ISAAC NEWTON

THE *PRINCIPIA*

Mathematical Principles of Natural Philosophy

The Authoritative Translation
by I. Bernard Cohen and Anne Whitman
assisted by Julia Budenz

UNIVERSITY OF CALIFORNIA PRESS

THE *PRINCIPIA*

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Portrait of Isaac Newton at about the age of sixty, a drawing presented by Newton to David Gregory. For details see the following page.

Portrait of Isaac Newton at about the age of sixty, presented by Newton to David Gregory (1661–1708). This small oval drawing (roughly 3¾ in. from top to bottom and 3¼ in. from left to right) is closely related to the large oval portrait in oils made by Kneller in 1702, which is considered to be the second authentic portrait made of Newton. The kinship between this drawing and the oil painting can be seen in the pose, the expression, and such unmistakable details as the slight cast in the left eye and the button on the shirt. Newton is shown in both this drawing and the painting of 1702 in his academic robe and wearing a luxurious wig, whereas in the previous portrait by Kneller (now in the National Portrait Gallery in London), painted in 1689, two years after the publication of the *Principia*, Newton is similarly attired but is shown with his own shoulder-length hair.

This drawing was almost certainly made after the painting, since Kneller's preliminary drawings for his paintings are usually larger than this one and tend to concentrate on the face rather than on the details of the attire of the subject. The fact that this drawing shows every detail of the finished oil painting is thus evidence that it was copied from the finished portrait. Since Gregory died in 1708, the drawing can readily be dated to between 1702 and 1708. In those days miniature portraits were commonly used in the way that we today would use portrait photographs. The small size of the drawing indicates that it was not a copy made in preparation for an engraved portrait but was rather made to be used by Newton as a gift.

The drawing captures Kneller's powerful representation of Newton, showing him as a person with a forceful personality, poised to conquer new worlds in his recently gained position of power in London. This high level of artistic representation and the quality of the drawing indicate that the artist responsible for it was a person of real talent and skill.

The drawing is mounted in a frame, on the back of which there is a longhand note reading: "This original drawing of Sir Isaac Newton, belonged formerly to Professor Gregory of Oxford; by him it was bequeathed to his youngest son (Sir Isaac's godson) who was later Secretary of Sion College; & by him left by Will to the Revd. Mr. Mence, who had the Goodness to give it to Dr. Douglas; March 8th 1870."

David Gregory first made contact with Newton in the early 1690s, and although their relations got off to a bad start, Newton did recommend Gregory for the Savilian Professorship of Astronomy at Oxford, a post which he occupied until his death in 1708. As will be evident to readers of the Guide, Gregory is one of our chief sources of information concerning Newton's intellectual activities during the 1690s and the early years of the eighteenth century, the period when Newton was engaged in revising and planning a reconstruction of his *Principia*. Gregory recorded many conversations with Newton in which Newton discussed his proposed revisions of the *Principia* and other projects and revealed some of his most intimate and fundamental thoughts about science, religion, and philosophy. So far as is known, the note on the back of the portrait is the only record that Newton stood godfather to Gregory's youngest son.

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This translation is
dedicated to

D. T. WHITESIDE

with respect and affection

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Contents

Publisher's Note

This volume contains I. Bernard Cohen and Anne Whitman's 1999 translation of Newton's *Principia*. In the preface and in the notes to the translation, Cohen refers to his "Guide to Newton's *Principia*" ("the Guide"). This volume contains an excerpt from the Guide ("A Brief History of the *Principia*"). The Guide appears in full in *The* Principia: *The Authoritative Translation and Guide*, also available from University of California Press.

Preface to the 1999 Edition

ALTHOUGH NEWTON'S *PRINCIPIA* has been translated into many languages, the last complete translation into English (indeed, the only such complete translation) was produced by Andrew Motte and published in London more than two and a half centuries ago. This translation was printed again and again in the nineteenth century and in the 1930s was modernized and revised as a result of the efforts of Florian Cajori. This latter version, with its partial modernization and partial revision, has become the standard English text of the *Principia*.

Motte's version is often almost as opaque to the modern reader as Newton's Latin original, since Motte used such older and unfamiliar expressions as "sub-sesquialterate" ratio. Additionally, there are statements in which the terms are no longer immediately comprehensible today, such as book 3, prop. 8, corol. 3, in which Motte writes that "the densities of dissimilar spheres are as those weights applied to the diameters of the spheres," a statement unaltered in the Motte-Cajori version. Of course, a little thought reveals that Newton was writing about the densities of nonhomogeneous spheres and was concluding with a reference to the weights divided by the diameters. The Motte-Cajori version, as explained in §2.3 of the Guide to the present translation, is also not satisfactory because it too is frequently difficult to read and, what is more important, does not always present an authentic rendition of Newton's original. The discovery of certain extraordinary examples in which scholars have been misled in this regard was a chief factor in our decision to produce the present translation.

When we completed our Latin edition, somewhat awed by the prospect of undertaking a wholly new translation, we thought of producing a new edition of Motte's English version, with notes that either would give the reader a modern equivalent of difficult passages in Motte's English prose or would contain some aids to help the reader with certain archaic mathematical expressions. That is, since Motte's text had been a chief means of disseminating Newton's science for over two centuries, we considered treating it as an important historical document in its own right. Such a plan was announced in our Latin edition, and we even prepared a special interleaved copy of the facsimile of the 1729 edition to serve as our working text.*

After the Latin edition appeared, however, many colleagues and some reviewers of that edition insisted that it was now our obligation to produce a completely new translation of the *Principia*, rather than confine our attentions to Motte's older pioneering work. We were at first reluctant to accept this assignment, not only because of the difficulty and enormous labor involved, but also because of our awareness that we ourselves would thereby become responsible for interpretations of Newton's thought for a long period of time.

Goaded by our colleagues and friendly critics, Anne Whitman and I finally agreed to produce a wholly new version of the *Principia*. We were fortunate in obtaining a grant from the National Science Foundation to support our efforts. Many scholars offered good advice, chief among them our good friends D. T. Whiteside and R. S. Westfall. In particular, Whiteside stressed for us that we should pay no attention to any existing translation, not even consulting any other version on occasions when we might be puzzled, until after our own assignment had been fully completed. Anyone who has had to translate a technical text will appreciate the importance of this advice, since it is all too easy to be influenced by other translations, even to the extent of unconsciously repeating their errors. Accordingly, during the first two or three rounds of translation and revision, we recorded puzzling or doubtful passages, and passages for which we hoped to produce a final version that would be less awkward than our preliminary efforts, reserving for some later time a possible comparison of our version with others. It should be noted that in the final two rounds of our revision, while checking some difficult passages and comparing some of our renditions with others, the most useful works for such purpose were Whiteside's own translation of an early draft of what corresponds to most of book 1 of the *Principia* and the French translation made in the

*An Index Verborum of the Latin edition of the *Principia* has been produced by Anne Whitman and I. Bernard Cohen in association with Owen Gingerich and Barbara Welther. This index includes the complete text of the third edition (1726) and also the variant readings as given in the Latin edition of the *Principia* (edited by Alexandre Koyré, I. Bernard Cohen, and Anne Whitman), published in 1972 by the Harvard University Press and the Cambridge University Press. Thus the Index includes the complete text of the three authorized Latin editions (1687, 1713, 1726) as well as the MS annotations in both Newton's "annotated" and "interleaved" personal copies of the first and second editions. The Index is on deposit in the Burndy Library of the Dibner Institute (Cambridge, Mass.), where it may be consulted. Microfilm copies can be purchased.

Very useful tools for scholars and students are the planned Octavo editions of the first and third Latin editions of Newton's *Principia*; the latter will include this English translation. The high-resolution facsimiles on CD-ROM allow readers to view the original book and search the complete Latin texts and translation. For publication information, see the Octavo web site: www.octavo.com.

mid-eighteenth century by the marquise du Châtelet. On some difficult points, we also profited from the exegeses and explanations in the Le Seur and Jacquier Latin edition and the Krylov Russian edition. While neither Anne Whitman nor I could read Russian, we did have the good fortune to have two former students, Richard Kotz and Dennis Brezina, able and willing to translate a number of Krylov's notes for us.

The translation presented here is a rendition of the third and final edition of Newton's *Principia* into present-day English with two major aims: to make Newton's text understandable to today's reader and yet to preserve Newton's form of mathematical expression. We have thus resisted any temptation to rewrite Newton's text by introducing equations where he expressed himself in words. We have, however, generally transmuted such expressions as "subsesquiplicate ratio" into more simply understandable terms. These matters are explained at length in §§10.3–10.5 of the Guide.

After we had completed our translation and had checked it against Newton's Latin original several times, we compared our version with Motte's and found many of our phrases to be almost identical, except for Motte's antique mathematical expressions. This was especially the case in the mathematical portions of books 1 and 2 and the early part of book 3. After all, there are not many ways of saying that a quantity A is proportional to another quantity B. Taking into account that Motte's phrasing represents the prose of Newton's own day (his translation was published in 1729) and that in various forms his rendition has been the standard for the English-reading world for almost three centuries, we decided that we would maintain some continuity with this tradition by making our phrasing conform to some degree to Motte's. This comparison of texts did show, however, that Motte had often taken liberties with Newton's text and had even expanded Newton's expressions by adding his own explanations—a result that confirmed the soundness of the advice that we not look at Motte's translation until *after* we had completed our own text.

