and the charge at the intersection of the four perpendiculars is

$$-\frac{1}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}}}.$$

*System of Four Spheres Intersecting at Right Angles under the Action of an Electrified Point.*

170.] Let the four spheres be $A, B, C, D$, and let the electrified point be $O$. Draw four spheres $A_1$, $B_1$, $C_1$, $D_1$, of which any one, $A_1$, passes through $O$ and cuts three of the spheres, in this case $B$, $C$, and $D$, at right angles. Draw six spheres $(ab)$, $(ac)$, $(ad)$, $(bc)$, $(bd)$, $(cd)$, of which each passes through $O$ and through the circle of intersection of two of the original spheres.

The three spheres $B_1$, $C_1$, $D_1$ will intersect in another point besides $O$. Let this point be called $A'$, and let $B'$, $C'$, and $D'$ be the intersections of $C_1$, $D_1$, $A_1$, of $D_1$, $A_1$, $B_1$, and of $A_1$, $B_1$, $C_1$ respectively. Any two of these spheres, $A_1$, $B_1$, will intersect one of the six $(cd)$ in a point $(a'b')$. There will be six such points.

Any one of the spheres, $A_1$, will intersect three of the six $(ab)$, $(ac)$, $(ad)$ in a point $a'$. There will be four such points. Finally, the six spheres $(ab)$, $(ac)$, $(ad)$, $(cd)$, $(db)$, $(bc)$, will intersect in one point $S$.

If we now invert the system with respect to a sphere of radius $R$ and centre $O$, the four spheres $A, B, C, D$ will be inverted into spheres, and the other ten spheres will become planes. Of the points of intersection the first four $A'$, $B'$, $C'$, $D'$ will become the centres of the spheres, and the others will correspond to the other eleven points in the preceding article. These fifteen points form the image of $O$ in the system of four spheres.

At the point $A'$, which is the image of $O$ in the sphere $A$, we must place a charge equal to the image of $O$, that is, $-\frac{\alpha}{a}$, where $\alpha$ is the radius of the sphere $A$, and $a$ is the distance of its centre from $O$. In the same way we must place the proper charges at $B'$, $C'$, $D'$.

The charges for each of the other eleven points may be found from the expressions in the last article by substituting $\alpha'$, $\beta'$, $\gamma'$, $\delta'$ for $\alpha$, $\beta$, $\gamma$, $\delta$, and multiplying the result for each point by the distance of the point from $O$, where

$$\alpha'=-\frac{\alpha}{a^2-\alpha^2}, \quad \beta'=-\frac{\beta}{b^2-\beta^2}, \quad \gamma'=-\frac{\gamma}{c^2-\gamma^2}, \quad \delta'=-\frac{\delta}{d^2-\delta^2}.$$

*Two Spheres not Intersecting.*

171.] When a space is bounded by two spherical surfaces which do not intersect, the successive images of an influencing point within this space form two infinite series, all of which lie beyond the spherical surfaces, and therefore fulfil the condition of the applicability of the method of electrical images.

Any two non-intersecting spheres may be inverted into two concentric spheres by assuming as the point of inversion either of the two common inverse points of the pair of spheres.

We shall begin, therefore, with the case of two uninsulated concentric spherical surfaces, subject to the induction of an electrified point placed between them.

Let the radius of the first be $b$, and that of the second $be^{\varpi}$, and let the distance of the influencing point from the centre be $r = be^{u}$.

Then all the successive images will be on the same radius as the influencing point.

Let $Q_0$, Fig. 14, be the image of $P$ in the first sphere, $P_1$ that of $Q_0$ in the second sphere, $Q_1$ that of $P_1$ in the first sphere, and so on; then

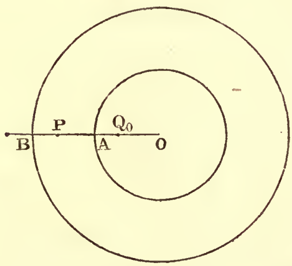


Fig. 14.

$$OP_s . OQ_s = b^2,$$

and $$OP_s . OQ_{s-1} = b^2 e^{2\varpi},$$

also $$OQ_0 = be^{-u},$$

$$OP_1 = be^{u+2\varpi},$$

$$OQ_1 = be^{-(u+2\varpi)}, \text{ \&c.}$$

Hence $$OP_s = be^{(u+2s\varpi)},$$

$$OQ_s = be^{-(u+2s\varpi)}.$$

If the charge of $P$ is denoted by $P$, then

$$P_s = Pe^{s\varpi}, \qquad Q_s = -Pe^{-(u+s\varpi)}.$$

Next, let $Q_1'$ be the image of $P$ in the second sphere, $P_1'$ that of $Q_1'$ in the first, &c.,

$$OQ_1' = be^{2\varpi-u}, \qquad OP_1' = be^{u-2\varpi},$$

$$OQ_2' = be^{4\varpi-u}, \qquad OP_2' = be^{u-4\varpi};$$

$$OP_s' = be^{u-2s\varpi}, \qquad OQ_s' = be^{2s\varpi-u},$$

$$P_s' = Pe^{-s\varpi}, \qquad Q_s' = Pe^{s\varpi-u}.$$

Of these images all the $P$'s are positive, and all the $Q$'s negative, all the $P'$'s and $Q$'s belong to the first sphere, and all the $P$'s and $Q'$'s to the second.

The images within the first sphere form a converging series, the sum of which is

$$-P\frac{e^{\varpi-u}-1}{e^{\varpi}-1}.$$

This therefore is the quantity of electricity on the first or interior sphere. The images outside the second sphere form a diverging series, but the surface-integral of each with respect to the spherical surface is zero. The charge of electricity on the exterior spherical surface is therefore

$$P\left(\frac{e^{\varpi-u}-1}{e^{\varpi}-1}-1\right)=-P\frac{e^{\varpi}-e^{\varpi-u}}{e^{\varpi}-1}.$$

If we substitute for these expressions their values in terms of $OA$, $OB$, and $OP$, we find

$$\text{charge on } A=-P\frac{OA}{OP}\frac{PB}{AB},$$

$$\text{charge on } B=-P\frac{OB}{OP}\frac{AP}{AB}.$$

If we suppose the radii of the spheres to become infinite, the case becomes that of a point placed between two parallel planes $A$ and $B$. In this case these expressions become

$$\text{charge on } A=-P\frac{PB}{AB},$$

$$\text{charge on } B=-P\frac{AP}{AB}.$$

172.] In order to pass from this case to that of any two spheres not intersecting each other, we begin by finding the two common inverse points $O$, $O'$ through which all circles pass that are orthogonal to both spheres. Then, inverting the system with respect to either of these points, the spheres become concentric, as in the first case.

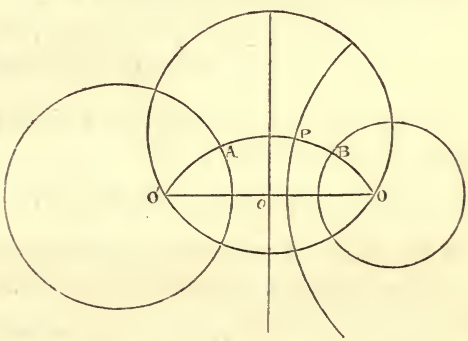

Fig. 15.

The radius $OAPB$ on which the successive images lie becomes an arc of a circle through $O$ and $O'$, and the ratio of $O'P$ to $OP$ is equal to $Ce^u$ where $C$ is a numerical quantity which for simplicity we may make equal to unity.

We therefore put

$$u = \log\frac{O'P}{OP}, \qquad \alpha = \log\frac{O'A}{OA}, \qquad \beta = \log\frac{O'B}{OB}.$$

Let $\qquad \beta - \alpha = \varpi, \qquad u - \alpha = \theta.$

Then all the successive images of $P$ will lie on the arc $OAPBO'$.

The position of the image of $P$ in $A$ is $Q_0$ where

$$u(Q_0) = \log\frac{O'Q}{OQ} = 2\alpha - u.$$

That of $Q_0$ in $B$ is $P_1$ where

$$u(P_1) = \log\frac{O'P_1}{OP_1} = u + 2\varpi.$$

Similarly

$$u(P_s) = u + 2s\varpi, \qquad u(Q_s) = 2\alpha - u - 2s\varpi.$$

In the same way if the successive images of $P$ in $B$, $A$, $B$, &c. are $Q_0'$, $P_1'$, $Q_1'$, &c.,

$$u(Q_0') = 2\beta - u, \qquad u(P_1') = u - 2\varpi;$$
$$u(P_s') = u - 2s\varpi, \qquad u(Q_s') = 2\beta - u + 2s\varpi.$$

To find the charge of any image $P_s$ we observe that in the inverted figure its charge is

$$P\sqrt{\frac{OP_s}{OP}}.$$

In the original figure we must multiply this by $O'P_s$. Hence the charge of $P_s$ in the dipolar figure is

$$P\sqrt{\frac{OP_s . O'P_s}{OP . O'P}}.$$

If we make $\xi = \sqrt{OP . O'P}$, and call $\xi$ the parameter of the point $P$, then we may write

$$P_s = \frac{\xi_s}{\xi}P,$$

or the charge of any image is proportional to its parameter.

If we make use of the curvilinear coordinates $u$ and $v$, such that

$$e^{u+\sqrt{-1}v} = \frac{x+\sqrt{-1}y-k}{x+\sqrt{-1}y+k},$$

then $\qquad x = -\frac{k\sin hu}{\cos hu - \cos v}, \qquad y = \frac{k\sin v}{\cos hu - \cos v};$

$$x^2+(y-k\cot v)^2 = k^2\operatorname{cosec}^2 v,$$
$$(x+k\cot hu)^2+y^2 = k^2\operatorname{cosec} h^2 u,$$
$$\cot v = \frac{x^2+y^2-k^2}{2ky}, \quad \cot hu = -\frac{x^2+y^2+k^2}{2kx};$$
$$\xi = \frac{\sqrt{2k}}{\sqrt{\cos hu-\cos v}}\,*.$$

Since the charge of each image is proportional to its parameter, $\xi$, and is to be taken positively or negatively according as it is of the form $P$ or $Q$, we find

$$P_s = \frac{P\sqrt{\cos hu-\cos v}}{\sqrt{\cos h(u+2s\varpi)-\cos v}},$$
$$Q_s = -\frac{P\sqrt{\cos hu-\cos v}}{\sqrt{\cos h(2\alpha-u-2s\varpi)-\cos v}},$$
$$P'_s = \frac{P\sqrt{\cos hu-\cos v}}{\sqrt{\cos h(u-2s\varpi)-\cos v}},$$
$$Q'_s = -\frac{P\sqrt{\cos hu-\cos v}}{\sqrt{\cos h(2\beta-u+2s\varpi)-\cos v}}.$$

We have now obtained the positions and charges of the two infinite series of images. We have next to determine the total charge on the sphere $A$ by finding the sum of all the images within it which are of the form $Q$ or $P'$. We may write this

$$P\sqrt{\cos hu-\cos v}\sum_{s=1}^{s=\infty}\frac{1}{\sqrt{\cos h(u-2s\varpi)-\cos v}},$$
$$-P\sqrt{\cos hu-\cos v}\sum_{s=0}^{s=\infty}\frac{1}{\sqrt{\cos h(2\alpha-u-2s\varpi)-\cos v}}.$$

In the same way the total induced charge on $B$ is

$$P\sqrt{\cos hu-\cos v}\sum_{s=1}^{s=\infty}\frac{1}{\sqrt{\cos h(u+2s\varpi)-\cos v}},$$
$$-P\sqrt{\cos hu-\cos v}\sum_{s=0}^{s=\infty}\frac{1}{\sqrt{\cos h(2\beta-u+2s\varpi)-\cos v}}.$$

* In these expressions we must remember that

$$2\cos hu = e^u+e^{-u}, \qquad 2\sin hu = e^u-e^{-u},$$

and the other functions of $u$ are derived from these by the same definitions as the corresponding trigonometrical functions.

The method of applying dipolar coordinates to this case was given by Thomson in *Liouville's Journal* for 1847. See Thomson's reprint of *Electrical Papers*, § 211, 212. In the text I have made use of the investigation of Prof. Betti, *Nuovo Cimento*, vol. xx, for the analytical method, but I have retained the idea of electrical images as used by Thomson in his original investigation, *Phil. Mag.*, 1853.

173.] We shall apply these results to the determination of the coefficients of capacity and induction of two spheres whose radii are $a$ and $b$, and the distance of whose centres is $c$.

In this case

$$k = \frac{\sqrt{a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2}}{2c};$$

$$\sin h.\alpha = \frac{k}{a}, \qquad \sin h.\beta = \frac{k}{b}.$$

Let the sphere $A$ be at potential unity, and the sphere $B$ at potential zero.

Then the successive images of a charge $a$ placed at the centre of the sphere $A$ will be those of the actual distribution of electricity. All the images will lie on the axis between the poles and the centres of the spheres.

The values of $u$ and $v$ for the centre of the sphere $A$ are

$$u = 2a, \qquad v = 0.$$

Hence we must substitute $a$ or $k\dfrac{1}{\sin h\,\alpha}$ for $P$, and $2a$ for $u$, and $v=0$ in the equations, remembering that $P$ itself forms part of the charge of $A$. We thus find for the coefficient of capacity of $A$

$$q_{aa} = \quad k\sum_{s=0}^{s=\infty}\frac{1}{\sin h(s\varpi-\alpha)},$$

for the coefficient of induction of $A$ on $B$ or of $B$ on $A$

$$q_{ab} = -k\sum_{s=1}^{s=\infty}\frac{1}{\sin h s\varpi},$$

and for the coefficient of capacity of $B$

$$q_{bb} = \quad k\sum_{s=0}^{s=\infty}\frac{1}{\sin h(\beta+s\varpi)}.$$

To calculate these quantities in terms of $a$ and $b$, the radii of the spheres, and of $c$ the distance between their centres, we make use of the following quantities

$$p = e^{\alpha} = \sqrt{\frac{k^2}{a^2}+1}-\frac{k}{a},$$

$$q = e^{\beta} = \sqrt{\frac{k^2}{b^2}+1}+\frac{k}{b},$$

$$\frac{q}{p} = r = e^{\varpi} = \left(\sqrt{\frac{k^2}{a^2}+1}+\frac{k}{a}\right)\left(\sqrt{\frac{k^2}{b^2}+1}+\frac{k}{b}\right).$$

We may now write the hyperbolic sines in terms of $p$, $q$, $r$; thus

$$q_{aa} = \sum_{s=0}^{s=\infty} \frac{2k}{\dfrac{r^s}{p} - \dfrac{p}{r^s}},$$

$$q_{ab} = -\sum_{s=1}^{s=\infty} \frac{2k}{r^s - \dfrac{1}{r^s}},$$

$$q_{bb} = \sum_{s=0}^{s=\infty} \frac{2k}{qr^s - \dfrac{1}{qr^s}}.$$

Proceeding to the actual calculation we find, either by this process or by the direct calculation of the successive images as shewn in Sir W. Thomson's paper, which is more convenient for the earlier part of the series,

$$q_{aa} = a + \frac{a^2 b}{c^2 - b^2} + \frac{a^3 b^2}{(c^2 - b^2 + ac)(c^2 - b^2 - ac)} + \&c.,$$

$$q_{ab} = -\frac{ab}{c} - \frac{a^2 b^2}{c(c^2 - a^2 - b^2)} - \frac{a^3 b^3}{c(c^2 - a^2 - b^2 + ab)(c^2 - a^2 - b^2 - ab)} - \&c.$$

$$q_{bb} = b + \frac{ab^2}{c^2 - a^2} + \frac{a^2 b^3}{(c^2 - a^2 + bc)(c^2 - a^2 - bc)} + \&c.$$

174.] We have then the following equations to determine the charges $E_a$ and $E_b$ of the two spheres when electrified to potentials $V_a$ and $V_b$ respectively,

$$E_a = V_a q_{aa} + V_b q_{ab},$$
$$E_b = V_a q_{ab} + V_b q_{bb}.$$

If we put $$q_{aa} q_{bb} - q_{ab}^2 = D = \frac{1}{D'},$$

and $$p_{aa} = q_{bb} D', \quad p_{ab} = -q_{ab} D', \quad p_{bb} = q_{aa} D',$$

whence $$p_{aa} p_{bb} - p_{ab}^2 = D';$$

then the equations to determine the potentials in terms of the charges are

$$V_a = p_{aa} E_a + p_{ab} E_b,$$
$$V_b = p_{ab} E_a + p_{bb} E_b,$$

and $p_{aa}$, $p_{ab}$, and $p_{bb}$ are the coefficients of potential.

The total energy of the system is, by Art. 85,

$$Q = \tfrac{1}{2}(E_a V_a + E_b V_b),$$
$$= \tfrac{1}{2}(V_a^2 q_{aa} + 2 V_a V_b q_{ab} + V_b^2 q_{bb}),$$
$$= \tfrac{1}{2}(E_a^2 p_{aa} + 2 E_a E_b p_{ab} + E_b^2 p_{bb}).$$

The repulsion between the spheres is therefore, by Arts. 92, 93,

$$F = \tfrac{1}{2}\left\{ V_a^2 \frac{dq_{aa}}{dc} + 2V_a V_b \frac{dq_{ab}}{dc} + V_b^2 \frac{dq_{bb}}{dc} \right\},$$

$$= -\tfrac{1}{2}\left\{ E_a^2 \frac{dp_{aa}}{dc} + 2E_a E_b \frac{dp_{ab}}{dc} + E_b^2 \frac{dp_{bb}}{dc} \right\},$$

where $c$ is the distance between the centres of the spheres.

Of these two expressions for the repulsion, the first, which expresses it in terms of the potentials of the spheres and the variations of the coefficients of capacity and induction, is the most convenient for calculation.

We have therefore to differentiate the $q$'s with respect to $c$. These quantities are expressed as functions of $k$, $\alpha$, $\beta$, and $\varpi$, and must be differentiated on the supposition that $a$ and $b$ are constant. From the equations

$$k = a \sin h\,\alpha = b \sin h\beta = c\,\frac{\sin h\,\alpha \sin h\beta}{\sin h\,\varpi},$$

we find

$$\frac{d\alpha}{dc} = \frac{\sin h\,\alpha \cos h\,\beta}{k \sin h\,\varpi},$$

$$\frac{d\beta}{dc} = \frac{\cos h\,\alpha \sin h\,\beta}{k \sin h\,\varpi},$$

$$\frac{d\varpi}{dc} = \frac{1}{k},$$

$$\frac{dk}{dc} = \frac{\cos h\,\alpha \cos h\,\beta}{\sin h\,\varpi};$$

whence we find

$$\frac{dq_{aa}}{dc} = \frac{\cos h\,\alpha \cos h\,\beta}{\sin h\,\varpi}\,\frac{q_{aa}}{k} - \sum_{s=0}^{s=\infty} \frac{(sc - a\cos h\,\beta)\cos h\,(s\varpi - \alpha)}{c\,(\sin h\,(s\varpi - \alpha))^2},$$

$$\frac{dq_{ab}}{dc} = \frac{\cos h\,\alpha \cos h\,\beta}{\sin h\,\varpi}\,\frac{q_{ab}}{k} + \sum_{s=1}^{s=\infty} \frac{s\cos h\,s\varpi}{(\sin h\,s\varpi)^2},$$

$$\frac{dq_{bb}}{dc} = \frac{\cos h\,\alpha \cos h\,\beta}{\sin h\,\varpi}\,\frac{q_{bb}}{k} - \sum_{s=0}^{s=\infty} \frac{(sc + b\cos h\,\alpha)\cos h\,(\beta + s\varpi)}{c\,(\sin h\,(\beta + s\varpi))^2}.$$

Sir William Thomson has calculated the force between two spheres of equal radius separated by any distance less than the diameter of one of them. For greater distances it is not necessary to use more than two or three of the successive images.

The series for the differential coefficients of the $q$'s with respect to $c$ are easily obtained by direct differentiation

$$\frac{dq_{aa}}{dc} = -\frac{2a^2bc}{(c^2-b^2)^2} - \frac{2a^3b^2c(2c^2-2b^2-a^2)}{(c^2-b^2+ac)^2(c^2-b^2-ac)^2} - \&c.,$$

$$\frac{dq_{ab}}{dc} = \frac{ab}{c^2} + \frac{a^2b^2(3c^2-a^2-b^2)}{c^2(c^2-a^2-b^2)}$$
$$+ \frac{a^3b^3\{(5c^2-a^2-b^2)(c^2-a^2-b^2)-a^2b^2\}}{c^2(c^2-a^2-b^2+ab)^2(c^2-a^2-b^2-ab)^2} - \&c.,$$

$$\frac{dq_{bb}}{dc} = -\frac{2ab^2c}{(c^2-a^2)^2} - \frac{2a^2b^3c(2c^2-2a^2-b^2)}{(c^2-a^2+bc)^2(c^2-a^2-bc)^2} - \&c.$$

*Distribution of Electricity on Two Spheres in Contact.*

175.] If we suppose the two spheres at potential unity and not influenced by any other point, then, if we invert the system with respect to the point of contact, we shall have two parallel planes, distant $\frac{1}{2a}$ and $\frac{1}{2b}$ from the point of inversion, and electrified by the action of a unit of electricity at that point.

There will be a series of positive images, each equal to unity, at distances $s\left(\frac{1}{a}+\frac{1}{b}\right)$ from the origin, where $s$ may have any integer value from $-\infty$ to $+\infty$.

There will also be a series of negative images each equal to $-1$, the distances of which from the origin, reckoned in the direction of $a$, are $\frac{1}{a}+s\left(\frac{1}{a}+\frac{1}{b}\right)$.

When this system is inverted back again into the form of the two spheres in contact, we have a corresponding series of negative images, the distances of which from the point of contact are of the form $\frac{1}{s\left(\frac{1}{a}+\frac{1}{b}\right)}$, where $s$ is positive for the sphere $A$ and negative for the sphere $B$. The charge of each image, when the potential of the spheres is unity, is numerically equal to its distance from the point of contact, and is always negative.

There will also be a series of positive images whose distances from the point of contact measured in the direction of the centre of $a$, are of the form $\frac{1}{\frac{1}{a}+s\left(\frac{1}{a}+\frac{1}{b}\right)}$.

When $s$ is zero, or a positive integer, the image is in the sphere $A$.

When $s$ is a negative integer the image is in the sphere $B$.

The charge of each image is measured by its distance from the origin and is always positive.

The total charge of the sphere $A$ is therefore

$$E_a = \sum_{s=0}^{s=\infty} \frac{1}{\frac{1}{a} + s\left(\frac{1}{a} + \frac{1}{b}\right)} - \frac{ab}{a+b}\sum_{s=1}^{s=\infty}\frac{1}{s}.$$

Each of these series is infinite, but if we combine them in the form

$$E_a = \sum_{s=1}^{s=\infty} \frac{a^2 b}{s(a+b)\,(s(a+b)-a)}$$

the series becomes converging.

In the same way we find for the charge of the sphere $B$,

$$E_b = \sum_{s=1}^{s=\infty} \frac{ab}{s(a+b)-b} - \frac{ab}{a+b}\sum_{s=-1}^{s=-\infty}\frac{1}{s},$$

$$= \sum_{s=1}^{s=\infty} \frac{ab^2}{s(a+b)\,\{s(a+b)-b\}}.$$

The values of $E_a$ and $E_b$ are not, so far as I know, expressible in terms of known functions. Their difference, however, is easily expressed, for

$$E_a - E_b = \sum_{s=-\infty}^{s=\infty} \frac{ab}{b+s(a+b)},$$

$$= \frac{\pi ab}{a+b}\cot\frac{\pi b}{a+b}.$$

When the spheres are equal the charge of each for potential unity is

$$E_a = a\sum_{s=1}^{s=\infty}\frac{1}{2s(2s-1)},$$

$$= a\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \&c.\right),$$

$$= a\log_e 2 = 1.0986\,a.$$

When the sphere $A$ is very small compared with the sphere $B$ the charge on $A$ is

$$E_a = \frac{a^2}{b}\sum_{s=1}^{s=\infty}\frac{1}{s^2} \text{ approximately};$$

$$\text{or} \quad E_a = \frac{\pi^2}{6}\frac{a^2}{b}.$$

The charge on $B$ is nearly the same as if $A$ were removed, or

$$E_b = b.$$

The mean density on each sphere is found by dividing the charge by the surface. In this way we get

$$\sigma_a = \frac{E_a}{4\pi a^2} = \frac{\pi}{24\,b},$$

$$\sigma_b = \frac{E_b}{4\pi b^2} = \frac{1}{4\pi b},$$

$$\sigma_a = \frac{\pi^2}{6}\sigma_b.$$

Hence, if a very small sphere is made to touch a very large one, the mean density on the small sphere is equal to that on the large sphere multiplied by $\frac{\pi^2}{6}$, or 1.644936.

*Application of Electrical Inversion to the case of a Spherical Bowl.*

176.] One of the most remarkable illustrations of the power of Sir W. Thomson's method of Electrical Images is furnished by his investigation of the distribution of electricity on a portion of a spherical surface bounded by a small circle. The results of this investigation, without proof, were communicated to M. Liouville and published in his *Journal* in 1847. The complete investigation is given in the reprint of Thomson's *Electrical Papers*, Article XV. I am not aware that a solution of the problem of the distribution of electricity on a finite portion of any curved surface has been given by any other mathematician.

As I wish to explain the method rather than to verify the calculation, I shall not enter at length into either the geometry or the integration, but refer my readers to Thomson's work.

*Distribution of Electricity on an Ellipsoid.*

177.] It is shewn by a well-known method * that the attraction of a shell bounded by two similar and similarly situated and concentric ellipsoids is such that there is no resultant attraction on any point within the shell. If we suppose the thickness of the shell to diminish indefinitely while its density increases, we ultimately arrive at the conception of a surface-density varying as the perpendicular from the centre on the tangent plane, and since the resultant attraction of this superficial distribution on any point within the ellipsoid is zero, electricity, if so distributed on the surface, will be in equilibrium.

Hence, the surface-density at any point of an ellipsoid undisturbed by external influence varies as the distance of the tangent plane from the centre.

* Thomson and Tait's *Natural Philosophy*, § 520, or Art. 150 of this book.

### *Distribution of Electricity on a Disk.*

By making two of the axes of the ellipsoid equal, and making the third vanish, we arrive at the case of a circular disk, and at an expression for the surface-density at any point $P$ of such a disk when electrified to the potential $V$ and left undisturbed by external influence. If $\sigma$ be the surface-density on one side of the disk, and if $KPL$ be a chord drawn through the point $P$, then

$$\sigma = \frac{V}{2\pi^2\sqrt{KP.PL}}.$$

### *Application of the Principle of Electric Inversion.*

178.] Take any point $Q$ as the centre of inversion, and let $R$ be the radius of the sphere of inversion. Then the plane of the disk becomes a spherical surface passing through $Q$, and the disk itself becomes a portion of the spherical surface bounded by a circle. We shall call this portion of the surface the *bowl.*

If $S'$ is the disk electrified to potential $V'$ and free from external influence, then its electrical image $S$ will be a spherical segment at potential zero, and electrified by the influence of a quantity $V'R$ of electricity placed at $Q$.

We have therefore by the process of inversion obtained the solution of the problem of the distribution of electricity on a bowl or a plane disk when under the influence of an electrified point in the surface of the sphere or plane produced.

### *Influence of an Electrified Point placed on the unoccupied part of the Spherical Surface.*

The form of the solution, as deduced by the principles already given and by the geometry of inversion, is as follows :

If $C$ is the central point or pole of the spherical bowl $S$, and if $a$ is the distance from $C$ to any point in the edge of the segment, then, if a quantity $q$ of electricity is placed at a point $Q$ in the surface of the sphere produced, and if the bowl $S$ is maintained at potential zero, the density $\sigma$ at any point $P$ of the bowl will be

$$\sigma = \frac{1}{2\pi^2}\,\frac{q}{QP^2}\sqrt{\frac{CQ^2-a^2}{a^2-CP^2}},$$

$CQ$, $CP$, and $QP$ being the straight lines joining the points, $C$, $Q$, and $P$.

It is remarkable that this expression is independent of the radius of the spherical surface of which the bowl is a part. It is therefore applicable without alteration to the case of a plane disk.

*Influence of any Number of Electrified Points.*

Now let us consider the sphere as divided into two parts, one of which, the spherical segment on which we have determined the electric distribution, we shall call the *bowl*, and the other the remainder, or unoccupied part of the sphere on which the influencing point $Q$ is placed.

If any number of influencing points are placed on the remainder of the sphere, the electricity induced by these on any point of the bowl may be obtained by the summation of the densities induced by each separately.

179.] Let the whole of the remaining surface of the sphere be uniformly electrified, the surface-density being $\rho$, then the density at any point of the bowl may be obtained by ordinary integration over the surface thus electrified.

We shall thus obtain the solution of the case in which the bowl is at potential zero, and electrified by the influence of the remaining portion of the spherical surface rigidly electrified with density $\rho$.

Now let the whole system be insulated and placed within a sphere of diameter $f$, and let this sphere be uniformly and rigidly electrified so that its surface-density is $\rho'$.

There will be no resultant force within this sphere, and therefore the distribution of electricity on the bowl will be unaltered, but the potential of all points within the sphere will be increased by a quantity $V$ where

$$V = \frac{2\pi\rho'}{f}.$$

Hence the potential at every point of the bowl will now be $V$.

Now let us suppose that this sphere is concentric with the sphere of which the bowl forms a part, and that its radius exceeds that of the latter sphere by an infinitely small quantity.

We have now the case of the bowl maintained at potential $V$ and influenced by the remainder of the sphere rigidly electrified with superficial density $\rho + \rho'$.

180.] We have now only to suppose $\rho + \rho' = 0$, and we get the case of the bowl maintained at potential $V$ and free from external influence.

If $\sigma$ is the density on either surface of the bowl at a given point when the bowl is at potential zero, and is influenced by the rest of the sphere electrified to density $\rho$, then, when the bowl is maintained at potential $V$, we must increase the density on the outside of the bowl by $\rho'$, the density on the supposed enveloping sphere.

The result of this investigation is that if $f$ is the diameter of the sphere, $a$ the chord of the radius of the bowl, and $r$ the chord of the distance of $P$ from the pole of the bowl, then the surface-density $\sigma$ on the *inside* of the bowl is

$$\sigma = \frac{V}{2\pi^2 f}\left\{\sqrt{\frac{f^2-a^2}{a^2-r^2}} - \tan^{-1}\sqrt{\frac{f^2-a^2}{a^2-r^2}}\right\},$$

and the surface-density on the outside of the bowl at the same point is

$$\sigma + \frac{V}{2\pi f}.$$

In the calculation of this result no operation is employed more abstruse than ordinary integration over part of a spherical surface. To complete the theory of the electrification of a spherical bowl we only require the geometry of the inversion of spherical surfaces.

181.] Let it be required to find the surface-density induced at any point of the bowl by a quantity $q$ of electricity placed at a point $Q$, not now in the spherical surface produced.

Invert the bowl with respect to $Q$, the radius of the sphere of inversion being $R$. The bowl $S$ will be inverted into its image $S'$, and the point $P$ will have $P'$ for its image. We have now to determine the density $\sigma'$ at $P'$ when the bowl $S'$ is maintained at potential $V'$, such that $q = V'R$, and is not influenced by any external force.

The density $\sigma$ at the point $P$ of the original bowl is then

$$\sigma = -\frac{\sigma' R^3}{QP^3},$$

this bowl being at potential zero, and influenced by a quantity $q$ of electricity placed at $Q$.

The result of this process is as follows:

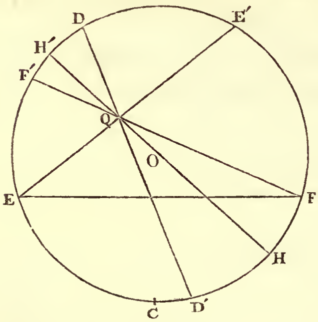

Fig. 16.

Let the figure represent a section through the centre, $O$, of the sphere, the pole, $C$, of the bowl, and the influencing point $Q$. $D$ is a point which corresponds in the inverted figure to the unoccupied pole of the rim of the bowl, and may be found by the following construction.

Draw through $Q$ the chords $EQE'$ and $FQF'$, then if we suppose the radius of the sphere of inversion to be a mean proportional between the segments into which a chord is divided at $Q$, $E'F'$ will be the image of $EF$. Bisect the arc $F'CE'$ in $D'$, so that $F'D'=D'E'$, and draw $D'QD$ to meet the sphere in $D$. $D$ is the point required. Also through $O$, the centre of the sphere, and $Q$ draw $HOQH'$ meeting the sphere in $H$ and $H'$. Then if $P$ be any point in the bowl, the surface-density at $P$ on the side which is separated from $Q$ by the completed spherical surface, induced by a quantity $q$ of electricity at $Q$, will be

$$\sigma = \frac{q}{2\pi^2}\frac{QH.QH'}{HH'.PQ^3}\left\{\frac{PQ}{DQ}\left(\frac{CD^2-a^2}{a^2-CP^2}\right)^{\frac{1}{2}} - \tan^{-1}\left[\frac{PQ}{DQ}\left(\frac{CD^2-a^2}{a^2-CP^2}\right)^{\frac{1}{2}}\right]\right\},$$

where $a$ denotes the chord drawn from $C$, the pole of the bowl, to the rim of the bowl.

On the side next to $Q$ the surface-density is

$$\sigma + \frac{q}{2\pi^2}\frac{QH.QH'}{HH'.PQ^3}.$$

# CHAPTER XII.

## THEORY OF CONJUGATE FUNCTIONS IN TWO DIMENSIONS.

182.] THE number of independent cases in which the problem of electrical equilibrium has been solved is very small. The method of spherical harmonics has been employed for spherical conductors, and the methods of electrical images and of inversion are still more powerful in the cases to which they can be applied. The case of surfaces of the second degree is the only one, as far as I know, in which both the equipotential surfaces and the lines of force are known when the lines of force are not plane curves.

But there is an important class of problems in the theory of electrical equilibrium, and in that of the conduction of currents, in which we have to consider space of two dimensions only.

For instance, if throughout the part of the electric field under consideration, and for a considerable distance beyond it, the surfaces of all the conductors are generated by the motion of straight lines parallel to the axis of $z$, and if the part of the field where this ceases to be the case is so far from the part considered that the electrical action of the distant part on the field may be neglected, then the electricity will be uniformly distributed along each generating line, and if we consider a part of the field bounded by two planes perpendicular to the axis of $z$ and at distance unity, the potential and the distribution of electricity will be functions of $x$ and $y$ only.

If $\rho\, dx\, dy$ denotes the quantity of electricity in an element whose base is $dx\, dy$ and height unity, and $\sigma\, ds$ the quantity on an element of area whose base is the linear element $ds$ and height unity, then the equation of Poisson may be written

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + 4\pi\rho = 0.$$

When there is no free electricity, this is reduced to the equation of Laplace,
$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} = 0.$$

The general problem of electric equilibrium may be stated as follows:—

A continuous space of two dimensions, bounded by closed curves $C_1$, $C_2$, &c. being given, to find the form of a function, $V$, such that at these boundaries its value may be $V_1$, $V_2$, &c. respectively, being constant for each boundary, and that within this space $V$ may be everywhere finite, continuous, and single valued, and may satisfy Laplace's equation.

I am not aware that any perfectly general solution of even this question has been given, but the method of transformation given in Art. 190 is applicable to this case, and is much more powerful than any known method applicable to three dimensions.

The method depends on the properties of conjugate functions of two variables.

### *Definition of Conjugate Functions.*

183.] Two quantities $\alpha$ and $\beta$ are said to be conjugate functions of $x$ and $y$, if $\alpha + \sqrt{-1}\,\beta$ is a function of $x + \sqrt{-1}\,y$.

It follows from this definition that
$$\frac{d\alpha}{dx} = \frac{d\beta}{dy}, \quad \text{and} \quad \frac{d\alpha}{dy} + \frac{d\beta}{dx} = 0\,; \tag{1}$$
$$\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} = 0, \qquad \frac{d^2\beta}{dx^2} + \frac{d^2\beta}{dy^2} = 0. \tag{2}$$

Hence both functions satisfy Laplace's equation. Also
$$\frac{d\alpha}{dx}\frac{d\beta}{dy} - \frac{d\alpha}{dy}\frac{d\beta}{dx} = \overline{\frac{d\alpha}{dx}}\Big|^2 + \overline{\frac{d\alpha}{dy}}\Big|^2 = \overline{\frac{d\beta}{dx}}\Big|^2 + \overline{\frac{d\beta}{dy}}\Big|^2 = R^2. \tag{3}$$

If $x$ and $y$ are rectangular coordinates, and if $ds_1$ is the intercept of the curve ($\beta$ = constant) between the curves $\alpha$ and $\alpha + d\alpha$, and $ds_2$ the intercept of $\alpha$ between the curves $\beta$ and $\beta + d\beta$, then
$$\frac{ds_1}{d\alpha} = \frac{ds_2}{d\beta} = \frac{1}{R}, \tag{4}$$
and the curves intersect at right angles.

If we suppose the potential $V = V_0 + k\alpha$, where $k$ is some constant, then $V$ will satisfy Laplace's equation, and the curves ($\alpha$) will be equipotential curves. The curves ($\beta$) will be lines of force, and the surface-integral of a surface whose projection on the plane of $xy$ is the curve $AB$ will be $k(\beta_B - \beta_A)$, where $\beta_A$ and $\beta_B$ are the values of $\beta$ at the extremities of the curve.

If a series of curves corresponding to values of $a$ in arithmetical progression is drawn on the plane, and another series corresponding to a series of values of $\beta$ having the same common difference, then the two series of curves will everywhere intersect at right angles, and, if the common difference is small enough, the elements into which the plane is divided will be ultimately little squares, whose sides, in different parts of the field, are in different directions and of different magnitude, being inversely proportional to $R$.

If two or more of the equipotential lines ($a$) are closed curves enclosing a continuous space between them, we may take these for the surfaces of conductors at potentials $(V_0 + k a_1)$, $(V_0 + k a_2)$, &c. respectively. The quantity of electricity upon any one of these between the lines of force $\beta_1$ and $\beta_2$ will be $\frac{k}{4\pi}(\beta_2 - \beta_1)$.

The number of equipotential lines between two conductors will therefore indicate their difference of potential, and the number of lines of force which emerge from a conductor will indicate the quantity of electricity upon it.

We must next state some of the most important theorems relating to conjugate functions, and in proving them we may use either the equations (1), containing the differential coefficients, or the original definition, which makes use of imaginary symbols.

184.] THEOREM I. *If $x'$ and $y'$ are conjugate functions with respect to $x$ and $y$, and if $x''$ and $y''$ are also conjugate functions with respect to $x$ and $y$, then the functions $x' + x''$ and $y' + y''$ will be conjugate functions with respect to $x$ and $y$.*

For $$\frac{dx'}{dx} = \frac{dy'}{dy}, \quad \text{and} \quad \frac{dx''}{dx} = \frac{dy''}{dy};$$

therefore $$\frac{d(x'+x'')}{dx} = \frac{d(y'+y'')}{dy}.$$

Also $$\frac{dx'}{dy} = -\frac{dy'}{dx}, \quad \text{and} \quad \frac{dx''}{dy} = -\frac{dy''}{dx};$$

therefore $$\frac{d(x'+x'')}{dy} = -\frac{d(y'+y'')}{dx};$$

or $x + x''$ and $y' + y''$ are conjugate with respect to $x$ and $y$.

*Graphic Representation of a Function which is the Sum of Two Given Functions.*

Let a function ($\alpha$) of $x$ and $y$ be graphically represented by a series of curves in the plane of $xy$, each of these curves corresponding to a value of $\alpha$ which belongs to a series of such values increasing by a common difference, $\delta$.

Let any other function, $\beta$, of $x$ and $y$ be represented in the same way by a series of curves corresponding to a series of values of $\beta$ having the same common difference as those of $\alpha$.

Then to represent the function $\alpha+\beta$ in the same way, we must draw a series of curves through the intersections of the two former series from the intersection of the curves ($\alpha$) and ($\beta$) to that of the curves ($\alpha+\delta$) and ($\beta-\delta$), then through the intersection of ($\alpha+2\delta$) and ($\beta-2\delta$), and so on. At each of these points the function will have the same value, namely $\alpha+\beta$. The next curve must be drawn through the points of intersection of $\alpha$ and $\beta+\delta$, of $\alpha+\delta$ and $\beta$, of $\alpha+2\delta$ and $\beta-\delta$, and so on. The function belonging to this curve will be $\alpha+\beta+\delta$.

In this way, when the series of curves ($\alpha$) and the series ($\beta$) are drawn, the series ($\alpha+\beta$) may be constructed. These three series of curves may be drawn on separate pieces of transparent paper, and when the first and second have been properly superposed, the third may be drawn.

The combination of conjugate functions by addition in this way enables us to draw figures of many interesting cases with very little trouble when we know how to draw the simpler cases of which they are compounded. We have, however, a far more powerful method of transformation of solutions, depending on the following theorem.

185.] THEOREM II. *If $x''$ and $y''$ are conjugate functions with respect to the variables $x'$ and $y'$, and if $x'$ and $y'$ are conjugate functions with respect to $x$ and $y$, then $x''$ and $y''$ will be conjugate functions with respect to $x$ and $y$.*

For
$$\begin{aligned}\frac{dx''}{dx} &= \frac{dx''}{dx'}\frac{dx'}{dx}+\frac{dx''}{dy'}\frac{dy'}{dx},\\ &= \frac{dy''}{dy'}\frac{dy'}{dy}+\frac{dy''}{dx'}\frac{dx'}{dy},\\ &= \frac{dy''}{dy};\end{aligned}$$

$$\text{and} \qquad \frac{dx''}{dy} = \frac{dx''}{dx'}\frac{dx'}{dy} + \frac{dx''}{dy'}\frac{dy'}{dy},$$

$$= -\frac{dy''}{dy'}\frac{dy'}{dx} - \frac{dy''}{dx'}\frac{dx'}{dx},$$

$$= -\frac{dy''}{dx};$$

and these are the conditions that $x''$ and $y''$ should be conjugate functions of $x$ and $y$.

This may also be shewn from the original definition of conjugate functions. For $x'' + \sqrt{-1}\,y''$ is a function of $x' + \sqrt{-1}\,y'$, and $x' + \sqrt{-1}\,y'$ is a function of $x + \sqrt{-1}\,y$. Hence, $x'' + \sqrt{-1}\,y''$ is a function of $x + \sqrt{-1}\,y$.

In the same way we may shew that if $x'$ and $y'$ are conjugate functions of $x$ and $y$, then $x$ and $y$ are conjugate functions of $x'$ and $y'$.

This theorem may be interpreted graphically as follows :—

Let $x'$, $y'$ be taken as rectangular coordinates, and let the curves corresponding to values of $x''$ and of $y''$ taken in regular arithmetical series be drawn on paper. A double system of curves will thus be drawn cutting the paper into little squares. Let the paper be also ruled with horizontal and vertical lines at equal intervals, and let these lines be marked with the corresponding values of $x'$ and $y'$.

Next, let another piece of paper be taken in which $x$ and $y$ are made rectangular coordinates and a double system of curves $x'$, $y'$ is drawn, each curve being marked with the corresponding value of $x'$ or $y'$. This system of curvilinear coordinates will correspond, point for point, to the rectilinear system of coordinates $x'$, $y'$ on the first piece of paper.

Hence, if we take any number of points on the curve $x''$ on the first paper, and note the values of $x'$ and $y'$ at these points, and mark the corresponding points on the second paper, we shall find a number of points on the transformed curve $x''$. If we do the same for all the curves $x''$, $y''$ on the first paper, we shall obtain on the second paper a double series of curves $x''$, $y''$ of a different form, but having the same property of cutting the paper into little squares.

186.] THEOREM III. *If $V$ is any function of $x'$ and $y'$, and if $x'$ and $y'$ are conjugate functions of $x$ and $y$, then*

$$\iint\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}\right)dx\,dy=\iint\left(\frac{d^2V}{dx'^2}+\frac{d^2V}{dy'^2}\right)dx'\,dy',$$

*the integration being between the same limits.*

For
$$\frac{dV}{dx}=\frac{dV}{dx'}\frac{dx'}{dx}+\frac{dV}{dy'}\frac{dy'}{dx},$$

$$\frac{d^2V}{dx^2}=\frac{d^2V}{dx'^2}\left(\frac{dx'}{dx}\right)^2+2\frac{d^2V}{dx'\,dy'}\frac{dx'}{dx}\frac{dy'}{dx}+\frac{d^2V}{dy'^2}\left(\frac{dy'}{dx}\right)^2+\frac{dV}{dx'}\frac{d^2x'}{dx^2}+\frac{dV}{dy'}\frac{d^2y'}{dx^2};$$

and
$$\frac{d^2V}{dy^2}=\frac{d^2V}{dx'^2}\left(\frac{dx'}{dy}\right)^2+2\frac{d^2V}{dx'\,dy'}\frac{dx'}{dy}\frac{dy'}{dy}+\frac{d^2V}{dy'^2}\left(\frac{dy'}{dy}\right)^2+\frac{dV}{dx'}\frac{d^2x'}{dy^2}+\frac{dV}{dy'}\frac{d^2y'}{dy^2}.$$

Adding the last two equations, and remembering the conditions of conjugate functions (1), we find

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=\frac{d^2V}{dx'^2}\left(\left(\frac{dx'}{dx}\right)^2+\left(\frac{dx'}{dy}\right)^2\right)+\frac{dV}{dy'^2}\left(\left(\frac{dy'}{dx}\right)^2+\left(\frac{dy'}{dy}\right)^2\right),$$

$$=\left(\frac{d^2V}{dx'^2}+\frac{d^2V}{dy'^2}\right)\left(\frac{dx'}{dx}\frac{dy'}{dy}-\frac{dx'}{dy}\frac{dy'}{dx}\right).$$

Hence

$$\iint\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}\right)dx\,dy=\iint\left(\frac{d^2V}{dx'^2}+\frac{d^2V}{dy'^2}\right)\left(\frac{dx'}{dx}\frac{dy'}{dy}-\frac{dx'}{dy}\frac{dy'}{dx}\right)dx\,dy,$$

$$=\iint\left(\frac{d^2V}{dx'^2}+\frac{d^2V}{dy'^2}\right)dx'\,dy'.$$

If $V$ is a potential, then, by Poisson's equation

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+4\pi\rho=0,$$

and we may write the result

$$\iint\rho\,dx\,dy=\iint\rho'\,dx'\,dy',$$

or the quantity of electricity in corresponding portions of two systems is the same if the coordinates of one system are conjugate functions of those of the other.

*Additional Theorems on Conjugate Functions.*

187.] THEOREM IV. *If $x_1$ and $y_1$, and also $x_2$ and $y_2$, are conjugate functions of $x$ and $y$, then, if*

$$X = x_1 x_2 - y_1 y_2, \quad \text{and} \quad Y = x_1 y_2 + x_2 y_1,$$

*$X$ and $Y$ will be conjugate functions of $x$ and $y$.*

For $$X + \sqrt{-1}\,Y = (x_1 + \sqrt{-1}\,y_1)(x_2 + \sqrt{-1}\,y_2).$$

THEOREM V. *If $\phi$ be a solution of the equation*

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0,$$

*and if* $$2R = \log\left(\left|\frac{d\phi}{dx}\right|^2 + \left|\frac{d\phi}{dy}\right|^2\right), \quad \text{and} \quad \Theta = \tan^{-1}\frac{\dfrac{d\phi}{dx}}{\dfrac{d\phi}{dy}},$$

*$R$ and $\Theta$ will be conjugate functions of $x$ and $y$.*

For $R$ and $\Theta$ are conjugate functions of $\dfrac{d\phi}{dx}$ and $\dfrac{d\phi}{dy}$, and these are conjugate functions of $x$ and $y$.

EXAMPLE I.—*Inversion.*

188.] As an example of the general method of transformation let us take the case of inversion in two dimensions.

If $O$ is a fixed point in a plane, and $OA$ a fixed direction, and if $r = OP = ae^\rho$, and $\theta = AOP$, and if $x$, $y$ are the rectangular coordinates of $P$ with respect to $O$,

$$\left.\begin{aligned} \rho &= \log\frac{1}{a}\sqrt{x^2+y^2}, & \theta &= \tan^{-1}\frac{y}{x}, \\ x &= ae^\rho\cos\theta, & y &= ae^\rho\sin\theta, \end{aligned}\right\} \tag{5}$$

$\rho$ and $\theta$ are conjugate functions of $x$ and $y$.

If $\rho' = n\rho$ and $\theta' = n\theta$, $\rho'$ and $\theta'$ will be conjugate functions of $\rho$ and $\theta$. In the case in which $n = -1$ we have

$$r' = \frac{a^2}{r}, \quad \text{and} \quad \theta' = -\theta, \tag{6}$$

which is the case of ordinary inversion combined with turning the figure 180° round $OA$.

*Inversion in Two Dimensions.*

In this case if $r$ and $r'$ represent the distances of corresponding points from $O$, $e$ and $e'$ the total electrification of a body, $S$ and $S'$ superficial elements, $V$ and $V'$ solid elements, $\sigma$ and $\sigma'$ surface-densities, $\rho'$ and $\rho'$ volume densities, $\phi$ and $\phi'$ corresponding potentials,

$$\left.\begin{array}{c} \dfrac{r'}{r} = \dfrac{S'}{S} = \dfrac{a^2}{r^2} = \dfrac{r'^2}{a^2}, \qquad \dfrac{V'}{V} = \dfrac{a^4}{r^4} = \dfrac{r'^4}{a^4}, \\ \dfrac{e'}{e} = 1, \qquad \dfrac{\sigma'}{\sigma} = \dfrac{r^2}{a^2} = \dfrac{a^2}{r'^2}, \qquad \dfrac{\rho'}{\rho} = \dfrac{r^4}{a^4} = \dfrac{a^4}{r'^4}, \\ \dfrac{\phi'}{\phi} = 1. \end{array}\right\} \qquad (7)$$

EXAMPLE II.—*Electric Images in Two Dimensions.*

189.] Let $A$ be the centre of a circle of radius $AQ = b$, and let $E$ be a charge at $A$, then the potential at any point $P$ is

$$\phi = 2E \log \frac{b}{AP}; \qquad (8)$$

and if the circle is a section of a hollow conducting cylinder, the surface-density at any point $Q$ is $-\dfrac{E}{2\pi b}$.

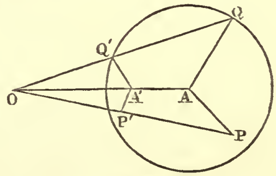


Fig. 17.

Invert the system with respect to a point $O$, making

$$AO = mb, \quad \text{and} \quad a^2 = (m^2 - 1)b^2;$$

then we have a charge at $A'$ equal to that at $A$, where $AA' = \dfrac{b}{m}$.

The density at $Q'$ is

$$-\frac{E}{2\pi b}\,\frac{b^2 - \overline{AA'}|^2}{A'Q'^2},$$

and the potential at any point $P'$ within the circle is

$$\begin{aligned} \phi' = \phi &= 2E(\log b - \log AP), \\ &= 2E(\log OP' - \log A'P' - \log m). \qquad (9) \end{aligned}$$

This is equivalent to a combination of a charge $E$ at $A'$, and a charge $-E$ at $O$, which is the image of $A'$, with respect to the circle. The imaginary charge at $O$ is equal and opposite to that at $A'$.

If the point $P'$ is defined by its polar coordinates referred to the centre of the circle, and if we put

$$\rho = \log r - \log b, \quad \text{and} \quad \rho_0 = \log AA' - \log b,$$

then $\qquad AP' = be^{\rho}, \qquad AA' = be^{\rho_0}, \qquad AO = be^{-\rho_0}; \qquad (10)$

and the potential at the point $(\rho, \theta)$ is

$$\phi = E \log (e^{-2\rho_0} - 2 e^{-\rho_0} e^{\rho} \cos \theta + e^{2\rho}) - E \log (e^{2\rho_0} - 2 e^{\rho_0} e^{\rho} \cos \theta + e^{2\rho}) + 2 E \rho_0. \quad (11)$$

This is the potential at the point $(\rho, \theta)$ due to a charge $E$, placed at the point $(\rho_0, 0)$, with the condition that when $\rho = 0$, $\phi = 0$.

In this case $\rho$ and $\theta$ are the conjugate functions in equations (5): $\rho$ is the logarithm of the ratio of the radius vector of a point to the radius of the circle, and $\theta$ is an angle.

The centre is the only singular point in this system of coordinates, and the line-integral of $\int \frac{d\theta}{ds} ds$ round a closed curve is zero or $2\pi$, according as the closed curve excludes or includes the centre.

### Example III.—*Neumann's Transformation of this Case* *.

190.] Now let $\alpha$ and $\beta$ be any conjugate functions of $x$ and $y$, such that the curves $(\alpha)$ are equipotential curves, and the curves $(\beta)$ are lines of force due to a system consisting of a charge of half a unit at the origin, and an electrified system disposed in any manner at a certain distance from the origin.

Let us suppose that the curve for which the potential is $\alpha_0$ is a closed curve, such that no part of the electrified system except the half-unit at the origin lies within this curve.

Then all the curves $(\alpha)$ between this curve and the origin will be closed curves surrounding the origin, and all the curves $(\beta)$ will meet in the origin, and will cut the curves $(\alpha)$ orthogonally.

The coordinates of any point within the curve $(\alpha_0)$ will be determined by the values of $\alpha$ and $\beta$ at that point, and if the point travels round one of the curves $\alpha$ in the positive direction, the value of $\beta$ will increase by $2\pi$ for each complete circuit.

If we now suppose the curve $(\alpha_0)$ to be the section of the inner surface of a hollow cylinder of any form maintained at potential zero under the influence of a charge of linear density $E$ on a line of which the origin is the projection, then we may leave the external electrified system out of consideration, and we have for the potential at any point $(\alpha)$ within the curve

$$\phi = 2E(\alpha - \alpha_0), \quad (12)$$

and for the quantity of electricity on any part of the curve $\alpha_0$ between the points corresponding to $\beta_1$ and $\beta_2$,

$$Q = 2E(\beta_1 - \beta_2). \quad (13)$$

* See Crelle's *Journal*, 1861.

If in this way, or in any other, we have determined the distribution of potential for the case of a given curve of section when the charge is placed at a given point taken as origin, we may pass to the case in which the charge is placed at any other point by an application of the general method of transformation.

Let the values of $\alpha$ and $\beta$ for the point at which the charge is placed be $\alpha_1$ and $\beta_1$, then substituting in equation (11) $\alpha-\alpha_0$ for $\rho$, and $\beta-\beta_1$ for $\theta$, we find for the potential at any point whose coordinates are $\alpha$ and $\beta$,

$$\phi = E\log\left(1-2e^{\alpha-\alpha_1}\cos(\beta-\beta_1)+e^{2(\alpha-\alpha_1)}\right)$$
$$-E\log\left(1-2e^{\alpha+\alpha_1-2\alpha_0}\cos(\beta-\beta_1)+e^{2(\alpha+\alpha_1-2\alpha_0)}\right)+2E(\alpha_1-\alpha_0). \quad (14)$$

This expression for the potential becomes zero when $\alpha=\alpha_0$, and is finite and continuous within the curve $\alpha_0$ except at the point $\alpha_1\beta_1$, at which point the first term becomes infinite, and in its immediate neighbourhood is ultimately equal to $2E\log r'$, where $r'$ is the distance from that point.

We have therefore obtained the means of deducing the solution of Green's problem for a charge at any point within a closed curve when the solution for a charge at any other point is known.

The charge induced upon an element of the curve $\alpha_0$ between the points $\beta$ and $\beta+d\beta$ by a charge $E$ placed at the point $\alpha_1\beta_1$ is

$$\frac{E}{2\pi}\,\frac{1-e^{2(\alpha_1-\alpha_0)}}{1-2e^{(\alpha_1-\alpha_0)}\cos(\beta-\beta_1)+e^{2(\alpha_1-\alpha_0)}}\,d\beta. \quad (15)$$

From this expression we may find the potential at any point $\alpha_1\beta_1$ within the closed curve, when the value of the potential at every point of the closed curve is given as a function of $\beta$, and there is no electrification within the closed curve.

For, by Theorem II of Chap. III, the part of the potential at $\alpha_1\beta_1$, due to the maintenance of the portion $d\beta$ of the closed curve at the potential $V$, is $nV$, where $n$ is the charge induced on $d\beta$ by unit of electrification at $\alpha_1\beta_1$. Hence, if $V$ is the potential at a point on the closed curve defined as a function of $\beta$, and $\phi$ the potential at the point $\alpha$, $\beta$, within the closed curve, there being no electrification within the curve,

$$\phi = \frac{1}{2\pi}\int_0^{2\pi}\frac{\left(1-e^{2(\alpha_1-\alpha_0)}\right)V\,d\beta}{1-2e^{(\alpha_1-\alpha_0)}\cos(\beta-\beta_1)+e^{2(\alpha_1-\alpha_0)}}. \quad (16)$$

### Example IV.—*Distribution of Electricity near an Edge of a Conductor formed by Two Plane Faces.*

191.] In the case of an infinite plane face of a conductor charged with electricity to the surface-density $\sigma_0$, we find for the potential at a distance $y$ from the plane

$$V = C - 4\pi\sigma_0 y,$$

where $C$ is the value of the potential of the conductor itself.

Assume a straight line in the plane as a polar axis, and transform into polar coordinates, and we find for the potential

$$V = C - 4\pi\sigma_0 a e^{\rho} \sin\theta,$$

and for the quantity of electricity on a parallelogram of breadth unity, and length $ae^{\rho}$ measured from the axis

$$E = \sigma_0 a e^{\rho}.$$

Now let us make $\rho = n\rho'$ and $\theta = n\theta'$, then, since $\rho'$ and $\theta'$ are conjugate to $\rho$ and $\theta$, the equations

$$V = C - 4\pi\sigma_0 a e^{n\rho'} \sin n\theta'$$

and

$$E = \sigma_0 a e^{n\rho'}$$

express a possible distribution of electricity and of potential.

