

Neural networks

Nando de Freitas



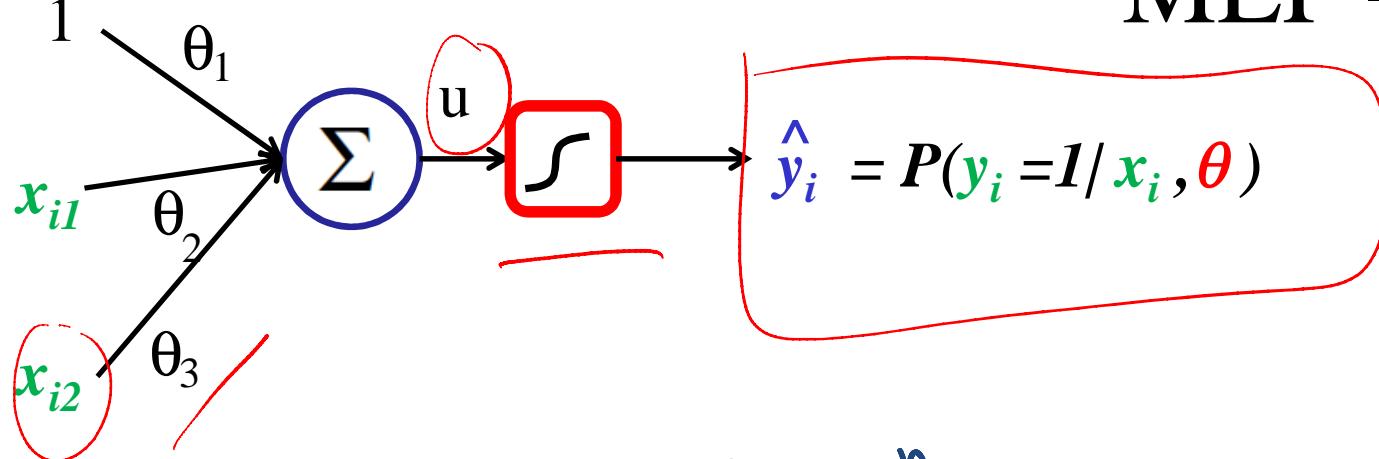
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Outline of the lecture

This lecture introduces you to the fascinating subject of classification and regression with artificial neural networks. In particular, it

- Introduces multi-layer perceptrons (MLPs)
- Teaches you how to combine probability with neural networks so that the nets can be applied to regression, binary classification and multivariate classification.
- Discusses the modular approach to backpropagation and neural network construction in Torch, which was introduced in the previous lecture.

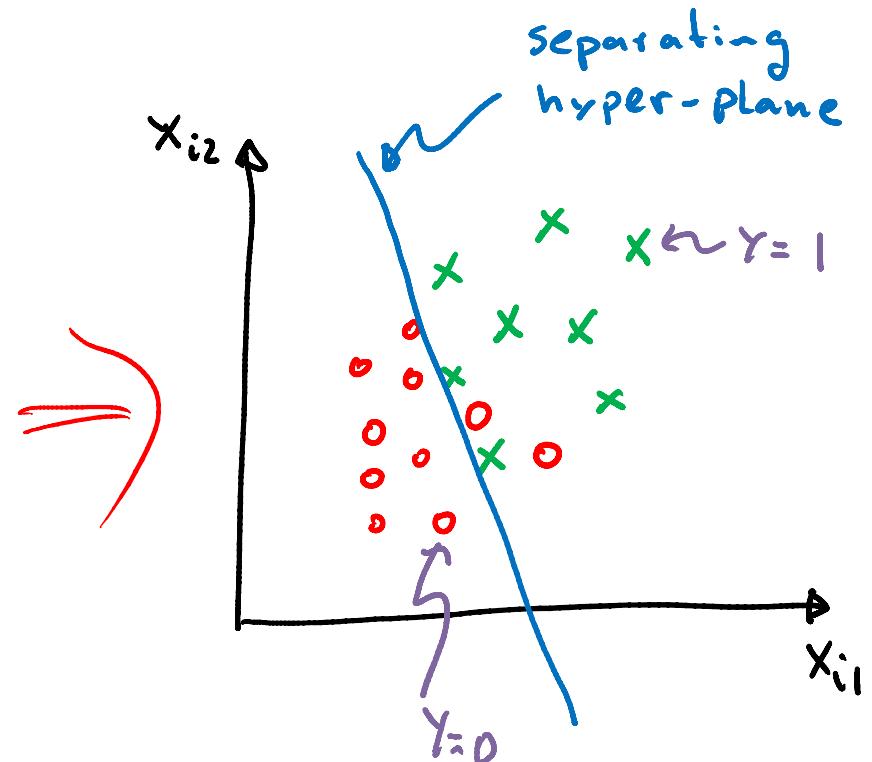
MLP – 1 neuron



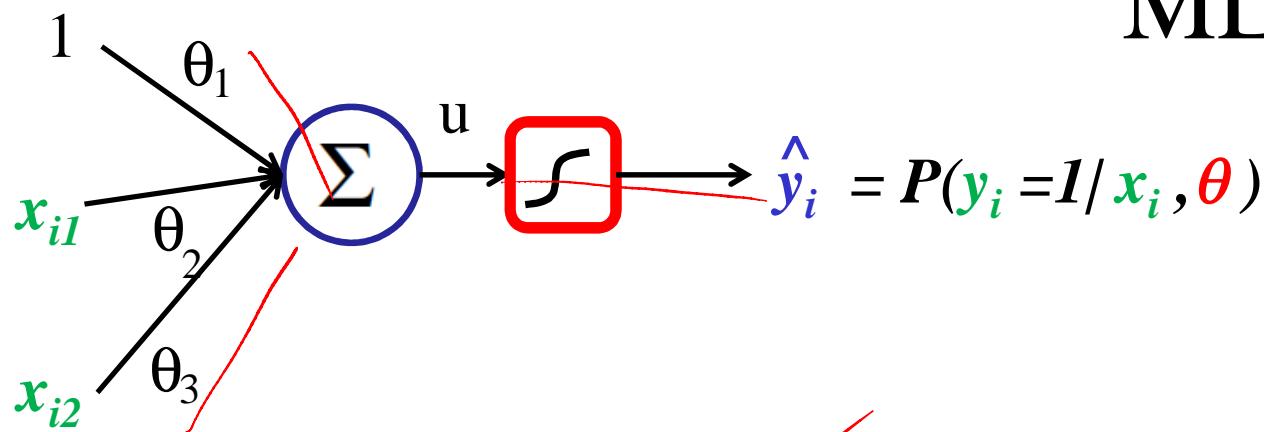
We are given the data $\{x_i, y_i\}_{i=1}^n$

e.g.

	x_{i1}	x_{i2}	y_i
$i=1$	0,2	6	0
$i=2$	0,3	22	-1
$i=3$	0,6	-0,6	-1
$i=4$	-0,4	58	0
:			



