A TREATISE

OF

CELESTIAL MECHANICS,

BY P. S. LAPLACE,

MEMBER OF THE NATIONAL INSTITUTE, AND OF THE COMMITTEE OF LONGITUDE, OF FRANCE; THE ROYAL SOCIETIES OF LONDON AND GOTTINGEN; OF THE ACADEMIES OF SCIENCES OF RUSSIA, DENMARK, AND PRUSSIA, &c.

PART THE FIRST—BOOK THE FIRST.

TRANSLATED FROM THE FRENCH, AND ELUCIDATED WITH

EXPLANATORY NOTES.

BY THE REV. HENRY H. HARTE, F.T.C.D. M.R.I.A.

DUBLIN:

PRINTED FOR RICHARD MILLIKEN, BOOKSELLER TO THE UNIVERSITY; AND FOR LONGMAN, HURST, REES, ORME AND BROWNE, LONDON.

1822.

D. GRAISBERRY,
PRINTER TO THE UNIVERSITY.

TO

THE REV. CHARLES WILLIAM WALL,

THIS TREATISE

IS DEDICATED,

BY

HIS FRIEND, AND FORMER PUPIL,

HENRY H. HARTE.

PREFACE.

IT has been made a matter of surprise, that considering the great capabilities of many individuals in these countries, so few are conversant with the contents of a work of such acknowledged eminence, as the Celestial Mechanics. Without adverting to other causes, it may be safely asserted, that the chief obstacle to a more general knowledge of the work, arises from the summary manner in which the Author passes over the intermediate steps in several of his most interesting investigations. To remove this obstacle, is the design of the present treatise, in which the translator endeavours to elucidate every difficulty in the text, and to expand the different operations which are taken for granted. He has not attempted to follow the principles into all their details; but he has occasionally adverted to some useful applications of them, which occur in different Authors. He is aware that those conversant with such subjects will find much observation that may be dispensed with; but when it is considered that his object was to render this work accessible to the general class of readers, he trusts that he will not be deemed unnecessarily diffuse, if he has insisted longer on some points than the experienced reader may think necessary. As many of the propositions which Newton announced *separately* are so many different results, which are all comprised under the same general law *analytically* investigated, he has also taken occasion to notice, in the notes, those propositions of Newton, which are embraced in the general analysis of the text, which he was induced to do, in order to show the great superiority of the analytic mode of investigating problems. The Work will be divided into five parts, which will be published in separate volumes. The first volume contains the first book, which treats of the general principles of the equilibrium and motion of bodies. The number of notes which was necessary for the elucidation of these principles is much greater than will be required in any of the subsequent volumes. The second volume will contain the second and third books of the original; the third volume, the fourth and fifth books; the fourth volume will contain the sixth, seventh, and eighth books; and the last volume will contain the ninth and tenth books, together with the supplement to the tenth book.

Trin. Coll.
April, 1822.

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BOOK I.

A TREATISE

OF

CELESTIAL MECHANICS,

&c. &c.

NEWTON published, towards the close of the seventeenth century, the discovery of universal gravitation. Since that period, Philosophers have reduced all the known phenomena of the system of the world to this great law of nature, and have thus succeeded in giving to the theories and astronomical tables a precision which could never have been anticipated. I propose in this present treatise to exhibit in one point of view, these theories which are scattered through a great number of works, of which the whole comprising the results of universal gravitation, on the equilibrium and motion of the bodies both solid and fluid, composing the solar and similar systems, constitutes *The Celestial Mechanics.* Astronomy, considered in the most general manner, is a great problem of Mechanics, of which the arbitrary quantities are the elements of the motions of the heavenly bodies; its solution depends, at the same time, on the precision of the observations, and on the perfection of analysis; and it is of the last importance to banish all empiricism, and to reduce it, so that it may borrow nothing from observation, but the indispensable data. The object of this work, is, as far as it is in my power, to accomplish this interesting end. I trust that, in consideration of the difficulties and importance of the subject, Philosophers and Mathematicians will receive it with indulgence, and that they will find the results sufficiently simple to be employed in their investigations. It will be divided into two parts. In the first, I will give the methods, and formulæ, for determining the motions of the centres of gravity of the heavenly bodies, their figures, the oscillations of the fluids which are spread over them, and their motions about their proper centres of gravity. In the second part, I will apply the formulæ which have been found in the first, to the planets, the satellites and the comets; and I will conclude with a discussion of several questions relative to the system of the world, and by a historical notice of the labours of Mathematicians on this subject. I will adopt the decimal division of the quadrant, and of the day, and I will refer the linear measures, to the length of the metre, determined by the arc of the terrestrial meridian comprised between Dunkirk and Barcelona.

ERRATA.

Page Line

3, 23, *for* the new forces, *read* these forces.

6, 12, *for* reluctant, *read* resultant.

12, 15, *for* (*c*) *read* (*b*).

17, 1, *for* equation, *read* equations.

23, 14, *for* $\varphi(f')$, *read* $\varphi'(f)$.

24, 14, *for* *a* and *b*, *read* *c* and *b*; and line 17, *for* angle, *read* angled.

32, 10, *for* $dy^2 dz^2$, *read* $dy^2 + dz^2$.

33, 2, from bottom, *after* A^2, *add* $=$; and last line, *after* the differential of the, add square of the, and *for* $s^n ds$, *read* $\frac{s^n ds}{n+1}$.

35, 2, from bottom, *for* first the order, *read* the first order.

40, 18, *after* centrifugal, *add* force.

47, last line, *for* dt constant, *read* dx constant.

49, 11, *for* $2h. \cos. \theta.$ *read* $2h. \cos.^2 \theta.$; and in lines 21, 22, *dele* the 2 which occur in the Den^r^.

50, 14, *dele* the 2 by which the values of dt, dz, dx, are multiplied.

51, 11, *for* $g2t$, *read* gt^2.

65, 1, *for* ds_2, *read* ds'^2, ; line 16, *for* $\log. n.(s+q)\frac{1}{n})$, *read* $\log. n.(s+q))\frac{1}{n}$; line 17, *for* $(s'+q^n)$ *read* $s'+q$.

72, 14, *for* $\frac{ds}{S\ ds}$ *read* $\frac{ds}{S.\ dt}$.

82, 4, *for* P, *read* $-P$.

83, 2, *for* they, *read* it.

84, 23, *for* k, k, k, *read* R, R, R.

86, 3 from bottom, *for* $\frac{\partial y}{\partial x}$ *read* $\frac{\partial s}{\partial x}$.

94, 2, *after* centre, *read* of grzvity.

99, 12, *for* figure, *read* figures.

105, *for* $\partial\varrho$, *read* ∂p; 20, *after* each, *read* other.

ERRATA.

Page *Line*

138, 16, *for* sin. θ. sin. ψ.+cos. ψ. sin. ϕ. *read* sin. θ. sin. ψ. cos. ψ. sin. ϕ; line 19, *for* makes *read* make.

142, 16, *for* Σmx, Σmy, Σmz, *read* $\Sigma mx_{,}$, $\Sigma my_{,}$, $\Sigma mz_{,}$.

148, 18, *for* $\Sigma m.(2\Sigma m.fm.\ Pdx+Qdy+Rdz)$, *read* $\Sigma m.\ 2\Sigma.fm,(Pdx+Qdy+Rdz$ and in line 19, *for* $fmm'_{,}fdf$, *read* $fmm'.\ Fdf$.

149, 2, *after* velocities, *read* of the bodies.

168, 15, to the second x''^{2} prefix + and line 19, *for* $+x''$ *read* $+x''^{2}$.

197, 20, *for* $+r'$, *read* $+r'^{2}$.

204, 21, *for* parallel, *read* perpendicular.

214, 2, *for* to coincide very nearly with the plane of x' and $y'_{,}$, *read* to be very nearly perpendicular to the plane of x' and y'.

217, 26, *for* dy, *read* dy'.

218, 19, *for* z''^{2} sin. θ. *read* z''^{2}. sin. $^{2}\theta$.

229, 2, *for* $\frac{dx}{db}$, *read* $\frac{dz}{dc}$.

230, 3, *for* dx, *read* dx, line 11, *for* the first $\frac{dy}{db}.\frac{dz}{da}$, *read* $\frac{dy}{da}.\frac{dz}{db}$.

231, 20, *for* $\frac{du}{dz}.\ V+$ *read* $\left(\frac{du}{dx}\right)u$.

233, 4, *for* $\frac{dy.+dw.dt}{dz}$ *read* $\frac{dy+dv.dt}{dx}$

234, 2, multiply the first member by dt, line 11, prefix — to dt, and *for* — V *read* — $V.dt$.

235, 17, *for* k *read* $\frac{1}{k}$.

236, 17, *for* δr, *read* $\delta\varrho$, and for homogenous, *read* homogeneous.

240, 16, *for* dz, *read* dz^{2}.

241, 12, *for* the second $\frac{d^{2}\phi}{dx^{2}}$, *read* $\frac{d^{2}\phi}{dz^{2}}$.

251, 16, *for* ra^{2}, *read* $r^{2}a$.

252, 8, *for* numbers, *read* members.

256, 17, *for* $r^{2}.(\sin.\ \theta+au.\ \cos.\ \theta)$; *read* $r'^{2}(\sin.\ \theta+au\ \cos.\ \theta)$; line 19, *for* $2as$, *read* $2ars$.

265, 17, *for* $\frac{n}{2}$, *read* $\frac{n^{2}}{2}$.

A

TREATISE

ON

CELESTIAL MECHANICS,

&c. &c.

PART I.—BOOK I.

IN this book, the general principles of the equilibrium and motion of bodies are established, and those problems in Mechanics are solved, the solution of which is indispensable in the theory of the system of the world.

CHAPTER I.

Of the equilibrium and of the composition of forces which act on a material point.

1. A body appears to us to move, when it changes its situation with respect to a system of bodies which we suppose to be at rest; but as all bodies, even those which seem to us to be in a state of the

most absolute rest, may be in motion; we, in imagination, refer the position of bodies to a space which is supposed to be boundless, immoveable, and penetrable to matter; and when they answer successively to different parts of this real or ideal space, we conceive them to be in motion.

The nature of that singular modification, in consequence of which a body is transferred from one place to another is, and always will be, unknown: we have designated it by the name force; but we can only determine its effects and the laws of its action. The effect of a force acting on a material point, is, if no obstacle opposes, to put it in motion; the direction of the force is the right line which it tends to make the point describe. It is evident that when two forces act in the same direction, their effect to move the point is the sum of the two forces, and that when they act in opposite directions, the point is moved by a force represented by their difference. If their directions form an angle with each other, a force results, the direction of which is intermediate between the directions of the composing forces. Let us investigate this resultant and its direction.

For this purpose, let us consider two forces x and y acting at the same time on the material point M, and forming a right angle with each other. Let z represent their resultant, and θ the angle which it makes with the direction of the force x; the two forces x and y being given, the angle θ will be determined, and also the resultant z, so that there exists between these three quantities x, y, z, a relation which it is required to ascertain.

Let us suppose at first the forces x and y infinitely small, and equal to the differentials dx and dy; let us suppose afterwards that x becoming successively dx, $2dx$, $3dx$, &c. y becomes dy, $2dy$, $3dy$, &c. it is evident that the angle θ will always remain the same, and that the resultant z will become successively dz, $2dz$, $3dz$, &c.; thus in the successive increments of the three forces x, y, and z, the ratio of x to z will be constant, and can be expressed by a function of θ which we will designate by $\varphi(\theta)$; therefore we shall have $x = z\,\varphi(\theta)$, in which equation x may be changed into y, provided that in like manner the angle θ is changed into $\frac{\pi}{2}-\theta$, π being the semi-circumference of a circle whose radius is equal to unity.

Now, we can consider the force x as the resultant of two forces x' and x'', of which the first x' is in the direction of the resultant z, the second x' being perpendicular to this resultant. The force x which results from these two new forces, forming the angle θ with the force x', and the angle $\frac{\pi}{2}-\theta$, with the force x'' we shall have

$$x' = x\varphi(\theta) = \frac{x^2}{y};\ x'' = x\varphi\left(\frac{\pi}{2}-\theta\right) = \frac{xy}{z}$$

therefore we can substitute these two forces, for the force x. In like manner we can substitute for the force y, two new forces y' and y'', of which the first is equal to $\frac{y^2}{z}$ and in the direction of z, and of which the second is equal to $\frac{xy}{z}$ and perpendicular to z, thus we shall have in place of the two forces x and y the four following,

$$\frac{x^2}{z},\ \frac{y^2}{z},\ \frac{xy}{z},\ \frac{xy}{z};$$

the two last acting in opposite directions, destroy each other; the two first acting in the same direction, when added together constitute the resultant z; we shall have therefore

$$x^2+y^2=z^2;$$

from which it follows that the resultant of the two forces x and y is represented in quantity by the diagonal of a rectangle, of which the sides represent the new forces.

Let us now proceed to determine the angle θ. If the force x is increased by its differential, without altering the force y,* this angle will be diminished by the indefinitely small quantity $d\theta$, but it is possible to suppose the force dx resolved into two, one dx' in the direction of z, the other dx'' perpendicular to z; the point M will then be acted on by the forces $z+dx'$ and dx'' perpendicular to each other, and the resultant of those two forces, which we represent by z', will make with dx'' the angle $\frac{\pi}{2}-d\theta$; therefore by what precedes we shall have $dx''=z'.\,\varphi\left(\frac{\pi}{2}-d\theta\right)$, consequently the function $\varphi\left(\frac{\pi}{2}-d\theta\right)$ is indefinitely small, and of the form $-Kd\theta$; K being a constant quantity independent of the angle θ; therefore we have $\frac{dx''}{z'}=-Kd\theta$; z' differing by an indefinitely small quantity from z; moreover as dx'' forms an angle with dx equal to $\frac{\pi}{2}-\theta$ we have

$$dx''=dx\,\varphi\left(\frac{\pi}{2}-\theta\right)=y.\,dx;$$

therefore

$$d\theta=\frac{-ydx}{Kz^2},$$

* Since the direction of the resultant depends on the relation which exists between composing forces, if one force be increased, while the other remains unaltered, the angle contained between the direction of the increased force and resultant, will be diminished by a quantity of the same order with that by which the force was increased. And when the force y receives the increase, the angle contained between the resultant and this increased force, will be diminished, therefore its complement, the angle θ, will be increased by the same quantity; and this is the reason why the expressions for the variations of θ corresponding to the respective variations of x and y are affected with contrary signs. If x and y are increased or diminished simultaneously, $d\theta$ will always vanish when dx, dy are respectively proportional to the quantities varied; this follows immediately from the expression for $d\theta$.

If the force y is varied by its differential dy, x being supposed to be constant, we shall have the corresponding variation of the angle θ, by changing x into y, y into x, and θ into $\frac{\pi}{2}-\theta$, in the preceding equation; which then gives

$$d\theta = \frac{xdy}{Kz^2}$$

therefore by making x and y to vary at the same time, the total variation of the angle θ will be $\frac{xdy-ydx}{Kz^2}$ and we shall have

$$\frac{xdy-ydx}{z^2} = Kd\theta$$

If we substitute for z^2 its value x^2+y^2, and then * integrate we shall have

$$\frac{y}{x} = \tan.\ (K\theta+\rho)$$

ρ being a constant arbitrary quantity. This equation being combined with the equation $x^2+y^2=z^2$ gives

$$x = z.\cos.\ (K\theta+\rho)$$

* $\frac{xdy-y\,dx}{x^2+y^2} = \frac{\frac{xdy-y\,dx}{x^2}}{1+\frac{y^2}{x^2}} = \frac{d.\left(\frac{y}{x}\right)}{1+\frac{y^2}{x^2}} = \frac{du}{1+u^2}$ (by putting $\left.\frac{y}{x}=u\right)$ therefore $\int\frac{du}{1+u^2}$ $(=\text{arc tang.}=u) = \int K\,d\theta =$ $K\theta+\varrho$ $\therefore u = \left(\frac{y}{x}\right) = \tan.\ (K\theta+\varrho) = \frac{\sin.\ K\theta+\varrho}{\cos.\ K\theta+\varrho}$ $\therefore y^2\ (=z^2-x^2) =$ $x^2\frac{\sin.^2(K\theta+\varrho)}{\cos.^2(K\theta+\varrho)}$ $\therefore z^2\cos.^2(K\theta+\varrho) = x^2\left((\sin.^2(K\theta+\varrho)+\cos.^2(K\theta+\varrho)\right) = x^2$

It is only now required to know the two constant quantities K and ρ; but if we suppose y to vanish we have evidently $z = x$, and $\theta = o$, therefore cos. $\rho = 1$ and $x = z$. cos. $K\theta$. If we suppose x to vanish, then $z = y$, and $\theta = \frac{1}{2}\varpi$; cos. $K\theta$ being then equal to nothing, K *must be equal to $2n+1$, n being an integral number; and in this case x will vanish as often as θ will be equal to $\frac{\frac{1}{2}\varpi}{2n+1}$; but x being nothing we have evidently $\theta = \frac{1}{2}\varpi$; therefore $2n+1 = 1$, or $n = o$, consequently

$$x = z. \cos. \theta.$$

From which it follows that the diagonal of a rectangle described on the right lines which represent the forces x and y, represents not only the quantity but also the direction of their reluctant. Thus we can substitute for any force whatever two other forces which form the sides of a rectangle, of which that force is the diagonal; and it is easy to infer from thence that it is possible to resolve a force into three others, which form the sides of a rectangular parallelipiped of which it is the diagonal.†

Let therefore a b and c represent the three rectangular coordinates of the extremity of a right line, which represents any force whatever, and of which the origin is that of the coordinates; this force will be represented by the function $\sqrt{a^2+b^2+c^2}$, and by resolving it

* In this case $K\theta$ is some odd multiple of $\frac{\pi}{2}$ and therefore K must be of the form $2n+1$.

† The given force being resolved into two, of which one is perpendicular to a plane given in position, the other being parallel to this plane, if this second partial force be decomposed into two others, parallel to two axes situated in this plane, and perpendicular to each other; it is evident that the three partial forces will be at right angles to each other, and that the sum of the squares of the lines representing these forces, will be equal to the square of the line representing the given force, therefore this last force is the diagonal of a rectangular parallelipiped, of which the partial forces constitute the sides.

parallel to the axes of a of b and of c, the partial forces will be expressed respectively by these coordinates.

Let a', b', c', be the coordinates of a second force; $a+a'$, $b+b'$, $c+c'$, will be the coordinates of the resultant of the two forces, and will represent the partial forces into which it can be resolved parallel to the three axes, from whence it is easy to conclude that this resultant is the diagonal of a parallelogram, of which the two forces are the sides.*

In general a, b, c; a', b', c'; a'', b'', c''; &c. being the coordinates of any number of forces; $a+a'+a''+$, &c. $b+b'+b''+$, &c. $c+c'+c''+$ &c. will be the coordinates of the resultant; the square of which will be equal to the sum of the squares of these last coordinates; thus we shall have both the quantity and the position of the resultant.†

* The coordinates of the extremity of this diagonal are evidently equal to $a+a'$, $b+b'$, $c+c'$, therefore this diagonal must be equal to the resultant of the two forces. We are enabled to derive an expression for the cosine of the angle, contained between the given forces, in terms of the cosines of the angles which these forces make with the coordinates, for calling the forces S and S', and the angles which S makes with the three axes, A, A', A'', and B, B', B'', the angles which S' makes with the same axes we have $a=S\cos. A$, $b=S\cos. A'$, $c=S\cos. A''$, $a'=S'\cos. B$, $c'=S'\cos. B'$, $c'=S'\cos. B''$; the square of the line connecting the extremities of S and $S'=S^2-2SS'.\cos.\Delta+S'^2$; Δ being the angle contained between S and S', the square of this line is also equal to

$$(S\cos. A-S'\cos. B)^2+(S\cos. A'-S'\cos. B')^2+(S\cos. A''-S'\cos. B'')^2;$$

$$=S^2+S'^2-2\,SS'\,(\cos. A.\cos. B+\cos. A'.\cos. B'+\cos. A''.\cos. B'',)$$

consequently we have

$$\cos.\Delta=\cos. A.\cos. B+\cos. A.\cos. B'+\cos. A''.\cos. B'',$$

therefore when the two forces are perpendicular to each other, the second member of this equation is equal to nothing.

† Let S S' S'', &c. represent the forces of which the coordinates are respectively a, b, c; a', b', c'; a'', b'', c'', &c. then by what precedes $a+a'$, $b+b'$, $c+c'$, are the coordinates of the resultant of S and S', $a+a'+a''$, $b+b'+b''$, $c+c'+c''$, are the coordinates of the resultant of this last force, and the force S'' &c.: therefore the resultant V of any number of forces is the diagonal of a rectangular parallelipiped of which

2. From any point whatever of the direction of a force S, which point we will take for the origin of this force, let us draw a right line, which we will call s, to the material point M; let x, y, z, be the three rectangular coordinates which determine the position of the point M, and a, b, c, the coordinates of the origin of the force; we shall have

$$S = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}.$$

If we resolve the force S parallel to the axes of x, of y, and of z, the corresponding partial forces will be by the preceding number

$$S.\frac{(x-a)}{s}; \quad S.\frac{(y-b)}{s}; \quad S.\frac{(z-c)}{s}; \quad \text{or } S.\left(\frac{\partial s}{\partial z}\right); \ S.\left(\frac{\partial s}{\partial y}\right); \ S.\left(\frac{\partial s}{\partial z}\right);^*$$

the coordinates are equal respectively to the sum of the coordinates of the composing forces,

$$\therefore V^2 = (a+a'+a''+\&c.)^2 + (b+b'+b''\ \&c.)^2 + (c+c+c'+\ \&c.)^2.$$

Let m, n, p = the angles which V makes with the rectangular axes

$$\cos. m = \frac{a+a'+a''+\ \&c.}{V} \quad \cos. m = \frac{b+b'+b''+\ \&c.}{V} \quad \cos. p = \frac{c+c'+c''+\&c.}{V}$$

$\therefore$ we have both the quantity and direction of the resultant.

From the preceding composition of forces it follows, that if a polygon is constructed, of which the sides, (which may be in different planes) are respectively proportional to these forces, and parallel to their directions, the last side of this polygon represents the resultant of all the forces in quantity and in direction.

* S being considered as a function of x, y, and z, $\partial s = \left(\frac{\partial s}{\partial x}\right)\partial x + \left(\frac{\partial s}{\partial y}\right)\partial y + \left(\frac{\partial z}{\partial z}\right)\partial z$

and when $s = \sqrt{(x-a)^2+(y-b)^2+(z-c^2} \quad \left(\frac{\partial s}{\partial x}\right) = \frac{x-a}{s}, \ \frac{\partial s}{\partial u} = \frac{y-b}{s}, \ \frac{\partial s}{\partial z} = \frac{z-c}{s}$

$\frac{x-a}{s}\ \frac{y-b}{s}$, &c. are evidently the expressions for the cosines of the angles which s makes with the coordinates x, y, and z, since

$$V.\left(\frac{\partial u}{\partial x}\right) = S\left(\frac{\partial s}{\partial x}\right) + S'\left(\frac{\partial s'}{\partial x}\right) + S'\left(\frac{\partial s''}{\partial x}\right) + \&c.;$$

$\left\{\frac{\partial s}{\partial x}\right\}$; $\left\{\frac{\partial s}{\partial y}\right\}$; $\left\{\frac{\partial s}{\partial z}\right\}$, expressing according to the received notation the coefficients of the variations of ∂x, ∂y, ∂z, in the variation of the preceding expression of s.

If, in like manner, we name s' the distance of M from any point in the direction of another force S', that point being taken for the origin of this force; $S'.\left\{\frac{\partial s'}{\partial x}\right\}$ will be this force resolved parallel to the axes of x, and just so the rest; therefore the sum of the forces S, S', S'', &c.

C

$$V.\left(\frac{\partial u}{\partial y}\right) = S.\left(\frac{\partial s}{\partial y}\right) + S'.\left(\frac{\partial s'}{\partial y}\right) + S'.\left(\frac{\partial s''}{\partial y}\right) + \&c.$$

$$V.\left(\frac{\partial u}{\partial z}\right) = S.\left(\frac{\partial s}{\partial z}\right) + S'.\left(\frac{\partial s'}{\partial z}\right) + S''.\left(\frac{\partial s''}{\partial z}\right) + \&c.$$

by multiplying these equations by ∂x, ∂y, ∂z, respectively, and adding them together, we get

$$V.\left(\left(\frac{\partial u}{\partial x}\right).\ \partial x + \left(\frac{\partial u}{\partial y}\right).\ \partial y + \left(\frac{\partial u}{dz}\right).\ \partial z\right) =$$

$$V.\ \partial u = S.\left(\left(\frac{\partial s}{\partial x}\right), \partial x + \left(\frac{\partial x}{\partial y}\right).\ \partial y + \left(\frac{\partial s}{\partial z}\right).\ \partial z\right)$$

$$+S'.\left(\left(\frac{\partial s'}{\partial x}\right).\ \partial x + \left(\frac{\partial s'}{\partial y}\right).\ \partial y + \left(\frac{\partial s'}{\partial z}\right)\partial z.\right)$$

$$+S''.\left(\left(\frac{\partial s''}{\partial x}\right)\partial x + \left(\frac{\partial s''}{\partial y}\right)\partial y + \left(\frac{\partial s''}{\partial z}\right)\right),\ \partial z + \&c. = S.\partial s + S'\partial s' + S''\partial s'' + \&c. = \Sigma.S.\partial s.$$

Now since these equation have place whatever be the variations ∂x, ∂y, ∂z, one of them may exist while the other two vanish, therefore the equation (a) is equivalent to the three equations which precede it. We shall see hereafter that the introduction of the coefficient $\left(\frac{\partial s}{\partial x}\right)$ is of the greatest consequence, for from the equation (b) which follows immediately from the equation (a), we deduce the equation (l) of No. 14, which involves the principle of vertual velocities, and this principle combined with that of D'Alembert, has given to Mechanics all the perfection of which it was susceptible, for by means of it the investigation of the motions of any system of bodies is reduced to the integration of differential equations. See No. 18.

resolved parallel to this axis will be $\Sigma.\ S.\left(\frac{\delta s}{\delta x}\right)$, the characteristic Σ of finite integrals expressing the sum of the terms $S.\left(\frac{\delta s}{\delta x}\right)$, $S'.\left\{\frac{\delta s'}{\delta x}\right\}$; &c.

Let V be the resultant of all the forces S, S', &c. and u the distance of the point M from any point in the direction of this resultant, which is taken for its origin; $V.\left\{\frac{\delta u}{\delta x}\right\}$ will be the expression of this resultant resolved parallel to the axis of x; therefore by the preceding number we shall have $V.\left\{\frac{\delta u}{\delta x}\right\} = \Sigma.\ S.\left\{\frac{\delta s}{\delta x}\right\}$,

we shall have in like manner

$$V.\left\{\frac{\delta u}{\delta y}\right\} = \Sigma.\ S.\left\{\frac{\delta s}{\delta y}\right\}; \quad V.\left\{\frac{\delta u}{\delta z}\right\} = \Sigma.\ S.\left\{\frac{\delta s}{\delta z}\right\}$$

from which we may obtain, by multiplying these equations respectively by δx, δy, δz, and adding them together

$$V.\delta u = \Sigma.\ S.\ \delta s;$$

As this last equation has place whatever be the variations δx, δy, δz it is equivalent to the three preceding. If its second member is an exact variation of a fuction φ, we shall have $V.\ \delta u = \delta\varphi$, and consequently

$$V.\left\{\frac{\delta u}{\delta x}\right\} = \frac{\delta\varphi}{\delta x}$$

which indicates that the sum of all the forces resolved parallel to the axis of x is equal to the partial difference $\left\{\frac{\delta\varphi}{\delta x}\right\}$. * This case ob-

* If we mult;ply δs the variation of any quantity by any function of that quantity, such as $\frac{g}{s.^m}$, $g.s^m$, &c. the product is evidently an exact variation, however this is not true of every species of function, for there are some transcendental and exponential functions, such as $\frac{\delta s}{\log.\ s.}$ which are not exact variations.

tains generally, when the forces are respectively functions of the distance of their origin from the point M. In order to have the resultant of all these forces resolved parallel to any right line whatever, we shall take the integral $\Sigma. \int S.\ \delta s$, and naming φ this integral, we shall consider it as a function of x, and of two right lines perpendicular to each other and to x; the partial difference $\left\{\frac{\delta\varphi}{\delta x}\right\}$ will be the resultant of the forces $S\ S'\ S''$, &c. resolved parallel to the right line x.

3. When the point M is in equilibrio, in consequence of the action of the forces which solicit it; their resultant vanishes, and the equation (a) becomes

$$O = \Sigma.\ S.\ \delta s \quad (b)$$

which indicates, that in the case of the equilibrium of a point acted on by any number of forces, the sum of the products of each force by the element of its direction is nothing.*

* Since the forces parallel to the coordinates x, y, z, are independant of each other, it follows from the notes to the preceding number, that when the point M is in equilibrio

$\Sigma.\ S \left\{\frac{\delta s}{\delta x}\right\}$ $\Sigma.\ S. \left\{\frac{\delta s}{\delta y}\right\}$ $\Sigma.\ S. \left\{\frac{\delta s}{\delta z}\right\}$ are = respectively to nothing.

$$i.\ e.\ S.\cos. A + S'.\cos. B + S''\cos. C + \&c. = 0$$

$$S.\cos. A' + S'\cos. B' + S''\cos. C' + \&c. = 0.$$

$$S.\cos. A'' + S'.\cos. B'' + S''\cos. C'' = 0.$$

(A, A', A''; B, B', B'', &c. are the angles which the directions of S, S', &c. make with x, y, z,); these are the equations of equilibrium of a system of forces applied to a material point which is entirely free. The independence which exists between these equations is extremely advantageous, it only obtains when the forces are resolved parallel to three rectangular coordinates. $\Sigma.\ S. \left\{\frac{\delta s}{\delta x}\right\} = 0$ indicates that M is at an invariable distance from the plane of y, z; in this case the forces are reducible to two rectangular ones, in the plane y, z.

When the point M is in equilibrio any one of the forces acting on it is equal and contrary to the resultant of all the remaining forces, for naming V' the resultant of the forces S', S''+&c. and a, b, c, the angles which it makes with the coordinates x, y, z, by

If the point M is forced to be on a curved surface, it will experience a reaction, which we will designate by R. This reaction is equal and directly contrary to the pressure with which the point presses on the surface; for by conceiving it acted on by two forces R and $-R$, it is possible to suppose that the force R is destroyed by the reaction of the surface, and that thus the point presses the surface with the force R; but the force of pressure of a point on a surface is perpendicular to it, otherwise it might be resolved into two, one perpendicular to the surface, which would be destroyed by it, the other parallel to the surface, in consequence of which the point would have no action on this surface, which is contrary to the hypothesis; consequently if r be the perpendicular drawn from the point M to the surface, and terminated in any point whatever of its direction, the force R will be directed along this perpendicular; therefore it will be necessary to add $R.\delta r$ to the second member of the equation (c) which thus becomes

$$0 = \Sigma.\ S.\ \delta s + R.\delta r \quad (c)$$

$-R$ being then the resultant of all the forces S, S', &c. it is perpendicular to the surface.

If we suppose that the arbitrary variations δx, δy, δz belong to the curved surface on which the point is subjected to remain, we shall have $\delta r = 0$, since r is perpendicular to the surface, therefore $R.\delta r$ vanishes from the preceding equation, in consequence of which the equation (b) obtains in this case, provided that one of the three variations δx, δy, δz, be eliminated by means of the equation to the surface; but then, the

what precedes we shall have $V'.\cos a = S' \cos. B + S'' \cos. C +$ &c. $V' \cos. c = S' \cos. B' + S''. \cos. C' +$ &c. and since $S. \cos. A + S'. \cos. B + S''. \cos. C +$ &c. $= 0$. We have $V'. \cos. a = -S. \cos. A$; in like manner it may be shewn that $V' \cos. b = -S. \cos. B$, and $V'. \cos. c = -S. \cos. C$: if we add together the squares of these equations we shall obtain $V'^2 = S^2$, because $\cos.^2 a + \cos.^2 b + {}^2 c = 1 = \cos.^2 A + \cos.^2 B + \cos.^2 C$.$\therefore$ we have $\cos. a = -\cos. A$ &c. $\because$ $a = 200' - A$, in like manner it follows, that $b = 200 - B$, $c = 200 - C$, $\therefore$ the forces S and V' are equal, and act in opposite directions.

equation (b) which in the general case is equivalent to three, is only equivalent to two distinct equations, which may be obtained by putting the coefficients of the two remaining differentials separately equal to nothing. Let $u = 0$ be the equation of the surface, the two equations $\delta r = 0$, and $\delta u = 0$ will have place at the same time; this requires that δr should be equal to $N\delta u$, N being a function of x, y, and z. Naming a, b, c, the coordinates of the origin of r we shall have to determine it

$$r = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$$

from which we may obtain $\left\{\frac{\delta r}{\delta x}\right\}^2 + \left\{\frac{\delta r}{\delta y}\right\}^2 + \left\{\frac{\delta r}{\delta z}\right\}^2 = 1$, and consequently

$$N^2.\left\{\left\{\frac{\delta u}{\delta x}\right\}^2 + \left\{\frac{\delta u}{\delta y}\right\}^2 + \left\{\frac{\delta u}{\delta z}\right\}^2\right\} = 1,$$

therefore by making

$$\lambda = \frac{R}{\sqrt{\left\{\frac{\delta u}{\delta x}\right\}^2 + \left\{\frac{\delta u}{\delta y}\right\}^2 + \left\{\frac{\delta u}{\delta z}\right\}^2}}$$

the term $R.\delta r$ of the equation (c) will be changed into $\lambda\delta u$, and this equation will become

$$0 = \Sigma.\ S.\ \delta s + \lambda\delta u$$

in which equation we ought to put the coefficients of the variations δx, δy, δz, separately equal to nothing, which gives three equations; but on account of the indeterminate quantity λ, which they contain, they are equivalent to only two between x, y, and z. Therefore instead of extracting from the equation (b) one of the variations δx, δy, δz, by means of the differential equation of the surface, we may add to it this equation multiplied by the indeterminate quantity λ, and then consider the variations δx, δy, and δz, as independant. This method, which also results from the theory of elimination combines the advantage of simplifying the calculation with that of indicating the force $-R$ with which the point M presses the surface.*

* When the point M is on a curved surface, then all that is required for its equilibrium is, that the direction of the resultant of all the forces which act on it should be perpendicular to this surface, but the intensity of this resultant is altogether undetermined, since the reaction is equal and contrary to the pressure of the point on the surface, by adding to $\Sigma.\ S.\ \delta s$ the quantity $R.\ \delta r$ we may consider the material point as entirely free.

δr vanishes because the perpendicular is the shortest line which can be drawn from a given point to the surface.

Since the same values of x, y, and z, satisfy the equations $\delta r = 0$ $\delta u = 0$, it follows from the theory of equations that $N = \frac{\delta r}{\delta u}$ is a function of x, y, and z,

$$\text{this function} = \frac{1}{\sqrt{\left\{\frac{\delta u}{\delta x}\right\}^2 + \left\{\frac{\delta u}{\delta y}\right\}^2 + \left\{\frac{\delta u}{\delta z}\right\}^2}}$$

it follows from the expression that is given for δr, that the cosines of the angles which the normal makes with the coordinates are equal respectively to $N.\left\{\frac{\delta u}{\delta x}\right\} N.\left\{\frac{\delta u}{\delta y}\right\} N.\left\{\frac{\delta z}{\delta u}\right\}$.

See notes to No. 9.

$$\text{Let } X = S.\left\{\frac{\delta s}{\delta x}\right\} + S'.\left\{\frac{\delta s'}{\delta y}\right\} + S''\left\{\frac{\delta s''}{\delta z}\right\} + \text{\&c.}$$

$$Y = S.\left\{\frac{\delta s}{\delta y}\right\} + S'\left\{\frac{\delta s'}{\delta y}\right\} + S''\left\{\frac{\delta s''}{\delta y}\right\} + \text{\&c.}$$

$$Z = S.\left\{\frac{\delta s}{\delta z}\right\} + S.\left\{\frac{\delta s'}{\delta z}\right\} + S''\left\{\frac{\delta s''}{\delta z}\right\} + \text{\&c.}$$

then $\Sigma.\ S.\ \delta s + \lambda\delta u = 0$ will be equal to $X.\ \delta x + Y.\ \delta y + Z.\ \delta z +$

$$\lambda\left\{\frac{\delta u}{\delta x}\right\}\delta x + \lambda\left\{\frac{\delta u}{\delta y}\right\}\delta y + \lambda\left\{\frac{\delta u}{\delta z}\right\}\delta z = 0.$$

and on account of the independance of the variables x, y, z, we shall have

$$X + \lambda\left\{\frac{\delta u}{\delta x}\right\} = 0, \quad Y + \lambda\left\{\frac{\delta u}{\delta y}\right\} = 0, \quad Z + \lambda\left\{\frac{\delta u}{\delta z}\right\} = 0,$$

eliminating λ we have the following equations:

$$Y.\frac{\delta u}{\delta x} - X.\frac{\delta u}{\delta y} = 0, \ Z.\frac{\delta u}{\delta z} - X.\frac{\delta u}{\delta z} = 0.$$

Let us conceive this point to be contained in a canal of simple or double curvature; the reaction of the canal which we will denote by k, will be equal and directly contrary to the pressure with which the point acts against the canal, the direction of which is perpendicular to its side; but the curve formed by this canal, is the intersection of two surfaces of which the equations express its nature, therefore we may consider the force k as the resultant of two forces R, R', which express the reactions of the two surfaces on the point M; since the directions of the three forces R, R', k, being respectively perpendicular to the side of the curve they are in the same plane, therefore by naming δr, $\delta r'$ the elements of the directions of the forces R, R', which directions are respectively perpendicular to each surface; we must add to the equation (b) the two terms $R\delta r$, $R'\delta r'$, which will change it into the following:

$$0=\Sigma.S\delta s+R.\delta r+R'.\delta r'. \quad (d)$$

These are the equations of equilibrium of a material point solicited by any number of forces S, S', S'', and constrained to move on a curved surface: if the position of M on the surface is not given, then the two equations, resulting from the elimination of λ, combined with the equation of the surface, $u=0$, are sufficient to determine the three coordinates of the point. When the forces and position of the point are given we obtain λ by means of one of the three preceding equations, from which we can collect immediately the value of R, and consequently the pressure; the investigation of R would be considerably abridged by making the axis of x to coincide with the normal, for then $\lambda.\left\{\frac{\delta u}{\delta y}\right\}$, $\lambda.\left\{\frac{\delta u}{\delta z}\right\}$, are equal respectively to nothing, and $\lambda\left\{\frac{\delta u}{\delta x}\right\} = RN.\left\{\frac{\delta u}{\delta x}\right\}=R$, for in this case $N.\left\{\frac{\delta u}{\delta x}\right\} = \left\{\frac{\delta r}{\delta x}\right\} = 1$; since $\lambda\left\{\frac{\delta u}{\delta y}\right\}$, $\lambda\left\{\frac{\delta u}{\delta z}\right\}$, are $=$ to nothing, we shall have $Y=0$, $Z=0$, which indicate that the forces resolved respectively parallel to two lines in the plane which touches the surface in the given point, are equal to nothing; this also follows from considering that the resultant of the forces is necessarily perpendicular to the surface. If the variations δx, δy, δz, are supposed to belong to the surface then we shall have $X\delta x+Y\delta y+Z\delta z=0$, and substituting for δz its value in terms of δx and δy, which we get by means of the equation $\left\{\frac{\delta u}{\delta x}\right\}.\delta x+\left\{\frac{\delta u}{\delta y}\right\}.\delta y+\left\{\frac{\delta u}{\delta z}\right\}.\delta z=0$, we can obtain *immediately* the equations of condition

$$Y.\frac{\delta u}{\delta x}-X.\frac{\delta u}{\delta y}=0. \quad Z.\frac{\delta u}{\delta x}-X.\frac{\delta u}{\delta z}=0.$$

If we determine the variations δx, δy, δz, so that they may appertain at the same time to the two surfaces, and consequently to the curve formed by the canal; δr and $\delta r'$ will vanish, and the preceding equation will be reduced to the equation (b) which therefore obtains in the case where the point is constrained to move in a canal; provided that we make two of the variations δx, δy, δz, to disappear by means of the two equations which express the nature of this canal.

Let us suppose that $u=0$, $u'=0$ are the equations of the two surfaces whose intersection forms the canal. If we make

$$\lambda = \frac{R}{\sqrt{\left(\frac{\delta u}{\delta z}\right)^2 + \left(\frac{\delta u}{\delta y}\right)^2 + \left(\frac{\delta u}{\delta z}\right)^2}}$$

$$\lambda' = \frac{R'}{\sqrt{\left(\frac{\delta u'}{\delta x}\right)^2 + \left(\frac{\delta u'}{\delta y}\right)^2 + \left(\frac{\delta u'}{\delta z}\right)^2}}$$

the equation (d) will become

$$0 = \Sigma.\, S.\, \delta s. + \lambda.\, \delta u + \lambda'.\delta u',$$

in which the coefficients of each of the variations δx, δy, δz, will be separately equal to nothing; thus three equations will be obtained, by means of which the values of λ and λ' may be determined, which will give R and R' the reaction of the two surfaces, and by composing them we shall have the reaction k of the canal on the point M, and consequently the pressure of this point against the canal. The reaction resolved parallel to the axis of x is equal to

$$R.\left(\frac{\delta r}{\delta x}\right) + R'.\left(\frac{\delta r'}{\delta x}\right), \text{ or to } \lambda.\left(\frac{\delta u}{\delta x}\right) + \lambda'.\left(\frac{\delta u'}{\delta x}\right):*$$

* When the point is forced to be on a canal of simple or double curvature there is only one equation of condition, which is obtained by eliminating λ and λ'; this equation combined with the equations $u = 0$, $u' = 0$ are sufficient to detrmine the coordinates of the

therefore the equation of condition $u=0$, $u'=0$, to which the motion of the point M is subjected, express by means of the partial differentials of functions, which are equal to nothing in consequence of these equations, the resistances with which the point is affected in consequence of the conditions of its motion.

It appears from what precedes that the equation (b) of equilibrium obtains universally, provided, that the variations δx, δy, δz, are subjected to the conditions of equilibrium. This equation may be made the foundation of the following principle.

" If an indefinitely small variation be made in the position of the " point M, so that it still remains on the curve or surface along which " it ought to move, if it is not entirely free; the sum of the forces " which solicit it, each multiplied by the space through which the " point moves in its direction, is equal to nothing, in the case of an " equilibrium."*

The variations δx, δy, δz, being supposed arbitrary and independant, it is possible to substitute for the coordinates x, y, z, in the equation (a), three other quantities which are functions of them, and to equal the coefficients of the variations of these quantities to nothing. Thus naming ρ the radius drawn from the origin of the coordinates, to the

point of the canal where the given forces constitute an equilibrium, in this case it is only required for the equilibrium of the point that the resultant of the forces should exist in a plane perpendicular to the element of the curve on which the point is situated, from whence it appears that the position of the resultant is more undetermined than when the point exists on a curved surface. See Notes to No. 9.

We might simplify the investigation of the pressures and obtain immediately the equation of equilibrium between the forces by taking two of the axes in the plane of the normals of the surfaces whose intersection constitutes the curve, for then we shall have at once $Z=0$, the third axis is in the direction of the tangent to the curve formed by the intersection of the two given surfaces.

* The equation (b) obtains universally, but under different circumstances, according as the point is free, or constrained to move on a surface; in the former case V the resultant of all the forces vanishes, and $\therefore \Sigma . S.\delta s. = V.\delta u$ must vanish; in the latter case V has a

projection of the point M, on the plane of x and y, and π the angle formed by ρ and the axis of x, we shall have

$$x = \rho.\cos.\pi;\ y = \rho.\sin.\pi.$$

If, therefore in the equation (a), we consider u, s, s' as functions of ρ, π, and z; and then compare the coefficients of $\delta\pi$, we shall have

$$V.\left\{\frac{\delta u}{\delta \varpi}\right\} = \Sigma.S.\left\{\frac{\delta s}{\delta \varpi}\right\};\quad (e)$$

$\frac{V.}{\rho}\left\{\frac{\delta u}{\delta \varpi}\right\}$ is the expression for the force V resolved in the direction of the element $\rho.\ \delta\varpi$. Let V' be this force resolved parallel to the plane of x and y, and P the perpendicular demitted from the axis of z on direction of V', parallel to the same plane; $\frac{PV'}{\rho}$ will be a second expression for the force V' resolved in the direction of the element $\rho\delta\varpi$; therefore we shall have

$$PV'. = V.\left\{\frac{\delta u}{\delta \varpi}\right\}.$$

If we conceive the force V' to be applied to the extremity of the perpendicular P, it will tend to make it turn about the axis of Z; the product of this force, by the perpendicular, is denominated the moment of the force V with respect to the axis of z; therefore this moment is equal to $V.\left\{\frac{\delta u}{\delta \varpi}\right\}$; and it appears from the equation (e), that the moment of the resultant of any number of forces is equal to the sum of the moments of these forces.*

finite value, but its direction being perpendicular to the surface δr the variation of this perpendicular must be equal to nothing, and consequently in this case also $\Sigma.S\delta s = V\delta u$ must vanish.

* The force V resolved parallel to the axis of $x = \frac{V.(x-a)}{u} =$, by substituting for

x its value $V.\frac{(\varrho.\ \cos.\ \pi-a)}{u}$, this last force resolved in the direction of the element $\varrho.\ \delta\pi$, i. e. perpendicular to $\varrho = V.\frac{(\varrho.\ \cos.\ \pi-a)}{u}.\frac{y}{\varrho}$ = (by substituting for y its value) $V.\frac{(\varrho.\ \cos.\ \pi-a)}{u}.$ sin. π in like manner if we resolve the force V parallel to the axis of y, and then this last force in the direction of $\varrho\ \delta\pi$, it will be equal to $V.\frac{(\varrho.\ \sin.\ \pi-b)}{u}.$ cos. π These forces in the direction of $\varrho.\ \delta\pi$ act in opposite directions, therefore their difference $=\frac{V.}{u}\Big((\varrho.\ \sin.\ \pi-b).\ \cos.\ \pi-(\varrho.\ \cos.\ \pi-a).\ \sin.\ \pi)\Big)$ is the expression for that part of the force V in the direction of the element $\varrho.\delta\pi$, which is really efficient, this expression $=\frac{V.}{\varrho}\left\{\frac{\delta u}{\delta\pi}\right\}$, for $u^2 = (\varrho.\ \cos.\ \pi-a)^2+(\varrho.\ \sin.\ \pi-b^2+(z-c)^2$ (by substituting for x and y their values); therefore taking the derivitive function, π being considered as the variable, we shall have, $u.\left\{\frac{\delta u}{\delta\pi}\right\} = -\varrho.\ \sin.\ \pi.\ (\varrho.\ \cos.\ \pi-a)+\varrho.\ \cos.\ \pi.\ (\varrho.\ \sin.\ \pi-b)$. $\because \frac{V.}{\varrho}\left\{\frac{\delta u}{\delta\pi}\right\} = \frac{V.}{u}\Big((\cos.\ \pi.\ (\varrho.\ \sin.\ \pi-b)-\sin.\ \pi.\ (\varrho.\ \cos.\ \pi-a)\Big), = \frac{PV'}{\varrho}$, for conceiving the force V' to be resolved into two, of which one is perpendicular to ϱ, the other being in the direction of ϱ, the triangle constituted by these forces will be similar to a triangle, two of whose sides are ϱ and P, and the third side $= V'$ produced to meet P, $\because$ that part of the force V' which is perpendicular to ϱ is to V' as P to ϱ $\because$ it is equal to $\frac{PV'}{\varrho}$.

From the definition that has been given in this No. of the moment of a force with respect to an axis, it appears that it can be geometrically exhibited by means of a triangle, whose vertex is in this axis, and whose base represents the intensity of the force, it vanishes when the resultant V vanishes, and also when P vanishes, i. e. when the resultant passes through the origin of the coordinates. See Notes to No. 6.

Let X and Y indicate, as in the preceding notes, the force V, resolved respectively parallel to the axes of x and y, $X = V.\frac{(x-a.)}{u}$, $Y = V.\frac{(y-b)}{u}$, the expression for these forces resolved perpendicular to $\varrho = V.\frac{(x-a)}{u}.\ \frac{y}{\varrho}$, $V.\frac{(y-b)}{u}.\ \frac{x}{\varrho}$, their difference $=\frac{PV'}{\varrho} = \frac{Yx-Xy}{\varrho}$; we are enabled by means of this expression to deduce the equation of the right line, along which the resultant is directed, for the equations of its projection V' on the plane of $x\,y$ is $y-b=\frac{Y'}{X}\cdot(x-a)$. $Xy-Xb=Yx-Y.a$. Let L be equal to $Yx-Xy$, and the preceding equation will become $b=\frac{Y}{X}.a-\frac{L}{X}$. we might derive similar expressions for the projection of V on the planes of x and z, and y and z, from whence it is easy to collect the equation of the right line along which V is directed, $-\frac{L}{X}$ indicates the distance of the origin of the coordinates from the intersection of V' with the axis of y, and $\frac{L}{Y}$ indicates the distance of the origin of the coordinates from the intersection of the resultant V' with the axis of x. $Yx-Xy=Ya-Xb$ shews that it is indifferent what point of the direction of V' is considered. $Yx-Xy=0$ when $V'=0$, and also when its direction passes through the axis o z.

CHAPTER II.

Of the motion of a material point.

4. A point in repose cannot excite any motion in itself, because there is nothing in its nature to determine it to move in one direction in preference to another. When solicited by any force, and then left to itself, it will move constantly, and uniformly in the direction of that force, if it meets with no resistance. This tendency of matter to persevere in its state of motion or rest, is what is termed its *inertia ;* it is the first law of the motion of bodies.

The direction of the motion in a right line follows necessarily from this, that there is no reason why the point should deviate to the right, rather than to the left of its primitive direction; but the uniformity of its motion is not equally evident. The nature of the moving force being unknown, it is impossible to know *a priori*, whether this force should continue without intermission or not. Indeed, as a body is incapable of exciting any motion in itself, it seems equally incapable of producing any change in that which it has received, so that the law of inertia is at least the most natural and the most simple which can be imagined; it is also confirmed by experience. In fact, we observe on the earth that the motions are perpetuated for a longer time, in proportion as the obstacles which oppose them are diminished; which induces us to think that if these obstacles were entirely removed, the motions would never cease. But the inertia of matter is most remarkable in the motions of the heavenly bodies, which for a great number of ages have not experienced any perceptible alteration. For these reasons we shall consider the inertia of bodies as a law of nature; and when we observe any change in the motion of a body we shall conclude that it arises from the action of some foreign cause.

In uniform motions the spaces described are proportional to the times. But the times employed in describing a given space are longer or shorter according to the magnitude of the moving force. From these differences has arisen the idea of *velocity*, which, in uniform motions is the ratio of the space to the time employed in describing it. Thus s representing the space, t the time, and v the velocity, we have $v = \frac{s}{t}$. Time and space being heterogeneal and consequently not comparable quantities, a determinate interval of time, such as a second, is taken for a unit of time, and in like manner a portion of space, such as a metre for an unit of space, and then time and space become abstract numbers, which express how often they contain units of their species, and thus they may be compared one with another. By this means the velocity becomes the ratio of two abstract numbers, and its unity is the velocity of a body which describes a metre in one second.

5. Force being only known to us by the space which it causes to be described in a given time, it is natural to take this space for its measure, but this supposes, that several forces acting in the same direction, would cause to be described in a second of time, a space equal to the sum of the spaces which each would have caused to be described separately in the same time, or in other words, that the force is proportional to the velocity; but of this we cannot be assured *a priori*, in consequence of our ignorance of the nature of the moving force. Therefore it is necessary on this subject also to have recourse to experience, for whatever is not a necessary consequence of the few data which we have on the nature of things, must be to us the result of observation.

Let us name v the velocity of the earth, which is common to all the bodies on its surface, let f be the force with which one of these bodies M is actuated in consequence of this velocity, and let us suppose that $v = f.\varphi(f)$ is the relation which exists between the velocity and the force, $\varphi(f)$ being a function of f which must be determined by experience. Let a, b, c. be the three partial forces into which the force f may be resolved parallel to three axes which are perpendicular to each other. Let us then suppose the moving body M to be solicited by a new force, f', which may be resolved into three others a', b', c', parallel to the same axis. The forces by which this body will be solicited parallel to these axis will be $a+a'$, $b+b'$, $c+c'$, naming F the sole resulting force, by what precedes we shall have

$$F = \sqrt{\overline{a+a'}|^2 + (b+b')^2 + (c+c')^2}$$

If the velocity corresponding to F be named U; * $\frac{(a+a').\ U}{F}$ will be this velocity resolved parallel to the axes of a, thus the relative velocity of the body on the earth parallel to this axis will be $\frac{(a+a')U}{F} - \frac{a}{f}$ or $(a+a').\ \varphi.(F) - a.\ \varphi.f.$ The most considerable forces which can be impressed on bodies at the surface of the earth being much smaller than those by which they are actuated in consequence of the motion of the earth, we may consider a', b', c', as indefinitely small quantities relative to f; therefore we shall have $F = f + \frac{aa' + bb' + cc'}{f}$ † and φ $(F) = \varphi.\ (f) + \frac{(aa' + bb' + cc').}{f}\ \varphi'(f)$; $\varphi.(f')$; being the differential

* The velocity of a body moving in a given direction is to *its* velocity, *estimated* in any other direction, as radius to the cosine of the angle which the two directions make with one another, that is, in this case as F to $a+a'$, therefore the velocity U resolved parallel to the axis of a will be equal to $U\ \frac{(a+a').}{F}$

† $F. = \sqrt{(a+a')^2+(b+b')^2+(c+c')^2} = \sqrt{a^2+b^2+c^2+2aa'+2bb'+2cc'}$, the squares of a', b', and c' being rejected as indefinitely small, if this radical is expanded by the binomial theorem (all the terms except the two first being neglected as involving the squares, products, and higher powers of a', b', c',) it will become

$$\sqrt{a^2+b^2+c^2} + \frac{2\,(aa'+bb'+cc')}{2\sqrt{a^2+b^2+c^2}}, = f + \frac{aa'bb'+cc'}{f},$$

and $\varphi\ (F) = \varphi.\left(f + \frac{aa'+bb'+cc'}{f}\right)$ equal by Taylor's theorem to

$\varphi(f) + \frac{aa'+bb'+cc'}{f}.\ \varphi'(f).$

of $\varphi.(f)$ divided by $d.f.$ The relative velocity of M in the direction of the axis of a will thus become

$$a'.\varphi.(f) + \frac{a}{f}\left\{ aa' + bb' + cc' \right\}.\varphi'.(f)$$

its relative velocities in the directions of b and c will be

$$b'.\varphi.(f) + \frac{b}{f}\left\{aa' + bb' + cc'\right\}.\varphi'.(f);$$

$$c'\varphi.(f) + \frac{c}{f}\left\{aa' + bb' + cc'\right\} \varphi'(f);$$

The position of the axes of a of b and of c being arbitrary, we may take the direction of the impressed force for the axis of a, and then b and c will vanish; the preceding relative velocities will be changed into the following

$$a\left\{ \varphi.(f) + \frac{a^2}{f}.\varphi'(f)\right\}; \frac{ab}{f}. a'.\varphi'(f); \frac{ac}{f}. a'\varphi'(f).$$

If $\varphi'(f)$ does not vanish, the moving body in consequence of the impressed force a' will have a relative velocity perpendicular to the direction of this force, provided that a and b do not vanish,—that is to say, provided that the direction of this force does not coincide with that of the motion of the earth. Thus, conceiving that a globe at rest upon a very smooth horizontal plane is struck by the base of a right angle cylinder, moving in the direction of its axis, which is supposed to be horizontal, the apparent relative motion of the globe will not be parallel to this axis in all positions of this axis relative to the horizon. We have thus an easy means of determining by experiment whether $\varphi'(f)$ has a perceptible value on the earth; but the most accurate experiments have not indicated in the apparent motion of the globe any deviation from the direction of the force impressed; from which it follows that on the earth $\varphi'(f)$ is very nearly nothing. If its value was at all perceptible, it would particularly be shewn in the duration of the oscillations of a pendulum, which duration would alter according as the position of the plane of its motion differed from the direction of the motion of the earth. As the most exact observations have not evinced any such difference, we ought to conclude that $\varphi'(f)$ is insensible, and at the surface of the earth ought to be supposed equal to nothing.*

If the equation $\varphi'(f) = 0$ has place whatever be the magnitude of the force f, $\varphi.(f)$ will be constant, and the velocity will be proportional to the force; it will be also proportional to it if the function $\varphi.(f)$ is composed of only one term, as otherwise $\varphi'.(f)$ would not vanish unless f did; therefore if the velocity is not proportional to the force, it is necessary to suppose that, in nature, the function of the velocity which expresses the force consists of several terms, which is very improbable; we must moreover suppose that the velocity of the earth is exactly such as corresponds to the equation $\varphi'(f) = 0$,† which is contrary to all probability. Besides, the velocity of the earth varies during the different seasons of the year; it is a thirtieth part greater in winter than in summer. This variation is even more considerable if, as every thing appears to indicate, the solar system be in motion in space; for according as this progressive motion conspires with that of the earth, or is contrary to it, there must result in the course of the year, very sensible variations in the absolute motion of the earth, which would alter the equation which we are considering, and the ratio of the force impressed to the absolute velocity which results from it, if this equation and this ratio were not independant of the motion of the earth. Nevertheless, the smallest difference has not been discovered by observation.