This translation was undertaken in order to provide a readable text for students of Newton's thought who are unable to penetrate the barrier of Newton's Latin. Following the advice of scholarly friends and counselors, we have not overloaded the translation with extensive notes and comments of the sort intended for specialists, rather allowing the text to speak for itself. Much of the kind of editorial comment and explanation that would normally appear in such notes may be found in the Guide. Similarly, information concerning certain important changes in the text from edition to edition is given in the Guide, as well as in occasional textual notes. The table of contents for the Guide, found on pages 3–7, will direct the reader to specific sections of the *Principia*, or even to particular propositions.

The Guide to the present translation is intended to be just that—a kind of extended road map through the sometimes labyrinthine pathways of the *Principia*. Some propositions, methods, and concepts are analyzed at length, and in some instances critical details of Newton's argument are presented and some indications are given of the alterations produced by Newton from one edition to the next. Sometimes reference is made to secondary works where particular topics are discussed, but no attempt has been made to indicate the vast range of scholarly information relating to this or that point. That is, I have tended to cite, in the text and in the footnote references, primarily those works that either have been of special importance for my understanding of some particular point or some sections of the *Principia* or that may be of help to the reader who wishes to gain a more extensive knowledge of some topic. As a result, I have not had occasion in the text to make public acknowledgment of all the works that have been important influences on my own thinking about the *Principia* and about the Newtonian problems associated with that work. In this rubric I would include, among others, the important articles by J. E. McGuire, the extremely valuable monographs on many significant aspects of Newton's science and philosophic background by Maurizio Mamiani (which have not been fully appreciated by the scholarly world because they are written in Italian), the two histories of mechanics by René Dugas and the antecedent documentary history by Léon Jouguet, the analysis of Newton's concepts and methods by Pierre Duhem and Ernst Mach, and monographic studies by Michel Blay, G. Bathélemy, Pierre Costabel, and A. Rupert Hall, and by François de Gandt.

I also fear that in the Guide I may not have sufficiently stressed how greatly my understanding of the *Principia* has profited from the researches of D. T. Whiteside and Curtis Wilson and from the earlier commentaries of David Gregory, of Thomas Le Seur and François Jacquier, and of Alexis Clairaut. The reader will find, as I have done, that R. S. Westfall's *Never at Rest* not only provides an admirable guide to the chronology of Newton's life and the development of his thought in general, but also analyzes the whole range of Newton's science and presents almost every aspect of the *Principia* in historical perspective.

All students of the *Principia* find a guiding beacon in D. T. Whiteside's essays and his texts and commentaries in his edition of Newton's *Mathematical Papers*, esp. vols. 6 and 8 (cited on p. 9 below). On Newton's astronomy, the concise analysis by Curtis Wilson (cited on p. 10 below) has been of enormous value. Many of the texts quoted in the Guide have been translated into English. It did not seem necessary to mention this fact again and again.

From the very start of this endeavor, Anne Whitman and I were continuously aware of the awesome responsibility that was placed on our shoulders, having in mind all too well the ways in which even scholars of the highest distinction have been misled by inaccuracies and real faults in the current twentieth-century English version. We recognized that no translator or editor could boast of having perfectly understood Newton's text and of having found the proper meaning of every proof and construction. We have ever been aware that a translation of a work as difficult as Newton's *Principia* will certainly contain some serious blunders or errors of interpretation. We were not so vain that we were always sure that we fully understood every level of Newton's meaning. We took comfort in noting that even Halley, who probably read the original *Principia* as carefully as anyone could, did not always fully understand the mathematical significance of Newton's text. We therefore, in close paraphrase of Newton's own preface to the first edition, earnestly ask that everything be read with an open mind and that the defects in a subject so difficult may be not so much reprehended as kindly corrected and improved by the endeavors of our readers.

I. B. C.

Some Acknowledgments

Anne Whitman died in 1984, when our complete text was all but ready for publication, being our fourth (and in some cases fifth and even sixth) version. It was her wish, as well as mine, that this translation be dedicated to the scholar whose knowledge of almost every aspect of Newton's mathematics, science, and life is unmatched in our time and whose own contributions to knowledge have raised the level of Newtonian scholarship to new heights.

We are fortunate that Julia Budenz has been able to help us with various aspects of producing our translation and especially in the final stages of preparing this work for publication.

It has been a continual joy to work with the University of California Press. I am especially grateful to Elizabeth Knoll, for her thoughtfulness with regard to every aspect of converting our work into a printed book, and to Rose Vekony, for the care and wisdom she has exercised in seeing this complex work through production, completing the assignment so skillfully begun by Rebecca Frazier. I have profited greatly from the many wise suggestions made by Nicholas Goodhue, whose command of Latin has made notable improvements in both the Guide and the translation. One of the fortunate aspects of having the translation published by the University of California Press is that we have been able to use the diagrams (some with corrections) of the older version.

I gladly acknowledge and record some truly extraordinary acts of scholarly friendship. Three colleagues—George E. Smith, Richard S. Westfall, and Curtis

Wilson—not only gave my Guide a careful reading, sending me detailed commentaries for its improvement; these three colleagues also checked our translation and sent me many pages of detailed criticisms and useful suggestions for its improvement. I am particularly indebted to George Smith of Tufts University for having allowed me to make use of his as yet unpublished *Companion to Newton's "Principia,"* a detailed analysis of the *Principia* proposition by proposition. Smith used the text of our translation in his seminar on the *Principia* at Tufts during the academic year 1993/94 and again during 1997/98. Our final version has profited greatly from the suggestions of the students, who were required to study the actual text of the *Principia* from beginning to end. I am happy to be able to include in the Guide a general presentation he was written (in his dual capacity as a philosopher of science and a specialist in fluid mechanics) on the contents of book 2 and also two longish notes, one on planetary perturbations, the other on the motion of the lunar apsis. I have also included a note by Prof. Michael Nauenberg of the University of California, Santa Cruz, on his current research into the origins of some of Newton's methods.

I am grateful to the University Library, Cambridge, for permission to quote extracts and translations of various Newton MSS. I gladly record here my deep gratitude to the staff and officers of the UL for their generosity, courtesy, kindness, and helpfulness over many years.

I am especially grateful to Robert S. Pirie for permission to reproduce the miniature portrait which serves as frontispiece to this work. The following illustrations are reproduced, with permission, from books in the Grace K. Babson Collection of the Works of Sir Isaac Newton, Burndy Library, Dibner Institute for the History of Science and Technology: the title pages of the first and second editions of the *Principia*; the half title, title page, and dedication of the third edition; and the diagrams for book 2, prop. 10, in the Jacquier and Le Seur edition of the *Principia*.

I gratefully record the continued and generous support of this project by the National Science Foundation, which also supported the prior production of our Latin edition with variant readings. Without such aid this translation and Guide would never have come into being.

Finally, I would like to thank the Alfred P. Sloan Foundation for a grant that made it possible to add an index to the second printing.

Addendum

I am particularly grateful to four colleagues who helped me read the proofs. Bruce Brackenridge checked the proofs of the Guide and shared with me many of his insights into the methods used by Newton in the *Principia*, while George Smith worked through the proofs of each section of the Guide and also helped me check the translation. Michael Nauenberg and William Harper helped me find errors in the Guide. A student, Luis Campos, gave me the benefit of his skill at proofreading.

I also gladly acknowledge the importance of correspondence with Mary Ann Rossi which helped me to clarify certain grammatical puzzles. Edmund J. Kelly kindly sent me the fruit of his detailed textual study of the Motte-Cajori version of the *Principia*.

Readers' attention is called to three collections of studies that are either in process of publication or appeared too late to be used in preparing the Guide: *Planetary Astronomy from the Renaissance to the Rise of Astrophysics, Part B: The Eighteenth and Nineteenth Centuries,* ed. René Taton and Curtis Wilson (Cambridge: Cambridge University Press, 1995); *Isaac Newton's Natural Philosophy,* ed. Jed Buchwald and I. B. Cohen (Cambridge: MIT Press, forthcoming); and *The Foundations of Newtonian Scholarship: Proceedings of the 1997 Symposium at the Royal Society,* ed. R. Dalitz and M. Nauenberg (Singapore: World Scientific, forthcoming). Some of the chapters in these collections, notably those by Michael Nauenberg, either suggest revisions of the interpretations set forth in the Guide or offer alternative interpretations. Other contributions of Nauenberg are cited in the notes to the Guide.

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A Brief History of the *Principia*

The Origins of the Principia

Isaac Newton's *Principia* was published in 1687. The full title is *Philosophiae Naturalis Principia Mathematica,* or *Mathematical Principles of Natural Philosophy.* A revised edition appeared in 1713, followed by a third edition in 1726, just one year before the author's death in 1727. The subject of this work, to use the name assigned by Newton in the first preface, is "rational mechanics." Later on, Leibniz introduced the name "dynamics." Although Newton objected to this name,[1] "dynamics" provides an appropriate designation of the subject matter of the *Principia,* since "force" is a primary concept of that work. Indeed, the *Principia* can quite properly be described as a study of a variety of forces and the different kinds of motions they produce. Newton's eventual goal, achieved in the third of the three "books" of which the *Principia* is composed, was to apply the results of the prior study to the system of the world, to the motions of the heavenly bodies. This subject is generally known today by the name used a century or so later by Laplace, "celestial mechanics."