If we write $r$ for $ae^{\rho'}$, $r$ will be the distance from the axis, and $\theta$ the angle, and we shall have

$$V = C - 4\pi\sigma_0 \frac{r^n}{a^{n-1}} \sin n\theta,$$

$$E = \sigma_0 \frac{r^n}{a^{n-1}}.$$

$V$ will be equal to $C$ whenever $n\theta = \pi$ or a multiple of $\pi$.

Let the edge be a salient angle of the conductor, the inclination of the faces being $\alpha$, then the angle of the dielectric is $2\pi - \alpha$, so that when $\theta = 2\pi - \alpha$ the point is in the other face of the conductor. We must therefore make

$$n(2\pi - \alpha) = \pi, \quad \text{or} \quad n = \frac{\pi}{2\pi - \alpha}.$$

Then

$$V = C - 4\pi\sigma_0 a \left(\frac{r}{a}\right)^{\frac{\pi}{2\pi-\alpha}} \sin\frac{\pi\theta}{2\pi - \alpha},$$

$$E = \sigma_0 a \left(\frac{r}{a}\right)^{\frac{\pi}{2\pi-\alpha}}.$$

The surface-density $\sigma$ at any distance $r$ from the edge is

$$\sigma = \frac{dE}{dr} = \frac{\pi}{2\pi - \alpha}\sigma_0 \left(\frac{r}{a}\right)^{\frac{\alpha-\pi}{2\pi-\alpha}}.$$

When the angle is a salient one $a$ is less than $\pi$, and the surface-density varies according to some inverse power of the distance from the edge, so that at the edge itself the density becomes infinite, although the whole charge reckoned from the edge to any finite distance from it is always finite.

Thus, when $a=0$ the edge is infinitely sharp, like the edge of a mathematical plane. In this case the density varies inversely as the square root of the distance from the edge.

When $a=\frac{\pi}{3}$ the edge is like that of an equilateral prism, and the density varies inversely as the $\frac{2}{5}$ power of the distance.

When $a=\frac{\pi}{2}$ the edge is a right angle, and the density is inversely as the cube root of the distance.

When $a=\frac{2\pi}{3}$ the edge is like that of a regular hexagonal prism, and the density is inversely as the fourth root of the distance.

When $a=\pi$ the edge is obliterated, and the density is constant.

When $a=\frac{4}{3}\pi$ the edge is like that in the inside of the hexagonal prism, and the density is *directly* as the square root of the distance from the edge.

When $a=\frac{3}{2}\pi$ the edge is a re-entrant right angle, and the density is directly as the distance from the edge.

When $a=\frac{5}{3}\pi$ the edge is a re-entrant angle of 60°, and the density is directly as the square of the distance from the edge.

In reality, in all cases in which the density becomes infinite at any point, there is a discharge of electricity into the dielectric at that point, as is explained in Art. 55.

### EXAMPLE V.—*Ellipses and Hyperbolas.* Fig. X.

192.] We have seen that if

$$x_1 = e^{\phi}\cos\psi, \qquad y_1 = e^{\phi}\sin\psi, \tag{1}$$

$x$ and $y$ will be conjugate functions of $\phi$ and $\psi$.

Also, if
$$x_2 = e^{-\phi}\cos\psi, \qquad y_2 = -e^{-\phi}\sin\psi, \tag{2}$$

$x_2$ and $y_2$ will be conjugate functions. Hence, if

$$2x = x_1 + x_2 = (e^{\phi}+e^{-\phi})\cos\psi, \qquad 2y = y_1 + y_2 = (e^{\phi}-e^{-\phi})\sin\psi, \tag{3}$$

$x$ and $y$ will also be conjugate functions of $\phi$ and $\psi$.

In this case the points for which $\phi$ is constant lie in the ellipse whose axes are $e^{\phi}+e^{-\phi}$ and $e^{\phi}-e^{-\phi}$.

The points for which $\psi$ is constant lie in the hyperbola whose axes are $2\cos\psi$ and $2\sin\psi$.

On the axis of $x$, between $x=-1$ and $x=+1$,

$$\phi = 0, \qquad \psi = \cos^{-1} x. \tag{4}$$

On the axis of $x$, beyond these limits on either side, we have

$$\begin{array}{lll} x > \ \ 1, & \psi = 0, & \phi = \log\left(x+\sqrt{x^2-1}\right), \\ x < -1, & \psi = \pi, & \phi = \log\left(\sqrt{x^2-1}-x\right). \end{array} \tag{5}$$

Hence, if $\phi$ is the potential function, and $\psi$ the function of flow, we have the case of electricity flowing from the negative to the positive side of the axis of $x$ through the space between the points $-1$ and $+1$, the parts of the axis beyond these limits being impervious to electricity.

Since, in this case, the axis of $y$ is a line of flow, we may suppose it also impervious to electricity.

We may also consider the ellipses to be sections of the equipotential surfaces due to an indefinitely long flat conductor of breadth 2, charged with half a unit of electricity per unit of length.

If we make $\psi$ the potential function, and $\phi$ the function of flow, the case becomes that of an infinite plane from which a strip of breadth 2 has been cut away and the plane on one side charged to potential $\pi$ while the other remains at zero.

These cases may be considered as particular cases of the quadric surfaces treated of in Chapter X. The forms of the curves are given in Fig. X.

### Example VI.—Fig. XI.

193.] Let us next consider $x'$ and $y'$ as functions of $x$ and $y$, where

$$x' = b\log\sqrt{x^2+y^2}, \qquad y' = b\tan^{-1}\frac{y}{x}, \tag{6}$$

$x'$ and $y'$ will be also conjugate functions of $\phi$ and $\psi$.

The curves resulting from the transformation of Fig. X with respect to these new coordinates are given in Fig. XI.

If $x'$ and $y'$ are rectangular coordinates, then the properties of the axis of $x$ in the first figure will belong to a series of lines parallel to $x'$ in the second figure for which $y' = bn'\pi$, where $n'$ is any integer.

The positive values of $x'$ on these lines will correspond to values of $x$ greater than unity, for which, as we have already seen,

$$\psi = n\pi, \qquad \phi = \log\left(x+\sqrt{x^2-1}\right) = \log\left(e^{\frac{x'}{b}} + \sqrt{e^{\frac{2x'}{b}}-1}\right). \tag{7}$$

The negative values of $x'$ on the same lines will correspond to values of $x$ less than unity, for which, as we have seen,

$$\phi = 0, \qquad \psi = \cos^{-1} x = \cos^{-1} e^{\frac{x'}{b}}. \tag{8}$$

The properties of the axis of $y$ in the first figure will belong to a series of lines in the second figure parallel to $x'$, for which

$$y' = b\pi\,(n' + \tfrac{1}{2}). \tag{9}$$

The value of $\psi$ along these lines is $\psi = \pi\,(n' + \frac{1}{2})$ for all points both positive and negative, and

$$\phi = \log\left(y + \sqrt{y^2+1}\right) = \log\left(e^{\frac{x'}{b}} + \sqrt{e^{\frac{2x'}{b}} + 1}\right). \tag{10}$$

194.] If we consider $\phi$ as the potential function, and $\psi$ as the function of flow, we may consider the case to be that of an indefinitely long strip of metal of breadth $\pi b$ with a non-conducting division extending from the origin indefinitely in the positive direction, and thus dividing the positive part of the strip into two separate channels. We may suppose this division to be a narrow slit in the sheet of metal.

If a current of electricity is made to flow along one of these divisions and back again along the other, the entrance and exit of the current being at an indefinite distance on the positive side of the origin, the distribution of potential and of current will be given by the functions $\phi$ and $\psi$ respectively.

If, on the other hand, we make $\psi$ the potential, and $\phi$ the function of flow, then the case will be that of a current in the general direction of $y$, flowing through a sheet in which a number of non-conducting divisions are placed parallel to $x$, extending from the axis of $y$ to an indefinite distance in the negative direction.

195.] We may also apply the results to two important cases in statical electricity.

(1) Let a conductor in the form of a plane sheet, bounded by a straight edge but otherwise unlimited, be placed in the plane of $xz$ on the positive side of the origin, and let two infinite conducting planes be placed parallel to it and at distances $\frac{1}{2}\pi b$ on either side. Then, if $\psi$ is the potential function, its value is 0 for the middle conductor and $\frac{1}{2}\pi$ for the two planes.

Let us consider the quantity of electricity on a part of the middle conductor, extending to a distance 1 in the direction of $z$, and from the origin to $x = a$.

The electricity on the part of this strip extending from $x_1$ to $x_2$ is $\dfrac{1}{4\pi}(\phi_2 - \phi_1)$.

Hence from the origin to $x' = a$ the amount is

$$E = \frac{1}{4\pi} \log\left(e^{\frac{a}{b}} + \sqrt{e^{\frac{2a}{b}} - 1}\right). \tag{11}$$

If $a$ is large compared with $b$, this becomes

$$E = \frac{1}{4\pi} \log 2 e^{\frac{a}{b}},$$

$$= \frac{a + b \log_e 2}{4\pi b}. \tag{12}$$

Hence the quantity of electricity on the plane bounded by the straight edge is greater than it would have been if the electricity had been uniformly distributed over it with the same density that it has at a distance from the boundary, and it is equal to the quantity of electricity having the same uniform surface-density, but extending to a breadth equal to $b \log_e 2$ beyond the actual boundary of the plate.

This imaginary uniform distribution is indicated by the dotted straight lines in Fig. XI. The vertical lines represent lines of force, and the horizontal lines equipotential surfaces, on the hypothesis that the density is uniform over both planes, produced to infinity in all directions.

196.] Electrical condensers are sometimes formed of a plate placed midway between two parallel plates extending considerably beyond the intermediate one on all sides. If the radius of curvature of the boundary of the intermediate plate is great compared with the distance between the plates, we may treat the boundary as approximately a straight line, and calculate the capacity of the condenser by supposing the intermediate plate to have its area extended by a strip of uniform breadth round its boundary, and assuming the surface-density on the extended plate the same as it is in the parts not near the boundary.

Thus, if $S$ be the actual area of the plate, $L$ its circumference, and $B$ the distance between the large plates, we have

$$b = \frac{1}{\pi} B, \tag{13}$$

and the breadth of the additional strip is

$$a = \frac{\log_e 2}{\pi} \cdot B, \tag{14}$$

so that the extended area is

$$S' = S + BL \frac{1}{\pi} \log_e 2. \tag{15}$$

The capacity of the middle plate is

$$\frac{1}{2\pi}\frac{S'}{B} = \frac{1}{2\pi}\left\{\frac{S}{B} + L\frac{1}{\pi}\log_e 2\right\}. \tag{16}$$

*Correction for the Thickness of the Plate.*

Since the middle plate is generally of a thickness which cannot be neglected in comparison with the distance between the plates, we may obtain a better representation of the facts of the case by supposing the section of the intermediate plate to correspond with the curve $\psi = \psi'$.

The plate will be of nearly uniform thickness, $\beta = 2b\psi'$, at a distance from the boundary, but will be rounded near the edge.

The position of the actual edge of the plate is found by putting $y' = 0$, whence

$$x' = b \log \cos \psi'. \tag{17}$$

The value of $\phi$ at this edge is 0, and at a point for which $x' = a$ it is

$$\frac{a + b\log_e 2}{b}.$$

Hence the quantity of electricity on the plate is the same as if a strip of breadth

$$a' = \frac{B}{\pi}\log_e\left(2\cos\frac{\pi\beta}{2B}\right) \tag{18}$$

had been added to the plate, the density being assumed to be everywhere the same as it is at a distance from the boundary.

*Density near the Edge.*

The surface-density at any point of the plate is

$$\frac{1}{4\pi}\frac{d\phi}{dx'} = \frac{1}{4\pi b}\frac{e^{\frac{x'}{b}}}{\sqrt{e^{\frac{2x'}{b}} - 1}}$$

$$= \frac{1}{4\pi b}\left(1 + \tfrac{1}{2}e^{-\frac{2x'}{b}} + \tfrac{3}{8}e^{-\frac{4x'}{b}} - \&c.\right). \tag{19}$$

The quantity within brackets rapidly approaches unity as $x'$ increases, so that at a distance from the boundary equal to $n$ times the breadth of the strip $a$, the actual density is greater than the normal density by about $\frac{1}{2^{2n+1}}$ of the normal density.

In like manner we may calculate the density on the infinite planes

$$= \frac{1}{4\pi b}\frac{e^{\frac{x}{b}}}{\sqrt{e^{\frac{2x'}{b}} + 1}}. \tag{20}$$

When $x' = 0$, the density is $2^{-\frac{1}{2}}$ of the normal density.

At $n$ times the breadth of the strip on the positive side, the density is less than the normal density by about $\frac{1}{2^{n+1}}$.

At $n$ times the breadth of the strip on the negative side, the density is about $\frac{1}{2^n}$ of the normal density.

These results indicate the degree of accuracy to be expected in applying this method to plates of limited extent, or in which irregularities may exist not very far from the boundary. The same distribution would exist in the case of an infinite series of similar plates at equal distances, the potentials of these plates being alternately $+V$ and $-V$. In this case we must take the distance between the plates equal to $B$.

197.] (2) The second case we shall consider is that of an infinite series of planes parallel to $xz$ at distances $B = \pi b$, and all cut off by the plane of $yz$, so that they extend only on the negative side of this plane. If we make $\phi$ the potential function, we may regard these planes as conductors at potential zero.

Let us consider the curves for which $\phi$ is constant.

When $y' = n\pi b$, that is, in the prolongation of each of the planes, we have

$$x' = b \log \tfrac{1}{2}(e^{\phi} + e^{-\phi}) \tag{21}$$

when $y' = (n+\frac{1}{2})b\pi$, that is, in the intermediate positions

$$x' = b \log \tfrac{1}{2}(e^{\phi} - e^{-\phi}). \tag{22}$$

Hence, when $\phi$ is large, the curve for which $\phi$ is constant is an undulating line whose mean distance from the axis of $y'$ is approximately

$$a = b(\phi - \log_e 2), \tag{23}$$

and the amplitude of the undulations on either side of this line is

$$\tfrac{1}{2} b \log \frac{e^{\phi} + e^{-\phi}}{e^{\phi} - e^{-\phi}}. \tag{24}$$

When $\phi$ is large this becomes $be^{-2\phi}$, so that the curve approaches to the form of a straight line parallel to the axis of $y'$ at a distance $a$ from $ab$ on the positive side.

If we suppose a plane for which $x' = a$, kept at a constant potential while the system of parallel planes is kept at a different potential, then, since $b\phi = a + b\log_e 2$, the surface-density of the electricity induced on the plane is equal to that which would have been induced on it by a plane parallel to itself at a potential equal to that of the series of planes, but at a distance greater than that of the edges of the planes by $b \log_e 2$.

If $B$ is the distance between two of the planes of the series, $B = \pi b$, so that the additional distance is

$$a = B\frac{\log_e 2}{\pi}. \tag{25}$$

198.] Let us next consider the space included between two of the equipotential surfaces, one of which consists of a series of parallel waves, while the other corresponds to a large value of $\phi$, and may be considered as approximately plane.

If $D$ is the depth of these undulations from the crest to the trough of each wave, then we find for the corresponding value of $\phi$,

$$\phi = \tfrac{1}{2}\log\frac{e^{\frac{D}{b}}+1}{e^{\frac{D}{b}}-1}. \tag{26}$$

The value of $x'$ at the crest of the wave is

$$b\log\tfrac{1}{2}(e^{\phi}+e^{-\phi}). \tag{27}$$

Hence, if $A$ is the distance from the crests of the waves to the opposite plane, the capacity of the system composed of the plane surface and the undulated surface is the same as that of two planes at a distance $A+a'$ where

$$a' = \frac{B}{\pi}\log_e\frac{2}{1+e^{-\pi\frac{D}{B}}}. \tag{28}$$

199.] If a single groove of this form be made in a conductor having the rest of its surface plane, and if the other conductor is a plane surface at a distance $A$, the capacity of the one conductor with respect to the other will be diminished. The amount of this diminution will be less than the $\frac{1}{n}$th part of the diminution due to $n$ such grooves side by side, for in the latter case the average electrical force between the conductors will be less than in the former case, so that the induction on the surface of each groove will be diminished on account of the neighbouring grooves.

If $L$ is the length, $B$ the breadth, and $D$ the depth of the groove, the capacity of a portion of the opposite plane whose area is $S$ will be

$$\frac{S}{4\pi A}-\frac{LB}{4\pi A}\frac{a'}{A+a'}. \tag{29}$$

If $A$ is large compared with $B$ or $a'$, the correction becomes

$$\frac{L}{4\pi^2}\frac{B^2}{A^2}\log_e\frac{2}{1+e^{-\pi\frac{D}{B}}}, \tag{30}$$

and for a slit of infinite depth, putting $D=\infty$, the correction is

$$\frac{L}{4\pi^2}\frac{B^2}{A^2}\log_e 2. \tag{31}$$

To find the surface-density on the series of parallel plates we must find $\sigma=\frac{1}{4\pi}\frac{d\psi}{dx'}$ when $\phi=0$. We find

$$\sigma=\frac{1}{4\pi b}\frac{1}{\sqrt{e^{-2\frac{x'}{b}}-1}}. \tag{32}$$

The average density on the plane plate at distance $A$ from the edges of the series of plates is $\bar{\sigma}=\frac{1}{4\pi b}$. Hence, at a distance from the edge of one of the plates equal to $na$ the surface-density is $\frac{1}{\sqrt{2^{2n}-1}}$ of this average density.

200.] Let us next attempt to deduce from these results the distribution of electricity in the figure formed by rotating the plane of the figure about the axis $y'=-R$. In this case, Poisson's equation will assume the form

$$\frac{d^2V}{dx'^2}+\frac{d^2V}{dy'^2}+\frac{1}{R+y'}\frac{dV}{dy'}+4\pi\rho=0. \tag{33}$$

Let us assume $V=\phi$, the function given in Art. 193, and determine the value of $\rho$ from this equation. We know that the first two terms disappear, and therefore

$$\rho=-\frac{1}{4\pi}\frac{1}{R+y'}\frac{d\phi}{dy'}. \tag{34}$$

If we suppose that, in addition to the surface-density already investigated, there is a distribution of electricity in space according to the law just stated, the distribution of potential will be represented by the curves in Fig. XI.

Now from this figure it is manifest that $\frac{d\phi}{dy'}$ is generally very small except near the boundaries of the plates, so that the new distribution may be approximately represented by what actually exists, namely a certain superficial distribution near the edges of the plates.

If therefore we integrate $\iint\rho\,dx'\,dy'$ between the limits $y'=0$ and $y'=\frac{\pi}{2}b$, and from $x'=-\infty$ to $x=+\infty$, we shall find the whole additional charge on one side of the plates due to the curvature.

Since $\frac{d\phi}{dy'} = \frac{d\psi}{dx'}$,

$$\int_{-\infty}^{+\infty} \rho\, dx' = \int_{-\infty}^{+\infty} \frac{1}{4\pi} \frac{1}{(R+y')} \frac{d\psi}{dx} dx,$$

$$= \frac{1}{8} \frac{1}{R+y'} \left(\frac{y'}{B} - 1\right). \tag{35}$$

Integrating with respect to $y'$, we find

$$\int_0^B \int_{-\infty}^{+\infty} \rho\, dx'\, dy' = \frac{1}{8} - \frac{1}{8} \frac{R+B}{B} \log \frac{R+B}{R}, \tag{36}$$

$$= \frac{1}{16} \frac{B}{R} + \frac{1}{48} \frac{B^2}{R^2} - \&c. \tag{37}$$

This is the total quantity of electricity which we must suppose distributed in space near the positive side of one of the cylindric plates per unit of circumference. Since it is only close to the edge of the plate that the density is sensible, we may suppose it all condensed on the surface of the plate without altering sensibly its action on the opposed plane surface, and in calculating the attraction between that surface and the cylindric surface we may suppose this electricity to belong to the cylindric surface.

The superficial charge on the positive surface of the plate per unit of length would have been $-\frac{1}{8}$, if there had been no curvature. Hence this charge must be multiplied by the factor $\left(1 + \frac{1}{2}\frac{B}{R}\right)$ to get the total charge on the positive side.

In the case of a disk of radius $R$ placed midway between two infinite parallel plates at a distance $B$, we find for the capacity of the disk

$$\frac{R^2}{B} + 2\frac{\log_e 2}{\pi} R + \frac{1}{2} B. \tag{38}$$

*Theory of Thomson's Guard-ring.*

201.] In some of Sir W. Thomson's electrometers, a large plane surface is kept at one potential, and at a distance $a$ from this surface is placed a plane disk of radius $R$ surrounded by a large plane plate called a Guard-ring with a circular aperture of radius $R'$ concentric with the disk. This disk and plate are kept at potential zero.

The interval between the disk and the guard-plate may be regarded as a circular groove of infinite depth, and of breadth $R'-R$, which we denote by $B$.

The charge on the disk due to unit potential of the large disk, supposing the density uniform, would be $\frac{R^2}{4A}$.

The charge on one side of a straight groove of breadth $B$ and length $L = 2\pi R$, and of infinite depth, would be

$$\frac{1}{4}\frac{RB}{A+a'}.$$

But since the groove is not straight, but has a radius of curvature $R$, this must be multiplied by the factor $\left(1+\frac{1}{2}\frac{B}{R}\right)$.

The whole charge on the disk is therefore

$$\frac{R^2}{4A}+\frac{1}{4}\frac{RB}{A+a'}\left(1+\frac{B}{2R}\right) \tag{39}$$

$$=\frac{R^2+R'^2}{8A}-\frac{R'^2-R^2}{8A}\cdot\frac{a'}{A+a'}. \tag{40}$$

The value of $a$ cannot be greater than

$$a' = \frac{B\log 2}{\pi}, \quad = 0.22\,B \text{ nearly.}$$

If $B$ is small compared with either $A$ or $R$ this expression will give a sufficiently good approximation to the charge on the disk due to unity of difference of potential. The ratio of $A$ to $R$ may have any value, but the radii of the large disk and of the guard-ring must exceed $R$ by several multiples of $A$.

### Example VII.—Fig. XII.

202.] Helmholtz, in his memoir on discontinuous fluid motion*, has pointed out the application of several formulae in which the coordinates are expressed as functions of the potential and its conjugate function.

One of these may be applied to the case of an electrified plate of finite size placed parallel to an infinite plane surface connected with the earth.

Since $x_1 = A\phi$ and $y_1 = A\psi$,

and also $x_2 = Ae^{\phi}\cos\psi$ and $y_2 = Ae^{\phi}\sin\psi$,

are conjugate functions of $\phi$ and $\psi$, the functions formed by adding $x_1$ to $x_2$ and $y_1$ to $y_2$ will be also conjugate. Hence, if

$$x = A\phi + Ae^{\phi}\cos\psi,$$
$$y = A\psi + Ae^{\phi}\sin\psi,$$

* *Königl. Akad. der Wissenschaften,* zu Berlin, April 23, 1868.

then $x$ and $y$ will be conjugate with respect to $\phi$ and $\psi$, and $\phi$ and $\psi$ will be conjugate with respect to $x$ and $y$.

Now let $x$ and $y$ be rectangular coordinates, and let $k\psi$ be the potential, then $k\phi$ will be conjugate to $k\psi$, $k$ being any constant.

Let us put $\psi = \pi$, then $y = A\pi$, $x = A(\phi - e^{\phi})$.

If $\phi$ varies from $-\infty$ to $0$, and then from $0$ to $+\infty$, $x$ varies from $-\infty$ to $-A$ and from $-A$ to $-\infty$. Hence the equipotential surface for which $k\psi = \pi$ is a plane parallel to $x$ at a distance $b = \pi A$ from the origin, and extending from $-\infty$ to $x = -A$.

Let us consider a portion of this plane, extending from

$$x = -(A+a) \text{ to } x = -A \text{ and from } z = 0 \text{ to } z = c,$$

let us suppose its distance from the plane of $xz$ to be $y = b = A\pi$, and its potential to be $V = k\psi = k\pi$.

The charge of electricity on any portion of this part of the plane is found by ascertaining the values of $\phi$ at its extremities.

If these are $\phi_1$ and $\phi_2$, the quantity of electricity is

$$\frac{1}{4\pi} ck(\phi_2 - \phi_1).$$

We have therefore to determine $\phi$ from the equation

$$x = -(A+a) = A(\phi - e^{\phi}),$$

$\phi$ will have a negative value $\phi_1$ and a positive value $\phi_2$ at the edge of the plane, where $x = -A$, $\phi = 0$.

Hence the charge on the negative side is $-ck\phi_1$, and that on the positive side is $ck\phi_2$.

If we suppose that $a$ is large compared with $A$,

$$\phi_1 = -\frac{a}{A} - 1 + e^{-\frac{a}{A} - 1 + e^{-\frac{a}{A} - 1 + \&c.}},$$

$$\phi_2 = \log\left\{\frac{a}{A} + 1 + \log\left(\frac{a}{A} + 1 + \&c.\right)\right\}.$$

If we neglect the exponential terms in $\phi_1$ we shall find that the charge on the negative surface exceeds that which it would have if the superficial density had been uniform and equal to that at a distance from the boundary, by a quantity equal to the charge on a strip of breadth $A = \frac{b}{\pi}$ with the uniform superficial density.

The total capacity of the part of the plane considered is

$$C = \frac{c}{4\pi^2}(\phi_2 - \phi_1).$$

The total charge is $CV$, and the attraction towards the infinite plane is

$$-\tfrac{1}{2}V^2\frac{dC}{db} = V^2\frac{ac}{4\pi b^2}\left(1 + \frac{\frac{A}{a}}{1+\frac{A}{a}\log\frac{a}{A}} + e^{-\frac{a}{A}} + \&c.\right)$$

$$= \frac{V^2 c}{4\pi b^2}\left\{a + \frac{b}{\pi} - \frac{b^2}{\pi^2 a}\log\frac{a}{A} + \&c.\right\}.$$

The equipotential lines and lines of force are given in Fig. XII.

EXAMPLE VIII.—*Theory of a Grating of Parallel Wires.* Fig. XIII.

203.] In many electrical instruments a wire grating is used to prevent certain parts of the apparatus from being electrified by induction. We know that if a conductor be entirely surrounded by a metallic vessel at the same potential with itself, no electricity can be induced on the surface of the conductor by any electrified body outside the vessel. The conductor, however, when completely surrounded by metal, cannot be seen, and therefore, in certain cases, an aperture is left which is covered with a grating of fine wire. Let us investigate the effect of this grating in diminishing the effect of electrical induction. We shall suppose the grating to consist of a series of parallel wires in one plane and at equal intervals, the diameter of the wires being small compared with the distance between them, while the nearest portions of the electrified bodies on the one side and of the protected conductor on the other are at distances from the plane of the screen, which are considerable compared with the distance between consecutive wires.

204.] The potential at a distance $r'$ from the axis of a straight wire of infinite length charged with a quantity of electricity $\lambda$ per unit of length is

$$V = -2\lambda\log r' + C. \tag{1}$$

We may express this in terms of polar coordinates referred to an axis whose distance from the wire is unity, in which case we must make

$$r'^2 = 1 + 2r\cos\theta + r^2, \tag{2}$$

and if we suppose that the axis of reference is also charged with the linear density $\lambda'$, we find

$$V = -\lambda\log(1-2r\cos\theta+r^2) - 2\lambda'\log r + C. \tag{3}$$

If we now make

$$r = e^{2\pi\frac{y}{a}}, \qquad \theta = \frac{2\pi x}{a}, \tag{4}$$

then, by the theory of conjugate functions,

$$V = -\lambda \log \left(1 - 2e^{\frac{2\pi y}{a}} \cos \frac{2\pi x}{a} + e^{\frac{4\pi y}{a}}\right) - 2\lambda' \log e^{\frac{2\pi y}{a}} + C, \qquad (5)$$

where $x$ and $y$ are rectangular coordinates, will be the value of the potential due to an infinite series of fine wires parallel to $z$ in the plane of $yz$, and passing through points in the axis of $x$ for which $x$ is a multiple of $a$.

Each of these wires is charged with a linear density $\lambda$.

The term involving $\lambda'$ indicates an electrification, producing a constant force $-\frac{4\pi\lambda'}{a}$ in the direction of $y$.

The forms of the equipotential surfaces and lines of force when $\lambda' = 0$ are given in Fig. XIII. The equipotential surfaces near the wires are nearly cylinders, so that we may consider the solution approximately true, even when the wires are cylinders of a diameter which is finite but small compared with the distance between them.

The equipotential surfaces at a distance from the wires become more and more nearly planes parallel to that of the grating.

If in the equation we make $y = b_1$, a quantity large compared with $a$, we find approximately,

$$V_1 = -\frac{4\pi b_1}{a}(\lambda + \lambda') + C \text{ nearly.} \qquad (6)$$

If we next make $y = -b_2$ where $b_2$ is a negative quantity large compared with $a$, we find approximately,

$$V_2 = -\frac{4\pi b_2}{a}(\lambda - \lambda') + C \text{ nearly.} \qquad (7)$$

If $c$ is the radius of the wires of the grating, $c$ being small compared with $a$, we may find the potential of the grating itself by supposing that the surface of the wire coincides with the equipotential surface which cuts the plane of $yz$ at a distance $c$ from the axis of $z$. To find the potential of the grating we therefore put $x = c$, and $y = 0$, whence

$$V = -2\lambda \log 2 \sin \frac{\pi c}{a} + C. \qquad (8)$$

205.] We have now obtained expressions representing the electrical state of a system consisting of a grating of wires whose diameter is small compared with the distance between them, and two plane conducting surfaces, one on each side of the grating, and at distances which are great compared with the distance between the wires.

The surface-density $\sigma_1$ on the first plane is got from the equation (6)

$$4\pi\sigma_1 = \frac{dV_1}{db_1} = -\frac{4\pi}{a}(\lambda+\lambda'). \tag{9}$$

That on the second plane $\sigma_2$ from the equation (7)

$$4\pi\sigma_2 = -\frac{dV_2}{db_2} = -\frac{4\pi}{a}(\lambda-\lambda'). \tag{10}$$

If we now write $$\alpha = -\frac{a}{2\pi}\log_e\left(2\sin\frac{\pi c}{a}\right), \tag{11}$$

and eliminate $\lambda$ and $\lambda'$ from the equations (6), (7), (8), (9), (10), we find

$$4\pi\sigma_1\left(b_1+b_2+\frac{2b_1b_2}{\alpha}\right) = V_1\left(1+2\frac{b_2}{\alpha}\right) - V_2 - V\frac{2b_2}{\alpha}, \tag{12}$$

$$4\pi\sigma_2\left(b_1+b_2+\frac{2b_1b_2}{\alpha}\right) = -V_1 + V_2\left(1+2\frac{b_1}{\alpha}\right) - V\frac{2b_1}{\alpha}. \tag{13}$$

When the wires are infinitely thin, $\alpha$ becomes infinite, and the terms in which it is the denominator disappear, so that the case is reduced to that of two parallel planes without a grating interposed.

If the grating is in metallic communication with one of the planes, say the first, $V = V_1$, and the right-hand side of the equation for $\sigma_1$ becomes $V_1 - V_2$. Hence the density $\sigma_1$ induced on the first plane when the grating is interposed is to that which would have been induced on it if the grating were removed, the second plane being maintained at the same potential, as 1 to $1+\dfrac{2b_1b_2}{\alpha(b_1+b_2)}$.

We should have found the same value for the effect of the grating in diminishing the electrical influence of the first surface on the second, if we had supposed the grating connected with the second surface. This is evident since $b_1$ and $b_2$ enter into the expression in the same way. It is also a direct result of the theorem of Art. 88.

The induction of the one electrified plane on the other through the grating is the same as if the grating were removed, and the distance between the planes increased from $b_1+b_2$ to

$$b_1+b_2+2\frac{b_1b_2}{\alpha}.$$

If the two planes are kept at potential zero, and the grating electrified to a given potential, the quantity of electricity on the grating will be to that which would be induced on a plane of equal area placed in the same position as

$$2b_1b_2 \text{ is to } 2b_1b_2+\alpha(b_1+b_2).$$

This investigation is approximate only when $b_1$ and $b_2$ are large compared with $a$, and when $a$ is large compared with $c$. The quantity $\alpha$ is a line which may be of any magnitude. It becomes infinite when $c$ is indefinitely diminished.

If we suppose $c = \frac{1}{2}a$ there will be no apertures between the wires of the grating, and therefore there will be no induction through it. We ought therefore to have for this case $\alpha = 0$. The formula (11), however, gives in this case

$$\alpha = -\frac{a}{2\pi}\log_e 2, \quad = -0.11\,a,$$

which is evidently erroneous, as the induction can never be altered in sign by means of the grating. It is easy, however, to proceed to a higher degree of approximation in the case of a grating of cylindrical wires. I shall merely indicate the steps of this process.

## *Method of Approximation.*

206.] Since the wires are cylindrical, and since the distribution of electricity on each is symmetrical with respect to the diameter parallel to $y$, the proper expansion of the potential is of the form

$$V = C_0 \log r + \Sigma\, C_i r^i \cos i\theta, \tag{14}$$

where $r$ is the distance from the axis of one of the wires, and $\theta$ the angle between $r$ and $y$, and, since the wire is a conductor, when $r$ is made equal to the radius $V$ must be constant, and therefore the coefficient of each of the multiple cosines of $\theta$ must vanish.

For the sake of conciseness let us assume new coordinates $\xi$, $\eta$, &c. such that

$$a\xi = 2\pi x, \quad a\eta = 2\pi y, \quad a\rho = 2\pi r, \quad a\beta = 2\pi b, \text{ \&c.} \tag{15}$$

and let
$$F_\beta = \log\left(\epsilon^{\eta+\beta} + \epsilon^{-(\eta+\beta)} - 2\cos\xi\right). \tag{16}$$

Then if we make

$$V = A_0 F + A_1\frac{dF}{d\eta} + A_2\frac{d^2F}{d\eta^2} + \text{\&c.} \tag{17}$$

by giving proper values to the coefficients $A$ we may express any potential which is a function of $\eta$ and $\cos\xi$, and does not become infinite except when $\eta + \beta = 0$ and $\cos\xi = 1$.

When $\beta = 0$ the expansion of $F$ in terms of $\rho$ and $\theta$ is

$$F_0 = 2\log\rho + \tfrac{1}{12}\rho^2\cos 2\theta - \tfrac{1}{1440}\rho^4\cos 4\theta + \text{\&c.} \tag{18}$$

For finite values of $\beta$ the expansion of $F$ is

$$F_\beta = \beta + 2\log(1 - e^{-\beta}) + \frac{1 + e^{-\beta}}{1 - e^{-\beta}}\rho\cos\theta - \frac{e^{-\beta}}{(1 - e^{-\beta})^2}\rho^2\cos 2\theta + \text{\&c.} \tag{19}$$

In the case of the grating with two conducting planes whose equations are $\eta = -\beta_1$ and $\eta = \beta_2$, that of the plane of the grating being $\eta = 0$, there will be two infinite series of images of the grating. The first series will consist of the grating itself together with an infinite series of images on both sides, equal and similarly electrified. The axes of these imaginary cylinders lie in planes whose equations are of the form

$$\eta = \pm 2n(\beta_1+\beta_2), \tag{20}$$

$n$ being an integer.

The second series will consist of an infinite series of images for which the coefficients $A_0$, $A_2$, $A_4$, &c. are equal and opposite to the same quantities in the grating itself, while $A_1$, $A_3$, &c. are equal and of the same sign. The axes of these images are in planes whose equations are of the form

$$\eta = 2\beta_2 \pm 2m(\beta_1+\beta_2), \tag{21}$$

$m$ being an integer.

The potential due to any finite series of such images will depend on whether the number of images is odd or even. Hence the potential due to an infinite series is indeterminate, but if we add to it the function $B\eta + C$, the conditions of the problem will be sufficient to determine the electrical distribution.

We may first determine $V_1$ and $V_2$, the potentials of the two conducting planes, in terms of the coefficients $A_0$, $A_1$, &c., and of $B$ and $C$. We must then determine $\sigma_1$ and $\sigma_2$, the surface-density at any point of these planes. The mean values of $\sigma_1$ and $\sigma_2$ are given by the equations

$$4\pi\sigma_1 = A_0 - B, \qquad 4\pi\sigma_2 = A_0 + B. \tag{22}$$

We must then expand the potentials due to the grating itself and to all the images in terms of $\rho$ and cosines of multiples of $\theta$, adding to the result $\qquad B\rho\cos\theta + C.$

The terms independent of $\theta$ then give $V$ the potential of the grating, and the coefficient of the cosine of each multiple of $\theta$ equated to zero gives an equation between the indeterminate coefficients.

In this way as many equations may be found as are sufficient to eliminate all these coefficients and to leave two equations to determine $\sigma_1$ and $\sigma_2$ in terms of $V_1$, $V_2$, and $V$.

These equations will be of the form

$$\begin{aligned} V_1 - V &= 4\pi\sigma_1(b_1+a-\gamma) + 4\pi\sigma_2(a+\gamma), \\ V_2 - V &= 4\pi\sigma_1(a+\gamma) + 4\pi\sigma_2(b_2+a-\gamma). \end{aligned} \tag{23}$$

The quantity of electricity induced on one of the planes protected by the grating, the other plane being at a given difference of potential, will be the same as if the plates had been at a distance

$$\frac{(\alpha-\gamma)(b_1+b_2)+\beta_1\beta_2-4\alpha\gamma}{\alpha+\gamma} \text{ instead of } b_1+b_2.$$

The values of $\alpha$ and $\gamma$ are approximately as follows,

$$\alpha = \frac{a}{2\pi}\left\{\log\frac{a}{2\pi c} - \frac{5}{3}\cdot\frac{\pi^4 c^4}{15a^4+\pi^4c^4} \right.$$
$$\left. + 2e^{-4\pi\frac{b_1+b_2}{a}}\left(1+e^{-4\pi\frac{b_1}{a}}+e^{-4\pi\frac{b_2}{a}}+\&\text{c.}\right)+\&\text{c.}\right\}, \tag{24}$$

$$\gamma = \frac{3\pi a c^2}{3a^2+\pi^2c^2}\left(\frac{e^{-4\pi\frac{b_1}{a}}}{1-e^{-4\pi\frac{b_1}{a}}} - \frac{e^{-4\pi\frac{b_2}{a}}}{1-e^{-4\pi\frac{b_2}{a}}}\right)+\&\text{c.} \tag{25}$$

# CHAPTER XIII.

## ELECTROSTATIC INSTRUMENTS.

### *On Electrostatic Instruments.*

THE instruments which we have to consider at present may be divided into the following classes :

(1) Electrical machines for the production and augmentation of electrification.

(2) Multipliers, for increasing electrification in a known ratio.

(3) Electrometers, for the measurement of electric potentials and charges.

(4) Accumulators, for holding large electrical charges.

### *Electrical Machines.*

207.] In the common electrical machine a plate or cylinder of glass is made to revolve so as to rub against a surface of leather, on which is spread an amalgam of zinc and mercury. The surface of the glass becomes electrified positively and that of the rubber negatively. As the electrified surface of the glass moves away from the negative electrification of the rubber it acquires a high positive potential. It then comes opposite to a set of sharp metal points in connexion with the conductor of the machine. The positive electrification of the glass induces a negative electrification of the points, which is the more intense the sharper the points and the nearer they are to the glass.

When the machine works properly there is a discharge through the air between the glass and the points, the glass loses part of its positive charge, which is transferred to the points and so to the insulated prime conductor of the machine, and to any other body with which it is in electric communication.

The portion of the glass which is advancing towards the rubber has thus a smaller positive charge than that which is leaving it at the same time, so that the rubber, and the conductors in communication with it, become negatively electrified.

The highly positive surface of the glass where it leaves the rubber is more attracted by the negative charge of the rubber than the partially discharged surface which is advancing towards the rubber. The electrical forces therefore act as a resistance to the force employed in turning the machine. The work done in turning the machine is therefore greater than that spent in overcoming ordinary friction and other resistances, and the excess is employed in producing a state of electrification whose energy is equivalent to this excess.

The work done in overcoming friction is at once converted into heat in the bodies rubbed together. The electrical energy may be also converted either into mechanical energy or into heat.

If the machine does not store up mechanical energy, all the energy will be converted into heat, and the only difference between the heat due to friction and that due to electrical action is that the former is generated at the rubbing surfaces while the latter may be generated in conductors at a distance *.

We have seen that the electrical charge on the surface of the glass is attracted by the rubber. If this attraction were sufficiently intense there would be a discharge between the glass and the rubber, instead of between the glass and the collecting points. To prevent this, flaps of silk are attached to the rubber. These become negatively electrified and adhere to the glass, and so diminish the potential near the rubber.

The potential therefore increases more gradually as the glass moves away from the rubber, and therefore at any one point there is less attraction of the charge on the glass towards the rubber, and consequently less danger of direct discharge to the rubber.

In some electrical machines the moving part is of ebonite instead of glass, and the rubbers of wool or fur. The rubber is then electrified positively and the prime conductor negatively.

### *The Electrophorus of Volta.*

208.] The electrophorus consists of a plate of resin or of ebonite backed with metal, and a plate of metal of the same size. An insulating handle can be screwed to the back of either of these plates. The ebonite plate has a metal pin which connects the metal

* It is probable that in many cases where dynamical energy is converted into heat by friction, part of the energy may be first transformed into electrical energy and then converted into heat as the electrical energy is spent in maintaining currents of short circuit close to the rubbing surfaces. See Sir W. Thomson. 'On the Electrodynamic Qualities of Metals.' *Phil. Trans.*, 1856, p. 650.

plate with the metal back of the ebonite plate when the two plates are in contact.

The ebonite plate is electrified negatively by rubbing it with wool or cat's skin. The metal plate is then brought near the ebonite by means of the insulating handle. No direct discharge passes between the ebonite and the metal plate, but the potential of the metal plate is rendered negative by induction, so that when it comes within a certain distance of the metal pin a spark passes, and if the metal plate be now carried to a distance it is found to have a positive charge which may be communicated to a conductor. The metal at the back of the ebonite plate is found to have a negative charge equal and opposite to the charge of the metal plate.

In using the instrument to charge a condenser or accumulator one of the plates is laid on a conductor in communication with the earth, and the other is first laid on it, then removed and applied to the electrode of the condenser, then laid on the fixed plate and the process repeated. If the ebonite plate is fixed the condenser will be charged positively. If the metal plate is fixed the condenser will be charged negatively.

The work done by the hand in separating the plates is always greater than the work done by the electrical attraction during the approach of the plates, so that the operation of charging the condenser involves the expenditure of work. Part of this work is accounted for by the energy of the charged condenser, part is spent in producing the noise and heat of the sparks, and the rest in overcoming other resistances to the motion.

*On Machines producing Electrification by Mechanical Work.*

209.] In the ordinary frictional electrical machine the work done in overcoming friction is far greater than that done in increasing the electrification. Hence any arrangement by which the electrification may be produced entirely by mechanical work against the electrical forces is of scientific importance if not of practical value. The first machine of this kind seems to have been Nicholson's Revolving Doubler, described in the *Philosophical Transactions* for 1788 as 'an instrument which by the turning of a Winch produces the two states of Electricity without friction or communication with the Earth.'

210.] It was by means of the revolving doubler that Volta succeeded in developing from the electrification of the pile an electrification capable of affecting his electrometer. Instruments on the same principle have been invented independently by Mr. C. F. Varley *, and Sir W. Thomson.

These instruments consist essentially of insulated conductors of various forms, some fixed and others moveable. The moveable conductors are called Carriers, and the fixed ones may be called Inductors, Receivers, and Regenerators. The inductors and receivers are so formed that when the carriers arrive at certain points in their revolution they are almost completely surrounded by a conducting body. As the inductors and receivers cannot completely surround the carrier and at the same time allow it to move freely in and out without a complicated arrangement of moveable pieces, the instrument is not theoretically perfect without a pair of regenerators, which store up the small amount of electricity which the carriers retain when they emerge from the receivers.

For the present, however, we may suppose the inductors and receivers to surround the carrier completely when it is within them, in which case the theory is much simplified.

We shall suppose the machine to consist of two inductors $A$ and $C$, and of two receivers $B$ and $D$, with two carriers $F$ and $G$.

Suppose the inductor $A$ to be positively electrified so that its potential is $A$, and that the carrier $F$ is within it and is at potential $F$. Then, if $Q$ is the coefficient of induction (taken positive) between $A$ and $F$, the quantity of electricity on the carrier will be $Q(F-A)$.

If the carrier, while within the inductor, is put in connexion with the earth, then $F=0$, and the charge on the carrier will be $-QA$, a negative quantity. Let the carrier be carried round till it is within the receiver $B$, and let it then come in contact with a spring so as to be in electrical connexion with $B$. It will then, as was shewn in Art. 32, become completely discharged, and will communicate its whole negative charge to the receiver $B$.

The carrier will next enter the inductor $C$, which we shall suppose charged negatively. While within $C$ it is put in connexion with the earth and thus acquires a positive charge, which it carries off and communicates to the receiver $D$, and so on.

In this way, if the potentials of the inductors remain always constant, the receivers $B$ and $D$ receive successive charges, which are the same for every revolution of the carrier, and thus every revolution produces an equal increment of electricity in the receivers.

* Specification of Patent, Jan. 27, 1860, No. 206.

But by putting the inductor $A$ in communication with the receiver $D$, and the inductor $C$ with the receiver $B$, the potentials of the inductors will be continually increased, and the quantity of electricity communicated to the receivers in each revolution will continually increase.

For instance, let the potential of $A$ and $O$ be $U$, and that of $B$ and $C$, $V$, and when the carrier is within $A$ let the charge on $A$ and $C$ be $x$, and that on the carrier $z$, then, since the potential of the carrier is zero, being in contact with earth, its charge is $z = -QU$. The carrier enters $B$ with this charge and communicates it to $B$. If the capacity of $B$ and $C$ is $B$, their potential will be changed from $V$ to $V - \frac{Q}{B}U$.

If the other carrier has at the same time carried a charge $-QV$ from $C$ to $D$, it will change the potential of $A$ and $O$ from $U$ to $U - \frac{Q'}{A}V$, if $Q'$ is the coefficient of induction between the carrier and $C$, and $A$ the capacity of $A$ and $D$. If, therefore, $U_n$ and $V_n$ be the potentials of the two inductors after $n$ half revolutions, and $U_{n+1}$ and $V_{n+1}$ after $n+1$ half revolutions,

$$U_{n+1} = U_n - \frac{Q'}{A}V_n,$$

$$V_{n+1} = V_n - \frac{Q}{B}U_n.$$

If we write $p^2 = \frac{Q}{B}$ and $q^2 = \frac{Q'}{A}$, we find

$$pU_{n+1} + qV_{n+1} = (pU_n + qV_n)(1 - pq) = (pU_0 + qV_0)(1 - pq)^{n+1},$$

$$pU_{n+1} - qV_{n+1} = (pU_n - qV_n)(1 + pq) = (pU_0 - qV_0)(1 + pq)^{n+1}.$$

Hence

$$U_n = U_0\left((1-pq)^n + (1+pq)^n\right) + \frac{q}{p}V_0\left((1-pq)^n - (1+pq)^n\right),$$

$$V_n = \frac{p}{q}U_0\left((1-pq)^n - (1+pq)^n\right) + V_0\left((1-pq)^n + (1+pq)^n\right).$$

It appears from these equations that the quantity $pU + qV$ continually diminishes, so that whatever be the initial state of electrification the receivers are ultimately oppositely electrified, so that the potentials of $A$ and $B$ are in the ratio of $p$ to $-q$.

On the other hand, the quantity $pU - qV$ continually increases, so that, however little $pU$ may exceed or fall short of $qV$ at first, the difference will be increased in a geometrical ratio in each revolution till the electromotive forces become so great that the insulation of the apparatus is overcome.

Instruments of this kind may be used for various purposes.

For producing a copious supply of electricity at a high potential, as is done by means of Mr. Varley's large machine.

For adjusting the charge of a condenser, as in the case of Thomson's electrometer, the charge of which can be increased or diminished by a few turns of a very small machine of this kind, which is then called a Replenisher.

For multiplying small differences of potential. The inductors may be charged at first to an exceedingly small potential, as, for instance, that due to a thermo-electric pair, then, by turning the machine, the difference of potentials may be continually multiplied till it becomes capable of measurement by an ordinary electrometer. By determining by experiment the ratio of increase of this difference due to each turn of the machine, the original electromotive force with which the inductors were charged may be deduced from the number of turns and the final electrification.

In most of these instruments the carriers are made to revolve about an axis and to come into the proper positions with respect to the inductors by turning an axle. The connexions are made by means of springs so placed that the carriers come in contact with them at the proper instants.

211.] Sir W. Thomson*, however, has constructed a machine for multiplying electrical charges in which the carriers are drops of water falling out of the inside of an inductor into an insulated receiver. The receiver is thus continually supplied with electricity of opposite sign to that of the inductor. If the inductor is electrified positively, the receiver will receive a continually increasing charge of negative electricity.

The water is made to escape from the receiver by means of a funnel, the nozzle of which is almost surrounded by the metal of the receiver. The drops falling from this nozzle are therefore nearly free from electrification. Another inductor and receiver of the same construction are arranged so that the inductor of the one system is in connexion with the receiver of the other. The rate of increase of charge of the receivers is thus no longer constant, but increases in a geometrical progression with the time, the charges of the two receivers being of opposite signs. This increase goes on till the falling drops are so diverted from their course by

* *Proc. R. S.*, June 20, 1867.

the electrical action that they fall outside of the receiver or even strike the inductor.

In this instrument the energy of the electrification is drawn from that of the falling drops.

212.] Several other electrical machines have been constructed in which the principle of electric induction is employed. Of these the most remarkable is that of Holtz, in which the carrier is a glass plate varnished with gum-lac and the inductors are pieces of pasteboard. Sparks are prevented from passing between the parts of the apparatus by means of two glass plates, one on each side of the revolving carrier plate. This machine is found to be very effective, and not to be much affected by the state of the atmosphere. The principle is the same as in the revolving doubler and the instruments developed out of the same idea, but as the carrier is an insulating plate and the inductors are imperfect conductors, the complete explanation of the action is more difficult than in the case where the carriers are good conductors of known form and are charged and discharged at definite points.

213.] In the electrical machines already described sparks occur whenever the carrier comes in contact with a conductor at a different potential from its own.

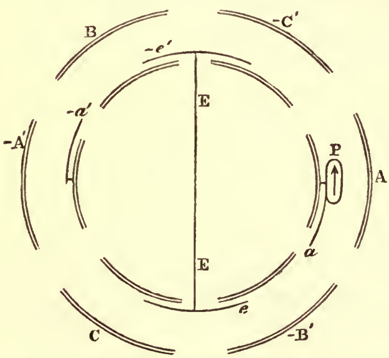

Fig. 17.

Now we have shewn that whenever this occurs there is a loss of energy, and therefore the whole work employed in turning the machine is not converted into electrification in an available form, but part is spent in producing the heat and noise of electric sparks.

I have therefore thought it desirable to shew how an electrical machine may be constructed which is not subject to this loss of efficiency. I do not propose it as a useful form of machine, but as an example of the method by which the contrivance called in heat-engines a regenerator may be applied to an electrical machine to prevent loss of work.

In the figure let $A$, $B$, $C$, $A'$, $B'$, $C'$ represent hollow fixed conductors, so arranged that the carrier $P$ passes in succession within each of them. Of these $A$, $A'$ and $B$, $B'$ nearly surround the carrier when it is at the middle point of its passage, but $C$, $C'$ do not cover it so much.

We shall suppose $A$, $B$, $C$ to be connected with a Leyden jar of great capacity at potential $V$, and $A'$, $B'$, $C'$ to be connected with another jar at potential $-V'$.

$P$ is one of the carriers moving in a circle from $A$ to $C'$, &c., and touching in its course certain springs, of which $a$ and $a'$ are connected with $A$ and $A'$ respectively, and $e$, $e'$ are connected with the earth.

Let us suppose that when the carrier $P$ is in the middle of $A$ the coefficient of induction between $P$ and $A$ is $-A$. The capacity of $P$ in this position is greater than $A$, since it is not completely surrounded by the receiver $A$. Let it be $A+a$.

Then if the potential of $P$ is $U$, and that of $A$, $V$, the charge on $P$ will be $(A+a)U-AV$.

Now let $P$ be in contact with the spring $a$ when in the middle of the receiver $A$, then the potential of $P$ is $V$, the same as that of $A$, and its charge is therefore $aV$.

If $P$ now leaves the spring $a$ it carries with it the charge $aV$. As $P$ leaves $A$ its potential diminishes, and it diminishes still more when it comes within the influence of $C'$, which is negatively electrified.

If when $P$ comes within $C$ its coefficient of induction on $C$ is $-C'$, and its capacity is $C'+c'$, then, if $U$ is the potential of $P$ the charge on $P$ is

$$(C'+c')U+C'V' = aV.$$

If $$C'V' = aV,$$

then at this point $U$ the potential of $P$ will be reduced to zero.

Let $P$ at this point come in contact with the spring $e'$ which is connected with the earth. Since the potential of $P$ is equal to that of the spring there will be no spark at contact.

This conductor $C'$, by which the carrier is enabled to be connected to earth without a spark, answers to the contrivance called a regenerator in heat-engines. We shall therefore call it a Regenerator.

Now let $P$ move on, still in contact with the earth-spring $e'$, till it comes into the middle of the inductor $B$, the potential of which is $V$. If $-B$ is the coefficient of induction between $P$ and $B$ at this point, then, since $U=0$ the charge on $P$ will be $-BV$.

When $P$ moves away from the earth-spring it carries this charge with it. As it moves out of the positive inductor $B$ towards the negative receiver $A'$ its potential will be increasingly negative. At the middle of $A'$, if it retained its charge, its potential would be

$$-\frac{A'V'+BV}{A'+a'},$$

and if $BV$ is greater than $a'V'$ its numerical value will be greater than that of $V'$. Hence there is some point before $P$ reaches the middle of $A'$ where its potential is $-V'$. At this point let it come in contact with the negative receiver-spring $a'$. There will be no spark since the two bodies are at the same potential. Let $P$ move on to the middle of $A'$, still in contact with the spring, and therefore at the same potential with $A'$. During this motion it communicates a negative charge to $A'$. At the middle of $A'$ it leaves the spring and carries away a charge $-a'V'$ towards the positive regenerator $C$, where its potential is reduced to zero and it touches the earth-spring $e$. It then slides along the earth-spring into the negative inductor $B'$, during which motion it acquires a positive charge $B'V'$ which it finally communicates to the positive receiver $A$, and the cycle of operations is repeated.

During this cycle the positive receiver has lost a charge $aV$ and gained a charge $B'V'$. Hence the total gain of positive electricity is

$$B'V'-aV.$$

Similarly the total gain of negative electricity is $BV-a'V'$.

By making the inductors so as to be as close to the surface of the carrier as is consistent with insulation, $B$ and $B'$ may be made large, and by making the receivers so as nearly to surround the carrier when it is within them, $a$ and $a'$ may be made very small, and then the charges of both the Leyden jars will be increased in every revolution.

The conditions to be fulfilled by the regenerators are

$$C'V'=aV, \quad \text{and} \quad CV=a'V'.$$

Since $a$ and $a'$ are small the regenerators do not require to be either large or very close to the carriers.

*On Electrometers and Electroscopes.*

214.] An electrometer is an instrument by means of which electrical charges or electrical potentials may be measured. Instruments by means of which the existence of electric charges or of differences of potential may be indicated, but which are not capable of affording numerical measures, are called Electroscopes.

An electroscope if sufficiently sensible may be used in electrical measurements, provided we can make the measurement depend on the absence of electrification. For instance, if we have two charged bodies $A$ and $B$ we may use the method described in Chapter I to determine which body has the greater charge. Let the body $A$ be carried by an insulating support into the interior of an insulated closed vessel $C$. Let $C$ be connected to earth and again insulated. There will then be no external electrification on $C$. Now let $A$ be removed, and $B$ introduced into the interior of $C$, and the electrification of $C$ tested by an electroscope. If the charge of $B$ is equal to that of $A$ there will be no electrification, but if it is greater or less there will be electrification of the same kind as that of $B$, or the opposite kind.

Methods of this kind, in which the thing to be observed is the non-existence of some phenomenon, are called *null* or *zero* methods. They require only an instrument capable of detecting the existence of the phenomenon.

In another class of instruments for the registration of phenomena the instrument may be depended upon to give always the same indication for the same value of the quantity to be registered, but the readings of the scale of the instrument are not proportional to the values of the quantity, and the relation between these readings and the corresponding value is unknown, except that the one is some continuous function of the other. Several electrometers depending on the mutual repulsion of parts of the instrument which are similarly electrified are of this class. The use of such instruments is to register phenomena, not to measure them. Instead of the true values of the quantity to be measured, a series of numbers is obtained, which may be used afterwards to determine these values when the scale of the instrument has been properly investigated and tabulated.

In a still higher class of instruments the scale readings are proportional to the quantity to be measured, so that all that is required for the complete measurement of the quantity is a knowledge of the coefficient by which the scale readings must be multiplied to obtain the true value of the quantity.

Instruments so constructed that they contain within themselves the means of independently determining the true values of quantities are called Absolute Instruments.

### *Coulomb's Torsion Balance.*

215.] A great number of the experiments by which Coulomb established the fundamental laws of electricity were made by measuring the force between two small spheres charged with electricity, one of which was fixed while the other was held in equilibrium by two forces, the electrical action between the spheres, and the torsional elasticity of a glass fibre or metal wire. See Art. 38.

The balance of torsion consists of a horizontal arm of gum-lac, suspended by a fine wire or glass fibre, and carrying at one end a little sphere of elder pith, smoothly gilt. The suspension wire is fastened above to the vertical axis of an arm which can be moved round a horizontal graduated circle, so as to twist the upper end of the wire about its own axis any number of degrees.

The whole of this apparatus is enclosed in a case. Another little sphere is so mounted on an insulating stem that it can be charged and introduced into the case through a hole, and brought so that its centre coincides with a definite point in the horizontal circle described by the suspended sphere. The position of the suspended sphere is ascertained by means of a graduated circle engraved on the cylindrical glass case of the instrument.

