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A TREATISE

ON

ELECTRICITY AND MAGNETISM

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A TREATISE

ON

ELECTRICITY AND MAGNETISM

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I. M
Irving Stringham

PREFACE.

THE fact that certain bodies, after being rubbed, appear to attract other bodies, was known to the ancients. In modern times, a great variety of other phenomena have been observed, and have been found to be related to these phenomena of attraction. They have been classed under the name of *Electric* phenomena, amber, ἤλεκτρον, having been the substance in which they were first described.

Other bodies, particularly the loadstone, and pieces of iron and steel which have been subjected to certain processes, have also been long known to exhibit phenomena of action at a distance. These phenomena, with others related to them, were found to differ from the electric phenomena, and have been classed under the name of *Magnetic* phenomena, the loadstone, μάγνης, being found in the Thessalian Magnesia.

These two classes of phenomena have since been found to be related to each other, and the relations between the various phenomena of both classes, so far as they are known, constitute the science of Electromagnetism.

In the following Treatise I propose to describe the most important of these phenomena, to shew how they may be subjected to measurement, and to trace the mathematical connexions of the quantities measured. Having thus obtained the data for a mathematical theory of electromagnetism, and having shewn how this theory may be applied to the calculation of phenomena, I shall endeavour to place in as clear a light as I can the relations between the mathematical form of this theory and that of the fundamental science of Dynamics, in order that we may be in some degree prepared to determine the kind of dynamical phenomena among which we are to look for illustrations or explanations of the electromagnetic phenomena.

In describing the phenomena, I shall select those which most clearly illustrate the fundamental ideas of the theory, omitting others, or reserving them till the reader is more advanced.

The most important aspect of any phenomenon from a mathematical point of view is that of a measurable quantity. I shall therefore consider electrical phenomena chiefly with a view to their measurement, describing the methods of measurement, and defining the standards on which they depend.

In the application of mathematics to the calculation of electrical quantities, I shall endeavour in the first place to deduce the most general conclusions from the data at our disposal, and in the next place to apply the results to the simplest cases that can be chosen. I shall avoid, as much as I can, those questions which, though they have elicited the skill of mathematicians, have not enlarged our knowledge of science.

The internal relations of the different branches of the science which we have to study are more numerous and complex than those of any other science hitherto developed. Its external relations, on the one hand to dynamics, and on the other to heat, light, chemical action, and the constitution of bodies, seem to indicate the special importance of electrical science as an aid to the interpretation of nature.

It appears to me, therefore, that the study of electromagnetism in all its extent has now become of the first importance as a means of promoting the progress of science.

The mathematical laws of the different classes of phenomena have been to a great extent satisfactorily made out.

The connexions between the different classes of phenomena have also been investigated, and the probability of the rigorous exactness of the experimental laws has been greatly strengthened by a more extended knowledge of their relations to each other.

Finally, some progress has been made in the reduction of electromagnetism to a dynamical science, by shewing that no electromagnetic phenomenon is contradictory to the supposition that it depends on purely dynamical action.

What has been hitherto done, however, has by no means exhausted the field of electrical research. It has rather opened up that field, by pointing out subjects of enquiry, and furnishing us with means of investigation.

It is hardly necessary to enlarge upon the beneficial results of magnetic research on navigation, and the importance of a knowledge of the true direction of the compass, and of the effect of the iron in a ship. But the labours of those who have endeavoured to render navigation more secure by means of magnetic observations have at the same time greatly advanced the progress of pure science.

Gauss, as a member of the German Magnetic Union, brought his powerful intellect to bear on the theory of magnetism, and on the methods of observing it, and he not only added greatly to our knowledge of the theory of attractions, but reconstructed the whole of magnetic science as regards the instruments used, the methods of observation, and the calculation of the results, so that his memoirs on Terrestrial Magnetism may be taken as models of physical research by all those who are engaged in the measurement of any of the forces in nature.

The important applications of electromagnetism to telegraphy have also reacted on pure science by giving a commercial value to accurate electrical measurements, and by affording to electricians the use of apparatus on a scale which greatly transcends that of any ordinary laboratory. The consequences of this demand for electrical knowledge, and of these experimental opportunities for acquiring it, have been already very great, both in stimulating the energies of advanced electricians, and in diffusing among practical men a degree of accurate knowledge which is likely to conduce to the general scientific progress of the whole engineering profession.

There are several treatises in which electrical and magnetic phenomena are described in a popular way. These, however, are not what is wanted by those who have been brought face to face with quantities to be measured, and whose minds do not rest satisfied with lecture-room experiments.

There is also a considerable mass of mathematical memoirs which are of great importance in electrical science, but they lie concealed in the bulky Transactions of learned societies; they do not form a connected system; they are of very unequal merit, and they are for the most part beyond the comprehension of any but professed mathematicians.

I have therefore thought that a treatise would be useful which should have for its principal object to take up the whole subject in a methodical manner, and which should also indicate how each part of the subject is brought within the reach of methods of verification by actual measurement.

The general complexion of the treatise differs considerably from that of several excellent electrical works, published, most of them, in Germany, and it may appear that scant justice is done to the speculations of several eminent electricians and mathematicians. One reason of this is that before I began the study of electricity I resolved to read no mathematics on the subject till I had first read through Faraday's *Experimental Researches on Electricity*. I was aware that there was supposed to be a difference between Faraday's way of conceiving phenomena and that of the mathematicians, so that neither he nor they were satisfied with each other's language. I had also the conviction that this discrepancy did not arise from either party being wrong. I was first convinced of this by Sir William Thomson *, to whose advice and assistance, as well as to his published papers, I owe most of what I have learned on the subject.

As I proceeded with the study of Faraday, I perceived that his method of conceiving the phenomena was also a mathematical one, though not exhibited in the conventional form of mathematical symbols. I also found that these methods were capable of being expressed in the ordinary mathematical forms, and thus compared with those of the professed mathematicians.

For instance, Faraday, in his mind's eye, saw lines of force traversing all space where the mathematicians saw centres of force attracting at a distance: Faraday saw a medium where they saw nothing but distance: Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids.

When I had translated what I considered to be Faraday's ideas into a mathematical form, I found that in general the results of the two methods coincided, so that the same phenomena were accounted for, and the same laws of action deduced by both methods, but that Faraday's methods resembled those

* I take this opportunity of acknowledging my obligations to Sir W. Thomson and to Professor Tait for many valuable suggestions made during the printing of this work.

in which we begin with the whole and arrive at the parts by analysis, while the ordinary mathematical methods were founded on the principle of beginning with the parts and building up the whole by synthesis.

I also found that several of the most fertile methods of research discovered by the mathematicians could be expressed much better in terms of ideas derived from Faraday than in their original form.

The whole theory, for instance, of the potential, considered as a quantity which satisfies a certain partial differential equation, belongs essentially to the method which I have called that of Faraday. According to the other method, the potential, if it is to be considered at all, must be regarded as the result of a summation of the electrified particles divided each by its distance from a given point. Hence many of the mathematical discoveries of Laplace, Poisson, Green and Gauss find their proper place in this treatise, and their appropriate expression in terms of conceptions mainly derived from Faraday.

Great progress has been made in electrical science, chiefly in Germany, by cultivators of the theory of action at a distance. The valuable electrical measurements of W. Weber are interpreted by him according to this theory, and the electromagnetic speculation which was originated by Gauss, and carried on by Weber, Riemann, J. and C. Neumann, Lorenz, &c. is founded on the theory of action at a distance, but depending either directly on the relative velocity of the particles, or on the gradual propagation of something, whether potential or force, from the one particle to the other. The great success which these eminent men have attained in the application of mathematics to electrical phenomena gives, as is natural, additional weight to their theoretical speculations, so that those who, as students of electricity, turn to them as the greatest authorities in mathematical electricity, would probably imbibe, along with their mathematical methods, their physical hypotheses.

These physical hypotheses, however, are entirely alien from the way of looking at things which I adopt, and one object which I have in view is that some of those who wish to study electricity may, by reading this treatise, come to see that there is another way of treating the subject, which is no less fitted to explain the phenomena, and which, though in some parts it may appear less definite, corresponds, as I think, more faithfully with our actual knowledge, both in what it affirms and in what it leaves undecided.

In a philosophical point of view, moreover, it is exceedingly important that two methods should be compared, both of which have succeeded in explaining the principal electromagnetic phenomena, and both of which have attempted to explain the propagation of light as an electromagnetic phenomenon, and have actually calculated its velocity, while at the same time the fundamental conceptions of what actually takes place, as well as most of the secondary conceptions of the quantities concerned, are radically different.

I have therefore taken the part of an advocate rather than that of a judge, and have rather exemplified one method than attempted to give an impartial description of both. I have no doubt that the method which I have called the German one will also find its supporters, and will be expounded with a skill worthy of its ingenuity.

I have not attempted an exhaustive account of electrical phenomena, experiments, and apparatus. The student who desires to read all that is known on these subjects will find great assistance from the *Traité d'Electricité* of Professor A. de la Rive, and from several German treatises, such as Wiedemann's *Galvanismus*, Riess' *Reibungselektricität*, Beer's *Einleitung in die Elektrostatik*, &c.

I have confined myself almost entirely to the mathematical treatment of the subject, but I would recommend the student, after he has learned, experimentally if possible, what are the phenomena to be observed, to read carefully Faraday's *Experimental Researches in Electricity.* He will there find a strictly contemporary historical account of some of the greatest electrical discoveries and investigations, carried on in an order and succession which could hardly have been improved if the results had been known from the first, and expressed in the language of a man who devoted much of his attention to the methods of accurately describing scientific operations and their results *.

It is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely assimilated when

* *Life and Letters of Faraday,* vol. i. p. 395.

it is in the nascent state, and in the case of Faraday's *Researches* this is comparatively easy, as they are published in a separate form, and may be read consecutively. If by anything I have here written I may assist any student in understanding Faraday's modes of thought and expression, I shall regard it as the accomplishment of one of my principal aims—to communicate to others the same delight which I have found myself in reading Faraday's *Researches*.

The description of the phenomena, and the elementary parts of the theory of each subject, will be found in the earlier chapters of each of the four Parts into which this treatise is divided. The student will find in these chapters enough to give him an elementary acquaintance with the whole science.

The remaining chapters of each Part are occupied with the higher parts of the theory, the processes of numerical calculation, and the instruments and methods of experimental research.

The relations between electromagnetic phenomena and those of radiation, the theory of molecular electric currents, and the results of speculation on the nature of action at a distance, are treated of in the last four chapters of the second volume.

Feb. 1, 1873.

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ERRATA. VOL. I.

Page 26, l. 3 from bottom, *dele* 'As we have made no assumption', &c. *down to* l. 7 of p. 27, 'the expression may then be written', and substitute as follows :—

Let us now suppose that the curves for which a is constant form a series of closed curves, surrounding the point on the surface for which a has its minimum value, a_0, the last curve of the series, for which $a = a_1$, coinciding with the original closed curve s.

Let us also suppose that the curves for which β is constant form a series of lines drawn from the point at which $a = a_0$ to the closed curve s, the first, β_0, and the last, β_1, being identical.

Integrating (8) by parts, the first term with respect to a and the second with respect to β, the double integrals destroy each other. The line integral,

$$\int_{\beta_0}^{\beta_1} \left(X\frac{dx}{d\beta}\right)_{a=a_0} d\beta,$$

is zero, because the curve $a = a_0$ is reduced to a point at which there is but one value of X and of x.

The two line integrals,

$$-\int_{a_0}^{a_1} \left(X\frac{dx}{da}\right)_{\beta=\beta_1} da + \int_{a_0}^{a_1} \left(X\frac{dx}{da}\right)_{\beta=\beta_0} da,$$

destroy each other, because the point (a, β_1) is identical with the point (a, β_0).

The expression (8) is therefore reduced to

$$\int_{\beta_0}^{\beta_1} \left(X\frac{dx}{d\beta}\right)_{a=a_1} d\beta. \tag{9}$$

Since the curve $a = a_1$ is identical with the closed curve s, we may write this expression

p. 80, in equations (3), (4), (6), (8), (17), (18), (19), (20), (21), (22), *for* R *read* N.

p. 82, l. 3, *for* Rl *read* Nl.

p. 83, in equations (28), (29), (30), (31), *for* $\frac{d^2 V_1}{dx^2}$ *read* $\frac{d^2 V'}{dx dv}$,

„ in equation (29), *insert* — *before* the second member.

p. 105, l. 2, *for* Q *read* $8\pi Q$.

p. 108, equation (1), *for* ρ *read* ρ'.

„ „ (2), *for* ρ' *read* ρ.

„ „ (3), *for* σ *read* σ'.

„ „ (4), *for* σ' *read* σ.

p. 113, l. 4, *for* KR *read* $\frac{1}{4\pi}KR$.

„ l. 5, *for* $KRR'\cos\epsilon$ *read* $\frac{1}{4\pi}KRR'\cos\epsilon$.

p. 114, l. 5, *for* S_1 *read* S.

p. 124, last line, *for* $e_1 + e_1$ *read* $e_1 + e_2$.

p. 125, lines 3 and 4, *transpose* within *and* without; l. 16, *for* v *read* V; and l. 18, *for* V *read* v.

p. 128, lines 11, 10, 8 from bottom, *for* dx *read* dz.

p. 149, l. 24, *for* equpotential *read* equipotential.

p. 159, l. 3, *for* F *read* f.
,, l. 2 from bottom, *for* M *read* M_2.
p. 163, l. 20, *for* λ_{i-s+1} *read* $\lambda_{i-\sigma+1}$.
p. 164, equation (34), *for* $(-1)^{i-s}\dfrac{\lfloor 2s}{2^{2s}\lfloor i \lfloor s}$ *read* $(-1)^{i-\sigma}\dfrac{\lfloor 2\sigma}{2^{2\sigma}\lfloor i \lfloor \sigma}$.
p. 179, equation (76), *for* $i+1$ *read* $2i+1$.
p. 185, equation (24), *for* $\dfrac{x^2}{b^2}-\dfrac{z^2}{c^2}=1$ *read* $\dfrac{x^2}{b^2}-\dfrac{z^2}{c^2-b^2}=1$.
p. 186, l. 5 from bottom, *for* 'The surface-density on the elliptic plate' *read* The surface-density on either side of the elliptic plate.
p. 186, equation (30), *for* 2π *read* 4π.
p. 188, equation (38), *for* π^2 *read* $2\pi^2$.
p. 196, l. 27, *for* $e\,..\,e$ *read* $e_1\,..\,e_2$.
p. 197, equation (10) should be $M=\dfrac{Ee}{f}-\frac{1}{2}\dfrac{e^2a^3}{f^2(f^2-a^2)}$.
p. 204, l. 15 from bottom, *dele* either.
p. 215, l. 4, *for* $\sqrt{2k}$ *read* $\sqrt{}2k$.
p. 234, equation (13), *for* $2E$ *read* $\dfrac{E}{2\pi}$.
p. 335, *dele* last 14 lines.
p. 336, l. 1, *dele* therefore.
,, l. 2, *for* 'the potential at C to exceed that at D by P,' *read* a current, C, from X to Y.
,, l. 4, *for* 'C to D will cause the potential at A to exceed that at B by the same quantity P,' *read* X to Y will cause an equal current C from A to B.
p. 351, l. 3, *for* $R_1{}^2u^2+R_2{}^2v^2+R_3{}^2w^2$ *read* $R_1u^2+R_2v^2+R_3w^2$.
,, l. 5, *read* $+2\iiint(u\dfrac{dV}{dx}+v\dfrac{dV}{dy}+w\dfrac{dV}{dz})\,dx\,dy\,dz$.
p. 355, last line, *for* S' *read* S.
p. 356, equation (12), *for* $\left.\overline{\dfrac{db}{d}}\right|^2$ *read* $\left.\overline{\dfrac{db}{dx}}\right|^2$.
p. 365, in equations (12), (15), (16), *for* A *read* Ar.
p. 366, equation (3), *for* $\dfrac{E_2}{r_1}$ *read* $\dfrac{E_2}{r_2}$.
p. 367, l. 5, *for* $2k_1S$ *read* $2k_2S$.
p. 368, equation (14), *for* J_2' *read* I_2'.
p. 397, l. 1, *for* $\dfrac{D'}{E}\delta'$ *read* $\dfrac{D'}{E'}\delta'$.
p. 404, at the end of Art. 350 insert as follows:—

When γ, the resistance to be measured, a, the resistance of the battery, and α, the resistance of the galvanometer, are given, the best values of the other resistances have been shewn by Mr. Oliver Heaviside (*Phil. Mag.*, Feb. 1873) to be

$$c=\sqrt{a\alpha},\quad b=\sqrt{a\gamma\frac{\alpha+\gamma}{a+\gamma}},\quad \beta=\sqrt{\alpha\gamma\frac{a+\gamma}{\alpha+\gamma}}.$$

ELECTRICITY AND MAGNETISM.

ELECTRICITY AND MAGNETISM.

PRELIMINARY.

ON THE MEASUREMENT OF QUANTITIES.

1.] EVERY expression of a Quantity consists of two factors or components. One of these is the name of a certain known quantity of the same kind as the quantity to be expressed, which is taken as a standard of reference. The other component is the number of times the standard is to be taken in order to make up the required quantity. The standard quantity is technically called the Unit, and the number is called the Numerical Value of the quantity.

There must be as many different units as there are different kinds of quantities to be measured, but in all dynamical sciences it is possible to define these units in terms of the three fundamental units of Length, Time, and Mass. Thus the units of area and of volume are defined respectively as the square and the cube whose sides are the unit of length.

Sometimes, however, we find several units of the same kind founded on independent considerations. Thus the gallon, or the volume of ten pounds of water, is used as a unit of capacity as well as the cubic foot. The gallon may be a convenient measure in some cases, but it is not a systematic one, since its numerical relation to the cubic foot is not a round integral number.

2.] In framing a mathematical system we suppose the fundamental units of length, time, and mass to be given, and deduce all the derivative units from these by the simplest attainable definitions.

The formulae at which we arrive must be such that a person

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of any nation, by substituting for the different symbols the numerical value of the quantities as measured by his own national units, would arrive at a true result.

Hence, in all scientific studies it is of the greatest importance to employ units belonging to a properly defined system, and to know the relations of these units to the fundamental units, so that we may be able at once to transform our results from one system to another.

This is most conveniently done by ascertaining the *dimensions* of every unit in terms of the three fundamental units. When a given unit varies as the *n*th power of one of these units, it is said to be of *n dimensions* as regards that unit.

For instance, the scientific unit of volume is always the cube whose side is the unit of length. If the unit of length varies, the unit of volume will vary as its third power, and the unit of volume is said to be of three dimensions with respect to the unit of length.

A knowledge of the dimensions of units furnishes a test which ought to be applied to the equations resulting from any lengthened investigation. The dimensions of every term of such an equation, with respect to each of the three fundamental units, must be the same. If not, the equation is absurd, and contains some error, as its interpretation would be different according to the arbitrary system of units which we adopt *.

The Three Fundamental Units.

3.] (1) *Length.* The standard of length for scientific purposes in this country is one foot, which is the third part of the standard yard preserved in the Exchequer Chambers.

In France, and other countries which have adopted the metric system, it is the mètre. The mètre is theoretically the ten millionth part of the length of a meridian of the earth measured from the pole to the equator; but practically it is the length of a standard preserved in Paris, which was constructed by Borda to correspond, when at the temperature of melting ice, with the value of the preceding length as measured by Delambre. The mètre has not been altered to correspond with new and more accurate measurements of the earth, but the arc of the meridian is estimated in terms of the original mètre.

* The theory of dimensions was first stated by Fourier, *Théorie de Chaleur*, § 160.

In astronomy the mean distance of the earth from the sun is sometimes taken as a unit of length.

In the present state of science the most universal standard of length which we could assume would be the wave length in vacuum of a particular kind of light, emitted by some widely diffused substance such as sodium, which has well-defined lines in its spectrum. Such a standard would be independent of any changes in the dimensions of the earth, and should be adopted by those who expect their writings to be more permanent than that body.

In treating of the dimensions of units we shall call the unit of length $[L]$. If l is the numerical value of a length, it is understood to be expressed in terms of the concrete unit $[L]$, so that the actual length would be fully expressed by $l\,[L]$.

4.] (2) *Time*. The standard unit of time in all civilized countries is deduced from the time of rotation of the earth about its axis. The sidereal day, or the true period of rotation of the earth, can be ascertained with great exactness by the ordinary observations of astronomers; and the mean solar day can be deduced from this by our knowledge of the length of the year.

The unit of time adopted in all physical researches is one second of mean solar time.

In astronomy a year is sometimes used as a unit of time. A more universal unit of time might be found by taking the periodic time of vibration of the particular kind of light whose wave length is the unit of length.

We shall call the concrete unit of time $[T]$, and the numerical measure of time t.

5.] (3) *Mass*. The standard unit of mass is in this country the avoirdupois pound preserved in the Exchequer Chambers. The grain, which is often used as a unit, is defined to be the 7000th part of this pound.

In the metrical system it is the gramme, which is theoretically the mass of a cubic centimètre of distilled water at standard temperature and pressure, but practically it is the thousandth part of a standard kilogramme preserved in Paris.

The accuracy with which the masses of bodies can be compared by weighing is far greater than that hitherto attained in the measurement of lengths, so that all masses ought, if possible, to be compared directly with the standard, and not deduced from experiments on water.

In descriptive astronomy the mass of the sun or that of the earth is sometimes taken as a unit, but in the dynamical theory of astronomy the unit of mass is deduced from the units of time and length, combined with the fact of universal gravitation. The astronomical unit of mass is that mass which attracts another body placed at the unit of distance so as to produce in that body the unit of acceleration.

In framing a universal system of units we may either deduce the unit of mass in this way from those of length and time already defined, and this we can do to a rough approximation in the present state of science; or, if we expect * soon to be able to determine the mass of a single molecule of a standard substance, we may wait for this determination before fixing a universal standard of mass.

We shall denote the concrete unit of mass by the symbol $[M]$ in treating of the dimensions of other units. The unit of mass will be taken as one of the three fundamental units. When, as in the French system, a particular substance, water, is taken as a standard of density, then the unit of mass is no longer independent, but varies as the unit of volume, or as $[L^3]$.

If, as in the astronomical system, the unit of mass is defined with respect to its attractive power, the dimensions of $[M]$ are $[L^3 T^{-2}]$.

For the acceleration due to the attraction of a mass m at a distance r is by the Newtonian Law $\frac{m}{r^2}$. Suppose this attraction to act for a very small time t on a body originally at rest, and to cause it to describe a space s, then by the formula of Galileo,

$$s = \tfrac{1}{2} f t^2 = \tfrac{1}{2} \frac{m}{r^2} t^2;$$

whence $m = 2\frac{r^2 s}{t^2}$. Since r and s are both lengths, and t is a time, this equation cannot be true unless the dimensions of m are $[L^3 T^{-2}]$. The same can be shewn from any astronomical equation in which the mass of a body appears in some but not in all of the terms †.

* See Prof. J. Loschmidt, 'Zur Grösse der Luftmolecule,' *Academy of Vienna*, Oct. 12, 1865; G. J. Stoney on 'The Internal Motions of Gases,' *Phil. Mag.*, Aug. 1868; and Sir W. Thomson on 'The Size of Atoms,' *Nature*, March 31, 1870.

† If a foot and a second are taken as units, the astronomical unit of mass would be about 932,000,000 pounds.

Derived Units.

6.] The unit of Velocity is that velocity in which unit of length is described in unit of time. Its dimensions are $[LT^{-1}]$.

If we adopt the units of length and time derived from the vibrations of light, then the unit of velocity is the velocity of light.

The unit of Acceleration is that acceleration in which the velocity increases by unity in unit of time. Its dimensions are $[LT^{-2}]$.

The unit of Density is the density of a substance which contains unit of mass in unit of volume. Its dimensions are $[ML^{-3}]$.

The unit of Momentum is the momentum of unit of mass moving with unit of velocity. Its dimensions are $[MLT^{-1}]$.

The unit of Force is the force which produces unit of momentum in unit of time. Its dimensions are $[MLT^{-2}]$.

This is the absolute unit of force, and this definition of it is implied in every equation in Dynamics. Nevertheless, in many text books in which these equations are given, a different unit of force is adopted, namely, the weight of the national unit of mass; and then, in order to satisfy the equations, the national unit of mass is itself abandoned, and an artificial unit is adopted as the dynamical unit, equal to the national unit divided by the numerical value of the force of gravity at the place. In this way both the unit of force and the unit of mass are made to depend on the value of the force of gravity, which varies from place to place, so that statements involving these quantities are not complete without a knowledge of the force of gravity in the places where these statements were found to be true.

The abolition, for all scientific purposes, of this method of measuring forces is mainly due to the introduction of a general system of making observations of magnetic force in countries in which the force of gravity is different. All such forces are now measured according to the strictly dynamical method deduced from our definitions, and the numerical results are the same in whatever country the experiments are made.

The unit of Work is the work done by the unit of force acting through the unit of length measured in its own direction. Its dimensions are $[ML^2T^{-2}]$.

The Energy of a system, being its capacity of performing work, is measured by the work which the system is capable of performing by the expenditure of its whole energy.

The definitions of other quantities, and of the units to which they are referred, will be given when we require them.

In transforming the values of physical quantities determined in terms of one unit, so as to express them in terms of any other unit of the same kind, we have only to remember that every expression for the quantity consists of two factors, the unit and the numerical part which expresses how often the unit is to be taken. Hence the numerical part of the expression varies inversely as the magnitude of the unit, that is, inversely as the various powers of the fundamental units which are indicated by the dimensions of the derived unit.

On Physical Continuity and Discontinuity.

7.] A quantity is said to vary continuously when, if it passes from one value to another, it assumes all the intermediate values.

We may obtain the conception of continuity from a consideration of the continuous existence of a particle of matter in time and space. Such a particle cannot pass from one position to another without describing a continuous line in space, and the coordinates of its position must be continuous functions of the time.

In the so-called 'equation of continuity,' as given in treatises on Hydrodynamics, the fact expressed is that matter cannot appear in or disappear from an element of volume without passing in or out through the sides of that element.

A quantity is said to be a continuous function of its variables when, if the variables alter continuously, the quantity itself alters continuously.

Thus, if u is a function of x, and if, while x passes continuously from x_0 to x_1, u passes continuously from u_0 to u_1, but when x passes from x_1 to x_2, u passes from u_1' to u_2, u_1' being different from u_1, then u is said to have a discontinuity in its variation with respect to x for the value $x = x_1$, because it passes abruptly from u_1 to u_1' while x passes continuously through x_1.

If we consider the differential coefficient of u with respect to x for the value $x = x_1$ as the limit of the fraction

$$\frac{u_2 - u_0}{x_2 - x_0},$$

when x_2 and x_0 are both made to approach x_1 without limit, then, if x_0 and x_2 are always on opposite sides of x_1, the ultimate value of the numerator will be $u_1' - u_1$, and that of the denominator will be zero. If u is a quantity physically continuous, the discontinuity can exist only with respect to the particular variable x. We must in this case admit that it has an infinite differential coefficient when $x = x_1$. If u is not physically continuous, it cannot be differentiated at all.

It is possible in physical questions to get rid of the idea of discontinuity without sensibly altering the conditions of the case. If x_0 is a very little less than x_1, and x_2 a very little greater than x_1, then u_0 will be very nearly equal to u_1 and u_2 to u_1'. We may now suppose u to vary in any arbitrary but continuous manner from u_0 to u_2 between the limits x_0 and x_2. In many physical questions we may begin with a hypothesis of this kind, and then investigate the result when the values of x_0 and x_2 are made to approach that of x_1 and ultimately to reach it. The result will in most cases be independent of the arbitrary manner in which we have supposed u to vary between the limits.

Discontinuity of a Function of more than One Variable.

8.] If we suppose the values of all the variables except x to be constant, the discontinuity of the function will occur for particular values of x, and these will be connected with the values of the other variables by an equation which we may write

$$\phi = \phi\,(x, y, z, \&c.) = 0.$$

The discontinuity will occur when $\phi = 0$. When ϕ is positive the function will have the form $F_2\,(x, y, z, \&c.)$. When ϕ is negative it will have the form $F_1\,(x, y, z, \&c.)$. There need be no necessary relation between the forms F_1 and F_2.

To express this discontinuity in a mathematical form, let one of the variables, say x, be expressed as a function of ϕ and the other variables, and let F_1 and F_2 be expressed as functions of ϕ, y, z, &c. We may now express the general form of the function by any formula which is sensibly equal to F_2 when ϕ is positive, and to F_1 when ϕ is negative. Such a formula is the following—

$$F = \frac{F_1 + e^{n\phi} F_2}{1 + e^{n\phi}}.$$

As long as n is a finite quantity, however great, F will be a continuous function, but if we make n infinite F will be equal to F_2 when ϕ is positive, and equal to F_1 when ϕ is negative.

Discontinuity of the Derivatives of a Continuous Function.

The first derivatives of a continuous function may be discontinuous. Let the values of the variables for which the discontinuity of the derivatives occurs be connected by the equation

$$\phi = \phi\,(x, y, z \ldots) = 0,$$

and let F_1 and F_2 be expressed in terms of ϕ and $n-1$ other variables, say $(y, z \ldots)$.

Then, when ϕ is negative, F_1 is to be taken, and when ϕ is positive F_2 is to be taken, and, since F is itself continuous, when ϕ is zero, $F_1 = F_2$.

Hence, when ϕ is zero, the derivatives $\frac{dF_1}{d\phi}$ and $\frac{dF_2}{d\phi}$ may be different, but the derivatives with respect to any of the other variables, such as $\frac{dF_1}{dy}$ and $\frac{dF_2}{dy}$, must be the same. The discontinuity is therefore confined to the derivative with respect to ϕ, all the other derivatives being continuous.

Periodic and Multiple Functions.

9.] If u is a function of x such that its value is the same for x, $x+a$, $x+na$, and all values of x differing by a, u is called a periodic function of x, and a is called its period.

If x is considered as a function of u, then, for a given value of u, there must be an infinite series of values of x differing by multiples of a. In this case x is called a multiple function of u, and a is called its cyclic constant.

The differential coefficient $\frac{dx}{du}$ has only a finite number of values corresponding to a given value of u.

On the Relation of Physical Quantities to Directions in Space.

10.] In distinguishing the kinds of physical quantities, it is of great importance to know how they are related to the directions of those coordinate axes which we usually employ in defining the positions of things. The introduction of coordinate axes into geometry by Des Cartes was one of the greatest steps in mathematical progress, for it reduced the methods of geometry to calculations performed on numerical quantities. The position of a point is made to depend on the length of three lines which are always drawn in determinate directions, and the line joining two points is in like manner considered as the resultant of three lines.

But for many purposes in physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing the Cartesian coordinates, and to fix the mind at once on a point of space instead of its three coordinates, and on the magnitude and direction of a force instead of its three components. This mode of contemplating geometrical and physical quantities is more primitive and more natural than the other, although the ideas connected with it did not receive their full development till Hamilton made the next great step in dealing with space, by the invention of his Calculus of Quaternions.

As the methods of Des Cartes are still the most familiar to students of science, and as they are really the most useful for purposes of calculation, we shall express all our results in the Cartesian form. I am convinced, however, that the introduction of the ideas, as distinguished from the operations and methods of Quaternions, will be of great use to us in the study of all parts of our subject, and especially in electrodynamics, where we have to deal with a number of physical quantities, the relations of which to each other can be expressed far more simply by a few words of Hamilton's, than by the ordinary equations.

11.] One of the most important features of Hamilton's method is the division of quantities into Scalars and Vectors.

A Scalar quantity is capable of being completely defined by a single numerical specification. Its numerical value does not in any way depend on the directions we assume for the coordinate axes.

A Vector, or Directed quantity, requires for its definition three numerical specifications, and these may most simply be understood as having reference to the directions of the coordinate axes.

Scalar quantities do not involve direction. The volume of a geometrical figure, the mass and the energy of a material body, the hydrostatical pressure at a point in a fluid, and the potential at a point in space, are examples of scalar quantities.

A vector quantity has direction as well as magnitude, and is such that a reversal of its direction reverses its sign. The displacement of a point, represented by a straight line drawn from its original to its final position, may be taken as the typical vector quantity, from which indeed the name of Vector is derived.

The velocity of a body, its momentum, the force acting on it, an electric current, the magnetization of a particle of iron, are instances of vector quantities.

There are physical quantities of another kind which are related to directions in space, but which are not vectors. Stresses and strains in solid bodies are examples of these, and the properties of bodies considered in the theory of elasticity and in the theory of double refraction. Quantities of this class require for their definition *nine* numerical specifications. They are expressed in the language of Quaternions by linear and vector functions of a vector.

The addition of one vector quantity to another of the same kind is performed according to the rule given in Statics for the composition of forces. In fact, the proof which Poisson gives of the 'parallelogram of forces' is applicable to the composition of any quantities such that a reversal of their sign is equivalent to turning them end for end.

When we wish to denote a vector quantity by a single symbol, and to call attention to the fact that it is a vector, so that we must consider its direction as well as its magnitude, we shall denote it by a German capital letter, as $\mathfrak{A}$, $\mathfrak{B}$, &c.

In the calculus of Quaternions, the position of a point in space is defined by the vector drawn from a fixed point, called the origin, to that point. If at that point of space we have to consider any physical quantity whose value depends on the position of the point, that quantity is treated as a function of the vector drawn from the origin. The function may be itself either scalar or vector. The density of a body, its temperature, its hydrostatic pressure, the potential at a point, are examples of scalar functions. The resultant force at the point, the velocity of a fluid at that point, the velocity of rotation of an element of the fluid, and the couple producing rotation, are examples of vector functions.

12.] Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area.

For instance, the resultant of an attractive force in any direction may be measured by finding the work which it would do on a body if the body were moved a short distance in that direction and dividing it by that short distance. Here the attractive force is defined with reference to a line.

On the other hand, the flux of heat in any direction at any point of a solid body may be defined as the quantity of heat which crosses a small area drawn perpendicular to that direction divided by that area and by the time. Here the flux is defined with reference to an area.

There are certain cases in which a quantity may be measured with reference to a line as well as with reference to an area.

Thus, in treating of the displacements of elastic solids, we may direct our attention either to the original and the actual position of a particle, in which case the displacement of the particle is measured by the line drawn from the first position to the second, or we may consider a small area fixed in space, and determine what quantity of the solid passes across that area during the displacement.

In the same way the velocity of a fluid may be investigated either with respect to the actual velocity of the individual particles, or with respect to the quantity of the fluid which flows through any fixed area.

But in these cases we require to know separately the density of the body as well as the displacement or velocity, in order to apply the first method, and whenever we attempt to form a molecular theory we have to use the second method.

In the case of the flow of electricity we do not know anything of its density or its velocity in the conductor, we only know the value of what, on the fluid theory, would correspond to the product of the density and the velocity. Hence in all such cases we must apply the more general method of measurement of the flux across an area.

In electrical science, electromotive force and magnetic force belong to the first class, being defined with reference to lines. When we wish to indicate this fact, we may refer to them as Forces.

On the other hand, electric and magnetic induction, and electric currents, belong to the second class, being defined with reference to areas. When we wish to indicate this fact, we shall refer to them as Fluxes.

Each of these forces may be considered as producing, or tending to produce, its corresponding flux. Thus, electromotive force produces electric currents in conductors, and tends to produce them in dielectrics. It produces electric induction in dielectrics, and probably in conductors also. In the same sense, magnetic force produces magnetic induction.

13.] In some cases the flux is simply proportional to the force and in the same direction, but in other cases we can only affirm that the direction and magnitude of the flux are functions of the direction and magnitude of the force.

The case in which the components of the flux are *linear* functions of those of the force is discussed in the chapter on the Equations of Conduction, Art. 296. There are in general nine coefficients which determine the relation between the force and the flux. In certain cases we have reason to believe that six of these coefficients form three pairs of equal quantities. In such cases the relation between the line of direction of the force and the normal plane of the flux is of the same kind as that between a diameter of an ellipsoid and its conjugate diametral plane. In Quaternion language, the one vector is said to be a linear and vector function of the other, and when there are three pairs of equal coefficients the function is said to be self-conjugate.

In the case of magnetic induction in iron, the flux, (the magnetization of the iron,) is not a linear function of the magnetizing force. In all cases, however, the product of the force and the flux resolved in its direction, gives a result of scientific importance, and this is always a scalar quantity.

14.] There are two mathematical operations of frequent occurrence which are appropriate to these two classes of vectors, or directed quantities.

In the case of forces, we have to take the integral along a line of the product of an element of the line, and the resolved part of the force along that element. The result of this operation is called the Line-integral of the force. It represents the work done on a body carried along the line. In certain cases in which the line-integral does not depend on the form of the line, but only on the position of its extremities, the line-integral is called the Potential.

In the case of fluxes, we have to take the integral, over a surface, of the flux through every element of the surface. The result of this operation is called the Surface-integral of the flux. It represents the quantity which passes through the surface.

There are certain surfaces across which there is no flux. If two of these surfaces intersect, their line of intersection is a line of flux. In those cases in which the flux is in the same direction as the force, lines of this kind are often called Lines of Force. It would be more correct, however, to speak of them in electrostatics and magnetics as Lines of Induction, and in electrokinematics as Lines of Flow.

15.] There is another distinction between different kinds of directed quantities, which, though very important in a physical point of view, is not so necessary to be observed for the sake of the mathematical methods. This is the distinction between longitudinal and rotational properties.

The direction and magnitude of a quantity may depend upon some action or effect which takes place entirely along a certain line, or it may depend upon something of the nature of rotation about that line as an axis. The laws of combination of directed quantities are the same whether they are longitudinal or rotational, so that there is no difference in the mathematical treatment of the two classes, but there may be physical circumstances which indicate to which class we must refer a particular phenomenon. Thus, electrolysis consists of the transfer of certain substances along a line in one direction, and of certain other substances in the opposite direction, which is evidently a longitudinal phenomenon, and there is no evidence of any rotational effect about the direction of the force. Hence we infer that the electric current which causes or accompanies electrolysis is a longitudinal, and not a rotational phenomenon.

On the other hand, the north and south poles of a magnet do not differ as oxygen and hydrogen do, which appear at opposite places during electrolysis, so that we have no evidence that magnetism is a longitudinal phenomenon, while the effect of magnetism in rotating the plane of polarized light distinctly shews that magnetism is a rotational phenomenon.

On Line-integrals.

16.] The operation of integration of the resolved part of a vector quantity along a line is important in physical science generally, and should be clearly understood.

Let x, y, z be the coordinates of a point P on a line whose length, measured from a certain point A, is s. These coordinates will be functions of a single variable s.

Let R be the value of the vector quantity at P, and let the tangent to the curve at P make with the direction of R the angle ϵ, then $R\cos\epsilon$ is the resolved part of R along the line, and the integral

$$L = \int_0^s R\cos\epsilon\, ds$$

is called the line-integral of R along the line s.

We may write this expression

$$L = \int_0^s \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds,$$

where X, Y, Z are the components of R parallel to x, y, z respectively.

This quantity is, in general, different for different lines drawn between A and P. When, however, within a certain region, the quantity

$$X\,dx + Y\,dy + Z\,dz = -D\Psi,$$

that is, is an exact differential within that region, the value of L becomes

$$L = \Psi_A - \Psi_P,$$

and is the same for any two forms of the path between A and P, provided the one form can be changed into the other by continuous motion without passing out of this region.

On Potentials.

The quantity Ψ is a scalar function of the position of the point, and is therefore independent of the directions of reference. It is called the Potential Function, and the vector quantity whose components are X, Y, Z is said to have a potential Ψ, if

$$X = -\left(\frac{d\Psi}{dx}\right), \quad Y = -\left(\frac{d\Psi}{dy}\right), \quad Z = -\left(\frac{d\Psi}{dz}\right).$$

When a potential function exists, surfaces for which the potential is constant are called Equipotential surfaces. The direction of R at any point of such a surface coincides with the normal to the surface, and if n be a normal at the point P, then $R = -\dfrac{d\Psi}{dn}$.

The method of considering the components of a vector as the first derivatives of a certain function of the coordinates with respect to these coordinates was invented by Laplace * in his treatment of the theory of attractions. The name of Potential was first given to this function by Green †, who made it the basis of his treatment of electricity. Green's essay was neglected by mathematicians till 1846, and before that time most of its important theorems had been rediscovered by Gauss, Chasles, Sturm, and Thomson ‡.

In the theory of gravitation the potential is taken with the opposite sign to that which is here used, and the resultant force in any direction is then measured by the rate of *increase* of the

* Méc. Céleste, liv. iii.

† Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism, Nottingham, 1828. Reprinted in *Crelle's Journal*, and in Mr. Ferrer's edition of Green's Works.

‡ Thomson and Tait, *Natural Philosophy*, § 483.

potential function in that direction. In electrical and magnetic investigations the potential is defined so that the resultant force in any direction is measured by the *decrease* of the potential in that direction. This method of using the expression makes it correspond in sign with potential energy, which always decreases when the bodies are moved in the direction of the forces acting on them.

17.] The geometrical nature of the relation between the potential and the vector thus derived from it receives great light from Hamilton's discovery of the form of the operator by which the vector is derived from the potential.

The resolved part of the vector in any direction is, as we have seen, the first derivative of the potential with respect to a coordinate drawn in that direction, the sign being reversed.

Now if i, j, k are three unit vectors at right angles to each other, and if X, Y, Z are the components of the vector $\mathfrak{F}$ resolved parallel to these vectors, then

$$\mathfrak{F} = iX + jY + kZ; \tag{1}$$

and by what we have said above, if Ψ is the potential,

$$\mathfrak{F} = -\left(i\frac{d\Psi}{dx} + j\frac{d\Psi}{dy} + k\frac{d\Psi}{dz}\right). \tag{2}$$

If we now write ∇ for the operator,

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}, \tag{3}$$

$$\mathfrak{F} = -\nabla\Psi. \tag{4}$$

The symbol of operation ∇ may be interpreted as directing us to measure, in each of three rectangular directions, the rate of increase of Ψ, and then, considering the quantities thus found as vectors, to compound them into one. This is what we are directed to do by the expression (3). But we may also consider it as directing us first to find out in what direction Ψ increases fastest, and then to lay off in that direction a vector representing this rate of increase.

M. Lamé, in his *Traité des Fonctions Inverses,* uses the term Differential Parameter to express the magnitude of this greatest rate of increase, but neither the term itself, nor the mode in which Lamé uses it, indicates that the quantity referred to has direction as well as magnitude. On those rare occasions in which I shall have to refer to this relation as a purely geometrical one, I shall call the vector $\mathfrak{F}$ the Slope of the scalar function Ψ, using the word Slope to indicate the direction, as well as the magnitude, of the most rapid decrease of Ψ.

18.] There are cases, however, in which the conditions

$$\frac{dZ}{dy}-\frac{dY}{dz}=0, \quad \frac{dX}{dz}-\frac{dZ}{dx}=0, \quad \text{and} \quad \frac{dY}{dx}-\frac{dX}{dy}=0,$$

which are those of $Xdx+Ydy+Zdz$ being a complete differential, are fulfilled throughout a certain region of space, and yet the line-integral from A to P may be different for two lines, each of which lies wholly within that region. This may be the case if the region is in the form of a ring, and if the two lines from A to P pass through opposite segments of the ring. In this case, the one path cannot be transformed into the other by continuous motion without passing out of the region.

We are here led to considerations belonging to the Geometry of Position, a subject which, though its importance was pointed out by Leibnitz and illustrated by Gauss, has been little studied. The most complete treatment of this subject has been given by J. B. Listing *.

Let there be p points in space, and let l lines of any form be drawn joining these points so that no two lines intersect each other, and no point is left isolated. We shall call a figure composed of lines in this way a Diagram. Of these lines, $p-1$ are sufficient to join the p points so as to form a connected system. Every new line completes a loop or closed path, or, as we shall call it, a Cycle. The number of independent cycles in the diagram is therefore $\kappa = l-p+1$.

Any closed path drawn along the lines of the diagram is composed of these independent cycles, each being taken any number of times and in either direction.

The existence of cycles is called Cyclosis, and the number of cycles in a diagram is called its Cyclomatic number.

Cyclosis in Surfaces and Regions.

Surfaces are either complete or bounded. Complete surfaces are either infinite or closed. Bounded surfaces are limited by one or more closed lines, which may in the limiting cases become finite lines or points.

A finite region of space is bounded by one or more closed surfaces. Of these one is the external surface, the others are

* *Der Census Raümlicher Complexe*, Gött. Abh., Bd. x. S. 97 (1861).

included in it and exclude each other, and are called internal surfaces.

If the region has one bounding surface, we may suppose that surface to contract inwards without breaking its continuity or cutting itself. If the region is one of simple continuity, such as a sphere, this process may be continued till it is reduced to a point; but if the region is like a ring, the result will be a closed curve; and if the region has multiple connexions, the result will be a diagram of lines, and the cyclomatic number of the diagram will be that of the region. The space outside the region has the same cyclomatic number as the region itself. Hence, if the region is bounded by internal as well as external surfaces, its cyclomatic number is the sum of those due to all the surfaces.

When a region encloses within itself other regions, it is called a Periphractic region.

The number of internal bounding surfaces of a region is called its periphractic number. A closed surface is also periphractic, its number being unity.

The cyclomatic number of a closed surface is twice that of the region which it bounds. To find the cyclomatic number of a bounded surface, suppose all the boundaries to contract inwards, without breaking continuity, till they meet. The surface will then be reduced to a point in the case of an acyclic surface, or to a linear diagram in the case of cyclic surfaces. The cyclomatic number of the diagram is that of the surface.

19.] THEOREM I. *If throughout any acyclic region*

$$X\,dx + Y\,dy + Z\,dz = -D\Psi,$$

the value of the line-integral from a point A to a point P taken along any path within the region will be the same.

We shall first shew that the line-integral taken round any closed path within the region is zero.

Suppose the equipotential surfaces drawn. They are all either closed surfaces or are bounded entirely by the surface of the region, so that a closed line within the region, if it cuts any of the surfaces at one part of its path, must cut the same surface in the opposite direction at some other part of its path, and the corresponding portions of the line-integral being equal and opposite, the total value is zero.

Hence if AQP and $AQ'P$ are two paths from A to P, the line-integral for $AQ'P$ is the sum of that for AQP and the closed path

$AQ'PQA$. But the line-integral of the closed path is zero, therefore those of the two paths are equal.

Hence if the potential is given at any one point of such a region, that at any other point is determinate.

20.] THEOREM II. *In a cyclic region in which the equation*

$$X\,dx + Y\,dy + Z\,dz = -D\Psi$$

is everywhere fulfilled, the line-integral from A to P, along a line drawn within the region, will not in general be determinate unless the channel of communication between A and P be specified.

Let K be the cyclomatic number of the region, then K sections of the region may be made by surfaces which we may call Diaphragms, so as to close up K of the channels of communication, and reduce the region to an acyclic condition without destroying its continuity.

The line-integral from A to any point P taken along a line which does not cut any of these diaphragms will be, by the last theorem, determinate in value.

Now let A and P be taken indefinitely near to each other, but on opposite sides of a diaphragm, and let K be the line-integral from A to P.

Let A' and P' be two other points on opposite sides of the same diaphragm and indefinitely near to each other, and let K' be the line-integral from A' to P'. Then $K' = K$.

For if we draw AA' and PP', nearly coincident, but on opposite sides of the diaphragm, the line-integrals along these lines will be equal. Suppose each equal to L, then the line-integral of $A'P'$ is equal to that of $A'A + AP + PP' = -L + K + L = K =$ that of AP.

Hence the line-integral round a closed curve which passes through one diaphragm of the system in a given direction is a constant quantity K. This quantity is called the Cyclic constant corresponding to the given cycle.

Let any closed curve be drawn within the region, and let it cut the diaphragm of the first cycle p times in the positive direction and p' times in the negative direction, and let $p - p' = n_1$. Then the line-integral of the closed curve will be $n_1 K_1$.

Similarly the line-integral of any closed curve will be

$$n_1 K_1 + n_2 K_2 + \ldots + n_K K_K;$$

where n_K represents the excess of the number of positive passages of the curve through the diaphragm of the cycle K over the number of negative passages.

If two curves are such that one of them may be transformed into the other by continuous motion without at any time passing through any part of space for which the condition of having a potential is not fulfilled, these two curves are called Reconcileable curves. Curves for which this transformation cannot be effected are called Irreconcileable curves *.

The condition that $Xdx + Ydy + Zdz$ is a complete differential of some function Ψ for all points within a certain region, occurs in several physical investigations in which the directed quantity and the potential have different physical interpretations.

In pure kinematics we may suppose X, Y, Z to be the components of the displacement of a point of a continuous body whose original coordinates are x, y, z, then the condition expresses that these displacements constitute a *non-rotational strain* †.

If X, Y, Z represent the components of the velocity of a fluid at the point x, y, z, then the condition expresses that the motion of the fluid is irrotational.

If X, Y, Z represent the components of the force at the point x, y, z, then the condition expresses that the work done on a particle passing from one point to another is the difference of the potentials at these points, and the value of this difference is the same for all reconcileable paths between the two points.

On Surface-Integrals.

21.] Let dS be the element of a surface, and ϵ the angle which a normal to the surface drawn towards the positive side of the surface makes with the direction of the vector quantity R, then $\iint R\cos\epsilon\, dS$ is called the *surface-integral of R over the surface S.*

THEOREM III. *The surface-integral of the flux through a closed surface may be expressed as the volume-integral of its convergence taken within the surface.* (See Art. 25.)

Let X, Y, Z be the components of R, and let l, m, n be the direction-cosines of the normal to S measured outwards. Then the surface-integral of R over S is

$$\iint R\cos\epsilon\, dS = \iint Xl\, dS + \iint Ym\, dS + \iint Zn\, dS$$
$$= \iint X\,dy\,dz + \iint Y\,dz\,dx + \iint Z\,dx\,dy; \qquad (1)$$

* See Sir W. Thomson 'On Vortex Motion,' *Trans. R. S. Edin.*, 1869.

† See Thomson and Tait's *Natural Philosophy*, § 190 (*i*).

the values of X, Y, Z being those at a point in the surface, and the integrations being extended over the whole surface.

If the surface is a closed one, then, when y and z are given, the coordinate x must have an even number of values, since a line parallel to x must enter and leave the enclosed space an equal number of times provided it meets the surface at all.

Let a point travelling from $x = -\infty$ to $x = +\infty$ first enter the space when $x = x_1$, then leave it when $x = x_2$, and so on; and let the values of X at these points be X_1, X_2, &c., then

$$\iint X\,dy\,dz = \iint \{(X_2 - X_1) + (X_4 - X_3) + \&\text{c.} + (X_{2n} - X_{2n-1})\}\,dy\,dz. \quad (2)$$

If X is a quantity which is continuous, and has no infinite values between x_1 and x_2, then

$$X_2 - X_1 = \int_{x_1}^{x_2} \frac{dX}{dx}\,dx; \quad (3)$$

where the integration is extended from the first to the second intersection, that is, along the first segment of x which is within the closed surface. Taking into account all the segments which lie within the closed surface, we find

$$\iint X\,dy\,dz = \iiint \frac{dX}{dx}\,dx\,dy\,dz, \quad (4)$$

the double integration being confined to the closed surface, but the triple integration being extended to the whole enclosed space. Hence, if X, Y, Z are continuous and finite within a closed surface S, the total surface-integral of R over that surface will be

$$\iint R\cos\epsilon\,dS = \iiint \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) dx\,dy\,dz, \quad (5)$$

the triple integration being extended over the whole space within S.

Let us next suppose that X, Y, Z are not continuous within the closed surface, but that at a certain surface $F(x, y, z) = 0$ the values of X, Y, Z alter abruptly from X, Y, Z on the negative side of the surface to X', Y', Z' on the positive side.

If this discontinuity occurs, say, between x_1 and x_2, the value of $X_2 - X_1$ will be

$$\int_{x_1}^{x_2} \frac{dX}{dx}\,dx + (X' - X), \quad (6)$$

where in the expression under the integral sign only the finite values of the derivative of X are to be considered.