* These experiments evince that the appearances of bodies in motion are independant of the direction of the motion of the earth; and from the preceding investigation it follows, that in order this should be the case, the small increase of the force by which the earth is actuated should be to the corresponding increase of the velocity, in the ratio of the quantities themselves; thus our experiments only prove the reality of this proportion, which if it had place, whatever the velocity of the earth might be, would give the law of the velocity proportional to the force.

† $\varphi'(f) = 0$, not only when $\varphi(f)$ is constant, but also in other cases, such as when $\varphi(f)$ is a maximum or minimum, in the former case the force f may be of any magnitude whatever; in the latter case the value of f is *unique*; but since the velocity

Thus we have two laws of motion; the law of inertia, and that of the force proportional to the velocity, which are both given by observation. They are the most natural and the most simple which can be imagined, and are, without doubt, derived from the nature itself of matter, but this nature being unknown, they are, with respect to us, solely the result of observation, and the only observed facts which the science of Mechanics borrows from experience.*

6. The velocity being proportional to the force, those two quantities may be represented one by the other, and we may apply to the composition of velocities all that has been previously established respecting the composition of forces.† Thus it follows, that the relative motions of a system of bodies actuated by any force whatever, are the same whatever be their common motion, for this last motion decomposed into three others, parallel to three fixed axes, only increases by the same quantity the partial velocities of each body parallel to these axes, and as their relative velocities only depend on the difference of these partial velocities, it will be the same whatever be the motion common to all bodies; it is therefore impossible to judge of the absolute motion of the system, of which we make a part by the appearances which can be observed, which circumstance characterises the law of the force proportional to the velocity.

of the earth is different in different points of its orbit, the value of f corresponding to this velocity must also vary.

If $\varphi\,(f)$ is an algebraic function of f, and consists of only one term, then $\varphi'\,(f)$ will not vanish unless f vanishes; but if φ was a transcendental function, then f might have a finite value, $\varphi'(f)$ vanisihing, or *vice versa.*

* In this respect, therefore, the theory of motion is less extensive than the theory of equilibrium, which does not involve any hypothesis whatever.

† Let v, v', v'', represent the uniform velocities parallel to the coordinates x, y, z, after any time t, $x = v\,t$, $y = v'.t$, $z = v''.t$, the resulting motion will be uniform, and its direction rectilinear, the equation of s, the line described, will be $s = t\sqrt{v^2+v'^2+v''^2}$, the velocity in the direction of $s = \sqrt{v^2+v'^2+v''^2}$, the cosines of the angles which this direction makes with x, y, and z, are equal respectively to

$$\frac{v}{\sqrt{v^2+v'^2+v'^2}}, \qquad \frac{v'}{\sqrt{v^2+v^2+v'^2}}, \qquad \frac{v''}{\sqrt{v^2+v'^2+v''^2}};$$

thus the composition and resolution of velocities are effected in the same manner as the composition and resolution of forces.

It follows also from No. 3, that, if we project each force and their resultant on a fixed plane, the sum of the moments of the composing forces thus projected with respect to a fixed point taken on the plane, is equal to the moment of the projection of the resultant; but if we draw from this point to the moving body a radius, which we shall call the *radius vector*, this radius projected on a fixed plane will trace, in consequence of each force acting separately, an area equal to the product of the projection of the line which the moving body is made to describe, into half the perpendicular let fall from the fixed point on this projection; therefore this area is proportional to the time; it is also in a given time* proportional to the moment of the projection of the force; thus the sum of the areas which the projection of the radius vector would describe, if each composing force acted separately, is equal to the area which the resultant makes this radius to describe. It follows from this, that if a body is first projected in a right line, and then solicited by any forces whatever, directed towards a fixed point, its radius vector will always describe about this point areas proportional to the times, because the areas which the new composing forces† make this radius to describe will vanish. It appears conversely, that if the moving body describes areas proportional to the times about the fixed point, the resultant of the new forces which solicit it is constantly directed towards this point.‡

E 2

* The area varies as the base multiplied into the altitude; the base varies as the time multiplied into the projection of the force; therefore the area varies as the continued product of the altitude, projection of force, and time, or (by substituting the moment for the altitude multiplied into the projection of the force.) as the moment multiplied into the time.

† If the forces directed to the fixed point did not act, the moving point would evidently describe areas proportional to the times; but these forces being supposed to act, the areas which are described about the fixed point, in consequence of the action of these forces, are nothing; for the perpendicular from the fixed point on the direction of the force in this case vanishes, consequently the proportionality of the areas to the times is not disturbed by the action of those forces.

‡ By means of the equations, $\frac{d^2x}{dt^2}=P$; $\frac{d^2y}{dt^2}=Q$. which are established in the subsequent number, we can exhibit immediately the relation which exists between the areas

7. Let us now consider the motion of a material point solicited by forces which seem to act continually, such as gravity. The causes of this and similar forces which have place in nature being unknown, it is impossible to know whether they act without interruption, or whether their successive actions are separated by imperceptible intervals of time ;

and moments; for if we multiply the first of these equations by y, and the second by x, and then subtract, we shall have, by concinnating $\frac{d^2y.x-d^2x.y}{dt^2} + yP-xQ) = 0$; if this equation be integrated, we shall obtain $\frac{xdy-ydx}{dt} + \int . dt\,(yP-xQ)=c$; $yP-xQ$. is the moment of the projection of the force on the plane x and y (see last note to No. 3); it vanishes when the force is directed to the origin of the coordinates, and also when P and Q vanish, that is when the point is not solicited by any accelerating force, consequently in both these cases, $xdy-ydx = cdt$ and is ∵ proportional to the time ; in the second case the origin of the coordinates may be any point whatever ; but in the first case, the origin must be in the *fixed* point, to which the forces soliciting the point are directed ; ($xdy-ydx$ = the elementary area which the projection of the radius vector on the plane $x\,y$ describes in dt; for $x = \varrho.\ \cos.\ \pi$, $y = \varrho.\ \sin.\ \pi$; therefore $dx = d\varrho.\ \cos.\ \pi - d\pi.\ \sin.\ \pi.\varrho.$ $dy = d\varrho.\ \sin.\ \pi + d\pi.\ \cos.\ \pi.\varrho.$ consequently $xdy-y.dx = d\varrho.\ \sin.\pi.\ \cos.\ \pi.\varrho + d\pi.\ \cos.^2\pi.\varrho^2 - d\varrho.\ \sin.\ \pi.\ \cos.\ \pi.\varrho + d\pi.\ \sin.\ ^2\pi.\varrho^2 = d\pi.\varrho^2$; but since $\varrho d\pi$ is the elementary arc described by the projection of the radius vector on the plane x, y, $\varrho.^2 d\pi$ will be the expression for the elementary area.) Since, when the areas are proportional to the times $yP-xQ=0$, it follows that the magnitude of the area described in a given time is not affected by the intensity of the accelerating force.

By a similar process of reasoning it may be shewn, that the projections of the elementary area on the plane x, z, y, z, which are equal to $xdz-zdx$, $ydz-zdy$ generally, are equal respectively to $c'.dt$, $c''.dt$. when the forces soliciting the point are directed towards the origin of the coordinates. When the areas are proportional to the times, the curve described is of single curvature ; for then we have $xdy-ydx=cdt$, $xdz-zdx=c'dt$, $ydz-zdy=c''dt$; if the first of these equations be multiplied by z, the second by y, and the third by x, we shall obtain, by adding them together, the equation $cz+c'y+c''x = 0$, which belongs to a plane.

The velocities are inversly as the perpendiculars when the areas are proportional to the times; for if we call the perpendicular p, and the elementary arc of the curve described ds, we will have $p.ds = x.dy-y.dx = c.dt$ ∵ $p = \frac{cdt}{ds} = \frac{c}{v}$.

The constant quantities c, c', c'', depend on the species of the curve described ; in conic sections when the force is directed to the focus, they are to the square roots of the parameters as the cosines of inclinations of the planes x,y, x,z, yz, to the plane of the section to radius. See No. 3, book 2.

but it is easy to be assured that the phenomena ought to be very nearly the same on the two hypotheses; for if we represent the velocity of a body solicited by a force whose action is continued by the ordinate of a curve of which the abscissa represents the time, this curve, on the second hypothesis will be changed into a polygon, having a great number of sides, which for this reason may be confounded with the curve. We shall, with geometers, adopt the first hypothesis, and suppose that the interval between two consecutive actions is equal to the element dt of the time, which we will denote by t. It is evident that the action of a force ought to be more considerable according as the interval is greater which separates its successive actions, in order that after the same time t the velocity may be always the same. Therefore the instantaneous action of a force ought to be supposed to be in the ratio of its intensity, and of the element of time during which it is supposed to act. Thus P, representing this intensity at the commencement of each instant, dt, the point, will be solicited by the force Pdt, and its motion will be uniform during this instant. This being agreed upon,

All the forces which solicit a point M may be reduced to three, P, Q, R, acting parallel to three rectangular coordinates x, y, z, which determine the position of this point;* we shall suppose these forces to act in a contrary direction from the origin of the coordinates, or to tend to increase them. At the commencement of a new instant dt, the moving point receives in the direction of each of its coordinates increments of force or velocity, Pdt, $Q.dt$, Rdt. The velocities of the point M, parallel to these coordinates, are $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$,† for during an inde-

* By thus referring the position of a point in space to rectangular coordinates, all curvilinear motion may be reduced to two or three rectilinear motions, according as the curve described is of simple or double curvature. For the position of the moving point is completely determined when we are able to assign the position of its projections on three rectangular axes, each coordinate represents the rectilinear space described by the point parallel to the axes to which it is referred, it will consequently be a given function of the time; and if we could determine these functions with respect to the three coordinates, the species of the curve described might be assigned by eliminating the time by means of the three equations between the coordinates and the time.

† The space being a function of the time, $dx = v.dt$ is the limit of the value of the incre-

finitely small portion of time, they may be considered as uniform, and therefore equal to the elementary spaces divided by the element of the time. Consequently the velocity with which the moving body is solicited at the commencement of a new instant, is

$$\frac{dx}{dt}+P.dt\,;\ \frac{dy}{dt}+Q.dt\,;\frac{dz}{dt}+Rdt\,;$$

or

$$\frac{dx}{dt}+d.\frac{dx}{dt}-d.\frac{dx}{dt}+P.dt\,;$$

$$\frac{dy}{dt}+d.\frac{dy}{dt}-d.\frac{dy}{dt}+Q.dt;$$

$$\frac{dz}{dt}+d.\frac{dz}{dt}-d.\frac{dz}{dt}+R.dt\,;$$

but in this new instant, the velocities with which the moving body is actuated parallel to the coordinates x, y, z, are evidently

$$\frac{dx}{dt}+d.\frac{dx}{dt}\,;\ \frac{dy}{dt}+d.\frac{dy}{dt};\ \frac{dz}{dt}+d.\frac{dz}{dt};$$

ment of the space, when dt becomes indefinitely small; we can assign the actual value by means of Taylor's theorem; for if t receive the increment dt, then ($x=f(t)$ becomes $x'=f(t+dt)$ $\because\ x'-x=f\,(t+dt)-f\,(t)=\frac{dx}{dt}.\,dt+\frac{d^2x}{dt^2}\cdot\frac{dt^2}{1.2}+\frac{d^3x}{dt^3}\cdot\frac{dt^3}{1.2.3}+$ &c. by making dt indefinitely small all the terms but the two first may be rejected; and since $\frac{dx}{dt}$ is the coefficient of dt it represents the velocity, and since $\frac{d^2x}{dt^2}$ is the coefficient of dt^2, it is proportional to the force; consequently if the action of the forces soliciting the point should cease suddenly $\frac{d^2x}{dt^2}$ would vanish, and the point would move with an uniform velocity, if instead of vanishing $\frac{d^2x}{dt^2}$ became constant, then $\frac{d^3x}{dt^3}$, and all subsequent coefficients would vanish, and the motion of the point would be composed of an uniform motion, and of one uniformly accelerated, both commencing at the same instant.

therefore the forces

$$-d.\frac{dx}{dt}+P.dt,\ -d.\frac{dy}{dt}+Q.dt,\ -d.\frac{dz}{dt}+R.dt,$$

must be destroyed, so that, if the point was actuated by these sole forces it would be in equilibrium. Thus if we denote by δx, δy, δz, any variations whatever of the three coordinates x, y, z, which variations are not necessarily the same with the differentials dx, dy, dz, that express the spaces described by the moving body parallel to the three coordinates during the instant dt, the equation (b) of No. 3, will become

$$0=\delta x.\left\{d.\frac{dx}{dt}-P.dt.\right\}+\delta y.\left\{d.\frac{dy}{dt}-Q.dt.\right\}+\delta z.\left\{d.\frac{dz}{dt}-R.dt\right\}.\ (f)^{*}$$

We may put the coefficients of δx, δy, δz, separately equal to nothing; if the point M be free, and the element dt of the time being supposed constant, the differential equations will become

$$\frac{d^2x}{dt^2}=P;\quad \frac{d^2y}{dt^2}=Q;\quad \frac{d^2z}{dt^2}=R.\dagger$$

* From the equation (f) it appears that the laws of the motion of a material point may be reduced to those of their equilibrium, we shall see in No. 18, that the laws of the motion of any system of bodies are reducible to the laws of their equilibrium.

† If P, Q, R, are given in functions of the coordinates, then by integrating twice we shall obtain the values of x, y, and z, in functions of the time; two constant quantities are introduced by these integrations, the first depends on the velocity of the point at a given instant, the second depends on the position of the point at the same instant.

If the values of the coordinates x, y, z, which are determined by these integrations, give equations of this form, $x=a.f(t)$, $y=b.f(t)$, $z=c.f(t)$, the point will move in a right line, the cosines of the angles which the direction of this line makes with x, y, and z, are respectively equal to $\frac{a}{\sqrt{a^2+b^2+c^2}}$, $\frac{b}{\sqrt{a^2+b^2+c^2}}$, $\frac{c}{\sqrt{a^2+b^2+c^2}}$, the constant quantities a, b, c, depend on the nature of the function $f(t)$, if $f(t)=t$; a, b, c, represent the uniform velocities parallel to x, y, and z, the uniform velocity of the point $=\sqrt{a^2+b^2+c^2}$, if $f(t)=t^2$, then a, b, c, are proportional to the accelerating forces parallel to x, y, z, and the point will be moved with a motion uniformly accelerated, repre-

If the point M be not free, but subjected to move on a curve or on a surface, then by means of the equations to the curve or surface, there must be eliminated from the equation (f) as many of the variations δx, δy, δz, as there are equations, and the coefficients of the remaining variations must be put separately equal to nothing.*

8. We may suppose the variations δx, δy, δz, in the equation (f) equal to the differentials dz, dy, dz, since these differentials are necessarily subjected to the conditions of the motion of the point M. By making this supposition, and then integrating the equation (f), we shall have†

$$\frac{dx^2 + dy^2 dz^2}{dt^2} = c + 2. \int (P.dx + Q.dy + Rdz,$$

sented by $\sqrt{a^2+b^2+c^2}$. If $x=a.f(t)+b\,f'(t)$, $y=c.f(t)+d.(f't)$, $z=\varepsilon.f(t)+g.f'(t)$, the point will move in a curved line; however, this curve is of single curvature; for by eliminating t we obtain an equation of the form $a'x+b'y+c'z=0$, which is the equation of a plane. The simplest case of this form is $x=a\,(t)+b\,(t^2)$, $y=b\,(t)+d\,(t^2)$, $z=\varepsilon.\,(t)+g\,(t^2)$, eliminating t between the two first equations we shall obtain an equation of the second order between x and y, and from the relation which exists between the coefficients of the three first terms of this equation, it is evident that the curve is a parabola. If $x=f(t)$, $y=F(t)$, $z=ff(t)$, all the points of the curve will not exist in the same plane.

* The law of the force being given, the investigation of the curve which this force makes the body describe, is much more difficult than the reverse problem of determining the velocity, and force the nature of the curve described being given; as the integrations which are required in the first case, are much more difficult than the differentials which determine the velocity and force in the second case.

† We have seen in No. 7, that when a point moves in a right line, its velocity is equa to the element of the space divided by the element of the time; this is also true when the motion is curvilinear; for if $P.Q.R$, the forces soliciting the point parallel to the three coordinates, should suddenly cease, then the velocity in the direction of each of the coordinates will be uniform, and equal to $\frac{dx}{dt'}$, $\frac{dy}{dt'}$, $\frac{dz}{dt'}$, respectively, (see second note to the preceding number) consequently the motion of the point will become uniform, and its direction rectilinear, ∵ if v express this velocity we will have, by first note to No. 6.

$$v. = \frac{\sqrt{dx^2+dy^2+dz^2}}{dt} = \frac{ds}{dt} \text{ for } ds = \sqrt{dx^2+dy^2+dz^2}.$$

c being a constant quantity. $\frac{dx^2+dy^2+dz^2}{dt^2}$ is the square of the velocity of M, which velocity we will denote by v; therefore if $Pdx, + Q.dy, + Rdz,$ is an exact differential of a function φ, we shall have

$$v^2 = c + 2\varphi. \quad (g)$$

This case obtains when the forces which solicit the point M are functions of the distances of their origins from this point. In fact, if S, S', &c.* represent these forces, s, s', being the distances of the point M

F

(See Lacroix Traite Elementaire, No 139.) The rectilinear direction is that of the tangent, for if A, B, C, denote the angles which this direction makes respectively with x, y, z, we shall have $v. \cos. A = \frac{dx}{dt}$, $v. \cos. B = \frac{dy}{dt}$, $v. \cos. C = \frac{dz}{dt}$, by substituting for v. its value, which has been given above, and then dividing we obtain $\cos. A = \frac{dx}{ds}$, $\cos. B = \frac{dy}{ds}$, $\cos. C = \frac{dz}{ds}$; but these are the cosines of the angles which the tangent makes with the coordinates $\therefore$ the tangent coincides with the line along which the point moves when the forces cease.

* If $P.dx + Q.dy + Rdz = f(x, y, z,)$ then $v^2 = c + 2.f(x, y, z,)$ let A be the velocity corresponding to the coordinates a, b, c; then $A^2 = c + 2.f(a, b, c,) \therefore v^2 - A^2 = 2.f(x, y, z) - 2.f(a, b, c,) \therefore$ the difference of the squares of the velocities depends only on the coordinates of the extreme points of the line described; consequently when the point describes a curve, the pressure of the moving point on the curve does not affect the velocity.

The constant quantity c depends on the values of v, and of x, y, z, at any given instant.

When the moving point describes a curve returning into itself, the velocity is always the same at the same point.

If the velocities of two points, of which one describes a curve, while the other describes a right line, are equal at equal distances from the centre of force in any one case, they will be equal at all other equal distances.

If the force varies as the n^{th} power of the distance from the centre, then s and s' being any two distances, φ or $f(x, y, z,) = s^{n+1}, \therefore v^2 - A^2, \; s^{n+1} - s'^{n+1}$.

In this case also the differential of the velocity $= s.^n ds$, therefore by erecting ordinates from their origins; the resultant of all these forces multiplied by the variation of its direction will, by No. 2, be equal to $\Sigma.S.\delta s$; it is also equal to $P.\delta x+Q.\delta y+R.\delta z$; therefore we have

$$P.\delta x+Q.\delta y+R.\delta z=\Sigma.S.\delta s.$$

and as the second member of this equation is an exact variation, the first will be so likewise.

From the equation (g)* it follows, 1st, that if the point M is not

proportional to s^n, we can exhibit the figure which represents the square of the velocity, when n is positive the figure is of the parabolic species, when negative it is hyperbolic.

If the distances increase in arithmetical progression, while the force decreases in geometric progression, the figure representing the square of the velocity will be the logarithmic curve. See Principia Matthematica, lib. 1, prop. 40, 39.

If $Pdx+Q.dy+Rdz$ be an exact differential, then $\frac{dP}{dy}=\frac{dQ}{dx}$; $\frac{dP}{dz}=\frac{dR}{dx}$ + &c. P, Q, R, must be functions of x, y, and z, independant of the time $\because$ if the centres to which the forces were directed had a motion in space, the time would be involved, and consequently $P.dx+Q.dy+R.dz$, would *not* be an exact differential, for then the equations $\frac{dP}{dz}=\frac{dR}{dx}$ + &c. would not obtain.

When the forces P, Q, R, arise from friction or the resistance of a fluid, the equation $P.dx+Q.dy+R.dz$, does not satisfy the preceding conditions of integrability, for since $P. Q, R$, depend on the velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ in this case; it is evident that $P.dx+Q.dy+Rdz$ cannot be an exact differential of a function of x, y, and z, considered as independant variables $\because$ to integrate $P.dx+Q.dy+R.dz$, we should substitute the values of these variables and their differentials in a function of the time, which supposes that we have solved the problem; consequently when the centre to which the force is directed is in motion, and when the force arises from friction or resistance, the velocity is not independant of the curve described.

* The velocity is constant when $f(x,y,z)$ is constant; and also when $f(x,y,z,)$ vanishes; when the point is put in motion by an initial impulse, the motion is uniform, and its direction rectilinear, and $v^2=A^2$, $\frac{dx}{dt}=c$, $\frac{dy}{dt}=c'$. $\frac{dz}{dt}=c''$, for then $\left\{\frac{d^2x}{dt^2}\right\}=P$, $\left\{\frac{d^2y}{dt^2}\right\}=Q$, $\left\{\frac{d^2z}{dt^2}\right\}=R$, are equal respectively to nothing.

The velocity lost by a body, in its passage from one plane to another, is proportional to solicited by any forces, its velocity is constant, because then $\varphi=0$. It is easy to be assured of this otherwise, by observing, that a body moving on a surface or on a curved line, looses, at each rencounter with the indefinitely small plane of the surface, or indefinitely small side of the curve, but an indefinitely small part of its velocity of the second order. 2dly. That the point M, in passing from a given point with a given velocity, will have, when it attains another point, the same velocity, whatever may be the curve which it shall have described.

But if the point is not constrained to move on a determined curve, then the curve described possesses a singular property, to which we have been led by metaphysical considerations, and which is, in fact, but a remarkable consequence of the preceding differential equations. It consists in this, that the integral $\int v.ds$ comprised between the two extreme points of the curve described, is less than on any other curve if the point is free, or than on any other curve subjected to the same surface if the point is not entirely free.

To make this appear we shall observe, that $P.dx+Q.dy+Rdz$ being supposed an exact differential, the equation (g) gives

$$v.\delta v = P.\delta x+Q.\delta y+R\delta z.$$

in like manner the equation (f) of the preceding number becomes,

$$0=\delta x.\, d.\frac{dx}{dt}+\delta y.d.\frac{dy}{dt}+\delta z.d.\frac{dz}{dt}-v.dt.\,\delta v.$$

naming ds the element of the curve described by the moving point, we shall have

$$v.dt=ds;\ ds=\sqrt{dx^2+dy^2+dz^2},$$

the difference between radius and cosine of the inclination of the planes, i. e. to the versed sine, or to the square of the sine; and when the curvature is continuous the sine is an indefinitely small quantity of first the order, ∵ the velocity lost, is an indefinitely small quantity of the second order.

consequently

$$0 = \delta x.d.\frac{dx}{dt} + \delta y.d.\frac{dy}{dt} + \delta z.d.\frac{dz}{dt} - ds.\delta v, \quad (h)$$

by differentiating with respect to δ, the expression for ds, we have

$$\frac{ds}{dt}.\,\delta.ds. = \frac{dx}{dt}.\,\delta.dx + \frac{dy}{dt}.\,\delta.dy + \frac{dz}{dt}.\,\delta.dz.$$

The characteristics d and δ being independant, it is indifferent which precedes the other; therefore the preceding equation may be made to assume the following form:

$$v.\delta.ds = d.\frac{(dx\,\delta x + dy.\delta y + dz.\delta z)}{dt} - \delta x.d.\frac{dx}{dt} - \delta y.d.\frac{dy}{dt} - \delta z.d.\frac{dz}{dt},$$

by substracting from the first member of this equation the second member of the equation (h) we shall have

$$\delta\,(vds) = \frac{d.(dx.\delta x + dy.\delta y + dz.\delta z)^*}{dt}$$

This last equation integrated with respect to the characteristic d, gives

$$\delta.\int v.ds = \text{const.} + \frac{dx.\delta x + dy.\delta y + dz.\delta z}{dt},$$

* For $d.\frac{(dx.\delta x + dy.\delta y + dz.\delta z)}{dt} = d.\frac{dx}{dt}\delta x. + d.\frac{dy}{dt}\delta y + d.\frac{dz}{dt}\delta z; + \frac{dx}{dt}d.\delta x + \frac{dy}{dt}d.\delta y$ $+ \frac{dz}{dt}d.\delta x \left\{ = \frac{dx}{dt}\delta.dx + \frac{dy}{dt}\delta.dy + \frac{dz}{dt}\delta.dz. \right\}$, $\because$ by performing the operations prescribed in the text, we obtain $v.\delta.ds + ds.\delta v = \delta.(v.ds) = d.\left\{\frac{dx.\delta x + dy.\delta y + dz.\delta z}{dt}\right\}$. This equation being integrated with respect to the characteristic d gives $\int.\delta.(v.ds.)$ const. $+ \frac{dx.\delta x + dy.\delta y + dz.\delta z}{dt}$; when the two extreme points of the curve are fixed, the variations δx, δy, δz, of the coordinates must be equal to nothing at these points; consequently the variation of $\int.(v.ds)$ is equal to nothing, and $\because$ $\int(v.ds)$ is either a maximum or minimum; but it is evident from the nature of function $\int.(v.ds.)$ that it does not admit a maximum.

If we extend this integral to the entire curve described by the moving point, and if we suppose the extreme points of this curve invariable, we will have $\delta.\int v.ds=0$, that is to say, of all the curves, which a point solicited by the forces P, Q, R, can describe in its passage from one given point to another, it describes that in which the variation of the integral $\int v.ds$, is equal to nothing, and in which, consequently, this integral is a *minimum*.

If the point moves on a given surface without being solicited by any force, its velocity is constant, and the integral $\int v.ds$ becomes $v.\int ds$. Therefore in this case the curve described by the moving point is the shortest which it is possible to trace on the surface from the point of departure to that of arrival.*

9. Let us determine the pressure of a point moving on a curved surface. Instead of eliminating from the equation (f) of No. 7, one of the variations δx, δy, δz, by means of the equation to the surface, we can by No. 3 add to this equation, the differential equation of the sur-

* When the velocity is constant the integral $\int v.ds$, becomes $v.\int.ds=v.s$; and since s is a minimum, the time of describing s, which is proportional to s in consequence of the uniformity of the motion, will be a minimum in like manner. Since the equation $\delta.\int.(v.ds.)=0$, has place when $Pdx+Qdy+R.dz$ is an exact differential, it belongs to all curves that are described by the actions of forces directed to *fixed* centres, the forces being functions of the distance from those centres; and if the form of these functions was given we could determine the species of the curve described, by substituting for v its value in terms of the force, (which we have by a preceding note), and then investigating by the calculus of variations, the relation existing between the coordinates of the curve which answers to the minimum of the expression $\int(v.ds)$. If S the force varied as $\frac{1}{s^2}$ by making use of Polar coordinates we would arrive at the polar equation of a conic section, in which the origin of the coordinates would be at the focus of the section; if S was proportional to s the resulting equation would be also that of a conic section, the origin of the coordinates being at the centre of the section. From the preceding property the known laws of refraction and reflection have been deduced. Mr. Laplace has also successfully applied it to the investigation of the law of double refraction of Iceland chrystal, which was first announced by Huyghens, and afterwards confirmed by the celebrated experiments of Malus on the polarization of light. See a paper of Laplace's in the volume of the Institute for the year 1809.

face multiplied by the indeterminate—λdt, and then consider the three variations δx, δy, δz, as independant quantities. Therefore let $u=0$ be the equation of the surface, by adding to the equation (f) the term $-\lambda\delta u.\,dt$. the pressure will, by No. 3, be equal to

$$\lambda.\sqrt{\left\{\frac{du}{dx}\right\}^2+\left\{\frac{du}{dy}\right\}^2+\left\{\frac{du}{dz}\right\}^2}$$

At first let us suppose that the point is not solicited by any force; its velocity v will be constant, and since $v.dt=ds$; the element of the time being supposed constant, the element ds of the curve will be so likewise, and by adding to the equation (f) the term $-\lambda.\delta u.dt$, we will obtain the three following:

$$0=v.^2\frac{d^2x}{ds^2}-\lambda.\left\{\frac{du}{dx}\right\};\quad 0=v^2\frac{d^2y}{ds^2}-\lambda.\left\{\frac{du}{dy}\right\};$$

$$0=v.^2\frac{d^2z}{ds^2}-\lambda.\left\{\frac{du}{dz}\right\},\,*$$

from which we may collect

$$\lambda.\sqrt{\left\{\frac{du}{dx}\right\}^2+\left\{\frac{du}{dy}\right\}^2+\left\{\frac{du}{dz}\right\}^2}=v.^2\frac{\sqrt{(d^2x)^2(+d^2y^2)+(d^2z^2)}}{ds^2},$$

but ds being constant, the radius of curvature of the curve described by the moving point is equal to

$$\frac{ds^2}{\sqrt{(d^2x)^2+(d^2y)^2+(d^2z)^2}}\;\dagger$$

* By substituting for dt^2 its value $\frac{ds^2}{dv^2}$, we eliminate the time t, if the resulting equations be squared, we obtain, by adding their corresponding members,

$$\frac{v.}{ds^4}\frac{(d^2x^2+d^2y^2+d^2z^2)}{}=\lambda^2\left\{\frac{du}{dx}\right\}^2+\left\{\frac{du}{dy}\right\}^2+\left\{\frac{du}{dz}\right\}^2.$$

† This expression for the radius of the osculating curve may be thus investigated: let a, b, c, express the coordinates of the centre of this circle, its radius being equal to r,

∴ by naming this radius r we shall have

$$\lambda.\sqrt{\left\{\frac{du}{dx}\right\}^2+\left\{\frac{du}{dy}\right\}^2+\left\{\frac{du}{dz}\right\}^2}=\frac{v^2}{r},$$

then $r^2=(x-a)^2+(y-b)^2+(z-c)^2$; $dx.\ (x-a)+dy.(y-b)+dz.\ (z-c)$, the differential of this equation is equal to nothing, as any one of these coordinates may be considered as a function of the two remaining, we can obtain the following equations of partial differences $dx.\ (x-a)+dz.\ (z-c)=0$, $dy.(y-b)+dz.\ (z-c)=0$, (the values of dz in these equations are evidently different,) consequently we have $d^2x.\ (x-a)+d^2z.\ (z-c)+dx^2+dz^2=0$; $d^2y\ (y-b)+d^2z.(z-c)+dy^2+dz^2=0$, ∵ $(x-a)=-\frac{dz}{dx}\ (z-c)$, $(y-b)=-\frac{dz}{dy}$, $(z-c)$, and since ds is supposed to be constant, we have $d^2x.dx+d^2y.dy+d^2z.dz=0$, ($d^2z$ in this equation refers to the entire variation of dz,) consequently z being considered as a function of x and y, we obtain

$$d^2x.\,dx+d^2z.\,dz=0;\ d^2y.\ dy+d^2z.dz=0;\ \because\ \frac{dz}{dx}=-\frac{d^2x}{d^2z};\ \frac{dz}{dy}=-\frac{d^2y}{d^2z};$$

these values being substituted in place of $\frac{dz}{dx}\frac{dz}{dy}$; in the preceding equations we shall have

$$x-a=\frac{d^2x}{d^2z}\ (z-c),\ (y-b)=\frac{d^2y}{d^2z}.\ (z-c),$$

∵ by adding together the two preceding differential equations of the second order, substituting for $(x-a)$ $(y-b)$ their values, and observing that the whole variation of z is equal to the sum of the partial ones in these equations, we obtain,

$$\frac{d^2x^2+d^2y^2+d^2z^2}{d^2z}\ .\ (z-c)+dx^2+dy^2+dz^2=0,\ \text{consequently}$$

$$(z-c)^2=\frac{(dx^2+dy^2+dz^2)^2}{d^2x^2+d^2y^2+d^2z^2}.\ d^2z^2;\ \text{and as } r^2=(x-a)^2+(y-b)^2+(z-c)^2.$$

by substituting for $(x-a)^2$ $(y-b^2$ their values $\frac{d^2x^2}{d^2z^2}.)z-c)^2$; $\frac{d^2y^2}{d^2z^2}.(z-c^2)$, which have been given, we obtain

$$(x-a)^2+(y-b)^2+(z-c)^2=\frac{(dx^2+dy^2+dz^2)^2}{d^2x^2+d^2y^2+d^2\rceil^2}\ .\ (d^2x^2+d^2y^2+d^2z^2)$$

$$\because\ r=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}=\frac{ds^2}{\sqrt{d^2x^2+d^2y^2+d^2z^2}}.$$

consequently the pressure which the point exercises against the surface is equal to the square of the velocity divided by the radius of curvature of the curve described.

If the point moves on a spheric surface,* it will describe the circumference of a great circle of the sphere, which passes through the primitive direction of its motion; since there is no reason why it should deviate to the right rather than to the left of the plane of this circle; therefore its pressure against the surface, or what amounts to the same, against the circumference which it describes, is equal to the square of the velocity divided by the radius of this circle.

If we conceive the point attached to the extremity of a thread destitute of mass, having the other extremity fastened to the centre of the surface, it is evident that the force with which the point presses the circumference is equal to the force with which the string would be tended if the point was retained by it alone. The effort which this point would make to tend the string, and to increase its distance from the centre of the circle, is denominated the centrifugal force; therefore the centrifugal is equal to the square of the velocity divided by the radius.

The centrifugal force† of a point moving on any curve whatever is

* If the point move on a spherical surface, the motion will be necessarily performed on a great circle, for the deflection can only take place in the direction of radius, and in the plane in which the body moves.

† If the body moves on any curve whatever, the centrifugal force $= \frac{v^2}{r}$, this force acts in the direction of a normal to the curve, and if all the accelerating forces which act on the point be resolved into two, of which one is in the direction of the normal, and the other in the direction of the tangent, the resultant of the centrifugal force, and of the former of these decomposed forces, is the entire pressure with which the point acts against the curve, and the resistance of the curve is an accelerating force equal and contrary to this resultant. If we denote this normal force by L, and if A, B, C, be the angles which it makes with the coordinates x, y, z, respectively, then by the equation (f) and No. 3, we have

$$\frac{d^2x}{dt^2}=P+L.\cos.A;\ \frac{d^2y}{dt^2}=Q+L.\cos.B;\ \frac{d^2z}{dt^2}=R+L.\cos.C;$$

equal to the square of the velocity divided by the radius of curvature of the curve; because the indefinitely small arc of this curve is confounded with the circumference of the osculating circle. Therefore we shall

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and since $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, express the cosines of the angles which the tangent makes with x, y, and z, $\frac{dx}{ds}.\cos.A+\frac{dy}{ds}.\cos.B+\frac{dz}{ds}.\cos.C.=0$; because the tangent is perpendicular to the normal. (See last note to No. 1). We have also $\cos{}^2A+\cos.{}^2B+\cos.{}^2C=1$, and the four undetermined quantities L, A, B, C, being eliminated between the five preceding equations, the resulting equation will be one of the second order between x, y, z, and t; this equation combined with the two equations of the trajectory which are given in each particular case, are sufficient to determine the coordinates in a function of the time. See notes to No. 3, and No. 7.

The elimination of L, A, B, C, might be effected by one operation; for multiplying the three preceding equations by dx, dy, dz, respectively, and adding them together, we obtain the following equation:

$$\frac{dx.d^2x+dy.d^2y+dz.d^2z}{dt^2}=P.dx+Q.dy.+R.dz+L.(\cos.A.dx+\cos.B.dy+\cos.C.dz.)$$

(the latter part of this second member is equal to nothing, as has been already remarked;) and since $ds^2=dx^2+dy^2+dz^2$, $d^2s.ds=d^2x.dx+d^2y.dy+d^2z.dz$, ∴ we shall have

$$\frac{d^2s}{dt^2}=P.\frac{dx}{ds}+Q.\frac{dy}{ds}+R.\frac{dz}{ds};$$

from this last equation it appears that the accelerating force resolved in the direction of the tangent, is equal to the second differential coefficient of the arc considered as a function of the time, ∵ this expression for the force has place whatever be the nature of the line along which the point moves. See Notes to No. 7. In like manner it appears that the expression for the force in the direction of the tangent is altogether independant of L. It is also evident, that when there is no accelerating force $\frac{d^2s}{dt^2}=0$, this also follows from the circumstance of the velocity being uniform when P, Q, R, are equal to nothing.

Let V denote the resultant of all the accelerating forces which act on the point, and θ the angle which this resultant makes with the normal, then $V.\cos.\theta$ will be the expression of the resultant resolved in the direction of the normal; and when all the points of the curve exist in the same plane, the entire pressure will be equal to the sum or difference of $\frac{v^2}{r}$, and $V.\cos.\theta$, according as these two forces act in the same or in conhave the pressure of the point on the curve which it describes by adding to the square of the velocity, divided by the radius of curvature, the pressure produced by the forces which solicit this point.*†

trary directions, $\because +L = \pm \frac{v^2}{r} + V.\cos.\theta$. We can express this pressure otherwise by means of the rectangular coordinates; for since P, Q, are the expressions for the force V resolved parallel to x and y, these forces resolved in the direction of the normal are equal respectively to $P.\frac{dy}{ds}$; $Q.\frac{dx}{ds}$, (the signs of $\frac{dx}{ds}$, and $\frac{dy}{ds}$, are evidently different) consequently we have

$$V.\cos.\theta = +P.\frac{dy}{ds} + Q.\frac{dx}{ds}, \text{ and } \because L = \frac{v^2}{r} + P.\frac{dy}{ds} + Q.\frac{dx}{ds},$$

therefore if we know the equation of the trajectory, and if we have also the values of P and Q in terms of the coordinates, we can determine the velocity, and consequently L, and substituting this value of L in the expressions for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, which have been given in the foregoing part of this note, we might by integrating determine the velocity in the direction of each of the coordinates, and also the position of the point at a given moment.

If the point be attached to one extremity of a thread supposed without mass, of which the other extremity is fixed in the evolute of the curve described, then the point receiving such an impulse, that the string remaining always tended, may unroll itself in the plane of the evolute, it will describe the given curve; the direction of the string is always perpendicular to the curve, and its tension is equal to the normal pressure on the trajectory, and consequently equal to $\frac{v^2}{r} + \frac{P.dy + Q\,dx}{ds}$. By equating this expression of L to nothing, we can derive the equation of those trajectories in which the motion is free, or in which the trajectory may be described freely, i. e. it is not necessary to retain the point on the curve by means of a thread, or a canal, or any perpendicular force.

* If the motion is performed in a resisting medium, this resistance may be considered as a force acting in a direction contrary to that of the motion of the body, consequently it must tend to some point in the tangent. If we denote this resistance by I its moment is equal $-I.\delta i$ ($i = \sqrt{(x-l)^2 + (y-m)^2 + (z-n)^2}$, l, m, n, are the coordinates of the centre of the force I, therefore $\delta i = \frac{(x-l)}{i}.\delta x + \frac{(y-m)}{i}.\delta y + \frac{(z-n)}{i}.\delta z$; if we suppose the centre of force in the tangent, then $i = \sqrt{dx^2 + dy^2 + dz^2} = ds \because \frac{x-l}{i} = \frac{dx}{ds}$;

$\frac{y-m}{i}=\frac{dy}{ds}$; $\frac{z-n}{i}=\frac{dz}{ds}$; and $\delta i=\frac{dx}{ds}.\ \delta x+\frac{dy}{ds}.\ \delta y+\frac{dz}{ds}.\ \delta z$; if the resisting medium was in motion, its motion must be compounded with the motion of the body, in order to have the direction of the resisting force. If da, db, dc, be the spaces described by the medium, while the body describes ds, these quantities must be added or subducted from dx, dy, dz, in order to have the relative motions, and as $ds=\sqrt{dx^2+dy^2+dz^2}$, if we make $d\sigma=\sqrt{(dx-da)^2+(dy-db)^2+(dx-dc)^2}$, we shall have $\delta i=\frac{dx-da}{d\sigma}.\ \delta x+\frac{dy-db}{d\sigma}.\ \delta y+\frac{dz-dz}{d\sigma}.\ \delta z$. The resistance I in general $=\psi\ (v)$, a function of the velocity, in this case it is a function of the relative velcoity.

By the preceding investigation we are enabled to apply our general formula to motions made in resisting mediums without entering into a *particular* consideration of this species of motion. However the analysis becomes very complicated when the forces which compose P, Q. R, exist in different planes, and as in this case, the causes on which the variation of the velocity depends, arise in some measure from the velocities themselves, we are not permitted to regard $P.dx+Q.dy+R.dz$, as an exact differential of three independant variables, which facilitates our investigations when the motion is performed in a vacuo. See Notes to Nos. 8.

We might also reduce to our general formula, the differential equations of motion, when the retardation arises from the friction against the sides of the canal.

† If the body moved on a surface we might, as before, abstract from the consideration of the surface, and consider the material point entirely free by adding to the given forces another accelerating force, of which the intensity is unknown, and of which the direction is normal to the surface, ∵ if this force be denoted by L we shall have, by the equation (f) of No. 7, and by No. 3, the following equations:

$$\frac{d^2x}{dt^2}=P+L.N.\left\{\frac{\delta u}{\delta x}\right\};\quad \frac{d^2y}{dt^2}=Q.+L.N.\left\{\frac{\delta u}{\delta y}\right\};\quad \frac{d^2z}{dt^2}=R.+L.N.\left\{\frac{\delta u}{\delta z}\right\};$$

($u=0$ is the equation of the suface. See Notes to No. 3).

If we eliminate L between these three equations, N will also disappear; and if the two differential equations of the second order, which result from this elimination, be combined with the equation $u=0$ of the surface, we can determine the three coordinates of the point in a function of the time. If we multiply the preceding equations by dx, dy, dz, respectively, and then add together the corresponding members, we will obtain

$$\frac{d^2x.dx+d^2y.dy+d^2z.dz}{dt^2}=P.dx+Q.dy+Rdz+N.L.\left\{\frac{\delta u}{\delta x}\right\}dx+\left\{\frac{\delta u}{\delta y}\right\}dy+$$

$\left\{\frac{\delta u}{\delta z}\right\}dz$; but the last part of the second member is = to nothing, When the point moves on a *surface*,* the pressure due to the centrifugal force, is equal to the square of the velocity, divided by the radius of the osculating circle, and multiplied by the sine of the inclination of the plane of this circle, to the plane which touches the surface; therefore, if we add to this pressure, that which arises from the action of the forces which solicit the point, we shall have the entire force with which the point presses the surface.

$\left\{\text{since } du = 0, \text{ and } \left\{\frac{du}{dx}\right\} = \left\{\frac{\partial u}{\partial x}\right\}\right.$ ∵ if $P.dx + Q.dy + R.dz$ is an exact differential, we shall have $\frac{d^2s}{dt^2} = P.\frac{dx}{ds} + Q.\frac{dy}{dt} + R.\frac{dz}{ds}$, as before, and $\frac{ds^2}{dt^2} = v^2 = C + \int(P.dx + Q.dy + R.dz)$, and if P, Q, R, and consequently v were given in terms of the coordinates, we might obtain immediately the differential equations of the trajectory by multiplying the equation $\frac{d^2x}{dt^2} = P + L.N.\left\{\frac{\partial u}{\partial x}\right\}$, by dy and dz successively, and then subducting it from the two remaining equations multiplied by dx; by concinnating the resulting equations, substituting for dt its value $\frac{ds}{v}$, and for v its value in a function of the coordinates, we obtain two differential equations of the second order, from which eliminating the quantities LN there results a differential equation of the second order between the three coordinates z, y, z, solely; this equation, and the equation $u=0$ of the surface will be the two equations of the *trajectory*.

* If a point moves on any curve the centrifugal force is always directed along the radius of the osculating circle; and since the pressure on the surface is always estimated in the direction of a normal to the surface, (see No. 3) if the plane of the trajectory is not at right angles to the surface, the radius of the osculating circle will not coincide with the normal to the surface, and consequently the part of the centrifugal which produces a pressure on the surface is equal to $\frac{v^2}{r}$, multiplied into the cosine of the angle which the radius makes with the normal, but this angle is evidently the complement of the angle which the plane of the osculating circle makes with the plane which touches the surface. If the forces soliciting the point are resolved into two, of which one is perpendicular to the trajectory, then the resultant of this last force, and of the centrifugal force, will express the whole force of pressure on the curve; if this curve was fixed, it would be sufficient for the pressure to be counteracted, that its direction was in a plane perpendicular to this curve, but if the curve be one traced on a given surface, then, in order that the pressure should be counteracted, it is necessary that the resultant of the forces should be in the direction of a *normal* to the surface. See note to page 16.

We have seen that when the point is not solicited by any forces, its pressure against the surface, is equal to the square of the velocity, divided by the radius of the osculating circle; therefore the plane of this circle, that is to say, the plane which passes through two consecutive sides of the curve described by the point is then perpendicular to the surface. This curve on the surface of the earth is called the perpendicular to the meridian; and it has been proved (in No. 8) that it is the shortest which can be drawn from one point to another on the surface.*

* If we make the axis of one of the coordinates to coincide with the normal to the surface, we can *immediately* determine the inclination of the plane of the osculating circle to the plane touching the surface; for if we denote by A, B, the angles which the radius of the osculating circle makes with the normal and with the coordinate which is in the plane of the tangent, and by m, n, l, the angles which the resultant V of all the forces makes with the three coordinates, the force $\frac{v^2}{r}$ resolved parallel to these coordinates is equal to $\frac{v^2}{r}.\cos. A$, $\frac{v^2}{r}.\cos. B$, $+\frac{v^2}{r}.\cos. 100°$, (because the angle between the radius and tangent to the curve is equal to 100°) in like manner the force V. resolved parallel to these coordinates equals $V.\cos. m$, $V.\cos. n$, $V.\cos. l$, since A and m denote the inclinations of the radius of curvature, and of the resultant to the normal, $\frac{v^2}{r}.\cos. A+V.\cos. m$, express the pressure of the point on the surface, $V.\cos. n+\frac{v^2}{r}.\cos. 100°$, or $V.\cos. n$ is the force by which the body is moved; and since this motion is performed in the direction of the tangent, $V.\cos. l+\frac{v^2}{r}.\cos. B$, which expresses the motion perpendicular to the tangent must vanish; consequently we have $V.\cos. l+\frac{v^2}{r}.\cos. B=0$, $\because$ if $V. l, v$, and r were given we might determine B, which is $=$ to the inclination of the plane of the osculating circle to the plane touching the surface, it also follows, that when the point is not solicited by any accelerating force, $\frac{v^2}{r}.\cos. B=0$, $\because B=100°$, or the plane of the osculating circle is perpendicular to the surface, which we have previously established from other considerations.

If the plane whose intersection with the surface produces the given curve is not *perpendicular* to the surface, then the radius of curvature is equal to the sine of the inclination of the cutting plane to the plane touching the surface, multiplied into the radius of curvature of the section made by a plane passing through the normal to the surface, and through the intersection of the plane touching the surface and the cutting plane. See Lacroix, No. 324. $\because$ the pressure is the same whether the point move in a greater or

10. Of all the forces that we observe on the earth, the most remarkable is gravity; it penetrates the most inward parts of bodies, and would make them all fall with equal velocities, if the resistance of the air was removed. Gravity is very nearly the same at the greatest heights to which we are able to ascend, and at the lowest depths to which we can descend; its direction is perpendicular to the horizon, but on account of the small extent of the curves which projectiles describe relatively to the circumference of the earth, we may, without sensible error, suppose that it is constant, and that it acts in parallel lines. These bodies being moved in a resisting fluid, we shall call b the resistance which they experience; it is directed along the side ds of the curve which they describe; moreover we will denote the gravity by g. This being premised, let us resume the equation (f) of No. 7, and suppose that the plane of x and y is horizontal, and that the origin of z is at the most elevated point; the force b will produce in the direction of the coordinates x, y, z, the three forces $-b.\frac{dx}{ds}$, $-b.\frac{dy}{ds}$, $-b.\frac{dz}{ds}$ $\because$ by No. 7 we shall have $P=-b.\frac{dx}{ds}$; $Q=-b.\frac{dy}{ds}$; $R=-b.\frac{dz}{ds}+g.$* and the equation (f) becomes

$$0=\delta x.\left\{d.\frac{dx}{dt}+b.\frac{dx}{ds}dt.\right\}+\delta y.\left\{d.\frac{dy}{dt}+b.\frac{dy}{ds}dt.\right\}$$
$$+\delta z.\left\{d.\frac{dz}{dt}+b.\frac{dz}{ds}dt.-g.dt.\right\}^{*}$$

less circle, for the sine of inclination occurs both in the numerator and denominator of the expression; this also follows from considering the proportion of the sagitta of curvature in a perpendicular and oblique plane.

The investigation of the shortest line which can be drawn between two given points on a curved surface, whose equation is $u=0$, by the method of variations, leads us to the same conclusions. See Lacroix. The consideration of the shortest line which can be traced on a spheroidical surface is of great importance in the theory of the figure of the earth. (See Book 3, No. 38.)

* Since the force b acts in the direction of the tangent or of the element ds of the curve (see note to No. 9,) this force resolved parallel to the three coordinates x, y, z, $= b.\frac{dx}{ds}, b.\frac{dy}{ds}, b.\frac{dz}{ds}$, for $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are = to the cosines of the angles

If the body be entirely free we shall have the three equations

$$0=d.\frac{dx}{dt}+b.\frac{dx}{ds}.dt;\ 0=d.\frac{dy}{dt}+b.\frac{dy}{ds}.dt;$$
$$0=d.\frac{dz}{dt}+b.\frac{dz}{ds}.dt-g.dt,$$

The two first give

$$\frac{dy}{dt}.\ d.\frac{dx}{dt}-\frac{dx}{dt}.\ d.\frac{dy}{dt}=0.$$

from which we obtain by integrating, $dx=fdy$, f being a constant arbitrary quantity. This equation belongs* to an horizontal right line, therefore the body moves in a vertical plane.

By taking this plane for that of x, z, we shall have $y=0$, the two equations,

$$0=d.\frac{dx}{dt}+b.\frac{dx}{ds}.dt;\ 0=d.\frac{dz}{dt}.+b.\frac{dz}{ds}.dt-g.dt,$$

will give, by making dx constant,

$$b=\frac{ds.d^2t}{dt^3},\ 0=\frac{d^2z}{dt}-\frac{dz.d^2t}{dt^2}+b.\frac{dz}{ds}.dt-g.dt.$$

From* which we obtain $g.dt^2=d^2z$, and by taking the differential

which the tangent makes with the three coordinates; they are affected with negative signs because they tend to diminish the coordinates.

* Dividing $\frac{dy}{dt}.d.\frac{dx}{dt}-\frac{dx}{dt}.d.\frac{dy}{dt}=0$, by $\frac{dy^2}{dt^2}$ it becomes

$d.\left\{\frac{\frac{dx}{dt}}{\frac{dy}{dt}}\right\}=0$, ∵ by integrating $\frac{\frac{dx}{dt}}{\frac{dy}{dt}}=f$ and $dx=f.dy$, since the equation of the projection of the line which the projectile describes on a horizontal plane, is that of a right line, the body must have moved in a vertical plane, otherwise its projection on an horizontal plane would not be a right line; this circumstance we might have anticipated from the manner in which the forces act on the body.

* If we make dt constant in the equation $d.\frac{dx}{dt}+b.\frac{dx}{ds}.dt=0$, we get $-\frac{dx.d^2t}{dt^2}$

$2g.dt.d^2t=d^3z$, if we substitute for d^2t its value $\frac{b.dt^3}{ds}$, and for dt^2 its value $\frac{d^2z}{g}$, we shall have

$$\frac{b}{g}=\frac{ds.d^3z}{2(d^2z)^2}.$$

This equation gives the law of the resistance b, which is necessary to make the projectile describe a given curve.

If the resistance be * proportional to the square of the velocity, b is equal to $h.\frac{ds^2}{dt^2}$, h being constant, when the density of the medium is uniform. We shall have then

$$\frac{b}{g}=\frac{h.ds^2}{g.dt^2}=\frac{h.ds^2}{d^2z},$$

therefore $h.ds=\frac{d^3z}{2.d^2z}$, which gives by integrating $\frac{d^2z}{dx^2}=2a.c^{2hs}$.†

$+b.\frac{dx}{ds}.dt=0$, $\therefore$ $b=\frac{ds.d^2t}{dt^3}$, by substituting this quantity in place of b, and differentiating, we get the expression

$$\frac{d^2z}{dt}-\frac{dz.d^2t}{dt^2}+\frac{ds.d^2t}{dt^3}\cdot\frac{dz}{ds}dt-g.dt=\frac{d^2z}{dt}-\frac{dz.d^2t}{dt^2}+\frac{dz.d^2t}{dt^2}-g.dt.=$$

$\frac{d^2z}{dt}-g.dt=0$, $\therefore$ by differentiating we obtain $d^3z=2g.dt.d^2t$, and substituting for d^2t its value $b.\frac{dt^3}{ds}$, and for dt^2 its value $\frac{d^2z}{g}$, we arrive at the following equation,

$$d^3z=\frac{2g.b.}{ds}\left\{\frac{d^2z}{g}\right\}^2\therefore\frac{b}{g}=\frac{d^3z}{2(d^2z)^2}.ds.$$

* The value of the constant coefficient h is obtained by experiment; it is different in different fluids, and when bodies of different figures move in the same fluid.