Now suppose both spheres charged, and the suspended sphere in equilibrium in a known position such that the torsion-arm makes an angle $\theta$ with the radius through the centre of the fixed sphere. The distance of the centres is then $2a\sin\frac{1}{2}\theta$, where $a$ is the radius of the torsion-arm, and if $F$ is the force between the spheres the moment of this force about the axis of torsion is $Fa\cos\frac{1}{2}\theta$.

Let both spheres be completely discharged, and let the torsion-arm now be in equilibrium at an angle $\phi$ with the radius through the fixed sphere.

Then the angle through which the electrical force twisted the torsion-arm must have been $\theta-\phi$, and if $M$ is the moment of the torsional elasticity of the fibre, we shall have the equation

$$Fa\cos\tfrac{1}{2}\theta = M(\theta-\phi).$$

Hence, if we can ascertain $M$, we can determine $F$, the actual force between the spheres at the distance $2a\sin\frac{1}{2}\theta$.

To find $M$, the moment of torsion, let $I$ be the moment of inertia of the torsion-arm, and $T$ the time of a double vibration of the arm under the action of the torsional elasticity, then

$$M = \frac{1}{4\pi^2}IT^2.$$

In all electrometers it is of the greatest importance to know what force we are measuring. The force acting on the suspended sphere is due partly to the direct action of the fixed sphere, but partly also to the electrification, if any, of the sides of the case.

If the case is made of glass it is impossible to determine the electrification of its surface otherwise than by very difficult measurements at every point. If, however, either the case is made of metal, or if a metallic case which almost completely encloses the apparatus is placed as a screen between the spheres and the glass case, the electrification of the inside of the metal screen will depend entirely on that of the spheres, and the electrification of the glass case will have no influence on the spheres. In this way we may avoid any indefiniteness due to the action of the case.

To illustrate this by an example in which we can calculate all the effects, let us suppose that the case is a sphere of radius $b$, that the centre of motion of the torsion-arm coincides with the centre of the sphere and that its radius is $a$; that the charges on the two spheres are $E_1$ and $E_2$, and that the angle between their positions is $\theta$; that the fixed sphere is at a distance $a_1$ from the centre, and that $r$ is the distance between the two small spheres.

Neglecting for the present the effect of induction on the distribution of electricity on the small spheres, the force between them will be a repulsion

$$= \frac{EE_1}{r^2},$$

and the moment of this force round a vertical axis through the centre will be

$$\frac{EE_1\, aa_1 \sin\theta}{r^3}.$$

The image of $E_1$ due to the spherical surface of the case is a point in the same radius at a distance $\frac{b^2}{a_1}$ with a charge $-E_1\frac{b}{a_1}$, and the moment of the attraction between $E$ and this image about the axis of suspension is

$$EE_1\frac{b}{a_1}\frac{a\dfrac{b^2}{a_1}\sin\theta}{\left\{a^2-2\dfrac{ab^2}{a_1}\cos\theta+\dfrac{b^4}{a_1{}^2}\right\}^{\frac{3}{2}}}$$

$$= EE_1\frac{aa_1\sin\theta}{b^3\left\{1-2\dfrac{aa_1}{b^2}\cos\theta+\dfrac{a^2a^2{}_1}{b^4}\right\}^{\frac{3}{2}}}.$$

If $b$, the radius of the spherical case, is large compared with $a$

and $a_1$, the distances of the spheres from the centre, we may neglect the second and third terms of the factor in the denominator. The whole moment tending to turn the torsion-arm may then be written

$$E E_1 a a_1 \sin\theta \left\{ \frac{1}{r^3} - \frac{1}{b^3} \right\} = M(\theta - \phi).$$

*Electrometers for the Measurement of Potentials.*

216.] In all electrometers the moveable part is a body charged with electricity, and its potential is different from that of certain of the fixed parts round it. When, as in Coulomb's method, an insulated body having a certain charge is used, it is the charge which is the direct object of measurement. We may, however, connect the balls of Coulomb's electrometer, by means of fine wires, with different conductors. The charges of the balls will then depend on the values of the potentials of these conductors and on the potential of the case of the instrument. The charge on each ball will be approximately equal to its radius multiplied by the excess of its potential over that of the case of the instrument, provided the radii of the balls are small compared with their distances from each other and from the sides or opening of the case.

Coulomb's form of apparatus, however, is not well adapted for measurements of this kind, owing to the smallness of the force between spheres at the proper distances when the difference of potentials is small. A more convenient form is that of the Attracted Disk Electrometer. The first electrometers on this principle were constructed by Sir W. Snow Harris *. They have since been brought to great perfection, both in theory and construction, by Sir W. Thomson †.

When two disks at different potentials are brought face to face with a small interval between them there will be a nearly uniform electrification on the opposite faces and very little electrification on the backs of the disks, provided there are no other conductors or electrified bodies in the neighbourhood. The charge on the positive disk will be approximately proportional to its area, and to the difference of potentials of the disks, and inversely as the distance between them. Hence, by making the areas of the disks large

* *Phil Trans.* 1834.

† See an excellent report on Electrometers by Sir W. Thomson. *Report of the British Association*, Dundee, 1867.

and the distance between them small, a small difference of potential may give rise to a measurable force of attraction.

The mathematical theory of the distribution of electricity over two disks thus arranged is given at Art. 202, but since it is impossible to make the case of the apparatus so large that we may suppose the disks insulated in an infinite space, the indications of the instrument in this form are not easily interpreted numerically.

217.] The addition of the guard-ring to the attracted disk is one of the chief improvements which Sir W. Thomson has made on the apparatus.

Instead of suspending the whole of one of the disks and determining the force acting upon it, a central portion of the disk is separated from the rest to form the attracted disk, and the outer ring forming the remainder of the disk is fixed. In this way the force is measured only on that part of the disk where it is most regular, and the want of uniformity of the electrification near the

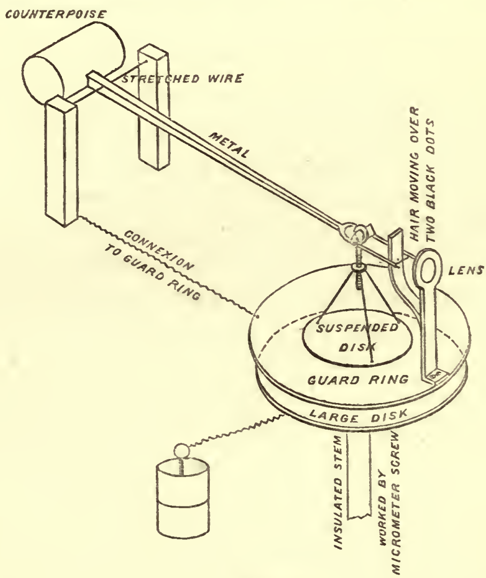


Fig. 18.

edge is of no importance, as it occurs on the guard-ring and not on the suspended part of the disk.

Besides this, by connecting the guard-ring with a metal case surrounding the back of the attracted disk and all its suspending apparatus, the electrification of the back of the disk is rendered impossible, for it is part of the inner surface of a closed hollow conductor all at the same potential.

Thomson's Absolute Electrometer therefore consists essentially of two parallel plates at different potentials, one of which is made so that a certain area, no part of which is near the edge of the plate, is moveable under the action of electric force. To fix our ideas we may suppose the attracted disk and guard-ring uppermost. The fixed disk is horizontal, and is mounted on an insulating stem which has a measurable vertical motion given to it by means of a micrometer screw. The guard-ring is at least as large as the fixed disk; its lower surface is truly plane and parallel to the fixed disk. A delicate balance is erected on the guard-ring to which is suspended a light moveable disk which almost fills the circular aperture in the guard-ring without rubbing against its sides. The lower surface of the suspended disk must be truly plane, and we must have the means of knowing when its plane coincides with that of the lower surface of the guard-ring, so as to form a single plane interrupted only by the narrow interval between the disk and its guard-ring.

For this purpose the lower disk is screwed up till it is in contact with the guard-ring, and the suspended disk is allowed to rest upon the lower disk, so that its lower surface is in the same plane as that of the guard-ring. Its position with respect to the guard-ring is then ascertained by means of a system of fiducial marks. Sir W. Thomson generally uses for this purpose a black hair attached to the moveable part. This hair moves up or down just in front of two black dots on a white enamelled ground and is viewed along with these dots by means of a plano convex lens with the plane side next the eye. If the hair as seen through the lens appears straight and bisects the interval between the black dots it is said to be in its *sighted position*, and indicates that the suspended disk with which it moves is in its proper position as regards height. The horizontality of the suspended disk may be tested by comparing the reflexion of part of any object from its upper surface with that of the remainder of the same object from the upper surface of the guard-ring.

The balance is then arranged so that when a known weight is placed on the centre of the suspended disk it is in equilibrium in its sighted position, the whole apparatus being freed from electrification by putting every part in metallic communication. A metal case is placed over the guard-ring so as to enclose the balance and suspended disk, sufficient apertures being left to see the fiducial marks.

The guard-ring, case, and suspended disk are all in metallic communication with each other, but are insulated from the other parts of the apparatus.

Now let it be required to measure the difference of potentials of two conductors. The conductors are put in communication with the upper and lower disks respectively by means of wires, the weight is taken off the suspended disk, and the lower disk is moved up by means of the micrometer screw till the electrical attraction brings the suspended disk down to its sighted position. We then know that the attraction between the disks is equal to the weight which brought the disk to its sighted position.

If $W$ be the numerical value of the weight, and $g$ the force of gravity, the force is $Wg$, and if $A$ is the area of the suspended disk, $D$ the distance between the disks, and $V$ the difference of the potentials of the disks,

$$Wg = \frac{V^2 A}{8\pi D^2},$$

$$\text{or} \quad V = D\sqrt{\frac{8\pi g W}{A}}.$$

If the suspended disk is circular, of radius $R$, and if the radius of the aperture of the guard-ring is $R'$, then

$$A = \tfrac{1}{2}\pi(R^2 + R'^2)\,*, \text{ and } V = 4D\sqrt{\frac{gW}{R^2 + R'^2}}.$$

218.] Since there is always some uncertainty in determining the micrometer reading corresponding to $D = 0$, and since any error

* Let us denote the radius of the suspended disk by $R$, and that of the aperture of the guard-ring by $R'$, then the breadth of the annular interval between the disk and the ring will be $B = R' - R$.

If the distance between the suspended disk and the large fixed disk is $D$, and the difference of potentials between these disks is $V$, then, by the investigation in Art. 201, the quantity of electricity on the suspended disk will be

$$Q = V\left\{\frac{R^2 + R'^2}{8D} - \frac{R'^2 - R^2}{8D}\frac{\alpha}{D + \alpha}\right\},$$

$$\text{where} \quad \alpha = B\frac{\log_e 2}{\pi}, \quad \text{or} \quad \alpha = 0.220635\,(R' - R).$$

If the surface of the guard-ring is not exactly in the plane of the surface of the suspended disk, let us suppose that the distance between the fixed disk and the guard-ring is not $D$ but $D + z = D'$, then it appears from the investigation in Art. 225 that there will be an additional charge of electricity near the edge of the disk on account of its height $z$ above the general surface of the guard-ring. The whole charge in this case is therefore

$$Q = V\left\{\frac{R^2 + R'^2}{8D} - \frac{R'^2 - R^2}{8D}\frac{\alpha}{D + \alpha} + \frac{R + R'}{D}(D' - D)\log_e\frac{4\pi(R + R')}{D' - D}\right\},$$

in the position of the suspended disk is most important when $D$ is small, Sir W. Thomson prefers to make all his measurements depend on differences of the electromotive force $V$. Thus, if $V$ and $V'$ are two potentials, and $D$ and $D'$ the corresponding distances,

$$V-V' = (D-D')\sqrt{\frac{8\pi g W}{A}}.$$

For instance, in order to measure the electromotive force of a galvanic battery, two electrometers are used.

By means of a condenser, kept charged if necessary by a replenisher, the lower disk of the principal electrometer is maintained at a constant potential. This is tested by connecting the lower disk of the principal electrometer with the lower disk of a secondary electrometer, the suspended disk of which is connected with the earth. The distance between the disks of the secondary electrometer and the force required to bring the suspended disk to its sighted position being constant, if we raise the potential of the condenser till the secondary electrometer is in its sighted position, we know that the potential of the lower disk of the principal electrometer exceeds that of the earth by a constant quantity which we may call $V$.

If we now connect the positive electrode of the battery to earth, and connect the suspended disk of the principal electrometer to the negative electrode, the difference of potentials between the disks will be $V+v$, if $v$ is the electromotive force of the battery. Let $D$ be the reading of the micrometer in this case, and let $D'$ be the reading when the suspended disk is connected with earth, then

$$v = (D-D')\sqrt{\frac{8\pi g W}{A}}.$$

In this way a small electromotive force $v$ may be measured by the electrometer with the disks at conveniently measurable distances. When the distance is too small a small change of absolute distance makes a great change in the force, since the

and in the expression for the attraction we must substitute for $A$, the area of the disk, the corrected quantity

$$A = \tfrac{1}{2}\pi\left\{R^2+R'^2-(R'^2-R^2)\frac{\alpha}{D+\alpha}+8(R+R')(D'-D)\log_e\frac{4\pi(R+R')}{D'-D}\right\},$$

where $R$ = radius of suspended disk,
$R'$ = radius of aperture in the guard-ring,
$D$ = distance between fixed and suspended disks,
$D'$ = distance between fixed disk and guard-ring,
$\alpha$ = $0.220635\,(R'-R)$.

When $\alpha$ is small compared with $D$ we may neglect the second term, and when $D'-D$ is small we may neglect the last term.

force varies inversely as the square of the distance, so that any error in the absolute distance introduces a large error in the result unless the distance is large compared with the limits of error of the micrometer screw.

The effect of small irregularities of form in the surfaces of the disks and of the interval between them diminish according to the inverse cube and higher inverse powers of the distance, and whatever be the form of a corrugated surface, the eminences of which just reach a plane surface, the electrical effect at any distance which is considerable compared to the breadth of the corrugations, is the same as that of a plane at a certain small distance behind the plane of the tops of the eminences. See Arts. 197, 198.

By means of the auxiliary electrification, tested by the auxiliary electrometer, a proper interval between the disks is secured.

The auxiliary electrometer may be of a simpler construction, in which there is no provision for the determination of the force of attraction in absolute measure, since all that is wanted is to secure a constant electrification. Such an electrometer may be called a *gauge* electrometer.

This method of using an auxiliary electrification besides the electrification to be measured is called the Heterostatic method of electrometry, in opposition to the Idiostatic method in which the whole effect is produced by the electrification to be measured.

In several forms of the attracted disk electrometer, the attracted disk is placed at one end of an arm which is supported by being attached to a platinum wire passing through its centre of gravity and kept stretched by means of a spring. The other end of the arm carries the hair which is brought to a sighted position by altering the distance between the disks, and so adjusting the force of the electric attraction to a constant value. In these electrometers this force is not in general determined in absolute measure, but is known to be constant, provided the torsional elasticity of the platinum wire does not change.

The whole apparatus is placed in a Leyden jar, of which the inner surface is charged and connected with the attracted disk and guard-ring. The other disk is worked by a micrometer screw and is connected first with the earth and then with the conductor whose potential is to be measured. The difference of readings multiplied by a constant to be determined for each electrometer gives the potential required.

219.] The electrometers already described are not self-acting, but require for each observation an adjustment of a micrometer screw, or some other movement which must be made by the observer. They are therefore not fitted to act as self-registering instruments, which must of themselves move into the proper position. This condition is fulfilled by Thomson's Quadrant Electrometer.

The electrical principle on which this instrument is founded may be thus explained :—

*A* and *B* are two fixed conductors which may be at the same or at different potentials. *C* is a moveable conductor at a high potential, which is so placed that part of it is opposite to the surface of *A* and part opposite to that of *B*, and that the proportions of these parts are altered as *C* moves.

For this purpose it is most convenient to make *C* moveable about an axis, and make the opposed surfaces of *A*, of *B*, and of *C* portions of surfaces of revolution about the same axis.

In this way the distance between the surface of *C* and the opposed surfaces of *A* or of *B* remains always the same, and the motion of *C* in the positive direction simply increases the area opposed to *B* and diminishes the area opposed to *A*.

If the potentials of *A* and *B* are equal there will be no force urging *C* from *A* to *B*, but if the potential of *C* differs from that of *B* more than from that of *A*, then *C* will tend to move so as to increase the area of its surface opposed to *B*.

By a suitable arrangement of the apparatus this force may be made nearly constant for different positions of *C* within certain limits, so that if *C* is suspended by a torsion fibre, its deflexions will be nearly proportional to the difference of potentials between *A* and *B* multiplied by the difference of the potential of *C* from the mean of those of *A* and *B*.

*C* is maintained at a high potential by means of a condenser provided with a replenisher and tested by a gauge electrometer, and *A* and *B* are connected with the two conductors the difference of whose potentials is to be measured. The higher the potential of *C* the more sensitive is the instrument. This electrification of *C*, being independent of the electrification to be measured, places this electrometer in the heterostatic class.

We may apply to this electrometer the general theory of systems of conductors given in Arts. 93, 127.

Let *A*, *B*, *C* denote the potentials of the three conductors respectively. Let $a$, $b$, $c$ be their respective capacities, $p$ the coefficient of induction between *B* and *C*, $q$ that between *C* and *A*, and $r$ that between $A$ and $B$. All these coefficients will in general vary with the position of $C$, and if $C$ is so arranged that the extremities of $A$ and $B$ are not near those of $C$ as long as the motion of $C$ is confined within certain limits, we may ascertain the form of these coefficients. If $\theta$ represents the deflexion of $C$ from $A$ towards $B$, then the part of the surface of $A$ opposed to $C$ will diminish as $\theta$ increases. Hence if $A$ is kept at potential 1 while $B$ and $C$ are kept at potential 0, the charge on $A$ will be $a = a_0 - \alpha\,\theta$, where $a_0$ and $\alpha$ are constants, and $a$ is the capacity of $A$.

If $A$ and $B$ are symmetrical, the capacity of $B$ is $b = b_0 + \alpha\,\theta$.

The capacity of $C$ is not altered by the motion, for the only effect of the motion is to bring a different part of $C$ opposite to the interval between $A$ and $B$. Hence $c = c_0$.

The quantity of electricity induced on $C$ when $B$ is raised to potential unity is $p = p_0 - \alpha\theta$.

The coefficient of induction between $A$ and $C$ is $q = q_0 + \alpha\theta$.

The coefficient of induction between $A$ and $B$ is not altered by the motion of $C$, but remains $r = r_0$.

Hence the electrical energy of the system is

$$Q = \tfrac{1}{2}A^2 a + \tfrac{1}{2}B^2 b + \tfrac{1}{2}C^2 c + BCp + CAq + ABr,$$

and if $\Theta$ is the moment of the force tending to increase $\theta$,

$$\Theta = \frac{dQ}{d\theta},\ A, B, C \text{ being supposed constant},$$

$$= \tfrac{1}{2}A^2\frac{da}{d\theta} + \tfrac{1}{2}B^2\frac{db}{d\theta} + \tfrac{1}{2}C^2\frac{dc}{d\theta} + BC\frac{dp}{d\theta} + CA\frac{dq}{d\theta} + AB\frac{dr}{d\theta},$$

$$= -\tfrac{1}{2}A^2\alpha + \tfrac{1}{2}B^2\alpha - BC\alpha + CA\alpha;$$

$$\text{or}\qquad \Theta = \alpha(A-B)\left(C - \tfrac{1}{2}(A+B)\right).$$

In the present form of Thomson's Quadrant Electrometer the conductors $A$ and $B$ are in the form of a cylindrical box completely divided into four quadrants, separately insulated, but joined by wires so that two opposite quadrants are connected with $A$ and the two others with $B$.

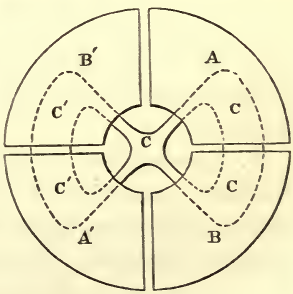

Fig. 19.

The conductor $C$ is suspended so as to be capable of turning about a vertical axis, and may consist of two opposite flat quadrantal arcs supported by their radii at their extremities. In the position of equilibrium these quadrants should be partly within $A$ and partly within $B$, and the supporting radii should be near the middle of the quadrants of the hollow base, so that the divisions of the box and the extremities and supports of $C$ may be as far from each other as possible.

The conductor $C$ is kept permanently at a high potential by being connected with the inner coating of the Leyden jar which forms the case of the instrument. $B$ and $A$ are connected, the first with the earth, and the other with the body whose potential is to be measured.

If the potential of this body is zero, and if the instrument be in adjustment, there ought to be no force tending to make $C$ move, but if the potential of $A$ is of the same sign as that of $C$, then $C$ will tend to move from $A$ to $B$ with a nearly uniform force, and the suspension apparatus will be twisted till an equal force is called into play and produces equilibrium. For deflexions within certain limits the deflexions of $C$ will be proportional to the product

$$(A-B)\left(C-\tfrac{1}{2}(A+B)\right).$$

By increasing the potential of $C$ the sensibility of the instrument may be increased, and for small values of $\frac{1}{2}(A+B)$ the force will be nearly proportional to $(A-B)\,C$.

### *On the Measurement of Electric Potential.*

220.] In order to determine large differences of potential in absolute measure we may employ the attracted disk electrometer, and compare the attraction with the effect of a weight. If at the same time we measure the difference of potential of the same conductors by means of the quadrant electrometer, we shall ascertain the absolute value of certain readings of the scale of the quadrant electrometer, and in this way we may deduce the value of the scale readings of the quadrant electrometer in terms of the potential of the suspended part, and the moment of torsion of the suspension apparatus.

To ascertain the potential of a charged conductor of finite size we may connect the conductor with one electrode of the electrometer, while the other is connected to earth or to a body of constant potential. The electrometer reading will give the potential of the conductor after the division of its electricity between it and the part of the electrometer with which it is put in contact. If $K$ denote the capacity of the conductor, and $K'$ that of this part of the electrometer, and if $V$, $V'$ denote the potentials of these bodies before making contact, then their common potential after making contact will be

$$\overline{V} = \frac{KV+K'V'}{K+K'}.$$

Hence the original potential of the conductor was

$$V = \overline{V} + \frac{K'}{K}(\overline{V} - V').$$

If the conductor is not large compared with the electrometer, $K'$ will be comparable with $K$, and unless we can ascertain the values of $K$ and $K'$ the second term of the expression will have a doubtful value. But if we can make the potential of the electrode of the electrometer very nearly equal to that of the body before making contact, then the uncertainty of the values of $K$ and $K'$ will be of little consequence.

If we know the value of the potential of the body approximately, we may charge the electrode by means of a 'replenisher' or otherwise to this approximate potential, and the next experiment will give a closer approximation. In this way we may measure the potential of a conductor whose capacity is small compared with that of the electrometer.

### *To Measure the Potential at any Point in the Air.*

221.] *First Method.* Place a sphere, whose radius is small compared with the distance of electrified conductors, with its centre at the given point. Connect it by means of a fine wire with the earth, then insulate it, and carry it to an electrometer and ascertain the total charge on the sphere.

Then, if $V$ be the potential at the given point, and $a$ the radius of the sphere, the charge on the sphere will be $-Va = Q$, and if $V'$ be the potential of the sphere as measured by an electrometer when placed in a room whose walls are connected with the earth, then

$$Q = V'a,$$

whence

$$V + V' = 0,$$

or the potential of the air at the point where the centre of the sphere was placed is equal but of opposite sign to the potential of the sphere after being connected to earth, then insulated, and brought into a room.

This method has been employed by M. Delmann of Creuznach in measuring the potential at a certain height above the earth's surface.

*Second Method.* We have supposed the sphere placed at the given point and first connected to earth, and then insulated, and carried into a space surrounded with conducting matter at potential zero.

Now let us suppose a fine insulated wire carried from the electrode of the electrometer to the place where the potential is to be measured. Let the sphere be first discharged completely. This may be done by putting it into the inside of a vessel of the same metal which nearly surrounds it and making it touch the vessel. Now let the sphere thus discharged be carried to the end of the wire and made to touch it. Since the sphere is not electrified it will be at the potential of the air at the place. If the electrode wire is at the same potential it will not be affected by the contact, but if the electrode is at a different potential it will by contact with the sphere be made nearer to that of the air than it was before. By a succession of such operations, the sphere being alternately discharged and made to touch the electrode, the potential of the electrode of the electrometer will continually approach that of the air at the given point.

222.] To measure the potential of a conductor without touching it, we may measure the potential of the air at any point in the neighbourhood of the conductor, and calculate that of the conductor from the result. If there be a hollow nearly surrounded by the conductor, then the potential at any point of the air in this hollow will be very nearly that of the conductor.

In this way it has been ascertained by Sir W. Thomson that if two hollow conductors, one of copper and the other of zinc, are in metallic contact, then the potential of the air in the hollow surrounded by zinc is positive with reference to that of the air in the hollow surrounded by copper.

*Third Method.* If by any means we can cause a succession of small bodies to detach themselves from the end of the electrode, the potential of the electrode will approximate to that of the surrounding air. This may be done by causing shot, filings, sand, or water to drop out of a funnel or pipe connected with the electrode. The point at which the potential is measured is that at which the stream ceases to be continuous and breaks into separate parts or drops.

Another convenient method is to fasten a slow match to the electrode. The potential is very soon made equal to that of the air at the burning end of the match. Even a fine metallic point is sufficient to create a discharge by means of the particles of the air when the difference of potentials is considerable, but if we wish to reduce this difference to zero, we must use one of the methods stated above.

If we only wish to ascertain the sign of the difference of the potentials at two places, and not its numerical value, we may cause drops or filings to be discharged at one of the places from a nozzle connected with the other place, and catch the drops or filings in an insulated vessel. Each drop as it falls is charged with a certain amount of electricity, and it is completely discharged into the vessel. The charge of the vessel therefore is continually accumulating, and after a sufficient number of drops have fallen, the charge of the vessel may be tested by the roughest methods. The sign of the charge is positive if the potential of the nozzle is positive relatively to that of the surrounding air.

## MEASUREMENT OF SURFACE-DENSITY OF ELECTRIFICATION.

### *Theory of the Proof Plane.*

223.] In testing the results of the mathematical theory of the distribution of electricity on the surface of conductors, it is necessary to be able to measure the surface-density at different points of the conductor. For this purpose Coulomb employed a small disk of gilt paper fastened to an insulating stem of gum-lac. He applied this disk to various points of the conductor by placing it so as to coincide as nearly as possible with the surface of the conductor. He then removed it by means of the insulating stem, and measured the charge of the disk by means of his electrometer.

Since the surface of the disk, when applied to the conductor, nearly coincided with that of the conductor, he concluded that the surface-density on the outer surface of the disk was nearly equal to that on the surface of the conductor at that place, and that the charge on the disk when removed was nearly equal to that on an area of the surface of the conductor equal to that of one side of the disk. This disk, when employed in this way, is called Coulomb's Proof Plane.

As objections have been raised to Coulomb's use of the proof plane, I shall make some remarks on the theory of the experiment.

The experiment consists in bringing a small conducting body into contact with the surface of the conductor at the point where the density is to be measured, and then removing the body and determining its charge.

We have first to shew that the charge on the small body when in contact with the conductor is proportional to the surface-density which existed at the point of contact before the small body was placed there.

We shall suppose that all the dimensions of the small body, and especially its dimension in the direction of the normal at the point of contact, are small compared with either of the radii of curvature of the conductor at the point of contact. Hence the variation of the resultant force due to the conductor supposed rigidly electrified within the space occupied by the small body may be neglected, and we may treat the surface of the conductor near the small body as a plane surface.

Now the charge which the small body will take by contact with a plane surface will be proportional to the resultant force normal to the surface, that is, to the surface-density. We shall ascertain the amount of the charge for particular forms of the body.

We have next to shew that when the small body is removed no spark will pass between it and the conductor, so that it will carry its charge with it. This is evident, because when the bodies are in contact their potentials are the same, and therefore the density on the parts nearest to the point of contact is extremely small. When the small body is removed to a very short distance from the conductor, which we shall suppose to be electrified positively, then the electrification at the point nearest to the small body is no longer zero but positive, but, since the charge of the small body is positive, the positive electrification close to the small body will be less than at other neighbouring points of the surface. Now the passage of a spark depends in general on the magnitude of the resultant force, and this on the surface-density. Hence, since we suppose that the conductor is not so highly electrified as to be discharging electricity from the other parts of its surface, it will not discharge a spark to the small body from a part of its surface which we have shewn to have a smaller surface-density.

224.] We shall now consider various forms of the small body.

Suppose it to be a small hemisphere applied to the conductor so as to touch it at the centre of its flat side.

Let the conductor be a large sphere, and let us modify the form of the hemisphere so that its surface is a little more than a hemisphere, and meets the surface of the sphere at right angles. Then we have a case of which we have already obtained the exact solution. See Art. 168.

If $A$ and $B$ be the centres of the two spheres cutting each other at right angles, $DD'$ a diameter of the circle of intersection, and $C$ the centre of that circle, then if $V$ is the potential of a conductor whose outer surface coincides with that of the two spheres, the quantity of electricity on the exposed surface of the sphere $A$ is

$$\tfrac{1}{2}V(AD+BD+AC-CD-BC),$$

and that on the exposed surface of the sphere $B$ is

$$\tfrac{1}{2}V(AD+BD+BC-CD-AC),$$

the total charge being the sum of these, or

$$V(AD+BD-CD).$$

If $a$ and $\beta$ are the radii of the spheres, then, when $a$ is large compared with $\beta$, the charge on $B$ is to that on $A$ in the ratio of

$$\frac{3}{4}\frac{\beta^2}{a^2}\left(1+\frac{1}{3}\frac{\beta}{a}+\frac{1}{6}\frac{\beta^2}{a^2}+\&c.\right) \text{ to } 1.$$

Now let $\sigma$ be the uniform surface-density on $A$ when $B$ is removed, then the charge on $A$ is

$$4\,\pi\,a^2\,\sigma,$$

and therefore the charge on $B$ is

$$3\,\pi\,\beta^2\,\sigma\left(1+\frac{1}{3}\frac{\beta}{a}+\&c.\right),$$

or, when $B$ is very small compared with $a$, the charge on the hemisphere $B$ is equal to three times that due to a surface-density $\sigma$ extending over an area equal to that of the circular base of the hemisphere.

It appears from Art. 175 that if a small sphere is made to touch an electrified body, and is then removed to a distance from it, the mean surface-density on the sphere is to the surface-density of the body at the point of contact as $\pi^2$ is to 6, or as 1.645 to 1.

225.] The most convenient form for the proof plane is that of a circular disk. We shall therefore shew how the charge on a circular disk laid on an electrified surface is to be measured.

For this purpose we shall construct a value of the potential function so that one of the equipotential surfaces resembles a circular flattened protuberance whose general form is somewhat like that of a disk lying on a plane.

Let $\sigma$ be the surface-density of a plane, which we shall suppose to be that of $xy$.

The potential due to this electrification will be

$$V = -4\pi\sigma z.$$

Now let two disks of radius $a$ be rigidly electrified with surface-densities $-\sigma'$ and $+\sigma'$. Let the first of these be placed on the plane of $xy$ with its centre at the origin, and the second parallel to it at the very small distance $c$.

Then it may be shewn, as we shall see in the theory of magnetism, that the potential of the two disks at any point is $\omega\sigma' c$, where $\omega$ is the solid angle subtended by the edge of either disk at the point. Hence the potential of the whole system will be

$$V = -4\pi\sigma z + \omega\sigma' c.$$

The forms of the equipotential surfaces and lines of induction are given on the left-hand side of Fig. XX, at the end of Vol. II.

Let us trace the form of the surface for which $V = 0$. This surface is indicated by the dotted line.

Putting the distance of any point from the axis of $z = r$, then, when $r$ is much less than $a$, and $z$ is small,

$$\omega = 2\pi - 2\pi\frac{z}{a} + \&c.$$

Hence, for values of $r$ considerably less than $a$, the equation of the zero equipotential surface is

$$0 = -4\pi\sigma z + 2\pi\sigma' c - 2\pi\sigma'\frac{zc}{a} + \&c.;$$

$$\text{or} \quad z_0 = \frac{\sigma' c}{2\sigma + \sigma'\dfrac{c}{a}}.$$

Hence this equipotential surface near the axis is nearly flat.

Outside the disk, where $r$ is greater than $a$, $\omega$ is zero when $z$ is zero, so that the plane of $xy$ is part of the equipotential surface.

To find where these two parts of the surface meet, let us find at what point of this plane $\dfrac{dV}{dz} = 0$.

When $r$ is very nearly equal to $a$

$$\frac{dV}{dz} = -4\pi\sigma + \frac{2\sigma' c}{r-a}.$$

Hence, when

$$\frac{dV}{dz} = 0, \qquad r_0 = a + \frac{\sigma' c}{2\pi\sigma}.$$

The equipotential surface $V = 0$ is therefore composed of a disk-like figure of radius $r_0$, and nearly uniform thickness $z_0$, and of the part of the infinite plane of $xy$ which lies beyond this figure.

The surface-integral over the whole disk gives the charge of electricity on it. It may be found, as in the theory of a circular current in Part IV, to be

$$Q = 4\pi a \sigma' c \left\{\log \frac{8a}{r_0 - a} - 2\right\} + \pi \sigma r_0^2.$$

The charge on an equal area of the plane surface is $\pi \sigma r_0^2$, hence the charge on the disk exceeds that on an equal area of the plane in the ratio of

$$1 + 8\frac{z}{r}\log\frac{8\pi r}{z} \text{ to unity,}$$

where $z$ is the thickness and $r$ the radius of the disk, $z$ being supposed small compared with $r$.

*On Electric Accumulators and the Measurement of Capacity.*

226.] An Accumulator or Condenser is an apparatus consisting of two conducting surfaces separated by an insulating dielectric medium.

A Leyden jar is an accumulator in which an inside coating of tinfoil is separated from the outside coating by the glass of which the jar is made. The original Leyden phial was a glass vessel containing water which was separated by the glass from the hand which held it.

The outer surface of any insulated conductor may be considered as one of the surfaces of an accumulator, the other being the earth or the walls of the room in which it is placed, and the intervening air being the dielectric medium.

The capacity of an accumulator is measured by the quantity of electricity with which the inner surface must be charged to make the difference between the potentials of the surfaces unity.

Since every electrical potential is the sum of a number of parts found by dividing each electrical element by its distance from a point, the ratio of a quantity of electricity to a potential must have the dimensions of a line. Hence electrostatic capacity is a linear quantity, or we may measure it in feet or metres without ambiguity.

In electrical researches accumulators are used for two principal purposes, for receiving and retaining large quantities of electricity in as small a compass as possible, and for measuring definite quantities of electricity by means of the potential to which they raise the accumulator.

For the retention of electrical charges nothing has been devised more perfect than the Leyden jar. The principal part of the loss arises from the electricity creeping along the damp uncoated surface of the glass from the one coating to the other. This may be checked in a great degree by artificially drying the air within the jar, and by varnishing the surface of the glass where it is exposed to the atmosphere. In Sir W. Thomson's electroscopes there is a very small percentage of loss from day to day, and I believe that none of this loss can be traced to direct conduction either through air or through glass when the glass is good, but that it arises chiefly from superficial conduction along the various insulating stems and glass surfaces of the instrument.

In fact, the same electrician has communicated a charge to sulphuric acid in a large bulb with a long neck, and has then hermetically sealed the neck by fusing it, so that the charge was completely surrounded by glass, and after some years the charge was found still to be retained.

It is only, however, when cold, that glass insulates in this way, for the charge escapes at once if the glass is heated to a temperature below 100°C.

When it is desired to obtain great capacity in small compass, accumulators in which the dielectric is sheet caoutchouc, mica, or paper impregnated with paraffin are convenient.

227.] For accumulators of the second class, intended for the measurement of quantities of electricity, all solid dielectrics must be employed with great caution on account of the property which they possess called Electric Absorption.

The only safe dielectric for such accumulators is air, which has this inconvenience, that if any dust or dirt gets into the narrow space between the opposed surfaces, which ought to be occupied only by air, it not only alters the thickness of the stratum of air, but may establish a connexion between the opposed surfaces, in which case the accumulator will not hold a charge.

To determine in absolute measure, that is to say in feet or metres, the capacity of an accumulator, we must either first ascertain its form and size, and then solve the problem of the distribution of electricity on its opposed surfaces, or we must compare its capacity with that of another accumulator, for which this problem has been solved.

As the problem is a very difficult one, it is best to begin with an accumulator constructed of a form for which the solution is known.

Thus the capacity of an insulated sphere in an unlimited space is known to be measured by the radius of the sphere.

A sphere suspended in a room was actually used by MM. Kohlrausch and Weber, as an absolute standard with which they compared the capacity of other accumulators.

The capacity, however, of a sphere of moderate size is so small when compared with the capacities of the accumulators in common use that the sphere is not a convenient standard measure.

Its capacity might be greatly increased by surrounding the sphere with a hollow concentric spherical surface of somewhat greater radius. The capacity of the inner surface is then a fourth proportional to the thickness of the stratum of air and the radii of the two surfaces.

Sir W. Thomson has employed this arrangement as a standard of capacity, but the difficulties of working the surfaces truly spherical, of making them truly concentric, and of measuring their distance and their radii with sufficient accuracy, are considerable.

We are therefore led to prefer for an absolute measure of capacity a form in which the opposed surfaces are parallel planes.

The accuracy of the surface of the planes can be easily tested, and their distance can be measured by a micrometer screw, and may be made capable of continuous variation, which is a most important property of a measuring instrument.

The only difficulty remaining arises from the fact that the planes must necessarily be bounded, and that the distribution of electricity near the boundaries of the planes has not been rigidly calculated. It is true that if we make them equal circular disks, whose radius is large compared with the distance between them, we may treat the edges of the disks as if they were straight lines, and calculate the distribution of electricity by the method due to Helmholtz, and described at Art. 202. But it will be noticed that in this case part of the electricity is distributed on the back of each disk, and that in the calculation it has been supposed that there are no conductors in the neighbourhood, which is not and cannot be the case in a small instrument.

228.] We therefore prefer the following arrangement, due to Sir W. Thomson, which we may call the Guard-ring arrangement, by means of which the quantity of electricity on an insulated disk may be exactly determined in terms of its potential.

*The Guard-ring Accumulator.*

$Bb$ is a cylindrical vessel of conducting material of which the outer surface of the upper face is accurately plane. This upper surface consists of two parts, a disk $A$, and a broad ring $BB$ surrounding the disk, separated from it by a very small interval all round, just sufficient to prevent sparks passing. The upper surface of the disk is accurately in the same plane with that of the guard-ring. The disk is supported by pillars of insulating material $GG$. $C$ is a metal disk, the under surface of which is accurately plane and parallel to $BB$. The disk $C$ is considerably larger than $A$. Its distance from $A$ is adjusted and measured by means of a micrometer screw, which is not given in the figure.

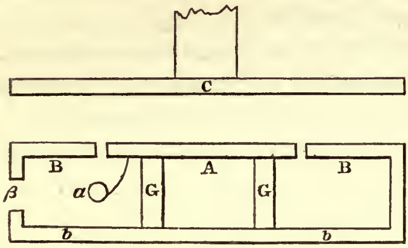

Fig. 20.

This accumulator is used as a measuring instrument as follows :—

Suppose $C$ to be at potential zero, and the disk $A$ and vessel $Bb$ both at potential $V$. Then there will be no electrification on the back of the disk because the vessel is nearly closed and is all at the same potential. There will be very little electrification on the edges of the disk because $BB$ is at the same potential with the disk. On the face of the disk the electrification will be nearly uniform, and therefore the whole charge on the disk will be almost exactly represented by its area multiplied by the surface-density on a plane, as given at Art. 124.

In fact, we learn from the investigation at Art. 201 that the charge on the disk is

$$V\left\{\frac{R^2+R'^2}{8A} - \frac{R'^2-R^2}{8A}\,\frac{a}{A+a}\right\},$$

where $R$ is the radius of the disk, $R'$ that of the hole in the guard-ring, $A$ the distance between $A$ and $C$, and $a$ a quantity which cannot exceed $(R'-R)\dfrac{\log_e 2}{\pi}$.

If the interval between the disk and the guard-ring is small compared with the distance between $A$ and $C$, the second term will be very small, and the charge on the disk will be nearly

$$V\frac{R^2+R'^2}{8A}.$$

Now let the vessel $Bb$ be put in connexion with the earth. The charge on the disk $A$ will no longer be uniformly distributed, but it will remain the same in quantity, and if we now discharge $A$ we shall obtain a quantity of electricity, the value of which we know in terms of $V$, the original difference of potentials and the measurable quantities $R$, $R'$ and $A$.

*On the Comparison of the Capacity of Accumulators.*

229.] The form of accumulator which is best fitted to have its capacity determined in absolute measure from the form and dimensions of its parts is not generally the most suitable for electrical experiments. It is desirable that the measures of capacity in actual use should be accumulators having only two conducting surfaces, one of which is as nearly as possible surrounded by the other. The guard-ring accumulator, on the other hand, has three independent conducting portions which must be charged and discharged in a certain order. Hence it is desirable to be able to compare the capacities of two accumulators by an electrical process, so as to test accumulators which may afterwards serve as secondary standards.

I shall first shew how to test the equality of the capacity of two guard-ring accumulators.

Let $A$ be the disk, $B$ the guard-ring with the rest of the conducting vessel attached to it, and $C$ the large disk of one of these accumulators, and let $A'$, $B'$, and $C'$ be the corresponding parts of the other.

If either of these accumulators is of the more simple kind, having only two conductors, we have only to suppress $B$ or $B'$, and to suppose $A$ to be the inner and $C$ the outer conducting surface. $C$ in this case being understood to surround $A$.

Let the following connexions be made.

Let $B$ be kept always connected with $C'$, and $B'$ with $C$, that is, let each guard-ring be connected with the large disk of the other condenser.

(1) Let $A$ be connected with $B$ and $C'$ and with $J$, the electrode of a Leyden jar, and let $A'$ be connected with $B'$ and $C$ and with the earth.

(2) Let $A$, $B$, and $C'$ be insulated from $J$.

(3) Let $A$ be insulated from $B$ and $C'$, and $A'$ from $B'$ and $C'$.

(4) Let $B$ and $C'$ be connected with $B'$ and $C$ and with the earth.

(5) Let $A$ be connected with $A'$.

(6) Let $A$ and $A'$ be connected with an electroscope $E$.

We may express these connexions as follows :—

$$
\begin{array}{lllll}
(1) & 0 = C = B' = A' & \mid & A = B = C' = J. \\
(2) & 0 = C = B' = A' & \mid & A = B = C' \mid J. \\
(3) & 0 = C = B' \mid A' & \mid & A \mid B = C'. \\
(4) & 0 = C = B' \mid A' & \mid & A \mid B = C' = 0. \\
(5) & 0 = C = B' \mid A' & = & A \mid B = C' = 0. \\
(6) & 0 = C = B' \mid A' & = E = & A \mid B = C' = 0.
\end{array}
$$

Here the sign of equality expresses electrical connexion, and the vertical stroke expresses insulation.

In (1) the two accumulators are charged oppositely, so that $A$ is positive and $A'$ negative, the charges on $A$ and $A'$ being uniformly distributed on the upper surface opposed to the large disk of each accumulator.

In (2) the jar is removed, and in (3) the charges on $A$ and $A'$ are insulated.

In (4) the guard-rings are connected with the large disks, so that the charges on $A$ and $A'$, though unaltered in magnitude, are now distributed over their whole surface.

In (5) $A$ is connected with $A'$. If the charges are equal and of opposite signs, the electrification will be entirely destroyed, and in (6) this is tested by means of the electroscope $E$.

The electroscope $E$ will indicate positive or negative electrification according as $A$ or $A'$ has the greater capacity.

By means of a key of proper construction, the whole of these operations can be performed in due succession in a very small fraction of a second, and the capacities adjusted till no electrification can be detected by the electroscope, and in this way the capacity of an accumulator may be adjusted to be equal to that of any other, or to the sum of the capacities of several accumulators, so that a system of accumulators may be formed, each of which has its capacity determined in absolute measure, i. e. in feet or in metres, while at the same time it is of the construction most suitable for electrical experiments.

This method of comparison will probably be found useful in determining the specific capacity for electrostatic induction of different dielectrics in the form of plates or disks. If a disk of the dielectric is interposed between $A$ and $C$, the disk being considerably larger than $A$, then the capacity of the accumulator will be altered and made equal to that of the same accumulator when $A$ and $C$ are nearer together. If the accumulator with the dielectric plate, and with $A$ and $C$ at distance $x$, is of the same capacity as the same accumulator without the dielectric, and with $A$ and $C$ at distance $x'$, then, if $a$ is the thickness of the plate, and $K$ its specific dielectric inductive capacity referred to air as a standard,

$$K = \frac{a}{a + x' - x}.$$

The combination of three cylinders, described in Art. 127, has been employed by Sir W. Thomson as an accumulator whose capacity may be increased or diminished by measurable quantities.

The experiments of MM. Gibson and Barclay with this apparatus are described in the *Proceedings of the Royal Society*, Feb. 2, 1871, and *Phil. Trans.*, 1871, p. 573. They found the specific inductive capacity of paraffin to be 1.975, that of air being unity.

# PART II.

## ELECTROKINEMATICS.

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## CHAPTER I.

### THE ELECTRIC CURRENT.

230.] We have seen, in Art. 45, that when a conductor is in electrical equilibrium the potential at every point of the conductor must be the same.

If two conductors $A$ and $B$ are charged with electricity so that the potential of $A$ is higher than that of $B$, then, if they are put in communication by means of a metallic wire $C$ touching both of them, part of the charge of $A$ will be transferred to $B$, and the potentials of $A$ and $B$ will become in a very short time equalized.

231.] During this process certain phenomena are observed in the wire $C$, which are called the phenomena of the electric conflict or current.

The first of these phenomena is the transference of positive electrification from $A$ to $B$ and of negative electrification from $B$ to $A$. This transference may be also effected in a slower manner by bringing a small insulated body into contact with $A$ and $B$ alternately. By this process, which we may call electrical convection, successive small portions of the electrification of each body are transferred to the other. In either case a certain quantity of electricity, or of the state of electrification, passes from one place to another along a certain path in the space between the bodies.

Whatever therefore may be our opinion of the nature of electricity, we must admit that the process which we have described constitutes a current of electricity. This current may be described as a current of positive electricity from $A$ to $B$, or a current of negative electricity from $B$ to $A$, or as a combination of these two currents.

According to Fechner's and Weber's theory it is a combination of a current of positive electricity with an exactly equal current of negative electricity in the opposite direction through the same substance. It is necessary to remember this exceedingly artificial hypothesis regarding the constitution of the current in order to understand the statement of some of Weber's most valuable experimental results.

If, as in Art. 36, we suppose $P$ units of positive electricity transferred from $A$ to $B$, and $N$ units of negative electricity transferred from $B$ to $A$ in unit of time, then, according to Weber's theory, $P=N$, and $P$ or $N$ is to be taken as the numerical measure of the current.

We, on the contrary, make no assumption as to the relation between $P$ and $N$, but attend only to the result of the current, namely, the transference of $P+N$ of positive electrification from $A$ to $B$, and we shall consider $P+N$ the true measure of the current. The current, therefore, which Weber would call 1 we shall call 2.

### *On Steady Currents.*

232.] In the case of the current between two insulated conductors at different potentials the operation is soon brought to an end by the equalization of the potentials of the two bodies, and the current is therefore essentially a Transient current.

But there are methods by which the difference of potentials of the conductors may be maintained constant, in which case the current will continue to flow with uniform strength as a Steady Current.

### *The Voltaic Battery.*

The most convenient method of producing a steady current is by means of the Voltaic Battery.

For the sake of distinctness we shall describe Daniell's Constant Battery :—

A solution of sulphate of zinc is placed in a cell of porous earthenware, and this cell is placed in a vessel containing a saturated solution of sulphate of copper. A piece of zinc is dipped into the sulphate of zinc, and a piece of copper is dipped into the sulphate of copper. Wires are soldered to the zinc and to the copper above the surface of the liquid. This combination is called a cell or element of Daniell's battery. See Art. 272.

233.] If the cell is insulated by being placed on a non-conducting stand, and if the wire connected with the copper is put in contact with an insulated conductor $A$, and the wire connected with the zinc is put in contact with $B$, another insulated conductor of the same metal as $A$, then it may be shewn by means of a delicate electrometer that the potential of $A$ exceeds that of $B$ by a certain quantity. This difference of potentials is called the Electromotive Force of the Daniell's Cell.

If $A$ and $B$ are now disconnected from the cell and put in communication by means of a wire, a transient current passes through the wire from $A$ to $B$, and the potentials of $A$ and $B$ become equal. $A$ and $B$ may then be charged again by the cell, and the process repeated as long as the cell will work. But if $A$ and $B$ be connected by means of the wire $C$, and at the same time connected with the battery as before, then the cell will maintain a constant current through $C$, and also a constant difference of potentials between $A$ and $B$. This difference will not, as we shall see, be equal to the whole electromotive force of the cell, for part of this force is spent in maintaining the current through the cell itself.

A number of cells placed in series so that the zinc of the first cell is connected by metal with the copper of the second, and so on, is called a Voltaic Battery. The electromotive force of such a battery is the sum of the electromotive forces of the cells of which it is composed. If the battery is insulated it may be charged with electricity as a whole, but the potential of the copper end will always exceed that of the zinc end by the electromotive force of the battery, whatever the absolute value of either of these potentials may be. The cells of the battery may be of very various construction, containing different chemical substances and different metals, provided they are such that chemical action does not go on when no current passes.

234.] Let us now consider a voltaic battery with its ends insulated from each other. The copper end will be positively or vitreously electrified, and the zinc end will be negatively or resinously electrified.

Let the two ends of the battery be now connected by means of a wire. An electric current will commence, and will in a very short time attain a constant value. It is then said to be a Steady Current.

*Properties of the Current.*

235.] The current forms a closed circuit in the direction from copper to zinc through the wires, and from zinc to copper through the solutions.

If the circuit be broken by cutting any of the wires which connect the copper of one cell with the zinc of the next in order, the current will be stopped, and the potential of the end of the wire in connexion with the copper will be found to exceed that of the end of the wire in connexion with the zinc by a constant quantity, namely, the total electromotive force of the circuit.

*Electrolytic Action of the Current.*

236.] As long as the circuit is broken no chemical action goes on in the cells, but as soon as the circuit is completed, zinc is dissolved from the zinc in each of the Daniell's cells, and copper is deposited on the copper.

The quantity of sulphate of zinc increases, and the quantity of sulphate of copper diminishes unless more is constantly supplied.

The quantity of zinc dissolved and also that of copper deposited is the same in each of the Daniell's cells throughout the circuit, whatever the size of the plates of the cell, and if any of the cells be of a different construction, the amount of chemical action in it bears a constant proportion to the action in the Daniell's cell. For instance, if one of the cells consists of two platinum plates dipped into sulphuric acid diluted with water, oxygen will be given off at the surface of the plate where the current enters the liquid, namely, the plate in metallic connexion with the copper of Daniell's cell, and hydrogen at the surface of the plate where the current leaves the liquid, namely, the plate connected with the zinc of Daniell's cell.

The volume of the hydrogen is exactly twice the volume of the oxygen given off in the same time, and the weight of the oxygen is exactly eight times the weight of the hydrogen.

In every cell of the circuit the weight of each substance dissolved, deposited, or decomposed is equal to a certain quantity called the electrochemical equivalent of that substance, multiplied by the strength of the current and by the time during which it has been flowing.

For the experiments which established this principle, see the seventh and eighth series of Faraday's *Experimental Researches;*

and for an investigation of the apparent exceptions to the rule, see Miller's *Chemical Physics* and Wiedemann's *Galvanismus*.

237.] Substances which are decomposed in this way are called Electrolytes. The process is called Electrolysis. The places where the current enters and leaves the electrolyte are called Electrodes. Of these the electrode by which the current enters is called the Anode, and that by which it leaves the electrolyte is called the Cathode. The components into which the electrolyte is resolved are called Ions: that which appears at the anode is called the Anion, and that which appears at the cathode is called the Cation.

Of these terms, which were, I believe, invented by Faraday with the help of Dr. Whewell, the first three, namely, electrode, electrolysis, and electrolyte have been generally adopted, and the mode of conduction of the current in which this kind of decomposition and transfer of the components takes place is called Electrolytic Conduction.

If a homogeneous electrolyte is placed in a tube of variable section, and if the electrodes are placed at the ends of this tube, it is found that when the current passes, the anion appears at the anode and the cation at the cathode, the quantities of these ions being electrochemically equivalent, and such as to be together equivalent to a certain quantity of the electrolyte. In the other parts of the tube, whether the section be large or small, uniform or varying, the composition of the electrolyte remains unaltered. Hence the amount of electrolysis which takes place across every section of the tube is the same. Where the section is small the action must therefore be more intense than where the section is large, but the total amount of each ion which crosses any complete section of the electrolyte in a given time is the same for all sections.

The strength of the current may therefore be measured by the amount of electrolysis in a given time. An instrument by which the quantity of the electrolytic products can be readily measured is called a Voltameter.

The strength of the current, as thus measured, is the same at every part of the circuit, and the total quantity of the electrolytic products in the voltameter after any given time is proportional to the amount of electricity which passes any section in the same time.

238.] If we introduce a voltameter at one part of the circuit of a voltaic battery, and break the circuit at another part, we may suppose the measurement of the current to be conducted thus.

Let the ends of the broken circuit be $A$ and $B$, and let $A$ be the anode and $B$ the cathode. Let an insulated ball be made to touch $A$ and $B$ alternately, it will carry from $A$ to $B$ a certain measurable quantity of electricity at each journey. This quantity may be measured by an electrometer, or it may be calculated by multiplying the electromotive force of the circuit by the electrostatic capacity of the ball. Electricity is thus carried from $A$ to $B$ on the insulated ball by a process which may be called Convection. At the same time electrolysis goes on in the voltameter and in the cells of the battery, and the amount of electrolysis in each cell may be compared with the amount of electricity carried across by the insulated ball. The quantity of a substance which is electrolysed by one unit of electricity is called an Electrochemical equivalent of that substance.

This experiment would be an extremely tedious and troublesome one if conducted in this way with a ball of ordinary magnitude and a manageable battery, for an enormous number of journeys would have to be made before an appreciable quantity of the electrolyte was decomposed. The experiment must therefore be considered as a mere illustration, the actual measurements of electrochemical equivalents being conducted in a different way. But the experiment may be considered as an illustration of the process of electrolysis itself, for if we regard electrolytic conduction as a species of convection in which an electrochemical equivalent of the anion travels with negative electricity in the direction of the anode, while an equivalent of the cation travels with positive electricity in the direction of the cathode, the whole amount of transfer of electricity being one unit, we shall have an idea of the process of electrolysis, which, so far as I know, is not inconsistent with known facts, though, on account of our ignorance of the nature of electricity and of chemical compounds, it may be a very imperfect representation of what really takes place.

### *Magnetic Action of the Current.*

239.] Oersted discovered that a magnet placed near a straight electric current tends to place itself at right angles to the plane passing through the magnet and the current. See Art. 475.

If a man were to place his body in the line of the current so that the current from copper through the wire to zinc should flow from his head to his feet, and if he were to direct his face towards the centre of the magnet, then that end of the magnet which tends to point to the north would, when the current flows, tend to point towards the man's right hand.

The nature and laws of this electromagnetic action will be discussed when we come to the fourth part of this treatise. What we are concerned with at present is the fact that the electric current has a magnetic action which is exerted outside the current, and by which its existence can be ascertained and its intensity measured without breaking the circuit or introducing anything into the current itself.

The amount of the magnetic action has been ascertained to be strictly proportional to the strength of the current as measured by the products of electrolysis in the voltameter, and to be quite independent of the nature of the conductor in which the current is flowing, whether it be a metal or an electrolyte.

240.] An instrument which indicates the strength of an electric current by its magnetic effects is called a Galvanometer.

Galvanometers in general consist of one or more coils of silk-covered wire within which a magnet is suspended with its axis horizontal. When a current is passed through the wire the magnet tends to set itself with its axis perpendicular to the plane of the coils. If we suppose the plane of the coils to be placed parallel to the plane of the earth's equator, and the current to flow round the coil from east to west in the direction of the apparent motion of the sun, then the magnet within will tend to set itself with its magnetization in the same direction as that of the earth considered as a great magnet, the north pole of the earth being similar to that end of the compass needle which points south.

The galvanometer is the most convenient instrument for measuring the strength of electric currents. We shall therefore assume the possibility of constructing such an instrument in studying the laws of these currents, reserving the discussion of the principles of the instrument for our fourth part. When therefore we say that an electric current is of a certain strength we suppose that the measurement is effected by the galvanometer.

# CHAPTER II.

## CONDUCTION AND RESISTANCE.

241.] If by means of an electrometer we determine the electric potential at different points of a circuit in which a constant electric current is maintained, we shall find that in any portion of the circuit consisting of a single metal of uniform temperature throughout, the potential at any point exceeds that at any other point farther on in the direction of the current by a quantity depending on the strength of the current and on the nature and dimensions of the intervening portion of the circuit. The difference of the potentials at the extremities of this portion of the circuit is called the External electromotive force acting on it. If the portion of the circuit under consideration is not homogeneous, but contains transitions from one substance to another, from metals to electrolytes, or from hotter to colder parts, there may be, besides the external electromotive force, Internal electromotive forces which must be taken into account.

The relations between Electromotive Force, Current, and Resistance were first investigated by Dr. G. S. Ohm, in a work published in 1827, entitled *Die Galvanische Kette Mathematisch Bearbeitet*, translated in Taylor's *Scientific Memoirs*. The result of these investigations in the case of homogeneous conductors is commonly called 'Ohm's Law.'

### *Ohm's Law.*

*The electromotive force acting between the extremities of any part of a circuit is the product of the strength of the current and the Resistance of that part of the circuit.*

Here a new term is introduced, the Resistance of a conductor, which is defined to be the ratio of the electromotive force to the strength of the current which it produces. The introduction of this term would have been of no scientific value unless Ohm had shewn, as he did experimentally, that it corresponds to a real physical quantity, that is, that it has a definite value which is altered only when the nature of the conductor is altered.