1. Newton's objections were not based on the name in terms of its Greek roots or its adequacy or inadequacy to describe the subject matter. Rather, he took umbrage at Leibniz's having devised a name as if he had been the inventor of the subject, whereas Newton believed that he himself had been the primary creator. In a private memorandum (my *Introduction*, p. 296, §6), Newton wrote that "Galileo began to consider the effect of Gravity upon Projectiles. Mr. Newton in his *Principia Philosophiae* improved that consideration into a large science. Mr. Leibnitz christened the child by [a] new name as if it had been his own, calling it *Dynamica.*" In another such memorandum (ibid., p. 297), he declared that Leibniz "changed the name of vis centripeta used by Newton into that of sollicitatio paracentrica, not because it is a fitter name, but to avoid being thought to build upon Mr. Newton's foundation." He also held that Leibniz "has set his mark upon this whole science of forces calling it Dynamick, as if he had invented it himself & is frequently setting his mark upon things by new names & new Notations."

The history of how the *Principia* came into being has been told and retold.[2] In the summer of 1684, the astronomer Edmond Halley visited Newton in order to find out whether he could solve a problem that had baffled Christopher Wren, Robert Hooke, and himself: to find the planetary orbit that would be produced by an inverse-square central force. Newton knew the answer to be an ellipse.[3] He had solved the problem of elliptical orbits earlier, apparently in the period 1679–1680 during the course of an exchange of letters with Hooke. When Halley heard Newton's reply, he urged him to write up his results. With Halley's prodding and encouragement, Newton produced a short tract which exists in several versions and will be referred to as *De Motu*[4] (On Motion), the common beginning of all the titles Newton gave to the several versions. Once started, Newton could not restrain the creative force of his genius, and the end product was the *Principia*. In his progress from the early versions of *De Motu* to the *Principia*, Newton's conception of what could be achieved by an empirically based mathematical science had become enlarged by several orders of magnitude.

2. E.g., my *Introduction*, Westfall's *Never at Rest*, Whiteside's introduction in *Math. Papers* (vol. 6), Herivel's *Background*, and more recently A. Rupert Hall, *Isaac Newton: Adventurer in Thought* (Oxford and Cambridge, Mass.: Blackwell, 1992).

3. Our source for this anecdote may be found in the notes accumulated by John Conduitt, husband of Newton's niece and Newton's successor at the Mint, for a proposed biography of Newton. Conduitt got the story from the mathematician Abraham de Moivre. The main lines of the story are undoubtedly correct, but we may doubt the accuracy of the details, since this is a secondhand record of an event that had happened about half a century earlier. What was the exact question that Halley would have asked Newton?

The question recorded by Conduitt has puzzled critical historians, because it does not have a simple answer. There has even been some speculation whether Halley may have asked Newton for the force acting in the case of an elliptical orbit rather than for the orbit produced by an inverse-square force. It is doubtful whether Conduitt knew enough mathematics to see the difference between the two. But, in fact, there is a real difference. As Newton shows in the *Principia*, in prop. 11, and as he proved in the drafts of *De Motu*, an elliptical orbit does imply an inverse-square force. Yet, as readers of the *Principia* would have been aware, an inverse-square force does not necessarily imply an elliptic orbit, rather a conic section (which can be an ellipse, a parabola, or a hyperbola).

Of course, Halley's question may have implied (or have been thought by Newton to have implied) an orbit of a planet or possibly a planetary satellite. Since such an orbit is a closed curve, and therefore not a parabola or a hyperbola, Halley's question to Newton would then have been, in effect, What is the planetary orbit (or closed orbit) produced by an inverse-square force? In this case, the answer would legitimately be the one recorded by Conduitt.

4. The several versions of *De Motu* may be found (with translations and commentary) in Whiteside's edition of *Math. Papers* 6:30–80; the Halls' *Unpublished Sci. Papers*, pp. 237–239, 243–292; Herivel's *Background*, pp. 256–303; and, earlier, in Rouse Ball's *Essay*, pp. 31–56, and in Stephen P. Rigaud, *Historical Essay on the First Publication of Sir Isaac Newton's "Principia"* (Oxford: Oxford University Press, 1838; reprint, with an introd. by I. B. Cohen, New York and London: Johnson Reprint Corp., 1972), appendix, no. 1, pp. 1–19. For a facsimile reprint of the MSS of *De Motu*, see n. 5 below.

As first conceived, the *Principia* consisted of two "books" and bore the simple title *De Motu Corporum* (On the Motion of Bodies).[5] This manuscript begins, as does the *Principia*, with a series of Definitions and Laws of Motion, followed by a book 1 whose subject matter more or less corresponds to book 1 of the *Principia*.[6] The subject matter of book 2 of this early draft is much the same as that of book 3 of the *Principia*. In revising this text for the *Principia*, Newton limited book 1 to the subject of forces and motion in free spaces, that is, in spaces devoid of any resistance. Book 2 of the *Principia* contains an expanded version of the analysis of motion in resisting mediums, plus discussions of pendulums,[7] of wave motion, and of the physics of vortices. In the *Principia*, the system of the world became the subject of what is there book 3, incorporating much that had been in the older book 2 but generally recast in a new form. As Newton explained in the final *Principia*, while introducing book 3, he had originally presented this subject in a popular manner, but then decided to recast it in a more mathematical form so that it would not be read by anyone who had not first mastered the principles of rational mechanics. Even so, whole paragraphs of the new book 3 were copied word for word from the old book 2.[8]

Steps Leading to the Composition and Publication of the Principia

The history of the development of Newton's ideas concerning mechanics, more specifically dynamics, has been explored by many scholars and is still the subject of active research and study.[9] The details of the early development of Newton's

5. On this first draft of book 1, see my *Introduction*, chap. 4 and suppl. 3, where it is referred to as Newton's Lucasian Lectures (LL) because Newton later deposited this MS in the University Library as if it were the text of his university lectures for 1684 and 1685. This text has been edited and translated by D. T. Whiteside in vol. 6 of *Math. Papers*, and Whiteside has also prepared a facsimile edition of the whole MS, together with the drafts of *De Motu*, under the general title *The Preliminary Manuscripts for Isaac Newton's 1687 "Principia," 1684–1685* (Cambridge and New York: Cambridge University Press, 1989).

6. This early book 1 concluded (as did *De Motu*) with a brief presentation of motion in resisting fluids, which was later considerably expanded so as to become the first sections of book 2 of the *Principia*.

7. Pendulums are also discussed in book 1.

8. A new translation of this early version of book 3, by Anne Whitman and I. Bernard Cohen, is scheduled for publication by the University of California Press. In order to distinguish this work from book 3 of the *Principia* (with its subtitle "De Systemate Mundi"), we have called this early version *Essay on the System of the World*. A list of the paragraphs that are the same in both versions may be found in a supplement to our edition of the *Principia* with variant readings, cited in n. 45 below.

9. The books and articles devoted to this topic are so numerous, and continue to appear at so rapid a rate, that it would hardly be practical to cite them all. The most accessible and authoritative presentations are to be found in Curtis Wilson's "Newt. Achievement" and especially in D. T. Whiteside, "Before the *Principia*: The Maturing of Newton's Thoughts on Dynamical Astronomy, 1664–84," *Journal for the History of Astronomy* 1 (1970): 5–19; "The Mathematical Principles Underlying Newton's *Principia*," ibid.,

ideas about force and motion, however interesting in their own right, are not directly related to the present assignment, which is to provide a reader's guide to the *Principia*. Nevertheless, some aspects of this prehistory should be of interest to every prospective reader of the *Principia*. In the scholium to book 1, prop. 4, Newton refers to his independent discovery (in the 1660s) of the $\frac{v^2}{r}$ rule for the force in uniform circular motion (at speed v along a circle of radius r), a discovery usually attributed to Christiaan Huygens, who formally announced it to the world in his *Horologium Oscillatorium* of 1673.[10] It requires only the minimum skill in algebraic manipulation to combine the $\frac{v^2}{r}$ rule with Kepler's third law in order to determine that in a system of bodies in uniform circular motion the force is proportional to $\frac{1}{r^2}$ or is inversely proportional to the square of the distance. Of course, this computation does not of itself specify anything about the nature of the force, whether it is a centripetal or a centrifugal force or whether it is a force in the sense of the later Newtonian dynamics or merely a Cartesian "conatus," or endeavor. In a manuscript note Newton later claimed that at an early date, in the 1660s, he had actually applied the $\frac{v^2}{r}$ rule to the moon's motion, much as he does later on in book 3, prop. 4, of the *Principia*, in order to confirm his idea of the force of "gravity extending to the Moon."[11] In this way he could counter Hooke's allegation that he had learned the concept of an inverse-square force of gravity from Hooke.

A careful reading of the documents in question shows that sometime in the 1660s, Newton made a series of computations, one of which was aimed at proving that what was later known as the outward or centrifugal force arising from the

116–138. See also my *Newt. Revolution*, chaps. 4 and 5; R. S. Westfall's *Never at Rest* and his earlier *Force in Newton's Physics* (London: Macdonald; New York: American Elsevier, 1971), chaps. 7 and 8; Herivel's *Background*. A splendid review of this topic is available in Hall, *Isaac Newton: Adventurer in Thought*, pp. 55–64. A list of other scholars who have made contributions to this subject would include, among others, Bruce Brackenridge, Herman Ehrlichson, J. E. McGuire, and Michael M. Nauenberg.

10. The text of Newton's early discovery of the $\frac{v^2}{r}$ rule is published (from Newton's "Waste Book") in *Background*, pp. 130–131.