In this case therefore the total surface-integral of R over the closed surface will be expressed by

$$\iint R\cos\epsilon\, dS = \iiint\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)dx\,dy\,dz + \iint(X'-X)\,dy\,dz$$
$$+\iint(Y'-Y)\,dz\,dx + \iint(Z'-Z)\,dx\,dy\,; \quad (7)$$

or, if l', m', n' are the direction-cosines of the normal to the surface of discontinuity, and dS' an element of that surface,

$$\iint R\cos\epsilon\, dS = \iiint\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)dx\,dy\,dz$$
$$+\iint\{l'(X'-X)+m'(Y'-Y)+n'(Z'-Z)\}\,dS', \quad (8)$$

where the integration of the last term is to be extended over the surface of discontinuity.

If at every point where X, Y, Z are continuous

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=0, \quad (9)$$

and at every surface where they are discontinuous

$$l'X'+m'Y'+n'Z' = l'X+m'Y+n'Z, \quad (10)$$

then the surface-integral over every closed surface is zero, and the distribution of the vector quantity is said to be Solenoidal.

We shall refer to equation (9) as the General solenoidal condition, and to equation (10) as the Superficial solenoidal condition.

22.] Let us now consider the case in which at every point within the surface S the equation

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=0 \quad (11)$$

is fulfilled. We have as a consequence of this the surface-integral over the closed surface equal to zero.

Now let the closed surface S consist of three parts S_1, S_0, and S_2. Let S_1 be a surface of any form bounded by a closed line L_1. Let S_0 be formed by drawing lines from every point of L_1 always coinciding with the direction of R. If l, m, n are the direction-cosines of the normal at any point of the surface S_0, we have

$$R\cos\epsilon = Xl+Ym+Zn = 0. \quad (12)$$

Hence this part of the surface contributes nothing towards the value of the surface-integral.

Let S_2 be another surface of any form bounded by the closed curve L_2 in which it meets the surface S_0.

Let Q_1, Q_0, Q_2 be the surface-integrals of the surfaces S_1, S_0, S_2, and let Q be the surface-integral of the closed surface S. Then

$$Q = Q_1+Q_0+Q_2 = 0\,; \quad (13)$$

and we know that $$Q_0 = 0\,; \tag{14}$$
therefore $$Q_2 = -Q_1\,; \tag{15}$$
or, in other words, the surface-integral over the surface S_2 is equal and opposite to that over S_1 whatever be the form and position of S_2, provided that the intermediate surface S_0 is one for which R is always tangential.

If we suppose L_1 a closed curve of small area, S_0 will be a tubular surface having the property that the surface-integral over every complete section of the tube is the same.

Since the whole space can be divided into tubes of this kind provided
$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0, \tag{16}$$
a distribution of a vector quantity consistent with this equation is called a Solenoidal Distribution.

On Tubes and Lines of Flow.

If the space is so divided into tubes that the surface-integral for every tube is unity, the tubes are called Unit tubes, and the surface-integral over any finite surface S bounded by a closed curve L is equal to the *number* of such tubes which pass through S in the positive direction, or, what is the same thing, the number which pass through the closed curve L.

Hence the surface-integral of S depends only on the form of its boundary L, and not on the form of the surface within its boundary.

On Periphractic Regions.

If, throughout the whole region bounded externally by the single closed surface S_1, the solenoidal condition
$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0$$
is fulfilled, then the surface-integral taken over any closed surface drawn within this region will be zero, and the surface-integral taken over a bounded surface within the region will depend only on the form of the closed curve which forms its boundary.

It is not, however, generally true that the same results follow if the region within which the solenoidal condition is fulfilled is bounded otherwise than by a single surface.

For if it is bounded by more than one continuous surface, one of these is the external surface and the others are internal surfaces, and the region S is a periphractic region, having within it other regions which it completely encloses.

If within any of these enclosed regions, S_1, the solenoidal condition is not fulfilled, let

$$Q_1 = \iint R \cos \epsilon \, dS_1$$

be the surface-integral for the surface enclosing this region, and let Q_2, Q_3, &c. be the corresponding quantities for the other enclosed regions.

Then, if a closed surface S' is drawn within the region S, the value of its surface-integral will be zero only when this surface S' does not include any of the enclosed regions S_1, S_2, &c. If it includes any of these, the surface-integral is the sum of the surface-integrals of the different enclosed regions which lie within it.

For the same reason, the surface-integral taken over a surface bounded by a closed curve is the same for such surfaces only bounded by the closed curve as are reconcileable with the given surface by continuous motion of the surface within the region S.

When we have to deal with a periphractic region, the first thing to be done is to reduce it to an aperiphractic region by drawing lines joining the different bounding surfaces. Each of these lines, provided it joins surfaces which were not already in continuous connexion, reduces the periphractic number by unity, so that the whole number of lines to be drawn to remove the periphraxy is equal to the periphractic number, or the number of internal surfaces. When these lines have been drawn we may assert that if the solenoidal condition is fulfilled in the region S, any closed surface drawn entirely within S, and not cutting any of the lines, has its surface-integral zero.

In drawing these lines we must remember that any line joining surfaces which are already connected does not diminish the periphraxy, but introduces cyclosis.

The most familiar example of a periphractic region within which the solenoidal condition is fulfilled is the region surrounding a mass attracting or repelling inversely as the square of the distance.

In this case we have

$$X = m\frac{x}{r^3}, \quad Y = m\frac{y}{r^3}, \quad Z = m\frac{z}{r^3};$$

where m is the mass supposed to be at the origin of coordinates.

At any point where r is finite

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0,$$

but at the origin these quantities become infinite. For any closed surface not including the origin, the surface-integral is zero. If a closed surface includes the origin, its surface-integral is $4\pi m$.

If, for any reason, we wish to treat the region round m as if it were not periphractic, we must draw a line from m to an infinite distance, and in taking surface-integrals we must remember to add $4\pi m$ whenever this line crosses from the negative to the positive side of the surface.

On Right-handed and Left-handed Relations in Space.

23.] In this treatise the motions of translation along any axis and of rotation about that axis, will be assumed to be of the same sign when their directions correspond to those of the translation and rotation of an ordinary or right-handed screw *.

For instance, if the actual rotation of the earth from west to east is taken positive, the direction of the earth's axis from south to north will be taken positive, and if a man walks forward in the positive direction, the positive rotation is in the order, head, right-hand, feet, left-hand.

If we place ourselves on the positive side of a surface, the positive direction along its bounding curve will be opposite to the motion of the hands of a watch with its face towards us.

This is the right-handed system which is adopted in Thomson and Tait's *Natural Philosophy*, § 243. The opposite, or left-handed system, is adopted in Hamilton's and Tait's *Quaternions.* The operation of passing from the one system to the other is called, by Listing, *Perversion.*

The reflexion of an object in a mirror is a perverted image of the object.

When we use the Cartesian axes of x, y, z, we shall draw them

* The combined action of the muscles of the arm when we turn the upper side of the right-hand outwards, and at the same time thrust the hand forwards, will impress the right-handed screw motion on the memory more firmly than any verbal definition. A common corkscrew may be used as a material symbol of the same relation.

Professor W. H. Miller has suggested to me that as the tendrils of the vine are right-handed screws and those of the hop left-handed, the two systems of relations in space might be called those of the vine and the hop respectively.

The system of the vine, which we adopt, is that of Linnæus, and of screw-makers in all civilized countries except Japan. De Candolle was the first who called the hop-tendril right-handed, and in this he is followed by Listing, and by most writers on the rotatory polarization of light. Screws like the hop-tendril are made for the couplings of railway-carriages, and for the fittings of wheels on the left side of ordinary carriages, but they are always called left-handed screws by those who use them.

so that the ordinary conventions about the cyclic order of the symbols lead to a right-handed system of directions in space. Thus, if x is drawn eastward and y northward, z must be drawn upward.

The areas of surfaces will be taken positive when the order of integration coincides with the cyclic order of the symbols. Thus, the area of a closed curve in the plane of xy may be written either

$$\int x\,dy \quad \text{or} \quad -\int y\,dx\,;$$

the order of integration being x, y in the first expression, and y, x in the second.

This relation between the two products $dx\,dy$ and $dy\,dx$ may be compared with that between the products of two perpendicular vectors in the doctrine of Quaternions, the sign of which depends on the order of multiplication, and with the reversal of the sign of a determinant when the adjoining rows or columns are exchanged.

For similar reasons a volume-integral is to be taken positive when the order of integration is in the cyclic order of the variables x, y, z, and negative when the cyclic order is reversed.

We now proceed to prove a theorem which is useful as establishing a connexion between the surface-integral taken over a finite surface and a line-integral taken round its boundary.

24.] THEOREM IV. *A line-integral taken round a closed curve may be expressed in terms of a surface-integral taken over a surface bounded by the curve.*

Let X, Y, Z be the components of a vector quantity $\mathfrak{A}$ whose line-integral is to be taken round a closed curve s.

Let S be any continuous finite surface bounded entirely by the closed curve s, and let ξ, η, ζ be the components of another vector quantity $\mathfrak{B}$, related to X, Y, Z by the equations

$$\xi = \frac{dZ}{dy} - \frac{dY}{dz}, \quad \eta = \frac{dX}{dz} - \frac{dZ}{dx}, \quad \zeta = \frac{dY}{dx} - \frac{dX}{dy}. \qquad (1)$$

Then the surface-integral of $\mathfrak{B}$ taken over the surface S is equal to the line-integral of $\mathfrak{A}$ taken round the curve s. It is manifest that ξ, η, ζ fulfil of themselves the solenoidal condition

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0.$$

Let l, m, n be the direction-cosines of the normal to an element of the surface dS, reckoned in the positive direction. Then the value of the surface-integral of $\mathfrak{B}$ may be written

$$\iint (l\xi + m\eta + n\zeta)\,dS. \tag{2}$$

In order to form a definite idea of the meaning of the element dS, we shall suppose that the values of the coordinates x, y, z for every point of the surface are given as functions of two independent variables α and β. If β is constant and α varies, the point (x, y, z) will describe a curve on the surface, and if a series of values is given to β, a series of such curves will be traced, all lying on the surface S. In the same way, by giving a series of constant values to α, a second series of curves may be traced, cutting the first series, and dividing the whole surface into elementary portions, any one of which may be taken as the element dS.

The projection of this element on the plane of y, z is, by the ordinary formula,

$$l\,dS = \left(\frac{dy}{d\alpha}\frac{dz}{d\beta} - \frac{dy}{d\beta}\frac{dz}{d\alpha}\right) d\beta\, d\alpha. \tag{3}$$

The expressions for $m\,dS$ and $n\,dS$ are obtained from this by substituting x, y, z in cyclic order.

The surface-integral which we have to find is

$$\iint (l\xi + m\eta + n\zeta)\,dS; \tag{4}$$

or, substituting the values of ξ, η, ζ in terms of X, Y, Z,

$$\iint \left(m\frac{dX}{dz} - n\frac{dX}{dy} + n\frac{dY}{dx} - l\frac{dY}{dz} + l\frac{dZ}{dy} - m\frac{dZ}{dx}\right) dS. \tag{5}$$

The part of this which depends on X may be written

$$\iint \left\{\frac{dX}{dz}\left(\frac{dz}{d\alpha}\frac{dx}{d\beta} - \frac{dz}{d\beta}\frac{dx}{d\alpha}\right) - \frac{dX}{dy}\left(\frac{dx}{d\alpha}\frac{dy}{d\beta} - \frac{dx}{d\beta}\frac{dy}{d\alpha}\right)\right\} d\beta\, d\alpha; \tag{6}$$

adding and subtracting $\dfrac{dX}{dx}\dfrac{dx}{d\alpha}\dfrac{dx}{d\beta}$, this becomes

$$\iint \left\{\frac{dx}{d\beta}\left(\frac{dX}{dx}\frac{dx}{d\alpha} + \frac{dX}{dy}\frac{dy}{d\alpha} + \frac{dX}{dz}\frac{dz}{d\alpha}\right)\right.$$

$$\left. - \frac{dx}{d\alpha}\left(\frac{dX}{dx}\frac{dx}{d\beta} + \frac{dX}{dy}\frac{dy}{d\beta} + \frac{dX}{dz}\frac{dz}{d\beta}\right)\right\} d\beta\, d\alpha; \tag{7}$$

$$= \iint \left(\frac{dX}{d\alpha}\frac{dx}{d\beta} - \frac{dX}{d\beta}\frac{dx}{d\alpha}\right) d\beta\, d\alpha. \tag{8}$$

Dele As we have made no assumption as to the form of the functions α and β, we may assume that α is a function of X, or, in other words, that the curves for which α is constant are those for which

X is constant. In this case $\frac{dX}{d\beta} = 0$, and the expression becomes by integration with respect to a,

$$\iint \frac{dX}{da}\frac{dx}{d\beta} d\beta\, da = \int X \frac{dx}{d\beta} d\beta; \tag{9}$$

Dele

where the integration is now to be performed round the closed curve. Since all the quantities are now expressed in terms of one variable β, we may make s, the length of the bounding curve, the independent variable, and the expression may then be written

$$\int X \frac{dx}{ds} ds, \tag{10}$$

where the integration is to be performed round the curve s. We may treat in the same way the parts of the surface-integral which depend upon Y and Z, so that we get finally,

$$\iint (l\xi + m\eta + n\zeta)\, dS = \int \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds\,; \tag{11}$$

where the first integral is extended over the surface S, and the second round the bounding curve s *.

On the effect of the operator ∇ on a vector function.

25.] We have seen that the operation denoted by ∇ is that by which a vector quantity is deduced from its potential. The same operation, however, when applied to a vector function, produces results which enter into the two theorems we have just proved (III and IV). The extension of this operator to vector displacements, and most of its further development, is due to Professor Tait †.

Let σ be a vector function of ρ, the vector of a variable point. Let us suppose, as usual, that

$$\rho = ix + jy + kz,$$

and

$$\sigma = iX + jY + kZ;$$

where X, Y, Z are the components of σ in the directions of the axes.

We have to perform on σ the operation

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}.$$

Performing this operation, and remembering the rules for the

* This theorem was given by Professor Stokes, *Smith's Prize Examination,* 1854, question 8. It is proved in Thomson and Tait's *Natural Philosophy*, § 190 (j).

† See *Proc. R. S. Edin.*, April 28, 1862. 'On Green's and other allied Theorems,' *Trans. R. S. Edin.*, 1869–70, a very valuable paper; and 'On some Quaternion Integrals,' *Proc. R. S. Edin.*, 1870–71.

multiplication of i, j, k, we find that $\nabla\sigma$ consists of two parts, one scalar and the other vector.

The scalar part is

$$S\nabla\sigma = -\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right), \text{ see Theorem III,}$$

and the vector part is

$$V\nabla\sigma = i\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + j\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + k\left(\frac{dY}{dx} - \frac{dX}{dy}\right).$$

If the relation between X, Y, Z and ξ, η, ζ is that given by equation (1) of the last theorem, we may write

$$V\nabla\sigma = i\xi + j\eta + k\zeta. \quad \text{See Theorem IV.}$$

It appears therefore that the functions of X, Y, Z which occur in the two theorems are both obtained by the operation ∇ on the vector whose components are X, Y, Z. The theorems themselves may be written

$$\iiint S\nabla\sigma\, ds = \iint S.\sigma\, U\nu\, ds, \quad \text{(III)}$$

and

$$\int S\sigma\, d\rho = \iint S.\nabla\sigma\, U\nu\, ds; \quad \text{(IV)}$$

where ds is an element of a volume, ds of a surface, $d\rho$ of a curve, and $U\nu$ a unit-vector in the direction of the normal.

To understand the meaning of these functions of a vector, let us suppose that σ_0 is the value of σ at a point P, and let us examine the value of $\sigma - \sigma_0$ in the neighbourhood of P. If we draw a closed surface round P, then, if the surface-integral of σ over this surface is directed inwards, $S\nabla\sigma$ will be positive, and the vector $\sigma - \sigma_0$ near the point P will be on the whole directed towards P, as in the figure (1).

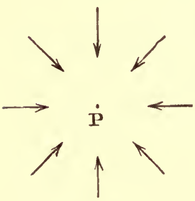


Fig. 1.

I propose therefore to call the scalar part of $\nabla\sigma$ the *convergence* of σ at the point P.

To interpret the vector part of $\nabla\sigma$, let us suppose ourselves to be looking in the direction of the vector whose components are ξ, η, ζ, and let us examine the vector $\sigma - \sigma_0$ near the point P. It will appear as in the figure (2), this vector being arranged on the whole tangentially in the direction opposite to the hands of a watch.

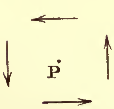


Fig. 2.

I propose (with great diffidence) to call the vector part of $\nabla\sigma$ the *curl*, or the *version* of σ at the point P.

At Fig. 3 we have an illustration of curl combined with convergence.

Let us now consider the meaning of the equation

$$V\nabla\sigma = 0.$$

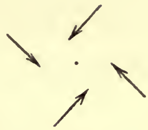


Fig. 3.

This implies that $\nabla\sigma$ is a scalar, or that the vector σ is the slope of some scalar function Ψ. These applications of the operator ∇ are due to Professor Tait *. A more complete development of the theory is given in his paper 'On Green's and other allied Theorems †,' to which I refer the reader for the purely Quaternion investigation of the properties of the operator ∇.

26.] One of the most remarkable properties of the operator ∇ is that when repeated it becomes

$$\nabla^2 = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right),$$

an operator occurring in all parts of Physics, which we may refer to as Laplace's Operator.

This operator is itself essentially scalar. When it acts on a scalar function the result is scalar, when it acts on a vector function the result is a vector.

If, with any point P as centre, we draw a small sphere whose radius is r, then if q_0 is the value of q at the centre, and $\bar{q}$ the mean value of q for all points within the sphere,

$$q_0 - \bar{q} = \tfrac{1}{10} r^2 \nabla^2 q;$$

so that the value at the centre exceeds or falls short of the mean value according as $\nabla^2 q$ is positive or negative.

I propose therefore to call $\nabla^2 q$ the *concentration* of q at the point P, because it indicates the excess of the value of q at that point over its mean value in the neighbourhood of the point.

If q is a scalar function, the method of finding its mean value is well known. If it is a vector function, we must find its mean value by the rules for integrating vector functions. The result of course is a vector.

* *Proceedings R. S. Edin.*, 1862.

† *Trans. R. S. Edin.*, 1869-70.

PART I.

ELECTROSTATICS.

CHAPTER I.

DESCRIPTION OF PHENOMENA.

Electrification by Friction.

27.] Experiment I *. Let a piece of glass and a piece of resin, neither of which exhibits any electrical properties, be rubbed together and left with the rubbed surfaces in contact. They will still exhibit no electrical properties. Let them be separated. They will now attract each other.

If a second piece of glass be rubbed with a second piece of resin, and if the pieces be then separated and suspended in the neighbourhood of the former pieces of glass and resin, it may be observed—

(1) That the two pieces of glass repel each other.

(2) That each piece of glass attracts each piece of resin.

(3) That the two pieces of resin repel each other.

These phenomena of attraction and repulsion are called Electrical phenomena, and the bodies which exhibit them are said to be *electrified*, or to be *charged with electricity*.

Bodies may be electrified in many other ways, as well as by friction.

The electrical properties of the two pieces of glass are similar to each other but opposite to those of the two pieces of resin, the glass attracts what the resin repels and repels what the resin attracts.

* See Sir W. Thomson 'On the Mathematical Theory of Electricity,' *Cambridge and Dublin Mathematical Journal*, March, 1848.

If a body electrified in any manner whatever behaves as the glass does, that is, if it repels the glass and attracts the resin, the body is said to be *vitreously* electrified, and if it attracts the glass and repels the resin it is said to be *resinously* electrified. All electrified bodies are found to be either vitreously or resinously electrified.

It is the established practice of men of science to call the vitreous electrification positive, and the resinous electrification negative. The exactly opposite properties of the two kinds of electrification justify us in indicating them by opposite signs, but the application of the positive sign to one rather than to the other kind must be considered as a matter of arbitrary convention, just as it is a matter of convention in mathematical diagrams to reckon positive distances towards the right hand.

No force, either of attraction or of repulsion, can be observed between an electrified body and a body not electrified. When, in any case, bodies not previously electrified are observed to be acted on by an electrified body, it is because they have become *electrified by induction.*

Electrification by Induction.

28.] EXPERIMENT II *. Let a hollow vessel of metal be hung up by white silk threads, and let a similar thread be attached to the lid of the vessel so that the vessel may be opened or closed without touching it.

Let the pieces of glass and resin be similarly suspended and electrified as before.

Let the vessel be originally unelectrified, then if an electrified piece of glass is hung up within it by its thread without touching the vessel, and the lid closed, the outside of the vessel will be found to be vitreously electrified, and it may be shewn that the electrification outside of the vessel is exactly the same in whatever part of the interior space the glass is suspended.

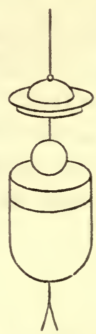


Fig. 4.

If the glass is now taken out of the vessel without touching it, the electrification of the glass will be the same as before it was put in, and that of the vessel will have disappeared.

This electrification of the vessel, which depends on the glass

* This, and several experiments which follow, are due to Faraday, 'On Static Electrical Inductive Action,' *Phil. Mag.*, 1843, or *Exp. Res.*, vol. ii. p. 279.

being within it, and which vanishes when the glass is removed, is called Electrification by induction.

Similar effects would be produced if the glass were suspended near the vessel on the outside, but in that case we should find an electrification vitreous in one part of the outside of the vessel and resinous in another. When the glass is inside the vessel the whole of the outside is vitreously and the whole of the inside resinously electrified.

Electrification by Conduction.

29.] EXPERIMENT III. Let the metal vessel be electrified by induction, as in the last experiment, let a second metallic body be suspended by white silk threads near it, and let a metal wire, similarly suspended, be brought so as to touch simultaneously the electrified vessel and the second body.

The second body will now be found to be vitreously electrified, and the vitreous electrification of the vessel will have diminished.

The electrical condition has been transferred from the vessel to the second body by means of the wire. The wire is called a *conductor* of electricity, and the second body is said to be *electrified by conduction.*

Conductors and Insulators.

EXPERIMENT IV. If a glass rod, a stick of resin or gutta-percha, or a white silk thread, had been used instead of the metal wire, no transfer of electricity would have taken place. Hence these latter substances are called Non-conductors of electricity. Non-conductors are used in electrical experiments to support electrified bodies without carrying off their electricity. They are then called Insulators.

The metals are good conductors; air, glass, resins, gutta-percha, vulcanite, paraffin, &c. are good insulators; but, as we shall see afterwards, all substances resist the passage of electricity, and all substances allow it to pass, though in exceedingly different degrees. This subject will be considered when we come to treat of the Motion of electricity. For the present we shall consider only two classes of bodies, good conductors, and good insulators.

In Experiment II an electrified body produced electrification in the metal vessel while separated from it by air, a non-conducting medium. Such a medium, considered as transmitting these electrical effects without conduction, has been called by Faraday a Dielectric medium, and the action which takes place through it is called Induction.

In Experiment III the electrified vessel produced electrification in the second metallic body through the medium of the wire. Let us suppose the wire removed, and the electrified piece of glass taken out of the vessel without touching it, and removed to a sufficient distance. The second body will still exhibit vitreous electrification, but the vessel, when the glass is removed, will have resinous electrification. If we now bring the wire into contact with both bodies, conduction will take place along the wire, and all electrification will disappear from both bodies, shewing that the electrification of the two bodies was equal and opposite.

30.] Experiment V. In Experiment II it was shewn that if a piece of glass, electrified by rubbing it with resin, is hung up in an insulated metal vessel, the electrification observed outside does not depend on the position of the glass. If we now introduce the piece of resin with which the glass was rubbed into the same vessel, without touching it or the vessel, it will be found that there is no electrification outside the vessel. From this we conclude that the electrification of the resin is exactly equal and opposite to that of the glass. By putting in any number of bodies, electrified in any way, it may be shewn that the electrification of the outside of the vessel is that due to the algebraic sum of all the electrifications, those being reckoned negative which are resinous. We have thus a practical method of adding the electrical effects of several bodies without altering the electrification of each.

31.] Experiment VI. Let a second insulated metallic vessel, B, be provided, and let the electrified piece of glass be put into the first vessel A, and the electrified piece of resin into the second vessel B. Let the two vessels be then put in communication by the metal wire, as in Experiment III. All signs of electrification will disappear.

Next, let the wire be removed, and let the pieces of glass and of resin be taken out of the vessels without touching them. It will be found that A is electrified resinously and B vitreously.

If now the glass and the vessel A be introduced together into a larger insulated vessel C, it will be found that there is no electrification outside C. This shews that the electrification of A is exactly equal and opposite to that of the piece of glass, and that of B may be shewn in the same way to be equal and opposite to that of the piece of resin.

We have thus obtained a method of charging a vessel with a quantity of electricity exactly equal and opposite to that of an electrified body without altering the electrification of the latter, and we may in this way charge any number of vessels with exactly equal quantities of electricity of either kind, which we may take for provisional units.

32.] EXPERIMENT VII. Let the vessel B, charged with a quantity of positive electricity, which we shall call, for the present, unity, be introduced into the larger insulated vessel C without touching it. It will produce a positive electrification on the outside of C. Now let B be made to touch the inside of C. No change of the external electrification will be observed. If B is now taken out of C without touching it, and removed to a sufficient distance, it will be found that B is completely discharged, and that C has become charged with a unit of positive electricity.

We have thus a method of transferring the charge of B to C.

Let B be now recharged with a unit of electricity, introduced into C already charged, made to touch the inside of C, and removed. It will be found that B is again completely discharged, so that the charge of C is doubled.

If this process is repeated, it will be found that however highly C is previously charged, and in whatever way B is charged, when B is first entirely enclosed in C, then made to touch C, and finally removed without touching C, the charge of B is completely transferred to C, and B is entirely free from electrification.

This experiment indicates a method of charging a body with any number of units of electricity. We shall find, when we come to the mathematical theory of electricity, that the result of this experiment affords an accurate test of the truth of the theory.

33.] Before we proceed to the investigation of the law of electrical force, let us enumerate the facts we have already established.

By placing any electrified system inside an insulated hollow conducting vessel, and examining the resultant effect on the outside of the vessel, we ascertain the character of the total electrification of the system placed inside, without any communication of electricity between the different bodies of the system.

The electrification of the outside of the vessel may be tested with great delicacy by putting it in communication with an electroscope.

We may suppose the electroscope to consist of a strip of gold leaf hanging between two bodies charged, one positively, and the other negatively. If the gold leaf becomes electrified it will incline towards the body whose electrification is opposite to its own. By increasing the electrification of the two bodies and the delicacy of the suspension, an exceedingly small electrification of the gold leaf may be detected.

When we come to describe electrometers and multipliers we shall find that there are still more delicate methods of detecting electrification and of testing the accuracy of our theorems, but at present we shall suppose the testing to be made by connecting the hollow vessel with a gold leaf electroscope.

This method was used by Faraday in his very admirable demonstration of the laws of electrical phenomena *.

34.] I. The total electrification of a body, or system of bodies, remains always the same, except in so far as it receives electrification from or gives electrification to other bodies.

In all electrical experiments the electrification of bodies is found to change, but it is always found that this change is due to want of perfect insulation, and that as the means of insulation are improved, the loss of electrification becomes less. We may therefore assert that the electrification of a body placed in a perfectly insulating medium would remain perfectly constant.

II. When one body electrifies another by conduction, the total electrification of the two bodies remains the same, that is, the one loses as much positive or gains as much negative electrification as the other gains of positive or loses of negative electrification.

For if the two bodies are enclosed in the hollow vessel, no change of the total electrification is observed.

III. When electrification is produced by friction, or by any other known method, equal quantities of positive and negative electrification are produced.

For the electrification of the whole system may be tested in the hollow vessel, or the process of electrification may be carried on within the vessel itself, and however intense the electrification of the parts of the system may be, the electrification of the whole, as indicated by the gold leaf electroscope, is invariably zero.

The electrification of a body is therefore a physical quantity capable of measurement, and two or more electrifications can be combined experimentally with a result of the same kind as when

* 'On Static Electrical Inductive Action,' *Phil. Mag.*, 1843, or *Exp. Res.*, vol. ii. p. 249.

two quantities are added algebraically. We therefore are entitled to use language fitted to deal with electrification as a quantity as well as a quality, and to speak of any electrified body as 'charged with a certain quantity of positive or negative electricity.'

35.] While admitting electricity, as we have now done, to the rank of a physical quantity, we must not too hastily assume that it is, or is not, a substance, or that it is, or is not, a form of energy, or that it belongs to any known category of physical quantities. All that we have hitherto proved is that it cannot be created or annihilated, so that if the total quantity of electricity within a closed surface is increased or diminished, the increase or diminution must have passed in or out through the closed surface.

This is true of matter, and is expressed by the equation known as the Equation of Continuity in Hydrodynamics.

It is not true of heat, for heat may be increased or diminished within a closed surface, without passing in or out through the surface, by the transformation of some other form of energy into heat, or of heat into some other form of energy.

It is not true even of energy in general if we admit the immediate action of bodies at a distance. For a body outside the closed surface may make an exchange of energy with a body within the surface. But if all apparent action at a distance is the result of the action between the parts of an intervening medium, and if the nature of this action of the parts of the medium is clearly understood, then it is conceivable that in all cases of the increase or diminution of the energy within a closed surface we may be able to trace the passage of the energy in or out through that surface.

There is, however, another reason which warrants us in asserting that electricity, as a physical quantity, synonymous with the total electrification of a body, is not, like heat, a form of energy. An electrified system has a certain amount of energy, and this energy can be calculated by multiplying the quantity of electricity in each of its parts by another physical quantity, called the Potential of that part, and taking half the sum of the products. The quantities 'Electricity' and 'Potential,' when multiplied together, produce the quantity 'Energy.' It is impossible, therefore, that electricity and energy should be quantities of the same category, for electricity is only one of the factors of energy, the other factor being 'Potential.'

Energy, which is the product of these factors, may also be considered as the product of several other pairs of factors, such as

A Force	× A distance through which the force is to act.
A Mass	× Gravitation acting through a certain height.
A Mass	× Half the square of its velocity.
A Pressure	× A volume of fluid introduced into a vessel at that pressure.
A Chemical Affinity	× A chemical change, measured by the number of electro-chemical equivalents which enter into combination.

If we obtain distinct mechanical ideas of the nature of electric potential, we may combine these with the idea of energy to determine the physical category in which 'Electricity' is to be placed.

36.] In most theories on the subject, Electricity is treated as a substance, but inasmuch as there are two kinds of electrification which, being combined, annul each other, and since we cannot conceive of two substances annulling each other, a distinction has been drawn between Free Electricity and Combined Electricity.

Theory of Two Fluids.

In the theory called that of Two Fluids, all bodies, in their unelectrified state, are supposed to be charged with equal quantities of positive and negative electricity. These quantities are supposed to be so great that no process of electrification has ever yet deprived a body of all the electricity of either kind. The process of electrification, according to this theory, consists in taking a certain quantity P of positive electricity from the body A and communicating it to B, or in taking a quantity N of negative electricity from B and communicating it to A, or in some combination of these processes.

The result will be that A will have $P+N$ units of negative electricity over and above its remaining positive electricity, which is supposed to be in a state of combination with an equal quantity of negative electricity. This quantity $P+N$ is called the Free electricity, the rest is called the Combined, Latent, or Fixed electricity.

In most expositions of this theory the two electricities are called 'Fluids,' because they are capable of being transferred from one body to another, and are, within conducting bodies, extremely mobile. The other properties of fluids, such as their inertia, weight, and elasticity, are not attributed to them by those who have used the theory for merely mathematical purposes; but the use of the word Fluid has been apt to mislead the vulgar, including many men of science who are not natural philosophers, and who have seized on the word Fluid as the only term in the statement of the theory which seemed intelligible to them.

We shall see that the mathematical treatment of the subject has been greatly developed by writers who express themselves in terms of the 'Two Fluids' theory.. Their results, however, have been deduced entirely from data which can be proved by experiment, and which must therefore be true, whether we adopt the theory of two fluids or not. The experimental verification of the mathematical results therefore is no evidence for or against the peculiar doctrines of this theory.

The introduction of two fluids permits us to consider the negative electrification of A and the positive electrification of B as the effect of *any one* of three different processes which would lead to the same result. We have already supposed it produced by the transfer of P units of positive electricity from A to B, together with the transfer of N units of negative electricity from B to A. But if $P+N$ units of positive electricity had been transferred from A to B, or if $P+N$ units of negative electricity had been transferred from B to A, the resulting 'free electricity' on A and on B would have been the same as before, but the quantity of 'combined electricity' in A would have been less in the second case and greater in the third than it was in the first.

It would appear therefore, according to this theory, that it is possible to alter not only the amount of free electricity in a body, but the amount of combined electricity. But no phenomena have ever been observed in electrified bodies which can be traced to the varying amount of their combined electricities. Hence either the combined electricities have no observable properties, or the amount of the combined electricities is incapable of variation. The first of these alternatives presents no difficulty to the mere mathematician, who attributes no properties to the fluids except those of attraction and repulsion, for in this point of view the two fluids simply annul one another, and their combination is a true mathematical zero. But to those who cannot use the word Fluid without thinking of a substance it is difficult to conceive that the combination of the two fluids shall have no properties at all, so that the addition of more or less of the combination to a body shall not in any way affect it, either by increasing its mass or its weight, or altering some of its other properties. Hence it has been supposed by some, that in every process of electrification exactly equal quantities of the two fluids are transferred in opposite directions, so that the total quantity of the two fluids in any body taken together remains always the same. By this new law they 'contrive to save appearances,' forgetting that there would have been no need of the law except to reconcile the 'two fluids' theory with facts, and to prevent it from predicting non-existent phenomena.

Theory of One Fluid.

37.] In the theory of One Fluid everything is the same as in the theory of Two Fluids except that, instead of supposing the two substances equal and opposite in all respects, one of them, generally the negative one, has been endowed with the properties and name of Ordinary Matter, while the other retains the name of The Electric Fluid. The particles of the fluid are supposed to repel one another according to the law of the inverse square of the distance, and to attract those of matter according to the same law. Those of matter are supposed to repel each other and attract those of electricity. The attraction, however, between units of the different substances at unit of distance is supposed to be a very little greater than the repulsion between units of the same kind, so that a unit of matter combined with a unit of electricity will exert a force of attraction on a similar combination at a distance, this force, however, being exceedingly small compared with the force between two uncombined units.

This residual force is supposed to account for the attraction of gravitation. Unelectrified bodies are supposed to be charged with as many units of electricity as they contain of ordinary matter. When they contain more electricity or less, they are said to be positively or negatively electrified.

This theory does not, like the Two-Fluid theory, explain too much. It requires us, however, to suppose the mass of the electric fluid so small that no attainable positive or negative electrification has yet perceptibly increased or diminished either the mass or the weight of a body, and it has not yet been able to assign sufficient reasons why the vitreous rather than the resinous electrification should be supposed due to an *excess* of electricity.

One objection has sometimes been urged against this theory by men who ought to have reasoned better. It has been said that the doctrine that the particles of matter uncombined with electricity *repel* one another, is in direct antagonism with the well-established fact that every particle of matter *attracts* every other particle throughout the universe. If the theory of One Fluid were true we should have the heavenly bodies repelling one another.

But it is manifest that the heavenly bodies, according to this theory, if they consisted of matter uncombined with electricity, would be in the highest state of negative electrification, and would repel each other. We have no reason to believe that they are in such a highly electrified state, or could be maintained in that state. The earth and all the bodies whose attraction has been observed are rather in an unelectrified state, that is, they contain the normal charge of electricity, and the only action between them is the residual force lately mentioned. The artificial manner, however, in which this residual force is introduced is a much more valid objection to the theory.

In the present treatise I propose, at different stages of the investigation, to test the different theories in the light of additional classes of phenomena. For my own part, I look for additional light on the nature of electricity from a study of what takes place in the space intervening between the electrified bodies. Such is the essential character of the mode of investigation pursued by Faraday in his *Experimental Researches*, and as we go on I intend to exhibit the results, as developed by Faraday, W. Thomson, &c., in a connected and mathematical form, so that we may perceive what phenomena are explained equally well by all the theories, and what phenomena indicate the peculiar difficulties of each theory.

Measurement of the Force between Electrified Bodies.

38.] Forces may be measured in various ways. For instance, one of the bodies may be suspended from one arm of a delicate balance, and weights suspended from the other arm, till the body, when unelectrified, is in equilibrium. The other body may then be placed at a known distance beneath the first, so that the attraction or repulsion of the bodies when electrified may increase or diminish the apparent weight of the first. The weight which must be added to or taken from the other arm, when expressed in dynamical measure, will measure the force between the bodies. This arrangement was used by Sir W. Snow Harris, and is that adopted in Sir W. Thomson's absolute electrometers. See Art. 217.

It is sometimes more convenient to use a torsion-balance in which a horizontal arm is suspended by a fine wire or fibre, so as to be capable of vibrating about the vertical wire as an axis, and the body is attached to one end of the arm and acted on by the force in the tangential direction, so as to turn the arm round the vertical axis, and so twist the suspension wire through a certain angle. The torsional rigidity of the wire is found by observing the time of oscillation of the arm, the moment of inertia of the arm being otherwise known, and from the angle of torsion and the torsional rigidity the force of attraction or repulsion can be deduced. The torsion-balance was devised by Michell for the determination of the force of gravitation between small bodies, and was used by Cavendish for this purpose. Coulomb, working independently of these philosophers, reinvented it, and successfully applied it to discover the laws of electric and magnetic forces; and the torsion-balance has ever since been used in all researches where small forces have to be measured. See Art. 215.

39.] Let us suppose that by either of these methods we can measure the force between two electrified bodies. We shall suppose the dimensions of the bodies small compared with the distance between them, so that the result may not be much altered by any inequality of distribution of the electrification on either body, and we shall suppose that both bodies are so suspended in air as to be at a considerable distance from other bodies on which they might induce electrification.

It is then found that if the bodies are placed at a fixed distance and charged respectively with e and e' of our provisional units of electricity, they will repel each other with a force proportional to the product of e and e'. If either e or e' is negative, that is, if one of the charges is vitreous and the other resinous, the force will be attractive, but if both e and e' are negative the force is again repulsive.

We may suppose the first body, A, charged with m units of vitreous and n units of resinous electricity, which may be conceived separately placed within the body, as in Experiment V.

Let the second body, B, be charged with m' units of positive and n' units of negative electricity.

Then each of the m positive units in A will repel each of the m' positive units in B with a certain force, say f, making a total effect equal to $mm'f$.

Since the effect of negative electricity is exactly equal and opposite to that of positive electricity, each of the m positive units in A will attract each of the n' negative units in B with the same force f, making a total effect equal to $mn'f$.

Similarly the n negative units in A will attract the m' positive units in B with a force $nm'f$, and will repel the n' negative units in B with a force $nn'f$.

The total repulsion will therefore be $(mm' + nn')f$; and the total attraction will be $(mn' + m'n)f$.

The resultant repulsion will be

$$(mm' + nn' - mn' - nm')f \quad \text{or} \quad (m-n)(m'-n')f.$$

Now $m - n = e$ is the algebraical value of the charge on A, and $m' - n' = e'$ is that of the charge on B, so that the resultant repulsion may be written $ee'f$, the quantities e and e' being always understood to be taken with their proper signs.

Variation of the Force with the Distance.

40.] Having established the law of force at a fixed distance, we may measure the force between bodies charged in a constant manner and placed at different distances. It is found by direct measurement that the force, whether of attraction or repulsion, varies inversely as the square of the distance, so that if f is the repulsion between two units at unit distance, the repulsion at distance r will be fr^{-2}, and the general expression for the repulsion between e units and e' units at distance r will be

$$fee'r^{-2}.$$

Definition of the Electrostatic Unit of Electricity.

41.] We have hitherto used a wholly arbitrary standard for our unit of electricity, namely, the electrification of a certain piece of glass as it happened to be electrified at the commencement of our experiments. We are now able to select a unit on a definite principle, and in order that this unit may belong to a general system we define it so that f may be unity, or in other words—

The electrostatic unit of electricity is that quantity of electricity which, when placed at unit of distance from an equal quantity, repels it with unit of force.

This unit is called the Electrostatic unit to distinguish it from the Electromagnetic unit, to be afterwards defined.

We may now write the general law of electrical action in the simple form $$F = ee'r^{-2}; \quad \text{or,}$$

The repulsion between two small bodies charged respectively with e and e′ units of electricity is numerically equal to the product of the charges divided by the square of the distance.

Dimensions of the Electrostatic Unit of Quantity.

42.] If $[Q]$ is the concrete electrostatic unit of quantity itself, and e, e' the numerical values of particular quantities; if $[L]$ is the unit of length, and r the numerical value of the distance; and if $[F]$ is the unit of force, and F the numerical value of the force, then the equation becomes

$$F[F] = ee'r^{-2}\,[Q^2]\,[L^{-2}];$$

whence

$$[Q] = [LF^{\frac{1}{2}}]$$
$$= [L^{\frac{3}{2}}\,T^{-1}\,M^{\frac{1}{2}}].$$

This unit is called the Electrostatic Unit of electricity. Other units may be employed for practical purposes, and in other departments of electrical science, but in the equations of electrostatics quantities of electricity are understood to be estimated in electrostatic units, just as in physical astronomy we employ a unit of mass which is founded on the phenomena of gravitation, and which differs from the units of mass in common use.

Proof of the Law of Electrical Force.

43.] The experiments of Coulomb with the torsion-balance may be considered to have established the law of force with a certain approximation to accuracy. Experiments of this kind, however, are rendered difficult, and in some degree uncertain, by several disturbing causes, which must be carefully traced and corrected for.

In the first place, the two electrified bodies must be of sensible dimensions relative to the distance between them, in order to be capable of carrying charges sufficient to produce measurable forces. The action of each body will then produce an effect on the distribution of electricity on the other, so that the charge cannot be considered as evenly distributed over the surface, or collected at the centre of gravity; but its effect must be calculated by an intricate investigation. This, however, has been done as regards two spheres by Poisson in an extremely able manner, and the investigation has been greatly simplified by Sir W. Thomson in his *Theory of Electrical Images*. See Arts. 172–174.

Another difficulty arises from the action of the electricity induced on the sides of the case containing the instrument. By making the inside of the instrument accurately cylindric, and making its inner surface of metal, this effect can be rendered definite and measurable.

An independent difficulty arises from the imperfect insulation of the bodies, on account of which the charge continually decreases. Coulomb investigated the law of dissipation, and made corrections for it in his experiments.

The methods of insulating charged conductors, and of measuring electrical effects, have been greatly improved since the time of Coulomb, particularly by Sir W. Thomson; but the perfect accuracy of Coulomb's law of force is established, not by any direct experiments and measurements (which may be used as illustrations of the law), but by a mathematical consideration of the phenomenon described as Experiment VII, namely, that an electrified conductor B, if made to touch the inside of a hollow closed conductor C and then withdrawn without touching C, is perfectly discharged, in whatever manner the outside of C may be electrified. By means of delicate electroscopes it is easy to shew that no electricity remains on B after the operation, and by the mathematical theory given at Art. 74, this can only be the case if the force varies inversely as the square of the distance, for if the law had been of any different form B would have been electrified.

The Electric Field.

44.] The Electric Field is the portion of space in the neighbourhood of electrified bodies, considered with reference to electric phenomena. It may be occupied by air or other bodies, or it may be a so-called vacuum, from which we have withdrawn every substance which we can act upon with the means at our disposal.

If an electrified body be placed at any part of the electric field it will be acted on by a force which will depend, in general, on the shape of the body and on its charge, if the body is so highly charged as to produce a sensible disturbance in the previous electrification of the other bodies.

But if the body is very small and its charge also very small, the electrification of the other bodies will not be sensibly disturbed, and we may consider the body as indicating by its centre of gravity a certain point of the field. The force acting on the body will then be proportional to its charge, and will be reversed when the charge is reversed.

Let e be the charge of the body, and F the force acting on the body in a certain direction, then when e is very small F is proportional to e, or

$$F = Re,$$

where R is a quantity depending on the other bodies in the field. If the charge e could be made equal to unity without disturbing the electrification of other bodies we should have $F = R$.

We shall call R the Resultant electric force at the given point of the field.

Electric Potential.

45.] If the small body carrying the small charge e be moved from the given point to an indefinite distance from the electrified bodies, it will experience at each point of its course a force Re, where R varies from point to point of the course. Let the whole work done on the body by these electrical forces be Ve, then V is the potential at the point of the field from which the body started. If the charge e could be made equal to unity without disturbing the electrification of other bodies, we might define the potential at any point as the work done on a body charged with unit of electricity in moving from that point to an infinite distance.

A body electrified positively tends to move from places of greater positive potential to places of smaller positive, or of negative potential, and a body negatively electrified tends to move in the opposite direction.

In a conductor the electrification is distributed exactly as if it were free to move in the conductor according to the same law. If therefore two parts of a conductor have different potentials, positive electricity will move from the part having greater potential to the part having less potential as long as that difference continues. A conductor therefore cannot be in electrical equilibrium unless every point in it has the same potential. This potential is called the Potential of the Conductor.

Equipotential Surfaces.

46.] If a surface described or supposed to be described in the electric field is such that the electric potential is the same at every point of the surface it is called an Equipotential surface.

An electrified point constrained to rest upon such a surface will have no tendency to move from one part of the surface to another, because the potential is the same at every point. An equipotential surface is therefore a surface of equilibrium or a level surface.

The resultant force at any point of the surface is in the direction of the normal to the surface, and the magnitude of the force is such that the work done on an electrical unit in passing from the surface V to the surface V' is $V-V'$.

No two equipotential surfaces having different potentials can meet one another, because the same point cannot have more than one potential, but one equipotential surface may meet itself, and this takes place at all points and lines of equilibrium.

The surface of a conductor in electrical equilibrium is necessarily an equipotential surface. If the electrification of the conductor is positive over the whole surface, then the potential will diminish as we move away from the surface on every side, and the conductor will be surrounded by a series of surfaces of lower potential.

But if (owing to the action of external electrified bodies) some regions of the conductor are electrified positively and others negatively, the complete equipotential surface will consist of the surface of the conductor itself together with a system of other surfaces, meeting the surface of the conductor in the lines which divide the positive from the negative regions. These lines will be lines of equilibrium, so that an electrified point placed on one of these lines will experience no force in any direction.

When the surface of a conductor is electrified positively in some parts and negatively in others, there must be some other electrified body in the field besides itself. For if we allow a positively electrified point, starting from a positively electrified part of the surface, to move always in the direction of the resultant force upon it, the potential at the point will continually diminish till the point reaches either a negatively electrified surface at a potential less than that of the first conductor, or moves off to an infinite distance. Since the potential at an infinite distance is zero, the latter case can only occur when the potential of the conductor is positive.

In the same way a negatively electrified point, moving off from a negatively electrified part of the surface, must either reach a positively electrified surface, or pass off to infinity, and the latter case can only happen when the potential of the conductor is negative.

Therefore, if both positive and negative electrification exists on a conductor, there must be some other body in the field whose potential has the same sign as that of the conductor but a greater numerical value, and if a conductor of any form is alone in the field the electrification of every part is of the same sign as the potential of the conductor.

Lines of Force.

47.] The line described by a point moving always in the direction of the resultant force is called a Line of force. It cuts the equipotential surfaces at right angles. The properties of lines of force will be more fully explained afterwards, because Faraday has expressed many of the laws of electrical action in terms of his conception of lines of force drawn in the electric field, and indicating both the direction and the magnitude of the force at every point.

Electric Tension.

48.] Since the surface of a conductor is an equipotential surface, the resultant force is normal to the surface, and it will be shewn in Art. 78 that it is proportional to the superficial density of the electrification. Hence the electricity on any small area of the surface will be acted on by a force tending *from* the conductor and proportional to the product of the resultant force and the density, that is, proportional to the square of the resultant force.

This force which acts outwards as a tension on every part of the conductor will be called electric Tension. It is measured like ordinary mechanical tension, by the force exerted on unit of area.

The word Tension has been used by electricians in several vague senses, and it has been attempted to adopt it in mathematical language as a synonym for Potential; but on examining the cases in which the word has been used, I think it will be more consistent with usage and with mechanical analogy to understand by tension a pulling force of so many pounds per square inch exerted on the surface of a conductor or elsewhere. We shall find that the conception of Faraday, that this electric tension exists not only at the electrified surface but all along the lines of force, leads to a theory of electric action as a phenomenon of stress in a medium.

Electromotive Force.

49.] When two conductors at different potentials are connected by a thin conducting wire, the tendency of electricity to flow along the wire is measured by the difference of the potentials of the two bodies. The difference of potentials between two conductors or two points is therefore called the Electromotive force between them.

Electromotive force may arise from other causes than difference of potential, but these causes are not considered in treating of statical electricity. We shall consider them when we come to chemical actions, motions of magnets, inequalities of temperature, &c.

Capacity of a Conductor.

50.] If one conductor is insulated while all the surrounding conductors are kept at the zero potential by being put in communication with the earth, and if the conductor, when charged with a quantity E of electricity, has a potential V, the ratio of E to V is called the Capacity of the conductor. If the conductor is completely enclosed within a conducting vessel without touching it, then the charge on the inner conductor will be equal and opposite to the charge on the inner surface of the outer conductor, and will be equal to the capacity of the inner conductor multiplied by the difference of the potentials of the two conductors.

Electric Accumulators.

A system consisting of two conductors whose opposed surfaces are separated from each other by a thin stratum of an insulating medium is called an electric Accumulator. Its capacity is directly proportional to the area of the opposed surfaces and inversely proportional to the thickness of the stratum between them. A Leyden jar is an accumulator in which glass is the insulating medium. Accumulators are sometimes called Condensers, but I prefer to restrict the term 'condenser' to an instrument which is used not to hold electricity but to increase its superficial density.

PROPERTIES OF BODIES IN RELATION TO STATICAL ELECTRICITY.

Resistance to the Passage of Electricity through a Body.

51.] When a charge of electricity is communicated to any part of a mass of metal the electricity is rapidly transferred from places of high to places of low potential till the potential of the whole mass becomes the same. In the case of pieces of metal used in ordinary experiments this process is completed in a time too short to be observed, but in the case of very long and thin wires, such as those used in telegraphs, the potential does not become uniform till after a sensible time, on account of the resistance of the wire to the passage of electricity through it.

The resistance to the passage of electricity is exceedingly different in different substances, as may be seen from the tables at

Arts. 362, 366, and 369, which will be explained in treating of Electric Currents.

All the metals are good conductors, though the resistance of lead is 12 times that of copper or silver, that of iron 6 times, and that of mercury 60 times that of copper. The resistance of all metals increases as their temperature rises.

Selenium in its crystalline state may also be regarded as a conductor, though its resistance is 3.7×10^{12} times that of a piece of copper of the same dimensions. Its resistance increases as the temperature rises. Selenium in the amorphous form is a good insulator, like sulphur.

Many liquids conduct electricity by electrolysis. This mode of conduction will be considered in Part II. For the present, we may regard all liquids containing water and all damp bodies as conductors, far inferior to the metals, but incapable of insulating a charge of electricity for a sufficient time to be observed.

On the other hand, the gases at the atmospheric pressure, whether dry or moist, are insulators so nearly perfect when the electric tension is small that we have as yet obtained no evidence of electricity passing through them by ordinary conduction. The gradual loss of charge by electrified bodies may in every case be traced to imperfect insulation in the supports, the electricity either passing through the substance of the support or creeping over its surface. Hence, when two charged bodies are hung up near each other, they will preserve their charges longer if they are electrified in opposite ways, than if they are electrified in the same way. For though the electromotive force tending to make the electricity pass through the air between them is much greater when they are oppositely electrified, no perceptible loss occurs in this way. The actual loss takes place through the supports, and the electromotive force through the supports is greatest when the bodies are electrified in the same way. The result appears anomalous only when we expect the loss to occur by the passage of electricity through the air between the bodies.