In the first place, then, the resistance of a conductor is independent of the strength of the current flowing through it.

In the second place the resistance is independent of the electric potential at which the conductor is maintained, and of the density of the distribution of electricity on the surface of the conductor.

It depends entirely on the nature of the material of which the conductor is composed, the state of aggregation of its parts, and its temperature.

The resistance of a conductor may be measured to within one ten thousandth or even one hundred thousandth part of its value, and so many conductors have been tested that our assurance of the truth of Ohm's Law is now very high. In the sixth chapter we shall trace its applications and consequences.

*Generation of Heat by the Current.*

242.] We have seen that when an electromotive force causes a current to flow through a conductor, electricity is transferred from a place of higher to a place of lower potential. If the transfer had been made by convection, that is, by carrying successive charges on a ball from the one place to the other, work would have been done by the electrical forces on the ball, and this might have been turned to account. It is actually turned to account in a partial manner in those dry pile circuits where the electrodes have the form of bells, and the carrier ball is made to swing like a pendulum between the two bells and strike them alternately. In this way the electrical action is made to keep up the swinging of the pendulum and to propagate the sound of the bells to a distance. In the case of the conducting wire we have the same transfer of electricity from a place of high to a place of low potential without any external work being done. The principle of the Conservation of Energy therefore leads us to look for internal work in the conductor. In an electrolyte this internal work consists partly of the separation of its components. In other conductors it is entirely converted into heat.

The energy converted into heat is in this case the product of the electromotive force into the quantity of electricity which passes. But the electromotive force is the product of the current into the resistance, and the quantity of electricity is the product of the current into the time. Hence the quantity of heat multiplied by the mechanical equivalent of unit of heat is equal to the square of the strength of the current multiplied into the resistance and into the time.

The heat developed by electric currents in overcoming the resistance of conductors has been determined by Dr. Joule, who first established that the heat produced in a given time is proportional to the square of the current, and afterwards by careful absolute measurements of all the quantities concerned, verified the equation

$$JH = C^2 Rt,$$

where $J$ is Joule's dynamical equivalent of heat, $H$ the number of units of heat, $C$ the strength of the current, $R$ the resistance of the conductor, and $t$ the time during which the current flows. These relations between electromotive force, work, and heat, were first fully explained by Sir W. Thomson in a paper on the application of the principle of mechanical effect to the measurement of electromotive forces *.

243.] The analogy between the theory of the conduction of electricity and that of the conduction of heat is at first sight almost complete. If we take two systems geometrically similar, and such that the conductivity for heat at any part of the first is proportional to the conductivity for electricity at the corresponding part of the second, and if we also make the temperature at any part of the first proportional to the electric potential at the corresponding point of the second, then the flow of heat across any area of the first will be proportional to the flow of electricity across the corresponding area of the second.

Thus, in the illustration we have given, in which flow of electricity corresponds to flow of heat, and electric potential to temperature, electricity tends to flow from places of high to places of low potential, exactly as heat tends to flow from places of high to places of low temperature.

244.] The theory of potential and that of temperature may therefore be made to illustrate one another; there is, however, one remarkable difference between the phenomena of electricity and those of heat.

Suspend a conducting body within a closed conducting vessel by a silk thread, and charge the vessel with electricity. The potential

* *Phil. Mag.*, Dec. 1851.

of the vessel and of all within it will be instantly raised, but however long and however powerfully the vessel be electrified, and whether the body within be allowed to come in contact with the vessel or not, no signs of electrification will appear within the vessel, nor will the body within shew any electrical effect when taken out.

But if the vessel is raised to a high temperature, the body within will rise to the same temperature, but only after a considerable time, and if it is then taken out it will be found hot, and will remain so till it has continued to emit heat for some time.

The difference between the phenomena consists in the fact that bodies are capable of absorbing and emitting heat, whereas they have no corresponding property with respect to electricity. A body cannot be made hot without a certain amount of heat being supplied to it, depending on the mass and specific heat of the body, but the electric potential of a body may be raised to any extent in the way already described without communicating any electricity to the body.

245.] Again, suppose a body first heated and then placed inside the closed vessel. The outside of the vessel will be at first at the temperature of surrounding bodies, but it will soon get hot, and will remain hot till the heat of the interior body has escaped.

It is impossible to perform a corresponding electrical experiment. It is impossible so to electrify a body, and so to place it in a hollow vessel, that the outside of the vessel shall at first shew no signs of electrification but shall afterwards become electrified. It was for some phenomenon of this kind that Faraday sought in vain under the name of an absolute charge of electricity.

Heat may be hidden in the interior of a body so as to have no external action, but it is impossible to isolate a quantity of electricity so as to prevent it from being constantly in inductive relation with an equal quantity of electricity of the opposite kind.

There is nothing therefore among electric phenomena which corresponds to the capacity of a body for heat. This follows at once from the doctrine which is asserted in this treatise, that electricity obeys the same condition of continuity as an incompressible fluid. It is therefore impossible to give a bodily charge of electricity to any substance by forcing an additional quantity of electricity into it. See Arts. 61, 111, 329, 334.

# CHAPTER III.

## ELECTROMOTIVE FORCE BETWEEN BODIES IN CONTACT.

### *The Potentials of Different Substances in Contact.*

246.] If we define the potential of a hollow conducting vessel as the potential of the air inside the vessel, we may ascertain this potential by means of an electrometer as described in Part I, Art. 222.

If we now take two hollow vessels of different metals, say copper and zinc, and put them in metallic contact with each other, and then test the potential of the air inside each vessel, the potential of the air inside the zinc vessel will be positive as compared with that inside the copper vessel. The difference of potentials depends on the nature of the surface of the insides of the vessels, being greatest when the zinc is bright and when the copper is coated with oxide.

It appears from this that when two different metals are in contact there is in general an electromotive force acting from the one to the other, so as to make the potential of the one exceed that of the other by a certain quantity. This is Volta's theory of Contact Electricity.

If we take a certain metal, say copper, as the standard, then if the potential of iron in contact with copper at the zero potential is $I$, and that of zinc in contact with copper at zero is $Z$, then the potential of zinc in contact with iron at zero will be $Z-I$.

It appears from this result, which is true of any three metals, that the differences of potential of any two metals at the same temperature in contact is equal to the difference of their potentials when in contact with a third metal, so that if a circuit be formed of any number of metals at the same temperature there will be electrical equilibrium as soon as they have acquired their proper potentials, and there will be no current kept up in the circuit.

247.] If, however, the circuit consist of two metals and an electrolyte, the electrolyte, according to Volta's theory, tends to reduce the potentials of the metals in contact with it to equality, so that the electromotive force at the metallic junction is no longer balanced, and a continuous current is kept up. The energy of this current is supplied by the chemical action which takes place between the electrolyte and the metals.

248.] The electric effect may, however, be produced without chemical action if by any other means we can produce an equalization of the potentials of two metals in contact. Thus, in an experiment due to Sir W. Thomson*, a copper funnel is placed in contact with a vertical zinc cylinder, so that when copper filings are allowed to pass through the funnel, they separate from each other and from the funnel near the middle of the zinc cylinder, and then fall into an insulated receiver placed below. The receiver is then found to be charged negatively, and the charge increases as the filings continue to pour into it. At the same time the zinc cylinder with the copper funnel in it becomes charged more and more positively.

If now the zinc cylinder were connected with the receiver by a wire, there would be a positive current in the wire from the cylinder to the receiver. The stream of copper filings, each filing charged negatively by induction, constitutes a negative current from the funnel to the receiver, or, in other words, a positive current from the receiver to the copper funnel. The positive current, therefore, passes through the air (by the filings) from zinc to copper, and through the metallic junction from copper to zinc, just as in the ordinary voltaic arrangement, but in this case the force which keeps up the current is not chemical action but gravity, which causes the filings to fall, in spite of the electrical attraction between the positively charged funnel and the negatively charged filings.

249.] A remarkable confirmation of the theory of contact electricity is supplied by the discovery of Peltier, that, when a current of electricity crosses the junction of two metals, the junction is heated when the current is in one direction, and cooled when it is in the other direction. It must be remembered that a current in its passage through a metal always produces heat, because it meets with resistance, so that the cooling effect on the whole conductor must always be less than the heating effect. We must therefore distinguish between the generation of heat in each metal,

* *North British Review*, 1864, p. 353; and *Proc. R. S.*, June 20, 1867.

due to ordinary resistance, and the generation or absorption of heat at the junction of two metals. We shall call the first the frictional generation of heat by the current, and, as we have seen, it is proportional to the square of the current, and is the same whether the current be in the positive or the negative direction. The second we may call the Peltier effect, which changes its sign with that of the current.

The total heat generated in a portion of a compound conductor consisting of two metals may be expressed by

$$H = \frac{R}{J} C^2 t - \Pi C t,$$

where $H$ is the quantity of heat, $J$ the mechanical equivalent of unit of heat, $R$ the resistance of the conductor, $C$ the current, and $t$ the time; $\Pi$ being the coefficient of the Peltier effect, that is, the heat absorbed at the junction due to the passage of unit of current for unit of time.

Now the heat generated is mechanically equivalent to the work done against electrical forces in the conductor, that is, it is equal to the product of the current into the electromotive force producing it. Hence, if $E$ is the external electromotive force which causes the current to flow through the conductor,

$$JH = CEt = RC^2 t - J\Pi C t,$$

whence

$$E = RC - J\Pi.$$

It appears from this equation that the external electromotive force required to drive the current through the compound conductor is less than that due to its resistance alone by the electromotive force $J\Pi$. Hence $J\Pi$ represents the electromotive contact force at the junction acting in the positive direction.

This application, due to Sir W. Thomson *, of the dynamical theory of heat to the determination of a local electromotive force is of great scientific importance, since the ordinary method of connecting two points of the compound conductor with the electrodes of a galvanometer or electroscope by wires would be useless, owing to the contact forces at the junctions of the wires with the materials of the compound conductor. In the thermal method, on the other hand, we know that the only source of energy is the current of electricity, and that no work is done by the current in a certain portion of the circuit except in heating that portion of the conductor. If, therefore, we can measure the amount of the

* *Proc. R. S. Edin.*, Dec. 15, 1851; and *Trans. R. S. Edin.*, 1854.

current and the amount of heat produced or absorbed, we can determine the electromotive force required to urge the current through that portion of the conductor, and this measurement is entirely independent of the effect of contact forces in other parts of the circuit.

The electromotive force at the junction of two metals, as determined by this method, does not account for Volta's electromotive force as described in Art. 246. The latter is in general far greater than that of this Article, and is sometimes of opposite sign. Hence the assumption that the potential of a metal is to be measured by that of the air in contact with it must be erroneous, and the greater part of Volta's electromotive force must be sought for, not at the junction of the two metals, but at one or both of the surfaces which separate the metals from the air or other medium which forms the third element of the circuit.

250.] The discovery by Seebeck of thermoelectric currents in circuits of different metals with their junctions at different temperatures, shews that these contact forces do not always balance each other in a complete circuit. It is manifest, however, that in a complete circuit of different metals at uniform temperature the contact forces must balance each other. For if this were not the case there would be a current formed in the circuit, and this current might be employed to work a machine or to generate heat in the circuit, that is, to do work, while at the same time there is no expenditure of energy, as the circuit is all at the same temperature, and no chemical or other change takes place. Hence, if the Peltier effect at the junction of two metals $a$ and $b$ be represented by $\Pi_{ab}$ when the current flows from $a$ to $b$, then for a circuit of two metals at the same temperature we must have

$$\Pi_{ab}+\Pi_{ba} = 0,$$

and for a circuit of three metals $a$, $b$, $c$, we must have

$$\Pi_{bc}+\Pi_{ca}+\Pi_{ab} = 0.$$

It follows from this equation that the three Peltier effects are not independent, but that one of them can be deduced from the other two. For instance, if we suppose $c$ to be a standard metal, and if we write $P_a = J\Pi_{ac}$ and $P_b = J\Pi_{bc}$, then

$$J\Pi_{ab} = P_a - P_b.$$

The quantity $P_a$ is a function of the temperature, and depends on the nature of the metal $a$.

251.] It has also been shewn by Magnus that if a circuit is formed of a single metal no current will be formed in it, however the section of the conductor and the temperature may vary in different parts.

Since in this case there is conduction of heat and consequent dissipation of energy, we cannot, as in the former case, consider this result as self-evident. The electromotive force, for instance, between two portions of a circuit might have depended on whether the current was passing from a thick portion of the conductor to a thin one, or the reverse, as well as on its passing rapidly or slowly from a hot portion to a cold one, or the reverse, and this would have made a current possible in an unequally heated circuit of one metal.

Hence, by the same reasoning as in the case of Peltier's phenomenon, we find that if the passage of a current through a conductor of one metal produces any thermal effect which is reversed when the current is reversed, this can only take place when the current flows from places of high to places of low temperature, or the reverse, and if the heat generated in a conductor of one metal in flowing from a place where the temperature is $x$ to a place where it is $y$, is $H$, then

$$JH = RC^2 t - S_{xy} Ct,$$

and the electromotive force tending to maintain the current will be $S_{xy}$.

If $x, y, z$ be the temperatures at three points of a homogeneous circuit, we must have

$$S_{yz} + S_{zx} + S_{xy} = 0,$$

according to the result of Magnus. Hence, if we suppose $z$ to be the zero temperature, and if we put

$$Q_x = S_{xz} \quad \text{and} \quad Q_y = S_{yz},$$

we find

$$S_{xy} = Q_x - Q_y,$$

where $Q_x$ is a function of the temperature $x$, the form of the function depending on the nature of the metal.

If we now consider a circuit of two metals $a$ and $b$ in which the temperature is $x$ where the current passes from $a$ to $b$, and $y$ where it passes from $b$ to $a$, the electromotive force will be

$$F = P_{ax} - P_{bx} + Q_{bx} - Q_{by} + P_{by} - P_{ay} + Q_{ay} - Q_{ax},$$

where $P_{ax}$ signifies the value of $P$ for the metal $a$ at the temperature $x$, or

$$F = P_{ax} - Q_{ax} - (P_{ay} - Q_{ay}) - (P_{bx} - Q_{bx}) + P_{by} - Q_{by}.$$

Since in unequally heated circuits of different metals there are in general thermoelectric currents, it follows that $P$ and $Q$ are in general different for the same metal and same temperature.

252.] The existence of the quantity $Q$ was first demonstrated by Sir W. Thomson, in the memoir we have referred to, as a deduction from the phenomenon of thermoelectric inversion discovered by Cumming *, who found that the order of certain metals in the thermoelectric scale is different at high and at low temperatures, so that for a certain temperature two metals may be neutral to each other. Thus, in a circuit of copper and iron if one junction be kept at the ordinary temperature while the temperature of the other is raised, a current sets from copper to iron through the hot junction, and the electromotive force continues to increase till the hot junction has reached a temperature $T$, which, according to Thomson, is about 284°C. When the temperature of the hot junction is raised still further the electromotive force is reduced, and at last, if the temperature be raised high enough, the current is reversed. The reversal of the current may be obtained more easily by raising the temperature of the colder junction. If the temperature of both junctions is above $T$ the current sets from iron to copper through the hotter junction, that is, in the reverse direction to that observed when both junctions are below $T$.

Hence, if one of the junctions is at the neutral temperature $T$ and the other is either hotter or colder, the current will set from copper to iron through the junction at the neutral temperature.

253.] From this fact Thomson reasoned as follows :—

Suppose the other junction at a temperature lower than $T$. The current may be made to work an engine or to generate heat in a wire, and this expenditure of energy must be kept up by the transformation of heat into electric energy, that is to say, heat must disappear somewhere in the circuit. Now at the temperature $T$ iron and copper are neutral to each other, so that no reversible thermal effect is produced at the hot junction, and at the cold junction there is, by Peltier's principle, an evolution of heat. Hence the only place where the heat can disappear is in the copper or iron portions of the circuit, so that either a current in iron from hot to cold must cool the iron, or a current in copper from cold to hot must cool the copper, or both these effects may take place. By an elaborate series of ingenious experiments Thomson succeeded in detecting the reversible thermal action of the current in passing between parts of different temperatures, and

* *Cambridge Transactions*, 1823.

he found that the current produced opposite effects in copper and in iron *.

When a stream of a material fluid passes along a tube from a hot part to a cold part it heats the tube, and when it passes from cold to hot it cools the tube, and these effects depend on the specific capacity for heat of the fluid. If we supposed electricity, whether positive or negative, to be a material fluid, we might measure its specific heat by the thermal effect on an unequally heated conductor. Now Thomson's experiments shew that positive electricity in copper and negative electricity in iron carry heat with them from hot to cold. Hence, if we supposed either positive or negative electricity to be a fluid, capable of being heated and cooled, and of communicating heat to other bodies, we should find the supposition contradicted by iron for positive electricity and by copper for negative electricity, so that we should have to abandon both hypotheses.

This scientific prediction of the reversible effect of an electric current upon an unequally heated conductor of one metal is another instructive example of the application of the theory of Conservation of Energy to indicate new directions of scientific research. Thomson has also applied the Second Law of Thermodynamics to indicate relations between the quantities which we have denoted by $P$ and $Q$, and has investigated the possible thermoelectric properties of bodies whose structure is different in different directions. He has also investigated experimentally the conditions under which these properties are developed by pressure, magnetization, &c.

254.] Professor Tait † has recently investigated the electromotive force of thermoelectric circuits of different metals, having their junctions at different temperatures. He finds that the electromotive force of a circuit may be expressed very accurately by the formula

$$E = a(t_1 - t_2)\left[t_0 - \tfrac{1}{2}(t_1 + t_2)\right],$$

where $t_1$ is the absolute temperature of the hot junction, $t_2$ that of the cold junction, and $t_0$ the temperature at which the two metals are neutral to each other. The factor $a$ is a coefficient depending on the nature of the two metals composing the circuit. This law has been verified through considerable ranges of temperature by Professor Tait and his students, and he hopes to make the thermoelectric circuit available as a thermometric instrument in his

* 'On the Electrodynamic Qualities of Metals.' *Phil. Trans.*, 1856.

† *Proc. R. S. Edin.*, Session 1870–71, p. 308, also Dec. 18, 1871.

experiments on the conduction of heat, and in other cases in which the mercurial thermometer is not convenient or has not a sufficient range.

According to Tait's theory, the quantity which Thomson calls the specific heat of electricity is proportional to the absolute temperature in each pure metal, though its magnitude and even its sign vary in different metals. From this he has deduced by thermodynamic principles the following results. Let $k_a t$, $k_b t$, $k_c t$ be the specific heats of electricity in three metals $a$, $b$, $c$, and let $T_{bc}$, $T_{ca}$, $T_{ab}$ be the temperatures at which pairs of these metals are neutral to each other, then the equations

$$(k_b - k_c) T_{bc} + (k_c - k_a) T_{ca} + (k_a - k_b) T_{ab} = 0,$$

$$J \Pi_{ab} = (k_a - k_b) t (T_{ab} - t),$$

$$E_{ab} = (k_a - k_b)(t_1 - t_2)\left[T_{ab} - \tfrac{1}{2}(t_1 + t_2)\right]$$

express the relation of the neutral temperatures, the value of the Peltier effect, and the electromotive force of a thermoelectric circuit.

# CHAPTER IV.

## ELECTROLYSIS.

### *Electrolytic Conduction.*

255.] I HAVE already stated that when an electric current in any part of its circuit passes through certain compound substances called Electrolytes, the passage of the current is accompanied by a certain chemical process called Electrolysis, in which the substance is resolved into two components called Ions, of which one, called the Anion, or the electronegative component, appears at the Anode, or place where the current enters the electrolyte, and the other, called the Cation, appears at the Cathode, or the place where the current leaves the electrolyte.

The complete investigation of Electrolysis belongs quite as much to Chemistry as to Electricity. We shall consider it from an electrical point of view, without discussing its application to the theory of the constitution of chemical compounds.

Of all electrical phenomena electrolysis appears the most likely to furnish us with a real insight into the true nature of the electric current, because we find currents of ordinary matter and currents of electricity forming essential parts of the same phenomenon.

It is probably for this very reason that, in the present imperfectly formed state of our ideas about electricity, the theories of electrolysis are so unsatisfactory.

The fundamental law of electrolysis, which was established by Faraday, and confirmed by the experiments of Beetz, Hittorf, and others down to the present time, is as follows :—

The number of electrochemical equivalents of an electrolyte which are decomposed by the passage of an electric current during a given time is equal to the number of units of electricity which are transferred by the current in the same time.

The electrochemical equivalent of a substance is that quantity of the substance which is electrolysed by a unit current passing through the substance for a unit of time, or, in other words, by the passage of a unit of electricity. When the unit of electricity is defined in absolute measure the absolute value of the electrochemical equivalent of each substance can be determined in grains or in grammes.

The electrochemical equivalents of different substances are proportional to their ordinary chemical equivalents. The ordinary chemical equivalents, however, are the mere numerical ratios in which the substances combine, whereas the electrochemical equivalents are quantities of matter of a determinate magnitude, depending on the definition of the unit of electricity.

Every electrolyte consists of two components, which, during the electrolysis, appear where the current enters and leaves the electrolyte, and nowhere else. Hence, if we conceive a surface described within the substance of the electrolyte, the amount of electrolysis which takes place through this surface, as measured by the electrochemical equivalents of the components transferred across it in opposite directions, will be proportional to the total electric current through the surface.

The actual transfer of the ions through the substance of the electrolyte in opposite directions is therefore part of the phenomenon of the conduction of an electric current through an electrolyte. At every point of the electrolyte through which an electric current is passing there are also two opposite material currents of the anion and the cation, which have the same lines of flow with the electric current, and are proportional to it in magnitude.

It is therefore extremely natural to suppose that the currents of the ions are convection currents of electricity, and, in particular, that every molecule of the cation is charged with a certain fixed quantity of positive electricity, which is the same for the molecules of all cations, and that every molecule of the anion is charged with an equal quantity of negative electricity.

The opposite motion of the ions through the electrolyte would then be a complete physical representation of the electric current. We may compare this motion of the ions with the motion of gases and liquids through each other during the process of diffusion, there being this difference between the two processes, that, in diffusion, the different substances are only mixed together and the mixture is not homogeneous, whereas in electrolysis they are chemically combined and the electrolyte is homogeneous. In diffusion the determining cause of the motion of a substance in a given direction is a diminution of the quantity of that substance per unit of volume in that direction, whereas in electrolysis the motion of each ion is due to the electromotive force acting on the charged molecules.

256.] Clausius *, who has bestowed much study on the theory of the molecular agitation of bodies, supposes that the molecules of all bodies are in a state of constant agitation, but that in solid bodies each molecule never passes beyond a certain distance from its original position, whereas in fluids a molecule, after moving a certain distance from its original position, is just as likely to move still farther from it as to move back again. Hence the molecules of a fluid apparently at rest are continually changing their positions, and passing irregularly from one part of the fluid to another. In a compound fluid he supposes that not only the compound molecules travel about in this way, but that, in the collisions which occur between the compound molecules, the molecules of which they are composed are often separated and change partners, so that the same individual atom is at one time associated with one atom of the opposite kind, and at another time with another. This process Clausius supposes to go on in the liquid at all times, but when an electromotive force acts on the liquid the motions of the molecules, which before were indifferently in all directions, are now influenced by the electromotive force, so that the positively charged molecules have a greater tendency towards the cathode than towards the anode, and the negatively charged molecules have a greater tendency to move in the opposite direction. Hence the molecules of the cation will during their intervals of freedom struggle towards the cathode, but will continually be checked in their course by pairing for a time with molecules of the anion, which are also struggling through the crowd, but in the opposite direction.

257.] This theory of Clausius enables us to understand how it is, that whereas the actual decomposition of an electrolyte requires an electromotive force of finite magnitude, the conduction of the current in the electrolyte obeys the law of Ohm, so that every electromotive force within the electrolyte, even the feeblest, produces a current of proportionate magnitude.

According to the theory of Clausius, the decomposition and recomposition of the electrolyte is continually going on even when there is no current, and the very feeblest electromotive force is

* Pogg. *Ann.* bd. ci. s. 338 (1857).

sufficient to give this process a certain degree of direction, and so to produce the currents of the ions and the electric current, which is part of the same phenomenon. Within the electrolyte, however, the ions are never set free in finite quantity, and it is this liberation of the ions which requires a finite electromotive force. At the electrodes the ions accumulate, for the successive portions of the ions, as they arrive at the electrodes, instead of finding molecules of the opposite ion ready to combine with them, are forced into company with molecules of their own kind, with which they cannot combine. The electromotive force required to produce this effect is of finite magnitude, and forms an opposing electromotive force which produces a reversed current when other electromotive forces are removed. When this reversed electromotive force, owing to the accumulation of the ions at the electrode, is observed, the electrodes are said to be Polarized.

258.] One of the best methods of determining whether a body is or is not an electrolyte is to place it between platinum electrodes and to pass a current through it for some time, and then, disengaging the electrodes from the voltaic battery, and connecting them with a galvanometer, to observe whether a reverse current, due to polarization of the electrodes, passes through the galvanometer. Such a current, being due to accumulation of different substances on the two electrodes, is a proof that the substance has been electrolytically decomposed by the original current from the battery. This method can often be applied where it is difficult, by direct chemical methods, to detect the presence of the products of decomposition at the electrodes. See Art. 271.

259.] So far as we have gone the theory of electrolysis appears very satisfactory. It explains the electric current, the nature of which we do not understand, by means of the currents of the material components of the electrolyte, the motion of which, though not visible to the eye, is easily demonstrated. It gives a clear explanation, as Faraday has shewn, why an electrolyte which conducts in the liquid state is a non-conductor when solidified, for unless the molecules can pass from one part to another no electrolytic conduction can take place, so that the substance must be in a liquid state, either by fusion or by solution, in order to be a conductor.

But if we go on, and assume that the molecules of the ions within the electrolyte are actually charged with certain definite quantities of electricity, positive and negative, so that the electrolytic current is simply a current of convection, we find that this tempting hypothesis leads us into very difficult ground.

In the first place, we must assume that in every electrolyte each molecule of the cation, as it is liberated at the cathode, communicates to the cathode a charge of positive electricity, the amount of which is the same for every molecule, not only of that cation but of all other cations. In the same way each molecule of the anion when liberated, communicates to the anode a charge of negative electricity, the numerical magnitude of which is the same as that of the positive charge due to a molecule of a cation, but with sign reversed.

If, instead of a single molecule, we consider an assemblage of molecules, constituting an electrochemical equivalent of the ion, then the total charge of all the molecules is, as we have seen, one unit of electricity, positive or negative.

260.] We do not as yet know how many molecules there are in an electrochemical equivalent of any substance, but the molecular theory of chemistry, which is corroborated by many physical considerations, supposes that the number of molecules in an electrochemical equivalent is the same for all substances. We may therefore, in molecular speculations, assume that the number of molecules in an electrochemical equivalent is $N$, a number unknown at present, but which we may hereafter find means to determine *.

Each molecule, therefore, on being liberated from the state of combination, parts with a charge whose magnitude is $\frac{1}{N}$, and is positive for the cation and negative for the anion. This definite quantity of electricity we shall call the molecular charge. If it were known it would be the most natural unit of electricity.

Hitherto we have only increased the precision of our ideas by exercising our imagination in tracing the electrification of molecules and the discharge of that electrification.

The liberation of the ions and the passage of positive electricity from the anode and into the cathode are simultaneous facts. The ions, when liberated, are not charged with electricity, hence, when they are in combination, they have the molecular charges as above described.

The electrification of a molecule, however, though easily spoken of, is not so easily conceived.

We know that if two metals are brought into contact at any

* See note to Art. 5.

point, the rest of their surfaces will be electrified, and if the metals are in the form of two plates separated by a narrow interval of air, the charge on each plate may become of considerable magnitude. Something like this may be supposed to occur when the two components of an electrolyte are in combination. Each pair of molecules may be supposed to touch at one point, and to have the rest of their surface charged with electricity due to the electromotive force of contact.

But to explain the phenomenon, we ought to shew why the charge thus produced on each molecule is of a fixed amount, and why, when a molecule of chlorine is combined with a molecule of zinc, the molecular charges are the same as when a molecule of chlorine is combined with a molecule of copper, although the electromotive force between chlorine and zinc is much greater than that between chlorine and copper. If the charging of the molecules is the effect of the electromotive force of contact, why should electromotive forces of different intensities produce exactly equal charges?

Suppose, however, that we leap over this difficulty by simply asserting the fact of the constant value of the molecular charge, and that we call this constant molecular charge, for convenience in description, *one molecule of electricity*.

This phrase, gross as it is, and out of harmony with the rest of this treatise, will enable us at least to state clearly what is known about electrolysis, and to appreciate the outstanding difficulties.

Every electrolyte must be considered as a binary compound of its anion and its cation. The anion or the cation or both may be compound bodies, so that a molecule of the anion or the cation may be formed by a number of molecules of simple bodies. A molecule of the anion and a molecule of the cation combined together form one molecule of the electrolyte.

In order to act as an anion in an electrolyte, the molecule which so acts must be charged with what we have called one molecule of negative electricity, and in order to act as a cation the molecule must be charged with one molecule of positive electricity.

These charges are connected with the molecules only when they are combined as anion and cation in the electrolyte.

When the molecules are electrolysed, they part with their charges to the electrodes, and appear as unelectrified bodies when set free from combination.

If the same molecule is capable of acting as a cation in one electrolyte and as an anion in another, and also of entering into compound bodies which are not electrolytes, then we must suppose that it receives a positive charge of electricity when it acts as a cation, a negative charge when it acts as an anion, and that it is without charge when it is not in an electrolyte.

Iodine, for instance, acts as an anion in the iodides of the metals and in hydriodic acid, but is said to act as a cation in the bromide of iodine.

This theory of molecular charges may serve as a method by which we may remember a good many facts about electrolysis. It is extremely improbable that when we come to understand the true nature of electrolysis we shall retain in any form the theory of molecular charges, for then we shall have obtained a secure basis on which to form a true theory of electric currents, and so become independent of these provisional theories.

261.] One of the most important steps in our knowledge of electrolysis has been the recognition of the secondary chemical processes which arise from the evolution of the ions at the electrodes.

In many cases the substances which are found at the electrodes are not the actual ions of the electrolysis, but the products of the action of these ions on the electrolyte.

Thus, when a solution of sulphate of soda is electrolysed by a current which also passes through dilute sulphuric acid, equal quantities of oxygen are given off at the anodes, and equal quantities of hydrogen at the cathodes, both in the sulphate of soda and in the dilute acid.

But if the electrolysis is conducted in suitable vessels, such as U-shaped tubes or vessels with a porous diaphragm, so that the substance surrounding each electrode can be examined separately, it is found that at the anode of the sulphate of soda there is an equivalent of sulphuric acid as well as an equivalent of oxygen, and at the cathode there is an equivalent of soda as well as two equivalents of hydrogen.

It would at first sight seem as if, according to the old theory of the constitution of salts, the sulphate of soda were electrolysed into its constituents sulphuric acid and soda, while the water of the solution is electrolysed at the same time into oxygen and hydrogen. But this explanation would involve the admission that the same current which passing through dilute sulphuric acid electrolyses one equivalent of water, when it passes through solution of sulphate of soda electrolyses one equivalent of the salt as well as one equivalent of the water, and this would be contrary to the law of electrochemical equivalents.

But if we suppose that the components of sulphate of soda are not $SO_3$ and $NaO$ but $SO_4$ and $Na$,—not sulphuric acid and soda but sulphion and sodium—then the sulphion travels to the anode and is set free, but being unable to exist in a free state it breaks up into sulphuric acid and oxygen, one equivalent of each. At the same time the sodium is set free at the cathode, and there decomposes the water of the solution, forming one equivalent of soda and two of hydrogen.

In the dilute sulphuric acid the gases collected at the electrodes are the constituents of water, namely one volume of oxygen and two volumes of hydrogen. There is also an increase of sulphuric acid at the anode, but its amount is not equal to an equivalent.

It is doubtful whether pure water is an electrolyte or not. The greater the purity of the water, the greater the resistance to electrolytic conduction. The minutest traces of foreign matter are sufficient to produce a great diminution of the electrical resistance of water. The electric resistance of water as determined by different observers has values so different that we cannot consider it as a determined quantity. The purer the water the greater its resistance, and if we could obtain really pure water it is doubtful whether it would conduct at all.

As long as water was considered an electrolyte, and was, indeed, taken as the type of electrolytes, there was a strong reason for maintaining that it is a binary compound, and that two volumes of hydrogen are chemically equivalent to one volume of oxygen. If, however, we admit that water is not an electrolyte, we are free to suppose that equal volumes of oxygen and of hydrogen are chemically equivalent.

The dynamical theory of gases leads us to suppose that in perfect gases equal volumes always contain an equal number of molecules, and that the principal part of the specific heat, that, namely, which depends on the motion of agitation of the molecules among each other, is the same for equal numbers of molecules of all gases. Hence we are led to prefer a chemical system in which equal volumes of oxygen and of hydrogen are regarded as equivalent, and in which water is regarded as a compound of two equivalents of hydrogen and one of oxygen, and therefore probably not capable of direct electrolysis.

While electrolysis fully establishes the close relationship between electrical phenomena and those of chemical combination, the fact that every chemical compound is not an electrolyte shews that chemical combination is a process of a higher order of complexity than any purely electrical phenomenon. Thus the combinations of the metals with each other, though they are good conductors, and their components stand at different points of the scale of electrification by contact, are not, even when in a fluid state, decomposed by the current. Most of the combinations of the substances which act as anions are not conductors, and therefore are not electrolytes. Besides these we have many compounds, containing the same components as electrolytes, but not in equivalent proportions, and these are also non-conductors, and therefore not electrolytes.

*On the Conservation of Energy in Electrolysis.*

262.] Consider any voltaic circuit consisting partly of a battery, partly of a wire, and partly of an electrolytic cell.

During the passage of unit of electricity through any section of the circuit, one electrochemical equivalent of each of the substances in the cells, whether voltaic or electrolytic, is electrolysed.

The amount of mechanical energy equivalent to any given chemical process can be ascertained by converting the whole energy due to the process into heat, and then expressing the heat in dynamical measure by multiplying the number of thermal units by Joule's mechanical equivalent of heat.

Where this direct method is not applicable, if we can estimate the heat given out by the substances taken first in the state before the process and then in the state after the process during their reduction to a final state, which is the same in both cases, then the thermal equivalent of the process is the difference of the two quantities of heat.

In the case in which the chemical action maintains a voltaic circuit, Joule found that the heat developed in the voltaic cells is less than that due to the chemical process within the cell, and that the remainder of the heat is developed in the connecting wire, or, when there is an electromagnetic engine in the circuit, part of the heat may be accounted for by the mechanical work of the engine.

For instance, if the electrodes of the voltaic cell are first connected by a short thick wire, and afterwards by a long thin wire, the heat developed in the cell for each grain of zinc dissolved is greater in the first case than the second, but the heat developed in the wire is greater in the second case than in the first. The sum of the heat developed in the cell and in the wire for each grain of zinc dissolved is the same in both cases. This has been established by Joule by direct experiment.

The ratio of the heat generated in the cell to that generated in the wire is that of the resistance of the cell to that of the wire, so that if the wire were made of sufficient resistance nearly the whole of the heat would be generated in the wire, and if it were made of sufficient conducting power nearly the whole of the heat would be generated in the cell.

Let the wire be made so as to have great resistance, then the heat generated in it is equal in dynamical measure to the product of the quantity of electricity which is transmitted, multiplied by the electromotive force under which it is made to pass through the wire.

263.] Now during the time in which an electrochemical equivalent of the substance in the cell undergoes the chemical process which gives rise to the current, one unit of electricity passes through the wire. Hence, the heat developed by the passage of one unit of electricity is in this case measured by the electromotive force. But this heat is that which one electrochemical equivalent of the substance generates, whether in the cell or in the wire, while undergoing the given chemical process.

Hence the following important theorem, first proved by Thomson (*Phil. Mag.* Dec. 1851):—

'The electromotive force of an electrochemical apparatus is in absolute measure equal to the mechanical equivalent of the chemical action on one electrochemical equivalent of the substance.'

The thermal equivalents of many chemical actions have been determined by Andrews, Hess, Favre and Silbermann, &c., and from these their mechanical equivalents can be deduced by multiplication by the mechanical equivalent of heat.

This theorem not only enables us to calculate from purely thermal data the electromotive force of different voltaic arrangements, and the electromotive force required to effect electrolysis in different cases, but affords the means of actually measuring chemical affinity.

It has long been known that chemical affinity, or the tendency which exists towards the going on of a certain chemical change, is stronger in some cases than in others, but no proper measure of this tendency could be made till it was shewn that this tendency in certain cases is exactly equivalent to a certain electromotive force, and can therefore be measured according to the very same principles used in the measurement of electromotive forces.

Chemical affinity being therefore, in certain cases, reduced to the form of a measurable quantity, the whole theory of chemical processes, of the rate at which they go on, of the displacement of one substance by another, &c., becomes much more intelligible than when chemical affinity was regarded as a quality *sui generis*, and irreducible to numerical measurement.

When the volume of the products of electrolysis is greater than that of the electrolyte, work is done during the electrolysis in overcoming the pressure. If the volume of an electrochemical equivalent of the electrolyte is increased by a volume $v$ when electrolysed under a pressure $p$, then the work done during the passage of a unit of electricity in overcoming pressure is $vp$, and the electromotive force required for electrolysis must include a part equal to $vp$, which is spent in performing this mechanical work.

If the products of electrolysis are gases which, like oxygen and hydrogen, are much rarer than the electrolyte, and fulfil Boyle's law very exactly, $vp$ will be very nearly constant for the same temperature, and the electromotive force required for electrolysis will not depend in any sensible degree on the pressure. Hence it has been found impossible to check the electrolytic decomposition of dilute sulphuric acid by confining the decomposed gases in a small space.

When the products of electrolysis are liquid or solid the quantity $vp$ will increase as the pressure increases, so that if $v$ is positive an increase of pressure will increase the electromotive force required for electrolysis.

In the same way, any other kind of work done during electrolysis will have an effect on the value of the electromotive force, as, for instance, if a vertical current passes between two zinc electrodes in a solution of sulphate of zinc a greater electromotive force will be required when the current in the solution flows upwards than when it flows downwards, for, in the first case, it carries zinc from the lower to the upper electrode, and in the second from the upper to the lower. The electromotive force required for this purpose is less than the millionth part of that of a Daniell's cell per foot.

# CHAPTER V.

## ELECTROLYTIC POLARIZATION.

264.] WHEN an electric current is passed through an electrolyte bounded by metal electrodes, the accumulation of the ions at the electrodes produces the phenomenon called Polarization, which consists in an electromotive force acting in the opposite direction to the current, and producing an apparent increase of the resistance.

When a continuous current is employed, the resistance appears to increase rapidly from the commencement of the current, and at last reaches a value nearly constant. If the form of the vessel in which the electrolyte is contained is changed, the resistance is altered in the same way as a similar change of form of a metallic conductor would alter its resistance, but an additional apparent resistance, depending on the nature of the electrodes, has always to be added to the true resistance of the electrolyte.

265.] These phenomena have led some to suppose that there is a finite electromotive force required for a current to pass through an electrolyte. It has been shewn, however, by the researches of Lenz, Neumann, Beetz, Wiedemann *, Paalzow †, and recently by those of MM. F. Kohlrausch and W. A. Nippoldt ‡, that the conduction in the electrolyte itself obeys Ohm's Law with the same precision as in metallic conductors, and that the apparent resistance at the bounding surface of the electrolyte and the electrodes is entirely due to polarization.

266.] The phenomenon called polarization manifests itself in the case of a continuous current by a diminution in the current, indicating a force opposed to the current. Resistance is also perceived as a force opposed to the current, but we can distinguish

* *Galvanismus*, bd. i.

† *Berlin Monatsbericht*, July, 1868.

‡ Pogg. *Ann.* bd. cxxxviii. s. 286 (October, 1869).

between the two phenomena by instantaneously removing or reversing the electromotive force.

The resisting force is always opposite in direction to the current, and the external electromotive force required to overcome it is proportional to the strength of the current, and changes its direction when the direction of the current is changed. If the external electromotive force becomes zero the current simply stops.

The electromotive force due to polarization, on the other hand, is in a fixed direction, opposed to the current which produced it. If the electromotive force which produced the current is removed, the polarization produces a current in the opposite direction.

The difference between the two phenomena may be compared with the difference between forcing a current of water through a long capillary tube, and forcing water through a tube of moderate length up into a cistern. In the first case if we remove the pressure which produces the flow the current will simply stop. In the second case, if we remove the pressure the water will begin to flow down again from the cistern.

To make the mechanical illustration more complete, we have only to suppose that the cistern is of moderate depth, so that when a certain amount of water is raised into it, it begins to overflow. This will represent the fact that the total electromotive force due to polarization has a maximum limit.

267.] The cause of polarization appears to be the existence at the electrodes of the products of the electrolytic decomposition of the fluid between them. The surfaces of the electrodes are thus rendered electrically different, and an electromotive force between them is called into action, the direction of which is opposite to that of the current which caused the polarization.

The ions, which by their presence at the electrodes produce the phenomena of polarization, are not in a perfectly free state, but are in a condition in which they adhere to the surface of the electrodes with considerable force.

The electromotive force due to polarization depends upon the density with which the electrode is covered with the ion, but it is not proportional to this density, for the electromotive force does not increase so rapidly as this density.

This deposit of the ion is constantly tending to become free, and either to diffuse into the liquid, to escape as a gas, or to be precipitated as a solid.

The rate of this dissipation of the polarization is exceedingly small for slight degrees of polarization, and exceedingly rapid near the limiting value of polarization.

268.] We have seen, Art. 262, that the electromotive force acting in any electrolytic process is numerically equal to the mechanical equivalent of the result of that process on one electrochemical equivalent of the substance. If the process involves a diminution of the intrinsic energy of the substances which take part in it, as in the voltaic cell, then the electromotive force is in the direction of the current. If the process involves an increase of the intrinsic energy of the substances, as in the case of the electrolytic cell, the electromotive force is in the direction opposite to that of the current, and this electromotive force is called polarization.

In the case of a steady current in which electrolysis goes on continuously, and the ions are separated in a free state at the electrodes, we have only by a suitable process to measure the intrinsic energy of the separated ions, and compare it with that of the electrolyte in order to calculate the electromotive force required for the electrolysis. This will give the maximum polarization.

But during the first instants of the process of electrolysis the ions when deposited at the electrodes are not in a free state, and their intrinsic energy is less than their energy in a free state, though greater than their energy when combined in the electrolyte. In fact, the ion in contact with the electrode is in a state which when the deposit is very thin may be compared with that of chemical combination with the electrode, but as the deposit increases in density, the succeeding portions are no longer so intimately combined with the electrode, but simply adhere to it, and at last the deposit, if gaseous, escapes in bubbles, if liquid, diffuses through the electrolyte, and if solid, forms a precipitate.

In studying polarization we have therefore to consider

(1) The superficial density of the deposit, which we may call $\sigma$. This quantity $\sigma$ represents the number of electrochemical equivalents of the ion deposited on unit of area. Since each electrochemical equivalent deposited corresponds to one unit of electricity transmitted by the current, we may consider $\sigma$ as representing either a surface-density of matter or a surface-density of electricity.

(2) The electromotive force of polarization, which we may call $p$. This quantity $p$ is the difference between the electric potentials of the two electrodes when the current through the electrolyte

is so feeble that the proper resistance of the electrolyte makes no sensible difference between these potentials.

The electromotive force $p$ at any instant is numerically equal to the mechanical equivalent of the electrolytic process going on at that instant which corresponds to one electrochemical equivalent of the electrolyte. This electrolytic process, it must be remembered, consists in the deposit of the ions on the electrodes, and the state in which they are deposited depends on the actual state of the surface of the electrodes, which may be modified by previous deposits.

Hence the electromotive force at any instant depends on the previous history of the electrode. It is, speaking very roughly, a function of $\sigma$, the density of the deposit, such that $p = 0$ when $\sigma = 0$, but $p$ approaches a limiting value much sooner than $\sigma$ does. The statement, however, that $p$ is a function of $\sigma$ cannot be considered accurate. It would be more correct to say that $p$ is a function of the chemical state of the superficial layer of the deposit, and that this state depends on the density of the deposit according to some law involving the time.

269.] (3) The third thing we must take into account is the dissipation of the polarization. The polarization when left to itself diminishes at a rate depending partly on the intensity of the polarization or the density of the deposit, and partly on the nature of the surrounding medium, and the chemical, mechanical, or thermal action to which the surface of the electrode is exposed.

If we determine a time $T$ such that at the rate at which the deposit is dissipated, the whole deposit would be removed in a time $T$, we may call $T$ the modulus of the time of dissipation. When the density of the deposit is very small, $T$ is very large, and may be reckoned by days or months. When the density of the deposit approaches its limiting value $T$ diminishes very rapidly, and is probably a minute fraction of a second. In fact, the rate of dissipation increases so rapidly that when the strength of the current is maintained constant, the separated gas, instead of contributing to increase the density of the deposit, escapes in bubbles as fast as it is formed.

270.] There is therefore a great difference between the state of polarization of the electrodes of an electrolytic cell when the polarization is feeble, and when it is at its maximum value. For instance, if a number of electrolytic cells of dilute sulphuric acid with platinum electrodes are arranged in series, and if a small electromotive force, such as that of one Daniell's cell, be made to act on the circuit, the electromotive force will produce a current of exceedingly short duration, for after a very short time the electromotive force arising from the polarization of the cell will balance that of the Daniell's cell.

The dissipation will be very small in the case of so feeble a state of polarization, and it will take place by a very slow absorption of the gases and diffusion through the liquid. The rate of this dissipation is indicated by the exceedingly feeble current which still continues to flow without any visible separation of gases.

If we neglect this dissipation for the short time during which the state of polarization is set up, and if we call $Q$ the total quantity of electricity which is transmitted by the current during this time, then if $A$ is the area of one of the electrodes, and $\sigma$ the density of the deposit, supposed uniform,

$$Q = A\sigma.$$

If we now disconnect the electrodes of the electrolytic apparatus from the Daniell's cell, and connect them with a galvanometer capable of measuring the whole discharge through it, a quantity of electricity nearly equal to $Q$ will be discharged as the polarization disappears.

271.] Hence we may compare the action of this apparatus, which is a form of Ritter's Secondary Pile, with that of a Leyden jar.

Both the secondary pile and the Leyden jar are capable of being charged with a certain amount of electricity, and of being afterwards discharged. During the discharge a quantity of electricity nearly equal to the charge passes in the opposite direction. The difference between the charge and the discharge arises partly from dissipation, a process which in the case of small charges is very slow, but which, when the charge exceeds a certain limit, becomes exceedingly rapid. Another part of the difference between the charge and the discharge arises from the fact that after the electrodes have been connected for a time sufficient to produce an apparently complete discharge, so that the current has completely disappeared, if we separate the electrodes for a time, and afterwards connect them, we obtain a second discharge in the same direction as the original discharge. This is called the residual discharge, and is a phenomenon of the Leyden jar as well as of the secondary pile.

The secondary pile may therefore be compared in several respects to a Leyden jar. There are, however, certain important differences. The charge of a Leyden jar is very exactly proportional to the electromotive force of the charge, that is, to the difference of potentials of the two surfaces, and the charge corresponding to unit of electromotive force is called the capacity of the jar, a constant quantity. The corresponding quantity, which may be called the capacity of the secondary pile, increases when the electromotive force increases.

The capacity of the jar depends on the area of the opposed surfaces, on the distance between them, and on the nature of the substance between them, but not on the nature of the metallic surfaces themselves. The capacity of the secondary pile depends on the area of the surfaces of the electrodes, but not on the distance between them, and it depends on the nature of the surface of the electrodes, as well as on that of the fluid between them. The maximum difference of the potentials of the electrodes in each element of a secondary pile is very small compared with the maximum difference of the potentials of those of a charged Leyden jar, so that in order to obtain much electromotive force a pile of many elements must be used.

On the other hand, the superficial density of the charge in the secondary pile is immensely greater than the utmost superficial density of the charge which can be accumulated on the surfaces of a Leyden jar, insomuch that Mr. C. F. Varley *, in describing the construction of a condenser of great capacity, recommends a series of gold or platinum plates immersed in dilute acid as preferable in point of cheapness to induction plates of tinfoil separated by insulating material.

The form in which the energy of a Leyden jar is stored up is the state of constraint of the dielectric between the conducting surfaces, a state which I have already described under the name of electric polarization, pointing out those phenomena attending this state which are at present known, and indicating the imperfect state of our knowledge of what really takes place. See Arts. 62, 111.

The form in which the energy of the secondary pile is stored up is the chemical condition of the material stratum at the surface of the electrodes, consisting of the ions of the electrolyte and the substance of the electrodes in a relation varying from chemical combination to superficial condensation, mechanical adherence, or simple juxtaposition.

The seat of this energy is close to the surfaces of the electrodes,

* Specification of C. F. Varley, 'Electric Telegraphs, &c.,' Jan. 1860.

and not throughout the substance of the electrolyte, and the form in which it exists may be called electrolytic polarization.

After studying the secondary pile in connexion with the Leyden jar, the student should again compare the voltaic battery with some form of the electrical machine, such as that described in Art. 211.

Mr. Varley has lately * found that the capacity of one square inch is from 175 to 542 microfarads and upwards for platinum plates in dilute sulphuric acid, and that the capacity increases with the electromotive force, being about 175 for 0.02 of a Daniell's cell, and 542 for 1.6 Daniell's cells.

But the comparison between the Leyden jar and the secondary pile may be carried still farther, as in the following experiment, due to Buff †. It is only when the glass of the jar is cold that it is capable of retaining a charge. At a temperature below 100°C the glass becomes a conductor. If a test-tube containing mercury is placed in a vessel of mercury, and if a pair of electrodes are connected, one with the inner and the other with the outer portion of mercury, the arrangement constitutes a Leyden jar which will hold a charge at ordinary temperatures. If the electrodes are connected with those of a voltaic battery, no current will pass as long as the glass is cold, but if the apparatus is gradually heated a current will begin to pass, and will increase rapidly in intensity as the temperature rises, though the glass remains apparently as hard as ever.

This current is manifestly electrolytic, for if the electrodes are disconnected from the battery, and connected with a galvanometer, a considerable reverse current passes, due to polarization of the surfaces of the glass.

If, while the battery is in action the apparatus is cooled, the current is stopped by the cold glass as before, but the polarization of the surfaces remains. The mercury may be removed, the surfaces may be washed with nitric acid and with water, and fresh mercury introduced. If the apparatus is then heated, the current of polarization appears as soon as the glass is sufficiently warm to conduct it.

We may therefore regard glass at 100°C, though apparently a solid body, as an electrolyte, and there is considerable reason to believe that in most instances in which a dielectric has a slight degree of conductivity the conduction is electrolytic. The

* *Proc. R. S.*, Jan. 12, 1871.

† *Annalen der Chemie und Pharmacie*, bd. xc. 257 (1854).

existence of polarization may be regarded as conclusive evidence of electrolysis, and if the conductivity of a substance increases as the temperature rises, we have good grounds for suspecting that it is electrolytic.

*On Constant Voltaic Elements.*

272.] When a series of experiments is made with a voltaic battery in which polarization occurs, the polarization diminishes during the time that the current is not flowing, so that when it begins to flow again the current is stronger than after it has flowed for some time. If, on the other hand, the resistance of the circuit is diminished by allowing the current to flow through a short shunt, then, when the current is again made to flow through the ordinary circuit, it is at first weaker than its normal strength on account of the great polarization produced by the use of the short circuit.

To get rid of these irregularities in the current, which are exceedingly troublesome in experiments involving exact measurements, it is necessary to get rid of the polarization, or at least to reduce it as much as possible.

It does not appear that there is much polarization at the surface of the zinc plate when immersed in a solution of sulphate of zinc or in dilute sulphuric acid. The principal seat of polarization is at the surface of the negative metal. When the fluid in which the negative metal is immersed is dilute sulphuric acid, it is seen to become covered with bubbles of hydrogen gas, arising from the electrolytic decomposition of the fluid. Of course these bubbles, by preventing the fluid from touching the metal, diminish the surface of contact and increase the resistance of the circuit. But besides the visible bubbles it is certain that there is a thin coating of hydrogen, probably not in a free state, adhering to the metal, and as we have seen that this coating is able to produce an electromotive force in the reverse direction, it must necessarily diminish the electromotive force of the battery.

Various plans have been adopted to get rid of this coating of hydrogen. It may be diminished to some extent by mechanical means, such as stirring the liquid, or rubbing the surface of the negative plate. In Smee's battery the negative plates are vertical, and covered with finely divided platinum from which the bubbles of hydrogen easily escape, and in their ascent produce a current of liquid which helps to brush off other bubbles as they are formed.

A far more efficacious method, however, is to employ chemical means. These are of two kinds. In the batteries of Grove and Bunsen the negative plate is immersed in a fluid rich in oxygen, and the hydrogen, instead of forming a coating on the plate, combines with this substance. In Grove's battery the plate is of platinum immersed in strong nitric acid. In Bunsen's first battery it is of carbon in the same acid. Chromic acid is also used for the same purpose, and has the advantage of being free from the acid fumes produced by the reduction of nitric acid.

A different mode of getting rid of the hydrogen is by using copper as the negative metal, and covering the surface with a coat of oxide. This, however, rapidly disappears when it is used as the negative electrode. To renew it Joule has proposed to make the copper plates in the form of disks, half immersed in the liquid, and to rotate them slowly, so that the air may act on the parts exposed to it in turn.

The other method is by using as the liquid an electrolyte, the cation of which is a metal highly negative to zinc.

In Daniell's battery a copper plate is immersed in a saturated solution of sulphate of copper. When the current flows through the solution from the zinc to the copper no hydrogen appears on the copper plate, but copper is deposited on it. When the solution is saturated, and the current is not too strong, the copper appears to act as a true cation, the anion $SO_4$ travelling towards the zinc.

When these conditions are not fulfilled hydrogen is evolved at the cathode, but immediately acts on the solution, throwing down copper, and uniting with $SO_4$ to form oil of vitriol. When this is the case, the sulphate of copper next the copper plate is replaced by oil of vitriol, the liquid becomes colourless, and polarization by hydrogen gas again takes place. The copper deposited in this way is of a looser and more friable structure than that deposited by true electrolysis.

To ensure that the liquid in contact with the copper shall be saturated with sulphate of copper, crystals of this substance must be placed in the liquid close to the copper, so that when the solution is made weak by the deposition of the copper, more of the crystals may be dissolved.

We have seen that it is necessary that the liquid next the copper should be saturated with sulphate of copper. It is still more necessary that the liquid in which the zinc is immersed should be free from sulphate of copper. If any of this salt makes its way to the surface of the zinc it is reduced, and copper is deposited on the zinc. The zinc, copper, and fluid then form a little circuit in which rapid electrolytic action goes on, and the zinc is eaten away by an action which contributes nothing to the useful effect of the battery.

To prevent this, the zinc is immersed either in dilute sulphuric acid or in a solution of sulphate of zinc, and to prevent the solution of sulphate of copper from mixing with this liquid, the two liquids are separated by a division consisting of bladder or porous earthenware, which allows electrolysis to take place through it, but effectually prevents mixture of the fluids by visible currents.

In some batteries sawdust is used to prevent currents. The experiments of Graham, however, shew that the process of diffusion goes on nearly as rapidly when two liquids are separated by a division of this kind as when they are in direct contact, provided there are no visible currents, and it is probable that if a septum is employed which diminishes the diffusion, it will increase in exactly the same ratio the resistance of the element, because electrolytic conduction is a process the mathematical laws of which have the same form as those of diffusion, and whatever interferes with one must interfere equally with the other. The only difference is that diffusion is always going on, while the current flows only when the battery is in action.

In all forms of Daniell's battery the final result is that the sulphate of copper finds its way to the zinc and spoils the battery. To retard this result indefinitely, Sir W. Thomson * has constructed Daniell's battery in the following form.

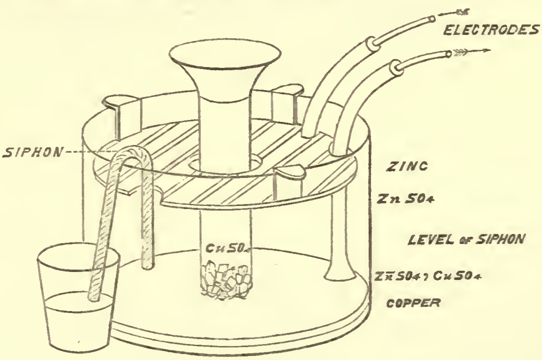



Fig. 21.

In each cell the copper plate is placed horizontally at the bottom

* *Proc. R. S.*, Jan. 19, 1871.

and a saturated solution of sulphate of zinc is poured over it. The zinc is in the form of a grating and is placed horizontally near the surface of the solution. A glass tube is placed vertically in the solution with its lower end just above the surface of the copper plate. Crystals of sulphate of copper are dropped down this tube, and, dissolving in the liquid, form a solution of greater density than that of sulphate of zinc alone, so that it cannot get to the zinc except by diffusion. To retard this process of diffusion, a siphon, consisting of a glass tube stuffed with cotton wick, is placed with one extremity midway between the zinc and copper, and the other in a vessel outside the cell, so that the liquid is very slowly drawn off near the middle of its depth. To supply its place, water, or a weak solution of sulphate of zinc, is added above when required. In this way the greater part of the sulphate of copper rising through the liquid by diffusion is drawn off by the siphon before it reaches the zinc, and the zinc is surrounded by liquid nearly free from sulphate of copper, and having a very slow downward motion in the cell, which still further retards the upward motion of the sulphate of copper. During the action of the battery copper is deposited on the copper plate, and $SO_4$ travels slowly through the liquid to the zinc with which it combines, forming sulphate of zinc. Thus the liquid at the bottom becomes less dense by the deposition of the copper, and the liquid at the top becomes more dense by the addition of the zinc. To prevent this action from changing the order of density of the strata, and so producing instability and visible currents in the vessel, care must be taken to keep the tube well supplied with crystals of sulphate of copper, and to feed the cell above with a solution of sulphate of zinc sufficiently dilute to be lighter than any other stratum of the liquid in the cell.