11. This celebrated autobiographical document was first printed in *A Catalogue of the Portsmouth Collection of Books and Papers Written by or Belonging to Sir Isaac Newton, the Scientific Portion of Which Has Been Presented by the Earl of Portsmouth to the University of Cambridge*, ed. H. R. Luard et al. (Cambridge: Cambridge University Press, 1888), and has been reprinted many times since. A corrected version, taken from the manuscript in the Cambridge University Library (ULC MS Add. 3968, §41, fol. 85) may be found in my *Introduction*, pp. 290–292.

earth's rotation is less than the earth's gravity, as it must be for the Copernican system to be possible. He then computed a series of forces. Cartesian vortical endeavors are not the kind of forces that, in the *Principia*, are exerted by the sun on the planets to keep them in a curved path or the similar force exerted by the earth on the moon. At this time, and for some years to come, Newton was deeply enmeshed in the Cartesian doctrine of vortices. He had no concept of a "force of gravity" acting on the moon in anything like the later sense of the dynamics of the *Principia*. These Cartesian "endeavors" (Newton used Descartes's own technical term, "conatus") are the magnitude of the planets' endeavors to fly out of their orbits. Newton concludes that since the cubes of the distances of the planets from the sun are "reciprocally as the squared numbers of their revolutions in a given time," their "*conatus* to recede from the Sun will be reciprocally as the squares of their distances from the Sun."[12]

Newton also made computations to show that the endeavor or "conatus" of receding from the earth's surface (caused by the earth's daily rotation) is 12½ times greater than the orbital endeavor of the moon to recede from the earth. He concludes that the force of receding at the earth's surface is "4000 and more times greater than the endeavor of the Moon to recede from the Earth."

In other words, "Newton had discovered an interesting mathematical correlation within the solar vortex,"[13] but he plainly had not as yet invented the radically new concept of a centripetal dynamical force, an attraction that draws the planets toward the sun and the moon toward the earth.[14] There was no "twenty years' delay" (from the mid-1660s to the mid-1680s) in Newton's publication of the theory of universal gravity, as was alleged by Florian Cajori.[15]

In 1679/80, Hooke initiated an exchange of correspondence with Newton on scientific topics. In the course of this epistolary interchange, Hooke suggested to Newton a "hypothesis" of his own devising which would account for curved orbital motion by a combination of two motions: an inertial or uniform linear component

12. *Corresp.* 1:300; see A. Rupert Hall, "Newton on the Calculation of Central Forces," *Annals of Science* 13 (1957): 62–71.

13. Hall, *Isaac Newton: Adventurer in Thought*, p. 62. This work gives an excellent critical summary of Newton's thoughts about celestial motions during the 1660s.

14. For the documents and an analysis, see Hall, "Newton on the Calculation of Central Forces," pp. 62–71; also *Background*, pp. 192–198, 68–69; and esp. *Never at Rest*, pp. 151–152. See, further, *Newt. Revolution*, esp. pp. 238–240. A splendid review of this subject is available in D. T. Whiteside, "The Prehistory of the *Principia* from 1664 to 1686," *Notes and Records of the Royal Society of London* 45 (1991): 11–61, esp. 18–22.

15. Florian Cajori, "Newton's Twenty Years' Delay in Announcing the Law of Gravitation," in *Sir Isaac Newton, 1727–1927: A Bicentenary Evaluation of His Work*, ed. Frederic E. Brasch (Baltimore: Williams and Wilkins, 1928), pp. 127–188.

along the tangent to the curve and a motion of falling inward toward a center. Newton told Hooke that he had never heard of this "hypothesis."[16] In the course of their letters, Hooke urged Newton to explore the consequences of his hypothesis, advancing the opinion or guess that in combination with the supposition of an inverse-square law of solar-planetary force, it would lead to the true planetary motions.[17] Hooke also wrote that the inverse-square law would lead to a rule for orbital speed being inversely proportional to the distance of a planet from the sun.[18] Stimulated by Hooke, Newton apparently then proved that the solar-planetary force is as the inverse square of the distance, a first step toward the eventual *Principia*.

We cannot be absolutely certain of exactly how Newton proceeded to solve the problem of motion in elliptical orbits, but most scholars agree that he more or less followed the path set forth in the tract *De Motu* which he wrote after Halley's visit a few years later in 1684.[19] Essentially, this is the path from props. 1 and 2 of book 1 to prop. 4, through prop. 6, to props. 10 and 11. Being secretive by nature, Newton didn't tell Hooke of his achievement. In any event, he would hardly have announced so major a discovery to a jealous professional rival, nor in a private letter. What may seem astonishing, in retrospect, is not that Newton did not reveal his discovery to Hooke, but that Newton was not at once galvanized into expanding his discovery into the eventual *Principia*.

Several aspects of the Hooke-Newton exchange deserve to be noted. First, Hooke was unable to solve the problem that arose from his guess or his intuition; he simply did not have sufficient skill in mathematics to be able to find the orbit produced by an inverse-square force. A few years later, Wren and Halley were equally baffled by this problem. Newton's solution was, as Westfall has noted, to invert the problem, to assume the path to be an ellipse and find the force rather

16. The Newton-Hooke correspondence during 1679/80 is to be found in *Corresp.*, vol. 2. See, in this regard, Alexandre Koyré, "An Unpublished Letter of Robert Hooke to Isaac Newton," *Isis* 43 (1952): 312–337, reprinted in Koyré's *Newtonian Studies* (Cambridge, Mass.: Harvard University Press, 1965), pp. 221–260. Also J. A. Lohne, "Hooke versus Newton: An Analysis of the Documents in the Case of Free Fall and Planetary Motion," *Centaurus* 7 (1960): 6–52.

17. Later, Newton quite correctly insisted that Hooke could not prove this assertion. In any event he himself had already been thinking of an inverse-square force.

18. Newton was to prove that this particular conclusion or guess of Hooke's was wrong. The force on a planet at a point P (see book 1, prop. 1, corol. 1) is inversely proportional to the perpendicular distance from the sun to the tangent to the curve at P. We shall take note, below, that Hooke's rule, previously stated by Kepler, is true only at the apsides.

19. There has, however, been some consideration given to the possibility that what Newton wrote up at this time was a prototype of the paper he later sent to John Locke.

This work is available, with a commentary by D. T. Whiteside, in *Math. Papers*, vol. 6, and in Herivel's *Background* and the Halls' edition of *Unpubl. Sci. Papers*.

than "investigating the path in an inverse-square force field."[20] Second, there is no certainty that the tract *De Motu* actually represents the line of Newton's thought after corresponding with Hooke; Westfall, for one, has argued that a better candidate would be an essay in English which Newton sent later to John Locke, a position he maintains in his biography of Newton.[21] A third point is that Newton was quite frank in admitting (in private memoranda) that the correspondence with Hooke provided the occasion for his investigations of orbital motion that eventually led to the *Principia*.[22] Fourth, as we shall see in §3.4 below, the encounter with Hooke was associated with a radical reorientation of Newton's philosophy of nature that is indissolubly linked with the *Principia*. Fifth, despite Newton's success in proving that an elliptical orbit implies an inverse-square force, he was not at that time stimulated—as he would be some four years later—to move ahead and to create modern rational mechanics. Sixth, Newton's solution of the problems of planetary force depended on both his own new concept of a dynamical measure of force (as in book 1, prop. 6) and his recognition of the importance of Kepler's law of areas.[23] A final point to be made is that most scholarly analyses of Newton's thoughts during this crucial period concentrate on conceptual formulations and analytical solutions, whereas we know that both Hooke[24] and Newton made

20. *Never at Rest*, p. 387. I have discussed this matter in my *Introduction*, pp. 49–52, in relation to the question of what Halley asked Newton on the famous visit in the summer of 1684 and what Newton would have replied.

21. Most scholars date the Locke paper after the *Principia*. An earlier dating was suggested by Herivel in 1961 and reaffirmed in his *Background*, pp. 108–117. This assigned date was then challenged by the Halls in 1963, and supported by Westfall in 1969, whose arguments were refuted by Whiteside in 1970. See the summary in Westfall's *Never at Rest*, pp. 387–388 n. 145.

An admirable discussion of the various attempts to date this work is given in Bruce Brackenridge, "The Critical Role of Curvature in Newton's Developing Dynamics," in *The Investigation of Difficult Things: Essays on Newton and the History of the Exact Sciences*, ed. P. M. Harman and Alan E. Shapiro (Cambridge: Cambridge University Press, 1992), pp. 231–260, esp. 241–242 and n. 35. Brackenridge concludes by agreeing with Whiteside that the date of a "prototype manuscript" on which this tract is based should be fixed at August 1684, shortly after Halley's visit.

22. Newton to Halley, 27 July 1686, *Corresp.* 2:447; my *Introduction*, suppl. I. My own awareness of the significance of the Hooke-Newton correspondence (in suggesting a fruitful way to analyze celestial orbital motions) derives from a pioneering study by R. S. Westfall, "Hooke and the Law of Universal Gravitation," *The British Journal for the History of Science* 3 (1967): 245–261.

23. On the problems of using Kepler's law of areas and the various approximations used by seventeenth-century astronomers in place of this law, see Curtis Wilson, "From Kepler's Laws, So-Called, to Universal Gravitation: Empirical Factors," *Archive for History of Exact Sciences* 6 (1970): 89–170; and my *Newt. Revolution*, pp. 224–229.