† Since the square of the velocity is equal to $\frac{ds^2}{dt^2}$, the resistance is expressed by $h.\frac{ds^2}{dt^2}$, $\therefore$ by substituting for dt^2 its value $\frac{d^2z}{g}$, $\frac{hds^2}{d^2z}=\frac{ds.d^3z}{2(d^2z)^2}$, $\therefore$ $2h.ds.=\frac{d^3z}{d^2z}$,

a being a constant arbitrary quantity, and c being the number whose hyperbolic logarithm is unity. If we suppose the resistance of the me-

H

$\therefore 2h.s = \log. d^2z + \log. F \therefore c^{2hs} = F.d^2z, \therefore \frac{c^{2hs}}{F.dx^2} = \frac{d^2z}{dx^2}$. (Let $2a = \frac{1}{F.dx^2}$) and we shall have $2ac^{2hs} = \frac{d^2z}{dx^2}$; dx being constant it is permitted to introduce dx^2 as a divisor. The constant quantity a depends on the velocity of projection, and on the angle which its direction makes with the horizon; for by substituting $-g.dt^2$ in place of d^2z we shall have $-\frac{dx^2}{dt^2} = \frac{g}{2a}.c^{-2hs}, \frac{dx^2}{dt^2}$ is the velocity of the body in the direction of the axis of x at the end of the time t; let u be the velocity of projection, and θ the angle which its direction makes with the horizon, we shall have at the same time $t = 0$, $x=0$, $z=0$, and $\frac{dx}{dt} = u.\cos.\theta$, $\therefore u^2.\cos.^2\theta = -\frac{g}{2a}$. Let h be the height due to the velocity u, $u^2 = 2gh$, $\therefore$ by substituting for u^2 its value, we deduce $a = \frac{-1}{4h}\frac{}{\cos.^2\theta}$

By making $dz = pdx$, ds becomes equal to $dx.\sqrt{1+p^2}$, $\therefore -c^{2hs}.\ ds = 2h.\cos.^2\theta.\ dp.\sqrt{1+p^2}, \left\{\text{for } dp = \frac{d^2z}{dx}\right\}$ $\therefore$ by integrating $\frac{-c^{2hs}.}{2h} + C = 2h.\cos.^2\theta.\int dp.\sqrt{1+p^2}$, $\left\{= 2h.\cos.^2\theta.\int\frac{dp}{\sqrt{1+p^2}} + 2h.\cos.\theta.\int\frac{dp.p^2}{\sqrt{1+p^2}}\right\} = h.\cos.^2\theta.\ \log.\ (p+\sqrt{1+p^2})$, $+h.\cos.^2\theta.\ p.\sqrt{1+p^2}$, the constant quantity C is easily found; for since p is the derivitive function of z considered as a function of x, at the commencement of the motion, when $s=0$, $p=$ the tangent of the angle of projection which is given, $\therefore C$ is equal to

$$h.\cos.^2\theta.\left\{(\log.(\tan.\theta+\sec.\theta)+\tan.\theta.\sec.\theta.)\right\} + \frac{1}{2h}.$$

By substituting for $-\frac{c^{2hs}}{2h}$ its value, which we obtain from the equation $-\frac{c^{2hs}}{2h} = 2..$ $\cos.^2\theta.\frac{dp}{dx}$, we deduce

$$dx = \frac{dp}{2h.\{(\log.(p+\sqrt{1+p^2})+p.\sqrt{1+p^2})-C\}},\ \text{and } dz =$$

$$p.dx = \frac{p.dp}{2h\{(\log.(p+\sqrt{1+p^2})+p.\sqrt{1+p^2})-C\}}$$

and since $g.dt^2 = d^2z = dp.dx$ we have $dt^2 = \frac{dp.dx}{g}$, dium to vanish, h is equal to 0; then by integrating* we will obtain the equation to the parabola $z=ax^2+bx+c$, b and c being constant arbitrary quantities.

The differential equation $d^2z=g.dt^2$, will give $dt^2=\frac{2a}{g}.dx^2$, from

$$\because dt=\frac{dp}{\{2gh\,(\log.\,(p+\sqrt{1+p^2})+p.\sqrt{1+p^2})-C\}^{\frac{1}{2}}}.$$

If the integrals for these values of dx, dy, dt, could be exhibited in a finite form, the problem would be completely solved, for the integrations of the two first equations would give the values of x and z in functions of p; and if p be eliminated between the resulting equations, the relation between x and y would be had; those integrations have hitherto baffled the skill of the most celebrated analysts. However by means of the expressions for dx and dz, we can describe the curve by a series of points, and the approximation will be always more accurate, according as we divide the interval between the extreme values of p into a greater number of parts. We might collect some of the remarkable properties of the curve described from the preceding values of dx, dz; for if p be very great, log. $(p+\sqrt{1+p^2})$ vanishes with respect to p, and $\because$ the limit of dx, dz, and dt are $\frac{dp}{2h.p^2}$, $\frac{dp}{2h.p}$, and $\frac{dp}{\sqrt{2gh.p}}$, $\because$ by integrating we get $x=a-\frac{1}{p}$, $z=a'+\log.\,p$, $t=a''+\frac{1}{2gh}\cdot\log.\,p$, the first equation indicates that x has a limit, the vertical ordinate increases indefinitely, but in a less ratio than p, therefore the descending branch has a vertical asymptote. By eliminating log. p in the expression for t we get an expression for z from which we may collect, that according as the direction of the motion approaches towards the vertical, the motion of the body tends to become uniform.

When the angle of projection is very small, we can find by approximation the relation which exists between x, and z, for that portion of the trajectory which is situated above the horizontal axis; in this case the tangent is very nearly horizontal, $\because$ p is very small, and $\sqrt{1+p^2}=1$, $q.p.$ $\because$ $ds=dx.\sqrt{1+p^2}=dx$, $q.p.$ and $s=x$, for they commence together, and substituting x in place of s, we have $\frac{dp}{dx}=-\frac{c^{2hx}}{2h\cos.^2\theta}$, but θ being by hypothesis very small, $\cos.^2\theta=1$, $\because$ $dp=-\frac{c^{2hx}}{2h}.\,dx$, by integrating this equation, when we know the value of the constant arbitrary quantity which is introduced by the integration we obtain the value of p and $\because$ of $z=\int p.dx$. See a memoir of Legendre's in the Transactions of the Academy of Berlin for the year 1782.

* In this case $\frac{d^2z}{dx^2}=2a$, $\because$ $\frac{dz}{dx}=2ax+b$, $\because$ $z=ax^2+bx+c$.

which we may obtain $t = x.\sqrt{\frac{2a}{g}} + f'$. If x, z, and t, commence together, we shall have $c=0$, $f'=0$, and consequently

$$t = x\sqrt{\frac{2a}{g}}\,;\ z = ax^2 + bx,$$

which gives

$$z = \frac{gt^2}{2} + b.t.\sqrt{\frac{g}{2a}}$$

These three equations contain the whole theory of projectiles in a vacuum; it follows, from what precedes, that the velocity is uniform in an horizontal direction,* and that in the vertical direction the velocity is the same as if the body fell down the vertical. If the body moves from a state of repose b will vanish, and we shall have

$$\frac{dz}{dt} = gt\,;\ z = \frac{1}{2}.g^2t\,;$$

therefore the velocity acquired increases as the time, and the space increases as the square of the time.

It is easy by means of these formulæ to compare the centrifugal force with that of gravity. For v being the velocity of a body moving in the circumference of a circle, of which the radius is r, it appears from No. 9, that its centrifugal force is equal to $\frac{v^2}{r}$. Let h be the height from which the body must fall to acquire the velocity v; by what precedes we shall have $v^2 = 2g.h$; from which we obtain $\frac{v^2}{r} = g.\frac{2h}{r}$. The centrifugal

* For $\frac{dx}{dt}$ = the velocity in an horizontal direction $=\sqrt{\frac{g}{2a}}$, and $\frac{dz}{dt}$ = the velocity in a vertical direction $= gt + b.\sqrt{\frac{g}{2a}}$.

force will be equal to the gravity g, if $h=\frac{r}{2}$. Therefore* a heavy body attached to the extremity of a thread, which is fixed at its other extremity, on an horizontal plane, will tend the string with the same force as if it was suspended vertically; provided that it moves on this plane, with a velocity equal to that which the body would acquire in falling down a height equal to half the length of the thread.

11. Let us consider the motion of a heavy body on a spherical surface, denoting its radius by r, and fixing the origin of the coordinates at its centre, we shall have $r^2-x^2-y^2-z^2=0$; this equation being compared with that of $u=0$, gives $u=r^2-x^2-y^2-z^2$; therefore if we add to the equation (f) of No. 7, the function δu multiplied by the indeterminate quantity $-\lambda.dt$. we shall have

$$0=\delta x.\left\{d.\frac{dx}{dt}+2\lambda x.dt.\right\}+\delta y.\left\{d.\frac{dy}{dt}+2\lambda.y.dt.\right\}$$

$$+\delta z.\left\{d.\frac{dz}{dt}+2.\lambda z.dt-g.dt.\right\}^{*}$$

In this equation we can put the coefficients of each of the variations δx, δy, δz, equal to nothing, which gives the three following equations:

$$0=d.\frac{dx}{dt}+2\lambda.xdt,$$

$$0=d.\frac{dy}{dt}+2\lambda.y.dt.$$

$$0=d.\frac{dz}{dt}+2\lambda.z.dt-g.dt.$$

* The plane of the motion being horizontal, the force with which the string is tended arises entirely from the centrifugal force.

† For $\left\{\frac{\delta u}{\delta x}\right\}=-2x$, $\left\{\frac{\delta u}{\delta y}\right\}=-2y$, $\left\{\frac{\delta u}{\delta z}\right\}=-2z$.

The indeterminate λ makes known the force with which the point presses on the surface. This pressure by No. 9 is equal to

$$\lambda.\sqrt{\left\{\frac{du}{dx}\right\}^2+\left\{\frac{du}{dy}\right\}^2+\left\{\frac{du}{dz}\right\}^2};$$

consequently it is equal to $2\lambda r$; but by No. 8 we have

$$c+2gz = \frac{dx^2+dy^2+dz^2}{dt^2},$$

c being a constant arbitrary quantity; by adding this equation to the equations (A) divided by dt, and multiplied respectively by x, y, z, and then observing that $x.dx+y.dy+z.dz = 0$, $x.d^2x+y.d^2y+z.d^2z+dx^2+dy^2+dz^2=0$, are the first and second differential equations of the surface, we shall obtain*

$$2\lambda.r = \frac{c+3gz}{r}.$$

* For performing these operations we get $c+2gz =$

$\frac{dx^2+dy^2+dz^2}{dt^2}+\frac{x.d^2x}{dt^2}+\frac{y.d^2y}{dt^2}+\frac{z.d^2z}{dt^2}+2\lambda.(x^2+y^2+z^2)-gz$, therefore we have

$2\lambda r^2 = c+3gz$, and $2\lambda r = \frac{c+3gz}{r}$, $\because$ the pressure is equal to $\frac{c+3gz}{r}$, when the initial velocity c vanishes, the tension of the pendulum vibrating in a quadrantal arc is, at the lowest point, = to three times the force of gravity; $\frac{z}{r}$ = the cosine of the angle which the radius r makes with the vertical, therefore it follows that when a body falls from a state of rest, the pressure on any point is proportional to the cosine of the distance from the lowest point, it is easy to collect, in like manner, that the accelerating force varies as the right sine of the angular distance from the lowest point; we might from the preceding expression for the pressure determine the point where this pressure is in a given ratio to the force of gravity.

If we multiply the first of the equations (A) by $-y$, and add it to the second, multiplied by x, and then integrate their sum, we shall have

$$\frac{x.dy-y.dx}{dt}=c'; \, *$$

c' being a new arbitrary quantity.

Thus the motion of the point is reduced to three differential equations of the first order,

$$x.dx+y.dy=-z.dz,$$

$$x.dy-y.dx=c'.dt,$$

$$\frac{dx^2+dy^2+dz^2}{dt^2}=c+2gz.$$

By squaring each member of the two first equations,† and then adding them together, we shall have

$$(x^2+y^2)\,(dx_2+dy^2)=c'^2dt^2+z^2dz^2.$$

* $x.dy-y.dx=c'.dt$ shews that the area described by a body moving on a spherical surface, and projected on the plane x, y, is proportional to the time; the same area projected on the plane x, z, or y, z, is not constant in a given time; for if we add to the first of the equations (A) multiplied by $-z$, the third multiplied by x, and then integrate their sum, it becomes equal to $\frac{x.dz-z.dx}{dt}=c''+f.(gx.dt)$, this might have been anticipated, as the force g does not pass perpetually through the origin of the coordinates, $\because$ $x.dz-z.dx$, $y.dz-z.dy$ are not proportional to the time, but as there is no force acting parallel to the horizontal plane, $x.dy-y.dx$ must be proportional to the time.

† For we have in this case

$$x^2.dx^2+y^2.dy^2+2x.y.dx.dy=z^2.dz^2.$$
$$x^2dy^2+y^2.dx^2-2x.y.dx.dy=c'^2dt^2.$$
$$\because (x^2+y^2)\,(dx^2+dy^2)=c'^2dt^2+z^2.dz^2.$$

$\because$ by substituting for x^2+y^2, and $\frac{dx^2+dy^2}{dt^2}$, their values we obtain (r^2-z^2).

If we substitute in place of x^2+y^2, and $\frac{dx^2+dy^2}{dt^2}$, their respective values r^2-z^2, and $c+2gz-\frac{dz^2}{dt^2}$; we shall have on the supposition that the body departs from the vertical

$$dt=\frac{-r.dz}{\sqrt{(r^2-z^2).(c+2gz)-c'^2}}.$$

The function* under the radical may be made to assume the form $(a-z).(b-z).(2gz+f)$; $a, b, f,$ being determined by the equations

$(cdt^2+2gz.dt^2-dz^2)=c'^2\,dt^2+z^2.dz^2$, therefore $(r^2-z^2).(c+2gz)-c'^2).\,dt^2=r^2.dz^2+z^2dz^2-z^2.dz^2$, consequently

$$dt=\frac{-r.dz}{\sqrt{(r^2-z^2).(c+2gz)-c'^2}},$$

dz is affected with a negative sign, because the motion commencing when the body is at the lowest point, z decreases according as t increases.

* If we multiply the factors of the expression, and range them according to the dimensions of z, we get $-2gz^3-cz^2+2r^2.gz+r^2c-c'^2$, if the same operation be performed on the expression $(a-z).(z-b).(2gz+f)$ we will obtain $-2gz^3+(2g(a+b)-f).\,z^2+(f.(a+b)-2g.ab)\,z-fab$, these two expressions being *always* equal, their corresponding terms must be *identical*, consequently, by comparing the coefficients of z, we have $f=2g.\frac{(r^2+ab)}{a+b}$, by comparing the coefficients of z^2, and substituting for f its value we get

$$2.g\left\{(a+b)-\frac{r^2+ab}{a+b}\right\}=-c \because \text{ by concinnating}$$

$$2g.\frac{a^2+2ab+b^2-r^2+ab}{a+b}=-c=2g.\left\{\frac{r^2-a^2-ab-b^2}{a+b}\right\}$$

the comparison of the absolute quantities, gives, by substituting for f and c their values, which have been already found,

$$2g.r^2\left\{\frac{r^2-a^2-ab-b^2}{a+b}\right\}-c'^2=-2g.ab.\left\{\frac{r^2+ab}{a+b}\right\}, \because c'^2=$$

$$2g.\left\{\frac{r^4-r^2.a^2-r^2ab-r^2.b^2+r^2.ab+a^2b^2}{a+b}\right\}=2g.\frac{(r^2-a^2).(r^2-b^2)}{a+b},$$

$$f = 2g.\frac{(r^2+ab)}{(a+b)}$$

$$c = 2g.\frac{(r^2-a^2-ab-b^2)}{a+b}$$

$$c'^2 = 2g.\frac{(r^2-a^2).(r^2-b^2)}{a+b}$$

We can thus substitute for the arbitrary quantities c and c', a and b, which are also arbitrary, of which the first is the greatest value of z, and the second the least. Then, by making

$$\sin.\theta = \sqrt{\frac{a-z}{a-b}},$$

the preceding differential equation will become

$$dt = \frac{r.\sqrt{2(a+b)}}{\sqrt{g\{(a+b)^2+r^2-b^2\}}} \cdot \frac{d\theta}{\sqrt{1-\gamma^2.\sin.^2\theta}}, \quad *$$

these values of f, c, c' being possible, we are permitted to substitute the expression $(a-z).(z-b).(2gz+f)$ in place of $(r^2-z^2).(c+2gz)-c'^2$, therefore $\frac{r.dz}{dt} = -\sqrt{(a-z).(z-b).(2gz+f)}$, z being a function of t, this differential coefficient vanishes when $a=z$, and also when $z=b$; $\frac{r\,z}{dt} = -\sqrt{(a-z).(z-b).(2gz+f)} = 0$, has at least *two* real roots; for as the point is constrained to move on the surface of the sphere, the trajectory has necessarily a maximum and a minimum; and as impossible roots enter equations by pairs, it follows that all the roots are real, moreover it is manifest from the variations of the signs, that one root is negative: $\frac{dz}{dt}$ expresses the velocity of the point in the direction of the vertical.

* The transformation $\sin.\theta = \sqrt{\frac{a-z}{a-b}}$ is made in order to facilitate the integration. $\sin.^2\theta = \frac{a-z}{a-b}$, and $\cos.^2\theta = \frac{z-b}{a-b}$ $\because z = a.\cos.^2\theta + b.(1-\cos.^2\theta) = a\cos.^2\theta + b\sin.^2\theta$,

γ^2 being equal to

$$\frac{a^2-b^2}{(a+b)^2+r^2-b^2},$$

The angle θ gives the coordinate z by means of the equation;

$$z=a.\cos.^2\theta+b.\sin.^2\theta,$$

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$$d\theta.\cos.\theta=\frac{-dz}{2.\sqrt{(a-z).(a-b)}}\ \therefore\ -dz=2d\theta.\sqrt{(a-z).(z-b)}$$

and $\dfrac{-r.dz}{\sqrt{(a-z).(z-b).(2gz+f)}}=$ (substituting for f its value)

$$\frac{2r.d\theta\sqrt{(a-z).(z-b)}}{\sqrt{2g.(a-z).(z-b)\left(z+\dfrac{r^2+ab}{a+b}\right\}}}=\frac{2r.d\theta}{\sqrt{2g.\left\{z+\dfrac{r^2+ab}{a+b}\right\}}};$$

(substituting for z its value $a.\cos.^2\theta+b.\sin.^2\theta$) we obtain

$$\frac{2r.d\theta\sqrt{a+b}}{\sqrt{2g.(a^2.\cos.^2\theta+ab.\cos.^2\theta+ab.\sin.^2\theta+b^2.\sin.^2\theta+r^2+ab}}$$

$$=\frac{2r.d\theta.\sqrt{a+b}}{\sqrt{2g.((a^2+2ab+b^2)+(r^2-b^2)+(b^2-a^2).)\sin.^2\theta}}$$

$$=\frac{2r.d\theta.\sqrt{a+b}}{\sqrt{2g.(a+b)^2+(r^2-b^2)+(b^2-a^2).\sin.^2\theta)}},$$

and if $\gamma^2=\dfrac{a^2-b^2}{(a+b)^2+r^2-b^2)}$, $b^2-a^2=-((a+b)^2+(r^2-b^2)).\gamma^2$, $\therefore$ substituting for b^2-a^2 in the preceding expression we shall have $dt=$

$$\frac{2r.d\theta.\sqrt{a+b}}{\sqrt{2g.((a+b^2)+(r^2-b^2)-((a+b)^2+(r^2-b^2)).\gamma^2.\sin.^2\theta.}}$$

$$=\frac{r.\sqrt{2.(a+b)}}{\sqrt{g.((a+b)^2+(r^2-b^2)}.}\times\frac{d\theta}{\sqrt{1-\gamma^2.\sin.^2\theta.}}$$

and the coordinate z divided by r, expresses the cosine of the angle which the radius r makes with the vertical.

Let ϖ be the angle which the vertical plane passing through the radius r, makes with the vertical plane which passes through the axis of x; we shall have

$$x=\sqrt{r^2-z^2}.\ \cos.\ \varpi;\ ^*\ y=\sqrt{r^2-z^2}.\sin.\ \varpi;$$

which give

$$xdy-ydx=(r^2-z^2).\ d\varpi,$$

$\therefore$ the equation $xdy-ydx=c'dt$ will give

$$d\varpi=\frac{c'.dt}{r^2-z^2},$$

we will obtain the angle ϖ in a function of θ, by substituting for z and dt their preceding values in terms of θ; thus we may know at any time whatever, the two angles θ and ϖ, which is sufficient to determine the position of the moving point.

Let us name, $\frac{T}{2}$, the time employed † in passing from the greatest

* $x=$ the product of the projection of r, on the plane x, y, into the cosine of the angle which x makes with the projected line, $\therefore$ as, $\sqrt{r^2-z^2}=r$ so projected, and $\varpi=$ the angle which x makes with $\sqrt{r^2-z^2}$, $x=\cos.\ \varpi.\ \sqrt{r^2-z^2}$, $dx=-\sqrt{r^2-z^2}.\ \sin.\ \varpi.$ $d\varpi.-\frac{zdz.\cos.\ \varpi}{2\sqrt{r^2-z^2}}$, $y=\sqrt{r^2-z^2}.\sin.\ \varpi$, $\therefore dy=\sqrt{r^2-z^2}.d\varpi.\ \cos.\ \varpi-\frac{zdz.\ \sin.\ \varpi}{2.\sqrt{r^2-z^2}}$,

$$xdy-ydx=(r^2-z^2)\,d\varpi\ \cos.^2\varpi.-\frac{zdz.\ \sin.\ \varpi.\ \cos.\ \varpi}{2}+(r^2-z^2)\,d\varpi.\ \sin.^2\varpi$$
$$+\frac{z.dz.\ \sin.\ \varpi\ \cos.\ \varpi.}{2}=(r^2-z^2).\ d\varpi.$$

† For evolving the expression for dt into a series, it becomes,

$$\frac{r.\sqrt{2(a+b)}}{\sqrt{g.\ ((a+b)^2+r^2-b^2}}.\ d\theta+\frac{1}{2}\gamma^2.\sin.^2\theta.\ d\theta+\frac{1.3}{2.4}\gamma^4.\sin.^4.\theta.d\theta+\frac{1.3.5}{2.4.6}\gamma^6.\ \sin.^6\theta d\theta+\&c.$$

$$\text{but } \sin.^2\theta=-\frac{\cos.\ 2\theta}{2}+\frac{1}{2},\ \sin.^4\theta=\frac{\cos.\ 4\theta}{8}-\frac{4.\cos.\ 2\theta}{8}+\frac{3}{2.4}$$

to the least value of z, a semi-oscillation. In order to determine it, we should integrate the preceding value of dt from $\theta=0$ to $\theta=\frac{1}{2}.\pi$, π

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$$\sin.^6\theta=-\frac{\cos.\,6\theta}{32}+\frac{6.\cos.\,4\theta}{32}-\frac{15.\cos.\,2\theta}{32}+\frac{10}{32}, \text{ \&c.}$$

$$\therefore \int \sin.^2\theta.\,d\theta=-\frac{\sin.\,2\theta}{4}+\frac{\theta}{2}; \int \sin.^4\theta.\,d\theta=\frac{\sin.\,4\theta}{32}-\frac{4\sin.\,2\theta}{16}+\frac{3\theta}{2.4},$$

$$\int \sin.^6\theta.\,d\theta=-\frac{\sin\,6\theta}{192}+\frac{6.\sin.\,4\theta}{128}-\frac{15.\sin.\,2\theta}{64}+\frac{10\,\theta}{32}, \text{ \&c.}$$

(See Lacroix, Traite Elementaire, No. 200.)

These quantities being integrated between the limits $\theta=0$, and $\theta=\frac{1}{2}.\,\pi$, or between sin. $\theta=0$, and sin. $\theta=1$, i. e. between the greatest and least values of z, become respectively $\frac{\pi}{2}.\frac{1}{2}, \frac{3\pi 1}{2.4.2}, \frac{10\pi}{32}.\frac{1}{2}=\left\{\frac{3.5.\pi}{2.4.6}.\frac{1}{2},\right\}$ &c. for the parts in which the sines of the multiple arcs occur, vanish, being respectively = to sin. (2π), sin. (4π), sin. (6π), the numeral coefficients of $\frac{\pi}{2}$ are equal to the corresponding coefficients in the expanded radical; $\therefore$ these integrals being substituted in the preceding series we obtain

$$\frac{T}{2}=\sqrt{\frac{r}{g}}.\sqrt{\frac{2r(a+b)}{(a+b)^2+(r^2-b^2)}}.\left\{\frac{\pi}{2}+\frac{1}{2}.\gamma^2\frac{1}{2}.\frac{\pi}{2}+\frac{1.3}{2.4}.\gamma^2.\left\{\frac{1.3}{2.4}.\frac{\pi}{2}\right\}\right.$$
$$\left.+\frac{1.3.5}{2.4.6}\gamma^6.\left(\frac{1.3.5}{2.4.6}\,\frac{\pi}{2}\right)+\text{\&c.}\right.$$

$$\therefore T=\pi.\sqrt{\frac{r}{g}}\sqrt{\frac{2r.(a+b)}{(a+b)^2+(r^2-b^2)}}\left\{1+\left\{\frac{1}{2}\right\}^2.\gamma^2+\left\{\frac{1.3}{2.4}\right\}^2.\gamma^4+\right.$$
$$\left.\left\{\frac{1.3.5}{2.4.6}\right\}^2.\gamma^6+\text{\&c.}\right.$$

If in the series, $d\theta+\frac{1}{2}.\gamma^2.\sin.^2\theta.d\theta+\frac{1.3}{2.4}.\gamma^4.\sin.^4\theta.\,d\theta+\frac{1.3.5}{2.4.6}.\gamma^6\sin.^6\theta.\,d\theta+$&c. the integrals being taken as above, between the limits sin. $\theta=0$, sin. $\theta=\pm1$; $\theta=k\pi$, $\theta=\frac{1}{2}(2n+1).\,\pi$, (where k and n are any numbers whatever) will satisfy these conditions; from which indetermination of k and n, it follows, that the vertical coordinate passes through its maximum and minimum an indefinite number of times, and consequently, when all obstacles are removed, the number of oscillations is infinite; we would obtain an expression for the time intervening between the commencement of the motion, and the successive transits through the greatest and least values of z, by taking θ successively $=\frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi$, these quantities differing by π, and as in the preceding integral, the first power of θ only occurs, it is evident that the times of all oscillations are equal.

being the semi-circumference of a circle, of which the radius is unity; we shall thus find

$$T = \pi.\sqrt{\frac{r}{g}}.\sqrt{\frac{2r.(a+b)}{(a+b)^2+r^2-b^2}}.\left\{1+\left(\frac{1}{2}\right)^2.\gamma^2+\left\{\frac{1.3}{2.4}\right\}^2.\gamma^4+\left\{\frac{1.3.5}{2.4.6}\right\}^2.\gamma^6+\&c\right.$$

Supposing the point suspended at the extremity of a thread without mass, of which the other extremity is firmly fixed; if the length of the thread is r, the motion of the point will be the same as in the interior of a spherical surface; it will constitute with the thread a pendulum, of which the cosine of the greatest deviation from the vertical will be $\frac{b}{r}$. If we suppose that in this state, the velocity of the point is nothing;* it will vibrate in a vertical plane, and in this case we shall

* $\frac{z}{r}$ expressing the cosine of the angle which the radius makes with the vertical, when the deviation from the vertical is the greatest, z is then least, and consequently it is equal to b, $\therefore \frac{b}{r}$ is the cosine of the greatest deviation, and as generally $\frac{\sin.^2 A}{2} = \frac{1-\cos. A}{2}$, in this case it is $=$ to $\frac{r-b}{2r}$, $\gamma^2 =$ this quantity, for making $a=r$ in the expression for γ^2, it becomes

$$\frac{r^2-b^2}{2r^2+2rb} = \frac{(r-b)(r+b)}{2r.(r+b)} = \frac{r-b}{2r}.$$

The pendulum described in the text is merely ideal, as every body has weight. However, philosophers have given a rule, by means of which we are able to determine the length of the imaginary pendulum, such as has been described, from the compound pendulum which is isochronous with it. (See No. 31 of this book.)

From the equation $d\varpi.(r^2-z^2)=c'.dt$ it follows that the angular velocity is inversly as the square of the distance; this is universally true, whenever the areas are proportional to the times, for we have then $\varrho^2.d\varpi = c'dt \therefore d\varpi = \frac{c.dt}{\varrho^2}$. See note to No. 6.

From the equation $c+2gz = \frac{dx^2+dy^2+dz^2}{dt} = \frac{ds^2}{dt^2}$, we derive $dt = \frac{ds}{\sqrt{c+2gz}}$, when the velocity $\frac{ds}{dt}$ vanishes before the tangent becomes a second time horizontal, have, $a=r$; $\gamma^2=\frac{r-b}{2r}$. The fraction $\frac{r-b}{2r}$ is the square of the sine of half the greatest angle which the thread makes with the vertical; the entire duration T of an oscillation of the pendulum will therefore be

$$T=\pi.\sqrt{\frac{r}{g}}.\left\{1+\left\{\frac{1}{2}\right\}^2.\left\{\frac{r-b}{2r}\right\}+\left\{\frac{1.3}{2.4}\right\}^2.\left\{\frac{r-b}{2r}\right\}^2+\left\{\frac{1.3.5}{2.4.6}\right\}.\left\{\frac{r-b}{2r}\right\}^3+\&c.\right\}$$

If the oscillation is very small, $\frac{r-b}{2r}$ is a very small fraction, which may be neglected, and then we shall have

$$T=\pi.\sqrt{\frac{r}{g}};$$

therefore the very small oscillations are isochronous, or of the same duration, whatever may be their extent; and by means of this duration, and of the corresponding length of the pendulum, we can easily determine the variations of the intensity of gravity, in different parts of the earth's surface.

Let z be the height through which a body would fall by the action of gravity in the time T; by No. 10 we shall have $2z=g\,T^2$, and consequently $z=\frac{1}{2}\pi.^2 r$; thus we can obtain with the greatest precision, by means of the length of a pendulum which vibrates seconds, the space through which bodies descend by the action of gravity in the first second of their fall. It appears from experiments, very accurately made,

the point describes only a part of a circle of the sphere, but if $\frac{ds}{dt}$ be finite, when the tangent becomes a second time horizontal, then the point describes the entire circumference. These circumstances may be determined by means of the equation

$$c+2gz=\frac{dx^2+dy^2+dz^2}{dt^2}.$$

that the length of the pendulum which vibrates seconds is the same, whatever may be the substances which are made to oscillate. From which it follows that gravity acts equally on all bodies, and that it tends, in the same place, to impress on them the same velocity, in the same time.

* When the oscillations are very small $T = \pi.\sqrt{\frac{r}{g}}$, and if a body vibrated in a cycloid whose length was equal to $2r$, the time of an entire vibration would be equal to $\pi.\sqrt{\frac{r}{g}}$, *whatever be the amplitude* of the arc, for the equation of this curve is $s^2 = az$. (See Lacroix Traite Elementaire, No. 102) $\therefore ds = \sqrt{a}\frac{dz}{\sqrt{z}}$, and $\sqrt{2g(h-z)} = -\frac{ds}{dt}$, ($h$ equal to the value of z when $t=0$) $\therefore dt = \frac{1}{\sqrt{2g}}.\frac{ds}{\sqrt{h-z}} = \frac{1}{2}.\sqrt{\left(\frac{2a}{g}\right)} \times -\frac{dz}{\sqrt{hz-z^2}}$, $\therefore t = \frac{1}{2}.\sqrt{\left(\frac{2a}{g}\right)}.$ arc cos.$\left(\frac{2z-h}{h}\right) + C$, if we take this integral between the limits $z=h$, $z=0$, $\frac{T}{2} = \frac{\pi}{2}.\sqrt{\frac{2a}{g}}$, $\therefore$ if $2a=r$, *i, e*, if the radius of the osculating circle be equal to $2a$, the small oscillations in this circle are equal to the oscillations in the cycloid, and since h does not occur in this integral, the time of describing all arcs of the cycloid are equal, provided one extremity of these arcs be at the *lowest* point.

It appears from the foregoing investigation, that the time of vibration in a cycloidal arc, is the limit to which the time in a circular arc approaches, when the latter becomes indefinitely small. When great accuracy is required, all the terms after the two first in the series expressing the time in a circular arch are rejected, and then the expression for $T = \pi.\sqrt{\frac{r}{g}}\left\{1+\left(\frac{1}{2}\right)^2.\left(\frac{r-b}{2r}\right);\right\}$ from which it appears that the aberration from isochronism varies, as the square of the sine of half the amplitude.

We might determine the time of describing any given arc of a circle, if we knew the coordinates a and b, and also z the coordinate of the extremity of the arc required, for then the angle θ would be determined. We might also, derive a general expression for the time of describing any given arc of a *cycloid*. For if in the initial velocity be such, as would be acquired in falling down a height equal to A, we shall have at any point in the cycloid $v^2 = 2g.(H+h-z)$ consequently $\frac{ds}{dt} = \sqrt{2g(H+h-z)}$ $\therefore dt = \frac{-ds}{\sqrt{2g.(H+h-z)}} =$ (by substituting for ds its value $\sqrt{a.}\left(\frac{dz}{\sqrt{z}}\right)$

12. The isochronism of the oscillations of the pendulum, being only an approximation; it is interesting to know the curve on which a heavy body ought to move, in order to arrive at the point where the motion ceases, in the same time, whatever may be the arc which it shall have described from the lowest point. But to solve this problem in the most general manner, we will suppose, conformably to what has place in nature, that the point moves in a resisting medium. Let s represent the arc described from the lowest point of the curve; z the vertical abscissa reckoned from this point; dt the element of the time, and g the gravity. The retarding force along the arc of the curve will be,

$$\sqrt{\frac{a}{2g}} \times \frac{dz}{\sqrt{z.(H+h-z)}} = \sqrt{\frac{a}{2g}} \frac{\left\{ \frac{dz}{\frac{1}{2}(H+h)} \right\}}{\sqrt{1-\left(\frac{z-\frac{1}{2}H+h}{\frac{1}{2}(H+h)}\right)^2}} =$$

$$\sqrt{\frac{a}{2g}}.d.\text{ arc.}\left\{\cos. = \frac{z-\frac{1}{2}(H+h)}{\frac{1}{2}(H+h)}\right\} \therefore t =$$

$\sqrt{\frac{a}{2g}}.\text{arc. cos.} = \frac{z-\frac{1}{2}(H+h)}{\frac{1}{2}(H+h)} + C$; we determine C by making $t = 0$, and $z = H$, we might deduce from this general expression, the time of describing the whole cycloidal arch; C is equal to $= \sqrt{\frac{a}{2g}}\left\{\text{arc}\cdot\left\{\cos. = \frac{h-H}{h+H}\right.\right.$, $\therefore$ when the initial velocity vanishes $C = 0$, for then H vanishes.

In the preceding investigations the motions are supposed to be performed in a *nonresisting* medium, but this is not essentially necessary, in order that the oscillations should be isochronous in the cycloid, or *nearly* so in the circle. For it is proved in No. 12, that the oscillations of a body moving in a medium, of which the resistance is as the velocity, are isochronous when the curve described is a cycloid, and it has been demonstrated by M. Poisson, that when a body describes a small *circular* arch, in a medium of which the resistance varies as the square of the velocity, or as the two first powers of the velocity, the oscillations are isochronous, the analytical expression indicates that the time of describing the first arc is as much lengthened by the resistance, as the time of describing the ascending arc is diminished, so that the time of the entire vibration remains the same as if the body moved in a vacuo, the amplitude of the arc perpetually lessens; and it may be proved, that if the intervals of time are taken in arithmetic progression, the amplitudes of the arcs described decrease in geometric proportion.

1st, the gravity resolved along the arc ds, which thus becomes equal to $g.\frac{dz}{ds}$; 2dly, the resistance of the medium, which we will express by $\varphi.\left\{\frac{ds}{dt}\right\}$, $\frac{ds}{dt}$ being the velocity of the point, and $\varphi.\left\{\frac{ds}{dt}\right\}$ being any function of this velocity. By No. 7 the differential of this velocity will be equal to $-g.\frac{dz}{ds}-\varphi.\left\{\frac{ds}{dt}\right\}$; therefore, by making dt constant we shall have

$$0=\frac{d^2s}{dt^2}+g.\frac{dz}{ds}+\varphi.\left\{\frac{ds}{dt}\right\}. \qquad (i)$$

Let us suppose that $\varphi.\left\{\frac{ds}{dt}\right\}=m.\frac{ds}{dt}+n.\frac{ds^2}{dt^2}$, and $s=\psi(s')$; denoting by $\psi'(s')$ the differential of $\psi(s')$ divided by ds'; and by $\psi''(s')$ the differential of $\psi'(s')$ divided by ds', we shall have

$$\frac{ds}{dt}=\frac{ds'}{dt}.\ \psi'(s')$$

$$\frac{d^2s}{dt^2}=\frac{d^2s'}{dt^2}.\ \psi'(s')+\frac{ds'^2}{dt^2}.\psi''(s');$$

the equation (i) will become

$$0=\frac{d^2s'}{dt^2}+m\,\frac{ds'}{dt}+\frac{ds'^2}{dt^2}.\left\{\frac{\psi''(s')+n\{\psi'(s')\}^2}{\psi'(s')}\right\}+\frac{g.dz}{ds'\{\psi'(s')\}^2};^{*}(l)$$

* Substituting for $\varphi.(ds)$ its value in the equation (i), it becomes

$$0=\frac{d^2s}{dt^2}+g.\frac{dz}{ds}+m.\frac{ds}{dt}+n.\frac{ds^2}{dt^2};\ \frac{d^2s}{dt^2}=\frac{d^2s'}{dt^2}.\ \psi'(s')+\frac{ds'^2}{dt^2}.\ \psi''(s'),\text{ and as}$$

$$ds=ds'.\psi'(s');\ \frac{ds^2}{dt^2}=\frac{ds'^2}{dt^2}.\ \psi'(s')^2$$

substituting these values for $\frac{ds^2}{dt^2}$, and $\frac{d^2s}{dt^2}$, we shall have

$$0=\frac{d^2s'}{dt^2}.\psi.(s')+\frac{ds'^2}{dt^2}.\psi''.(s')+n.\frac{ds'^2}{dt^2}.\psi'(s')^2+m.\frac{ds'}{dt}.\psi.(s')+\frac{g.dz}{ds'.\psi'.(s')}$$

We make the term multiplied by $\frac{ds^2}{dt^2}$, to disappear by means of the equation

$$0 = \psi''.(s') + [\psi'(s')]^2;$$

which gives by integrating

$$\psi\,[s') = \log.\ \{(h(s'+q)^{\frac{1}{n}}\} = s;$$

h and q being arbitrary quantities. By making s' commence with s we shall have $hq^{\frac{1}{n}} = 1$, and if, for greater simplicity we make, $h = 1$, we shall have $s' = c^{ns} - 1$.*

$\therefore$ dividing all the terms by $\psi'.(s')$ and concinnating we obtain

$$0 = \frac{d^2s'}{dt^2} + m.\frac{ds'}{dt} + \frac{ds'^2}{dt^2}.\left\{\frac{\psi''.(s') + n.(\psi'.(s'))^2}{\psi'.(s')}\right\} + \frac{g.dz}{ds'(.\psi(s'))^2}.$$

* From the value of d^2s which has been already given, we get

$$\psi''.(s') = \frac{d^2s}{ds'^2} - \frac{d^2s'.ds}{ds'^3}$$

$$\therefore 0 = \psi''.(s') + n\psi'.(s')^2 = \frac{d^2s}{ds'^2} - \frac{d^2s'}{ds'^3}.\,ds + n.\frac{ds^2}{ds'^2} = \frac{d^2s}{ds} - \frac{d^2s'}{ds'} + n.ds,$$

and by integrating we obtain, $\log.\ ds - \log.\ ds' + ns = e$ or $\log.\ \frac{ds}{ds'} = e - ns$;

$\therefore \frac{ds}{ds'} = \frac{c^e}{c^{ns}}$, and $ds' = \frac{ds.c^{ns}}{c^e}$, integrating again we shall have $s' + q = \frac{c^{ns-e}}{n}$,

$\therefore \log.(n.(s'+q)) = ns - e$ or dividing both sides by n; $\frac{\log(n(s'+q))}{n} = \left((\log.(n.(s+q)^{\frac{1}{n}})\right)$ $= s - \frac{e}{n}$, and if $\frac{e}{n}$, be made equal to $\frac{h}{n^{\frac{1}{n}}}$ we obtain $\log.((h.(s'+q^n))^{\frac{1}{n}} = s$. If we suppose s' to commence with s, they are $=$ to 0 at the same instant, $\therefore \log.\ h.q^{\frac{1}{n}} = 0$, at this instant, and consequently $h.q^{\frac{1}{n}} = 1$, q must be equal to unity since n is a constant indetermined coefficient, $\therefore \log.\ (s'+1)^{\frac{1}{n}} = s = \psi(s')$, and $s' = c^{ns} - 1$.

c being the number whose hyperbolic logarithm is unity; the differential equation (l) becomes then

$$0 = \frac{d^2s'}{dt^2} + m.\frac{ds'}{dt} + n^2g.\frac{dz}{ds'}.(1+s')^2.$$

By supposing s' very small, we may develope the last term of this equation into a series ascending according to the powers of s' which will be of this form, $ks'+ls'^i+$, &c.; i being greater than unity; the last equation then becomes

$$0 = \frac{d^2s'}{dt^2} + m.\frac{ds'}{dt} + ks' + ls'_i + \text{ \&c.*}$$

This equation multiplied by $c^{\frac{mt}{2}}.(\cos.\gamma t+\sqrt{-1}.\sin.\gamma t)$, and then integrated, becomes (γ being supposed equal to $\sqrt{k-\frac{m^2}{4}}$)

$$c^{\frac{mt}{2}}.\left\{\cos.\gamma t+\sqrt{-1}.\sin.\gamma t\right\}.\left\{\frac{ds'}{dt}+\left(\frac{m}{2}-\gamma.\sqrt{-1}.s'\right\}\right. =$$

$$-l.\int s'^i dt.\, c^{\frac{mt}{2}}\left\{\cos.\gamma t+\sqrt{-1}.\sin.\gamma t.\right\} - \text{\&c.}\dagger$$

* For since $s' = c^{ns}-1$, $\frac{ds}{ds'} = \psi'(s') = \frac{1}{n.c^{ns}} = \frac{1}{n.(1+s')}$ $\therefore \psi'(s')^2 = \frac{1}{n^2.(1+s')^2}$, $\therefore$ the equation (l) becomes $\frac{d^2s'}{dt^2} + m.\frac{ds'}{dt} + n^2g.\frac{dz}{ds'}.(1+s')^2$, when s' is very small the variable part of the last term of this equation may be expanded into a series proceeding according to the ascending powers of s', for substituting in place of s' it becomes $= \frac{dz}{ds}.c^{ns}$, when s is very small s' is also very small, as is evident from the equation $s' = c^{ns}-1$ $\therefore$ $\frac{dz}{ds}$ = the sine of the inclination of the tangent to the horizon is very small, and as all the terms which occur in the expression $\frac{dz}{ds}.(1+s')^2$ are very small it can be developed in a series of the form given in the text.

† $\text{Cos.}\,\gamma t+\sqrt{-1}.\sin.\gamma t = c^{\gamma t\sqrt{-1}}$. See Lacroix Traite Elementaire, No. 164.) $\therefore$ by substituting $c^{\gamma t\sqrt{-1}}$ in place of the circular function, we obtain

By comparing separately the real and imaginary parts, we will have two equations by means of which we can eliminate $\frac{ds'}{dt}$; but it will be

$$c^{(\frac{m}{2}+\gamma\sqrt{-1})t}.\frac{d^2s}{dt^2}+m\,c^{(\frac{m}{2}+\gamma\sqrt{-1})t}.\frac{ds'}{dt}+ks'c^{(\frac{m}{2}\gamma+\sqrt{-1})t}=-l.s'^i.\,c^{(\frac{m}{2}+\gamma\sqrt{-1})t}.$$

&c. If we multiply both sides of this equation by dt, and then partially integrate, we shall have

$$c^{(\frac{m}{2}+\gamma\sqrt{-1})t}.\frac{ds'}{dt}-\left(\frac{m}{2}+\gamma.\sqrt{-1}\right)\int ds'.c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}+ms'.\,c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}$$

$$-m\left(\frac{m}{2}+\gamma.\sqrt{-1}\right).\int s'\,c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}.\,dt.$$

$$+k.\int s'dt.\,c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}=-\,l.\int s'^i dt.c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}\ \text{\&c.}$$

(the integral of $ds'c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}=s'c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}$

$$-\left(\frac{m}{2}+\gamma.\sqrt{-1}\right).\int s'c^{(\frac{m}{2}+\gamma\sqrt{-1})}dt,)$$

substituting this value of $\int ds'.c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}$ in the second term of the preceding integral, and for k its value $\gamma^2+\frac{m^2}{4}$, we obtain

$$c^{(\frac{m}{2}+\gamma.\sqrt{-1}).t}.\frac{ds'}{dt}+\left(\frac{m}{2}-\gamma.\sqrt{-1}\right)\,s'.c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}+\left(\frac{m^2}{4}+m.\gamma.\sqrt{-1}-\gamma^2\right)\times$$

$$\int(s'dt.c^{(\frac{m}{2}+\gamma\sqrt{-1})t}\quad\left(-\frac{m^2}{2}-m\gamma.\sqrt{-1}+\frac{m^2}{4}+\gamma^2\right).\int s'dt.c^{(\frac{m}{2}+\gamma.\sqrt{-1}t}$$

$$=\left(-\frac{m}{4}-m\gamma.\sqrt{-1}.+\gamma^2.\int s'.dt.\,c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}\right)=-\,l.\int s'^i c^{(\frac{m}{2}+\gamma.\sqrt{-1})t}\ \text{\&c.}$$

If we substitute for $c^{\gamma\sqrt{-1}\,t}$ its value cos. $\gamma t+\sqrt{-1}$ sin. γt, and concinnate, we will obtain

$$c^{\frac{mt}{2}}\left(\cos.\gamma t+\sqrt{-1}.\sin.\gamma t\right)\left(\frac{ds'}{dt}+\left(\frac{m}{2}-\gamma.\sqrt{-1}\right).s'\right)=-\,l\int s'^i.\,dt.\,c^{(\frac{mt}{2}}$$

$$(\cos.\gamma t.+\sqrt{-1}.\sin.\gamma t.)\ \text{\&c.}$$

sufficient to consider here the following*

$$c^{\frac{mt}{2}}.\frac{ds'}{dt}.\ \sin.\gamma t + c^{\frac{mt}{2}}.s'.\left\{\frac{m}{2}.\ \sin.\ \gamma t - \gamma.\cos.\gamma t.\right\}$$

$$= -\ l.\textstyle\int s'^{i}\, dt.c^{\frac{mt}{2}}.\sin.\ \gamma t\text{—\&c.}$$

the integrals of the second member being supposed to commence with t. Naming T the value of t at the end of the motion, when $\frac{ds}{dt}$ vanishes, at that instant we shall have

$$c^{\frac{mT}{}}.\ s'.\left\{\frac{m}{2}.\ \sin.\ \gamma T - \gamma.\ \cos.\ \gamma T.\right\} = -\ l.\textstyle\int s'^{i}.\ dt.\ c^{\frac{mt}{2}}.\sin.\gamma t\text{—\&c.}$$

When s' is indefinitely small, the second member of this equation vanishes, when compared with the first, and we shall have ;

$$0 = \frac{m}{2}.\ \sin.\ \gamma T - \gamma.\cos.\ \gamma T,^{*}$$

* As the imaginary parts of this equation cannot be equated with the real parts, the real and imaginary parts must be compared separately, which gives two distinct equations, the part of this equation which is considered, is the part which was multiplied by $\sqrt{-1}$

† Partially integrating the expression $-l.\int s'^{i}.c^{\frac{mt}{2}}.\sin.\ \gamma t.dt +$ &c. we obtain

$$\frac{l.}{\gamma}.\ c^{\frac{mt}{2}}.\cos.\gamma t.\ s'^{i} - \frac{lm}{2\gamma}\textstyle\int.c^{\frac{mt}{2}}.\ dt.\cos.\ \gamma t.\ s'^{i} - \frac{li}{\gamma}\int.c^{\frac{mt}{2}}.\ \cos.\gamma t.\ s'^{i-1}ds',$$

if we integrate the second term of this expression, as before, we shall have

$$-\frac{lm}{2\gamma^{2}}.\ c^{\frac{t\,m}{2}}.\ \sin.\ \gamma t.\ s'^{i} + \frac{lm^{2}}{4\gamma^{2}}\textstyle\int.c^{\frac{mt}{2}}\ dt.\sin.\gamma t.\ s'^{i} + \frac{lmi}{2\gamma^{2}}\int c^{\frac{mt}{2}}.\sin.\ \gamma t.\ s'^{i.-1}\, ds',$$

in like manner the integration of the term in this last expression, which contains dt, would give terms of the same form as in the preceding integral; consequently the value of $-l\int.s'^{i}\ dt.c^{\frac{mt}{2}}.\sin.\ \gamma t +$&c. cannot be exhibited in a finite number of terms; but if the preceding intervals are taken from $t = 0$ to $t = T$, then the value of $-l\int s'.c^{\frac{mt}{2}}.\ dt.$ sin.

consequently

$$\text{tang. } \gamma T = \frac{2\gamma}{m},$$

and as the time T is, by hypothesis independant of the arc described,

$\gamma t = 0$, for by substituting in place of cos. γT its value $\frac{m}{2\gamma}$ sin. γT, in the terms where ds occurs, these terms in two succeeding expressions will be equal, and affected with contrary signs, consequently they destroy each other; with respect to those terms which are free from the sign of integration $\int$, we may remark that they resolve themselves into two decreasing geometric series, which are respectively of the following forms

$$\frac{l}{\gamma}.\cos.\gamma t.c^{\frac{mt}{2}}.s'^{i} - \frac{lm^2}{4\gamma^3}.\cos.\gamma t.c^{\frac{mt}{2}}.s'^{i} + \frac{lm^4}{16.\gamma^5}.\cos.\gamma t.c^{\frac{mt}{2}}.s'^{i}, \text{ \&c. ad infinitum,}$$

$$-\frac{lm}{2\gamma^2}.\sin.\gamma t.c.^{\frac{mt}{2}}.s'_{i} + \frac{lm^3}{8\gamma^4}.\sin.\gamma t.c^{\frac{mt}{2}}.s'^{i} - \frac{lm^5}{32\gamma^6}.\sin.\gamma t.c^{\frac{mt}{2}}.s'^{i} + \text{\&c. ad infinitum,}$$

by summing these series they come out equal respectively to

$$\frac{\frac{l}{\gamma}}{1+\frac{m^2}{4\gamma^2}}.\cos.\gamma t.c^{\frac{mt}{2}}.s'^{i} - \frac{\frac{lm}{2\gamma^2}}{1+\frac{m^2}{4\gamma^2}}.\sin.\gamma t.c^{\frac{mt}{2}}.s'_{i}, \text{ by substituting}$$

for cos. γT its value $\frac{m}{2\gamma}$. sin. γT, the first expression becomes

$$\frac{\frac{lm}{2\gamma^2}}{1+\frac{m^2}{4\gamma^2}}.\sin.\gamma T.c^{\frac{mT}{2}}.s'_{i}$$ which is equal to the second with a contrary sign, consequently

it follows that whatever be the magnitude s';—$l.\int s'_{i}.dt.c^{\frac{mt}{2}}$. sin. $\gamma t = 0$, when the integral is taken from $t=0$ to $t=T$. The same reasoning applies to the other terms of the series. which contain powers of s' superior to i.

l being independant of s', if it is equal to nothing when s' is very small it will be always equal to nothing; and since neither sin. γt, nor $c^{\frac{mt}{2}}$ change their signs from $t=0$, to $t=T$, it is evident that the evanescence of $\int s'^{i}.c^{\frac{mt}{2}}$. sin. γt can only arise from l being equal to nothing, in this case also the coefficients of the powers of s' greater than s'^{i}. i. e. the subseqnent terms of the series vanish.

this value of tang. γT has place for any arc whatever, therefore whatever be the value of s', we have

$$0 = l.\int s'^{i}.\ dt.c^{\frac{mt}{2}}.\ \sin.\ \gamma t + \&c.$$

the integral being taken from $t=0$ to $t=T$. If we suppose s' very small the second member of this equation will be reduced to its first term, and it can only be satisfied by making $l=0$; for the factor $c^{\frac{mt}{2}}$. sin. γt, being constantly positive from $t=0$ to $t=T$, the preceding integral is necessarily positive in this interval. Therefore the tautochronism is only possible on the supposition of

$$n^2.g.\frac{dz}{ds'}.\ (1+s')^2 = ks',^{*}$$

which gives for the equation of the tautochronous curve

$$g.dz = \frac{k.ds}{n}.\ (1-c^{-ns})$$

In a vacuum, and when the resistance is proportional to the velocity, n

* Substituting for $1+s'$ its value c^{ns}, and ds' its value $n.ds.c^{ns}$, we obtain

$$\frac{n^2g.dz.}{n.ds.c^{ns}}\ c^{2ns} = k(c^{ns}-1)\ \because g.dz = \frac{k.ds}{n}\left\{1-c^{-ns}\right\},\ \because$$

when the body moves in a vacuo, or in a medium of which the resistance is proportional to the velocity, $n=o\ \because gdz = \frac{k.ds}{0}\ (1-c^{-0s})\ = ks.\frac{0}{0}$, but if we express c^{-ns} in a series it becomes $= 1-\frac{ns}{1}+\frac{n^2s^2}{1.2}$, &c. $\because$ the general expression for

$$g.dz = \frac{k.ds}{n}\left(1-1+\frac{ns}{1}-\frac{n^2s^2}{1.2}+\frac{n^3s^3}{1.2.3}-\&c.\right)=,$$

when $n=0$, $k.ds.s.$ From this equation it follows that $k = \frac{g.dz}{ds.s}$, this is also $q.p$ true, when n has a finite value, if s be taken very small, as is evident from the preceding series.

is nothing; and this equation becomes $g.dz = ks.ds$; which is the equation of the cycloid.