Daniell's battery is by no means the most powerful in common use. The electromotive force of Grove's cell is 192,000,000, of Daniell's 107,900,000 and that of Bunsen's 188,000,000.

The resistance of Daniell's cell is in general greater than that of Grove's or Bunsen's of the same size.

These defects, however, are more than counterbalanced in all cases where exact measurements are required, by the fact that Daniell's cell exceeds every other known arrangement in constancy of electromotive force. It has also the advantage of continuing in working order for a long time, and of emitting no gas.

# CHAPTER VI.

## LINEAR ELECTRIC CURRENTS.

### *On Systems of Linear Conductors.*

273.] Any conductor may be treated as a linear conductor if it is arranged so that the current must always pass in the same manner between two portions of its surface which are called its electrodes. For instance, a mass of metal of any form the surface of which is entirely covered with insulating material except at two places, at which the exposed surface of the conductor is in metallic contact with electrodes formed of a perfectly conducting material, may be treated as a linear conductor. For if the current be made to enter at one of these electrodes and escape at the other the lines of flow will be determinate, and the relation between electromotive force, current and resistance will be expressed by Ohm's Law, for the current in every part of the mass will be a linear function of $E$. But if there be more possible electrodes than two, the conductor may have more than one independent current through it, and these may not be conjugate to each other. See Art. 282.

### *Ohm's Law.*

274.] Let $E$ be the electromotive force in a linear conductor from the electrode $A_1$ to the electrode $A_2$. (See Art. 69.) Let $C$ be the strength of the electric current along the conductor, that is to say, let $C$ units of electricity pass across every section in the direction $A_1 A_2$ in unit of time, and let $R$ be the resistance of the conductor, then the expression of Ohm's Law is

$$E = CR. \tag{1}$$

### *Linear Conductors arranged in Series.*

275.] Let $A_1$, $A_2$ be the electrodes of the first conductor and let the second conductor be placed with one of its electrodes in contact with $A_2$, so that the second conductor has for its electrodes $A_2$, $A_3$. The electrodes of the third conductor may be denoted by $A_3$ and $A_4$.

Let the electromotive force along each of these conductors be denoted by $E_{12}$, $E_{23}$, $E_{34}$, and so on for the other conductors.

Let the resistance of the conductors be

$$R_{12}, \quad R_{23}, \quad R_{34}, \text{ \&c.}$$

Then, since the conductors are arranged in series so that the same current $C$ flows through each, we have by Ohm's Law,

$$E_{12} = CR_{12}, \quad E_{23} = CR_{23}, \quad E_{34} = CR_{34}. \tag{2}$$

If $E$ is the resultant electromotive force, and $R$ the resultant resistance of the system, we must have by Ohm's Law,

$$E = CR. \tag{3}$$

Now $$E = E_{12}+E_{23}+E_{34}, \tag{4}$$

the sum of the separate electromotive forces,

$$= C(R_{12}+R_{23}+R_{34}) \quad \text{by equations (2).}$$

Comparing this result with (3), we find

$$R = R_{12}+R_{23}+R_{34}. \tag{5}$$

Or, *the resistance of a series of conductors is the sum of the resistances of the conductors taken separately.*

*Potential at any Point of the Series.*

Let $A$ and $C$ be the electrodes of the series, $B$ a point between them, $a$, $c$, and $b$ the potentials of these points respectively. Let $R_1$ be the resistance of the part from $A$ to $B$, $R_2$ that of the part from $B$ to $C$, and $R$ that of the whole from $A$ to $C$, then, since

$$a-b = R_1 C, \quad b-c = R_2 C, \quad \text{and} \quad a-c = RC,$$

the potential at $B$ is

$$b = \frac{R_2 a + R_1 c}{R}, \tag{6}$$

which determines the potential at $B$ when those at $A$ and $C$ are given.

*Resistance of a Multiple Conductor.*

276.] Let a number of conductors $ABZ$, $ACZ$, $ADZ$ be arranged side by side with their extremities in contact with the same two points $A$ and $Z$. They are then said to be arranged in multiple arc.

Let the resistances of these conductors be $R_1$, $R_2$, $R_3$ respectively, and the currents $C_1$, $C_2$, $C_3$, and let the resistance of the multiple conductor be $R$, and the total current $C$. Then, since the potentials at $A$ and $Z$ are the same for all the conductors, they have the same difference, which we may call $E$. We then have

$$E = C_1 R_1 = C_2 R_2 = C_3 R_3 = CR,$$

but
$$C = C_1 + C_2 + C_3,$$

whence
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}. \tag{7}$$

Or, *the reciprocal of the resistance of a multiple conductor is the sum of the reciprocals of the component conductors.*

If we call the reciprocal of the resistance of a conductor the conductivity of the conductor, then we may say that *the conductivity of a multiple conductor is the sum of the conductivities of the component conductors.*

### *Current in any Branch of a Multiple Conductor.*

From the equations of the preceding article, it appears that if $C_1$ is the current in any branch of the multiple conductor, and $R_1$ the resistance of that branch,

$$C_1 = C\frac{R}{R_1}, \tag{8}$$

where $C$ is the total current, and $R$ is the resistance of the multiple conductor as previously determined.

### *Longitudinal Resistance of Conductors of Uniform Section.*

277.] Let the resistance of a cube of a given material to a current parallel to one of its edges be $\rho$, the side of the cube being unit of length, $\rho$ is called the 'specific resistance of that material for unit of volume.'

Consider next a prismatic conductor of the same material whose length is $l$, and whose section is unity. This is equivalent to $l$ cubes arranged in series. The resistance of the conductor is therefore $l\rho$.

Finally, consider a conductor of length $l$ and uniform section $s$. This is equivalent to $s$ conductors similar to the last arranged in multiple arc. The resistance of this conductor is therefore

$$R = \frac{l\rho}{s}.$$

When we know the resistance of a uniform wire we can determine the specific resistance of the material of which it is made if we can measure its length and its section.

The sectional area of small wires is most accurately determined by calculation from the length, weight, and specific gravity of the specimen. The determination of the specific gravity is sometimes inconvenient, and in such cases the resistance of a wire of unit length and unit mass is used as the 'specific resistance per unit of weight.'

If $r$ is this resistance, $l$ the length, and $m$ the mass of a wire, then

$$R = \frac{l^2 r}{m}.$$

*On the Dimensions of the Quantities involved in these Equations.*

278.] The resistance of a conductor is the ratio of the electromotive force acting on it to the current produced. The conductivity of the conductor is the reciprocal of this quantity, or in other words, the ratio of the current to the electromotive force producing it.

Now we know that in the electrostatic system of measurement the ratio of a quantity of electricity to the potential of the conductor on which it is spread is the capacity of the conductor, and is measured by a line. If the conductor is a sphere placed in an unlimited field, this line is the radius of the sphere. The ratio of a quantity of electricity to an electromotive force is therefore a line, but the ratio of a quantity of electricity to a current is the time during which the current flows to transmit that quantity. Hence the ratio of a current to an electromotive force is that of a line to a time, or in other words, it is a velocity.

The fact that the conductivity of a conductor is expressed in the electrostatic system of measurement by a velocity may be verified by supposing a sphere of radius $r$ charged to potential $V$, and then connected with the earth by the given conductor. Let the sphere contract, so that as the electricity escapes through the conductor the potential of the sphere is always kept equal to $V$. Then the charge on the sphere is $rV$ at any instant, and the current is $\frac{d}{dt}(rV)$, but, since $V$ is constant, the current is $\frac{dr}{dt}V$, and the electromotive force through the conductor is $V$.

The conductivity of the conductor is the ratio of the current to the electromotive force, or $\frac{dr}{dt}$, that is, the *velocity* with which the radius of the sphere must diminish in order to maintain the potential constant when the charge is allowed to pass to earth through the conductor.

In the electrostatic system, therefore, the conductivity of a conductor is a velocity, and of the dimensions $[LT^{-1}]$.

The resistance of the conductor is therefore of the dimensions $[L^{-1}T]$.

The specific resistance per unit of volume is of the dimension of $[T]$, and the specific conductivity per unit of volume is of the dimension of $[T^{-1}]$.

The numerical magnitude of these coefficients depends only on the unit of time, which is the same in different countries.

The specific resistance per unit of weight is of the dimensions $[L^{-3}MT]$.

279.] We shall afterwards find that in the electromagnetic system of measurement the resistance of a conductor is expressed by a velocity, so that in this system the dimensions of the resistance of a conductor are $[LT^{-1}]$.

The conductivity of the conductor is of course the reciprocal of this.

The specific resistance per unit of volume in this system is of the dimensions $[L^2T^{-1}]$, and the specific resistance per unit of weight is of the dimensions $[L^{-1}T^{-1}M]$.

### *On Linear Systems of Conductors in general.*

280.] The most general case of a linear system is that of $n$ points, $A_1, A_2, \dots A_n$, connected together in pairs by $\frac{1}{2}n(n-1)$ linear conductors. Let the conductivity (or reciprocal of the resistance) of that conductor which connects any pair of points, say $A_p$ and $A_q$, be called $K_{pq}$, and let the current from $A_p$ to $A_q$ be $C_{pq}$. Let $P_p$ and $P_q$ be the electric potentials at the points $A_p$ and $A_q$ respectively, and let the internal electromotive force, if there be any, along the conductor from $A_p$ to $A_q$ be $E_{pq}$.

The current from $A_p$ to $A_q$ is, by Ohm's Law,

$$C_{pq} = K_{pq}(P_p - P_q + E_{pq}). \tag{1}$$

Among these quantities we have the following sets of relations :

The conductivity of a conductor is the same in either direction,

or $$K_{pq} = K_{qp}. \tag{2}$$

The electromotive force and the current are directed quantities , so that

$$E_{pq} = -E_{qp}, \quad \text{and} \quad C_{pq} = -C_{qp}. \tag{3}$$

Let $P_1, P_2, \ldots P_n$ be the potentials at $A_1, A_2, \ldots A_n$ respectively, and let $Q_1, Q_2, \ldots Q_n$ be the quantities of electricity which enter the system in unit of time at each of these points respectively. These are necessarily subject to the condition of 'continuity'

$$Q_1 + Q_2 \cdots + Q_n = 0, \tag{4}$$

since electricity can neither be indefinitely accumulated nor produced within the system.

The condition of 'continuity' at any point $A_p$ is

$$Q_p = C_{p1} + C_{p2} + \&\text{c.} + C_{pn}. \tag{5}$$

Substituting the values of the currents in terms of equation (1), this becomes

$$Q_p = (K_{p1} + K_{p2} + \&\text{c.} + K_{pn})\, P_p - (K_{p1} P_1 + K_{p2} P_2 + \&\text{c.} + K_{pn} P_n) \\ + (K_{pq}\, E_{p1} + \&\text{c.} + K_{pn}\, E_{pn}). \tag{6}$$

The symbol $K_{pp}$ does not occur in this equation. Let us therefore give it the value

$$K_{pp} = -(K_{p1} + K_{p2} + \&\text{c.} + K_{pn}); \tag{7}$$

that is, let $K_{pp}$ be a quantity equal and opposite to the sum of all the conductivities of the conductors which meet in $A_p$. We may then write the condition of continuity for the point $A_p$,

$$K_{p1} P_1 + K_{p2} P_2 + \&\text{c.} + K_{pp} P_p + \&\text{c.} + K_{pn} P_n \\ = K_{p1} E_{p1} + \&\text{c.} + K_{pn} E_{pn} - Q_p. \tag{8}$$

By substituting 1, 2, &c. $n$ for $p$ in this equation we shall obtain $n$ equations of the same kind from which to determine the $n$ potentials $P_1$, $P_2$, &c., $P_n$.

Since, however, there is a necessary condition, (4), connecting the values of $Q$, there will be only $n-1$ independent equations. These will be sufficient to determine the differences of the potentials of the points, but not to determine the absolute potential of any. This, however, is not required to calculate the currents in the system.

If we denote by $D$ the determinant

$$D = \begin{vmatrix} K_{11}, & K_{12}, & \ldots\ldots K_{1(n-1)}, \\ K_{21}, & K_{22}, & \ldots\ldots K_{2(n-1)}, \\ - - - & - - - & - - - - \\ K_{(n-1)1}, & K_{(n-1)2}, & \ldots\ldots K_{(n-1)(n-1)}; \end{vmatrix} \tag{9}$$

and by $D_{pq}$, the minor of $K_{pq}$, we find for the value of $P_p - P_n$,

$$(P_p - P_n)\, D = (K_{12} E_{12} + \&\text{c.} - Q_1) D_{p1} + (K_{21} E_{21} + \&\text{c.} - Q_2) D_{p2} + \&\text{c.} \\ + (K_{q1}\, E_{q1} + \&\text{c.} + K_{qn}\, E_{qn} - Q_q)\, D_{pq} + \&\text{c.} \tag{10}$$

In the same way the excess of the potential of any other point, say $A_q$, over that of $A_n$ may be determined. We may then determine the current between $A_p$ and $A_q$ from equation (1), and so solve the problem completely.

281.] We shall now demonstrate a reciprocal property of any two conductors of the system, answering to the reciprocal property we have already demonstrated for statical electricity in Art. 88.

The coefficient of $Q_q$ in the expression for $P_p$ is $\frac{D_{pq}}{D}$. That of $Q_p$ in the expression for $P_q$ is $\frac{D_{qp}}{D}$.

Now $D_{pq}$ differs from $D_{qp}$ only by the substitution of the symbols such as $K_{qp}$ for $K_{pq}$. But, by equation (2), these two symbols are equal, since the conductivity of a conductor is the same both ways. Hence

$$D_{pq} = D_{qp}. \tag{11}$$

It follows from this that the part of the potential at $A_p$ arising from the introduction of a unit current at $A_q$ is equal to the part of the potential at $A_q$ arising from the introduction of a unit current at $A_p$.

We may deduce from this a proposition of a more practical form.

Let $A$, $B$, $C$, $D$ be any four points of the system, and let the effect of a current $Q$, made to enter the system at $A$ and leave it at $B$, be to make the potential at $C$ exceed that at $D$ by $P$. Then, if an equal current $Q$ be made to enter the system at $C$ and leave it at $D$, the potential at $A$ will exceed that at $B$ by the same quantity $P$.

We may also establish a property of a similar kind relating to the effect of the internal electromotive force $E_{rs}$, acting along the conductor which joins the points $A_r$ and $A_s$ in producing an external electromotive force on the conductor from $A_p$ to $A_q$, that is to say, a difference of potentials $P_p - P_q$. For since

$$E_{rs} = -E_{sr},$$

the part of the value of $P_p$ which depends on this electromotive force is

$$\frac{1}{D}(D_{pr} - D_{ps})E_{rs},$$

and the part of the value of $P_q$ is

$$\frac{1}{D}(D_{qr} - D_{qs})E_{rs}.$$

Therefore the coefficient of $E_{rs}$ in the value of $P_p - P_q$ is

$$\frac{1}{D}\{D_{pr} + D_{qs} - D_{ps} - D_{qr}\}. \tag{12}$$

This is identical with the coefficient of $E_{pq}$ in the value of $P_r - P_s$.

If therefore an electromotive force $E$ be introduced, acting in the conductor from $A$ to $B$, and if this causes the potential at $C$ to exceed that at $D$ by $P$, then the same electromotive force $E$ introduced into the conductor from $C$ to $D$ will cause the potential at $A$ to exceed that at $B$ by the same quantity $P$.

The electromotive force $E$ may be that of a voltaic battery introduced between the points named, care being taken that the resistance of the conductor is the same before and after the introduction of the battery.

282.] If $$D_{pr}+D_{qs}-D_{ps}-D_{qr} = 0, \tag{13}$$
the conductor $A_p A_q$ is said to be *conjugate* to $A_r A_s$, and we have seen that this relation is reciprocal.

An electromotive force in one of two conjugate conductors produces no electromotive force or current along the other. We shall find the practical application of this principle in the case of the electric bridge.

The theory of conjugate conductors has been investigated by Kirchhoff, who has stated the conditions of a linear system in the following manner, in which the consideration of the potential is avoided.

(1) (Condition of 'continuity.') At any point of the system the sum of all the currents which flow towards that point is zero.

(2) In any complete circuit formed by the conductors the sum of the electromotive forces taken round the circuit is equal to the sum of the products of the current in each conductor multiplied by the resistance of that conductor.

We obtain this result by adding equations of the form (1) for the complete circuit, when the potentials necessarily disappear.

### *Heat Generated in the System.*

283.] The mechanical equivalent of the quantity of heat generated in a conductor whose resistance is $R$ by a current $C$ in unit of time is, by Art. 242, $$JH = RC^2. \tag{14}$$

We have therefore to determine the sum of such quantities as $RC^2$ for all the conductors of the system.

For the conductor from $A_p$ to $A_q$ the conductivity is $K_{pq}$, and the resistance $R_{pq}$, where $$K_{pq}.R_{pq} = 1. \tag{15}$$

The current in this conductor is, according to Ohm's Law,
$$C_{pq} = K_{pq}(P_p-P_q). \tag{16}$$

We shall suppose, however, that the value of the current is not that given by Ohm's Law, but $X_{pq}$, where

$$X_{pq} = C_{pq} + Y_{pq}. \qquad (17)$$

To determine the heat generated in the system we have to find the sum of all the quantities of the form

$$R_{pq} X^2_{pq},$$

or
$$JH = \Sigma \{R_{pq} C^2_{pq} + 2 R_{pq} C_{pq} Y_{pq} + R_{pq} Y^2_{pq}\}. \qquad (18)$$

Giving $C_{pq}$ its value, and remembering the relation between $K_{pq}$ and $R_{pq}$, this becomes

$$\Sigma (P_p - P_q)(C_{pq} + 2 Y_{pq}) + R_{pq} Y^2_{pq}. \qquad (19)$$

Now since both $C$ and $X$ must satisfy the condition of continuity at $A_p$, we have

$$Q_p = C_{p1} + C_{p2} + \&c. + C_{pn}, \qquad (20)$$

$$Q_p = X_{p1} + X_{p2} + \&c. + X_{pn}, \qquad (21)$$

therefore
$$0 = Y_{p1} + Y_{p2} + \&c. + Y_{pn}. \qquad (22)$$

Adding together therefore all the terms of (19), we find

$$\Sigma (R_{pq} X^2_{pq}) = \Sigma P_p Q_p + \Sigma R_{pq} Y^2_{pq}. \qquad (23)$$

Now since $R$ is always positive and $Y^2$ is essentially positive, the last term of this equation must be essentially positive. Hence the first term is a minimum when $Y$ is zero in every conductor, that is, when the current in every conductor is that given by Ohm's Law.

Hence the following theorem:

284.] In any system of conductors in which there are no internal electromotive forces the heat generated by currents distributed in accordance with Ohm's Law is less than if the currents had been distributed in any other manner consistent with the actual conditions of supply and outflow of the current.

The heat actually generated when Ohm's Law is fulfilled is mechanically equivalent to $\Sigma P_p Q_q$, that is, to the sum of the products of the quantities of electricity supplied at the different external electrodes, each multiplied by the potential at which it is supplied.

# CHAPTER VII.

## CONDUCTION IN THREE DIMENSIONS.

### *Notation of Electric Currents.*

285.] At any point let an element of area $dS$ be taken normal to the axis of $x$, and let $Q$ units of electricity pass across this area from the negative to the positive side in unit of time, then, if $\frac{Q}{dS}$ becomes ultimately equal to $u$ when $dS$ is indefinitely diminished, $u$ is said to be the Component of the electric current in the direction of $x$ at the given point.

In the same way we may determine $v$ and $w$, the components of the current in the directions of $y$ and $z$ respectively.

286.] To determine the component of the current in any other direction $OR$ through the given point $O$.

Let $l$, $m$, $n$ be the direction-cosines of $OR$, then cutting off from the axes of $x$, $y$, $z$ portions equal to

$$\frac{r}{l}, \ \frac{r}{m}, \text{ and } \frac{r}{n}$$

respectively at $A$, $B$ and $C$, the triangle $ABC$ will be normal to $OR$.

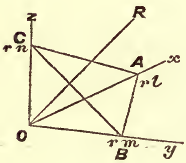



Fig. 22.

The area of this triangle $ABC$ will be

$$dS = \tfrac{1}{2}\frac{r^2}{lmn},$$

and by diminishing $r$ this area may be diminished without limit.

The quantity of electricity which leaves the tetrahedron $ABCO$ by the triangle $ABC$ must be equal to that which enters it through the three triangles $OBC$, $OCA$, and $OAB$.

The area of the triangle $OBC$ is $\frac{1}{2}\frac{r^2}{mn}$, and the component of the current normal to its plane is $u$, so that the quantity which enters through this triangle is $\frac{1}{2} r^2 \frac{u}{mn}$.

The quantities which enter through the triangles $OCA$ and $OAB$ respectively are

$$\frac{1}{2} r^2 \frac{v}{nl}, \quad \text{and} \quad \frac{1}{2} r^2 \frac{w}{lm}.$$

If $\gamma$ is the component of the velocity in the direction $OR$, then the quantity which leaves the tetrahedron through $ABC$ is

$$\frac{1}{2} r \frac{\gamma}{lmn}.$$

Since this is equal to the quantity which enters through the three other triangles,

$$\frac{1}{2} \frac{r^2 \gamma}{lmn} = \frac{1}{2} r^2 \left\{ \frac{u}{mn} + \frac{v}{nl} + \frac{w}{lm} \right\};$$

multiplying by $\frac{2\,lmn}{r^2}$, we get

$$\gamma = lu + mv + nw. \tag{1}$$

If we put $$u^2 + v^2 + w^2 = \Gamma^2,$$

and make $l'$, $m'$, $n'$ such that

$$u = l'\Gamma, \qquad v = m'\Gamma, \quad \text{and} \quad w = n'\Gamma;$$

then $$\gamma = \Gamma (ll' + mm' + nn'). \tag{2}$$

Hence, if we define the resultant current as a vector whose magnitude is $\Gamma$, and whose direction-cosines are $l'$, $m'$, $n'$, and if $\gamma$ denotes the current resolved in a direction making an angle $\theta$ with that of the resultant current, then

$$\gamma = \Gamma \cos\theta; \tag{3}$$

shewing that the law of resolution of currents is the same as that of velocities, forces, and all other vectors.

287.] To determine the condition that a given surface may be a surface of flow.

Let $$F(x, y, z) = \lambda \tag{4}$$

be the equation of a family of surfaces any one of which is given by making $\lambda$ constant, then, if we make

$$\overline{\frac{d\lambda}{dx}}\Big|^2 + \overline{\frac{d\lambda}{dy}}\Big|^2 + \overline{\frac{d\lambda}{dz}}\Big|^2 = \frac{1}{N^2}, \tag{5}$$

the direction-cosines of the normal, reckoned in the direction in which $\lambda$ increases, are

$$l = N\frac{d\lambda}{dx}, \qquad m = N\frac{d\lambda}{dy}, \qquad n = N\frac{d\lambda}{dz}. \tag{6}$$

Hence, if $\gamma$ is the component of the current normal to the surface,

$$\gamma = N\left\{u\frac{d\lambda}{dx} + v\frac{d\lambda}{dy} + w\frac{d\lambda}{dz}\right\}. \tag{7}$$

If $\gamma = 0$ there will be no current through the surface, and the surface may be called a Surface of Flow, because the lines of motion are in the surface.

288.] The equation of a surface of flow is therefore

$$u\frac{d\lambda}{dx} + v\frac{d\lambda}{dy} + w\frac{d\lambda}{dz} = 0. \tag{8}$$

If this equation is true for all values of $\lambda$, all the surfaces of the family will be surfaces of flow.

289.] Let there be another family of surfaces, whose parameter is $\lambda'$, then, if these are also surfaces of flow, we shall have

$$u\frac{d\lambda'}{dx} + v\frac{d\lambda'}{dy} + w\frac{d\lambda'}{dz} = 0. \tag{9}$$

If there is a third family of surfaces of flow, whose parameter is $\lambda''$, then

$$u\frac{d\lambda''}{dx} + v\frac{d\lambda''}{dy} + w\frac{d\lambda''}{dz} = 0. \tag{10}$$

Eliminating between these three equations, $u$, $v$, and $w$ disappear together, and we find

$$\begin{vmatrix} \frac{d\lambda}{dx}, & \frac{d\lambda}{dy}, & \frac{d\lambda}{dz} \\ \frac{d\lambda'}{dx}, & \frac{d\lambda'}{dy}, & \frac{d\lambda'}{dz} \\ \frac{d\lambda''}{dx}, & \frac{d\lambda''}{dy}, & \frac{d\lambda''}{dz} \end{vmatrix} = 0; \tag{11}$$

or $$\lambda'' = \phi(\lambda, \lambda'); \tag{12}$$

that is, $\lambda''$ is some function of $\lambda$ and $\lambda'$.

290.] Now consider the four surfaces whose parameters are $\lambda$, $\lambda+\delta\lambda$, $\lambda'$, and $\lambda'+\delta\lambda'$. These four surfaces enclose a quadrilateral tube, which we may call the tube $\delta\lambda.\delta\lambda'$. Since this tube is bounded by surfaces across which there is no flow, we may call it a Tube of Flow. If we take any two sections across the tube, the quantity which enters the tube at one section must be equal to the quantity which leaves it at the other, and since this quantity is therefore the same for every section of the tube, let us call it $L\,\delta\lambda.\delta\lambda'$ where $L$ is a function of $\lambda$ and $\lambda'$, the parameters which determine the particular tube.

291.] If $\delta S$ denotes the section of a tube of flow by a plane normal to $x$, we have by the theory of the change of the independent variables,

$$\delta\lambda.\delta\lambda' = \delta S\left(\frac{d\lambda}{dy}\frac{d\lambda'}{dz} - \frac{d\lambda}{dz}\frac{d\lambda'}{dy}\right), \tag{13}$$

and by the definition of the components of the current

$$u\,\delta S = L\,\delta\lambda.\delta\lambda'. \tag{14}$$

$$\left.\begin{aligned} &\text{Hence} & u &= L\left(\frac{d\lambda}{dy}\frac{d\lambda'}{dz} - \frac{d\lambda}{dz}\frac{d\lambda'}{dy}\right). \\ &\text{Similarly} & v &= L\left(\frac{d\lambda}{dz}\frac{d\lambda'}{dx} - \frac{d\lambda}{dx}\frac{d\lambda'}{dz}\right), \\ & & w &= L\left(\frac{d\lambda}{dx}\frac{d\lambda'}{dy} - \frac{d\lambda}{dy}\frac{d\lambda'}{dx}\right). \end{aligned}\right\} \tag{15}$$

292.] It is always possible when one of the functions $\lambda$ or $\lambda'$ is known, to determine the other so that $L$ may be equal to unity. For instance, let us take the plane of $yz$, and draw upon it a series of equidistant lines parallel to $y$, to represent the sections of the family $\lambda'$ by this plane. In other words, let the function $\lambda'$ be determined by the condition that when $x = 0$ $\lambda' = z$. If we then make $L = 1$, and therefore (when $x = 0$)

$$\lambda = \int u\,dy\,;$$

then in the plane ($x = 0$) the amount of electricity which passes through any portion will be

$$\iint u\,dy\,dz = \iint d\lambda\,d\lambda'. \tag{16}$$

Having determined the nature of the sections of the surfaces of flow by the plane of $yz$, the form of the surfaces elsewhere is determined by the conditions (8) and (9). The two functions $\lambda$ and $\lambda'$ thus determined are sufficient to determine the current at every point by equations (15), unity being substituted for $L$.

*On Lines of Flow.*

293.] Let a series of values of $\lambda$ and of $\lambda'$ be chosen, the successive differences in each series being unity. The two series of surfaces defined by these values will divide space into a system of quadrilateral tubes through each of which there will be a unit current. By assuming the unit sufficiently small, the details of the current may be expressed by these tubes with any desired amount of minuteness. Then if any surface be drawn cutting the system of tubes, the quantity of the current which passes through this surface will be expressed by the *number* of tubes which cut it, since each tube carries unity of current.

The actual intersections of the surfaces may be called Lines of Flow. When the unit is taken sufficiently small, the number of lines of flow which cut a surface is approximately equal to the number of tubes of flow which cut it, so that we may consider the lines of flow as expressing not only the *direction* of the current but its *strength*, since each line of flow through a given section corresponds to a unit current.

*On Current-Sheets and Current-Functions.*

294.] A stratum of a conductor contained between two consecutive surfaces of flow of one system, say that of $\lambda'$, is called a Current-Sheet. The tubes of flow within this sheet are determined by the function $\lambda$. If $\lambda_A$ and $\lambda_P$ denote the values of $\lambda$ at the points $A$ and $P$ respectively, then the current from right to left across any line drawn on the sheet from $A$ to $P$ is $\lambda_P - \lambda_A$. If $AP$ be an element, $ds$, of a curve drawn on the sheet, the current which crosses this element from right to left is

$$\frac{d\lambda}{ds}\, ds.$$

This function $\lambda$, from which the distribution of the current in the sheet can be completely determined, is called the Current-Function.

Any thin sheet of metal or conducting matter bounded on both sides by air or some other non-conducting medium may be treated as a current-sheet, in which the distribution of the current may be expressed by means of a current-function. See Art. 647.

*Equation of 'Continuity.'*

295.] If we differentiate the three equations (15) with respect to $x$, $y$, $z$ respectively, remembering that $L$ is a function of $\lambda$ and $\lambda'$, we find

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \qquad (17)$$

The corresponding equation in Hydrodynamics is called the Equation of 'Continuity.' The continuity which it expresses is the continuity of existence, that is, the fact that a material substance cannot leave one part of space and arrive at another, without going through the space between. It cannot simply vanish in the one place and appear in the other, but it must travel along a continuous path, so that if a closed surface be drawn, including the one place and excluding the other, a material substance in passing from the one place to the other must go through the closed surface. The most general form of the equation in hydrodynamics is

$$\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} + \frac{d\rho}{dt} = 0; \qquad (18)$$

where $\rho$ signifies the ratio of the quantity of the substance to the volume it occupies, that volume being in this case the differential element of volume, and $(\rho u)$, $(\rho v)$, and $(\rho w)$ signify the ratio of the quantity of the substance which crosses an element of area in unit of time to that area, these areas being normal to the axes of $x$, $y$, and $z$ respectively. Thus understood, the equation is applicable to any material substance, solid or fluid, whether the motion be continuous or discontinuous, provided the existence of the parts of that substance is continuous. If anything, though not a substance, is subject to the condition of continuous existence in time and space, the equation will express this condition. In other parts of Physical Science, as, for instance, in the theory of electric and magnetic quantities, equations of a similar form occur. We shall call such equations 'equations of continuity' to indicate their form, though we may not attribute to these quantities the properties of matter, or even continuous existence in time and space.

The equation (17), which we have arrived at in the case of electric currents, is identical with (18) if we make $\rho = 1$, that is, if we suppose the substance homogeneous and incompressible. The equation, in the case of fluids, may also be established by either of the modes of proof given in treatises on Hydrodynamics. In one of these we trace the course and the deformation of a certain element of the fluid as it moves along. In the other, we fix our attention on an element of space, and take account of all that enters or leaves it. The former of these methods cannot be applied to electric currents, as we do not know the velocity with which the electricity passes through the body, or even whether it moves in the positive or the negative direction of the current. All that we know is the algebraical value of the quantity which crosses unit of area in unit of time, a quantity corresponding to $(\rho u)$ in the equation (18). We have no means of ascertaining the value of either of the factors $\rho$ or $u$, and therefore we cannot follow a particular portion of electricity in its course through the body. The other method of investigation, in which we consider what passes through the walls of an element of volume, is applicable to electric currents, and is perhaps preferable in point of form to that which we have given, but as it may be found in any treatise on Hydrodynamics we need not repeat it here.

*Quantity of Electricity which passes through a given Surface.*

296.] Let $\Gamma$ be the resultant current at any point of the surface. Let $dS$ be an element of the surface, and let $\epsilon$ be the angle between $\Gamma$ and the normal to the surface, then the total current through the surface will be

$$\iint \Gamma \cos \epsilon \, dS,$$

the integration being extended over the surface.

As in Art. 21, we may transform this integral into the form

$$\iint \Gamma \cos \epsilon \, dS = \iiint \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) dx\, dy\, dz \tag{19}$$

in the case of any closed surface, the limits of the triple integration being those included by the surface. This is the expression for the total efflux from the closed surface. Since in all cases of steady currents this must be zero whatever the limits of the integration, the quantity under the integral sign must vanish, and we obtain in this way the equation of continuity (17).

# CHAPTER VIII.

## RESISTANCE AND CONDUCTIVITY IN THREE DIMENSIONS.

### *On the most General Relations between Current and Electromotive Force.*

297.] LET the components of the current at any point be $u, v, w$.

Let the components of the electromotive force be $X, Y, Z$.

The electromotive force at any point is the resultant force on a unit of positive electricity placed at that point. It may arise (1) from electrostatic action, in which case if $V$ is the potential,

$$X = -\frac{dV}{dx}, \quad Y = -\frac{dV}{dy}, \quad Z = -\frac{dV}{dz}; \tag{1}$$

or (2) from electromagnetic induction, the laws of which we shall afterwards examine; or (3) from thermoelectric or electrochemical action at the point itself, tending to produce a current in a given direction.

We shall in general suppose that $X, Y, Z$ represent the components of the actual electromotive force at the point, whatever be the origin of the force, but we shall occasionally examine the result of supposing it entirely due to variation of potential.

By Ohm's Law the current is proportional to the electromotive force. Hence $X, Y, Z$ must be linear functions of $u, v, w$. We may therefore assume as the equations of Resistance,

$$\left.\begin{aligned} X &= R_1 u + Q_3 v + P_2 w, \\ Y &= P_3 u + R_2 v + Q_1 w, \\ Z &= Q_2 u + P_1 v + R_3 w. \end{aligned}\right\} \tag{2}$$

We may call the coefficients $R$ the coefficients of longitudinal resistance in the directions of the axes of coordinates.

The coefficients $P$ and $Q$ may be called the coefficients of transverse resistance. They indicate the electromotive force in one direction required to produce a current in a different direction.

If we were at liberty to assume that a solid body may be treated as a system of linear conductors, then, from the reciprocal property (Art. 281) of any two conductors of a linear system, we might shew that the electromotive force along $z$ required to produce a unit current parallel to $y$ must be equal to the electromotive force along $y$ required to produce a unit current parallel to $z$. This would shew that $P_1 = Q_1$, and similarly we should find $P_2 = Q_2$, and $P_3 = Q_3$. When these conditions are satisfied the system of coefficients is said to be Symmetrical. When they are not satisfied it is called a Skew system.

We have great reason to believe that in every actual case the system is symmetrical, but we shall examine some of the consequences of admitting the possibility of a skew system.

298.] The quantities $u, v, w$ may be expressed as linear functions of $X, Y, Z$ by a system of equations, which we may call Equations of Conductivity,

$$\left.\begin{aligned} u &= r_1 X + p_3 Y + q_2 Z, \\ v &= q_3 X + r_2 Y + p_1 Z, \\ w &= p_2 X + q_1 Y + r_3 Z; \end{aligned}\right\} \tag{3}$$

we may call the coefficients $r$ the coefficients of Longitudinal conductivity, and $p$ and $q$ those of Transverse conductivity.

The coefficients of resistance are inverse to those of conductivity. This relation may be defined as follows:

Let $[PQR]$ be the determinant of the coefficients of resistance, and $[pqr]$ that of the coefficients of conductivity, then

$$[PQR] = P_1 P_2 P_3 + Q_1 Q_2 Q_3 + R_1 R_2 R_3 - P_1 Q_1 R_1 - P_2 Q_2 R_2 - P_3 Q_3 R_3, \tag{4}$$

$$[pqr] = p_1 p_2 p_3 + q_1 q_2 q_3 + r_1 r_2 r_3 - p_1 q_1 r_1 - p_2 q_2 r_2 - p_3 q_3 r_3, \tag{5}$$

$$[PQR]\,[pqr] = 1, \tag{6}$$

$$[PQR]\,p_1 = (P_2 P_3 - Q_1 R_1), \quad [pqr]\,P_1 = (p_2 p_3 - q_1 r_1), \tag{7}$$

&c. &c.

The other equations may be formed by altering the symbols $P, Q, R, p, q, r$, and the suffixes 1, 2, 3 in cyclical order.

### *Rate of Generation of Heat.*

299.] To find the work done by the current in unit of time in overcoming resistance, and so generating heat, we multiply the components of the current by the corresponding components of the electromotive force. We thus obtain the following expressions for $W$, the quantity of work expended in unit of time:

$$W = Xu + Yv + Zw; \tag{8}$$
$$= R_1 u^2 + R_2 v^2 + R_3 w^2 + (P_1 + Q_1) vw + (P_2 + Q_2) wu + (P_3 + Q_3) uv; \tag{9}$$
$$= r_1 X^2 + r_2 Y^2 + r_3 Z^2 + (p_1 + q_1) YZ + (p_2 + q_2) ZX + (p_3 + q_3) XY. \tag{10}$$

By a proper choice of axes, either of the two latter equations may be deprived of the terms involving the products of $u, v, w$ or of $X, Y, Z$. The system of axes, however, which reduces $W$ to the form

$$R_1 u^2 + R_2 v^2 + R_3 w^2$$

is not in general the same as that which reduces it to the form

$$r_1 X^2 + r_2 Y^2 + r_3 Z^2.$$

It is only when the coefficients $P_1, P_2, P_3$ are equal respectively to $Q_1, Q_2, Q_3$ that the two systems of axes coincide.

If with Thomson * we write

$$\left.\begin{array}{lll} & P = S + T, & Q = S - T; \\ \text{and} & p = s + t, & q = s - t; \end{array}\right\} \tag{11}$$

then we have

$$\left.\begin{array}{l} [PQR] = R_1 R_2 R_3 + 2 S_1 S_2 S_3 - S_1^2 R_1 - S_2^2 R_2 - S_3^2 R_3 \\ \qquad + 2 (S_1 T_2 T_3 + S_2 T_3 T_1 + S_3 T_1 T_2) + R_1 T_1^2 + R_2 T_2^2 + R_3 T_3^2; \end{array}\right\} \tag{12}$$

and

$$\left.\begin{array}{l} [PQR]\, r_1 = R_2 R_3 - S_1^2 + T_1^2, \\ [PQR]\, s_1 = T_2 T_3 + S_2 S_3 - R_1 S_1, \\ [PQR]\, t_1 = -R_1 T_1 + S_2 T_3 + S_3 T_2. \end{array}\right\} \tag{13}$$

If therefore we cause $S_1, S_2, S_3$ to disappear, $s_1$ will not also disappear unless the coefficients $T$ are zero.

### *Condition of Stability.*

300.] Since the equilibrium of electricity is stable, the work spent in maintaining the current must always be positive. The conditions that $W$ must be positive are that the three coefficients $R_1, R_2, R_3$, and the three expressions

$$\left.\begin{array}{l} 4 R_2 R_3 - (P_1 + Q_1)^2, \\ 4 R_3 R_1 - (P_2 + Q_2)^2, \\ 4 R_1 R_2 - (P_3 + Q_3)^2, \end{array}\right\} \tag{14}$$

must all be positive.

There are similar conditions for the coefficients of conductivity.

* *Trans. R. S. Edin.*, 1853–4, p. 165.

*Equation of Continuity in a Homogeneous Medium.*

301.] If we express the components of the electromotive force as the derivatives of the potential $V$, the equation of continuity

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \tag{15}$$

becomes in a homogeneous medium

$$r_1\frac{d^2V}{dx^2}+r_2\frac{d^2V}{dy^2}+r_3\frac{d^2V}{dz^2}+2s_1\frac{d^2V}{dy\,dz}+2s_2\frac{d^2V}{dz\,dx}+2s_3\frac{d^2V}{dx\,dy}=0. \tag{16}$$

If the medium is not homogeneous there will be terms arising from the variation of the coefficients of conductivity in passing from one point to another.

This equation corresponds to Laplace's equation in an isotropic medium.

302.] If we put

$$[rs] = r_1r_2r_3+2s_1s_2s_3-r_1s_1^2-r_2s_2^2-r_3s_3^2, \tag{17}$$

and $$[AB] = A_1A_2A_3+2B_1B_2B_3-A_1B_1^2-A_2B_2^2-A_3B_3^2, \tag{18}$$

where $$\left.\begin{aligned}[rs]A_1 &= r_2r_3-s_1^2,\\ [rs]B_1 &= s_2s_3-r_1s_1,\\ &\text{- - - - - - -}\end{aligned}\right\} \tag{19}$$

and so on, the system $A$, $B$ will be inverse to the system $r$, $s$, and if we make

$$A_1x^2+A_2y^2+A_3z^2+2B_1yz+2B_2zx+2B_3xy = [AB]\rho^2, \tag{20}$$

we shall find that

$$V=\frac{c}{4\pi}\frac{1}{\rho} \tag{21}$$

is a solution of the equation.

In the case in which the coefficients $T$ are zero, the coefficients $A$ and $B$ become identical with $R$ and $S$. When $T$ exists this is not the case.

In the case therefore of electricity flowing out from a centre in an infinite homogeneous, but not isotropic, medium, the equipotential surfaces are ellipsoids, for each of which $\rho$ is constant. The axes of these ellipsoids are in the directions of the principal axes of conductivity, and these do not coincide with the principal axes of resistance unless the system is symmetrical.

By a transformation of this equation we may take for the axes of $x$, $y$, $z$ the principal axes of conductivity. The coefficients of the forms $s$ and $B$ will then be reduced to zero, and each coefficient of the form $A$ will be the reciprocal of the corresponding coefficient of the form $r$. The expression for $\rho$ will be

$$\frac{x^2}{r_1} + \frac{y^2}{r_2} + \frac{z^2}{r_3} = \frac{\rho^2}{r_1 r_2 r_3}. \tag{22}$$

303.] The theory of the complete system of equations of resistance and of conductivity is that of linear functions of three variables, and it is exemplified in the theory of Strains *, and in other parts of physics. The most appropriate method of treating it is that by which Hamilton and Tait treat a linear and vector function of a vector. We shall not, however, expressly introduce Quaternion notation.

The coefficients $T_1, T_2, T_3$ may be regarded as the rectangular components of a vector $T$, the absolute magnitude and direction of which are fixed in the body, and independent of the direction of the axes of reference. The same is true of $t_1, t_2, t_3$, which are the components of another vector $t$.

The vectors $T$ and $t$ do not in general coincide in direction.

Let us now take the axis of $z$ so as to coincide with the vector $T$, and transform the equations of resistance accordingly. They will then have the form

$$\left.\begin{aligned} X &= R_1 u + S_3 v + S_2 w - Tv, \\ Y &= S_3 u + R_2 v + S_1 w + Tu, \\ Z &= S_2 u + S_1 v + R_3 w. \end{aligned}\right\} \tag{23}$$

It appears from these equations that we may consider the electromotive force as the resultant of two forces, one of them depending only on the coefficients $R$ and $S$, and the other depending on $T$ alone. The part depending on $R$ and $S$ is related to the current in the same way that the perpendicular on the tangent plane of an ellipsoid is related to the radius vector. The other part, depending on $T$, is equal to the product of $T$ into the resolved part of the current perpendicular to the axis of $T$, and its direction is perpendicular to $T$ and to the current, being always in the direction in which the resolved part of the current would lie if turned 90° in the positive direction round $T$.

Considering the current and $T$ as vectors, the part of the electromotive force due to $T$ is the vector part of the product, $T \times$ current.

The coefficient $T$ may be called the Rotatory coefficient. We

* See Thomson and Tait's *Natural Philosophy*, § 154.

have reason to believe that it does not exist in any known substance. It should be found, if anywhere, in magnets, which have a polarization in one direction, probably due to a rotational phenomenon in the substance.

304.] Let us next consider the general characteristic equation of $V$,

$$\frac{d}{dx}\left(r_1\frac{dV}{dx}+p_3\frac{dV}{dy}+q_2\frac{dV}{dz}\right)+\frac{d}{dy}\left(q_3\frac{dV}{dx}+r_2\frac{dV}{dy}+p_1\frac{dV}{dz}\right)$$
$$+\frac{d}{dz}\left(p_2\frac{dV}{dx}+q_1\frac{dV}{dy}+r_3\frac{dV}{dz}\right)+4\pi\rho=0, \quad (24)$$

where the coefficients of conductivity $p, q, r$ may have any positive values, continuous or discontinuous, at any point of space, and $V$ vanishes at infinity.

Also, let $a, b, c$ be three functions of $x, y, z$ satisfying the condition

$$\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}+4\pi\rho=0; \quad (25)$$

and let

$$\left.\begin{aligned} a &= r_1\frac{dV}{dx}+p_3\frac{dV}{dy}+q_2\frac{dV}{dz}+u,\\ b &= q_3\frac{dV}{dx}+r_2\frac{dV}{dy}+p_1\frac{dV}{dz}+v,\\ c &= p_2\frac{dV}{dx}+q_1\frac{dV}{dy}+r_3\frac{dV}{dz}+w.\end{aligned}\right\} \quad (26)$$

Finally, let the triple-integral

$$W=\iiint\{R_1a^2+R_2b^2+R_3c^2$$
$$+(P_1+Q_1)\,bc+(P_2+Q_2)\,ca+(P_3+Q_3)\,ab\}\,dx\,dy\,dz \quad (27)$$

be extended over spaces bounded as in the enunciation of Art. 97, where the coefficients $P, Q, R$ are the coefficients of resistance.

Then $W$ will have a unique minimum value when $a, b, c$ are such that $u, v, w$ are each everywhere zero, and the characteristic equation (24) will therefore, as shewn in Art. 98, have one and only one solution.

In this case $W$ represents the mechanical equivalent of the heat generated by the current in the system in unit of time, and we have to prove that there is one way, and one only, of making this heat a minimum, and that the distribution of currents $(abc)$ in that case is that which arises from the solution of the characteristic equation of the potential $V$.

The quantity $W$ may be written in terms of equations (25) and (26),

$$W = \iiint \Big\{ r_1 \overline{\frac{dV}{dx}}\Big|^2 + r_2 \overline{\frac{dV}{dy}}\Big|^2 + r_3 \overline{\frac{dV}{dz}}\Big|^2$$
$$+ (p_1+q_1)\frac{dV}{dy}\frac{dV}{dz} + (p_2+q_2)\frac{dV}{dz}\frac{dV}{dx} + (p_3+q_3)\frac{dV}{dx}\frac{dV}{dy} \Big\} dx\,dy\,dz$$
$$+ \iiint \{ R_1^2 u^2 + R_2^2 v^2 + R_3^2 w^2$$
$$+ (P_1+Q_1)vw + (P_2+Q_2)wu + (P_3+Q_3)uv \}\, dx\,dy\,dz$$
$$+ \iiint \Big( u\frac{dV}{dx} + v\frac{dV}{dy} + w\frac{dV}{dz} \Big) dx\,dy\,dz. \qquad (28)$$

Since $$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \qquad (29)$$

the third term of $W$ vanishes within the limits.

The second term, being the rate of conversion of electrical energy into heat, is also essentially positive. Its minimum value is zero, and this is attained only when $u$, $v$, and $w$ are everywhere zero.

The value of $W$ is in this case reduced to the first term, and is then a minimum and a unique minimum.

305.] As this proposition is of great importance in the theory of electricity, it may be useful to present the following proof of the most general case in a form free from analytical operations.

Let us consider the propagation of electricity through a conductor of any form, homogeneous or heterogeneous.

Then we know that

(1) If we draw a line along the path and in the direction of the electric current, the line must pass from places of high potential to places of low potential.

(2) If the potential at every point of the system be altered in a given uniform ratio, the currents will be altered in the same ratio, according to Ohm's Law.

(3) If a certain distribution of potential gives rise to a certain distribution of currents, and a second distribution of potential gives rise to a second distribution of currents, then a third distribution in which the potential is the sum or difference of those in the first and second will give rise to a third distribution of currents, such that the total current passing through a given finite surface in the third case is the sum or difference of the currents passing through it in the first and second cases. For, by Ohm's Law, the additional current due to an alteration of potentials is independent of the original current due to the original distribution of potentials.

(4) If the potential is constant over the whole of a closed surface,

and if there are no electrodes or intrinsic electromotive forces within it, then there will be no currents within the closed surface, and the potential at any point within it will be equal to that at the surface.

If there are currents within the closed surface they must either be closed curves, or they must begin and end either within the closed surface or at the surface itself.

But since the current must pass from places of high to places of low potential, it cannot flow in a closed curve.

Since there are no electrodes within the surface the current cannot begin or end within the closed surface, and since the potential at all points of the surface is the same, there can be no current along lines passing from one point of the surface to another.

Hence there are no currents within the surface, and therefore there can be no difference of potential, as such a difference would produce currents, and therefore the potential within the closed surface is everywhere the same as at the surface.

(5) If there is no electric current through any part of a closed surface, and no electrodes or intrinsic electromotive forces within the surface, there will be no currents within the surface, and the potential will be uniform.

We have seen that the currents cannot form closed curves, or begin or terminate within the surface, and since by the hypothesis they do not pass through the surface, there can be no currents, and therefore the potential is constant.

(6) If the potential is uniform over part of a closed surface, and if there is no current through the remainder of the surface, the potential within the surface will be uniform for the same reasons.

(7) If over part of the surface of a body the potential of every point is known, and if over the rest of the surface of the body the current passing through the surface at each point is known, then only one distribution of potentials at points within the body can exist.

For if there were two different values of the potential at any point within the body, let these be $V_1$ in the first case and $V_2$ in the second case, and let us imagine a third case in which the potential of every point of the body is the excess of potential in the first case over that in the second. Then on that part of the surface for which the potential is known the potential in the third case will be zero, and on that part of the surface through which the currents are known the currents in the third case will be zero, so that by (6) the potential everywhere within the surface will be zero, or there is no excess of $V_1$ over $V_2$, or the reverse. Hence there is only one possible distribution of potentials. This proposition is true whether the solid be bounded by one closed surface or by several.

*On the Approximate Calculation of the Resistance of a Conductor of a given Form.*

306.] The conductor here considered has its surface divided into three portions. Over one of these portions the potential is maintained at a constant value. Over a second portion the potential has a constant value different from the first. The whole of the remainder of the surface is impervious to electricity. We may suppose the conditions of the first and second portions to be fulfilled by applying to the conductor two electrodes of perfectly conducting material, and that of the remainder of the surface by coating it with perfectly non-conducting material.

Under these circumstances the current in every part of the conductor is simply proportional to the difference between the potentials of the electrodes. Calling this difference the electromotive force, the total current from the one electrode to the other is the product of the electromotive force by the conductivity of the conductor as a whole, and the resistance of the conductor is the reciprocal of the conductivity.

It is only when a conductor is approximately in the circumstances above defined that it can be said to have a definite resistance, or conductivity as a whole. A resistance coil, consisting of a thin wire terminating in large masses of copper, approximately satisfies these conditions, for the potential in the massive electrodes is nearly constant, and any differences of potential in different points of the same electrode may be neglected in comparison with the difference of the potentials of the two electrodes.

A very useful method of calculating the resistance of such conductors has been given, so far as I know, for the first time, by the Hon. J. W. Strutt, in a paper on the Theory of Resonance*.

It is founded on the following considerations.

If the specific resistance of any portion of the conductor be changed, that of the remainder being unchanged, the resistance of

* *Phil. Trans.*, 1871, p. 77. See Art. 102.

the whole conductor will be increased if that of the portion is increased, and diminished if that of the portion be diminished.

This principle may be regarded as self-evident, but it may easily be shewn that the value of the expression for the resistance of a system of conductors between two points selected as electrodes, increases as the resistance of each member of the system increases.

It follows from this that if a surface of any form be described in the substance of the conductor, and if we further suppose this surface to be an infinitely thin sheet of a perfectly conducting substance, the resistance of the conductor as a whole will be diminished unless the surface is one of the equipotential surfaces in the natural state of the conductor, in which case no effect will be produced by making it a perfect conductor, as it is already in electrical equilibrium.

If therefore we draw within the conductor a series of surfaces, the first of which coincides with the first electrode, and the last with the second, while the intermediate surfaces are bounded by the non-conducting surface and do not intersect each other, and if we suppose each of these surfaces to be an infinitely thin sheet of perfectly conducting matter, we shall have obtained a system the resistance of which is certainly not greater than that of the original conductor, and is equal to it only when the surfaces we have chosen are the natural equipotential surfaces.

To calculate the resistance of the artificial system is an operation of much less difficulty than the original problem. For the resistance of the whole is the sum of the resistances of all the strata contained between the consecutive surfaces, and the resistance of each stratum can be found thus:

Let $dS$ be an element of the surface of the stratum, $\nu$ the thickness of the stratum perpendicular to the element, $\rho$ the specific resistance, $E$ the difference of potential of the perfectly conducting surfaces, and $dC$ the current through $dS$, then

$$dC = E\frac{1}{\rho\nu}dS, \tag{1}$$

and the whole current through the stratum is

$$C = E\iint\frac{1}{\rho\nu}dS, \tag{2}$$

the integration being extended over the whole stratum bounded by the non-conducting surface of the conductor.

Hence the conductivity of the stratum is

$$\frac{C}{E} = \iint \frac{1}{\rho \nu}\, dS, \tag{3}$$

and the resistance of the stratum is the reciprocal of this quantity.

If the stratum be that bounded by the two surfaces for which the function $F$ has the values $F$ and $F + dF$ respectively, then

$$\frac{dF}{\nu} = \nabla F = \left[ \left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2 \right]^{\frac{1}{2}}, \tag{4}$$

and the resistance of the stratum is

$$\frac{dF}{\iint \frac{1}{\rho} \nabla F\, dS}. \tag{5}$$

To find the resistance of the whole artificial conductor, we have only to integrate with respect to $F$, and we find

$$R_1 = \int \frac{dF}{\iint \frac{1}{\rho} \nabla F\, dS}. \tag{6}$$

The resistance $R$ of the conductor in its natural state is greater than the value thus obtained, unless all the surfaces we have chosen are the natural equipotential surfaces. Also, since the true value of $R$ is the absolute maximum of the values of $R_1$ which can thus be obtained, a small deviation of the chosen surfaces from the true equipotential surfaces will produce an error of $R$ which is comparatively small.

This method of determining a lower limit of the value of the resistance is evidently perfectly general, and may be applied to conductors of any form, even when $\rho$, the specific resistance, varies in any manner within the conductor.

The most familiar example is the ordinary method of determining the resistance of a straight wire of variable section. In this case the surfaces chosen are planes perpendicular to the axis of the wire, the strata have parallel faces, and the resistance of a stratum of section $S$ and thickness $ds$ is

$$dR_1 = \frac{\rho\, ds}{S}, \tag{7}$$

and that of the whole wire of length $s$ is

$$dR_1 = \int \frac{\rho\, ds}{S}, \tag{8}$$

where $S'$ is the transverse section and is a function of $s$.

This method in the case of wires whose section varies slowly with the length gives a result very near the truth, but it is really only a lower limit, for the true resistance is always greater than this, except in the case where the section is perfectly uniform.

307.] To find the higher limit of the resistance, let us suppose a surface drawn in the conductor to be rendered impermeable to electricity. The effect of this must be to increase the resistance of the conductor unless the surface is one of the natural surfaces of flow. By means of two systems of surfaces we can form a set of tubes which will completely regulate the flow, and the effect, if there is any, of this system of impermeable surfaces must be to increase the resistance above its natural value.

The resistance of each of the tubes may be calculated by the method already given for a fine wire, and the resistance of the whole conductor is the reciprocal of the sum of the reciprocals of the resistances of all the tubes. The resistance thus found is greater than the natural resistance, except when the tubes follow the natural lines of flow.

In the case already considered, where the conductor is in the form of an elongated solid of revolution, let us measure $x$ along the axis, and let the radius of the section at any point be $b$. Let one set of impermeable surfaces be the planes through the axis for each of which $\phi$ is constant, and let the other set be surfaces of revolution for which

$$y^2 = \psi b^2, \tag{9}$$

where $\psi$ is a numerical quantity between 0 and 1.

Let us consider a portion of one of the tubes bounded by the surfaces $\phi$ and $\phi + d\phi$, $\psi$ and $\psi + d\psi$, $x$ and $x + dx$.

The section of the tube taken perpendicular to the axis is

$$y\,dy\,d\phi = \tfrac{1}{2} b^2\,d\psi\,d\phi. \tag{10}$$

If $\theta$ be the angle which the tube makes with the axis

$$\tan\theta = \psi^{\frac{1}{2}} \frac{db}{dx}. \tag{11}$$

The true length of the element of the tube is $dx \sec\theta$, and its true section is

$$\tfrac{1}{2} b^2\,d\psi\,d\phi \cos\theta,$$

so that its resistance is

$$2\rho \frac{dx}{b^2 d\psi\,d\phi} \sec^2\theta = 2\rho \frac{dx}{b^2\,d\psi\,d\phi} \left(1 + \psi \overline{\frac{db}{d}}\Big|^2\right). \tag{12}$$

Let $$A = \int \rho \frac{dx}{b^2}, \quad \text{and} \quad B = \int \rho \frac{dx}{b^2} \overline{\frac{db}{dx}}\Big|^2, \tag{13}$$

the integration being extended over the whole length, $x$, of the conductor, then the resistance of the tube $d\psi\, d\phi$ is

$$\frac{2}{d\psi\, d\phi}(A+\psi B),$$

and its conductivity is

$$\frac{d\psi\, d\phi}{2(A+\psi B)}.$$

To find the conductivity of the whole conductor, which is the sum of the conductivities of the separate tubes, we must integrate this expression between $\phi = 0$ and $\phi = 2\pi$, and between $\psi = 0$ and $\psi = 1$. The result is

$$\frac{1}{R'} = \frac{\pi}{B}\log\left(1+\frac{B}{A}\right), \tag{14}$$

which may be less, but cannot be greater, than the true conductivity of the conductor.