24. Patri Pugliese, "Robert Hooke and the Dynamics of Motion in the Curved Path," in *Robert Hooke: New Studies*, ed. Michael Hunter and Simon Schaffer (London: Boydell Press, 1989), pp. 181–205. See, further, Michael Nauenberg, "Hooke, Orbital Motion, and Newton's *Principia*," *American Journal of Physics* 62 (1994): 331–350.

important use of graphical methods, a point rightly stressed by Curtis Wilson.[25] Newton, in fact, in an early letter to Hooke, wrote of Hooke's "acute Letter having put me upon considering . . . the species of this curve," saying he might go on to "add something about its description quam proxime,"[26] or by graphic methods. The final proposition in the *Principia* (book 3, prop. 42) declared its subject (in the first edition) to be: "To correct a comet's trajectory found graphically." In the second and third editions, the text of the demonstration was not appreciably altered, but the statement of the proposition now reads: "To correct a comet's trajectory that has been found."

When Newton wrote up his results for Halley (in the tract *De Motu*) and proved (in the equivalent of book 1, prop. 11) that an elliptical orbit implies an inverse-square central force, he included in his text the joyous conclusion: "Therefore the major planets revolve in ellipses having a focus in the center of the sun; and the radii to the sun describe areas proportional to the times, exactly ["omnino"] as Kepler supposed."[27] But after some reflection, Newton recognized that he had been considering a rather artificial situation in which a body moves about a mathematical center of force. In nature, bodies move about other bodies, not about mathematical points. When he began to consider such a two-body system, he came to recognize that in this case each body must act on the other. If this is true for one such pair of bodies, as for the sun-earth system, then it must be so in all such systems. In this way he concluded that the sun (like all the planets) is a body on which the force acts and also a body that gives rise to the force. It follows at once that each planet must exert a perturbing force on every other planet in the solar system. The consequence must be, as Newton recognized almost at once, that "the displacement of the sun from the center of gravity" may have the effect that "the centripetal force does not always tend to" an "immobile center" and that "the planets neither revolve exactly in ellipses nor revolve twice in the same orbit." In other words, "Each time a planet revolves it traces a fresh orbit, as happens also with the motion of the moon, and each orbit is dependent upon the combined motions of all the planets, not to mention their actions upon each other."[28] This led him to the melancholy conclusion: "Unless I am much mistaken, it would exceed the force of human wit to consider so many causes of motion at the same time, and to define the motions by exact laws which would allow of any easy calculation."

25. "Newt. Achievement," pp. 242–243.

26. Newton to Hooke, 13 December 1679, *Corresp.* 2:308. Wilson's suggested reconstruction occurs in "Newt. Achievement," p. 243.

27. *Unpubl. Sci. Papers*, pp. 253, 277.

28. Ibid., pp. 256, 281.

We don't know exactly how Newton reached this conclusion, but a major factor may have been the recognition of the need to take account of the third law, that to every action there must be an equal and opposite reaction. Yet, in the texts of *De Motu*, the third law does not appear explicitly among the "laws" or "hypotheses." We do have good evidence, however, that Newton was aware of the third law long before writing *De Motu*.[29] In any event, the recognition that there must be interplanetary perturbations was clearly an essential step on the road to universal gravity and the *Principia*.[30]

In reviewing this pre-*Principia* development of Newton's dynamics, we should take note that by and large, Newton has been considering exclusively the motion of a particle, of unit mass. Indeed, a careful reading of the *Principia* will show that even though mass is the subject of the first definition at the beginning of the *Principia*, mass is not a primary variable in Newton's mode of developing his dynamics in book 1. In fact, most of book 1 deals exclusively with particles. Physical bodies with significant dimensions or shapes do not appear until sec. 12, "The attractive forces of spherical bodies."

Newton's concept of mass is one of the most original concepts of the *Principia*. Newton began thinking about mass some years before Halley's visit. Yet, in a series of definitions which he wrote out some time after *De Motu* and before composing the *Principia*, mass does not appear as a primary entry. We do not have documents that allow us to trace the development of Newton's concept of mass with any precision. We know, however, that two events must have been important, even though we cannot tell whether they initiated Newton's thinking about mass or reinforced ideas that were being developed by Newton. One of these was the report of the Richer expedition, with evidence that indicated that weight is a variable quantity, depending on the terrestrial latitude. Hence weight is a "local" property and cannot be used as a universal measure of a body's quantity of matter. Another was Newton's study of the comet of 1680. After he recognized that the comet turned around the sun and after he concluded that the sun's action

29. See D. T. Whiteside's notes in *Math. Papers* 5:148–149 n. 152; 6:98–99 n. 16.

30. A quite different reconstruction of Newton's path to universal gravity has been proposed by George Smith. He suggests: "The 'one-body' solutions of the tract 'De Motu' expressly entail that the a^3/T^2 value associated with each celestial central body is a measure of the centripetal tendency toward it. The known values for the Sun, Jupiter, Saturn, and the Earth can then be used, in conjunction with the principle that the center of gravity of the system remains unaffected (corollary 4 of the Laws of Motion), first to conclude independently of any explicit reference to mass that the Copernican system is basically correct (as in the 'Copernican scholium' of the revised version of 'De Motu'), and *then* to infer that the gravitational force acting celestially is proportional to the masses of the central and orbiting bodies. The final step to universal gravitation then follows along the lines of Propositions 8 and 9 of Book III of the *Principia*." See, also, Wilson's "From Kepler's Laws" (n. 23 above).

on the comet cannot be magnetic, he came to believe that Jupiter must also exert an influence on the comet. Clearly, this influence must derive from the matter in Jupiter, Jupiter acting on the comet just as it does on its satellites.

Once Newton had concluded that planets are centers of force because of their matter or mass, he sought some kind of empirical confirmation of so bold a concept. Since Jupiter is by far the most massive of all the planets, it was obvious that evidence of a planetary force would be most manifest in relation to the action of Jupiter on a neighboring planet. In happened that in 1684/85 the orbital motions of Jupiter and Saturn were bringing these two planets to conjunction. If Newton's conclusion were correct, then the interactions of these two giant planets should show the observable effects of an interplanetary force. Newton wrote to the astronomer John Flamsteed at the Royal Observatory at Greenwich for information on this point. Flamsteed reported that Saturn's orbital speed in the vicinity of Jupiter did not exactly follow the expected path, but he could not detect the kind of effect or perturbation that Newton had predicted.[31] As we shall see, the effect predicted by Newton does occur, but its magnitude is so tiny that Flamsteed could never have observed it. Newton needed other evidence to establish the validity of his force of universal gravity.

Newton's discovery of interplanetary forces as a special instance of universal gravity enables us to specify two primary goals of the *Principia*. The first is to show the conditions under which Kepler's laws of planetary motion are exactly or accurately true; the second is to explore how these laws must be modified in the world of observed nature by perturbations, to show the effects of perturbations on the motions of planets and their moons.[32]

It is well known that after the *Principia* was presented to the Royal Society, Hooke claimed that he should be given credit for having suggested to Newton the idea of universal gravity. We have seen that Hooke did suggest to Newton that the sun exerts an inverse-square force on the planets, but Newton insisted that he didn't need Hooke to suggest to him that there is an inverse-square relation. Furthermore, Newton said that this was but one of Hooke's guesses. Newton again and again asserted that Hooke didn't know enough mathematics to substantiate his guess, and he was right. As the mathematical astronomer Alexis Clairaut said

31. *Corresp.* 2:419–420.

32. These two goals are discussed in my "Newton's Theory vs. Kepler's Theory and Galileo's Theory: An Example of a Difference between a Philosophical and a Historical Analysis of Science," in *The Interaction between Science and Philosophy*, ed. Yehuda Elkana (Atlantic Highlands, N.J.: Humanities Press, 1974), pp. 299–388.

of Hooke's claim, a generation later, it serves "to show what a distance there is between a truth that is glimpsed and a truth that is demonstrated."[33]

In explaining his position with respect to Hooke's guess, Newton compared his own work with that of Hooke and Kepler. Newton evidently believed that he himself had "as great a right" to the inverse-square law "as to the ellipsis." For, just "as Kepler knew the orb to be not circular but oval, and guessed it to be elliptical, so Mr. Hooke, without knowing what I have found out since his letters to me,"[34] knew only "that the proportion was duplicata quam proxime at great distances from the centre," and "guessed it to be so accurately, and guessed amiss in extending that proportion down to the very centre." But, unlike Hooke, "Kepler guessed right at the ellipsis," so that "Mr. Hooke found less of the proportion than Kepler of the ellipsis."[35] Newton believed that he himself deserved credit for the law of elliptical orbits, as well as the law of the inverse square, on the grounds that he had proved both in their generality.[36] In the *Principia* (e.g., in the "Phenomena" in book 3), Newton gave Kepler credit only for the third or harmonic law. At the time that Newton was writing his *Principia*, there were alternatives to the area law that were in use in making tables of planetary motion. Newton proposed using the eclipses of Jupiter's satellites (and later of those of Saturn) to show that this law holds to a high degree. But the law of elliptical orbits was of a different sort because there was no observational evidence that would distinguish between an ellipse and other ovals. Thus there may have been very different reasons for not giving Kepler credit for these two laws.

At one point during the exchange of letters with Halley on Hooke's claims to recognition, Newton—in a fit of pique—threatened to withdraw book 3 altogether.[37] We do not know how serious this threat was, but Halley was able to explain matters and to calm Newton's rage. Halley deserves much praise for his services as midwife to Newton's brainchild. Not only was he responsible for goading Newton into writing up his preliminary results; he encouraged Newton to produce the *Principia*. At an early stage of composition of the *Principia*, as I discovered while preparing the Latin edition with variant readings, Halley even

33. "Exposition abregée du système du monde, et explication des principaux phénomènes astronomiques tirée des *Principes* de M. Newton," suppl. to the marquise du Châtelet's translation of the *Principia* (Paris: chez Desaint & Saillant [&] Lambert, 1756), 2:6.