It is remarkable * that the coefficient n of the part of the resistance, which is proportional to the square of the velocity, does not enter into the expression of the time T; and it is evident from the preceding analysis that this expression will be the same, even though we should add to the expression for the law of the resistance, which has been given above, the terms,

$$p.\frac{ds^3}{dt^3} + q.\frac{ds^4}{dt^4} + \&c.$$

If in general, R represents the retarding force along the curve, we shall have

$$0 = \frac{d^2s}{dt^2} + R.$$

s being a function of t, and of the entire arc described, which consequently, is a function of t and of s. By differentiating this last function, we obtain a differential equation of this form,

$$\frac{ds}{dt} = V.$$

V being a function of t and of s, which, by the conditions of the problem must vanish, when t has a determinate value, which is independant of the whole arc described. Suppose, for example, $V = S.T$, S

* Since the value of T is the same when the terms $P.\frac{ds^3}{dt^3} + q.\frac{ds^4}{dt^4} + \&c.$ are added to $m.\frac{ds}{dt} + n.\frac{ds^2}{dt^2}$, it follows that the generality of the conclusion is not affected by substituting $m.\frac{ds}{dt} + n.\frac{ds^2}{dt^2}$ in place of $\varphi\left\{\frac{ds}{dt}\right\}$.

being a function of s only, and T being a function of t only; we shall have

$$\frac{d^2s}{dt^2} = T.\frac{dS}{ds}\cdot\frac{ds}{dt} + S.\frac{dT}{dt} = \frac{dS}{S.ds}\cdot\frac{ds^2}{dt^2} + S.\frac{dT}{dt};*$$

but the equation $\frac{ds}{dt} = ST$, gives t, and consequently $\frac{dT}{dt}$ equal to a function of $\frac{ds}{S.dt}$, which function we will denote by $\frac{ds^2}{S^2dt^2}.\psi\left\{\frac{ds}{S.dt}\right\}$ therefore we shall have

$$\frac{d^2s}{dt^2} = \frac{ds^2}{S.dt^2}\left\{\frac{dS}{ds} + \psi\left(\frac{ds}{S.dt}\right)\right\} = -R.$$

Such is the expression for the resistance which corresponds to the differential equation $\frac{ds}{dt} = ST$; and it is easy to perceive that it involves the case of the resistance proportional to the two first powers of the velocity, multiplied respectively by constant coefficients. Other differential equations would give different laws of resistance.†

* S being a function of s, which is a function of t, the differential coefficient of S, with respect to $t = \frac{dS}{ds}\cdot\frac{ds}{dt}$, and substituting for T its value $\frac{ds}{S.ds}$ we obtain

$$\frac{d^2s}{dt^2} = \frac{dS}{S.ds}\cdot\frac{ds^2}{dt^2} + S.\frac{dT}{dt}.$$

† In the preceding investigation the body is supposed to ascend from the lowest point, and the curve which then satisfies the condition of tautochronism is *unique* in a given medium; but if the body descended from the highest point, then it would oscillate at the other side of the point where the tangent was horizontal, and the problem becomes somewhat more indeterminate, in this case it may be announced more generally thus; to find the lines, the time of describing which will be given, whatever be the amplitude of the arch described; the discussion of this problem is too long to be inserted here, the reader will find a complete investigation of it by Euler in the Transactions of the Academy of Petersburgh for the years 1764 and 1734, he demonstrates that the arcs at each side of the lowest point are not necessarily equal and similar, however, the sum of these arcs

is proportional to the square root of the vertical coordinate, $\because$ the curve whose length is equal to the sum of these arcs will be the common cycloid, in like manner, if we have the differential equation of one of these arcs, we can determine the differential equation of the other; if the first arc be a cycloid, the second will also be the arc of a cycloid; in this case the time of describing *each* of the cycloidal arcs will be constant, however the generating circle of the second cycloid is not necessarily equal to that of the first. If we combine the condition of tautochronism, with the condition of the two branches at each side of the lowest point, being equal and similar, the curve will be then the vulgar cycloid, therefore this is the only *plane* curve in which the sum of the times of the ascent and descent is always the same in a vacuo; but this property belongs to an indefinite number of curves of double curvature which are formed by applying the cycloid to a vertical cylinder of any base, the altitude of the curve above the horizon remaining the same as before, for $v^2 = \frac{ds^2}{dt^2} = c - 2gz$, $\because dt = \frac{\pm ds}{\sqrt{c-2gz}}$, consequently the value of t depends on the initial velocity, and on the relation between the vertical ordinates and arc of the curve $\because$ whatever changes are made in the curve compatible with the continuity, the value of dt will not be changed, provided the preceding relation remains; and it follows conversely, that the projection of any tautochronous curve of double curvature, on a vertical plane, will be a cycloid with a horizontal base.

In the cycloid, if a body falls freely, the accelerating force along the tangent varies as the distance from the lowest point, for $s^2 = 4az$, $\because g.\frac{dz}{ds}$ (= accelerating force $= \frac{gs}{2a}$,) the pressure arising from gravity $= g.\frac{\sqrt{4a^2 - z^2}}{2a}$, and the pressure which is produced by the centrifugal force $= \frac{2.g.(a-z)}{2.\sqrt{a(a-z)}}$, for radius of curvature $= 2.\sqrt{a(a-z)}$, and the square of the velocity $= 2.g.(a-z)$, see No. 9, (the coordinates of z are reckoned from the lowest point;) it follows from the preceding expression that the *whole* pressure at the lowest point, and consequently the tension at this point of a body vibrating in a cycloid is = to twice the gravity.

When a body describes a cycloid, the accelerating force varies as the distance from the lowest point, as has been stated above; and if a body was solicited by a force varying according to this law, the time of falling to the centre will be given, for we have

$$\frac{dv}{dt} = -As \because \frac{dvv}{dt} = -As.\frac{ds}{dt}, \because v^2 = -As^2 + C, \; v=0, \; s = S, \because C = AS^2, \because$$

$$v = A.^{\frac{1}{2}}\sqrt{S^2 - s^2} \;\&\; A^{\frac{1}{2}}.dt = \frac{ds}{\sqrt{S^2 - s^2}}, \because A^{\frac{1}{2}}t = \text{arc. cos.}\frac{s}{S}, \text{ and when } s=0, \; t=T$$

$= \frac{\pi}{2A^{\frac{1}{2}}}$, consequently the time of descent to the centre, is the same from whatever point, the body begins to fall. From the preceding expression, it follows, that the time of describing any space s, varies as the arc, and the velocity acquired varies as the right sine. See Princip. Mat. Prop. 38, Book 1st.

CHAPTER III.

Of the equilibrium of a system of bodies.

13. The simplest case of the equilibrium of several bodies, is that of two material points meeting each other with equal and directly contrary velocities; their mutual impenetrability evidently annihilates their motion, and reduces them to a state of rest.

Let us now suppose a number m of contiguous material points, arranged in a right line, and moving in its direction with the velocity u, and also another number m' of contiguous points, disposed in the same line, and moving with the velocity u', directly contrary to u, so that the two systems may strike each other; there must exist a certain relation between u and u', when both the systems remain at rest after the shock.

In order to determine this condition, it may be observed that the system m, moving with the velocity u, will constitute an equilibrium with a single material point, moving in a contrary direction with the velocity mu; for every point of the system would destroy in this last point, a velocity equal to u, and consequently the m points would destroy the whole velocity mu; we may therefore substitute for this system a single point, moving with the velocity mu. In like manner we may substitute for the system m', a single point moving with the velocity $m'u'$; now* the two systems being supposed to constitute an equilibrium, the two points which are substituted in their place, ought to be also in equilibrio, therefore their velocities must be equal; consequently we

* These two systems of contiguous material points, may be supposed to represent two bodies M, M', of different masses, equal respectively to the sum of all the *ms*, and m',*s*.

have for the condition of the equilibrium of the two systems, $mu = m'u'$.

The mass of a body is the number of its material points, and the product of the mass by the velocity, is what is termed its *quantity of motion*; this is also what we understand by the force of a body in motion. In order that the two bodies, or two systems of points which strike each in contrary directions, may be in equilibrio, the quantities of motion or the opposite forces must be equal, and consequently the velocities must be inversely as the masses.

The density of bodies depends on the number of material points which they contain in a given volume. In order to determine their absolute density, we should compare their masses with that of a body † which has no pores; but as we know no such body, we can only determine the relative density of bodies, that is to say, the ratio of their density, to that of a given substance. It is evident that the mass is in the ratio of the volume and density; therefore, if we denote the mass of the body by M, its volume by U, and its density by D, we shall have generally $M = U.D$; in this equation the quantities M, D, U, relate to the units of their respective species.

In what precedes, we suppose that bodies are composed of similar material points, and that they only differ in the relative situation of these points. But the intimate nature of matter being unknown, this supposition is at least very precarious, and it is possible that there may be essential differences‡ in their integrant molecules. Fortunately, the truth of this hypothesis is of no consequence to the science of mechanics, and we may adopt it without any apprehension of

† Distilled water, at its greatest density, is the substance which has been selected for the term of comparison, as being one of the most homogeneous substances, and that which may be readily reduced to a pure state.

‡ By the integrant molecules of bodies, as contradistinguished from their constituent parts, we understand those which arise from the subdivision of the body, into minuter portions; by the constituent parts are understood the elementary substances of which a body is composed.

error, provided that *by similar material points,* we understand points which, when they meet with equal and opposite velocities, mutually constitute equilibrium, whatever their nature may be.*

14. Two material points, of which the masses are m and m', can only act on each other in the direction of the line joining them. Indeed, if the two points are connected by a thread passing over a fixed pully, their reciprocal action cannot be directed along this line ; but the fixed pully may be considered as having at its centre a mass of infinite density, which reacts on the two bodies, so that their mutual action may be considered as indirect.

Let p denote the action which is exerted by m on m' by means of the right line which joins them, which line we suppose to be inflexible and without mass. Conceive this line to be actuated by two equal and opposite forces p and $-p$; the force $-p$ will destroy in the body m a force equal to p, and the force p of the right line will be communicated entirely to the body m'. This loss of force in m, occasioned by its action on m', is termed the *reaction* of m' ; therefore in the communication of motions, *the reaction is always equal and contrary to the action.* It appears from observation that this principle obtains for all the forces of nature.†

* If there be actually *essential* differences in the integrant molecules, then it is not inconsistent to suppose, with some philosophers, that the planetary regions are filled with a very subtle fluid destitute of pores, and of such a nature as not to oppose any resistance to the motions of the planets. We can thus reconcile the permanency of these motions, which is evinced by observation, with the opinion of those philosophers who regard a vacuum as an impossibility ; however the plenum, for which De-Cartes contended, is not confirmed by the preceding hypothesis, as he held that all matter was homogeneous, and that the ether, which, according to him filled the planetary regions, differed from other substances only in *the form of the matter.* See Princip. Math. Book 2, Prop. 40 ; Exper. 14, and Book 3, Prop. 6, Cor. 2 and 3 ; Newton's Optics, Query 18 ; and Systeme de Monde, page 166. However, as extension and motion are the only properties which are taken into account in Mechanics, it is indifferent whether matter be considered as homogeneous or not.

† This equality does not suppose any particular force inherent in matter, it follows necessarily from this, that a body cannot be moved by another body, without depriving this body of the quantity of motion which is acquired by the first body, in the same manner as when two vessels communicate with each other, one cannot be filled but at the expense of the other.

Let us suppose two heavy bodies m and m' attached to the extremities of an horizontal right line, supposed to be inflexible and without mass, which can turn freely about a point assumed in this right line. In order to conceive the action of those bodies on each other, when they are in equilibrio, we must suppose the right line to be bent by an indefinitely small quantity at the assumed point, so as to be formed of two right lines, constituting at this point an angle, which differs from two right angles by an indefinitely small quantity ω. Let f and f' represent the distances of m and m' from the fixed point; if we resolve the weight of m into two forces, one acting on the fixed point, and the other directed towards m', this last force will be represented by $\frac{mg.(f+f')}{\omega f'}$*, g being the force of gravity. In like manner the action of m' on m will be represented by $\frac{m'g.(f+f')}{\omega f}$, the two bodies constituting an equilibrium, these two expressions will be equal, consequently we will have $mf=m'f'$; this gives the known law of the equilibrium of the lever, and at the same time, enables us to conceive the reciprocal action of parallel forces.

Let us now consider the equilibrium of a system of points actuated by any forces whatever, and reacting on each other. Let f represent the distance of m from m'; f' the distance of m from m'', f'' the distance of m' from m'', &c.

* Gravity must be distinguished from weight; the weight of a body is the product of the gravity of a single particle, by the number of particles.

If we conceive a line drawn from the fixed point, parallel to the direction of gravity, meeting a line connecting m and m', this last line will be $q.p.$ horizontal, and therefore perpendicular to the vertical line, which will ∵ be equal to f multiplied into the sine of the angle which f makes with the horizontal line, but as the sides are as the sines of the opposite angles, we have the sine of the angle which f makes with the horizontal line, to the sine of ω, or its supplement, as $f':f+f'$ ∴ it is equal to $\frac{f'.\sin.\omega}{f+f'}=q.p.\frac{f'\omega}{f+f'}$, now if the weight be represented by the vertical line, then mg divided by sine of the angle which f makes with the horizontal line, *i. e.* $\frac{m.g.(f+f')}{\omega f'}$ will be the force in the direction of f.

also let p be the reciprocal action of m on m'; p' that of m on m''; p'' that of m' on m'', &c. and lastly, let mS, $m'S'$, $m''S''$, be the forces which act on m, m', m''; &c. s, s', s'', lines drawn from any fixed points in the direction of these forces, to the bodies m, m', m'', &c.; this being premised, we may consider the point m as perfectly free, and in equilibrio in consequence of the action of the force mS, and of the forces, which the bodies m, m', m'', communicate to it; but if it was subjected to move on a curve or on a surface, it would be necessary to add to these forces, the reaction of the curve or of the surface. Therefore, let δs be the variation of s, and let $\delta_{,} f$, denote the variation of f, taken on the supposition that m' is fixed. In like manner let $\delta_{,} f'$, be the variation of f', on the supposition that m'' is fixed, &c. Let R, R', represent the reactions of the two surfaces, which form by their intersection the curve on which the point is constrained to move, and let δr, $\delta r'$ be the variations of the directions of these last forces. The equation (d) of No. 3, will give:

$$0 = mS.\delta s + p.\delta_{,} f + p'.\delta_{,} f' + \&c. + R\delta r + R'\delta r' + \&c.$$

In the same manner m' may be considered as a point perfectly free, retained in equilibrio by means of the force $m'S'$, of the actions of the bodies m, m', m'', and of the reactions of the surfaces on which m' is constrained to move, which reactions we will denote by R'', and R'''. Let, therefore, the variation of s' be called $\delta s'$, and the variations of f, and f'', taken on the supposition that m and m'' are fixed, be respectively $\delta_{,,} f$, $\delta_{,} f''$; in like manner, let $\delta r''$, $\delta r'''$, be the respective variations of the directions of R'', R''', and we shall have for the equilibrium of m'

$$0 = m'S'.\delta s' + p.\delta_{,,} f + p''.\delta_{,} f'' + \&c. + R''.\delta r'' + R'''.\delta r'''.$$

If we form similar equations relative to the equilibrium of m'', and m''', &c. by adding them together, and observing that $\delta f = \delta_{,} f + \delta_{,,} f$; $\delta f' = \delta_{,} f' + \delta_{,,} f'$; * &c. δf, and $\delta f'$, being the total

* $\delta f = \delta_{,} f + \delta_{,,} f$; $\delta f' = \delta_{,} f' + \delta_{,,} f'$ + &c.; for f and f' are respectively functions of the coordinates of their extreme points, and when these are moved by an indefinitely small quantity, all the powers of the increments of the coordinates, after the first may be rejected, and then the entire increment of f is equal to the sum of the partial increments

variations of f and $f'+$ &c. we shall have

$$0=\Sigma.m.S.\delta s+\Sigma p.\delta f+\Sigma R.\delta r\,; \qquad (k)$$

in this equation, the variations of the coordinates of the different points of the system are entirely arbitrary. It should be observed here, that in consequence of the equation (a) of No. 2, we may substitute in place of $mS.\delta s$, the sum of the products of all the partial forces by which m is actuated, multiplied by the respective variations of their directions. The same may be observed of the products $m'S\delta s'$; $m''.S''.\delta s''$; $+$ &c.*

If the distances of the bodies from each other be invariable, *i. e.* if $f, f', f'', +$ &c. are constant, this condition may be expressed by making $\delta f=0$, $\delta f'=0$, &c. The variations of the coordinates in the equation (k) being arbitrary, they may be subjected to satisfy these last equations, and then the forces p, p', p'', &c. which depend on the reciprocal action of the bodies composing the system, will disappear from this equation; we can also make the terms $R.\delta r$, $R'.\delta r'$. $+$ &c. † to disappear, by subjecting the variations of the coordinates to satisfy the equations of the surfaces, on which the body is constrained to move. The equation (k) will then become

$$0=\Sigma.mS.\delta s\,; \qquad (l)$$

from which it follows that in case of equilibrium, the sum of the varia-

which are due to the separate variation of each coordinate, $\because$ the entire variation of f is equal to the sum of the partial variations, which correspond to the characteristics $\delta_{,}$ and $\delta_{,,}$.

* From this it appears, that the conditions of the equilibrium of a system of bodies, may be always determined by the law of the composition of forces; for we can conceive the force by which each point is actuated to be applied to the point in its direction, where all the forces concurring, constitute an equilibrium when the point is entirely free, or which constitute a resultant, which is destroyed by the fixed points of the system, when the point is not altogether free.

† See Notes to No. 3.

The equation (l) obtains, whether the points are all free, or are subjected to move on tions of the products of the forces, into the elementary variations of their directions will be equal to nothing, whatever changes be made in the position of the system compatible with the conditions of the connection of the parts of the system.

We have arrived at this theorem, on the particular supposition of the parts of the system being at invariable distances from each other; however it is true whatever may be the conditions of the connection of the parts of the system. In order to prove this, it will be sufficient to shew that when the variations, of the coordinates, are subjected to those conditions, we have in the equation (k)

$$0 = \Sigma.p.\delta f + \Sigma.R.\delta r\,;$$

but it is evident that δr, $\delta r'$, &c. are equal to nothing, when these conditions are satisfied; therefore it is only necessary to prove that in the same circumstances we have

$$0 = \Sigma.p.\delta f.$$

Let us therefore suppose the system actuated by the sole forces p, p', p, &c. and that the bodies are subjected to move on the curves, which they can describe in consequence of the same conditions; these forces may be resolved into others, some of which q, q', q'', &c. acting in the direction of f, f', f'', &c. will mutually destroy each other, without producing any action on the curves described; others will be perpendicular to those curves; and others again will act in the direction of tangents to those curves, by the action of which the bodies may be moved; but it is easy to perceive that the sum of these last forces ought to be equal to nothing; since the system being by hypothesis at liberty to move in their directions, they are not able to produce either pressure on the curves described, or reaction between the bodies;

curved surfaces; in the former case, the forces S, S', S'', constitute an equilibrium; in the latter case, these forces have a resultant, of which the direction is perpendicular to the surface. (See Note to page 17.)

consequently they cannot constitute an equilibrium with the forces $-p, -p', -p''$, &c. q, q', q'', &c. T, T', T''; therefore they must vanish, and the system must be in equilibrio in consequence of the sole forces $p, -p', -p''$, &c.; q, q', q'', &c.; T, T', &c. Now, if $\delta i, \delta i'$, &c. represent the variations of the directions of the forces T, T', &c. we shall have in consequence of the equation (k)

$$0 = \Sigma.(q-p).\delta f + \Sigma.T.\delta i;$$

but the system being supposed to be at rest, in consequence of the sole action of the forces q, q', &c. without any action being produced on the curves described, the equation (k) gives us also $0 = \Sigma.q.\delta f$;* consequently we have

$$0 = \Sigma.p.\delta f - \Sigma.T.\delta i;$$

but as the variations of the coordinates are subjected to satisfy the conditions of the curves described, we have $\delta i, = 0, \delta i', = 0$, &c.; therefore the preceding equation becomes

$$0 = \Sigma.p.\delta f;\dagger$$

as the curves described are themselves arbitrary, and are only subjected to the conditions of the connection of the system, the preceding equation obtains, provided that we satisfy these conditions, and then the equation (k) will be changed into the equation (l). The following principle, known by the name of the principle of virtual velocities, when analytically expressed, is represented by this equation. It is thus an-

* $0 = \Sigma\, q.\delta f$, for q, q', q'', are directed along the lines f, f', f''; and are supposed to *destroy* each other without producing *any* action on the curves described.

† The object of the second part of this demonstration is to shew, that if the system is at rest, and acted on by the sole forces p, p', p'', these forces may be so decomposed as to afford forces equivalent to the reciprocal actions of the respective bodies, and that the remaining portions of the forces, as well as these reciprocal actions, will balance each other, in case of equilibrium, according to the terms of the proposition.

Since the equation (k) is reduced to the equation (l), when we subject the variations of the coordinates to satisfy the equations of the surfaces, on which the bodies are constrained to move, it follows that it is not necessary to compute the forces p, p', &c. in order to derive the equations of equilibrium in each particular case.

nounced: "If we make an indefinitely small variation in the position* of a system of bodies, which are subjected to the conditions they ought to fulfil, the sum of the forces which solicit it, multiplied respectively by the space that the body to which it is applied, moves along its direction, should be equal to nothing in the case of the equilibrium of the system."

This principle not only obtains in the case of equilibrium, but it also insures its existence. Let us suppose, in fact, that whilst the equation (l) obtains, the points m, m', &c. acquire the velocities v, v', in consequence of the action of the forces mS, $m'S'$, which are applied to them. The system will be in equilibrio in consequence of the action of these forces, and of $-mv$, $-m'v'$, &c.; denoting by δv, $\delta v'$, &c. the variations of the directions of these new forces, we shall have in consequence of the principle of virtual velocities

$$0 = \Sigma.mS.\delta s - \Sigma.mv.\delta v,$$

but by hypothesis $\Sigma.mS.\delta s. = 0$, therefore we have $0 = \Sigma.mv.\delta v$. We may suppose the variations δv, $\delta v'$, &c. equal to $v.dt$, $v'dt$, &c. since they are necessarily subjected to the conditions of the system, and then we have $0 = \Sigma.mv^2$, and consequently $v = 0$, $v' = 0$, &c. that is to say, the system is in equilibrio in consequence of the sole forces mS, m',S', &c.

The conditions of the connection of the parts of the system may be always reduced to equations between the coordinates of the several bodies. Let $u = 0$, $u' = 0$, &c. be these different equations, by No. 3, we can add to the equation (l), the function $\lambda\delta u$, $\lambda'\delta u'$, &c. or $\Sigma\lambda\delta u$; λ, λ', being indeterminate functions of the coordinates of the bodies, the

M 2

* When an indefinitely small change is made in the position of the system, so that the conditions of the connections of the points of the system may be preserved, each point advances in the direction of the force which solicits it by a quantity equal to a part of this direction, contained between the first position of this point, and a perpendicular demitted from the second position on this direction; these indefinitely small lines are termed the virtual velocities; they have been denominated vertual, because the system being in equilibrio, these changes may obtain without the equilibrium being disturbed.

equation will then become

$$0 = \Sigma.mS.\delta s + \Sigma.\lambda\delta u\,;^{*}$$

in this case the variations of all the coordinates are arbitrary, and we may equal their coefficients to nothing; which will give as many equations, by means of which we can determine the functions λ, λ'. If we compare this equation with the equation (k) we shall have

$$\Sigma.\lambda.\delta u = \Sigma.p.\delta f + \Sigma.R.\delta r\,;$$

by means of which we can easily determine the reciprocal actions of the bodies m, m', &c. on each other, and also the forces $-R, -R'$, with which they press against the surfaces on which they are constrained to move.

15. If all the bodies of the system are firmly united to each other, its position will be determined by that of three of its points which are not in the same right line; the position of each of these points depends on three coordinates; this produces nine indeterminate quantities; but we can reduce them to six others, because the mutual distances of the three points are given and invariable; these being substituted in the equation (l), will introduce six arbitrary variations; by supposing their coefficients to vanish, we shall obtain six equations, which will contain all the conditions of the equilibrium of the system: let us proceed to develope these equations.†

* By means of the formulæ which are given in the notes to No. 3, page 14 and 15, we can determine λ, λ', &c. when S, S', S'', are given for each individual point; and therefore p, p', p'', k, k', k'', by means of the equation $\Sigma.\lambda.\delta u = \Sigma.p.\delta f + \Sigma.R.\delta r$; in the equation $\Sigma\, m.S\delta s + \Sigma.\lambda.\delta u$, m, m', m'', &c. may be considered as entirely free; and if we put the coefficients of the variation of each variable equal to nothing, and then eliminate the indeterminate quantities, λ, λ', λ'', &c. between these equations, the expressions which result, will give the relations which must exist bstween S, S', S'', &c. and the coordinates, when the system is in equilibrio.

† It follows immediately, from the demonstration of the principle of virtual velocities, that it has place for all the indefinitely small motions which can be given to a solid body, which is either free or constrained to certain conditions, for in all these motions the respective distances of the points of the body remain the same.

For this purpose, let x, y, z, be the coordinates of m; x', y', z', those of m'; x'', y'', z'', those of m''; &c.; we shall have then

$$f = \sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$$

$$f' = \sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}$$

$$f'' = \sqrt{(x''-x')^2+(y''-y')^2+(z''-z')^2} \quad \text{\&c.}$$

and if we suppose

$$\delta x = \delta x' = \delta x'' = \text{\&c.}$$

$$\delta y = \delta y' = \delta y'' = \text{\&c.};$$

$$\delta z = \delta z' = \delta z'' = \text{\&c.};$$

we shall have $\delta f = 0$, $\delta f' = 0$, $\delta f'' = 0$, &c.;* the required conditions will therefore be satisfied, and from the equation (l) we may infer

$$0 = \Sigma.m.S.\left\{\frac{\delta s}{\delta x}\right\}; \quad 0 = \Sigma.mS.\left\{\frac{\delta s}{\delta y}\right\}; \quad 0 = \Sigma.m.S.\left\{\frac{\delta s}{\delta z}\right\}; \quad (m)$$

we have thus obtained three of the six equations, which contain the conditions of the equilbrium of the system. The second members of these equations are the sum of the forces of the system, resolved parallel to the three axes of x, y, and z, therefore each of these sums must vanish in the case of equilibrium.

And as the number of the equations of equilibrium, which are derived from the principle of virtual velocities, is always equal to the number of possible motions, this number being equal to six, in the case of a solid body, or of a body whose parts are invariably connected, the number of equations of equilibrium will be six in like manner.

$$* \ \delta f = \frac{(x'-x).(\delta' x-\delta x)+(y'-y)(\delta y'-\delta y)+(z'-z)(\delta z'-\delta z)}{f} + \text{\&c.}$$

consequently when $\delta x'=\delta x$, $\delta y'=\delta y$, $\delta z'=\delta z$, &c. $\delta f=0$, therefore $\Sigma m.S.\left\{\frac{\delta s}{\delta x}\right\}=0$, $\Sigma m.S.\left\{\frac{\delta s}{\delta y}\right\} = 0$. &c.; for when $\delta x=\delta x'=\delta x''$; $\delta y=\delta y'=\delta y''$; $\delta z = \delta z' = \delta z'' =$ &c.;

The equations $\delta f, =0$, $\delta f' = 0$, $\delta f'' = 0$, &c. will be also satisfied, if we suppose, z, z', z'', invariable, and then make

$$\delta x = y\delta\omega\,; \qquad \delta y = -x.\delta\omega\,;$$

$$\delta x' = y'.\delta\omega, \text{ \&c.} \qquad \delta y' = -x'.\delta\omega, \text{ \&c.}$$

$\delta\omega$ being any variation whatever. By substituting these values in the equation (l), we shall have

$$*0 = \Sigma.mS.\left\{y.\left(\frac{\delta s}{\delta x}\right) - x.\left(\frac{\delta s}{\delta y}\right)\right\}.$$

It is evident that we may, in this equation, change either the coordinates x, x', x'', &c. or the coordinates y, y', y'', &c, into z, z', z'', which will give two other equations, and these reunited with the preceding equation, will constitute the following system of equations :

$$0 = \Sigma mS.\left\{y.\left(\frac{\delta s}{\delta x}\right) - x\left(\frac{\delta s}{\delta y}\right)\right\};$$

$$0 = \Sigma mS.\left\{z.\left(\frac{\delta s}{\delta x}\right) - x\left(\frac{\delta s}{\delta z}\right)\right\}; \quad (n)$$

$$0 = \Sigma mS.\left\{y.\left(\frac{\delta s}{\delta z}\right) - z.\left(\frac{\delta s}{\delta y}\right)\right\};$$

$\Sigma m.S.\delta s = 0$, is equivalent to $\Sigma mS.\left\{\frac{\delta s}{\delta x}\right\}.\delta x = 0$, $\Sigma mS.\left\{\frac{\delta s}{\delta y}\right\}.\delta y = 0$, $\Sigma.m.S.\left\{\frac{\delta s}{\delta z}\right\}.\delta z = 0$. See Note to No. 2, page 9.

* In like manner, if we suppose, $\delta x = y.\delta\omega$, $\delta x' = y'.\delta\omega$, $\delta y = -x\delta\omega$, $\delta y' = -x'\delta\omega$, δf, $\delta f'$, &c. $= 0$, for substituting in the preceding expression for δf, which has been given, for δx, $\delta x'$, δy, $\delta y'$, and it becomes

$$= \frac{(x'-x).(y'-y) + (y'-y).(x-x')}{f} + \text{\&c.} = 0, \ \delta z = 0.$$

By substituting in the equation, $\Sigma m.S\delta s = 0$, for δx, δy, &c. their values it becomes $\Sigma m.S.\left\{y.\frac{\delta y}{\delta x}\right\} - x\left\{\frac{\delta s}{\delta y}\right\}\delta\omega = 0.$

When all the forces are applied to the same point, the three first equations suffice for the equilibrium; but when these forces act in different points of space, or when they are by No. 3, the function $\Sigma mSy.\left\{\frac{\delta s}{\delta x}\right\}$ is the sum of the moments of all the forces, parallel to the axes of x, which would cause the system to revolve about the axis of z. In like manner, the function $\Sigma m.Sx.\left\{\frac{\delta s}{\delta y}\right\}$ is the sum of the moments of all the forces parallel to the axes of y, which would cause the system to revolve round the axis of z, but in a direction contrary to that of the former forces; therefore the first of the equations (n) indicates that the sum of the moments of the forces is nothing with respect to the axis of z. The second and third equations indicate, in a similar manner, that the sum of the moments of the forces is nothing with respect to the axes of y and x, respectively. If we combine these three conditions with those, in which the sum of the forces parallel to those axes, was nothing with respect to each of them; we shall have the six conditions of the equilibrium of a system of bodies invariably connected together.*

If the origin of the coordinates is fixed, and firmly attached to the system, it will destroy the forces parallel to the three axes, and the conditions of the equilibrium of the system about this origin, will be reduced to the following, that the sum of the moments of the forces which would make it turn about the three axes, be equal to nothing, with respect to each of them.†

applied to different parts of the same solid body, it is also requisite that the moments of the forces with respect to axis of x, y, and z, should be respectively equal to nothing.

* If all the points exist in the plane of x, y, then δz, $\delta z'$, $\delta z''$, are equal respectively to nothing, consequently the equations of equilibrium are reduced to the three following:

$$\Sigma.m\,S.\left\{\frac{\delta s}{\delta x}\right\}=0,\quad \Sigma.m.S.\left\{\frac{\delta s}{\delta y}\right\}=0,\quad \Sigma.m.S.\left\{y.\left(\frac{\delta s}{\delta x}\right)\right\}-x.\left(\frac{\delta s}{\delta y}\right)\Big\}$$

† When the origin of the coordinates is fixed and invariably attached to the system, the number of possible motions is reduced to three, therefore the number of equations of equilibrium will be three; this also appears from considering that the number of indeterminate quantities may be reduced to three, because the distances of any three assumed points in the system, not existing in the same right line, from the fixed origin of the coordinates, are given.

† In *this* case, the resultant of all the forces which act on the body passes through the fixed point, which resultant is therefore destroyed by the resistance of the fixed point, and it expresses the force with which this point is pressed. (See last note to No. 3.) When there are two points of the system fixed and invariable, then the only possible motion, which can be impressed on the body, is that of rotation, about the line joining the given points, consequently if this line be taken for the axis of z, there will be but one equation of equilibrium, *i. e.* $\Sigma.mS.\left\{ y.\left(\frac{\delta s}{\delta x}\right) - x.\left(\frac{\delta s}{\delta y}\right)\right\} = 0$, this is also manifest from the circumstances of the indeterminate quantities, which were six in number when there was no fixed point, being reducible to one, when the origin of the coordinates, and also another point of the system, were fixed and invariable. The forces parallel to the axes of z cannot produce any motion in the system, ∵ it is only necessary to consider those which exist in the plane of x, y; and as to those, it is evident, from the equation $\Sigma m.S.\left\{ y.\left(\frac{\delta s}{\delta x}\right) - x.\left(\frac{\delta s}{\delta y}\right)\right\} = 0$; that their resultant passes through the origin of the coordinates, its direction will be perpendicular to the axis of z, and its intensity will express the force with which it presses on this axis. When the number of fixed points is three, there is evidently no equation of equilibrium.

If the forces S, S', S'', &c. do not constitute an equilibrium, in order to reduce them to the least possible number, we should resolve them into three systems of forces, parallel respectively to the axes of x, of y, and of z, then reducing the forces parallel to the axes of x, and y, to forces $=$ to them respectively, but acting in the same plane, which is always possible, if this last system of forces, and also the forces parallel to the axis of z, have separately unique resultants; and if these resultants exist in the same plane, we can compose them into one sole force, which will be the resultant of the given forces; but if the forces directed in the plane x, y, can only be reduced to two parallel forces, not reducible into one, then if we combine them with the force parallel to the axis of z, the entire system of forces, will be reduced to two parallel ones acting in different planes, consequently irreducible into a unique force. Denoting

$\Sigma.m.S.\left\{\frac{\delta s}{\delta x}\right\}$; $\Sigma.m.S.\left\{\frac{\delta s}{\delta y}\right\}$; $\Sigma.m.S.\left\{\frac{\delta s}{\delta z}\right\}$, by P, Q, R; respectively, and $\Sigma.m.S.\left\{ y.\left\{\frac{\delta s}{\delta x}\right\} - x.\left\{\frac{\delta s}{\delta y}\right\}\right\}$; $\Sigma.m.S.\left\{ z.\left\{\frac{\delta s}{\delta x}\right\} - x.\left\{\frac{\delta s}{\delta z}\right\}\right\}$; $\Sigma.m.S.\left\{ y.\left\{\frac{\delta s}{\delta x}\right\} - z.\left\{\frac{\delta s}{\delta y}\right\}\right\}$, by L, M, N; if $x_{\prime}$, $y_{\prime}$, be the coordinates of that point in which the resultant of all the forces meets the plane of the axes of x, y, we shall have by the last note to No. 3, $P.y_{\prime}, -Q.x_{\prime} = L$; $R.x_{\prime} = M$; $-Q.y_{\prime} = N$; therefore $x_{\prime} = \frac{M}{R}$; $y_{\prime} = -\frac{N}{R}$, substituting these expressions for $x_{\prime}$ and $y_{\prime}$ in the equation $P.y_{\prime} - Q.x_{\prime} = L$, we will obtain the equation $L.R+M.Q+N.P=0$, which may be considered as an equation of condition which must be satisfied, when the forces which act on the different points of the system, have an unique resultant. We must however except the case where P, Q, R, are respectively equal to nothing; for then the forces are reducible to two parallel forces $=$, but not *directly* opposed to each other. If only P, and Q vanish, then in order that the preceding equation may be satisfied, it is necessary that L should vanish, consequently since P, Q, and L vanish, the forces which are directed in the plane, x, y, constitute an equilibrium, $\because$ the unique resultant of the forces S, S', S'', &c. must be the same with the resultant R, of the forces parallel to the axes of z, $\because$ we conclude that if L does not vanish when P and Q vanish, the forces have not an unique resultant, since the forces in the plane of x, y, are in this case evidently irreducible to one sole force; if however only one of the three sums P, Q, R, vanish, then the forces in the plane x, y, and those parallel to the axes of z, would have respectively unique resultants, consequently the preceding equation of condition would apply to this case.

When the forces have an unique resultant, it is very easy to determine its position with respect to the coordinates, for if we denote this resultant by V, we shall have $V^2=P^2+Q^2+R^2$, and $\frac{P}{V}$, $\frac{Q}{V}$, $\frac{R}{V}$, $=$ the cosines of the angles which V makes with the axes of x, y, and z, respectively, and $\frac{M}{R}$, $-\frac{N}{R}$, are the distances of the intersection of V with the plane of x, y, from the axes of x and y, respectively.

Supposing the system to revolve round the axis of z, the elementary variations of x and y, &c. are $=$ respectively to $y\delta\omega$, $-x\delta\omega$; if y be made the axis of rotation, and $\delta\varphi$ the variation of the angle, then we shall have $\delta x=-z.\delta\varphi$;$\delta z=+x.\delta\varphi$; in like manner, x being the axis of rotation, and $\delta\psi$ the corresponding variation of the angle, $\delta y=+z.\delta\psi$; $\delta z=-y.\delta\psi$; &c.; now if the three rotations be supposed to take place together, we shall have the entire variation of $x=y.\delta\omega-z.\delta\varphi$, of $y=z.\delta\psi-x.\delta\omega$ of $z=x.\delta\varphi-y.\delta\psi$, and similar expressions may be derived for the variations of x', y', z', x'', &c.; now if we substitute these values for δx, δy,+ &c. in the equation *(l)*, we shall have the equation $L.\delta\varphi+M.\delta\psi, N.\delta\omega=0$, L, M, N, indicating the same quantities as before; this equation is evidently equivalent to the equation (n); when the coordinates x, y, z, of any point of the system are proportional to the elementary variations $\delta\psi$, $\delta\varphi$, $\delta\omega$, $z\,\delta\varphi=y.\delta\omega$, $z.\delta\psi=x\delta\omega$, $x\delta\varphi=y\delta\psi$.

And consequently $\delta x=0$, $\delta y=0$, $\delta z=0$; $\because$ this point and all others which have the same property are immoveable, during the instant the point describes the angles $\delta\varphi$, $\delta\psi$, $\delta\omega$, by turning round the axes of x, y, and z; all points possessing this property exist in a right line passing through the origin of the coordinates, see No. 28, as the cosines of the angles m, n, l, which this line make with the axes of x, y, and z, are

$$\frac{x}{\sqrt{x^2+y^2+z^2}}=\text{in this case }\frac{\frac{z.\delta\psi}{\delta\omega}}{\sqrt{\frac{z^2.\delta\psi^2}{\delta\omega^2}+\frac{z^2.\delta\varphi^2.}{\delta\omega^2}+z^2.}}$$

$$=\left\{\frac{d\psi}{\sqrt{\delta\varphi^2+\delta\psi^2+\delta\omega)^2}}\right\};\ \frac{y}{\sqrt{x^2+y^2+z^2}}=\left\{\frac{\delta\varphi}{\sqrt{\delta\psi^2+\delta\omega^2+\delta\varphi^2}}\right\};$$

Let us suppose that the bodies m, m', m'', are subject to the sole force

$$\frac{z}{\sqrt{x^2+y^2+z^2}}=\left\{\frac{\partial\psi}{\sqrt{(\partial\psi^2+\partial\omega^2+\partial\varphi^2)}}\right\};$$

$\because$ the right line which makes with the axes, angles whose cosines are equal to those expressions, is the locus of all the points, which are quiescent during the instantaneous rotation of the system. Making $\partial\theta=\sqrt{\partial\psi^2+\partial\varphi^2+\partial\omega^2}$, we obtain $\partial\psi=\partial\theta.\cos. m$; $\partial\varphi=\partial\theta.\cos. n$; $\partial\omega=\partial\theta.\cos. l$; consequently $\partial x=(y.\cos. l-z.\cos. n).\partial\theta$; $\partial y=(z.\cos. m-x.\cos. l)\,\partial\theta$; $\partial z.=(x.\cos. n-y.\cos. m.)\,\partial\theta$, substituting for ∂x, ∂y, ∂z, these values in the expression $\partial x^2+\partial y^2+\partial z^2$, which is equal to the indefinitely small space described by the point whose coordinates are x, y, z, and observing that $\cos.^2 l+\cos.^2 m+\cos.^2 n=1$, it becomes equal to $(x^2+y^2+z^2-(x.\cos. m+y\ \cos. n+z.\cos. l.)^2).\,\partial\theta^2$; $x.\cos. l+y.\cos. m+z.\cos. n.$ is proportional to the cosine of the angle which the line whose coordinates are x, y, z, makes with the right line which makes the angles l, m, n, with the axes of x, y, z, $\because$ when the line drawn from the origin of the coordinates to the point whose coordinates are x, y, z, is perpendicular, to the instantaneous axis of rotation, the elementary space described by a point so circumstanced $=\sqrt{x^2+y^2+z^2}.\,\partial\theta$, this agrees with what is demonstrated in No. 28. If we suppose $\partial\psi$, $\partial\varphi$, $\partial\omega$, proportional to L, M, N, and make $H=\sqrt{L^2+M^2+N^2}$, then

$$\frac{L}{H}=\frac{\partial\psi}{\partial\theta}=\cos. m;\quad \frac{M}{H}=\frac{\partial\varphi}{\partial\theta}=\cos.\ n.\quad \frac{N}{H}=\frac{\partial\omega}{\partial\theta}=\cos l.$$

$\because L=H.\cos m$; $M=H.\cos. n$; $N=H.\cos. l$; $\because$ if $H=L$, $m=0$, $n=100°$, $l=100°$; $\because$ L, the moment of the force is a maximum when $=H$, and the moments whose axes are perpendicular to the axis of H, will be equal to nothing. This will be more fully explained in Nos. 21, and 28, it is mentioned here in order to shew how the conditions of the equilibrium of a solid body may be expressed by means of the greatest moment, and unique resultant; if this resultant, and this moment respectively vanish, then $R=0$, $H=0$, i. e. P,Q,R; L,M,N, which are equivalent to the equations $(m)\,(n)$, are equal respectively to nothing; consequently the evanescence of H and R contains the six equations of the equilibrium of a system, whose parts are invariably connected; and as by No. 3, the sum of the moments of the composing forces with respect to an axis, is equal to the moment of the projection of the resultant of these forces; this resultant must necessarily exist in that plane, in which the moment is the greatest possible, $\because$ the perpendicular to this plane must be at right angles to the resultant, consequently, as $\frac{L}{H}$, $\frac{M}{H}$, $\frac{N}{H}$, are equal to the cosines of the angles which the axis of the greatest moment make with the axis of x, y, and z, and as $\frac{P}{V}$, $\frac{Q}{V}$, $\frac{R}{V}$, are equal to the cosines of the angles which V, the unique resultant makes with the same axes; by note to No. 2, page 7, we have $LR+MQ+NP=0$, which is the equation indicating that the forces have an unique resultant of gravity, as its acts equally on all bodies; and as we may conceive, that its direction is the same, for all the bodies of the system, we shall have

$$S, = S', = S'', = \&c.;$$

$$\left\{\frac{\partial s}{\partial x}\right\} = \left\{\frac{\partial s}{\partial x'}\right\} = \left\{\frac{\partial s}{\partial x''}\right\} = \&c.;$$

$$\left\{\frac{\partial s}{\partial y}\right\} = \left\{\frac{\partial s}{\partial y'}\right\} = \left\{\frac{\partial s}{\partial y''}\right\} = \&c.;$$

$$\left\{\frac{\partial s}{\partial z}\right\} = \left\{\frac{\partial s}{\partial z'}\right\} = \left\{\frac{\partial s}{\partial z''}\right\} = \&c.;$$

whatever may be supposed the direction of s, or of the gravity, we shall satisfy the three equations (n), by means of the three following:*

$$0 = \Sigma.m.x;\ 0 = \Sigma.m.y;\ 0 = \Sigma.m.z;\ (o)$$

* The force of gravity being uniform, and the direction of its action being always the same, $S = S' = S'' = \&c.$; $\left\{\frac{\partial s}{\partial x}\right\} = \left\{\frac{\partial s'}{\partial x'}\right\} = \&c. \left\{\frac{\partial s}{\partial y}\right\} = \left\{\frac{\partial s'}{\partial y'}\right\}$, (for these quantities $\left\{\frac{\partial s}{\partial x}\right\}$ &c. indicate the cosines of the angles which the directions of gravity makes with the three coordinates,) the three equations (n) may be made to assume the following form:

$$0 = S.\left\{\left\{\frac{\partial s}{\partial x}\right\}.\ \Sigma.my - \left\{\frac{\partial s}{\partial x}\right\}\Sigma.mx.\right\};\ 0 = S.\left\{\left\{\frac{\partial s}{\partial s}\right\}.\ \Sigma.mz - \left\{\frac{\partial s}{\partial z}\right\}.\ \Sigma.mx.\right\};$$

$$0 = S.\left\{\left\{\frac{\partial s}{\partial z}\right\}.\ \Sigma.my - \left\{\frac{\partial s}{\partial y}\right\}.\ \Sigma.mz.\right\};$$

they are satisfied by means of the three following: $0 = \Sigma.mx$; $0 = \Sigma.my$; $0 = \Sigma.mz$. The equations (m) will be reduced to the following

$$0 = S.\left\{\frac{\partial s}{\partial x}\right\}.\ \Sigma m;\ 0 = S.\left\{\frac{\partial s}{\partial y}\right\}.\ \Sigma m;\ 0 = S.\left\{\frac{\partial s}{\partial z}\right\}.\ \Sigma m;$$

The origin of the coordinates, being supposed fixed, it will destroy parallel to each of the three axes, the forces

$$S. \left\{\frac{\partial s}{\partial x}\right\} \Sigma m; \quad S. \left\{\frac{\partial s}{\partial y}\right\} \Sigma.m; \quad S. \left\{\frac{\partial s}{\partial z}\right\}. \ \Sigma m;$$

by composing these three forces, we shall obtain an unique force, equal to $S.\Sigma.m.$ i. e. to the weight of the system.

This origin of the coordinates about which we suppose the system in equilibrio, is a very remarkable point in it, on this account, that being supported, the system actuated by the sole force of gravity remains in equilibrio, whatever position it may be made to assume about this point, which is from thence denominated the *centre of gravity* of the system. Its position may be determined by this property, that if we make any plane whatever pass through this point, the sum of the products of each body,* by its distance from this plane, is equal to nothing; for this

these forces admit a resultant, see note to page 89, and as $\frac{\partial s}{\partial x}$, $\frac{\partial s}{\partial y}$, $\frac{\partial s}{\partial z}$, are equal to the cosines of the angles which its direction makes with the axes of x, of y, and of z, combining those three expressions, the resultant is evidently $=$ to $S\Sigma m$; consequently the force with which the fixed origin is pressed, in this case equals the weight of the bodies composing the systems. $S.\Sigma m.$ answers to the expression $m\,g.$ in the first note to page 78.

It follows from note to page 88, that the resultant of all the forces must pass through the origin for $\Sigma.mx$; $\Sigma.my$; $\Sigma.mz$; are equal respectively to nothing. If another point in the system besides the centre of gravity was fixed, then $0 = S.\left\{\left\{\frac{\partial s}{\partial z}\right\} \Sigma.my - \frac{\partial s}{\partial y}. \Sigma.mz.\right\}$ is the sole equation of equilibrium; in this case the fixed axis of rotation must be vertical.

* If $Ax' + By' + Cz' = 0$, be the equation of a plane passing through the centre of gravity, the cosines of the angles which this plane makes with the plane of the axes $x\,y$, of $x\,z$, and of $y\,z$, respectively, i. e. the cosines of the angles which a perpendicular to this plane makes with the axis of x, and of y, of z =

$$\frac{A}{\sqrt{A^2+B^2+C^2}}, \quad \frac{B}{\sqrt{A^2+B^2+C^2}}, \quad \frac{C}{\sqrt{A^2+B^2+C^2}};$$

see Lacroix, tom. 1. No. 269,

in like manner the cosines of the angles, which lines drawn, from the point, whose coordinates are x, y, z, make with the axes of x, of y, and of z, distance is a linear function of the coordinates x, y, z, of the body; consequently by multiplying it by the mass of the body, the sum of these products will be equal to nothing in consequence of the equations. *(o)*

In order to determine the position of the centre of gravity, let X, Y, Z, represent its three coordinates with respect to a given origin; let x, y, z, be the coordinates of m with respect to the same point; x', y', z', those of m', &c. the equations *(o)* will then give

$$0 = \Sigma.m.(x-X.)$$

but we have $\Sigma.m.X = X.\Sigma.m$, $\Sigma.m$ being being the entire mass of the system, therefore we have

$$X = \frac{\Sigma.m.x}{\Sigma.m},$$

we shall have in like manner

$$Y = \frac{\Sigma.m.y}{\Sigma.m}; \quad Z = \frac{\Sigma.m.z}{\Sigma.m};$$

$$= \frac{x}{\sqrt{z^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}},$$

∵ by note to No. 2, page 7, the cosine of the angle which the perpendicular to the given plane, makes with the line whose coordinates are x, y, z,

$$= \frac{xA+yB+zC}{\sqrt{A^2+B^2+C^2} \times \sqrt{x^2+y^2+z^2}},$$

let this angle $= a$ and $\sqrt{x^2+y^2+z^2} \times \cos.\ a = \dfrac{xA+yB+zC}{\sqrt{A^2+B^2+C^2}}$ = the distance of the point from the given plane, consequently, the sum of all the distances multiplied respectively into their masses

$$= \frac{A.\Sigma.mx + B.\Sigma.my + C.\Sigma.mz}{\sqrt{A^2+B^2+C^2}} = 0,$$

in consequence of the equation (o).

thus, as the coordinates X, Y, Z, determine only one point, it follows that the centre of a system of bodies is an unique point.

The three preceding equations give

$$X^2+Y^2+Z^2 = \frac{(\Sigma\, m.x)^2+(\Sigma.m.y)^2+(\Sigma\, m.z)^2}{(\Sigma.m)^2}$$

this equation may oe made to assume the following form :*

$$X^2+Y^2+Z^2 = \frac{\Sigma.m\,(x^2+y^2+z^2)}{\Sigma.m} - \frac{\Sigma mm'\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}{(\Sigma.m)^2};$$

the finite integral $\Sigma mm'\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}$ expresses the sum of all the products similar to that, which is contained under the characteristic Σ, and which is formed by considering all the combinations of

* The square of the sum of any number of quantities, being equal to the sum of the squares of those quantities, and twice the sum of the products of all the binary combinations of the different quantities, we have

$$(\Sigma(mx))^2 = \Sigma(m^2x^2)+2\,\Sigma(mm'.xx');\ \Sigma mm'.(x-x')^2)$$

denotes the products which are obtained, by taking on one part all the binary combinations of the bodies mm', &c. in which the quantities mm' are affected with different accents, and then multiplying these by the square of $(x-x')$, in which the terms have respectively the same accents as the bodies which they are multiplied by, thus $\Sigma.(x-x')^2 = x^2+x'^2+x''^2+$ &c.$-2xx'-2xx''-2x'x''-$&c. and $\Sigma(mm'\,x-x')^2) = mm'x^2+mm'x'^2+mm''.x^2+mm''.x''^2+m'm''x'^2+m'm''.x''^2+$&c. $-2mm'xx'\ -2mm''xx''\ -2m'm''\,x'x''$; &c. $=\Sigma(mm'.x^2)$ $-2\Sigma(mm'.(xx'))$ and as $\Sigma(mx^2) =,\ mx^2+m'x'^2+m''x''^2+$&c. $\Sigma(m.x^2).\Sigma m.$ $=(mx^2+m'x'^2+m''^2x''^2+$ &c.$).(m+m'+m''+m'''+$&c.$) = m^2x^2+m'^2x'^2+m''^2x''^2+$&c. $+mm\,x^2+mm'x^2+mm'x'^2+m''m'.x'^2+mm''.x''^2+m'm''.x''^2+$&c. $=\Sigma(m^2x^2)+\Sigma(mm'x^2)$ ∴ $(\Sigma mx)^2 = \Sigma(m^2x^2)+2\Sigma(mm'xx') = \Sigma(mx^2).\ \Sigma m -\Sigma(mm'x^2)-\Sigma.mm'.(x-x')^2 +\Sigma mm')x^2) = \Sigma(mx^2).\Sigma m -\Sigma mm'(x-x)^2$, (by substituting for $\Sigma(m^2x^2)$ its value $\Sigma(mx^2)\,\Sigma m -\Sigma mm'.(x^2)$, and for $2\Sigma(mm'.xx')$. its value $\Sigma(mm'.(x^2),) -\Sigma(mm'.(x-x')^2)$ ∴ the value of X^2

$$\frac{(\Sigma mx)^2}{(\Sigma m)^2} = \frac{(\Sigma mx^2)}{\Sigma m} - \frac{\Sigma mm'(x-x)^2}{(\Sigma m)^2},$$

we might derive corresponding expressions for Y^2, and Z^2.

This method gives the position of the centre of gravity of any body of a given form without being obliged, to refer the position of its molecules to coordinate planes.

the different bodies of the system. We shall thus obtain the distance of the centre of gravity from any fixed point, by means of the distances of the bodies of the system, from the same fixed axis, and of their mutual distances. By determining in this manner the distance of the centre of gravity from any three fixed points, we shall have its position in space; which suggests a new way of determining this point.

The denomination of centre of gravity has been extended to that point, of any system of bodies, either with or without weight, which is determined by the three coordinates X, Y, Z.

16. Is is easy to apply the preceding results to the equilibrium of a solid body of any figure, by conceiving it made up of an indefinite number of points, firmly united together. Therefore let dm be one of these points, or an indefinitely small molecule of the body, and let x, y, z, be the rectangular coordinates of this molecule; also let P, Q, R, represent the forces by which it is actuated parallel to the axis of x, of y, and of z, the equations (m) and (n) of the preceding number will be changed into the following:

$$0=\int P.dm;\quad 0=\int Q.dm;\quad 0=\int R.dm;*$$

$$0=\int(Py-Qx).\,dm;\quad 0=\int(Pz-Rx).\,dm;\quad 0=\int(Ry-Qz).\,dm;$$

The sign of integration $\int$ is relative to the molecule dm, and ought to be extended to the entire mass of the solid.

* $\left\{\frac{\partial s}{\partial x}\right\}$ being the cosine of the angle which the direction of the force S makes with the axis of x, $S.\left\{\frac{\partial s}{\partial x}\right\}$ = the force resolved parallel to the axis of x, $\because$ it is equal to P; and as $\Sigma.m=\int dm$, $\Sigma m.\ S.\left\{\frac{\partial s}{\partial x}\right\}=\int P.dm$, and since $\Sigma.S.\left\{\frac{\partial s}{\partial x}\right\}.ym=\int Py.dm$; $\Sigma.m.S.\left\{y.\left\{\frac{\partial s}{\partial x}\right\}-x.\left\{\frac{\partial s}{\partial y}\right\}\right\}=\int(Py-Qx)\,dm$, &c.

From the values which have been given in the text for the coordinates of the centre of gravity, it is manifest that the position of this centre remains unaltered, whatever change may take place in the absolute force of gravity, $\because$ when bodies are transferred from one latitude to another on the surface of the earth, though the absolute weight varies, still the position of the centre of gravity is fixed.

If the body could only turn about the origin of the coordinates, the three last equations will be sufficient for its equilibrium.*

* When any system of homogeneous bodies is in equilibrio, the centre of gravity is then the highest or lowest possible; this is immediately evident from the principle of virtual velocities, for let the weights of any number of bodies m, m', m'', be denoted by S, S', S'', &c. and let s, s', s'', &c. represent lines demitted from the centres of the several bodies m, m', m'', &c. on the horizontal plane; now if the position of the system be disturbed in an indefinitely small degree, we shall have, when the bodies of the system are in equilibrio, the equation of virtual velocities

$$S\delta s + S'.\delta s' + S.\delta s'' + \&c. = 0,$$

consequently the quantity of which this expression is the variation, i. e. $Ss + S's' + S''s'' +$ &c. (= the entire weight of all the bodies composing the system, multiplied by the distance of the centre of gravity of the system from the horizontal plane, $= s_{,}.S.\Sigma m.$) is a maximum or minimum, and as the weight of all the bodies of the system is always given, the distance of the centre of gravity of the system from the horizontal plane must be either a maximum or a minimum when the system is in equilibrio; this being established, it is interesting to know the equation of the curve, in which the centre of gravity is lower than in any other curve whose points of suspension and length are given; the investigation of this curve, which is termed the catenary, is very easy, it occurs in all the elementary treatises, the differential equation is of the following form $(y+g).dx = g.\ \cos.\ c.\sqrt{dx^2+y^2}$.

It might be proved conversely, that when the distance of the centre of gravity from an horizontal plane is the greatest or least possible, the system is in equilibrio, for we shall have $S.\delta s + S'.\delta s' + S''\delta s'' + \&c. = G.\delta s_{,} = 0$, however there is an essential difference between their states of equilibrium; in the first case, the equilibrio is denominated instable, in the second, it is termed stable, in order to determine these two different states, we should attend to the species of the motion when the centre deviates by an indefinitely small quantity from the vertical, see No. 30.

* In Physical and Astronomical problems, the method that is generally employed, to determine the mean value between several observed ones, of which some are greater, and some less than the true one, is to divide the sum of all the observed values by their number. This comes, in fact, to determine the distance of the centre of gravity from a given plane. For if z, z', z'', &c. represent the observed quantities, then $\frac{z+z'+z''+}{n}$, &c. is the expression for the mean value, but if z, z', z'', denote the distances of the centres of gravity of n masses, equa each to m from the plane, then $\frac{zm+z'm'+z''m''+}{nm}$, &c. =

the distance of the centre of gravity of the system of m masses from this plane $= \frac{z+z'+z''+}{m}$, &c. = the required mean value.