MLP – 1 neuron



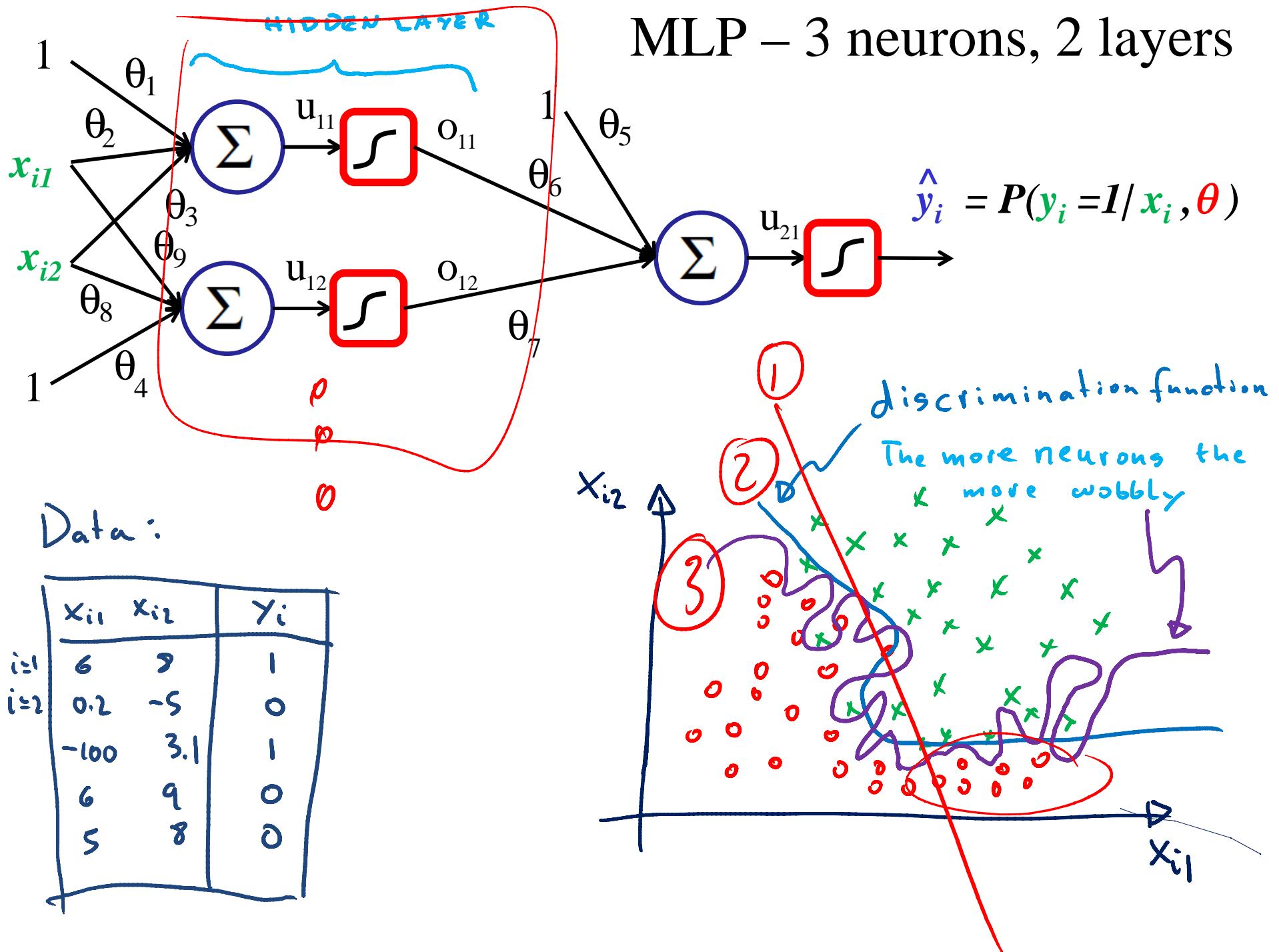
$$u = \theta_0 + \theta_1 x_{i1} + \theta_2 x_{i2}$$

$$\hat{y}_i = \frac{1}{1+e^{-u}} = \frac{1}{1+e^{-\theta_0 - \theta_1 x_{i1} - \theta_2 x_{i2}}} = P(y_i = 1 | x_i, \theta)$$

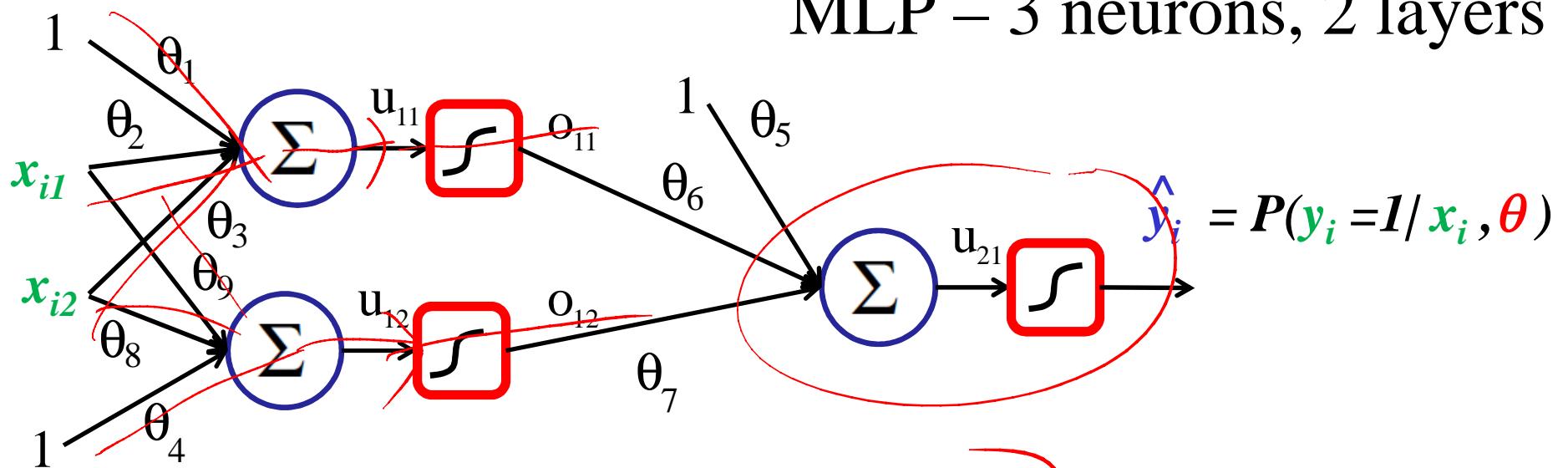
$$P(y_i = 1 | x_i, \theta) = \hat{y}_i^{y_i} (1 - \hat{y}_i)^{1-y_i} = \begin{cases} \hat{y}_i & \text{When } y_i = 1 \\ 1 - \hat{y}_i & \text{Otherwise} \end{cases}$$

For n independent observations (Bernoulli)

