^{-1}(\beta))$.

∴ $\beta \in$ range $(T)$.

$\Rightarrow W \subset \text{range}(T)$.

But $\text{range}(T) \subseteq W$.

∴ Range $= W$ .... surjective.

× ——— ×

**Eg 2:** Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined as :-

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_1 - x_2 \end{bmatrix}$$

(a) $B = B' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. $[T]_B = ?$

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix}_B = \begin{matrix} 1+2.0 \\ 1-0 \end{matrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix}_B = \begin{matrix} 0+2 \\ 0-1 \end{matrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$$[T]_B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$ Transformation WRT B.

**eg 3:** $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as :-

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_1 - x_2 \end{bmatrix}.$$

$$B = B' = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}.$$

$$T\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_B = \begin{bmatrix} 1+4 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$5 = x_1 + 3x_2.$$
$$-1 = 2x_1 - x_2.$$

$$10 = 2x_1 + 6x_2$$
$$-1 = 2x_1 - x_2.$$

$$11 = 7x_2 \qquad x_2 = \frac{11}{7}.$$

$$x_1 = 5 - \frac{33}{7} = \frac{2}{7}.$$

$$\therefore\ T\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_B = \begin{bmatrix} 2/7 \\ 11/7 \end{bmatrix}$$

$$T\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

$$x_1 + 3x_2 = 1 \qquad\qquad 2x_1 + 6x_2 = 2$$
$$2x_1 - x_2 = 4 \qquad\qquad 2x_1 - x_2 = 4$$

$$7x_2 = -2$$
$$x_2 = -2/7$$

$$x_1 = 1 - 3\cdot\left(\frac{-2}{7}\right) = \frac{13}{7}$$

$$\therefore\ [T]_{B'} = \left[[T\alpha_1]_B \ [T\alpha_2]_B\right]$$

$$= \cancel{\begin{bmatrix} & \\ 11/7 & 13/7 \end{bmatrix}}$$

$$= \begin{bmatrix} 2/7 & 13/7 \\ 11/7 & -2/7 \end{bmatrix}$$

$$\bar{\alpha} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

using the normal transformation :-

$$T[\bar{\alpha}] = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$[T\alpha]_B = [T_B][\alpha]_B = [$$

$$x_1 + 3x_2 = 3$$
$$2x_1 - x_2 = 2.$$

$$2x_1 + 6x_2 = 6$$
$$2x_1 - x_2 = 2.$$
$$7x_2 = 4$$
$$x_2 = 4/7.$$

$$x_1 = 3 - \frac{12}{7} = \frac{9}{7}$$

$$\therefore [T\alpha]_B = \begin{bmatrix} 2/7 & 13/7 \\ 11/7 & -2/7 \end{bmatrix} \begin{bmatrix} 9/7 \\ 4/7 \end{bmatrix}.$$

$$= \begin{bmatrix} 10/7 \\ 13/7 \end{bmatrix}.$$

$$\cancel{x_1 + 3x_2 = \frac{10}{7}} \qquad \cancel{3x_1 + 9x_2 = \frac{30}{7}}$$
$$\cancel{3x_1 - x_2 = \frac{13}{7}} \qquad \cancel{3x_1 - x_2 = \frac{13}{7}}$$
$$\cancel{10x_2 =}$$

$$\frac{10}{7}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{13}{7}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{7} + \frac{39}{7} \\ \frac{20}{7} - \frac{13}{7} \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

eg:- $[T_{BB'}] = ?$

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \qquad B' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}.$$

$$[T]_{BB'} = \left[ T\begin{bmatrix} 1 \\ -1 \end{bmatrix}_{B'} \quad T\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B'} \right].$$

$$T\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}_{B'} \quad \Bigg| \quad T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}_{B'}$$

$$[T]_{BB'} = \begin{bmatrix} -1 & 3 \\ -2 & 0 \end{bmatrix}.$$

Verify: $[\bar{\alpha}] = \begin{bmatrix} \frac{3}{2} \end{bmatrix}$ $\quad T[\alpha] = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$$[T\alpha]_{B'} = [T_{BB'}][\alpha]_B$$

$$= \begin{bmatrix} -1 & 3 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1/2 \\ 5/2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} + \frac{15}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$[T\bar{\alpha}] \quad 7\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

- THEOREM:

Let $T$ be a linear transformation from $V \to W$
$T$ is invertible iff it is surjective & injective

Pf: Let $T$ be invertible then :-
$\exists \; T^{-1} \; W \to V$ such that
$T^{-1}T \; V \to V$ is identity transformation
and $TT^{-1} \; W \to W$ is identity trans.

a) We have to show it is injective :-
Let $\alpha_1, \alpha_2 \in V$ ST:
$T(\alpha_1) = T(\alpha_2)$.
but $\alpha_1 = I\alpha_1$
$= T^{-1}T\bar{\alpha}_1$
$= T^{-1}(T\bar{\alpha}_1)$
$= T^{-1}(T\alpha_2)$
$= T^{-1}T\alpha_2$
$= I\alpha_2$
$= \underline{\underline{\alpha_2}}$. Injective

• $T(U(\alpha)) = (TU)(\alpha)$



b). To show that $T$ is surjective :-
Let $\bar{\beta} \in W$.

$$\bar{\beta} = (TT^{-1})\beta.$$
$$= T(T^{-1}(\beta)).$$

But $T^{-1}(\beta) \in V$.

So for every $\bar{\beta} \in W$ $\exists$ $T^{-1}(\bar{\beta}) \in V$ such that $\bar{\beta}$ is an image of $T^{-1}(\beta)$ under $T$

$\therefore$ surjective.

Reverse Assume $T$ is injective & surjective :-
To show $T$ is invertible :-

Let $\bar{\beta} \in W$. but there exists unique $S\bar{\beta} \in V$
ST. $T(S\bar{\beta}) = \beta$
unique because $T$ is injective & surjective.

$TS(\bar{\beta}) \Rightarrow TS : W \to W$. Identity map.

Now we have to show that :-
$ST : V \to V$ identity map.

Let $\alpha \in V$.
$\therefore$ $T(ST(\alpha)) = TS(T\bar{\alpha}). = T\bar{\alpha}$ [$\because ST$ id map]
$ST(\bar{\alpha}) = \bar{\alpha}$ $\Rightarrow$ $ST$ is id. map. $V \to V$.

To complete the proof we still need to show that $S$ is a lin. transformation.
Let $\bar{\beta_1}, \bar{\beta_2} \in W$.

$$T(S\bar{\beta_1} + S\bar{\beta_2}) = TS\bar{\beta_1} + TS\bar{\beta_2}$$
$$= \bar{\beta_1} + \bar{\beta_2}.$$

Thus $S\bar{\beta_1} + S\bar{\beta_2}$ is unique el. in $V$ s.t. it gets mapped to $\bar{\beta_1} + \bar{\beta_2}$ in $W$

$T(CS\beta_1) = C\,T(\beta_1)$
$T(CS\beta_1) = C\beta_1$

$C(S\beta_1) = S(C\beta_1)$

$T(CS\beta_1)$
$= C(TS\beta_1) - (C\beta_1) =$

Let $\alpha_1, \alpha_2 \in V$ s.t :-

$T(\bar{\alpha}_1) = \bar{\beta}_1$ , $T(\bar{\alpha}_2) = \bar{\beta}_2$.

$S(\bar{\beta}_1 + \bar{\beta}_2) = S(T(\bar{\alpha}_1) + T(\bar{\alpha}_2))$

$= ST(\bar{\alpha}_1 + \bar{\alpha}_2)$ identity map.

$= \bar{\alpha}_1 + \bar{\alpha}_2$

But $\bar{\alpha}_1 = S\bar{\beta}_1$ , $\bar{\alpha}_2 = S\bar{\beta}_2$ because

$T(S\bar{\beta}_1) = \bar{\beta}_1$

$T(\bar{\alpha}_1) = \beta_1$

Similarly:-

$\alpha_2 = S\beta_2$.

$S(\bar{\beta}_1 + \bar{\beta}_2) = \bar{\alpha}_1 + \bar{\alpha}_2 = S\bar{\beta}_1 + S\bar{\beta}_2$.

*

- Matrix Rep. for a linear transformation :-

Let $V$ be a finite dim vector space.
$T : V \to V$ be a lin. operator
Let $B$ & $B'$ be two bases for $V$
$[T]_B$ & $[T]_{B'}$ are they related? YES!

Theorem:

Let $V$ be a finite dimensional vector space
Let $B = [\bar{\alpha}_1, \dots \bar{\alpha}_n]$ & $B' = [\bar{\beta}_1 \dots \bar{\beta}_n]$ be two different basis for $V$.
If $P = [\bar{P}_1, \bar{P}_2 \dots \bar{P}_n]$ is an $n \times n$ matrix with
$\bar{P}_j = [\bar{\beta}_j]_B$
Let $T : V \to V$ be a lin. operator
$[T]_{B'} = \cancel{P^{-1}[B]P}\ P^{-1}[T]_B P$

Alternatively if $U$ is an invertible operator defined as:

$$U\bar{\alpha}_j = \bar{\beta}_j \qquad j = 1 \dots n.$$

then $$[T]_{B'} = [U]^{-1}_B \; [T]_B \; [U]_B$$

Proof: We have already seen that there exists a unique matrix $P$ such that :-

Part I:

$$P_j = [\bar{\beta}_j]_B$$

& $$[\bar{\alpha}]_B = P[\bar{\alpha}]_{B'} \quad \text{——} \; (1) \quad \forall \; \alpha \in V.$$

we also know that :-

$$[T\bar{\alpha}]_B = [T]_B \; [\bar{\alpha}]_B \quad \text{——} \; (2).$$

(Lin. operator).

(1) in (2).

$$[T\bar{\alpha}]_B = [T]_B \, P \, [\bar{\alpha}]_{B'} \quad \text{——} \; (3)$$

Let $\bar{\gamma} = T\bar{\alpha}$.

then

$$[T\bar{\alpha}]_B \leftarrow [\bar{\gamma}]_B = P[\gamma]_{B'} = P[T\bar{\alpha}]_{B'}. \quad \text{——} \; (4)$$

Using (3) in (4) :-

$$P[T\bar{\alpha}]_{B'} = [T]_B P[\bar{\alpha}]_B.$$

$$[T\bar{\alpha}]_{B'} = P^{-1} \, [T]_B \, P \, [\bar{\alpha}]_B$$

but $$[T\bar{\alpha}]_{B'} = [T]_{B'} \; [\bar{\alpha}]_{B'}$$

using this :-

$$[T]_{B'} [\bar{\alpha}]_{B'} = P^{-1} \, [T]_B \, P[\bar{\alpha}]_B. \quad \text{——} \; (5)$$

Equ (5) holds for all $\bar{\alpha} \in V$. Thus :-

$$[T]_{B'} = P^{-1} \; [T]_B \, P.$$

Part II: If $U\bar{\alpha}_j = \bar{\beta}_j \qquad j = 1, 2 \dots n$

$$= [U]_B.$$

$$= [\,[U\bar{\alpha}_1]_B \dots [U\bar{\alpha}_2]_B \dots\dots [U\alpha_n]_B.$$

$$= [\;[\beta_1]_B \; [\beta_2]_B \; \dots\dots \; [\beta_n]_B\,]_B.$$

$$= [\bar{P}_1 \; \bar{P}_2 \; \dots\dots \; \bar{P}_n] = P$$

$\Rightarrow$ $P$ is matrix rep. of linear op. $U$ with respect to $B$.

$$[U]_B = P$$

$\Rightarrow$ Using eqn. (6) we get.

$$[T]_{B'} = [U^{-1}]_B \, [T]_B \, [U]_B$$

Example:

Let $V = P_3(\mathbb{R})$.

Let $D$ be the differentiation operator

Let $B = \{f_1, f_2, f_3, f_4\}$ $\quad f_i = x^{i-1}$

Let $g_i(x) = (x+t)^{i-1}$.

$g(1) = f_1$.

$g(2) = (x+t) = f_2 + tf_1$.

$g(3) = (x+t)^2 = x^2 + 2xt + t^2$

$= f_3 + \cancel{2f_2} \; 2f_2t + f_1t^2$

$g(4) = (x+t)^3 = x^3 + 3tx^2 + 3t^2x + t^3$.