If several forces concurring in a point constitute an equilibrium, then supposing that, at the extremities of lines, in the directions of these forces, and respectively proportional to them, we place the centres of gravity of bodies equal to each other, the common centre of gravity of these masses will be the point where all the forces concur. For since the forces are by hypothesis represented by lines taken their direction, and concurring in one point, it is evident that by making this point the origin of the coordinates, we shall have the sum of the forces parallel to the three rectangular axes proportional to $\Sigma(x)$, $\Sigma(y)$, $\Sigma(z)$, these sums are $\because$ by the conditions of the problem = to nothing, see note to page 11; and since the masses are all equal we shall have $\Sigma(x).m = \Sigma(mx) = 0$, this also obtains for the other axes, consequently we shall have $\Sigma(m.x) = 0$, $\Sigma(my) = 0$, $\Sigma(mz) = 0$, $\therefore$ the origin of the coordinates coincides with the centre of gravity of the system of masses respectively equal to m.

The centre of gravity of a body, or system of bodies, is that point in space from which if lines be drawn to the molecules of the body, the sum of their squares is the least possible. For if X, Y, Z, represent the coordinates of such a point, then the sum of the squares of the distances of all the molecules of the system from this point is equal to $\Sigma((x-X)^2+(y-Y)^2(z-Z)^2)$, if we take the differential of this expression with respect to each of the coordinates, and multiply each of the terms of the sums which are respectively equal to nothing, by the element of the mass, we shall have $\Sigma.m.(x-X) = 0$, $\Sigma.m.(y-Y)=0$, $\Sigma.m.(z-Z,) = 0$,

$$\therefore X = \frac{\Sigma(mx)}{\Sigma m}; \; Y = \frac{\Sigma my}{\Sigma m}; \; Z = \frac{\Sigma mz}{\Sigma m};$$

and from what has been demonstrated in the preceding note it follows, that if we apply to all the points of the system, forces directed towards the centre of gravity, and proportional to the distances between those points and the centre of gravity, these forces will constitute an equilibrium; consequently when several forces constitute an equilibrium, the sum of the squares of the distances of the point of concourse of these forces, from the extremities of lines representing these forces, *i.e.* the sum of the squares of these lines, is a minimum.

From the preceding property it appears, that if several observations give different values for the position of a point in space, the mean position, *i.e*, the position which deviates the least from the observed positions, is that in which the sum of the squares of its distances from the observed positions is the least possible. The problem is altogether similar when we wish to combine several observations of *any* kind whatever; for the distances of the points correspond to the differences between the particular results and their mean value; and since it is impossible entirely to exterminate these differences, we are obliged to select a mean result, such that the sums of the squares of these differences may be a minimum; this is the principal of the method of the least squares, which was devised by Le Gendre to combine the equations of conditions between the errors deduced from a comparison of the astronomical tables with observation; it comes in fact to find the centre of gravity of the observations which we compare together.

The general form of the equations of condition is as follows:

$0 = a + bx + cy + dz + \&c.$ when we pass into one member all the terms which compose them, a, b, c, are given numerical coefficients, if all these equations could be satisfied exactly, by the values of x, y, z, their first members would be necessarily reduced to nothing by substituting for x, y, z, their values, but as this substitution does not render them accnrately equal to nothing, let E, E', E'', represent the errors which remain, then we shall have $E = a + bx + cy + dz + \&c.$; $E' = a' + b'x + c'y + d'z + \&c.$; $E'' = a'' + b''x + c''y + \&c.$ the quantities x, y, z', &c. are to be determined by the condition that the values E, E', are either nothing, or very small; the sum of the squares of the errors =

$$E^2 + E^2 + E^2 + \&c. = (a^2 + a'^2 + a''^2 + \&c.) + (b^2 + b'^2 + b''^2).x^2 + (c^2 + c'^2 + c''^2 + \&c.)y^2$$
$$+ (d^2 + d'^2 + d''^2 + \&c.)z^2 + ;$$

$$2\,(ab + a'b' + a''b'' + \&c.)\,x + 2\,(ac + a'c' + a''c'')\,y + 2\,(ad + a'd' + a''d'' + \&c.)z\,;$$

$$+ 2(bc + b'c' + b''c'' + \&c.)\,xy + 2(bd + b'd' + b''d'') + 4z + \&c.$$

the minimum of this expression, with respect to x, will be 0

$$= \Sigma.ab + x\,\Sigma.b^2 + y\,\Sigma.bc + z.\,\Sigma.bd + \&c.$$

the minimum with respect to $y = \Sigma.ac + x\Sigma.bc + y\Sigma c^2 + x.\Sigma.dc = 0$, we derive a corresponding value for the minimum of z, hence in order to form the equation of the minimum with respect to one of the unknown quantities, we must multiply all the terms of each proposed equation by the coefficient of the unknown term in that equation, and then put the sum of the products equal to nothing. Though this method requires more numerical calculations, in order to form the particular equation relative to each unknown quantity, than the method suggested by Mayer; it is more direct in its application, and requires no tentation on the resulting equations. Laplace has shewn in his Theory of Probabilities, that when we would take the mean between a great number of observations of the same quantity, obtained by *different* means, this is the only method which the theory permits us to employ, see Le Gendres Memoir on the determination of the orbits of the comets, and Biot's Astronomie Physique, tome 2. page 200.

CHAPTER IV.

Of the equilibrium of fluids.

17. In order to determine the laws of the equilibrium, and of the motion of *each* of the molecules of a fluid, it would be necessary to ascertain their figure, which is impossible; but we have no occasion to determine these laws, except for fluids* considered in a mass, and for this purpose the knowledge of the figures of their molecules is useless. Whatever may be the nature of these figures, and the properties which depend on them in the integrant molecules, all fluids, considered in the aggregate, ought to exhibit the same phenomena in their equilibrium, and also in their motions, so that from the observation of these phenomena, we are not able to discover any thing respecting the configuration of the fluid molecules. These general phenomena depend on

* Although the figure of the molecules of fluids are unknown to us, still there can be no question but that they are material, and consequently that the general laws of the equilibrium and motion of solid bodies are applicable to them. If we were able analytically to express their characteristic property, to wit, extreme smallness, and perfect mobility, no particular theory would be required in order to determine the laws of their equilibrium and motion; they would be then only a particular case of the general laws of Statics and Dynamics. But as we are not able to effect this, it is proposed to derive the theory of their equilibrium and motion from the property which is peculiar to them, of transmitting equally, and in every direction, the pressure to which their surface is subjected; this property is a necessary consequence of the perfect mobility of the molecules of the fluids. In the definition which has been given in the text there is no account made of the tenacity or adhesion of the molecules, which is an obstacle to this free separation; this adhesion exists however between the molecules of most of the fluids with which we are acquainted.

the perpect mobility of these molecules, which are thus able to yield to the slightest force. This mobility is the characteristic property of fluids; it distinguishes them from solid bodies, and serves to define them. It follows from this, that when a fluid mass is in equilibrio each molecule must be in equilibrio in consequence of the forces which * solicit it, and of the pressures to which it is subjected by the action of the surrounding particles. Let us proceed to develope the equations which may be deduced from this property.

For this purpose, let us consider a system of fluid molecules, constituting an indefinitely small rectangular parallelepiped. Let x, y, z, denote the three rectangular coordinates of that angle of the parallelopiped, which is nearest to the origin of the coordinates. Let dx, dy, dz, represent the three dimensions of this parallelepiped; let p repre-

* When a fluid is contained in a vessel, the pressure to which it is subjected at its surface is transmitted in every direction, as has been just stated, but since the molecules are material, they must have weight, therefore it also presses the sides of the vessel with a force arising from the weight of the molecules, and different in every point of the sides; and if the fluid is contained in a vessel closed in every side, when the molecules are solicited by any given accelerating forces, then the pressure is different for every particular point, its direction is always perpendicular to the surface, since by No. 3, when the resistance of a surface destroys the pressure on it, the direction of this pressure must be normal to the surface. The intensity of this pressure depends on the given forces, and on the position of the point.

Therefore it appears, that in the equilibrium of a fluid contained in a vessel, the entire pressure in each point of the sides is the sum of two pressures altogether distinct; one of which arises from the pressure, exerted on the surface, and is the same on all the points; the other is owing to the motive forces of the particles of the fluids, and varies from one point to another.

Fluids are generally distinguished into two classes, incompressible, and elastic; with respect to the last class, they may press against the sides of the vessel in which they are enclosed, although no motive forces act on the particles, or without any pressure urging the surface of the fluid. For from their elasticity they tend perpetually to dilate themselves, which gives rise to a pressure on the sides of the vessel; however this is a constant pressure in the same fluid; it depends on the matter of the fluid, its density and temperature.

sent the mean of all the pressures, to which the different points of the side $dy.\,dz$ of the parallelepiped, which is nearest to the origin of the coordinates, is subjected; and let p' be the corresponding quantity on the opposite side. The parallelepiped, in consequence of the pressure to which it is subjected, will be urged in the direction of x, by a force equal to $(p-p').\,dy.dz$; $p'-p$ is the difference of p, taken on the hypothesis that x alone is variable; for although the pressure p' acts in a direction contrary to p, nevertheless the pressure to which a point is subject being the same in every direction, $p'-p$ may be considered as the difference of two forces infinitely near, and acting in the same direction; consequently we have*

$$p'-p=\left\{\frac{dp}{dx}\right\}.dx, \text{ and } (p-p').\,dy.\,dz=-\left\{\frac{dp}{dx}\right\}.\,dx.\,dy.\,dz.$$

Let P, Q, R, be the three accelerating forces which solicit the molecules of the fluid, independently of their connexion, parallel to the axes of x, of y, and of z; if the density of the parallelepiped be denoted by ρ, its mass will be equal to $\rho.\,dx.\,dy.\,dz.$ and the product of the force P by this mass, will represent the whole motive force, which is derived from

* Since p, ρ, P,Q,R, generally vary from one point to another of the fluid mass, they must be considered as functions of x, y, z. We distribute the fluid into parallelepipeds, in order more easily to express in analytical language the fact of the equality of pressure, which, as has been stated, is the fundamental principle from which we deduce the whole theory of their equilibrium, and by supposing these parallelepipeds indefinitely small, we are permitted to consider all the points of the same side as equally pressed, and also ρ, P, Q, R, as constant for each side respectively, by means of which we are able to determine the pressure p. x, y, z, being the coordinates of the angular point next the origin, and p being a function of these coordinates, we shall have

$$\partial p=\left\{\frac{dp}{dx}\right\}.\,\partial x+\left\{\frac{dp}{dy}\right\}.\,\partial y+\left\{\frac{dp}{dz}\right\}.\,\partial z;$$

the coefficient $\left\{\frac{dp}{dx}\right\}=\left\{\frac{\partial p}{\partial x}\right\}$ &c. they are taken negatively because they tend to diminish the coordinates.

it; consequently this mass will be solicited parallel to the axes of x, by the force $\left\{\rho P - \left\{\frac{dp}{dx}\right\}\right\}. dx.dy.dz.$ For similar reasons it will be solicited parallel to the axes of y, and of z, by the forces

$$\left\{\rho.Q - \left\{\frac{dp}{dy}\right\}\right\}. dx.dy.dz. \text{ and } \left\{\varrho.R - \left\{\frac{dp}{dz}\right\}\right\}. dx. dy. dz. \text{ \&c.}$$

therefore, by the equation (b) of No. 3, we shall have

$$0 = \left\{\rho P - \left\{\frac{dp}{dx}\right\}\right\}.\delta x + \left\{\rho Q - \left\{\frac{dp}{dy}\right\}\right\}.\delta y + \left\{\rho R - \left\{\frac{dp}{dz}\right\}\right\}.\delta z;$$

$$\text{or } \delta p = \rho(P.\delta x + Q.\delta y + R.\delta z).$$

The first member of this equation being an exact variation, the second must be so likewise; from which we may deduce the following equation of partial differentials,*

$$\left\{\frac{d.\rho P}{dy}\right\} = \left\{\frac{d.\rho Q}{dx}\right\}; \left\{\frac{d.\rho P}{dz}\right\} = \left\{\frac{d.\rho R}{dx}\right\}; \left\{\frac{d.\rho Q}{dz}\right\} = \left\{\frac{d.\rho R}{dy}\right\};$$

* When $\varrho\ (P.\delta x + Q.\delta y + R.\delta z.)$ is an exact differential, $\left\{\frac{d.\varrho P}{dy}\right\} = \left\{\frac{d.\varrho Q}{dx}\right\}$ &c. (see Lacroix Traite Elementaire, Calcul. Differential and Integral, No. 261.)

$$\because \frac{\varrho.dP}{dy} + \frac{P.d\varrho}{dy} = \frac{\varrho.dQ}{dx} + \frac{Q.d\varrho}{dx}; \quad \frac{\varrho.dP}{dz} + \frac{P.d\varrho}{dz} = \frac{\varrho.dR}{dx} + \frac{R.d\varrho}{dx};$$

$\frac{\varrho.dQ}{dz} + \frac{Q.d\varrho}{dz} = \frac{\varrho.dR}{dy} + \frac{R.d\varrho}{dy}$, if we multiply the first equation by R, the second by $-Q$, and the third by P, we shall obtain,

$$\frac{\varrho.R.dP}{dy} + \frac{R.P.d\varrho}{dy} = \frac{R.\varrho.dQ}{dx} + \frac{R.Q.d\varrho}{dx}; \quad -\frac{\varrho.Q.dP}{dz} - \frac{Q.P.d\varrho}{dz} = -\frac{\varrho.Q.dR}{dx}$$

$$-\frac{R.Q.d\varrho}{dx}; \quad \frac{\varrho.P.dQ}{dz} + \frac{P.Q.d\varrho}{dz} = \frac{\varrho.P.dR}{dy} + \frac{R.P.d\varrho}{dy},$$

from which we may obtain

$$0 = P.\left\{\frac{dQ}{dz}\right\} - Q.\left\{\frac{dP}{dz}\right\} + R.\left\{\frac{dP}{dy}\right\} -$$

$$P.\left\{\frac{dR}{dy}\right\} + Q.\left\{\frac{dR}{dx}\right\} - R.\left\{\frac{dQ}{dx}\right\}.$$

This equation expresses the relation which must exist between the forces P, Q, and R, in order that the equilibrium may be possible.

If the fluid be free at its surface, or in certain parts of this surface, the value of p will be equal to nothing in those parts; therefore we shall have $\delta p = 0$, provided that the variations δx, δy, δz, appertain to this surface; consequently when these conditions are satisfied, we shall have

$$0 = P.\delta x + Q.\delta y + R.\delta z.$$

If $\delta u = 0$, be the differential equation of the surface, we shall have

$$P.\delta x + Q.\delta y + R.\delta z = \lambda.\delta u,$$

λ being a function of x, y, z; from which it follows, by No. 3, that

by reducing all the terms in which $\delta\rho$ is involved to one side, and then adding them together, we get

$$\rho.\left\{\frac{R.dP}{dy} - \frac{R.dQ}{dx} - \frac{Q.dP}{dz} + \frac{Q.dR}{dx} + \frac{P.dQ}{dz} - \frac{P.dR}{dy}\right. =$$

$$-\frac{RP.\delta\rho}{dy} + \frac{RQ.d\rho}{dx} + \frac{QP.d\rho}{dz} - \frac{RQ.d\rho}{dx} - \frac{PQ.\delta\rho}{dz} + \frac{RP.d\rho}{dy} = 0.$$

by concinnating

$$P.\left\{\frac{dQ}{dz}\right. - Q.\left\{\frac{dP}{dz}\right\} + R.\left\{\frac{dP}{dy}\right\} - P.\left\{\frac{dR}{dy}\right\} + Q.\left\{\frac{dR}{dx}\right\} - R.\left\{\frac{dQ}{dx}\right\} = 0.$$

This equation shews whether the equilibrium is possible, though we are unable to ascertain the density ρ.

the resultant of the forces $P,Q,R,$* must be a perpendicular to those parts of the surface, in which the fluid is free.

Let us suppose that the variation $P\delta x + Q.\delta y + R.\delta z$ is exact, this is the case when $P,Q,R,$ are the result of attractive forces. Denoting this variation by $\delta\varphi$, we shall have $\delta p = \rho\delta\varphi$; therefore ρ must be a function of p and of φ, and as the integration of this differential equation gives φ

If the relation indicated by this equation does not obtain between the forces P, Q, R, the fluid will be in a perpetual state of agitation, whatever figure it may be made to assume; but when this relation is satisfied, the equilibrium will be *possible*, and *vice versa*; and as $P, Q, R,$ are functions of the coordinates, we can integrate the expression $\rho.(P.\delta x + Q.\delta y + R.\delta z.)$ by the method of quadratures, by means of which we can find the value of the pressure for any given place of the fluid; consequently we can obtain the force with which any side of the vessel in which the fluid is enclosed is pressed. But though the relation which exists between the forces must be such as to satisfy the preceding equation, when there is an equilibrium, still this is not sufficient, in most cases, to insure the equilibrium, for the fluid must also assume a determined figure, depending on the nature of the forces P, Q, R, which solicit the molecules.

* When an incompressible fluid is free at its surface, and in a state of equilibrium, p must vanish, $\because$ $\delta p = 0$, if the fluid is elastic this condition can never be satisfied, because ρ being proportional to p, whilst the density has a finite value, p can never vanish. When p vanishes, $0 = \delta p = P.\delta x + Q.\delta y + R.\delta z$, $\because$ when δx, δy, δz, appertain to the surface, by substituting for P, Q, R, their values, the resulting expression will be the equation of the surface. It follows from No. 3, that the resultant of the forces P, Q, R, must be perpendicular to the surface; it may be proved directly thus:

$$\frac{P}{\sqrt{P^2+Q^2+R^2}}, \quad \frac{Q}{\sqrt{P^2+Q^2+R^2}}, \quad \frac{R}{\sqrt{P^2+Q^2+R^2}},$$

are equal to the cosines of the angles, which the resultant makes with the axes of x, of y, and of z, but since $P.\delta x + Q.\delta y + R.\delta z$, is the equation of the surface, they also express the cosines of the angles which the normal make with the same axes respectively; see Notes to page 14; consequently the normal coincides with the resultant. This coincidence of the resultant with the normal is the second condition, which must be satisfied, in order, as has been stated above, to insure the equilibrium; and it is this condition which enables us in each particular case to determine the figure corresponding to the equilibrium of the fluid, and if there be one only attractive force directed towards a fixed point, then the surface will be of a spherical form, the fixed point being the centre of the sphere; if this point in a function of p, we shall have p in a function of ρ. Therefore the pressure is the same, for all molecules whose density is the same; thus δp must vanish with respect to those strata of the fluid, in which the density is constant, and with regard to these surfaces, we have,

$$0 = P.\delta x + Q.\delta y + R.\delta z.^*$$

consequently, the resultant of the forces, which solicit each molecule

P

be at an infinite distance the surface will degenerate into a plane, ∵ if the planets were originally fluid, and if their molecules attracted each other with forces, varying as $\frac{1}{d^2}$ they would assume a spherical form. See No. 12, Book 2[d].

* If $P.\delta x + Q.\delta y + R.\delta z$ is an exact variation, $\delta\varphi$, $\delta p = \rho\delta\varphi$, ∵ ρ must be some function of φ, otherwise it would not be an exact variation; however, the form of this function is undetermined, see note to page 10, consequently p will be a function of φ, and p and ρ will be the same for all those molecules in which the value of φ is given, i. e. for the molecules in the same strata of level, therefore when the density varies, an equilibrium cannot subsist unless each stratum is homogeneous during its entire extent; for when this is the case, ρ, and consequently p is the same; ∵ $\delta p = 0$, for the surfaces in which ρ is constant, ∵ for such surfaces $0 = P.\delta x + Q.\delta y + R.\delta z$, and the resultant coincides with the normal. If we integrate the preceding equation, by putting φ equal to a constant arbitrary quantity, we derive an equation which appertains to an indefinite number of surfaces, differing from each only by the value of this constant arbitrary quantity. If we make this quantity increase by insensible gradations, we will have an infinite series of surfaces, distributing the entire mass into an indefinite number of strata, and constituting between any two successive surfaces, what have been denominated *strata of Level*. The law of the variation of the density ρ, in the transit from one strata to another, is altogether arbitrary, as it depends on what function of φ, ρ is, but this is undetermined. It appears from what precedes, that there are two cases, in which $\delta p = 0$, when it is at the free surface, in which case p must vanish of itself, and also when p is constant, i. e. for all surfaces of the same level, consequently when the fluid is homogeneous, the strata to which the resultant of the forces is perpendicular, are then necessarily of the same density.

When the fluid is contained in a vessel, closed in on every side, it is only necessary that all strata of the same level must have the same density; in elastic fluids, the first condition to wit, that p should vanish, or that $P.\delta x + Q.\delta y + R.\delta z = 0$, can never obtain, ∵ unless this fluid extends indefinitely into space, so that ρ may be altogether insensible it cannot be in equilibrium, except in a vessel closed in on every side.

of the fluid, is in the state of equilibrium, perpendicular to the surfaces of these strata, and on this account they have been termed strata of level. This condition is *always* satisfied, if the fluid is homogeneous, and incompressible, because then the strata, to which this resultant is perpendicular, are all of the same density.

For the equilibrium of an homogeneous fluid mass, of which the extreme surface is free, and covers a fixed solid nucleus of any figure whatever, it is necessary and sufficient, first, that the variation $P\delta x + Q.\delta y + R + .\delta z$ be exact; secondly, that the resultant of the forces at the exterior surface be directed perpendicularly *towards* this surface.*

* If two different fluids are in equilibrio, then the surface which separates them must be horizontal; if the denser fluid is superior, the centre of gravity of *all* the molecules will be highest; if it be inferior, then the centre will be lower than in any other position, ∴ that the equilibrium may be stable, the denser strata should be inferior. See Notes to No. 15.

When p is constant, the equation $\varphi = C$, gives the relation which must exist for each stratum of level between the coordinates of the different molecules of the surface which answers to the preceding equation; in this case $\delta\varphi = 0$, which shews that φ is either a maximum or minimum, and generally when $P.\delta x + Q.\delta y + R.\delta z$ is an exact variation, ρ is a function of φ, ∴ the equation of equilibrium $\delta p - \rho.\delta\varphi = 0$, shews that in the state of equilibrium there is a function of p and of x, y, z, which is either a maximum or a minimum. Though in the state of equilibrium all the molecules in the same strata of level have necessarily the *same* density, and experience the same pressure, still the converse is not true, for in homogeneous incompressible fluids, ρ is constant in those sections of the fluid in which neither $\delta\varphi$, nor $\delta p = 0$.

In elastic fluids, the density ρ is observed to be proportional to the compressing force, ∴ $p = k.\rho$; k depends on the temperature and matter of the fluid, by substituting for ρ, in the equation $\delta p = \rho\delta\varphi$, we obtain $\delta p = \frac{p}{k}.\ \delta\varphi$, ∴ by integrating we get $\log. p + C = \frac{\varphi}{k}$, because when the matter and temperature are given, k will be constant, ∴ by making $C = -\log. E$, we obtain $p = Ec^{\frac{\varphi}{k}}$, ∴ since p and ρ, $= \left\{\frac{\varphi}{k}\right\}$, are respectively functions of φ, the pressure and density will be constant for each stratum of level, but the law of the variation of the density is not arbitrary, as in the case of incompressible fluids, for the equation $\rho = \frac{p}{k} = \frac{E}{k}.\ c^{\frac{\varphi}{k}}$, *determines* the law. If the matter of the fluid remaining homogeneous, the temperature undergoes any alteration, k will be a function of the variable temperature, but in order that the equation $\frac{\partial p}{p}=\frac{\partial\varphi}{k}$ may be an exact variation, it is necessary that k, and ∴ the temperature should be functions of φ, these functions are altogether arbitrary; consequently we conclude, that when the fluid is in a state of equilibrium, the temperature of each stratum is uniform, and that the law of the variation of temperature is arbitrary; but this law being given, we are able to integrate the expression $\frac{\partial\varphi}{k}$, from which integral we can conclude the law of the densities and pressures by means of the equations $p=E.\,c^{\int\frac{\partial\varphi}{k}}$; $\varrho=\frac{E}{k}.\,c^{\int\left\{\frac{\partial\varphi}{k}\right\}}$.

In incompressible fluids, if the force varies as the n^{th} power of the distance from the centre, by fixing the origin of the coordinates at this point, we have $P=A\varrho r^{n-1}.\,x$, $Q=A\varrho r^{n-1}\,y$, $R=A\varrho r^{n-1}.\,z$, ∴ $P.\partial x+Q.\partial y+R.\partial z=A\varrho r^{n-1}.\,(x.\partial x+y.\partial y+z.\partial z)=A\varrho r^{n}.\,\partial r,=\partial p$, ∴ $\frac{A\varrho r^{n+1}}{n+1}$ when ϱ is given, $=\varphi=p+C$, when $n=-2$, $\frac{A\varrho r^{n+1}}{n+1}=p=-\frac{A\varrho}{r}$, if gravity is the sole force acting on the molecules, by making the axis of z vertical, P and Q, will vanish, and $R=g$, ∴ $P.\partial x+Q.\partial y+R.\partial z$ is reduced to the equation $g.\partial z=0$, ∴ $gz=C$, consequently the surface is horizontal, since $R=(A\varrho r^{n-1}.\,z)=g$, $\int(g.\partial z)=p$ ∴ the pressure varies as the height. Since when the force varies as the n^{th} power of the distance from the centre $\partial\varphi=Ar^{n}.\partial r$, by substituting in the equation of elastic fluids $\frac{\partial p}{p}=\frac{\partial\varphi}{k}$ for $\partial\varphi$, and integrating, we get log. $p=\frac{Ar^{n+1}}{k.(n+1)}$, consequently, if the $(n+1)^{\text{th}}$ powers of the distance be taken in arithmetic progression, the pressures and the densities proportional to them, will be in geometric progression, ∴ if n is negative, and if in the radius, ordinates be erected proportional to the pressures or densities ,the locus of their extremities will be a curve of the hyperbolic species, and the radius produced, will be an asymptote to the curve, if n is positive, the locus of the extremities of the coordinates, will be a curve of the parabolic species, if $n=0$, i. e. if the force is constant, the locus will be the logarithmic curve. See Princip. Matth. Liber 2. Prop. 22, et Scholium.

CHAPTER V.

The general principles of the motions of a system of bodies.

18. We have, in No. 7,* reduced the laws of the motion of a point, to those of its equilibrium, by resolving the instantaneous motion into two others, of which one remains, while the other is destroyed by the action of the forces which solicit the point; we have derived the differential equations of its motion, from the equilibrium which subsists between these forces, and the motion lost by the body. We now proceed to employ the same method, in order to determine the motion of a system of bodies m, m', m'', &c. Thus, let mP, mQ, Rm, be the forces which solicit m parallel to the axes of the rectangular coordinates x, y, z; let $m'P'$, $m'Q'$, $m'R'$, be the forces which solicit m', parallel to the same axes, and so on of the rest; and let us denote the time by t. The partial forces $m.\frac{dx}{dt}, m.\frac{dy}{dt}, m.\frac{dz}{dt}$ of the body m at any instant whatever will become in the following:†

* The principle established in this number, has been termed *the principle of D'Alembert*, by it the laws of the motion of a system are reducible to one sole principle, in the same manner as the laws of the equilibrium of bodies have been reduced to the equation (l) of No. 14.

† In consequence of the mutual connection which subsists between the different bodies of the system, the effect, which the forces immediately applied to the respective bodies would produce, is somewhat modified, so that their velocities, and the directions of their motions, are different from what would take place, if the bodies composing the system were altogether free; consequently, if at any point of time we compute the motions which

$$m.\frac{dx}{dt} + m.\,d.\,\frac{dx}{dt} - m.\,d.\frac{dx}{dt} + mP.dt;$$

$$m.\frac{dy}{dt} + m.\,d.\,\frac{dy}{dt} - m.\,d.\frac{dy}{dt} + mQ.dt;$$

$$m.\frac{dz}{dt} + m.\,d.\,\frac{dz}{dt} - m.\,d.\frac{dz}{dt} + mR.dt;$$

the bodies would have at the subsequent instant, if they were not subjected to their mutual action; and if we also compute the motions, which they have in the subsequent instant, in consequence of their mutual action, the motions which must be compounded with the first of these, in order to produce the second, are such as if they acted on the system alone, would constitute an equilibrium between the bodies of the system; for if not, the second of the abovementioned motions would not be those which actually obtain, contrary to the hypothesis. But as these motions, which must be compounded with the motions which actually have place. in order to produce the first, are altogether unknown; in the analytical expressions, we substitute expressions equivalent to them, i. e. the quantities of motion which have actually place, taken in a direction contrary to their true one, and the motions which would take place, taken in the true direction, by means of this we are able to establish immediately equations of equilibrium between the first and second of the abovementioned species of motion, and also to determine the velocities which would take place, if the bodies composing the system were altogether free. Now if we suppose the preceding motions, resolved respectively into three others parallel to three rectangular coordinates, mP, mQ, mR, $m'P'$, &c. will represent the motions parallel to the three axes which the bodies would assume, if they were altogether free.

$$m.\frac{d^2x}{dt^2},\quad m.\frac{d^2y}{dt^2},\quad m.\frac{d^2z}{dt^2},\quad m'.\frac{d^2x'}{dt^2},\ \&c.$$

represent the motions parallel to the same axes, which the bodies actually have, at the commencement of the secondinstant. Since the motions which actually take place, are to be taken in a direction contrary to their true one, they are affected with negative signs.

We might by means of this principle, without introducing the consideration of virtual velocities, derive several important consequences; but it is the combination of this principle with that of virtual velocities, which has contributed so much to the perfection of Mechanics; this combination was first suggested by L'Agrange, who by this means has reduced the investigation of the motion of any system of bodies, to the integration of differential equations; thus we can reduce into an equation every problem relating to Dynamics, and it belongs to pure analysis to complete the solution; so that it appears that the only bar to the complete solution of every problem of Mechanics, arises from the imperfection of the analysis.

It is manifest from the introduction of the expression $\frac{d^2x}{dt^2}$, in place of the increase and as the forces

$$m.\frac{dx}{dt}+m.d.\frac{dx}{dt}\,;\ m.\frac{dy}{dt}+m.d.\frac{dy}{dt}\,;\ m.\frac{dz}{dt}+m.d.\frac{dz}{dt}\,;$$

of the velocity, that the changes in the motions of the body are made by insensible degrees.

The inspection of the equation (P) shews that it consists of two parts entirely distinct, of which one is the quantity which we ought to put equal to nothing, when the forces P, Q, R, P', &c. which are applied to the different points of the system, constitute an equilibrium, the other part arises from the motion which is produced by the forces P, Q, R, P', &c. when they do not constitute an equilibrium; therefore we may express the equation (P) in this manner:

$$0=\Sigma.(m.(P.\delta x+Q.\delta y+R.\delta z)-\Sigma m.\left\{\frac{d^2x}{dt^2}.\delta x+\frac{d^2y}{dt^2}.\delta y+\frac{d^2z}{dt^2}.\delta z.\right\}$$

and the equation (l) of No. 14, is only a particular case of the equation (P); thus the principle of virtual velocities may be considered as an universal instrument which is necessary for the solution of all problems relating to Mechanics. The expression

$$m.\frac{d^2x}{dt^2}.\ \delta x+m.\frac{d^2y}{dt^2}.\ \delta y+m.\frac{d^2z}{dt^2}.\ \delta z.$$

by which the equation (P) differs from the equation (l) is entirely independant of the *position* of the axes of the coordinates; for by substituting the coordinates x', y', z', in place of the preceding coordinates x, y, z, by the known formulæ we have

$$x=ax'+by'+cz',$$

$$y=a'x'+b'y'+c'z',$$

$$z=a''x'+b''y'+c''z',$$

the origin being the same, by differentiating the preceding expression twice, the coefficients a, b, c, a', &c. being constant, we obtain

$$d^2x=a.d^2x'+b.d^2y'+c.d^2z',$$

$$d^2y=a'.d^2x'+b'.d^2y'+c'.d^2z',$$

$$d^2z=a''.d^2x'+b''.d^2y'+c''.d^2z';$$

only remain; the forces

$$-m.d.\frac{dx}{dt}+P.dt;-m.d.\frac{dy}{dt}+Q.dt;\ -m.d.\frac{dz}{dt}+R.dt,$$

will be destroyed.

By distinguishing, in this expression, the characters m, x, y, z, P, Q, R, by one, two, marks, &c. successively, we shall have an expression for the forces destroyed in the bodies m', m'', &c. This being premised, if we multiply these forces by the respective variations of their directions δx, δy, δz, &c. we shall obtain, by means of the principle of virtual velocities, laid down in No. 14, the following equation, in which dt is supposed to be constant.

$$\left.\begin{array}{l} 0=m.\delta x.\left\{\frac{d^2x}{dt^2}-P.\right\}+m.\delta y.\left\{\frac{d^2y}{dt^2}-Q.\right\}+m.\delta z.\left\{\frac{d^2z}{dt^2}-R.\right\} \\ +m'.\delta x'\left\{\frac{d^2x'}{dt^2}-P'.\right\}+m'.\delta y'.\left\{\frac{d^2y'}{dt^2}-Q'.\right\}+m'\delta z'.\left\{\frac{d^2z'}{dt^2}-R'.\right\} \end{array}\right\};(P)$$

From this equation we may eliminate, by means of the particular conditions of the system, as many variations as we have conditions; and then by making the coefficients of the remaining variations separately

and also,

$$\partial x=a.\partial x'+b.\partial y'+c.\partial z',$$

$$\partial y=a'.\partial x'+b'.\partial y+c'.\partial z',$$

$$\partial z=a''.\partial x'+b''.\partial y''+c''.\partial z';$$

$\because$ by substituting for these expressions in the expression

$m.\frac{d^2x}{dt^2}.\partial x+m.\frac{d^2y}{dt^2}.+m.\frac{d^2z}{dt^2}$. we get $m.\frac{d^2x'}{dt^2}.\partial x'+m.\frac{d^2y'}{dt^2}.\partial y'+m.\frac{d^2z'}{dt^2}.\partial z'$.

for $a^2+a'^2+a''^2=1$, $ab+ac+bc=0$, &c. see Notes to page 7; the same substitutions being made in the expressions of the mutual distances between the bodies, the coefficients a, b, c, a', &c. will disappear for the same reasons.

equal to nothing, we shall obtain all the equations necessary for determining the motions of the several bodies of the system.

19. The equation (p) involves several general principles of motion, which we shall examine in detail. The variations δx, δy, δz, will be subjected to all the conditions of the connection of the * parts of the forces, by supposing them equal to the differentials dz, dy, dz, dx', &c.

* If the equation of condition involves the time explicitly, then we are not permitted to suppose the variations δx, δy, δz, equal to the differentials dx, dy, dz, as for instance, if one of the bodies composing the system, always existed on a given surface, which surface moved according to a given law; or if the body moved in a resisting medium, which medium was in motion, then there will exist an equation between the coordinates of the body and the time which will also be at any instant, the differential equation of the surface, the most general equation expressing the preceding condition, is of the following form:

$$\varphi.(x, y, z;\ x', y', z', \text{ \&c. } t) = 0,$$

at the following instant the coordinates will be varied by the quantities δx, δy, δz : $\delta x'$, $\delta y'$, &c and the equation of condition will become

$$\varphi.(x+\delta x,\ y+\delta y,\ z\,\delta z;\ x'+\delta x',\ y'+\delta y',\ z'+\delta z', \text{ \&c. } t) = F = 0,$$

∵ the difference of these two expressions, i. e.

$$\left\{\frac{dF}{dx}\right\}.\,\delta x+\left\{\frac{dF}{dy}\right\}.\,\delta y+\left\{\frac{dF}{dz}\right\}.\,\delta z+\left\{\frac{dF}{\delta x}\right\}.\,\delta x+\text{ \&c.} = 0,$$

but the complete differential of the preceding function =

$$\left\{\frac{dF}{dx}\right\}.\,dx+\left\{\frac{dF}{dy}\right\}.\,dy+\left\{\frac{dF}{dz}\right\}.\,dz+\left\{\frac{dF}{dx'}\right\}.\,dx', \text{ \&c.} + T.dt:$$

T is the differential coefficient of F, taken on the hypothesis that the time varies, consequently, if F involves the time explicitly, when we subject the variations δx, δy, &c. to satisfy the conditions of the connection of the parts of the system, we are not permitted to regard the expression

$$\left\{\frac{dF}{dx}\right\}.\,dx+\left\{\frac{dF}{dy}\right\}.\,dy+\left\{\frac{dF}{dz}\right\}.\,dz+\left\{\frac{dF}{dx'}\right\}.\,dx'+\text{\&c.}$$

as equal to nothing.

This supposition is consequently permitted, and then the integration of the equation (P) gives

$$\Sigma.m.\frac{(dx^2+dy^2+dz^2)}{dt^2}=c+2.\Sigma.\textstyle\int m.(P.dx+Q.dy+R.dz)\,; \quad (Q)$$

c being a constant arbitrary quantity introduced by the integration.

If the forces P, Q, R, are the results of attractive forces, directed towards fixed centres, and of a mutual attraction between the bodies; the function $\Sigma.\int m.(P.dx+Q.dy+Rdz)$* is an exact integral. For the

Q

* In fact, the accelerating force of m, produced by the action of m' in the direction of the line f, $=m'F$, (F is always a given function of f,) $\because$ the components of this force parallel to the axes of x, y, z, are $m'F.\frac{(x'-x)}{f}$, $m'F.\frac{(y'-y)}{f}$, $m'F.\frac{(z'-z)}{f}$, $\because$ the parts of $P.dx+Q.dy+R.dz$, which answers to this force alone are

$m'F.((x'-x).dx+(y'-y).dy+(z'-z).dz)$, and as the accelerating force of m', arising from the action of m, resolved parallel to the coordinates x, y, z, respectively $=$ $m.F.\frac{(x-x')}{f}+m.F.\frac{(y-y')}{f}+m.F.\frac{(z-z')}{f}$, the corresponding part of $P'.dx'+Q'.dy'+R'.dz'$, is, $F.m.\left\{\frac{(x-x')}{f}.dx'+\frac{(y-y')}{f}.dy'+\frac{(z-z')}{f}.dz'\right\}$, therefore in order to have the motive force arising from the mutual action of the bodies m and m' we must multiply the first expression by m, and the second by m', and adding them together, they will become

$$mm'.F.\,(x'-x).dx+(y'-y).dy+(z-z).dz+(x-x').dx'+(y-y').dy'+(z-z'),dz')=$$

$$mm'.F.fdf, \text{ for as } f^2=(x-x')^2+(y-y')^2+(z-z')^2,\ fdf=$$

$$(x-x').(dx-dx')+(y-y).(dy-dy')+(z-z').(dz-dz'),$$

consequently as F is given to be a function of f, $Ff.df.$ is an exact differential. If the centres to which the forces are directed have a motion in space, then $P.dx+Q.dy+Rdz$, is not an exact differential, though the law according to which the forces vary should be a function of the distance, see Note to page 34.

The sum of the living forces at any instant will be given by the equation (Q), when we know the value of this sum at a determined instant, and the coordinates of the bodies composing the system in the two positions of the system. And when the system returns to the same position, the living forces will be the same as before.

part which depends on the attractions directed towards fixed points, are exact integrals by No. 8. This is equally the case, with respect to those parts, which depend on the mutual attractions of the bodies composing the system; for if we name f, the distance of m from m', $m'F$, the attraction of m' on m; the part of $m(P.dx+Q.dy+R.dz)$ which arises from the attraction of m' on m, will be, by the above cited No. equal to $-mm'Fdf$, the differential df being taken on the supposition, that the coordinates x, y, z, only vary. But reaction being equal and contrary to action, the part of $m'(P'.dx'+Q'.dy'+R'dz')$ which is due to the attraction of m on m', is equal to $-mm'.Fdf$, the coordinates x', y', z', being the only quantities which are supposed to vary, consequently df being the differential of f on the supposition that both the coordinates x, y, z, and x', y', z', vary simultaneously, the part of the function $\Sigma.m(P.dx+Q.dy+R.dz)$ which depends on the reciprocal action of m on m' is equal to $-mm'.F.df$. Therefore this quantity is an exact differential when F is a function of f, or when the attraction varies as some function of the distance, which we shall always suppose; consequently the function $\Sigma.m.(P.dx+Q.dy+R.dz)$ is an exact differential, as often as the forces which act on the different bodies of the system, are the result of their mutual attraction, or of attractive forces directed towards fixed points. Let then $d\varphi$ represent this differential, and naming v the velocity of m, v' the velocity of m', &c. we shall have

$$\Sigma.mv^2 = c+2\varphi. \quad (R)$$

This equation corresponds to the equation (g) of No. 8, it is the analytical expression of the principle of the conservation of living forces. The product of the mass of a body by the square of its velocity, is termed the living force, or the vis viva of a body. The principle just announced consists in this, that the sum of the living forces, or the entire living force of the system is constant, if the system is not solicited by any forces; and if the bodies are actuated by any forces whatever, the sum of the increments of the entire living force is the same whatever may be the nature of the curves described, provided that their points of departure and arrival be the same.*

However this principle is only applicable, when the motions of the bodies change by imperceptible gradations.† If these motions undergo abrupt changes, the living force is diminished by a quantity which may be thus determined. The analysis which has conducted us to the equation (P) of the preceding number, gives us in this case, instead of that equation, the following:

$$0 = \Sigma.m. \left\{ \frac{\delta x}{dt}.\Delta.\frac{dx}{dt} + \frac{\delta y}{dt}.\Delta.\frac{dy}{dt} + \frac{\delta z}{dt}.\Delta.\frac{dz}{dt} \right\};*$$

* What has been demonstrated respecting the mutual attraction of the bodies of the system, is equally true respecting repulsive forces which vary as some function of the distance; it is true also when the repulsions are produced by the action of springs interposed between the bodies; for the force of the spring must vary as some function of the distance between the points, ∵ in the impact of perfectly elastic bodies though the quantity of motion communicated may be increased indefinitely, still the living force after the impact is the same as before; indeed *during* the impact, the vis viva varies as the coordinates of the respective points vary, but after its completion, from the nature of perfectly elastic bodies they resume their original position, and consequently the value of the vis viva will be the same as before, but if the elasticity is not perfect, in order to have the value of the vis viva at any instant, we should know the law of the elasticity, or the relation which exists between the compressive and restitutive force.

‡ When the motions of the bodies of the system, are modified by friction, or the resistance of the medium in which the motion is performed, the expression $P.dx+Q.dy+R.dz$ is not an exact differential, see note to page 34, and the living forces must be diminished. This is indeed evident of itself, for when the bodies of the system are actuated by no other forces but those of resistance, the sum of the living forces must be *gradually* diminished, in order to determine the actual loss experienced after any time, we should know the law according to which the resistance varies, which is very difficult to be determined; but there is another cause of diminution of the living force, in which we are able to determine accurately the loss sustained, to wit, the case adverted to in the text, when the bodies undergo an *abrupt* change in their motions.

‡ The characteristic Δ designates according to the received notation, the *difference* which exists between two consecutive states of the same quantity.

$$-\Sigma.m.(P.\delta x+Q.\delta y+R.\delta z);$$

$\Delta.\frac{dx}{dt}$, $\Delta.\frac{dy}{dt}$, $\Delta.\frac{dz}{dz}$, being the differences of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, from one instant to another; differences which become finite, when the motions of the bodies undergo finite alterations in an instant. In this

The equation (P) may be made to assume the following form:

$$\Sigma.m.\left\{\frac{d^2x}{dt^2}.\delta x+\frac{d^2y}{dt^2}.\delta y+\frac{d^2z}{dt^2}.\delta z.\right\}-\Sigma m.(P.\delta x+Q.\delta y+Q.\delta z).$$

in which the changes that are produced in the motions of the bodies composing the system, are made by insensible degrees, as is evident from the circumstance, that the differential of the velocities is expressed by $\frac{d^2x}{dt^2}$, see note to page 30; now, if instead of this gradual diminution, bodies experience abrupt changes in their motions $\Delta.\frac{dx}{dt}.\Delta.\frac{dy}{dt}$, &c. expressing those changes, the preceding expression will be changed into the following:

$$\Sigma.m.\left\{\left\{\Delta.\frac{dx}{dt}.\frac{\delta x}{dt}\right\}+\left\{\Delta.\frac{dy}{dt}.\frac{\delta y}{dt}\right\}+\left\{\Delta.\frac{dz}{dt}.\frac{\delta z}{dt}\right\}\right\}$$

$$-\Sigma.m.(P.\delta x+Q.\delta y+R.\delta y;$$

and as in this case $m\Delta.\frac{dx}{dt}$ is the variation of the force of the body, on the supposition that it is entirely free, and $m.P.dt$ is the variation which actually takes place in consequence of the action of the bodies of the system, the reasoning in No. 18 is applicable to this case, consequently the preceding expression may be put equal to nothing; and since the values of dx, dy, dz, are changed in the following instant into $dx+\Delta.dx$, $dy+\Delta.dy$. $dz+\Delta dz$, we shall satisfy the conditions of the connection of the parts of the system, by making the variations δx, δy, δz, equal to these expressions respectively; and then the preceding equation will assume this form

$$\Sigma.m.\left\{\Delta.\frac{dx}{at}+\frac{dx}{dt}\right\}\Delta.\frac{dx}{dt}+\left\{\Delta.\frac{dy}{dt}+\frac{dy}{dt}\right\}\Delta.\frac{dy}{dt}+$$

$$\left\{\Delta.\frac{dz}{dt}+\frac{dz}{dt}\right\}\Delta.\frac{dz}{dt}-$$

$$\Sigma.m.(P.(dx+\Delta.dx)+Q.(dy+\Delta.dy)+R.(dz+\Delta.dz),=0,$$

equation we may suppose

$$\delta x = dx + \Delta . dx \,;\ \delta y = dy + \Delta . dy \,;\ \delta z = dz + \Delta .\ dz \,;$$

because the values of dx, dy, dz, being changed in the following instant into $dx + \Delta . dx$, $dy + \Delta . dy$, $dz + \Delta . dz$, these values of δx, δy, δz, satisfy the conditions the connection of the parts of the system; therefore we shall have

$$0 = \Sigma . m. \left\{ \left\{ \frac{dx}{dt} + \Delta . \frac{dx}{dt} \right\} \Delta . \frac{dx}{dt} + \left\{ \frac{dy}{dt} + \Delta . \frac{dy}{dt} \right\} \Delta . \frac{dy}{dt} + \left\{ \frac{dz}{dt} + \Delta . \frac{dz}{dt} \right\} . \Delta \frac{dz}{dt} \right\}$$

$$- \Sigma . m . (P . (dx + \Delta . dx) + Q . (dy + \Delta . dy + R . (dz + \Delta . dz))$$

This equation should be integrated as an equation of finite differences relative to the time t, of which the variations are infinitely small, as well as the variations of x, y, z, x', &c. Let $\Sigma_{,}$ denote the finite integrals resulting from this integration, in order to distinguish them from the preceding finite integrals, which refer to the aggregate of all the bodies of the system. The integral of $mP.(dx + \Delta . dx)$ is evidently equal to $\int mP.dx$; therefore we shall have const. $=$

$$\Sigma . m. \frac{dx^2 + dy^2 + dz^2}{dt^2} + \Sigma_{,} . \Sigma m. \left\{ \left(\Delta . \frac{dx^2}{dt^2} \right) + \left(\Delta \frac{dy^2}{dt^2} \right) + \left(\Delta . \frac{dz^2}{dt^2} \right) \right\} *$$

$$-2\Sigma . \int m . (P.dx + Q.dp + R.dz)\,;$$

* In this equation, though the value of $\Delta . \frac{dx}{dt}$ may be finite, still $dx + \Delta . dx$, and the variation of the time may be indefinitely small, and ∵ integrating with respect to *this* quantity, $\Sigma_{,} \Sigma . m. \left\{ \frac{dx}{dt} . \Delta . \frac{dx}{dt} \right\} = \Sigma . m. \frac{dx^2}{dt^2}$, or it may be otherwise expressed thus, $\Delta . (x^2) = $ (see Lacroix No. 344) $2xh + h^2$, and if h be made equal to Δx, it becomes $2x . \Delta x + (\Delta . x)^2$, ∵ $2. \Sigma_{,} (x \Delta . x + (\Delta . x)^2) = \Sigma_{,} (2x . \Delta . x + (\Delta . x)^2) + \Sigma_{,} . (\Delta x)^2 = x^2 + \Sigma_{,} . (\Delta x^2)$, consequently, if we multiply the preceding equation by two, and substitute dx in place of x, and then integrate, we obtain the expression which has been given in the text.

therefore v, v, v'' denoting the velocities of m, m', m'', &c. we shall have

$$\Sigma.mv^2 = \text{Const.} - \Sigma_{\prime}\Sigma.m.\left\{\left\{\Delta.\frac{dx}{dt}\right\}^2 + \left\{\Delta.\frac{dy}{dt}\right\}^2 + \left\{\Delta.\frac{dz}{dt}\right\}^2\right\}$$

$$+ 2\Sigma.\textstyle\int m.(P.dx + Q.dy + R.dz).$$

The quantity contained under the sign $\Sigma_{\prime}$ being necessarily positive, we may perceive that the living force of the system is diminished by the mutual action of the bodies, as often as during the motion, any of the variations $\Delta.\frac{dx}{dt}$, $\Delta.\frac{dy}{dt}$, &c. are finite. Moreover, the preceding equation affords a simple means of determining the quantity of this diminution.

At each abrupt variation of the motion of the system,* the velocity

* At every abrupt change in the motion of the system, the velocity is not always diminished for *every* body, but the expression which is here given may be considered as general, by supposing that when the velocity is increased, a negative portion of it has been destroyed, and the square of the velocity after the shock is equal to

$$\Sigma.m.\frac{(dx^2 + 2dx\Delta.dx + (\Delta.dx)^2 + dy^2 + 2dy.\Delta.dy + (\Delta.dy)^2 + dz^2 + 2dz\Delta.dz + (\Delta dz.)^2}{dt^2}$$

and as

$$\Sigma m.\frac{2.dx\Delta.dx + 2(\Delta.dx)^2 + 2.dy\Delta.dy + 2(\Delta.dy)^2 + 2dz\Delta.dz + 2(\Delta.dz)^2}{dt^2},$$

$=0$, by subtracting this equation from the preceding, we obtain the square of the velocity after the shock, equal to

$$\Sigma.m.\frac{(dx^2 + dy^2 + dz^2)}{dt^2} - \Sigma.m.\frac{(\Delta.dx)^2 + (\Delta.dy)^2 + (\Delta.dz)^2}{dt^2}$$

and as the square of the velocity before the shock is equal to $\Sigma.mv^2 = \Sigma.m.\frac{dx^2 + dy^2 + dz^2}{dt^2}$, the square of the velocity lost by the shock $= \Sigma.m.V^2$ $= \Sigma m.\frac{(\Delta.dx)^2 + (\Delta.dy)^2 + (\Delta.dz)^2}{dt^2}$;

consequently the loss which the living forces experience, is equal to the sum of the living forces, which would belong to the system, if each body was actuated by that velocity which it loses by the shock.

of m, may be conceived to be resolved into two others, of which one v subsists in the following instant, the other V being destroyed by the action of the other bodies, but the velocity of m before the decomposition being $\frac{\sqrt{dx^2+dy^2+dz^2}}{dt}$, and changing afterwards into

$$\frac{\sqrt{(dx+\Delta.dx)^2+(dy^2+\Delta.dy)^2+(dz+\Delta.dz)^2}}{dt}$$

it is easy to perceive that

$$V^2=\left\{\Delta.\frac{dx}{dt}\right\}^2+\left\{\Delta.\frac{dy}{dt}\right\}^2+\left\{\Delta.\frac{dz}{dt}\right\}^2;$$

consequently the preceding equation may be made to assume the following form,

$$\Sigma.mv^2=\text{const.}-\Sigma_{,}.\Sigma.m.V^2-2.\Sigma.\textstyle\int m.(P.dx+Q.dy+.dz),^*$$

* The variation of the vis viva of the system, is equal to $2\Sigma m.(P.dx+Q.dy+R.dz)$ consequently when this expression vanishes, i. e. when $d.\Sigma.(mv^2)$ vanishes, the vis viva of the system, equal to $\Sigma.(mv^2)$, is a maximum, or a minimum; but it appears from the principle of virtual velocities, that $\Sigma m.(P.\delta x+Q.\delta y+R.\delta z)$ is equal to nothing, when the forces P, Q, R, P', constitute an equilibrium; and since the differentials dx, dy, dz, may be substituted for the variations δx, δy, δz, when they are subjected to satisfy the conditions of the connection of the parts of the system, $\Sigma m.(P.dx+Q.dy+R.dz)$ is equal to nothing, in the same circumstances; $\because$ when the forces P, Q, R, P', constitute an equilibrium, the vis viva of the system is a maximum or a minimum.

And as it appears from note to page 96, that the positions of equilibrium of a system of heavy bodies, correspond to the instants, when the centre of gravity is the highest or lowest possible, the sum of the living forces is always a maximum or a minimum when the centre ceases to ascend, and commences to descend, and when it ceases to descend and commences to ascend. The value of the vis viva is a minimum in the first case, and a maximum in the second, for $\Sigma m.(P.dx+Q.dy+R.dz)$ corresponds to the expression $S.\delta s+S'.\delta s'+S''\delta s''+\&c.$ in page 96, and $\because$ by substitution we have $\Sigma mv^2=c+s_{,}.S\Sigma m.$ consequently $\Sigma.mv^2$ is a maximum or minimum, when s, is a maximum or minimum. When $\Sigma.mv^2$. is a maximum, the equilibrium is stable; when a minimum, the equilibrium is instable. For from the definition of stability, (see No. 28) it appears that if the system is only agitated by one sole species of simple oscillation, the bodies composing it will perpe-

20. If in the equation (P) of No. 18, we suppose,

$$\delta x' = \delta x + \delta x_{\prime}' ; \quad \delta y' = \delta y + \delta y_{\prime}' ; \quad \delta z' = \delta z + \delta z_{\prime}' ;$$

tually tend to revert to the position of equilibrium, consequently their velocities will diminish according as their distance from the position of equilibrium is increased, and ∵ the sign of the second differential of φ will be negative, consequently Σmv^2. will be a maximum in this case; and it may be shewn by a like process of reasoning, that the vis viva of the system is a minimum, when the equilibrium is instable.

From a comparison of this observation with the note to page 96, it appears that in a system of heavy bodies, when the vis viva is a maximum, the centre of gravity is the lowest possible, and highest when the vis viva is a minimum.