Certain kinds of glass when cold are marvelously perfect insulators, and Sir W. Thomson has preserved charges of electricity for years in bulbs hermetically sealed. The same glass, however, becomes a conductor at a temperature below that of boiling water.

Gutta-percha, caoutchouc, vulcanite, paraffin, and resins are good insulators, the resistance of gutta-percha at 75° F. being about 6×10^{19} times that of copper.

Ice, crystals, and solidified electrolytes, are also insulators.

Certain liquids, such as naphtha, turpentine, and some oils, are insulators, but inferior to most of the solid insulators.

The resistance of most substances, except the metals, and selenium and carbon, seems to diminish as the temperature rises.

DIELECTRICS.

Specific Inductive Capacity.

52.] All bodies whose insulating power is such that when they are placed between two conductors at different potentials the electromotive force acting on them does not immediately distribute their electricity so as to reduce the potential to a constant value, are called by Faraday Dielectrics.

Faraday discovered that the capacity of an accumulator depends on the nature of the insulating medium between the two conductors, as well as on the dimensions and relative position of the conductors themselves. By substituting other insulating media for air as the dielectric of the accumulator, without altering it in any other respect, he found that when air and other gases were employed as the insulating medium the capacity of the accumulator remained the same, but that when shell-lac, sulphur, glass, &c., were substituted for air, the capacity was increased in a ratio which was different for each substance.

The ratio of the capacity of an accumulator formed of any dielectric medium to the capacity of an accumulator of the same form and dimensions filled with air, was named by Faraday the Specific Inductive Capacity of the dielectric medium. It is equal to unity for air and other gases at all pressures, and probably at all temperatures, and it is greater than unity for all other liquid or solid dielectrics which have been examined.

If the dielectric is not a good insulator, it is difficult to measure its inductive capacity, because the accumulator will not hold a charge for a sufficient time to allow it to be measured; but it is certain that inductive capacity is a property not confined to good insulators, and it is probable that it exists in all bodies.

Absorption of Electricity.

53.] It is found that when an accumulator is formed of certain dielectrics, the following phenomena occur.

When the accumulator has been for some time electrified and is then suddenly discharged and again insulated, it becomes recharged in the same sense as at first, but to a smaller degree, so that it may be discharged again several times in succession, these discharges always diminishing. This phenomenon is called that of the Residual Discharge.

The instantaneous discharge appears always to be proportional to the difference of potentials at the instant of discharge, and the ratio of these quantities is the true capacity of the accumulator; but if the contact of the discharger is prolonged so as to include some of the residual discharge, the apparent capacity of the accumulator, calculated from such a discharge, will be too great.

The accumulator if charged and left insulated appears to lose its charge by conduction, but it is found that the proportionate rate of loss is much greater at first than it is afterwards, so that the measure of conductivity, if deduced from what takes place at first, would be too great. Thus, when the insulation of a submarine cable is tested, the insulation appears to improve as the electrification continues.

Thermal phenomena of a kind at first sight analogous take place in the case of the conduction of heat when the opposite sides of a body are kept at different temperatures. In the case of heat we know that they depend on the heat taken in and given out by the body itself. Hence, in the case of the electrical phenomena, it has been supposed that electricity is absorbed and emitted by the parts of the body. We shall see, however, in Art. 329, that the phenomena can be explained without the hypothesis of absorption of electricity, by supposing the dielectric in some degree heterogeneous.

That the phenomenon called Electric Absorption is not an actual absorption of electricity by the substance may be shewn by charging the substance in any manner with electricity while it is surrounded by a closed metallic insulated vessel. If, when the substance is charged and insulated, the vessel be instantaneously discharged and then left insulated, no charge is ever communicated to the vessel by the gradual dissipation of the electrification of the charged substance within it.

54.] This fact is expressed by the statement of Faraday that it is impossible to charge matter with an absolute and independent charge of one kind of electricity *.

In fact it appears from the result of every experiment which has been tried that in whatever way electrical actions may take

* *Exp. Res.*, vol. i. series xi. ¶ ii. 'On the Absolute Charge of Matter,' and (1244).

place among a system of bodies surrounded by a metallic vessel, the charge on the outside of that vessel is not altered.

Now if any portion of electricity could be forced into a body so as to be absorbed in it, or to become latent, or in any way to exist in it, without being connected with an equal portion of the opposite electricity by lines of induction, or if, after having being absorbed, it could gradually emerge and return to its ordinary mode of action, we should find some change of electrification in the surrounding vessel.

As this is never found to be the case, Faraday concluded that it is impossible to communicate an absolute charge to matter, and that no portion of matter can by any change of state evolve or render latent one kind of electricity or the other. He therefore regarded induction as 'the essential function both in the first development and the consequent phenomena of electricity.' His 'induction' is (1298) a polarized state of the particles of the dielectric, each particle being positive on one side and negative on the other, the positive and the negative electrification of each particle being always exactly equal.

Disruptive Discharge.*

55.] If the electromotive force acting at any point of a dielectric is gradually increased, a limit is at length reached at which there is a sudden electrical discharge through the dielectric, generally accompanied with light and sound, and with a temporary or permanent rupture of the dielectric.

The intensity of the electromotive force when this takes place depends on the nature of the dielectric. It is greater, for instance, in dense air than in rare air, and greater in glass than in air, but in every case, if the electromotive force be made great enough, the dielectric gives way and its insulating power is destroyed, so that a current of electricity takes place through it. It is for this reason that distributions of electricity for which the electric resultant force becomes anywhere infinite cannot exist in nature.

The Electric Glow.

Thus, when a conductor having a sharp point is electrified, the theory, based on the hypothesis that it retains its charge, leads to the conclusion that as we approach the point the superficial density of the electricity increases without limit, so that at the point itself the surface-density, and therefore the resultant

* See Faraday, *Exp. Res.*, vol. i., series xii. and xiii.

electrical force, would be infinite. If the air, or other surrounding dielectric, had an invincible insulating power, this result would actually occur; but the fact is, that as soon as the resultant force in the neighbourhood of the point has reached a certain limit, the insulating power of the air gives way, so that the air close to the point becomes a conductor. At a certain distance from the point the resultant force is not sufficient to break through the insulation of the air, so that the electric current is checked, and the electricity accumulates in the air round the point.

The point is thus surrounded by particles of air charged with electricity of the same kind with its own. The effect of this charged air round the point is to relieve the air at the point itself from part of the enormous electromotive force which it would have experienced if the conductor alone had been electrified. In fact the surface of the electrified body is no longer pointed, because the point is enveloped by a rounded mass of electrified air, the surface of which, rather than that of the solid conductor, may be regarded as the outer electrified surface.

If this portion of electrified air could be kept still, the electrified body would retain its charge, if not on itself at least in its neighbourhood, but the charged particles of air being free to move under the action of electrical force, tend to move away from the electrified body because it is charged with the same kind of electricity. The charged particles of air therefore tend to move off in the direction of the lines of force and to approach those surrounding bodies which are oppositely electrified. When they are gone, other uncharged particles take their place round the point, and since these cannot shield those next the point itself from the excessive electric tension, a new discharge takes place, after which the newly charged particles move off, and so on as long as the body remains electrified.

In this way the following phenomena are produced:—At and close to the point there is a steady glow, arising from the constant discharges which are taking place between the point and the air very near it.

The charged particles of air tend to move off in the same general direction, and thus produce a current of air from the point, consisting of the charged particles, and probably of others carried along by them. By artificially aiding this current we may increase the glow, and by checking the formation of the current we may prevent the continuance of the glow.

The electric wind in the neighbourhood of the point is sometimes very rapid, but it soon loses its velocity, and the air with its charged particles is carried about with the general motions of the atmosphere, and constitutes an invisible electric cloud. When the charged particles come near to any conducting surface, such as a wall, they induce on that surface an electrification opposite to their own, and are then attracted towards the wall, but since the electromotive force is small they may remain for a long time near the wall without being drawn up to the surface and discharged. They thus form an electrified atmosphere clinging to conductors, the presence of which may sometimes be detected by the electrometer. The electrical forces, however, acting between charged portions of air and other bodies are exceedingly feeble compared with the forces which produce winds arising from inequalities of density due to differences of temperature, so that it is very improbable that any observable part of the motion of ordinary thunder clouds arises from electrical causes.

The passage of electricity from one place to another by the motion of charged particles is called Electrical Convection or Convective Discharge.

The electrical glow is therefore produced by the constant passage of electricity through a small portion of air in which the tension is very high, so as to charge the surrounding particles of air which are continually swept off by the electric wind, which is an essential part of the phenomenon.

The glow is more easily formed in rare air than in dense air, and more easily when the point is positive than when it is negative. This and many other differences between positive and negative electrification must be studied by those who desire to discover something about the nature of electricity. They have not, however, been satisfactorily brought to bear upon any existing theory.

The Electric Brush.

56.] The electric brush is a phenomenon which may be produced by electrifying a blunt point or small ball so as to produce an electric field in which the tension diminishes, but in a less rapid manner, as we leave the surface. It consists of a succession of discharges, ramifying as they diverge from the ball into the air, and terminating either by charging portions of air or by reaching some other conductor. It is accompanied by a sound, the pitch of which depends on the interval between the successive discharges, and there is no current of air as in the case of the glow.

The Electric Spark.

57.] When the tension in the space between two conductors is considerable all the way between them, as in the case of two balls whose distance is not great compared with their radii, the discharge, when it occurs, usually takes the form of a spark, by which nearly the whole electrification is discharged at once.

In this case, when any part of the dielectric has given way, the parts on either side of it in the direction of the electric force are put into a state of greater tension so that they also give way, and so the discharge proceeds right through the dielectric, just as when a little rent is made in the edge of a piece of paper a tension applied to the paper in the direction of the edge causes the paper to be torn through, beginning at the rent, but diverging occasionally where there are weak places in the paper. The electric spark in the same way begins at the point where the electric tension first overcomes the insulation of the dielectric, and proceeds from that point, in an apparently irregular path, so as to take in other weak points, such as particles of dust floating in air.

On the Electric Force required to produce a Spark in Air.

In the experiments of Sir W. Thomson * the electromotive force required to produce a spark across strata of air of various thicknesses was measured by means of an electrometer.

The sparks were made to pass between two surfaces, one of which was plane, and the other only sufficiently convex to make the sparks occur always at the same place.

The difference of potential required to cause a spark to pass was found to increase with the distance, but in a less rapid ratio, so that the electric force at any point between the surfaces, which is the quotient of the difference of potential divided by the distance, can be raised to a greater value without a discharge when the stratum of air is thin.

When the stratum of air is very thin, say .00254 of a centimètre, the resultant force required to produce a spark was found to be 527.7, in terms of centimètres and grammes. This corresponds to an electric tension of 11.29 grammes weight per square centimètre.

When the distance between the surfaces is about a millimètre the electric force is about 130, and the electric tension .68 grammes weight per square centimètre. It is probable that the value for

* *Proc. R. S.*, 1860 ; or, Reprint, chap. xix.

greater distances is not much less than this. The ordinary pressure of the atmosphere is about 1032 grammes per square centimètre.

It is difficult to explain why a thin stratum of air should require a greater force to produce a disruptive discharge across it than a thicker stratum. Is it possible that the air very near to the surface of dense bodies is condensed, so as to become a better insulator? or does the potential of an electrified conductor differ from that of the air in contact with it by a quantity having a maximum value just before discharge, so that the observed difference of potential of the conductors is in every case greater than the difference of potentials on the two sides of the stratum of air by a constant quantity equivalent to the addition of about .005 of an inch to the thickness of the stratum? See Art. 370.

All these phenomena differ considerably in different gases, and in the same gas at different densities. Some of the forms of electrical discharge through rare gases are exceedingly remarkable. In some cases there is a regular alternation of luminous and dark strata, so that if the electricity, for example, is passing along a tube containing a very small quantity of gas, a number of luminous disks will be seen arranged transversely at nearly equal intervals along the axis of the tube and separated by dark strata. If the strength of the current be increased a new disk will start into existence, and it and the old disks will arrange themselves in closer order. In a tube described by Mr. Gassiot* the light of each of the disks is bluish on the negative and reddish on the positive side, and bright red in the central stratum.

These, and many other phenomena of electrical discharge, are exceedingly important, and when they are better understood they will probably throw great light on the nature of electricity as well as on the nature of gases and of the medium pervading space. At present, however, they must be considered as outside the domain of the mathematical theory of electricity.

Electric Phenomena of Tourmaline.

58.] Certain crystals of tourmaline, and of other minerals, possess what may be called Electric Polarity. Suppose a crystal of tourmaline to be at a uniform temperature, and apparently free from electrification on its surface. Let its temperature be now raised, the crystal remaining insulated. One end will be found positively

* *Intellectual Observer*, March, 1866.

and the other end negatively electrified. Let the surface be deprived of this apparent electrification by means of a flame or otherwise, then if the crystal be made still hotter, electrification of the same kind as before will appear, but if the crystal be cooled the end which was positive when the crystal was heated will become negative.

These electrifications are observed at the extremities of the crystallographic axis. Some crystals are terminated by a six-sided pyramid at one end and by a three-sided pyramid at the other. In these the end having the six-sided pyramid becomes positive when the crystal is heated.

Sir W. Thomson supposes every portion of these and other hemihedral crystals to have a definite electric polarity, the intensity of which depends on the temperature. When the surface is passed through a flame, every part of the surface becomes electrified to such an extent as to exactly neutralize, for all external points, the effect of the internal polarity. The crystal then has no external electrical action, nor any tendency to change its mode of electrification. But if it be heated or cooled the interior polarization of each particle of the crystal is altered, and can no longer be balanced by the superficial electrification, so that there is a resultant external action.

Plan of this Treatise.

59.] In the following treatise I propose first to explain the ordinary theory of electrical action, which considers it as depending only on the electrified bodies and on their relative position, without taking account of any phenomena which may take place in the surrounding media. In this way we shall establish the law of the inverse square, the theory of the potential, and the equations of Laplace and Poisson. We shall next consider the charges and potentials of a system of electrified conductors as connected by a system of equations, the coefficients of which may be supposed to be determined by experiment in those cases in which our present mathematical methods are not applicable, and from these we shall determine the mechanical forces acting between the different electrified bodies.

We shall then investigate certain general theorems by which Green, Gauss, and Thomson have indicated the conditions of solution of problems in the distribution of electricity. One result of these theorems is, that if Poisson's equation is satisfied by any function, and if at the surface of every conductor the function has the value of the potential of that conductor, then the function expresses the actual potential of the system at every point. We also deduce a method of finding problems capable of exact solution.

In Thomson's theorem, the total energy of the system is expressed in the form of the integral of a certain quantity extended over the whole space between the electrified bodies, and also in the form of an integral extended over the electrified surfaces only. The equation between these two expressions may be thus interpreted physically. We may conceive the relation into which the electrified bodies are thrown, either as the result of the state of the intervening medium, or as the result of a direct action between the electrified bodies at a distance. If we adopt the latter conception, we may determine the law of the action, but we can go no further in speculating on its cause. If, on the other hand, we adopt the conception of action through a medium, we are led to enquire into the nature of that action in each part of the medium.

It appears from the theorem, that if we are to look for the seat of the electric energy in the different parts of the dielectric medium, the amount of energy in any small part must depend on the square of the intensity of the resultant electromotive force at that place multiplied by a coefficient called the specific inductive capacity of the medium.

It is better, however, in considering the theory of dielectrics in the most general point of view, to distinguish between the electromotive force at any point and the electric polarization of the medium at that point, since these directed quantities, though related to one another, are not, in some solid substances, in the same direction. The most general expression for the electric energy of the medium per unit of volume is half the product of the electromotive force and the electric polarization multiplied by the cosine of the angle between their directions.

In all fluid dielectrics the electromotive force and the electric polarization are in the same direction and in a constant ratio.

If we calculate on this hypothesis the total energy residing in the medium, we shall find it equal to the energy due to the electrification of the conductors on the hypothesis of direct action at a distance. Hence the two hypotheses are mathematically equivalent.

If we now proceed to investigate the mechanical state of the medium on the hypothesis that the mechanical action observed between electrified bodies is exerted through and by means of the medium, as in the familiar instances of the action of one body on another by means of the tension of a rope or the pressure of a rod, we find that the medium must be in a state of mechanical stress.

The nature of this stress is, as Faraday pointed out*, a tension along the lines of force combined with an equal pressure in all directions at right angles to these lines. The magnitude of these stresses is proportional to the energy of the electrification, or, in other words, to the square of the resultant electromotive force multiplied by the specific inductive capacity of the medium.

This distribution of stress is the only one consistent with the observed mechanical action on the electrified bodies, and also with the observed equilibrium of the fluid dielectric which surrounds them. I have therefore thought it a warrantable step in scientific procedure to assume the actual existence of this state of stress, and to follow the assumption into its consequences. Finding the phrase *electric tension* used in several vague senses, I have attempted to confine it to what I conceive to have been in the mind of some of those who have used it, namely, the state of stress in the dielectric medium which causes motion of the electrified bodies, and leads, when continually augmented, to disruptive discharge. Electric tension, in this sense, is a tension of exactly the same kind, and measured in the same way, as the tension of a rope, and the dielectric medium, which can support a certain tension and no more, may be said to have a certain strength in exactly the same sense as the rope is said to have a certain strength. Thus, for example, Thomson has found that air at the ordinary pressure and temperature can support an electric tension of 9600 grains weight per square foot before a spark passes.

60.] From the hypothesis that electric action is not a direct action between bodies at a distance, but is exerted by means of the medium between the bodies, we have deduced that this medium must be in a state of stress. We have also ascertained the character of the stress, and compared it with the stresses which may occur in solid bodies. Along the lines of force there is tension, and perpendicular to them there is pressure, the numerical magnitude of these forces being equal, and each proportional to the square of the resultant force at the point. Having established these results, we are prepared to take another step, and to form

* *Exp. Res.*, series xi. 1297.

an idea of the nature of the electric polarization of the dielectric medium.

An elementary portion of a body may be said to be polarized when it acquires equal and opposite properties on two opposite sides. The idea of internal polarity may be studied to the greatest advantage as exemplified in permanent magnets, and it will be explained at greater length when we come to treat of magnetism.

The electric polarization of an elementary portion of a dielectric is a forced state into which the medium is thrown by the action of electromotive force, and which disappears when that force is removed. We may conceive it to consist in what we may call an electrical displacement, produced by the electromotive force. When the electromotive force acts on a conducting medium it produces a current through it, but if the medium is a non-conductor or dielectric, the current cannot flow through the medium, but the electricity is displaced within the medium in the direction of the electromotive force, the extent of this displacement depending on the magnitude of the electromotive force, so that if the electromotive force increases or diminishes the electric displacement increases and diminishes in the same ratio.

The amount of the displacement is measured by the quantity of electricity which crosses unit of area, while the displacement increases from zero to its actual amount. This, therefore, is the measure of the electric polarization.

The analogy between the action of electromotive force in producing electric displacement and of ordinary mechanical force in producing the displacement of an elastic body is so obvious that I have ventured to call the ratio of the electromotive force to the corresponding electric displacement the *coefficient of electric elasticity* of the medium. This coefficient is different in different media, and varies inversely as the specific inductive capacity of each medium.

The variations of electric displacement evidently constitute electric currents. These currents, however, can only exist during the variation of the displacement, and therefore, since the displacement cannot exceed a certain value without causing disruptive discharge, they cannot be continued indefinitely in the same direction, like the currents through conductors.

In tourmaline, and other pyro-electric crystals, it is probable that a state of electric polarization exists, which depends upon temperature, and does not require an external electromotive force to produce it. If the interior of a body were in a state of permanent electric polarization, the outside would gradually become charged in such a manner as to neutralize the action of the internal electrification for all points outside the body. This external superficial charge could not be detected by any of the ordinary tests, and could not be removed by any of the ordinary methods for discharging superficial electrification. The internal polarization of the substance would therefore never be discovered unless by some means, such as change of temperature, the amount of the internal polarization could be increased or diminished. The external electrification would then be no longer capable of neutralizing the external effect of the internal polarization, and an apparent electrification would be observed, as in the case of tourmaline.

If a charge e is uniformly distributed over the surface of a sphere, the resultant force at any point of the medium surrounding the sphere is numerically equal to the charge e divided by the square of the distance from the centre of the sphere. This resultant force, according to our theory, is accompanied by a displacement of electricity in a direction outwards from the sphere.

If we now draw a concentric spherical surface of radius r, the whole displacement, E, through this surface will be proportional to the resultant force multiplied by the area of the spherical surface. But the resultant force is directly as the charge e and inversely as the square of the radius, while the area of the surface is directly as the square of the radius.

Hence the whole displacement, E, is proportional to the charge e, and is independent of the radius.

To determine the ratio between the charge e, and the quantity of electricity, E, displaced outwards through the spherical surface, let us consider the work done upon the medium in the region between two concentric spherical surfaces, while the displacement is increased from E to $E+\delta E$. If V_1 and V_2 denote the potentials at the inner and the outer of these surfaces respectively, the electromotive force by which the additional displacement is produced is V_1-V_2, so that the work spent in augmenting the displacement is $(V_1-V_2)\delta E$.

If we now make the inner surface coincide with that of the electrified sphere, and make the radius of the other infinite, V_1 becomes V, the potential of the sphere, and V_2 becomes zero, so that the whole work done in the surrounding medium is $V\delta E$.

But by the ordinary theory, the work done in augmenting the charge is $V\delta e$, and if this is spent, as we suppose, in augmenting the displacement, $\delta E = \delta e$, and since E and e vanish together, $E = e$, or—

The displacement outwards through any spherical surface concentric with the sphere is equal to the charge on the sphere.

To fix our ideas of electric displacement, let us consider an accumulator formed of two conducting plates A and B, separated by a stratum of a dielectric C. Let W be a conducting wire joining A and B, and let us suppose that by the action of an electromotive force a quantity Q of positive electricity is transferred along the wire from B to A. The positive electrification of A and the negative electrification of B will produce a certain electromotive force acting from A towards B in the dielectric stratum, and this will produce an electric displacement from A towards B within the dielectric. The amount of this displacement, as measured by the quantity of electricity forced across an imaginary section of the dielectric dividing it into two strata, will be, according to our theory, exactly Q. See Arts. 75, 76, 111.

It appears, therefore, that at the same time that a quantity Q of electricity is being transferred along the wire by the electromotive force from B towards A, so as to cross every section of the wire, the same quantity of electricity crosses every section of the dielectric from A towards B by reason of the electric displacement.

The reverse motions of electricity will take place during the discharge of the accumulator. In the wire the discharge will be Q from A to B, and in the dielectric the displacement will subside, and a quantity of electricity Q will cross every section from B towards A.

Every case of electrification or discharge may therefore be considered as a motion in a closed circuit, such that at every section of the circuit the same quantity of electricity crosses in the same time, and this is the case, not only in the voltaic circuit where it has always been recognised, but in those cases in which electricity has been generally supposed to be accumulated in certain places.

61.] We are thus led to a very remarkable consequence of the theory which we are examining, namely, that the motions of electricity are like those of an *incompressible* fluid, so that the total quantity within an imaginary fixed closed surface remains always the same. This result appears at first sight in direct contradiction to the fact that we can charge a conductor and then introduce it into the closed space, and so alter the quantity of electricity within that space. But we must remember that the ordinary theory takes no account of the electric displacement in the substance of dielectrics which we have been investigating, but confines its attention to the electrification at the bounding surfaces of the conductors and dielectrics. In the case of the charged conductor let us suppose the charge to be positive, then if the surrounding dielectric extends on all sides beyond the closed surface there will be electric polarization, accompanied with displacement from within outwards all over the closed surface, and the surface-integral of the displacement taken over the surface will be equal to the charge on the conductor within.

Thus when the charged conductor is introduced into the closed space there is immediately a displacement of a quantity of electricity equal to the charge through the surface from within outwards, and the whole quantity within the surface remains the same.

The theory of electric polarization will be discussed at greater length in Chapter V, and a mechanical illustration of it will be given in Art. 334, but its importance cannot be fully understood till we arrive at the study of electromagnetic phenomena.

62.] The peculiar features of the theory as we have now developed them are :—

That the energy of electrification resides in the dielectric medium, whether that medium be solid, liquid, or gaseous, dense or rare, or even deprived of ordinary gross matter, provided it be still capable of transmitting electrical action.

That the energy in any part of the medium is stored up in the form of a state of constraint called electric polarization, the amount of which depends on the resultant electromotive force at the place.

That electromotive force acting on a dielectric produces what we have called electric displacement, the relation between the force and the displacement being in the most general case of a kind to be afterwards investigated in treating of conduction, but in the most important cases the force is in the same direction as the displacement, and is numerically equal to the displacement multiplied by a quantity which we have called the coefficient of electric elasticity of the dielectric.

That the energy per unit of volume of the dielectric arising from the electric polarization is half the product of the electromotive force and the electric displacement multiplied, if necessary, by the cosine of the angle between their directions.

That in fluid dielectrics the electric polarization is accompanied by a tension in the direction of the lines of force combined with an equal pressure in all directions at right angles to the lines of force, the amount of the tension or pressure per unit of area being numerically equal to the energy per unit of volume at the same place.

That the surfaces of any elementary portion into which we may conceive the volume of the dielectric divided must be conceived to be electrified, so that the surface-density at any point of the surface is equal in magnitude to the displacement through that point of the surface *reckoned inwards*, so that if the displacement is in the positive direction, the surface of the element will be electrified negatively on the positive side and positively on the negative side. These superficial electrifications will in general destroy one another when consecutive elements are considered, except where the dielectric has an internal charge, or at the surface of the dielectric.

That whatever electricity may be, and whatever we may understand by the movement of electricity, the phenomenon which we have called electric displacement is a movement of electricity in the same sense as the transference of a definite quantity of electricity through a wire is a movement of electricity, the only difference being that in the dielectric there is a force which we have called electric elasticity which acts against the electric displacement, and forces the electricity back when the electromotive force is removed; whereas in the conducting wire the electric elasticity is continually giving way, so that a current of true conduction is set up, and the resistance depends, not on the total quantity of electricity displaced from its position of equilibrium, but on the quantity which crosses a section of the conductor in a given time.

That in every case the motion of electricity is subject to the same condition as that of an incompressible fluid, namely, that at every instant as much must flow out of any given closed space as flows into it.

It follows from this that every electric current must form a closed circuit. The importance of this result will be seen when we investigate the laws of electro-magnetism.

Since, as we have seen, the theory of direct action at a distance is mathematically identical with that of action by means of a medium, the actual phenomena may be explained by the one theory as well as by the other, provided suitable hypotheses be introduced when any difficulty occurs. Thus, Mossotti has deduced the mathematical theory of dielectrics from the ordinary theory of attraction by merely giving an electric instead of a magnetic interpretation to the symbols in the investigation by which Poisson has deduced the theory of magnetic induction from the theory of magnetic fluids. He assumes the existence within the dielectric of small conducting elements, capable of having their opposite surfaces oppositely electrified by induction, but not capable of losing or gaining electricity on the whole, owing to their being insulated from each other by a non-conducting medium. This theory of dielectrics is consistent with the laws of electricity, and may be actually true. If it is true, the specific inductive capacity of a dielectric may be greater, but cannot be less, than that of air or vacuum. No instance has yet been found of a dielectric having an inductive capacity less than that of air, but if such should be discovered, Mossotti's theory must be abandoned, although his formulae would all remain exact, and would only require us to alter the sign of a coefficient.

In the theory which I propose to develope, the mathematical methods are founded upon the smallest possible amount of hypothesis, and thus equations of the same form are found applicable to phenomena which are certainly of quite different natures, as, for instance, electric induction through dielectrics; conduction through conductors, and magnetic induction. In all these cases the relation between the force and the effect produced is expressed by a set of equations of the same kind, so that when a problem in one of these subjects is solved, the problem and its solution may be translated into the language of the other subjects and the results in their new form will also be true.

CHAPTER II.

ELEMENTARY MATHEMATICAL THEORY OF STATICAL ELECTRICITY.

Definition of Electricity as a Mathematical Quantity.

63.] We have seen that the actions of electrified bodies are such that the electrification of one body may be equal to that of another, or to the sum of the electrifications of two bodies, and that when two bodies are equally and oppositely electrified they have no electrical effect on external bodies when placed together within a closed insulated conducting vessel. We may express all these results in a concise and consistent manner by describing an electrified body as *charged* with a certain *quantity of electricity*, which we may denote by e. When the electrification is positive, that is, according to the usual convention, vitreous, e will be a positive quantity. When the electrification is negative or resinous, e will be negative, and the quantity $-e$ may be interpreted either as a negative quantity of vitreous electricity or as a positive quantity of resinous electricity.

The effect of adding together two equal and opposite charges of electricity, $+e$ and $-e$, is to produce a state of no electrification expressed by zero. We may therefore regard a body not electrified as virtually charged with equal and opposite charges of indefinite magnitude, and an electrified body as virtually charged with unequal quantities of positive and negative electricity, the algebraic sum of these charges constituting the observed electrification. It is manifest, however, that this way of regarding an electrified body is entirely artificial, and may be compared to the conception of the velocity of a body as compounded of two or more different velocities, no one of which is the actual velocity of the body. When we speak therefore of a body being charged with a quantity e of electricity we mean simply that the body is electrified, and that the electrification is vitreous or resinous according as e is positive or negative.

ON ELECTRIC DENSITY.

Distribution in Three Dimensions.

64.] *Definition.* The electric volume-density at a given point in space is the limiting ratio of the quantity of electricity within a sphere whose centre is the given point to the volume of the sphere, when its radius is diminished without limit.

We shall denote this ratio by the symbol ρ, which may be positive or negative.

Distribution on a Surface.

It is a result alike of theory and of experiment, that, in certain cases, the electrification of a body is entirely on the surface. The density at a point on the surface, if defined according to the method given above, would be infinite. We therefore adopt a different method for the measurement of surface-density.

Definition. The electric density at a given point on a surface is the limiting ratio of the quantity of electricity within a sphere whose centre is the given point to the area of the surface contained within the sphere, when its radius is diminished without limit.

We shall denote the surface-density by the symbol σ.

Those writers who supposed electricity to be a material fluid or a collection of particles, were obliged in this case to suppose the electricity distributed on the surface in the form of a stratum of a certain thickness θ, its density being ρ_0, or that value of ρ which would result from the particles having the closest contact of which they are capable. It is manifest that on this theory

$$\rho_0 \theta = \sigma.$$

When σ is negative, according to this theory, a certain stratum of thickness θ is left entirely devoid of positive electricity, and filled entirely with negative electricity.

There is, however, no experimental evidence either of the electric stratum having any thickness, or of electricity being a fluid or a collection of particles. We therefore prefer to do without the symbol for the thickness of the stratum, and to use a special symbol for surface-density.

Distribution along a Line.

It is sometimes convenient to suppose electricity distributed on a line, that is, a long narrow body of which we neglect the thickness. In this case we may define the line-density at any point to be the limiting ratio of the electricity on an element of the line to the length of that element when the element is diminished without limit.

If λ denotes the line-density, then the whole quantity of electricity on a curve is $e = \int \lambda \, ds$, where ds is the element of the curve. Similarly, if σ is the surface-density, the whole quantity of electricity on the surface is

$$e = \iint \sigma \, dS,$$

where dS is the element of surface.

If ρ is the volume-density at any point of space, then the whole electricity within a certain volume is

$$e = \iiint \rho \, dx \, dy \, dz,$$

where $dx\, dy\, dz$ is the element of volume. The limits of integration in each case are those of the curve, the surface, or the portion of space considered.

It is manifest that e, λ, σ and ρ are quantities differing in kind, each being one dimension in space lower than the preceding, so that if a be a line, the quantities e, $a\lambda$, $a^2\sigma$, and $a^3\rho$ will be all of the same kind, and if a be the unit of length, and λ, σ, ρ each the unit of the different kinds of density, $a\lambda$, $a^2\sigma$, and $a^3\rho$ will each denote one unit of electricity.

Definition of the Unit of Electricity.

65.] Let A and B be two points the distance between which is the unit of length. Let two bodies, whose dimensions are small compared with the distance AB, be charged with equal quantities of positive electricity and placed at A and B respectively, and let the charges be such that the force with which they repel each other is the unit of force, measured as in Art. 6. Then the charge of either body is said to be the unit of electricity. If the charge of the body at B were a unit of negative electricity, then, since the action between the bodies would be reversed, we should have an attraction equal to the unit of force.

If the charge of A were also negative, and equal to unity, the force would be repulsive, and equal to unity.

Since the action between any two portions of electricity is not affected by the presence of other portions, the repulsion between e units of electricity at A and e' units at B is ee', the distance AB being unity. See Art. 39.

Law of Force between Electrified Bodies.

66.] Coulomb shewed by experiment that the force between electrified bodies whose dimensions are small compared with the distance between them, varies inversely as the square of the distance. Hence the actual repulsion between two such bodies charged with quantities e and e' and placed at a distance r is

$$\frac{ee'}{r^2}.$$

We shall prove in Art. 74 that this law is the only one consistent with the observed fact that a conductor, placed in the inside of a closed hollow conductor and in contact with it, is deprived of all electrical charge. Our conviction of the accuracy of the law of the inverse square of the distance may be considered to rest on experiments of this kind, rather than on the direct measurements of Coulomb.

Resultant Force between Two Bodies.

67.] In order to find the resultant force between two bodies we might divide each of them into its elements of volume, and consider the repulsion between the electricity in each of the elements of the first body and the electricity in each of the elements of the second body. We should thus get a system of forces equal in number to the product of the numbers of the elements into which we have divided each body, and we should have to combine the effects of these forces by the rules of Statics. Thus, to find the component in the direction of x we should have to find the value of the sextuple integral

$$\iiiiiint \frac{\rho\rho'(x-x')\,dx\,dy\,dz\,dx'\,dy'\,dz'}{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{\frac{3}{2}}},$$

where x, y, z are the coordinates of a point in the first body at which the electrical density is ρ, and x', y', z', and ρ' are the corresponding quantities for the second body, and the integration is extended first over the one body and then over the other.

Resultant Force at a Point.

68.] In order to simplify the mathematical process, it is convenient to consider the action of an electrified body, not on another body of any form, but on an indefinitely small body, charged with an indefinitely small amount of electricity, and placed at any point of the space to which the electrical action extends. By making the charge of this body indefinitely small we render insensible its disturbing action on the charge of the first body.

Let e be the charge of this body, and let the force acting on it when placed at the point (x, y, z) be Re, and let the direction-cosines of the force be l, m, n, then we may call R the resultant force at the point (x, y, z).

In speaking of the resultant electrical force at a point, we do not necessarily imply that any force is actually exerted there, but only that if an electrified body were placed there it would be acted on by a force Re, where e is the charge of the body.

Definition. The Resultant electrical force at any point is the force which would be exerted on a small body charged with the unit of positive electricity, if it were placed there without disturbing the actual distribution of electricity.

This force not only tends to move an electrified body, but to move the electricity within the body, so that the positive electricity tends to move in the direction of R and the negative electricity in the opposite direction. Hence the force R is also called the Electromotive Force at the point (x, y, z).

When we wish to express the fact that the resultant force is a vector, we shall denote it by the German letter $\mathfrak{E}$. If the body is a dielectric, then, according to the theory adopted in this treatise, the electricity is displaced within it, so that the quantity of electricity which is forced in the direction of $\mathfrak{E}$ across unit of area fixed perpendicular to $\mathfrak{E}$ is

$$\mathfrak{D} = \frac{1}{4\pi} K\mathfrak{E};$$

where $\mathfrak{D}$ is the displacement, $\mathfrak{E}$ the resultant force, and K the specific inductive capacity of the dielectric. For air, $K = 1$.

If the body is a conductor, the state of constraint is continually giving way, so that a current of conduction is produced and maintained as long as the force $\mathfrak{E}$ acts on the medium.

Components of the Resultant Force.

If X, Y, Z denote the components of R, then

$$X = Rl, \qquad Y = Rm, \qquad Z = Rn;$$

where l, m, n are the direction-cosines of R.

Line-Integral of Electric Force, or Electromotive Force along an Arc of a Curve.

69.] The Electromotive force along a given arc AP of a curve is numerically measured by the work which would be done on a unit of positive electricity carried along the curve from the beginning, A, to P, the end of the arc.

If s is the length of the arc, measured from A, and if the resultant force R at any point of the curve makes an angle ϵ with the tangent drawn in the positive direction, then the work done on unit of electricity in moving along the element of the curve ds will be

$$R \cos \epsilon \, ds,$$

and the total electromotive force V will be

$$V = \int_0^s R \cos \epsilon \, ds,$$

the integration being extended from the beginning to the end of the arc.

If we make use of the components of the force R, we find

$$V = \int_0^s \left(X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} \right) ds.$$

If X, Y, and Z are such that $X dx + Y dy + Z dz$ is a complete differential of a function of x, y, z, then

$$V = \int_A^P (X dx + Y dy + Z dz) = V_A - V_P;$$

where the integration is performed in any way from the point A to the point P, whether along the given curve or along any other line between A and P.

In this case V is a scalar function of the position of a point in space, that is, when we know the coordinates of the point, the value of V is determinate, and this value is independent of the position and direction of the axes of reference. See Art. 16.

On Functions of the Position of a Point.

In what follows, when we describe a quantity as a function of the position of a point, we mean that for every position of the point the function has a determinate value. We do not imply that this value can always be expressed by the same formula for all points of space, for it may be expressed by one formula on one side of a given surface and by another formula on the other side.

On Potential Functions.

70.] The quantity $Xdx+Ydy+Zdz$ is an exact differential whenever the force arises from attractions or repulsions whose intensity is a function of the distance only from any number of points. For if r_1 be the distance of one of the points from the point (x, y, z), and if R_1 be the repulsion, then

$$X_1 = R_1 l = R_1 \frac{dr_1}{dx},$$

with similar expressions for Y_1 and Z_1, so that

$$X_1 dx + Y_1 dy + Z_1 dz = R_1 dr_1;$$

and since R_1 is a function of r_1 only, $R_1 dr_1$ is an exact differential of some function of r_1, say V_1.

Similarly for any other force R_2, acting from a centre at distance r_2,

$$X_2 dx + Y_2 dy + Z_2 dz = R_2 dr_2 = dV_2.$$

But $X = X_1 + X_2 + \&c.$ and Y and Z are compounded in the same way, therefore

$$Xdx + Ydy + Zdz = dV_1 + dV_2 + \&c. = dV.$$

V, the integral of this quantity, under the condition that $V = 0$ at an infinite distance, is called the Potential Function.

The use of this function in the theory of attractions was introduced by Laplace in the calculation of the attraction of the earth. Green, in his essay 'On the Application of Mathematical Analysis to Electricity,' gave it the name of the Potential Function. Gauss, working independently of Green, also used the word Potential. Clausius and others have applied the term Potential to the work which would be done if two bodies or systems were removed to an infinite distance from one another. We shall follow the use of the word in recent English works, and avoid ambiguity by adopting the following definition due to Sir W. Thomson.

Definition of Potential. The Potential at a Point is the work which would be done on a unit of positive electricity by the electric forces if it were placed at that point without disturbing the electric distribution, and carried from that point to an infinite distance.

71.] Expressions for the Resultant Force and its components in terms of the Potential.

Since the total electromotive force along any arc AB is

$$V_A - V_B,$$

if we put ds for the arc AB we shall have for the force resolved in the direction of ds,

$$R\cos\epsilon = -\frac{dV}{ds};$$

whence, by assuming ds parallel to each of the axes in succession, we get

$$X = -\frac{dV}{dx}, \qquad Y = -\frac{dV}{dy}, \qquad Z = -\frac{dV}{dz};$$

$$R = \left(\overline{\frac{dV}{dx}}\Big|^2 + \overline{\frac{dV}{dy}}\Big|^2 + \overline{\frac{dV}{dz}}\Big|^2\right)^{\frac{1}{2}}.$$

We shall denote the force itself, whose magnitude is R and whose components are X, Y, Z, by the German letter $\mathfrak{E}$, as in Arts. 17 and 68.

The Potential at all Points within a Conductor is the same.

72.] A conductor is a body which allows the electricity within it to move from one part of the body to any other when acted on by electromotive force. When the electricity is in equilibrium there can be no electromotive force acting within the conductor. Hence $R = 0$ throughout the whole space occupied by the conductor. From this it follows that

$$\frac{dV}{dx} = 0, \quad \frac{dV}{dy} = 0, \quad \frac{dV}{dz} = 0;$$

and therefore for every point of the conductor

$$V = C,$$

where C is a constant quantity.

Potential of a Conductor.

Since the potential at all points within the substance of the conductor is C, the quantity C is called the Potential of the conductor. C may be defined as the work which must be done by external agency in order to bring a unit of electricity from an infinite distance to the conductor, the distribution of electricity being supposed not to be disturbed by the presence of the unit.

If two conductors have equal potentials, and are connected by a wire so fine that the electricity on the wire itself may be neglected, the total electromotive force along the wire will be zero, and no electricity will pass from the one conductor to the other.

If the potentials of the conductors A and B be V_A and V_B, then the electromotive force along any wire joining A and B will be

$$V_A - V_B$$

in the direction AB, that is, positive electricity will tend to pass from the conductor of higher potential to the other.

Potential, in electrical science, has the same relation to Electricity that Pressure, in Hydrostatics, has to Fluid, or that Temperature, in Thermodynamics, has to Heat. Electricity, Fluids, and Heat all tend to pass from one place to another, if the Potential, Pressure, or Temperature is greater in the first place than in the second. A fluid is certainly a substance, heat is as certainly not a substance, so that though we may find assistance from analogies of this kind in forming clear ideas of formal electrical relations, we must be careful not to let the one or the other analogy suggest to us that electricity is either a substance like water, or a state of agitation like heat.

Potential due to any Electrical System.

73.] Let there be a single electrified point charged with a quantity e of electricity, and let r be the distance of the point x', y', z' from it,

then
$$V = \int_r^\infty R\, dr = \int_r^\infty \frac{e}{r^2}\, dr = \frac{e}{r}.$$

Let there be any number of electrified points whose coordinates are (x_1, y_1, z_1), (x_2, y_2, z_2), &c. and their charges e_1, e_2, &c., and let their distances from the point (x', y', z') be r_1, r_2, &c., then the potential of the system at x', y', z' will be

$$V = \Sigma \left(\frac{e}{r}\right).$$

Let the electric density at any point (x, y, z) within an electrified body be ρ, then the potential due to the body is

$$V = \iiint \frac{\rho}{r}\, dx\, dy\, dz\,;$$

where
$$r = \{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}},$$

the integration being extended throughout the body.

On the Proof of the Law of the Inverse Square.

74.] The fact that the force between electrified bodies is inversely as the square of the distance may be considered to be established by direct experiments with the torsion-balance. The results, however, which we derive from such experiments must be regarded as affected by an error depending on the probable error of each experiment, and unless the skill of the operator be very great, the probable error of an experiment with the torsion-balance is considerable. As an argument that the attraction is really, and not merely as a rough approximation, inversely as the square of the distance, Experiment VII (p. 34) is far more conclusive than any measurements of electrical forces can be.

In that experiment a conductor B, charged in any manner, was enclosed in a hollow conducting vessel C, which completely surrounded it. C was also electrified in any manner.

B was then placed in electric communication with C, and was then again insulated and removed from C without touching it, and examined by means of an electroscope. In this way it was shewn that a conductor, if made to touch the inside of a conducting vessel which completely encloses it, becomes completely discharged, so that no trace of electrification can be discovered by the most delicate electrometer, however strongly the conductor or the vessel has been previously electrified.

The methods of detecting the electrification of a body are so delicate that a millionth part of the original electrification of B could be observed if it existed. No experiments involving the direct measurement of forces can be brought to such a degree of accuracy.

It follows from this experiment that a non-electrified body in the inside of a hollow conductor is at the same potential as the hollow conductor, in whatever way that conductor is charged. For if it were not at the same potential, then, if it were put in electric connexion with the vessel, either by touching it or by means of a wire, electricity would pass from the one body to the other, and the conductor, when removed from the vessel, would be found to be electrified positively or negatively, which, as we have already stated, is not the case.

Hence the whole space inside a hollow conductor is at the same potential as the conductor if no electrified body is placed within it. If the law of the inverse square is true, this will be the case whatever be the form of the hollow conductor. Our object at present, however, is to ascertain from this fact the form of the law of attraction.

For this purpose let us suppose the hollow conductor to be a thin spherical shell. Since everything is symmetrical about its centre, the shell will be uniformly electrified at every point, and we have to enquire what must be the law of attraction of a uniform spherical shell, so as to fulfil the condition that the potential at every point within it shall be the same.

Let the force at a distance r from a point at which a quantity e of electricity is concentrated be R, where R is some function of r.

All central forces which are functions of the distance admit of a potential, let us write $\frac{f(r)}{r}$ for the potential function due to a unit of electricity at a distance r.

Let the radius of the spherical shell be a, and let the surface-density be σ. Let P be any point within the shell at a distance p from the centre. Take the radius through P as the axis of spherical coordinates, and let r be the distance from P to an element dS of the shell. Then the potential at P is

$$V = \iint \sigma \frac{f(r)}{r} dS,$$

$$V = \int_0^{2\pi}\int_0^{\pi} \sigma \frac{f(r)}{r} a^2 \sin\theta \, d\theta \, d\phi.$$

Now $$r^2 = a^2 - 2ap\cos\theta + p^2,$$

$$r\,dr = ap\sin\theta\, d\theta.$$

Hence $$V = 2\pi\sigma\frac{a}{p}\int_{a-p}^{a+p} f(r)\,dr;$$

and V must be constant for all values of p less than a.

Multiplying both sides by p and differentiating with respect to p,

$$V = 2\pi\sigma a\,\{f(a+p)+f(a-p)\}.$$

Differentiating again with respect to p,

$$0 = f'(a+p) - f'(a-p).$$

Since a and p are independent,

$$f'(r) = C, \text{ a constant.}$$

Hence $$f(r) = Cr + C',$$

and the potential function is

$$\frac{f(r)}{r} = C + \frac{C'}{r}.$$

The force at distance r is got by differentiating this expression with respect to r, and changing the sign, so that

$$R = \frac{C'}{r^2};$$

or the force is inversely as the square of the distance, and this therefore is the only law of force which satisfies the condition that the potential within a uniform spherical shell is constant*. Now

* See Pratt's *Mechanical Philosophy*, p. 144.

this condition is shewn to be fulfilled by the electric forces with the most perfect accuracy. Hence the law of electric force is verified to a corresponding degree of accuracy.

Surface-Integral of Electric Induction, and Electric Displacement through a Surface.

75.] Let R be the resultant force at any point of the surface, and ϵ the angle which R makes with the normal drawn towards the positive side of the surface, then $R\cos\epsilon$ is the component of the force normal to the surface, and if dS is the element of the surface, the electric displacement through dS will be, by Art. 68,

$$\frac{1}{4\pi} KR\cos\epsilon\, dS.$$

Since we do not at present consider any dielectric except air, $K=1$.

We may, however, avoid introducing at this stage the theory of electric displacement, by calling $R\cos\epsilon\, dS$ the Induction through the element dS. This quantity is well known in mathematical physics, but the name of induction is borrowed from Faraday. The surface-integral of induction is

$$\iint R\cos\epsilon\, dS,$$

and it appears by Art. 21, that if X, Y, Z are the components of R, and if these quantities are continuous within a region bounded by a closed surface S, the induction reckoned from within outwards is

$$\iint R\cos e\, dS = \iiint \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) dx\, dy\, dz,$$

the integration being extended through the whole space within the surface.

Induction through a Finite Closed Surface due to a Single Centre of Force.

76.] Let a quantity e of electricity be supposed to be placed at a point O, and let r be the distance of any point P from O, the force at that point is $R = \frac{e}{r^2}$ in the direction OP.

Let a line be drawn from O in any direction to an infinite distance. If O is without the closed surface this line will either not cut the surface at all, or it will issue from the surface as many times as it enters. If O is within the surface the line must first issue from the surface, and then it may enter and issue any number of times alternately, ending by issuing from it.

Let ϵ be the angle between OP and the normal to the surface drawn outwards where OP cuts it, then where the line issues from the surface $\cos\epsilon$ will be positive, and where it enters $\cos\epsilon$ will be negative.

Now let a sphere be described with centre O and radius unity, and let the line OP describe a conical surface of small angular aperture about O as vertex.

This cone will cut off a small element $d\omega$ from the surface of the sphere, and small elements dS_1, dS_2, &c. from the closed surface at the various places where the line OP intersects it.

Then, since any one of these elements dS intersects the cone at a distance r from the vertex and at an obliquity ϵ,

$$dS = r^2 \sec\epsilon\, d\omega\,;$$

and, since $R = er^{-2}$, we shall have

$$R\cos\epsilon\, dS = \pm\, e\, d\omega\,;$$

the positive sign being taken when r issues from the surface, and the negative where it enters it.

If the point O is without the closed surface, the positive values are equal in number to the negative ones, so that for any direction of r,

$$\Sigma R\cos\epsilon\, dS = 0,$$

and therefore

$$\iint R\cos\epsilon\, dS = 0,$$

the integration being extended over the whole closed surface.

If the point O is within the closed surface the radius vector OP first issues from the closed surface, giving a positive value of $e\,d\omega$, and then has an equal number of entrances and issues, so that in this case

$$\Sigma R\cos\epsilon\, dS = e\, d\omega.$$

Extending the integration over the whole closed surface, we shall include the whole of the spherical surface, the area of which is 4π, so that

$$\iint R\cos\epsilon\, dS = e\iint d\omega = 4\pi e.$$

Hence we conclude that the total induction outwards through a closed surface due to a centre of force e placed at a point O is zero when O is without the surface, and $4\pi e$ when O is within the surface.

Since in air the displacement is equal to the induction divided by 4π, the displacement through a closed surface, reckoned outwards, is equal to the electricity within the surface.

Corollary. It also follows that if the surface is not closed but is bounded by a given closed curve, the total induction through it is ωe, where ω is the solid angle subtended by the closed curve at O. This quantity, therefore, depends only on the closed curve, and not on the form of the surface of which it is the boundary.

On the Equations of Laplace and Poisson.

77.] Since the value of the total induction of a single centre of force through a closed surface depends only on whether the centre is within the surface or not, and does not depend on its position in any other way, if there are a number of such centres e_1, e_2, &c. within the surface, and e_1', e_2', &c. without the surface, we shall have

$$\iint R \cos\epsilon \, dS = 4\pi e;$$

where e denotes the algebraical sum of the quantities of electricity at all the centres of force within the closed surface, that is, the total electricity within the surface, resinous electricity being reckoned negative.

If the electricity is so distributed within the surface that the density is nowhere infinite, we shall have by Art. 64,

$$4\pi e = 4\pi \iiint \rho \, dx \, dy \, dz,$$

and by Art. 75,

$$\iint R \cos\epsilon \, dS = \iiint \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) dx \, dy \, dz.$$

If we take as the closed surface that of the element of volume $dx\,dy\,dz$, we shall have, by equating these expressions,

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 4\pi\rho;$$

and if a potential V exists, we find by Art. 71,

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} + 4\pi\rho = 0.$$

This equation, in the case in which the density is zero, is called Laplace's Equation. In its more general form it was first given by Poisson. It enables us, when we know the potential at every point, to determine the distribution of electricity.

We shall denote, as at Art. 26, the quantity

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2} \text{ by } -\nabla^2 V,$$

and we may express Poisson's equation in words by saying that the electric density multiplied by 4π is the concentration of the potential. Where there is no electrification, the potential has no concentration, and this is the interpretation of Laplace's equation.

If we suppose that in the superficial and linear distributions of electricity the volume-density ρ remains finite, and that the electricity exists in the form of a thin stratum or narrow fibre, then, by increasing ρ and diminishing the depth of the stratum or the section of the fibre, we may approach the limit of true superficial or linear distribution, and the equation being true throughout the process will remain true at the limit, if interpreted in accordance with the actual circumstances.

On the Conditions to be fulfilled at an Electrified Surface.

78.] We shall consider the electrified surface as the limit to which an electrified stratum of density ρ and thickness ν approaches when ρ is increased and ν diminished without limit, the product $\rho\nu$ being always finite and equal to σ the surface-density.