$= f_4 + 3tf_3 + 3t^2f_2 + t^3f_1$

Show that $B' = \{g_1, g_2, g_3, g_4\}$ forms basis.

$$P = \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 1 & 3t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & -t & t^2 & -t^3 \\ 0 & 1 & -2t & 3t^2 \\ 0 & 0 & 1 & -3t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[D]_{B'} = P^{-1}[D]_B P$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

## COMPOSITION OF LINEAR TRANSFORMATION:

Let $V_1, V_2, V_3$ be finite dim vector spaces over $F$

Let $B = \{\bar{\alpha}_1 \dots \bar{\alpha}_p\}$.

$B' = \{\bar{\beta}_1 \dots \beta_q\}$ and

$B'' = \{\bar{\gamma}_1 \dots \bar{\gamma}_r\}$ be the basis for

$V_1, V_2, V_3$ respectively

Let $T_1 : V_1 \to V_2$.

$T_2 : V_2 \to V_3$. be linear trans.

Let $A = [T_1]_{BB'}$

$B = [T_2]_{B',B''}$

then $[T_2 T_1]_{BB''}$ ?

Pf: because. $A = [T]_{BB'}$ that means...

$[T_1 \bar{\alpha}]_{B'} = A[\bar{\alpha}]_B$ — ①.

Similarly $B = [T_2]_{B'B''}$.

$[T_2 \beta]_{B''} = B[\bar{\beta}]_{B'}$. — ②.

$$[T_2 T_1 \bar{\alpha}]_{B''} = [T_2(T_1 \bar{\alpha})]_{B''}$$

$= B[T_1 \bar{\alpha}]_{B'}$ — ③

(obtained using eqn ② assuming)
$\bar{\beta} = T\bar{\alpha}$

$[T_2 T_1 \alpha]_{B''} = B[T_1 \alpha]_{B'}$.

using ① $= BA[\bar{\alpha}]_B$.

## ISOMORPHISM :

Let $V$ & $W$ be 2 vector spaces over $F$
Any invertible linear transformation from $V$ into $W$ is Isomorphism from $V$ onto $W$.

— $V$ is iso to ~~itself~~ $V$ because id. transf. being an isomorphism of $V$ onto $V$ (reflexive)

— If $V$ is iso to $W$ via an iso $T$ then $W$ is also iso to $V$ via $T$ (transitivity).

— If $T$ is iso to $W$ via $T$ $W$ is iso to $Z$ via $U$ then $V$ is iso to $Z$ via $U$ (symmetric).

→ **Theorem:**

Two fin. dim vector spaces are isomorphic if and only if they have same dimension.

Pf: Let $V$ & $W$ be 2 iso vector spaces
thus there exist $T : V \to W$ which is invertible
Thus

$null(T) = \{0\}$ & $Range(T) = W$.
$\dim(null) = 0$  $\dim(Range) = \dim(W)$.

$\dim V = \dim(null) + \cancel{\dim(T)}\ Range(T)$
$= 0 + \dim W$.
$\dim V = \dim W$.

Now assume $V$ & $W$ have same dim
Let $B = \{\bar{\alpha}_1, \bar{\alpha}_2 \dots \bar{\alpha}_n\}$ be basis of $V$
$B' = \{\bar{\beta}_1, \bar{\beta}_2 \dots \bar{\beta}_n\}$ basis of $W$.

Similar & Eigen matrices

Define a linear map as follows
$T\overline{\alpha_j} = \overline{\beta_j}$ $j = 1 \dots n$.

To show that T is invertible we need to ~~invertible~~ show T is sur & inj.

Let $r_1, r_2 \in V$ ~~the~~ $T(r_1) = T(r_2)$

$r_1 = \sum_{i=1}^{n} a_i \overline{\alpha_i}$

$r_2 = \sum_{i=1}^{n} b_i \overline{\alpha_i}$

~~Then~~ $T(\overline{r_1} - \overline{r_2}) = \overline{0}$.

$\Rightarrow T\left(\sum_{i=1}^{n} (a_i - b_i)\overline{\alpha_i}\right) = \overline{0}$.

$\Rightarrow \sum_{i=1}^{n} (a_i - b_i)(T\alpha_i) = \overline{0}$.

$\Rightarrow \sum_{i=1}^{n} (a_i - b_i)\overline{\beta_i} = 0$.

$a_i - b_i = 0$ $i = 1 \dots n$.
because $\{\beta_1, \dots \beta_n\}$ basis of W.
$\Rightarrow a_i = b_i$ $i = 1 \dots n$.
$\overline{r_1} = \overline{r_2}$.

HW Show surjective.

Let $\{\beta_1, \beta_2 \dots \beta_n\}$ be basis of W
Let $y \in W$ $y = c_1\beta_1 + c_2\beta_2 + \dots + c_n\beta_n$.
Also, $y = T(x)$ $x \in V$.
$\therefore T(x) = c_1\beta_1 + c_2\beta_2 + \dots + c_n\beta_n$
$= c_1T(\alpha_1) + c_2T(\alpha_2) + \dots + c_nT(\alpha_n)$

$PBP^{-1} = A$ Eigen value

$\therefore \quad PBP^{-1} X = \lambda_x X \qquad B(P^{-1}x) = \lambda_x (P^{-1}x)$

same char eqns.

## SIMILAR MATRICES :-

Let A & B be two $m \times n$ matrices over $\mathbb{F}$. We say that A is similar to B, if there exists an inv. matrix P such that

$$B = P^{-1} A P.$$

Eg: Let B & B' be the basis for V. Let $T : V \to V$ be a lin. operator on V then

$$[T]_{B'} = P^{-1} [T]_B P$$

where P is such that

$$\overline{P_j} = [\beta_j]_B$$

Eg: Let B is similar to A $\exists$ P such that

$$B = P^{-1} A P.$$

Let C is similar to B $\exists$ Q ST

$$C = Q^{-1} B Q.$$

Sub :-
$$C = Q^{-1} P^{-1} A P Q.$$
$$= (PQ)^{-1} A PQ.$$

$\Rightarrow$ This implies C is similar to A.

## SYSTEM OF LINEAR EQUATIONS :-

Let $(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$

$$A_{11} x_1 + A_{12} x_2 + \ldots + A_{1n} x_n = y_1$$
$$A_{21} x_1 + A_{22} x_2 + \ldots + A_{2n} x_n = y_2$$
$$\vdots$$
$$A_{m1} x_1 + A_{m2} x_2 + \ldots + A_{mn} x_n = y_m$$

This can be written as :-

$$A\overline{X} = \overline{Y}$$

where $A \in \mathbb{F}^{m \times n}$

$x \in F^n \qquad \overline{Y} \in \mathbb{F}^m$

$A^2 = -I$

$AA = -I$

$A^{-1} = -A$

$A^2 = -I$ $\quad -A^{-1}$

$AA = -I$

$A = A^{-1}(-I)$



Goal: To find a tuple $(x_1, x_2, \dots, x_n) \in \mathbb{F}^n$ which satisfies all the $m$ equations simultaneously

Eg:

$2x_1 - x_2 + x_3 = 0 \quad — ①$

$x_1 + 3x_2 + 4x_3 = 0 \quad — ②$

eqn ① − 2 eqn ②:-

$-7x_2 - 7x_3 = 0. \Rightarrow x_2 = -x_3.$

Also $7x_1 + 7x_3 = 0$

$x_1 = -x_3.$

$\therefore (-x_3, -x_3, x_3)$

- ELEMENTARY ROW TRANSF.

(a) Multiplication of $r^{th}$ row by a const. $c \in \mathbb{F}$.

$$E_1 \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{r-1} \\ z_r \\ \vdots \\ z_m \end{bmatrix} \longrightarrow \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{r-1} \\ c z_r \\ \vdots \\ z_m \end{bmatrix}$$

$$[E_1]_B = \begin{array}{c} \\ \\ r-1 \rightarrow \\ r \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & & 0 & 0 & & 0 \\ 0 & 0 & & 1 & 0 & & 0 \\ 0 & 0 & & 0 & c & & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$E_1^{i,j} = \begin{cases} \delta_{ij}; & i \neq r. \\ c\,\delta_{ij} & i = r. \end{cases} \qquad i, j \in \{1, 2, \dots, m\}$$

b) Replacing $r^{th}$ row by row $r$ + $c$. row $s$.

$$E_2 \quad \begin{bmatrix} z_1 \\ \vdots \\ z_r \\ z_s \\ \vdots \\ z_m \end{bmatrix} \longrightarrow \begin{bmatrix} z_1 \\ \vdots \\ z_r + c z_s \\ z_s \\ \vdots \\ z_m \end{bmatrix}$$

$$[E_2]_B = \begin{array}{c} \\ \\ \\ r \rightarrow \\ s \rightarrow \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccc} & & & r^{th} & & s^{th} & & \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ 0 & 1 & & 0 & \cdots & 0 & \cdots \\ 0 & 0 & & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & \cdots & c & \cdots \\ 0 & 0 & & 0 & & 1 & \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \end{bmatrix} \end{array}$$

c). Interchanging row $r$ & $s$.

$$E_3 \quad \begin{bmatrix} z_1 \\ \vdots \\ z_r \\ z_s \\ \vdots \\ z_m \end{bmatrix} \longrightarrow \begin{bmatrix} z_1 \\ \vdots \\ z_s \\ z_r \\ \vdots \\ z_m \end{bmatrix}$$

$$[E_3]_B = \begin{array}{c} \\ \\ \\ \\ r \rightarrow \\ s \rightarrow \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccc} & & & r^{th} & & s^{th} & & \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ 0 & 1 & & 0 & & \vdots & \\ 0 & 0 & & 0 & & \vdots & \\ \vdots & \vdots & & \vdots & & 0 & \\ 0 & \vdots & & 0 & \cdots & 1 & \\ 0 & 0 & & 1 & & 0 & \cdots \\ \vdots & 0 & & 0 & & \vdots & \\ 0 & 0 & & 0 & \cdots & 0 & \cdots \end{bmatrix} \end{array}$$

→ Row equivalence of 2 system of linear eqns
Start with : $A\bar{x} = \bar{y}$
after applying a seq. of el. row op. we get $B\bar{x} = \bar{z}$

Homogenous system of L.E.

Th. If $A$ & $B$ are row equivalent ~~m×n~~ m×n matrices
$A\bar{X} = 0$ & $B\bar{X} = 0$ will have the same solutions.

Pf: $A = A_0 \rightarrow A_1 \rightarrow A_2$
$B = A_k$.

To show that $A\bar{X} = \bar{0}$ & $B\bar{X} = \bar{0}$ have the same solutions, it is sufficient to show that any single elementary row op. does not change the solution.

- Let us achieve $B$ by a single el. row op. on $A$.

Case 1: Row interchange
$B = E_3 A$
Because it just changes the order in which the eqns. appear thus soln of $A\bar{X} = \bar{0}$ & $B\bar{X} = \bar{0}$ will be same.

Case 2: Multiplying an eqn of $A\bar{X} = \bar{0}$ by $c \in \mathbb{F}$.
$B = E_1 A$.
$B\bar{X} = 0$.
$E_1 A\bar{X} = 0$ $\quad E_1$ is invertible.
$A_1 \bar{Y} = E_1^{-1} 0$.
$A_1 \bar{X} = \bar{0}$.

$\Rightarrow \{\bar{X} \in \mathbb{R}^n \mid B\bar{X} = 0\} \subseteq \{\bar{X} \in \mathbb{R}^n \mid A\bar{X} = 0\}$

Now because $E_1$ is invertible.
$A = E_1^{-1} B$.
$A\bar{X} = \bar{0}$

when

$$\Rightarrow E_1^{-1} BX = \bar{0}$$

$$\Rightarrow BX = E_1 \bar{0}$$

$$B\bar{x} = 0$$

$$\{ \bar{x} \mid A\bar{x} = 0 \} \subseteq \{ \bar{x} \mid B\bar{x} = 0 \}.$$

$$\{ \bar{x} \mid A\bar{x} = 0 \} = \{ \bar{x} \mid B\bar{x} = 0 \}.$$

<u>Case 3:</u> Replace $r^{th}$ row by row $r$ plus $c$ times rows.

$$B = E_2 A.$$

Same procedure.