$$\underline{P(Y | X, \theta)} = \prod_{i=1}^n P(y_i | x_i, \theta)$$



MLP – 3 neurons, 2 layers



$$u_{11} = \theta_1 + \theta_2 x_{i1} + \theta_3 x_{i2}$$

$$u_{12} = \theta_4 + \theta_5 x_{i1} + \theta_6 x_{i2}$$

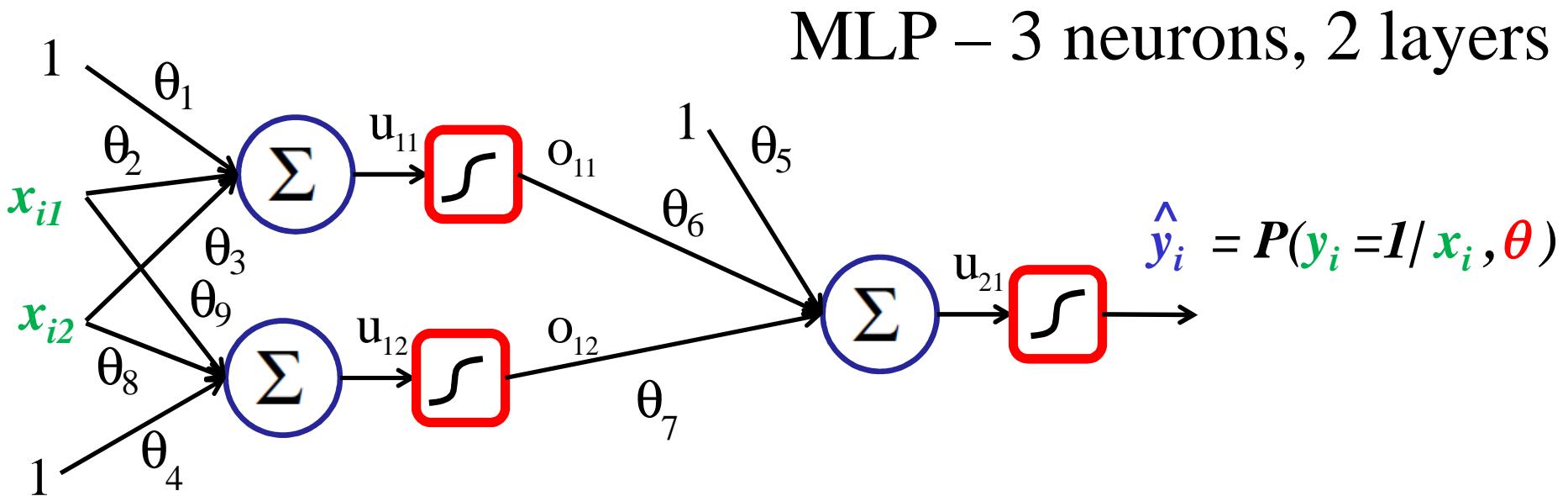
$$o_{11} = \frac{1}{1+e^{-u_{11}}}$$

$$o_{12} = \frac{1}{1+e^{-u_{12}}}$$

$$\hat{y}_i = \frac{1}{1+e^{-u_{21}}}$$

$$u_{21} = \theta_5 + \theta_6 o_{11} + \theta_7 o_{12}$$

$$P(y_i | x_i, \Theta) = \hat{y}_i^{y_i} (1 - \hat{y}_i)^{1-y_i}$$



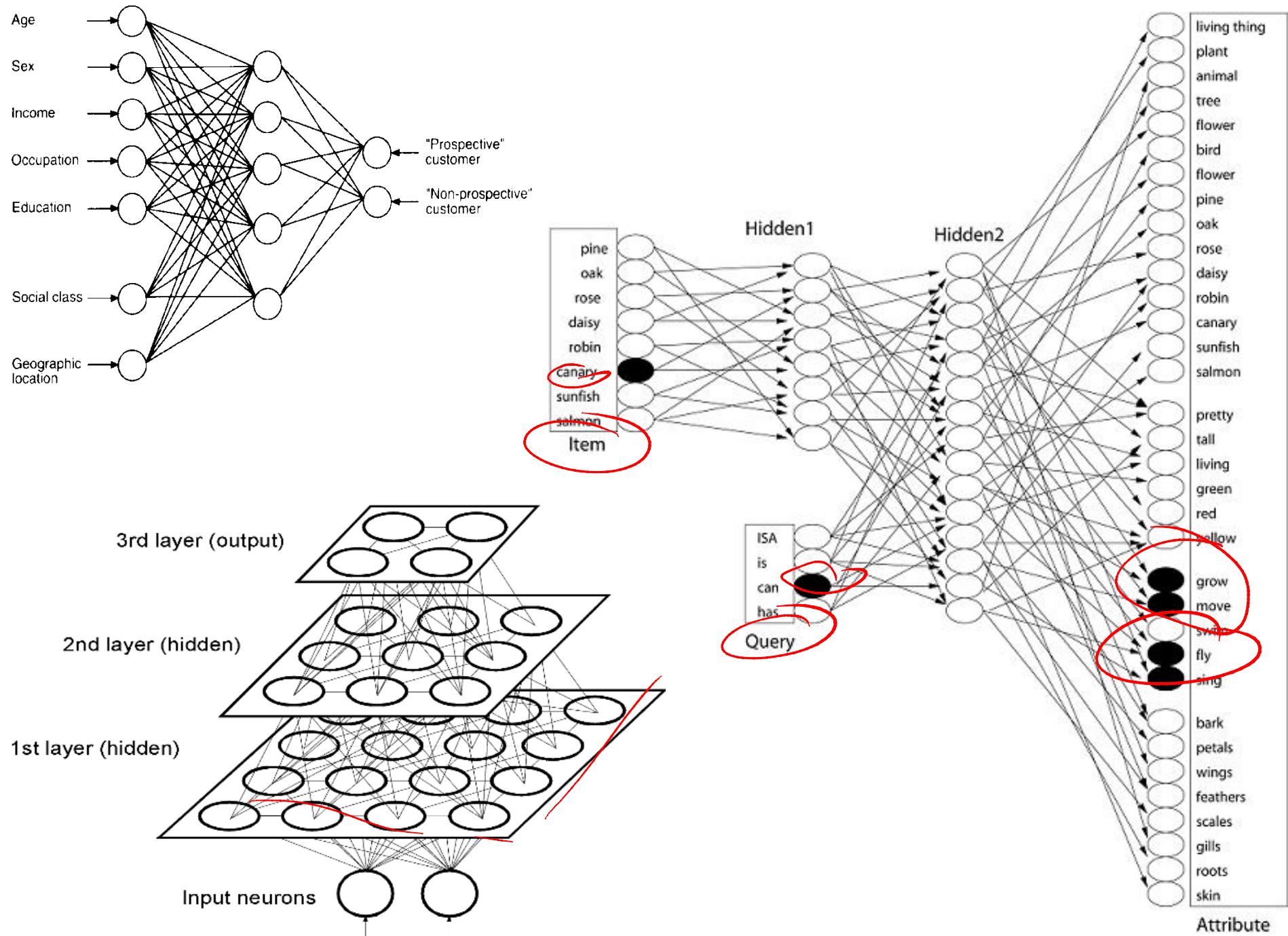
For n independent observations.

$$P(Y|X, \Theta) = \prod_{i=1}^n \hat{y}_i^{y_i} (1-\hat{y}_i)^{1-y_i} = \prod_{i=1}^n P(Y_i|X_i, \Theta)$$

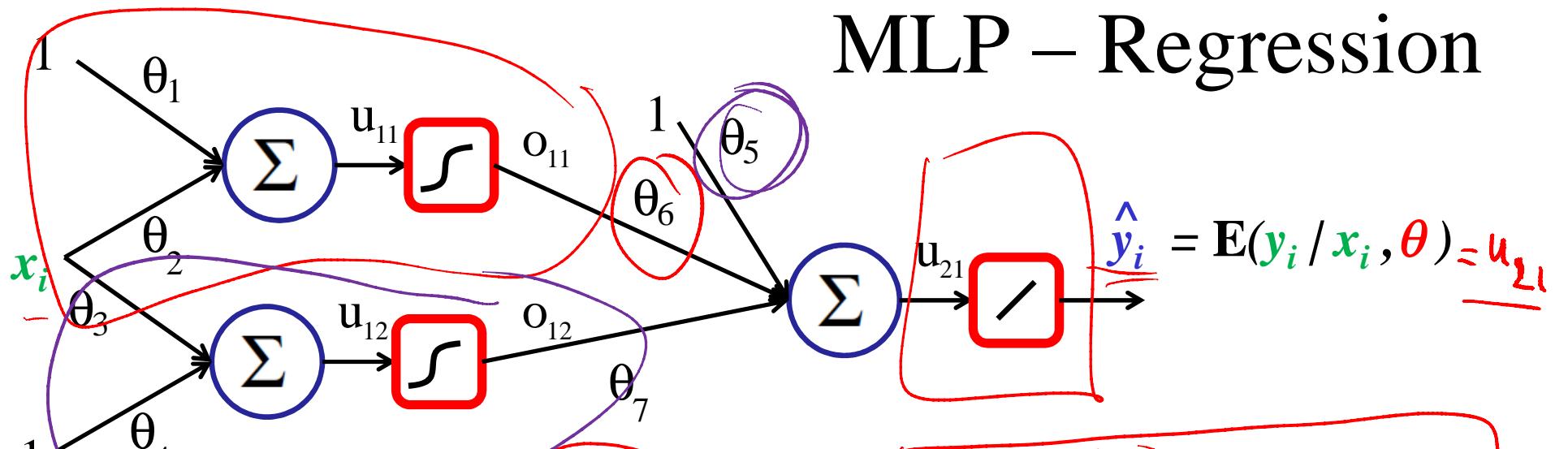
Cost:

$$C(\Theta) = -\log P(Y|X, \Theta) = -\sum_{i=1}^n y_i \log \hat{y}_i + (1-y_i) \log (1-\hat{y}_i)$$

i.e. minimize the cross-entropy error.