This may be more strictly demonstrated thus: if the system be disturbed by an indefinitely small quantity from the position of equilibrium, by substituting for P, Q, R, P', &c. their values in terms of the coordinates, and then expanding the resulting expressioninto a series ascending according to the variations of these coordinates, the first term of the series will be the value of φ, when the system is in equilibrio; and since it is given, it may be made to coalesce with the constant quantity c, which was introduced by the integration; the second term vanishes by the conditions of the problem; and when $\Sigma.mv^2$. is a maximum, the theory of maxima and minima shews that the third term of the expansion may be made to assume the form of a sum of squares, affected with a negative sign, see Locroix, No. 134; the number of terms in this sum, being equal to the number of variations, or independant variables; the terms whose squares we have assumed, are linear functions of the variations of the coordinates, and vanish at the same time with them; they are therefore greater than the sum of all the remaining terms of the expansion. The constant quantity being equal to the sum of c, and of the value of $\Sigma.mv^2$. when the forces P, Q, R, P', &c. constitute an equilibrium, it is necessarily positive, and may be rendered as small as small as we please, by diminishing the velocities; but it is always greater than the greatest of the quantities whose squares have been substituted in place of the variations of the coordinates; for if it were less, this quantity being negative, would exceed the constant quantity, and therefore render the value of $\Sigma.mv^2$. negative, consequently these squares, and the variations of the coordinates, of which they are linear functions, must always remain very small, ∵ the system will always *oscillate* about the position of equilibrium, and this equilibrium will be stable. But in the case of a minimum it is not requisite that the variations should be always constrained to be very small, in order to satisfy the equation of living forces when φ is a minimum; this, indeed, does not prove that there is no limit then to these variations which is necessary, in order that the equilibrium may be instable; in order to shew this we should substitute for these variations, their values in functions of the time, and then shew from the form of these functions, that they increase indefinitely with the time, however small the primi-

$$\delta x'' = \delta x + \delta x_{,}'' \,; \quad \delta y'' = \delta y + \delta y_{,}'' \,; \quad \delta z'' = \delta z + \delta z_{,}'' \,;^{*}$$

&c. by substituting these variations, in the expressions of the variations δf, $\delta f'$, $\delta f''$, &c. of the mutual distances of the bodies composing the system, the values of which have been given in No. 15; we shall find that the variations δx, δy, δz, will disappear from those expressions. If the system be free, that is, if it have none of its parts connected with foreign bodies, the conditions relative to the mutual connection of the bodies, will only depend on their mutual distances, and therefore the variations δx, δy, δz, will be independent of these conditions; consequently when we substitute in place of $\delta x'$, $\delta y'$, $\delta z'$, $\delta x_{,}''$, &c. their preceding values in the equation (P), we should put the coefficients of the

R

tive velocities may be. For a complete solution of the problem of the small oscillations of a system, the reader is referred to the Mechanique Analytique of Lagrange, 5th and 6th section, seconde partie, where the important problem of coexisting oscillations is discussed in all its generality, and all difficulties are cleared up; see also Notes to No. 23 and 30, of this book.

* It is always possible to make these substitutions, for it in fact comes to transferring the origin of the coordinates to a point of which the coordinates are equal to x, y, z, respectively; as the expression for δf,

$$= \frac{(x'-x).(\delta x'-\delta x)+(y'-y).(\delta y'-\delta y)+(z'-z).(\delta z'-\delta z)}{f}$$

equal by substituting for x', y', z', $\delta x'$, $\delta y'$, $\delta z'$, their values,

$$\frac{(x+x_{,}'-x).(\delta x+\delta x_{,}'-\delta x)+(y+y_{,}'-y).(\delta y+\delta y_{,}'-\delta y)+(z+z_{,}'-z).(\delta z+\delta z_{,}'-\delta z)}{f} +$$

$$= \frac{x_{,}'\delta x_{,}'+y_{,}'.\delta y_{,}'+z_{,}'.\delta z_{,}'}{f}$$

consequently as δx, δy, δz, disappear from the expressions of the variations δf, $\delta f'$, and as when the system is at liberty, the conditions relating to the mutual connexion of its parts, depend only on their distance from each other, the variations δx, δy, δz, will be independent of these conditions, ∴ substituting for $\delta x'$, $\delta y'$, $\delta z'$ in the equation (P), the values which have been just given for them, the coefficients δx, δy, δz, must be put equal to nothing.

variations δx, δy, δz, separately equal to nothing; which gives the three following equations:

$$0=\Sigma m.\left\{\frac{d^2x}{dt^2}-P\right\};\quad 0=\Sigma m.\left\{\frac{d^2y}{dt^2}-Q\right\};\quad 0=\Sigma m.\left\{\frac{d^2z}{dt^2}-R\right\}*$$

Let us suppose that X, Y, Z, are the three coordinates of the centre of gravity of the system; by No. 15 we shall have,

$$X=\frac{\Sigma.mx}{\Sigma.m};\quad Y=\frac{\Sigma.my}{\Sigma.m};\quad Z=\frac{\Sigma.mz}{\Sigma.m};$$

consequently

$$0=\frac{d^2X}{dt^2}-\frac{\Sigma.mP}{\Sigma.m};\quad 0=\frac{d^2Y}{dt^2}-\frac{\Sigma.mQ}{\Sigma.m};\quad 0=\frac{d^2Z}{dt^2}-\frac{\Sigma.mR}{\Sigma.m},\dagger$$

therefore the motion of the centre of gravity of the system is the same

* By actually substituting for $\delta x'$, $\delta y'$, $\delta z'$, $\delta z''$, &c. in the equation (P) we obtain $0=$.

$$m.\delta x.\left\{\frac{d^2x}{dt^2}-P\right\}+m.\delta y.\left\{\frac{d^2y}{dt^2}-Q\right\}+m.\delta z.\left\{\frac{d^2z}{dt^2}-R\right\}$$

$$+m'.\delta x.\left\{\frac{d^2x'}{dt^2}-P'\right\}+m'.\delta x'_{,}.\left\{\frac{d^2x'}{dt^2}-P'\right\}$$

$$+m'\delta y.\left\{\frac{d^2y'}{dt^2}-Q'\right\}+m'\delta y'_{,}.\left\{\frac{d^2y'}{dt^2}-Q'\right\};$$

the terms in this expression which are multiplied by δx, δy, δz, respectively, are by adding them together

$$\Sigma.m.\left\{\frac{d^2x}{dt^2}-P\right\};\ \Sigma.m.\left\{\frac{d^2y}{dt^2}-Q\right\};\ \Sigma.m.\left\{\frac{d^2z}{dt^2}-R\right\}$$

and being independent of the conditions of the connection of the system, they must be put severally equal to nothing.

† Since $X=\frac{\Sigma.m.x}{\Sigma.m}$, $Y=\frac{\Sigma.my}{\Sigma.m}$, &c. $\frac{d^2X}{dt^2}=\frac{\Sigma.m.\frac{d^2x}{dt^2}}{\Sigma m}=\frac{\Sigma.mP}{\Sigma.m}$,

because

$$\Sigma.m.\frac{d^2x}{dt^2}-\Sigma m.P=0.$$

as if all the bodies m, m', &c. were concentrated in this point, the forces which solicit the system being applied to it.

If the system is only subjected to the mutual action of the bodies which compose it, and to their reciprocal attractions, we shall have

$$0 = \Sigma.mP;\ 0 = \Sigma.mQ;\ 0 = \Sigma.mR;$$

for p designating the reciprocal action of m on m', whatever its nature may be, and f denoting the mutual distance of these two bodies; we shall have, in consequence of this sole action,

$$mP = p.\frac{(x-x')}{f};\ mQ = p.\frac{(y-y')}{f}; mR = p.\frac{(z-z')}{f};$$

$$m'P' = p.\frac{(x'-x)}{f};\ m'Q' = p.\frac{(y'-y)}{f};\ m'R' = p.\frac{(z'-z)}{f};$$

from which we collect

$$0 = mP + m'P';\ 0 = mQ + m'Q';\ 0 = mR + m'R';\ *$$

and it is evident that these equations obtain, even in the case in

* $x'-x$, $y'-y$, $z'-z$, being the coordinates of m relative to the new origin of the forces, and the action of p being directed along the line

$$f\left(=\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}\right),$$

the part of mP, which corresponds to the force p resolved parallel to the axis of $x = p\,\frac{(x-x')}{f}$, the analogous parts of mQ, and mR, are $p.\frac{(y-y')}{f}$, $p.\frac{(z-z')}{f}$ respectively, in like manner the forces soliciting m' parallel to the coordinates, arising from the action of p,

$$= p.\frac{(x'-x)}{f}, p.\frac{(y'-y)}{f}, p.\frac{(z'-z)}{f};$$

$\therefore$ when the sole force soliciting m and m' arises from p, which expresses the reciprocal action of m on m', we have $mP + m'P', = p\,\frac{(x-x'+x'-x)}{f} = 0$.

Action being equal to reaction, and its direction being contrary thereto, when two bowhich the bodies exercise on each other, a finite action in an instant. Their reciprocal action disappears from the integrals $\Sigma.mP$, $\Sigma.mQ$, $\Sigma.mR$, and consequently, these expressions vanish, when the system is not solicited by any extraneous forces. In this case we have

$$0=\frac{d^2X}{dt^2};\ 0=\frac{d^2Y}{dt^2};\ 0=\frac{d^2Z}{dt^2};$$

and by integrating

$$X=a+bt:\ Y=a'+b't;\ Z=a''+b''t;^{*}$$

a, b, a', b', a'', b'', being constant arbitrary quantities. By eliminating the time t, we shall have an equation of the first order, between either X and Y, or X and Z; consequently the motion of the centre of gravity is rectilinear. Moreover, its velocity being equal to

$$\sqrt{\left\{\frac{dX}{dt}\right\}^2+\left\{\frac{dY}{dt}\right\}^2+\left\{\frac{dZ}{dt}\right\}^2};$$

or to $\sqrt{b^2+b'^2+b''^2}$, it is constant, and the motion is uniform.

It is manifest, from the preceding analysis, that this invariability of the motion of the centre of gravity of a system of bodies, whatever their mutual action may be,† subsists even in the case in which any one

dies concurring, exercise on each other a finite action in an instant, their reciprocal action will disappear in the expressions $\Sigma.mP$, $\Sigma.mQ$, &c. in fact, as we can always suppose the action of the bodies to be effected by means of a spring, interposed between them, which endeavours to restore itself after the shock, the effect of the shock will be produced by forces of the same nature with p, which, as we have seen, disappear in the expressions $\Sigma.mP$, $\Sigma.mQ$, $\Sigma.mR$.

* By integrating once we get $\frac{dX}{dt}=b$, $\therefore dX=bdt$, and $X=bt+a$; the constant quantities a, a', a'', are equal to the coordinates of the centre of gravity when $t=0$, and b, b', b'', are equal to the velocity of the centre of gravity resolved parallel to the coordinates. See notes to page 31.

† In fact, from what has been observed, in the note to page 116, it is evident that the principle of D'Alembert is true, whether the velocities acquired by the bodies be finite, after a given time, or indefinitely small, or whether the velocities be partly finite, and partly infinitely small, such as arise from the action of accelerating forces, and both

of the bodies loses in an instant, by this action, a finite quantity of motion.*

21. If we make

$$\delta x' = \frac{y'.\delta x}{y} + \delta x_{,}'\,;\ \delta x'' = \frac{y''.\delta x}{y} + \delta x_{,}''\,;\ \&c.$$

$$\delta y = -\frac{x.\delta x}{y} + \delta y_{,}\,;\ \delta y' = -\frac{x'.\delta x}{y} + \delta y_{,}'\,;\ \delta y'' = -\frac{x''.\delta x}{y} + \delta y_{,}''\,;\ \&c.\ \dagger$$

the variation δx will again disappear from the expressions δf, $\delta f'$, $\delta f''$, &c.; therefore, by supposing the system free, the conditions relative

before and after the impact, we have $0 = \frac{d^2 X}{dt^2}$, $0 = \frac{d^2 Y}{dt^2}$, &c. and also $\frac{dX}{dt}.\ \Sigma.m = \Sigma.m.\frac{dx}{dt}$, &c. = the quantity of motion, and since by hypothesis the quantity of motion lost, equal to the difference between $\Sigma m.\frac{dx}{dt}$ before and after impact, should be = to nothing, such as would cause an equilibrium in the system, it follows that $\frac{dX}{dt}.\Sigma.m.$ before and after impact must be the same, but $\Sigma.m$ being given, $\frac{dX}{dt}$ equal to the velocity of the centre of gravity, will be the same before and after impact.

* As the centre of gravity of a system, moves in the same manner as a body equal to the sum of the bodies would move, if placed in the centre of gravity, provided that the same momenta were communicated to it, which are impressed on the respective bodies of the system, the motion and direction of the centre of gravity, may be always determined by the law of composition of forces.

If the several bodies of a system were only subjected to their mutual action, then they would meet in the centre of gravity, for the bodies must meet, and the centre of gravity remains at rest.

† The fractional part of these expressions for $\delta x'$, $\delta x''$, δy, $\delta y'$, $\delta y''$, &c. arises from the rotatory motion of the system about an axis parallel to z, for it appears from Nos. 22 and 25, that when the direction of the impulse does not pass through the centre of gravity, the body acquires both a rotatory and rectilinear motion, now if the only motion impressed on the system was that of rotation, then the element of the angle described by the body m, is equal to the variation of the sine divided by the cosine $= \frac{\sqrt{x^2 + y^2}}{y}.\ \delta x$, the elementary angle described by

to the connection of the parts of the system will only influence the variations $\delta f'$, $\delta f''$ &c.; the variation δx is independent of them, and entirely arbitrary; thus by substituting in the equation (P) of No. 18, in the place of $\delta x'$, $\delta x''$, $\delta x'''$, &c. $\delta y'$, $\delta y''$, $\delta y'''$, &c. their preceding values,

$$m' = \frac{\sqrt{x^2+y^2}}{y} . \frac{\sqrt{x'^2+y'^2}}{\sqrt{x^2+y^2}} . \delta x = \frac{\sqrt{x'^2+y'^2}}{y} . \delta x,$$

$\therefore$ the variation of x' will be equal to

$\frac{\sqrt{x'^2+y'^2}}{y} . \frac{y'}{\sqrt{x'^2+y'^2}} . \delta x . = \frac{y'}{y} . \delta x$ the same may be proved of the other variations $\delta x'$, $\delta x''$,

$\sqrt{x^2+y^2}$ = the distance of m from the axis of z, $\therefore \frac{x}{\sqrt{x^2+y^2}}$ is equal to the sine of the angle which $\sqrt{x^2+y^2}$ makes with y. If the expression $\frac{\delta x \sqrt{x^2+y^2}}{y}$ be considered with respect to the cosine y, the variation $\delta y = -\delta x . \frac{\sqrt{x^2+y^2}}{y} . \frac{x}{\sqrt{x^2+y^2}}$ $= -\frac{\delta x . x}{y}$, for the variation of the cosine is equal to the variation of the arc affected with a negative sign, and divided by the sine, and as the variation of the angle described by $m' = \frac{\sqrt{x'^2+y'^2}}{y} . \delta x$, this expression being referred to the cosine is equal to $-\frac{\sqrt{x'^2+y'^2}}{y} . \frac{x'}{\sqrt{x'^2+y'^2}} . \delta x . = -\frac{x'}{y} . \delta x .$

If in the expression

$$\delta f = \frac{(x'-x).(\delta x'-\delta x)+(y'-y).(\delta y'-\delta y)+(z'-z).(\delta z'-\delta z)}{f}$$

we substitute for $\delta x'$, $\delta x''$, δy. $\delta y'$, $\delta y''$, &c. their values, it becomes

$$\frac{(x'-x).\left(\frac{y'.\delta x}{y}+\delta x_{,}'-\delta x\right)+(y'-y).\left(-\frac{x'.\delta x}{y}+\delta y_{,}'+\frac{x.\delta x}{y}-\delta y_{,}\right)}{f} =$$

$$\frac{x'y'\delta x}{y} - \frac{xy'.\delta x}{y} - x'.\delta x + x.\delta x + x'.\delta x_{,}' - x.\delta x_{,}' - \frac{y'x'.\delta x}{y} + \frac{y.x'\delta x}{y} + \frac{y'x.\delta x}{y}$$

$$-\frac{yx\,\delta x}{y} + y'\delta y_{,}' - y.\delta y_{,}' - y'\delta y_{,} + y\delta y_{,} \div f = \frac{(x'-x).\delta x_{,}'+(y'-y).(\delta y_{,}'-\delta y_{,})}{f}$$

therefore the variation δx disappears from the expressions δf, $\delta f'$, &c.

Making the same substitutions in the equation (P) it becomes we should put the coefficient of δx separately equal to nothing, which gives

$$0 = \Sigma.m.\frac{(xd^2y - yd^2x)}{dt^2} + \Sigma.m.\ (Py - Qx);$$

from which we deduce by integrating with respect to the time t,

$$c = \Sigma.m.\frac{(xdy - ydx)}{dt} + \Sigma.\textstyle\int.m.(Py - Qx).\ dt;$$

c being a constant arbitrary quantity.

In this integral, we may change the coordinates y, y', &c. into z, z', provided that we substitute in place of the forces Q, Q', &c. parallel to the axis of y, the forces R, R', parallel to the axis of z, which gives,

$$c' = \Sigma.m.\frac{(ydz - zdy)}{dt} + \Sigma.\textstyle\int.m.(Pz - Rx).\ dt;$$

c' being a new arbitrary quantity. In like manner we shall have

$$c'' = \Sigma.m.\frac{(ydz - zdy)}{dt} + \Sigma.\textstyle\int.m.(Qz - Ry).\ dt;$$

c'' being a third arbitrary quantity.

$$0 = \delta x.\left\{ m.\left\{\frac{d^2x}{dt_2} - P\right\} - \frac{x}{y}.m.\left\{\frac{d^2y}{dt^2} - Q\right\} + \frac{y'}{y}m'.\left\{\frac{d^2x'}{dt^2} - P'\right\}\right.$$

$$\left. - \frac{x'}{y}.m'.\left\{\frac{d^2y'}{dt^2} - Q'\right\}\right\}$$

$$+ m.dy_{,}\left\{\frac{d^2y}{dt^2} - Q\right\} + m'.\delta x'_{,}\left\{\frac{d^2x'}{dt^2} - P'\right\} \pm m'.\delta y'_{,}.\left\{\frac{d^2y'}{dt^2} - Q'\right\}\text{\&c.} =$$

$$\frac{\delta x}{y}.\left\{ m.\left\{\frac{y.d^2x - x.d^2y}{dt^2}\right\} + m'.\left\{\frac{y'\ d^2x' - x'd^2y'}{dt}\right\} - mPy + mQx - m'P'y' + m'Q'x',\ \text{\&c.}\right.$$

therefore if this expression is extended to all the coordinates, it will become

$$= \frac{\delta x}{y}.\left\{ \Sigma.m.\ \frac{y.d^2x - xd^2y}{dt^2} + \Sigma.m.(Q.x - P.y)\right\} + m.\delta y_{,}.\left\{\frac{d^2y}{dt^2} - Q\right\}$$

$$\left. + m'.\delta x'_{,}\left(\frac{d^2x'}{dt^2} - P'\right) + (m'.\delta y'_{,}(.\frac{d^2y'}{dt^2} - Q')\right\}$$

Let us suppose, that the bodies of the system are only subjected to their mutual action, and to a force directed towards the origin of the coordinates. Let p denote, as before, the reciprocal action of m on m', we shall have in consequence of this sole action,

$$0 = m.(Py - Qx) + m'.(P'y' - Q'x');$$

thus the mutual action of the bodies disappears from the finite integral $\Sigma.m.(Py - Qx)$. Let S be the force which solicits m towards the origin of the coordinates; in consequence of this sole force, we shall have

$$P = \frac{-S.x}{\sqrt{x^2+y^2+z^2}}; \quad Q = \frac{-S.y}{\sqrt{x^2+y^2+z^2}};$$

consequently the force S disappears from the expression $Py - Qx$, thus, in the case in which the different bodies composing the system are only solicited by their action and mutual attraction, and by forces directed towards the origin of the coordinates, we have

$$c = \Sigma.m.\frac{(xdy - ydx)}{dt}; \quad c' = \Sigma.m.\frac{(xdz - zdx)}{dt}; \quad c'' = \Sigma.m.\frac{(ydz - zdy)}{dt}$$

If we project the body m, on the plane of x and of y, the differential $\frac{xdy - ydx}{2}$, will represent the area which the radius vector, drawn from the origin of the coordinates to the projection of m, describes in the time dt; consequently the sum of the areas, multiplied respectively by the masses of the bodies, is proportional to the element of the time, from which it follows, that in a finite time, it is proportional to the time. It is this which constitutes the principle of the conservation of areas.*

* When the bodies are only subjected to their reciprocal action,

$$\Sigma.m.(Py - Qx) = m.(Py - Qx) + m'.(P'y' - Q'x') + \&c. =$$

by substituting for mP, mQ, their values, given in page 122,

$$p.\left\{\frac{(xy - x'y - yx + xy')}{f} + \frac{x'y' - xy' - y'x' + x'y}{f}\right\} = 0,$$

The fixed plane of x and of y being arbitrary, this principle obtains for any plane whatever, and if the force S vanishes, i. e. if the bodies are only subjected to their reciprocal action and mutual attraction, the origin of the coordinates is arbitrary, and may be in any point whatever. Finally, it is evident from what precedes, that this principle subsists, even when by the mutual action of the bodies composing the system, they undergo sudden changes in their motions.

There exists a plane, with respect to which c' and c'' vanish, and which, for this reason, it is interesting to know, for it is manifest that

see preceding number. If the bodies are solicited by forces directed towards a fixed point, then making this point the origin of the coordinates,

$$P = \frac{-Sx}{\sqrt{x^2+y^2+z^2}},\ Q = \frac{-Sy}{\sqrt{x^2+y^2+z^2}},\ \therefore Py - Qx = S.\ \frac{(xy-xy)}{\sqrt{x^2+y^2+z^2}} = 0,$$

consequently this force will also disappear from the expression $Py-Qx$, $\therefore$ in these two cases we have $c = \Sigma\, m.\,\frac{xdy-ydx}{dt}$, &c; $\frac{xdy-ydx}{2}$ = the area which the projection of the radius vector on the plane of x, y, describes in the time dt, see notes to No. 6, page 27. $\Sigma.m.(Py-Qx) = 0$, also when P and Q, &c. vanish, i. e. when the system is not actuated by any accelerating force, but only moved by an initial impulse; $\therefore$ the principle of the conservation of the areas obtains in these three cases; 1st. when the forces are only the result of the mutual action of the bodies composing the system; 2ndly, when the forces pass through the origin of the coordinates; and 3dly, when the system is moved by a primitive impulse. In the first and last case, the origin of the coordinates may be any point whatever. If there is a *fixed* point in the system, the equations (Z) are only true when this point is made the origin of the coordinates, any other point being made the origin, the moment $Py-Qx$ will not disappear, see notes to No. 3, page 12; if $\because$ in these circumstances the bodies are solicited by forces directed towards a given centre, this centre coincides with the fixed point of the system, when the equations (Z) obtain; if there are *two fixed* points in the system, only one of the equations (Z) will subsist, to wit, that which contains those coordinates, the plane of which is perpendicular to line joining the given points, the origin of the coordinates may be any point whatever in this line, see notes to No. 15, page 88.

The constant quantities c, c', c'', may be determined at any instant, when the velocities and the coordinates of the bodies of the system, are given at that instant.

*

the equality of c' and c'' to nothing, ought to simplify considerably the investigation of the motion of a system of bodies. In order to determine this plane, we must refer the coordinates $x,y,z,$ to three other axes having the same origin as the preceding. Let therefore θ represent the inclination of the required plane, formed by two of the new axes, with the plane of x and of y, and ψ the angle which the axis of x constitutes with the intersection of these two planes, so that $\frac{\pi}{2} - \theta$ may be the inclination of the third new axis with the plane of x and of y, and $\frac{\pi}{2} - \psi$ may represent the angle which its projection on the same plane, makes with the axis of x, π being the semi periphery.

In order to assist the imagination, let us suppose the origin of the coordinates to be at the centre of the earth; and that the plane of x and of y coincides with the plane of the ecliptic, and that the axis of z is the line drawn from the centre of the earth to the north pole of the ecliptic: moreover, let us suppose that the required plane is that of the equator, and that the third new axis, is the axis of rotation of the earth, directed towards the north pole; θ will represent the obliquity of the ecliptic, and ψ will be the longitude of the fixed axis of x, relative to the moveable equinox of spring. The two first new axes will be in the plane of the equator, and by calling φ, the angular distance of the first of those axes from this equinox, φ will represent the rotation of the earth reckoned from the same equinox, and $\frac{\pi}{2} + \varphi$ will be the angular distance of the second of these axes from the same equinox. We will name these three new axes, *principal axes.*

Let $x_{,}, y_{,}, z_{,}$ represent the coordinates of m referred, first to the line drawn from the origin of the coordinates, to the equinox of spring; $x_{,}$ being reckoned positive on this side of the equinox; 2dly, to the projection of the third principal axis on the plane of x and of y; 3dly to the axis of z, we shall have

$$x = x_{,}.\cos.\psi + y_{,}.\sin.\psi;$$

$$y = y_{,}.\cos.\psi - x_{,}.\sin.\psi;*$$

$$z = z_{,}.$$

Let $x_{,,}$, $y_{,,}$, $z_{,,}$, be the coordinates referred, 1st to the line of the equinox of spring; 2dly, to the perpendicular to this line in the plane of the equator; 3dly, to the third principal axis; we shall have

$$x_{,} = x_{,,};$$

$$y_{,} = y_{,,}.\cos.\theta + z_{,,}.\sin.\theta;$$

$$z_{,} = z_{,,}.\cos.\theta - y_{,,}.\sin.\theta.$$

Finally, let $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, be the cooordinates of m, referred to the first,

* As the axes of the coordinates $x_{,}$, $y_{,}$, exist in the plane of x, y, and as the angle which the axis of x makes with the axis of $x_{,}$ is equal to ψ, we have by the known formulæ for the transformation of one system of rectangular coordinates, into another system existing in the same plane,

$x = x_{,}.\cos.\psi + y_{,}.\sin.\psi$; $y = y_{,}.\cos.\psi - x_{,}.\sin.\psi$; and because the axis of z coincides with the axis of $z_{,}$, we have $z = z_{,}$. Comparing the coordinates, $x_{,}, y_{,}, z_{,}$, with the coordinates $x_{,,}, y_{,,}, z_{,,}$, it appears that the axis of $x_{,}$ coincides with the axis of $x_{,,}$, and consequently $x_{,} = x_{,,}$; and as the axis of $y_{,}$ is in the plane of the ecliptic, perpendicular to the line of equinox of spring, and as the axis of $y_{,,}$ exists in the plane of the equator perpendicular to the same line, it is manifest that the angle formed by these axes is equal to the angle θ, the inclination of the two planes, and that these two lines and the axes of $z_{,}$ and $z_{,,}$, which are respectively perpendicular to those planes, exist in the same plane, consequently we have, as before, $y_{,} = y_{,,}.\cos.\theta. + z_{,,}.\sin.\theta$, $z_{,} = z_{,,}.\cos.\theta. - y_{,,}\sin.\theta$. Lastly, it appears that the axis of $z_{,,}$ coincides with the axis of $z_{,,,}$, and consequently that $z_{,,} = z_{,,,}$; and as the axis of $x_{,,}$ and $y_{,,}$, and of $x_{,,,}$ and $y_{,,,}$ are in the plane of equator; and as by hypothesis, φ is equal to the angle which the axis of $x_{,,}$ makes with the line of equinox of spring, which line is supposed to coincide with the axis of $x_{,,}$, we have $x_{,,} = x_{,,,}.\cos.\varphi - y_{,,,}.\sin.\varphi$; $y_{,,} = y_{,,,}.\cos.\varphi + x_{,,,}.\sin.\varphi$. By substituting for $x_{,}$, $y_{,}$, $x_{,,}$, $y_{,,}$, their values, we obtain $x = x_{,}.\cos.\psi + y_{,}\sin.\psi = (x_{,,}.\cos.\psi + y_{,,}.\cos.\theta.\sin.\psi + z_{,,}.\sin.\theta.\sin.\psi) = (x_{,,,}.\cos.\varphi.\cos.\psi - y_{,,,}.\cos.\psi.\sin.\varphi + y_{,,,}.\cos.\theta.\sin.\psi.\cos.\varphi. + x_{,,,}.\cos.\theta.\sin.\psi.\sin.\varphi + z_{,,,}.\sin\theta.\sin.\psi)$, ∴ by concinnating we obtain $x = x_{,,,}(\cos.\theta.\sin.\psi.\sin.\varphi + \cos.\varphi.\cos.\psi) + y_{,,,}(\cos.\theta.\sin.\psi.\cos.\varphi. - \cos.\psi.\sin.\varphi) + z_{,,,}.\sin.\theta.\sin.\psi$, which is the expression given in the text; by a similar process we could derive values for y and z.

second, and third principal axes ; we shall have

$$x_{,,} = x_{,,,}.\cos.\varphi - y_{,}.\sin.\varphi;$$

$$y_{,,} = y_{,,,}.\cos.\varphi + x_{,,,}.\sin.\varphi;$$

$$z_{,,} = z_{,,,}.$$

From which it is easy to deduce

$$x = x_{,,,}.(\cos.\theta.\sin.\psi.\sin.\varphi + \cos.\psi.\cos.\varphi) +$$
$$y_{,,,}.(\cos.\theta.\sin.\psi.\cos.\varphi - \cos.\psi.\sin.\varphi)$$
$$z_{,,,}.(\sin.\theta.\sin.\psi);$$

$$y = x_{,,,}.(\cos.\theta.\cos.\psi.\sin.\varphi - \sin.\psi.\cos.\varphi) +$$
$$y_{,,,}.(\cos.\theta.\cos.\psi.\cos.\varphi + \sin.\psi.\sin.\varphi)$$
$$+ z_{,,,}(\sin.\theta.\cos.\psi);$$

$$z = z_{,,,}.\cos.\theta - y_{,,,}.\sin.\theta.\cos.\varphi - x_{,,,}\sin.\theta.\sin.\varphi.^{*}$$

If we multiply these values of x, y, z, by the respective coefficients of

* If any line x is drawn from the origin of the coordinates $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, and if A, B, C, represent the cosines of the angles which x makes with $z_{,,,}$, $y_{,,,}$, $z_{,,,}$, respectively, then $x = Ax_{,,,} + By_{,,,} + Cz_{,,,}$, for if a perpendicular erected from the extremity of x meets a line r, whose coordinates are $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, then $\frac{x}{r}$ is equal to the cosine of the angle which x makes with r, and $\frac{x_{,,,}}{r}$, $\frac{y_{,,,}}{r}$, $\frac{z_{,,,}}{r}$, are equal to the cosines of the angles which r makes with $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, $\because$ we have by note to page 7, $\frac{x}{r} = A.\frac{x_{,,,}}{r} + B.\frac{y_{,,,}}{r} + C.\frac{z_{,,,}}{r}$, $\because x = Ax_{,,,} + By_{,,,} + Cz_{,,,}$. Consequently we infer that the coefficients of $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, in the expression given in the text for x, y, z, are equal to the cosines of the angles which the axis of x, y, z, make with the principal axes respectively; therefore sin. θ. sin. ψ, is equal to the cosine of the angle which the axis of $z_{,,,}$, makes with the axis of

$x_{,,,}$, in the preceding expressions; we shall have, by adding them together,

$$x_{,,,} = x.(\cos.\theta.\sin.\psi.\sin.\varphi+\cos.\psi.\cos.\varphi) +$$

$$y.(\cos.\theta.\cos.\psi.\sin.\varphi-\sin.\psi.\cos.\varphi)-z.\sin.\theta.\sin.\varphi.$$

By multiplying in like manner the values of x, y, z, by the respective coefficients of $y_{,,,}$ in the same expression, and afterwards by the coefficients of $z_{,,,}$, we shall have

$$y_{,,,} = x.(\cos.\theta.\sin.\psi.\cos.\varphi-\cos.\psi.\sin.\varphi.)+$$

x, ∵ equal to the cosine of the angle which the plane of $y_{,,,}$, $x_{,,,}$ makes with the plane y, z; in like manner $\sin.\theta.\cos.\psi$, is equal to the cosine of the angle contained between the axis of $z_{,,,}$ and of y, = the cosine of the inclination of the plane $x_{,,,}$, $y_{,,,}$ to the plane x, z, also $\sin.\theta.\sin.\varphi$, $\sin.\theta.\cos.\varphi$, $\cos.\theta$. are equal to the cosines of the angles which the axis of z makes with the axes of $x_{,,,}$, $y_{,,,}$, and $z_{,,,}$, respectively, see No. 27.

We may observe that in the general expressions for the transformations of one system to another of rectangular coordinates, which are of the following form:

$$x = Ax_{,,,}+By_{,,,}+Cz_{,,,},$$
$$y = A_{,}.x_{,,,}+B_{,}.y_{,,,}+C_{,}.z_{,,,},$$
$$z = A_{,,}.x_{,,,}+B_{,,}.y_{,,,}+C_{,,}.z_{,,,},$$

there are six equations of condition, i. e.

$$A^2+A_{,}^2+A_{,,}^2, =1 \qquad AB+A_{,}B_{,}+A_{,,}B_{,,}=0,$$
$$B^2+B_{,}^2+B_{,,}^2, =1 \qquad AC+A_{,}C_{,}+A_{,,}C_{,,}=0,$$
$$C^2+C_{,}^2+C_{,,,}^2, =1 \qquad BC+B_{,}C_{,}+B_{,,}C_{,,}=0,$$

which are derived from the identity between the expressions $x^2+y^2+z^2$, and $x_{,,,}^2+y_{,,,}^2+z_{,,,}^2$, for they are respectively equal to the square of the distance of the same point, from the common origin of the coordinates, ∵ three of the nine coefficients which are introduced by the transformation, may be regarded as undetermined; these three undetermined quantities are, in fact, the angles θ, ψ, and φ; for, by substituting in the six preceding equations of condition for A, B, C, $A_{,}$, &c. their values in functions of the angles θ, ψ, and φ, the resulting equations will become identical, and there arises no relation between θ, ψ, and φ.

$$y.(\cos.\theta.\cos.\psi.\cos.\varphi+\sin.\psi.\sin.\varphi)-z.\sin.\theta.\cos.\varphi;$$

$$z_{,,,}=x.\sin.\theta.\sin.\psi+y.\sin.\theta.\cos.\psi.+z.\cos.\theta.$$

These different transformations will be very useful hereafter, we will obtain the coordinates corresponding to the bodies m', m'', &c.: by placing one, two, &c. marks above the coordinates $x_{,}$, $y_{,}$, $z_{,,,}$, $y_{,,,}$, $z_{,,}$.*

* If we actually perform this operation we shall obtain

$$x.(\cos.\theta.\sin.\psi.\sin.\varphi+\cos.\psi\cos.\varphi)=x_{,,,}.(\cos.^2\theta.\sin.^2\psi.\sin.^2\varphi+\cos.^2\psi.\cos.^2\varphi+2\cos.\theta.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi.)$$

$$+y_{,,,}.(\cos.^2\theta.\sin.^2\psi.\sin.\varphi.\cos.\varphi+\cos.\theta.\sin.\psi.\cos\psi.\cos.^2\varphi-\cos.\theta.\sin.\psi.\cos.\psi.\sin.^2\varphi-\cos.^2\psi.\sin.\varphi.\cos.\varphi.)$$

$$+z_{,,,}.(\sin.\theta.\cos.\theta.\sin.^2\psi.\sin.\varphi+\sin.\theta.\sin.\psi.\cos.\psi.\cos.\varphi);$$

$$y.(\cos.\theta.\cos.\psi.\sin.\varphi-\sin.\psi.\cos.\varphi.)$$

$$=x_{,,,}.(\cos.^2\theta.\cos.^2\psi.\sin.^2\varphi+\sin.^2\psi.\cos.^2\varphi-2(\cos.\theta.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi)$$

$$+y_{,,,}.(\cos.^2\theta.\cos.^2\psi.\sin.\varphi.\cos.\varphi+\cos.\theta.\sin.\psi.\cos.\psi.\sin.^2\varphi-\cos.\theta.\sin.\psi.\cos.\psi.\cos.^2\varphi-\sin.^2\psi.\sin.\varphi.\cos.\varphi.)$$

$$+z_{,,,}.(\sin.\theta.\cos.\theta.\cos.^2\psi.\sin.\varphi-\sin.\theta.\sin.\psi.\cos.\psi.\cos.\varphi)$$

$$-x.\sin.\theta.\sin.\varphi=-z_{,,,}.\sin.\theta.\cos.\theta.\sin.\varphi+y_{,,,}.\sin.^2\theta.\sin.\varphi.\cos.\varphi+x_{,,,}.\sin.^2\theta.\sin.^2\varphi;$$

adding these three equations together, and making the terms which are at the right hand side to coalesce, we shall get the coefficients of $x_{,,,}=$ to $\cos.^2\theta.\sin.^2\varphi+\cos^2\varphi.+\sin.^2\theta.\sin.^2\varphi$, $(=\sin.^2\varphi-\sin.^2\theta.\sin.^2\varphi+\cos.^2\varphi+\sin.^2\theta.\sin.^2\varphi)=1$, the coefficients of $y_{,,,}$ will be equal to $\cos.^2\theta.\sin.\varphi.\cos.\varphi,-\sin.\varphi.\cos.\varphi+\sin.^2\theta.\sin.\varphi.\cos.\varphi=0$, in like manner the coefficient of $z_{,,}=\sin.\theta.\cos.\theta.\sin.\varphi-\sin.\theta.\cos.\theta.\sin.\varphi=0$; the terms at the other side are those which have been given in the text. In like manner to obtain the value of $y_{,,,}$, a corresponding multiplication gives

$$x.(\cos.\theta.\sin.\psi.\cos.\varphi-\cos.\psi.\sin.\varphi)=$$

$$x_{,,,}.(\cos.^2\theta.\sin.^2\psi.\sin.\varphi.\cos.\varphi+\cos.\theta.\sin.\psi.\cos.\psi.\cos.^2\varphi-\cos.\theta.\sin.\psi.\cos.\psi.\sin.^2\varphi-\cos.^2\psi.\sin.\varphi.\cos.\varphi)$$

$$+y_{,,,}(\cos.^2\theta.\sin.^2\psi.\cos.^2\varphi+\cos.^2\psi.\sin.^2\varphi-2.\cos.\theta.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi)$$

$$+z_{,,,}(\sin.\theta.\cos.\theta.\sin.^2\psi.\cos.\varphi-\sin\theta.\sin.\psi.\cos.\psi.\sin.\varphi)$$

From what precedes, it is easy to conclude, by substituting c, c', c'', in place of

$$\Sigma.m.\frac{(xdy-ydx)}{dt}, \Sigma.m.\frac{(xdz-zdx)}{dt}, \Sigma m.\frac{(ydz-zdy)}{dt},$$

$$y.(\cos.\theta.\cos.\psi.\cos.\varphi+\sin.\psi.\sin.\varphi)$$

$$=x_{,,,}(.\cos.^2\theta.\cos.^2\psi.\sin.\varphi.\cos.\varphi-\cos.\theta.\sin.\psi.\cos.\psi.\cos.^2\varphi+\cos.\theta.\sin.\psi.\cos.\psi.\sin.^2\varphi-\sin.^2\psi.\sin.\varphi.\cos.\varphi)$$

$$+y_{,,,}(\cos.^2\theta.\cos.^2\psi.\cos.^2\varphi+\sin.^2\psi.\sin.^2\varphi+2.\cos.\theta.\sin.\varphi.\cos.\varphi.\sin.\psi.\cos.\psi)$$

$$+z_{,,,}.\sin.\theta.\cos.\theta.\cos.^2\psi.\cos.\varphi+\sin.\theta.\sin.\psi.\cos.\psi.\sin.\varphi.)$$

$$-z.\sin.\theta.\cos.\varphi=$$

$$-z_{,,,}\sin.\theta.\cos.\theta.\cos.\varphi+y_{,,,}.\sin.^2\theta.\cos.^2\varphi+x_{,,,}.\sin.^2\theta.\sin.\varphi.\cos.\varphi,$$

adding those quantities together, and concinnating as before, we obtain

$$x.(\cos.\theta.\sin.\psi.\cos.\varphi-\cos.\psi.\sin.\varphi)+y.(\cos.\theta.\cos.\psi.\cos.\varphi+\sin.\psi.\sin.\varphi)$$

$$-z.\sin.\theta.\cos.\varphi=$$

$$x_{,,,}.(\cos.^2\theta.\sin.\varphi.\cos.\varphi-\sin.\varphi.\cos.\varphi+\sin.^2\theta.\sin.\varphi.\cos.\varphi)=0,$$

$$+y_{,,,}(\cos.^2\theta.\cos.^2\varphi+\sin.^2\varphi+\sin.^2\theta.\cos.^2\varphi)=y_{,,,},$$

$$+z_{,,,}.(\sin.\theta.\cos.\theta.\cos.\varphi-\sin.\theta.\cos.\theta.\cos.\varphi)=0.$$

For the value of $z_{,,,}$, by performing similar operations, we obtain $x.\sin.\theta.\sin.\psi=$

$$x_{,,,}.(\sin.\theta.\cos.\theta.\sin.^2\psi.\sin.\varphi+\sin.\theta.\sin.\psi.\cos.\psi.\cos.\varphi)$$

$$+y_{,,,}.(\sin.\theta.\cos.\theta.\sin.^2\psi.\cos.\varphi-\sin.\theta.\sin.\psi.\cos.\psi.\sin.\varphi)$$

$$+z_{,,,}.(\sin.^2\theta.\sin.^2\psi.$$

$$y.\sin.\theta.\cos.\psi=$$

$$x_{,,,}.(\sin.\theta.\cos.\theta.\cos.^2\psi.\sin.\varphi-\sin.\theta.\sin.\psi.\cos.\psi.\cos.\varphi)$$

$$+y_{,,,}.(\sin.\theta.\cos.\theta.\cos.^2\psi.\cos.\varphi+\sin.\theta.\sin.\psi.\cos.\psi.\sin.\varphi)+z_{,,,}(\sin.^2\theta.\cos.^2\psi.)$$

$$z.\cos.\theta=-x_{,,,}.\sin.\theta.\cos.\theta.\sin.\varphi-y_{,,,}.(\sin.\theta.\cos.\theta.\cos.\varphi+z_{,,,}.(\cos.^2\theta),$$

∴ adding the corresponding quantities together, we obtain that $\Sigma m.\frac{(x_{,,,}dy_{,,,}-y_{,,,}dx_{,,,})}{dt}=c.\cos.\theta-c'.\sin.\theta.\cos\ \psi+c''.\sin.\theta.\sin\psi$;

$$\Sigma m.\ \frac{x_{,,,}.dz_{,,,}.-z_{,,,}.dx_{,,,}}{dt}=c.\sin.\theta.\cos.\varphi\ *$$

$$x.(\sin.\theta.\sin.\psi)+y.\sin.\theta.\cos.\psi+z.\cos.\theta=$$

$$x_{,,,}.(\sin.\theta.\cos.\theta.\sin.\varphi-\sin.\theta.\cos.\theta.\sin.\varphi.)=0,$$

$$+y_{,,,}.(\sin.\theta.\cos.\theta.\cos.\varphi-\sin.\theta.\cos.\theta.\cos.\varphi)=0,\ +z_{,,,}.(\sin.^2\theta.+\cos.^2\theta)=z_{,,,}.$$

* When we substitute for the expression $x_{,,,}dy_{,,,}-y_{,,,}dx_{,,,}$, the respective values of $x_{,,,}$, $dx_{,,,}$, $y_{,,,}$, $dy_{,,,}$, in functions of x, dx, y, dy, and of the angles θ, ψ, and φ, it is not necessary to take into account any expression, in which the variable part is the product of a coordinate into its own differential, because this expression occurs again, affected with a sign, the opposite to that, with which it was affected before. By performing the prescribed multiplication of the value of $x_{,,,}$ into the value of $dy_{,,,}$ of $y_{,,,}$ into $dx_{,,,}$ we obtain

$$x_{,,,}.dy_{,,,}=x.dy.(\cos.^2\theta.\sin.\psi.\cos\ \psi.\sin.\varphi.\cos.\varphi+\cos.\theta.\sin.^2\psi.\sin.^2\varphi+\cos.\theta.\cos.^2\psi.\cos.^2\varphi+\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi),$$

$$+y.dx.(\cos.^2\theta.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi-\cos.\theta.\cos.^2\psi.\sin.^2\varphi-\cos.\theta.\sin.^2\psi.\cos.^2\varphi+\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi),$$

$$-z.dx.(\sin.\theta.\cos.\theta.\sin.\psi.\sin.\varphi.\cos.\varphi-\sin.\theta.\cos.\psi.\sin.^2\varphi),$$

$$-z.dy.(\sin.\theta.\cos.\theta.\cos.\psi.\sin.\varphi.\cos.\varphi+\sin.\theta.\sin.\psi.\sin.^2\varphi),$$

$$-x.dz.(\sin.\theta.\cos.\theta.\sin.\psi.\sin.\varphi.\cos.\varphi+\sin.\theta.\cos.\psi.\cos.^2\varphi),$$

$$-y.dz.(\sin.\theta.\cos.\theta.\cos.\psi.\sin.\varphi.\cos.\varphi-\sin.\theta.\sin.\psi.\cos.^2\varphi),$$

$$y_{,,,}.dx_{,,,}=x.dy.(\cos.^2\theta.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi-\cos.\theta.\sin.^2\psi.\cos.^2\varphi-\cos.\theta.\cos.^2\psi.\sin.^2\varphi+\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi),$$

$$+y.dx.(\cos.^2\theta.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi+\cos.\theta.\cos.^2\psi.\cos.^2\varphi+\cos.\theta.\sin.^2\psi.\sin.^2\varphi+\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi),$$

$$-z.dx.(\sin.\theta.\cos.\theta.\sin.\psi.\sin.\varphi.\cos.\varphi+\sin.\theta.\cos.\psi.\cos.^2\varphi),$$

$$-z.dy.(\sin.\theta.\cos.\theta.\cos.\psi.\sin.\varphi.\cos.\varphi-\sin.\theta.\sin.\psi.\cos.^2\varphi),$$

$$-x.dz.(\sin.\theta.\cos.\theta.\sin.\psi.\sin.\varphi.\cos.\varphi-\sin.\theta.\cos.\psi.\sin.^2\varphi),$$

$$-y.dz.(\sin.\theta.\cos.\theta.\cos.\psi.\sin.\varphi.\cos.\varphi+\sin.\theta.\sin.\psi.\sin.^2\varphi);$$

$c'.(\sin.\psi.\sin.\varphi+\cos.\theta.\cos.\psi.\cos.\varphi)+c''.(\cos.\psi.\sin.\varphi-\cos.\theta.\sin.\psi.$
$$\cos.\varphi);\ \frac{\Sigma m.y_{,,,}.dz_{,,,}-z_{,,,}.dy_{,,,}}{dt}=-c.\sin.\theta.\sin.\varphi.$$

$$+c'.(\sin.\psi.\cos.\varphi-\cos.\theta.\cos.\psi.\sin.\varphi)$$
$$+c''.\cos.\psi.\cos.\varphi+\cos.\theta.\sin.\psi.\sin.\varphi).$$

If we determine ψ and θ, so that we may have $\sin.\theta.\sin.\psi$

$$=\frac{c''}{\sqrt{c^2+c'^2+c''^2}};\ \sin.\theta.\cos.\psi=\frac{-c'}{\sqrt{c^2+c'^2+c''^2}},$$ which gives

$\cos.\theta=\dfrac{c}{\sqrt{c^2+c'^2+c''^2}}$ we shall have *

$$\Sigma m.\frac{x_{,,,}.dy_{,,,}-y_{,,,}.dx_{,,,}}{dt}=\sqrt{c^2+c'^2+c''^2}\ \dagger$$

∵ subducting $x_{,,,}.dy_{,,,}$ from $y_{,,,}.dx_{,,,}$, and making the terms whose variable parts are the same coalese, we obtain $x_{,,,}.dy_{,,,}-y_{,,,}.dx_{,,,}=(x.dy-y.dx).\cos.\theta+(x.dz-z.dx).\sin.\theta.\cos.\psi+(y.dz-z.dy).\sin.\theta.\sin.\psi$; and substituting for $x.dy-y.dx$, $x.dz-z.dx$, &c. their values c', c'', we obtain $c.\cos.\theta.-c'.\sin.\theta.\cos.\psi,+c''.\sin.\theta.\sin.\psi$; $=x_{,,,}.dy_{,,,}-y_{,,,}.dx_{,,,}$, by a similar analysis we arrive at the expressions for $x_{,,,}.dz_{,,,}-z_{,,,}.dx_{,,,}$, $y_{,,,}.dz_{,,,}-z_{,,,}.dy_{,,,}$, which are given in the text.

* For $\sin.^2\theta.\sin.^2\psi+\sin.^2\theta.\cos.^2\psi=\sin.^2\theta=\dfrac{c'^2+c'^2}{c^2+c'^2+c''^2}$ ∴ $\cos.^2\theta.=1-\sin.^2\theta$

$$=\frac{c^2}{c^2+c'^2+c''^2}.$$

† For substituting in place of $\cos.\theta$, $\sin.\theta.\cos.\psi$, $\sin.\theta.\sin.\psi$, these values, we shall have

$$\Sigma.m.\frac{x_{,,,}.dy_{,,,}-y_{,,,}.dx_{,,,}}{dt}=\frac{c^2}{\sqrt{c^2+c'^2+c''^2}},+\frac{c'^2}{\sqrt{c^2+c'^2+c''^2}},+\frac{c''^2}{\sqrt{c^2+c'^2+c''^2}},$$

$=\sqrt{c^2+c'^2+c''^2}$, and if we substitute for c, c', c'', their values, $\sqrt{c^2+c'^2+c''^2}$, $\cos.\theta$, $-\sqrt{c^2+c'^2+c''^2}$, $\sin.\theta.\cos.\psi.$ $+\sqrt{c^2+'c^2+''c^2}$, $\sin.\theta.\sin.\psi$, the expression $\Sigma.m.\dfrac{x_{,,,}.dz_{,,,}-z_{,,,}.dx_{,,,}}{dt}$ will become $\sqrt{c^2+c^2+c^2}$, $(\sin.\theta.\cos.\theta.\cos.\varphi,-\sin.\theta.\sin.\psi,$

$$\Sigma m.\frac{x_{,,,}.dz_{,,,}-z_{,,,}.dx_{,,,}}{dt}=0;\ \Sigma m.\frac{y_{,,,}.dz_{,,,}-z_{,,,}.dy_{,,,}}{dt}=0;$$

$\therefore$ the values of c' and c'' vanish with respect to the plane of $x_{,,,}$ and $y_{,,,}$, determined in this manner. There exists only one plane, which possesses this property, for supposing that it is the plane of x and y, we shall have

$$\Sigma m.\frac{x_{,,,}.dz_{,,,}-z_{,,,}.dx_{,,,}}{dt}=c.\sin.\theta.\cos.\varphi;\ \Sigma m.\frac{y_{,,,}.dz_{,,,}-z_{,,,}.dy_{,,,}}{dt}=-c.\sin.\theta.\sin.\varphi;$$

If these two functions are put equal to nothing, we shall have $\sin.\theta=0$, which shews that the plane of $x_{,,,}$ and $y_{,,,}$, then coincides with the plane of x and y. Since the value of $\Sigma m.\frac{x_{,,,}.dy_{,,,}-y_{,,,}.dx_{,,,}}{dt}$ is equal to $\sqrt{c^2+c'^2+c''^2}$, whatever may be the plane of x and y, it follows that the quantity $c^2+c'^2+c''^2$ is the same, whatever this plane may be, and that the plane of $x_{,,,}$ and $y_{,,,}$, determined by the preceding analysis is that, with respect to which the function $\Sigma m.\frac{x_{,,,}.dy_{,,,}-y_{,,,}.dx_{,,,}}{dt}$ is a maximum; therefore, this plane * possesses these remarkable pro-

$\cos.\psi.\sin.\varphi-\sin.\theta.\cos.\theta.\cos.^2\psi.\cos.\varphi+\sin.\theta.\sin.\psi+\cos.\psi.\sin.\varphi-\sin.\theta.\cos.\theta.\sin.^2\psi.\cos.\varphi)=\sqrt{c^2+c'^2+c''^2},(\sin.\theta.\cos.\theta.\cos.\varphi-\sin.\theta.\cos.\theta.\cos.\varphi)=0$ the same is true respecting the expression $\Sigma.m.\frac{y_{,,,}.dz_{,,,}-z_{,,,}.dy_{,,,}}{dt}$.

* As the cosines of the angles which the axes of $z_{,,,}$ makes with the axes z, y, x, i.e. the cosines of the angles which the plane $x_{,,,}.y_{,,,}$, makes with the planes x, y; x, z; y, z, (see note to page 133,) are equal to $\cos.\theta$, $\sin.\theta.\cos.\psi$, $\sin.\theta.\sin.\psi$, it follows that when we have the projections c, c', c'', of any area on three coordinate planes, we have its projection $\Sigma m.(x_{,,,}.dy_{,,,}-y_{,,,}.dx_{,,,})$ on the plane $x_{,,,}.y_{,,,}$ whose position, with respect to the three planes x,y; x,z; y,z, is given. In like manner it follows from the exression, $\Sigma.m.\left\{\frac{x_{,,,}.dy_{,,,}-y_{,,,}.dx_{,,,}}{dt}\right\}$, which has been given in the text, that for all planes equally inclined to the plane on which the projection is the greatest, the values of the projection of the area are equal, for supposing the plane of x, y to be the invariable plane, then $\Sigma.m.\left\{\frac{x.dy-y.dx}{dt}\right\}$, will be the greatest possible, $\Sigma.m.\left\{\frac{x.dz-z.dx}{dt}\right\}$,

perties—first, that the sum of the areas traced by the projection of the radii vectores of the respective bodies, and multiplied by their masses, is the greatest possible; secondly, that the same sum, vanishes relative to any plane, which is perpendicular to it, because the angle φ is undetermined. By means of these properties, we shall be able to find this plane at any instant, whatever variations may be induced in the respective positions of the bodies by their mutual action; we can, in like manner very easily

$\Sigma.m. \left\{\frac{y.dz - z.dy}{dt}\right\}$, are respectively equal to nothing, $\because$ $\Sigma.m. \left\{\frac{x_{,,,}.dy_{,,,} - y_{,,,}.dx_{,,,}}{dt}\right\}$ $= \Sigma.m. \left\{\frac{x.dy - y\,dx}{dt}\right\}. \cos.\theta.$

Since c, c', c'', are constant quantities, and proportional to the cosines of the angles which the plane on which the projection of the area is a maximum, makes with the coordinate planes, it follows, that the position of *this* plane is always fixed and *invariable;* and as the quantities c, c', c'', depend on the coordinates of the bodies at any instant, and on the velocities $\frac{dx}{dt}$, &c. parallel to the coordinates, when these quantities are given, we can determine the position of this invariable plane; we have termed this plane invariable, because it depends on the quantities c, c', c'', which are constant during the motion of the system, provided that the bodies composing it are only subjected to this mutual action, and to the action of forces directed towards a fixed point. (See page, 128.)

Since the plane x, y is indetermined in the text, we conclude, that the sum of the squares of the projections of any area, existing in the invariable plane, on any three coordinate planes passing through the same point in space is constant; consequently, if we take on the axes to any coordinate planes y, z; x, z; x, y, lines proportional to c, c', c'', then the diagonal of a parallepiped, whose sides are proportional to those lines, will represent the quantity and direction of the greatest moment, and this direction is the same whatever three coordinate planes be assumed, but the *position* in absolute space is undetermined, for the projections on all *parallel* planes are evidently the same. The conclusions to which we have arrived, respecting the projections of areas on coordinate planes, are in like manner applicable to the projections of moments, since as has been observed in Note, page 28, these moments are geometrically exhibited by triangles of which the bases represent the projected force, the altitudes being equal to perpendiculars let fall from the point to which the moments are referred, on the direction of the bases.