When $\frac{db}{dx}$ is always a small quantity $\frac{B}{A}$ will also be small, and we may expand the expression for the conductivity, thus

$$\frac{1}{R'} = \frac{\pi}{A}\left(1 - \tfrac{1}{2}\frac{B}{A} + \tfrac{1}{3}\frac{B^2}{A^2} - \tfrac{1}{4}\frac{B^3}{A^3} + \&c.\right). \tag{15}$$

The first term of this expression, $\frac{\pi}{A}$, is that which we should have found by the former method as the superior limit of the conductivity. Hence the true conductivity is less than the first term but greater than the whole series. The superior value of the resistance is the reciprocal of this, or

$$R' = \frac{A}{\pi}\left(1 + \tfrac{1}{2}\frac{B}{A} - \frac{1}{12}\frac{B^2}{A^2} + \frac{1}{24}\frac{B^3}{A^3} - \&c.\right). \tag{16}$$

If, besides supposing the flow to be guided by the surfaces $\phi$ and $\psi$, we had assumed that the flow through each tube is proportional to $d\psi\, d\phi$, we should have obtained as the value of the resistance under this additional constraint

$$R'' = \frac{1}{\pi}(A + \tfrac{1}{2}B), \tag{17}$$

which is evidently greater than the former value, as it ought to be, on account of the additional constraint. In Mr. Strutt's paper this is the supposition made, and the superior limit of the resistance there given has the value (17), which is a little greater than that which we have obtained in (16).

308.] We shall now apply the same method to find the correction which must be applied to the length of a cylindrical conductor of radius $a$ when its extremity is placed in metallic contact with a massive electrode, which we may suppose of a different metal.

For the lower limit of the resistance we shall suppose that an infinitely thin disk of perfectly conducting matter is placed between the end of the cylinder and the massive electrode, so as to bring the end of the cylinder to one and the same potential throughout. The potential within the cylinder will then be a function of its length only, and if we suppose the surface of the electrode where the cylinder meets it to be approximately plane, and all its dimensions to be large compared with the diameter of the cylinder, the distribution of potential will be that due to a conductor in the form of a disk placed in an infinite medium. See Arts. 152, 177.

If $E$ is the difference of the potential of the disk from that of the distant parts of the electrode, $C$ the current issuing from the surface of the disk into the electrode, and $\rho'$ the specific resistance of the electrode,

$$\rho' C = 4aE. \tag{18}$$

Hence, if the length of the wire from a given point to the electrode is $L$, and its specific resistance $\rho$, the resistance from that point to any point of the electrode not near the junction is

$$R = \rho\frac{L}{\pi a^2} + \frac{\rho'}{4a},$$

and this may be written

$$R = \frac{\rho}{\pi a^2}\left(L + \frac{\rho'}{\rho}\frac{\pi a}{4}\right), \tag{19}$$

where the second term within brackets is a quantity which must be added to the length of the cylinder or wire in calculating its resistance, and this is certainly too small a correction.

To understand the nature of the outstanding error we may observe, that whereas we have supposed the flow in the wire up to the disk to be uniform throughout the section, the flow from the disk to the electrode is not uniform, but is at any point inversely proportional to the minimum chord through that point. In the actual case the flow through the disk will not be uniform, but it will not vary so much from point to point as in this supposed case. The potential of the disk in the actual case will not be uniform, but will diminish from the middle to the edge.

309.] We shall next determine a quantity greater than the true resistance by constraining the flow through the disk to be uniform at every point. We may suppose electromotive forces introduced for this purpose acting perpendicular to the surface of the disk.

The resistance within the wire will be the same as before, but in the electrode the rate of generation of heat will be the surface-integral of the product of the flow into the potential. The rate of flow at any point is $\frac{C}{\pi a^2}$, and the potential is the same as that of an electrified surface whose surface-density is $\sigma$, where

$$2\pi\sigma = \frac{C\rho'}{\pi a^2}, \tag{20}$$

$\rho'$ being the specific resistance.

We have therefore to determine the potential energy of the electrification of the disk with the uniform surface-density $\sigma$.

The potential at the edge of a disk of uniform density $\sigma$ is easily found to be $4a\sigma$. The work done in adding a strip of breadth $da$ at the circumference of the disk is $2\pi a\sigma da \,.\, 4a\sigma$, and the whole potential energy of the disk is the integral of this,

$$\text{or} \qquad P = \frac{8\pi}{3} a^3 \sigma^2. \tag{21}$$

In the case of electrical conduction the rate at which work is done in the electrode whose resistance is $R'$ is

$$C^2 R' = \frac{4\pi}{\rho'} P, \tag{22}$$

whence, by (20) and (21),

$$R' = \frac{8\rho'}{3\pi^2 a},$$

and the correction to be added to the length of the cylinder is

$$\frac{\rho'}{\rho}\frac{8}{3\pi}a,$$

this correction being greater than the true value. The true correction to be added to the length is therefore $\frac{\rho'}{\rho}an$, where $n$ is a number lying between $\frac{\pi}{4}$ and $\frac{8}{3\pi}$, or between 0.785 and 0.849.

Mr. Strutt, by a second approximation, has reduced the superior limit of $n$ to 0.8282.

# CHAPTER IX.

## CONDUCTION THROUGH HETEROGENEOUS MEDIA.

### *On the Conditions to be Fulfilled at the Surface of Separation between Two Conducting Media.*

310.] There are two conditions which the distribution of currents must fulfil in general, the condition that the potential must be continuous, and the condition of 'continuity' of the electric currents.

At the surface of separation between two media the first of these conditions requires that the potentials at two points on opposite sides of the surface, but infinitely near each other, shall be equal. The potentials are here understood to be measured by an electrometer put in connexion with the given point by means of an electrode of a given metal. If the potentials are measured by the method described in Arts. 222, 246, where the electrode terminates in a cavity of the conductor filled with air, then the potentials at contiguous points of different metals measured in this way will differ by a quantity depending on the temperature and on the nature of the two metals.

The other condition at the surface is that the current through any element of the surface is the same when measured in either medium.

Thus, if $V_1$ and $V_2$ are the potentials in the two media, then at any point in the surface of separation

$$V_1 = V_2, \tag{1}$$

and if $u_1, v_1, w_1$ and $u_2, v_2, w_2$ are the components of currents in the two media, and $l, m, n$ the direction-cosines of the normal to the surface of separation,

$$u_1 l + v_1 m + w_1 n = u_2 l + v_2 m + w_2 n. \tag{2}$$

In the most general case the components $u, v, w$ are linear functions of the derivatives of $V$, the forms of which are given in the equations

$$\left.\begin{aligned} u &= r_1 X + p_3 Y + q_2 Z, \\ v &= q_3 X + r_2 Y + p_1 Z, \\ w &= p_2 X + q_1 Y + r_3 Z, \end{aligned}\right\} \tag{3}$$

where $X, Y, Z$ are the derivatives of $V$ with respect to $x, y, z$ respectively.

Let us take the case of the surface which separates a medium having these coefficients of conduction from an isotropic medium having a coefficient of conduction equal to $r$.

Let $X', Y', Z'$ be the values of $X, Y, Z$ in the isotropic medium, then we have at the surface

$$V = V', \tag{4}$$

or $$X\,dx + Y\,dy + Z\,dz = X'dx + Y'dy + Z'dz, \tag{5}$$

when $$l\,dx + m\,dy + n\,dz = 0. \tag{6}$$

This condition gives

$$X' = X + 4\pi\sigma l, \qquad Y' = Y + 4\pi\sigma m, \qquad Z' = Z + 4\pi\sigma n, \tag{7}$$

where $\sigma$ is the surface-density.

We have also in the isotropic medium

$$u' = rX', \qquad v' = rY', \qquad w' = rZ', \tag{8}$$

and at the boundary the condition of flow is

$$u'l + v'm + w'n = ul + vm + wn, \tag{9}$$

or $r(lX + mY + nZ + 4\pi\sigma)$

$$= l(r_1 X + p_3 Y + q_2 Z) + m(q_3 X + r_2 Y + p_1 Z) + n(p_2 X + q_1 Y + r_3 Z), \tag{10}$$

whence

$$4\pi\sigma r = (l(r_1 - r) + mq_3 + np_2)X + (lp_3 + m(r_2 - r) + nq_1)Y \\ + (lq_2 + mp_1 + n(r_3 - r))Z. \tag{11}$$

The quantity $\sigma$ represents the surface-density of the charge on the surface of separation. In crystallized and organized substances it depends on the direction of the surface as well as on the force perpendicular to it. In isotropic substances the coefficients $p$ and $q$ are zero, and the coefficients $r$ are all equal, so that

$$4\pi\sigma = \left(\frac{r_1}{r} - 1\right)(lX + mY + nZ), \tag{12}$$

where $r_1$ is the conductivity of the substance, $r$ that of the external medium, and $l, m, n$ the direction-cosines of the normal drawn towards the medium whose conductivity is $r$.

When both media are isotropic the conditions may be greatly simplified, for if $k$ is the specific resistance per unit of volume, then

$$u = -\frac{1}{k}\frac{dV}{dx}, \qquad v = -\frac{1}{k}\frac{dV}{dy}, \qquad w = -\frac{1}{k}\frac{dV}{dz}, \tag{13}$$

and if $\nu$ is the normal drawn at any point of the surface of separation from the first medium towards the second, the conduction of continuity is

$$\frac{1}{k_1}\frac{dV_1}{d\nu} = \frac{1}{k_2}\frac{dV_2}{d\nu}. \tag{14}$$

If $\theta_1$ and $\theta_2$ are the angles which the lines of flow in the first and second media respectively make with the normal to the surface of separation, then the tangents to these lines of flow are in the same plane with the normal and on opposite sides of it, and

$$k_1 \tan\theta_1 = k_2 \tan\theta_2. \tag{15}$$

This may be called the law of refraction of lines of flow.

311.] As an example of the conditions which must be fulfilled when electricity crosses the surface of separation of two media, let us suppose the surface spherical and of radius $a$, the specific resistance being $k_1$ within and $k_2$ without the surface.

Let the potential, both within and without the surface, be expanded in solid harmonics, and let the part which depends on the surface harmonic $S_i$ be

$$V_1 = (A_1 r^i + B_1 r^{-(i+1)}) S_i, \tag{1}$$

$$V_2 = (A_2 r^i + B_2 r^{-(i+1)}) S_i \tag{2}$$

within and without the sphere respectively.

At the surface of separation where $r = a$ we must have

$$V_1 = V_2 \quad \text{and} \quad \frac{1}{k_1}\frac{dV_1}{dr} = \frac{1}{k_2}\frac{dV_2}{dr}. \tag{3}$$

From these conditions we get the equations

$$\left.\begin{aligned} (A_1 - A_2)\, a^{2i+1} + B_1 - B_2 &= 0, \\ \left(\frac{1}{k_1}A_1 - \frac{1}{k_2}A_2\right) i a^{2i+1} - \left(\frac{1}{k_1}B_1 - \frac{1}{k_2}B_2\right)(i+1) &= 0. \end{aligned}\right\} \tag{4}$$

These equations are sufficient, when we know two of the four quantities $A_1$, $A_2$, $B_1$, $B_2$, to deduce the other two.

Let us suppose $A_1$ and $B_1$ known, then we find the following expressions for $A_2$ and $B_2$,

$$\left.\begin{aligned} A_2 &= \frac{(k_1(i+1) + k_2 i) A_1 + (k_1 - k_2)(i+1) B_1 a^{-(2i+1)}}{k_1 (2i+1)}, \\ B_2 &= \frac{(k_1 - k_2)\, i A_1 a^{2i+1} + (k_1 i + k_2 (i+1)) B_1}{k_1 (2i+1)}. \end{aligned}\right\} \tag{5}$$

In this way we can find the conditions which each term of the harmonic expansion of the potential must satisfy for any number of strata bounded by concentric spherical surfaces.

312.] Let us suppose the radius of the first spherical surface to be $a_1$, and let there be a second spherical surface of radius $a_2$ greater than $a_1$, beyond which the specific resistance is $k_3$. If there are no sources or sinks of electricity within these spheres there will be no infinite values of $V$, and we shall have $B_1 = 0$.

We then find for $A_3$ and $B_3$, the coefficients for the outer medium,

$$\left.\begin{aligned} A_3 k_1 k_2 (2i+1)^2 &= \Big[\{k_1(i+1)+k_2 i\}\{k_2(i+1)+k_3 i\} \\ &\qquad + i(i+1)(k_1-k_2)(k_2-k_3)\Big(\frac{a_1}{a_2}\Big)^{2i+1}\Big] A_1, \\ B_3 k_1 k_2 (2i+1)^2 &= \big[i\{k_1(i+1)+k_2 i\}(k_2-k_3)\, a_2^{2i+1} \\ &\qquad + i(k_1-k_2)\{k_2 i+k_3(i+1)\}\, a_1^{2i+1}\big] A_1. \end{aligned}\right\} \tag{6}$$

The value of the potential in the outer medium depends partly on the external sources of electricity, which produce currents independently of the existence of the sphere of heterogeneous matter within, and partly on the disturbance caused by the introduction of the heterogeneous sphere.

The first part must depend on solid harmonics of positive degrees only, because it cannot have infinite values within the sphere. The second part must depend on harmonics of negative degrees, because it must vanish at an infinite distance from the centre of the sphere.

Hence the potential due to the external electromotive forces must be expanded in a series of solid harmonics of positive degree. Let $A_3$ be the coefficient of one these, of the form

$$A_3 S_i r^i.$$

Then we can find $A_1$, the corresponding coefficient for the inner sphere by equation (6), and from this deduce $A_2$, $B_2$, and $B_3$. Of these $B_3$ represents the effect on the potential in the outer medium due to the introduction of the heterogeneous spheres.

Let us now suppose $k_3 = k_1$, so that the case is that of a hollow shell for which $k = k_2$, separating an inner from an outer portion of the same medium for which $k = k_1$.

If we put

$$C = \frac{1}{(2i+1)^2 k_1 k_2 + i(i+1)(k_2-k_1)^2\Big(1-\Big(\frac{a_1}{a_2}\Big)^{2i+1}\Big)},$$

then
$$\left.\begin{aligned}
A_1 &= k_1 k_2 (2i+1)^2 C A_3,\\
A_2 &= k_2 (2i+1)\left(k_1(i+1)+k_2 i\right) C A_3,\\
B_2 &= k_2 i (2i+1)(k_1-k_2)\, a_1^{2i+1} C A_3,\\
B_3 &= i(k_2-k_1)\left(k_1(i+1)+k_2 i\right)\left(a_2^{2i+1}-a_1^{2i+1}\right) C A_3.
\end{aligned}\right\} \tag{7}$$

The difference between $A_3$ the undisturbed coefficient, and $A_1$ its value in the hollow within the spherical shell, is

$$A_3 - A_1 = (k_2-k_1)^2\, i(i+1)\left(1-\left(\frac{a_1}{a_2}\right)^{2i+1}\right) C A_3. \tag{8}$$

Since this quantity is always positive whatever be the values of $k_1$ and $k_2$, it follows that, whether the spherical shell conducts better or worse than the rest of the medium, the electrical action within the shell is less than it would otherwise be. If the shell is a better conductor than the rest of the medium it tends to equalize the potential all round the inner sphere. If it is a worse conductor, it tends to prevent the electrical currents from reaching the inner sphere at all.

The case of a solid sphere may be deduced from this by making $a_1 = 0$, or it may be worked out independently.

313.] The most important term in the harmonic expansion is that in which $i = 1$, for which

$$\left.\begin{aligned}
C &= \frac{1}{9k_1k_2 + 2(k_1-k_2)^2\left(1-\left(\frac{a_1}{a_2}\right)^3\right)},\\
A_1 &= 9k_1k_2 C A_3, \qquad A_2 = 3k_2(2k_1+k_2) C A_3,\\
B_2 &= 3k_2(k_1-k_2)a_1^3 C A_3, \quad B_3 = (k_2-k_1)(2k_1+k_2)(a_2^3-a_1^3) C A_3.
\end{aligned}\right\} \tag{9}$$

The case of a solid sphere of resistance $k_2$ may be deduced from this by making $a_1 = 0$. We then have

$$\left.\begin{aligned}
A_2 &= \frac{3k_2}{k_1+2k_2} A_3, \quad B_2 = 0,\\
B_3 &= \frac{k_2-k_1}{k_1+2k_2} a_2^3 A_3.
\end{aligned}\right\} \tag{10}$$

It is easy to shew from the general expressions that the value of $B_3$ in the case of a hollow sphere having a nucleus of resistance $k_1$, surrounded by a shell of resistance $k_2$, is the same as that of a uniform solid sphere of the radius of the outer surface, and of resistance $K$, where

$$K = \frac{(2k_1+k_2)a_2^3 + (k_1-k_2)a_1^3}{(2k_1+k_2)a_2^3 - 2(k_1-k_2)a_1^3} k_2. \tag{11}$$

314.] If there are $n$ spheres of radius $a_1$ and resistance $k_1$, placed in a medium whose resistance is $k_2$, at such distances from each other that their effects in disturbing the course of the current may be taken as independent of each other, then if these spheres are all contained within a sphere of radius $a_2$, the potential at a great distance from the centre of this sphere will be of the form

$$V = \left(A r + nB \frac{1}{r^2}\right) \cos\theta, \tag{12}$$

where the value of $B$ is

$$B = \frac{k_1 - k_2}{2k_1 + k_2} a_1^{\,3} A. \tag{13}$$

The ratio of the volume of the $n$ small spheres to that of the sphere which contains them is

$$p = \frac{n a_1^{\,3}}{a_2^{\,3}}. \tag{14}$$

The value of the potential at a great distance from the sphere may therefore be written

$$V = \left(A r + p a_2^{\,3} \frac{k_1 - k_2}{2k_1 + k_2} \frac{1}{r^2}\right) \cos\theta. \tag{15}$$

Now if the whole sphere of radius $a_2$ had been made of a material of specific resistance $K$, we should have had

$$V = \left\{A r + a_2^{\,3} \frac{K - k_2}{2K + k_2} \frac{1}{r^2}\right\} \cos\theta. \tag{16}$$

That the one expression should be equivalent to the other,

$$K = \frac{2k_1 + k_2 + p(k_1 - k_2)}{2k_1 + k_2 - 2p(k_1 - k_2)} k_2. \tag{17}$$

This, therefore, is the specific resistance of a compound medium consisting of a substance of specific resistance $k_2$, in which are disseminated small spheres of specific resistance $k_1$, the ratio of the volume of all the small spheres to that of the whole being $p$. In order that the action of these spheres may not produce effects depending on their interference, their radii must be small compared with their distances, and therefore $p$ must be a small fraction.

This result may be obtained in other ways, but that here given involves only the repetition of the result already obtained for a single sphere.

When the distance between the spheres is not great compared with their radii, and when $\frac{k_1 - k_2}{2k_1 + k_2}$ is considerable, then other terms enter into the result, which we shall not now consider. In consequence of these terms certain systems of arrangement of the spheres cause the resistance of the compound medium to be different in different directions.

*Application of the Principle of Images.*

315.] Let us take as an example the case of two media separated by a plane surface, and let us suppose that there is a source $S$ of electricity at a distance $a$ from the plane surface in the first medium, the quantity of electricity flowing from the source in unit of time being $S$.

If the first medium had been infinitely extended the current at any point $P$ would have been in the direction $SP$, and the potential at $P$ would have been $\frac{E}{r_1}$ where $E = \frac{Sk_1}{4\pi}$ and $r_1 = SP$.

In the actual case the conditions may be satisfied by taking a point $I$, the image of $S$ in the second medium, such that $IS$ is normal to the plane of separation and is bisected by it. Let $r_2$ be the distance of any point from $I$, then at the surface of separation

$$r_1 = r_2, \tag{1}$$

$$\frac{dr_1}{dv} = -\frac{dr_2}{dv}. \tag{2}$$

Let the potential $V_1$ at any point in the first medium be that due to a quantity of electricity $E$ placed at $S$, together with an imaginary quantity $E_2$ at $I$, and let the potential $V_2$ at any point of the second medium be that due to an imaginary quantity $E_1$ at $S$, then if

$$V_1 = \frac{E}{r_1} + \frac{E_2}{r_1} \quad \text{and} \quad V_2 = \frac{E_1}{r_1}, \tag{3}$$

the superficial condition $V_1 = V_2$ gives

$$E + E_2 = E_1, \tag{4}$$

and the condition

$$\frac{1}{k_1}\frac{dV_1}{dv} = \frac{1}{k_2}\frac{dV_2}{dv} \tag{5}$$

gives

$$\frac{1}{k_1}(E - E_2) = \frac{1}{k_2}E_1, \tag{6}$$

whence

$$E_1 = \frac{2k_2}{k_1 + k_2}E, \qquad E_2 = \frac{k_2 - k_1}{k_1 + k_2}E. \tag{7}$$

The potential in the first medium is therefore the same as would be produced in air by a charge $E$ placed at $S$, and a charge $E_1$ at $I$ on the electrostatic theory, and the potential in the second medium is the same as that which would be produced in air by a charge $E_1$ at $S$.

The current at any point of the first medium is the same as would have been produced by the source $S$ together with a source $\frac{k_2 - k_1}{k_1 + k_2} S$ placed at $I$ if the first medium had been infinite, and the current at any point of the second medium is the same as would have been produced by a source $\frac{2\,k_1 S}{(k_1 + k_2)}$ placed at $S$ if the second medium had been infinite.

We have thus a complete theory of electrical images in the case of two media separated by a plane boundary. Whatever be the nature of the electromotive forces in the first medium, the potential they produce in the first medium may be found by combining their direct effect with the effect of their image.

If we suppose the second medium a perfect conductor, then $k_2 = 0$, and the image at $I$ is equal and opposite to the course at $S$. This is the case of electric images, as in Thomson's theory in electrostatics.

If we suppose the second medium a perfect insulator, then $k_2 = \infty$, and the image at $I$ is equal to the source at $S$ and of the same sign. This is the case of images in hydrokinetics when the fluid is bounded by a rigid plane surface.

316.] The method of inversion, which is of so much use in electrostatics when the bounding surface is supposed to be that of a perfect conductor, is not applicable to the more general case of the surface separating two conductors of unequal electric resistance. The method of inversion in two dimensions is, however, applicable, as well as the more general method of transformation in two dimensions given in Art. 190 *.

*Conduction through a Plate separating Two Media.*

317.] Let us next consider the effect of a plate of thickness $AB$ of a medium whose resistance is $k_2$, and separating two media whose resistances are $k_1$ and $k_3$, in altering the potential due to a source $S$ in the first medium.

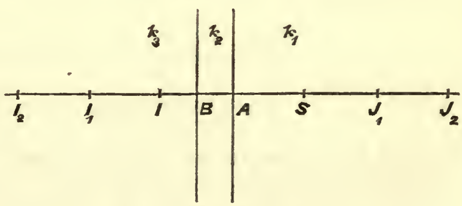


Fig. 23.

The potential will be

* See Kirchhoff, Pogg. *Ann.* lxiv. 497, and lxvii. 344 ; Quincke, Pogg. xcvii. 382; and Smith, *Proc. R. S. Edin.*, 1869–70, p. 79.

equal to that due to a system of charges placed in air at certain points along the normal to the plate through $S$.

Make

$$AI = SA, \quad BI_1 = SB, \quad AJ_1 = I_1A, \quad BI_2 = J_1B, \quad AJ_2 = I_2A, \text{ \&c.};$$

then we have two series of points at distances from each other equal to twice the thickness of the plate.

318.] The potential in the first medium at any point $P$ is equal to

$$\frac{E}{PS} + \frac{I}{PI} + \frac{I_1}{PI_1} + \frac{I_2}{PI_2} + \text{\&c.}, \tag{8}$$

that at a point $P'$ in the second

$$\frac{E'}{P'S} + \frac{I'}{P'I} + \frac{I_1'}{P'I_1} + \frac{I_2'}{P'I_2} + \text{\&c.}$$
$$+ \frac{J_1'}{P'J_1} + \frac{J_2'}{P'J_2} + \text{\&c.}, \tag{9}$$

and that at a point $P''$ in the third

$$\frac{E''}{P''S} + \frac{J_1}{P''J_1} + \frac{J_2}{P''J_2} + \text{\&c.}, \tag{10}$$

where $I$, $I'$, &c. represent the imaginary charges placed at the points $I$, &c., and the accents denote that the potential is to be taken within the plate.

Then, by the last Article, for the surface through $A$ we have,

$$I = \frac{k_2 - k_1}{k_2 + k_1} E, \quad E' = \frac{2\,k_2}{k_2 + k_1} E. \tag{11}$$

For the surface through $B$ we find

$$I_1' = \frac{k_3 - k_2}{k_3 + k_2} E', \quad E'' = \frac{2\,k_3}{k_2 + k_1} E'. \tag{12}$$

Similarly for the surface through $A$ again,

$$J_1' = \frac{k_1 - k_2}{k_1 + k_2} I_1', \quad I_1 = \frac{2\,k_1}{k_1 + k_2} I_1', \tag{13}$$

and for the surface through $B$,

$$J_2' = \frac{k_3 - k_2}{k_3 + k_2} J_1', \quad J_1 = \frac{2\,k_3}{k_3 + k_2} J_1'. \tag{14}$$

If we make $\rho = \dfrac{k_1 - k_2}{k_1 + k_2}$ and $\rho' = \dfrac{k_3 - k_2}{k_3 + k_2}$,

we find for the potential in the first medium,

$$V = \frac{E}{PS} - \rho\,\frac{E}{PI} + (1-\rho^2)\rho'\,\frac{E}{PI_1} + \rho'(1-\rho^2)\,\rho\rho'\,\frac{E}{PI_2} + \text{\&c.}$$
$$+ \rho'(1-\rho^2)(\rho\rho')^{n-1}\frac{E}{PI_n}. \tag{15}$$

For the potential in the third medium we find

$$V = (1+\rho')(1-\rho)E\left\{\frac{1}{PS}+\frac{\rho\rho'}{PJ_1}+\&c.+\frac{(\rho\rho')^n}{PJ_n}\right\}. \tag{16}$$

If the first medium is the same as the third, then $k_1 = k_3$ and $\rho = \rho'$, and the potential on the other side of the plate will be

$$V = (1-\rho^2)\,E\left\{\frac{1}{PS}+\frac{\rho^2}{PJ_1}+\&c.+\frac{\rho^{2n}}{PJ_n}\right\}. \tag{17}$$

If the plate is a very much better conductor than the rest of the medium, $\rho$ is very nearly equal to 1. If the plate is a nearly perfect insulator, $\rho$ is nearly equal to $-1$, and if the plate differs little in conducting power from the rest of the medium, $\rho$ is a small quantity positive or negative.

The theory of this case was first stated by Green in his 'Theory of Magnetic Induction' (*Essay*, p. 65). His result, however, is correct only when $\rho$ is nearly equal to 1 *. The quantity $g$ which he uses is connected with $\rho$ by the equations

$$g = \frac{2\rho}{3-\rho} = \frac{k_1-k_2}{k_1+2k_2}, \quad \rho = \frac{3g}{2+g} = \frac{k_1-k_2}{k_1+k_2}.$$

If we put $\rho = \dfrac{2\pi\kappa}{1+2\pi\kappa}$, we shall have a solution of the problem of the magnetic induction excited by a magnetic pole in an infinite plate whose coefficient of magnetization is $\kappa$.

*On Stratified Conductors.*

319.] Let a conductor be composed of alternate strata of thickness $c$ and $c'$ of two substances whose coefficients of conductivity are different. Required the coefficients of resistance and conductivity of the compound conductor.

Let the plane of the strata be normal to $Z$. Let every symbol relating to the strata of the second kind be accented, and let every symbol relating to the compound conductor be marked with a bar thus, $\overline{X}$. Then

$$\overline{X} = X = X', \qquad (c+c')\,\overline{u} = cu + c'u',$$
$$\overline{Y} = Y = Y', \qquad (c+c')\,\overline{v} = cv + c'v';$$
$$(c+c')\,\overline{Z} = cZ + c'Z', \qquad \overline{w} = w = w'.$$

We must first determine $u$, $u'$, $v$, $v'$, $Z$ and $Z'$ in terms of $\overline{X}$, $\overline{Y}$ and $\overline{w}$ from the equations of resistance, Art. 297, or those

* See Sir W. Thomson's 'Note on Induced Magnetism in a Plate,' *Camb. and Dub. Math. Journ.*, Nov. 1845, or *Reprint*, art. ix. § 156.

of conductivity, Art. 298. If we put $D$ for the determinant of the coefficients of resistance, we find

$$\begin{aligned} u\,r_3\,D &= R_2\,\overline{X} - Q_3\,\overline{Y} + \overline{w}\,q_2\,D, \\ v\,r_3\,D &= R_1\,\overline{Y} - P_3\,\overline{X} + \overline{w}\,p_1\,D, \\ Z\,r_3 \quad &= -p_2\,\overline{X} - q_1\,\overline{Y} + \overline{w}. \end{aligned}$$

Similar equations with the symbols accented give the values of $u'$, $v'$ and $z'$. Having found $\overline{u}$, $\overline{v}$ and $\overline{w}$ in terms of $\overline{X}$, $\overline{Y}$ and $\overline{Z}$, we may write down the equations of conductivity of the stratified conductor. If we make $h = \frac{c}{r_3}$ and $h' = \frac{c'}{r_3'}$, we find

$$\overline{p}_1 = \frac{hp_1 + h'p_1'}{h+h'}, \qquad \overline{q}_1 = \frac{hq_1 + h'q_1'}{h+h'},$$

$$\overline{p}_2 = \frac{hp_2 + h'p_2'}{h+h'}, \qquad \overline{q}_2 = \frac{hq_2 + h'q_2'}{h+h'},$$

$$\overline{p}_3 = \frac{cp_3 + c'p_3'}{c+c'} - \frac{hh'(q_1 - q_1')(q_2 - q_2')}{(h+h')(c+c')},$$

$$\overline{q}_3 = \frac{cq_3 + c'q_3'}{c+c'} - \frac{hh'(p_1 - p_1')(p_2 - p_2')}{(h+h')(c+c')},$$

$$\overline{r}_1 = \frac{cr_1 + c'r_1'}{c+c'} - \frac{hh'(p_2 - p_2')(q_2 - q_2')}{(h+h')(c+c')},$$

$$\overline{r}_2 = \frac{cr_2 + c'r_2'}{c+c'} - \frac{hh'(p_1 - p_1')(q_1 - q_1')}{(h+h')(c+c')},$$

$$\overline{r}_3 = \frac{c+c'}{h+h'}.$$

320.] If neither of the two substances of which the strata are formed has the rotatory property of Art. 303, the value of any $P$ or $p$ will be equal to that of its corresponding $Q$ or $q$. From this it follows that in the stratified conductor also

$$\overline{p}_1 = \overline{q}_1, \qquad \overline{p}_2 = \overline{q}_2, \qquad \overline{p}_3 = \overline{q}_3,$$

or there is no rotatory property developed by stratification, unless it exists in the materials.

321.] If we now suppose that there is no rotatory property, and also that the axes of $x$, $y$ and $z$ are the principal axes, then the $p$ and $q$ coefficients vanish, and

$$\overline{r}_1 = \frac{cr_1 + c'r_1'}{c+c'}, \qquad \overline{r}_2 = \frac{cr_2 + c'r_2'}{c+c'}, \qquad \overline{r}_3 = \frac{c+c'}{\frac{c}{r_3} + \frac{c'}{r_3'}}.$$

If we begin with both substances isotropic, but of different conductivities, then the result of stratification will be to make the resistance greatest in the direction of a normal to the strata, and the resistance in all directions in the plane of the strata will be equal.

322.] Take an isotropic substance of conductivity $r$, cut it into exceedingly thin slices of thickness $a$, and place them alternately with slices of a substance whose conductivity is $s$, and thickness $k_1 a$.

Let these slices be normal to $x$. Then cut this compound conductor into thicker slices, of thickness $b$, normal to $y$, and alternate these with slices whose conductivity is $s$ and thickness $k_2 b$.

Lastly, cut the new conductor into still thicker slices, of thickness $c$, normal to $z$, and alternate them with slices whose conductivity is $s$ and thickness $k_3 c$.

The result of the three operations will be to cut the substance whose conductivity is $r$ into rectangular parallelepipeds whose dimensions are $a$, $b$ and $c$, where $b$ is exceedingly small compared with $c$, and $a$ is exceedingly small compared with $b$, and to embed these parallelepipeds in the substance whose conductivity is $s$, so that they are separated from each other $k_1 a$ in the direction of $x$, $k_2 b$ in that of $y$, and $k_3 c$ in that of $z$. The conductivities of the conductor so formed in the directions of $x$, $y$ and $z$ are

$$r_1 = \frac{\{1+k_1(1+k_2)(1+k_3)\}\,r+(k_2+k_3+k_2k_3)\,s}{(1+k_2)(1+k_3)(k_1r+s)}\,s,$$

$$r_2 = \frac{(1+k_2+k_2k_3)\,r+(k_1+k_3+k_1k_2+k_1k_3+k_1k_2k_3)\,s}{(1+k_3)\{k_2r+(1+k_1+k_1k_2)s\}}\,s,$$

$$r_3 = \frac{(1+k_3)\,(r+(k_1+k_2+k_1k_2)\,s)}{k_3r+(1+k_1+k_2+k_2k_3+k_3k_1+k_1k_2+k_1k_2k_3)\,s}\,s.$$

The accuracy of this investigation depends upon the three dimensions of the parallelepipeds being of different orders of magnitude, so that we may neglect the conditions to be fulfilled at their edges and angles. If we make $k_1$, $k_2$ and $k_3$ each unity, then

$$r_1 = \frac{5r+3s}{4r+4s}\,s, \qquad r_2 = \frac{3r+5s}{2r+6s}\,s, \qquad r_3 = \frac{2r+6s}{r+7s}\,s.$$

If $r=0$, that is, if the medium of which the parallelepipeds are made is a perfect insulator, then

$$r_1 = \tfrac{3}{4}s, \qquad r_2 = \tfrac{5}{6}s, \qquad r_3 = \tfrac{6}{7}s.$$

If $r = \infty$, that is, if the parallelepipeds are perfect conductors,

$$r_1 = \tfrac{5}{4}s, \qquad r_2 = \tfrac{3}{2}s, \qquad r_3 = 2s.$$

In every case, provided $k_1 = k_2 = k_3$, it may be shewn that $r_1$, $r_2$ and $r_3$ are in ascending order of magnitude, so that the greatest conductivity is in the direction of the longest dimensions of the parallelepipeds, and the greatest resistance in the direction of their shortest dimensions.

323.] In a rectangular parallelepiped of a conducting solid, let there be a conducting channel made from one angle to the opposite, the channel being a wire covered with insulating material, and let the lateral dimensions of the channel be so small that the conductivity of the solid is not affected except on account of the current conveyed along the wire.

Let the dimensions of the parallelepiped in the directions of the coordinate axes be $a$, $b$, $c$, and let the conductivity of the channel, extending from the origin to the point $(abc)$, be $abcK$.

The electromotive force acting between the extremities of the channel is

$$aX + bY + cZ,$$

and if $C'$ be the current along the channel

$$C' = Kabc(aX + bY + cZ).$$

The current across the face $bc$ of the parallelepiped is $bcu$, and this is made up of that due to the conductivity of the solid and of that due to the conductivity of the channel, or

$$bcu = bc(r_1 X + p_3 Y + q_2 Z) + Kabc(aX + bY + cZ),$$

or
$$u = (r_1 + Ka^2)X + (p_3 + Kab)Y + (q_2 + Kca)Z.$$

In the same way we may find the values of $v$ and $w$. The coefficients of conductivity as altered by the effect of the channel will be

$$\begin{array}{lll} r_1 + Ka^2, & r_2 + Kb^2, & r_3 + Kc^2, \\ p_1 + Kbc, & p_2 + Kca, & p_3 + Kab, \\ q_1 + Kbc, & q_2 + Kca, & q_3 + Kab. \end{array}$$

In these expressions, the additions to the values of $p_1$, &c., due to the effect of the channel, are equal to the additions to the values of $q_1$, &c. Hence the values of $p_1$ and $q_1$ cannot be rendered unequal by the introduction of linear channels into every element of volume of the solid, and therefore the rotatory property of Art. 303, if it does not exist previously in a solid, cannot be introduced by such means.

324.] *To construct a framework of linear conductors which shall have any given coefficients of conductivity forming a symmetrical system.*

Let the space be divided into equal small cubes, of which let the figure represent one. Let the coordinates of the points $O, L, M, N$, and their potentials be as follows:

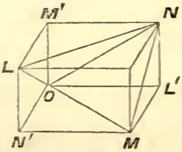


Fig. 24.

| | $x$ | $y$ | $z$ | *Potential.* |
|---|---|---|---|---|
| $O$ | 0 | 0 | 0 | 0 |
| $L$ | 0 | 1 | 1 | $0+Y+Z$, |
| $M$ | 1 | 0 | 1 | $0+Z+X$, |
| $N$ | 1 | 1 | 0 | $0+X+Y$. |

Let these four points be connected by six conductors,

$$OL, \quad OM, \quad ON, \quad MN, \quad NL, \quad LM,$$

of which the conductivities are respectively

$$A, \quad B, \quad C, \quad P, \quad Q, \quad R.$$

The electromotive forces along these conductors will be

$$Y+Z, \quad Z+X, \quad X+Y, \quad Y-Z, \quad Z-X, \quad X-Y,$$

and the currents

$$A(Y+Z), \; B(Z+X), \; C(X+Y), \; P(Y-Z), \; Q(Z-X), \; R(X-Y).$$

Of these currents, those which convey electricity in the positive direction of $x$ are those along $LM$, $LN$, $OM$ and $ON$, and the quantity conveyed is

$$u = (B+C+Q+R)X+(C-R)Y \qquad +(B-Q)Z.$$

Similarly

$$v = (C-R)X \qquad +(C+A+R+P)Y+(A-P)Z,$$

$$w = (B-Q)X \qquad +(A-P)Y \qquad +(A+B+P+Q)Z;$$

whence we find by comparison with the equations of conduction, Art. 298,

$$4A = r_2+r_3-r_1+2p_1, \qquad 4P = r_2+r_3-r_1-2p_1,$$

$$4B = r_3+r_1-r_2+2p_2, \qquad 4Q = r_3+r_1-r_2-2p_2,$$

$$4C = r_1+r_2-r_3+2p_3, \qquad 4R = r_1+r_2-r_3-2p_3.$$

# CHAPTER X.

## CONDUCTION IN DIELECTRICS.

325.] We have seen that when electromotive force acts on a dielectric medium it produces in it a state which we have called electric polarization, and which we have described as consisting of electric displacement within the medium in a direction which, in isotropic media, coincides with that of the electromotive force, combined with a superficial charge on every element of volume into which we may suppose the dielectric divided, which is negative on the side towards which the force acts, and positive on the side from which it acts.

When electromotive force acts on a conducting medium it also produces what is called an electric current.

Now dielectric media, with very few, if any, exceptions, are also more or less imperfect conductors, and many media which are not good insulators exhibit phenomena of dielectric induction. Hence we are led to study the state of a medium in which induction and conduction are going on at the same time.

For simplicity we shall suppose the medium isotropic at every point, but not necessarily homogeneous at different points. In this case, the equation of Poisson becomes, by Art. 83,

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right) + \frac{d}{dy}\left(K\frac{dV}{dy}\right) + \frac{d}{dz}\left(K\frac{dV}{dz}\right) + 4\pi\rho = 0, \tag{1}$$

where $K$ is the 'specific inductive capacity.'

The 'equation of continuity' of electric currents becomes

$$\frac{d}{dx}\left(\frac{1}{r}\frac{dV}{dx}\right) + \frac{d}{dy}\left(\frac{1}{r}\frac{dV}{dy}\right) + \frac{d}{dz}\left(\frac{1}{r}\frac{dV}{dz}\right) - \frac{d\rho}{dt} = 0, \tag{2}$$

where $r$ is the specific resistance referred to unit of volume.

When $K$ or $r$ is discontinuous, these equations must be transformed into those appropriate to surfaces of discontinuity.

In a strictly homogeneous medium $r$ and $K$ are both constant, so that we find

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = -4\pi \frac{\rho}{K} = r\frac{d\rho}{dt}, \tag{3}$$

whence
$$\rho = Ce^{-\frac{4\pi}{Kr}t}; \tag{4}$$

or, if we put
$$T = \frac{Kr}{4\pi}, \qquad \rho = Ce^{-\frac{t}{T}}. \tag{5}$$

This result shews that under the action of any external electric forces on a homogeneous medium, the interior of which is originally charged in any manner with electricity, the internal charges will die away at a rate which does not depend on the external forces, so that at length there will be no charge of electricity within the medium, after which no external forces can either produce or maintain a charge in any internal portion of the medium, provided the relation between electromotive force, electric polarization and conduction remains the same. When disruptive discharge occurs these relations cease to be true, and internal charge may be produced.

*On Conduction through a Condenser.*

326.] Let $C$ be the capacity of a condenser, $R$ its resistance, and $E$ the electromotive force which acts on it, that is, the difference of potentials of the surfaces of the metallic electrodes.

Then the quantity of electricity on the side from which the electromotive force acts will be $CE$, and the current through the substance of the condenser in the direction of the electromotive force will be $\frac{E}{R}$.

If the electrification is supposed to be produced by an electromotive force $E$ acting in a circuit of which the condenser forms part, and if $\frac{dQ}{dt}$ represents the current in that circuit, then

$$\frac{dQ}{dt} = \frac{E}{R} + C\frac{dE}{dt}. \tag{6}$$

Let a battery of electromotive force $E_0$ and resistance $r_1$ be introduced into this circuit, then

$$\frac{dQ}{dt} = \frac{E_0 - E}{r_1} = \frac{E}{R} + C\frac{dE}{dt}. \tag{7}$$

Hence, at any time $t_1$,

$$E\,(=E_1) = E_0\frac{R}{R+r_1}\left(1 - e^{-\frac{t_1}{T_1}}\right) \text{ where } T_1 = \frac{CRr_1}{R+r_1}. \tag{8}$$

Next, let the circuit $r_1$ be broken for a time $t_2$,

$$E(=E_2)=E_1 e^{-\frac{t_2}{T_2}} \text{ where } T_2 = CR. \tag{9}$$

Finally, let the surfaces of the condenser be connected by means of a wire whose resistance is $r_3$ for a time $t_3$,

$$E(=E_3)=E_2 e^{-\frac{t_3}{T_3}} \text{ where } T_3 = \frac{CRr_3}{R+r_3}. \tag{10}$$

If $Q_3$ is the total discharge through this wire in the time $t_3$,

$$Q_3 = E_0 \frac{CR^2}{(R+r_1)(R+r_3)} \left(1-e^{-\frac{t_1}{T_1}}\right) e^{-\frac{t_2}{T_2}} \left(1-e^{-\frac{t_3}{T_3}}\right). \tag{11}$$

In this way we may find the discharge through a wire which is made to connect the surfaces of a condenser after being charged for a time $t_1$, and then insulated for a time $t_2$. If the time of charging is sufficient, as it generally is, to develope the whole charge, and if the time of discharge is sufficient for a complete discharge, the discharge is

$$Q_3 = E_0 \frac{CR^2}{(R+r_1)(R+r_3)} e^{-\frac{t_2}{CR}}. \tag{12}$$

327.] In a condenser of this kind, first charged in any way, next discharged through a wire of small resistance, and then insulated, no new electrification will appear. In most actual condensers, however, we find that after discharge and insulation a new charge is gradually developed, of the same kind as the original charge, but inferior in intensity. This is called the residual charge. To account for it we must admit that the constitution of the dielectric medium is different from that which we have just described. We shall find, however, that a medium formed of a conglomeration of small pieces of different simple media would possess this property.

### *Theory of a Composite Dielectric.*

328.] We shall suppose, for the sake of simplicity, that the dielectric consists of a number of plane strata of different materials and of area unity, and that the electric forces act in the direction of the normal to the strata.

Let $a_1$, $a_2$, &c. be the thicknesses of the different strata.

$X_1$, $X_2$, &c. the resultant electrical force within each stratum.

$p_1, p_2$, &c. the current due to conduction through each stratum.

$f_1, f_2$, &c. the electric displacement.

$u_1$, $u_2$, &c. the total current, due partly to conduction and partly to variation of displacement.

$r_1$, $r_2$, &c. the specific resistance referred to unit of volume.

$K_1$, $K_2$, &c. the specific inductive capacity.

$k_1$, $k_2$, &c. the reciprocal of the specific inductive capacity.

$E$ the electromotive force due to a voltaic battery, placed in the part of the circuit leading from the last stratum towards the first, which we shall suppose good conductors.

$Q$ the total quantity of electricity which has passed through this part of the circuit up to the time $t$.

$R_0$ the resistance of the battery with its connecting wires.

$\sigma_{12}$ the surface-density of electricity on the surface which separates the first and second strata.

Then in the first stratum we have, by Ohm's Law,

$$X_1 = r_1 p_1. \tag{1}$$

By the theory of electrical displacement,

$$X_1 = 4\pi k_1 f_1. \tag{2}$$

By the definition of the total current,

$$u_1 = p_1 + \frac{df_1}{dt}, \tag{3}$$

with similar equations for the other strata, in each of which the quantities have the suffix belonging to that stratum.

To determine the surface-density on any stratum, we have an equation of the form

$$\sigma_{12} = f_2 - f_1, \tag{4}$$

and to determine its variation we have

$$\frac{d\sigma_{12}}{dt} = p_1 - p_2. \tag{5}$$

By differentiating (4) with respect to $t$, and equating the result to (5), we obtain

$$p_1 + \frac{df_1}{dt} = p_2 + \frac{df_2}{dt} = u, \text{ say}, \tag{6}$$

or, by taking account of (3),

$$u_1 = u_2 = \&\text{c.} = u. \tag{7}$$

That is, the total current $u$ is the same in all the strata, and is equal to the current through the wire and battery.

We have also, in virtue of equations (1) and (2),

$$u = \frac{1}{r_1} X_1 + \frac{1}{4\pi k_1} \frac{dX_1}{dt}, \tag{8}$$

from which we may find $X_1$ by the inverse operation on $u$,

$$X_1 = \left(\frac{1}{r_1} + \frac{1}{4\pi k} \frac{d}{dt}\right)^{-1} u. \tag{9}$$

The total electromotive force $E$ is

$$E = a_1 X_1 + a_2 X_2 + \&c., \tag{10}$$

or
$$E = \left\{ a_1 \left(\frac{1}{r_1} + \frac{1}{4\pi k_1}\frac{d}{dt}\right)^{-1} + a_2 \left(\frac{1}{r_2} + \frac{1}{4\pi k_2}\frac{d}{dt}\right)^{-1} + \&c. \right\} u, \tag{11}$$

an equation between $E$, the external electromotive force, and $u$, the external current.

If the ratio of $r$ to $k$ is the same in all the strata, the equation reduces itself to

$$E + \frac{r}{4\pi k}\frac{dE}{dt} = (a_1 r_1 + a_2 r_2 + \&c.)\, u, \tag{12}$$

which is the case we have already examined, and in which, as we found, no phenomenon of residual charge can take place.

If there are $n$ substances having different ratios of $r$ to $k$, the general equation (11), when cleared of inverse operations, will be a linear differential equation, of the $n$th order with respect to $E$ and of the $(n-1)$th order with respect to $u$, $t$ being the independent variable.

From the form of the equation it is evident that the order of the different strata is indifferent, so that if there are several strata of the same substance we may suppose them united into one without altering the phenomena.

329.] Let us now suppose that at first $f_1, f_2$, &c. are all zero, and that an electromotive force $E$ is suddenly made to act, and let us find its instantaneous effect.

Integrating (8) with respect to $t$, we find

$$Q = \int u\, dt = \frac{1}{r_1}\int X_1\, dt + \frac{1}{4\pi k_1} X_1 + \text{const.} \tag{13}$$

Now, since $X_1$ is always in this case finite, $\int X_1\, dt$ must be insensible when $t$ is insensible, and therefore, since $X_1$ is originally zero, the instantaneous effect will be

$$X_1 = 4\pi k_1 Q. \tag{14}$$

Hence, by equation (10),

$$E = 4\pi (k_1 a_1 + k_2 a_2 + \&c.)\, Q, \tag{15}$$

and if $C$ be the electric capacity of the system as measured in this instantaneous way,

$$C = \frac{Q}{E} = \frac{1}{4\pi (k_1 a_1 + k_2 a_2 + \&c.)}. \tag{16}$$

This is the same result that we should have obtained if we had neglected the conductivity of the strata.

Let us next suppose that the electromotive force $E$ is continued uniform for an indefinitely long time, or till a uniform current of conduction equal to $p$ is established through the system.

We have then $X_1 = r_1 p$, and therefore

$$E = (r_1 a_1 + r_2 a_2 + \&c.)\, p. \tag{17}$$

If $R$ be the total resistance of the system,

$$R = \frac{E}{p} = r_1 a_1 + r_2 a_2 + \&c. \tag{18}$$

In this state we have by (2),

$$f_1 = \frac{r_1}{4\pi k_1} p,$$

so that

$$\sigma_{12} = \left(\frac{r_2}{4\pi k_2} - \frac{r_1}{4\pi k_1}\right) p. \tag{19}$$

If we now suddenly connect the extreme strata by means of a conductor of small resistance, $E$ will be suddenly changed from its original value $E_0$ to zero, and a quantity $Q$ of electricity will pass through the conductor.

To determine $Q$ we observe that if $X_1'$ be the new value of $X_1$, then by (13),

$$X_1' = X_1 + 4\pi k_1 Q. \tag{20}$$

Hence, by (10), putting $E = 0$,

$$0 = a_1 X_1 + \&c. + 4\pi (a_1 k_1 + a_2 k_2 + \&c.)\, Q, \tag{21}$$

or

$$0 = E_0 + \frac{1}{C} Q. \tag{22}$$

Hence $Q = -CE_0$ where $C$ is the capacity, as given by equation (16). The instantaneous discharge is therefore equal to the instantaneous charge.

Let us next suppose the connexion broken immediately after this discharge. We shall then have $u = 0$, so that by equation (8),

$$X_1 = X' e^{-\frac{4\pi k_1}{r_1} t}, \tag{23}$$

where $X'$ is the initial value after the discharge.

Hence, at any time $t$,

$$X_1 = E_0 \left\{ \frac{r_1}{R} - 4\pi k_1 C \right\} e^{-\frac{4\pi k_1}{r_1} t}.$$

The value of $E$ at any time is therefore

$$E = E_0 \left\{ \left(\frac{a_1 r_1}{R} - 4\pi a_1 k_1 C\right) e^{-\frac{4\pi k_1}{r_1} t} + \left(\frac{a_2 r_2}{R} - 4\pi a_2 k_2 C\right) e^{-\frac{4\pi k_2}{r_2} t} + \&c. \right\}, \tag{24}$$

and the instantaneous discharge after any time $t$ is $EC$. This is called the residual discharge.

If the ratio of $r$ to $k$ is the same for all the strata, the value of $E$ will be reduced to zero. If, however, this ratio is not the same, let the terms be arranged according to the values of this ratio in descending order of magnitude.

The sum of all the coefficients is evidently zero, so that when $t=0$, $E=0$. The coefficients are also in descending order of magnitude, and so are the exponential terms when $t$ is positive. Hence, when $t$ is positive, $E$ will be positive, so that the residual discharge is always of the same sign as the primary discharge.

When $t$ is indefinitely great all the terms disappear unless any of the strata are perfect insulators, in which case $r_1$ is infinite for that stratum, and $R$ is infinite for the whole system, and the final value of $E$ is not zero but

$$E = E_0 (1 - 4\pi a_1 k_1 C). \tag{25}$$

Hence, when some, but not all, of the strata are perfect insulators, a residual discharge may be permanently preserved in the system.

330.] We shall next determine the total discharge through a wire of resistance $R_0$ kept permanently in connexion with the extreme strata of the system, supposing the system first charged by means of a long-continued application of the electromotive force $E$.

At any instant we have

$$E = a_1 r_1 p_1 + a_2 r_2 p_2 + \&c. + R_0 u = 0, \tag{26}$$

and also, by (3),
$$u = p_1 + \frac{df_1}{dt}. \tag{27}$$

Hence
$$(R+R_0)\, u = a_1 r_1 \frac{df_1}{dt} + a_2 r_2 \frac{df_2}{dt} + \&c. \tag{28}$$

Integrating with respect to $t$ in order to find $Q$, we get

$$(R+R_0)\, Q = a_1 r_1 (f_1' - f_1) + a_2 r_2 (f_2' - f_2) + \&c., \tag{29}$$

where $f_1$ is the initial, and $f_1'$ the final value of $f_1$.

In this case $f_1' = 0$, and $f_1 = E_0 \left(\frac{r_1}{4\pi k_1 R} - C\right)$.

Hence
$$(R+R_0)\, Q = \frac{E_0}{4\pi R}\left(\frac{a_1 r_1^2}{k_1} + \frac{a_2 r_2^2}{k_2} + \&c.\right) - E_0 C R, \tag{30}$$

$$= -\frac{CE_0}{R} \Sigma\Sigma \left[a_1 a_2 k_1 k_2 \left(\frac{r_1}{k_1} - \frac{r_2}{k_2}\right)^2\right], \tag{31}$$

where the summation is extended to all quantities of this form belonging to every pair of strata.

It appears from this that $Q$ is always negative, that is to say, in the opposite direction to that of the current employed in charging the system.

This investigation shews that a dielectric composed of strata of different kinds may exhibit the phenomena known as electric absorption and residual discharge, although none of the substances of which it is made exhibit these phenomena when alone. An investigation of the cases in which the materials are arranged otherwise than in strata would lead to similar results, though the calculations would be more complicated, so that we may conclude that the phenomena of electric absorption may be expected in the case of substances composed of parts of different kinds, even though these individual parts should be microscopically small.

It by no means follows that every substance which exhibits this phenomenon is so composed, for it may indicate a new kind of electric polarization of which a homogeneous substance may be capable, and this in some cases may perhaps resemble electrochemical polarization much more than dielectric polarization.

The object of the investigation is merely to point out the true mathematical character of the so-called electric absorption, and to shew how fundamentally it differs from the phenomena of heat which seem at first sight analogous.

331.] If we take a thick plate of any substance and heat it on one side, so as to produce a flow of heat through it, and if we then suddenly cool the heated side to the same temperature as the other, and leave the plate to itself, the heated side of the plate will again become hotter than the other by conduction from within.

Now an electrical phenomenon exactly analogous to this can be produced, and actually occurs in telegraph cables, but its mathematical laws, though exactly agreeing with those of heat, differ entirely from those of the stratified condenser.

In the case of heat there is true absorption of the heat into the substance with the result of making it hot. To produce a truly analogous phenomenon in electricity is impossible, but we may imitate it in the following way in the form of a lecture-room experiment.

Let $A_1$, $A_2$, &c. be the inner conducting surfaces of a series of condensers, of which $B_0$, $B_1$, $B_2$, &c. are the outer surfaces.

Let $A_1$, $A_2$, &c. be connected in series by connexions of resistance $R$, and let a current be passed along this series from left to right.

Let us first suppose the plates $B_0$, $B_1$, $B_2$, each insulated and free from charge. Then the total quantity of electricity on each of the plates $B$ must remain zero, and since the electricity on the plates $A$ is in each case equal and opposite to that of the opposed

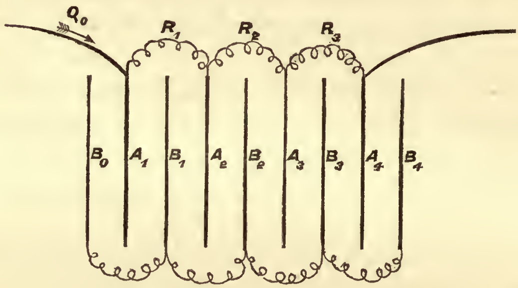



Fig. 25.

surface they will not be electrified, and no alteration of the current will be observed.

But let the plates $B$ be all connected together, or let each be connected with the earth. Then, since the potential of $A_1$ is positive, while that of the plates $B$ is zero, $A_1$ will be positively electrified and $B_1$ negatively.

If $P_1$, $P_2$, &c. are the potentials of the plates $A_1$, $A_2$, &c., and $C$ the capacity of each, and if we suppose that a quantity of electricity equal to $Q_0$ passes through the wire on the left, $Q_1$ through the connexion $R_1$, and so on, then the quantity which exists on the plate $A_1$ is $Q_0 - Q_1$, and we have

$$Q_0 - Q_1 = C_1 P_1.$$

Similarly

$$Q_1 - Q_2 = C_2 P_2,$$

and so on.

But by Ohm's Law we have

$$P_1 - P_2 = R_1 \frac{dQ_1}{dt},$$

$$P_2 - P_3 = R_2 \frac{dQ_2}{dt}.$$

If we suppose the values of $C$ the same for each plate, and those of $R$ the same for each wire, we shall have a series of equations of the form

$$Q_0 - 2\,Q_1 + Q_2 = RC\frac{dQ_1}{dt},$$

$$Q_1 - 2\,Q_2 + Q_3 = RC\frac{dQ_2}{dt}.$$

If there are $n$ quantities of electricity to be determined, and if either the total electromotive force, or some other equivalent conditions be given, the differential equation for determining any one of them will be linear and of the $n$th order.

By an apparatus arranged in this way, Mr. Varley succeeded in imitating the electrical action of a cable 12,000 miles long.

When an electromotive force is made to act along the wire on the left hand, the electricity which flows into the system is at first principally occupied in charging the different condensers beginning with $A_1$, and only a very small fraction of the current appears at the right hand till a considerable time has elapsed. If galvanometers be placed in circuit at $R_1$, $R_2$, &c. they will be affected by the current one after another, the interval between the times of equal indications being greater as we proceed to the right.

332.] In the case of a telegraph cable the conducting wire is separated from conductors outside by a cylindrical sheath of gutta-percha, or other insulating material. Each portion of the cable thus becomes a condenser, the outer surface of which is always at potential zero. Hence, in a given portion of the cable, the quantity of free electricity at the surface of the conducting wire is equal to the product of the potential into the capacity of the portion of the cable considered as a condenser.