34. Newton was referring to the problem of the gravitational action of a homogeneous sphere on an external particle; see Whiteside's note in *Math. Papers* 6:19 n. 59.

35. Newton to Halley, 20 June 1686, *Corresp.*, vol. 2.

36. Ibid. "I do pretend [i.e., claim]," Newton wrote, "to have done as much for the proportion [of the inverse square] as for the ellipsis, and to have as much right to the one from Mr. Hooke and all men, as to the other from Kepler."

37. See §3.1 below.

helped Newton by making suggestive comments on an early draft of book 1, the manuscript of which no longer exists.[38]

Although publication of the *Principia* was sponsored by the Royal Society, there were no funds available for the costs of printing, and so Halley had to assume those expenses.[39] Additionally, he edited the book for the printer, saw to the making of the woodcuts of the diagrams, and read the proofs. He wrote a flattering ode to Newton that introduces the *Principia* in all three editions,[40] and he also wrote a book review that was published in the Royal Society's *Philosophical Transactions*.[41]

Revisions and Later Editions

Within a decade of publication of the *Principia*, Newton was busy with a number of radical revisions, including an extensive restructuring of the opening sections.[42] He planned to remove secs. 4 and 5, which are purely geometrical and not necessary to the rest of the text, and to publish them separately.[43] He also developed plans to include a mathematical supplement on his methods of the calculus, his treatise *De Quadratura*. Many of the proposed revisions and restructurings of the 1690s are recorded in Newton's manuscripts; others were reported in some detail by David Gregory.[44] When Newton began to produce a second edition, however, with the aid of Roger Cotes, the revisions were of a quite different sort. Some of the major or most interesting alterations are given in the notes to the present translation. The rest are to be found in the *apparatus criticus* of our Latin edition of the *Principia* with variant readings.[45]

There were a number of truly major emendations that appeared in the second edition, some of which involved a complete replacement of the original text. One

38. See my *Introduction*, suppl. 7.

39. A. N. L. Munby estimated the size of the first edition at some 300 or 400 copies, but this number has recently been increased to perhaps 500. See Whiteside, "The Prehistory of the *Principia*" (n. 14 above), esp. p. 34. Whiteside reckons that, granting this larger size of the edition, Halley would not have suffered financially by paying the printing costs of the *Principia* and would even have made not "less than £10 in pocket for all his time and trouble."

40. On the alterations in the poem in successive editions of the *Principia*, see our Latin edition, cited in n. 45 below.

41. *Philosophical Transactions* 16, no. 186 (Jan.-Feb.-March 1687): 291–297, reprinted in *Isaac Newton's Papers and Letters on Natural Philosophy*, ed. I. B. Cohen and Robert E. Schofield, 2d ed. (Cambridge, Mass.: Harvard University Press, 1978), pp. 405–411.

42. See my *Introduction*, chap. 7, and esp. *Math. Papers*, vol. 6.

43. *Introduction*, p. 193.

44. Ibid., pp. 188–198.

45. *Isaac Newton's "Philosophiae Naturalis Principia Mathematica": The Third Edition (1726) with Variant Readings*, assembled and edited by Alexandre Koyré, I. Bernard Cohen, and Anne Whitman, 2 vols. (Cambridge: Cambridge University Press; Cambridge, Mass.: Harvard University Press, 1972).

of these was the wholly new proof of book 2, prop. 10, a last-minute alteration in response to a criticism made by Johann Bernoulli.[46] Another occurred in book 2, sec. 7, on the motion of fluids and the resistance encountered by projectiles, where most of the propositions and their proofs are entirely different in the second edition from those of the first edition. That is, the whole set of props. 34–40 of the first edition were cast out and replaced.[47] This complete revision of sec. 7 made it more appropriate to remove to the end of sec. 6 the General Scholium on pendulum experiments which originally had been at the end of sec. 7. This was a more thorough revision of the text than occurred in any other part of the *Principia*.

Another significant novelty of the second edition was the introduction of a conclusion to the great work, the celebrated General Scholium that appears at the conclusion of book 3. The original edition ended rather abruptly with a discussion of the orbits of comets, a topic making up about a third of book 3. Newton had at first essayed a conclusion, but later changed his mind. His intentions were revealed in 1962 by A. Rupert Hall and Marie Boas Hall, who published the original drafts. In these texts, Newton shows that he intended to conclude the *Principia* with a discussion of the forces between the particles of matter, but then thought better of introducing so controversial a topic. While preparing the second edition, Newton thought once again of an essay on "the attraction of the small particles of bodies," but on "second thought" he chose "rather to add but one short Paragraph about that part of Philosophy."[48] The conclusion he finally produced is the celebrated General Scholium, with its oft-quoted slogan "Hypotheses non fingo." This General Scholium ends with a paragraph about a "spirit" which has certain physical properties, but whose laws have not as yet been determined by experiment. Again thanks to the researches of A. Rupert Hall and Marie Boas Hall, we now know that while composing this paragraph, Newton was thinking about the new phenomena of electricity.[49]

Another change that occurs in the second edition is in the beginning of book 3. In the first edition, book 3 opened with a preliminary set of *Hypotheses*.[50] Perhaps

46. See D. T. Whiteside's magisterial discussion of this episode, together with all the relevant documents concerning the stages of alteration of book 2, prop. 10, in *Math. Papers* 8:50–53, esp. nn. 175, 180, and esp. §6, appendix 2.1.52 in that same volume. See also §7.3 below and my *Introduction*, §9.4.

47. These props. 34–40 of the first edition (translated by I. Bernard Cohen and Anne Whitman) will be published, together with a commentary by George Smith, in *Newton's Natural Philosophy*, ed. Jed Buchwald and I. Bernard Cohen (Cambridge: MIT Press, forthcoming).

48. *Unpubl. Sci. Papers*, pp. 320–347 (see §9.3 below); Newton to Cotes, 2 Mar. 1712/13. On the production of the second edition, see the texts, notes, and commentaries in *Correspondence of Sir Isaac Newton and Professor Cotes*, ed. J. Edleston (London: John W. Parker; Cambridge: John Deighton, 1850).

49. See §9.3 below.

50. See §8.2 below.

in reply to the criticism in the *Journal des Sçavans*,[51] Newton now renamed the "hypotheses" and divided them into several classes. Some became *Regulae Philosophandi*, or "Rules for Natural Philosophy," with a new rule (no. 3). Others became "Phenomena," with new numerical data. Yet another was transferred to a later place in book 3, where it became "hypothesis 1."

Newton also made a slight modification in the scholium following lem. 2 (book 2, sec. 2), in reference to Leibniz's method of the calculus. He had originally written that Leibniz's method "hardly differed from mine except in the forms of words and notations." In the second edition Newton altered this statement by adding that there was another difference between the two methods, namely, in "the generation of quantities." This scholium and its successive alterations attracted attention because of the controversy over priority in the invention of the calculus. In the third edition, Newton eliminated any direct reference to Leibniz.

Critical readers of the *Principia* paid close attention to the alteration in the scholium following book 3, prop. 35. In the second edition, the original short text was replaced by a long discussion of Newton's attempts to apply the theory of gravity to some inequalities of the moon's motion.[52] Much of the text of this scholium had been published separately by David Gregory.[53]

Many of Newton's plans for the actual revisions of the first edition, in order to produce a second edition, were entered in two personal copies of the *Principia*. One of these was specially bound and interleaved. Once the second edition had been published, Newton again prepared an interleaved copy and kept track of proposed alterations or emendations in his interleaved copy and in an annotated copy. These four special copies of the *Principia* have been preserved among Newton's books, and their contents have been noted in our Latin edition with variant readings.[54]

Soon after the appearance of the second edition, Newton began planning for yet another revision. The preface which he wrote for this planned edition of the late 1710s is of great interest in that it tells us in Newton's own words about some of the features of the *Principia* he believed to be most significant. It is printed below in §3.2. Newton at this time once again planned to have a treatise on the calculus published together with the *Principia*. In the end he abandoned this effort. Later on, when he was in his eighties, he finally decided to produce a new edition. He

51. See my *Introduction*, chap. 6, sec. 6.

52. See §8.14 below.

53. For details see *Isaac Newton's Theory of the Moon's Motion (1702)*, introd. I. Bernard Cohen (Folkestone: Dawson, 1975).

54. These four special copies are described in my *Introduction*; Newton's MS notes appear in our edition with variant readings (n. 45 above).

chose as editor Dr. Henry Pemberton, a medical doctor and authority on pharmacy and an amateur mathematician.

The revisions in the third edition were not quite as extensive as those in the second edition.[55] A new rule 4 was added on the subject of induction, and there were other alterations, some of which may be found in the notes to the present translation. An important change was made in the "Leibniz Scholium" in book 2, sec. 2. The old scholium was replaced by a wholly different one. Newton now boldly asserted his own claims to be the primary inventor of the calculus, referring to some correspondence to prove the point.[56] Even though Leibniz had been dead for almost a decade, Newton still pursued his rival with dogged obstinacy. Another innovation in the third edition appeared in book 3, where Newton inserted (following prop. 33) two propositions by John Machin, astronomy professor at Gresham College, whose academic title would later lead to the invention of a fictitious scientist in the Motte-Cajori edition.[57]

By the time of the third edition, Newton seems to have abandoned his earlier attempts to explain the action of gravity by reference to electrical phenomena and had come rather to hope that an explanation might be found in the actions of an "aethereal medium" of varying density.[58] In his personal copy of the *Principia*, in which he recorded his proposed emendations and revisions, he at first had entered an addition to specify that the "spirit" to which he had referred in the final paragraph of the General Scholium was "electric and elastic."[59] Later on, he apparently decided that since he no longer believed in the supreme importance of the electrical theory, he would cancel the whole paragraph. Accordingly, he drew a line through the text, indicating that this paragraph should be omitted. It is one of the oddities of history that Andrew Motte should have learned of Newton's planned insertion of the modifier "electricus et elasticus" but not of Newton's proposed elimination of the paragraph. Without comment, Motte entered "electric and elastic" into his English version of 1729. These words were in due course preserved in the Motte-Cajori version and have been quoted in the English-speaking world ever since.