<u>Example:</u> $A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & \cancel{4}5 \end{bmatrix}$

$R_1 \to R_1 - 2R_2$. $\begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}$

$R_3 = R_3 - 2R_2$. $\begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & -2 & -1 & \cancel{4}7 \end{bmatrix}$

$R_3 \times -1/2$ = $\begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & -1 & -1/2 & 7/2 \end{bmatrix}$

$$\xrightarrow{R_2 - 4R_3} \xrightarrow{R_1 + 9R_3} \xrightarrow{R_1 \times 2/15} \xrightarrow{R_3 + 2R_1} \xrightarrow{R_3 - 1/2 R_1}$$

$$\begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix}.$$

$$B\bar{X} = \bar{O}$$

Let :- $x_4 = c \in \mathbb{R}.$

then

$$\bar{X} = \begin{bmatrix} 17/3\, c \\ -5/3\, c \\ -11/3\, c \\ c \end{bmatrix}$$

→ $\boxed{\text{TUTORIAL :}}$

When :- $L(X)_{n\times 1} = \lambda x_{n\times 1}.$

$\lambda$ :- Eigan matrix. $X \to$ Eigan vector.

$$LX - \lambda X = 0.$$

$$(L - \lambda I) = 0.$$

$$\det(M - \lambda I) = 0$$

Eg :- $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad (\lambda-2)^2 - 1 = 0$

$$\lambda^2 - 4\lambda + 4 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

Q: Using Gaussian Elimination :-

$$2x_2 + x_3 = -8$$

$$x_1 - 2x_2 - 3x_3 = 0$$

$$-x_1 + x_2 + 2x_3 = 3$$

$(15, 11, -14)$

$$\begin{bmatrix} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{bmatrix}$$

$R_2 \to R_2 + R_1$ $\begin{bmatrix} 0 & 2 & 1 & -8 \\ 1 & 0 & -2 & -8 \\ -1 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_3} \begin{bmatrix} 0 & 2 & 1 & 8 \\ 0 & 1 & 0 & -5 \\ -1 & 1 & 2 & 3 \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 - 2R_2}$ $\begin{bmatrix} 0 & 0 & 1 & -14 \\ 0 & 1 & 0 & 11 \\ 1 & -1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -14 \\ 0 & 1 & 0 & 11 \\ 1 & 0 & 0 & 15 \end{bmatrix}$

Complex Eigen values.

- Find Eigan value of A :-

$$A^2 = -I$$

$$Ax = \lambda x$$

$$A^2 x = \lambda(Ax)$$

$$-x = \lambda(\lambda x)$$

$$\lambda^2 = -1$$ Imaginary values

- Prove that if $A \in \mathbb{R}^{n \times n}$, $n$ is even

$$\det(A^2) = \det(-I)$$

$$(\det A)^2 = \det(-1) = (-1)^n$$

- For Solution of ~~the~~ Linear Eqns:

If $AX = b$

&

$$\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= x \begin{bmatrix} 2 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

∴ LHS is a linear combination of the columns of A. That is B lies in the column space of A.

→ There exists one unique solution iff columns of A are linearly independent.

→ Solⁿ doesn't exist iff $\bar{b} \notin col(A)$.

→ ∞ sol. if columns of A are linearly dependent

→ For $A\bar{x} = \bar{B}$. soln exists if rank of A is equal to rank of $[A \ \bar{B}]$

Note: Matrix B is ~~called~~ row equivalent to matrix A if $B = E_k E_{k-1} \ldots E_1(A)$

Defn: Row reduced matrix:

An $m \times n$ matrix R is called row reduced if

a) the first non-zero entry in each non-zero row is equal to 1.

b) Each column of R which contains the leading non-zero entry of some row has all its other entries zero.

Eg: Identity:-

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & & & 1 \end{bmatrix}$$

Non example:-

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

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**Defn**: Row Reduced Echelon matrix :-
An $m \times n$ matrix $R$ is called row reduced echelon if

1). $R$ is row reduced.
2). Every row of $R$ which has all its entries 0 occurs below every row which has a non-zero entry
3). If rows $1, 2, \ldots, r$ are the non-zero rows of $R$, & if the leading non-zero entry of row $i$ occurs at col. $k$ then $k_1 < k_2 < \ldots < k_r$.

## SOLUTION OF SYSTEM OF HOM. LINEAR EQN.

$A\bar{X} = \bar{O}$.

→ By applying a seq of el. row operations, one can convert matrix $A$ to a row reduced echelon matrix $R$. Thus $R$ is row equivalent to $A$

→ We have shown that $R\bar{X} = \bar{O}$ will have same set of sol. as $A\bar{X} = \bar{O}$.

→ Now assume that mat $R$ has $r$ non-zero rows & $m - r$ zero rows.

→ First leading entry of $i^{th}$ non-zero row happens at col. $k_i$.

→ Thus the unknown variable $x_{k_i}$ will appear only in the $i^{th}$ eqn

→ Let $u_1 \dots u_{n-r}$ denote the other $n-r$ variables. these are different from $x_{k_1}, x_{k_2}, \dots x_{k_r}$

The $r$ non-zero eqns of $R\bar{X} = 0$ will be of the following form

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$x_{kr} + \sum_{j=1}^{n-r} C_{rj} u_j = 0 .$$

Example:

$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \bar{0}$$

$$x_2 - 3x_3 + 1/2\, x_5 = 0 .$$
$$x_4 + 2x_5 = 0 .$$

soln of $R\bar{x} = \bar{0}$ will have the form :-

Let $x_1 = a$, $x_3 = b$, $x_5 = c$.

$(a, 3b - 1/2c, b, -2c, c)$.

▲ SOLUTION OF NON-HOM SYS. OF EQNS :-

$A\bar{X} = \bar{Y}$  $\bar{Y} \neq \bar{0}$.

Eg: $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 1 & y_1 \\ 2 & 1 & 1 & y_2 \\ 0 & 5 & -1 & y_3 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & y_2 - 2y_1 \\ 0 & 5 & -1 & y_3 \end{bmatrix}$$

must & should

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & (y_2 - 2y_1) \\ 0 & 0 & 0 & y_3 - y_2 + 2y_1 \end{bmatrix} \xrightarrow{R_2 \times \frac{1}{5}}$$

Matrix C

$$\begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 1 & -1/5 & (y_2 - 2y_1)/5 \\ 0 & 0 & 0 & y_3 - y_2 + 2y_1 \end{bmatrix} \xrightarrow{R_1 + 2R_2}$$

$$\begin{bmatrix} 1 & 0 & 3/5 & y_1 + \frac{2}{5}(y_2 - 2y_1) \\ 0 & 1 & -1/5 & (y_2 - 2y_1)/5 \\ 0 & 0 & 0 & y_3 - y_2 + 2y_1 \end{bmatrix}$$

For the solution to exist $y_3 - y_2 + 2y_1 = 0$

**Th.** Consider a system of linear eqns $A\bar{X} = \bar{b}$ where $A \in \mathbb{R}^{m \times n}$ $X \in \mathbb{R}^n$ $b \in \mathbb{R}^m$

let $r = \text{rank}(A)$ &

$r_a = \text{rank}([A \ \bar{b}])$

Then exactly one of the following statements will hold.

(1) (i). If $r = r_a < n$ the set of solutions is is an infinite set as follows.

$$\{ \bar{u_0} + a_1\bar{u_1} + \dots + a_{n-r}\bar{u}_{n-r} : a_i \in \mathbb{R}\ i = 1 \dots n-r \}$$

where $\bar{u_0}, \bar{u_1} \dots u_{n-r} \in \mathbb{R}^{n \times 1}$.

s.t. $A\bar{u_0} = \bar{b}$ & $A\bar{u_i} = 0$ $i = 1 \dots n-r$.

(2). If $r = r_a = n$ unique sol. $\bar{x}_0$ will exist such that $A\bar{x}_0 = \bar{b}$

(3). If $r < r_a$ then no solution exists

[A b].

~~Let [C d] be the corresponding row reduced echelon matrix~~

Let C be the row reduced echelon matr corr to A.

Thus the resulting sys of eqns will be

$$C\bar{x} = \bar{d}.$$

- in C there are r non-zero rows & remaining m − r are zero rows.

- Let $k_i$ be the index of first non-zero entry in the ith row of C

- $x_{k_1}, \ldots, x_{k_r}$ be column indices corr to leading non-zero entries of C.

- The $C\bar{x} = \bar{d}$ will have following form

$$x_{k_1} + \sum_{j=1}^{n-r} c_{ij} v_j = d$$

$\vdots$

$x_{k_r}$

The remaining m−r are the free variables.

$\therefore$ $C\bar{x} = \bar{d}$ will have the form:

$$x_{k_1} + \sum_{j=1}^{m-r} c_{ij} z_j = d_1$$

$$x_{k_2} + \sum_{j=1}^{m-r} c_{ij} z_j = d_2$$

$\vdots$

$$x_{k_r} + \sum_{j=1}^{m-r} c_{rj} z_j = d_r$$

→ If $r(A) = r([A \ b]) = n$
There will be unique solutions.
& $d_{r+1}$ will be $= 0$.

→ If $z_1 = z_2 = \dots z_{n-r} = 0$
then ~~$x_{k1} = x_{k2} = \dots$~~

$$\left.\begin{aligned} x_{k1} &= d_1 \\ x_{k2} &= d_2 \\ &\vdots \\ x_{kr} &= d_r \end{aligned}\right\} \quad \overline{u_0} = \begin{bmatrix} x_{k1} \\ x_{k2} \\ \vdots \\ x_{kr} \end{bmatrix}$$

ie: $C\overline{u_0} = \overline{d}$

Hence $A\overline{u_0} = \overline{b}$ [∵ row equivalent].

## • EIGEN VALUES & VECTORS:

Eigen values → singular / characteristic values.

Let $T$ be some lin. operator on $V$. (over $\mathbb{F}$).
A scalar $\lambda \in \mathbb{F}$ is said to be an Eigen value of $T$ if

$\exists$ some non-zero vector $\overline{v}$ ST:-

$$T\overline{v} = \lambda \overline{v}.$$

And any vector $v \in V$ which satisfies $T\overline{v} = \lambda\overline{v}$ is said to be an Eigen vector corresponding to $\lambda$.

Suppose $\lambda$ is an Eigen value of $T$: then
$S = \{ \overline{v} \in V : T\overline{v} = \lambda \overline{v} \}$ is a subspace of V

Pf: Let $\overline{v_1}$, $\overline{v_2}$ $\in S$.

$T(\overline{v_1}) = \lambda v_1$ $\quad$ $T(\overline{v_2}) = \lambda v_2$.

$cT(\overline{v_1}) = c\lambda(\overline{v_1})$

$\therefore T(c\overline{v_1} + \overline{v_2}) = c.\lambda\overline{v_1} + \lambda\overline{v_2}$

NOTE :- If $\lambda$ is eigen value then $T\bar{v} = \lambda \bar{v}$ (for atleast One $\bar{x} \neq \bar{0}$)

ie: $(T - \lambda I)\bar{v} = \bar{0}$

$U\bar{v} = \bar{0}$

$\lambda(\underbrace{c\bar{v}_1 + \bar{v}_2}) \in S$ $\therefore$ S is a subspace.

$\therefore$, This subspace is called the Eigen space corresponding to $\lambda$.

★ For eigen spaces, they are T-invariant.

ie:- For any $\bar{v} \in S$ $T\bar{v} \in S$

$\because$ scaling $\bar{v}$ by any constant $\in S$.

$\Rightarrow$ block diagonal rep possible.

Eg: ~~Consider~~ $\cancel{A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}$ ~~from~~ $\cancel{\mathbb{R}^{2\times1} \to \mathbb{R}^{2\times1}}$

defined as $T(\bar{x})$

NOTE: If $\lambda$ is eigen val. then $T\bar{v} = \lambda\bar{v}$ for at least one $\bar{v} \neq 0$

ie: $(T - \lambda I)\bar{v} = 0$

$U\bar{v} = 0$. $T-\lambda I$.

Eigen space of T corresponding to $\lambda$

$=$ N.S $(T - \lambda I)$.

If V is Finite dim (n) in standard basis we can write

$([T]_B - \lambda [I]_B)[v]_B = 0$.

$[U]_B [\bar{v}]_B = 0$

$\downarrow$ at least one co.. $\neq 0$.

since $[\bar{v}]_B \neq 0$

$\det([U]_B) = 0$

Eg: Consider $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $T : \mathbb{R}^{2\times1} \to \mathbb{R}^{2\times1}$

defined as $T(\bar{x}) = \underset{2\times2}{A}\ \underset{2\times1}{\bar{x}}$.

Finding Eigen vals:-

$\det(T - \lambda I)\cancel{x} = 0$ ← characteristic

$$\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0.$$

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0.$$

$$(2-\lambda)^2 - 1 = 0$$

$$4 + \lambda^2 - 4\lambda - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0.$$

$$\lambda^2 - 3\lambda - \lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

→ $\lambda = 3$, $\lambda = 1$

Take $\lambda = 1$.