Let the stratum be that included between the surfaces

$$F(x, y, z) = F = a \tag{1}$$

and

$$F = a+h. \tag{2}$$

If we put

$$R^2 = \left.\overline{\frac{dF}{dx}}\right|^2 + \left.\overline{\frac{dF}{dy}}\right|^2 + \left.\overline{\frac{dF}{dz}}\right|^2, \tag{3}$$

and if l, m, n are the direction-cosines of the normal to the surface,

$$Rl = \frac{dF}{dx}, \qquad Rm = \frac{dF}{dy}, \qquad Rn = \frac{dF}{dz}. \tag{4}$$

Now let V_1 be the value of the potential on the negative side of the surface $F = a$, V' its value between the surfaces $F = a$ and $F = a+h$, and V_2 its value on the positive side of $F = a+h$.

Also, let ρ_1, ρ', and ρ_2 be the values of the density in these three portions of space. Then, since the density is everywhere finite, the second derivatives of V are everywhere finite, and the first derivatives, and also the function itself, are everywhere continuous and finite.

At any point of the surface $F = a$ let a normal be drawn of length ν, till it meets the surface $F = a+h$, then the value of F at the extremity of the normal is

$$a+\nu\left(l\frac{dF}{dx}+m\frac{dF}{dy}+n\frac{dF}{dz}\right)+\&c., \tag{5}$$

$$\text{or} \qquad a+h = a+\nu R+\&c. \qquad /N \tag{6}$$

The value of V at the same point is

$$V_2 = V_1+\nu\left(l\frac{dV'}{dx}+m\frac{dV'}{dy}+n\frac{dV'}{dz}\right)+\&c., \tag{7}$$

$$\text{or} \qquad V_2-V_1 = \frac{h}{R}\frac{dV'}{d\nu}+\&c. \qquad /N \tag{8}$$

Since the first derivatives of V continue always finite, the second side of the equation vanishes when h is diminished without limit, and therefore if V_2 and V_1 denote the values of V on the outside and inside of an electrified surface at the point x, y, z,

$$V_1 = V_2. \tag{9}$$

If $x+dx$, $y+dy$, $z+dz$ be the coordinates of another point on the electrified surface, $F=a$ and $V_1=V_2$ at this point also; whence

$$0 = \frac{dF}{dx}dx+\frac{dF}{dy}dy+\frac{dF}{dz}dz+\&c., \tag{10}$$

$$0=\left(\frac{dV_2}{dx}-\frac{dV_1}{dx}\right)dx+\left(\frac{dV_2}{dy}-\frac{dV_1}{dy}\right)dy+\left(\frac{dV_2}{dz}-\frac{dV_1}{dz}\right)dz+\&c.; \tag{11}$$

and when dx, dy, dz vanish, we find the conditions

$$\left.\begin{aligned} \frac{dV_2}{dx}-\frac{dV_1}{dx} &= Cl,\\ \frac{dV_2}{dy}-\frac{dV_1}{dy} &= Cm,\\ \frac{dV_2}{dz}-\frac{dV_1}{dz} &= Cn, \end{aligned}\right\} \tag{12}$$

where C is a quantity to be determined.

Next, let us consider the variation of F and $\dfrac{dV}{dx}$ along the ordinate parallel to x between the surfaces $F=a$ and $F=a+h$.

We have
$$F = a+\frac{dF}{dx}dx+\tfrac{1}{2}\frac{d^2F}{dx^2}(dx)^2+\&c., \tag{13}$$

and
$$\frac{dV}{dx}=\frac{dV_1}{dx}+\frac{d^2V'}{dx^2}dx+\tfrac{1}{2}\frac{d^3V'}{dx^3}(dx)^2+\&c. \tag{14}$$

Hence, at the second surface, where $F=a+h$, and V becomes V_2,

$$\frac{dV_2}{dx}=\frac{dV_1}{dx}+\frac{d^2V'}{dx^2}dx+\&c.; \tag{15}$$

whence $$\frac{d^2 V'}{dx^2}\,dx + \&\text{c.} = Cl, \tag{16}$$

by the first of equations (12).

Multiplying by Rl, and remembering that at the second surface

$$Rl\,dx = h \tag{17}$$

we find $$\frac{d^2 V'}{dx^2}\,h = CRl^2. \tag{18}$$

Similarly $$\frac{d^2 V'}{dy^2}\,h = CRm^2; \tag{19}$$

and $$\frac{d^2 V'}{dz^2}\,h = CRn^2. \tag{20}$$

Adding $$\left(\frac{d^2 V'}{dx^2} + \frac{d^2 V'}{dy^2} + \frac{d^2 V'}{dz^2}\right)h = CR; \tag{21}$$

but $$\frac{d^2 V'}{dx^2} + \frac{d^2 V'}{dy^2} + \frac{d^2 V'}{dz^2} = -4\pi\rho' \text{ and } h = \nu R; \tag{22}$$

hence $$C = -4\pi\rho'\nu = -4\pi\sigma, \tag{23}$$

where σ is the surface-density; or, multiplying the equations (12) by l, m, n respectively, and adding,

$$l\left(\frac{dV_2}{dx} - \frac{dV_1}{dx}\right) + m\left(\frac{dV_2}{dy} - \frac{dV_1}{dy}\right) + n\left(\frac{dV_2}{dz} - \frac{dV_1}{dz}\right) + 4\pi\sigma = 0. \tag{24}$$

This equation is called the *characteristic equation* of V at a surface.

This equation may also be written

$$\frac{dV_1}{d\nu_1} + \frac{dV_2}{d\nu_2} + 4\pi\sigma = 0; \tag{25}$$

where ν_1, ν_2 are the normals to the surface drawn towards the first and the second medium respectively, and V_1, V_2 the potentials at points on these normals. We may also write it

$$R_2 \cos\epsilon_2 + R_1 \cos\epsilon_1 + 4\pi\sigma = 0; \tag{26}$$

where R_1, R_2 are the resultant forces, and ϵ_1, ϵ_2 the angles which they make with the normals drawn *from* the surface on either side.

79.] Let us next determine the total mechanical force acting on an element of the electrified surface.

The general expression for the force parallel to x on an element whose volume is $dx\,dy\,dz$, and volume-density ρ, is

$$dX = -\frac{dV}{dx}\rho\,dx\,dy\,dz. \tag{27}$$

In the present case we have for any point on the normal ν

$$\frac{dV}{dx} = \frac{dV_1}{dx} + \nu \frac{d^2 V_1}{dx^2} + \&c.; \tag{28}$$

also, if the element of surface is dS, that of the volume of the element of the stratum may be written $dS\,d\nu$; and if X is the whole force on a stratum of thickness ν,

$$X = \iiint \left(\frac{dV_1}{dx} + \nu \frac{d^2 V_1}{dx^2} + \&c.\right) \rho'\, dS\, d\nu. \tag{29}$$

Integrating with respect to ν, we find

$$X = -\iint \rho'\, dS \left(\nu \frac{dV_1}{dx} + \frac{\nu^2}{2} \frac{d^2 V_1}{dx^2} + \&c.\right); \tag{30}$$

or, since

$$\frac{dV_2}{dx} = \frac{dV_1}{dx} + \nu \frac{d^2 V'}{dx^2} + \&c.; \tag{31}$$

$$X = -\iint \tfrac{1}{2} \rho' \nu\, dS \left(\frac{dV_1}{dx} + \frac{dV_2}{dx}\right) + \&c. \tag{32}$$

When ν is diminished and ρ' increased without limit, the product $\rho'\nu$ remaining always constant and equal to σ, the expression for the force in the direction of x on the electricity $\sigma\, dS$ on the element of surface dS is

$$X = -\sigma\, dS \tfrac{1}{2} \left(\frac{dV_1}{dx} + \frac{dV_2}{dx}\right); \tag{33}$$

that is, the force acting on the electrified element $\sigma\, dS$ in any given direction is the arithmetic mean of the forces acting on equal quantities of electricity placed one just inside the surface and the other just outside the surface close to the actual position of the element, and therefore the resultant mechanical force on the electrified element is equal to the resultant of the forces which would act on two portions of electricity, each equal to half that on the element, and placed one on each side of the surface and infinitely near to it.

80.] *When a conductor is in electrical equilibrium, the whole of the electricity is on the surface.*

We have already shewn that throughout the substance of the conductor the potential V is constant. Hence $\nabla^2 V$ is zero, and therefore by Poisson's equation, ρ is zero throughout the substance of the conductor, and there can be no electricity in the interior of the conductor.

Hence a superficial distribution of electricity is the only possible one in the case of conductors in equilibrium. A distribution throughout the mass can only exist in equilibrium when the body is a non-conductor.

Since the resultant force within a conductor is zero, the resultant force just outside the conductor is along the normal and is equal to $4\pi\sigma$, acting outwards from the conductor.

81.] If we now suppose an elongated body to be electrified, we may, by diminishing its lateral dimensions, arrive at the conception of an electrified line.

Let ds be the length of a small portion of the elongated body, and let c be its circumference, and σ the superficial density of the electricity on its surface; then, if λ is the electricity per unit of length, $\lambda = c\sigma$, and the resultant electrical force close to the surface will be

$$4\pi\sigma = 4\pi\frac{\lambda}{c}.$$

If, while λ remains finite, c be diminished indefinitely, the force at the surface will be increased indefinitely. Now in every dielectric there is a limit beyond which the force cannot be increased without a disruptive discharge. Hence a distribution of electricity in which a finite quantity is placed on a finite portion of a line is inconsistent with the conditions existing in nature.

Even if an insulator could be found such that no discharge could be driven through it by an infinite force, it would be impossible to charge a linear conductor with a finite quantity of electricity, for an infinite electromotive force would be required to bring the electricity to the linear conductor.

In the same way it may be shewn that a point charged with a finite quantity of electricity cannot exist in nature. It is convenient, however, in certain cases, to speak of electrified lines and points, and we may suppose these represented by electrified wires, and by small bodies of which the dimensions are negligible compared with the principal distances concerned.

Since the quantity of electricity on any given portion of a wire diminishes indefinitely when the diameter of the wire is indefinitely diminished, the distribution of electricity on bodies of considerable dimensions will not be sensibly affected by the introduction of very fine metallic wires into the field, so as to form electrical connexions between these bodies and the earth, an electrical machine, or an electrometer.

On Lines of Force.

82.] If a line be drawn whose direction at every point of its course coincides with that of the resultant force at that point, the line is called a Line of Force.

If lines of force be drawn from every point of a line they will form a surface such that the force at any point is parallel to the tangent plane at that point. The surface-integral of the force with respect to this surface or any part of it will therefore be zero.

If lines of force are drawn from every point of a closed curve L_1 they will form a tubular surface S_0. Let the surface S_1, bounded by the closed curve L_1, be a section of this tube, and let S_2 be any other section of the tube. Let Q_0, Q_1, Q_2 be the surface-integrals over S_0, S_1, S_2, then, since the three surfaces completely enclose a space in which there is no attracting matter, we have

$$Q_0 + Q_1 + Q_2 = 0.$$

But $Q_0 = 0$, therefore $Q_2 = -Q_1$, or the surface-integral over the second section is equal and opposite to that over the first: but since the directions of the normal are opposite in the two cases, we may say that the surface-integrals of the two sections are equal, the direction of the line of force being supposed positive in both.

Such a tube is called a Solenoid *, and such a distribution of force is called a Solenoidal distribution. The velocities of an incompressible fluid are distributed in this manner.

If we suppose any surface divided into elementary portions such that the surface-integral of each element is unity, and if solenoids are drawn through the field of force having these elements for their bases, then the surface-integral for any other surface will be represented by the number of solenoids which it cuts. It is in this sense that Faraday uses his conception of lines of force to indicate not only the direction but the amount of the force at any place in the field.

We have used the phrase Lines of Force because it has been used by Faraday and others. In strictness, however, these lines should be called Lines of Electric Induction.

In the ordinary cases the lines of induction indicate the direction and magnitude of the resultant electromotive force at every point, because the force and the induction are in the same direction and in a constant ratio. There are other cases, however, in which it is important to remember that these lines indicate the induction, and that the force is indicated by the equipotential surfaces, being normal to these surfaces and inversely proportional to the distances of consecutive surfaces.

* From σωλήν, a tube. Faraday uses (3271) the term 'Sphondyloid' in the same sense.

On Specific Inductive Capacity.

83.] In the preceding investigation of surface-integrals I have adopted the ordinary conception of direct action at a distance, and have not taken into consideration any effects depending on the nature of the dielectric medium in which the forces are observed.

But Faraday has observed that the quantity of electricity induced by a given electromotive force on the surface of a conductor which bounds a dielectric is not the same for all dielectrics. The induced electricity is greater for most solid and liquid dielectrics than for air and gases. Hence these bodies are said to have a greater specific inductive capacity than air, which is the standard medium.

We may express the theory of Faraday in mathematical language by saying that in a dielectric medium the induction across any surface is the product of the normal electric force into the coefficient of specific inductive capacity of that medium. If we denote this coefficient by K, then in every part of the investigation of surface-integrals we must multiply X, Y, and Z by K, so that the equation of Poisson will become

$$\frac{d}{dx}\cdot K\frac{dV}{dx}+\frac{d}{dy}\cdot K\frac{dV}{dy}+\frac{d}{dz}\cdot K\frac{dV}{dz}+4\pi\rho=0.$$

At the surface of separation of two media whose inductive capacities are K_1 and K_2, and in which the potentials are V_1 and V_2, the characteristic equation may be written

$$K_2\frac{dV_2}{d\nu}-K_1\frac{dV_1}{d\nu}+4\pi\sigma=0;$$

where ν is the normal drawn from the first medium to the second, and σ is the true surface-density on the surface of separation; that is to say, the quantity of electricity which is actually on the surface in the form of a charge, and which can be altered only by conveying electricity to or from the spot. This true electrification must be distinguished from the apparent electrification σ', which is the electrification as deduced from the electrical forces in the neighbourhood of the surface, using the ordinary characteristic equation

$$\frac{dV_2}{d\nu}-\frac{dV_1}{d\nu}+4\pi\sigma'=0.$$

If a solid dielectric of any form is a perfect insulator, and if its surface receives no charge, then the true electrification remains zero, whatever be the electrical forces acting on it.

Hence $\dfrac{dV_2}{dv} = \dfrac{K_1}{K_2}\dfrac{dV_1}{dv}$, and $\dfrac{K_1 - K_2}{K_2}\dfrac{dV_1}{dv} + 4\pi\sigma' = 0$,

$$\frac{dV_1}{dv} = \frac{4\pi\sigma' K_2}{K_1 - K_2}, \qquad \frac{dV_2}{dv} = \frac{4\pi\sigma' K_1}{K_1 - K_2}.$$

The surface-density σ' is that of the apparent electrification produced at the surface of the solid dielectric by induction. It disappears entirely when the inducing force is removed, but if during the action of the inducing force the apparent electrification of the surface is discharged by passing a flame over the surface, then, when the inducing force is taken away, there will appear an electrification opposite to σ' *.

In a heterogeneous dielectric in which K varies continuously, if ρ' be the apparent volume-density,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho' = 0.$$

Comparing this with the equation above, we find

$$4\pi(\rho - K\rho') + \frac{dK}{dx}\frac{dV}{dx} + \frac{dK}{dy}\frac{dV}{dy} + \frac{dK}{dz}\frac{dV}{dz} = 0.$$

The true electrification, indicated by ρ, in the dielectric whose variable inductive capacity is denoted by K, will produce the same potential at every point as the apparent electrification, indicated by ρ', would produce in a dielectric whose inductive capacity is everywhere equal to unity.

* See Faraday's 'Remarks on Static Induction,' *Proceedings of the Royal Institution*, Feb. 12, 1858.

CHAPTER III.

SYSTEMS OF CONDUCTORS.

On the Superposition of Electrical Systems.

84.] Let E_1 be a given electrified system of which the potential at a point P is V_1, and let E_2 be another electrified system of which the potential at the same point would be V_2 if E_1 did not exist. Then, if E_1 and E_2 exist together, the potential of the combined system will be $V_1 + V_2$.

Hence, if V be the potential of an electrified system E, if the electrification of every part of E be increased in the ratio of n to 1, the potential of the new system nE will be nV.

Energy of an Electrified System.

85.] Let the system be divided into parts, A_1, A_2, &c. so small that the potential in each part may be considered constant throughout its extent. Let e_1, e_2, &c. be the quantities of electricity in each of these parts, and let V_1, V_2, &c. be their potentials.

If now e_1 is altered to ne_1, e_2 to ne_2, &c., then the potentials will become nV_1, nV_2, &c.

Let us consider the effect of changing n into $n+dn$ in all these expressions. It will be equivalent to charging A_1 with a quantity of electricity $e_1\,dn$, A_2 with $e_2 dn$, &c. These charges must be supposed to be brought from a distance at which the electrical action of the system is insensible. The work done in bringing $e_1\,dn$ of electricity to A_1, whose potential before the charge is nV_1, and after the charge $(n+dn)\,V_1$, must lie between

$$nV_1\,e_1\,dn \quad \text{and} \quad (n+dn)V_1\,e_1\,dn.$$

In the limit we may neglect the square of dn, and write the expression
$$V_1\,e_1\,n\,dn.$$

Similarly the work required to increase the charge of A_2 is $V_2 e_2 n\, dn$, so that the whole work done in increasing the charge of the system is

$$(V_1 e_1 + V_2 e_2 + \&c.)\, n\, dn.$$

If we suppose this process repeated an indefinitely great number of times, each charge being indefinitely small, till the total effect becomes sensible, the work done will be

$$\Sigma(Ve)\int n\, dn = \tfrac{1}{2}\Sigma(Ve)(n_1^2 - n_0^2);$$

where $\Sigma(Ve)$ means the sum of all the products of the potential of each element into the quantity of electricity in that element when $n = 1$, and n_0 is the initial and n_1 the final value of n.

If we make $n_0 = 0$ and $n_1 = 1$, we find for the work required to charge an unelectrified system so that the electricity is e and the potential V in each element,

$$Q = \tfrac{1}{2}\Sigma(Ve).$$

General Theory of a System of Conductors.

86.] Let $A_1, A_2, \ldots A_n$ be any number of conductors of any form. Let the charge or total quantity of electricity on each of these be $E_1, E_2, \ldots E_n$, and let their potentials be $V_1, V_2, \ldots V_n$ respectively.

Let us suppose the conductors to be all insulated and originally free of charge, and at potential zero.

Now let A_1 be charged with unit of electricity, the other bodies being without charge. The effect of this charge on A_1 will be to raise the potential of A_1 to p_{11}, that of A_2 to p_{12}, and that of A_n to p_{1n}, where p_{11}, &c. are quantities depending on the form and relative position of the conductors. The quantity p_{11} may be called the Potential Coefficient of A_1 on itself, and p_{12} may be called the Potential Coefficient of A_1 on A_2, and so on.

If the charge upon A_1 is now made E_1, then, by the principle of superposition, we shall have

$$V_1 = p_{11} E_1 \;\ldots\ldots\; V_n = p_{1n} E_1.$$

Now let A_1 be discharged, and A_2 charged with unit of electricity, and let the potentials of $A_1, A_2, \ldots A_n$ be $p_{21}, p_{22}, \ldots p_{2n}$, then the potentials due to E_2 on A_2 will be

$$V_1 = p_{21} E_2 \;\ldots\ldots\; V_n = p_{2n} E_2.$$

Similarly let us denote the potential of A_s due to a unit charge on A_r by p_{rs}, and let us call p_{rs} the Potential Coefficient of A_r on A_s, then we shall have the following equations determining the potentials in terms of the charges:

$$
\begin{aligned}
V_1 &= p_{11} E_1 \ldots + p_{r1} E_r \ldots + p_{n1} E_n, \\
&- \; - \; - \; - \; - \; - \; - \\
V_s &= p_{1s} E_1 \ldots + p_{rs} E_r \ldots + p_{ns} E_n, \\
&- \; - \; - \; - \; - \; - \; - \\
V_n &= p_{1n} E_1 \ldots + p_{rn} E_r \ldots + p_{nn} E_n.
\end{aligned} \tag{1}
$$

We have here n linear equations containing n^2 coefficients of potential.

87.] By solving these equations for E_1, E_2, &c. we should obtain n equations of the form

$$
\begin{aligned}
E_1 &= q_{11} V_1 \ldots + q_{1s} V_s \ldots + q_{1n} V_n, \\
&- \; - \; - \; - \; - \; - \; - \\
E_r &= q_{r1} V_1 \ldots + q_{rs} V_s \ldots + q_{rn} V_n, \\
&- \; - \; - \; - \; - \; - \; - \\
E_n &= q_{n1} V_1 \ldots + q_{ns} V_s \ldots + q_{nn} V_n.
\end{aligned} \tag{2}
$$

The coefficients in these equations may be obtained directly from those in the former equations. They may be called Coefficients of Induction.

Of these q_{11} is numerically equal to the quantity of electricity on A_1 when A_1 is at potential unity and all the other bodies are at potential zero. This is called the Capacity of A_1. It depends on the form and position of all the conductors in the system.

Of the rest q_{rs} is the charge induced on A_r when A_s is maintained at potential unity and all the other conductors at potential zero. This is called the Coefficient of Induction of A_s on A_r.

The mathematical determination of the coefficients of potential and of capacity from the known forms and positions of the conductors is in general difficult. We shall afterwards prove that they have always determinate values, and we shall determine their values in certain special cases. For the present, however, we may suppose them to be determined by actual experiment.

Dimensions of these Coefficients.

Since the potential of an electrified point at a distance r is the charge of electricity divided by the distance, the ratio of a quantity of electricity to a potential may be represented by a line. Hence all the coefficients of capacity and induction (q) are of the nature of lines, and the coefficients of potential (p) are of the nature of the reciprocals of lines.

88.] THEOREM I. *The coefficients of A_r relative to A_s are equal to those of A_s relative to A_r.*

If E_r, the charge on A_r, is increased by δE_r, the work spent in bringing δE_r from an infinite distance to the conductor A_r whose potential is V_r, is by the definition of potential in Art. 70,

$$V_r \delta E_r,$$

and this expresses the increment of the electric energy caused by this increment of charge.

If the charges of the different conductors are increased by δE_1, &c., the increment of the electric energy of the system will be

$$\delta Q = V_1 \delta E_1 + \&c. + V_r \delta E_r + \&c.$$

If, therefore, the electric energy Q is expressed as a function of the charges E_1, E_2, &c., the potential of any conductor may be expressed as the partial differential coefficient of this function with respect to the charge on that conductor, or

$$V_r = \left(\frac{dQ}{dE_r}\right) \cdots\cdots V_s = \left(\frac{dQ}{dE_s}\right).$$

Since the potentials are linear functions of the charges, the energy must be a quadratic function of the charges. If we put

$$C E_r E_s$$

for the term in the expansion of Q which involves the product $E_r E_s$, then, by differentiating with respect to E_s, we find the term of the expansion of V_s which involves E_r to be CE_r.

Differentiating with respect to E_r, we find the term in the expansion of V_r which involves E_s to be CE_s.

Comparing these results with equations (1), Art. 86, we find

$$p_{rs} = C = p_{sr},$$

or, interpreting the symbols p_{rs} and p_{sr} :—

The potential of A_s due to a unit charge on A_r is equal to the potential of A_r due to a unit charge on A_s.

This reciprocal property of the electrical action of one conductor on another was established by Helmholtz and Sir W. Thomson.

If we suppose the conductors A_r and A_s to be indefinitely small, we have the following reciprocal property of any two points :—

The potential at any point A_s, due to unit of electricity placed at A_r in presence of any system of conductors, is a function of the positions of A_r and A_s in which the coordinates of A_r and of A_s enter in the same manner, so that the value of the function is unchanged if we exchange A_r and A_s.

This function is known by the name of Green's Function.

The coefficients of induction q_{rs} and q_{sr} are also equal. This is easily seen from the process by which these coefficients are obtained from the coefficients of potential. For, in the expression for q_{rs}, p_{rs} and p_{sr} enter in the same way as p_{sr} and p_{rs} do in the expression for q_{sr}. Hence if all pairs of coefficients p_{rs} and p_{sr} are equal, the pairs q_{rs} and q_{sr} are also equal.

89.] THEOREM II. *Let a charge E_r be placed on A_r, and let all the other conductors be at potential zero, and let the charge induced on A_s be $-n_{rs}E_r$, then if A_r is discharged and insulated, and A_s brought to potential V_s, the other conductors being at potential zero, then the potential of A_r will be $+n_{rs}V_s$.*

For, in the first case, if V_r is the potential of A_r, we find by equations (2),

$$E_s = q_{rs}V_r, \quad \text{and} \quad E_r = q_{rr}V_r.$$

$$\text{Hence} \qquad E_s = \frac{q_{rs}}{q_{rr}}E_r, \quad \text{and} \quad n_{rs} = -\frac{q_{rs}}{q_{rr}}.$$

In the second case, we have

$$E_r = 0 = q_{rr}V_r + q_{rs}V_s.$$

$$\text{Hence} \qquad V_r = -\frac{q_{rs}}{q_{rr}}V_s = n_{rs}V_s.$$

From this follows the important theorem, due to Green:—

If a charge unity, placed on the conductor A_0 in presence of conductors A_1, A_2, &c. at potential zero induces charges $-n_1$, $-n_2$, &c. in these conductors, then, if A_0 is discharged and insulated, and these conductors are maintained at potentials V_1, V_2, &c., the potential of A_0 will be

$$n_1V_1 + n_2V_2 + \&c.$$

The quantities (n) are evidently numerical quantities, or ratios.

The conductor A_0 may be supposed reduced to a point, and A_1, A_2, &c. need not be insulated from each other, but may be different elementary portions of the surface of the same conductor. We shall see the application of this principle when we investigate Green's Functions.

90.] THEOREM III. *The coefficients of potential are all positive, but none of the coefficients p_{rs} is greater than p_{rr} or p_{ss}.*

For let a charge unity be communicated to A_r, the other conductors being uncharged. A system of equipotential surfaces will be formed. Of these one will be the surface of A_r, and its potential will be p_{rr}. If A_s is placed in a hollow excavated in A_r so as to be completely enclosed by it, then the potential of A_s will also be p_{rr}.

If, however, A_s is outside of A_r its potential p_{rs} will lie between p_{rr} and zero.

For consider the lines of force issuing from the charged conductor A_r. The charge is measured by the excess of the number of lines which issue from it over those which terminate in it. Hence, if the conductor has no charge, the number of lines which enter the conductor must be equal to the number which issue from it. The lines which enter the conductor come from places of greater potential, and those which issue from it go to places of less potential. Hence the potential of an uncharged conductor must be intermediate between the highest and lowest potentials in the field, and therefore the highest and lowest potentials cannot belong to any of the uncharged bodies.

The highest potential must therefore be p_{rr}, that of the charged body A_r, and the lowest must be that of space at an infinite distance, which is zero, and all the other potentials such as p_{rs} must lie between p_{rr} and zero.

If A_s completely surrounds A_t, then $p_{rs} = p_{rt}$.

91.] THEOREM IV. *None of the coefficients of induction are positive, and the sum of all those belonging to a single conductor is not numerically greater than the coefficient of capacity of that conductor, which is always positive.*

For let A_r be maintained at potential unity while all the other conductors are kept at potential zero, then the charge on A_r is q_{rr}, and that on any other conductor A_s is q_{rs}.

The number of lines of force which issue from A_r is p_{rr}. Of these some terminate in the other conductors, and some may proceed to infinity, but no lines of force can pass between any of the other conductors or from them to infinity, because they are all at potential zero.

No line of force can issue from any of the other conductors such as A_s, because no part of the field has a lower potential than A_s. If A_s is completely cut off from A_r by the closed surface of one of the conductors, then q_{rs} is zero. If A_s is not thus cut off, q_{rs} is a negative quantity.

If one of the conductors A_t completely surrounds A_r, then all the lines of force from A_r fall on A_t and the conductors within it, and the sum of the coefficients of induction of these conductors with respect to A_r will be equal to q_{rr} with its sign changed. But if A_r is not completely surrounded by a conductor the arithmetical sum of the coefficients of induction q_{rs}, &c. will be less than q_{rr}.

We have deduced these two theorems independently by means of electrical considerations. We may leave it to the mathematical student to determine whether one is a mathematical consequence of the other.

Resultant Mechanical Force on any Conductor in terms of the Charges.

92.] Let $\delta\phi$ be any mechanical displacement of the conductor, and let Φ be the the component of the force tending to produce that displacement, then $\Phi\,\delta\phi$ is the work done by the force during the displacement. If this work is derived from the electrification of the system, then if Q is the electric energy of the system,

$$\Phi\,\delta\phi + \delta Q = 0, \tag{3}$$

or
$$\Phi = -\frac{\delta Q}{\delta\phi}. \tag{4}$$

Here
$$Q = \tfrac{1}{2}(E_1 V_1 + E_2 V_2 + \&c.) \tag{5}$$

If the bodies are insulated, the variation of Q must be such that E_1, E_2, &c. remain constant. Substituting therefore for the values of the potentials, we have

$$Q = \tfrac{1}{2}\Sigma_r\,\Sigma_s\,(E_r E_s p_{rs}), \tag{6}$$

where the symbol of summation Σ includes all terms of the form within the brackets, and r and s may each have any values from 1 to n. From this we find

$$\Phi = -\frac{dQ}{d\phi} = -\tfrac{1}{2}\Sigma_r\,\Sigma_s\left(E_r E_s \frac{dp_{rs}}{d\phi}\right) \tag{7}$$

as the expression for the component of the force which produces variation of the generalized coordinate ϕ.

Resultant Mechanical Force in terms of the Potentials.

93.] The expression for Φ in terms of the charges is

$$\Phi = -\tfrac{1}{2}\Sigma_r\,\Sigma_s\left(E_r E_s \frac{dp_{rs}}{d\phi}\right), \tag{8}$$

where in the summation r and s have each every value in succession from 1 to n.

Now $E_r = \Sigma_1^t (V_t\, q_{rt})$ where t may have any value from 1 to n, so that

$$\Phi = -\tfrac{1}{2}\Sigma_r \Sigma_s \Sigma_t \left(E_s V_t q_{rt} \frac{dp_{rs}}{d\phi}\right). \tag{9}$$

Now the coefficients of potential are connected with those of induction by n equations of the form

$$\Sigma_r (p_{ar} q_{ar}) = 1, \tag{10}$$

and $\frac{1}{2}n(n-1)$ of the form

$$\Sigma_r (p_{ar} q_{br}) = 0. \tag{11}$$

Differentiating with respect to ϕ we get $\frac{1}{2}n(n+1)$ equations of the form

$$\Sigma_r \left(p_{ar} \frac{dq_{br}}{d\phi}\right) + \Sigma_r \left(q_{br} \frac{dp_{ar}}{d\phi}\right) = 0, \tag{12}$$

where a and b may be the same or different.

Hence, putting a and b equal to r and s,

$$\Phi = \tfrac{1}{2}\Sigma_r \Sigma_s \Sigma_t \left(E_s V_t p_{rs} \frac{dq_{rt}}{d\phi}\right), \tag{13}$$

but $\Sigma_s (E_s p_{rs}) = V_r$, so that we may write

$$\Phi = \tfrac{1}{2}\Sigma_r \Sigma_t \left(V_r V_t \frac{dq_{rt}}{d\phi}\right), \tag{14}$$

where r and t may have each every value in succession from 1 to n. This expression gives the resultant force in terms of the potentials.

If each conductor is connected with a battery or other contrivance by which its potential is maintained constant during the displacement, then this expression is simply

$$\Phi = \frac{dQ}{d\phi}, \tag{15}$$

under the condition that all the potentials are constant.

The work done in this case during the displacement $\delta\phi$ is $\Phi\,\delta\phi$, and the electrical energy of the system of conductors is increased by δQ; hence the energy spent by the batteries during the displacement is

$$\Phi\,\delta\phi + \delta Q = 2\Phi\,\delta\phi = 2\,\delta Q. \tag{16}$$

It appears from Art. 92, that the resultant force Φ is equal to $-\frac{dQ}{d\phi}$, under the condition that the charges of the conductors are constant. It is also, by Art. 93, equal to $\frac{dQ}{d\phi}$, under the condition that the potentials of the conductors are constant. If the conductors are insulated, they tend to move so that their energy is diminished, and the work done by the electrical forces during the displacement is equal to the diminution of energy.

If the conductors are connected with batteries, so that their potentials are maintained constant, they tend to move so that the energy of the system is increased, and the work done by the electrical forces during the displacement is equal to the increment of the energy of the system. The energy spent by the batteries is equal to double of either of these quantities, and is spent half in mechanical, and half in electrical work.

On the Comparison of Similar Electrified Systems.

94.] If two electrified systems are similar in a geometrical sense, so that the lengths of corresponding lines in the two systems are as L to L', then if the dielectric which separates the conducting bodies is the same in both systems, the coefficients of induction and of capacity will be in the proportion of L to L'. For if we consider corresponding portions, A and A', of the two systems, and suppose the quantity of electricity on A to be E, and that on A' to be E', then the potentials V and V' at corresponding points B and B', due to this electrification, will be

$$V = \frac{E}{AB}, \text{ and } V' = \frac{E'}{A'B'}.$$

But AB is to $A'B'$ as L to L', so that we must have

$$E : E' :: LV : L'V'.$$

But if the inductive capacity of the dielectric is different in the two systems, being K in the first and K' in the second, then if the potential at any point of the first system is to that at the corresponding point of the second as V to V', and if the quantities of electricity on corresponding parts are as E to E', we shall have

$$E : E' :: LVK : L'V'K'.$$

By this proportion we may find the relation between the total electrification of corresponding parts of two systems, which are in the first place geometrically similar, in the second place composed of dielectric media of which the dielectric inductive capacity at corresponding points is in the proportion of K to K', and in the third place so electrified that the potentials of corresponding points are as V to V'.

From this it appears that if q be any coefficient of capacity or induction in the first system, and q' the corresponding one in the second,

$$q : q' :: LK : L'K';$$

and if p and p' denote corresponding coefficients of potential in the two systems,

$$p : p' :: \frac{1}{LK} : \frac{1}{L'K'}.$$

If one of the bodies be displaced in the first system, and the corresponding body in the second system receive a similar displacement, then these displacements are in the proportion of L to L', and if the forces acting on the two bodies are as F to F', then the work done in the two systems will be as FL to $F'L'$.

But the total electrical energy is half the sum of the quantities of electricity multiplied each by the potential of the electrified body, so that in the similar systems, if Q and Q' be the total electrical energy,

$$Q : Q' :: EV : E'V',$$

and the difference of energy after similar displacements in the two systems will be in the same proportion. Hence, since FL is proportional to the electrical work done during the displacement,

$$FL : F'L' :: EV : E'V'.$$

Combining these proportions, we find that the ratio of the resultant force on any body of the first system to that on the corresponding body of the second system is

$$F : F' :: V^2K : V'^2K',$$

or

$$F : F' :: \frac{E^2}{L^2K} : \frac{E'^2}{L'^2K'}.$$

The first of these proportions shews that in similar systems the force is proportional to the square of the electromotive force and to the inductive capacity of the dielectric, but is independent of the actual dimensions of the system.

Hence two conductors placed in a liquid whose inductive capacity is greater than that of air, and electrified to given potentials, will attract each other more than if they had been electrified to the same potentials in air.

The second proportion shews that if the quantity of electricity on each body is given, the forces are proportional to the squares of the electrifications and inversely to the squares of the distances, and also inversely to the inductive capacities of the media.

Hence, if two conductors with given charges are placed in a liquid whose inductive capacity is greater than that of air, they will attract each other less than if they had been surrounded with air and electrified with the same charges of electricity.

CHAPTER IV.

GENERAL THEOREMS.

95.] In the preceding chapter we have calculated the potential function and investigated its properties on the hypothesis that there is a direct action at a distance between electrified bodies, which is the resultant of the direct actions between the various electrified parts of the bodies.

If we call this the direct method of investigation, the inverse method will consist in assuming that the potential is a function characterised by properties the same as those which we have already established, and investigating the form of the function.

In the direct method the potential is calculated from the distribution of electricity by a process of integration, and is found to satisfy certain partial differential equations. In the inverse method the partial differential equations are supposed given, and we have to find the potential and the distribution of electricity.

It is only in problems in which the distribution of electricity is given that the direct method can be used. When we have to find the distribution on a conductor we must make use of the inverse method.

We have now to shew that the inverse method leads in every case to a determinate result, and to establish certain general theorems deduced from Poisson's partial differential equation

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} + 4\pi\rho = 0.$$

The mathematical ideas expressed by this equation are of a different kind from those expressed by the equation

$$V = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho}{r}\, dx'\, dy'\, dz'.$$

In the differential equation we express that the values of the second derivatives of V in the neighbourhood of any point, and the density at that point are related to each other in a certain manner, and no relation is expressed between the value of V at that point and the value of ρ at any point at a sensible distance from it.

In the second expression, on the other hand, the distance between the point (x', y', z') at which ρ exists from the point (x, y, z) at which V exists is denoted by r, and is distinctly recognised in the expression to be integrated.

The integral, therefore, is the appropriate mathematical expression for a theory of action between particles at a distance, whereas the differential equation is the appropriate expression for a theory of action exerted between contiguous parts of a medium.

We have seen that the result of the integration satisfies the differential equation. We have now to shew that it is the only solution of that equation fulfilling certain conditions.

We shall in this way not only establish the mathematical equivalence of the two expressions, but prepare our minds to pass from the theory of direct action at a distance to that of action between contiguous parts of a medium.

Characteristics of the Potential Function.

96.] The potential function V, considered as derived by integration from a known distribution of electricity either in the substance of bodies with the volume-density ρ or on certain surfaces with the surface-density σ, ρ and σ being everywhere finite, has been shewn to have the following characteristics :—

(1) V is finite and continuous throughout all space.

(2) V vanishes at an infinite distance from the electrified system.

(3) The first derivatives of V are finite throughout all space, and continuous except at the electrified surfaces.

(4) At every point of space, except on the electrified surfaces, the equation of Poisson

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} + 4\pi\rho = 0$$

is satisfied. We shall refer to this equation as the General Characteristic equation.

At every point where there is no electrification this equation becomes the equation of Laplace,

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0.$$

(5) At any point of an electrified surface at which the surface-density is σ, the first derivative of V, taken with respect to the normal to the surface, changes its value abruptly at the surface, so that

$$\frac{dV'}{d\nu'} + \frac{dV}{d\nu} + 4\pi\sigma = 0,$$

where ν and ν' are the normals on either side of the surface, and V and V' are the corresponding potentials. We shall refer to this equation as the Superficial Characteristic equation.

(6) If V denote the potential at a point whose distance from any fixed point in a finite electrical system is r, then the product Vr, when r increases indefinitely, is ultimately equal to E, the total charge in the finite system.

97.] *Lemma.* Let V be any continuous function of x, y, z, and let u, v, w be functions of x, y, z, subject to the general solenoidal condition

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \tag{1}$$

where these functions are continuous, and to the superficial solenoidal condition

$$l(u_1 - u_2) + m(v_1 - v_2) + n(w_1 - w_2) = 0, \tag{2}$$

at any surface at which these functions become discontinuous, l, m, n being the direction-cosines of the normal to the surface, and u_1, v_1, w_1 and u_2, v_2, w_2 the values of the functions on opposite sides of the surface, then the triple integral

$$M = \iiint \left(u\frac{dV}{dx} + v\frac{dV}{dy} + w\frac{dV}{dz}\right) dx\, dy\, dz \tag{3}$$

vanishes when the integration is extended over a space bounded by surfaces at which either V is constant, or

$$lu + mv + nw = 0, \tag{4}$$

l, m, n, being the direction-cosines of the surface.

Before proceeding to prove this theorem analytically we may observe, that if u, v, w be taken to represent the components of the velocity of a homogeneous incompressible fluid of density unity, and if V be taken to represent the potential at any point of space of forces acting on the fluid, then the general and superficial equations of continuity ((1) and (2)) indicate that every part of the space is, and continues to be, full of the fluid, and equation (4) is the condition to be fulfilled at a surface through which the fluid does not pass.

The integral M represents the work done by the fluid against the forces acting on it in unit of time.

Now, since the forces which act on the fluid are derived from the potential function V, the work which they do is subject to the law of conservation of energy, and the work done on the whole fluid within a certain space may be found if we know the potential at the points where each line of flow enters the space and where it issues from it. The excess of the second of these potentials over the first, multiplied by the quantity of fluid which is transmitted along each line of flow, will give the work done by that portion of the fluid, and the sum of all such products will give the whole work.

Now, if the space be bounded by a surface for which $V = C$, a constant quantity, the potential will be the same at the place where any line of flow enters the space and where it issues from it, so that in this case no work will be done by the forces on the fluid within the space, and $M = 0$.

Secondly, if the space be bounded in whole or in part by a surface satisfying equation (4), no fluid will enter or leave the space through this surface, so that no part of the value of M can depend on this part of the surface.

The quantity M is therefore zero for a space bounded externally by the closed surface $V = C$, and it remains zero though any part of this space be cut off from the rest by surfaces fulfilling the condition (4).

The analytical expression of the process by which we deduce the work done in the interior of the space from that which takes place at the bounding surface is contained in the following method of integration by parts.

Taking the first term of the integral M,

$$\iiint u \frac{dV}{dx}\, dx\, dy\, dz = \iint \Sigma (uV)\, dy\, dz - \iiint V \frac{du}{dx}\, dx\, dy\, dz,$$

where $$\Sigma (uV) = u_1 V_1 - u_2 V_2 + u_3 V_3 - u_4 V_4 + \&c.;$$

and where $u_1 V_1$, $u_2 V_2$, &c. are the values of u and v at the points whose coordinates are (x_1, y, z), (x_2, y, z), &c., x_1, x_2, &c. being the values of x where the ordinate cuts the bounding surface or surfaces, arranged in descending order of magnitude.

Adding the two other terms of the integral M, we find

$$M = \iint \Sigma (uV)\, dy\, dz + \iint \Sigma (vV)\, dz\, dx + \iint \Sigma (wV)\, dx\, dy$$
$$- \iiint V \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) dx\, dy\, dz.$$

If l, m, n are the direction-cosines of the normal drawn inwards from the bounding surface at any point, and dS an element of that surface, then we may write

$$M = -\iint V(lu+mv+nw)\,dS - \iiint V\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)dx\,dy\,dz;$$

the integration of the first term being extended over the bounding surface, and that of the second throughout the entire space.

For all spaces within which u, v, w are continuous, the second term vanishes in virtue of equation (1). If for any surface within the space u, v, w are discontinuous but subject to equation (2), we find for the part of M depending on this surface,

$$M_1 = -\iint V_1(l_1 u_1 + m_1 v_1 + n_1 w_1)\,dS_1,$$

$$M_2 = -\iint V_2(l_2 u_2 + m_2 v_2 + n_2 w_2)\,dS_2;$$

where the suffixes $_1$ and $_2$, applied to any symbol, indicate to which of the two spaces separated by the surface the symbol belongs.

Now, since V is continuous, we have at every point of the surface,

$$V_1 = V_2 = V;$$

we have also

$$dS_1 = dS_2 = dS;$$

but since the normals are drawn in opposite directions, we have

$$l_1 = -l_2 = l, \qquad m_1 = -m_2 = m, \qquad n_1 = -n_2 = n;$$

so that the total value of M, so far as it depends on the surface of discontinuity, is

$$M_1 + M_2 = -\iint V\left(l(u_1-u_2) + m(v_1-v_2) + n(w_1-w_2)\right)dS.$$

The quantity under the integral sign vanishes at every point in virtue of the superficial solenoidal condition or characteristic (2).

Hence, in determining the value of M, we have only to consider the surface-integral over the actual bounding surface of the space considered, or

$$M = -\iint V(lu+mv+nw)\,dS.$$

Case 1. If V is constant over the whole surface and equal to C,

$$M = -C\iint (lu+mv+nw)\,dS.$$

The part of this expression under the sign of double integration represents the surface-integral of the flux whose components are u, v, w, and by Art. 21 this surface-integral is zero for the closed surface in virtue of the general and superficial solenoidal conditions (1) and (2).

Hence $M = 0$ for a space bounded by a single equipotential surface.

If the space is bounded externally by the surface $V = C$, and internally by the surfaces $V = C_1$, $V = C_2$, &c., then the total value of M for the space so bounded will be

$$M - M_1 - M_2 \text{ \&c.},$$

where M is the value of the integral for the whole space within the surface $V = C$, and M_1, M_2 are the values of the integral for the spaces within the internal surfaces. But we have seen that M, M_1, M_2, &c. are each of them zero, so that the integral is zero also for the periphractic region between the surfaces.

Case 2. If $lu + mv + nw$ is zero over any part of the bounding surface, that part of the surface can contribute nothing to the value of M, because the quantity under the integral sign is everywhere zero. Hence M will remain zero if a surface fulfilling this condition is substituted for any part of the bounding surface, provided that the remainder of the surface is all at the same potential.

98.] We are now prepared to prove a theorem which we owe to Sir William Thomson *.

As we shall require this theorem in various parts of our subject, I shall put it in a form capable of the necessary modifications.

Let a, b, c be any functions of x, y, z (we may call them the components of a flux) subject only to the condition

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} + 4\pi\rho = 0, \tag{5}$$

where ρ has given values within a certain space. This is the general characteristic of a, b, c.

Let us also suppose that at certain surfaces (S) a, b, and c are discontinuous, but satisfy the condition

$$l(a_1 - a_2) + m(b_1 - b_2) + n(c_1 - c_2) + 4\pi\sigma = 0; \tag{6}$$

where l, m, n are the direction-cosines of the normal to the surface, a_1, b_1, c_1 the values of a, b, c on the positive side of the surface, and a_2, b_2, c_2 those on the negative side, and σ a quantity given for every point of the surface. This condition is the superficial characteristic of a, b, c.

Next, let us suppose that V is a continuous function of x, y, z, which either vanishes at infinity or whose value at a certain point is given, and let V satisfy the general characteristic equation

* *Cambridge and Dublin Mathematical Journal*, February, 1848.

$$\frac{d}{dx}K\frac{dV}{dx}+\frac{d}{dy}K\frac{dV}{dy}+\frac{d}{dz}K\frac{dV}{dz}+4\pi\rho=0; \qquad (7)$$

and the superficial characteristic at the surfaces (S),

$$l\left(K_1\frac{dV_1}{dx}-K_2\frac{dV_2}{dx}\right)+m\left(K_1\frac{dV_1}{dy}-K_2\frac{dV_2}{dy}\right)$$
$$+n\left(K_1\frac{dV_1}{dz}-K_2\frac{dV_2}{dz}\right)+4\pi\sigma=0, \qquad (8)$$

K being a quantity which may be positive or zero but not negative, given at every point of space.

Finally, let $8\pi Q$ represent the triple integral

$$8\pi Q=\iiint\frac{1}{K}(a^2+b^2+c^2)\,dx\,dy\,dz, \qquad (9)$$

extended over a space bounded by surfaces, for each of which either

$$V = \text{constant},$$

or
$$la+mb+nc = Kl\frac{dV}{dx}+Km\frac{dV}{dy}+Kn\frac{dV}{dz}=q, \qquad (10)$$

where the value of q is given at every point of the surface; then, if a, b, c be supposed to vary in any manner, subject to the above conditions, the value of Q will be a *unique minimum*, when

$$a=K\frac{dV}{dx}, \quad b=K\frac{dV}{dy}, \quad c=K\frac{dV}{dz}. \qquad (11)$$

Proof.

If we put for the general values of a, b, c,

$$a=K\frac{dV}{dx}+u, \quad b=K\frac{dV}{dy}+v, \quad c=K\frac{dV}{dz}+w; \qquad (12)$$

then, by substituting these values in equations (5) and (7), we find that u, v, w satisfy the general solenoidal condition

$$(1) \quad \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

We also find, by equations (6) and (8), that at the surfaces of discontinuity the values of u_1, v_1, w_1 and u_2, v_2, w_2 satisfy the superficial solenoidal condition

$$(2) \quad l(u_1-u_2)+m(v_1-v_2)+n(w_1-w_2)=0.$$

The quantities u, v, w, therefore, satisfy at every point the solenoidal conditions as stated in the preceding lemma.

We may now express Q in terms of u, v, w and V,

$$\pi\, Q = \iiint K \left(\left.\overline{\frac{dV}{dx}}\right|^2 + \left.\overline{\frac{dV}{dy}}\right|^2 + \left.\overline{\frac{dV}{dz}}\right|^2 \right) dx\, dy\, dz + \iiint \frac{1}{K} (u^2+v^2+w^2)\, dx\, dy\, dz$$
$$+ 2 \iiint \left(u \frac{dV}{dx} + v \frac{dV}{dy} + w \frac{dV}{dz} \right) dx\, dy\, dz. \quad (13)$$

The last term of Q may be written $2M$, where M is the quantity considered in the lemma, and which we proved to be zero when the space is bounded by surfaces, each of which is either equipotential or satisfies the condition of equation (10), which may be written

$$(4) \quad lu + mv + nw = 0.$$

Q is therefore reduced to the sum of the first and second terms.

In each of these terms the quantity under the sign of integration consists of the sum of three squares, and is therefore essentially positive or zero. Hence the result of integration can only be positive or zero.

Let us suppose the function V known, and let us find what values of u, v, w will make Q a minimum.

If we assume that at every point $u = 0$, $v = 0$, and $w = 0$, these values fulfil the solenoidal conditions, and the second term of Q is zero, and Q is then a minimum as regards the variation of u, v, w.

For if any of these quantities had at any point values differing from zero, the second term of Q would have a positive value, and Q would be greater than in the case which we have assumed.

But if $u = 0$, $v = 0$, and $w = 0$, then

$$(11) \quad a = K\frac{dV}{dx}, \quad b = K\frac{dV}{dy}, \quad c = K\frac{dV}{dz}.$$

Hence these values of a, b, c make Q a minimum.

But the values of a, b, c, as expressed in equations (12), are perfectly general, and include all values of these quantities consistent with the conditions of the theorem. Hence, no other values of a, b, c can make Q a minimum.

Again, Q is a quantity essentially positive, and therefore Q is always capable of a minimum value by the variation of a, b, c. Hence the values of a, b, c which make Q a minimum must have a real existence. It does not follow that our mathematical methods are sufficiently powerful to determine them.

Corollary I. If a, b, c and K are given at every point of space, and if we write

$$(12)\quad a = K\frac{dV}{dx}+u,\quad b = K\frac{dV}{dy}+v,\quad c = K\frac{dV}{dz}+w,$$

with the condition (1)

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0,$$

then V, u, v, w can be found without ambiguity from these four equations.

Corollary II. The general characteristic equation

$$\frac{d}{dx}K\frac{dV}{dx}+\frac{d}{dy}K\frac{dV}{dy}+\frac{d}{dz}K\frac{dV}{dz}+4\pi\rho=0,$$

where V is a finite quantity of single value whose first derivatives are finite and continuous except at the surface S, and at that surface fulfil the superficial characteristic

$$l\left(K_1\frac{dV_1}{dx}-K_2\frac{dV_2}{dx}\right)+m\left(K_1\frac{dV_1}{dy}-K_2\frac{dV_2}{dy}\right)$$
$$+n\left(K_1\frac{dV_1}{dz}-K_2\frac{dV_2}{dz}\right)+4\pi\sigma=0,$$

can be satisfied by one value of V, and by one only, in the following cases.

Case 1. When the equations apply to the space within any closed surface at every point of which $V=C$.

For we have proved that in this case a, b, c have real and unique values which determine the first derivatives of V, and hence, if different values of V exist, they can only differ by a constant. But at the surface V is given equal to C, and therefore V is determinate throughout the space.

As a particular case, let us suppose a space within which $\rho=0$ bounded by a closed surface at which $V=C$. The characteristic equations are satisfied by making $V=C$ for every point within the space, and therefore $V=C$ is the only solution of the equations.

Case 2. When the equations apply to the space within any closed surface at every point of which V is given.