If $a_1$, $a_2$ are the outer and inner radii of the insulating sheath, and if $K$ is its specific dielectric capacity, the capacity of unit of length of the cable is, by Art. 126,

$$c = \frac{K}{2\log\frac{a_1}{a_2}}. \tag{1}$$

Let $v$ be the potential at any point of the wire, which we may consider as the same at every part of the same section.

Let $Q$ be the total quantity of electricity which has passed through that section since the beginning of the current. Then the quantity which at the time $t$ exists between sections at $x$ and at $x+\delta x$, is

$$Q - \left(Q + \frac{dQ}{dx}\delta x\right), \quad \text{or} \quad -\frac{dQ}{dx}\delta x,$$

and this is, by what we have said, equal to $cv\delta x$.

Hence $$cv = -\frac{dQ}{dx}. \tag{2}$$

Again, the electromotive force at any section is $-\frac{dv}{dx}$, and by Ohm's Law,
$$-\frac{dv}{dx} = k\frac{dQ}{dt}, \tag{3}$$
where $k$ is the resistance of unit of length of the conductor, and $\frac{dQ}{dt}$ is the strength of the current. Eliminating $Q$ between (2) and (3), we find
$$ck\frac{dv}{dt} = \frac{d^2v}{dx^2}. \tag{4}$$

This is the partial differential equation which must be solved in order to obtain the potential at any instant at any point of the cable. It is identical with that which Fourier gives to determine the temperature at any point of a stratum through which heat is flowing in a direction normal to the stratum. In the case of heat $c$ represents the capacity of unit of volume, or what Fourier calls $CD$, and $k$ represents the reciprocal of the conductivity.

If the sheath is not a perfect insulator, and if $k_1$ is the resistance of unit of length of the sheath to conduction through it in a radial direction, then if $\rho_1$ is the specific resistance of the insulating material,
$$k_1 = 2\rho_1 \log_e \frac{r_1}{r_2}. \tag{5}$$

The equation (2) will no longer be true, for the electricity is expended not only in charging the wire to the extent represented by $cv$, but in escaping at a rate represented by $\frac{v}{k_1}$. Hence the rate of expenditure of electricity will be
$$-\frac{d^2Q}{dx\,dt} = c\frac{dv}{dt} + \frac{1}{k_1}v, \tag{6}$$
whence, by comparison with (3), we get
$$ck\frac{dv}{dt} = \frac{d^2v}{dx^2} - \frac{k}{k_1}v, \tag{7}$$
and this is the equation of conduction of heat in a rod or ring as given by Fourier *.

333.] If we had supposed that a body when raised to a high potential becomes electrified throughout its substance as if electricity were compressed into it, we should have arrived at equations of this very form. It is remarkable that Ohm himself,

* *Théorie de la Chaleur*, art. 105.

misled by the analogy between electricity and heat, entertained an opinion of this kind, and was thus, by means of an erroneous opinion, led to employ the equations of Fourier to express the true laws of conduction of electricity through a long wire, long before the real reason of the appropriateness of these equations had been suspected.

*Mechanical Illustration of the Properties of a Dielectric.*

334.] Five tubes of equal sectional area $A$, $B$, $C$, $D$ and $P$ are arranged in circuit as in the figure. $A$, $B$, $C$ and $D$ are vertical and equal, and $P$ is horizontal.

The lower halves of $A$, $B$, $C$, $D$ are filled with mercury, their upper halves and the horizontal tube $P$ are filled with water.

A tube with a stopcock $Q$ connects the lower part of $A$ and $B$ with that of $C$ and $D$, and a piston $P$ is made to slide in the horizontal tube.

Let us begin by supposing that the level of the mercury in the four tubes is the same, and that it is indicated by $A_0$, $B_0$, $C_0$, $D_0$, that the piston is at $P_0$, and that the stopcock $Q$ is shut.

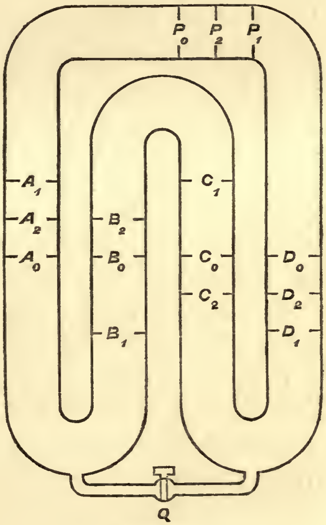


Fig. 26.

Now let the piston be moved from $P_0$ to $P_1$, a distance $a$. Then, since the sections of all the tubes are equal, the level of the mercury in $A$ and $C$ will rise a distance $a$, or to $A_1$ and $C_1$, and the mercury in $B$ and $D$ will sink an equal distance $a$, or to $B_1$ and $D_1$.

The difference of pressure on the two sides of the piston will be represented by $4a$.

This arrangement may serve to represent the state of a dielectric acted on by an electromotive force $4a$.

The excess of water in the tube $D$ may be taken to represent a positive charge of electricity on one side of the dielectric, and the excess of mercury in the tube $A$ may represent the negative charge on the other side. The excess of pressure in the tube $P$ on the side of the piston next $D$ will then represent the excess of potential on the positive side of the dielectric.

If the piston is free to move it will move back to $P_0$ and be in equilibrium there. This represents the complete discharge of the dielectric.

During the discharge there is a reversed motion of the liquids throughout the whole tube, and this represents that change of electric displacement which we have supposed to take place in a dielectric.

I have supposed every part of the system of tubes filled with incompressible liquids, in order to represent the property of all electric displacement that there is no real accumulation of electricity at any place.

Let us now consider the effect of opening the stopcock $Q$ while the piston $P$ is at $P_1$.

The level of $A_1$ and $D_1$ will remain unchanged, but that of $B$ and $C$ will become the same, and will coincide with $B_0$ and $C_0$.

The opening of the stopcock $Q$ corresponds to the existence of a part of the dielectric which has a slight conducting power, but which does not extend through the whole dielectric so as to form an open channel.

The charges on the opposite sides of the dielectric remain insulated, but their difference of potential diminishes.

In fact, the difference of pressure on the two sides of the piston sinks from $4a$ to $2a$ during the passage of the fluid through $Q$.

If we now shut the stopcock $Q$ and allow the piston $P$ to move freely, it will come to equilibrium at a point $P_2$, and the discharge will be apparently only half of the charge.

The level of the mercury in $A$ and $B$ will be $\frac{1}{2}a$ above its original level, and the level in the tubes $C$ and $D$ will be $\frac{1}{2}a$ below its original level. This is indicated by the levels $A_2$, $B_2$, $C_2$, $D_2$.

If the piston is now fixed and the stopcock opened, mercury will flow from $B$ to $C$ till the level in the two tubes is again at $B_0$ and $C_0$. There will then be a difference of pressure $= a$ on the two sides of the piston $P$. If the stopcock is then closed and the piston $P$ left free to move, it will again come to equilibrium at a point $P_3$, half way between $P_2$ and $P_0$. This corresponds to the residual charge which is observed when a charged dielectric is first discharged and then left to itself. It gradually recovers part of its charge, and if this is again discharged a third charge is formed, the successive charges diminishing in quantity. In the case of the illustrative experiment each charge is half of the preceding, and the discharges, which are $\frac{1}{2}$, $\frac{1}{4}$, &c. of the original charge, form a series whose sum is equal to the original charge.

If, instead of opening and closing the stopcock, we had allowed it to remain nearly, but not quite, closed during the whole experiment, we should have had a case resembling that of the electrification of a dielectric which is a perfect insulator and yet exhibits the phenomenon called 'electric absorption.'

To represent the case in which there is true conduction through the dielectric we must either make the piston leaky, or we must establish a communication between the top of the tube $A$ and the top of the tube $D$.

In this way we may construct a mechanical illustration of the properties of a dielectric of any kind, in which the two electricities are represented by two real fluids, and the electric potential is represented by fluid pressure. Charge and discharge are represented by the motion of the piston $P$, and electromotive force by the resultant force on the piston.

# CHAPTER XI.

## THE MEASUREMENT OF ELECTRIC RESISTANCE.

335.] In the present state of electrical science, the determination of the electric resistance of a conductor may be considered as the cardinal operation in electricity, in the same sense that the determination of weight is the cardinal operation in chemistry.

The reason of this is that the determination in absolute measure of other electrical magnitudes, such as quantities of electricity, electromotive forces, currents, &c., requires in each case a complicated series of operations, involving generally observations of time, measurements of distances, and determinations of moments of inertia, and these operations, or at least some of them, must be repeated for every new determination, because it is impossible to preserve a unit of electricity, or of electromotive force, or of current, in an unchangeable state, so as to be available for direct comparison.

But when the electric resistance of a properly shaped conductor of a properly chosen material has been once determined, it is found that it always remains the same for the same temperature, so that the conductor may be used as a standard of resistance, with which that of other conductors can be compared, and the comparison of two resistances is an operation which admits of extreme accuracy.

When the unit of electrical resistance has been fixed on, material copies of this unit, in the form of 'Resistance Coils,' are prepared for the use of electricians, so that in every part of the world electrical resistances may be expressed in terms of the same unit. These unit resistance coils are at present the only examples of material electric standards which can be preserved, copied, and used for the purpose of measurement. Measures of electrical capacity, which are also of great importance, are still defective, on account of the disturbing influence of electric absorption.

336.] The unit of resistance may be an entirely arbitrary one, as in the case of Jacobi's Etalon, which was a certain copper wire of 22.4932 grammes weight, 7.61975 metres length, and 0.667

millimetres diameter. Copies of this have been made by Leyser of Leipsig, and are to be found in different places.

According to another method the unit may be defined as the resistance of a portion of a definite substance of definite dimensions. Thus, Siemens' unit is defined as the resistance of a column of mercury of one metre long, and one square millimetre section, at the temperature 0°C.

337.] Finally, the unit may be defined with reference to the electrostatic or the electromagnetic system of units. In practice the electromagnetic system is used in all telegraphic operations, and therefore the only systematic units actually in use are those of this system.

In the electromagnetic system, as we shall shew at the proper place, a resistance is a quantity homogeneous with a velocity, and may therefore be expressed as a velocity. See Art. 628.

338.] The first actual measurements on this system were made by Weber, who employed as his unit one millimetre per second. Sir W. Thomson afterwards used one foot per second as a unit, but a large number of electricians have now agreed to use the unit of the British Association, which professes to represent a resistance which, expressed as a velocity, is ten millions of metres per second. The magnitude of this unit is more convenient than that of Weber's unit, which is too small. It is sometimes referred to as the B.A. unit, but in order to connect it with the name of the discoverer of the laws of resistance, it is called the Ohm.

339.] To recollect its value in absolute measure it is useful to know that ten millions of metres is professedly the distance from the pole to the equator, measured along the meridian of Paris. A body, therefore, which in one second travels along a meridian from the pole to the equator would have a velocity which, on the electromagnetic system, is professedly represented by an Ohm.

I say professedly, because, if more accurate researches should prove that the Ohm, as constructed from the British Association's material standards, is not really represented by this velocity, electricians would not alter their standards, but would apply a correction. In the same way the metre is professedly one ten-millionth of a certain quadrantal arc, but though this is found not to be exactly true, the length of the metre has not been altered, but the dimensions of the earth are expressed by a less simple number.

According to the system of the British Association, the absolute value of the unit is *originally chosen* so as to represent as nearly as possible a quantity derived from the electromagnetic absolute system.

340.] When a material unit representing this abstract quantity has been made, other standards are constructed by copying this unit, a process capable of extreme accuracy—of much greater accuracy than, for instance, the copying of foot-rules from a standard foot.

These copies, made of the most permanent materials, are distributed over all parts of the world, so that it is not likely that any difficulty will be found in obtaining copies of them if the original standards should be lost.

But such units as that of Siemens can without very great labour be reconstructed with considerable accuracy, so that as the relation of the Ohm to Siemens unit is known, the Ohm can be reproduced even without having a standard to copy, though the labour is much greater and the accuracy much less than by the method of copying.

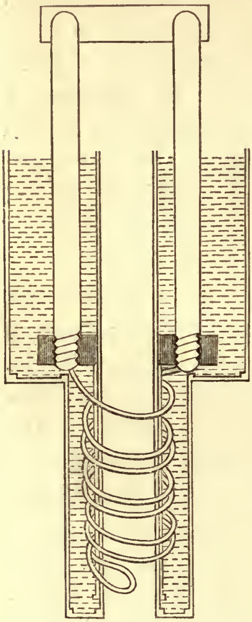


Fig. 27.

Finally, the Ohm may be reproduced by the electromagnetic method by which it was originally determined. This method, which is considerably more laborious than the determination of a foot from the seconds pendulum, is probably inferior in accuracy to that last mentioned. On the other hand, the determination of the electromagnetic unit in terms of the Ohm with an amount of accuracy corresponding to the progress of electrical science, is a most important physical research and well worthy of being repeated.

The actual resistance coils constructed to represent the Ohm were made of an alloy of two parts of silver and one of platinum in the form of wires from .5 millimetres to .8 millimetres diameter, and from one to two metres in length. These wires were soldered to stout copper electrodes. The wire itself was covered with two layers of silk, imbedded in solid paraffin, and enclosed in a thin brass case, so that it can be easily brought to a temperature at which its resistance is accurately one Ohm. This temperature is marked on the insulating support of the coil. (See Fig. 27.)

### *On the Forms of Resistance Coils.*

341.] A Resistance Coil is a conductor capable of being easily placed in the voltaic circuit, so as to introduce into the circuit a known resistance.

The electrodes or ends of the coil must be such that no appreciable error may arise from the mode of making the connexions. For resistances of considerable magnitude it is sufficient that the electrodes should be made of stout copper wire or rod well amalgamated with mercury at the ends, and that the ends should be made to press on flat amalgamated copper surfaces placed in mercury cups.

For very great resistances it is sufficient that the electrodes should be thick pieces of brass, and that the connexions should be made by inserting a wedge of brass or copper into the interval between them. This method is found very convenient.

The resistance coil itself consists of a wire well covered with silk, the ends of which are soldered permanently to the electrodes.

The coil must be so arranged that its temperature may be easily observed. For this purpose the wire is coiled on a tube and covered with another tube, so that it may be placed in a vessel of water, and that the water may have access to the inside and the outside of the coil.

To avoid the electromagnetic effects of the current in the coil the wire is first doubled back on itself and then coiled on the tube, so that at every part of the coil there are equal and opposite currents in the adjacent parts of the wire.

When it is desired to keep two coils at the same temperature the wires are sometimes placed side by side and coiled up together. This method is especially useful when it is more important to secure equality of resistance than to know the absolute value of the resistance, as in the case of the equal arms of Wheatstone's Bridge, (Art. 347).

When measurements of resistance were first attempted, a resistance coil, consisting of an uncovered wire coiled in a spiral groove round a cylinder of insulating material, was much used. It was called a Rheostat. The accuracy with which it was found possible to compare resistances was soon found to be inconsistent with the use of any instrument in which the contacts are not more perfect than can be obtained in the rheostat. The rheostat, however, is still used for adjusting the resistance where accurate measurement is not required.

Resistance coils are generally made of those metals whose resistance is greatest and which vary least with temperature. German silver fulfils these conditions very well, but some specimens are found to change their properties during the lapse of years. Hence for standard coils, several pure metals, and also an alloy of platinum and silver, have been employed, and the relative resistance of these during several years has been found constant up to the limits of modern accuracy.

342.] For very great resistances, such as several millions of Ohms, the wire must be either very long or very thin, and the construction of the coil is expensive and difficult. Hence tellurium and selenium have been proposed as materials for constructing standards of great resistance. A very ingenious and easy method of construction has been lately proposed by Phillips *. On a piece of ebonite or ground glass a fine pencil-line is drawn. The ends of this filament of plumbago are connected to metallic electrodes, and the whole is then covered with insulating varnish. If it should be found that the resistance of such a pencil-line remains constant, this will be the best method of obtaining a resistance of several millions of Ohms.

343.] There are various arrangements by which resistance coils may be easily introduced into a circuit.

For instance, a series of coils of which the resistances are 1, 2, 4, 8, 16, &c., arranged according to the powers of 2, may be placed in a box in series.

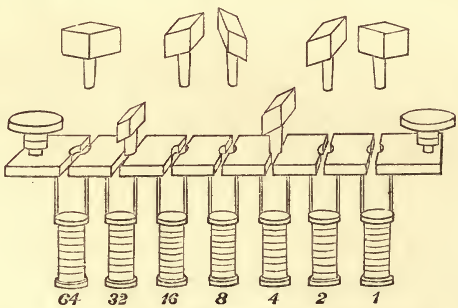



Fig. 28.

The electrodes consist of stout brass plates, so arranged on the outside of the box that by inserting a brass plug or wedge between

* *Phil. Mag.*, July, 1870.

two of them as a shunt, the resistance of the corresponding coil may be put out of the circuit. This arrangement was introduced by Siemens.

Each interval between the electrodes is marked with the resistance of the corresponding coil, so that if we wish to make the resistance box equal to 107 we express 107 in the binary scale as $64+32+8+2+1$ or 1101011. We then take the plugs out of the holes corresponding to 64, 32, 8, 2 and 1, and leave the plugs in 16 and 4.

This method, founded on the binary scale, is that in which the smallest number of separate coils is needed, and it is also that which can be most readily tested. For if we have another coil equal to 1 we can test the equality of 1 and $1'$, then that of $1+1'$ and 2, then that of $1+1'+2$ and 4, and so on.

The only disadvantage of the arrangement is that it requires a familiarity with the binary scale of notation, which is not generally possessed by those accustomed to express every number in the decimal scale.

344.] A box of resistance coils may be arranged in a different way for the purpose of measuring conductivities instead of resistances.

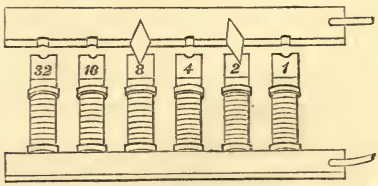


Fig. 29.

The coils are placed so that one end of each is connected with a long thick piece of metal which forms one electrode of the box, and the other end is connected with a stout piece of brass plate as in the former case.

The other electrode of the box is a long brass plate, such that by inserting brass plugs between it and the electrodes of the coils it may be connected to the first electrode through any given set of coils. The conductivity of the box is then the sum of the conductivities of the coils.

In the figure, in which the resistances of the coils are 1, 2, 4, &c., and the plugs are inserted at 2 and 8, the conductivity of the box is $\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$, and the resistance of the box is therefore $\frac{8}{5}$ or 1.6.

This method of combining resistance coils for the measurement of fractional resistances was introduced by Sir W. Thomson under the name of the method of multiple arcs. See Art. 276.

*On the Comparison of Resistances.*

345.] If $E$ is the electromotive force of a battery, and $R$ the resistance of the battery and its connexions, including the galvanometer used in measuring the current, and if the strength of the current is $I$ when the battery connexions are closed, and $I_1$, $I_2$ when additional resistances $r_1$, $r_2$ are introduced into the circuit, then, by Ohm's Law,

$$E = IR = I_1(R+r_1) = I_2(R+r_2).$$

Eliminating $E$, the electromotive force of the battery, and $R$ the resistance of the battery and its connexions, we get Ohm's formula

$$\frac{r_1}{r_2} = \frac{(I-I_1)\,I_2}{(I-I_2)\,I_1}.$$

This method requires a measurement of the ratios of $I$, $I_1$ and $I_2$, and this implies a galvanometer graduated for absolute measurements.

If the resistances $r_1$ and $r_2$ are equal, then $I_1$ and $I_2$ are equal, and we can test the equality of currents by a galvanometer which is not capable of determining their ratios.

But this is rather to be taken as an example of a faulty method than as a practical method of determining resistance. The electromotive force $E$ cannot be maintained rigorously constant, and the internal resistance of the battery is also exceedingly variable, so that any methods in which these are assumed to be even for a short time constant are not to be depended on.

346.] The comparison of resistances can be made with extreme

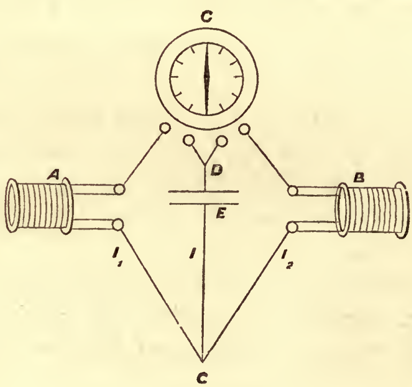



Fig. 30.

accuracy by either of two methods, in which the result is independent of variations of $R$ and $E$.

The first of these methods depends on the use of the differential galvanometer, an instrument in which there are two coils, the currents in which are independent of each other, so that when the currents are made to flow in opposite directions they act in opposite directions on the needle, and when the ratio of these currents is that of $m$ to $n$ they have no resultant effect on the galvanometer needle.

Let $I_1, I_2$ be the currents through the two coils of the galvanometer, then the deflexion of the needle may be written

$$\delta = m I_1 - n I_2.$$

Now let the battery current $I$ be divided between the coils of the galvanometer, and let resistances $A$ and $B$ be introduced into the first and second coils respectively. Let the remainder of the resistance of their coils and their connexions be $\alpha$ and $\beta$ respectively, and let the resistance of the battery and its connexions between $C$ and $D$ be $r$, and its electromotive force $E$.

Then we find, by Ohm's Law, for the difference of potentials between $C$ and $D$,

$$C-D = I_1(A+\alpha) = I_2(B+\beta) = E-Ir,$$

and since $$I_1+I_2=I,$$

$$I_1 = E\frac{B+\beta}{D}, \qquad I_2 = E\frac{A+\alpha}{D}, \qquad I = E\frac{A+\alpha+B+\beta}{D},$$

where $$D = (A+\alpha)(B+\beta)+r(A+\alpha+B+\beta).$$

The deflexion of the galvanometer needle is therefore

$$\delta = \frac{E}{D}\{m(B+\beta)-n(A+\alpha)\},$$

and if there is no observable deflexion, then we know that the quantity enclosed in brackets cannot differ from zero by more than a certain small quantity, depending on the power of the battery, the suitableness of the arrangement, the delicacy of the galvanometer, and the accuracy of the observer.

Suppose that $B$ has been adjusted so that there is no apparent deflexion.

Now let another conductor $A'$ be substituted for $A$, and let $A'$ be adjusted till there is no apparent deflexion. Then evidently to a first approximation $A' = A$.

To ascertain the degree of accuracy of this estimate, let the altered quantities in the second observation be accented, then

$$m(B+\beta)-n(A+a)=\frac{D}{E}\delta,$$

$$m(B+\beta)-n(A'+a)=\frac{D'}{E'}\delta'.$$

Hence $$n(A'-A)=\frac{D}{E}\delta-\frac{D'}{E'}\delta'.$$

If $\delta$ and $\delta'$, instead of being both apparently zero, had been only observed to be equal, then, unless we also could assert that $E=E'$, the right-hand side of the equation might not be zero. In fact, the method would be a mere modification of that already described.

The merit of the method consists in the fact that the thing observed is the absence of any deflexion, or in other words, the method is a Null method, one in which the non-existence of a force is asserted from an observation in which the force, if it had been different from zero by more than a certain small amount, would have produced an observable effect.

Null methods are of great value where they can be employed, but they can only be employed where we can cause two equal and opposite quantities of the same kind to enter into the experiment together.

In the case before us both $\delta$ and $\delta'$ are quantities too small to be observed, and therefore any change in the value of $E$ will not affect the accuracy of the result.

The actual degree of accuracy of this method might be ascertained by taking a number of observations in each of which $A'$ is separately adjusted, and comparing the result of each observation with the mean of the whole series.

But by putting $A'$ out of adjustment by a known quantity, as, for instance, by inserting at $A$ or at $B$ an additional resistance equal to a hundredth part of $A$ or of $B$, and then observing the resulting deviation of the galvanometer needle, we can estimate the number of degrees corresponding to an error of one per cent. To find the actual degree of precision we must estimate the smallest deflexion which could not escape observation, and compare it with the deflexion due to an error of one per cent.

*If the comparison is to be made between $A$ and $B$, and if the positions of $A$ and $B$ are exchanged, then the second equation becomes

* This investigation is taken from Weber's treatise on Galvanometry. *Göttingen Transactions*, x. p. 65.

$$m(A+\beta)-n(B+a) = \frac{D'}{E}\delta',$$

whence $$(m+n)(B-A) = \frac{D}{E}\delta - \frac{D'}{E'}\delta'.$$

If $m$ and $n$, $A$ and $B$, $a$ and $\beta$ are approximately equal, then

$$B-A = \frac{1}{2nE}(A+a)(A+a+2r)(\delta-\delta').$$

Here $\delta-\delta'$ may be taken to be the smallest observable deflexion of the galvanometer.

If the galvanometer wire be made longer and thinner, retaining the same total mass, then $n$ will vary as the length of the wire and $a$ as the square of the length. Hence there will be a minimum value of $\frac{(A+a)(A+a+2r)}{n}$ when

$$a = \tfrac{1}{3}(A+r)\left\{2\sqrt{1-\frac{3}{4}\frac{r^2}{(A+r)^2}}-1\right\}.$$

If we suppose $r$, the battery resistance, small compared with $A$, this gives
$$a = \tfrac{1}{3}A;$$

or, *the resistance of each coil of the galvanometer should be one-third of the resistance to be measured.*

We then find $$B-A = \frac{8}{9}\frac{A^2}{nE}(\delta-\delta').$$

If we allow the current to flow through one only of the coils of the galvanometer, and if the deflexion thereby produced is $\Delta$ (supposing the deflexion strictly proportional to the deflecting force), then

$$\Delta = \frac{mE}{A+a+r} = \frac{3}{4}\frac{nE}{A} \text{ if } r = 0 \text{ and } a = \frac{1}{3}A.$$

Hence $$\frac{B-A}{A} = \frac{2}{3}\frac{\delta-\delta'}{\Delta}.$$

In the differential galvanometer two currents are made to produce equal and opposite effects on the suspended needle. The force with which either current acts on the needle depends not only on the strength of the current, but on the position of the windings of the wire with respect to the needle. Hence, unless the coil is very carefully wound, the ratio of $m$ to $n$ may change when the position of the needle is changed, and therefore it is necessary to determine this ratio by proper methods during each course of experiments if any alteration of the position of the needle is suspected.

The other null method, in which Wheatstone's Bridge is used, requires only an ordinary galvanometer, and the observed zero deflexion of the needle is due, not to the opposing action of two currents, but to the non-existence of a current in the wire. Hence we have not merely a null deflexion, but a null current as the phenomenon observed, and no errors can arise from want of regularity or change of any kind in the coils of the galvanometer. The galvanometer is only required to be sensitive enough to detect the existence and direction of a current, without in any way determining its value or comparing its value with that of another current.

347.] Wheatstone's Bridge consists essentially of six conductors connecting four points. An electromotive force $E$ is made to act between two of the points by means of a voltaic battery introduced between $B$ and $C$. The current between the other two points $O$ and $A$ is measured by a galvanometer.

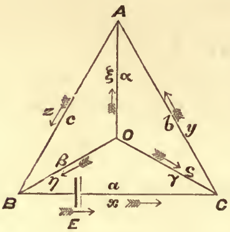



Fig. 31.

Under certain circumstances this current becomes zero. The conductors $BC$ and $OA$ are then said to be *conjugate* to each other, which implies a certain relation between the resistances of the other four conductors, and this relation is made use of in measuring resistances.

If the current in $OA$ is zero, the potential at $O$ must be equal to that at $A$. Now when we know the potentials at $B$ and $C$ we can determine those at $O$ and $A$ by the rule given at Art. 274, provided there is no current in $OA$,

$$O = \frac{B\gamma + C\beta}{\beta + \gamma}, \quad A = \frac{Bb + Cc}{b + c},$$

whence the condition is

$$b\beta = c\gamma,$$

where $b$, $c$, $\beta$, $\gamma$ are the resistances in $CA$, $AB$, $BO$ and $OC$ respectively.

To determine the degree of accuracy attainable by this method we must ascertain the strength of the current in $OA$ when this condition is not fulfilled exactly.

Let $A$, $B$, $C$ and $O$ be the four points. Let the currents along $BC$, $CA$ and $AB$ be $x$, $y$ and $z$, and the resistances of these conductors $a$, $b$ and $c$. Let the currents along $OA$, $OB$ and $OC$ be $\xi$, $\eta$, $\zeta$ and the resistances $\alpha$, $\beta$ and $\gamma$. Let an electromotive force $E$ act along $BC$. Required the current $\xi$ along $OA$.

Let the potentials at the points $A$, $B$, $C$ and $O$ be denoted by the symbols $A$, $B$, $C$ and $O$. The equations of conduction are

$$\begin{aligned} ax &= B-C+E, & \alpha\xi &= O-A,\\ by &= C-A, & \beta\eta &= O-B,\\ cz &= A-B, & \gamma\zeta &= O-C; \end{aligned}$$

with the equations of continuity

$$\begin{aligned} \xi+y-z &= 0,\\ \eta+z-x &= 0,\\ \zeta+x-y &= 0. \end{aligned}$$

By considering the system as made up of three circuits $OBC$, $OCA$ and $OAB$ in which the currents are $x$, $y$, $z$ respectively, and applying Kirchhoff's rule to each cycle, we eliminate the values of the potentials $O$, $A$, $B$, $C$, and the currents $\xi$, $\eta$, $\zeta$, and obtain the following equations for $x$, $y$ and $z$,

$$\begin{aligned} (a+\beta+\gamma)x-\gamma y \quad -\beta z \quad &= E,\\ -\gamma x \quad +(b+\gamma+\alpha)y-\alpha z \quad &= 0,\\ -\beta x \quad -\alpha y \quad +(c+\alpha+\beta)z &= 0. \end{aligned}$$

Hence, if we put

$$D = \begin{vmatrix} a+\beta+\gamma & -\gamma & -\beta \\ -\gamma & b+\gamma+\alpha & -\alpha \\ -\beta & -\alpha & c+\alpha+\beta \end{vmatrix},$$

we find $$\xi = \frac{E}{D}(b\beta-c\gamma),$$

and $$x = \frac{E}{D}\{(b+\gamma)(c+\beta)+\alpha(b+c+\beta+\gamma)\}.$$

348.] The value of $D$ may be expressed in the symmetrical form,

$$D = abc+bc(\beta+\gamma)+ca(\gamma+\alpha)+ab(\alpha+\beta)+(a+b+c)(\beta\gamma+\gamma\alpha+\alpha\beta)$$

or, since we suppose the battery in the conductor $a$ and the galvanometer in $\alpha$, we may put $B$ the battery resistance for $a$ and $G$ the galvanometer resistance for $\alpha$. We then find

$$D = BG(b+c+\beta+\gamma)+B(b+\gamma)(c+\beta) \\ +G(b+c)(\beta+\gamma)+bc(\beta+\gamma)+\beta\gamma(b+c).$$

If the electromotive force $E$ were made to act along $OA$, the resistance of $OA$ being still $\alpha$, and if the galvanometer were placed in $BC$, the resistance of $BC$ being still $a$, then the value of $D$ would remain the same, and the current in $BC$ due to the electromotive force $E$ acting along $OA$ would be equal to the current in $OA$ due to the electromotive force $E$ acting in $BC$.

But if we simply disconnect the battery and the galvanometer, and without altering their respective resistances connect the battery to $O$ and $A$ and the galvanometer to $B$ and $C$, then in the value of $D$ we must exchange the values of $B$ and $G$. If $D'$ be the value of $D$ after this exchange, we find

$$D'-D = (G-B)\{(b+c)(\beta+\gamma)-(b+\gamma)(\beta+c)\},$$
$$= (B-G)\{(b-\beta)(c-\gamma)\}.$$

Let us suppose that the resistance of the galvanometer is greater than that of the battery.

Let us also suppose that in its original position the galvanometer connects the junction of the two conductors of least resistance $\beta$, $\gamma$ with the junction of the two conductors of greatest resistance $b$, $c$, or, in other words, we shall suppose that if the quantities $b$, $c$, $\gamma$, $\beta$ are arranged in order of magnitude, $b$ and $c$ stand together, and $\gamma$ and $\beta$ stand together. Hence the quantities $b-\beta$ and $c-\gamma$ are of the same sign, so that their product is positive, and therefore $D'-D$ is of the same sign as $B-G$.

If therefore the galvanometer is made to connect the junction of the two greatest resistances with that of the two least, and if the galvanometer resistance is greater than that of the battery, then the value of $D$ will be less, and the value of the deflexion of the galvanometer greater, than if the connexions are exchanged.

The rule therefore for obtaining the greatest galvanometer deflexion in a given system is as follows:

Of the two resistances, that of the battery and that of the galvanometer, connect the greater resistance so as to join the two greatest to the two least of the four other resistances.

349.] We shall suppose that we have to determine the ratio of the resistances of the conductors $AB$ and $AC$, and that this is to be done by finding a point $O$ on the conductor $BOC$, such that when the points $A$ and $O$ are connected by a wire, in the course of which a galvanometer is inserted, no sensible deflexion of the galvanometer needle occurs when the battery is made to act between $B$ and $C$.

The conductor $BOC$ may be supposed to be a wire of uniform resistance divided into equal parts, so that the ratio of the resistances of $BO$ and $OC$ may be read off at once.

Instead of the whole conductor being a uniform wire, we may make the part near $O$ of such a wire, and the parts on each side may be coils of any form, the resistance of which is accurately known.

We shall now use a different notation instead of the symmetrical notation with which we commenced.

Let the whole resistance of $BAC$ be $R$.

Let $c = mR$ and $b = (1-m)R$.

Let the whole resistance of $BOC$ be $S$.

Let $\beta = nS$ and $\gamma = (1-n)S$.

The value of $n$ is read off directly, and that of $m$ is deduced from it when there is no sensible deviation of the galvanometer.

Let the resistance of the battery and its connexions be $B$, and that of the galvanometer and its connexions $G$.

We find as before

$$D = G\{BR+BS+RS\}+m(1-m)R^2(B+S)+n(1-n)S^2(B+R) \\ +(m+n-2mn)BRS,$$

and if $\xi$ is the current in the galvanometer wire

$$\xi = \frac{ERS}{D}(n-m).$$

In order to obtain the most accurate results we must make the deviation of the needle as great as possible compared with the value of $(n-m)$. This may be done by properly choosing the dimensions of the galvanometer and the standard resistance wire.

It will be shewn, when we come to Galvanometry, Art. 716, that when the form of a galvanometer wire is changed while its mass remains constant, the deviation of the needle for unit current is proportional to the length, but the resistance increases as the square of the length. Hence the maximum deflexion is shewn to occur when the resistance of the galvanometer wire is equal to the constant resistance of the rest of the circuit.

In the present case, if $\delta$ is the deviation,

$$\delta = C\sqrt{G}\,\xi,$$

where $C$ is some constant, and $G$ is the galvanometer resistance which varies as the square of the length of the wire. Hence we find that in the value of $D$, when $\delta$ is a maximum, the part involving $G$ must be made equal to the rest of the expression.

If we also put $m = n$, as is the case if we have made a correct observation, we find the best value of $G$ to be

$$G = n(1-n)(R+S).$$

This result is easily obtained by considering the resistance from $A$ to $O$ through the system, remembering that $BC$, being conjugate to $AO$, has no effect on this resistance.

In the same way we should find that if the total area of the acting surfaces of the battery is given, the most advantageous arrangement of the battery is when

$$B = \frac{RS}{R+S}.$$

Finally, we shall determine the value of $S$ such that a given change in the value of $n$ may produce the greatest galvanometer deflexion. By differentiating the expression for $\xi$ we find

$$S^2 = \frac{BR}{B+R}\left(R + \frac{G}{n(1-n)}\right).$$

If we have a great many determinations of resistance to make in which the actual resistance has nearly the same value, then it may be worth while to prepare a galvanometer and a battery for this purpose. In this case we find that the best arrangement is

$$S = R, \qquad B = \tfrac{1}{2}R, \qquad G = 2n(1-n)R,$$

and if $n = \frac{1}{2}$, $G = \frac{1}{2}R$.

*On the Use of Wheatstone's Bridge.*

350.] We have already explained the general theory of Wheatstone's Bridge, we shall now consider some of its applications.

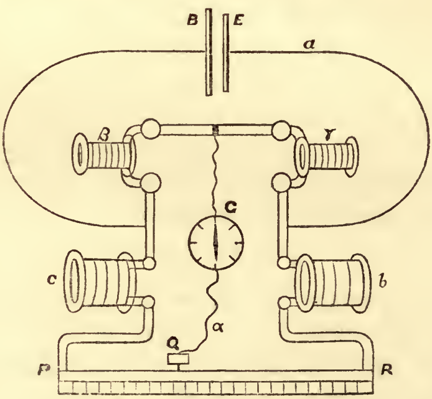



Fig. 32.

The comparison which can be effected with the greatest exactness is that of two equal resistances.

Let us suppose that $\beta$ is a standard resistance coil, and that we wish to adjust $\gamma$ to be equal in resistance to $\beta$.

Two other coils, $b$ and $c$, are prepared which are equal or nearly equal to each other, and the four coils are placed with their electrodes in mercury cups so that the current of the battery is divided between two branches, one consisting of $\beta$ and $\gamma$ and the other of $b$ and $c$. The coils $b$ and $c$ are connected by a wire $PR$, as uniform in its resistance as possible, and furnished with a scale of equal parts.

The galvanometer wire connects the junction of $\beta$ and $\gamma$ with a point $Q$ of the wire $PR$, and the point of contact at $Q$ is made to vary till on closing first the battery circuit and then the galvanometer circuit, no deflexion of the galvanometer needle is observed.

The coils $\beta$ and $\gamma$ are then made to change places, and a new position is found for $Q$. If this new position is the same as the old one, then we know that the exchange of $\beta$ and $\gamma$ has produced no change in the proportions of the resistances, and therefore $\gamma$ is rightly adjusted. If $Q$ has to be moved, the direction and amount of the change will indicate the nature and amount of the alteration of the length of the wire of $\gamma$, which will make its resistance equal to that of $\beta$.

If the resistances of the coils $b$ and $c$, each including part of the wire $PR$ up to its zero reading, are equal to that of $b$ and $c$ divisions of the wire respectively, then, if $x$ is the scale reading of $Q$ in the first case, and $y$ that in the second,

$$\frac{c+x}{b-x} = \frac{\beta}{\gamma}, \qquad \frac{c+y}{b-y} = \frac{\gamma}{\beta},$$

whence

$$\frac{\gamma^2}{\beta^2} = 1 + \frac{(b+c)(y-x)}{(c+x)(b-y)}.$$

Since $b-y$ is nearly equal to $c+x$, and both are great with respect to $x$ or $y$, we may write this

$$\frac{\gamma^2}{\beta^2} = 1 + 4\frac{y-x}{b+c},$$

and

$$\gamma = \beta\left(1 + 2\frac{y-x}{b+c}\right).$$

When $\gamma$ is adjusted as well as we can, we substitute for $b$ and $c$ other coils of (say) ten times greater resistance.

The remaining difference between $\beta$ and $\gamma$ will now produce a ten times greater difference in the position of $Q$ than with the original coils $b$ and $c$, and in this way we can continually increase the accuracy of the comparison.

The adjustment by means of the wire with sliding contact piece is more quickly made than by means of a resistance box, and it is capable of continuous variation.

The battery must never be introduced instead of the galvanometer into the wire with a sliding contact, for the passage of a powerful current at the point of contact would injure the surface of the wire. Hence this arrangement is adapted for the case in which the resistance of the galvanometer is greater than that of the battery.

*On the Measurement of Small Resistances.*

351.] When a short and thick conductor is introduced into a circuit its resistance is so small compared with the resistance occasioned by unavoidable faults in the connexions, such as want of contact or imperfect soldering, that no correct value of the resistance can be deduced from experiments made in the way described above.

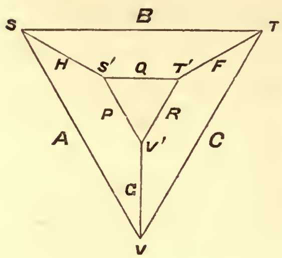



Fig. 33.

The object of such experiments is generally to determine the specific resistance of the substance, and it is resorted to in cases when the substance cannot be obtained in the form of a long thin wire, or when the resistance to transverse as well as to longitudinal conduction has to be measured.

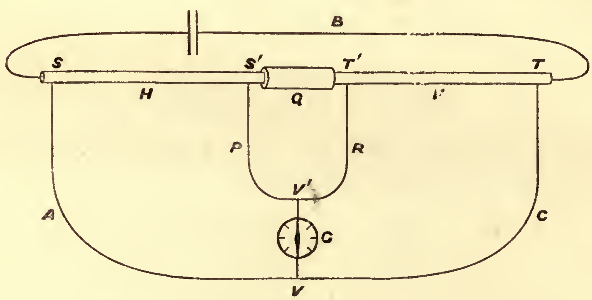



Fig. 34.

Sir W. Thomson,* has described a method applicable to such cases, which we may take as an example of a system of nine conductors.

* *Proc. R. S.*, June 6, 1861.

The most important part of the method consists in measuring the resistance, not of the whole length of the conductor, but of the part between two marks on the conductor at some little distance from its ends.

The resistance which we wish to measure is that experienced by a current whose intensity is uniform in any section of the conductor, and which flows in a direction parallel to its axis. Now close to the extremities, when the current is introduced by means of electrodes, either soldered, amalgamated, or simply pressed to the ends of the conductor, there is generally a want of uniformity in the distribution of the current in the conductor. At a short distance from the extremities the current becomes sensibly uniform. The student may examine for himself the investigation and the diagrams of Art. 193, where a current is introduced into a strip of metal with parallel sides through one of the sides, but soon becomes itself parallel to the sides.

The resistance of the conductors between certain marks $S$, $S'$ and $TT'$ is to be compared.

The conductors are placed in series, and with connexions as perfectly conducting as possible, in a battery circuit of small resistance. A wire $SVT$ is made to touch the conductors at $S$ and $T$, and $S'V'T'$ is another wire touching them at $S'$ and $T'$.

The galvanometer wire connects the points $V$ and $V'$ of these wires.

The wires $SVT$ and $S'V'T'$ are of resistance so great that the resistance due to imperfect connexion at $S$, $T$, $S'$ or $T'$ may be neglected in comparison with the resistance of the wire, and $V$, $V'$ are taken so that the resistance in the branches of either wire leading to the two conductors are nearly in the ratio of the resistances of the two conductors.

Calling $H$ and $F$ the resistances of the conductors $SS'$ and $T'T$.
" $A$ and $C$ those of the branches $SV$ and $VT$.
" $P$ and $R$ those of the branches $S'V'$ and $V'T'$.
" $Q$ that of the connecting piece $S'T'$.
" $B$ that of the battery and its connexions.
" $G$ that of the galvanometer and its connexions.

The symmetry of the system may be understood from the skeleton diagram. Fig. 33.

The condition that $B$ the battery and $G$ the galvanometer may be conjugate conductors is, in this case,

$$\frac{F}{C} - \frac{H}{A} + \left(\frac{R}{C} - \frac{P}{A}\right) \frac{Q}{P+Q+R} = 0.$$

Now the resistance of the connector $Q$ is as small as we can make it. If it were zero this equation would be reduced to

$$\frac{F}{C} = \frac{H}{A},$$

and the ratio of the resistances of the conductors to be compared would be that of $C$ to $A$, as in Wheatstone's Bridge in the ordinary form.

In the present case the value of $Q$ is small compared with $P$ or with $R$, so that if we assume the points $V$, $V'$ so that the ratio of $R$ to $C$ is nearly equal to that of $P$ to $A$, the last term of the equation will vanish, and we shall have

$$F : H :: C : A.$$

The success of this method depends in some degree on the perfection of the contact between the wires and the tested conductors at $SS'$, $T'$ and $T$. In the following method, employed by Messrs. Matthiessen and Hockin *, this condition is dispensed with.

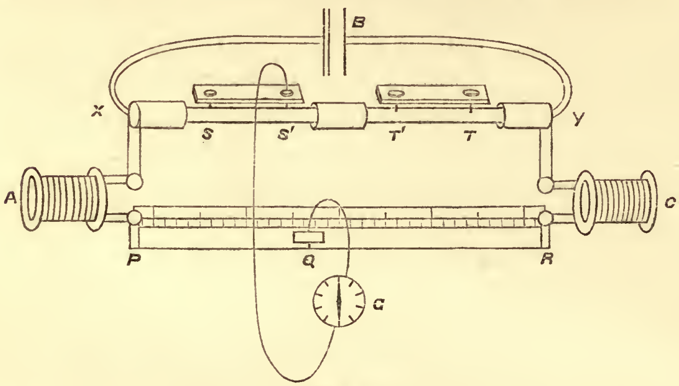



Fig. 35.

352.] The conductors to be tested are arranged in the manner already described, with the connexions as well made as possible, and it is required to compare the resistance between the marks $SS'$ on the first conductor with the resistance between the marks $T'T$ on the second.

Two conducting points or sharp edges are fixed in a piece of insulating material so that the distance between them can be accurately measured. This apparatus is laid on the conductor to be tested, and the points of contact with the conductor are then at a known distance $SS'$. Each of these contact pieces is connected

* *Laboratory.* Matthiessen and Hockin on Alloys.

with a mercury cup, into which one electrode of the galvanometer may be plunged.

The rest of the apparatus is arranged, as in Wheatstone's Bridge, with resistance coils or boxes $A$ and $C$, and a wire $PR$ with a sliding contact piece $Q$, to which the other electrode of the galvanometer is connected.

Now let the galvanometer be connected to $S$ and $Q$, and let $A_1$ and $C_1$ be so arranged, and the position of $Q$ so determined, that there is no current in the galvanometer wire.

Then we know that
$$\frac{XS}{SY} = \frac{A_1+PQ}{C_1+QR}$$
where $XS$, $PQ$, &c. stand for the resistances in these conductors.

From this we get
$$\frac{XS}{XY} = \frac{A_1+PQ_1}{A_1+C_1+PR}.$$

Now let the electrode of the galvanometer be connected to $S'$, and let resistance be transferred from $C$ to $A$ (by carrying resistance coils from one side to the other) till electric equilibrium of the galvanometer wire can be obtained by placing $Q$ at some point of the wire, say $Q_2$. Let the values of $C$ and $A$ be now $C_2$ and $A_2$, and let
$$A_2+C_2+PR = A_1+C_1+PR = R.$$

Then we have, as before,
$$\frac{XS'}{XY} = \frac{A_2+PQ_2}{R}.$$

Whence
$$\frac{SS'}{XY} = \frac{A_2-A_1+Q_1Q_2}{R}.$$

In the same way, placing the apparatus on the second conductor at $TT'$ and again transferring resistance, we get, when the electrode is in $T'$,
$$\frac{XT'}{XY} = \frac{A_3+PQ_3}{R},$$
and when it is in $T$,
$$\frac{XT}{XY} = \frac{A_4+PQ_4}{R}.$$

Whence
$$\frac{T'T}{XY} = \frac{A_4-A_3+Q_3Q_4}{R}.$$

We can now deduce the ratio of the resistances $SS'$ and $T'T$, for
$$\frac{SS'}{T'T} = \frac{A_2-A_1+Q_1Q_2}{A_4-A_3+Q_3Q_4}.$$

When great accuracy is not required we may dispense with the resistance coils $A$ and $C$, and we then find

$$\frac{SS'}{T'T} = \frac{Q_1 Q_2}{Q_3 Q_4}.$$

The readings of the position of $Q$ on a wire of a metre in length cannot be depended on to less than a tenth of a millimetre, and the resistance of the wire may vary considerably in different parts owing to inequality of temperature, friction, &c. Hence, when great accuracy is required, coils of considerable resistance are introduced at $A$ and $C$, and the ratios of the resistances of these coils can be determined more accurately than the ratio of the resistances of the parts into which the wire is divided at $Q$.

It will be observed that in this method the accuracy of the determination depends in no degree on the perfection of the contacts at $SS'$ or $TT'$.

This method may be called the differential method of using Wheatstone's Bridge, since it depends on the comparison of observations separately made.

An essential condition of accuracy in this method is that the resistance of the connexions should continue the same during the course of the four observations required to complete the determination. Hence the series of observations ought always to be repeated in order to detect any change in the resistances.

### *On the Comparison of Great Resistances.*

353.] When the resistances to be measured are very great, the comparison of the potentials at different points of the system may be made by means of a delicate electrometer, such as the Quadrant Electrometer described in Art. 219.

If the conductors whose resistance is to be measured are placed in series, and the same current passed through them by means of a battery of great electromotive force, the difference of the potentials at the extremities of each conductor will be proportional to the resistance of that conductor. Hence, by connecting the electrodes of the electrometer with the extremities, first of one conductor and then of the other, the ratio of their resistances may be determined.

This is the most direct method of determining resistances. It involves the use of an electrometer whose readings may be depended on, and we must also have some guarantee that the current remains constant during the experiment.

Four conductors of great resistance may also be arranged as in Wheatstone's Bridge, and the bridge itself may consist of the electrodes of an electrometer instead of those of a galvanometer. The advantage of this method is that no permanent current is required to produce the deviation of the electrometer, whereas the galvanometer cannot be deflected unless a current passes through the wire.

354.] When the resistance of a conductor is so great that the current which can be sent through it by any available electromotive force is too small to be directly measured by a galvanometer, a condenser may be used in order to accumulate the electricity for a certain time, and then, by discharging the condenser through a galvanometer, the quantity accumulated may be estimated. This is Messrs. Bright and Clark's method of testing the joints of submarine cables.

355.] But the simplest method of measuring the resistance of such a conductor is to charge a condenser of great capacity and to connect its two surfaces with the electrodes of an electrometer and also with the extremities of the conductor. If $E$ is the difference of potentials as shewn by the electrometer, $S$ the capacity of the condenser, and $Q$ the charge on either surface, $R$ the resistance of the conductor and $x$ the current in it, then, by the theory of condensers,

$$Q = SE.$$

By Ohm's Law, $$E = Rx,$$

and by the definition of a current,

$$x = -\frac{dQ}{dt}.$$

Hence $$Q = RS\frac{dQ}{dt},$$

and $$Q = Q_0 e^{-\frac{t}{RS}},$$

where $Q_0$ is the charge at first when $t = 0$.

Similarly $$E = E_0 e^{-\frac{t}{RS}}$$

where $E_0$ is the original reading of the electrometer, and $E$ the same after a time $t$. From this we find

$$R = \frac{t}{S\{\log_e E_0 - \log_e E\}},$$

which gives $R$ in absolute measure. In this expression a knowledge of the value of the unit of the electrometer scale is not required.

If $S$, the capacity of the condenser, is given in electrostatic measure as a certain number of metres, then $R$ is also given in electrostatic measure as the reciprocal of a velocity.

If $S$ is given in electromagnetic measure its dimensions are $\frac{T^2}{L}$, and $R$ is a velocity.

Since the condenser itself is not a perfect insulator it is necessary to make two experiments. In the first we determine the resistance of the condenser itself, $R_0$, and in the second, that of the condenser when the conductor is made to connect its surfaces. Let this be $R'$. Then the resistance, $R$, of the conductor is given by the equation

$$\frac{1}{R} = \frac{1}{R'} - \frac{1}{R_0}.$$

This method has been employed by MM. Siemens.

*Thomson's * Method for the Determination of the Resistance of the Galvanometer.*

356.] An arrangement similar to Wheatstone's Bridge has been employed with advantage by Sir W. Thomson in determining the

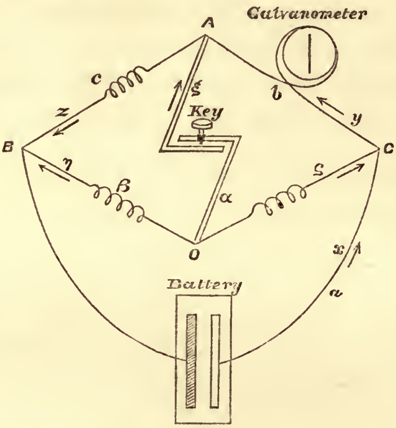



Fig. 36.

resistance of the galvanometer when in actual use. It was suggested to Sir W. Thomson by Mance's Method. See Art. 357.

Let the battery be placed, as before, between $B$ and $C$ in the figure of Article 347, but let the galvanometer be placed in $CA$ instead of in $OA$. If $b\beta - c\gamma$ is zero, then the conductor $OA$ is conjugate to $BC$, and, as there is no current produced in $OA$ by the battery in $BC$, the strength of the current in any other conductor

* *Proc. R. S.*, Jan. 19, 1871.

is independent of the resistance in $OA$. Hence, if the galvanometer is placed in $CA$ its deflexion will remain the same whether the resistance of $OA$ is small or great. We therefore observe whether the deflexion of the galvanometer remains the same when $O$ and $A$ are joined by a conductor of small resistance, as when this connexion is broken, and if, by properly adjusting the resistances of the conductors, we obtain this result, we know that the resistance of the galvanometer is

$$b = \frac{c\gamma}{\beta},$$

where $c$, $\gamma$, and $\beta$ are resistance coils of known resistance.

It will be observed that though this is not a null method, in the sense of there being no current in the galvanometer, it is so in the sense of the fact observed being the negative one, that the deflexion of the galvanometer is not changed when a certain contact is made. An observation of this kind is of greater value than an observation of the equality of two different deflexions of the same galvanometer, for in the latter case there is time for alteration in the strength of the battery or the sensitiveness of the galvanometer, whereas when the deflexion remains constant, in spite of certain changes which we can repeat at pleasure, we are sure that the current is quite independent of these changes.

The determination of the resistance of the coil of a galvanometer can easily be effected in the ordinary way of using Wheatstone's Bridge by placing another galvanometer in $OA$. By the method now described the galvanometer itself is employed to measure its own resistance.

### *Mance's* Method of determining the Resistance of the Battery.*

357.] The measurement of the resistance of a battery when in action is of a much higher order of difficulty, since the resistance of the battery is found to change considerably for some time after the strength of the current through it is changed. In many of the methods commonly used to measure the resistance of a battery such alterations of the strength of the current through it occur in the course of the operations, and therefore the results are rendered doubtful.

In Mance's method, which is free from this objection, the battery is placed in $BC$ and the galvanometer in $CA$. The connexion between $O$ and $B$ is then alternately made and broken.

* *Proc. R. S.*, Jan. 19, 1871.

If the deflexion of the galvanometer remains unaltered, we know that $OB$ is conjugate to $CA$, whence $c\gamma = a\alpha$, and $a$, the resistance of the battery, is obtained in terms of known resistances $c$, $\gamma$, $\alpha$.

When the condition $c\gamma = a\alpha$ is fulfilled, then the current through the galvanometer is

$$y = \frac{Ea}{ba + c(b + \alpha + \gamma)},$$

and this is independent of the resistance $\beta$ between $O$ and $B$. To test the sensibility of the method let us suppose that the condition $c\gamma = a\alpha$ is nearly, but not accurately, fulfilled, and that $y_0$ is the

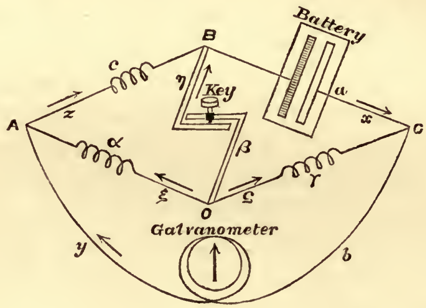



Fig. 37.

current through the galvanometer when $O$ and $B$ are connected by a conductor of no sensible resistance, and $y_1$ the current when $O$ and $B$ are completely disconnected.

To find these values we must make $\beta$ equal to 0 and to $\infty$ in the general formula for $y$, and compare the results.

In this way we find

$$\frac{y_0 - y_1}{y} = \frac{\alpha}{\gamma} \frac{c\gamma - a\alpha}{(c + a)(\alpha + \gamma)},$$

where $y_0$ and $y_1$ are supposed to be so nearly equal that we may, when their difference is not in question, put either of them equal to $y$, the value of the current when the adjustment is perfect.

The resistance, $c$, of the conductor $AB$ should be equal to $a$, that of the battery, $\alpha$ and $\gamma$, should be equal and as small as possible, and $b$ should be equal to $a + \gamma$.

Since a galvanometer is most sensitive when its deflexion is small, we should bring the needle nearly to zero by means of fixed magnets before making contact between $O$ and $B$.

In this method of measuring the resistance of the battery, the current in the battery is not in any way interfered with during the operation, so that we may ascertain its resistance for any given strength of current, so as to determine how the strength of current effects the resistance.

If $y$ is the current in the galvanometer, the actual current through the battery is $x_0$ with the key down and $x_1$ with the key up, where

$$x_0 = y\left(1+\frac{b}{a+\gamma}\right), \qquad x_1 = y\left(1+\frac{b}{\gamma}+\frac{ac}{\gamma(a+c)}\right),$$

the resistance of the battery is

$$a = \frac{c\gamma}{a},$$

and the electromotive force of the battery is

$$E = y\left(b+c+\frac{c}{a}(b+\gamma)\right).$$

The method of Art. 356 for finding the resistance of the galvanometer differs from this only in making and breaking contact between $O$ and $A$ instead of between $O$ and $B$, and by exchanging $a$ and $\beta$ we obtain for this case

$$\frac{y_0-y_1}{y} = \frac{\beta}{\gamma}\,\frac{c\gamma-b\beta}{(c+\beta)(\beta+\gamma)}.$$

*On the Comparison of Electromotive Forces.*

358.] The following method of comparing the electromotive forces of voltaic and thermoelectric arrangements, when no current passes through them, requires only a set of resistance coils and a constant battery.

Let the electromotive force $E$ of the battery be greater than that of either of the electromotors to be compared, then, if a sufficient

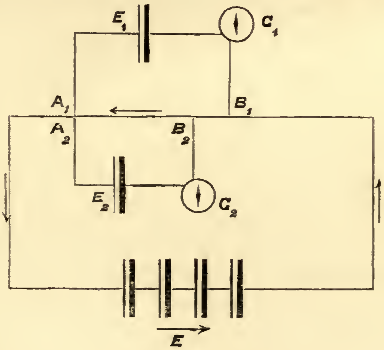



Fig. 38.

resistance, $R_1$, be interposed between the points $A_1$, $B_1$ of the primary circuit $EB_1A_1E$, the electromotive force from $B_1$ to $A_1$

may be made equal to that of the electromotor $E_1$. If the electrodes of this electromotor are now connected with the points $A_1$, $B_1$ no current will flow through the electromotor. By placing a galvanometer $G_1$ in the circuit of the electromotor $E_1$, and adjusting the resistance between $A_1$ and $B_1$, till the galvanometer $G_1$ indicates no current, we obtain the equation

$$E_1 = R_1 C,$$

where $R_1$ is the resistance between $A_1$ and $B_1$, and $C$ is the strength of the current in the primary circuit.