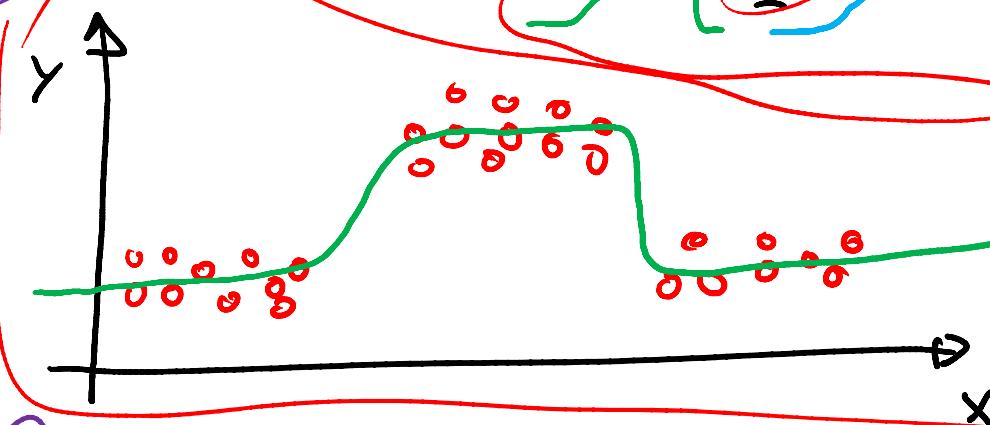


MLP – Regression



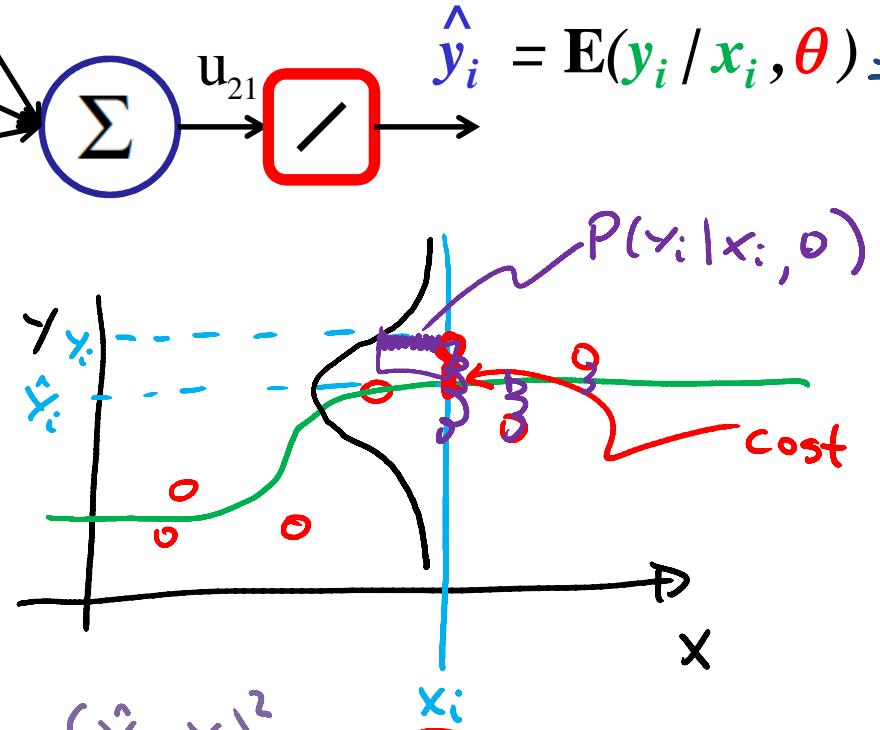
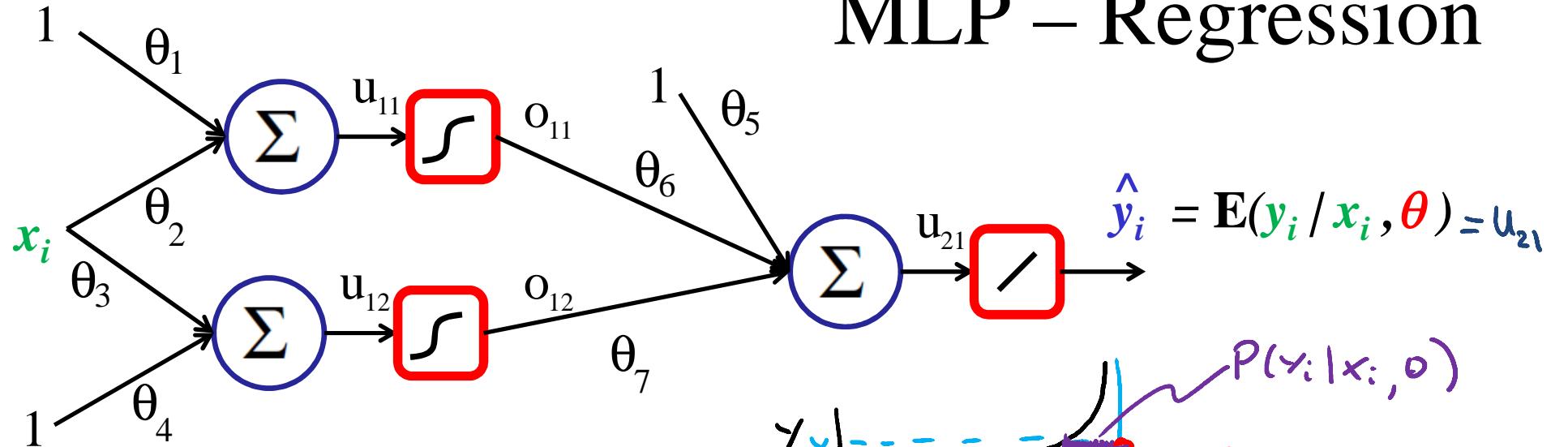
Data:

x_i	y_i
0.2	0.6
0.9	0.4
-0.6	5
-0.3	6.2



$$\hat{y}_i = \Theta_5 + \frac{\Theta_6}{1 + e^{-\Theta_1 - \Theta_2 x_i}} + \frac{\Theta_7}{1 + e^{-\Theta_4 - \Theta_3 x_i}}$$