For if in this case the characteristic equations could be satisfied by two different values of V, say V and V', put $U=V-V'$, then subtracting the characteristic equation in V' from that in V, we find a characteristic equation in U. At the closed surface $U=0$ because at the surface $V=V'$, and within the surface the density is zero because $\rho=\rho'$. Hence, by Case 1, $U=0$ throughout the enclosed space, and therefore $V=V'$ throughout this space.

Case 3. When the equations apply to a space bounded by a closed surface consisting of two parts, in one of which V is given at every point, and in the other

$$Kl\frac{dV}{dx}+Km\frac{dV}{dy}+Kn\frac{dV}{dz}=q,$$

where q is given at every point.

For if there are two values of V, let U' represent, as before, their difference, then we shall have the equation fulfilled within a closed surface consisting of two parts, in one of which $U'=0$, and in the other

$$l\frac{dU'}{dx}+m\frac{dU'}{dy}+n\frac{dU'}{dz}=0;$$

and since $U'=0$ satisfies the equation it is the only solution, and therefore there is but one value of V possible.

Note.—The function V in this theorem is restricted to one value at each point of space. If multiple values are admitted, then, if the space considered is a cyclic space, the equations may be satisfied by values of V containing terms with multiple values. Examples of this will occur in Electromagnetism.

99.] To apply this theorem to determine the distribution of electricity in an electrified system, we must make $K=1$ throughout the space occupied by air, and $K=\infty$ throughout the space occupied by conductors. If any part of the space is occupied by dielectrics whose inductive capacity differs from that of air, we must make K in that part of the space equal to the specific inductive capacity.

The value of V, determined so as to fulfil these conditions, will be the only possible value of the potential in the given system.

Green's Theorem shews that the quantity Q, when it has its minimum value corresponding to a given distribution of electricity, represents the potential energy of that distribution of electricity. See Art. 100, equation (11).

In the form in which Q is expressed as the result of integration over every part of the field, it indicates that the energy due to the electrification of the bodies in the field may be considered as the result of the summation of a certain quantity which exists in every part of the field where electrical force is in action, whether electrification be present or not in that part of the field.

The mathematical method, therefore, in which Q, the symbol of electrical energy, is made an object of study, instead of ρ, the symbol of electricity itself, corresponds to the method of physical speculation, in which we look for the seat of electrical action in every part of the field, instead of confining our attention to the electrified bodies.

The fact that Q attains a minimum value when the components of the electric force are expressed in terms of the first derivatives of a potential, shews that, if it were possible for the electric force to be distributed in any other manner, a mechanical force would be brought into play tending to bring the distribution of force into its actual state. The actual state of the electric field is therefore a state of stable equilibrium, considered with reference to all variations of that state consistent with the actual distribution of free electricity.

Green's Theorem.

100.] The following remarkable theorem was given by George Green in his essay 'On the Application of Mathematics to Electricity and Magnetism.'

I have made use of the coefficient K, introduced by Thomson, to give greater generality to the statement, and we shall find as we proceed that the theorem may be modified so as to apply to the most general constitution of crystallized media.

We shall suppose that U and V are two functions of x, y, z, which, with their first derivatives, are finite and continuous within the space bounded by the closed surface S.

We shall also put for conciseness

$$\frac{d}{dx} K \frac{dU}{dx} + \frac{d}{dy} K \frac{dU}{dy} + \frac{d}{dz} K \frac{dU}{dz} = -4\pi\rho, \tag{1}$$

and

$$\frac{d}{dy} K \frac{dV}{dx} + \frac{d}{dy} K \frac{dV}{dy} + \frac{d}{dz} K \frac{dV}{dz} = -4\pi\rho', \tag{2}$$

where K is a real quantity, given for each point of space, which may be positive or zero but not negative. The quantities ρ and ρ' correspond to volume-densities in the theory of potentials, but in this investigation they are to be considered simply as abbreviations for the functions of U and V to which they are here equated.

In the same way we may put

$$lK \frac{dU}{dx} + mK \frac{dU}{dy} + nK \frac{dU}{dz} = 4\pi\sigma, \tag{3}$$

and

$$lK \frac{dV}{dx} + mK \frac{dV}{dy} + nK \frac{dV}{dz} = 4\pi\sigma', \tag{4}$$

where l, m, n are the direction-cosines of the normal drawn inwards from the surface S. The quantities σ and σ' correspond to superficial densities, but at present we must consider them as defined by the above equations.

Green's Theorem is obtained by integrating by parts the expression

$$4\pi M = \iiint K\left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz}\right) dx\, dy\, dz \tag{5}$$

throughout the space within the surface S.

If we consider $\frac{dV}{dx}$ as a component of a force whose potential is V, and $K\frac{dU}{dx}$ as a component of a flux, the expression will give the work done by the force on the flux.

If we apply the method of integration by parts, we find

$$4\pi M = \iint VK\left(l\frac{dU}{dx} + m\frac{dU}{dy} + n\frac{dU}{dz}\right) dS$$
$$-\iiint V\left(\frac{d}{dx}K\frac{dU}{dx} + \frac{d}{dy}K\frac{dU}{dy} + \frac{d}{dz}K\frac{dU}{dz}\right) dx\,dy\,dz\,; \tag{6}$$

$$\text{or} \quad 4\pi M = \iint 4\pi\sigma' V\, dS + \iiint 4\pi\rho' V\, dx\, dy\, dz. \tag{7}$$

In precisely the same manner by exchanging U and V, we should find

$$4\pi M = +\iint 4\pi\sigma U\, dS + \iiint 4\pi\rho U\, dx\, dy\, dz. \tag{8}$$

The statement of Green's Theorem is that these three expressions for M are identical, or that

$$M = \iint \sigma' V\, dS + \iiint \rho' V\, dx\, dy\, dz = \iint \sigma U\, dS + \iiint \rho U\, dx\, dy\, dz$$
$$= \frac{1}{4\pi}\iiint K\left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz}\right) dx\, dy\, dz. \tag{9}$$

Correction of Green's Theorem for Cyclosis.

There are cases in which the resultant force at any point of a certain region fulfils the ordinary condition of having a potential, while the potential itself is a many-valued function of the coordinates. For instance, if

$$X = \frac{y}{x^2+y^2}, \quad Y = -\frac{x}{x^2+y^2}, \quad Z = 0,$$

we find $V = \tan^{-1}\frac{y}{x}$, a many-valued function of x and y, the values of V forming an arithmetical series whose common difference is 2π, and in order to define which of these is to be taken in any particular case we must make some restriction as to the line along which we are to integrate the force from the point where $V = 0$ to the required point.

In this case the region in which the condition of having a potential is fulfilled is the cyclic region surrounding the axis of z, this axis being a line in which the forces are infinite and therefore not itself included in the region.

The part of the infinite plane of xz for which x is positive may be taken as a diaphragm of this cyclic region. If we begin at a point close to the positive side of this diaphragm, and integrate along a line which is restricted from passing through the diaphragm, the line-integral will be restricted to that value of V which is positive but less than 2π.

Let us now suppose that the region bounded by the closed surface S in Green's Theorem is a cyclic region of any number of cycles, and that the function V is a many-valued function having any number of cyclic constants.

The quantities $\frac{dV}{dx}$, $\frac{dV}{dy}$, and $\frac{dV}{dz}$ will have definite values at all points within S, so that the volume-integral

$$\iiint K\left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz}\right)$$

has a definite value, σ and ρ have also definite values, so that if U is a single valued function, the expression

$$\iint \sigma U\, dS + \iiint \rho U\, dx\, dy\, dz$$

has also a definite value.

The expression involving V has no definite value as it stands, for V is a many-valued function, and any expression containing it is many-valued unless some rule be given whereby we are directed to select one of the many values of V at each point of the region.

To make the value of V definite in a region of n cycles, we must conceive n diaphragms or surfaces, each of which completely shuts one of the channels of communication between the parts of the cyclic region. Each of these diaphragms reduces the number of cycles by unity, and when n of them are drawn the region is still a connected region but acyclic, so that we can pass from any one point to any other without cutting a surface, but only by reconcileable paths.

Let S_1 be the first of these diaphragms, and let the line-integral of the force for a line drawn in the acyclic space from a point on the positive side of this surface to the contiguous point on the negative side be κ_1, then κ_1 is the first cyclic constant.

Let the other diaphragms, and their corresponding cyclic constants, be distinguished by suffixes from 1 to n, then, since the region is rendered acyclic by these diaphragms, we may apply to it the theorem in its original form.

We thus obtain for the complete expression for the first member of the equation

$$\iiint \rho' V\,dx\,dy\,dz + \iint \sigma' V\,dS + \iint \sigma_1'\kappa_1\,dS_1 + \iint \sigma_2'\kappa_2\,dS_2 + \&c. + \iint \sigma_n'\kappa_n\,dS_n.$$

The addition of these terms to the expression of Green's Theorem, in the case of many-valued functions, was first shewn to be necessary by Helmholtz *, and was first applied to the theorem by Thomson.

Physical Interpretation of Green's Theorem.

The expressions $\sigma\,dS$ and $\rho\,dx\,dy\,dz$ denote the quantities of electricity existing on an element of the surface S and in an element of volume respectively. We may therefore write for either of these quantities the symbol e, denoting a quantity of electricity. We shall then express Green's Theorem as follows—

$$M = \Sigma\,(Ve') = \Sigma\,(V'e);$$

where we have two systems of electrified bodies, e standing in succession for e_1, e_2, &c., any portions of the electrification of the first system, and V denoting the potential at any point due to all these portions, while e' stands in succession for e_1', e_2', &c., portions of the second system, and V' denotes the potential at any point due to the second system.

Hence Ve' denotes the product of a quantity of electricity at a point belonging to the second system into the potential at that point due to the first system, and $\Sigma\,(Ve')$ denotes the sum of all such quantities, or in other words, $\Sigma\,(Ve')$ represents that part of the energy of the whole electrified system which is due to the action of the second system on the first.

In the same way $\Sigma\,(V'e)$ represents that part of the energy of

* 'Ueber Integrale der Hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen,' *Crelle*, 1858. Translated by Tait in *Phil. Mag.*, 1867, (*i*).

† 'On Vortex Motion,' *Trans. R. S. Edin.*, xxv. part i. p. 241 (1868).

the whole system which is due to the action of the first system on the second.

If we define V as $\Sigma\left(\frac{e}{r}\right)$, where r is the distance of the quantity e of electricity from the given point, then the equality between these two values of M may be obtained as follows, without Green's Theorem—

$$\Sigma'(Ve') = \Sigma'\left(\Sigma\left(\frac{e}{r}\right)e'\right) = \Sigma\Sigma\left(\frac{ee'}{r}\right) = \Sigma\left(\Sigma'\left(\frac{e'}{r}\right)e\right) = \Sigma(V'e).$$

This mode of regarding the question belongs to what we have called the direct method, in which we begin by considering certain portions of electricity, placed at certain points of space, and acting on one another in a way depending on the distances between these points, no account being taken of any intervening medium, or of any action supposed to take place in the intervening space.

Green's Theorem, on the other hand, belongs essentially to what we have called the inverse method. The potential is not supposed to arise from the electrification by a process of summation, but the electrification is supposed to be deduced from a perfectly arbitrary function called the potential by a process of differentiation.

In the direct method, the equation is a simple extension of the law that when any force acts directly between two bodies, action and reaction are equal and opposite.

In the inverse method the two quantities are not proved directly to be equal, but each is proved equal to a third quantity, a triple integral which we must endeavour to interpret.

If we write R for the resultant electromotive force due to the potential V, and l, m, n for the direction-cosines of R, then, by Art. 71,

$$-\frac{dV}{dx} = Rl, \quad -\frac{dV}{dy} = Rm, \quad -\frac{dV}{dz} = Rn.$$

If we also write R' for the force due to the second system, and l', m', n' for its direction-cosines,

$$-\frac{dV'}{dx} = R'l', \quad -\frac{dV'}{dy} = R'm', \quad -\frac{dV'}{dz} = R'n';$$

and the quantity M may be written

$$M = \frac{1}{4\pi}\iiint (KRR'\cos\epsilon)\,dx\,dy\,dz, \tag{10}$$

where $$\cos\epsilon = ll' + mm' + nn',$$

ϵ being the angle between the directions of R and R'.

Now if K is what we have called the coefficient of electric inductive capacity, then KR will be the electric displacement due to the electromotive force R, and the product $KRR'\cos\epsilon$ will $\frac{1}{4\pi}$ represent the work done by the force R' on account of the displacement caused by the force R, or in other words, the amount of intrinsic energy in that part of the field due to the mutual action of R and R'.

We therefore conclude that the physical interpretation of Green's theorem is as follows:

If the energy which is known to exist in an electrified system is due to actions which take place in all parts of the field, and not to direct action at a distance between the electrified bodies, then that part of the intrinsic energy of any part of the field upon which the mutual action of two electrified systems depends is $KRR'\cos\epsilon$ per unit of volume.

The energy of an electrified system due to its action on itself is, by Art. 85, $$\tfrac{1}{2}\Sigma(eV),$$

which is by Green's theorem, putting $U = V$,

$$Q = \frac{1}{8\pi}\iiint K\left(\overline{\frac{dV}{dx}}\Big|^2 + \overline{\frac{dV}{dy}}\Big|^2 + \overline{\frac{dV}{dz}}\Big|^2\right) dx\,dy\,dz\,; \tag{11}$$

and this is the unique minimum value of the integral considered in Thomson's theorem.

Green's Function.

101.] Let a closed surface S be maintained at potential zero. Let P and Q be two points on the positive side of the surface S (we may suppose either the inside or the outside positive), and let a small body charged with unit of electricity be placed at P; the potential at the point Q will consist of two parts, of which one is due to the direct action of the electricity on P, while the other is due to the action of the electricity induced on S by P. The latter part of the potential is called Green's Function, and is denoted by G_{pq}.

This quantity is a function of the positions of the two points P and Q, the form of which depends on that of the surface S. It has been determined in the case in which S is a sphere, and in a very few other cases. It denotes the potential at Q due to the electricity induced on S by unit of electricity at P.

The actual potential at any point Q due to the electricity at P and on S is

$$\frac{1}{r_{pq}} + G_{pq},$$

where r_{pq} denotes the distance between P and Q.

At the surface S and at all points on the negative side of S, the potential is zero, therefore

$$G_{pa} = -\frac{1}{r_{pa}}, \tag{1}$$

where the suffix $_a$ indicates that a point A on the surface S is taken instead of Q.

Let $\sigma_{pa'}$ denote the surface-density induced by P at a point A' of the surface S, then, since G_{pq} is the potential at Q due to the superficial distribution,

$$G_{pq} = \iint \frac{\sigma_{pa'}}{r_{qa'}}\, dS', \tag{2}$$

where dS' is an element of the surface S at A', and the integration is to be extended over the whole surface S.

But if unit of electricity had been placed at Q, we should have had by equation (1),

$$\frac{1}{r_{qa'}} = -G_{qa'} \tag{3}$$

$$= -\iint \frac{\sigma_{qa}}{r_{aa'}}\, dS; \tag{4}$$

where σ_{qa} is the density induced by Q on an element dS at A, and $r_{aa'}$ is the distance between A and A'. Substituting this value of $\frac{1}{r_{qa'}}$ in the expression for G_{pq}, we find

$$G_{pq} = -\iiiint \frac{\sigma_{qa}\,\sigma_{pa'}}{r_{aa'}}\, dS\, dS'. \tag{5}$$

Since this expression is not altered by changing $_p$ into $_q$ and $_q$ into $_p$, we find that

$$G_{pq} = G_{qp}; \tag{6}$$

a result which we have already shewn to be necessary in Art. 88, but which we now see to be deducible from the mathematical process by which Green's function may be calculated.

If we assume any distribution of electricity whatever, and place in the field a point charged with unit of electricity, and if the surface of potential zero completely separates the point from the assumed distribution, then if we take this surface for the surface S, and the point for P, Green's function, for any point on the same side of the surface as P, will be the potential of the assumed distribution on the other side of the surface. In this way we may construct any number of cases in which Green's function can be found for a particular position of P. To find the form of the function when the form of the surface is given and the position of P is arbitrary, is a problem of far greater difficulty, though, as we have proved, it is mathematically possible.

Let us suppose the problem solved, and that the point P is taken within the surface. Then for all external points the potential of the superficial distribution is equal and opposite to that of P. The superficial distribution is therefore *centrobaric* *, and its action on all external points is the same as that of a unit of negative electricity placed at P.

Method of Approximating to the Values of Coefficients of Capacity, &c.

102.] Let a region be completely bounded by a number of surfaces S_0, S_1, S_2, &c., and let K be a quantity, positive or zero but not negative, given at every point of this region. Let V be a function subject to the conditions that its values at the surfaces S_1, S_2, &c. are the constant quantities C_1, C_2, &c., and that at the surface S_0

$$\frac{dV}{d\nu} = 0, \tag{1}$$

where ν is a normal to the surface S_0. Then the integral

$$Q = \frac{1}{8\pi}\iiint K\left(\overline{\frac{dV}{dx}}\bigg|^2 + \overline{\frac{dV}{dy}}\bigg|^2 + \overline{\frac{dV}{dz}}\bigg|^2\right) dx\,dy\,dz, \tag{2}$$

taken over the whole region, has a unique minimum when V satisfies the equation

$$\frac{d}{dx}K\frac{dV}{dx} + \frac{d}{dy}K\frac{dV}{dy} + \frac{d}{dz}K\frac{dV}{dz} = 0 \tag{3}$$

throughout the region, as well as the original conditions.

We have already shewn that a function V exists which fulfils the conditions (1) and (3), and that it is determinate in value. We have next to shew that of all functions fulfilling the surface-conditions it makes Q a minimum.

Let V_0 be the function which satisfies (1) and (3), and let

$$V = V_0 + U \tag{4}$$

be a function which satisfies (1).

It follows from this that at the surfaces S_1, S_2, &c. $U = 0$.

The value of Q becomes

$$Q = \frac{1}{8\pi}\iiint\left\{K\left(\overline{\frac{dV_0}{dx}}\bigg|^2 + \&c.\right) + K\left(\overline{\frac{dU}{dx}}\bigg|^2 + \&c.\right)\right.$$
$$\left. + 2K\left(\frac{dV_0}{dx}\frac{dU}{dx} + \&c.\right)\right\} dx\,dy\,dz. \tag{5}$$

* Thomson and Tait's *Natural Philosophy*, § 526.

Let us confine our attention to the last of these three groups of terms, merely observing that the other groups are essentially positive. By Green's theorem

$$\iiint K\left(\frac{dV_0}{dx}\frac{dU}{dx}+\frac{dV_0}{dy}\frac{dU}{dy}+\frac{dV_0}{dz}\frac{dU}{dz}\right)dx\,dy\,dz=\iint KU\frac{dV_0}{d\nu}dS$$
$$-\iiint U\left(\frac{d}{dx}K\frac{dV_0}{dx}+\frac{d}{dy}K\frac{dV_0}{dy}+\frac{d}{dz}K\frac{dV_0}{dz}\right)dx\,dy\,dz\,; \quad (6)$$

the first integral of the second member being extended over the surface of the region and the second throughout the enclosed space. But on the surfaces S_1, S_2, &c. $U=0$, so that these contribute nothing to the surface-integral.

Again, on the surface S_0, $\frac{dV_0}{d\nu}=0$, so that this surface contributes nothing to the integral. Hence the surface-integral is zero.

The quantity within brackets in the volume-integral also disappears by equation (3), so that the volume-integral is also zero. Hence Q is reduced to

$$Q=\frac{1}{8\pi}\iiint K\left(\left|\overline{\frac{dV_0}{dx}}\right|^2+\text{\&c.}\right)dx\,dy\,dz+\frac{1}{8\pi}\iiint K\left(\left|\overline{\frac{dU}{dx}}\right|^2+\text{\&c.}\right)dx\,dy\,dz. \quad (7)$$

Both these quantities are essentially positive, and therefore the minimum value of Q is when

$$\frac{dU}{dx}=\frac{dU}{dy}=\frac{dU}{dz}=0, \quad (8)$$

or when U is a constant. But at the surfaces S, &c. $U=0$. Hence $U=0$ everywhere, and V_0 gives the unique minimum value of Q.

Calculation of a Superior Limit of the Coefficients of Capacity.

The quantity Q in its minimum form can be expressed by means of Green's theorem in terms of V_1, V_2, &c., the potentials of S_1, S_2, and E_1, E_2, &c., the charges of these surfaces,

$$Q=\tfrac{1}{2}(V_1E_1+V_2E_2+\text{\&c.})\,; \quad (9)$$

or, making use of the coefficients of capacity and induction as defined in Article 87,

$$Q=\tfrac{1}{2}(V_1^2q_{11}+V_2^2q_{22}+\text{\&c.})+V_1V_2q_{12}+\text{\&c.} \quad (10)$$

The accurate determination of the coefficients q is in general difficult, involving the solution of the general equation of statical electricity, but we make use of the theorem we have proved to determine a superior limit to the value of any of these coefficients.

To determine a superior limit to the coefficient of capacity q_{11}, make $V_1 = 1$, and V_2, V_3, &c. each equal to zero, and then take any function V which shall have the value 1 at S_1, and the value 0 at the other surfaces.

From this trial value of V calculate Q by direct integration, and let the value thus found be Q'. We know that Q' is not less than the absolute minimum value Q, which in this case is $\frac{1}{2} q_{11}$.

Hence $\qquad q_{11}$ is not greater than $2\,Q'$. $\qquad$ (11)

If we happen to have chosen the right value of the function V, then $q_{11} = 2\,Q'$, but if the function we have chosen differs slightly from the true form, then, since Q is a minimum, Q' will still be a close approximation to the true value.

Superior Limit of the Coefficients of Potential.

We may also determine a superior limit to the coefficients of potential defined in Article 86 by means of the minimum value of the quantity Q in Article 98, expressed in terms of a, b, c.

By Thomson's theorem, if within a certain region bounded by the surfaces S_0, S_1, &c. the quantities a, b, c are subject to the condition

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0; \tag{12}$$

and if

$$la + mb + nc = q \tag{13}$$

be given all over the surface, where l, m, n are the direction-cosines of the normal, then the integral

$$Q = \frac{1}{8\pi}\iiint \frac{1}{K}\,(a^2+b^2+c^2)\,dx\,dy\,dz \tag{14}$$

is an absolute and unique minimum when

$$a = K\frac{dV}{dx}, \quad b = K\frac{dV}{dy}, \quad c = K\frac{dV}{dz}. \tag{15}$$

When the minimum is attained Q is evidently the same quantity which we had before.

If therefore we can find any form for a, b, c which satisfies the condition (12) and at the same time makes

$$\iint q\,dS_1 = E_1, \quad \iint q\,dS_2 = E_2 \text{ \&c.}; \tag{16}$$

and if Q'' be the value of Q calculated by (14) from these values of a, b, c, then Q'' is not less than

$$\tfrac{1}{2}\,(E_1{}^2\,p_{11} + E_2{}^2\,p_{22}) + E_1\,E_2\,p_{12}. \tag{17}$$

If we take the case in which one of the surfaces, say S_2, surrounds the rest at an infinite distance, we have the ordinary case of conductors in an infinite region; and if we make $E_2 = -E_1$, and $E = 0$ for all the other surfaces, we have $V_2 = 0$ at infinity, and p_{11} is not greater than $\frac{2Q''}{E_1}$.

In the very important case in which the electrical action is entirely between two conducting surfaces S_1 and S_2, of which S_2 completely surrounds S_1 and is kept at potential zero, we have $E_1 = -E_2$ and $q_{11} p_{11} = 1$.

Hence in this case we have

$$q_{11} \text{ not less than } \frac{E_1}{2Q''}; \tag{18}$$

and we had before

$$q_{11} \text{ not greater than } 2Q'; \tag{19}$$

so that we conclude that the true value of q_{11}, the capacity of the internal conductor, lies between these values.

This method of finding superior and inferior limits to the values of these coefficients was suggested by a memoir 'On the Theory of Resonance,' by the Hon. J. W. Strutt, *Phil. Trans.*, 1871. See Art. 308.

CHAPTER V.

MECHANICAL ACTION BETWEEN ELECTRIFIED BODIES.

103.] Let $V = C$ be any closed equipotential surface, C being a particular value of a function V, the form of which we suppose known at every point of space. Let the value of V on the outside of this surface be V_1, and on the inside V_2. Then, by Poisson's equation

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} + 4\pi\rho = 0, \tag{1}$$

we can determine the density ρ_1 at every point on the outside, and the density ρ_2 at every point on the inside of the surface. We shall call the whole electrified system thus explored on the outside E_1, and that on the inside E_2. The actual value of V arises from the combined action of both these systems.

Let R be the total resultant force at any point arising from the action of E_1 and E_2, R is everywhere normal to the equipotential surface passing through the point.

Now let us suppose that on the equipotential surface $V = C$ electricity is distributed so that at any point of the surface at which the resultant force due to E_1 and E_2 reckoned outwards is R, the surface-density is σ, with the condition

$$R = 4\pi\sigma; \tag{2}$$

and let us call this superficial distribution the electrified surface S, then we can prove the following theorem relating to the action of this electrified surface.

If any equipotential surface belonging to a given electrified system be coated with electricity, so that at each point the surface-density $\sigma = \dfrac{R}{4\pi}$, where R is the resultant force, due to the original electrical system, acting outwards from that point of the surface, then the potential due to the electrified surface at any point on the outside of that surface will be equal to the potential at the same point due to that part of the original system which was on the inside of the surface, and the potential due to the electrified surface at any point on the inside added to that due to the part of the original system on the outside will be equal to C, the potential of the surface.

For let us alter the original system as follows:

Let us leave everything the same on the outside of the surface, but on the inside let us make V_2 everywhere equal to C, and let us do away with the electrified system E_2 on the inside of the surface, and substitute for it a surface-density σ at every point of the surface S, such that

$$R = 4\pi\sigma. \tag{3}$$

Then this new arrangement will satisfy the characteristics of V at every point.

For on the outside of the surface both the distribution of electricity and the value of V are unaltered, therefore, since V originally satisfied Laplace's equation, it will still satisfy it.

On the inside V is constant and ρ zero. These values of V and ρ also satisfy the characteristic equations.

At the surface itself, if V_1 is the potential at any point on the outside and V_2 that on the inside, then, if l, m, n are the direction-cosines of the normal to the surface reckoned outwards,

$$l\frac{dV_1}{dx} + m\frac{dV_1}{dy} + n\frac{dV_1}{dz} = -R = -4\pi\sigma; \tag{4}$$

and on the inside the derivatives of V vanish, so that the superficial characteristic

$$l\left(\frac{dV_1}{dx} - \frac{dV_2}{dx}\right) + m\left(\frac{dV_1}{dy} - \frac{dV_2}{dy}\right) + n\left(\frac{dV_1}{dz} - \frac{dV_2}{dz}\right) + 4\pi\sigma = 0 \tag{5}$$

is satisfied at every point of the surface.

Hence the new distribution of potential, in which it has the old value on the outside of the surface and a constant value on the inside, is consistent with the new distribution of electricity, in which the electricity in the space within the surface is removed and a distribution of electricity on the surface is substituted for it. Also, since the original value of V_1 vanishes at infinity, the new value, which is the same outside the surface, also fulfils this condition, and therefore the new value of V is the sole and only value of V belonging to the new arrangement of electricity.

On the Mechanical Action and Reaction of the Systems E_1 and E_2.

104.] If we now suppose the equipotential surface $V = C$ to become rigid and capable of sustaining the action of forces, we may prove the following theorem.

If on every element dS of an equipotential surface a force $\frac{1}{8\pi} R^2 \, dS$ be made to act in the direction of the normal reckoned outwards, where R is the 'electrical resultant force' along the normal, then the total statical effect of these forces on the surface considered as a rigid shell will be the same as the total statical effect of the electrical action of the electrified system E_1 outside the shell on the electrified system E_2 inside the shell, the parts of the interior system E_2 being supposed rigidly connected together.

We have seen that the action of the electrified surface in the last theorem on any external point was equal to that of the internal system E_2, and, since action and reaction are equal and opposite, the action of any external electrified body on the electrified surface, considered as a rigid system, is equal to that on the internal system E_2. Hence the statical action of the external system E_1 on the electrified surface is equal in all respects to the action of E_1 on the internal system E_2.

But at any point just outside the electrified surface the resultant force is R in a direction normal to the surface, and reckoned positive when it acts outwards. The resultant inside the surface is zero, therefore, by Art. 79, the resultant force acting on the element dS of the electrified surface is $\frac{1}{2} R \sigma \, dS$, where σ is the surface-density.

Substituting the value of σ in terms of R from equation (2), and denoting by $p \, dS$ the resultant force on the electricity spread over the element dS, we find

$$p \, dS = \frac{1}{8\pi} R^2 \, dS.$$

This force always acts along the normal and outwards, whether R be positive or negative, and may be considered as equal to a pressure $p = \frac{1}{8\pi} R^2$ acting on the surface from within, or to a tension of the same numerical value acting from without.

* See Sir W. Thomson 'On the Attractions of Conducting and Non-conducting Electrified Bodies,' *Cambridge Mathematical Journal*, May 1843, and Reprint, Art. VII, § 147.

Now R is the resultant due to the combined action of the external system E_1 and the electrification of the surface S. Hence the effect of the pressure p on each element of the inside of the surface considered as a rigid body is equivalent to this combined action.

But the actions of the different parts of the surface on each other form a system in equilibrium, therefore the effect of the pressure p on the rigid shell is equivalent in all respects to the electric attraction of E_1 on the shell, and this, as we have before shewn, is equivalent to the electric attraction of E_1 on E_2 considered as a rigid system.

If we had supposed the pressure p to act on the outside of the shell, the resultant effect would have been equal and opposite, that is, it would have been statically equivalent to the action of the internal system E_2 on the external system E_1.

Let us now take the case of two electrified systems E_1 and E_2, such that two equipotential surfaces $V = C_1$ and $V = C_2$, which we shall call S_1 and S_2 respectively, can be described so that E_1 is exterior to S_1, and S_1 surrounds S_2, and E_2 lies within S_2.

Then if R_1 and R_2 represent the resultant force at any point of S_1 and S_2 respectively, and if we make

$$p_1 = \frac{1}{8\pi} R_1^2 \text{ and } p_2 = \frac{1}{8\pi} R_2^2,$$

the mechanical action between E_1 and E_2 is equivalent to that between the shells S_1 and S_2, supposing every point of S_1 pressed inwards, that is, towards S_2 with a pressure p_1, and every point of S_2 pressed outwards, that is, towards S_1 with a pressure p_2.

105.] According to the theory of action at a distance the action between E_1 and E_2 is really made up of a system of forces acting in straight lines between the electricity in E_1 and that in E_2, and the actual mechanical effect is in complete accordance with this theory.

There is, however, another point of view from which we may examine the action between E_1 and E_2. When we see one body acting on another at a distance, before we assume that the one acts directly on the other we generally inquire whether there is any material connexion between the two bodies, and if we find strings, or rods, or framework of any kind, capable of accounting for the observed action between the bodies, we prefer to explain the action by means of the intermediate connexions, rather than admit the notion of direct action at a distance.

Thus when two particles are connected by a straight or curved rod, the action between the particles is always along the line joining them, but we account for this action by means of a system of internal forces in the substance of the rod. The existence of these internal forces is deduced entirely from observation of the effect of external forces on the rod, and the internal forces themselves are generally assumed to be the resultants of forces which act between particles of the rod. Thus the observed action between two distant particles is, in this instance, removed from the class of direct actions at a distance by referring it to the intervention of the rod; the action of the rod is explained by the existence of internal forces in its substance; and the internal forces are explained by means of forces assumed to act between the particles of which the rod is composed, that is, between bodies at distances which though small must be finite.

The observed action at a considerable distance is therefore explained by means of a great number of forces acting between bodies at very small distances, for which we are as little able to account as for the action at any distance however great.

Nevertheless, the consideration of the phenomenon, as explained in this way, leads us to investigate the properties of the rod, and to form a theory of elasticity which we should have overlooked if we had been satisfied with the explanation by action at a distance.

106.] Let us now examine the consequence of assuming that the action between electrified bodies can be explained by the intermediate action of the medium between them, and let us ascertain what properties of the medium will account for the observed action.

We have first to determine the internal forces in the medium, and afterwards to account for them if possible.

In order to determine the internal forces in any case we proceed as follows:

Let the system M be in equilibrium under the action of the system of external forces F. Divide M by an imaginary surface into two parts, M_1 and M_2, and let the systems of external forces acting on these parts respectively be F_1 and F_2. Also let the internal forces acting on M_1 in consequence of its connexion with M_2 be called the system I.

Then, since M_1 is in equilibrium under the action of F_1 and I, it follows that I is statically equivalent to F_1 reversed.

In the case of the electrical action between two electrified systems E_1 and E_2, we described two closed equipotential surfaces entirely surrounding E_2 and cutting it off from E_1, and we found that the application of a certain normal pressure at every point of the inner side of the inner surface, and on the outer side of the outer surface, would, if these surfaces were each rigid, act on the outer surface with a resultant equal to that of the electrical forces on the outer system E_1, and on the inner surface with a resultant equal to that of the electrical forces on the inner system.

Let us now consider the space between the surfaces, and let us suppose that at every point of this space there is a tension in the direction of R and equal to $\frac{1}{8\pi}R^2$ per unit of area. This tension will act on the two surfaces in the same way as the pressures on the other side of the surfaces, and will therefore account for the action between E_1 and E_2, so far as it depends on the internal force in the space between S_1 and S_2.

Let us next investigate the equilibrium of a portion of the shell bounded by these surfaces and separated from the rest by a surface everywhere perpendicular to the equipotential surfaces. We may suppose this surface generated by describing any closed curve on S_1, and drawing from every point of this curve lines of force till they meet S_2.

The figure we have to consider is therefore bounded by the two equipotential surfaces S_1 and S_2, and by a surface through which there is no induction, which we may call S_0.

Let us first suppose that the area of the closed curve on S_1 is very small, call it dS_1, and that $C_2 = C_1 + dV$.

The portion of space thus bounded may be regarded as an element of volume. If ν is the normal to the equipotential surface, and dS the element of that surface, then the volume of this element is ultimately $dS\,d\nu$.

The induction through dS_1 is $R\,dS_1$, and since there is no induction through S_0, and no free electricity within the space considered, the induction through the opposite surface dS_2 will be equal and opposite, considered with reference to the space within the closed surface.

There will therefore be a quantity of electricity

$$e_1 = -\frac{1}{4\pi} R_1\, dS_1$$

on the first equipotential surface, and a quantity

$$e_2 = \frac{1}{4\pi} R_2\, dS_2$$

on the second equipotential surface, with the condition

$$e_1 + e_2 = 0.$$

Let us next consider the resultant force due to the action of the electrified systems on these small electrified surfaces.

The potential within the surface S_1 is constant and equal to C_1, and without the surface S_2 it is constant and equal to C_2. In the shell between these surfaces it is continuous from C_1 to C_2.

Hence the resultant force is zero except within the shell.

The electrified surface of the shell itself will be acted on by forces which are the arithmetical means of the forces just within and just without the surface, that is, in this case, since the resultant force outside is zero, the force acting on the superficial electrification is one-half of the resultant force just within the surface.

Hence, if $X\,dS\,dv$ be the total moving force resolved parallel to x, due to the electrical action on both the electrified surfaces of the element $dS\,dv$,

$$X\,dS\,dv = -\tfrac{1}{2}\left(e_1\frac{dV_1}{dx} + e_2\frac{dV_2}{dx}\right),$$

where the suffixes denote that the derivatives of V are to be taken at dS_1 and dS_2 respectively.

Let l, m, n be the direction-cosines of ν, the normal to the equipotential surface, then making

$$dx = l\,dv, \quad dy = m\,dv, \quad \text{and} \quad dz = n\,dv,$$

$$\left(\frac{dV}{dx}\right)_2 = \left(\frac{dV}{dx}\right)_1 + \left(l\frac{d^2V}{dx^2} + m\frac{d^2V}{dx\,dy} + n\frac{d^2V}{dx\,dz}\right)dv + \&c.;$$

and since $e_2 = -e_1$, we may write the value of X

$$X\,dS\,dv = \tfrac{1}{2}e_1\frac{d}{dx}\left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}\right)dv.$$

But $\quad e_1 = -\dfrac{1}{4\pi}R\,dS \quad$ and $\quad \left(l\dfrac{dV}{dx} + m\dfrac{dV}{dy} + n\dfrac{dV}{dz}\right) = -R;$

therefore $\quad X\,dS\,dv = \dfrac{1}{8\pi}R\dfrac{dR}{dx}\,dS\,dv;$

or, if we write

$$p = \frac{1}{8\pi}R^2 = \frac{1}{8\pi}\left(\left|\frac{dV}{dx}\right|^2 + \left|\frac{dV}{dy}\right|^2 + \left|\frac{dV}{dz}\right|^2\right),$$

then $\quad X = \tfrac{1}{2}\dfrac{dp}{dx}, \qquad Y = \tfrac{1}{2}\dfrac{dp}{dy}, \qquad Z = \tfrac{1}{2}\dfrac{dp}{dz};$

or the force in any direction on the element arising from the action of the electrified system on the two electrified surfaces of the element is equal to half the rate of increase of p in that direction multiplied by the volume of the element.

This result is the same if we substitute for the forces acting on the electrified surfaces an imaginary force whose potential is $-\frac{1}{2}p$, acting on the whole volume of the element and soliciting it to move so as to increase $\frac{1}{2}p$.

If we now return to the case of a figure of finite size, bounded by the equipotential surfaces S_1 and S_2 and by the surface of no induction S_0, we may divide the whole space into elements by a series of equipotential surfaces and two series of surfaces of no induction. The charges of electricity on those faces of the elements which are in contact will be equal and opposite, so that the total effect will be that due to the electrical forces acting on the charges on the surfaces S_1 and S_2, and by what we have proved this will be the same as the action on the whole volume of the figure due to a system of forces whose potential is $-\frac{1}{2}p$.

But we have already shewn that these electrical forces are equivalent to a tension p applied at all points of the surfaces S_1 and S_2. Hence the effect of this tension is to pull the figure in the direction in which p increases. The figure therefore cannot be in equilibrium unless some other forces act on it.

Now we know that if a hydrostatic pressure p is applied at every point of the surface of any closed figure, the effect is equal to that of a system of forces acting on the whole volume of the figure and having a potential p. In this case the figure is pushed in the direction in which p diminishes.

We can now arrange matters so that the figure shall be in equilibrium.

At every point of the two equipotential surfaces S_1 and S_2, let a *tension* $= p$ be applied, and at every point of the surface of no induction S_0 let a *pressure* $= p$ be applied. These forces will keep the figure in equilibrium.

For the tension p may be considered as a pressure p combined with a tension $2p$. We have then a hydrostatic pressure p acting at every point of the surface, and a tension $2p$ acting on S_1 and S_2 only.

The effect of the tension $2p$ at every point of S_1 and S_2 is double of that which we have just calculated, that is, it is equal to that of forces whose potential is $-p$ acting on the whole volume of the figure. The effect of the pressure p acting on the whole surface is by hydrostatics equal and opposite to that of this system of forces, and will keep the figure in equilibrium.

107.] We have now determined a system of internal forces in the medium which is consistent with the phenomena so far as we have examined them. We have found that in order to account for the electric attraction between distant bodies without admitting direct action, we must assume the existence of a *tension* p at every point of the medium in the direction of the resultant force R at that point. In order to account for the equilibrium of the medium itself we must further suppose that in every direction perpendicular to R there is a *pressure* p *.

By establishing the necessity of assuming these internal forces in the theory of an electric medium, we have advanced a step in that theory which will not be lost though we should fail in accounting for these internal forces, or in explaining the mechanism by which they can be maintained in air, glass, and other dielectric media.

We have seen that the internal stresses in solid bodies can be ascertained with precision, though the theories which account for these stresses by means of molecular forces may still be doubtful. In the same way we may estimate these internal electrical forces before we are able to account for them.

In order, however, that it may not appear as if we had no explanation of these internal forces, we shall shew that on the ordinary theory they must exist in a shell bounded by two equipotential surfaces, and that the attractions and repulsions of the electricity on the surfaces of the shell are sufficient to account for them.

Let the first surface S_1 be electrified so that the surface-density is

$$\sigma_1 = -\frac{1}{4\pi} R_1,$$

and the second surface S_2 so that the surface-density is

$$\sigma_2 = \frac{1}{4\pi} R_2;$$

then, if we suppose that the value of V is C_1 at every point within S_1, and C_2 at every point outside of S_2, the value of V between these surfaces remaining as before, the characteristic equation of V will be satisfied everywhere, and V is therefore the true value of the potential.

We have already shewn that the outer and inner surfaces of the shell will be pulled towards each other with a force the value of which referred to unit of surface is p, or in other words, there is a tension p in the substance of the shell in the direction of the lines of force.

* See Faraday, *Exp. Res.* (1224) and (1297).

If we now conceive the shell divided into two segments by a surface of no induction, the two parts will experience electrical forces the resultants of which will tend to separate the parts with a force equivalent to the resultant force due to a pressure p acting on every part of the surface of no induction which divides them.

This illustration is to be taken merely as an explanation of what is meant by the tension and pressure, not as a physical theory to account for them.

108.] We have next to consider whether these internal forces are capable of accounting for the observed electrical forces in every case, as well as in the case where a closed equipotential surface can be drawn surrounding one of the electrified systems.

The statical theory of internal forces has been investigated by writers on the theory of elasticity. At present we shall require only to investigate the effect of an oblique tension or pressure on an element of surface.

Let p be the value of a tension referred to unit of a surface to which it is normal, and let there be no tension or pressure in any direction normal to p. Let the direction-cosines of p be l, m, n. Let $dy\,dz$ be an element of surface normal to the axis of x, and let the effect of the internal force be to urge the parts on the positive side of this element with a force whose components are

$$\begin{aligned} &p_{xx}\,dy\,dz \text{ in the direction of } x,\\ &p_{xy}\,dy\,dz \;\ldots\ldots\ldots\ldots\; y, \text{ and}\\ &p_{xz}\,dy\,dz \;\ldots\ldots\ldots\ldots\; z. \end{aligned}$$

From every point of the boundary of the element $dy\,dz$ let lines be drawn parallel to the direction of the tension p, forming a prism whose axis is in the line of tension, and let this prism be cut by a plane normal to its axis.

z The area of this section will be $l\,dy\,d\cancel{x}$, and the whole tension
z upon it will be $p\,l\,dy\,d\cancel{x}$, and since there is no action on the sides of the prism, which are normal to p, the force on the base $dy\,dz$
z must be equivalent to the force $p\,l\,dy\,d\cancel{x}$ acting in the direction (l, m, n). Hence the component in the direction of x,

$$p_{xx}\,dy\,dz = pl^2\,dy\,dz\,; \quad \text{or}$$

$$p_{xx} = pl^2.$$

Similarly

$$p_{xy} = plm, \qquad (1)$$

$$p_{xz} = pln.$$

If we now combine with this tension two tensions p' and p'' in directions (l', m', n') and (l'', m'', n'') respectively, we shall have

$$\begin{aligned} p_{xx} &= pl^2 + p'l'^2 + p''l''^2, \\ p_{xy} &= plm + p'l'm' + p''l''m'', \\ p_{xz} &= pln + p'l'n' + p''l''n''. \end{aligned} \tag{2}$$

In the case of the electrical tension and pressure the pressures are numerically equal to the tension at every point, and are in directions at right angles to the tension and to each other. Hence, putting

$$p' = p'' = -p, \tag{3}$$

$$l^2 + l'^2 + l''^2 = 1, \quad lm + l'm' + l''m'' = 0, \quad ln + l'n' + l''n'' = 0, \tag{4}$$

we find

$$\begin{aligned} p_{xx} &= (2l^2 - 1)p, \\ p_{xy} &= 2lmp, \\ p_{xz} &= 2lnp, \end{aligned} \tag{5}$$

for the action of the combined tension and pressures.

Also, since $p = \frac{1}{8\pi} R^2$, where R denotes the resultant force, and since $Rl = X$, $Rm = Y$, $Rn = Z$,

$$\begin{aligned} p_{xx} &= \frac{1}{8\pi}(X^2 - Y^2 - Z^2), \\ p_{xy} &= \frac{1}{8\pi} 2XY = p_{yx}, \\ p_{xz} &= \frac{1}{8\pi} 2XZ = p_{zx}; \end{aligned} \tag{6}$$

where X, Y, Z are the components of R, the resultant electromotive force.

The expressions for the component internal forces on surfaces normal to y and z may be written down from symmetry.

To determine the conditions of equilibrium of the element dxdydz.

This element is bounded by the six planes perpendicular to the axes of coordinates passing through the points (x, y, z) and $(x+dx, y+dy, z+dz)$.

The force in the direction of x which acts on the first face $dy\,dz$ is $-p_{xx}\,dy\,dz$, tending to draw the element towards the negative side. On the second face $dy\,dz$, for which x has the value $x+dx$, the tension p_{xx} has the value

$$p_{xx}\,dy\,dz + \left(\frac{d}{dx} p_{xx}\right) dx\,dy\,dz,$$

and this tension tends to draw the element in the positive direction.

If we next consider the two faces $dz\,dx$ with respect to the tangential forces urging them in the direction of x, we find the force on the first face $-p_{yx}\,dz\,dx$, and that on the second

$$p_{yx}\,dz\,dx + \left(\frac{d}{dy}p_{yx}\right)dz\,dx\,dy.$$

Similarly for the faces $dx\,dy$, we find that a force $-p_{zx}\,dx\,dy$ acts on the first face, and

$$p_{zx}\,dx\,dy + \left(\frac{d}{dz}p_{zx}\right)dx\,dy\,dz$$

on the second in the direction of x.

If $\xi\,dx\,dy\,dz$ denotes the total effect of all these internal forces acting parallel to the axis of x on the six faces of the element, we find

$$\xi\,dx\,dy\,dz = \left(\frac{d}{dx}p_{xx} + \frac{d}{dy}p_{yx} + \frac{d}{dz}p_{zx}\right)dx\,dy\,dz;$$

or, denoting by ξ the internal force, referred to unit of volume, and resolved parallel to the axis of x,

$$\xi = \frac{d}{dx}p_{xx} + \frac{d}{dy}p_{yx} + \frac{d}{dz}p_{zx}, \tag{7}$$

with similar expressions for η and ζ, the component forces in the other directions *.

Differentiating the values of p_{xx}, p_{yx}, and p_{zx} given in equations (6), we find

$$\xi = \frac{1}{4\pi}X\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right). \tag{8}$$

But by Art. 77

$$\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) = 4\pi\rho. \tag{9}$$

Hence $\qquad \xi = \rho X.$

Similarly $\qquad \eta = \rho Y, \qquad (10)$

$$\zeta = \rho Z.$$

Thus, the resultant of the tensions and pressures which we have supposed to act upon the surface of the element is a force whose components are the same as those of the force, which, in the ordinary theory, is ascribed to the action of electrified bodies on the electricity within the element.

If, therefore, we admit that there is a medium in which there is maintained at every point a tension p in the direction of the

* This investigation may be compared with that of the 'equation of continuity in hydrodynamics,' and with others in which the effect on an element of volume is deduced from the values of certain quantities at its bounding surface.

resultant electromotive force R, and such that $R^2 = 8\pi p$, combined with an equal pressure p in every direction at right angles to the resultant R, then the mechanical effect of these tensions and pressures on any portion of the medium, however bounded, will be identical with the mechanical effect of the electrical forces according to the ordinary theory of direct action at a distance.

109.] This distribution of stress is precisely that to which Faraday was led in his investigation of induction through dielectrics. He sums up in the following words :—

'(1297) The direct inductive force, which may be conceived to be exerted in lines between the two limiting and charged conducting surfaces, is accompanied by a lateral or transverse force equivalent to a dilatation or repulsion of these representative lines (1224.); or the attracting force which exists amongst the particles of the dielectric in the direction of the induction is accompanied by a repulsive or a diverging force in the transverse direction.

'(1298) Induction appears to consist in a certain polarized state of the particles, into which they are thrown by the electrified body sustaining the action, the particles assuming positive and negative points or parts, which are symmetrically arranged with respect to each other and the inducting surfaces or particles. The state must be a forced one, for it is originated and sustained only by force, and sinks to the normal or quiescent state when that force is removed. It can be *continued* only in insulators by the same portion of electricity, because they only can retain this state of the particles.'

This is an exact account of the conclusions to which we have been conducted by our mathematical investigation. At every point of the medium there is a state of stress such that there is tension along the lines of force and pressure in all directions at right angles to these lines, the numerical magnitude of the pressure being equal to that of the tension, and both varying as the square of the resultant force at the point.

The expression 'electric tension' has been used in various senses by different writers. I shall always use it to denote the tension along the lines of force, which, as we have seen, varies from point to point, and is always proportional to the square of the resultant force at the point.

110.] The hypothesis that a state of stress of this kind exists in a fluid dielectric, such as air or turpentine, may at first sight appear at variance with the established principle that at any point in a fluid the pressures in all directions are equal. But in the deduction of this principle from a consideration of the mobility and equilibrium of the parts of the fluid it is taken for granted that no action such as that which we here suppose to take place along the lines of force exists in the fluid. The state of stress which we have been studying is perfectly consistent with the mobility and equilibrium of the fluid, for we have seen that, if any portion of the fluid is devoid of electric charge, it experiences no resultant force from the stresses on its surface, however intense these may be. It is only when a portion of the fluid becomes charged, that its equilibrium is disturbed by the stresses on its surface, and we know that in this case it actually tends to move. Hence the supposed state of stress is not inconsistent with the equilibrium of a fluid dielectric.

The quantity Q, which was investigated in Thomson's theorem, Art. 98, may be interpreted as the energy in the medium due to the distribution of stress. It appears from that theorem that the distribution of stress which satisfies the ordinary conditions also makes Q an absolute minimum. Now when the energy is a minimum for any configuration, that configuration is one of equilibrium, and the equilibrium is stable. Hence the dielectric, when subjected to the inductive action of electrified bodies, will of itself take up a state of stress distributed in the way we have described.

It must be carefully borne in mind that we have made only one step in the theory of the action of the medium. We have supposed it to be in a state of stress, but we have not in any way accounted for this stress, or explained how it is maintained. This step, however, seems to me to be an important one, as it explains, by the action of the consecutive parts of the medium, phenomena which were formerly supposed to be explicable only by direct action at a distance.

111.] I have not been able to make the next step, namely, to account by mechanical considerations for these stresses in the dielectric. I therefore leave the theory at this point, merely stating what are the other parts of the phenomenon of induction in dielectrics.

I. Electric Displacement. When induction takes place in a dielectric a phenomenon takes place which is equivalent to a displacement of electricity in the direction of the induction. For instance, in a Leyden jar, of which the inner coating is charged positively and the outer coating negatively, the displacement in the substance of the glass is from within outwards.

Any increase of this displacement is equivalent, during the time of increase, to a current of positive electricity from within outwards, and any diminution of the displacement is equivalent to a current in the opposite direction.

The whole quantity of electricity displaced through any area of a surface fixed in the dielectric is measured by the quantity which we have already investigated (Art. 75) as the surface-integral of induction through that area, multiplied by $\frac{1}{4\pi}K$, where K is the specific inductive capacity of the dielectric.

II. Superficial Electrification of the Particles of the Dielectric. Conceive any portion of the dielectric, large or small, to be separated (in imagination) from the rest by a closed surface, then we must suppose that on every elementary portion of this surface there is an electrification measured by the total displacement of electricity through that element of surface *reckoned inwards.*

In the case of the Leyden jar of which the inner coating is charged positively, any portion of the glass will have its inner side charged positively and its outer side negatively. If this portion be entirely in the interior of the glass, its superficial electrification will be neutralized by the opposite electrification of the parts in contact with it, but if it be in contact with a conducting body which is incapable of maintaining in itself the inductive state, the superficial electrification will not be neutralized, but will constitute that apparent electrification which is commonly called the Electrification of the Conductor.

The electrification therefore at the bounding surface of a conductor and the surrounding dielectric, which on the old theory was called the electrification of the conductor, must be called in the theory of induction the superficial electrification of the surrounding dielectric.