For any $\bar{v}$ to be eigen vector.

$$A\bar{v} = \lambda\bar{v} = \bar{v}.$$

$$\underset{2\times2}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\begin{bmatrix} x + y \\ x + y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} x \\ -x \end{bmatrix}$

$W_1$ ∴, N.S of $((A-I)) = \{ (x, -x) \quad x \in \mathbb{R} \}$.

∴, T invariant [any $\bar{v}$ $T\bar{v} = \bar{v}$]

$\lambda = 3$ :

$$(A - \lambda I) \begin{bmatrix} x \\ y \end{bmatrix} = \left( \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\mathbb{R}^{2\times 1}$

$W_2$ : NS. $(A - 3I) = \{ (x, x), \quad x \in \mathbb{R} \}$.
$\begin{bmatrix} x \\ x \end{bmatrix}$

$T \begin{bmatrix} x \\ x \end{bmatrix} = 3 \begin{bmatrix} x \\ x \end{bmatrix}$

Claim $V = W_1 \oplus W_2 = \mathbb{R}^{2\times 1}$

dim $(W_1) = 1$ dim $(W_2) = 1$.

dim $(W_1 \oplus W_2) = 2$ [ ∴, $W_1 \oplus W_2 = V$.]

Diagonal matrix as a rep for a L.T is the best

If ∃ $D = P^{-1}AP$ then the L.T is diagonizable

$$[T]_B = \begin{bmatrix} a_1 & O \\ O & a_n \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x + y = 0$

$x = -y$

1

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 7 \\ 1 & 2 & 5 \end{bmatrix}_{3\times 3} \xrightarrow{R_1 \to R_1 - R_2} \quad R_3 \to R_3 - R_1$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 7 \\ 0 & 0 & -2 \end{bmatrix} \quad R_2 \to R_2 - R_1 \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 8 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_2 \to R_2 + 4R_2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad R_1 \to R_1$$

x. ——— x

LA Basic Q:

1) Vectors.

2). Scalars.

3) The sets $Z$ - integers $Q$ - Rational no:s $R$ :: real nos

$C$ complex integers, $F_p$ int modulo $p$ $p \to$ prime.

| | Add | | Mult | Non-example |
|---|---|---|---|---|
| $Z$: | $1 \in Z$ | $-1 \in Z$ | $1 \in Z$ | $1 \in Z$ |
| $Q$: | $1 \in Q$ | $-1 \in Q$ | $2 \in Q$ | $1/2 \in Q$ |
| $R$ | $1 \in Q$ | $-1 \in Q$ | $\sqrt{3} \in R$ | $1/\sqrt{3} \in R$ |
| $C$ | $i \in C$ | $-i \in C$ | $i \in C$ | $-i \in C$ |
| $F_p$ | when $p = 5$: $1 \in F_p$ | $4 \in F_p$ | $4 \in F_p$ | $4 \in F_p$ |

$\{0, 1, 2, 3, 4\}$

$\frac{\pi}{7}$

4). $Q$ is a field as all the properties of V.S

ie:- if $\bar{\alpha}, \bar{\beta} \in Q$.

$\bar{\alpha} + \bar{\beta} \in Q$ $\quad \bar{\alpha} \cdot \bar{\beta} \in Q$.

if $\bar{\alpha} \in Q$, $\frac{1}{\bar{\alpha}} \in Q$ & $-\bar{\alpha} \in Q$.

But for $Z$ field

eg: $2 \in Z$ but $\frac{1}{2} \notin Z$

$\therefore$ Not a field

Fermat's th. $a^{p-1} \equiv 1 \bmod p$.

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5). Arbitrary element :- $[a_1, a_2 \ldots\ldots a_n]$ $a_i \in \mathbb{F}^n_{\cancel{?}}$.

Example Element :- $[1, 1, 1, 1, 1 \ldots\ldots 1]$

6) $m$ vectors in $V$, $n$ vectors in $W$.

$\therefore$ in $V + W$ :- $mn$ vectors since

for each vector $v \in V$, there are ~~n~~ $n$

vectors to choose from in $W$.

7). $S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R} : x_1 = 0, x_2 + 2x_3 + 3x_4 = 0\}$

Arbitrary vectors :-

$(0, 2x_3 + 3x_4, x_3, x_4)$

Eg:

$(0, 5, 1, 1)$.

Subspace proof:-

~~$cx_1 + cx_2 + cx_3 +$~~ $(x_1, x_2, x_3, x_4)$, $(y_1, y_2, y_3, y_4)$

II. 7

$B \to$ basis for $V$.

T.P. $B'$ is basis for $V$

iff: $B'$ spans $B$ & ~~is~~ $B'$ is L. I.

$B = \{\alpha_1, \alpha_2 \ldots\ldots \alpha_n\}$

$B' = \{\beta_1, \beta_2 \ldots\ldots \beta_n\}$

For any $\alpha \in V$

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots + c_n\alpha_n$

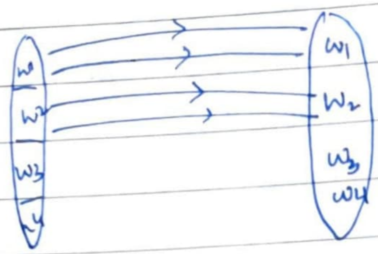
$\alpha = c_1[d_1\beta_1 + \ldots d_n\beta_n] + c_2[d_1\beta_1 \ldots + d_n\beta_n] + \ldots$

→ Sparse representation

Rep in the form of block diagonal matrix :-

$$[T]_B = \left[\begin{array}{c|c|c} \eta & & \\ \hline & \eta & \\ \hline & & \eta \end{array}\right]$$

where $\dim(W_i) \times \dim(W_i$ is the size of each el in diagonal.




! Eigen spaces are T-invariant

- Let $V$ (fdvs) $T : V \to V$
  $n$ $M = [T]_B$ .
  $M$ $n \times 2$.

If $\lambda$ is an eigen value then $[T - \lambda I]_B$
$= (M - \lambda I)$ should be 'non-singular'

ie :

$\Rightarrow$ cols. of $M - \lambda I$ should be lin. dependent.

$\Rightarrow$ rank $(M - \lambda I) < n$

$\Rightarrow$ $\det(M - \lambda I) = 0$

If $\lambda$ is some scalar satisfying $\det(M - \lambda I) = 0$
then $\lambda$ is an eigen value of $M$
& $[T]_B = M$

$$(M - \lambda I)[v]_B = 0$$

If there is at least one non-zero vector satisfying this then that is the eigen vector.

Let $W_i$ be the eigen space associated with eigen value $(\lambda_i)$

Let $V = \bigoplus_{i=1} W_i$ $\quad$ $w_i$ = subspace $(\lambda_i)$.

had $\lambda_i$ not been unique.

$\rightarrow W_i$

$$[T]_B = \left[\begin{array}{ccc|cc} \lambda_1 & & & & \\ 0 & \lambda_1 & & & \\ 0 & & \lambda_1 & & \\ \hline 0 & & & \lambda_2 & 0 \\ 0 & & & 0 & \lambda_2 \end{array}\right]$$

$\Downarrow$

$[T w_i]_B$.

$[\lambda_1 w_1]_B$

where $w_{11}, w_{12}, w_{13} \in W_1$ $\quad$ [basis] $\rightarrow$ 3D

$w_{21}, w_{22} \in W_2$

Eg: $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$F^{2\times 1} \rightarrow F^{2\times 1}$

$Mx \mapsto M\bar{x}$ $\quad$ $\lambda_1 = 3, \lambda_2 = 1$

Eigen vectors :-

$\lambda = 3 \rightarrow W_1 = \{(x, x) : x \in \mathbb{R}\}$

$\lambda = 1 \rightarrow W_2 = \{(x, -x) \quad x \in \mathbb{R}\}$

$V = W_1 \oplus W_2$.

Picking basis :-

$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$[T]_B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

(1). Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ representing lin transf.

from $\mathbb{R}^{2\times1} \to \mathbb{R}^{2\times1}$

a) Find Eigen values for A. Does A have diag rep

b) If $A : \mathbb{C}^{2\times1} \to \mathbb{C}^{2\times1}$ " ?

(2). Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ over $L(\mathbb{R})$. Check the same.

A: (1) a) $(T - \lambda I)\bar{v} = 0.$

$$\left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0.$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1 = 0 \qquad \lambda^2 = 1 \qquad \lambda = \pm 1$$

$\therefore$ Eigenvalues $= +1, -1$

For $\lambda = 1$ :-

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -x + y \\ x - y \end{bmatrix} = 0. \quad \Rightarrow x = y$$

$$\lambda = -1 : \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x + y \end{bmatrix} = 0 \qquad x = -y$$

$\therefore$ $W_1 = \{ \begin{bmatrix} x \\ x \end{bmatrix} \; x \in \mathbb{R} \}.$

$W_2 = \{ \begin{bmatrix} x \\ -x \end{bmatrix} \; x \in \mathbb{R} \}.$

$\therefore$ $B = \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}(-1)$

$\lambda^2 - 1 = 0$

$\therefore$ YES diagonal rep :-

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

b) Complex no: still same.

(2) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$|T - \lambda I| = 0.$

$\left| \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0.$

$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0.$

$(1-\lambda)^2 = 0.$

$1 - \lambda = 0 \qquad \lambda = 1.$

~~$A\lambda x$~~ .

$\left[\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix} \leftarrow$

Dimension's dont add up
Hence NO! cannot have a diagonal rep.
with the eigen spaces, it is not diagonizable

Q: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ under $\mathbb{R}, \mathbb{C}$.

$\det(A - \lambda I) = 0. \rightarrow$ characteristic equ.

ie: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = 0$

$\lambda^2 + 1 = 0 \qquad \lambda^2 = -1 \qquad \lambda^2 = i^2 \qquad \lambda = \pm i$

Under $\mathbb{R}$, it doesn't have any eigen values.

Under $\mathbb{C}$ $\lambda = \pm i$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

$$\begin{bmatrix} -xi + y \\ -x - iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$-xi + y = 0$ $\quad$ $x = 0,\ y = 0.$ , $x = 1,\ y = i$

$-x - iy = 0$ $\quad$ $x = -i,\ y = 1$

$W_i = \text{span}\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right).$

~~$W_2 = \text{span}\left(\begin{bmatrix} i \\ 1 \end{bmatrix}\right)$~~

$$\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} xi + y \\ x + iy \end{bmatrix} = 0$$

$W_2 = \text{span}\left(\begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$

Using dimensionality arguments,
$\dim(W_1) = 1 = \dim(W_2)$ & say diagonalizable matrix exists.

[Q:] When can we express a linear operator $T: V \to V$ using a diagonal matrix?

Theorem: $T: V \to V$ (fdvs)
is representable using a diagonal matrix
iff $\exists\, W_i$ $i = 1\ldots r$, ST $\bigoplus_{i=1}^{n} W_i = V$

where $W_i$ is the eigenspace of $T$ $\forall i$

$$[T]_B = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_r \end{bmatrix}.$$

where $B$ is union of basis of all $W_i$'s.

Pf: Given: $T \rightarrow$ diagonalizable.

To prove: $\exists W_i$ (eigen spaces)

ST: $\bigoplus W_i = V$

$\exists$ some basis $B = \{ \bar{v}_{11}, \ldots \underbrace{\ldots\ldots}\, v_{1l_1},$

$\bar{v}_{21}, \ldots\ldots v_{2l_2}$

$\vdots$

$v_{r1} \ldots\ldots v_{rl_r} \}$.

ST:- $$[T]_B = \begin{bmatrix} \overbrace{a_1 \ldots a_1}^{l_1} & & \\ & \ddots & \\ & & \overbrace{a_r \ldots a_r}^{l_r} \end{bmatrix}.$$

Taking a particular vector.

$$[\bar{v}_{11}]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$[T]_B [\bar{v}_{11} B] = [T]_B \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [T\bar{v}_{11}]_B.$$

$$= \begin{bmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [a_1 \bar{v}_{11}]_B$$

$$\Rightarrow T\bar{v}_{11} = a_1 \bar{v}_{11}.$$

$\Rightarrow a_1$ is an eigen value & $v_{11}$ is one of its eigen vectors ($\because \bar{v}_{11} \neq 0$)

Generalizing further:

$$T v_{ij} = a_i v_{ij} \qquad \forall\ i = 1 \dots r$$
$$j = 1 \dots l_i$$

$$W_i = \text{span}\left(\{ v_{ij} : j = 1 \dots l_i \}\right)$$

Clearly $W_i$ is the eigen space of $T$ corresponding to $a_i$

Further

$$W_i \cap W_{i'} = \{0\} \quad \forall\ i, i' \quad i \neq i'$$

since the basis $B$ contains independent vectors & $W_i$, $W_{i'}$ are spans of non-intersecting subsets of $B$.

also $\displaystyle\bigcup_{i=1}^{r} \{ v_{ij} : j = 1 \dots l_i \} = B$

Hence $\displaystyle\bigoplus_{i=1}^{r} W_i = V$

Corollary: If a linear operator $T$ has a diagonal representation, then the basis corresponding to which $T$ is diagonalized consists of eigen vectors & the diagonal entries are eigen values.