When the forces applied to the different points of the system have an unique resultant, V; then since the sum of the moments of any forces projected on a plane is equal to the moment of the projection of their resultant, it follows necessarily, that find at all times the position of the centre of gravity of the system, and for this reason it is as natural to refer the position of the coordinates x and y to this plane, as to refer their origin to the centre of gravity.*

22. The principles of the preservation of living forces, and of areas, obtain when the origin of the coordinates has a uniform rectilinear motion in space. To demonstrate this, let X, Y, Z, represent the coordinates of this origin, supposed to be in motion, with reference to a fixed point, and let us suppose

$$x = X + x_{,}; \; y = Y + y_{,}; \; z = Z + z_{,};$$

$$x' = X + x'_{,}; \; y' = Y + y'_{,}; \; z = Z + z'_{,}; \text{ \&c.}$$

$x_{,}$, $y_{,}$, $z_{,}$; $x'_{,}$, &c. will be the coordinates of m, m', &c. relative to the

the unique resultant V and the point to which the moments are referred, must exist in the invariable plane; ∴ the axis of this plane must be perpendicular to this resultant, and as $\frac{P}{V}$, $\frac{Q}{V}$, $\frac{R}{V}$, are equal to the cosines of the angles which V makes with the coordinates, and as

$$\frac{c}{\sqrt{c^2+c'^2+c''^2}}, \; \frac{c'}{\sqrt{c^2+c'^2+c''^2}}, \; \frac{c''}{\sqrt{c^2+c'^2+c''^2}},$$

are equal to the cosines of the angles which the axes to the invariable plane makes with the same coordinates, we have

$$\frac{cP+c'Q+c''R}{\sqrt{c^2+c'^2+c''^2}} = 0, \; \therefore \; cP+c'Q+c''R = 0. \text{ (See note to No. 1, page 7.)}$$

* Besides the advantages adverted to in the text, it may be observed, that our investigations are considerably simplified by the circumstance of two of the constant arbitrary quantities c, c', c'', vanishing when we make the plane of projection the invariable plane. It may also be remarked that this plane always subsists when the bodies composing the system are not solicited by any forces beside those of mutual attraction, and of forces directed towards fixed points; nor is the position of this plane affected in any respect when two or any number of bodies impinge on each other; for as we have before observed, these impacts dont cause any change in the expressions $Py-Qx$, &c.—on the equality of which to nothing depends, the principle of the conservation of areas, and the

moveable origin. We shall have by hypothesis,

$$d^2X = 0;\ d^2Y = 0;\ d^2Z = 0;$$

but we have by the nature of the centre of gravity, when the system is free

$$0 = \Sigma m.(d^2X + d^2x_{,}) - \Sigma m.P.dt^2;*$$

$$0 = \Sigma m.(d^2Y + d^2y_{,}) - \Sigma m.Q.dt^2;$$

$$0 = \Sigma m.(d^2Z + d^2z_{,}) - \Sigma m.R.dt^2;$$

position of this invariable plane. The practical rule for the determination of this plane is given in the exposition, Du Systeme du Monde, page 207, the investigation of this rule will be given in No. 62, of the second book.

We shall see in No. 26, chapter 7, that the consideration of this plane is of great service in the determinations of the motions of a body of any figure whatever.

* $0 = \Sigma.m.d^2x - \Sigma.m.P.dt^2$; $0 = \Sigma.m.d^2y - \Sigma.m.Q.dt^2$; $0 = \Sigma.m.dz^2 - \Sigma.m.R.dt^2$; substituting in place of d^2x, d^2y, d^2z, we obtain the expression in the text; and since d^2X is by hypothesis equal to 0, the expression, $0 = \Sigma.m.(d^2x_{,} + d^2X - \Sigma.P.dt^2) = \Sigma.m.d^2x_{,} + d^2X.\ \Sigma.m. - \Sigma.m.P.dt^2 = \Sigma.m.dx_{,}^2 - \Sigma.m.Pdt_{,}^2$, &c.; in like manner, substituting for δx, δy, &c. in the equation (P), we obtain

$$0 = \Sigma.m. \left\{ \delta X + \delta x_{,} \right\} \left\{ \frac{d^2x}{dt^2} - P. \right\} + \Sigma.m.(\delta Y + \delta y_{,} \left\{ \frac{d^2y}{dt^2} - Q. \right\} + \&c. =$$

$$\Sigma.m\delta x_{,}. \left\{ \frac{d^2x}{dt^2} - P. \right\} + \Sigma.m. \left\{ \frac{d^2x}{dt^2} - P. \right\}.\ \delta X + \Sigma.m.\delta y_{,}. \left\{ \frac{dy^2}{dt^2} - Q \right\}$$

$$\left\{ + \Sigma.m. \left\{ \frac{d^2y}{dt^2} - Q. \right\} \delta Y + \&c. \right.$$

but as by the nature of the centre of gravity

$\Sigma.m. \left\{ \frac{d^2x}{dt^2} - P. \right\}$, $\Sigma.m. \left\{ \frac{d^2y}{dt^2} - Q. \right\}$, &c. are respectively equal to nothing, and also $d^2x = d^2x_{,}$, $d^2y = d^2y_{,}$, &c. the preceding expression becomes

$$0 = \Sigma.m.\delta x_{,}. \left\{ \frac{d^2x_{,}}{dt^2} - P. \right\} + \Sigma.m.\delta y_{,}. \left\{ \frac{d^2y_{,}}{dt^2} - Q. \right\} \&c.$$

and by substituting $\delta X+\delta x_{,}$ $\delta Y+\delta y_{,}$ $\delta Z+\delta z_{,}$ &c. in place of δx, δy, δz, &c. the equation (P) of No. 18, will also become

$$0=\Sigma m.\delta x_{,}.\left\{\frac{d^2x_{,}}{dt^2}-P.\right\}+\Sigma m.\delta y_{,}.\left\{\frac{d^2y_{,}}{dt^2}-Q.\right\}$$

$$+\Sigma m.\delta z_{,}.\left\{\frac{d^2z_{,}}{dt^2}-R\right\};$$

which is precisely of the same same form as the equation (P), if the forces P, Q, R, P', depend only on the coordinates $x_{,}$, $y_{,}$, $z_{,}$, $x'_{,}$, &c. Therefore by applying to it the preceding analysis, we can obtain the principle of the preservation of living forces and of areas, relative to the moveable origin of the coordinates.

If the system is not acted on by any extraneous forces, its centre of gravity will move uniformly in a rectilinear direction in space as we have seen in No. 20; therefore, by fixing the origin of the coordinates x, y and z at this centre these principles will always have place, X, Y, and Z, being in this case the coordinates of the centre of gravity, by the nature of this point, we shall have

$$0=\Sigma.mx;\ 0=\Sigma m.y;\ 0=\Sigma m.z;$$

consequently we have

$$\Sigma.m.\left\{\frac{x.dy-y.dx}{dt}\right\}=\frac{X.dY-Y.dX}{dt}.\ \Sigma.m+\Sigma m.\ \frac{x_{,}.dy_{,}-y_{,}.dx_{,}}{dt};*$$

$$*\ \Sigma.m.\left\{\frac{X+x_{,}).dY+dy_{,})-(Y+y_{,})(dX+dx_{,})}{dt}\right\}=$$

$$\frac{\Sigma.m.X.dY+\Sigma mx_{,}dY+\Sigma.m.Xd.y_{,}+\Sigma.m.x_{,}dy_{,}}{dt}$$

$-\dfrac{\Sigma.m.Y.dX-\Sigma.m.y_{,}dX-\Sigma m.Y.dx_{,}-\Sigma.my_{,}.dx_{,}}{dt}$, and as $\Sigma.mx_{,}$, $\Sigma.my_{,}$, $\Sigma m.dx_{,}$, $\Sigma.m.dy_{,}$ are respectively equal to nothing by the nature of the centre of gravity, the preceding

$$\Sigma.m. \frac{dx^2+dy^2+dz^2}{dt^2}=\frac{dX^2+dY^2+dZ^2}{dt^2}\Sigma.m.$$

$$+\Sigma.m.\left\{\frac{dx_{,}^2+dy_{,}^2+dx_{,}^2}{dt}\right\};*$$

thus the quantities which result from the preceding principles are com-

expression becomes equal to

$$\frac{X.dY-Y.dX.}{dt}\Sigma m.+\Sigma.m.\frac{x_{,}dy_{,}-y_{,}.dx_{,}}{dt}=c, \text{\&c.}$$

* $\Sigma.m.dx^2=\Sigma.m.dX^2+2\Sigma.m.dx_{,}dX+\Sigma.m.dx_{,}^2$, and as $2dX.\ \Sigma.m.dx_{,}=0$, we have $\Sigma.m.dx^2=dX^2.\ \Sigma.m+\Sigma.m.dx_{,}^2$, $\therefore$

$$\Sigma.m.\frac{dx^2+dy^2+dz^2}{dt^2}=\frac{dX^2+dY^2+dZ^2}{dt^2},\ \Sigma.m+\Sigma.m.\frac{dx_{,}^2+dy_{,}^2+dz_{,}^2}{dt^2}\ \text{\&c.}=c+2\varphi.$$

If all the bodies were concentrated in their common centre of gravity, $x_{,}, y_{,}$; $dx_{,}, dy_{,}$; would vanish, therefore the second part of the first members of the preceding equation would vanish, and we would have $\frac{X.dY-Y.dX}{dt}\Sigma.m.=c$, $\frac{dX^2+dY^2+dZ^2}{dt^2}.\Sigma m=c+2\varphi$. Consequently, it appears from what has been established in this number, that when the bodies composing the system are not acted on by foreign forces, the quantities which are concerned in the principles of living forces, and of areas are composed of quantities which would have existed, if all the bodies of the system were concentrated in their centre of gravity; and 2dly, of quantities which would obtain if the centre of gravity quiesced, the former description of quantities are represented respectively by $\frac{dX^2+dY^2+dZ^2}{dt^2}\Sigma.m$, $\Sigma m.\frac{XdY-YdX}{dt}$, and the latter by $\Sigma m.\frac{dx_{,}^2+dy_{,}^2+dz_{,}^2}{at^2}$, $\Sigma m.\frac{x_{,}dy-y_{,}.dx_{,}}{dt}$, the first indicates what obtains in consequence of the progressive motion of the system, the second what arises from a rotatory motion, about an axis passing through the centre of gravity. (See No. 25.)

If the origin of the coordinates x, y, z, be transferred to a point of which the coordinates are A, B, C, the expression for the projection of area on the plane x, y, becomes $\Sigma m.\ \frac{(x-A)\,dy-(y-B).\,dx}{dt}=\Sigma.m\left\{\frac{xdy-y.dx}{dt}\right\}-\frac{A.\Sigma m.\ dy+B.\Sigma m\ dx}{dt}$, but as $\Sigma m.\,dy$, $\Sigma m.\ dx=dY\Sigma m$, $dX.\Sigma m$; $-\frac{A\Sigma m\ dy+B\Sigma m.dx}{dt}$ becomes $-\frac{A.dY+B.\,dX}{dt}.\Sigma.m.$

posed, 1st of quantities which would obtain, if all the bodies of the system were concentrated in the centre of gravity; 2dly, of quantities relative to the centre of gravity supposed immoveable; and since the first described quantities are constant, we may perceive the reason why the principles in question have place with respect to the centre of gravity. Therefore if we place the origin of the coordinates at this point, the equation Z, of the preceding number will always subsist; from which it follows that the plane which constantly passes through this centre, and with respect to which the function $\Sigma.m.\left\{\frac{x.dy-y.dx}{dt}\right\}$

∴ the projection of the area on the plane x, y, with respect to the new origin becomes equal to $c+\frac{B.\,dX-A.\,dY,}{dt}.\Sigma m$, and similar expressions may be derived for the projections on the planes x,z,y,z, From this it appears, that for all points in which $\frac{B.dX-A.dY}{dt}.\Sigma m=0$ the value of c will remain constantly the same, but it is evident that this equation will be satisfied, if the locus of the origin of the coordinates be either the right line described by the centre of gravity, or any line parallel to this line, consequently for all such lines the position of the invariable plane will remain constantly parallel to itself; however, though for all points of the *same* parallel the position of the invariable plane is constant, yet in the transit from one parallel to another the *direction* of this plane changes.

If the forces which act on the several points of the system are reducible to an unique resultant, by making the origin of the coordinates any point in this resultant, the quantities c, c', c'', and therefore the plane with respect to which the projection of the areas is a maximum, will vanish, if the locus of the origin of the coordinates be a line parallel to this resultant, the value of the projection of the area with respect to this line on the plane x, y, will be constant and equal to $\frac{B.\,dX-A.\,dY}{dt}.\Sigma m$ for c in this case vanishes, if the locus of the origin of the coordinates be a right line *diverging* from this resultant, the expression $\frac{Bd\,X-A\,dY}{dt}.\Sigma m$ is susceptible of perpetual increase. From these observations it appears that when the forces admit an unique resultant, that point with respect to which the value of $\sqrt{c^2+c'^2\;c''^2}$ is least of all is a point so circumstanced, that the axis or perpendicular to the plane of greatest projection passing through this point, is parallel to the direction of the unique resultant;

is a maximum, remains always parallel to itself, during the motion of the system, and that the same function relative to every other plane which is perpendicular to it, is equal to nothing.

The principles of the conservation of areas, and of living forces, may be reduced to certain relations between the coordinates of the mutual distances of the bodies composing the system. In fact, the origin of the coordinates x, y, z, being supposed always to be at the centre of gravity; the equations (Z) of the preceding number, may be made to assume the following form

$$c.\Sigma.m = \Sigma mm'.\left\{\frac{(x'-x).)dy'-dy)-(y'-y).(dx'-dx)}{dt}\right\};$$

$$c'.\Sigma m = \Sigma.mm'.\left\{\frac{(x'-x).(dz'-dz)-(z'-z).)dx'-dx)}{dt}\right\};$$

$$c''.\Sigma.m = \Sigma.mm'.\left\{\frac{(y'-y).dz'-dz)-(z'-z).(dy'-dy)}{dt}\right\}.*$$

It may be remarked, that the second members of these equations

* This expression is proved to be true with respect to three bodies in the following manner and as the same reasoning is applicable to any number of bodies whatever, it may be considered as a general proof

$$\Sigma mm'\left\{\frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt}\right\} = mm'x'\frac{dy'}{dt} - mm'x\frac{dy'}{dt} - mm'x'.\frac{dy}{dt}$$

$$+ mm'.x\frac{dy}{dt} - mm'y'.\frac{dx'}{dt} + mm'y'.\frac{dx}{dt} + mm'.y\frac{dx'}{dt} - mm'y.\frac{dx}{dt} + mm''.x''\frac{dy''}{dt}$$

$$-mm''.x.\frac{dy''}{dt} - mm''.x''.\frac{dy}{dt} + mm''.x\frac{dy}{dt} - mm''.y''.\frac{dx''}{dt} + mm''.y''.\frac{dx}{dt} + mm''.y.\frac{dx''}{dt}.$$

$$-mm''.y\frac{dx}{dt} + m''.m'.x''.\frac{dy''}{dt} - m''.m'.x'\frac{dy''}{dt} - m''m'x''\frac{dy'}{dt} + m'm''.x'.\frac{dy'}{dt}$$

$$-m''m'.y''.\frac{dx''}{dt} + m''m'.y''.\frac{dx'}{dt} + m''m'y'.\frac{dx''}{dt} - m'm''.y'.\frac{dx'}{dt}$$

multiplied by dt, express the sum of the projections of the elementary areas, traced by each line which joins the two bodies of the system, of which one is supposed to move round the other, considered as immoveable, each area being multiplied by the product of the two masses, which are connected by the right line.

and by concinnating it comes out equal to

$$mm'.\left\{\frac{x'dy'-y'.dx'}{dt}\right\}+mm'.\left\{\frac{xdy-y.dx}{dt}\right\}-mm'.\left\{\frac{x'.dy-y'.dx}{dt}\right\}$$

$$-mm'.\left\{\frac{xdy'-y.dx'}{dt}\right\}+mm''.\left\{\frac{x''.dy''-y''.dx''}{dt}\right\}+mm''.\left\{\frac{xdy-y.dx}{dt}\right\}$$

$$-mm''.\left\{\frac{xdy''-y.dx''}{dt}\right\}-mm''.\left\{\frac{x''.dy-y''.dx}{dt}\right\}+m''.m'.\left\{\frac{x''.dy''-y''.dx''}{dt}\right\}$$

$$+m'm''.\left\{\frac{x'dy'-y'.dx'}{dt}\right\}-m''m'.\left\{\frac{x'.dy''-y'.dx''}{dt}\right\}-m''.m'.\left\{\frac{x''.dy'-y''dx'}{dt}\right\}$$

and as in the case of three bodies

$$c=\Sigma m.\left\{\frac{xdy-y.dx}{dt}\right\}=m.\left\{\frac{xdy-y.dx}{dt}\right\}+m'.\left\{\frac{x'dy'-y'dx'}{dt}\right\}$$

$$+m''.\left\{\frac{x''.dy''-y''.dx''}{dt}\right\}\therefore c.\Sigma m=c(m+m'+m'')=m^2\left\{\frac{xdy-y.dx}{dt}\right\}$$

$$+m'.^2\left\{\frac{x'.dy'-y'.dx'}{dt}\right\}+m''.^2\left\{\frac{x''.dy''-y''.dx''}{dt}\right\}+mm'\left\{\frac{xdy-y.dx}{dt}\right\}$$

$$+mm''\left\{\frac{xdy-y.dx}{dt}\right\}+mm'\left\{\frac{x'dy'-y'.dx'}{dt}\right\}+m'm''.\left\{\frac{x'dy'-y'.dx'}{dt}\right\}$$

$$+mm''\left\{\frac{x''.dy''-y''.dx''}{dt}\right\}+m'm''\left\{\frac{x''.dy''-y''.dx''}{dt}\right\}.$$

By the nature of the centre of gravity we have $mx+m'x'+m''x''=0$ and also $mdy+m'dy'+m''.dy''=0$ $\therefore$ their product vanishes *i, e,* $m^2xdy+m'^2x'dy'+m''^2x''.dy''+mm'x'.dy+mm''.x''.dy+mm'.xdy'+m'm''.x''dy'+m.m''.xdy''+m''m'.x'.dy''=0$ $\therefore$ we have $m^2xdy+m'^2x'.dy'+m''.^2x''.dy''=-mm'.x'dy-mm''.x''.dy-mm'.xdy'-m'm''.x''dy-mm''.xdy''.-m''.m'.x'dy''$, and by multiplying $my+m'y'+m''y''$, into $mdx+m'dx'+m''dx''$;$-m^2ydx$

By applying to the preceding equations, the analysis of No. 21, it will appear, that the plane passing constantly through any of the bodies of the system, and with respect to which the function

$$\Sigma.mm'.\left\{\frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt}\right\}$$

is a *maximum*, remains always parallel to itself, during the motion of the system, and that this plane is parallel to the plane passing through

U 2

$-m'^2\ y'.dx'-m''^2\,y''.\,dx''=+mm'.y'.\,dx+mm''\,y''.\,dx+mm'y.dx'\ +m'm''y''dx'$ $+mm''.y.dx''+m''.m'.y'.dx''$, $\therefore$ adding these quantities together we obtain

$$m^2\left\{\frac{xdy-ydx}{dt}\right\}+m'.^2\left\{\frac{x'.dy'-y.'dx'}{dt}\right\}+m''.^2\left\{\frac{x''dy''-y''.dx''}{dt}\right\}$$

$$=-mm'.\left\{\frac{x'.dy-y'.dx}{dt}\right\}-mm''\left\{\frac{x''.dy-y''.dx}{dt}\right\}-mm'.\left\{\frac{x.dy'-y.dx'}{dt}\right\}$$

$$-mm''.\left\{\frac{xdy''-ydx''}{dt}\right\}-m''m'.\left\{\frac{x'dy''-y'.dx''}{dt}\right\}-m''m'.\left\{\frac{x''.dy'-y''.dx'}{dt}\right\}$$

$\therefore$ if in the expression for $c(m+m'+m'')$ we substitute in place of the sum of the functions which are multiplied by the squares of the masses, the quantities which are equivalent to them we shall obtain $c\ (m+m'+m'')=$

$$mm'\left\{\frac{xdy-y.dx.}{dt}\right\}+mm''.\left\{\frac{xdy-y.dx}{dt}\right\}+mm'.\left\{\frac{x'dy'-y'.dx'}{dt}\right\}$$

$$+m'm''.\left\{\frac{x'dy'-y'.dx'}{dt}\right\}+mm''\left\{\frac{x''dy''-y''.dx''}{dt}\right\}+m'.m''.\left\{\frac{x''.dy''-y''.dx''}{dt}\right\}$$

$$-mm'\left\{\frac{x'dy-y'.dx}{dt}\right\}-mm''\left\{\frac{x''.dy-y''dx}{dt}\right\}-mm'.\left\{\frac{xdy'-y.dx'}{dt}\right\}$$

$$-m''m'.\left\{\frac{x''.dy'-y''dx'}{dt}\right\}-m''m.\left\{\frac{xdy''-ydx''}{dt}\right\}-m''.m'\left\{\frac{x'.dy''-y'.dx''}{dt}\right\}$$

which is equal to to the expression which has been given above for the value of $\Sigma m.m'.$ $\left\{\frac{(x'-x)\ dy'-dy)--y'-y\ (dx'-dx}{dt}\right\}$

the centre of gravity, and relatively to which, the function $\Sigma.m. \dfrac{(x.dy-y.dx)}{dt}$ is a *maximum.* It will also appear that the second members of the preceding equations vanish with respect to all planes passing through the same body, and perpendicular to the plane in question.†

The equation (Q) of No. 19, can be made to assume the form*

$$\Sigma.mm'. \left\{\frac{(dx'-dx)^2+(dy'-dy)^2+(dz'-dz)^2}{dt}\right. = \text{const.} - 2\Sigma.m.$$

$\Sigma.\int mm'. Fdf$; this equation respects solely the coordinates of the mu-

* When there are but three bodies $\Sigma.m.dx^2 = mdx^2+m'dx'^2+m''.dx'',^2$ but by the nature of the centre of gravity we have $mdx+m'dx'+m''.dx''=0$ and $\therefore m^2 dx^2 + m'.^2 dx'^2 + m''.^2 dx''^2 + 2m.m'. dx.dx' + 2mm''. dx.dx'' + 2m'm''. dx'.dx'', = 0$, and multiplying $\Sigma.m.dx^2$ by $\Sigma m.$ we obtain, $m^2 dx^2 + m'.^2 dx'^2 + m''.^2 dx''^2 + mm' dx'^2 + m'm''. dx'^2 + m'm.dx^2 + m''.m\, dx^2 + mm''.dx''^2 + m'm''.dx'',^2$ if we substract the previous equation from this we get, $mm'.dx'^2 + m'm''.dx'^2 + m'm.dx^2 + m''.m.dx^2 + mm''.dx''.^2 + m'm''.dx''^2 - 2mm'. dx\, dx' - 2mm''.dx. dx'' - 2m'. m''.dx'.dx'' = mm'. (dx'-dx)^2 + m'm'' (dx''-dx')^2 + mm'' (dx''-dx)^2 = \Sigma.mm'. (dx'-dx)^2 = \Sigma.m. (\Sigma m. (dx^2)$, and in like manner we derive $\Sigma.mm' (dy'-dy)^2 = \Sigma m. (\Sigma m. dy^2), \Sigma.mm'. (dz'-dz)^2 = \Sigma m. (\Sigma m. dz^2)$, $\therefore$ we have $\Sigma mm' \left\{\dfrac{dx'-dx)^2 + (dy'-dy)^2 + (dz'-dz^2}{dt^2}\right\} = c. \Sigma m + \Sigma.m. (2. \Sigma m.\int m. (P.dx + Qdx + Qdy) = \text{const.} - 2\, \Sigma m.\, \Sigma.\int mm' fdf$, (substituting $- \Sigma \int mm'. fdf$ in place of $\Sigma.\int m. (Pdx + Q.dy + Rdz)$. (See No. 19, page 113.)

† When the origin of the coordinates is in the centre of gravity of the system, the quantities $c\, c'\, c''$, are constant and $\therefore$ the position of the plane, with respect to which the function $\Sigma m. \left\{\dfrac{xdy-y.dx}{dt}\right\}$ is a maximum, remains the same during the motion of the system, $\therefore$ as the quantity Σm, would occur both in the numerator and denominator of the expression for the cosines of the angles which the plane with respect to which the function $\Sigma.m.m' \left\{\dfrac{(x'-x((dy'-dy-(y'-y) (dx'-dx)}{dt}\right\}$ is a maximum, makes with the three coordinate planes, it is evident that the values of the angles which the invariable plane makes with three coordinate planes, is the same in both cases, from these considerations it appears that the invariable plane may be determined at each instant by means of the *relative* velocities of the system, without a knowledge of their *absolute* velocities in space. (See Notes to page 139.)

tual distances of the bodies, in which the first member expresses the sum of the squares of the relative velocities of the system about each other, considering them two by two, and supposing at the same time that one of them is immoveable, each square being multiplied by the product of the two masses which are considered.

23. If we resume the equation (R) of No. 19, and differentiate it with respect to the characteristic δ, we shall have

$$\Sigma.m\ v.\ \delta v = \Sigma m.\ (P.\delta x + Q.\delta y + R\delta z)\ ;$$

and the equation (P) of No. 18, will then become

$$0 = \Sigma.m.\left\{\delta x.\ d.\frac{dx}{dt} + \delta y.\ d.\ \frac{dy}{dt} + \delta z.\ d.\ \frac{dz}{dt}\right\} - \Sigma.m.dt.v\delta v.$$

Denoting by $ds,\ ds'$ &c. the elements of the curves described by $m,\ m'$ &c.; we shall have

$$vdt = ds\ ;\ v'dt = ds'\ ;\ \text{\&c.}$$

$$ds = \sqrt{dx^2 + dy^2 + dz^2}\ ;\ \text{\&c.}$$

from which we can obtain, by following the same process as in the analysis of No. 8,

$$\Sigma.m.\delta.\ (vds) = \Sigma.m.\ d.\left(\frac{dx.\ \delta x + dy.\ \delta y + dz.\ \delta z}{dt}\right).$$

By integrating with respect to the differential characteristic d, and making the integrals extend to the entire curves described by the bodies m, m', &c. we shall have

$$\Sigma.\delta.\textstyle\int mvds = \text{const.} + \Sigma.m.\ \left(\frac{dx.\delta x + dy.\delta y + dz.\ \delta z.}{dt}\right);$$

in this equation the variations $\delta x, \delta y, \delta z$, &c and also that part of its second member, which is constant, refer to the extreme points of the curves described by m, m', &c.

From which it appears that when these points are invariable, we shall have

$$0 = \Sigma.\delta.\int mvds\,;^*$$

which indicates that the function $\Sigma.\int mvds$ is a minimum. It is in this, that the principle of the least action, in the motion of a system of bodies, consists; a principle, which, as we have seen, is only a mathematical result of the primitive laws of the equilibrium and motion of bodies. It is also apparent, that this principle combined with the principle of living forces, gives the equation (P) of No. 18, which contains all that is necessary for the the determination of the motions of the system. Finally, it appears from No. 22, that this principle obtains, even when the origin of the coordinates is in motion; provided that the motion is uniform, its direction rectilinear and the system entirely free.†

* By substituting for ds, ds' their values $v.dt$, $v'dt$, the expression $\Sigma.\int v.ds$ will become $\Sigma.\int mv.^2\, dt$, and as $\int mv.^2 dt$ is the sum of the living forces of the body m during the motion; $\Sigma.\int mv.^2 dt$ will express the sum of the living forces of *all* the bodies of the system during the same time; therefore the principle of the least action, in fact indicates, that the sum of the living forces of the system, during its transit from one given position to another, is a minimum, and when the bodies are not actuated by any accelerating forces, the velocities v, v', and the sum of the living forces at each instant, are constant, (see No. 18, page 114.) $\therefore$ $\Sigma.\int mv.^2 dt = \Sigma mv.^2 \int dt$, and the sum of the living forces for any interval of time is proportional to this time, consequently in this case the system passes from one position to another in the shortest time. Since therefore the expression $\Sigma \int v.ds$ is the same as $\Sigma.\int mv^2 dt$ La Grange proposed to alter the denomination of the principle of least action, and to term it the principle of the greatest or least living force, for by contemplating in this manner, it is equally applicable to the states of equilibrium and motion, since it has been demonstrated in the notes page 119, that in case of equilibrium the vis viva is either a maximum or a minimum; from what precedes it appears that, as La Place observes in his Systeme du Monde, the true economy of nature is that of the living force, and it is this economy which we should always have in view in the construction of machines, which are always more perfect according as less living force is consumed in producing a given effect.

† With respect to the extent of the different principles which are treated of in this fifth chapter, it is important to remark, that the principles of the conservation of the motion of the centre of gravity, and of the conservation of areas subsist, even when by the mutual action of the bodies, they undergo sudden changes in their motions, which renders these principles extremely useful in several circumstances, but the principles of the conservation

of the vis viva, and of the least action, require that the variations of the motion of the system, be made by imperceptible gradations.

The principle of the least action differs from the other principles in this, that the other principles are the *real integrals* of the differential equations of the motion of bodies, whereas this of the least action is only a singular combination of these equations, in fact it being established that $\Sigma . \int mv . ds$ is a minimum by seeking by the known rules, the conditions which render it such, and making use of the general equation of the conservation of living forces, we should find all the equations which are necessary to determine the motion of each body.

The principle established in this number was first *assumed* as a metaphysical truth, and was applied by Maupertius to the discovery of the laws of reflection and refraction, however it ought not to be deemed a *final* cause, for we can infer analogous results from all relations mathematically possible between the force and the velocity, provided that we substitute in this principle, in place of the velocity, that *function* of the velocity by which the force is expressed, (see next chapter, page 154,) and so far from having been the origin of the laws of motion, it has not even contributed to their discovery, without which we should be still debating what was to be understood by the least action of nature.

CHAPTER VI.

Of the laws of motion of a system of bodies, in all the relations mathematically possible between the force and the velocity.

24. It has been already remarked in No. 5, that there are an infinite number of ways of expressing the relation between force and velocity, which do not imply a contradiction. The simplest of all these relations is that of the force proportional to the velocity, which as we have observed, is the law of nature. It is from this law that we have derived, in the preceding chapter, the differential equations of the motion of a system of bodies; but it is easy to apply the same analysis, to all relations mathematically possible, which may exist, between the force and the velocity, and thus to exhibit under a new point of view the general principles of motion. For this purpose, let F represent the force and v the velocity, we have $F = \varphi\,(v)$; $\varphi\,(v)$ being any function whatever of v; let $\varphi'\,(v)$ denote the difference of $\varphi(v)$ divided by dv. The denominations of the preceding numbers always remaining, the body m will be solicited parallel to the axis of x by the force $\varphi\,(v).\frac{dx}{ds}$. *

In the following instant, this force will become $\varphi\,(v).\frac{dx}{ds} + d.\left\{\varphi\,(v).\frac{dx}{ds}\right\}$

* ds being the differential of the line described by the body, the cosine of the angle which the direction of the motion makes with the axis of x is equal to $\frac{dx}{ds}$, $\therefore$ the force F or $\varphi\,(v)$ resolved in the direction of the axis of $x = \varphi\,(v).\frac{dx}{ds}$.

or $\varphi(v) . \frac{dx}{ds} + d. \left(\frac{\varphi(v).}{v} . \frac{dx}{dt}\right)$, because $\frac{ds}{dt} = v$. Moreover, P, Q, R, being the forces which solicit the body m parallel to the axes of the co-ordinates; the system will, by No. 18, be in equilibrio in consequence of these forces, and of the differentials,

$$d. \left\{\frac{dx}{dt} . \frac{\varphi(v)}{v}\right\}, \; d. \left\{\frac{dy}{dt} . \frac{\varphi(v)}{v}\right\}, \; d. \left\{\frac{dz}{dt} . \frac{\varphi(v)}{v}\right\},$$

taken with a contrary sign; therefore in place of the equation (P) of the same number we shall have the following:

$$0 = \Sigma.m. \left\{ \delta x. \, d. \left\{\frac{dx}{dt} . \frac{\varphi(v)}{v} - Pdt\right\} + \delta y. \, d. \left\{\frac{dy}{dt} . \frac{\varphi.(v)}{v}\right.\right.$$

$$\left.\left. - Q.dt\right\} + \delta z. \, d. \left\{\frac{dz}{dt} . \frac{\varphi(v)}{v} - Rdt\right\}\right\}; \quad (S)$$

which only differs from it in this respect, that $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, are multiplied by the function $\frac{\varphi(v)}{v}$, which in the case of the force proportional to the velocity, may be assumed equal to unity. However this difference renders the solution of the problems of mechanics very difficult. Notwithstanding, we can obtain from the equation (S), principles analogous to those of the conservation of living forces, of areas, and of the centre of gravity.

By changing δx into dx, δy into dy, δz into dz, &c., we shall have

$$\Sigma.m. \left\{\delta x. d. \left\{\frac{dx}{dt} . \frac{\varphi(v)}{v}\right\} + \delta y. \; d. \left\{\frac{dy}{dt} . \frac{\varphi(v)}{v}\right\} + \delta z. \; d. \left\{\frac{dz}{dt} . \frac{\varphi(v)}{v}\right\}\right\} =$$

$$\Sigma.m. v. dv. dt. \varphi'(v); *$$

X

* Substituting ds in place of $v.dt$, the expression

$$\Sigma.m. \left\{ dx. d. \left\{\frac{dx}{dt} . \frac{\varphi(v)}{v}\right\} + dy. d. \left\{\frac{dy}{dt} . \frac{\varphi(v)}{v}\right\} + dz. d. \left\{\frac{dz}{dt} . \frac{\varphi(v)}{v}\right\}\right.$$

and consequently

$$\Sigma.\int mv.dv.\varphi'(v) = \text{const.} + \Sigma.\int m.\ (P.dx + Q.dy + R.dz),$$

If we suppose that $\Sigma m.(P.dx + Q.dy + R.dz)$ is an exact differential equal to $d\lambda$, we shall have

$$\Sigma.\int mv.dv.\varphi'(v) = \text{const.} + \lambda;\ (T)$$

which equation is analogous to the equation (R) of No. 19, into which it is changed in the case of the law of nature, or of $\varphi'(v) = 1$. Therefore, the principle of the conservation of living forces obtains in all laws mathematically possible between force and velocity, provided that we understand by the living force of a body, the product of its mass by double the integral of its velocity, multiplied by the differential of the function of the velocity which expresses the force.

If in the equation (S), we make $\delta x' = \delta x + \delta x'_{,}$, $\delta y' = \delta y + \delta y'_{,}$, $\delta z' = \delta z + \delta z'_{,}$, $\delta x'' = \delta x + \delta x''_{,}$; &c. we shall have by putting the coefficients of δx, δy, δz, respectively equal to nothing

$$0 = \Sigma.m.\left\{d.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right) - Pdt\right\},\ 0 = \Sigma.m.\left\{d.\left(\frac{dy}{dt}.\frac{\varphi(v)}{v}\right) - Qdt\right\},$$

$$0 = \Sigma m.\left\{d.\left(\frac{dz}{dt}.\frac{\varphi(v)}{v}\right) - Rdt\right\},$$

becomes

$$\Sigma.m.\left\{dx.d.\left\{\frac{dx}{ds}.\varphi(v)\right\} + dy.d.\left\{\frac{dy}{ds}.\varphi(v)\right\} + dz.d.\left\{\frac{dz}{ds}.\varphi(v)\right\}\right\}$$

and by taking the differential it becomes.

$$\Sigma.m.\left\{\frac{dx.d^2x + dy.d^2y + dz.d^2z}{ds}\right\}.\varphi(v) - \Sigma.m.\left\{\frac{dx^2 + dy^2 + dz^2}{ds^2}\right\}.d^2s.\varphi(v) +$$

$$\Sigma.m.\left\{\frac{dx^2 + dy^2 + dz^2}{ds}\right\}.d.\varphi(v) = \Sigma.m.d^2s.\varphi(v) - \Sigma.m.d^2s.\varphi(v) + \Sigma.m.ds.d.\varphi(v)$$

and this last quantity is equal by substitution to $\Sigma.m.\ v.dt.dv.\varphi'(v)$.

These three equations are analogous to those of No. 20, from which we have inferred, the conservation of the motion of the centre of gravity, in the case of nature, when the system is not subjected to any forces but those of the mutual action and attraction of the bodies of the system. In this case $\Sigma.m.P$, $\Sigma.m.Q$, $\Sigma.m.R$, vanish, and we have

$$\text{const.} = \Sigma.m.\frac{dx}{dt}.\frac{\varphi(v)}{v};\ \text{const.} = \Sigma m.\frac{dy}{dt}.\frac{\varphi(v)}{v};$$

$$\text{const.} = \Sigma m.\frac{dz}{dt}.\frac{\varphi(v)}{v};\ m.\frac{dx}{dt}.\frac{\varphi(v)}{v} \text{ is } = m.\varphi(v).\frac{dx}{ds}.$$

and this last quantity is the finite force of the body, resolved parallel to to the axis of x; the force of a body being the product of its mass by the function of the velocity which expresses the force. Therefore in this case the sum of the finite forces of the system, resolved parallel to any axis, is constant whatever may be the relation between the force and the velocity, and what distinguishes the state of motion from that of repose, is, that in this last state, the same sum vanishes. These results are common to all laws mathematically possible between the force and the velocity; but it is only in the law of nature, that the centre of gravity moves with an uniform motion in a rectilinear direction. *

Again let us make in the equation (S)

$$\delta x' = \frac{y'.\delta x}{y} + \delta x'_{\prime};\ \delta x'' = \frac{y''.\delta x}{y} + \delta x''_{\prime}\ \ \&c.$$

$$\delta y = -\frac{xdx}{y} + \delta y_{\prime}\ \delta y' = -\frac{x'.\delta x}{y} + \delta y'_{\prime};\ \&c.$$

the variation δx will disappear from the variations of the mutual distances

x 2

* It is evident that the centre of gravity does not move uniformly in a right line when P, Q, R, vanish, except when $\frac{\varphi(v)}{v}$ is equal to unity, for it is only in this case that we could prove from the expression, const. $= \Sigma.m.\frac{dx}{dt}.\frac{\varphi v}{v}$, that dX the differential of the co-ordinate of the centre of gravity is constant.

f, f', &c. of the bodies composing the system, and of the forces which depend on these quantities. If the system is not affected by extraneous obstacles, we shall have, by putting the coefficient of δx equal to nothing

$$0 = \Sigma.m. \left\{ x.d. \left(\frac{dy}{dt} . \frac{\varphi(v)}{v}\right) - y.\, d. \left(\frac{dx}{dt} . \frac{\varphi(v)}{v}\right) \right\} +$$

$\Sigma m.(Py - Q.x)dt$, from which we deduce by integrating,

$$c = \Sigma.m. \left(\frac{xdy - ydx}{dt}\right) . \frac{\varphi(v)}{v} + \Sigma. \int m\, (Py - Qx).dt, \text{*}$$

we shall have in like manner

$$c' = \Sigma.m. \left(\frac{xdz - zdx}{dt}\right) . \frac{\varphi(v)}{v} + \Sigma. \int m\, (Pz - Rx).dt;$$

$$c'' = \Sigma.m. \left(\frac{ydz - zdy}{dt}\right) . \frac{\varphi(v)}{v} + \Sigma \int m\, (Qz - Ry).dt;$$

c, c', c'', being constant arbitrary quantities.

If the system is only subjected to the mutual action of its component parts, we have, by No. 21, $\Sigma m.\ (Py - Qx) = 0$; $\Sigma m.\ (Pz - Rx) = 0$ $\Sigma m.\ (Qz - Ry) = 0$; also $m \left\{ x \dfrac{dy}{dt} - y . \dfrac{dx}{dt} \right\} . \dfrac{\varphi(v)}{v}$ is the moment of the finite force by which the body is actuated, resolved parallel to the plane of x and y, which tends to make the system turn about the axis of z; therefore the finite integral $\Sigma.m. \left\{ \dfrac{xdy - ydx}{dt} \right\} . \dfrac{\varphi(v)}{v}$ is equal to the sum of the moments of all the finite forces of the bodies of the system

* The integral of this expression is equal to $\Sigma.m \left\{ x . \frac{dy}{dt} . \frac{\varphi(v)}{v} - \int dx . \left(\frac{dy}{dt} . \frac{\varphi(v)}{v}\right) - y . \frac{dx}{dt} . \frac{\varphi(v)}{v} + \int \left(\frac{dy.dx}{dt} . \frac{\varphi(v)}{v} \right) \right\} = \Sigma.m. \frac{xdy - y.dx}{dt} . \frac{\varphi(v)}{v}$.

to make it revolve round the same axis; consequently this sum is constant. It vanishes in the case of equilibrium; therefore, there is the same difference between these two states as there is relatively to the sum of the forces parallel to any axis. In the law of nature, this property indicates that the sum of the areas described about a fixed point, by the projections of the radii vectores of the bodies is constant in a given time, but this constancy of the areas described does not obtain in any other law.*

By differentiating with respect to the characteristic δ, the function $\Sigma.\int m.\ \varphi\ (v).\ ds$;

we shall obtain

$$\delta.\ \Sigma.\int m.\varphi(v)\ ds = \Sigma.\int m.\varphi(v).\delta.ds + \Sigma.\int m.\delta v.\varphi'(v).ds;$$

but we have

$$\delta ds = \frac{dx.\delta dx + dy.\delta dy + dz.\delta dz}{ds} = \frac{1}{v}\left\{\frac{dx}{dt}.\ d.\delta x + \frac{dy}{dt}.d.\ \delta y + \frac{dz}{dt}.\ d.\ \delta z\right\};$$

therefore by partial integration we shall obtain

$$\delta.\Sigma.\int m.\varphi(v).ds = \Sigma.\ \frac{m\varphi(v)}{v}\left\{\frac{dx}{dt}.\delta x + \frac{dy}{dt}.\ \delta y + \frac{dz}{dt}.\delta z\right\}$$

$$-\Sigma.\int m\left\{\delta x.d.\ \left(\frac{dx}{dt}.\ \frac{\varphi(v)}{v}\right) + \delta y.d.\ \left(\frac{dy}{dt}.\ \frac{\varphi(v)}{v}\right) + \delta z.d.\ \left(\frac{dz}{dt}.\ \frac{\varphi(v)}{v}\right)\right\}$$

$$+\Sigma.\int m.\delta v.\varphi'(v).ds.$$

The extreme points of the curves described by the bodies of the system

* As the factor $\frac{\varphi(v)}{v}$ is variable in every other case beside that of nature, it follows that though the quantity $\Sigma.m.\ \left\{\frac{x\,dy - y\,dx}{dt}\right\}.\frac{\varphi(v)}{v}$ is constant and equal to c, still that part of it $\Sigma.m.\ \left\{\frac{xdy - y.dx}{dt}\right\}$ is not constant.

being supposed fixed, the term which is not affected by the sign $\int$ must disappear in this equation; therefore we shall have in consequence of the equation (S),

$$\delta.\,\Sigma.\int m.\varphi(v).ds = \Sigma.\int m.\delta v.\varphi'(v).ds - \Sigma.\int mdt(P\delta x + Q.\delta y + R.\delta z)$$

but the equation (T) differentiated with respect to δ gives

$$\Sigma.\int m.\delta v.\varphi'(v).ds = \Sigma.\int mdt\,(P\delta x + Q.\delta y + R.\delta z)\ ;$$

therefore we have

$$0 = \delta.\,\Sigma.\int m.\varphi(v).ds.$$

This equation corresponds to the principle of the least action in the law of nature. $m.\varphi(v)$ is the entire force of the body m, thus the principle comes to this, that the sum of the integrals of the finite forces of the bodies of the system, respectively multiplied by the elements of their directions, is a minimum, presented in this manner, it answers to all laws mathematically possible between the force and velocity. In the state of equilibrium the sum of the forces multiplied by the elements of their directions vanishes, in consequence of the principle of virtual velocities, what therefore in this respect distinguishes the state of equilibrium, from that of motion is that the same differential function, which in the state of equilibrium vanishes, gives in a state of motion by its integration a minimum.

CHAPTER VII.

Of the motions of a solid body of any figure whatever.

25. The differential equations of the motions of translation and rotation of a solid body, may be easily deduced from those which have been given in the fifth chapter; but from their importance in the theory of the system of the world we are induced to develope them in detail.

Let us suppose a solid body of which all the parts are solicited by any forces whatever. Let x, y, z, represent the orthogonal coordinates of its centre of gravity, and let $x+x'$, $y+y'$, $z+z'$, be the coordinates of any molecule dm of the body, then x', y', z', will be the coordinates of this molecule with respect to the centre of gravity of the body. Moreover, let P, Q, R, be the forces which solicit the molecule parallel to the axes of x, of y, and of z,. The forces destroyed at each instant in the molecule, parallel to these axes, will be by No. 18,

$$-\left\{\frac{d^2x+d^2x'}{dt}\right\}.\,dm+P.dt.\,dm;$$

$$-\left\{\frac{d^2y+d^2y'}{dt}\right\}.\,dm+Q.dt.\,dm;$$

$$-\left\{\frac{d^2z+d^2z'}{dt}\right\}.\,dm+R.dt.dm;$$

(the element dt of the time being considered as constant.)

Therefore it follows that all the molecules actuated by similar forces should mutually constitute an equilibrium. We have seen in No. 15, that for this purpose, it is necessary that the sum of the forces parallel to the same axes, should vanish which gives the three following equations

$$S.\left\{\frac{d^2x+d^2x'}{dt^2}\right\}.dm = S.Pdm;$$

$$S.\left\{\frac{d^2y+d^2y'}{dt^2}\right\}.dm = S.Qdm;$$

$$S.\left\{\frac{d^2z+d^2z'}{dt^2}\right\}.dm = S.Rdm;$$

the letter S being here a sign of integration relative to the moelcule dm, which we should extend to the entire mass of the body. The variables x, y, z, are the same for all the molecules, therefore we can bring them from under the sign S; thus, denoting the mass of the body by m, we shall have

$$S.\frac{d^2x}{dt^2}.dm = m.\frac{d^2x}{dt^2}; \; S.\frac{d^2y}{dt^2}.dm = m.\frac{d^2y}{dt^2}; \; S.\frac{d^2z}{dt^2}.dm = m.\frac{d^2z}{dt^2}.$$

Moreover by the nature of the centre of gravity, we have,

$$S.x'.dm = 0; \; S.y'.dm = 0; \; Sz'.dm = 0\,*$$

therefore

$$S.\frac{d^2x'}{dt^2}.dm = 0; \; S.\frac{d^2y'}{dt^2}.dm = 0; \; S.\frac{d^2z'}{dt^2}.dm = 0;$$

$$* \; S.\frac{d^2x}{dt^2}.dm = \frac{d^2x}{dt^2}.\,Sdm = \frac{d^2x}{dt^2}.m \;\&c.$$

$S.x'.dm = 0$ $S.y'.dm = 0$ because x', y', &c. are the coordinates of the body referred to the centre of gravity, see No. 15, page 91.

consequently we shall have

$$\left.\begin{aligned} m.\frac{d^2x}{dt^2}&=S.Pdm\,; \\ m.\frac{d^2y}{dt^2}&=S.Qdm\,; \\ m.\frac{d^2z}{dt^2}&=S.Rdm\,; \end{aligned}\right\}; \quad (A)$$

these three equations determine the motion of the centre of gravity of the body ; they correspond to the equations of No. 20, which relate to the motion of the centre of gravity of a system of bodies.

We have seen in No. 15, that for the equilibrium of a solid body the sum of the forces parallel to the axis of x, multiplied by their distances from the axis of z, minus the sum of the forces parallel to the axis of y, multiplied by their distances from the axis of z, should be equal to nothing; thus we shall have

$$S.\left\{(x+x').\left(\frac{d^2y+d^2y'}{dt^2}\right)-(y+y').\left(\frac{d^2x+d^2x'}{dt^2}\right)\right\}.\,dm$$

$$=S.\,\{(x+x')\,Q-(y+y').\,P.\}.dm\,; \qquad (1\cdot)$$

but we have

$$S.\,(x.d^2y-y.d^2x).\,dm=m.(x.d^2y-y.d^2x)\,;$$

in like manner we have

$$S.\,(Qx-Py).\,dm=x.\textstyle\int Qdm-y.\int Pdm$$

finally we have

$$S.\,(x'.d^2y+x.d^2y'-y'.d^2x-y.d^2x').\,dm=d^2y.\,S.x'dm-d^2x.\,Sy'dm$$

$$+x.S.d^2y'.dm-y.\ S.d^2x'.dm\,;\ ^*$$

by the nature of the centre of gravity, each of the terms of the second member of this equation vanishes; therefore the equation (1) will become in consequence of the equations A,

$$S.\left\{\frac{x'.d^2y'-y'.d^2x'}{dt^2}\right\}.dm=S.\ (Qx'-Py').dm\,;$$

* By performing the multiplication,

$$S.\left\{(x+x').\frac{(d^2y+d^2y')}{dt^2}-(y+y').\frac{(d^2x+d^2x')}{dt^2}.\right\}.dm$$

$$=S.\left\{(x+x').\ Q-(y+y').\ P\right\}.dm\,;=S.\left\{\frac{xd^2y-y.d^2x}{dt^2}\right\}.dm+$$

$$S.\left\{\frac{x'.d^2y+xd^2y'-y'.d^2x-y.d^2x'}{dt^2}\right\}.dm+S.\left\{\frac{x'.d^2y'-y'.d^2x'}{dt^2}\right\}.dm=$$

$.\,(Qx-Py).dm+S.\,(Q.x'-Py').dm$, $\therefore$ by substituting for

$$S.\frac{d^2y}{dt^2}.dm=\frac{d^2y}{dt^2}.\ m,\quad S.\frac{d^2x}{dt^2}.dm=\frac{d^2x}{dt^2}.\ m$$

the expressions $S.\ P.\ dm$, $S.\ Q.dm$, to which they are respectively equal as appears from the equations (A), and freeing the quantities d^2y, d^2x, x, y, from the sign S, the preceding equation will be changed into the following

$$x.\ S.\ Q.\ dm-y.\ SP.\ dm+\frac{d^2y}{dt^2}.\ Sx'.dm+x.\ S.\frac{d^2y'}{dt^2}.dm-\frac{d^2x}{dt^2}.\ Sy'.dm$$

$$-y.S.\frac{d^2x'}{dt^2}.dm+S\left\{\frac{x'\,d^2y'-y'.d^2x'}{dt^2}\right\}.\ dm=x.\ S.\ Q.\ dm-y.\ S.P.dm+$$

$S\,(Q.x'-Py').\ dm$, and omitting quantities which destroy each other, and also those which by the nature of the centre of gravity, vanish, we will obtain the equation

$$S.\left\{\frac{x'd^2y'-y'.d^2x'}{dt^2}\right\}.dm=S.\ (Qx'-Py').dm\,;$$

this equation involves the principle of the conservation of areas, for if the forces which solicit the molecules arise from their mutual action, and from the action of forces directed towards fixed points, $S(Qx'.-Py'.)\ dm=0$.

this equation integrated with respect to the time, gives

$$S.\left\{\frac{x'dy'-y'.dx'}{dt}\right\}.\,dm=S.\int(Qx'-Py').\,dt.\,dm\,;$$

the sign of integration $\int$ being relative to the time t.

From what precedes it is easy to infer that if we make

$$S.\int(Q.x'-Py').\,dt.\,dm=N\,;$$

$$S.\int(Rx'-Pz').\,dt.\,dm=N'\,;$$

$$S.\int(Ry'-Q.z').\,dt.\,dm=N''\,;$$

we shall obtain the three following equations

$$\left.\begin{aligned}&S.\left\{\frac{x'.dy'-y'.dx'}{dt}\right\}.dm=N;\\&S.\left\{\frac{x'dz'-z'.dx'}{dt}\right\}.dm=N'\,;\\&S.\left\{\frac{y'dz'-z'dy'}{dt}\right\}.dm=N''\,;\end{aligned}\right\}\;;\quad(B)$$

these three equations contain the principle of the conservation of areas; they are sufficient to determine * the motion of rotation of a body about its centre of gravity; combined with the equations (A), they completely determine the motions of translation and rotation of a body.

* In our investigations relative to the invariable plane in the 5th chapter, we have seen that when a body or system of bodies are not solicited by any extraneous forces, the motion may be distinguished into two others, of which one is progressive and the same for all points, the other is rotatory about a point in the body or system, the first determined by the equation (A), and the second by the equation(B); by thus distinguishing the motion into two others, we can represent with more clearness the motion of a solid body in space,for these two motions are entirely independent of each other, as is evident from the inspection of the equations which indicate them, so that the equations (A) may vanish, while the equations (B) have a finite

If the body is constrained to turn about a fixed point; it follows from No. 15, that the equations (B) are sufficient for this purpose; but then it is necessary to fix the origin of the coordinates x', y', z', at this point.*

value or vice versa. The centre of the rotatory motion may be any point whatever, however when we would wish to determine these two kind of motions it is advantageous to assume for this point, the centre of gravity of the body, because in most cases its motion may be determined directly, and independently of that of the other points of the body.

Dividing the equations (A) by m, we may perceive by a comparison of the resulting expressions, with the equations of the motion of a material point, which have been given in No. 7, page 31, that the motion of the centre of gravity is the same, as if the entire mass of the body was concentrated in it, and the forces of all the points and in their respective directions were applied to it; this rectilineal motion is common to all the points of the body, and the same as the motion of *translation*.

* If a solid body is acted on by forces which act *instantaneously*, in general it acquires the two kinds of motions, of translation and of rotation; which are respectively determined by the equations (A) and (B); when the equations (A) vanish, the forces are reducible to two parallel forces, equal, and acting in opposite directions, when the rotatory motion vanishes the instantaneous forces have an unique resultant passing through the centre of gravity, see notes to page 143, when the molecules of the body are solicited by accelerating forces, their action in general will alter the two motions which have been produced by initial impulse, however if the resultant of the accelerating forces passes through the centre of gravity of the body, the rotatory motion will not be affected by the action of these forces, this is the case of a sphere acted on by forces which vary as the distance, or in the inverse square of the distance from the molecules, see Newton prin. Vol. 1. Section 12, or Book 2, No. 12, of this work, consequently if the planets were spherical bodies, the motive force arising from the mutual action of the sun and planets would pass through the centre of gravity, and the rotatory motion would not be affected, but the direction of this force does not always pass accurately through this centre, in consequence of the oblateness of the planets, therefore the axis of rotation does not remain accurately parallel to itself, however the *velocity* of rotation is not sensibly affected, see Systeme du Monde, Chapter 14, Book 4, and Book 5, No. 7 and 8. It is in this slight oscillation of the axis of the earth arising principally from the attractions of the sun and moon, that the phenomena of the precession of the equinoxes and of the nutation of the earths axis consist. (See Nos. 28, 29.

If the body be moved in consequence of initial impulses, the directions of the forces, their intensities and points of application been given, we might by the formula of No. 21, determine the principal moment of the forces with respect to the centre of gravity, and the direction of the plane to which this moment is referred, which would completely determine the moment of rotation obout the centre of gravity, and it is evident that the same data would be sufficient to determine the rectilinear motion of the centre of gravity, and consequently the motion of translation of the system, see No. 29.

26. Let us attentively consider these equations, the origin of the coordinates being supposed fixed at any point, the same or different from the centre of gravity. Let us refer the position of each molecule to three axes perpendicular to each other, fixed in the body, but moveable in space. Let θ be the inclination of the plane formed by the two first axes to the plane of x, y, ; let φ be the angle formed by the line of intersection of these two planes and by the first axis ; finally, let ψ be the complement of the angle which the projection of the third axis on the plane of x, y, makes with the axis of x. We will term these three axes principal axes, and we will denote the three coordinates of the molecule dm, referred to those axes by x'', y'', z'', ; then by No. 21, the following equations will obtain

$$x'=x''.\ (\cos.\theta.\sin.\psi.\sin.\varphi+\cos.\psi.\cos.\varphi)+$$

$$y''.\ (\cos.\theta.\sin.\psi.\cos.\varphi-\cos.\psi.\sin.\varphi)+z''.\sin.\theta.\sin.\psi\,;$$

$$y'=x''.\ (\cos.\theta.\cos.\psi.\sin.\varphi-\sin.\psi.\cos.\varphi)+$$

$$y''.\ (\cos.\theta.\cos.\psi.\cos.\varphi+\sin.\psi.\sin.\varphi)+z''.\sin.\theta.\cos.\psi\,;$$

$$z'=z''.\cos.\theta-y''.\sin.\theta.\cos.\varphi-x''.\sin.\theta.\sin.\varphi.$$

By means of these equations, we are enabled to develop the the first members of the equations (B) in functions of θ, ψ, φ and their differentials. But this investigation will be considerably simplified, by observing that the position of the three principal axes depends on three arbitrary quantities, which we can always determine so as to satisfy these three equations.

$$S.x''y''.dm=0\,;\ S.x''z''.dm=0\,;\ S.y''z''.dm=0.\ *$$

* In deducing the values of $-$N, $-$N′, in functions of θ, ψ, φ, and the coordinates x'', y'', z'', it is assumed that there are three axes possessing this property of having $Sy''z''.dm$, $S\,x''y''.\,dm=0$, $S\,x''z'''.\,dm=0$. However it is afterwards *demonstrated* that there exists three such axes in every body.