55. See my *Introduction*, chap. 11.

56. See A. Rupert Hall, *Philosophers at War: The Quarrel between Newton and Leibniz* (Cambridge, London, New York: Cambridge University Press, 1980), and especially *Math. Papers*, vol. 8.

57. See §2.3 below.

58. See the later Queries of the *Opticks* and the discussion by Betty Jo Dobbs, *Janus Faces* (§3.1, n. 10 below).

59. See §9.3 below.

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THE *PRINCIPIA*

Translated by I. Bernard Cohen and Anne Whitman
with the assistance of Julia Budenz

N E W T O N I

P R I N C I P I A

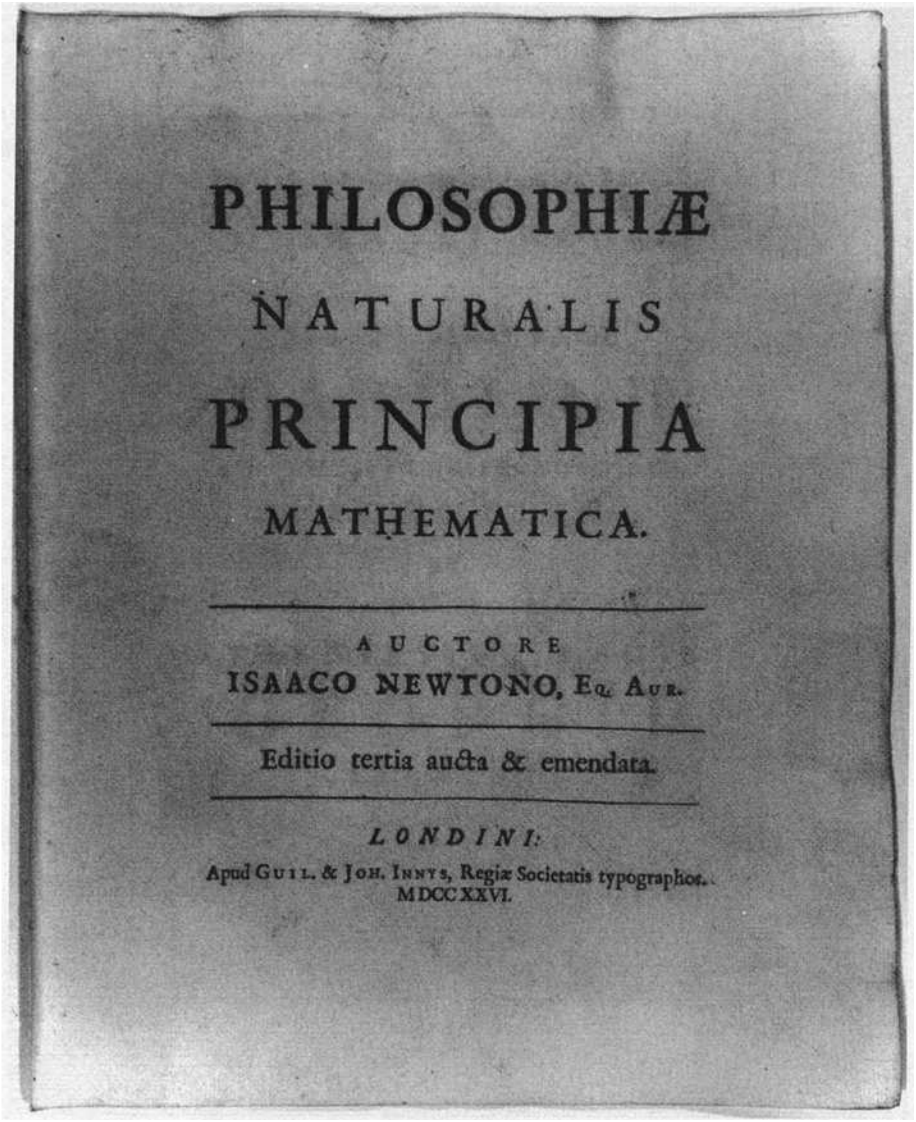
P H I L O S O P H I Æ.

The third edition of Newton's *Principia* begins with this half title followed by two inserted leaves. One is a portrait of Newton engraved by George Vertue from a painting by John Vanderbank, the other the "privilege" or license for publication, dated 25 March 1726. (Grace K. Babson Collection, Burndy Library)

NEWTON'S

PRINCIPLES

OF PHILOSOPHY


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

AUCTORE
ISAACO NEWTONO, Eq. Aur.

Editio tertia aucta & emendata.

LONDINI:
Apud Guil. & Joh. Innys, Regiæ Societatis typographos.
MDCCXXVI.

Title page of the third edition of Newton's *Principia*. In the original, the words PHILOSOPHIAE and PRINCIPIA are printed in red, as are ISAACO NEWTONO and LONDINI. (Grace K. Babson Collection, Burndy Library)

MATHEMATICAL

PRINCIPLES

OF NATURAL

PHILOSOPHY.

WRITTEN BY

Sir ISAAC NEWTON.

Third edition, enlarged & revised.

LONDON:

WILL. & JNO. INNYS, printers to the Royal Society.

M DCC XXVI.

ILLUSTRISSIMÆ

SOCIETATI REGALI

A

SERENISSIMO REGE

CAROLO II

AD PHILOSOPHIAM PROMOVENDAM

FUNDATÆ,

ET

AUSPICIIS

SERENISSIMI REGIS

GEORGII

FLORENTI

TRACTATUM HUNC D.D.D.

IS. NEWTON.

TO THE MOST ILLUSTRIOUS

ROYAL SOCIETY,

FOUNDED

FOR THE PROMOTION OF PHILOSOPHY

BY

HIS MOST SERENE MAJESTY

CHARLES II,

AND

FLOURISHING

UNDER THE PATRONAGE OF

HIS MOST SERENE MAJESTY

GEORGE,

THIS TREATISE IS DEDICATED.

IS. NEWTON.

The Latin dedication to the third edition (*opposite*; Grace K. Babson Collection, Burndy Library) describes the Royal Society as "ad philosophiam promovendam," in the sense of the promotion of natural philosophy or science. In this expression, Newton was producing a variant of the official name, "The Royal Society of London for Promoting Natural Knowledge." The Latin original of this English version, however, is "Regalis Societas Londini pro scientia naturali promovenda," as stated in the third charter. In the first edition of the *Principia*, the latter part of the dedication reads: "and flourishing under the patronage of the Most Powerful Monarch James II"; additionally, it is said that this treatise is "most humbly" ("humillime") dedicated. In the second edition, the latter part of the dedication reads: "and flourishing under the patronage of the Most August Queen Anne."

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Ode on This Splendid Ornament of Our Time and Our Nation, the Mathematico-Physical Treatise by the Eminent Isaac Newton

Behold the pattern of the heavens, and the balances of the divine structure;
Behold Jove's calculation and the laws
That the creator of all things, while he was setting the beginnings of the world,
 would not violate;
Behold the foundations he gave to his works.
Heaven has been conquered and its innermost secrets are revealed;
The force that turns the outermost orbs around is no longer hidden.
The Sun sitting on his throne commands all things
To tend downward toward himself, and does not allow the chariots of the
 heavenly bodies to move
Through the immense void in a straight path, but hastens them all along
In unmoving circles around himself as center.
Now we know what curved path the frightful comets have;
No longer do we marvel at the appearances of a bearded star.
From this treatise we learn at last why silvery Phoebe moves at an unequal pace,
Why, till now, she has refused to be bridled by the numbers of any astronomer,
Why the nodes regress, and why the upper apsides move forward.
We learn also the magnitude of the forces with which wandering Cynthia
Impels the ebbing sea, while its weary waves leave the seaweed far behind
And the sea bares the sands that sailors fear, and alternately beat high up on
 the shores.
The things that so often vexed the minds of the ancient philosophers
And fruitlessly disturb the schools with noisy debate
We see right before our eyes, since mathematics drives away the cloud.

Error and doubt no longer encumber us with mist;
For the keenness of a sublime intelligence has made it possible for us to enter
The dwellings of the gods above and to climb the heights of heaven.

Mortals arise, put aside earthly cares,
And from this treatise discern the power of a mind sprung from heaven,
Far removed from the life of beasts.
He who commanded us by written tablets to abstain from murder,
Thefts, adultery, and the crime of bearing false witness,
Or he who taught nomadic peoples to build walled cities, or he who enriched the nations with the gift of Ceres,
Or he who pressed from the grape a solace for cares,
Or he who with a reed from the Nile showed how to join together
Pictured sounds and to set spoken words before the eyes,
Exalted the human lot less, inasmuch as he was concerned with only a few comforts of a wretched life,
And thus did less than our author for the condition of mankind.
But we are now admitted to the banquets of the gods;
We may deal with the laws of heaven above; and we now have
The secret keys to unlock the obscure earth; and we know the immovable order of the world
And the things that were concealed from the generations of the past.