According to this theory, all electrification is the residual effect of the polarization of the dielectric. This polarization exists throughout the interior of the substance, but it is there neutralized by the juxtaposition of oppositely electrified parts, so that it is only at the surface of the dielectric that the effects of the electrification become apparent.

The theory completely accounts for the theorem of Art. 77, that the total induction through a closed surface is equal to the total quantity of electricity within the surface multiplied by 4π. For what we have called the induction through the surface is simply the electric displacement multiplied by 4π, and the total displacement outwards is necessarily equal to the total electrification within the surface.

The theory also accounts for the impossibility of communicating an absolute charge to matter. For every particle of the dielectric is electrified with equal and opposite charges on its opposite sides, if it would not be more correct to say that these electrifications are only the manifestations of a single phenomenon, which we may call Electric Polarization.

A dielectric medium, when thus polarized, is the seat of electrical energy, and the energy in unit of volume of the medium is numerically equal to the electric tension on unit of area, both quantities being equal to half the product of the displacement and the resultant electromotive force, or

$$p = \tfrac{1}{2}\mathfrak{D}\mathfrak{E} = \frac{1}{8\pi}K\mathfrak{E}^2 = \frac{2\pi}{K}\mathfrak{D}^2,$$

where p is the electric tension, $\mathfrak{D}$ the displacement, $\mathfrak{E}$ the electromotive force, and K the specific inductive capacity.

If the medium is not a perfect insulator, the state of constraint, which we call electric polarization, is continually giving way. The medium yields to the electromotive force, the electric stress is relaxed, and the potential energy of the state of constraint is converted into heat. The rate at which this decay of the state of polarization takes place depends on the nature of the medium. In some kinds of glass, days or years may elapse before the polarization sinks to half its original value. In copper, this change may occupy less than the billionth of a second.

We have supposed the medium after being polarized to be simply left to itself. In the phenomenon called the electric current the constant passage of electricity through the medium tends to restore the state of polarization as fast as the conductivity of the medium allows it to decay. Thus the external agency which maintains the current is always doing work in restoring the polarization of the medium, which is continually becoming relaxed, and the potential energy of this polarization is continually becoming transformed into heat, so that the final result of the energy expended in maintaining the current is to raise the temperature of the conductor.

CHAPTER VI.

ON POINTS AND LINES OF EQUILIBRIUM.

112.] If at any point of the electric field the resultant force is zero, the point is called a Point of equilibrium.

If every point on a certain line is a point of equilibrium, the line is called a Line of equilibrium.

The conditions that a point shall be a point of equilibrium are that at that point

$$\frac{dV}{dx} = 0, \quad \frac{dV}{dy} = 0, \quad \frac{dV}{dz} = 0.$$

At such a point, therefore, the value of V is a maximum, or a minimum, or is stationary, with respect to variations of the coordinates. The potential, however, can have a maximum or a minimum value only at a point charged with positive or with negative electricity, or throughout a finite space bounded by a surface electrified positively or negatively. If, therefore, a point of equilibrium occurs in an unelectrified part of the field it must be a stationary point, and not a maximum or a minimum.

In fact, the first condition of a maximum or minimum is that

$$\frac{d^2 V}{dx^2}, \quad \frac{d^2 V}{dy^2}, \quad \text{and} \quad \frac{d^2 V}{dz^2}$$

must be all negative or all positive, if they have finite values.

Now, by Laplace's equation, at a point where there is no electrification, the sum of these three quantities is zero, and therefore this condition cannot be fulfilled.

Instead of investigating the analytical conditions for the cases in which the components of the force simultaneously vanish, we shall give a general proof by means of the equipotential surfaces.

If at any point, P, there is a true maximum value of V, then, at all other points in the immediate neighbourhood of P, the value of V is less than at P. Hence P will be surrounded by a series of closed equipotential surfaces, each outside the one before it, and at all points of any one of these surfaces the electrical force will be directed outwards. But we have proved, in Art. 76, that the surface-integral of the electrical force taken over any closed surface gives the total electrification within that surface multiplied by 4π. Now, in this case the force is everywhere outwards, so that the surface-integral is necessarily positive, and therefore there is positive electrification within the surface, and, since we may take the surface as near to P as we please, there is positive electrification at the point P.

In the same way we may prove that if V is a minimum at P, then P is negatively electrified.

Next, let P be a point of equilibrium in a region devoid of electrification, and let us describe a very small closed surface round P, then, as we have seen, the potential at this surface cannot be everywhere greater or everywhere less than at P. It must therefore be greater at some parts of the surface and less at others. These portions of the surface are bounded by lines in which the potential is equal to that at P. Along lines drawn from P to points at which the potential is less than that at P the electrical force is from P, and along lines drawn to points of greater potential the force is towards P. Hence the point P is a point of stable equilibrium for some displacements, and of unstable equilibrium for other displacements.

113.] To determine the number of the points and lines of equilibrium, let us consider the surface or surfaces for which the potential is equal to C, a given quantity. Let us call the regions in which the potential is less than C the negative regions, and those in which it is greater than C the positive regions. Let V_0 be the lowest, and V_1 the highest potential existing in the electric field. If we make $C = V_0$, the negative region will include only the electrified point or conductor of lowest potential, and this is necessarily electrified negatively. The positive region consists of the rest of space, and since it surrounds the negative region it is periphractic. See Art. 18.

If we now increase the value of C the negative region will expand, and new negative regions will be formed round negatively electrified bodies. For every negative region thus formed the surrounding positive region acquires one degree of periphraxy.

As the different negative regions expand, two or more of them may meet in a point or a line. If $n+1$ negative regions meet, the positive region loses n degrees of periphraxy, and the point or the line in which they meet is a point or line of equilibrium of the nth degree.

When C becomes equal to V_1 the positive region is reduced to the electrified point or conductor of highest potential, and has therefore lost all its periphraxy. Hence, if each point or line of equilibrium counts for one, two, or n according to its degree, the number so made up by the points or lines now considered will be one less than the number of negatively electrified bodies.

There are other points or lines of equilibrium which occur where the positive regions become separated from each other, and the negative region acquires periphraxy. The number of these, reckoned according to their degrees, is one less than the number of positively electrified bodies.

If we call a point or line of equilibrium positive when it is the meeting-place of two or more positive regions, and negative when the regions which unite there are negative, then, if there are p bodies positively and n bodies negatively electrified, the sum of the degrees of the positive points and lines of equilibrium will be $p-1$, and that of the negative ones $n-1$.

But, besides this definite number of points and lines of equilibrium arising from the junction of different regions, there may be others, of which we can only affirm that their number must be even. For if, as the negative region expands, it meets itself, it becomes a cyclic region, and it may acquire, by repeatedly meeting itself, any number of degrees of cyclosis, each of which corresponds to the point or line of equilibrium at which the cyclosis was established. As the negative region continues to expand till it fills all space, it loses every degree of cyclosis it has acquired, and becomes at last acyclic. Hence there is a set of points or lines of equilibrium at which cyclosis is lost, and these are equal in number of degrees to those at which it is acquired.

If the form of the electrified bodies or conductors is arbitrary, we can only assert that the number of these additional points or lines is even, but if they are electrified points or spherical conductors, the number arising in this way cannot exceed $(n-1)(n-2)$, where n is the number of bodies.

114.] The potential close to any point P may be expanded in the series

$$V = V_0 + H_1 + H_2 + \&c.;$$

where H_1, H_2, &c. are homogeneous functions of x, y, z, whose dimensions are 1, 2, &c. respectively.

Since the first derivatives of V vanish at a point of equilibrium, $H_1 = 0$, if P be a point of equilibrium.

Let H_i be the first function which does not vanish, then close to the point P we may neglect all functions of higher degrees as compared with H_i.

Now
$$H_i = 0$$
is the equation of a cone of the degree i, and this cone is the cone of closest contact with the equipotential surface at P.

It appears, therefore, that the equipotential surface passing through P has, at that point, a conical point touched by a cone of the second or of a higher degree.

If the point P is not on a line of equilibrium this cone does not intersect itself, but consists of i sheets or some smaller number.

If the nodal line intersects itself, then the point P is on a line of equilibrium, and the equipotential surface through P cuts itself in that line.

If there are intersections of the nodal line not on opposite points of the sphere, then P is at the intersection of three or more lines of equilibrium. For the equipotential surface through P must cut itself in each line of equilibrium.

115.] If two sheets of the same equipotential surface intersect, they must intersect at right angles.

For let the tangent to the line of intersection be taken as the axis of z, then $\dfrac{d^2 V}{dz^2} = 0$. Also let the axis of x be a tangent to one of the sheets, then $\dfrac{d^2 V}{dx^2} = 0$. It follows from this, by Laplace's equation, that $\dfrac{d^2 V}{dy^2} = 0$, or the axis of y is a tangent to the other sheet.

This investigation assumes that H_2 is finite. If H_2 vanishes, let the tangent to the line of intersection be taken as the axis of z, and let $x = r\cos\theta$, and $y = r\sin\theta$, then, since

$$\frac{d^2 V}{dz^2} = 0, \quad \frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} = 0;$$

or
$$\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} + \frac{1}{r^2}\frac{d^2 V}{d\theta^2} = 0;$$

the solution of which equation in ascending powers of r is

$$V = V_0 + A_1 r\cos(\theta + a) + A_2 r^2 \cos(2\theta + a_2) + \&c. + A_i r^i \cos(i\theta + a_i).$$

At a point of equilibrium A_1 is zero. If the first term that does not vanish is that in r^i, then

$$V - V_0 = A_i r^i \cos(i\theta + a_i) + \text{terms in higher powers of } r.$$

This gives i sheets of the equipotential surface $V = V_0$, intersecting at angles each equal to $\frac{\pi}{i}$. This theorem was given by Rankine*.

It is only under certain conditions that a line of equilibrium can exist in free space, but there must be a line of equilibrium on the surface of a conductor whenever the electrification of the conductor is positive in one portion and negative in another.

In order that a conductor may be oppositely electrified in different portions of its surface, there must be in the field some places where the potential is higher than that of the body and others where it is lower. We must remember that at an infinite distance the potential is zero.

Let us begin with two conductors electrified positively to the same potential. There will be a point of equilibrium between the two bodies. Let the potential of the first body be gradually raised. The point of equilibrium will approach the other body, and as the process goes on it will coincide with a point on its surface. If the potential of the first body be now increased, the equipotential surface round the first body which has the same potential as the second body will cut the surface of the second body at right angles in a closed curve, which is a line of equilibrium.

Earnshaw's Theorem.

116.] An electrified body placed in a field of electric force cannot be in stable equilibrium.

First, let us suppose the electricity of the moveable body (A), and also that of the system of surrounding bodies (B), to be fixed in those bodies.

Let V be the potential at any point of the moveable body due to the action of the surrounding bodies (B), and let e be the electricity on a small portion of the moveable body A surrounding this point. Then the potential energy of A with respect to B will be

$$M = \Sigma(Ve),$$

where the summation is to be extended to every electrified portion of A.

* 'Summary of the Properties of certain Stream Lines,' *Phil. Mag.*, Oct. 1864. See also, Thomson and Tait's *Natural Philosophy*, § 780; and Rankine and Stokes, in the *Proc. R. S.*, 1867, p. 468; also W. R. Smith, *Proc. R. S. Edin.*, 1869–70, p. 79.

Let a, b, c be the coordinates of any electrified part of A with respect to axes fixed in A, and parallel to those of x, y, z. Let the coordinates of the point fixed in the body through which these axes pass be ξ, η, ζ.

Let us suppose for the present that the body A is constrained to move parallel to itself, then the absolute coordinates of the point a, b, c will be

$$x = \xi + a, \qquad y = \eta + b, \qquad z = \zeta + c.$$

The potential of the body A with respect to B may now be expressed as the sum of a number of terms, in each of which V is expressed in terms of a, b, c and ξ, η, ζ and the sum of these terms is a function of the quantities a, b, c, which are constant for each point of the body, and of ξ, η, ζ, which vary when the body is moved.

Since Laplace's equation is satisfied by each of these terms it is satisfied by their sum, or

$$\frac{d^2M}{d\xi^2} + \frac{d^2M}{d\eta^2} + \frac{d^2M}{d\zeta^2} = 0.$$

Now let a small displacement be given to A, so that

$$d\xi = l\,dr, \qquad d\eta = m\,dr, \qquad d\zeta = n\,dr;$$

then $\frac{dM}{dr}dr$ will be the increment of the potential of A with respect to the surrounding system B.

If this be positive, work will have to be done to increase r, and there will be a force $\frac{dM}{dr}$ tending to diminish r and to restore A to its former position, and for this displacement therefore the equilibrium will be stable. If, on the other hand, this quantity is negative, the force will tend to increase r, and the equilibrium will be unstable.

Now consider a sphere whose centre is the origin and whose radius is r, and so small that when the point fixed in the body lies within this sphere no part of the moveable body A can coincide with any part of the external system B. Then, since within the sphere $\nabla^2 M = 0$, the surface-integral

$$\iint \frac{dM}{dr}\,dS = 0,$$

taken over the surface of the sphere.

Hence, if at any part of the surface of the sphere $\frac{dM}{dr}$ is positive, there must be some other part of the surface where it is negative, and if the body A be displaced in a direction in which $\frac{dM}{dr}$ is negative, it will tend to move from its original position, and its equilibrium is therefore necessarily unstable.

The body therefore is unstable even when constrained to move parallel to itself, *à fortiori* it is unstable when altogether free.

Now let us suppose that the body A is a conductor. We might treat this as a case of equilibrium of a system of bodies, the moveable electricity being considered as part of that system, and we might argue that as the system is unstable when deprived of so many degrees of freedom by the fixture of its electricity, it must *à fortiori* be unstable when this freedom is restored to it.

But we may consider this case in a more particular way, thus—

First, let the electricity be fixed in A, and let it move through the small distance dr. The increment of the potential of A due to this cause is $\frac{dM}{dr}dr$.

Next, let the electricity be allowed to move within A into its position of equilibrium, which is always stable. During this motion the potential will necessarily be *diminished* by a quantity which we may call $C\,dr$.

Hence the total increment of the potential when the electricity is free to move will be

$$\left(\frac{dM}{dr} - C\right)dr;$$

and the force tending to bring A back towards its original position will be

$$\frac{dM}{dr} - C,$$

where C is always positive.

Now we have shewn that $\frac{dM}{dr}$ is negative for certain directions of r, hence when the electricity is free to move the instability in these directions will be increased.

CHAPTER VII.

FORMS OF THE EQUIPOTENTIAL SURFACES AND LINES OF INDUCTION IN SIMPLE CASES.

117.] WE have seen that the determination of the distribution of electricity on the surface of conductors may be made to depend on the solution of Laplace's equation

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0,$$

V being a function of x, y, and z, which is always finite and continuous, which vanishes at an infinite distance, and which has a given constant value at the surface of each conductor.

It is not in general possible by known mathematical methods to solve this equation so as to fulfil arbitrarily given conditions, but it is always possible to assign various forms to the function V which shall satisfy the equation, and to determine in each case the forms of the conducting surfaces, so that the function V shall be the true solution.

It appears, therefore, that what we should naturally call the inverse problem of determining the forms of the conductors from the potential is more manageable than the direct problem of determining the potential when the form of the conductors is given.

In fact, every electrical problem of which we know the solution has been constructed by an inverse process. It is therefore of great importance to the electrician that he should know what results have been obtained in this way, since the only method by which he can expect to solve a new problem is by reducing it to one of the cases in which a similar problem has been constructed by the inverse process.

This historical knowledge of results can be turned to account in two ways. If we are required to devise an instrument for making electrical measurements with the greatest accuracy, we may select those forms for the electrified surfaces which correspond to cases of which we know the accurate solution. If, on the other hand, we are required to estimate what will be the electrification of bodies whose forms are given, we may begin with some case in which one of the equipotential surfaces takes a form somewhat resembling the given form, and then by a tentative method we may modify the problem till it more nearly corresponds to the given case. This method is evidently very imperfect considered from a mathematical point of view, but it is the only one we have, and if we are not allowed to choose our conditions, we can make only an approximate calculation of the electrification. It appears, therefore, that what we want is a knowledge of the forms of equipotential surfaces and lines of induction in as many different cases as we can collect together and remember. In certain classes of cases, such as those relating to spheres, we may proceed by mathematical methods. In other cases we cannot afford to despise the humbler method of actually drawing tentative figures on paper, and selecting that which appears least unlike the figure we require.

This latter method I think may be of some use, even in cases in which the exact solution has been obtained, for I find that an eye-knowledge of the forms of the equipotential surfaces often leads to a right selection of a mathematical method of solution.

I have therefore drawn several diagrams of systems of equipotential surfaces and lines of force, so that the student may make himself familiar with the forms of the lines. The methods by which such diagrams may be drawn will be explained as we go on, as they belong to questions of different kinds.

118.] In the first figure at the end of this volume we have the equipotential surfaces surrounding two points electrified with quantities of electricity of the same kind and in the ratio of 20 to 5.

Here each point is surrounded by a system of equipotential surfaces which become more nearly spheres as they become smaller, but none of them are accurately spheres. If two of these surfaces, one surrounding each sphere, be taken to represent the surfaces of two conducting bodies, nearly but not quite spherical, and if these bodies be charged with the same kind of electricity, the charges being as 4 to 1, then the diagram will represent the equipotential surfaces, provided we expunge all those which are drawn inside the two bodies. It appears from the diagram that the action between the bodies will be the same as that between two points having the same charges, these points being not exactly in the middle of the axis of each body, but somewhat more remote than the middle point from the other body.

The same diagram enables us to see what will be the distribution of electricity on one of the oval figures, larger at one end than the other, which surround both centres. Such a body, if electrified with a charge 25 and free from external influence, will have the surface-density greatest at the small end, less at the large end, and least in a circle somewhat nearer the smaller than the larger end.

There is one equipotential surface, indicated by a dotted line, which consists of two lobes meeting at the conical point P. That point is a point of equilibrium, and the surface-density on a body of the form of this surface would be zero at this point.

The lines of force in this case form two distinct systems, divided from one another by a surface of the sixth degree, indicated by a dotted line, passing through the point of equilibrium, and somewhat resembling one sheet of the hyperboloid of two sheets.

This diagram may also be taken to represent the lines of force and equipotential surfaces belonging to two spheres of gravitating matter whose masses are as 4 to 1.

119.] In the second figure we have again two points whose charges are as 4 to 1, but the one positive and the other negative. In this case one of the equipotential surfaces, that, namely, corresponding to potential zero, is a sphere. It is marked in the diagram by the dotted circle Q. The importance of this spherical surface will be seen when we come to the theory of Electrical Images.

We may see from this diagram that if two round bodies are charged with opposite kinds of electricity they will attract each other as much as two points having the same charges but placed somewhat nearer together than the middle points of the round bodies.

Here, again, one of the equipotential surfaces, indicated by a dotted line, has two lobes, an inner one surrounding the point whose charge is 5 and an outer one surrounding both bodies, the two lobes meeting in a conical point P which is a point of equilibrium.

If the surface of a conductor is of the form of the outer lobe, a roundish body having, like an apple, a conical dimple at one end of its axis, then, if this conductor be electrified, we shall be able to determine the superficial density at any point. That at the bottom of the dimple will be zero.

Surrounding this surface we have others having a rounded dimple which flattens and finally disappears in the equipotential surface passing through the point marked M.

The lines of force in this diagram form two systems divided by a surface which passes through the point of equilibrium.

If we consider points on the axis on the further side of the point B, we find that the resultant force diminishes to the double point P, where it vanishes. It then changes sign, and reaches a maximum at M, after which it continually diminishes.

This maximum, however, is only a maximum relatively to other points on the axis, for if we draw a surface perpendicular to the axis, M is a point of minimum force relatively to neighbouring points on that surface.

120.] Figure III represents the equipotential surfaces and lines of force due to an electrified point whose charge is 10 placed at A, and surrounded by a field of force, which, before the introduction of the electrified point, was uniform in direction and magnitude at every part. In this case, those lines of force which belong to A are contained within a surface of revolution which has an asymptotic cylinder, having its axis parallel to the undisturbed lines of force.

The equipotential surfaces have each of them an asymptotic plane. One of them, indicated by a dotted line, has a conical point and a lobe surrounding the point A. Those below this surface have one sheet with a depression near the axis. Those above have a closed portion surrounding A and a separate sheet with a slight depression near the axis.

If we take one of the surfaces below A as the surface of a conductor, and another a long way below A as the surface of another conductor at a different potential, the system of lines and surfaces between the two conductors will indicate the distribution of electric force. If the lower conductor is very far from A its surface will be very nearly plane, so that we have here the solution of the distribution of electricity on two surfaces, both of them nearly plane and parallel to each other, except that the upper one has a protuberance near its middle point, which is more or less prominent according to the particular equipotential line we choose for the surface.

121.] Figure IV represents the equipotential surfaces and lines of force due to three electrified points A, B and C, the charge of A being 15 units of positive electricity, that of B 12 units of negative electricity, and that of C 20 units of positive electricity. These points are placed in one straight line, so that

$$AB = 9, \quad BC = 16, \quad AC = 25.$$

In this case, the surface for which the potential is unity consists of two spheres whose centres are A and C and their radii 15 and 20.

These spheres intersect in the circle which cuts the plane of the paper in D and D', so that B is the centre of this circle and its radius is 12. This circle is an example of a line of equilibrium, for the resultant force vanishes at every point of this line.

If we suppose the sphere whose centre is A to be a conductor with a charge of 3 units of positive electricity, and placed under the influence of 20 units of positive electricity at C, the state of the case will be represented by the diagram if we leave out all the lines within the sphere A. The part of this spherical surface within the small circle DD' will be negatively electrified by the influence of C. All the rest of the sphere will be positively electrified, and the small circle DD' itself will be a line of no electrification.

We may also consider the diagram to represent the electrification of the sphere whose centre is C, charged with 8 units of positive electricity, and influenced by 15 units of positive electricity placed at A.

The diagram may also be taken to represent the case of a conductor whose surface consists of the larger segments of the two spheres meeting in DD', charged with 23 units of positive electricity.

We shall return to the consideration of this diagram as an illustration of Thomson's *Theory of Electrical Images*. See Art. 168.

122.] I am anxious that these diagrams should be studied as illustrations of the language of Faraday in speaking of 'lines of force,' the 'forces of an electrified body,' &c.

In strict mathematical language the word Force is used to signify the supposed cause of the tendency which a material body is found to have towards alteration in its state of rest or motion. It is indifferent whether we speak of this observed tendency or of its immediate cause, since the cause is simply inferred from the effect, and has no other evidence to support it.

Since, however, we are ourselves in the practice of directing the motion of our own bodies, and of moving other things in this way, we have acquired a copious store of remembered sensations relating to these actions, and therefore our ideas of force are connected in our minds with ideas of conscious power, of exertion, and of fatigue, and of overcoming or yielding to pressure. These ideas, which give a colouring and vividness to the purely abstract idea of force, do not in mathematically trained minds lead to any practical error.

But in the vulgar language of the time when dynamical science was unknown, all the words relating to exertion, such as force, energy, power, &c., were confounded with each other, though some of the schoolmen endeavoured to introduce a greater precision into their language.

The cultivation and popularization of correct dynamical ideas since the time of Galileo and Newton has effected an immense change in the language and ideas of common life, but it is only within recent times, and in consequence of the increasing importance of machinery, that the ideas of force, energy, and power have become accurately distinguished from each other. Very few, however, even of scientific men, are careful to observe these distinctions; hence we often hear of the force of a cannon-ball when either its energy or its momentum is meant, and of the force of an electrified body when the quantity of its electrification is meant.

Now the quantity of electricity in a body is measured, according to Faraday's ideas, by the *number* of lines of force, or rather of induction, which proceed from it. These lines of force must all terminate somewhere, either on bodies in the neighbourhood, or on the walls and roof of the room, or on the earth, or on the heavenly bodies, and wherever they terminate there is a quantity of electricity exactly equal and opposite to that on the part of the body from which they proceeded. By examining the diagrams this will be seen to be the case. There is therefore no contradiction between Faraday's views and the mathematical results of the old theory, but, on the contrary, the idea of lines of force throws great light on these results, and seems to afford the means of rising by a continuous process from the somewhat rigid conceptions of the old theory to notions which may be capable of greater expansion, so as to provide room for the increase of our knowledge by further researches.

123.] These diagrams are constructed in the following manner:—

First, take the case of a single centre of force, a small electrified body with a charge E. The potential at a distance r is $V = \frac{E}{r}$; hence, if we make $r = \frac{E}{V}$, we shall find r, the radius of the sphere for which the potential is V. If we now give to V the values 1, 2, 3, &c., and draw the corresponding spheres, we shall obtain a series of equipotential surfaces, the potentials corresponding to which are measured by the natural numbers. The sections of these spheres by a plane passing through their common centre will be circles, which we may mark with the number denoting the potential of each. These are indicated by the dotted circles on the right hand of Fig. 6.

If there be another centre of force, we may in the same way draw the equipotential surfaces belonging to it, and if we now wish to find the form of the equipotential surfaces due to both centres together, we must remember that if V_1 be the potential due to one centre, and V_2 that due to the other, the potential due to both will be $V_1 + V_2 = V$. Hence, since at every intersection of the equipotential surfaces belonging to the two series we know both V_1 and V_2, we also know the value of V. If therefore we draw a surface which passes through all those intersections for which the value of V is the same, this surface will coincide with a true equipotential surface at all these intersections, and if the original systems of surfaces be drawn sufficiently close, the new surface may be drawn with any required degree of accuracy. The equipotential surfaces due to two points whose charges are equal and opposite are represented by the continuous lines on the right hand side of Fig. 6.

This method may be applied to the drawing of any system of equipotential surfaces when the potential is the sum of two potentials, for which we have already drawn the equipotential surfaces.

The lines of force due to a single centre of force are straight lines radiating from that centre. If we wish to indicate by these lines the intensity as well as the direction of the force at any point, we must draw them so that they mark out on the equipotential surfaces portions over which the surface-integral of induction has definite values. The best way of doing this is to suppose our plane figure to be the section of a figure in space formed by the revolution of the plane figure about an axis passing through the centre of force. Any straight line radiating from the centre and making an angle θ with the axis will then trace out a cone, and the surface-integral of the induction through that part of any surface which is cut off by this cone on the side next the positive direction of the axis, is $2\pi E(1-\cos\theta)$.

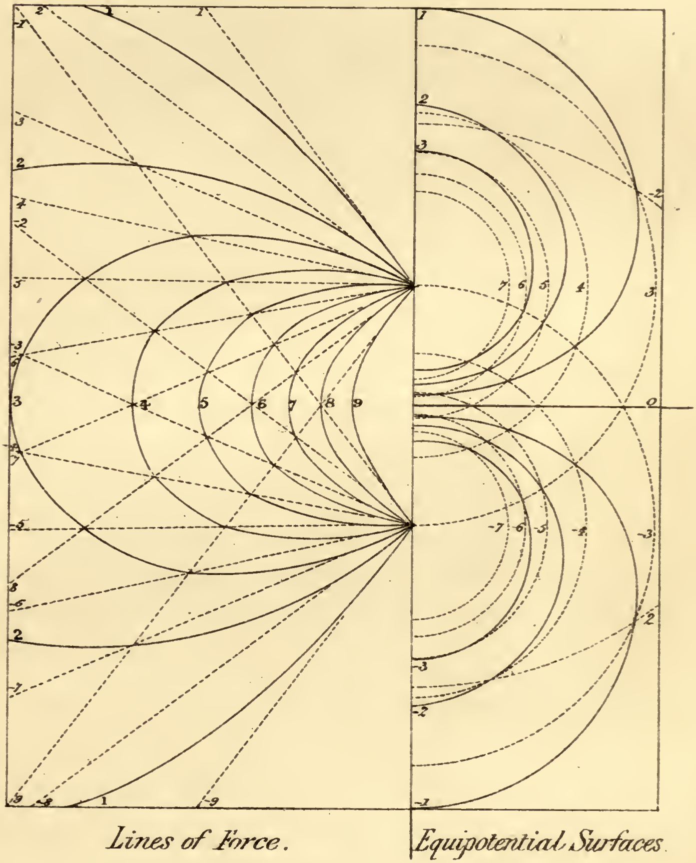
If we further suppose this surface to be bounded by its intersection with two planes passing through the axis, and inclined at the angle whose arc is equal to half the radius, then the induction through the surface so bounded is

$$E(1-\cos\theta) = 2\Psi, \text{ say;}$$

$$\text{and} \quad \theta = \cos^{-1}\left(1 - 2\frac{\Psi}{E}\right).$$

If we now give to Ψ a series of values 1, 2, 3 ... E, we shall find

Fig. 6.



Method of drawing
Lines of Force and Equipotential Surfaces.

For the Delegates of the Clarendon Press.

a corresponding series of values of θ, and if E be an integer, the number of corresponding lines of force, including the axis, will be equal to E.

We have therefore a method of drawing lines of force so that the charge of any centre is indicated by the number of lines which converge to it, and the induction through any surface cut off in the way described is measured by the number of lines of force which pass through it. The dotted straight lines on the left hand side of Fig. 6 represent the lines of force due to each of two electrified points whose charges are 10 and -10 respectively.

If there are two centres of force on the axis of the figure we may draw the lines of force for each axis corresponding to values of Ψ_1 and Ψ_2, and then, by drawing lines through the consecutive intersections of these lines, for which the value of $\Psi_1+\Psi_2$ is the same, we may find the lines of force due to both centres, and in the same way we may combine any two systems of lines of force which are symmetrically situated about the same axis. The continuous curves on the left hand side of Fig. 6 represent the lines of force due to the two electrified points acting at once.

After the equipotential surfaces and lines of force have been constructed by this method the accuracy of the drawing may be tested by observing whether the two systems of lines are everywhere orthogonal, and whether the distance between consecutive equipotential surfaces is to the distance between consecutive lines of force as half the distance from the axis is to the assumed unit of length.

In the case of any such system of finite dimensions the line of force whose index number is Ψ has an asymptote which passes through the centre of gravity of the system, and is inclined to the axis at an angle whose cosine is $1-2\frac{\Psi}{E}$, where E is the total electrification of the system, provided Ψ is less than E. Lines of force whose index is greater than E are finite lines.

The lines of force corresponding to a field of uniform force parallel to the axis are lines parallel to the axis, the distances from the axis being the square roots of an arithmetical series.

The theory of equipotential surfaces and lines of force in two dimensions will be given when we come to the theory of conjugate functions *.

* See a paper 'On the Flow of Electricity in Conducting Surfaces,' by Prof. W. R. Smith, *Proc. R. S. Edin.*, 1869-70, p. 79.

CHAPTER VIII.

SIMPLE CASES OF ELECTRIFICATION.

Two Parallel Planes.

124.] We shall consider, in the first place, two parallel plane conducting surfaces of infinite extent, at a distance c from each other, maintained respectively at potentials A and B.

It is manifest that in this case the potential V will be a function of the distance z from the plane A, and will be the same for all points of any parallel plane between A and B, except near the boundaries of the electrified surfaces, which by the supposition are at an infinitely great distance from the point considered.

Hence, Laplace's equation becomes reduced to

$$\frac{d^2 V}{dz^2} = 0,$$

the integral of which is

$$V = C_1 + C_2 z;$$

and since when $z = 0$, $V = A$, and when $z = c$, $V = B$,

$$V = A + (B - A)\frac{z}{c}.$$

For all points between the planes, the resultant electrical force is normal to the planes, and its magnitude is

$$R = \frac{A - B}{c}.$$

In the substance of the conductors themselves, $R = 0$. Hence the distribution of electricity on the first plane has a surface-density σ, where

$$4\pi\sigma = R = \frac{A - B}{c}.$$

On the other surface, where the potential is B, the surface-density σ' will be equal and opposite to σ, and

$$4\pi\sigma' = -R = \frac{B - A}{c}.$$

Let us next consider a portion of the first surface whose area is S, taken so that no part of S is near the boundary of the surface.

The quantity of electricity on this surface is $E_1 = S\sigma$, and, by Art. 79, the force acting on every unit of electricity is $\frac{1}{2}R$, so that the whole force acting on the area S, and attracting it towards the other plane, is

$$F = \tfrac{1}{2}RS\sigma = \frac{1}{8\pi}R^2 S = \frac{S}{8\pi}\,\frac{(B-A)^2}{c^2}.$$

Here the attraction is expressed in terms of the area S, the difference of potentials of the two surfaces $(A-B)$, and the distance between them c. The attraction, expressed in terms of the charge E_1 on the area S, is

$$F = \frac{2\pi}{S}E_1^2.$$

The electrical energy due to the distribution of electricity on the area S, and that on an area S' on the surface B defined by projecting S on the surface B by a system of lines of force, which in this case are normals to the planes, is

$$\begin{aligned} Q &= \tfrac{1}{2}(E_1 A + E_2 B), \\ &= \tfrac{1}{2}\left(\frac{S}{4\pi}\,\frac{(A-B)^2}{c}\right), \\ &= \frac{R^2}{8\pi}Sc, \\ &= \frac{2\pi}{S}E_1^2 c, \\ &= Fc. \end{aligned}$$

The first of these expressions is the general expression of electrical energy.

The second gives the energy in terms of the area, the distance, and the difference of potentials.

The third gives it in terms of the resultant force R, and the volume Sc included between the areas S and S', and shews that the energy in unit of volume is p where $8\pi p = R^2$.

The attraction between the planes is pS, or in other words, there is an electrical tension (or negative pressure) equal to p on every unit of area.

The fourth expression gives the energy in terms of the charge.

The fifth shews that the electrical energy is equal to the work which would be done by the electric force if the two surfaces were to be brought together, moving parallel to themselves, with their electric charges constant.

To express the charge in terms of the difference of potentials, we have

$$E_1 = \frac{1}{4\pi}\frac{S}{c}(B-A) = q\,(B-A).$$

The coefficient $\frac{1}{4\pi}\frac{S}{c} = q$ represents the charge due to a difference of potentials equal to unity. This coefficient is called the Capacity of the surface S, due to its position relatively to the opposite surface.

Let us now suppose that the medium between the two surfaces is no longer air but some other dielectric substance whose specific inductive capacity is K, then the charge due to a given difference of potentials will be K times as great as when the dielectric is air, or

$$E_1 = \frac{KS}{4\pi c}(B-A).$$

The total energy will be

$$Q = \frac{KS}{8\pi c}(B-A)^2,$$

$$= \frac{2\pi}{KS}E_1^2 c.$$

The force between the surfaces will be

$$F = pS = \frac{KS}{8\pi}\frac{(B-A)^2}{c^2},$$

$$= \frac{2\pi}{KS}E_1^2.$$

Hence the force between two surfaces kept at given potentials varies directly as K, the specific capacity of the dielectric, but the force between two surfaces charged with given quantities of electricity varies inversely as K.

Two Concentric Spherical Surfaces.

125.] Let two concentric spherical surfaces of radii a and b, of which b is the greater, be maintained at potentials A and B respectively, then it is manifest that the potential V is a function of r the distance from the centre. In this case, Laplace's equation becomes

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$$

The integral of this is

$$V = C_1 + C_2 r^{-1};$$

and the condition that $V = A$ when $r = a$, and $V = B$ when $r = b$, gives for the space between the spherical surfaces,

$$V = \frac{Aa - Bb}{a-b} + \frac{A-B}{a^{-1}-b^{-1}}r^{-1};$$

$$R = -\frac{dV}{dr} = \frac{A-B}{a^{-1}-b^{-1}}r^{-2}.$$

If σ_1, σ_2 are the surface-densities on the opposed surfaces of a solid sphere of radius a, and a spherical hollow of radius b, then

$$\sigma_1 = \frac{1}{4\pi a^2}\frac{A-B}{a^{-1}-b^{-1}}, \quad \sigma_2 = \frac{1}{4\pi b^2}\frac{B-A}{a^{-1}-b^{-1}}.$$

If E_1 and E_2 be the whole charges of electricity on these surfaces,

$$E_1 = 4\pi a^2 \sigma_1 = \frac{A-B}{a^{-1}-b^{-1}} = -E_2.$$

The capacity of the enclosed sphere is therefore $\frac{ab}{b-a}$.

If the outer surface of the shell be also spherical and of radius c, then, if there are no other conductors in the neighbourhood, the charge on the outer surface is

$$E_3 = Bc.$$

Hence the whole charge on the inner sphere is

$$E_1 = \frac{ab}{b-a}(A-B),$$

and that of the outer

$$E_2 + E_3 = \frac{ab}{b-a}(B-A) + Bc.$$

If we put $b = \infty$, we have the case of a sphere in an infinite space. The electric capacity of such a sphere is a, or it is numerically equal to its radius.

The electric tension on the inner sphere per unit of area is

$$p = \frac{1}{8\pi}\frac{b^2}{a^2}\frac{(A-B)^2}{(b-a)^2}.$$

The resultant of this tension over a hemisphere is $\pi a^2 p = F$ normal to the base of the hemisphere, and if this is balanced by a surface tension exerted across the circular boundary of the hemisphere, the tension on unit of length being T, we have

$$F = 2\pi a T.$$

Hence
$$F = \frac{b^2}{8}\frac{(A-B)^2}{(b-a)^2} = \frac{E_1^2}{8a^2},$$

$$T = \frac{b^2}{16\pi a}\frac{(A-B)^2}{(b-a)^2}.$$

If a spherical soap bubble is electrified to a potential A, then, if its radius is a, the charge will be Aa, and the surface-density will be

$$\sigma = \frac{1}{4\pi}\frac{A}{a}.$$

The resultant electrical force just outside the surface will be $4\pi\sigma$, and inside the bubble it is zero, so that by Art. 79 the electrical force on unit of area of the surface will be $2\pi\sigma^2$, acting outwards. Hence the electrification will diminish the pressure of the air within the bubble by $2\pi\sigma^2$, or

$$\frac{1}{8\pi}\frac{A^2}{a^2}.$$

But it may be shewn that if T is the tension which the liquid film exerts across a line of unit length, then the pressure from within required to keep the bubble from collapsing is $2\frac{T}{a}$. If the electrical force is just sufficient to keep the bubble in equilibrium when the air within and without is at the same pressure

$$A^2 = 16\pi a T.$$

Two Infinite Coaxal Cylindric Surfaces.

126.] Let the radius of the outer surface of a conducting cylinder be a, and let the radius of the inner surface of a hollow cylinder, having the same axis with the first, be b. Let their potentials be A and B respectively. Then, since the potential V is in this case a function of r, the distance from the axis, Laplace's equation becomes

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0,$$

whence $$V = C_1 + C_2 \log r.$$

Since $V = A$ when $r = a$, and $V = B$ when $r = b$,

$$V = \frac{A\log\frac{b}{r} + B\log\frac{r}{a}}{\log\frac{b}{a}}.$$

If σ_1, σ_2 are the surface-densities on the inner and outer surfaces,

$$4\pi\sigma_1 = \frac{A-B}{a\log\frac{b}{a}}, \qquad 4\pi\sigma_2 = \frac{B-A}{b\log\frac{b}{a}}.$$

If E_1 and E_2 are the charges on a portion of the two cylinders of length l, measured along the axis,

$$E_1 = 2\pi a l \sigma_1 = \frac{1}{2}\frac{A-B}{\log\frac{b}{a}}\,l = -E_2.$$

The capacity of a length l of the interior cylinder is therefore

$$\frac{1}{2}\frac{l}{\log\frac{b}{a}}.$$

If the space between the cylinders is occupied by a dielectric of specific capacity K instead of air, then the capacity of the inner cylinder is

$$\frac{1}{2}\frac{lK}{\log\frac{b}{a}}.$$

The energy of the electrical distribution on the part of the infinite cylinder which we have considered is

$$\frac{1}{4}\frac{lK(A-B)^2}{\log\frac{b}{a}}.$$

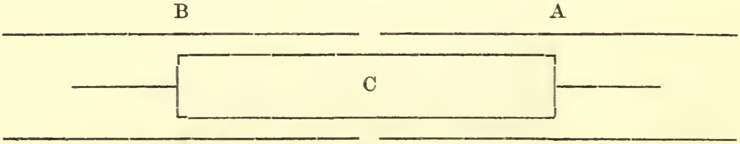


Fig. 5.

127.] Let there be two hollow cylindric conductors A and B, Fig. 5, of indefinite length, having the axis of x for their common axis, one on the positive and the other on the negative side of the origin, and separated by a short interval near the origin of coordinates.

Let a hollow cylinder C of length $2l$ be placed with its middle point at a distance x on the positive side of the origin, so as to extend into both the hollow cylinders.

Let the potential of the positive hollow cylinder be A, that of the negative one B, and that of the internal one C, and let us put α for the capacity per unit of length of C with respect to A, and β for the same quantity with respect to B.

The capacities of the parts of the cylinders near the origin and near the ends of the inner cylinder will not be affected by the value of x provided a considerable length of the inner cylinder enters each of the hollow cylinders. Near the ends of the hollow cylinders, and near the ends of the inner cylinder, there will be distributions of electricity which we are not yet able to calculate, but the distribution near the origin will not be altered by the motion of the inner cylinder provided neither of its ends comes near the origin, and the distributions at the ends of the inner cylinder will move with it, so that the only effect of the motion will be to increase or diminish the length of those parts of the inner cylinder where the distribution is similar to that on an infinite cylinder.

Hence the whole energy of the system will be, so far as it depends on x,

$$Q = \tfrac{1}{2}\alpha(l+x)(C-A)^2 + \tfrac{1}{2}\beta(l-x)(C-B)^2 + \text{quantities independent of } x;$$

and the resultant force parallel to the axis of the cylinders will be

$$X = \frac{dQ}{dx} = \tfrac{1}{2}\alpha(C-A)^2 - \tfrac{1}{2}\beta(C-B)^2.$$

If the cylinders A and B are of equal section, $\alpha = \beta$, and

$$X = \alpha(B-A)(C-\tfrac{1}{2}(A+B)).$$

It appears, therefore, that there is a constant force acting on the inner cylinder tending to draw it into that one of the outer cylinders from which its potential differs most.

If C be numerically large and $A+B$ comparatively small, then the force is approximately $X = \alpha(B-A)C$;

so that the difference of the potentials of the two cylinders can be measured if we can measure X, and the delicacy of the measurement will be increased by raising C, the potential of the inner cylinder.

This principle in a modified form is adopted in Thomson's Quadrant Electrometer, Art. 219.

The same arrangement of three cylinders may be used as a measure of capacity by connecting B and C. If the potential of A is zero, and that of B and C is V, then the quantity of electricity on A will be

$$E_3 = (q_{13} + \alpha(l+x))V;$$

so that by moving C to the right till x becomes $x+\xi$ the capacity of the cylinder C becomes increased by the definite quantity $\alpha\xi$, where

$$\alpha = \frac{1}{2\log\frac{b}{a}},$$

a and b being the radii of the opposed cylindric surfaces.

CHAPTER IX.

SPHERICAL HARMONICS.

On Singular Points at which the Potential becomes Infinite.

128.] We have already shewn that the potential due to a quantity of electricity e, condensed at a point whose coordinates are (a, b, c), is

$$V = \frac{e}{r}; \tag{1}$$

where r is the distance from the point (a, b, c) to the point (x, y, z), and V is the potential at the point (x, y, z).

At the point (a, b, c) the potential and all its derivatives become infinite, but at every other point they are finite and continuous, and the second derivatives of V satisfy Laplace's equation.

Hence, the value of V, as given by equation (1), may be the actual value of the potential in the space outside a closed surface surrounding the point (a, b, c), but we cannot, except for purely mathematical purposes, suppose this form of the function to hold up to and at the point (a, b, c) itself. For the resultant force close to the point would be infinite, a condition which would necessitate a discharge through the dielectric surrounding the point, and besides this it would require an infinite expenditure of work to charge a point with a finite quantity of electricity.

We shall call a point of this kind an infinite point of degree zero. The potential and all its derivatives at such a point are infinite, but the product of the potential and the distance from the point is ultimately a finite quantity e when the distance is diminished without limit. This quantity e is called the *charge* of the infinite point.

This may be shewn thus. If V' be the potential due to other electrified bodies, then near the point V' is everywhere finite, and the whole potential is

$$V = V' + \frac{e}{r},$$

whence

$$Vr = V'r + e.$$

When r is indefinitely diminished V' remains finite, so that ultimately
$$Vr = e.$$

129.] There are other kinds of singular points, the properties of which we shall now investigate, but, before doing so, we must define some expressions which we shall find useful in emancipating our ideas from the thraldom of systems of coordinates.

An *axis* is any definite direction in space. We may suppose it defined in Cartesian coordinates by its three direction-cosines l, m, n, or, better still, we may suppose a mark made on the surface of a sphere where the radius drawn *from* the centre in the direction of the axis meets the surface. We may call this point the Pole of the axis. An axis has therefore one pole only, not two.

If through any point x, y, z a plane be drawn perpendicular to the axis, the perpendicular from the origin on the plane is
$$p = lx+my+nz. \tag{2}$$

The operation
$$\frac{d}{dh} = l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}, \tag{3}$$
is called Differentiation with respect to an axis h whose direction-cosines are l, m, n.

Different axes are distinguished by different suffixes.

The cosine of the angle between the vector r and any axis h_i is denoted by λ_i, and the vector resolved in the direction of the axis by p_i, where
$$\lambda_i r = l_i x + m_i y + n_i z = p_i. \tag{4}$$

The cosine of the angle between two axes h_i and h_j is denoted by μ_{ij} where
$$\mu_{ij} = l_i l_j + m_i m_j + n_i n_j. \tag{5}$$

From these definitions it is evident that
$$\frac{dr}{dh_i} = \frac{p_i}{r} = \lambda_i, \tag{6}$$
$$\frac{dp_j}{dh_i} = \mu_{ij} = \frac{dp_i}{dh_j}, \tag{7}$$
$$\frac{d\lambda_i}{dh_j} = \frac{\mu_{ij} - \lambda_i \lambda_j}{r}. \tag{8}$$

Now let us suppose that the potential at the point (x, y, z) due to a singular point of any degree placed at the origin is
$$Mf(x, y, z).$$

If such a point be placed at the extremity of the axis h, the potential at (x, y, z) will be
$$Mf((x-lh), (y-mh), (z-nh));$$

and if a point in all respects equal and of opposite sign be placed at the origin, the potential due to the pair of points will be

$$V = Mf\{(x-lh), (y-mh), (z-nh)\} - Mf(x, y, z),$$

$$= -Mh\frac{d}{dh}F(x, y, z) + \text{terms containing } h^2.$$

f

If we now diminish h and increase M without limit, their product Mh remaining constant and equal to M', the ultimate value of the potential of the pair of points will be

$$V' = -M'\frac{d}{dh}f(x, y, z). \tag{9}$$

If $f(x, y, z)$ satisfies Laplace's equation, then V', which is the difference of two functions, each of which separately satisfies the equation, must itself satisfy it.

If we begin with an infinite point of degree zero, for which

$$V_0 = M_0\frac{1}{r}, \tag{10}$$

we shall get for a point of the first degree

$$V_1 = -M_1\frac{d}{dh_1}\frac{1}{r},$$

$$= M_1\frac{p_1}{r^3} = M_1\frac{\lambda_1}{r^2}. \tag{11}$$

A point of the first degree may be supposed to consist of two points of degree zero, having equal and opposite charges M_0 and $-M_0$, and placed at the extremities of the axis h. The length of the axis is then supposed to diminish and the magnitude of the charges to increase, so that their product $M_0 h$ is always equal to M_1. The ultimate result of this process when the two points coincide is a point of the first degree, whose moment is M_1 and whose axis is h_1. A point of the first degree may therefore be called a Double point.

By placing two equal and opposite points of the first degree at the extremities of the second axis h_2, and making $M_1 h_2 = M_2$, we get by the same process a point of the second degree whose potential is

$$V_2 = -h_2\frac{d}{dh_2}V_1,$$

$$= M_2\frac{d^2}{dh_1\,dh_2}\frac{1}{r},$$

$$= M_2\frac{3\lambda_1\lambda_2 - \mu_{12}}{r^3}. \tag{12}$$

We may call a point of the second degree a Quadruple point, because it is constructed by making four points approach each other. It has two axes, h_1 and h_2, and a moment M_2. The *directions* of these two axes and the *magnitude* of the moment completely define the nature of the point.

130.] Let us now consider an infinite point of degree i having i axes, each of which is defined by a mark on a sphere or by two angular coordinates, and having also its moment M_i, so that it is defined by $2i+1$ independent quantities. Its potential is obtained by differentiating V_0 with respect to the i axes in succession, so that it may be written

$$V_i = (-1)^i M_i \frac{d^i}{dh_1 \dots dh_i} \cdot \frac{1}{r}. \tag{13}$$

The result of the operation is of the form

$$V_i = \underline{|i}\, M_i \frac{Y_i}{r^{i+1}}, \tag{14}$$

where Y_i, which is called the Surface Harmonic, is a function of the i cosines, $\lambda_1 \dots \lambda_i$ of the angles between r and the i axes, and of the $\frac{1}{2} i(i-1)$ cosines, μ_{12}, &c. of the angles between the different axes themselves. In what follows we shall suppose the moment M_i unity.

Every term of Y_i consists of products of these cosines of the form

$$\mu_{12} \cdot \mu_{34} \cdots \mu_{2s-1 \cdot 2s} \lambda_{2s+1} \cdots \lambda_i,$$

in which there are s cosines of angles between two axes, and $i-2s$ cosines of angles between the axes and the radius vector. As each axis is introduced by one of the i processes of differentiation, the symbol of that axis must occur once and only once among the suffixes of these cosines.

Hence in every such product of cosines all the indices occur once, and none is repeated.

The number of different products of s cosines with double suffixes, and $i-2s$ cosines with single suffixes, is

$$N = \frac{\underline{|i}}{2^s \underline{|s} \underline{|i-2s}}. \tag{15}$$

For if we take any one of the N different terms we can form from it 2^s arrangements by altering the order of the suffixes of the cosines with double suffixes. From any one of these, again, we can form $\underline{|s}$ arrangements by altering the order of these cosines, and from any one of these we can form $\underline{|i-2s}$ arrangements by altering the order of the cosines with single suffixes. Hence, without altering the value of the term we may write it in $2^s \underline{|s} \underline{|i-2s}$

different ways, and if we do so to all the terms, we shall obtain the whole permutations of i symbols, the number of which is $\underline{|i}$.

Let the sum of all terms of this kind be written in the abbreviated form

$$\Sigma\,(\lambda^{i-2s}\,\mu^s).$$

If we wish to express that a particular symbol j occurs among the λ's only, or among the μ's only, we write it as a suffix to the λ or the μ. Thus the equation

$$\Sigma\,(\lambda^{i-2s}\,\mu^s) = \Sigma\,(\lambda_j^{\,i-2s}\,\mu^s) + \Sigma\,(\lambda^{i-2s}\,\mu_j^{\,s}) \tag{16}$$

expresses that the whole system of terms may be divided into two portions, in one of which the symbol j occurs among the direction-cosines of the radius vector, and in the other among the cosines of the angles between the axes.

Let us now assume that up to a certain value of i

$$\begin{aligned} Y_i = A_{i.0}\,\Sigma\,(\lambda^i) + A_{i.1}\,\Sigma\,(\lambda^{i-2}\,\mu^1) + \&\text{c}. \\ + A_{i.s}\,\Sigma\,(\lambda^{i-2s}\,\mu^s) + \&\text{c}. \end{aligned} \tag{17}$$

This is evidently true when $i=1$ and when $i=2$. We shall shew that if it is true for i it is true for $i+1$. We may write the series

$$Y_i = S\,\{A_{i.s}\,\Sigma\,(\lambda^{i-2s}\,\mu^s)\}, \tag{18}$$

where S indicates a summation in which all values of s not greater than $\frac{1}{2}\,i$ are to be taken.