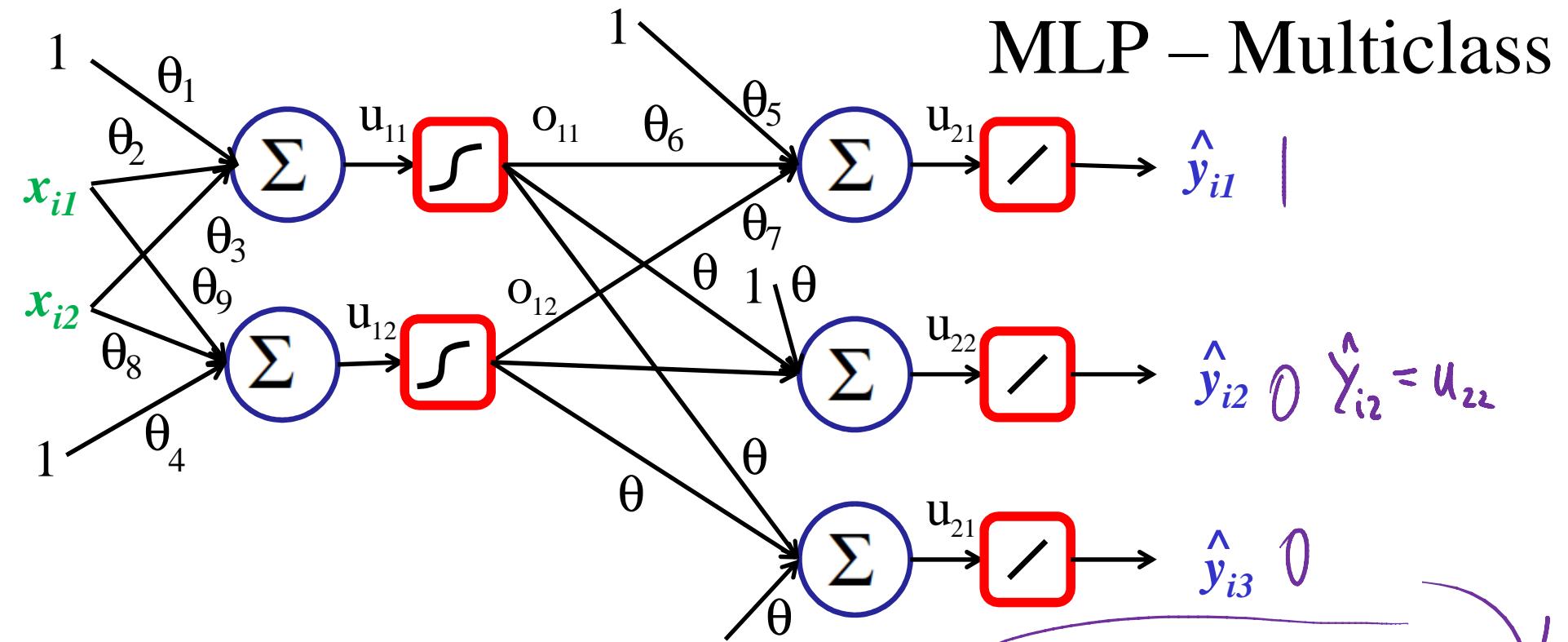
MLP – Regression



$$P(y_i | x_i, \theta) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} (\hat{y}_i - y_i)^2}$$

$$C(\theta) = \sum_{i=1}^n (\hat{y}_i - y_i)^2$$

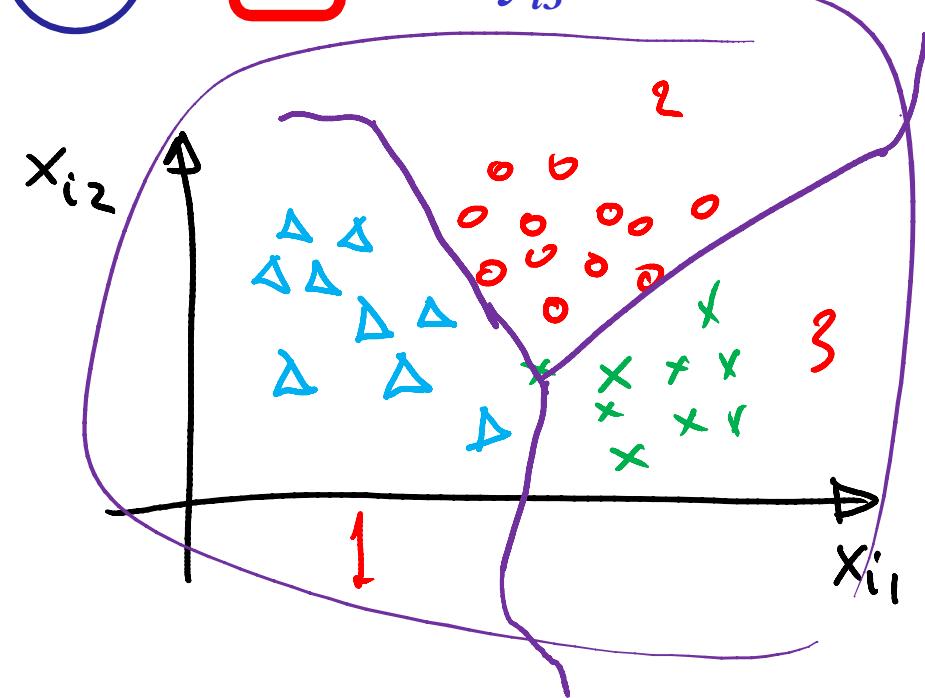
$$\hat{y}_i = f(\theta, x_i)$$

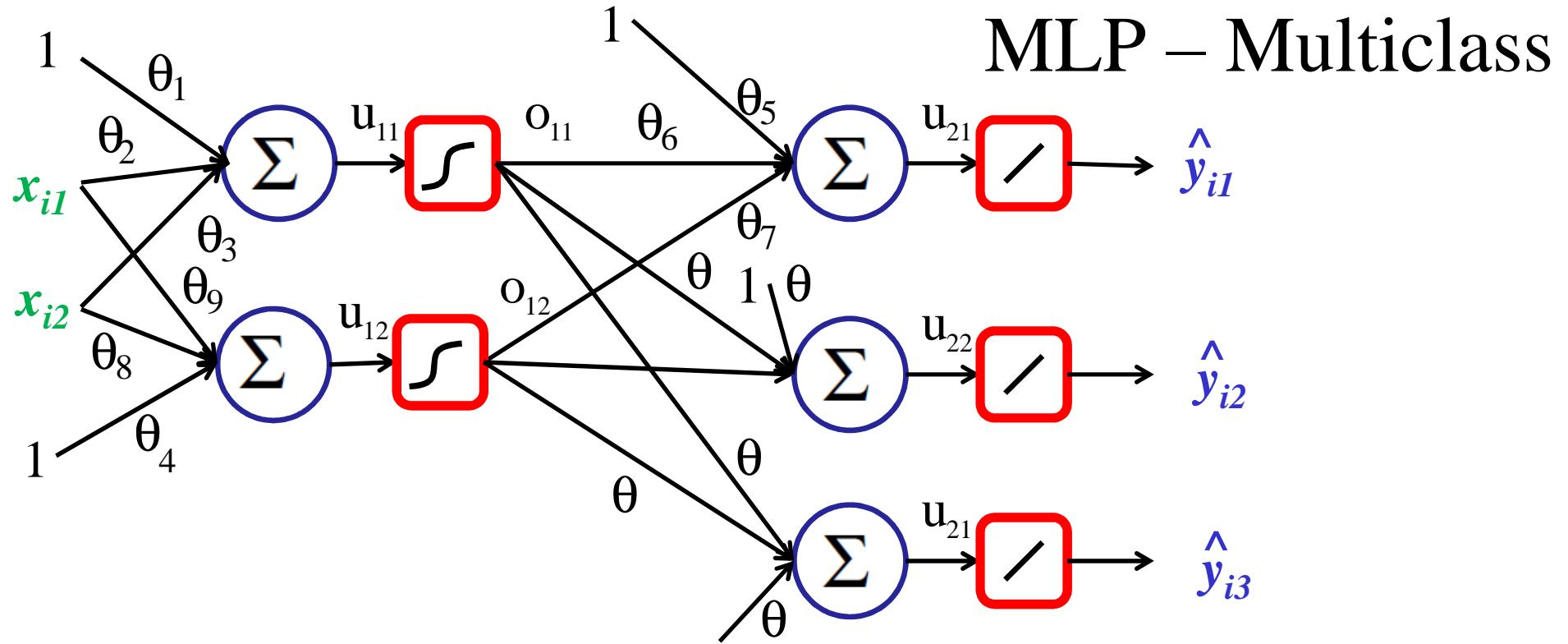


Data :

x_{i1}	x_{i2}	y_{i1}	y_{i2}	y_{i3}
0.2	0.3	0	1	0
-5	-6	1	0	0
-20	4	1	0	0
42	6.8	0	0	1