Since by hypothesis the principal axes preserve their initial positions, being moveable in space though fixed in the body, while the axes of x', y', and z', are fixed *in space*, it follows then let us make

$S.\,(y''^2+z''^2).\,dm=A$; $S.\,(x''^2+z''^2).\,dm=B$; $S.\,(x''^2+y''^2)\,dm\ \ C$;

and in order to abridge let us make

$$d\varphi - d\psi.\cos.\theta = p.dt;$$

$$d\psi.\sin.\theta.\sin.\varphi - d\theta.\cos.\varphi = q.dt;$$

$$d\psi.\sin.\theta.\cos.\varphi + d\theta.\sin.\varphi = r.dt.$$

The equations (B) will, after all reductions, be changed into the three following;

$$\left.\begin{array}{l} A.q.\sin.\theta.\sin.\varphi + Br.\sin.\theta.\cos.\varphi - Cp.\cos.\theta = -N; \\ \text{Cos.}\,\psi.\,\{Aq.\cos.\theta.\sin.\varphi + Br.\cos.\theta.\cos.\varphi + Cp.\sin.\theta\} \\ +\sin.\psi.\,\{Br.\sin.\varphi - Aq.\cos.\varphi\} = -N'; \\ \text{Cos.}\,\psi.\,\{Br.\sin.\varphi - Aq.\cos.\varphi\} \\ -\sin.\psi.\{Aq.\cos.\theta.\sin.\varphi + Br.\cos.\theta.\cos.\varphi + Cp.\sin.\theta\} = -N'' \end{array}\right\}; (C)\ ^*$$

that the coordinates x'', y'', z'', are constantly the same for the same molecule, and vary only in passing from one molecule to another, but the coordinates $x'\ y'\ z'$ vary with the time ∴ they are functions of the time, as are also the angles θ, ψ, φ, since they depend on the position of the principal axes with respect to the fixed axes ∴ when we take the differential of x', y', and z', with respect to the time in terms of $x''\ y''\ z''$ and the angles θ, ψ, φ, we should not take the differentials of x'', y'', z'', it may likewise be observed that we can omit the consideration of those quantities of which one of the factors is the product of two different coordinates, for such quantities disappear from the expression $x'dy - y'dx'$, as they occur in the two parts of it affected with contrary signs, these considerations enable us to abridge considerably the investigation of the value of $\frac{x'dy' - y'dx'}{dt}$ in terms of $x''y''z''$ and functions of the angles θ, ψ, φ, for we shall not take into account, those terms which would eventually disappear in the expression $\frac{x'dy' - y'.dx'}{dt}$.

* $dx' = = x''.(-d\theta.\sin.\theta.\sin.\psi\sin.\varphi + d\psi.\cos.\psi.\cos.\theta.\sin.\varphi$
$+ d\varphi.\cos.\varphi.\sin.\psi.\cos.\theta - d\psi.\sin.\psi.\cos.\varphi - d\varphi.\sin.\varphi.\cos.\psi)$

these three equations give by differentiating them and then supposing $\psi=0$, after the differentiations, which is equivalent to assuming the

$$+y''(-d\theta.\sin.\theta.\sin.\psi.\cos.\varphi+d\psi.\cos.\psi.\cos.\varphi.\cos.\theta$$

$$-d\varphi.\sin.\varphi.\sin.\psi.\cos.\theta+d\psi.\sin.\psi.\sin.\varphi-d\varphi.\cos.\varphi.\cos.\psi)$$

$$+z''.(d\theta.\cos.\theta.\sin.\psi+d\psi.\cos.\psi\sin.\theta);$$

$$dy'=x''.(-d\theta.\sin.\theta.\cos.\psi.\sin.\varphi-d\psi.\sin.\psi.\sin.\varphi.\cos.\theta$$

$$+d\varphi.\cos.\varphi.\cos.\psi.\cos.\theta-d\psi.\cos.\psi.\cos.\varphi+d\varphi.\sin.\varphi.\sin.\psi)$$

$$+y''(-d\theta.\sin.\theta.\cos.\psi.\cos.\varphi-d\psi.\sin.\psi.\cos.\varphi.\cos.\theta$$

$$-d\varphi.\sin.\varphi.\cos.\psi.\cos.\theta+d\psi.\cos.\psi.\sin.\varphi+d\varphi.\cos.\varphi.\sin.\psi)$$

$$+z''.(d\theta.\cos.\theta.\cos.\psi-d\psi.\sin.\psi.\sin.\theta)$$

$$dz'=-z''.d\theta.\sin.\theta-y''.d\theta.\cos.\theta.\cos.\varphi+y''.d\varphi.\sin.\varphi.\sin.\theta$$

$$-x''.d\theta.\cos.\theta.\sin.\varphi-x''.d\varphi.\cos.\varphi.\sin.\theta$$

$$\therefore x'dy'=$$

$$(x''.\cos.\theta.\sin.\psi.\sin.\varphi+x''\cos.\psi.\cos.\varphi+y''.\cos.\theta.\sin.\psi.\cos.\varphi$$

$$-y''.\cos.\psi.\sin.\varphi+z''.\sin.\theta.\sin.\psi)\times$$

$$(-x''.d\theta.\sin.\theta.\cos.\psi.\sin.\varphi-x''.d\psi.\sin.\psi.\sin.\varphi.\cos.\theta+x''d\varphi.\cos.\varphi.\cos.\psi.\cos.\theta$$

$$-x''d\psi.\cos.\psi.\cos.\varphi+x''d\varphi.\sin.\varphi.\sin.\psi$$

$$-y''.d\theta.\sin.\theta.\cos.\psi.\cos.\varphi-y''.d\psi.\sin.\psi.\cos.\varphi.\cos.\theta$$

$$-y''.d\varphi.\sin.\varphi.\cos.\psi.\cos.\theta+y''.d\psi.\cos.\psi.\sin.\varphi+y''.d\varphi.\cos.\varphi,\sin.\psi$$

$$+z''.d\theta.\cos.\theta.\cos.\psi-z''.d\psi.\sin.\psi.\sin.\theta)=$$

$$-x''.^2d\theta.\sin.\theta.\cos.\theta.\sin.\psi.\cos.\psi.\sin.^2\varphi-x''.^2d\theta.\sin.\theta.\cos.^2\psi.\sin.\varphi.\cos.\varphi$$

$$-x''.^2d\psi.\sin.^2\psi.\sin.^2\varphi.\cos.^2\theta.-x''.^2d\psi.\sin.\psi\cos.\psi.\sin.\varphi.\cos.\varphi.\cos.\theta$$

$$+x''.^2d\varphi.\sin.\varphi.\cos.\varphi.\sin.\psi.\cos.\psi.\cos.^2\theta+x''.^2d\varphi.\cos.^2\varphi.\cos.^2\psi.\cos.\theta$$

$$-x''.^2d\psi.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi.\cos.\theta-x''.^2d\psi.\cos.^2\psi.\cos.^2\varphi$$

$$+x''.^2d\varphi.\sin.^2\varphi.\sin.^2\psi.\cos.\theta+x''.^2d\varphi.\sin.\varphi.\cos.\varphi.\sin.\psi.\cos.\psi.$$

$$-y''^2.d\theta.\sin.\theta.\cos.\theta.\sin.\psi.\cos.\psi.\cos.^2\varphi+y''.^2d\theta.\sin.\theta.\cos.^2\psi.\sin.\varphi.\cos.\varphi$$

$$-y''.^2d\psi.\sin.^2\psi.\cos.^2\varphi\cos.^2\theta.+y''.^2d\psi.\sin.\psi.\cos.\psi.\sin.\varphi,\cos.\varphi.\cos.\theta$$

axis of x' indefinitely near the line of intersection of the plane of x' and y', with that of x'' and y'',

$$-y''.^2\, d\varphi.\sin.\varphi.\cos.\varphi.\sin.\psi.\cos.\psi.\cos.^2\theta.+y''.^2 d\varphi.\sin.^2\varphi\cos.^2\psi.\cos.\theta$$

$$+y''.^2 d\psi.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi.\cos.\theta-y''.^2 d\psi.\cos.^2\psi.\sin.^2\varphi$$

$$+y''.^2\, d\varphi.\cos.^2\varphi.\sin.^2\psi.\cos.\theta-y''.^2 d\varphi.\sin.\varphi.\cos.\varphi.\sin.\psi.\cos.\psi$$

$$+z''.^2 d\theta.\sin.\theta.\cos.\theta.\sin.\psi.\cos.\psi-z''.^2 d\psi.\sin.^2\psi.\sin.^2\theta$$

$$y'.dx'.=$$

$$(x''.\cos.\theta.\cos.\psi.\sin.\varphi-x''\sin.\psi.\cos.\varphi+y''.\cos.\theta.\cos.\psi.\cos.\varphi$$

$$+y''.\sin.\psi.\sin.\varphi+z''.\sin.\theta.\cos.\psi)\times$$

$$(-x''.d\theta.\sin.\theta.\sin.\psi.\sin.\varphi+x''.d\psi.\cos.\psi.\sin.\varphi.\cos.\theta+x''.d\varphi.\cos.\varphi.\sin.\psi.\cos.\theta$$

$$-x''.d\psi.\sin.\psi.\cos.\varphi-x''.d\varphi.\sin.\varphi.\cos.\psi$$

$$-y''.d\theta.\sin.\theta.\sin.\psi.\cos.\varphi+y''.d\psi.\cos.\psi.\cos.\varphi.\cos.\theta-y''.d\varphi.\sin.\varphi.\sin.\psi.\cos.\theta$$

$$y''.d\psi.\sin.\psi.\sin.\varphi-y''.d\varphi.\cos.\varphi.\cos.\psi.$$

$$+z''.d\theta.\cos.\theta.\sin.\psi.+z''\,d\psi.\cos.\psi.\sin.\theta)=$$

$$-x''.^2 d\theta.\sin.\theta.\cos.\theta.\sin.\psi.\cos.\psi.\sin.^2\varphi-x''.^2 d\theta.\sin.\theta.\sin.^2\psi.\sin.\varphi.\cos.\varphi$$

$$+x''.^2\, d\psi.\cos.^2\psi.\sin.^2\varphi.\cos.^2\theta-x''.^2 d\psi.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi.\cos.\theta$$

$$+x''.^2 d\varphi.\sin.\varphi.\cos.\varphi.\sin.\psi.\cos.\psi.\cos.^2\theta.-x''.^2 d\varphi.\cos.^2\varphi.\sin.^2\psi.\cos.\theta$$

$$-x''.^2 d\psi.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi.\cos.\theta+x''.^2 d\psi.\sin.^2\psi.\cos.^2\varphi$$

$$-x''.^2 d\varphi.\sin.^2\varphi.\cos.^2\psi.\cos.\theta+x''.d\varphi.\sin.\varphi.\cos.\varphi.\sin.\psi.\cos.\psi.$$

$$-y''.^2 d\theta.\sin.\theta.\cos.\theta.\sin.\psi.\cos.\psi.\cos.^2\varphi-y''.^2 d\theta.\sin.\theta.\sin.^2\psi.\sin.\varphi.\cos.\varphi$$

$$+y''.^2\, d\psi.\cos.^2\psi.\cos.^2\theta.\cos.^2\varphi.+y''.^2 d\psi.\sin\psi.\cos.\psi.\sin.\varphi.\cos.\varphi.\cos.\theta.$$

$$-y''.^2 d\varphi.\sin.\varphi.\cos.\varphi.\sin.\psi.\cos.\psi.\cos.^2\theta.-y''.^2 d\varphi.\sin.^2\varphi\sin.^2\psi.\cos.\theta$$

$$+y''.^2 d\psi.\sin.\psi.\cos.\psi.\sin.\varphi.\cos.\varphi.\cos.\theta+y''.^2 d\psi.\sin.^2\psi.\sin.^2\varphi$$

$$-y''.^2 d\varphi.\cos.^2\varphi.\cos.^2\psi.\cos.\theta-y''.^2 d\varphi.\sin.\varphi.\cos.\varphi.\sin.\psi.\cos.\psi,$$

$$+z''.^2 d\theta.\sin.\theta.\cos.\theta.\sin.\psi.\cos.\psi.+z''^2 d\psi.\cos.^2\psi.\sin.^2\theta.$$

$$d\theta.\cos.\theta.(Br.\cos.\varphi+Aq.\sin.\varphi)+\sin.\theta.d.(Br.\cos.\varphi+Aq.\sin.\varphi)$$
$$-d.(Cp.\cos.\theta)=-dN;$$

$$d\psi.(Br.\sin.\varphi-Aq.\cos.\varphi)-d\theta.\sin.\theta.(Br.\cos.\varphi+Aq.\sin.\varphi)+\cos\theta.$$
$$d.(Br\cos.\varphi+Aq.\sin.\varphi)+d.(Cp.\sin.\theta)=-dN';$$

$$d.(Br\sin.\varphi-Aq.\cos.\varphi)-d\psi.\cos.\theta.(Br\cos.\varphi+Aq.\sin.\varphi)$$
$$-Cp.d\psi.\sin.\theta=-dN''$$

making

$$Cp=p';\ Aq=q';\ Br=r';$$

Z

∴ observing the terms which coalesce and those which destroy each other in the expression for $x'dy'-y'dx'$, this function becomes equal to

$$-x''.^2\,d\theta.\sin.\theta.\sin.\varphi.\cos.\varphi-x''.^2d\psi.\sin.^2\varphi.\cos.^2\theta-x''.^2d\psi.\cos.^2\varphi$$
$$+(x''.^2\,d\varphi.\cos.^2\varphi.\cos.\theta+x''.^2\,d\varphi.\sin.^2\varphi.\cos.\theta)=(x''.^2d\varphi.\cos.\theta.)$$
$$+y''.^2d\theta.\sin.\theta.\sin.\varphi.\cos.\varphi-y''.^2d\psi.\cos.^2\varphi.\cos.^2\theta-y''.^2d\psi.\sin.^2\varphi$$
$$+(y''.^2d\varphi.\sin.^2\varphi.\cos.\theta+y''.^2d\varphi.\cos.^2\varphi.\cos.\theta)=(y''.^2d\varphi.\cos.\theta).$$
$$-z''.^2d\psi.\sin.^2\theta.$$

This equation when extended to all the molecules of the body is identical with the equation,

$$A.q.\sin.\theta.\sin.\varphi+Br.\sin.\theta.\cos.\varphi-C.p.\cos.\theta.=-N;$$

taken with a contrary sign, for substituting in place of A, B, C, p, r, q, their values, in this equation, it becomes for one molecule

$$(y''+z'')^2\left\{\frac{d\psi.\sin.^2\theta.\sin.^2\varphi-d\theta.\sin.\theta.\sin.\varphi.\cos.\varphi}{dt}\right\}+(x''^2+z''^2).$$
$$\frac{(d\psi.\sin.^2\theta.\cos.^2\varphi+d\theta.\sin.\theta.\sin.\varphi.\cos.\varphi)-(x''^2+y''^2)(d\varphi.\cos.\theta-d\psi.\cos.^2\theta),}{dt}$$

equal by making all the quantities by which y''^2, z''^2, x''^2 are respectively multiplied coalesce so that they may be respectively factors of these coordinates these three differential equations give the following ones *

$$\left.\begin{array}{l} dp' + \left\{\dfrac{B-A}{AB}\right\} q'r'.dt = dN.\cos.\theta - dN'.\sin.\theta; \\[2ex] dq' + \left\{\dfrac{C-B}{CB}\right\}.r'p'.dt = -(dN.\sin.\theta + dN'.\cos.\theta).\sin.\varphi \\ \qquad\qquad + dN''.\cos.\varphi; \\[2ex] dr' + \left\{\dfrac{A-C}{CA}\right\}.p'q'.dt = -(dN.\sin.\theta + dN'.\cos.\theta).\cos.\varphi. \\ \qquad\qquad - dN''.\sin.\varphi. \end{array}\right\}; (D)$$

$$y''.^2 d\psi\,(\sin.^2\theta.\sin.^2\varphi + \cos.^2\theta) - y''.^2 d\theta.\sin.\theta.\sin.\varphi.\cos.\varphi - y''.^2 d\varphi.\cos.\theta$$

$$= y''^2.d\psi.\cos.^2\varphi.\cos.^2\theta + y''.^2 d\psi.\sin.^2\varphi - y''^2.d\theta.\sin.\theta.\sin.\varphi.\cos.\varphi - y''^2.d\varphi.\cos.\theta$$

$$z''^2.d\psi.\sin.^2\theta.\sin.^2\varphi - z''^2.d\theta.\sin.\theta.\sin.\varphi.\cos.\varphi + z''^2.d\psi.\sin.^2\theta.\cos.^2\varphi$$

$$+ z''.^2 d\theta.\sin.\theta.\sin.\varphi.\cos.\varphi = z''.^2 d\psi.\sin.^2\theta$$

$$+ x''^2.d\psi.\sin.^2\theta.\cos.^2\varphi + x''^2.d\theta.\sin.\theta.\sin.\varphi.\cos.\varphi - x''^2.d\varphi.\cos.\theta + x''^2.d\psi.\cos.^2\theta$$

$$= x''^2.d\psi.(\sin.^2\varphi.\cos.^2\theta) + x''^2.d\psi.\cos.^2\varphi - x''^2.d\varphi.\cos.\theta + x''^2.d\theta.\sin.\theta.\sin.\varphi.\cos.\varphi.$$

Since the angle ψ vanishes after the differentiations, wherever sin. ψ occcurs as a factor this quantity must be rejected, and wherever cos. ψ occurs it becomes equal to unity, keeping these circumstances in view it will immediately appear that the expressions for $-dN$ $-dN'$ $-dN''$ should be such as are given in the text.

* The first differential equation being multiplied by $-\cos.\theta$ becomes equal to

$$-d\theta.\cos.^2\theta\,(Br.\cos.\varphi + Aq.\sin.\varphi) - \sin.\theta.\cos.\theta.d.(Br.\cos.\varphi + Aq.\sin.\varphi)$$

$$+ \cos.\theta.d.(Cp.\cos.\theta) = dN.\cos.\theta$$

and multiplying the second equation by sin. θ, we have

$$d\psi.\sin.\theta.(Br.\sin.\varphi - Aq.\cos.\varphi) - d\theta.\sin.^2\theta.(Br.\cos.\varphi + Aq.\sin.\varphi) + \sin.\theta.\cos.\theta.$$

$$d.(Br.\cos.\varphi + Aq.\sin.\varphi) + \sin.\theta\,d.(Cp.\sin.\theta) = -dN'.\sin.\theta$$

$$\therefore dN.\cos.\theta - d.N'.\sin.\theta = -d\theta.(Br.\cos.\varphi + Aq.\sin.\varphi) + d\psi.\sin.\theta.(Br.\sin.\varphi - Aq.\cos.\varphi)$$

$$+ \cos.^2\theta.d.(C.p) - d\theta.\sin.\theta.\cos.\theta.(Cp) + \sin.^2\theta.d(Cp) + d\theta\sin.\theta.\cos.\theta.(Cp); =$$

these three equations are very convenient for determining the motion of rotation of a body, when it turns very nearly about one of the principal axes, which is the case of the celestial bodies.

27. The three principal axes to which we have referred the angles

by substituting for r and q their values

$$=\frac{-B.(d\theta.d\psi.\sin.\theta.\cos.^2\varphi+d\theta.^2\sin.\varphi.\cos.\varphi)-A.(d\theta.d\psi.\sin.\theta.\sin.^2\varphi-d\theta.^2\sin.\varphi.\cos.\varphi)}{dt}$$

$$\frac{+B.(d\psi.^2\sin.^2\theta.\sin.\varphi.\cos.\varphi+d\psi.d\theta.\sin.\theta.\sin.^2\varphi)-A.(d\psi.^2\sin.^2\theta.\sin.\varphi.\cos.\varphi}{dt}$$

$$\frac{-d\psi.d\theta.\sin.\theta.\cos.^2\varphi)}{dt}+d.(C.p.)=$$

$$\frac{(B-A.).(d\psi.^2\sin.^2\theta.\sin.\varphi.\cos.\varphi)-d\theta.^2\sin.\varphi.\cos.\varphi+d\psi.d\theta.(\sin.\theta.\sin.^2\varphi-\sin.\theta.\cos.^2\varphi))}{dt}$$

$$+d.(C.p.)=(B-A).\ q.r.dt+dp'=\frac{B-A}{AB}.\ q'.r'.dt+dp'$$

in like manner, multiplying the first of the differential equations by $\sin.\theta.\sin.\varphi$, the second $\cos.\theta.\sin.\varphi.$ and the third by $-\cos.\varphi$, and then adding them together we obtain

$$-dN.\sin.\theta.\sin.\varphi-dN'.\cos.\theta.\sin.\varphi-dN''.\cos.\varphi=\text{ to}$$

$$d\theta.\sin.\theta.\cos.\theta.\sin.\varphi.(Br.\cos.\varphi+Aq.\sin.\varphi)+\sin.^2\theta.\sin.\varphi.d.(Br.\cos.\varphi+Aq.\sin.\varphi)$$

$$-\sin.\theta.\sin.\varphi.d.(C.p.\cos.\theta)$$

$$+d\psi.\cos.\theta.\sin.\varphi(Br.\sin.\varphi-Aq.\cos.\varphi)-d\theta.\sin.\theta.\cos.\theta.\sin.\varphi(Br.\cos.\varphi+Aq.\sin.\varphi)$$

$$+\cos.^2\theta.\sin.\varphi.d.(Br.\cos.\varphi+Aq.\sin.\varphi)+\cos.\theta.\sin.\varphi.d.(Cp.\sin.\theta)$$

$$-\cos.\varphi.d.(Br.\sin.\varphi-Aq.\cos.\varphi)+d\psi.\cos.\theta.\cos.\varphi.(Br.\cos.\varphi+Aq.\sin.\varphi)$$

$$+Cp.d\psi.\sin.\theta.\cos.\varphi=\text{ by concinnating}$$

$$\sin.\varphi.d.(Br.\cos.\varphi+Aq.\sin.\varphi)+d\psi.\cos.\theta.Br-\cos.\varphi.d.(Br.\sin.\varphi-Aq.\cos.\varphi)$$

$$-\sin.\theta.\cos.\theta.\sin.\varphi.d.(Cp)+d\theta.\sin.^2\theta.\sin.\varphi.(Cp)+\sin.\theta.\cos.\theta.\sin.\varphi.d.(Cp)$$

$$+d\theta.\cos.^2\theta.\sin.\varphi.(Cp.)+(Cp.)d\psi.\sin.\theta.\cos.\varphi;$$

$$=\sin.\varphi.\cos.\varphi.d.(Br)+\sin.^2\varphi.d.(Aq)-Br.d\varphi.\sin.^2\varphi.+Aq.d\varphi.\sin.\varphi.\cos.\varphi.$$

$$+d\psi.\cos.\theta.Br-\sin.\varphi.\cos.\varphi.d.(Br)+\cos.^2\varphi.d.(Aq)-Br.d\varphi.\cos.^2\varphi-Aq.d\varphi.\sin.\varphi.\cos.\varphi.$$

$$+d\theta.\sin.\varphi.(C.p.)+(Cp).d\psi.\sin.\theta.\cos.\varphi=d.(Aq)-Br.d\varphi+d\psi.\cos.\theta.Br$$

$$+d\theta.\sin.\varphi.(Cp),+d\psi.\ Cp.\sin.\theta.\cos.\varphi)$$

θ, φ, ψ, deserve particular consideration; we now proceed to determine their position in any solid whatever. From the values of x' y' z', which have been given in the preceding number we may obtain the following expressions by No. 21.

$$x'' = x'(\cos.\theta.\sin.\psi.\sin.\varphi + \cos.\psi.\cos.\varphi) + y'.(\cos.\theta.\cos.\psi.\sin.\varphi$$
$$-\sin.\psi.\cos.\varphi) - z'.\sin.\theta.\sin.\varphi;$$

$$y'' = x'(\cos.\theta.\sin.\psi.\cos.\varphi - \cos.\psi.\sin.\varphi) + y'.(\cos.\theta.\cos.\psi.\cos.\varphi$$
$$+\sin.\psi.\sin.\varphi) - z'.\sin.\theta.\cos.\varphi;$$

$$z'' = x'.\sin.\theta.\sin.\psi + y'.\sin.\theta.\cos.\psi + z'.\cos.\theta;$$

From which may be obtained,

$$x''.\cos.\varphi - y''.\sin.\varphi = x'.\cos.\psi - y'\sin.\psi;$$

$$x''.\sin.\varphi + y''.\cos.\varphi = x'.\cos.\theta.\sin.\psi + y'.\cos.\theta.\cos.\psi - z'.\sin.\theta;$$

and making

$$S.x'.^2 dm = a^2;\ S.y'.^2 dm = b^2;\ S.z'.^2 dm = c^2;$$

$$S.x'y'.dm = f;\ S.x'z'.dm = g;\ S.y'z'.dm = h;$$

we shall have

$$\cos.\varphi.S.x''z''.dm - \sin.\varphi.S.y''z''.dm = (a^2 - b^2)\sin.\theta.\sin.\psi.\cos.\psi$$

but by substitution $d.(Aq) + Br.(-d\varphi + d\psi.\cos.\theta) + d\theta.\sin.\varphi.Cp. + d\psi.(Cp.)\sin.\theta.\cos.\varphi. =$

$$d.(Aq)\frac{-Bd\psi.d\varphi.\sin.\theta.\cos.\varphi - d\varphi.d\theta.\sin.\varphi + d\psi.^2\sin.\theta.\cos.\theta.\cos.\varphi + d\psi.d\theta.\cos.\theta.\sin.\varphi.}{dt}$$

$$\frac{+C.(d\theta.d\varphi.\sin.\varphi - d\theta.d\psi.\cos.\theta.\sin.\varphi) + C.d\psi.d\varphi.\sin.\theta.\cos.\varphi - C.d\psi.^2\sin.\theta.\cos.\theta.\cos.\varphi.}{dt}$$

$$\frac{=(C-B).d\varphi.d\psi.(\sin.\theta.\cos.\varphi.) + (d\varphi.d\theta.\sin.\varphi) - d\psi.^2\sin.\theta.\cos.\theta.\cos.\varphi - d\theta.d\psi.\cos.\theta\sin.\varphi)}{dt}$$

$$+d(Aq.) = (C-B).p.r.dt + d.(Aq.) = \frac{C-B}{CB}p'.r'dt + dq'$$

by a similar process we might deduce the value of the last differential equation.

$$+f.\ \sin.\ \theta.\ (\cos.^2\psi-\sin.^2\psi)$$

$$+\cos.\ \theta.\ (g.\ \cos.\psi-h.\ \sin.\ \psi);$$

$$\sin.\ \varphi.\ S.\ x''z''dm+\cos.\ \varphi.\ S.\ y''z''.dm=$$

$$\sin.\ \theta.\cos.\ \theta.\ (a.^2\sin.^2\psi+b.^2\cos.^2\psi-c^2+2f.\ \sin.\ \psi.\ \cos.\ \psi)^*$$

$$+(\cos.^2\theta-\sin.^2\theta).\ (g.\ \sin.\ \psi+h.\ \cos.\ \psi).$$

$$*\ x''.\ \cos.\ \varphi=x'.\ (\cos.\ \theta.\ \sin.\ \psi.\ \sin.\ \varphi.\ \cos.\ \varphi+\cos.\ \psi.\ \cos.^2\varphi)$$

$$+y'.(\cos.\ \theta.\ \cos.\ \psi.\ \sin.\ \varphi.\ \cos.\ \varphi-\sin.\ \psi.\ \cos.^2\varphi)-z'.\ \sin.\ \theta.\ \sin.\ \varphi.\ \cos.\ \varphi.$$

$$y''.\ \sin.\ \varphi=x'.(\cos.\ \theta.\ \sin.\ \psi.\ \sin.\ \varphi.\ \cos.\ \varphi-\cos.\ \psi.\ \sin.^2\varphi)$$

$$+y'.\ (\cos.\ \theta.\cos.\psi.\ \sin.\ \Phi.\ \cos.\ \varphi+\sin.\psi.\ \sin.^2\varphi)-z'.\ \sin.\ \theta.\ \sin.\ \varphi.\cos.\ \varphi,$$

$$\therefore x''.\ \cos.\ \varphi-y''.\ \sin.\ \varphi=x'.\ \cos.\ \psi-y'.\ \sin.\ \psi$$

$$x''.\ \sin.\ \varphi=x'.(\cos.\ \theta.\ \sin.\ \psi.\ \sin.^2\varphi+\cos.\psi.\ \sin.\ \varphi.\ \cos.\ \varphi)$$

$$+y'\ (\cos.\ \theta.\ \cos.\ \psi.\ \sin.^2\varphi-\sin.\ \psi.\ \sin.\ \varphi.\ \cos.\varphi)-z'.\ \sin.\ \theta.\ \sin.^2\varphi.$$

$$y''.\ \cos.\ \varphi=x'\ (\cos.\ \theta.\ \sin.\ \psi.\ \cos.^2\varphi-\cos.\ \psi.\ \sin.\ \varphi.\ \cos.\ \varphi)$$

$$+y'.\ (\cos.\ \theta.\ \cos.\psi.\ \cos.^2\varphi+\sin.\ \psi.\ \sin.\ \varphi.\ \cos.\ \varphi)-z'.\ \sin.\ \theta.\ \cos.^2\varphi$$

$$\therefore x''.\ \sin.\ \varphi+y''.\ \cos.\ \varphi=x'.\ \cos.\ \theta.\ \sin.\ \psi+y'.\ \cos.\ \theta.\ \cos.\ \psi-z'.\ \sin.\ \theta;$$

multiplying the first member of the equation $x''.\ \cos.\ \varphi-y''.\ \sin.\ \varphi=x'.\ \cos.\ \psi-y'.\ \sin.\ \psi.$ by z'' and the second member by the value of z'' we obtain

$$\cos.\ \varphi.\ x''z''-\sin.\ \varphi.\ y''.z''=x'.^2\ \sin.\ \theta.\ \sin.\ \psi.\ \cos.\ \psi-x'y'.\ \sin.\ \theta.\ \sin.^2\psi$$

$$+x'y'.\ \sin.\ \theta.\ \cos.^2\psi-y'.^2\ \sin.\ \theta.\ \sin.\ \psi.\ \cos.\ \psi.$$

$$+z'.x'.\ \cos.\ \theta.\ \cos.\ \psi-z'y'.\ \cos.\ \theta.\ \sin.\ \psi,$$

substituting for $x',^2\ y',^2\ x'y',\ z'y',\ z'x'$, their values and concinnating we obtain

$$\cos.\ \varphi.\ x''z''-\sin.\ \varphi.\ y''\ z''=(x'^2-y'^2).\ \sin.\ \theta.\ \sin.\ \psi.\ \cos.\ \psi+x'y'.\ \sin.\ \theta.\ (\cos.^2\psi-\sin.^2\psi.)$$

$$+z'x'.\ \cos.\ \theta.\ \cos.\ \psi-z'y'.\ \cos.\ \theta.\ \sin.\ \psi,$$

this expression being extended to all the molecules of the body, will give by substituting for $Sx'^2dm\ Sy'.^2dm$, &c. their respective values a^2, b^2, f, g, h, &c. the expression in the text, in like manner $\sin.\ \varphi.\ x''\ z''$

by equalling the second members of these two equations to nothing, we shall obtain

$$\tan.\theta = \frac{h.\sin.\psi - g.\cos.\psi}{(a^2-b^2).\sin.\psi.\cos.\psi + f.(\cos.^2\psi - \sin.^2\psi)};$$

$$\tfrac{1}{2}\tan.2\theta = \frac{g.\sin.\psi + h.\cos.\psi}{c^2 - a^2.\sin.^2\psi - b^2.\cos.^2.\psi - 2f.\sin.\psi.\cos.\psi};$$

but we have always

$$\tfrac{1}{2}\tan.2\theta = \frac{\tan.\theta}{1-\tan.^2\theta};$$

by equalling these two values of tan. 2θ, and substituting in the last expression, in place of tan. θ. its value, which has been given in a function of ψ; and then in order to abridge, making tan. $\psi = u$; we shall obtain after all reductions, the following equation of the third order.*

$$0 = (gu+h).(hu-g)^2$$
$$+\{(a^2-b^2).u + f.(1-u^2)\}.\{hc^2 - ha^2 + fg).u + gb^2 - gc^2 - hf\}.$$

$$+\cos.\varphi.y'.z'' = x'^2\sin.\theta.\cos.\theta.\sin.^2\psi + x'y'.\sin.\theta.\cos.\theta.\sin.\psi.\cos.\psi - z'x'.\sin.^2\theta.\sin.\psi$$
$$+x'y'.\sin.\theta.\cos.\theta.\sin.\psi.\cos.\psi + y'.^2\sin.\theta.\cos.\theta.\cos.^2\psi - z'y'\sin.^2\theta.\cos.\psi$$
$$+z'x'.\cos.^2\theta.\sin.\psi + z'y'.\cos.^2\theta.\cos.\psi - z'.^2\sin.\theta.\cos.\theta =$$
$$\sin.\theta.\cos.\theta.(x'.^2\sin.^2\psi + y'.^2\cos.^2\psi - z'^2) + 2x'y'.\sin.\theta.\cos.\theta.\sin.\psi.\cos.\psi)$$
$$+(\cos.^2\theta - \sin.^2\theta.)(z'x'.\sin.\psi. + z'y'.\cos.\psi.)$$

by extending this expression to all the molecules and substituting $a^2, b^2, c^2, h, f\,g$, &cæt. for Sx'^2dm and Sy'^2dm &c. we shall obtain the expression which has been given in the text.

* The second members are put equal to nothing because by the conditions of the problem, the first members respectively vanish, consequently we have 0 =

$$((a^2-b^2).\sin.\psi.\cos.\psi + f.(\cos^2\psi - \sin.^2\psi)).\sin.\theta + (g.\cos.\psi - h.\sin.\psi).\cos.\theta;$$

$$0 = \sin.\theta.\cos.\theta.(a^2\sin.^2\psi + b.^2\cos.^2\psi - c^2 + 2f.\sin.\psi.\cos.\psi)$$
$$+(\cos.^2\theta - \sin.^2\theta).(g.\sin.\psi + h.\cos.\psi);$$

As this equation has at least one real root we may perceive that it is always possible to make these two subsequent expressions, and consequently the sum of their squares, to vanish at the same time

$$\therefore \frac{\sin.\,\theta}{\cos.\,\theta} = \tan.\,\theta = \frac{h\sin.\,\psi - g.\cos.\,\psi}{(a^2-b^2).\sin.\,\psi.\cos.\,\psi + f.(\cos.^2\psi - \sin.^2\psi)},$$

$$\frac{\sin.\,\theta.\cos.\,\theta}{\cos.^2\theta - \sin.^2\theta} = \frac{\dfrac{\sin.\,\theta}{\cos.\,\theta}}{1-\dfrac{\sin.^2\theta}{\cos.^2\theta}} = \frac{\tan.\,\theta}{1-\tan.^2\theta} = \tfrac{1}{2}\tan.\,2\theta =$$

$$\frac{g.\sin.\,\psi. + h.\cos.\,\psi}{c^2 - a.^2\sin.^2\psi - b.^2\cos.^2\psi - 2f.\sin.\,\psi.\cos.\,\psi}$$

these fractions being divided cos. ψ, become by substituting u in place of

$$\frac{\sin.\,\psi}{\cos.\,\psi},\ \frac{hu-g}{((a^2-b^2).u + f.(1-u^2)).\cos.\,\psi},\ \frac{gu+h}{((c.^2(1+u^2) - a^2u^2 - b^2 - 2fu).\cos.\,\psi}$$

if we call the factors of cos. ψ in the denominators of these respective fractions m and n we shall have

$$\tan.\,\theta. = \frac{hu-g}{m.\cos.\,\psi} \therefore \tfrac{1}{2}.\tan.\,2\theta =$$

$$\frac{\dfrac{hu-g}{m.\cos.\,\psi}}{1-\left\{\dfrac{hu-g}{m.\cos.\,\psi}\right\}^2} = \frac{(hu-g).m.\cos.\,\psi}{\overline{m.\cos.\,\psi}^2 - \overline{hu-g}^2} = \frac{gu+h}{n.\cos.\,\psi} \therefore$$

by reducing we obtain

$$(hu-g).mn.\cos.^2\psi = (gu+h).(.(m\cos\psi)^2 - (hu-g)^2)$$

and consequently $0 =$

$$\cos.^2\psi.(m.(hu-g).n - (gu+h).m) + (hu-g).^2(gu+h,) \text{ now } (hu-g)n. =$$

by substituting for n, $(c.^2(1+u^2) - a^2u^2 - b^2 - 2fu)$ and then multiplying

$$hc^2u + hc^2u^3 - ha^2u^3 - hb^2u - 2fhu^2 - gc^2 - gc^2u^2 + ga^2u^2 + gb^2 + 2fgu,$$

in like manner

$$\cos.\ \varphi.\ S.\ x''z''.dm - \sin.\ \varphi.\ S.\ y'.z''.dm\ ;$$

$$\sin.\ \varphi.\ S.\ x''z''.dm + \cos.\ \varphi.\ S.\ y''z''.dm\ ;$$

and this requires that we should have $S.\ x''z''.dm$; $S\ y''z''.dm$ separately equal to nothing.

The value of u gives that of the angle ψ, and consequently the value of tang. θ, and of the angle θ. It is only now required to determine the angle φ and this will be effected by means of the condition $S.\ x''y''.dm = 0$, which we have yet to satisfy. For this purpose it may be observed, that if we substitute in $S.x''y'.dm$ * in place of x'', y'',

$$-(gu+h).m = (-(gu+h).(a^2-b^2).\ u+f(1-u^2) =$$

$$-a^2gu^2+gb^2u^2-gfu+gfu^3-ha^2u+hb^2u-hf+hfu^2\ \therefore$$

the preceding equation becomes, to by making the similar factors of u and its powers to coalesce, equal to,

$$(hc^2\ u.(1+u^2)-ha^2\ u.\ (1+u^2)-fh.\ (1+u^2)-gc.^2\ (1+u^2)+fgu.(1+u^2)$$

$$+gb.^2(1+u^2)).\left\{\frac{(a^2-b^2)u+f.(1-u^2)}{1+u^2}\right\}$$

$$\text{for } \cos.\ ^2\psi.\ m.(hu-g) = \frac{(a^2-b^2).u+f.(1-u^2)}{1+u_2}$$

$$+(hu-g)2.\ (gu+h) =$$

$$(\ (a^2-b.)^2\ u+f.(1-u^2)).\ ((hc^2-ha^2+fg).u-fh-gc^2+gb^2)\ )+(hu-g).^2(gu+h) = 0,$$

which is the expression given in the text.

$$*\ x''.\ \cos.\ \varphi - y''.\ \sin.\ \varphi = x'.\ \cos.\ \psi - y'.\ \sin.\ \psi. = P$$

$$x''.\ \sin.\ \varphi + y''.\ \cos.\ \varphi = x'.\ \cos.\ \theta.\ \sin.\ \psi + y'.\ \cos.\ \theta.\ \cos.\ \psi - z.\ \sin.\ \theta = Q$$

$$\therefore x''.\ \cos.\ ^2\varphi - y''.\ \sin.\ \varphi.\ \cos.\ \varphi = x'.\cos.\ \psi.\ \cos.\ \varphi - y'.\ \sin.\ \psi.\ \cos.\ \varphi = P.\cos.\ \varphi$$

$$x''.\ \sin.\ ^2\varphi + y''.\sin.\ \varphi.\ \cos.\ \varphi = x'.\ \cos.\ \theta.\ \sin.\ \psi.\ \sin.\ \varphi + y'.\cos.\ \theta.\cos.\ \psi.\sin.\ \varphi$$

$$-z'\ \sin.\ \theta.\sin.\ \varphi = Q.\sin.\ \varphi$$

$$\therefore x''. = x'.(\cos.\ \theta.\ \sin.\psi.\ \sin.\ \varphi + \cos.\ \psi.\ \cos.\ \varphi) + y'(\cos.\ \theta.\ \cos.\psi.\ \sin.\ \varphi - \sin.\ \psi.\ \cos.\ \varphi)$$

$$-z'.\ \sin.\ \theta.\sin.\ \varphi = P.\cos.\ \varphi + Q.\ \sin.\ \varphi$$

their preceding values, this function will assume this form, $H.\sin 2\varphi + L.\cos. 2\varphi$; H and L being functions of the angles θ and ψ, and of the constant quantities a^2, b^2, c^2, f, g, h, by putting this expression equal to nothing, we shall obtain $\tan. 2\varphi = \frac{-L}{H}$.

The three axes determined by means of the preceding values of θ, ψ, and φ, satisfy the three equations,

also

$$x''.\sin.\varphi.\cos.\varphi - y''.\sin.^2\varphi = x'.\cos.\psi.\sin.\varphi - y'.\sin.\psi\sin.\varphi = P.\sin.\varphi$$

$$x''.\sin.\varphi.\cos.\varphi + y''\cos.^2\varphi = x'.\cos.\theta.\sin.\psi.\cos.\varphi + y'.\cos.\theta.\cos.\psi.\cos.\varphi -$$

$$z'\sin.\theta.\cos.\varphi = Q.\cos.\varphi$$

$$\therefore y'' = x'.(\cos.\theta.\sin.\psi.\cos.\varphi - \cos.\psi.\sin.\varphi) + y'(\cos.\theta.\cos.\psi.\cos.\varphi + \sin.\psi.\sin.\varphi)$$

$$+ z'\cdot\sin.\theta.\cos.\varphi = Q.\cos.\varphi - P.\sin.\varphi \;\therefore$$

$x''y'' = PQ.\cos.^2\varphi - PQ.\sin.^2\varphi + Q.^2\sin.\varphi.\cos.\varphi - P.^2\sin.\varphi.\cos.\varphi$ $\therefore$ if $Sx''y''.dm = 0$, we shall have

$$SPQ.dm(\cos.^2\varphi - \sin.^2\varphi) + S(Q^2 - P^2).dm\sin.\varphi.\cos.\varphi = 0. \text{ and } \frac{S.PQdm}{S(Q^2 - P^2)dm} = \frac{\sin.\varphi.\cos.\varphi}{\cos.^2\varphi - \sin.^2\varphi},$$

making $H = S(Q^2 - P^2)dm$ and $\frac{L}{2H} = S.PQ\,dm$ we shall have $-\frac{L}{2L} =$

$$\frac{\sin. 2\varphi.}{2.\cos. 2\varphi} \therefore \frac{-L}{H} = \tan. 2\varphi;$$

this equation determines a real value for, $\tan. 2\varphi$ and $\therefore$ for φ, and as the equation which determines the value of u has at least one real root, $\tan.\psi$ and $\therefore \tan.\theta.$ are real, consequently we are justified in assuming as we have done

$$Sx''y''.dm, \quad Sy''z''.dm, \quad Sx''z''.dm,$$

respectively equal to nothing, and therefore we shall have at least one system of principal axes existing in every body.

$$S.x''y''.dm=0\,;\ S.x''z''.dm=0\,;\ S.y''z''.dm=0.*$$

The equation of the third order in u, seems to indicate three systems of principal axes, similar to the preceding; but it ought to be observed

* All the roots in the equation which determines the value of u are real, and this equation must be of the third dimension, for in the investigation of the angles θ, ψ, ϕ, there is no difference between the principal axes, nor is there any condition to determine which of the three principal planes we assume, $\therefore$ the solution must be applicable equally to the angle contained between the axes of x', and either of the three intersections formed by the plane of x', y', with the three principal planes of the body respectively, consequently the roots of the equation must be all real, it also follows that there is only one system of principal axes in every body, for as each system would give three values of u, the dimension of the resulting equation which determines the value of u, should be equal to three multiplied into the number of systems, but the equation does not transcend the third order, $\therefore$the number of systems is only one, indeed if the equations which give the values of θ ψ and φ are identical, the number of principal axes is infinite, this will evidently be the case where the terms which compose the equation in u vanish without supposing any relations existing between the terms i, e, when $a^2=b^2=c^2$, and f, g, h, respectively vanish we shall have for the coordinates x', y', z', $S.x'y'.dm=0$, $S.x'z'dm=0$, $S.y'z'.dm=0$ $\therefore$ they are principal axes, and as in this case tan. $\theta=\frac{0}{0}$, the position of these axes is entirely undetermined $\therefore$ all systems of rectangular axes are principal axes and their number is infinite; from the expression for tan. θ it appears in like manner, that this angle is 100°, when $a^2=b^2$ and $f=0$, and consequently that the plane of the axes of y' and x' must pass through the axis of z''.

For all bodies symmetrically constituted, one of the principal axes, is the axis of the figure i, e, a line perpendicular to the plane dividing the bodies into two parts perfectly equal and similar, for supposing this plane to be that of x, y, then if we take two equal molecules, similarly situated with respect to this plane, it is evident that if the coordinates of one molecule be x, y, z, the coordinates of the other will be $x, y, -z$, and the indefinitely small elements of the integrals $S.xz.dm$, $S.yz.dm$, which correspond to these molecules will be $xz.dm$, $-xz\ dm$, $yz.dm$ $-yz.dm$, $\therefore$ the sum of all the indefinitely small quantities $xz.dm$, $-xz.dm$, $yz.dm-yz.dm$, at one side of the plane will be equal to the sum of the indefinitely small quantities at the other side affected with a contrary sign, $\therefore$ their respective aggregates $S.xzdm$, $S.yz.dm$ are equal to nothing, $\therefore$ the axis of z is a principal axis, and if the molecules of the body be symmetrically arranged with respect to a plane passing through the axis of z' perpendicular to the first mentioned plane, we shall have $S.xy.dm=0$ $\therefore$ the axes of x, y, z, will be principal axes.

What has been established in the preceding note is of great importance, as the investithat u is the tangent of the angle formed by the axis of x', and by the intersection of the plane of x' and y' with the plane of x'' and y'', and it is evident that one of the three axes of x'', of y'', and of z'' may be changed in another, since the three preceding equations will be always satisfied; therefore the equation in u ought to determine indifferently, the tangent of the angle formed by the axis of x', with the intersection of the plane x', y', either with the plane x'' y'', or the plane x'', z'', or finally with the plane y'', z'',. Thus the three roots of the equation in u are real, and they belong to the same system of axes.

It follows from what precedes, that generally a solid has only one system of axes, which possess the property in question. These axes

gation of the position of the principal axes is considerably facilitated by making one of them to coincide with one of three coordinates x' y' z', whose position is entirely arbitrary, for supposing the axis of x'' to coincide with the axis of x', then since φ=the angle which the intersection of the plane of x'' and y'', with the plane x', y', makes with the axis of x', and since ψ = the complement of the angle, which the projection of the third axis on the plane of x' and y' makes with the axes of x', these angles are severally equal to nothing

$$\therefore \text{tan.}\,\theta = \frac{h.\,\text{sin.}\psi - g.\,\text{cos.}\,\psi}{(a^2-b^2)\,\text{sin.}\,\psi.\,\text{cos.}\psi + f.(\text{cos.}^2\psi - \text{sin.}^2\psi)}$$

becomes equal to

$$-\frac{g'}{f'} \text{ and } \tfrac{1}{2}.\,\text{tan.}\,2\theta = \frac{g.\,\text{sin.}\,\psi + h.\,\text{cos.}\psi}{c^2 - a^2.\,\text{cos.}^2\psi - b^2.\,\text{sin.}^2\psi - 2f.\,\text{sin.}\,\psi.\,\text{cos.}\,\psi} = \frac{h'^2}{c'^2 - b'^2},$$

n which c', b', f', g', h' indicate what c, b, f, g, h, become when x' coincides with x'', and as tan. $2(\theta + 100)$=tan. $(2\theta + 200)$ = tan. 2θ, it follows that the other two axes must be taken in the plane y', z', one making the angle θ and the other the angle $\theta + 100$ with the axis of y', now if we made the axes of y'', and z'', to coincide with the axes of y' and z' respectively, θ, and $\therefore$ h' would vanish, and consequently $S(y'z'.)dm$ would be equal to nothing. But if h' remaining equal to nothing, b' and c' would be equal to each other then $\text{tan.}2\theta = \frac{h'}{c'^2 - b'^2}$ would be equal to $\frac{0}{0}$ $\therefore$ θ would be indeterminate and every line in the plane y' z', and passing through the origin of the coordinates would be a principal axis, see notes to page 184.

have been named principal axes of rotation, on account of a property which is peculiar to them and which will be noticed in the sequel.

The sum of the products of each molecule of the body, into the square of its distance, from an axis, is called the *moment of inertia* of a body with respect to this axis. Thus the quantities A, B, C, are the moments of inertia of the solid, which we have considered, with respect to axis of x'', of y'', and of z''. Naming C' the moment of inertia of the same solid with respect to the axis of z', by means of the values of x', y', and z', which are given in the preceding number, we shall find

$$C' = A.\ \sin.^2\theta.\ \sin.^2\varphi + B.\ \sin.^2\theta.\ \cos.^2\varphi + C.\ \cos.^2\theta. \ *$$

The quantities $\sin.^2\theta.\ \sin.^2\varphi$, $\sin.^2\theta, \cos.^2\varphi$, $\cos.^2\theta$, are the squares of the cosines of the angles, which the axes of x'', of y'', and of z'', make with the axis of z'; hence it follows in general that, if we multiply the moment of inertia relative to each principal axis of rotation,

* Since $S.(x''^2+y''^2+z''^2).dm = S.(x'^2+y'^2+z'^2).\ dm$ by substituting the value of $z',^2$ in terms of $x'',^2\ y'',^2\ z''.^2$ and observing that $S.x''y''.dm$, $Sx''z''.dm$, $Sy''z''.dm$, are equal to nothing, we have

$$Sx''.^2dm + Sy''.^2dm + Sz''.^2dm = S.x'.^2dm + S.y'.^2dm + S.x''.^2\ \sin.^2\theta.\sin.^2\varphi\ dm$$

$$+ S.y''.^2\ \sin.^2\theta.\ \cos.^2\varphi.dm + Sz''.^2\cos.^2\theta.dm \ \therefore\ Sx''^2(1-\sin.^2\theta.\ \sin.^2\varphi)dm$$

$$+ Sy^{2''}.(1-\sin.^2\theta.\ \cos.^2\varphi).dm +$$

$$S.z''.^2(1-\cos.^2\theta).dm = S(x'^2+y'^2).dm\ .\because. = S.x''.^2(\cos.^2\theta+\sin.^2\theta.\ \cos.^2\varphi).dm$$

$$+ S'y'.^2(\cos.^2\theta+\sin.^2\theta.\ \sin.^2\varphi)dm$$

$$+ S.z''.^2\sin.^2\theta.\ \sin.^2\varphi\ dm + S.z''.^2\sin.^2\theta.\ \cos.^2\theta.dm$$

and making the like factors coalesce we obtain C'

$$S.(y''.^2+z''^2).\ \sin.^2\theta.\sin.^2\varphi.dm + S.(x''^2+z''^2).\ \sin.^2\theta.\ \cos.^2\varphi.dm + S\ (x''^2+y''^2).\cos.^2\theta.\ dm \quad \text{i, e,}$$

$$C' = A.\ \sin.^2\theta.\ \sin.^2\varphi + B.\ \sin.^2\theta.\ \cos.^2\varphi + C.\ \cos.^2\theta.;$$

$\sin.\ \theta.\ \sin.\ \varphi$, $\sin.\ \theta.\ \cos.\ \varphi$, $\cos.\ \theta$, are equal to the cosines of the angles which the axes of x'', of y'' and of z'' make with the axes of z', see Note, page 132.

by the square of the cosine of the angle which it makes with any axis, the sum of the three products, will be the moment of inertia of the solid, relative to this last axis.

The quantity C' is less than the greatest, and greater than the least of the three quantities * A, B, C,; therefore the greatest and least moments of inertia appertain to the principal axes. †

* Let A be the greatest and C the least moment of inertia, the value of C' may be made to assume the following form

$$C' = A + (B-A).\sin.^2\theta.\cos.^2\varphi + (C-A).\cos.^2\theta,$$

$\therefore$ since the moments of inertia are always affirmative, the two last terms of the second member of this equation will be negative, consequently C' is less than A, let C be the greatest moment of inertia and the expression for C' will become

$$C + (A-C).\sin.^2\theta.\sin.^2\varphi + (B-C).\sin.^2\theta.\cos.^2\varphi,$$

in this case also the two last terms of the second member are negative, $\therefore$ C' is less than C; the moment of inertia C' is greater than the least of the three principal moments, for if A be the least of the three moments which refer to the principal axes, we have as before

$$C' = A + (B-A).\sin.^2\theta.\cos.^2\varphi + (C-A).\cos.^2\theta,$$

and as the differences are on the present hypothesis affirmative, C' is greater than A, let C be the least of the three moments, and we have

$$C' = C + (A-C).\sin.^2\theta.\sin.^2\varphi + (B-A).\sin.^2\theta.\cos.^2\varphi,$$

the terms which compose the second members are always affirmative, $\therefore$ we conclude that C' is greater than the least of the three moments, A, B, C,

From what has been established in the preceding note, it appears that when the three principal moments of inertia are unequal there is only one system of principal axes, for let there be another system and make A', B', C', the moments of inertia relative to these axes, then we shall have at the same time $A > A'$ and $A' > A$ which is impossible, see note to page 178.

† For $S(x'-X)^2.dm = Sx'.^2dm - 2X.Sx'.dm + X^2m = Sx'.^2dm - 2X^2m + X^2m$, for $Sx'.dm = X.m.$ and as the quantity $-m.(X^2 + Y^2)$ is essentially negative, the moment of inertia with respect to the centre of gravity must be less than the corresponding moment for any axis not passing through the centre of gravity. If the moments are referred to an axis passing through a point different from the centre of gravity and of which the coordinates are a, b, c,

Let X, Y, Z, be the coordinates of the centre of gravity of the solid, relatively to the origin of the coordinates which we fix at the point about which the body is subjected to revolve, if it is not free; $x'-X$, $y'-Y$, $z'-Z$, will be coordinates of the molecule of the body, with respect to the centre of gravity; therefore the moment of inertia, relative to an axis passing through the centre of gravity, and parallel to the axis of z' will be

$$S.\left\{(x'-X)^2+(y'-Y).^2\right\}.dm;$$

but from the nature of the centre of gravity, we have $S.\,x'.dm=m\,X$, $S.\,y'.dm=mY$; $\therefore$ the preceding expression will be reduced to

$$-m.\,(X \quad Y^2)+S.(x'^2+y'^2).\,dm.$$

Consequently we shall have the moments of inertia of the solid, with respect to an axis passing through any point whatever; when these moments are known for axes passing through the centre of gravity. At the same time it appears that the minimum minimorum of the moments of inertia appertains to one of the three principal axes, passing through this centre.

Let us suppose the nature of the body to be such, that the two moments of inertia A and B are equal, then we shall have

$$C'=A.\,\sin.\,^2\theta+C.\,\cos.\,^2\theta;\ *$$

the value of the moment of inertia with respect to this point is equal to

$$A-2.(aX+bY).\,m+(a^2+b^2).m$$

It is evident from an inspection of their values, that the greatest moment of inertia with respect to any point, is less than the sum of the other two moments.

* When $A=B$ the moment of inertia with respect to any other axis $=A.\,\sin,^2\theta+C.\,\cos.^2\theta$, and as neither ψ or φ occur in this expression, the moment of inertia for all axes making the same angle, with the axis of z' are equal, and if θ be a right angle $C'=A$, therefore in this case there is an indefinite number of principal axes, but they have all a common axis z'', when $\theta=100°$ we have $a^2=b^2$ and $f=0$ i, e, $Sx'.^2dm=Sy'.^2dm$ and $Sx'y'.dm=0$ this also