- Let $T : V \to V$.

$$[T]_{B_1} = [P^{-1}]_{B_2}^{B_1} \ [T]_{B_2} \ [P]_{B_1}^{B_2}$$

where $[T]_{B_1}$, $[T]_{B_2}$ are similar matrices

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(diff rep of same operator)
$\Updownarrow$

Theorem: Any two similar matrices have the same eigen values

Pf: Let A & C be two similar matrices

[To check if 2 matrices have the same eigen values, solns of the characteristic eqns must be the same]

ie: $\det(A - \lambda I) = 0$

$\because$ $A \sim C$, $A = PCP^{-1}$

$\therefore$ $\det(PCP^{-1} - \lambda I) = 0$

$\Rightarrow$ $\det(P(C - \lambda I)P^{-1}) = 0$

$\Rightarrow$ $\det(P) \times \det(C - \lambda I) \times \det(P^{-1}) = 0$

But $\det(P) \neq 0$ & $\det(P^{-1}) \neq 0$

$\Rightarrow$ $\det(C - \lambda I) = 0$

$\Rightarrow$ If $\lambda$ is a root of characteristic eqn of A then it is a root of char. eqn of C (& vice versa) $\Rightarrow$ Eigen values of A = Eigen val of C.

Corrollary: Suppose $T: V \to V$ is a lin. operator on V. For any 2 basis $B_1, B_2$, $[T]_{B_1}$ & $[T]_{B_2}$ have the same Eigen values & the eigen vectors of T are unique representation

Pf: Eigen vector :-
suppose $[\bar{v}]_{B_1}$ is an eigen vector of $[T]_{B_1}$ corr. to $\lambda_1$. $[\bar{v}]_{B_2}$ is an eigen vector of $[T]_{B_2}$

Now: $[T]_{B_1}[v_1]_{B_1} = \lambda_1 [v_1]_{B_1}$.

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$$[T]_{B_1} [\bar{v}_1]_{B_1} = \lambda_1 [\bar{v}_1]_{B_1}.$$

$$\downarrow$$

$$[P]_{B_2}^{B_1} [T]_{B_2} \cancel{[P^{-1}]_{B_1}^{B_2} [P]_{B_2}^{B_1}} [v_2]_{B_2} = \lambda_1 [P]_{B_2}^{B_1} [v_1]_{B_2}$$

Pre-multiply by $[P^{-1}]_{B_1}^{B_2}$

$$\therefore \quad [T]_{B_2} = \lambda_1 [v_1]_{B_2}.$$

<u>Theorem</u>: Let $\lambda_1, \dots, \lambda_r$ be distinct eigen values of an operator $T: V \to V$. Let $\underline{v_1}, \dots, \underline{v_r}$ etc be corresponding eigen vectors.
then $\{\bar{v}_1 \dots \bar{v}_r\}$ are linearly independent

★ Corr: If $T$ has $\dim(V)$ distinct eigen values then $T$ is diagonalizable.

<u>Pf</u>: Let if possible $\{\bar{v}_1, \dots \bar{v}_r\}$ are dependent
let $k$ be the first positive integer ST:

$$v_k = \sum_{i=1}^{k-1} a_i \bar{v}_i \qquad (2 \leq k \leq r) \quad - (1)$$

Applying $T$:-

$$T\bar{v}_k = \sum_{i=1}^{k-1} a_i T \bar{v}_i$$

$$\lambda_k \bar{v}_k = \sum_{i=1}^{k-1} a_i \lambda_i \bar{v}_i \qquad - (2)$$

(2) - $\lambda_k$(1) :-

$$\lambda_k \bar{v}_k = \sum_{i=1}^{k-1} a_i \lambda_i \bar{v}_i$$

$$\lambda_k v_k = \sum_{i=1}^{k-1} a_i v_i \lambda_k$$

$$0 = \sum_{i=1}^{k-1} a_i (\lambda_i - \lambda_k) v_i$$

But $\{\bar{v}_1, \bar{v}_2 \dots v_{k-1}\}$ are linearly independent.

$\therefore \quad a_i(\lambda_i - \lambda_k) = 0.$

but $\lambda_i \neq \lambda_k \quad \because$ distinct.

$\therefore \quad a_i = 0$

Sub in ① :-

$\bar{v}_k = 0$ # contradiction

$\because \quad \bar{v}_k$ is eigen vector.

$\therefore$ Not possible ; must be lin indep.

- Linear operator over fdvs over $\mathbb{C}$ :-

Th. Let $p(x)$ is a polyⁿ with complex co-efficient

$p(x) = p_0 + p_1 x + \dots + p_n x^n$

then $p(x)$ has a root in $\mathbb{C}$

$(\exists \ a \in \mathbb{C}$ ST. $p(a) = 0)$.

Corollary :- Any $p(x) \in \mathbb{C}[x]$ (set of all polyⁿ over $\mathbb{C}$) has a linear factorization $p(x) = c(x-a_1)\dots(x-a_n)$

$\mathbb{C}$ is algebraically closed.

→ Polynomials are a field with operators as the indeterminals.

$T: V \to V$ (V fdvs over $\mathbb{C}$)

Corr to $p(x)$

define $P(T) = p_0 I + p_1 T + p_2 T^2 + \dots + p_n T^n$

$\downarrow$

is also an operator from $V \to V$.

→ what is the operator $T^i$ $(i \geq 0)$ [Define $T^0 = I$]

[Require the image of the basis]

$$T^i : \underset{\in V}{v} \longmapsto \underbrace{T(T \dots T(\bar{v}))}_{i \text{ times.}}$$

(HW: show that $T^i$ is a lin. op (induction)).

Now, $cT^i : v \to c.(T^i(\bar{v}))$

Doubt!

$$U = \cancel{A} \sum_{i=0}^{n} p_i T^i : v \longmapsto \sum_{i=0}^{n} p_i (T^i(v)) = pT(\bar{v})$$

Let $p(x) = c(x-a_1) \dots (x-a_n) \in \mathbb{C}[x]$

$$p(T) = c(T-a_1 I) \dots \underbrace{(T - a_n I)}_{\text{recall eigen values.}}$$

[$T^i$ & $I$ commute $\forall i$].

Theorem: Any lin. operator over a fdvs $V$ over $\mathbb{C}$ has an eigen value & an associated vector over $\mathbb{C}$

Proof: Surely $\det([T - \lambda I]_B) = 0$ has atleast one root say $\lambda_1$ (fund. th. of alg.)

$$(T - \lambda I)\bar{v} = 0$$

Nullity $(T - \lambda I) \geq 1$.

Linear operator.

Let $\bar{\omega} \in V$ be any non-zero vector ~~bhz~~ let $\dim(V) = n$

$$S = \{ \underset{u}{\bar{\omega}}, T\bar{\omega}, T^2\bar{\omega}, \ldots, T^n\bar{\omega} \}$$

$S$ is a linearly dept. set since $|S| > \dim(V)$

Since non-zero lin. operator.

$$\sum_{i=0}^{n} a_i T^i \omega = \bar{0} \quad \text{for not all } a_i = 0$$

$a(T)\bar{\omega}$, where $a(T) = \sum_{i=1}^{n} a_i T^i$

By collorary :-

$$a(T) = \underset{\Vert a}{c}(T - c_1 I_1)(T - c_2 I) \ldots (T - c_n I)\bar{\omega} = 0 \quad \text{— (1)}$$

$c_i, c \in \mathbb{C}$

$c \in \mathbb{C} \setminus 0$

consider

If $(T - c_n I)\underline{\omega} = 0$

then $T\underline{\omega} = c_n \omega$ then $T$ has eigen value $c_n$ & eigen vector $\bar{\omega}$.

Suppose $(T - c_n I)\underline{\omega} \neq 0$ let $\omega' = (T - c_n I)\omega$

check if

$$(T - c_{n-1} I)\bar{\omega}' = \bar{0}$$

If true $c_{n-1}$ eigen value $\bar{\omega}'$ vector

We keep doing this until we see for some $i^{th}$ stage

$$(T - c_i I)w_i = \underline{0} \qquad w_i \neq 0$$

(This will happen for sure since (1) is true)

where

$$\bar{w_i} = (T - c_{i+1}) \ldots (T - c_n I)\bar{\omega}$$

thus $\bar{w_i}$ is an eigen vector corr to value $c_i$

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## Triangularizability of Operators :-

$$[T]_B = \begin{bmatrix} \ddots & \cdots \\ 0 & \ddots \end{bmatrix}$$

**Th:** Every linear operator on a fdvs over $\mathbb{C}$ is triangularizable (upper triangular rep)

**Pf:** $T : V \to V$, $\dim(V) = n$

Need to show basis $B$ ST :-
$$[T]_B = \begin{bmatrix} \times & \times & - \\ & \times & - \\ 0 & & \times \end{bmatrix}$$

(if you pick a basis of eigen vectors you are done)

x —— x

**Q:** Diagonalize if you can :-

① $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$.

$T - \lambda I = 0$ ie: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 1 & -\lambda & 1 \\ 2 & 1 & 4-\lambda \end{bmatrix} = 0.$$

$$\det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 1 & -\lambda & 1 \\ 2 & 1 & 4-\lambda \end{bmatrix} = 0$$

$\therefore \; (1-\lambda)(-4) \quad - 2(4-\lambda-2) \quad + 3(1+2\lambda) = 0$

$4\lambda - 4 \quad - 4 + 2\lambda \quad + 3 + 6\lambda = 0$

$12\lambda - 5 = 0$

$\lambda = 5/12$

$(1-\lambda)[-4\lambda+\lambda^2-1] - 2[4-\lambda-2] + 3(1+2\lambda) = 0$

$(1-\lambda)(\lambda^2-4\lambda-1) - 2[2-\lambda] + 3(1+2\lambda) = 0$

$\cancel{\lambda^2} - \cancel{4\lambda} - 1 - \lambda^3 + \cancel{4\lambda^2} + \cancel{\lambda} - 4 + \cancel{2\lambda} + 3 + \cancel{6\lambda} = 0$

$-\lambda^3 + 5\lambda^2 + 5\lambda - 2 = 0$

$\lambda^3 - 5\lambda^2 - 5\lambda + 2 = 0$ Solve

Step 1: form $|(A-\lambda I)| = 0$.
Step 2: Find eigen values by solving for $\bar{v}$ s.t $A\bar{v} = \lambda\bar{v}$

(If eigen values form a basis for $V \Rightarrow$ diag.)

(2)
$$\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

$T - \lambda I = 0$ ie: $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$

$$\begin{bmatrix} 6-\lambda & 3 & -8 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{bmatrix} = 0$$

$(6-\lambda)(-2-\lambda)(-3-\lambda) - 3(0) + 8(-2-\lambda) = 0$

$(6-\lambda)(-2-\lambda)(-3-\lambda) - 8(2+\lambda) = 0$

$(\lambda-6)(2+\lambda)(-3-\lambda) - 8(2+\lambda) = 0$

$(2+\lambda)[(\lambda-6)(-3-\lambda) - 8] = 0$

$(2+\lambda)[3\lambda - \lambda^2 - 18 + 6\lambda + 8] = 0$

$\cancel{(2+\lambda)[-\lambda^2+9\lambda-10] = 0}$

$\cancel{(2+\lambda)[\lambda^2-9\lambda+10] = 0}$

$\frac{a \pm \sqrt{\Delta}}{2}$

$$(2+\lambda)\left[(6-\lambda)(3+\lambda)-8\right]=0$$

$$(2+\lambda)\left[18+6\lambda-3\lambda-\lambda^2-8\right]=0$$

$$(2+\lambda)\left[-\lambda^2+3\lambda+10\right]=0$$

$$(2+\lambda)(\lambda^2-3\lambda-10)=0$$

$$(2+\lambda)(\lambda^2+5\lambda-2\lambda-10)=0$$

$$(2+\lambda)(\lambda-5)(\lambda+2)=0$$

$$(\lambda+2)^2(\lambda-5)=0$$

$$\lambda = -2,\ 5$$

Algebraic multiplicity of $-2 = 2$

$5 = 1$

solving,

$$(A-\lambda I)\bar{v}=0$$

$$\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 3 & -8 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

rank of matrix $= 2$

nullity $= 3-2 = 1$.