Multiplying by $\underline{|i}\; r^{-(i+1)}$, and remembering that $p_i = r\,\lambda_i$, we obtain by (14), for the value of the solid harmonic of negative degree, and moment unity,

$$V_i = \underline{|i}\; S\,\{A_{i,s}\,r^{2s-2i-1}\,\Sigma\,(p^{i-2s}\,\mu^s)\}. \tag{19}$$

Differentiating V_i with respect to a new axis whose symbol is j, we should obtain V_{i+1} with its sign reversed,

$$\begin{aligned} -V_{i+1} = \underline{|i}\; S\,\{A_{i,s}(2s-2i-1)r^{2s-2i-3}\,\Sigma\,(p_j^{\,i-2s+1}\,\mu^s) \\ + A_{i,s}\,r^{2s-2i-1}\,\Sigma\,(p^{i-2s-1}\,\mu_j^{\,s+1})\}. \end{aligned} \tag{20}$$

If we wish to obtain the terms containing s cosines with double suffixes we must diminish s by unity in the second term, and we find

$$\begin{aligned} -V_{i+1} = \underline{|i}\; S\,\{r^{2s-2i-3}\,[A_{i,s}(2s-2i-1)\,\Sigma\,(p_j^{\,i-2s+1}\,\mu^s) \\ + A_{i,s-1}\,\Sigma\,(p^{i-2s+1}\,\mu_j^{\,s})]\}. \end{aligned} \tag{21}$$

If we now make

$$A_{i,s}\,(2s-2i-1) = A_{i,s-1} = -(i+1)\,A_{i+1.s} \tag{22}$$

then
$$V_{i+1} = \underline{|i+1}\; S\,\{A_{i+1.s}\,r^{2s-2(i+1)-1}\,\Sigma\,(p^{i+1-2s}\,\mu^s)\}, \tag{23}$$

and this value of V_{i+1} is the same as that obtained by changing i

into $i+1$ in the assumed expression, equation (19), for Y_i. Hence the assumed form of Y_i, in equation (19), if true for any value of i, is true for the next higher value.

To find the value of $A_{i.s}$, put $s=0$ in equation (22), and we find

$$A_{i+1.0} = \frac{2i+1}{i+1} A_{i.0}; \tag{24}$$

and therefore, since $A_{1.0}$ is unity,

$$A_{i.0} = \frac{\lfloor 2i}{2^i \lfloor i \lfloor i}; \tag{25}$$

and from this we obtain, by equation (22), for the general value of the coefficient

$$A_{i.s} = (-1)^s \frac{\lfloor 2i-2s}{2^{i-s} \lfloor i \lfloor i-s}; \tag{26}$$

and finally, the value of the trigonometrical expression for Y_i is

$$Y_i = S\left\{(-1)^s \frac{\lfloor 2i-2s}{2^{i-s} \lfloor i \lfloor i-s} \Sigma\left(\lambda^{i-2s}\mu^s\right)\right\}. \tag{27}$$

This is the most general expression for the spherical surface-harmonic of degree i. If i points on a sphere are given, then, if any other point P is taken on the sphere, the value of Y_i for the point P is a function of the i distances of P from the i points, and of the $\frac{1}{2}i(i-1)$ distances of the i points from each other. These i points may be called the Poles of the spherical harmonic. Each pole may be defined by two angular coordinates, so that the spherical harmonic of degree i has $2i$ independent constants, exclusive of its moment, M_i.

131.] The theory of spherical harmonics was first given by Laplace in the third book of his *Mécanique Celeste*. The harmonics themselves are therefore often called Laplace's Coefficients.

They have generally been expressed in terms of the ordinary spherical coordinates θ and ϕ, and contain $2i+1$ arbitrary constants. Gauss appears* to have had the idea of the harmonic being determined by the position of its poles, but I have not met with any development of this idea.

In numerical investigations I have often been perplexed on account of the apparent want of definiteness of the idea of a Laplace's Coefficient or spherical harmonic. By conceiving it as derived by the successive differentiation of $\frac{1}{r}$ with respect to i axes, and as expressed in terms of the positions of its i poles on a sphere, I

* Gauss. *Werke*, bd. v. s. 361.

have made the conception of the general spherical harmonic of any integral degree perfectly definite to myself, and I hope also to those who may have felt the vagueness of some other forms of the expression.

When the poles are given, the value of the harmonic for a given point on the sphere is a perfectly definite numerical quantity. When the form of the function, however, is given, it is by no means so easy to find the poles except for harmonics of the first and second degrees and for particular cases of the higher degrees.

Hence, for many purposes it is desirable to express the harmonic as the sum of a number of other harmonics, each of which has its axes disposed in a symmetrical manner.

Symmetrical System.

132.] The particular forms of harmonics to which it is usual to refer all others are deduced from the general harmonic by placing $i-\sigma$ of the poles at one point, which we shall call the Positive Pole of the sphere, and the remaining σ poles at equal distances round one half of the equator.

In this case $\lambda_1, \lambda_2, \dots \lambda_{i-\sigma}$ are each of them equal to $\cos\theta$, and $\lambda_{i-s+1} \dots \lambda_i$ are of the form $\sin\theta\cos(\phi-\beta)$. We shall write μ for $\cos\theta$ and ν for $\sin\theta$.

Also the value of $\mu_{jj'}$ is unity if j and j' are both less than $i-\sigma$, zero when one is greater and the other less than this quantity, and $\cos n\dfrac{\pi}{\sigma}$ when both are greater.

When all the poles are concentrated at the pole of the sphere, the harmonic becomes a zonal harmonic for which $\sigma = 0$. As the zonal harmonic is of great importance we shall reserve for it the symbol Q_i.

We may obtain its value either from the trigonometrical expression (27), or more directly by differentiation, thus

$$Q_i = (-1)^i \frac{r^{i+1}}{\lfloor i} \frac{d^i}{dz^i}\left(\frac{1}{r}\right), \tag{28}$$

$$Q_i = \frac{1.3.5\dots(2i-1)}{1.2.3\dots i}\left\{\mu^i - \frac{i(i-1)}{2.(2i-1)}\mu^{i-2} + \frac{i(i-1)(i-2)(i-3)}{2.4.(2i-1)(2i-3)}\mu^{i-4} - \&c.\right\}$$

$$= \Sigma_n \left\{(-1)^n \frac{\lfloor 2i-2n}{2^i \lfloor n \lfloor i-n \lfloor i-2n}\mu^{i-2n}\right\}. \tag{29}$$

It is often convenient to express Q_i as a homogeneous function of $\cos\theta$ and $\sin\theta$, which we shall write μ and ν respectively,

$$Q_i = \mu^i - \frac{i(i-1)}{2.2}\mu^{i-2}\nu^2 + \frac{i(i-1)(i-2)(i-3)}{2.2\ .\ 4.4}\mu^{i-4}\nu^4 - \&c.$$

$$= \Sigma_n \left\{ (-1)^n \frac{\lfloor i}{2^{2n} \lfloor n \lfloor n \lfloor i-2n} \mu^{i-2n}\nu^{2n} \right\}. \tag{30}$$

In this expansion the coefficient of μ_i is unity, and all the other terms involve ν. Hence at the pole, where $\mu = 1$ and $\nu = 0$, $Q_i = 1$.

It is shewn in treatises on Laplace's Coefficients that Q_i is the coefficient of h^i in the expansion of $(1-2\mu h + h^2)^{-\frac{1}{2}}$.

The other harmonics of the symmetrical system are most conveniently obtained by the use of the imaginary coordinates given by Thomson and Tait, *Natural Philosophy*, vol. i. p. 148,

$$\xi = x + \sqrt{-1}\,y, \qquad \eta = x - \sqrt{-1}\,y. \tag{31}$$

The operation of differentiating with respect to σ axes in succession, whose directions make angles $\frac{\pi}{\sigma}$ with each other in the plane of the equator, may then be written

$$\frac{d^\sigma}{dh_1 \ldots dh_\sigma} = \frac{d^\sigma}{d\xi^\sigma} + \frac{d^\sigma}{d\eta^\sigma}. \tag{32}$$

The surface harmonic of degree i and type σ is found by differentiating $\frac{1}{r}$ with respect to i axes, σ of which are at equal intervals in the plane of the equator, while the remaining $i-\sigma$ coincide with that of z, multiplying the result by r^{i+1} and dividing by $\lfloor i$. Hence

$$Y_i^{(\sigma)} = (-1)^i \frac{r^{i+1}}{\lfloor i} \frac{d^{i-\sigma}}{dz^{i-\sigma}} \left(\frac{d^\sigma}{d\xi^\sigma} + \frac{d^\sigma}{d\eta^\sigma}\right)\left(\frac{1}{r}\right), \tag{33}$$

$$= (-1)^{i-s} \frac{\lfloor 2s}{2^{2s} \lfloor i \lfloor s} (\xi^\sigma + \eta^\sigma)\, r^{i+1} \frac{d^{i-\sigma}}{dz^{i-\sigma}} \frac{1}{r^{2\sigma+1}}. \tag{34}$$

Now $$\xi^\sigma + \eta^\sigma = 2\, r^\sigma \nu^\sigma \cos(\sigma\phi + \beta), \tag{35}$$

and $$\frac{d^{i-\sigma}}{dz^{i-\sigma}} \frac{1}{r^{2\sigma+1}} = (-1)^{i-\sigma} \frac{\lfloor i+\sigma}{\lfloor 2\sigma} \frac{1}{r^{i+\sigma+1}} \vartheta_i^{(\sigma)}. \tag{36}$$

Hence $$Y_i^{(\sigma)} = 2 \frac{\lfloor i+\sigma}{2^{2\sigma} \lfloor i \lfloor \sigma} \vartheta_i^{(\sigma)} \cos(\sigma\phi + \beta), \tag{37}$$

where the factor 2 must be omitted when $\sigma = 0$.

The quantity $\vartheta_i^{(\sigma)}$ is a function of θ, the value of which is given in Thomson and Tait's *Natural Philosophy*, vol. i. p. 149.

It may be derived from Q_i by the equation

$$\vartheta_i^{(\sigma)} = 2^\sigma \frac{\lfloor i-\sigma \lfloor \sigma}{\lfloor i+\sigma} \nu^\sigma \frac{d^\sigma}{d\mu^\sigma} Q_i, \tag{38}$$

where Q_i is expressed as a function of μ only.

Performing the differentiations on Q_i as given in equation (29), we obtain

$$\vartheta_i^{(\sigma)} = \nu^\sigma \Sigma \left\{ (-1)^n \frac{|\underline{i-\sigma}\; |\underline{\sigma}\; |\underline{2i-2n}}{2^{i-\sigma}\, |\underline{i+\sigma}\; |\underline{n}\; |\underline{i-n}\; |\underline{i-\sigma-2n}} \mu^{i-\sigma-2n} \right\}. \quad (39)$$

We may also express it as a homogeneous function of μ and ν,

$$\vartheta_i^{(\sigma)} = \nu^\sigma \Sigma \left\{ (-1)^n \frac{|\underline{i-\sigma}\; |\underline{\sigma}}{2^{2\sigma}\, |\underline{n}\; |\underline{\sigma+n}\; |\underline{i-\sigma-2n}} \mu^{i-\sigma-2n} \nu^{2n} \right\}. \quad (40)$$

In this expression the coefficient of the first term is unity, and the others may be written down in order by the application of Laplace's equation.

The following relations will be found useful in Electrodynamics. They may be deduced at once from the expansion of Q_i.

$$\mu\, Q_i - Q_{i+1} = \frac{1}{i+1} \nu^2 \frac{dQ_i}{d\mu} = \frac{i}{2} \nu\, \vartheta_i^1, \quad (41)$$

$$Q_{i-1} - \mu\, Q_i = \frac{1}{i} \nu^2 \frac{dQ_i}{d\mu} = \frac{i+1}{2} \nu\, \vartheta_i^1. \quad (42)$$

On Solid Harmonics of Positive Degree.

133.] We have hitherto considered the spherical surface harmonic Y_i as derived from the solid harmonic

$$V_i = |\underline{i}\, M_i \frac{Y_i}{r^{i+1}}.$$

This solid harmonic is a homogeneous function of the coordinates of the negative degree $-(i+1)$. Its values vanish at an infinite distance and become infinite at the origin.

We shall now shew that to every such function there corresponds another which vanishes at the origin and has infinite values at an infinite distance, and is the corresponding solid harmonic of positive degree i.

A solid harmonic in general may be defined as a homogeneous function of x, y, and z, which satisfies Laplace's equation

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0.$$

Let H_i be a homogeneous function of the degree i, such that

$$H_i = |\underline{i}\, M_i r^i Y_i = r^{2i+1} V_i. \quad (43)$$

Then
$$\frac{dH_i}{dx} = (2i+1) r^{2i-1} x V_i + r^{2i+1} \frac{dV_i}{dx},$$

$$\frac{d^2 H_i}{dx^2} = (2i+1)\left((2i-1)x^2 + r^2\right) r^{2i-3} V_i + 2(2i+1) r^{2i-1} x \frac{dV_i}{dx} + r^{2i+1} \frac{d^2 V_i}{dx^2}.$$

Hence

$$\frac{d^2 H_i}{dx^2} + \frac{d^2 H_i}{dy^2} + \frac{d^2 H_i}{dz^2} = (2i+1)(2i+2) r^{2i-1} V_i$$

$$+ 2(2i+1) r^{2i-1} \left(x\frac{dV_i}{dx} + y\frac{dV_i}{dy} + z\frac{dV_i}{dz}\right) + r^{2i+1}\left(\frac{d^2 V_i}{dx^2} + \frac{d^2 V_i}{dy^2} + \frac{d^2 V_i}{dz^2}\right). \quad (44)$$

Now, since V_i is a homogeneous function of negative degree $i+1$,

$$x\frac{dV_i}{dx} + y\frac{dV_i}{dy} + z\frac{dV_i}{dz} = -(i+1)V_i. \quad (45)$$

The first two terms therefore of the right hand member of equation (44) destroy each other, and, since V_i satisfies Laplace's equation, the third term is zero, so that H_i also satisfies Laplace's equation, and is therefore a solid harmonic of degree i.

We shall next shew that the value of H_i thus derived from V_i is of the most general form.

A homogeneous function of x, y, z of degree i contains

$$\tfrac{1}{2}(i+1)(i+2)$$

terms. But

$$-\nabla^2 H_i = \frac{d^2 H_i}{dx^2} + \frac{d^2 H_i}{dy^2} + \frac{d^2 H_i}{dz^2}$$

is a homogeneous function of degree $i-2$, and therefore contains $\frac{1}{2}i(i-1)$ terms, and the condition $\nabla^2 H_i = 0$ requires that each of these must vanish. There are therefore $\frac{1}{2}i(i-1)$ equations between the coefficients of the $\frac{1}{2}(i+1)(i+2)$ terms of the homogeneous function, leaving $2i+1$ independent constants in the most general form of H_i.

But we have seen that V_i has $2i+1$ independent constants, therefore the value of H_i is of the most general form.

Application of Solid Harmonics to the Theory of Electrified Spheres.

134.] The function V_i satisfies the condition of vanishing at infinity, but does not satisfy the condition of being everywhere finite, for it becomes infinite at the origin.

The function H_i satisfies the condition of being finite and continuous at finite distances from the origin, but does not satisfy the condition of vanishing at an infinite distance.

But if we determine a closed surface from the equation

$$V_i = H_i, \quad (46)$$

and make H_i the potential function within the closed surface and

V_i the potential outside it, then by making the surface-density σ satisfy the characteristic equation

$$\frac{dH_i}{dr} - \frac{dV_i}{dr} + 4\pi\sigma = 0, \tag{47}$$

we shall have a distribution of potential which satisfies all the conditions.

It is manifest that if H_i and V_i are derived from the same value of Y_i, the surface $H_i = V_i$ will be a spherical surface, and the surface-density will also be derived from the same value of Y_i.

Let a be the radius of the sphere, and let

$$H_i = Ar^i Y_i, \qquad V_i = B\frac{Y_i}{r^{i+1}}, \qquad \sigma = CY_i. \tag{48}$$

Then at the surface of the sphere, where $r = a$,

$$Aa^i = \frac{B}{a^{i+1}},$$

and
$$\frac{dV}{dr} - \frac{dH}{dr} = -4\pi\sigma;$$

or
$$(i+1)\frac{B}{a^{i+2}} + ia^{i-1}A = 4\pi C;$$

whence we find H_i and V_i in terms of C,

$$H_i = \frac{4\pi C}{2i+1}\frac{r^i}{a^{i-1}}Y_i, \qquad V_i = \frac{4\pi C}{2i+1}\frac{a^{i+2}}{r^{i+1}}Y_i. \tag{49}$$

We have now obtained an electrified system in which the potential is everywhere finite and continuous. This system consists of a spherical surface of radius a, electrified so that the surface-density is everywhere CY_i, where C is some constant density and Y_i is a surface harmonic of degree i. The potential inside this sphere, arising from this electrification, is everywhere H_i, and the potential outside the sphere is V_i.

These values of the potential within and without the sphere might have been obtained in any given case by direct integration, but the labour would have been great and the result applicable only to the particular case.

135.] We shall next consider the action between a spherical surface, rigidly electrified according to a spherical harmonic, and an external electrified system which we shall call E.

Let V be the potential at any point due to the system E, and V_i that due to the spherical surface whose surface-density is σ.

Then, by Green's theorem, the potential energy of E on the electrified surface is equal to that of the electrified surface on E, or

$$\iint V\sigma\, dS = \Sigma V_i\, dE, \tag{50}$$

where the first integration is to be extended over every element dS of the surface of the sphere, and the summation Σ is to be extended to every part dE of which the electrified system E is composed.

But the same potential function V_i may be produced by means of a combination of 2^i electrified points in the manner already described. Let us therefore find the potential energy of E on such a compound point.

If M_0 is the charge of a single point of degree zero, then $M_0 V$ is the potential energy of V on that point.

If there are two such points, a positive and a negative one, at the positive and negative ends of a line h_1, then the potential energy of E on the double point will be

$$-M_0 V + M_0\left(V + h_1 \frac{dV}{dh_1} + \tfrac{1}{2} h^2 \frac{d^2 V}{dh_1^{\,2}} + \&c.\right);$$

and when M_0 increases and h_1 diminishes indefinitely, but so that

$$M_0 h_1 = M_1,$$

the value of the potential energy will be for a point of the first degree

$$M_1 \frac{dV}{dh_1}.$$

Similarly for a point of degree i the potential energy with respect to E will be

$$M_i \frac{d^i V}{dh_1\, dh_2 \ldots dh_i}.$$

This is the value of the potential energy of E upon the singular point of degree i. That of the singular point on E is $\Sigma V_i\, dE$, and, by Green's theorem, these are equal. Hence, by equation (50),

$$\iint V\sigma\, dS = M_i \frac{d^i V}{dh_1 \ldots dh_i}.$$

If $\sigma = CY_i$ where C is a constant quantity, then, by equations (49) and (14),

$$M_i = \frac{4\pi C}{\lfloor i} \frac{a^{i+2}}{2i+1}. \tag{51}$$

Hence, if V is any potential function whatever which satisfies Laplace's equation within the spherical surface of radius a, then the integral of $VY_i\,dS$, extended over every element dS_1 of the surface of a sphere of radius a, is given by the equation

$$\iint VY_i\,dS = \frac{4\pi}{\lfloor i} \frac{a^{i+2}}{2i+1} \frac{d^i V}{dh_1 \dots dh_i}; \tag{52}$$

where the differentiations of V are taken with respect to the axes of the harmonic Y_i, and the value of the differential coefficient is that at the centre of the sphere.

136.] Let us now suppose that V is a solid harmonic of positive degree j of the form

$$V = \frac{r^j}{a^j} Y_j. \tag{53}$$

At the spherical surface, $r = a$, the value of V is the surface harmonic Y_j, and equation (52) becomes

$$\iint Y_i Y_j\,dS = \frac{4\pi}{\lfloor i} \frac{a^{i-j+2}}{2i+1} \frac{d^i (r^j Y_j)}{dh_1 \dots dh_i}, \tag{54}$$

where the value of the differential coefficient is that at the centre of the sphere.

When i is numerically different from j, the surface-integral of the product $Y_i\,Y_j$ vanishes. For, when i is less than j, the result of the differentiation in the second member of (54) is a homogeneous function of x, y, and z, of degree $j-i$, the value of which at the centre of the sphere is zero. If i is equal to j the result is a constant, the value of which will be determined in the next article. If the differentiation is carried further, the result is zero. Hence the surface-integral vanishes when i is greater than j.

137.] The most important case is that in which the harmonic $r^j Y_j$ is differentiated with respect to i new axes in succession, the numerical value of j being the same as that of i, but the directions of the axes being in general different. The final result in this case is a constant quantity, each term being the product of i cosines of angles between the different axes taken in pairs. The general form of such a product may be written symbolically

$$\mu_{ii}{}^s\, \mu_{jj}{}^s\, \mu_{ij}{}^{i-2s},$$

which indicates that there are s cosines of angles between pairs of axes of the first system and s between axes of the second system, the remaining $i-2s$ cosines being between axes one of which belongs to the first and the other to the second system.

In each product the suffix of every one of the $2i$ axes occurs once, and once only.

The number of different products for a given value of s is

$$N = \frac{(\lfloor i)^2}{2^{2s}(\lfloor s)^2 \lfloor i-2s}. \tag{55}$$

The final result is easily obtained by the successive differentiation of

$$r_j Y_j = \frac{1}{\lfloor j} S\left\{(-1)^s \frac{\lfloor 2j-2s}{2^{j-s}\lfloor j-s} r^{2s} \Sigma\left(p^{j-2s}\mu^s\right)\right\}.$$

Differentiating this i times in succession with respect to the new axes, so as to obtain any given combination of the axes in pairs, we find that in differentiating r^{2s} with respect to s of the new axes, which are to be combined with other axes of the new system, we introduce the numerical factor $2s(2s-2)\ldots 2$, or $2^s \lfloor s$. In continuing the differentiation the p's become converted into μ's, but no numerical factor is introduced. Hence

$$\frac{d^i}{dh_1 \ldots dh_i} r^j Y_j = \frac{1}{\lfloor i} S\left\{(-1)^s \frac{\lfloor 2i-2s \lfloor s}{2^{i-2s}\lfloor i-s} \Sigma\left(\mu_{ii}{}^s \mu_{jj}{}^s \mu_{ij}{}^{i-2s}\right)\right\}. \tag{56}$$

Substituting this result in equation (54) we find for the value of the surface-integral of the product of two surface harmonics of the same degree, taken over the surface of a sphere of radius a,

$$\iint Y_i Y_j dS = \frac{4\pi a^2}{(2i+1)(\lfloor i)^2} S\left\{(-1)^s \frac{\lfloor 2i-2s \lfloor s}{2^{i-2s}\lfloor i-s} \Sigma\left(\mu_{ii}{}^s \mu_{jj}{}^s \mu_{ij}{}^{i-2s}\right)\right\}. \tag{57}$$

This quantity differs from zero only when the two harmonics are of the same degree, and even in this case, when the distribution of the axes of the one system bears a certain relation to the distribution of the axes of the other, this integral vanishes. In this case, the two harmonics are said to be conjugate to each other.

On Conjugate Harmonics.

138.] If one harmonic is given, the condition that a second harmonic of the same degree may be conjugate to it is expressed by equating the right hand side of equation (57) to zero.

If a third harmonic is to be found conjugate to both of these there will be two equations which must be satisfied by its $2i$ variables.

If we go on constructing new harmonics, each of which is conjugate to all the former harmonics, the variables will be continually more and more restricted, till at last the $(2i+1)$th harmonic will have all its variables determined by the $2i$ equations, which must be satisfied in order that it may be conjugate to the $2i$ preceding harmonics.

Hence a system of $2i+1$ harmonics of degree i may be constructed, each of which is conjugate to all the rest. Any other harmonic of the same degree may be expressed as the sum of this system of conjugate harmonics each multiplied by a coefficient.

The system described in Art. 132, consisting of $2i+1$ harmonics symmetrical about a single axis, of which the first is zonal, the next $i-1$ pairs tesseral, and the last pair sectorial, is a particular case of a system of $2i+1$ harmonics, all of which are conjugate to each other. Sir W. Thomson has shewn how to express the conditions that $2i+1$ perfectly general harmonics, each of which, however, is expressed as a linear function of the $2i+1$ harmonics of this symmetrical system, may be conjugate to each other. These conditions consist of $i(2i+1)$ linear equations connecting the $(2i+1)^2$ coefficients which enter into the expressions of the general harmonics in terms of the symmetrical harmonics.

Professor Clifford has also shewn how to form a conjugate system of $2i+1$ sectorial harmonics having different poles.

Both these results were communicated to the British Association in 1871.

139.] If we take for Y_j the zonal harmonic Q_j, we obtain a remarkable form of equation (57).

In this case all the axes of the second system coincide with each other.

The cosines of the form μ_{ij} will assume the form λ where λ is the cosine of the angle between the common axis of Q_j and an axis of the first system.

The cosines of the form μ_{jj} will all become equal to unity.

The number of combinations of s symbols, each of which is distinguished by two out of i suffixes, no suffix being repeated, is

$$N = \frac{\lfloor i}{2^s \lfloor s \lfloor i-2s}; \tag{58}$$

and when each combination is equal to unity this number represents the sum of the products of the cosines μ_{jj}, or $\Sigma(\mu_{jj}{}^s)$.

The number of permutations of the remaining $i-2s$ symbols of the second set of axes taken all together is $\lfloor i-2s$. Hence

$$\Sigma(\mu_{ij}{}^{i-2s}) = \lfloor i-2s \, \Sigma \lambda^{i-2s}. \tag{59}$$

Equation (57) therefore becomes, when Y_j is the zonal harmonic,

$$\iint Y_i Q_j \, dS = \frac{4\pi a^2}{(2i+1)\,\underline{|i}} \, S\left\{(-1)^s \frac{\underline{|2i-2s}}{2^{i-s}\,\underline{|i-s}} \, \Sigma\left(\lambda^{i-2s}\mu^s\right)\right\},$$

$$= \frac{4\pi a^2}{2i+1} Y_{i(j)}; \tag{60}$$

where $Y_{i(j)}$ denotes the value of Y_i in equation (27) at the common pole of all the axes of Q_j.

140.] This result is a very important one in the theory of spherical harmonics, as it leads to the determination of the form of a series of spherical harmonics, which expresses a function having any arbitrarily assigned value at each point of a spherical surface.

For let F be the value of the function at any given point of the sphere, say at the centre of gravity of the element of surface dS, and let Q_i be the zonal harmonic of degree i whose pole is the point P on the sphere, then the surface-integral

$$\iint F Q_i \, dS$$

extended over the spherical surface will be a spherical harmonic of degree i, because it is the sum of a number of zonal harmonics whose poles are the various elements dS, each being multiplied by $F\,dS$. Hence, if we make

$$A_i Y_i = \frac{2i+1}{4\pi a^2} \iint F Q_i \, dS, \tag{61}$$

we may expand F in the form

$$F = A_0 Y_0 + A_1 Y_1 + \&c. + A_i Y_i, \tag{62}$$

or

$$F = \frac{1}{4\pi a^2}\left\{\iint F Q_0 \, dS + 3\iint F Q_1 \, dS + \&c. + (2i+1)\iint F Q_i \, dS\right\}. \tag{63}$$

This is the celebrated formula of Laplace for the expansion in a series of spherical harmonics of any quantity distributed over the surface of a sphere. In making use of it we are supposed to take a certain point P on the sphere as the pole of the zonal harmonic Q_i, and to find the surface-integral

$$\iint F Q_i \, dS$$

over the whole surface of the sphere. The result of this operation when multiplied by $2i+1$ gives the value of $A_i Y_i$ at the point P, and by making P travel over the surface of the sphere the value of $A_i Y_i$ at any other point may be found.

But $A_i Y_i$ is a general surface harmonic of degree i, and we wish to break it up into the sum of a series of multiples of the $2i+1$ conjugate harmonics of that degree.

Let P_i be one of these conjugate harmonics of a particular type, and let $B_i P_i$ be the part of $A_i Y_i$ belonging to this type.

We must first find
$$M = \iint P_i P_i \, dS, \tag{64}$$
which may be done by means of equation (57), making the second set of poles the same, each to each, as the first set.

We may then find the coefficient B_i from the equation
$$B_i = \frac{1}{M} \iint F P_i \, dS. \tag{65}$$

For suppose F expanded in terms of spherical harmonics, and let $B_j P_j$ be any term of this expansion. Then, if the degree of P_j is different from that of P_i, or if, the degree being the same, P_j is conjugate to P_i, the result of the surface-integration is zero. Hence the result of the surface-integration is to select the coefficient of the harmonic of the same type as P_i.

The most remarkable example of the actual development of a function in a series of spherical harmonics is the calculation by Gauss of the harmonics of the first four degrees in the expansion of the magnetic potential of the earth, as deduced from observations in various parts of the world.

He has determined the twenty-four coefficients of the three conjugate harmonics of the first degree, the five of the second, seven of the third, and nine of the fourth, all of the symmetrical system. The method of calculation is given in his *General Theory of Terrestrial Magnetism.*

141.] When the harmonic P_i belongs to the symmetrical system we may determine the surface-integral of its square extended over the sphere by the following method.

The value of $r^i Y_i^\sigma$ is, by equations (34) and (36),
$$r^i Y_i^{(\sigma)} = \frac{\lfloor i+\sigma}{2^\sigma \lfloor i \lfloor \sigma} (\xi^\sigma + \eta^\sigma) \left(z^{i-\sigma} - \frac{(i-\sigma)(i-\sigma-1)}{4(\sigma+1)} z^{i-\sigma-2} \xi\eta + \&c. \right);$$
and by equations (33) and (54),
$$\iint (Y_i^{(\sigma)})^2 dS = \frac{4\pi}{\lfloor i} \frac{a^2}{2i+1} \cdot \frac{d^{i-\sigma}}{dz^{i-\sigma}} \left(\frac{d^\sigma}{d\xi^\sigma} + \frac{d^\sigma}{d\eta^\sigma} \right) (r_i Y_i^{(\sigma)}).$$

Performing the differentiations, we find that the only terms which do not disappear are those which contain $z^{i-\sigma}$. Hence

$$\iint (Y_i^{(\sigma)})^2\, dS = \frac{8\pi a^2}{2i+1} \cdot \frac{\underline{|i+\sigma}\; \underline{|i-\sigma}}{2^{2\sigma} \underline{|i}\; \underline{|i}}, \tag{66}$$

except when $\sigma = 0$, in which case we have, by equation (60),

$$\iint (Q_i)^2\, dS = \frac{4\pi a^2}{2i+1}. \tag{67}$$

These expressions give the value of the surface-integral of the square of any surface harmonic of the symmetrical system.

We may deduce from this the value of the integral of the square of the function $\vartheta_i^{(\sigma)}$, given in Art. 132,

$$\int_{-1}^{+1} (\vartheta_i^{(\sigma)})^2\, d\mu = \frac{2}{2i+1}\; \frac{2^{2\sigma}\, \underline{|i-\sigma}\, (\underline{|\sigma})^2}{\underline{|i+\sigma}}. \tag{68}$$

This value is identical with that given by Thomson and Tait, and is true without exception for the case in which $\sigma = 0$.

142.] The spherical harmonics which I have described are those of integral degrees. To enter on the consideration of harmonics of fractional, irrational, or impossible degrees is beyond my purpose, which is to give as clear an idea as I can of what these harmonics are. I have done so by referring the harmonic, not to a system of polar coordinates of latitude and longitude, or to Cartesian coordinates, but to a number of points on the sphere, which I have called the Poles of the harmonic. Whatever be the type of a harmonic of the degree i, it is always mathematically possible to find i points on the sphere which are its poles. The actual calculation of the position of these poles would in general involve the solution of a system of $2i$ equations of the degree i. The conception of the general harmonic, with its poles placed in any manner on the sphere, is useful rather in fixing our ideas than in making calculations. For the latter purpose it is more convenient to consider the harmonic as the sum of $2i+1$ conjugate harmonics of selected types, and the ordinary symmetrical system, in which polar coordinates are used, is the most convenient. In this system the first type is the zonal harmonic Q_i, in which all the axes coincide with the axis of polar coordinates. The second type is that in which $i-1$ of the poles of the harmonic coincide at the pole of the sphere, and the remaining one is on the equator at the origin of longitude. In the third type the remaining pole is at 90° of longitude.

In the same way the type in which $i-\sigma$ poles coincide at the pole of the sphere, and the remaining σ are placed with their axes at equal intervals $\frac{\pi}{\sigma}$ round the equator, is the type 2σ, if one of the poles is at the origin of longitude, or the type $2\sigma+1$ if it is at longitude $\frac{\pi}{2\sigma}$.

143.] It appears from equation (60) that it is always possible to express a harmonic as the sum of a system of zonal harmonics of the same degree, having their poles distributed over the surface of the sphere. The simplification of this system, however, does not appear easy. I have however, for the sake of exhibiting to the eye some of the features of spherical harmonics, calculated the zonal harmonics of the third and fourth degrees, and drawn, by the method already described for the addition of functions, the equipotential lines on the sphere for harmonics which are the sums of two zonal harmonics. See Figures VI to IX at the end of this volume.

Fig. VI represents the sum of two zonal harmonics of the third degree whose axes are inclined 120° in the plane of the paper, and the sum is the harmonic of the second type in which $\sigma = 1$, the axis being perpendicular to the paper.

In Fig. VII the harmonic is also of the third degree, but the axes of the zonal harmonics of which it is the sum are inclined 90°, and the result is not of any type of the symmetrical system. One of the nodal lines is a great circle, but the other two which are intersected by it are not circles.

Fig. VIII represents the difference of two zonal harmonics of the fourth degree whose axes are at right angles. The result is a tesseral harmonic for which $i = 4$, $\sigma = 2$.

Fig. IX represents the sum of the same zonal harmonics. The result gives some notion of one type of the more general harmonic of the fourth degree. In this type the nodal line on the sphere consists of six ovals not intersecting each other. Within these ovals the harmonic is positive, and in the sextuply connected part of the spherical surface which lies outside the ovals, the harmonic is negative.

All these figures are orthogonal projections of the spherical surface.

I have also drawn in Fig. V a plane section through the axis of a sphere, to shew the equipotential surfaces and lines of force due to a spherical surface electrified according to the values of a spherical harmonic of the first degree.

Within the sphere the equipotential surfaces are equidistant planes, and the lines of force are straight lines parallel to the axis, their distances from the axis being as the square roots of the natural numbers. The lines outside the sphere may be taken as a representation of those which would be due to the earth's magnetism if it were distributed according to the most simple type.

144.] It appears from equation (52), by making $i = 0$, that if V satisfies Laplace's equation throughout the space occupied by a sphere of radius a, then the integral

$$\iint V\,dS = 4\pi a^2 V_0, \tag{69}$$

where the integral is taken over the surface of the sphere, dS being an element of that surface, and V_0 is the value of V at the centre of the sphere. This theorem may be thus expressed.

The value of the potential at the centre of a sphere is the mean value of the potential for all points of its surface, provided the potential be due to an electrified system, no part of which is within the sphere.

It follows from this that if V satisfies Laplace's equation throughout a certain continuous region of space, and if, throughout a finite portion, however small, of that space, V is constant, it will be constant throughout the whole continuous region.

If not, let the space throughout which the potential has a constant value C be separated by a surface S from the rest of the region in which its values differ from C, then it will always be possible to find a finite portion of space touching S and outside of it in which V is either everywhere greater or everywhere less than C.

Now describe a sphere with its centre within S, and with part of its surface outside S, but in a region throughout which the value of V is everywhere greater or everywhere less than C.

Then the mean value of the potential over the surface of the sphere will be greater than its value at the centre in the first case and less in the second, and therefore Laplace's equation cannot be satisfied throughout the space occupied by the sphere, contrary to our hypothesis. It follows from this that if $V=C$ throughout any portion of a connected region, $V=C$ throughout the whole of the region which can be reached in any way by a body oi finite size without passing through electrified matter. (We suppose the body to be of finite size because a region in which V is constant may be separated from another region in which it is variable by an electrified surface, certain points or lines of which are not electrified, so that a mere point might pass out of the region through one of these points or lines without passing through electrified matter.) This remarkable theorem is due to Gauss. See Thomson and Tait's *Natural Philosophy*, § 497.

It may be shewn in the same way that if throughout any finite portion of space the potential has a value which can be expressed by a continuous mathematical formula satisfying Laplace's equation, the potential will be expressed by the same formula throughout every part of space which can be reached without passing through electrified matter.

For if in any part of this space the value of the function is V', different from V, that given by the mathematical formula, then, since both V and V' satisfy Laplace's equation, $U = V' - V$ does. But within a finite portion of the space $U = 0$, therefore by what we have proved $U = 0$ throughout the whole space, or $V' = V$.

145.] Let Y_i be a spherical harmonic of i degrees and of any type. Let any line be taken as the axis of the sphere, and let the harmonic be turned into n positions round the axis, the angular distance between consecutive positions being $\frac{2\pi}{n}$.

If we take the sum of the n harmonics thus formed the result will be a harmonic of i degrees, which is a function of θ and of the sines and cosines of $n\phi$.

If n is less than i the result will be compounded of harmonics for which s is zero or a multiple of n less than i, but if n is greater than i the result is a zonal harmonic. Hence the following theorem :

Let any point be taken on the general harmonic Y_i, and let a small circle be described with this point for centre and radius θ, and let n points be taken at equal distances round this circle, then if Q_i is the value of the zonal harmonic for an angle θ, and if Y_i' is the value of Y_i at the centre of the circle, then the mean of the n values of Y_i round the circle is equal to $Q_i\,Y_i'$ provided n is greater than i.

If n is greater than $i+s$, and if the value of the harmonic at each point of the circle be multiplied by $\sin s\phi$ or $\cos s\phi$ where s is less than i, and the arithmetical mean of these products be A_s, then if $\vartheta_i'^{(s)}$ is the value of $\vartheta_i^{(s)}$ for the angle θ, the coefficient of $\sin s\phi$ or $\cos s\phi$ in the expansion of Y_i will be

$$2\,A_s\,\frac{\vartheta_i^{(s)}}{\vartheta_i'^{(s)}}.$$

In this way we may analyse Y_i into its component conjugate harmonics by means of a finite number of ascertained values at selected points on the sphere.

Application of Spherical Harmonic Analysis to the Determination of the Distribution of Electricity on Spherical and nearly Spherical Conductors under the Action of known External Electrical Forces.

146.] We shall suppose that every part of the electrified system which acts on the conductor is at a greater distance from the centre of the conductor than the most distant part of the conductor itself, or, if the conductor is spherical, than the radius of the sphere.

Then the potential of the external system, at points within this distance, may be expanded in a series of solid harmonics of positive degree

$$V = A_0 + A_1 r Y_1 + \&c + A_i Y_i r^i. \tag{70}$$

The potential due to the conductor at points outside it may be expanded in a series of solid harmonics of the same type, but of negative degree

$$U = B_0 \frac{1}{r} + B_1 Y_1 \frac{1}{r^2} + \&c. + B_i Y_i \frac{1}{r^{i+1}}. \tag{71}$$

At the surface of the conductor the potential is constant and equal, say, to C. Let us first suppose the conductor spherical and of radius a. Then putting $r = a$, we have $U + V = C$, or, equating the coefficients of the different degrees,

$$\begin{aligned} B_0 &= a(C - A_0), \\ B_1 &= -a^3 A_1, \\ &\cdots \\ B_i &= -a^{2i+1} A_i. \end{aligned} \tag{72}$$

The total charge of electricity on the conductor is B_0.

The surface-density at any point of the sphere may be found from the equation

$$\begin{aligned} 4\pi\sigma &= \frac{dV}{dr} - \frac{dU}{dr} \\ &= \frac{B_0}{a^2} - 3a^3 A_1 r Y_1 - \&c. - (2i+1) a^{2i+1} A_i Y_i. \end{aligned} \tag{73}$$

Distribution of Electricity on a nearly Spherical Conductor.

Let the equation of the surface of the conductor be

$$r = a(1 + F), \tag{74}$$

where F is a function of the direction of r, and is a numerical quantity the square of which may be neglected.

Let the potential due to the external electrified system be expressed, as before, in a series of solid harmonics of positive degree, and let the potential U be a series of solid harmonics of negative degree. Then the potential at the surface of the conductor is obtained by substituting the value of r from equation (74) in these series.

Hence, if C is the value of the potential of the conductor and B_0 the charge upon it,

$$\begin{aligned} C = {} & A_0 + A_1 a Y_1 + \ldots + A_i a^i Y_i, \\ & + A_1 a F Y_1 + \ldots + i A_i a^i F Y_i, \\ & + B_0 \frac{1}{a} + B_1 \frac{1}{a^2} Y_1 + \ldots + B_i a^{-(i+1)} Y_i + \ldots + B_j a^{-j+1} Y_j, \\ & - B_0 \frac{1}{a} F - 2 B_1 \frac{1}{a^2} F Y_1 + \ldots - (i+1) B_i a^{-(i+1)} F Y_i + \ldots \\ & \ldots - (j+1) B_j a^{-(j+1)} F Y_j. \end{aligned} \tag{75}$$

Since F is very small compared with unity, we have first a set of equations of the form (72), with the additional equation

$$\begin{aligned} 0 = & - B_0 \frac{1}{a} F + 3 A_1 a F Y_1 + \&c. + (i+1) A_i a^i F Y_i \\ & + \Sigma (B_j a^{-(j+1)} Y_j) - \Sigma ((j+1) B_j a^{-(j+1)} F Y_j). \end{aligned} \tag{76}$$

To solve this equation we must expand $F, FY_1 \ldots FY_i$ in terms of spherical harmonics. If F can be expanded in terms of spherical harmonics of degrees lower than k, then FY_i can be expanded in spherical harmonics of degrees lower than $i+k$.

Let therefore

$$B_0 \frac{1}{a} F - 3 A_1 a F Y_1 - \ldots - (2i+1) A_i a^i F Y_i = \Sigma (B_j a^{-(j+1)} Y_j), \tag{77}$$

then the coefficients B_j will each of them be small compared with the coefficients $B_0 \ldots B_i$ on account of the smallness of F, and therefore the last term of equation (76), consisting of terms in $B_j F$, may be neglected.

Hence the coefficients of the form B_j may be found by expanding equation (76) in spherical harmonics.

For example, let the body have a charge B_0, and be acted on by no external force.

Let F be expanded in a series of the form

$$F = S_1 Y_1 + \&c. + S_k Y_k. \tag{78}$$

Then $$B_0 \frac{1}{a} S_1 Y_1 + \&c. + B_0 \frac{1}{a} S_k Y_k = \Sigma (B_j a^{-(j+1)} Y_j), \tag{79}$$

or the potential at any point outside the body is

$$U = \frac{1}{a} B_0 \left(\frac{a}{r} + \frac{a^2}{r^2} S_1 Y_1 + \ldots + \frac{a^{k+1}}{r^{k+1}} S_k Y_k \right); \tag{80}$$

and if σ is the surface-density at any point

$$4\pi\sigma = -\frac{dU}{dr},$$

or
$$4\pi a\sigma = B_0 (1 + S_2 Y_2 + \ldots + (k-1) S_k Y_k). \tag{81}$$

Hence, if the surface differs from that of a sphere by a thin stratum whose depth varies according to the values of a spherical harmonic of degree k, the ratio of the difference of the superficial densities at any two points to their sum will be $k-1$ times the ratio of the difference of the radii of the same two points to their sum.

CHAPTER X.

CONFOCAL QUADRIC SURFACES *.

147.] Let the general equation of a confocal system be

$$\frac{x^2}{\lambda^2-a^2}+\frac{y^2}{\lambda^2-b^2}+\frac{z^2}{\lambda^2-c^2}=1, \tag{1}$$

where λ is a variable parameter, which we shall distinguish by the suffix λ_1 for the hyperboloids of two sheets, λ_2 for the hyperboloids of one sheet, and λ_3 for the ellipsoids. The quantities

$$a,\ \lambda_1,\ b,\ \lambda_2,\ c,\ \lambda_3$$

are in ascending order of magnitude. The quantity a is introduced for the sake of symmetry, but in our results we shall always suppose $a=0$.

If we consider the three surfaces whose parameters are λ_1, λ_2, λ_3, we find, by elimination between their equations, that the value of x^2 at their point of intersection satisfies the equation

$$x^2(b^2-a^2)(c^2-a^2)=(\lambda_1{}^2-a^2)(\lambda_2{}^2-a^2)(\lambda_3{}^2-a^2). \tag{2}$$

The values of y^2 and z^2 may be found by transposing a, b, c symmetrically.

Differentiating this equation with respect to λ_1, we find

$$\frac{dx}{d\lambda_1}=\frac{\lambda_1}{\lambda_1{}^2-a^2}x. \tag{3}$$

If ds_1 is the length of the intercept of the curve of intersection of λ_2 and λ_3 cut off between the surfaces λ_1 and $\lambda_1+d\lambda_1$, then

$$\overline{\frac{ds_1}{d\lambda_1}}\Big|^2=\overline{\frac{dx}{d\lambda_1}}\Big|^2+\overline{\frac{dy}{d\lambda_1}}\Big|^2+\overline{\frac{dz}{d\lambda_1}}\Big|^2=\frac{\lambda_1{}^2(\lambda_2{}^2-\lambda_1{}^2)(\lambda_3{}^2-\lambda_1{}^2)}{(\lambda_1{}^2-a^2)(\lambda_1{}^2-b^2)(\lambda_1{}^2-c^2)}. \tag{4}$$

* This investigation is chiefly borrowed from a very interesting work,—*Leçons sur les Fonctions Inverses des Transcendantes et les Surfaces Isothermes.* Par G. Lamé. Paris, 1857.

The denominator of this fraction is the product of the squares of the semi-axes of the surface λ_1.

If we put

$$D_1^2 = \lambda_3^2 - \lambda_2^2, \quad D_2^2 = \lambda_3^2 - \lambda_1^2, \quad \text{and} \quad D_3^2 = \lambda_2^2 - \lambda_1^2, \tag{5}$$

and if we make $a = 0$, then

$$\frac{ds_1}{d\lambda_1} = \frac{D_2 D_3}{\sqrt{b^2 - \lambda_1^2}\sqrt{c^2 - \lambda_1^2}}. \tag{6}$$

It is easy to see that D_2 and D_3 are the semi-axes of the central section of λ_1 which is conjugate to the diameter passing through the given point, and that D_2 is parallel to ds_2, and D_3 to ds_3.

If we also substitute for the three parameters λ_1, λ_2, λ_3 their values in terms of three functions a, β, γ, defined by the equations

$$\begin{aligned}
\frac{da}{d\lambda_1} &= \frac{c}{\sqrt{b^2 - \lambda_1^2}\sqrt{c^2 - \lambda_1^2}}, & \lambda_1 &= 0 \text{ when } a = 0, \\
\frac{d\beta}{d\lambda_2} &= \frac{c}{\sqrt{\lambda_2^2 - b^2}\sqrt{c^2 - \lambda_2^2}}, & \lambda_2 &= b \text{ when } \beta = 0, \\
\frac{d\gamma}{d\lambda_3} &= \frac{c}{\sqrt{\lambda_3^2 - b^2}\sqrt{\lambda_3^2 - c^2}}, & \lambda_3 &= c \text{ when } \gamma = 0;
\end{aligned} \tag{7}$$

then $ds_1 = \frac{1}{c} D_2 D_3 \, da, \quad ds_2 = \frac{1}{c} D_3 D_1 \, d\beta, \quad ds_3 = \frac{1}{c} D_1 D_2 \, d\gamma.$ (8)

148.] Now let V be the potential at any point a, β, γ, then the resultant force in the direction of ds_1 is

$$R_1 = -\frac{dV}{ds_1} = -\frac{dV}{da}\frac{da}{ds_1} = -\frac{dV}{da}\frac{c}{D_2 D_3}. \tag{9}$$

Since ds_1, ds_2, and ds_3 are at right angles to each other, the surface-integral over the element of area $ds_2\, ds_3$ is

$$\begin{aligned}
R_1\, ds_2\, ds_3 &= \frac{dV}{da}\frac{c}{D_2 D_3} \cdot \frac{D_3 D_1}{c} \cdot \frac{D_1 D_2}{c} \cdot d\beta\, d\gamma \\
&= \frac{dV}{da}\frac{D_1^2}{c}\, d\beta\, d\gamma.
\end{aligned} \tag{10}$$

Now consider the element of volume intercepted between the surfaces a, β, γ, and $a + da$, $\beta + d\beta$, $\gamma + d\gamma$. There will be eight such elements, one in each octant of space.

We have found the surface-integral for the element of surface intercepted from the surface a by the surfaces β and $\beta + d\beta$, γ and $\gamma + d\gamma$.

The surface-integral for the corresponding element of the surface $a + da$ will be

$$\frac{dV}{da}\frac{D_1^2}{c}\,d\beta\,d\gamma + \frac{d^2V}{da^2}\frac{D_1^2}{c}\,da\,d\beta\,d\gamma$$

since D_1 is independent of a. The surface-integral for the two opposite faces of the element of volume, taken with respect to the interior of that volume, will be the difference of these quantities, or

$$\frac{d^2V}{da^2}\frac{D_1^2}{c}\,da\,d\beta\,d\gamma.$$

Similarly the surface-integrals for the other two pairs of forces will be

$$\frac{d^2V}{d\beta^2}\frac{D_2^2}{c}\,da\,d\beta\,d\gamma \quad \text{and} \quad \frac{d^2V}{d\gamma^2}\frac{D_3^2}{c}\,da\,d\beta\,d\gamma.$$

These six faces enclose an element whose volume is

$$ds_1\,ds_2\,ds_3 = \frac{D_1^2 D_2^2 D_3^2}{c^3}\,da\,d\beta\,d\gamma,$$

and if ρ is the volume-density within that element, we find by Art. 77 that the total surface-integral of the element, together with the quantity of electricity within it, multiplied by 4π is zero, or, dividing by $da\,d\beta\,d\gamma$,

$$\frac{d^2V}{da^2}D_1^2 + \frac{d^2V}{d\beta^2}D_2^2 + \frac{d^2V}{d\gamma^2}D_3^2 + 4\pi\rho\frac{D_1^2 D_2^2 D_3^2}{c^2} = 0, \qquad (11)$$

which is the form of Poisson's extension of Laplace's equation referred to ellipsoidal coordinates.

If $\rho = 0$ the fourth term vanishes, and the equation is equivalent to that of Laplace.

For the general discussion of this equation the reader is referred to the work of Lamé already mentioned.

149.] To determine the quantities a, β, γ, we may put them in the form of ordinary elliptic functions by introducing the auxiliary angles θ, ϕ, and ψ, where

$$\lambda_1 = b\sin\theta, \qquad (12)$$

$$\lambda_2 = \sqrt{c^2\sin^2\phi + b^2\cos^2\phi}, \qquad (13)$$

$$\lambda_3 = \frac{c}{\sin\psi}. \qquad (14)$$

If we put $b = kc$, and $k^2 + k'^2 = 1$, we may call k and k' the two complementary moduli of the confocal system, and we find

$$a = \int_0^\theta \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}, \qquad (15)$$

an elliptic integral of the first kind, which we may write according to the usual notation $F(k\theta)$.

In the same way we find

$$\beta = \int_0^{\phi} \frac{d\phi}{\sqrt{1-k'^2\cos^2\phi}} = F(k') - F(k'\phi), \tag{16}$$

where Fk' is the complete function for modulus k',

$$\gamma = \int_0^{\psi} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} = F(k\psi). \tag{17}$$

Here α is represented as a function of the angle θ, which is a function of the parameter λ_1, β as a function of ϕ and thence of λ_2, and γ as a function of ψ and thence of λ_3.

But these angles and parameters may be considered as functions of α, β, γ. The properties of such inverse functions, and of those connected with them, are explained in the treatise of M. Lamé on that subject.

It is easy to see that since the parameters are periodic functions of the auxiliary angles, they will be periodic functions of the quantities α, β, γ: the periods of λ_1 and λ_3 are $4\,F(k)$ and that of λ_2 is $2\,F(k')$.

Particular Solutions.

150.] If V is a linear function of α, β, or γ, the equation is satisfied. Hence we may deduce from the equation the distribution of electricity on any two confocal surfaces of the same family maintained at given potentials, and the potential at any point between them.