In the same way, by taking a second electromotor $E_2$ and placing its electrodes at $A_2$ and $B_2$, so that no current is indicated by the galvanometer $G_2$,

$$E_2 = R_2 C,$$

where $R_2$ is the resistance between $A_2$ and $B_2$. If the observations of the galvanometers $G_1$ and $G_2$ are simultaneous, the value of $C$, the current in the primary circuit, is the same in both equations, and we find

$$E_1 : E_2 :: R_1 : R_2.$$

In this way the electromotive force of two electromotors may be compared. The absolute electromotive force of an electromotor may be measured either electrostatically by means of the electrometer, or electromagnetically by means of an absolute galvanometer.

This method, in which, at the time of the comparison, there is no current through either of the electromotors, is a modification of Poggendorff's method, and is due to Mr. Latimer Clark, who has deduced the following values of electromotive forces:

| | | | | Concentrated solution of | | Volts. |
|---|---|---|---|---|---|---|
| *Daniell* | I. | Amalgamated Zinc | $HSO_4$ + 4 aq. | $Cu\,SO_4$ | Copper | = 1.079 |
| | II. | ,, | $HSO_4$ + 12 aq. | $Cu\,SO_4$ | Copper | = 0.978 |
| | III. | ,, | $HSO_4$ + 12 aq. | $Cu\,NO_6$ | Copper | = 1.00 |
| *Bunsen* | I. | ,, | ,, ,, | $H\,NO_6$ | Carbon | = 1.964 |
| | II. | ,, | ,, ,, | sp. g. 1.38 | Carbon | = 1.888 |
| *Grove* | | ,, | $HSO_4$ + 4 aq. | $H\,NO_6$ | Platinum | = 1.956 |

*A Volt is an electromotive force equal to* 100,000,000 *units of the centimetre-gramme-second system.*

# CHAPTER XII.

## ON THE ELECTRIC RESISTANCE OF SUBSTANCES.

359.] THERE are three classes in which we may place different substances in relation to the passage of electricity through them.

The first class contains all the metals and their alloys, some sulphurets, and other compounds containing metals, to which we must add carbon in the form of gas-coke, and selenium in the crystalline form.

In all these substances conduction takes place without any decomposition, or alteration of the chemical nature of the substance, either in its interior or where the current enters and leaves the body. In all of them the resistance increases as the temperature rises.

The second class consists of substances which are called electrolytes, because the current is associated with a decomposition of the substance into two components which appear at the electrodes. As a rule a substance is an electrolyte only when in the liquid form, though certain colloid substances, such as glass at 100°C, which are apparently solid, are electrolytes. It would appear from the experiments of Sir B. C. Brodie that certain gases are capable of electrolysis by a powerful electromotive force.

In all substances which conduct by electrolysis the resistance diminishes as the temperature rises.

The third class consists of substances the resistance of which is so great that it is only by the most refined methods that the passage of electricity through them can be detected. These are called Dielectrics. To this class belong a considerable number of solid bodies, many of which are electrolytes when melted, some liquids, such as turpentine, naphtha, melted paraffin, &c., and all gases and vapours. Carbon in the form of diamond, and selenium in the amorphous form, belong to this class.

The resistance of this class of bodies is enormous compared with that of the metals. It diminishes as the temperature rises. It is difficult, on account of the great resistance of these substances, to determine whether the feeble current which we can force through them is or is not associated with electrolysis.

*On the Electric Resistance of Metals.*

360.] There is no part of electrical research in which more numerous or more accurate experiments have been made than in the determination of the resistance of metals. It is of the utmost importance in the electric telegraph that the metal of which the wires are made should have the smallest attainable resistance. Measurements of resistance must therefore be made before selecting the materials. When any fault occurs in the line, its position is at once ascertained by measurements of resistance, and these measurements, in which so many persons are now employed, require the use of resistance coils, made of metal the electrical properties of which have been carefully tested.

The electrical properties of metals and their alloys have been studied with great care by MM. Matthiessen, Vogt, and Hockin, and by MM. Siemens, who have done so much to introduce exact electrical measurements into practical work.

It appears from the researches of Dr. Matthiessen, that the effect of temperature on the resistance is nearly the same for a considerable number of the *pure* metals, the resistance at 100°C being to that at 0°C in the ratio of 1.414 to 1, or of 1 to 70.7. For pure iron the ratio is 1.645, and for pure thallium 1.458.

The resistance of metals has been observed by Dr. C.W. Siemens* through a much wider range of temperature, extending from the freezing point to 350°C, and in certain cases to 1000°C. He finds that the resistance increases as the temperature rises, but that the rate of increase diminishes as the temperature rises. The formula, which he finds to agree very closely both with the resistances observed at low temperatures by Dr. Matthiessen and with his own observations through a range of 1000°C, is

$$r = aT^{\frac{1}{2}} + \beta T + \gamma,$$

where $T$ is the absolute temperature reckoned from $-273$°C, and $a$, $\beta$, $\gamma$ are constants. Thus, for

Platinum...... $r = 0.039369T^{\frac{1}{2}} + 0.00216407T - 0.2413$,

Copper......... $r = 0.026577T^{\frac{1}{2}} + 0.0031443T - 0.22751$,

Iron............ $r = 0.072545T^{\frac{1}{2}} + 0.0038133T - 1.23971$.

* *Proc. R. S.*, April 27, 1871.

From data of this kind the temperature of a furnace may be determined by means of an observation of the resistance of a platinum wire placed in the furnace.

Dr. Matthiessen found that when two metals are combined to form an alloy, the resistance of the alloy is in most cases greater than that calculated from the resistance of the component metals and their proportions. In the case of alloys of gold and silver, the resistance of the alloy is greater than that of either pure gold or pure silver, and, within certain limiting proportions of the constituents, it varies very little with a slight alteration of the proportions. For this reason Dr. Matthiessen recommended an alloy of two parts by weight of gold and one of silver as a material for reproducing the unit of resistance.

The effect of change of temperature on electric resistance is generally less in alloys than in pure metals.

Hence ordinary resistance coils are made of German silver, on account of its great resistance and its small variation with temperature.

An alloy of silver and platinum is also used for standard coils.

361.] The electric resistance of some metals changes when the metal is annealed; and until a wire has been tested by being repeatedly raised to a high temperature without permanently altering its resistance, it cannot be relied on as a measure of resistance. Some wires alter in resistance in course of time without having been exposed to changes of temperature. Hence it is important to ascertain the specific resistance of mercury, a metal which being fluid has always the same molecular structure, and which can be easily purified by distillation and treatment with nitric acid. Great care has been bestowed in determining the resistance of this metal by W. and C. F. Siemens, who introduced it as a standard. Their researches have been supplemented by those of Matthiessen and Hockin.

The specific resistance of mercury was deduced from the observed resistance of a tube of length $l$ containing a weight $w$ of mercury, in the following manner.

No glass tube is of exactly equal bore throughout, but if a small quantity of mercury is introduced into the tube and occupies a length $\lambda$ of the tube, the middle point of which is distant $x$ from one end of the tube, then the area $s$ of the section near this point will be $s = \dfrac{C}{\lambda}$, where $C$ is some constant.

The weight of mercury which fills the whole tube is

$$w = \rho \int s\, dx = \rho C \Sigma \left(\frac{1}{\lambda}\right) \frac{l}{n},$$

where $n$ is the number of points, at equal distances along the tube, where $\lambda$ has been measured, and $\rho$ is the mass of unit of volume.

The resistance of the whole tube is

$$R = \int \frac{r}{s}\, dx = \frac{r}{C} \Sigma (\lambda) \frac{l}{n},$$

where $r$ is the specific resistance per unit of volume.

Hence $$wR = r\rho \Sigma (\lambda) \Sigma \left(\frac{1}{\lambda}\right) \frac{l^2}{n^2},$$

and $$r = \frac{wR}{\rho l^2} \frac{n^2}{\Sigma (\lambda) \Sigma \left(\frac{1}{\lambda}\right)}$$

gives the specific resistance of unit of volume.

To find the resistance of unit of length and unit of mass we must multiply this by the density.

It appears from the experiments of Matthiessen and Hockin that the resistance of a uniform column of mercury of one metre in length, and weighing one gramme at 0°C, is 13.071 Ohms, whence it follows that if the specific gravity of mercury is 13.595, the resistance of a column of one metre in length and one square millimetre in section is 0.96146 Ohms.

362.] In the following table $R$ is the resistance in Ohms of a column one metre long and one gramme weight at 0°C, and $r$ is the resistance in centimetres per second of a cube of one centimetre, according to the experiments of Matthiessen *.

| | Specific gravity | | $R$ | $r$ | Percentage increment of resistance for 1°C at 20°C. |
|---|---|---|---|---|---|
| Silver ........ | 10.50 | hard drawn | 0.1689 | 1609 | 0.377 |
| Copper ...... | 8.95 | hard drawn | 0.1469 | 1642 | 0.388 |
| Gold ........ | 19.27 | hard drawn | 0.4150 | 2154 | 0.365 |
| Lead ........ | 11.391 | pressed | 2.257 | 19847 | 0.387 |
| Mercury ...... | 13.595 | liquid | 13.071 | 96146 | 0.072 |
| Gold 2, Silver 1.. | 15.218 | hard or annealed | 1.668 | 10988 | 0.065 |
| Selenium at 100°C | | Crystalline form | | $6 \times 10^{13}$ | 1.00 |

* *Phil. Mag.*, May, 1865.

*On the Electric Resistance of Electrolytes.*

363.] The measurement of the electric resistance of electrolytes is rendered difficult on account of the polarization of the electrodes, which causes the observed difference of potentials of the metallic electrodes to be greater than the electromotive force which actually produces the current.

This difficulty can be overcome in various ways. In certain cases we can get rid of polarization by using electrodes of proper material, as, for instance, zinc electrodes in a solution of sulphate of zinc. By making the surface of the electrodes very large compared with the section of the part of the electrolyte whose resistance is to be measured, and by using only currents of short duration in opposite directions alternately, we can make the measurements before any considerable intensity of polarization has been excited by the passage of the current.

Finally, by making two different experiments, in one of which the path of the current through the electrolyte is much longer than in the other, and so adjusting the electromotive force that the actual current, and the time during which it flows, are nearly the same in each case, we can eliminate the effect of polarization altogether.

364.] In the experiments of Dr. Paalzow * the electrodes were in the form of large disks placed in separate flat vessels filled with the electrolyte, and the connexion was made by means of a long siphon filled with the electrolyte and dipping into both vessels. Two such siphons of different lengths were used.

The observed resistances of the electrolyte in these siphons being $R_1$ and $R_2$, the siphons were next filled with mercury, and their resistances when filled with mercury were found to be $R_1'$ and $R_2'$.

The ratio of the resistance of the electrolyte to that of a mass of mercury at 0°C of the same form was then found from the formula

$$\rho = \frac{R_1 - R_2}{R_1' - R_2'}.$$

To deduce from the values of $\rho$ the resistance of a centimetre in length having a section of a square centimetre, we must multiply them by the value of $r$ for mercury at 0°C. See Art. 361.

* *Berlin Monatsbericht*, July, 1868.

The results given by Paalzow are as follow :—

*Mixtures of Sulphuric Acid and Water.*

| | Temp. | Resistance compared with mercury. |
|---|---|---|
| $H_2SO_4$ .... | 15°C | 96950 |
| $H_2SO_4 + 14\ H^2O$ .... | 19°C | 14157 |
| $H_2SO_4 + 13\ H^2O$ .... | 22°C | 13310 |
| $H_2SO_4 + 499\ H^2O$ .... | 22°C | 184773 |

*Sulphate of Zinc and Water.*

| | | |
|---|---|---|
| $ZnSO_4 + 23\ H^2O$ .... | 23°C | 194400 |
| $ZnSO_4 + 24\ H^2O$ .... | 23°C | 191000 |
| $ZnSO_4 + 105\ H^2O$ .... | 23°C | 354000 |

*Sulphate of Copper and Water.*

| | | |
|---|---|---|
| $CuSO_4 + 45\ H^2O$ .... | 22°C | 202410 |
| $CuSO_4 + 105\ H^2O$ .... | 22°C | 339341 |

*Sulphate of Magnesium and Water.*

| | | |
|---|---|---|
| $MgSO_4 + 34\ H^2O$ .... | 22°C | 199180 |
| $MgSO_4 + 107\ H^2O$ .... | 22°C | 324600 |

*Hydrochloric Acid and Water.*

| | | |
|---|---|---|
| $HCl + 15\ H^2O$ .... | 23°C | 13626 |
| $HCl + 500\ H^2O$ .... | 23°C | 86679 |

365.] MM. F. Kohlrausch and W. A. Nippoldt* have determined the resistance of mixtures of sulphuric acid and water. They used alternating magneto-electric currents, the electromotive force of which varied from $\frac{1}{2}$ to $\frac{1}{74}$ of that of a Grove's cell, and by means of a thermoelectric copper-iron pair they reduced the electromotive force to $\frac{1}{429000}$ of that of a Grove's cell. They found that Ohm's law was applicable to this electrolyte throughout the range of these electromotive forces.

The resistance is a minimum in a mixture containing about one-third of sulphuric acid.

The resistance of electrolytes diminishes as the temperature increases. The percentage increment of conductivity for a rise of 1°C is given in the following table.

* Pogg., *Ann.* cxxxviii, p. 286, Oct. 1869.

*Resistance of Mixtures of Sulphuric Acid and Water at* 22°C *in terms of Mercury at* 0°C. MM. Kohlrausch and Nippoldt.

| Specific gravity at 18°5 | Percentage of $H_2SO_4$ | Resistance at 22°C (Hg=1) | Percentage increment of conductivity for 1°C. |
|---|---|---|---|
| 0.9985 | 0.0 | 746300 | 0.47 |
| 1.00 | 0.2 | 465100 | 0.47 |
| 1.0504 | 8.3 | 34530 | 0.653 |
| 1.0989 | 14.2 | 18946 | 0.646 |
| 1.1431 | 20.2 | 14990 | 0.799 |
| 1.2045 | 28.0 | 13133 | 1.317 |
| 1.2631 | 35.2 | 13132 | 1.259 |
| 1.3163 | 41.5 | 14286 | 1.410 |
| 1.3547 | 46.0 | 15762 | 1.674 |
| 1.3994 | 50.4 | 17726 | 1.582 |
| 1.4482 | 55.2 | 20796 | 1.417 |
| 1.5026 | 60.3 | 25574 | 1.794 |

### *On the Electrical Resistance of Dielectrics.*

366.] A great number of determinations of the resistance of gutta-percha, and other materials used as insulating media, in the manufacture of telegraphic cables, have been made in order to ascertain the value of these materials as insulators.

The tests are generally applied to the material after it has been used to cover the conducting wire, the wire being used as one electrode, and the water of a tank, in which the cable is plunged, as the other. Thus the current is made to pass through a cylindrical coating of the insulator of great area and small thickness.

It is found that when the electromotive force begins to act, the current, as indicated by the galvanometer, is by no means constant. The first effect is of course a transient current of considerable intensity, the total quantity of electricity being that required to charge the surfaces of the insulator with the superficial distribution of electricity corresponding to the electromotive force. This first current therefore is a measure not of the conductivity, but of the capacity of the insulating layer.

But even after this current has been allowed to subside the residual current is not constant, and does not indicate the true conductivity of the substance. It is found that the current continues to decrease for at least half an hour, so that a determination of the resistance deduced from the current will give a greater value if a certain time is allowed to elapse than if taken immediately after applying the battery.

Thus, with Hooper's insulating material the apparent resistance at the end of ten minutes was four times, and at the end of nineteen hours twenty-three times that observed at the end of one minute. When the direction of the electromotive force is reversed, the resistance falls as low or lower than at first and then gradually rises.

These phenomena seem to be due to a condition of the gutta-percha, which, for want of a better name, we may call polarization, and which we may compare on the one hand with that of a series of Leyden jars charged by cascade, and, on the other, with Ritter's secondary pile, Art. 271.

If a number of Leyden jars of great capacity are connected in series by means of conductors of great resistance (such as wet cotton threads in the experiments of M. Gaugain), then an electromotive force acting on the series will produce a current, as indicated by a galvanometer, which will gradually diminish till the jars are fully charged.

The apparent resistance of such a series will increase, and if the dielectric of the jars is a perfect insulator it will increase without limit. If the electromotive force be removed and connexion made between the ends of the series, a reverse current will be observed, the total quantity of which, in the case of perfect insulation, will be the same as that of the direct current. Similar effects are observed in the case of the secondary pile, with the difference that the final insulation is not so good, and that the capacity per unit of surface is immensely greater.

In the case of the cable covered with gutta-percha, &c., it is found that after applying the battery for half an hour, and then connecting the wire with the external electrode, a reverse current takes place, which goes on for some time, and gradually reduces the system to its original state.

These phenomena are of the same kind with those indicated by the 'residual discharge' of the Leyden jar, except that the amount of the polarization is much greater in gutta-percha, &c. than in glass.

This state of polarization seems to be a directed property of the material, which requires for its production not only electromotive force, but the passage, by displacement or otherwise, of a considerable quantity of electricity, and this passage requires a considerable time. When the polarized state has been set up, there is an internal electromotive force acting in the substance in the reverse direction, which will continue till it has either produced a reversed current equal in total quantity to the first, or till the state of polarization has quietly subsided by means of true conduction through the substance.

The whole theory of what has been called residual discharge, absorption of electricity, electrification, or polarization, deserves a careful investigation, and will probably lead to important discoveries relating to the internal structure of bodies.

367.] The resistance of the greater number of dielectrics diminishes as the temperature rises.

Thus the resistance of gutta-percha is about twenty times as great at 0°C as at 24°C. Messrs. Bright and Clark have found that the following formula gives results agreeing with their experiments. If $r$ is the resistance of gutta-percha at temperature $T$ centigrade, then the resistance at temperature $T+t$ will be

$$R = r \times 0.8878^{t},$$

the number varies between 0.8878 and 0.9.

Mr. Hockin has verified the curious fact that it is not until some hours after the gutta-percha has taken its temperature that the resistance reaches its corresponding value.

The effect of temperature on the resistance of india-rubber is not so great as on that of gutta-percha.

The resistance of gutta-percha increases considerably on the application of pressure.

The resistance, in Ohms, of a cubic metre of various specimens of gutta-percha used in different cables is as follows *.

| Name of Cable. | |
|---|---|
| Red Sea | $.267 \times 10^{12}$ to $.362 \times 10^{12}$ |
| Malta-Alexandria | $1.23 \times 10^{12}$ |
| Persian Gulf | $1.80 \times 10^{12}$ |
| Second Atlantic | $3.42 \times 10^{12}$ |
| Hooper's Persian Gulf Core | $74.7 \times 10^{12}$ |
| Gutta-percha at 24°C | $3.53 \times 10^{12}$ |

368.] The following table, calculated from the experiments of

* Jenkin's *Cantor Lectures.*

M. Buff, described in Art. 271, shews the resistance of a cubic metre of glass in Ohms at different temperatures.

| Temperature. | Resistance. |
|---|---|
| 200°C | 227000 |
| 250° | 13900 |
| 300° | 1480 |
| 350° | 1035 |
| 400° | 735 |

369.] Mr. C. F. Varley * has recently investigated the conditions of the current through rarefied gases, and finds that the electromotive force $E$ is equal to a constant $E_0$ together with a part depending on the current according to Ohm's Law, thus

$$E = E_0 + RC.$$

For instance, the electromotive force required to cause the current to begin in a certain tube was that of 323 Daniell's cells, but an electromotive force of 304 cells was just sufficient to maintain the current. The intensity of the current, as measured by the galvanometer, was proportional to the number of cells above 304. Thus for 305 cells the deflexion was 2, for 306 it was 4, for 307 it was 6, and so on up to 380, or $304+76$ for which the deflexion was 150, or $76 \times 1.97$.

From these experiments it appears that there is a kind of polarization of the electrodes, the electromotive force of which is equal to that of 304 Daniell's cells, and that up to this electromotive force the battery is occupied in establishing this state of polarization. When the maximum polarization is established, the excess of electromotive force above that of 304 cells is devoted to maintaining the current according to Ohm's Law.

The law of the current in a rarefied gas is therefore very similar to the law of the current through an electrolyte in which we have to take account of the polarization of the electrodes.

In connexion with this subject we should study Thomson's results, described in Art. 57, in which the electromotive force required to produce a spark in air was found to be proportional not to the distance, but to the distance together with a constant quantity. The electromotive force corresponding to this constant quantity may be regarded as the intensity of polarization of the electrodes.

370.] MM. Wiedemann and Rühlmann have recently † investi-

* *Proc. R. S.*, Jan. 12, 1871.

† *Berichte der Königl. Sächs. Gesellschaft*, Oct. 20, 1871.

gated the passage of electricity through gases. The electric current was produced by Holtz's machine, and the discharge took place between spherical electrodes within a metallic vessel containing rarefied gas. The discharge was in general discontinuous, and the interval of time between successive discharges was measured by means of a mirror revolving along with the axis of Holtz's machine. The images of the series of discharges were observed by means of a heliometer with a divided object-glass, which was adjusted till one image of each discharge coincided with the other image of the next discharge. By this method very consistent results were obtained. It was found that the quantity of electricity in each discharge is independent of the strength of the current and of the material of the electrodes, and that it depends on the nature and density of the gas, and on the distance and form of the electrodes.

These researches confirm the statement of Faraday * that the electric tension (see Art. 48) required to cause a disruptive discharge to begin at the electrified surface of a conductor is a little less when the electrification is negative than when it is positive, but that when a discharge does take place, much more electricity passes at each discharge when it begins at a positive surface. They also tend to support the hypothesis stated in Art. 57, that the stratum of gas condensed on the surface of the electrode plays an important part in the phenomenon, and they indicate that this condensation is greatest at the positive electrode.

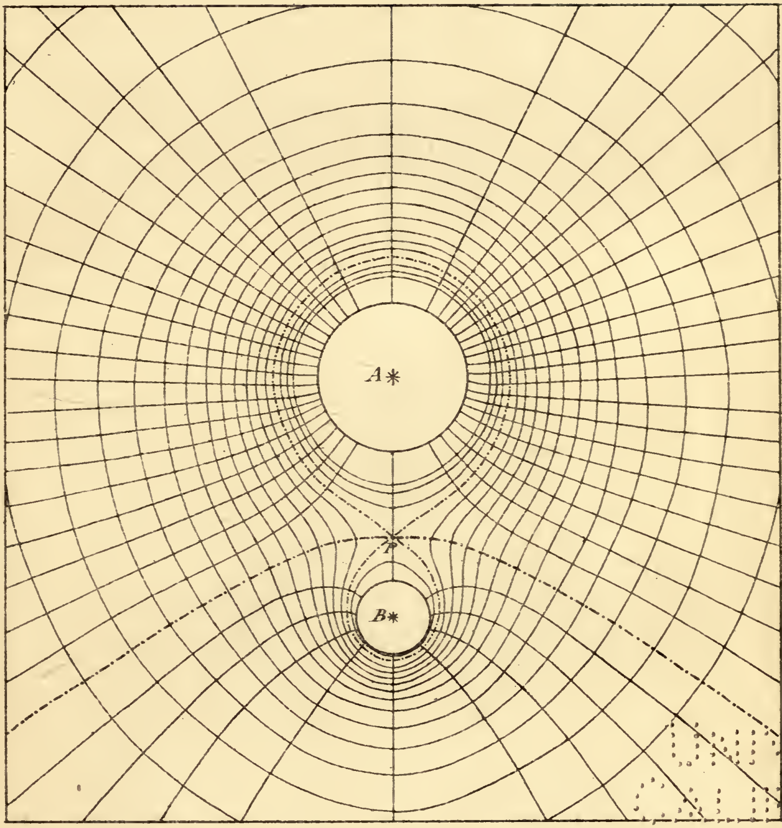
* *Exp. Res.*, 1501.

# PLATES.

Vol. I.

FIG. I.

Art. 118.

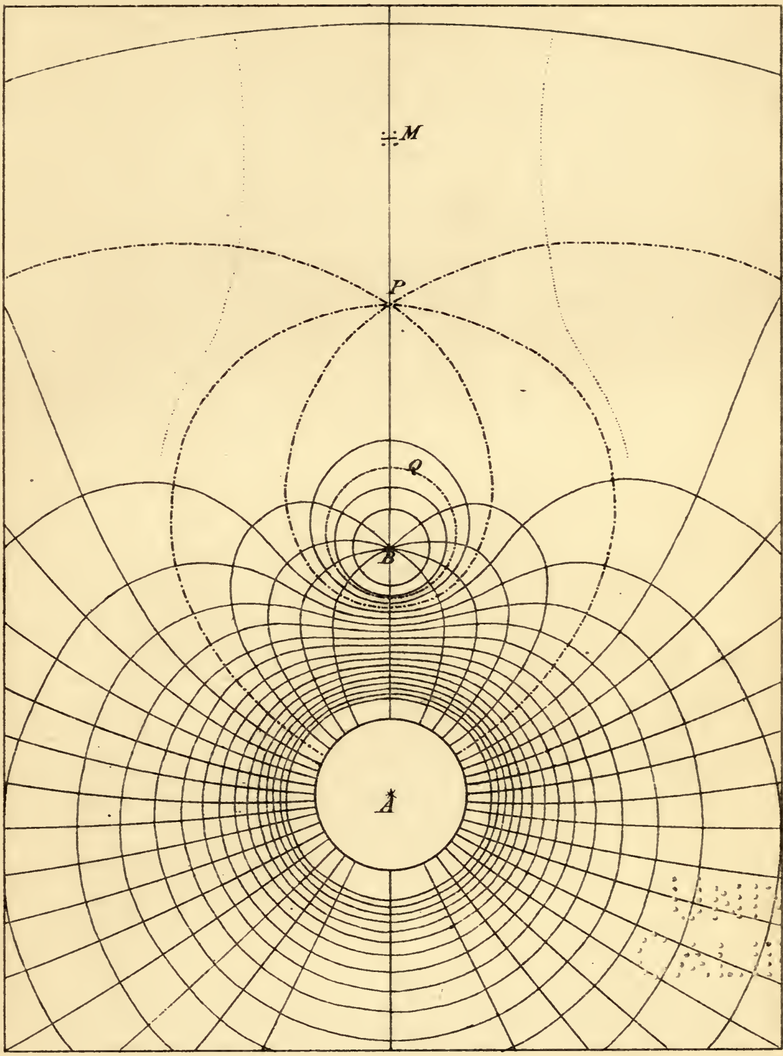



*Lines of Force and Equipotential Surfaces.*

$A = 20$. $B = 5$. $P$, *Point of Equilibrium.* $AP = \frac{2}{3} AB$.

*For the Delegates of the Clarendon Press.*

FIG. II.
Art. 119.




*Lines of Force and Equipotential Surfaces.*

*A = 20.* *B = −5* *P, Point of Equilibrium.* *AP = 2 AB*

*Q, Spherical surface of Zero potential.*

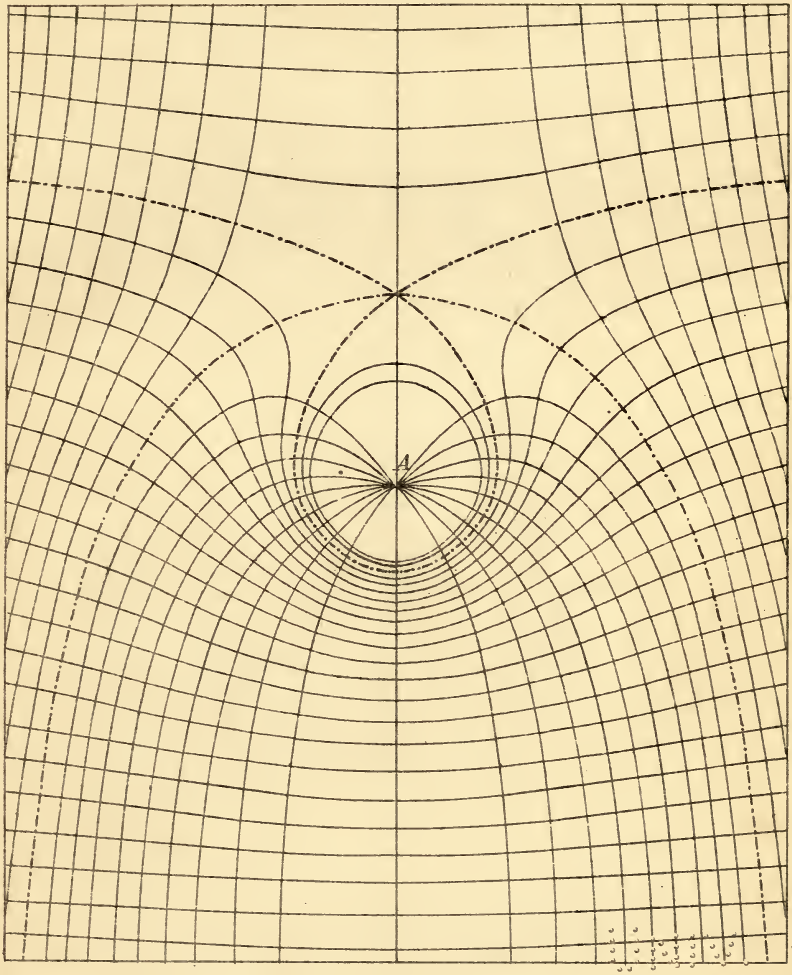
*M, Point of Maximum Force along the axis.*

*The dotted line is the Line of Force* $\Psi = 0.1$ *thus* ..........

For the Delegates of the Clarendon Press

FIG III.

Art. 120.

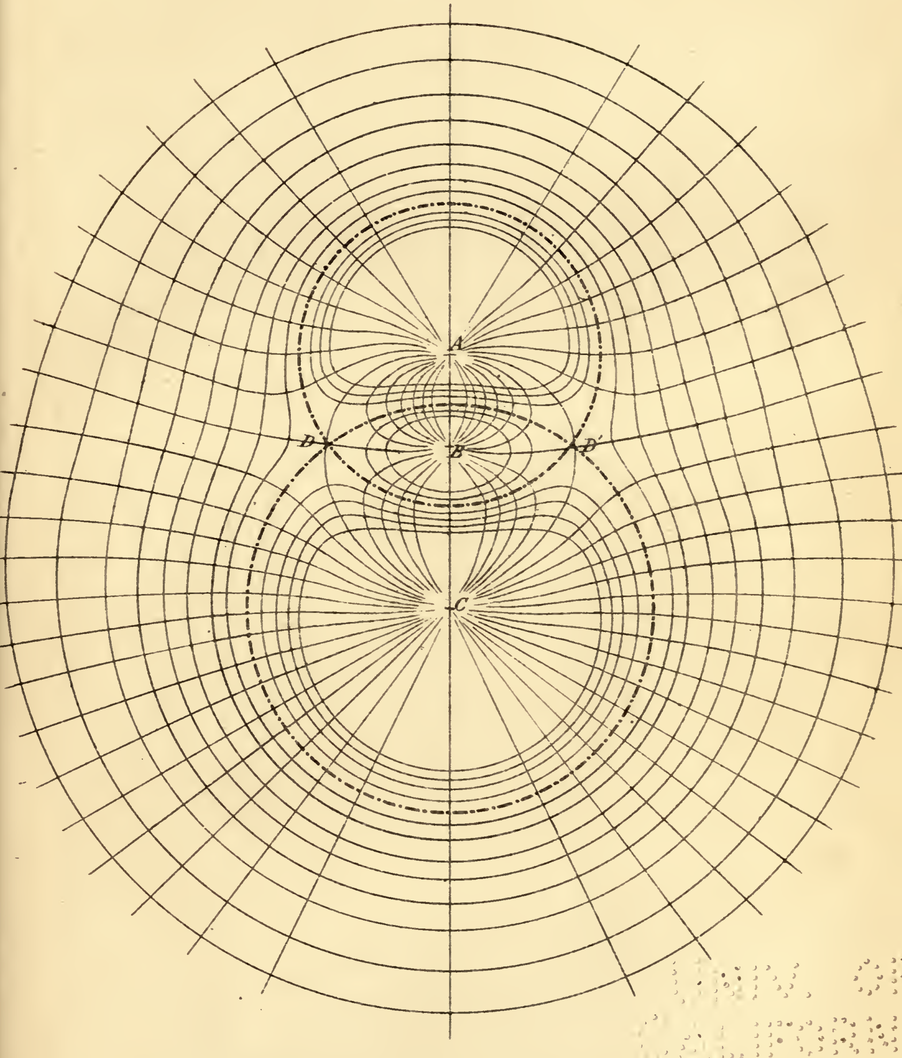



*Lines of Force and Equipotential Surfaces.*

*A = 10.*

*For the Delegates of the Clarendon Press.*

FIG. IV.
Art. 121.



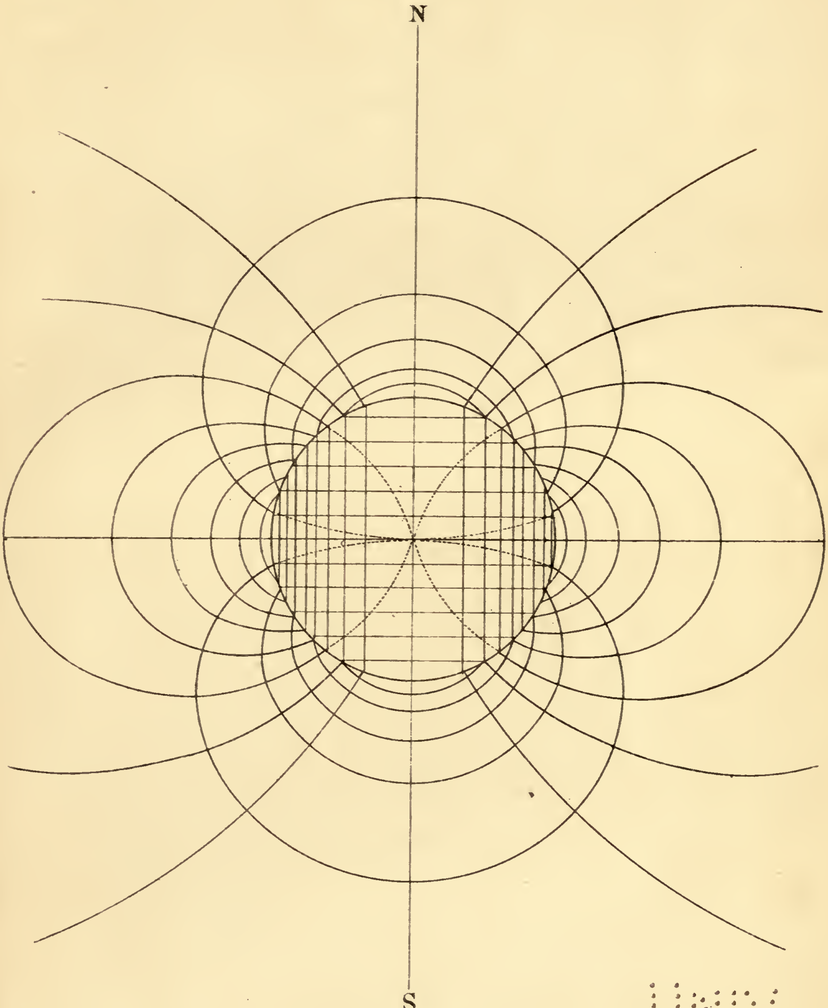

*Lines of Force and Equipotential Surfaces.*

$A = 15.$ $B = -12.$ $C = 20.$

For the Delegates of the Clarendon Press.

FIG. V.

Art. 143.



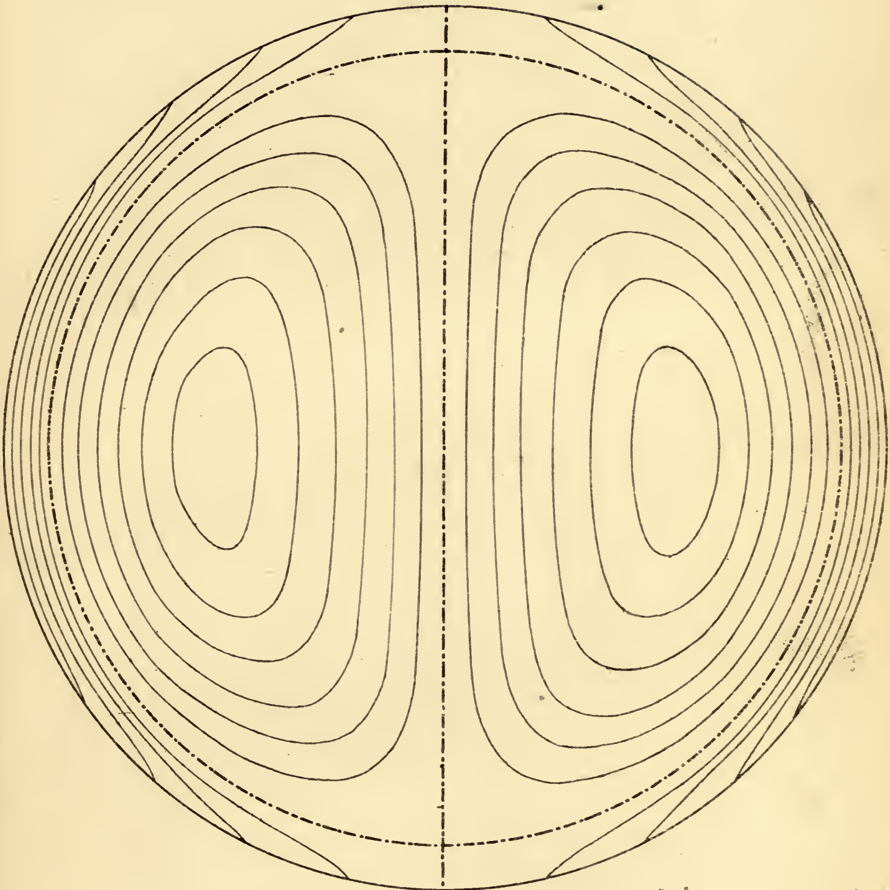



*Lines of Force and Equipotential Surfaces in a diametral section of a spherical Surface in which the superficial density is a harmonic of the first degree.*

For the Delegates of the Clarendon Press.

FIG. VI.

Art. 143.





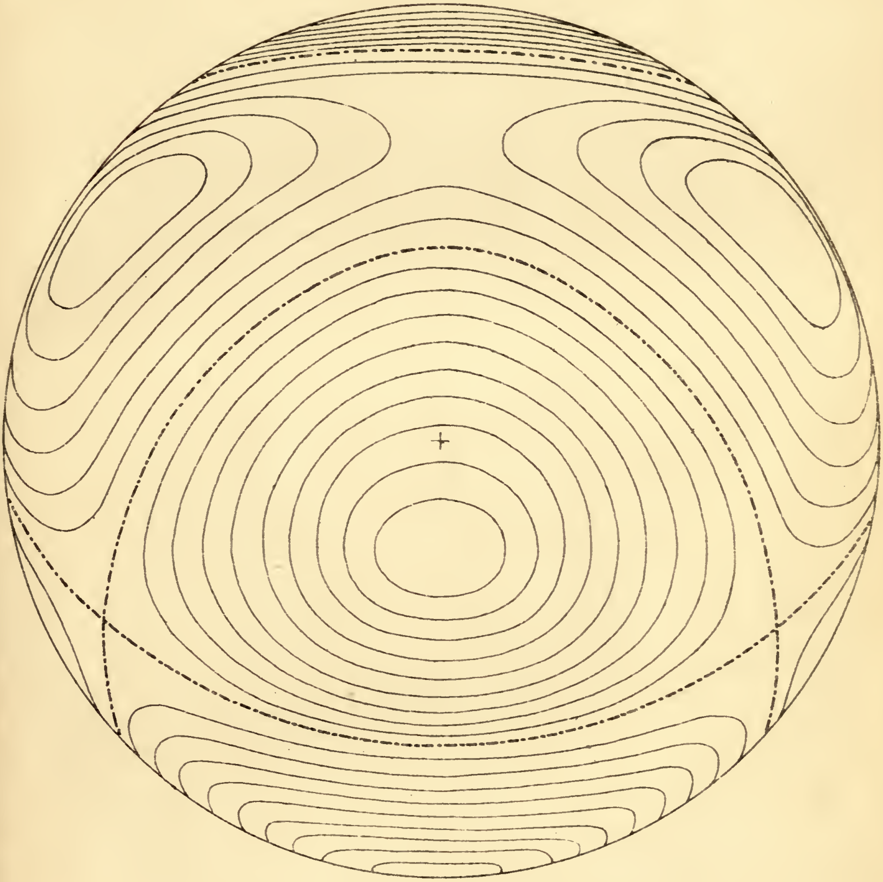
*Spherical Harmonic of the third degree.*

$i = 3$. $s = 1$.

*For the Delegates of the Clarendon Press.*

FIG. VII.

Art. 143.





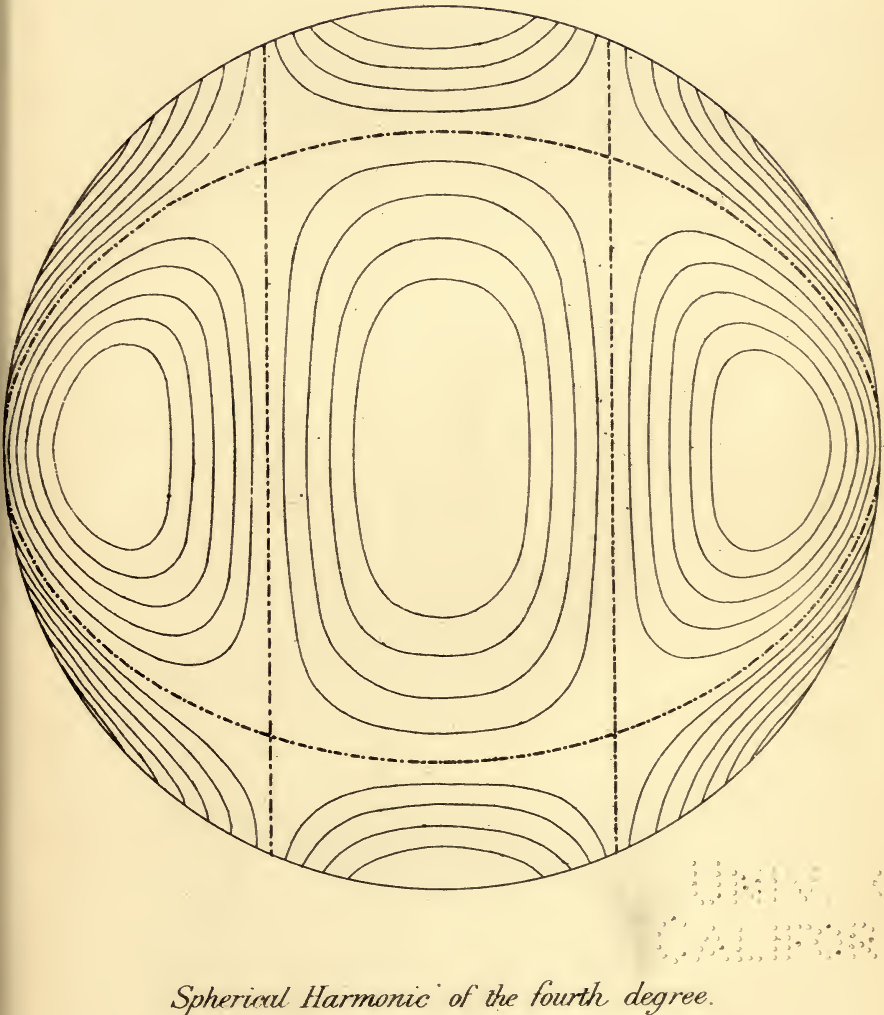
*Spherical Harmonic of the third degree.*

$i = 3$.

*For the Delegates of the Clarendon Press.*

FIG. VIII.

Art. 143.

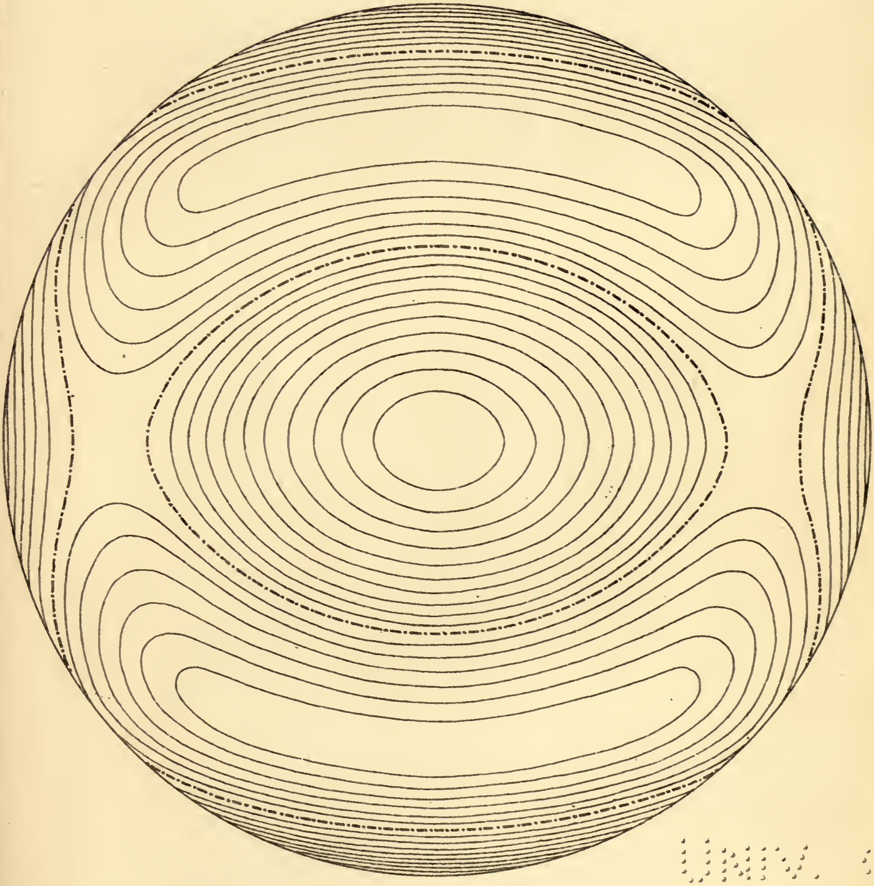


*Spherical Harmonic of the fourth degree.*

$i = 4 \qquad s = 2$.

FIG. IX.

Art. 143.

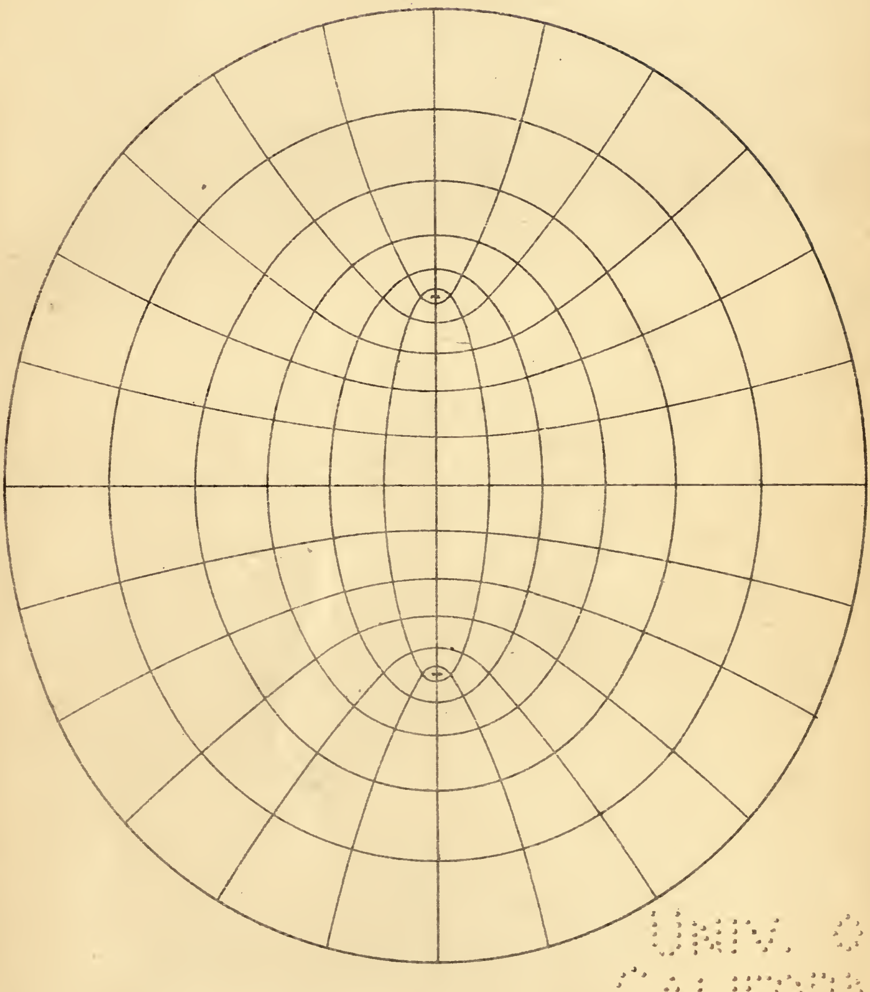




*Spherical Harmonic of the fourth degree.*

FIG. X.

Art. 192.



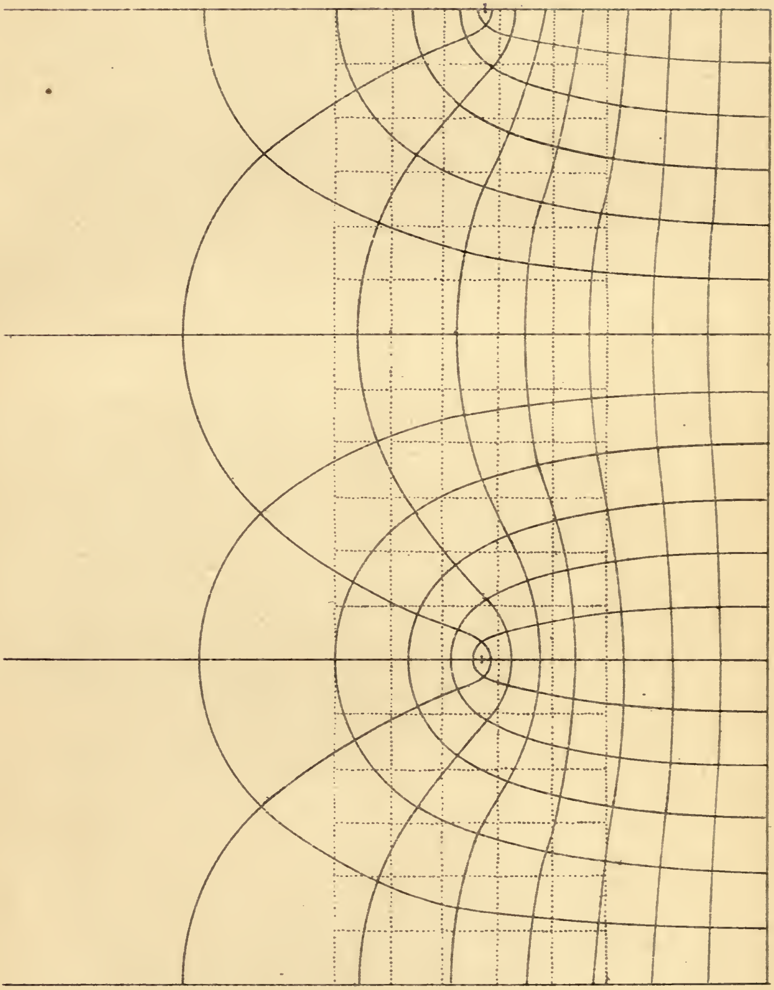


*Confocal Ellipses and Hyperbolas.*

For the Delegates of the Clarendon Press.

FIG. XI.

Art. 193.

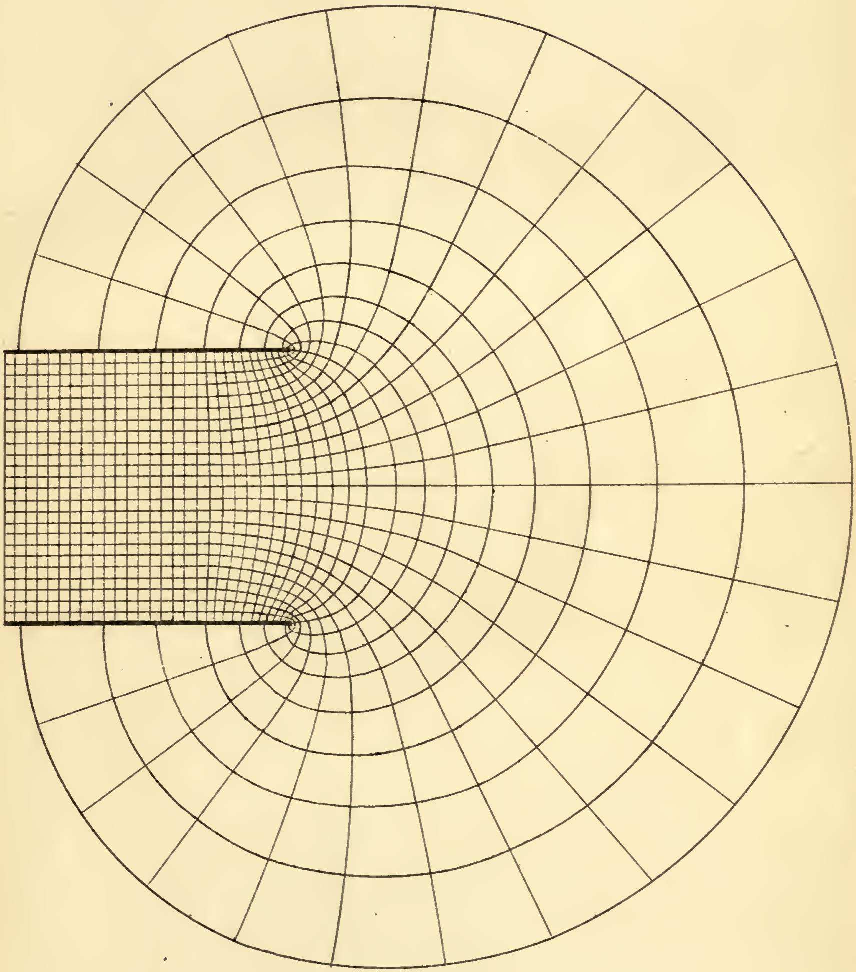


*Lines of Force near the edge of a Plate.*

For the Delegates of the Clarendon Press.

FIG. XII.

Art. 202.

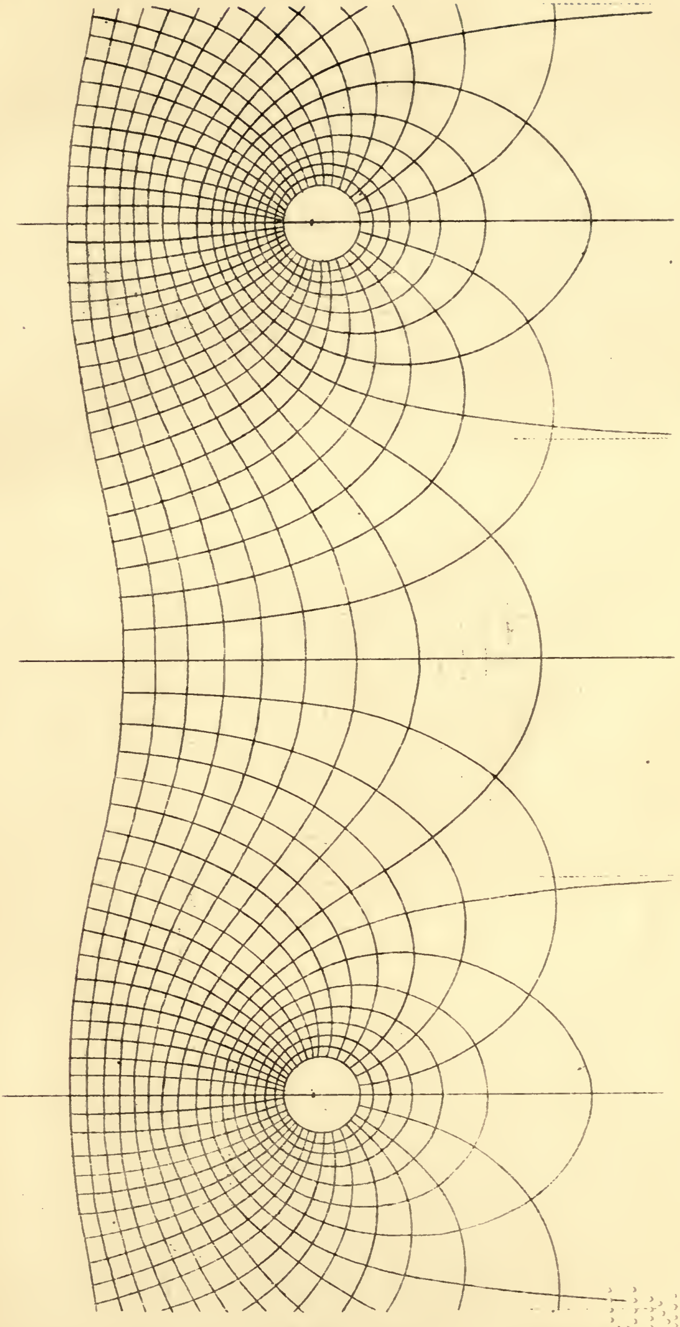


*Lines of Force between two Plates.*

For the Delegates of the Clarendon Press.

FIG. XIII

Art. 203



*Lines of Force near a Grating.*

For the Delegates of the Clarendon Press.