$\Rightarrow$ only one eigen vector ie $\in$ nullspace

**Defn:** Let $\lambda$ be eigen values of an operator $T: V \to V$. $\dim(NS(T-\lambda I))$ is called the geometric multiplicity of $\lambda$.

**Th:** Geometric multiplicity of $\lambda$ $\leq$ Alg. multiplicity of $\lambda$

Prove!

Proof:

If Geometric multiplicity $= k$ then there are $k$ L.I vectors S.T. $\{v_1, v_2, \ldots, v_k\}$ extending to basis

Char eqn: Alg. mul. $\geq k$

✓ Prove

Theorem: An operator $T: V \to V$ (fdvs) is diagonalizable Iff for every value $\lambda$,

geo mult = alg mult.

* Had the matrix itself been,

$$\begin{bmatrix} -2 & & \\ & -2 & \\ & & 5 \end{bmatrix}$$

$$(A+2I)\bar{x} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$NS(A+2I) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

NOTE: $\det(A-\lambda I) = \det(A)$ $\lambda = 0$.

$(\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda) = 0$.

where $\lambda_i$'s are the eigen values.

$$= \boxed{\prod_{i=1}^{n} \lambda_i = \det(A)}$$

- Theorem:

Let $T$ be a lin. operator over a fdvns over $\mathbb{C}$ then $T$ is triangularizable.

then there exists some basis $B$ for $V$ s.t.

$$[T]_B = \begin{bmatrix} \diagdown & \times \\ 0 & \diagdown \end{bmatrix}.$$

$B = \{\bar{v}_1, \bar{v}_2 \dots \bar{v}_n\}$.
be the basis which triangularizes $T$.

$$[T]_B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots \\ 0 & a_{22} & a_{23} & \dots \\ 0 & 0 & a_{33} & \dots \\ 0 & 0 & \vdots & \\ 0 & 0 & \vdots & a_{nn} \end{bmatrix}$$

$$[T]_B [v_i] = [v_i]_B$$

$\therefore$ $$\begin{bmatrix} a_{11} \\ 0 \\ 0 \\ 0 \end{bmatrix} = a_{11}[v_1]_B.$$

$$\begin{bmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \end{bmatrix} = a_{12}[v_1]_B + a_{22}[v_2]_B$$

$\Rightarrow T\bar{v}_2$ lies in the span $(v_1, v_2)$

$\Rightarrow T\bar{v}_i \in \text{span}(\{v_1, v_2 \dots v_i\})$

The basis B which triangularises T satisfies

holds for diagonalizable also
$\therefore T\bar{v}_i = \lambda \bar{v}_i \to$ span of $\{v_i\}$

Proof: Since $V$ is a vs. of over $\mathbb{C}$
$\exists$ some eigen value $\lambda_i \in \mathbb{C}$ for

$\Rightarrow \dim(NS(T-\lambda I)) \geq 1$.
$\Rightarrow \dim(\text{range}(T-\lambda I)) < n$
$\Downarrow$
$n - r_i$

where $r_i = \dim(NS(T-\lambda I))$

Claim: Range $(T - \lambda_i I)$ is $T$ invariant.

To prove claim :-

Let $\bar{\omega} \in$ Range $(T-\lambda I)$ then want to show
$T\bar{\omega} \in$ Range $(T-\lambda_2 I)$.

If $\omega \in$ range $(T-\lambda_1 I)$ then
$\bar{\omega} = (T-\lambda_1 I)\bar{v}$ for some $\bar{v} \in V$.

then
$$T\bar{\omega} = T(T-\lambda \bar{I})\bar{v} = (T-\lambda_1) I\, v' \quad \text{Range}$$
where $v' = T\bar{v}$

since $T$ & $(T-\lambda I)$ commute.

Let $U_1 =$ Range $(T-\lambda_1 I)$
$\dim(U_1) = n - r_1$ $\quad r_1 \geq 1$.

Consider a basis for $U_1$
$\{u_1, u_2 \dots u_{n-r_1}\}$. extend to basis for entire space.
$\{u_1, u_2, \dots u_{n-r_1}\} \cup \{v_1, v_2 \dots v_r\}$.

$$\therefore \quad [T]_{B1} = \left[\begin{array}{c|cc} (n-r_1 \times n-r_1) & \times & \times \\ & \vdots & \vdots \\ & \times & \times \\ \hline 0 & \lambda_1 & 0 \\ & 0 & \lambda_1 \\ & \vdots & \ddots \end{array}\right]$$

no. of times the eigen value occurs = algb. mult

$$T v_1 = \underbrace{(T-\lambda I)\bar{v}_1}_{\in U} + \lambda_1 \bar{v}.$$
$\{u_1, u_2 \dots u_{n-r}\}$,

$Tv =$
$Tv - Tv + \lambda v - \lambda v$
$= (T-\lambda I)v + \lambda v$

If the left $(n-r_1) \times (n-r_1)$ matrix has a $\Delta$ form then $T$ is $\Delta$ izable.

Observation :-

① $Tu_i \in \text{range}(T-\lambda_1 I)$

$\downarrow$ linear combination of $\{u_1, \dots u_{n-r}\}$

②. $Tw_i = \underbrace{(T-\lambda_1 I) w_i}_{\text{lin. comb of } \{u_1 \dots u_{n-r}\}} + \lambda_1 w_i$

$$[T]_B = \left[\begin{array}{ccc|ccc} \times & & \times & & \times & \times \\ \vdots & & \vdots & \Big\}(n-r) & & \vdots \\ \times & & \times & & \lambda & \\ \hline & & & \lambda_1 & & \times \\ & O & & & \lambda_1 \dots & \\ & & & & & \lambda_1 \end{array}\right]$$

Define $T|_{U_1} : U_1 \to U_1$

For $u \in U_1$, $T|_{U_1} : u \mapsto T\bar{u}$ $(\in U_1)$ since $U_1$ is T-invariant

$\Downarrow$

'Restricting $T$ to $U_1$'

Basis of $U_1 = \{u_1, u_2 \dots, u_{n-r}\} = B_{U_1}$

$[T|_{U_1}]_{B_{U_1}} = \Big[\, [T|_{U_1} u_i]_{B_{U_1}} = [Tu_i]_{B_{U_1}}$ first col of $[T]_{B_1}$ .... except for last $r$ entries

→ top left $(n-r) \times (n-r)$ submatrix of $[T]_{B_1}$

Proceed in a recursive fashion

Problem size reduced at least by 1. *

Keep calculating until the problem is solved

* ∵ at least one eigen value exists.

$$\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \rightarrow \text{Triangularize.}$$

$$|(T - \lambda I)| = 0.$$

$$\begin{vmatrix} 6-\lambda & 3 & -8 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{vmatrix} = 0.$$

$$(6-\lambda)(2+\lambda)(3+\lambda) - 3(0) - 8(2+\lambda) = 0.$$

$$\cancel{(6-\lambda)(\lambda^2+5\lambda+6) - 8(2+\lambda)}$$

$$\cancel{6\lambda^2 + 30\lambda + 36 - \lambda^3 - 5\lambda^2 - 6\lambda + 16 - 8\lambda = 0}.$$

$$\cancel{-\lambda^3 + \lambda^2}$$

$$(2+\lambda)\left[(6-\lambda)(3+\lambda) - 8\right] = 0.$$

$$(2+\lambda)\left[18 + 6\lambda - 3\lambda - \lambda^2 - 8\right] = 0.$$

$$(2+\lambda)\left[-\lambda^2 + 3\lambda + 10\right] = 0.$$

$$(2+\lambda)\left[\lambda^2 - 3\lambda - 10\right] = 0.$$

$$(2+\lambda)\left[\lambda - 5\right]\left[\lambda + 2\right] = 0.$$

$\lambda = -2$ or $\lambda = 5$.  $-2, 5$

$\lambda_1 = -2$

Range of $(T - \lambda_1 I) = U_1$

$$\begin{bmatrix} 8 & 3 & -8 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \text{col. space of } \{T - \lambda_1 I\}.$$

Basis of range = first 2 cols. $\begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ ok

Let $w_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$U_{11} = $ Range$((T+2I))$

range

$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Range of V $\left\{ \cancel{\begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \right\} = B_1$

$T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \cancel{\begin{bmatrix} -8 \\ 0 \\ -3 \end{bmatrix}}. \begin{bmatrix} -8 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -8 \\ -3 \\ 0 \end{bmatrix}_{B_1}$

$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}_{B_1}.$

$T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}_{B_1}.$

$[T]_{B_1} = \begin{bmatrix} 6 & -8 & 3 \\ 1 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$

$T' = \begin{bmatrix} 6 & -8 \\ 1 & -3 \end{bmatrix}$

Now $T_{|U_1} = \begin{bmatrix} 6 & -8 \\ 1 & -3 \end{bmatrix}$ $\quad u \in U_1 \longrightarrow T\bar{u}$

$\therefore\ [T_{|U_1}]_{B_{U_1}} = \begin{bmatrix} 6 & -8 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}_{B_{U_1}} = \begin{bmatrix} 6a & -8b \\ a & -3b \end{bmatrix}_{B_{U_1}}$

$B_{U_1} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Eigen vals:- $\det(T_{|U_1} - \lambda I) = 0$

$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 6-\lambda & -8 \\ 1 & -3-\lambda \end{bmatrix} = 0$

$(\lambda-6)(\lambda+3) + 8 = 0$

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$\lambda^2 - 3\lambda - 10 = 0.$

$(\lambda - 5)(\lambda + 2) = 0 \qquad \lambda = 5, -2.$

$U_2 = \text{Range}\ (T|_{U_1} + 2I) = \begin{bmatrix} 8 & -8 \\ 1 & -1 \end{bmatrix}.$

$= \text{span}\left\{\begin{bmatrix} 8 \\ 1 \end{bmatrix}\right\}.$

basis. of $U_2$ $= \left\{\begin{bmatrix} 8 \\ 1 \end{bmatrix}\right\}.$

extend basis of $U_1 = \left\{\begin{bmatrix} 8 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}. = B_{U_1}'.$

$[T|_{U_1}]_{B_{U_1}'} = ?$

$T|_{U_1}\begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 40 \\ 5 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$

ie: $8x + y = 40.$

$x = 5$

$[T|_{U_1}]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$

$8x + y = 6 \qquad y = -2.$

$x = 1.$

$\therefore [T|_{U_1}]_{B_{U_1}'} = \begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix}.$

Now represent the basis of $B_{U_1}'$ WRT to basis $B_{U_1}$.

$\therefore B_1' = \left\{\begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

$\therefore T = \begin{bmatrix} 5 & 1 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$

# INNER PRODUCT ON VECTOR SPACES:

Consider 2 vectors in $\mathbb{R}^m$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\bar{a}\cdot\bar{b} = \sum_{i=1}^{n} a_i b_i$$


An inner product on $V$ over $\mathbb{F}$ is a function from $V \times V \to \mathbb{F}$

$\langle \bar{\alpha}, \bar{\beta} \rangle$. where $(\bar{\alpha}, \bar{\beta}) \in V \times V$

such that

a) $\langle \bar{\alpha}, \bar{\beta} \rangle = \overline{\langle \beta, \alpha \rangle}$ (conjugate symmetry)

b) $\langle \bar{\alpha}, \bar{\alpha} \rangle \geq 0$ $(\in \mathbb{R})$ $\forall\, \alpha \in \bar{V}$ (non-negativity)

c) $\langle \bar{\alpha}, \bar{\alpha} \rangle = 0$ iff $\bar{\alpha} = 0$ (non degeneracy)

d) $\langle c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2, \beta \rangle = c_1\langle \bar{\alpha}_1, \bar{\beta}_1 \rangle + c_2\langle \bar{\alpha}_2, \bar{\beta}_2 \rangle$
$\forall\, c_1, c_2 \in \mathbb{F}$. $\forall\, \alpha_i\ \beta \in V$ (Linearity)

( d) applies to $\bar{\beta}$ also using a) with conj
is: $\langle \bar{\alpha}, c_1\bar{\beta}_1 + c_2\bar{\beta}_2 \rangle = \bar{c}_1\langle \alpha, \beta_1 \rangle + \bar{c}_2\langle \alpha, \beta_2 \rangle$

$\boxed{\text{Eg:}}$ Let $V = \mathbb{R}^n$

Define $\langle \alpha, \beta \rangle = \sum_{i=1}^{n} \alpha_i \beta_i$

$(\alpha_1 \cdots \alpha_n)$ $(\beta_1 \cdots \beta_n)$

a) $\langle \beta, \alpha \rangle = \sum_{i=1}^{n} \beta_i \alpha_i$

$= \sum_{i=1}^{n} \alpha_i \beta_i$ (commutative)

$= \langle \alpha, \beta \rangle$ over $\mathbb{R}$.