class 2
class 1
" 1
class 3





To get a probabilistic model, define: SOFTMAX

$$P(\underline{y_i = (0 \ 1 \ 0)} | x_i, \theta) = P(\underline{y_i = 2} | x_i, \theta) =$$

$$\frac{e^{\hat{y}_2}}{e^{\hat{y}_1} + e^{\hat{y}_2} + e^{\hat{y}_3}}$$

$$\mathbb{I}_2(y_i) = \begin{cases} 1 & y_i = 2 \\ 0 & \text{o.w.} \end{cases}$$

MLP – Multiclass

Then,

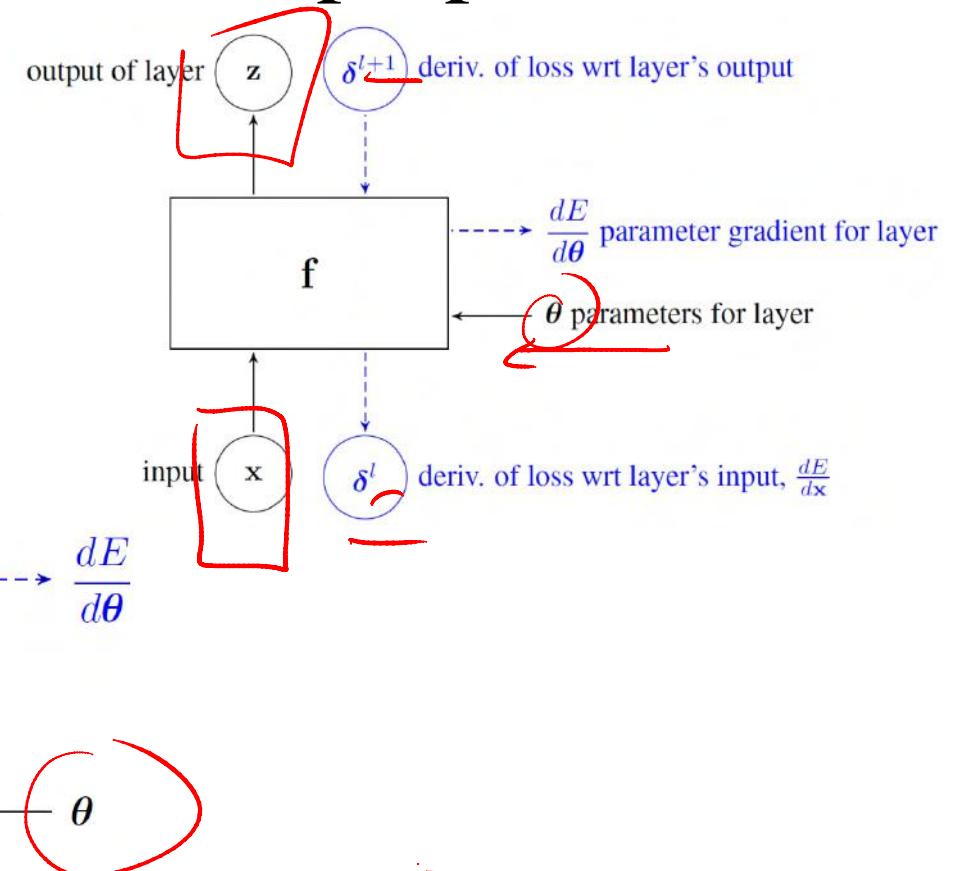
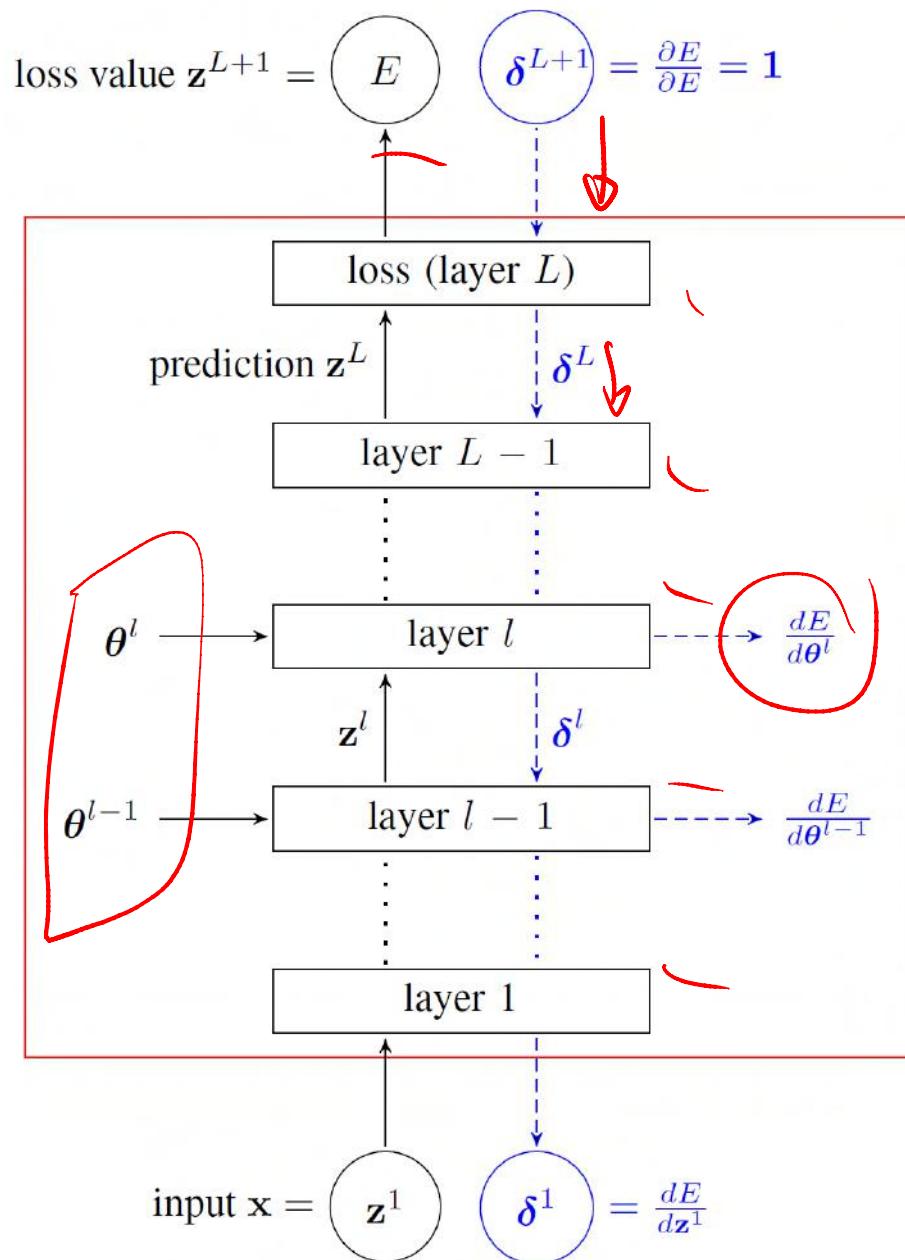
$$P(y_i | x_i, \theta) = \left[\frac{e^{\hat{y}_{i1}}}{e^{\hat{y}_{i1}} + e^{\hat{y}_{i2}} + e^{\hat{y}_{i3}}} \right] \mathbb{I}_1(y_i) + \left[\frac{e^{\hat{y}_{i2}}}{e^{\hat{y}_{i1}} + e^{\hat{y}_{i2}} + e^{\hat{y}_{i3}}} \right] \mathbb{I}_2(y_i) + \left[\frac{e^{\hat{y}_{i3}}}{e^{\hat{y}_{i1}} + e^{\hat{y}_{i2}} + e^{\hat{y}_{i3}}} \right] \mathbb{I}_3(y_i)$$

$$= \begin{cases} e^{\hat{y}_{i1}} / \text{sum} & y_i = 1 \\ e^{\hat{y}_{i2}} / \text{sum} & y_i = 2 \\ e^{\hat{y}_{i3}} / \text{sum} & y_i = 3 \end{cases}$$