The Hyperboloids of Two Sheets.

When α is constant the corresponding surface is a hyperboloid of two sheets. Let us make the sign of α the same as that of x in the sheet under consideration. We shall thus be able to study one of these sheets at a time.

Let α_1, α_2 be the values of α corresponding to two single sheets, whether of different hyperboloids or of the same one, and let V_1, V_2 be the potentials at which they are maintained. Then, if we make

$$V = \frac{\alpha_1 V_2 - \alpha_2 V_1 + \alpha(V_1 - V_2)}{\alpha_1 - \alpha_2}, \tag{18}$$

the conditions will be satisfied at the two surfaces and throughout the space between them. If we make V constant and equal to V_1 in the space beyond the surface α_1, and constant and equal to V_2

in the space beyond the surface a_2, we shall have obtained the complete solution of this particular case.

The resultant force at any point of either sheet is

$$R_1 = -\frac{dV}{ds_1} = -\frac{dV}{da}\frac{da}{ds_1}, \tag{19}$$

or $$R_1 = \frac{V_1 - V_2}{a_1 - a_2}\frac{c}{D_2 D_3}. \tag{20}$$

If p_1 be the perpendicular from the centre on the tangent plane at any point, and P_1 the product of the semi-axes of the surface, then $p_1 D_2 D_3 = P_1$.

Hence we find $$R_1 = \frac{V_1 - V_2}{a_1 - a_2}\frac{c p_1}{P_1}, \tag{21}$$

or the force at any point of the surface is proportional to the perpendicular from the centre on the tangent plane.

The surface-density σ may be found from the equation

$$4\pi\sigma = R_1. \tag{22}$$

The total quantity of electricity on a segment cut off by a plane whose equation is $x = a$ from one sheet of the hyperboloid is

$$Q = \frac{c}{2}\frac{V_1 - V_2}{a_1 - a_2}\left(\frac{a}{\lambda_1} - 1\right). \tag{23}$$

The quantity on the whole infinite sheet is therefore infinite.

The limiting forms of the surface are :—

(1) When $a = F_{(k)}$ the surface is the part of the plane of xz on the positive side of the positive branch of the hyperbola whose equation is

$$\frac{x^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{24}$$

(2) When $a = 0$ the surface is the plane of yz.

(3) When $a = -F_{(k)}$ the surface is the part of the plane of xz on the negative side of the negative branch of the same hyperbola.

The Hyperboloids of One Sheet.

By making β constant we obtain the equation of the hyperboloid of one sheet. The two surfaces which form the boundaries of the electric field must therefore belong to two different hyperboloids. The investigation will in other respects be the same as for the hyperboloids of two sheets, and when the difference of potentials is given the density at any point of the surface will be proportional to the perpendicular from the centre on the tangent plane, and the whole quantity on the infinite sheet will be infinite.

Limiting Forms.

(1) When $\beta = 0$ the surface is the part of the plane of xz between the two branches of the hyperbola whose equation is written above, (24).

(2) When $\beta = F(k')$ the surface is the part of the plane of xy which is on the outside of the focal ellipse whose equation is

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - b^2} = 1. \tag{25}$$

The Ellipsoids.

For any given ellipsoid γ is constant. If two ellipsoids, γ_1 and γ_2, be maintained at potentials V_1 and V_2, then, for any point γ in the space between them, we have

$$V = \frac{\gamma_1 V_2 - \gamma_2 V_1 + \gamma (V_1 - V_2)}{\gamma_1 - \gamma_2}. \tag{26}$$

The surface-density at any point is

$$\sigma = -\frac{1}{4\pi} \frac{V_1 - V_2}{\gamma_1 - \gamma_2} \frac{c p_3}{P_3}, \tag{27}$$

where p_3 is the perpendicular from the centre on the tangent plane, and P_3 is the product of the semi-axes.

The whole charge of electricity on either surface is

$$Q_2 = c \frac{V_1 - V_2}{\gamma_1 - \gamma_2} = -Q_1, \tag{28}$$

a finite quantity.

When $\gamma = F(k)$ the surface of the ellipsoid is at an infinite distance in all directions.

If we make $V_2 = 0$ and $\gamma_2 = F(k)$, we find for the quantity of electricity on an ellipsoid maintained at potential V in an infinitely extended field,

$$Q = c \frac{V}{F(k) - \gamma}. \tag{29}$$

The limiting form of the ellipsoids occurs when $\gamma = 0$, in which case the surface is the part of the plane of xy within the focal ellipse, whose equation is written above, (25).

The surface-density on the elliptic plate whose equation is (25), and whose eccentricity is k, is

$$\sigma = \frac{V}{2\pi\sqrt{c^2 - b^2}} \frac{1}{F(k)} \frac{1}{\sqrt{1 - \frac{x^2}{c^2} - \frac{y^2}{c^2 - b^2}}}, \tag{30}$$

and its charge is

$$Q = c \frac{V}{F(k)}. \tag{31}$$

Particular Cases.

151.] If k is diminished till it becomes ultimately zero, the system of surfaces becomes transformed in the following manner :—

The real axis and one of the imaginary axes of each of the hyperboloids of two sheets are indefinitely diminished, and the surface ultimately coincides with two planes intersecting in the axis of z.

The quantity a becomes identical with θ, and the equation of the system of meridional planes to which the first system is reduced is

$$\frac{x^2}{(\sin a)^2} - \frac{y^2}{(\cos a)^2} = 0. \tag{32}$$

The quantity β is reduced to

$$\beta = \int \frac{d\phi}{\sin\phi} = \log\tan\frac{\phi}{2}, \tag{33}$$

whence we find

$$\sin\phi = \frac{2}{e^\beta + e^{-\beta}}, \qquad \cos\phi = \frac{e^\beta - e^{-\beta}}{e^\beta + e^{-\beta}}. \tag{34}$$

If we call the exponential quantity $\frac{1}{2}(e^\beta + e^{-\beta})$ the hyperbolic cosine of β, or more concisely the hypocosine of β, or $\cos h\,\beta$, and if we call $\frac{1}{2}(e^\beta - e^{-\beta})$ the hyposine of β, or $\sin h\,\beta$, and if by the same analogy we call

$\dfrac{1}{\cos h\,\beta}$ the hyposecant of β, or $\sec h\,\beta$,

$\dfrac{1}{\sin h\,\beta}$ the hypocosecant of β, or $\operatorname{cosec} h\,\beta$,

$\dfrac{\sin h\,\beta}{\cos h\,\beta}$ the hypotangent of β, or $\tan h\,\beta$,

and $\dfrac{\cos h\,\beta}{\sin h\,\beta}$ the hypocotangent of β, or $\cot h\,\beta$;

then $\lambda_2 = c\sec h\,\beta$, and the equation of the system of hyperboloids of one sheet is

$$\frac{x^2+y^2}{(\sec h\,\beta)^2} - \frac{z^2}{(\tan h\,\beta)^2} = c^2. \tag{35}$$

The quantity γ is reduced to ψ, so that $\lambda_3 = c\operatorname{cosec}\gamma$, and the equation of the system of ellipsoids is

$$\frac{x^2+y^2}{(\sec\gamma)^2} + \frac{z^2}{(\tan\gamma)^2} = c^2. \tag{36}$$

Ellipsoids of this kind, which are figures of revolution about their conjugate axes, are called Planetary ellipsoids.

The quantity of electricity on a planetary ellipsoid maintained at potential V in an infinite field, is

$$Q = c\frac{V}{\frac{\pi}{2}-\gamma}, \tag{37}$$

where $c \sec\gamma$ is the equatorial radius, and $c\tan\gamma$ is the polar radius.

If $\gamma = 0$, the figure is a circular disk of radius c, and

$$\sigma = \frac{V}{\pi^2\sqrt{c^2-r^2}}, \tag{38}$$

$$Q = c\frac{V}{\frac{\pi}{2}}. \tag{39}$$

152.] *Second Case.* Let $b = c$, then $k = 1$ and $k' = 0$,

$$a = \log\tan\frac{\pi-2\theta}{4}, \text{ whence } \lambda_1 = c\tan h\, a, \tag{40}$$

and the equation of the hyperboloids of revolution of two sheets becomes

$$\frac{x^2}{(\tan h\, a)^2} - \frac{y^2+z^2}{(\sec h\, a)^2} = c^2. \tag{41}$$

The quantity β becomes reduced to ϕ, and each of the hyperboloids of one sheet is reduced to a pair of planes intersecting in the axis of x whose equation is

$$\frac{y^2}{(\sin\beta)^2} - \frac{z^2}{(\cos\beta)^2} = 0. \tag{42}$$

This is a system of meridional planes in which β is the longitude.

The quantity γ becomes $\log\tan\frac{\pi-2\psi}{4}$, whence $\lambda_3 = c\cot h\,\gamma$, and the equation of the family of ellipsoids is

$$\frac{x^2}{(\cot h\gamma)^2} + \frac{y^2+z^2}{(\operatorname{cosec} h\,\gamma)^2} = c^2. \tag{43}$$

These ellipsoids, in which the transverse axis is the axis of revolution, are called Ovary ellipsoids.

The quantity of electricity on an ovary ellipsoid maintained at a potential V in an infinite field is

$$Q = c\frac{V}{\gamma}. \tag{44}$$

If the polar radius is $A = c\cot h\,\gamma$, and the equatorial radius is $B = c \operatorname{cosec} h\,\gamma$,

$$\gamma = \log\frac{A+\sqrt{A^2-B^2}}{2B}. \tag{45}$$

If the equatorial radius is very small compared to the polar radius, as in a wire with rounded ends,

$$\gamma = \log \frac{A}{B}, \quad \text{and} \quad Q = \frac{AV}{\log A - \log B}. \tag{46}$$

When both b and c become zero, their ratio remaining finite, the system of surfaces becomes two systems of confocal cones, and a system of spherical surfaces of which the radius is inversely proportional to γ.

If the ratio of b to c is zero or unity, the system of surfaces becomes one system of meridian planes, one system of right cones having a common axis, and a system of concentric spherical surfaces of which the radius is inversely proportional to γ. This is the ordinary system of spherical polar coordinates.

Cylindric Surfaces.

153.] When c is infinite the surfaces are cylindric, the generating lines being parallel to z. One system of cylinders is elliptic, with the equation

$$\frac{x^2}{(\cos h\, a)^2} + \frac{y^2}{(\sin h\, a)^2} = b^2. \tag{47}$$

The other is hyperbolic, with the equation

$$\frac{x^2}{(\cos \beta)^2} - \frac{y^2}{(\sin \beta)^2} = b^2. \tag{48}$$

This system is represented in Fig. X, at the end of this volume.

Confocal Paraboloids.

154.] If in the general equations we transfer the origin of coordinates to a point on the axis of x distant t from the centre of the system, and if we substitute for x, λ, b, and c, $t+x$, $t+\lambda$, $t+b$, and $t+c$ respectively, and then make t increase indefinitely, we obtain, in the limit, the equation of a system of paraboloids whose foci are at the points $x = b$ and $x = c$,

$$4(x-\lambda) + \frac{y^2}{\lambda - b} + \frac{z^2}{\lambda - c} = 0. \tag{49}$$

If the variable parameter is λ for the first system of elliptic paraboloids, μ for the hyperbolic paraboloids, and ν for the second system of elliptic paraboloids, we have λ, b, μ, c, ν in ascending order of magnitude, and

$$\left.\begin{aligned} x &= \lambda+\mu+\nu-c-b,\\ y^2 &= 4\,\frac{(b-\lambda)(\mu-b)(\nu-b)}{c-b},\\ z^2 &= 4\,\frac{(c-\lambda)(c-\mu)(\nu-c)}{c-b}; \end{aligned}\right\} \tag{50}$$

$$\left.\begin{aligned} \lambda &= \tfrac{1}{2}(b+c)-\tfrac{1}{2}(c-b)\cos h\,\alpha,\\ \mu &= \tfrac{1}{2}(b+c)-\tfrac{1}{2}(c-b)\cos\beta,\\ \nu &= \tfrac{1}{2}(b+c)+\tfrac{1}{2}(c-b)\cos h\,\gamma; \end{aligned}\right\} \tag{51}$$

$$\left.\begin{aligned} x &= \tfrac{1}{2}(b+c)+\tfrac{1}{2}(c-b)(\cos h\,\gamma-\cos\beta-\cos h\,\alpha),\\ y &= 2(c-b)\sin h\,\frac{\alpha}{2}\sin\frac{\beta}{2}\cos h\,\frac{\gamma}{2},\\ z &= 2(c-b)\cos h\,\frac{\alpha}{2}\cos\frac{\beta}{2}\sin h\,\frac{\gamma}{2}. \end{aligned}\right\} \tag{52}$$

When $b=c$ we have the case of paraboloids of revolution about the axis of x, and

$$\begin{aligned} x &= a\,(e^{2\alpha}-e^{2\gamma}),\\ y &= 2\,a\,e^{\alpha+\gamma}\cos\beta,\\ z &= 2\,a\,e^{\alpha+\gamma}\sin\beta. \end{aligned} \tag{53}$$

The surfaces for which β is constant are planes through the axis, β being the angle which such a plane makes with a fixed plane through the axis.

The surfaces for which α is constant are confocal paraboloids. When $\alpha=0$ the paraboloid is reduced to a straight line terminating at the origin.

We may also find the values of α, β, γ in terms of r, θ, and ϕ, the spherical polar coordinates referred to the focus as origin, and the axis of the parabolas as axis of the sphere,

$$\begin{aligned} \alpha &= \log(r^{\frac{1}{2}}\cos\tfrac{1}{2}\theta),\\ \beta &= \phi,\\ \gamma &= \log(r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta). \end{aligned} \tag{54}$$

We may compare the case in which the potential is equal to α, with the zonal solid harmonic $r_i\,Q_i$. Both satisfy Laplace's equation, and are homogeneous functions of x, y, z, but in the case derived from the paraboloid there is a discontinuity at the axis, and i has a value not differing by any finite quantity from zero.

The surface-density on an electrified paraboloid in an infinite field (including the case of a straight line infinite in one direction) is inversely as the distance from the focus, or, in the case of the line, from the extremity of the line.

CHAPTER XI.

THEORY OF ELECTRIC IMAGES AND ELECTRIC INVERSION.

155.] We have already shewn that when a conducting sphere is under the influence of a known distribution of electricity, the distribution of electricity on the surface of the sphere can be determined by the method of spherical harmonics.

For this purpose we require to expand the potential of the influencing system in a series of solid harmonics of positive degree, having the centre of the sphere as origin, and we then find a corresponding series of solid harmonics of negative degree, which express the potential due to the electrification of the sphere.

By the use of this very powerful method of analysis, Poisson determined the electrification of a sphere under the influence of a given electrical system, and he also solved the more difficult problem to determine the distribution of electricity on two conducting spheres in presence of each other. These investigations have been pursued at great length by Plana and others, who have confirmed the accuracy of Poisson.

In applying this method to the most elementary case of a sphere under the influence of a single electrified point, we require to expand the potential due to the electrified point in a series of solid harmonics, and to determine a second series of solid harmonics which express the potential, due to the electrification of the sphere, in the space outside.

It does not appear that any of these mathematicians observed that this second series expresses the potential due to an imaginary electrified point, which has no physical existence as an electrified point, but which may be called an electrical image, because the action of the surface on external points is the same as that which would be produced by the imaginary electrified point if the spherical surface were removed.

This discovery seems to have been reserved for Sir W. Thomson, who has developed it into a method of great power for the solution of electrical problems, and at the same time capable of being presented in an elementary geometrical form.

His original investigations, which are contained in the *Cambridge and Dublin Mathematical Journal*, 1848, are expressed in terms of the ordinary theory of attraction at a distance, and make no use of the method of potentials and of the general theorems of Chapter IV, though they were probably discovered by these methods. Instead, however, of following the method of the author, I shall make free use of the idea of the potential and of equipotential surfaces, whenever the investigation can be rendered more intelligible by such means.

Theory of Electric Images.

156.] Let A and B, Figure 7, represent two points in a uniform dielectric medium of infinite extent. Let the charges of A and B be e_1 and e_2 respectively. Let P be any point in space whose distances from A and B are r_1 and r_2 respectively. Then the value of the potential at P will be

$$V = \frac{e_1}{r_1} + \frac{e_2}{r_2}. \tag{1}$$

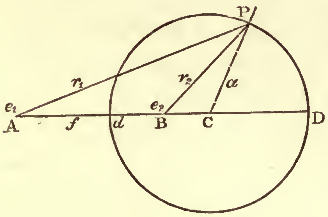


Fig. 7.

The equipotential surfaces due to this distribution of electricity are represented in Fig. I (at the end of this volume) when e_1 and e_2 are of the same sign, and in Fig. II when they are of opposite signs. We have now to consider that surface for which $V = 0$, which is the only spherical surface in the system. When e_1 and e_2 are of the same sign, this surface is entirely at an infinite distance, but when they are of opposite signs there is a plane or spherical surface at a finite distance for which the potential is zero.

The equation of this surface is

$$\frac{e_1}{r_1} + \frac{e_2}{r_2} = 0. \tag{2}$$

Its centre is at a point C in AB produced, such that

$$AC : BC :: {e_1}^2 : {e_2}^2,$$

and the radius of the sphere is

$$AB \frac{e_1 e_2}{{e_1}^2 - {e_2}^2}.$$

The two points A and B are inverse points with respect to this sphere, that is to say, they lie in the same radius, and the radius is a mean proportional between their distances from the centre.

Since this spherical surface is at potential zero, if we suppose it constructed of thin metal and connected with the earth, there will be no alteration of the potential at any point either outside or inside, but the electrical action everywhere will remain that due to the two electrified points A and B.

If we now keep the metallic shell in connexion with the earth and remove the point B, the potential within the sphere will become everywhere zero, but outside it will remain the same as before. For the surface of the sphere still remains at the same potential, and no change has been made in the exterior electrification.

Hence, if an electrified point A be placed outside a spherical conductor which is at potential zero, the electrical action at all points outside the sphere will be that due to the point A together with another point B within the sphere, which we may call the electrical image of A.

In the same way we may shew that if B is a point placed inside the spherical shell, the electrical action within the sphere is that due to B, together with its image A.

157.] *Definition of an Electrical Image.* An electrical image is an electrified point or system of points on one side of a surface which would produce on the other side of that surface the same electrical action which the actual electrification of that surface really does produce.

In Optics a point or system of points on one side of a mirror or lens which if it existed would emit the system of rays which actually exists on the other side of the mirror or lens, is called a *virtual* image.

Electrical images correspond to virtual images in optics in being related to the space on the other side of the surface. They do not correspond to them in actual position, or in the merely approximate character of optical foci.

There are no *real* electrical images, that is, imaginary electrified points which would produce, in the region on the same side of the electrified surface, an effect equivalent to that of the electrified surface.

For if the potential in any region of space is equal to that due to a certain electrification in the same region it must be actually produced by that electrification. In fact, the electrification at any point may be found from the potential near that point by the application of Poisson's equation.

Let a be the radius of the sphere.

Let f be the distance of the electrified point A from the centre C.

Let e be the charge of this point.

Then the image of the point is at B, on the same radius of the sphere at a distance $\frac{a^2}{f}$, and the charge of the image is $-e\frac{a}{f}$.

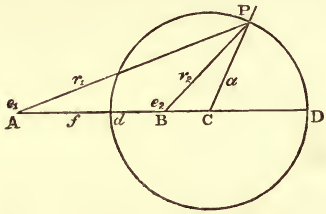


Fig. 7.

We have shewn that this image will produce the same effect on the opposite side of the surface as the actual electrification of the surface does. We shall next determine the surface-density of this electrification at any point P of the spherical surface, and for this purpose we shall make use of the theorem of Coulomb, Art. 80, that if R is the resultant force at the surface of a conductor, and σ the superficial density,

$$R = 4\pi\sigma,$$

R being measured away from the surface.

We may consider R as the resultant of two forces, a repulsion $\frac{e}{AP^2}$ acting along AP, and an attraction $e\frac{a}{f}\frac{1}{PB^2}$ acting along PB.

Resolving these forces in the directions of AC and CP, we find that the components of the repulsion are

$$\frac{ef}{AP^3} \text{ along } AC, \text{ and } \frac{ea}{AP^3} \text{ along } CP.$$

Those of the attraction are

$$-e\frac{a}{f}\frac{1}{BP^3}BC \text{ along } AC, \text{ and } -e\frac{a^2}{f}\frac{1}{BP^3} \text{ along } CP.$$

Now $BP = \frac{a}{f}AP$, and $BC = \frac{a^2}{f}$, so that the components of the attraction may be written

$$-ef\frac{1}{AP^3} \text{ along } AC, \text{ and } -e\frac{f^2}{a}\frac{1}{AP^3} \text{ along } CP.$$

The components of the attraction and the repulsion in the direction of AC are equal and opposite, and therefore the resultant force is entirely in the direction of the radius CP. This only confirms what we have already proved, that the sphere is an equipotential surface, and therefore a surface to which the resultant force is everywhere perpendicular.

The resultant force measured along CP, the normal to the surface in the direction towards the side on which A is placed, is

$$R = -e\frac{f^2-a^2}{a}\frac{1}{AP^3}. \tag{3}$$

If A is taken inside the sphere f is less than a, and we must measure R inwards. For this case therefore

$$R = -e\frac{a^2-f^2}{a}\frac{1}{AP^3}. \tag{4}$$

In all cases we may write

$$R = -e\frac{AD.Ad}{CP}\frac{1}{AP^3}, \tag{5}$$

where AD, Ad are the segments of any line through A cutting the sphere, and their product is to be taken positive in all cases.

158.] From this it follows, by Coulomb's theorem, Art. 80, that the surface-density at P is

$$\sigma = -e\frac{AD.Ad}{4\pi.CP}\frac{1}{AP^3}. \tag{6}$$

The density of the electricity at any point of the sphere varies inversely as the cube of its distance from the point A.

The effect of this superficial distribution, together with that of the point A, is to produce on the same side of the surface as the point A a potential equivalent to that due to e at A, and its image $-e\frac{a}{f}$ at B, and on the other side of the surface the potential is everywhere zero. Hence the effect of the superficial distribution by itself is to produce a potential on the side of A equivalent to that due to the image $-e\frac{a}{f}$ at B, and on the opposite side a potential equal and opposite to that of e at A.

The whole charge on the surface of the sphere is evidently $-e\frac{a}{f}$ since it is equivalent to the image at B.

We have therefore arrived at the following theorems on the action of a distribution of electricity on a spherical surface, the surface-density being inversely as the cube of the distance from a point A either without or within the sphere.

Let the density be given by the equation

$$\sigma = \frac{C}{AP^3}, \tag{7}$$

where C is some constant quantity, then by equation (6)

$$C = -e\frac{AD.Ad}{4\pi a}. \tag{8}$$

The action of this superficial distribution on any point separated from A by the surface is equal to that of a quantity of electricity $-e$, or

$$\frac{4\pi a C}{AD\,.\,Ad}$$

concentrated at A.

Its action on any point on the same side of the surface with A is equal to that of a quantity of electricity

$$\frac{4\pi C a^2}{f\,AD\,.\,Ad}$$

concentrated at B the image of A.

The whole quantity of electricity on the sphere is equal to the first of these quantities if A is within the sphere, and to the second if A is without the sphere.

These propositions were established by Sir W. Thomson in his original geometrical investigations with reference to the distribution of electricity on spherical conductors, to which the student ought to refer.

159.] If a system in which the distribution of electricity is known is placed in the neighbourhood of a conducting sphere of radius a, which is maintained at potential zero by connexion with the earth, then the electrifications due to the several parts of the system will be superposed.

Let A_1, A_2, &c. be the electrified points of the system, f_1, f_2, &c. their distances from the centre of the sphere, e_1, e_2, &c. their charges, then the images B_1, B_2, &c. of these points will be in the same radii as the points themselves, and at distances $\frac{a^2}{f_1}, \frac{a^2}{f_2}$ &c. from the centre of the sphere, and their charges will be

$$-e\frac{a}{f_1},\ -e\frac{a}{f_2}\ \&c.$$

The potential on the outside of the sphere due to the superficial electrification will be the same as that which would be produced by the system of images B_1, B_2, &c. This system is therefore called the electrical image of the system A_1, A_2, &c.

If the sphere instead of being at potential zero is at potential V, we must superpose a distribution of electricity on its outer surface having the uniform surface-density

$$\sigma = \frac{V}{4\pi a}.$$

The effect of this at all points outside the sphere will be equal to that of a quantity Va of electricity placed at its centre, and at all points inside the sphere the potential will be simply increased by V.

The whole charge on the sphere due to an external system of influencing points A_1, A_2, &c. is

$$E = Va - e_1\frac{a}{f_1} - e_2\frac{a}{f_2} - \&c., \tag{9}$$

from which either the charge E or the potential V may be calculated when the other is given.

When the electrified system is within the spherical surface the induced charge on the surface is equal and of opposite sign to the inducing charge, as we have before proved it to be for every closed surface, with respect to points within it.

160.] The energy due to the mutual action between an electrified point e, at a distance f from the centre of the sphere greater than a the radius, and the electrification of the spherical surface due to the influence of the electrified point and the charge of the sphere, is

$$M = e\left(\frac{Va}{f} - \frac{ea}{f^2-a^2}\right) = \frac{e}{f}\left(E - \frac{e^2a^3}{f(f^2-a^2)}\right), \tag{10}$$

where V is the potential, and E the charge of the sphere.

The repulsion between the electrified point and the sphere is therefore, by Art. 92,

$$F = ea\left(\frac{V}{f^2} - \frac{ef}{(f^2-a^2)^2}\right)$$

$$= \frac{e}{f^2}\left(E - e\frac{a^3(2f^2-a^2)}{f(f^2-a^2)^2}\right). \tag{11}$$

Hence the force between the point and the sphere is always an attraction in the following cases—

(1) When the sphere is uninsulated.

(2) When the sphere has no charge.

(3) When the electrified point is very near the surface.

In order that the force may be repulsive, the potential of the sphere must be positive and greater than $e\dfrac{f^3}{(f^2-a^2)^2}$, and the charge of the sphere must be of the same sign as e and greater than $e\dfrac{a^3(2f^2-a^2)}{f(f^2-a^2)^2}$.

At the point of equilibrium the equilibrium is unstable, the force being an attraction when the bodies are nearer and a repulsion when they are farther off.

When the electrified point is within the spherical surface the force on the electrified point is always away from the centre of the sphere, and is equal to

$$\frac{e^2 af}{(a^2-f^2)^2}.$$

The surface-density at the point of the sphere nearest to the electrified point where it lies outside the sphere is

$$\begin{aligned}\sigma_1 &= \frac{1}{4\pi a^2}\left\{Va - e\frac{a(f+a)}{(f-a)^2}\right\}\\ &= \frac{1}{4\pi a^2}\left\{E - e\frac{a^2(3f-a)}{f(f-a)^2}\right\}.\end{aligned} \tag{12}$$

The surface-density at the point of the sphere farthest from the electrified point is

$$\begin{aligned}\sigma_2 &= \frac{1}{4\pi a^2}\left\{Va - e\frac{a(f-a)}{(f+a)^2}\right\}\\ &= \frac{1}{4\pi a^2}\left\{E + e\frac{a^2(3f+a)}{f(f+a)^2}\right\}.\end{aligned} \tag{13}$$

When E, the charge of the sphere, lies between

$$e\frac{a^2(3f-a)}{f(f-a)^2} \quad \text{and} \quad -e\frac{a^2(3f+a)}{f(f+a)^2}$$

the electrification will be negative next the electrified point and positive on the opposite side. There will be a circular line of division between the positively and the negatively electrified parts of the surface, and this line will be a line of equilibrium.

If $$E = ea\left(\frac{1}{\sqrt{f^2-a^2}} - \frac{1}{f}\right), \tag{14}$$

the equipotential surface which cuts the sphere in the line of equilibrium is a sphere whose centre is the electrified point and whose radius is $\sqrt{f^2-a^2}$.

The lines of force and equipotential surfaces belonging to a case of this kind are given in Figure IV at the end of this volume.

Images in an Infinite Plane Conducting Surface.

161.] If the two electrified points A and B in Art. 156 are electrified with equal charges of electricity of opposite signs, the surface of zero potential will be the plane, every point of which is equidistant from A and B.

Hence, if A be an electrified point whose charge is e, and AD a perpendicular on the plane, produce AD to B so that $DB = AB$, and place at B a charge equal to $-e$, then this charge at B will be the image of A, and will produce at all points on the same side of the plane as A, an effect equal to that of the actual electrification of the plane. For the potential on the side of A due to A and B fulfils the conditions that $\nabla^2 V = 0$ everywhere except at A, and that $V = 0$ at the plane, and there is only one form of V which can fulfil these conditions.

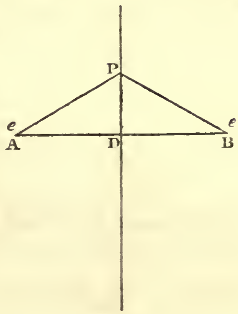


Fig. 8.

To determine the resultant force at the point P of the plane, we observe that it is compounded of two forces each equal to $\frac{e}{AP^2}$, one acting along AP and the other along PB. Hence the resultant of these forces is in a direction parallel to AB and equal to

$$\frac{e}{AP^2} \cdot \frac{AB}{AP}.$$

Hence R, the resultant force measured from the surface towards the space in which A lies, is

$$R = -\frac{2eAD}{AP^3}, \tag{15}$$

and the density at the point P is

$$\sigma = -\frac{eAD}{2\pi AP^3}. \tag{16}$$

On Electrical Inversion.

162.] The method of electrical images leads directly to a method of transformation by which we may derive from any electrical problem of which we know the solution any number of other problems with their solutions.

We have seen that the image of a point at a distance r from the centre of a sphere of radius R is in the same radius and at a distance r' such that $rr' = R^2$. Hence the image of a system of points, lines, or surfaces is obtained from the original system by the method known in pure geometry as the method of inversion, and described by Chasles, Salmon, and other mathematicians.

If A and B are two points, A' and B' their images, O being the centre of inversion, and R the radius of the sphere of inversion,

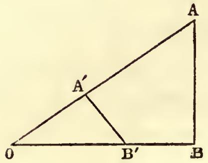


Fig. 9.

$$OA.OA' = R^2 = OB.OB'.$$

Hence the triangles OAB, $OB'A'$ are similar, and $AB : A'B' :: OA : OB' :: OA.OB : R^2$.

If a quantity of electricity e be placed at A, its potential at B will be

$$V = \frac{e}{AB}.$$

If e' be placed at A' its potential at B' will be

$$V' = \frac{e'}{A'B'}.$$

In the theory of electrical images

$$e : e' :: OA : R :: R : OA'.$$

Hence $$V : V' :: R : OB, \tag{17}$$

or the potential at B due to the electricity at A is to the potential at the image of B due to the electrical image of A as R is to OB.

Since this ratio depends only on OB and not on OA, the potential at B due to any system of electrified bodies is to that at B' due to the image of the system as R is to OB.

If r be the distance of any point A from the centre, and r' that of its image A', and if e be the electrification of A, and e' that of A', also if L, S, K be linear, superficial, and solid elements at A, and L', S', K' their images at A', and λ, σ, ρ, λ', σ', ρ' the corresponding line-surface and volume-densities of electricity at the two points, V the potential at A due to the original system, and V' the potential at A' due to the inverse system, then

$$\left.\begin{array}{c}
\dfrac{r'}{r} = \dfrac{L'}{L} = \dfrac{R^2}{r^2} = \dfrac{r'^2}{R^2}, \quad \dfrac{S'}{S} = \dfrac{R^4}{r^4} = \dfrac{r'^4}{R^4}, \quad \dfrac{K'}{K} = \dfrac{R^6}{r^6} = \dfrac{r'^6}{R^6}; \\
\dfrac{e'}{e} = \dfrac{R}{r} = \dfrac{r'}{R}, \qquad \dfrac{\lambda'}{\lambda} = \dfrac{r}{R} = \dfrac{R}{r'}, \\
\dfrac{\sigma'}{\sigma} = \dfrac{r^3}{R^3} = \dfrac{R^3}{r'^3}, \qquad \dfrac{\rho'}{\rho} = \dfrac{r^5}{R^5} = \dfrac{R^5}{r'^5}, \\
\dfrac{V'}{V} = \dfrac{r}{R} = \dfrac{R}{r'}.
\end{array}\right\} *(18)$$

If in the original system a certain surface is that of a conductor,

* See Thomson and Tait's *Natural Philosophy*, § 515.

and has therefore a constant potential P, then in the transformed system the image of the surface will have a potential $P\frac{R}{r'}$. But by placing at O, the centre of inversion, a quantity of electricity equal to $-PR$, the potential of the transformed surface is reduced to zero.

Hence, if we know the distribution of electricity on a conductor when insulated in open space and charged to the potential P, we can find by inversion the distribution on a conductor whose form is the image of the first under the influence of an electrified point with a charge $-PR$ placed at the centre of inversion, the conductor being in connexion with the earth.

163.] The following geometrical theorems are useful in studying cases of inversion.

Every sphere becomes, when inverted, another sphere, unless it passes through the centre of inversion, in which case it becomes a plane.

If the distances of the centres of the spheres from the centre of inversion are a and a', and if their radii are α and α', and if we define the *power* of the sphere with respect to the centre of inversion to be the product of the segments cut off by the sphere from a line through the centre of inversion, then the power of the first sphere is $a^2-\alpha^2$, and that of the second is $a'^2-\alpha'^2$. We have in this case

$$\frac{a'}{a}=\frac{\alpha'}{\alpha}=\frac{R^2}{a^2-\alpha^2}=\frac{a'^2-\alpha'^2}{R^2}, \tag{19}$$

or the ratio of the distances of the centres of the first and second spheres is equal to the ratio of their radii, and to the ratio of the power of the sphere of inversion to the power of the first sphere, or of the power of the second sphere to the power of the sphere of inversion.

The centre of either sphere corresponds to the inverse point of the other with respect to the centre of inversion.

In the case in which the inverse surfaces are a plane and a sphere, the perpendicular from the centre of inversion on the plane is to the radius of inversion as this radius is to the diameter of the sphere, and the sphere has its centre on this perpendicular and passes through the centre of inversion.

Every circle is inverted into another circle unless it passes through the centre of inversion, in which case it becomes a straight line.

The angle between two surfaces, or two lines at their intersection, is not changed by inversion.

Every circle which passes through a point, and the image of that point with respect to a sphere, cuts the sphere at right angles.

Hence, any circle which passes through a point and cuts the sphere at right angles passes through the image of the point.

164.] We may apply the method of inversion to deduce the distribution of electricity on an uninsulated sphere under the influence of an electrified point from the uniform distribution on an insulated sphere not influenced by any other body.

If the electrified point be at A, take it for the centre of inversion, and if A is at a distance f from the centre of the sphere whose radius is a, the inverted figure will be a sphere whose radius is a' and whose centre is distant f', where

$$\frac{a'}{a} = \frac{f'}{f} = \frac{R^2}{f^2 - a^2}. \tag{20}$$

The centre of either of these spheres corresponds to the inverse point of the other with respect to A, or if C is the centre and B the inverse point of the first sphere, C' will be the inverse point, and B' the centre of the second.

Now let a quantity e' of electricity be communicated to the second sphere, and let it be uninfluenced by external forces. It will become uniformly distributed over the sphere with a surface-density

$$\sigma' = \frac{e'}{4\pi a'^2}. \tag{21}$$

Its action at any point outside the sphere will be the same as that of a charge e' placed at B' the centre of the sphere.

At the spherical surface and within it the potential is

$$P' = \frac{e'}{a'}, \tag{22}$$

a constant quantity.

Now let us invert this system. The centre B' becomes in the inverted system the inverse point B, and the charge e' at B' becomes $e'\frac{AB}{R}$ at B, and at any point separated from B by the surface the potential is that due to this charge at B.

The potential at any point P on the spherical surface, or on the same side as B, is in the inverted system

$$\frac{e'}{a'}\frac{R}{AP}.$$

If we now superpose on this system a charge e at A, where

$$e = -\frac{e'}{a'}R, \tag{23}$$

the potential on the spherical surface, and at all points on the same side as B, will be reduced to zero. At all points on the same side as A the potential will be that due to a charge e at A, and a charge $e'\frac{R}{f'}$ at B.

But
$$e'\frac{R}{f'} = -e\frac{a'}{f'} = -e\frac{a}{f}, \tag{24}$$

as we found before for the charge of the image at B.

To find the density at any point of the first sphere we have

$$\sigma = \sigma'\frac{R^3}{AP^3}. \tag{25}$$

Substituting for the value of σ' in terms of the quantities belonging to the first sphere, we find the same value as in Art. 158,

$$\sigma = -\frac{e(f^2-a^2)}{4\pi a AP^3}. \tag{26}$$

On Finite Systems of Successive Images.

165.] If two conducting planes intersect at an angle which is a submultiple of two right angles, there will be a finite system of images which will completely determine the electrification.

For let AOB be a section of the two conducting planes perpendicular to their line of intersection, and let the angle of intersection $AOB = \frac{\pi}{n}$, let P be an electrified point, and let $PO = r$, and $POB = \theta$. Then, if we draw a circle with centre O and radius OP, and find points which are the successive images of P in the two planes beginning with OB, we shall find Q_1 for the image of P in OB, P_2 for the image of Q_1 in OA, Q_3 for that of P_2 in OB, P_3 for that of Q_3 in OA, and Q_2 for that of P_3 in OB.

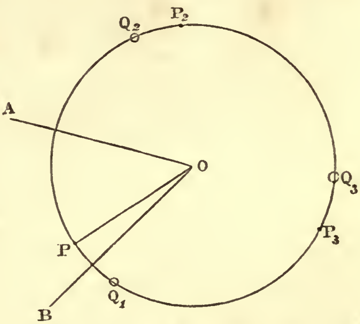


Fig. 10.

If we had begun with the image of P in AO we should have found the same points in the reverse order Q_2, P_3, Q_3, P_2, Q_1, provided AOB is a submultiple of two right angles.

For the alternate images P_1, P_2, P_3 are ranged round the circle at angular intervals equal to $2AOB$, and the intermediate images Q_1, Q_2, Q_3 are at intervals of the same magnitude. Hence, if $2AOB$ is a submultiple of 2π, there will be a finite number of images, and none of these will fall within the angle AOB. If, however, AOB is not a submultiple of π, it will be impossible to represent the actual electrification as the result of a finite series of electrified points.

If $AOB = \frac{\pi}{n}$, there will be n negative images Q_1, Q_2, &c., each equal and of opposite sign to P, and $n-1$ positive images P_2, P_3, &c., each equal to P, and of the same sign.

The angle between successive images of the same sign is $\frac{2\pi}{n}$. If we consider either of the conducting planes as a plane of symmetry, we shall find the positive and negative images placed symmetrically with regard to that plane, so that for every positive image there is a negative image in the same normal, and at an equal distance on the opposite side of the plane.

If we now invert this system with respect to any point, the two planes become two spheres, or a sphere and a plane intersecting at an angle $\frac{\pi}{n}$, the influencing point P being within this angle.

The successive images lie on the circle which passes through P and intersects both spheres at right angles.

To find the position of the images we may either make use of the principle that a point and its image are in the same radius of the sphere, and draw successive chords of the circle beginning at P and passing through the centres of the two spheres alternately.

To find the charge which must be attributed to each image, take any point in the circle of intersection, then the charge of each image is proportional to its distance from this point, and its sign is positive or negative according as it belongs to the first or the second system.

166.] We have thus found the distribution of the images when any space bounded by a conductor consisting of two spherical surfaces meeting at an angle $\frac{\pi}{n}$, and kept at potential zero, is influenced by an electrified point.

We may by inversion deduce the case of a conductor consisting of two spherical segments meeting at a re-entering angle $\frac{\pi}{n}$, charged to potential unity and placed in free space.

For this purpose we invert the system with respect to P. The circle on which the images formerly lay now becomes a straight line through the centres of the spheres.

If the figure (11) represents a section through the line of centres AB, and if D, D' are the points where the circle of intersection cuts the plane of the paper, then, to find the successive images, draw DA a radius of the first circle, and draw DC, DB, &c., making angles $\frac{\pi}{n}$, $\frac{2\pi}{n}$, &c. with DA.

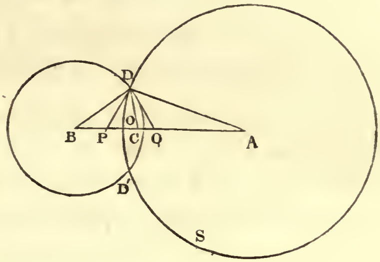


Fig. 11.

The points C, B, &c. at which they cut the line of centres will be the positions of the positive images, and the charge of each will be represented by its distances from D. The last of these images will be at the centre of the second circle.

To find the negative images draw DP, DQ, &c., making angles $\frac{\pi}{n}$, $\frac{2\pi}{n}$, &c. with the line of centres. The intersections of these lines with the line of centres will give the positions of the negative images, and the charge of each will be represented by its distance from D.

The surface-density at any point of either sphere is the sum of the surface-densities due to the system of images. For instance, the surface-density at any point S of the sphere whose centre is A, is

$$\sigma = \frac{1}{4\pi DA}\left\{1 + (AD^2 - AB^2)\frac{DB}{BS^3} + (AD^2 - AC^2)\frac{DC}{CS^3} + \&c.\right\},$$

where A, B, C, &c. are the positive series of images.

When S is on the circle of intersection the density is zero.

To find the total charge on each of the spherical segments, we may find the surface-integral of the induction through that segment due to each of the images.

The total charge on the segment whose centre is A due to the image at A whose charge is DA is

$$DA\frac{DA+OA}{2\,DA} = \tfrac{1}{2}(DA+OA),$$

where O is the centre of the circle of intersection.

In the same way the charge on the same segment due to the image at B is $\frac{1}{2}(DB+OB)$, and so on, lines such as OB measured from O to the left being reckoned negative.

Hence the total charge on the segment whose centre is A is

$$\tfrac{1}{2}(DA+DB+DC+\&c.)+\tfrac{1}{2}(OA+OB+OC+\&c.),$$
$$-\tfrac{1}{2}(DP+DQ+\&c.)-\tfrac{1}{2}(OP+OQ+\&c.).$$

167.] The method of electrical images may be applied to any space bounded by plane or spherical surfaces all of which cut one another in angles which are submultiples of two right angles.

In order that such a system of spherical surfaces may exist, every solid angle of the figure must be trihedral, and two of its angles must be right angles, and the third either a right angle or a submultiple of two right angles.

Hence the cases in which the number of images is finite are—

(1) A single spherical surface or a plane.

(2) Two planes, a sphere and a plane, or two spheres intersecting at an angle $\frac{\pi}{n}$.

(3) These two surfaces with a third, which may be either plane or spherical, cutting both orthogonally.

(4) These three surfaces with a fourth cutting the first two orthogonally and the third at an angle $\frac{\pi}{n'}$. Of these four surfaces one at least must be spherical.

We have already examined the first and second cases. In the first case we have a single image. In the second case we have $2n-1$ images arranged in two series in a circle which passes through the influencing point and is orthogonal to both surfaces. In the third case we have, besides these images, their images with respect to the third surface, that is, $4n-1$ images in all besides the influencing point.

In the fourth case we first draw through the influencing point a circle orthogonal to the first two surfaces, and determine on it the positions and magnitudes of the n negative images and the $n-1$ positive images. Then through each of these $2n$ points, including the influencing point, we draw a circle orthogonal to the third and fourth surfaces, and determine on it two series of images, n' in each series. We shall obtain in this way, besides the influencing point, $2nn'-1$ positive and $2nn'$ negative images. These $4nn'$ points are the intersections of n circles with n' other circles, and these circles belong to the two systems of lines of curvature of a cyclide.

If each of these points is charged with the proper quantity of electricity, the surface whose potential is zero will consist of $n+n'$ spheres, forming two series of which the successive spheres of the first set intersect at angles $\frac{\pi}{n}$, and those of the second set at angles $\frac{\pi}{n'}$, while every sphere of the first set is orthogonal to every sphere of the second set.

Case of Two Spheres cutting Orthogonally. See Fig. IV at the end of this volume.

168.] Let A and B, Fig. 12, be the centres of two spheres cutting each other orthogonally in D and D', and let the straight line DD' cut the line of centres in C. Then C is the image of A with respect to the sphere B, and also the image of B with respect to the sphere whose centre is A. If $AD = a$, $BD = \beta$, then $AB = \sqrt{a^2+\beta^2}$, and if we place at A, B, C quantities of electricity equal to a, β, and $-\frac{a\beta}{\sqrt{a^2+\beta^2}}$ respectively, then both spheres will be equipotential surfaces whose potential is unity.

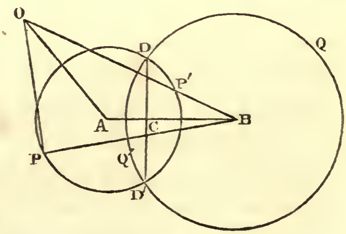


Fig. 12.

We may therefore determine from this system the distribution of electricity in the following cases:

(1) On the conductor $PDQD'$ formed of the larger segments of both spheres. Its potential is 1, and its charge is

$$a+\beta-\frac{a\beta}{\sqrt{a^2+\beta^2}} = AD+BD-CD.$$

This quantity therefore measures the capacity of such a figure when free from the inductive action of other bodies.

The density at any point P of the sphere whose centre is A, and the density at any point Q of the sphere whose centre is B, are respectively

$$\frac{1}{4\pi a}\left(1-\left(\frac{\beta}{BP}\right)^3\right) \text{ and } \frac{1}{4\pi\beta}\left(1-\left(\frac{a}{AQ}\right)^3\right).$$

At the points of intersection, D, D', the density is zero.

If one of the spheres is very much larger than the other, the density at the vertex of the smaller sphere is ultimately three times that at the vertex of the larger sphere.

(2) The lens $P'DQ'D'$ formed by the two smaller segments of the spheres, charged with a quantity of electricity $= -\dfrac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}}$, and acted on by points A and B, charged with quantities α and β, is also at potential unity, and the density at any point is expressed by the same formulae.

(3) The meniscus $DPD'Q'$ formed by the difference of the segments charged with a quantity α, and acted on by points B and C, charged respectively with quantities β and $\dfrac{-\alpha\beta}{\sqrt{\alpha^2+\beta^2}}$, is also in equilibrium at potential unity.

(4) The other meniscus $QDP'D'$ under the action of A and C.

We may also deduce the distribution of electricity on the following internal surfaces.

The hollow lens $P'DQ'D$ under the influence of the internal electrified point C at the centre of the circle DD'.

The hollow meniscus under the influence of a point at the centre of the concave surface.

The hollow formed of the two larger segments of both spheres under the influence of the three points A, B, C.

But, instead of working out the solutions of these cases, we shall apply the principle of electrical images to determine the density of the electricity induced at the point P of the external surface of the conductor $PDQD'$ by the action of a point at O charged with unit of electricity.

Let $OA = a,\quad OB = b,\quad OP = r,\quad BP = p,$

$AD = \alpha,\quad BD = \beta,\quad AB = \sqrt{\alpha^2+\beta^2}.$

Invert the system with respect to a sphere of radius unity and centre O.

The two spheres will remain spheres, cutting each other orthogonally, and having their centres in the same radii with A and B. If we indicate by accented letters the quantities corresponding to the inverted system,

$$a' = \frac{a}{a^2-\alpha^2},\quad b' = \frac{b}{b^2-\beta^2},\quad \alpha' = \frac{\alpha}{a^2-\alpha^2},\quad \beta' = \frac{\beta}{b^2-\beta^2},$$

$$r' = \frac{1}{r},\quad p'^2 = \frac{\beta^2 r^2 + (b^2-\beta^2)(p^2-\beta^2)}{r^2(b^2-\beta^2)^2}.$$

If, in the inverted system, the potential of the surface is unity, then the density at the point P' is

$$\sigma' = \frac{1}{4\pi a'}\left(1 - \left(\frac{\beta'}{p'}\right)^3\right).$$

If, in the original system, the density at P is σ, then

$$\frac{\sigma}{\sigma'} = \frac{1}{r^3},$$

and the potential is $\frac{1}{r}$. By placing at O a negative charge of electricity equal to unity, the potential will become zero over the surface, and the density at P will be

$$\sigma = \frac{1}{4\pi}\frac{a^2 - \alpha^2}{\alpha r^3}\left(1 - \frac{\beta^3 r^3}{(\beta^2 r^2 + (b^2 - \beta^2)(p^2 - \beta^2))^{\frac{3}{2}}}\right).$$

This gives the distribution of electricity on one of the spherical surfaces due to a charge placed at O. The distribution on the other spherical surface may be found by exchanging a and b, α and β, and putting q or AQ instead of p.

To find the total charge induced on the conductor by the electrified point at O, let us examine the inverted system.

In the inverted system we have a charge α' at A', and β' at B', and a negative charge $\frac{\alpha'\beta'}{\sqrt{\alpha'^2 + \beta'^2}}$ at a point C' in the line $A'B'$, such that $AC : CB :: \alpha'^2 : \beta'^2$.

If $OA' = a'$, $OB' = b'$, $OC' = c'$, we find

$$c'^2 = \frac{a'^2\beta'^2 + b'^2\alpha'^2 - \alpha'^2\beta'^2}{\alpha'^2 + \beta'^2}.$$

Inverting this system the charges become

$$\frac{\alpha'}{a'} = \frac{\alpha}{a}, \quad \frac{\beta'}{b'} = \frac{\beta}{b};$$

and

$$-\frac{\alpha'\beta'}{\sqrt{\alpha^2 + \beta^2}}\frac{1}{c'} = -\frac{\alpha\beta}{\sqrt{a^2\beta^2 + b^2\alpha^2 - \alpha^2\beta^2}}.$$

Hence the whole charge on the conductor due to a unit of negative electricity at O is

$$\frac{\alpha}{a} + \frac{\beta}{b} - \frac{\alpha\beta}{\sqrt{a^2\beta^2 + b^2\alpha^2 - \alpha^2\beta^2}}.$$

Distribution of Electricity on Three Spherical Surfaces which Intersect at Right Angles.

169.] Let the radii of the spheres be α, β, γ, then

$$BC = \sqrt{\beta^2+\gamma^2}, \quad CA = \sqrt{\gamma^2+\alpha^2}, \quad AB = \sqrt{\alpha^2+\beta^2}.$$

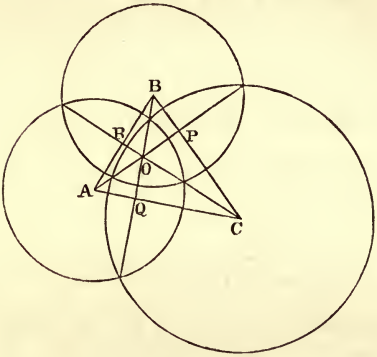


Fig. 13.

Let PQR, Fig. 13, be the feet of the perpendiculars from ABC on the opposite sides of the triangle, and let O be the intersection of perpendiculars.

Then P is the image of B in the sphere γ, and also the image of C in the sphere β. Also O is the image of P in the sphere α.

Let charges α, β, and γ be placed at A, B, and C.

Then the charge to be placed at P is

$$-\frac{\beta\gamma}{\sqrt{\beta^2+\gamma^2}} = -\frac{1}{\sqrt{\frac{1}{\beta^2}+\frac{1}{\gamma^2}}}.$$

Also $AP = \dfrac{\sqrt{\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2}}{\sqrt{\beta^2+\gamma^2}}$, so that the charge at O, considered as the image of P, is

$$\frac{\alpha\beta\gamma}{\sqrt{\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2}} = \frac{1}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}}.$$

In the same way we may find the system of images which are electrically equivalent to four spherical surfaces at potential unity intersecting at right angles.

If the radius of the fourth sphere is δ, and if we make the charge at the centre of this sphere $= \delta$, then the charge at the intersection of the line of centres of any two spheres, say α and β, with their plane of intersection, is $\quad -\dfrac{1}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}}}$.

The charge at the intersection of the plane of any three centres ABC with the perpendicular from D is

$$+\frac{1}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}},$$