Linearity cond: matrix operations

b). $\langle \bar{\alpha}, \bar{\alpha} \rangle = \sum_{i=1}^{n} \alpha_i \alpha_i = \sum_{i=1}^{n} (\alpha_i)^2$

where $(\alpha_i)^2 \geq 0$ for all $\alpha_i \in V$.

$\therefore \sum_{i=1}^{n} (\alpha_i)^2 \geq 0$

$\Rightarrow \langle \bar{\alpha}, \bar{\alpha} \rangle \geq 0$.

c) $\langle \bar{\alpha}, \bar{\alpha} \rangle = 0$ iff $\bar{\alpha} = 0$.

$\langle \bar{\alpha}, \bar{\alpha} \rangle = \sum_{i=1}^{n} (\alpha_i)^2 = 0$ for $\alpha_i = 0$

d). $\langle c_1 \alpha_{1i} + c_2 \alpha_{2i}, \beta_i \rangle$

$= \sum_{i=1}^{n} (c_1 \alpha_{1i} + c_2 \alpha_{2i}) \cdot \beta_i$

$= \sum_{i=1}^{n} c_1 \alpha_{1i} \beta_i + \sum c_2 \alpha_{2i} \beta_i$

$= c_1 \langle \alpha_{1i}, \beta_i \rangle + c_2 \langle \alpha_{2i}, \beta_i \rangle$.

$\boxed{\text{Eg 2}}$ $V = \mathbb{C}^n$.

b) Let $\alpha_i = (i, 0, 0 \dots 0)$

$\langle \alpha_i \; \alpha_i \rangle = -1$ false!

To overcome this, redefine

$\langle \alpha_i, \bar{\beta_i} \rangle = \sum_{i=1}^{n} (\alpha_i \bar{\beta_i})$

a) $\langle \bar{\alpha}, \bar{\beta} \rangle = \sum_{i=1}^{n} \alpha_i \overline{\beta_i}$

$= \sum_{i=1}^{n} \overline{\beta_i} \alpha_i = \overline{\left( \sum_{i=1}^{n} \beta_i \bar{\alpha_i} \right)}$

$= \overline{\langle \bar{\beta}, \bar{\alpha} \rangle}$ True.

$\rightarrow$ For any $V = \mathbb{R}^n$

$\bar{\alpha} = (\alpha_1 \dots \alpha_n)$

$\bar{\beta} = (\beta_1 \dots \beta_n)$.

$$\langle \bar{\alpha}, \bar{\beta} \rangle \triangleq k \sum_{i=1}^{n} \alpha_i \beta_i \qquad k > 0.$$

$$\langle \bar{\alpha}, \bar{\beta} \rangle \triangleq \sum_{i=1}^{n} k\alpha_i \beta_i, \qquad \forall\, k_i > 0$$

$\rightarrow$ For $\mathbb{C}^n$

$$\langle \alpha, \beta \rangle \triangleq \sum_{i=1}^{n} \alpha_i \overline{\beta_i}$$

- Vector spaces endowed with inner product are called inner product spaces.

Norm of a vector in an inner product space.

For a $\bar{v} \in \mathbb{R}^n$. or $\bar{v} \in V$.

In general norm of any vector is defined as:-

$$\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}.$$

$$= \sqrt{\langle \bar{v}, \bar{v} \rangle} \qquad \text{where } \langle v, w \rangle = \text{inner prod.}$$

$$\therefore \|\bar{v}\|^2 = \langle \bar{v}, \bar{v} \rangle \,!$$

- CAUCHY SCHWARZ INEQUALITY :

$$|\langle \bar{v}, \bar{w} \rangle| \leq \|v\|\,\|w\|$$

magnitude.

Proof: Start with

$\| \bar{v} - \lambda \bar{w} \|^2 = \langle \bar{v} - \lambda \bar{w}, \bar{v} - \lambda \bar{w} \rangle$

Expanding:-

$\langle \bar{v}, \bar{v} - \lambda \bar{w} \rangle - \lambda \langle \bar{w}, \bar{v} - \lambda \bar{w} \rangle$.

$\langle \bar{v}, \bar{v} \rangle - \bar{\lambda} \langle \bar{v}, \bar{w} \rangle - \lambda [\langle \bar{w}, \bar{v} \rangle - \bar{\lambda} \langle \bar{w}, \bar{w} \rangle]$

$\| \bar{v} \|^2 - \bar{\lambda} \langle \bar{v}, \bar{w} \rangle - \lambda \langle \bar{w}, \bar{v} \rangle + |\lambda|^2 \| \bar{w} \|^2$

& use $\lambda = \dfrac{\langle \bar{v}, \bar{w} \rangle}{\langle \bar{w}, \bar{w} \rangle} = \dfrac{\langle \bar{v}, \bar{w} \rangle}{\| \bar{w} \|^2}$ at some point

~ Triangular inequality

Lemma:- $\| \bar{v} + \bar{w} \| \leq \| v \| + \| w \|$

Proof: $\| v + w \|^2 = \langle \bar{v} + \bar{w}, \bar{v} + \bar{w} \rangle$

$\langle \bar{v} + \bar{w}, \bar{v} + \bar{w} \rangle \leq \langle v, v \rangle + \langle w, w \rangle + 2\|v\|\|w\|$

$\bar{0}$ is trivially orthogonal to every vect
$\langle \bar{v}, \bar{0} \rangle = \langle \bar{v}, \bar{v} - \bar{v} \rangle = \langle \bar{v}, \bar{v} \rangle - \langle \bar{v}, \bar{v} \rangle$



**Def:** Orthogonality :

$\bar{v} \in V$ is said to be orthogonal to $\bar{w} \in V$
if $\langle \bar{v}, \bar{w} \rangle = 0 \in \mathbb{F}$

$$\bar{v} \perp \bar{w}$$

**Lemma:** Let $\bar{v}_i = i = 1, 2, \ldots n,$ be mutually orthogonal. non-zero vectors then $v_i$ $i = 1 \ldots n$ are lin. independent.

Let if possible $\bar{v}_1, \bar{v}_2 \ldots \ldots \bar{v}_n$ not be linearly independent

$$c_1\bar{v}_1 + c_2\bar{v}_2 + \ldots + c_n\bar{v}_n = 0$$

Inner prod with $\bar{v}_1$.

$$\bar{c_1} \langle \bar{v}_1, \bar{v}_1 \rangle + \bar{c_2} \langle \bar{v}_1, \bar{v}_2 \rangle + \ldots + \bar{c_n} \langle \bar{v}_1, \bar{v}_n \rangle$$
$$= \langle \bar{v}, 0 \rangle.$$

$$c_1 \langle \bar{v}_1, \bar{v}_1 \rangle = 0.$$
But $\langle \bar{v}_1, \bar{v}_1 \rangle \neq 0$
$\therefore$ $c_1 = 0$.

Similarly extending to $c_2, c_3 \ldots c_n$.
$c_1 = c_2 = \ldots = c_n = 0$. # contradiction

$\therefore$ $\bar{v}_1, \bar{v}_2 \ldots$

Suppose $\Sigma c_i v_i = 0$ where $c_j \neq 0$.
$0 = \langle v_j, \Sigma c_i v_i \rangle$.
$= c_j \langle v_j, v_j \rangle$.

Show that $\langle v, \bar{0} \rangle = \bar{0} \quad \forall v$

Pf: take $\langle v, \bar{v} - \bar{v} \rangle = \langle \bar{v}, \bar{v} \rangle - \langle \bar{v}, \bar{v} \rangle$

→ Suppose we have an n-dim inner prod space V any collection of n mutually orthogonal non-zero vectors in V form a basis

Suppose $\{v_1, \dots v_n\}$ is a basis for V then

any $\bar{v} = \Sigma c_i \bar{v_i}$ for some unique $c$

If this is an orthogonal basis, co-efficients $c_i$s can be easily determined.

ie: $\langle v_i, v_j \rangle = 0 \quad i \neq j$

inner prod of ① with $\bar{v}_j$

$$\langle v, v_j \rangle = c_j \langle v_j, v_j \rangle$$

$$\Rightarrow \quad c_j = \frac{\langle \bar{v}, \bar{v}_j \rangle}{\langle v_j, v_j \rangle}$$

Defn:- Let V be an inner product space. Let W be a subspace of V then,

A vector $\bar{v}$ is said to be orthogonal to W if

$$\bar{v} \perp \bar{w} \quad \forall \bar{w} \in W.$$

Lemma:- Suppose $\{\bar{w}_1 \dots \bar{w}_r\}$ be an orthogonal basis for W then $\bar{v} \perp W$ iff $\bar{v} \perp \bar{w}_i \quad \forall i = \{1, 2, \dots r\}$

→ If $\bar{v} \perp \bar{W}$ it is perp to every vector $\bar{w} \in W$ in particular also the basis vectors.

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→ If $\bar{v} \perp \bar{w_i}$ $\{w_i \in \text{basis of } W\}$

To show $\bar{v} \perp \bar{w}$ $\forall \bar{w} \in W$.

$$\bar{w} = c_1\bar{w_1} + c_2\bar{w_2} + \dots + c_r\bar{w}_r.$$

$$\langle \bar{v}, \bar{w} \rangle = \langle \bar{v}, c_1\bar{w_1} \rangle + \langle \bar{v}, c_2\bar{w_2} \rangle \dots$$

$$= \bar{c_1}\underbrace{\langle v, \bar{w_1} \rangle}_{0} + \bar{c_2}\underbrace{\langle v, \bar{w_2} \rangle}_{0} \dots$$

$$= 0$$

Proved.

- Projection of $\bar{v}$ on $\bar{w}$ $(\neq \bar{0})$.

$$\triangleq \frac{\langle \bar{v}, \bar{w} \rangle}{\langle \bar{w}, \bar{w} \rangle} \bar{w} = \frac{\langle v, \bar{w} \rangle}{\| \bar{w} \|^2} \bar{w} \in \text{span}(\bar{w})$$

(Check projection of $\bar{v}$ on itself $= \bar{v}$).

Orth. proj. of $\bar{v}$ on a subspace $W$ is a vector $\bar{w} \in W$.

such that $(\underset{\text{given vector}}{\bar{v}} - \underset{\text{orthogonal projection}}{\bar{w}}) \perp W$. remaining component

$$\left\{ \left\langle \bar{v} - \frac{\langle \bar{v}, \bar{w} \rangle}{\langle w, \bar{w} \rangle} \bar{w}, \bar{w} \right\rangle \right\}.$$

??

Theorem: Let $\{w_1 \dots w_r\}$ be the orth. basis for $W$

the orth. proj of $\bar{v}$ on $W$ is

precisely the vector

$$\bar{w} = \sum c_i w_i$$

where $$c_i = \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle}$$

Clearly $\bar{w} \in W$.

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Now to show $(\bar{v}-\bar{w}) \perp w_i \ \forall i = 1, 2, \ldots n$

Theorem: Any basis $\{\bar{v}_1 \ldots \bar{v}_n\}$ of $V$ can be used to obtain orthogonal basis $\{w_1 \ldots w_n\}$ such that $\text{span}(\{v_1 \ldots v_n\}) = \text{span}(\{w_1 \ldots w_n\})$
$\forall i = 1 \ldots n$.

Pf: Let $v_1 = w_1$.

Let $w_2 = \underbrace{v_2 - \{\text{orth projection of } v_2 \text{ on span}(w_1)\}}_{\text{orthogonal to all vectors in span}(w_1)}$

$\text{span}(\{w_1, w_2\}) = \text{span}(\{v_1, v_2\})$

Further let: $\bar{w}_3 = \bar{v}_3$ − proj of $v_3$ on span$(w_1, w_2)$

Continuing like so:-

$w_i = v_i - \{\text{proj of } v_i \text{ on span}(w_1 \cdots w_{i-1})\}$

- Computing the projection of $\bar{v}$ on $W$:-

Pick some orthogonal basis for $W$
$\{w_1 \ldots w_r\}$

Projection vector $\bar{w}$ should be such that $\bar{w} = \text{Span}(w_1 \ldots w_r)$ ①

And:- $\bar{v} - \bar{w} \perp W$. ②.