Cost:

$$C(\theta) = -\log P(y_i | x_i, \theta) = -\sum_{i=1}^n \sum_{j=1}^3 \mathbb{I}_j(y_i) \log \frac{e^{\hat{y}_{ij}}}{\text{sum}}$$

Deep learning & backprop



```

1 | model = nn.Sequential()
2 | model.add( nn.Linear(2,3) )
3 | model.add( nn.LogSoftMax() )

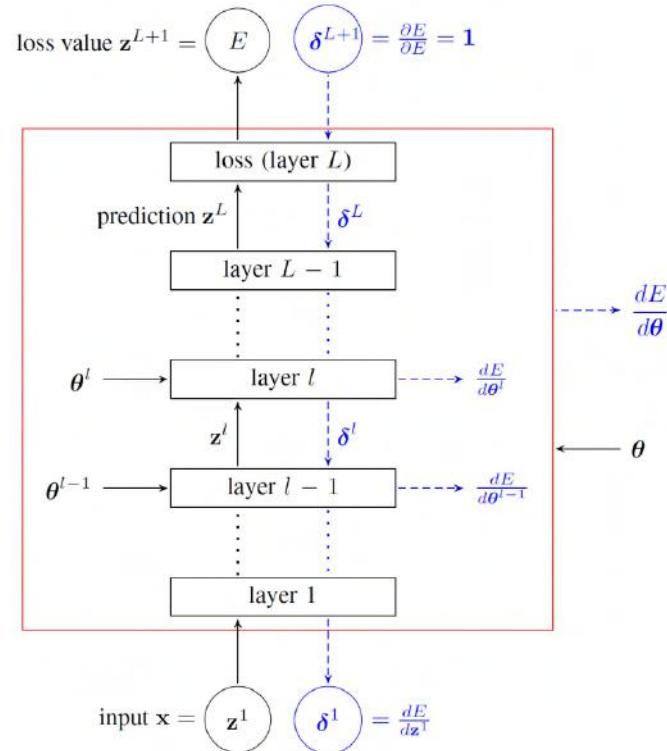
```

```

1 | -- params/gradients
2 | x, dl_dx = model.getParameters()

```

Deep learning & backprop



Forward pass:

$$\mathbf{z}^{l+1} = \mathbf{f}^l(\mathbf{z}^l; \theta^l)$$

Backpropagation (Jacobians):

$$\delta^l := \frac{\partial E}{\partial \mathbf{z}^l} = \frac{\partial E}{\partial \mathbf{z}^{l+1}} \frac{\partial \mathbf{z}^{l+1}}{\partial \mathbf{z}^l} = \delta^{l+1} \frac{\partial \mathbf{f}^l(\mathbf{z}^l; \theta^l)}{\partial \mathbf{z}^l}$$

$$\delta_i^l = \sum_j \frac{\partial E}{\partial z_j^{l+1}} \frac{\partial z_j^{l+1}}{\partial z_i^l} = \sum_j \delta_j^{l+1} \frac{\partial f_j^l(\mathbf{z}^l; \theta^l)}{\partial z_i^l}$$

$$\frac{\partial E}{\partial \theta^l} = \frac{\partial E}{\partial \mathbf{z}^{l+1}} \frac{\partial \mathbf{z}^{l+1}}{\partial \theta^l} = \delta^{l+1} \frac{\partial \mathbf{f}^l(\mathbf{z}^l; \theta^l)}{\partial \theta^l}$$

$$\frac{\partial E}{\partial \theta_i^l} = \sum_j \frac{\partial E}{\partial z_j^{l+1}} \frac{\partial z_j^{l+1}}{\partial \theta_i^l} = \sum_j \delta_j^{l+1} \frac{\partial f_j^l(\mathbf{z}^l; \theta^l)}{\partial \theta_i^l}$$

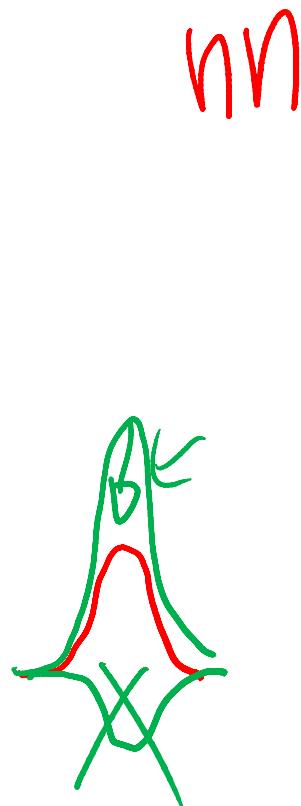
Reverse auto-diff: why it works

$$\underline{f^L}(\underline{\mathbf{f}}^{L-1}(\dots \underline{\mathbf{f}}^3(\underline{\mathbf{f}}^1(\mathbf{x}))))$$

$$\frac{\partial f^L(\mathbf{f}^{L-1}(\dots \mathbf{f}^2(\mathbf{f}^1(\mathbf{x}))))}{\partial \mathbf{x}} = \frac{\cancel{\partial f^L}}{\cancel{\partial \mathbf{f}^{L-1}}} \cdot \frac{\cancel{\partial \mathbf{f}^{L-1}}}{\cancel{\partial \mathbf{f}^{L-2}}} \cdots \frac{\cancel{\partial \mathbf{f}^2}}{\cancel{\partial \mathbf{f}^1}} \cdot \frac{\cancel{\partial \mathbf{f}^1}}{\partial \mathbf{x}}$$

$$\frac{\partial f^L(\mathbf{f}^{L-1}(\dots \mathbf{f}^2(\mathbf{f}^1(\mathbf{x}))))}{\partial \mathbf{x}} = \left(\left(\left(\left(\frac{\partial f^L}{\partial \mathbf{f}^{L-1}} \cdot \frac{\partial \mathbf{f}^{L-1}}{\partial \mathbf{f}^{L-2}} \right) \cdots \right) \frac{\partial \mathbf{f}^2}{\partial \mathbf{f}^1} \right) \cdot \frac{\partial \mathbf{f}^1}{\partial \mathbf{x}} \right)$$

Deep learning: linear layer



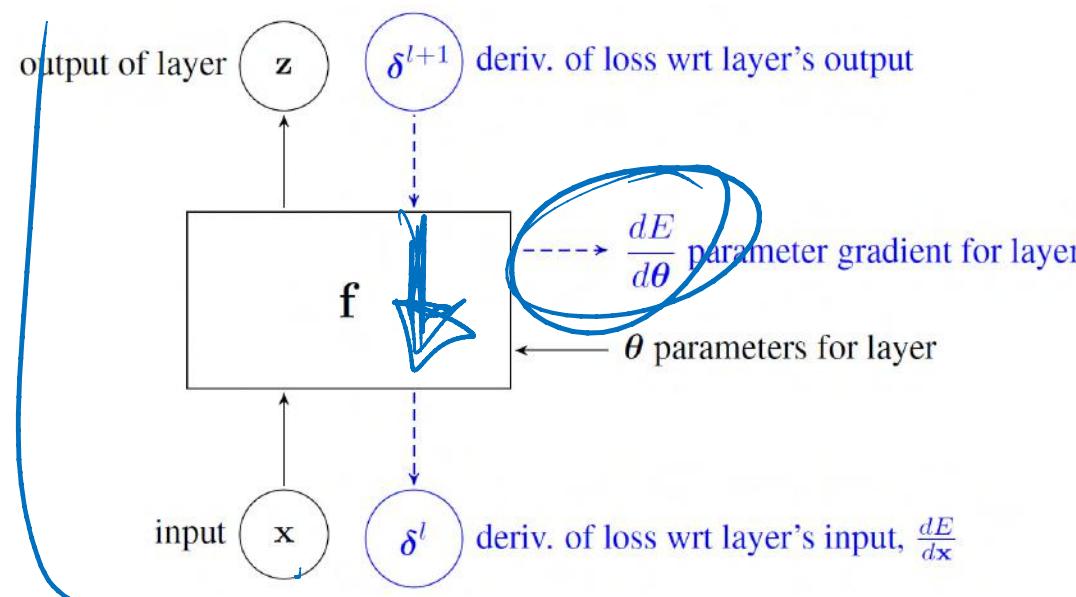
nn

$$z_j = f_j(\mathbf{x}; \boldsymbol{\theta}_j) = \sum_i x_i \theta_{ij}$$

$$\delta_i^l = \sum_j \delta_{j+1}^{l+1} \frac{\partial f_j(\mathbf{x}; \boldsymbol{\theta}_j)}{\partial x_i} = \sum_j \delta_{j+1}^{l+1} \theta_{ij}$$