Prop ①: $\bar{w} = \sum_{i=1}^{r} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \bar{w}_i$

?? Prop ②: $\left\langle v - \sum_{i=1}^{r} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \bar{w}_i, \bar{W} \right\rangle$

→ Orthogonal basis satisfies :-
$\{w_1 \ldots w_n\}$ $\langle w_i, w_j \rangle = 0$ iff $i \neq j$

→ Orthonormal basis
above prop + $\|w_i\| = \sqrt{\langle w_i, w_i \rangle} = 1$

• Let $V = \mathbb{C}^{n\times 1}$ ($\mathbb{R}^{n\times 1}$
consider the standard inner product

$$\langle \bar{\alpha}, \bar{\beta} \rangle = \sum_{i=1}^{n} \alpha_i \bar{\beta}_i$$

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \quad \Downarrow = [\bar{\beta}_1 \ldots \bar{\beta}_n] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

$$= \bar{\beta}^T \bar{\alpha}$$

$\bar{\beta}^T \Rightarrow$ Hermitian op [taking transpose & conjugate].

→ Suppose $\{u_1, \ldots, u_n\}$ is an orthogonal basis of $V$ $\mathbb{C}^{n\times 1}$

$$\underset{1\times n}{u_i^H} \; \underset{n\times 1}{u_j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \qquad \underset{1\times n}{[\quad]} \underset{n\times 1}{[\quad]}$$

Converting into a matrix:-

$$\underset{n\times n}{\begin{bmatrix} u_1^H \\ u_2^H \\ \vdots \\ u_n^H \end{bmatrix}} \underbrace{\underset{n\times n}{\begin{bmatrix} u_1 \ldots u_n \end{bmatrix}}}_{U} = I_{n\times n}.$$

$U^H U = I$. Here $U$ is called a unitary matrix.

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- **Hermetian matrices or self adjoint matrices :-**

A matrix is said to be hermetian if $A = A^H$

Over realss $A = A^T \rightarrow$ hermetian. (symmetric)

→ Lemma:
The eigen values of Hermitian matrix are real

Pf: ~~Suppose $\lambda$ is an eigen value~~
consider $\langle\ \rangle$ std inner product.

$$\langle \underline{x}, A\underline{y} \rangle = \underset{1\times n}{(A\bar{y})^H} \underset{n\times 1}{x}.$$

$$= \underset{1\times n}{\bar{y}^H} \underset{n\times n}{A^H} \underset{n\times 1}{\bar{x}} \quad — \ (1)$$

$$= \langle A^H x, \bar{y} \rangle$$

Given $A = A^H$ Let $\lambda$ be any eigen value.
& $\bar{x}$ be an eigen vector

$$\langle \bar{x}, A\bar{x} \rangle = \langle \bar{x}, \lambda \bar{x} \rangle = \overline{\langle \lambda x, x \rangle}$$

$$= \bar{\lambda} \langle \bar{x}, \bar{x} \rangle \qquad \langle \bar{\lambda} x, x \rangle$$

$$= \bar{\lambda} \|x\|^2 \quad — \ (2) \qquad \bar{\lambda} \langle x, x \rangle$$

From (1) :-

$$\langle A^H x, \bar{x} \rangle = \langle Ax, x \rangle \qquad A = A^H$$

$$= \langle \lambda x, x \rangle$$

$$= \lambda \langle x, x \rangle$$

$$= \lambda \|x\|^2 \quad — \ (3)$$

(2) = (3) :-

$$(\lambda - \bar{\lambda}) \|x\|^2 = 0.$$

$\bar{x}$ is an eigen vector $\therefore \|x\|^2 \neq 0$

$\therefore \quad \underline{\lambda = \bar{\lambda}} \qquad \bar{\lambda} = \lambda$

Theorem set:-

1. To prove subspace
   - Prove subset.
   - Prove $cx+y \in W$.

2. Sum of subsets.
Sum of subrspaces of V is a subspace of V.
Sum of subsets of V is a subset.

3. Intersection of subspaces is a subspace.

4. Spanning set:- Linear comb of vectors $\in$ S.S. spans vector space [co-efficients $\in \mathbb{F}$].

5. Linearly Independent set:-
If $c_1\bar{x}_1 + c_2\bar{x}_2 + \dots + c_n\bar{x}_n = 0$
$\Rightarrow c_1 = c_2 = \dots c_n = 0$.

6. Basis is:-
   - Maximal L.I set
   - Minimal spanning set.

7. Dimension of ~~Basis~~ V.S.: Cardinality of basis

8. Linear Transformations.
Claim:- If ~~IF~~ $T: V \to W$
T is injective iff $null(T) = \{\bar{0}\}$.

Injective ~~Surjective~~: $T(v_1) = T(v_2) \Rightarrow v_1 = v_2$.
Surjective: For every $\alpha \in W$ $\exists$ a $v \in V$
ST. $T(\bar{v}) = \alpha$.

9. Range of T is a subspace of W
★ dim of V = dim (null T) + ~~dim~~ range T

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Exercises :

Ex. 2.1 :
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Linear Transformation .
Q.4

→ Row rank = col. rank.
→ For 2 basis :- ~~[P]~~ $[\alpha]_B = P[\alpha]_{B'}$
→ Set of all $L : V \to W$ over $\mathbb{F}$ is a Vector Space.
→ Dimension of L = Dim V x Dim W.
→ Non-zero rows of rref form basis
→ ~~If~~ A & B are row equivalent iff they have the same row space.
→ linear operator is defined as a tran from $V \to V$

→ $[T\alpha]_B$ = ~~$A[T$~~ $A[\bar{\alpha}]_{B'}$.
→ $[T\alpha]_B = [T]_B [\alpha]_B$ → Linear operator.

→ If T is lin. trans from $V \to W$. $(\dim(V) = \dim(W))$
then.
T is inv ⟺ T is non singular ⟺ T is onto

→ If T is a linear operator. then :-
$[T]_{B'} = P^{-1}[T]_B P$.

→ V is isomorphic to W if there exists a one-one linear tranformation from $V \to W$.
$V \to V$ is iso, $V \to W$ is iso & $W \to T$ is iso ⇒ $V \to T$ is iso
Also, $V \to W$ ⇒ $W \to V$ is iso.

→ 2 finite dim V.S are isomorphic if same dimension

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→ For composition of 2 linear transformations :-

$[T_2 T_1]_B = C[\alpha]_B$

$C \to BA$, $B \to T_2$, $A \to T_1$

→ Similar matrices :- $A \sim B \Rightarrow B = P^{-1}AP$

Also, $A \sim B$ & $B \sim C \Rightarrow A \sim C$

→ Solutions of linear Eqns.

$$\overset{A}{\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \cancel{\phi} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

$$x\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + y\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

$\therefore \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ lies in the col. space of A

Cases :-

- One sol. iff columns of A are L.I.
- sol doesn't exist if $b \notin$ colspace(A)
- $\infty$ sol. if columns are linearly dependent

→ Row reduced

All entries before leading 1 are 0.

All entries in column of leading 1 are 0.

→ Row reduced echelon :-

- Row reduced
- All non-zero rows come above zero rows

## Eigen values & vectors:

→ Any scalar $\lambda \in \mathbb{F}$ is said to be an eigen value if $T\bar{v} = \lambda \bar{v}$ for ~~x~~ $\bar{v} \neq \bar{0}$

→ Corresponding vector $\bar{v}$ is : eigen vector
Subspace $\{\bar{v} \mid T\bar{v} = \lambda\bar{v}\}$ eigen space of $\lambda$

→ To find eigen vectors of $\lambda$, compute $\bar{v} \in$ nullspace of $(T - \lambda I)$.
ie:- $\det(T - \lambda I) = 0$ characteristic eqn.

→ Sparse representation : Block diagonal rep.

→ Th: $T: V \to V$ can be represented as block diagonal matrix iff $V = \bigoplus_{i=1}^{r} W_i$ $i = 1, 2, \ldots r$.

ST :- $$[T] = \begin{bmatrix} \underbrace{\begin{matrix}\lambda_1 & & \\ & \lambda_1 & \\ & & \ddots \; \lambda\end{matrix}}_{\dim W_1} & & \\ & \ddots & \\ & & \underbrace{\lambda_r \lambda_r \lambda_r \ldots}_{\dim W_r} \end{bmatrix}$$

→ Corrolary: If a lin op. has a blk. diagonal rep then its basis comprises of eigen vectors & the diagonal entries are eigen values

→ Any 2 similar matrices have the same eigen values

→ For any 2 basis $B_1, B_2$ for a linear operator $T$ $[T]_{B_1}, [T]_{B_2}$ have the same eigen values & the representation is unique for eigen vector

→ If $\lambda_1, \ldots \lambda_r$ are distinct eigen values &

$\bar{v}_1, \bar{v}_2 \ldots \bar{v}_r$ are distinct eigen vectors then $\{\bar{v}_1 \ldots \bar{v}_r\}$ L.I.

→ If T has dim(V) distinct eigen values, T is diagonalizable.

→ Every lin. op. on a fdvs over $\mathbb{C}$ is triangularizable

→ For any eigen value
geo multiplicity <= algebraic multiplicity.

→ Iff Geo mult = Alg mult then
it ~~T~~ is diagonizable.

→ For an n-dim space any collection of n orthogonal vectors forms a basis

→ Polynomials too can be written ~~as~~ with T as an indeterminant.

→ For every linear operator there exists an eigen value & corresponding eigen vector.

→ Prop: • If T is upper triangularizable :- $\{v_1 \ldots v_n\}$ basis
• $Tv_k$ is spanned by $\{v_1 \ldots v_k\}$
• Span of $\{v_1 \ldots v_k\}$ is invariant.

→ Suppose T: upper $\Delta$ rep then it is invertible iff all entries on the diagonal are non-zero

→ An operator has a diagonal matrix rep iff basis vectors are eigen values

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→ For any lin. op. nullspace(T) & range(T) are invariant

→ Every lin. operator on a finite dim. $\mathbb{R}$ vector space has an invariant subspace of dimension 1 or 2.

→ Every operator on an odd dimensional real vector space has eigen values

→ Inner product spaces satisfy 4 properties
- $\langle \bar{\alpha}, \bar{\beta} \rangle = \overline{\langle \bar{\beta}, \bar{\alpha} \rangle}$
- $\langle \bar{\alpha}, \bar{\alpha} \rangle \geq 0$
- $\langle \bar{\alpha}, \bar{\alpha} \rangle = 0$ iff $\bar{\alpha} = \bar{0}$
- $\langle c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2, \bar{\beta} \rangle = c_1\langle \bar{\alpha}_1, \bar{\beta} \rangle + c_2\langle \bar{\alpha}_2, \bar{\beta} \rangle$.

→ Norm of a vector $= \|x\| = \sqrt{x_1^2 + x_2^2}$

→ Cauch Shwarez inequality: $|\langle \bar{v}, \bar{w} \rangle| <= \|v\| \, \|W\|$.
→ Triangular inequality : $\|\bar{v} + \bar{w}\| <= \|v\| + \|W\|$.
→ Pythagorean th. $\|\bar{u} + \bar{v}\|^2 = \|u\|^2 + \|v\|^2$ if $u \perp v$.

→ Cauchy Shwartz inequality holds iff $\bar{u} = \lambda \bar{v}$
→ Parallelogram inequality
$$\|\bar{u}+\bar{v}\|^2 + \|\bar{u} - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

→ Gram Schmidh ~~th~~ eqn if $(v_1 \dots v_n)$ basis then there exists an orthonormal basis such that span $(v_1, \dots v_n)$ = span $(w_1 \dots w_n)$

→ Corollary from Gram Schmidth: Every fd ip space has an orthonormal basis

→ The eigen values of hermitian matrices are real nos.

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→ Solving a system of eqns
RREF (A).
Pivot entries → solve for them.
Remaining → Free variable

→ To find the basis for the range of a lin transformation, find the no. of lin independent columns.
So if A → $T: V \to V$.
rref ($A^T$) then calculate rank
and those rows will be the basis

→ If $\lambda$ is an eigen value, the
($T - \lambda I$) → columns dependent
→ not invertible
→ det = 0

→ Every finite dimensional inner product space has an orthonormal basis

→ If T is a triangularized matrix/linear transformation it has an orthonormal basis

→ ~~Property~~ Fact: only $\{\bar{0}\}$ & T are invariant under every linear operator.

→ For any linear operator over $\mathbb{C}$ $\exists$ invariant subspaces of dimensions ~~of~~ 1, 2... dim ($V$)

→ $P_{U,W}(\bar{v}) = \bar{u}$ & ~~P~~ $P_{W,U}(\bar{v}) = \bar{w}$
where $V = U \oplus W$
Range $P_{U,W} = U$ Null space $= W$.