$$\frac{\partial E}{\partial \theta_{ij}} = \sum_j \delta_{j+1}^{l+1} \frac{\partial f_j(\mathbf{x}; \boldsymbol{\theta}_j)}{\partial \theta_{ij}} = \delta_{j+1}^{l+1} x_i$$

Diagram illustrating a neural network layer. Inputs x_i are multiplied by weights θ_{ij} to produce outputs z_j . Red annotations indicate $j=1, j=2, j=k$ and i . Blue annotations indicate o_1, o_2, o_n and Q_j .

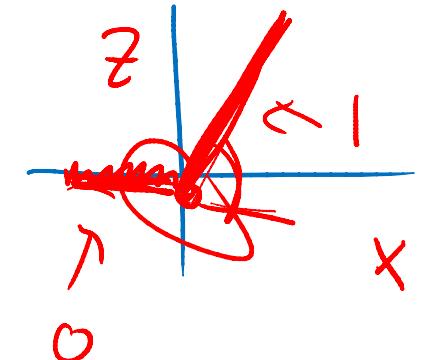


Deep learning: ReLU layer

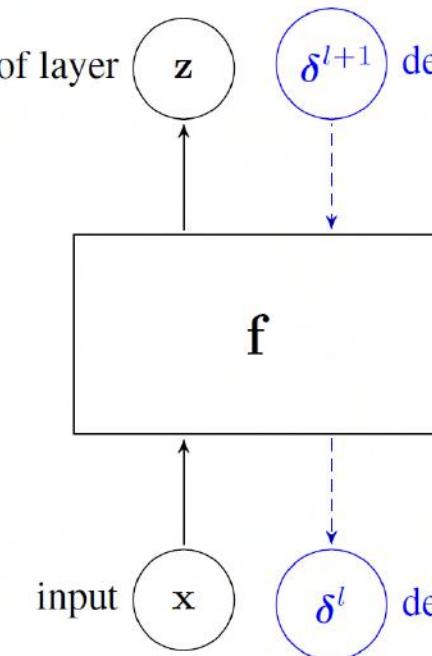
Fukushima

$$z_j = f_j(x_j) = \max(0, x_j)$$

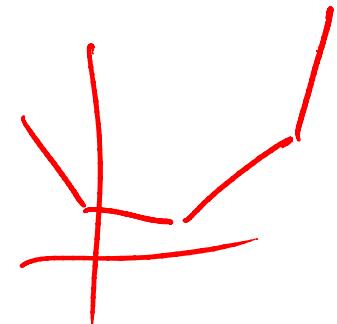
$$\delta_i^l = \sum_j \delta_{j+1}^{l+1} \frac{\partial f_j(x_j)}{\partial x_i} = \delta_{i+1}^{l+1} \mathbb{I}_{[x_i > 0]}$$



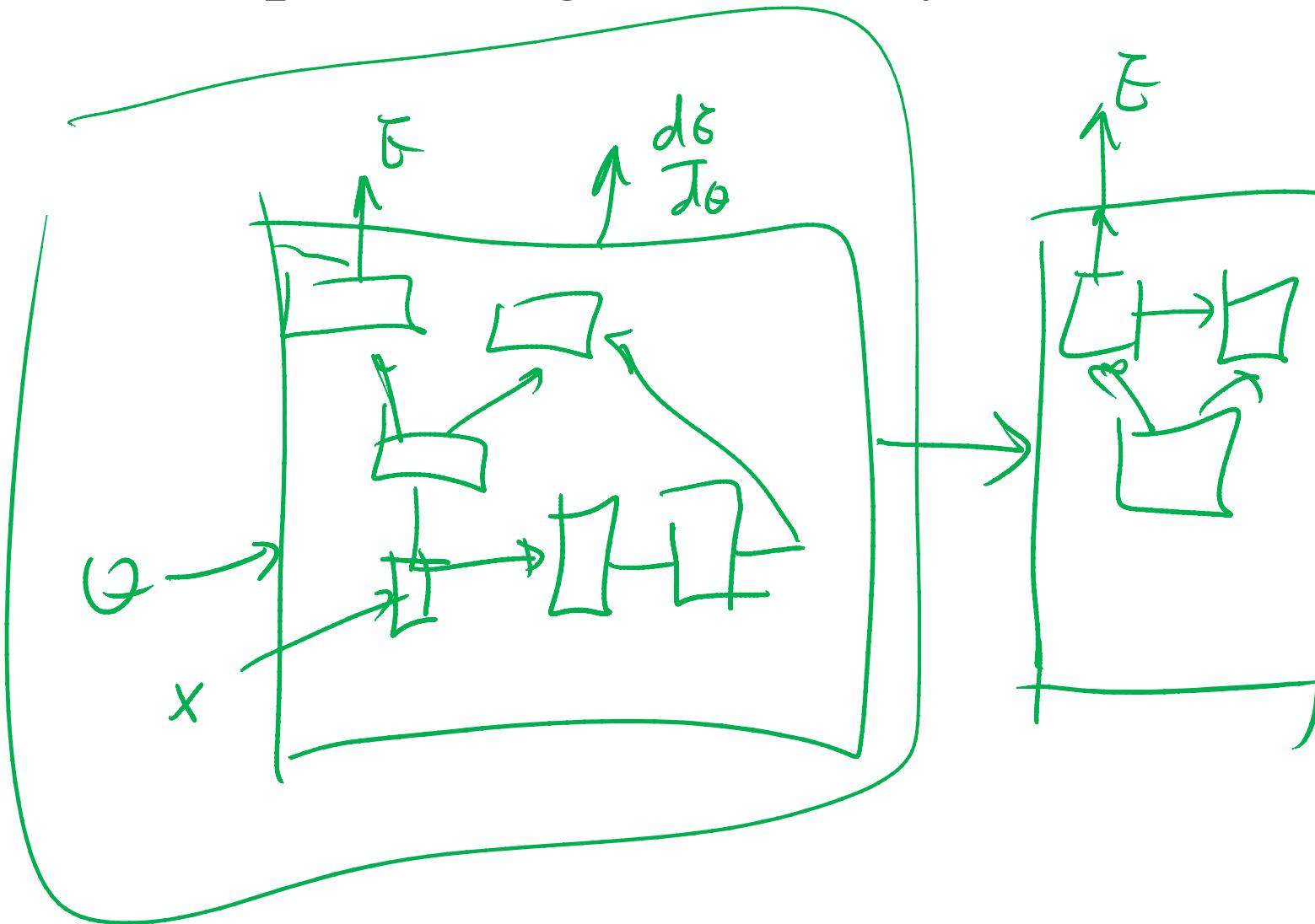
output of layer z δ^{l+1} deriv. of loss wrt layer's output



maxout



Deep learning: extremely flexible!



Next lecture

In the next lecture, we will look at a successful type of neural network that is very popular in speech and object recognition, known as a convolutional neural network.