O you who rejoice in feeding on the nectar of the gods in heaven,
Join me in singing the praises of NEWTON, who reveals all this,
Who opens the treasure chest of hidden truth,
NEWTON, dear to the Muses,
The one in whose pure heart Phoebus Apollo dwells and whose mind he has filled with all his divine power;
No closer to the gods can any mortal rise.

Edm. Halley

Author's Preface to the Reader

SINCE THE ANCIENTS (according to Pappus) considered *mechanics* to be of the greatest importance in the investigation of nature and science and since the moderns—rejecting substantial forms and occult qualities—have undertaken to reduce the phenomena of nature to mathematical laws, it has seemed best in this treatise to concentrate on *mathematics* as it relates to natural philosophy. The ancients divided *mechanics* into two parts: the *rational*, which proceeds rigorously through demonstrations, and the *practical*.[a] *Practical mechanics* is the subject that comprises all the manual arts, from which the subject of *mechanics* as a whole has adopted its name. But since those who practice an art do not generally work with a high degree of exactness, the whole subject of *mechanics* is distinguished from *geometry* by the attribution of exactness to *geometry* and of anything less than exactness to *mechanics*. Yet the errors do not come from the art but from those who practice the art. Anyone who works with less exactness is a more imperfect mechanic, and if anyone could work with the greatest exactness, he would be the most perfect mechanic of all. For the description of straight lines and circles, which is the foundation of *geometry*, appertains to *mechanics*. *Geometry*

All notes to the translation are keyed to the text by superscript letters. When a note is introduced by two letters, such as "aa," it refers to that part of the text enclosed between an opening superscript "a" and a final or closing "a."

These notes are, for the most part, extracts from variant passages or expressions as found in the first two editions. The glosses and explanations of the text are to be found in the Guide, the text of which follows the order of Newton's presentation in the *Principia*.

a. Newton's comparison and contrast between the subject of rational or theoretical mechanics and practical mechanics was a common one at the time of the *Principia*. Thus John Harris in his Newtonian *Lexicon Technicum* (London, 1704), citing the authority of John Wallis, made a distinction between the two as follows. One was a "Geometry of Motion," a "Mathematical Science which shews the Effects of *Powers*, or moving Forces," and "demonstrates the Laws of Motion." The other is "commonly taken for those *Handy-crafts*, which require as well the Labour of the Hands, as the Study of the Brain." The subject of the *Principia* became generally known as "rational mechanics" following Newton's use of that name in his Preface.

does not teach how to describe these straight lines and circles, but postulates such a description. For *geometry* postulates that a beginner has learned to describe lines and circles exactly before he approaches the threshold of *geometry*, and then it teaches how problems are solved by these operations. To describe straight lines and to describe circles are problems, but not problems in *geometry. Geometry* postulates the solution of these problems from *mechanics* and teaches the use of the problems thus solved. And *geometry* can boast that with so few principles obtained from other fields, it can do so much. Therefore *geometry* is founded on mechanical practice and is nothing other than that part of *universal mechanics* which reduces the art of measuring to exact propositions and demonstrations. But since the manual arts are applied especially to making bodies move, *geometry* is commonly used in reference to magnitude, and *mechanics* in reference to motion. In this sense *rational mechanics* will be the science, expressed in exact propositions and demonstrations, of the motions that result from any forces whatever and of the forces that are required for any motions whatever. The ancients studied this part of *mechanics* in terms of the *five powers* that relate to the manual arts [i.e., the five mechanical powers] and paid hardly any attention to gravity (since it is not a manual power) except in the moving of weights by these powers. But since we are concerned with natural philosophy rather than manual arts, and are writing about natural rather than manual powers, we concentrate on aspects of gravity, levity, elastic forces, resistance of fluids, and forces of this sort, whether attractive or impulsive. And therefore our present work sets forth mathematical principles of natural philosophy. For the basic problem [*lit.* whole difficulty[b]] of philosophy seems to be to discover the forces of nature from the phenomena of motions and then to demonstrate the other phenomena from these forces. It is to these ends that the general propositions in books 1 and 2 are directed, while in book 3 our explanation of the system of the world illustrates these propositions. For in book 3, by means of propositions demonstrated mathematically in books 1 and 2, we derive from celestial phenomena the gravitational forces by which bodies tend toward the sun and toward the individual planets. Then the motions of the planets, the comets, the moon, and the sea are deduced from these forces by propositions that are also mathematical. If only we could derive the other phenomena of nature from mechanical principles by the same kind of reasoning! For many things lead me to have a suspicion that all phenomena may depend on certain forces by which the particles of bodies, by causes not yet known, either are impelled toward one another and cohere in regular figures, or are repelled

b. Newton would seem to be expressing in Latin more or less the same concept that later appears in English (in query 28 of the *Opticks*) as "the main Business of natural Philosophy."

from one another and recede. Since these forces are unknown, philosophers have hitherto made trial of nature in vain. But I hope that the principles set down here will shed some light on either this mode of philosophizing or some truer one.

In the publication of this work, Edmond Halley, a man of the greatest intelligence and of universal learning, was of tremendous assistance; not only did he correct the typographical errors and see to the making of the woodcuts, but it was he who started me off on the road to this publication. For when he had obtained my demonstration of the shape of the celestial orbits, he never stopped asking me to communicate it to the Royal Society, whose subsequent encouragement and kind patronage made me begin to think about publishing it. But after I began to work on the inequalities of the motions of the moon, and then also began to explore other aspects of the laws and measures of gravity and of other forces, the curves that must be described by bodies attracted according to any given laws, the motions of several bodies with respect to one another, the motions of bodies in resisting mediums, the forces and densities and motions of mediums, the orbits of comets, and so forth, I thought that publication should be put off to another time, so that I might investigate these other things and publish all my results together. I have grouped them together in the corollaries of prop. 66 the inquiries (which are imperfect) into lunar motions, so that I might not have to deal with these things one by one in propositions and demonstrations, using a method more prolix than the subject warrants, which would have interrupted the sequence of the remaining propositions. There are a number of things that I found afterward which I preferred to insert in less suitable places rather than to change the numbering of the propositions and the cross-references. I earnestly ask that everything be read with an open mind and that the defects in a subject so difficult may be not so much reprehended as investigated, and kindly supplemented, by new endeavors of my readers.

Trinity College, Cambridge Is. Newton
8 May 1686

Author's Preface to the Second Edition

IN THIS SECOND EDITION of the *Principles,* many emendations have been made here and there, and some new things have been added. In sec. 2 of book 1, the finding of forces by which bodies could revolve in given orbits has been made easier and has been enlarged. In sec. 7 of book 2, the theory of the resistance of fluids is investigated more accurately and confirmed by new experiments. In book 3 the theory of the moon and the precession of the equinoxes are deduced more fully from their principles; and the theory of comets is confirmed by more examples of their orbits, calculated with greater accuracy.

London
28 March 1713

Is. Newton

Editor's Preface to the Second Edition

THE LONG-AWAITED NEW EDITION of Newton's *Principles of Natural Philosophy* is presented to you, kind reader, with many corrections and additions. The main topics of this celebrated work are listed in the table of contents and the index prepared for this edition. The major additions or changes are indicated in the author's preface. Now something must be said about the method of this philosophy.

Those who have undertaken the study of natural science can be divided into roughly three classes. There have been those who have endowed the individual species of things with specific occult qualities, on which—they have then alleged—the operations of individual bodies depend in some unknown way. The whole of Scholastic doctrine derived from Aristotle and the Peripatetics is based on this. Although they affirm that individual effects arise from the specific natures of bodies, they do not tell us the causes of those natures, and therefore they tell us nothing. And since they are wholly concerned with the names of things rather than with the things themselves, they must be regarded as inventors of what might be called philosophical jargon, rather than as teachers of philosophy.

Therefore, others have hoped to gain praise for greater carefulness by rejecting this useless hodgepodge of words. And so they have held that all matter is homogeneous, and that the variety of forms that is discerned in bodies all arises from certain very simple and easily comprehensible attributes of the component particles. And indeed they are right to set up a progression from simpler things to more compounded ones, so long as they do not give those primary attributes of the particles any characteristics other than those given by nature itself. But when they take the liberty of imagining that the unknown shapes and sizes of the particles are whatever they please, and of assuming their uncertain positions and motions, and even further of feigning certain occult fluids that permeate the pores of bodies very freely, since they are endowed with an omnipotent subtlety and are acted on by occult motions: when they do this, they are drifting off into dreams, ignoring the true constitution of things, which is obviously to be sought in vain from false conjectures, when it can scarcely be found out even by the most certain observations. Those who take the foundation of their speculations from hypotheses, even if they then proceed most rigorously according to mechanical laws, are merely putting together a romance, elegant perhaps and charming, but nevertheless a romance.

There remains then the third type, namely, those whose natural philosophy is based on experiment. Although they too hold that the causes of all things are to be derived from the simplest possible principles, they assume nothing as a principle that has not yet been thoroughly proved from phenomena. They do not contrive hypotheses, nor do they admit them into natural science otherwise than as questions whose truth may be discussed. Therefore they proceed by a twofold method, analytic and synthetic. From certain selected phenomena they deduce by analysis the forces of nature and the simpler laws of those forces, from which they then give the constitution of the rest of the phenomena by synthesis. This is that incomparably best way of philosophizing which our most celebrated author thought should be justly embraced in preference to all others. This alone he judged worthy of being cultivated and enriched by the expenditure of his labor. Of this therefore he has given a most illustrious example, namely, the explication of the system of the world most successfully deduced from the theory of gravity. That the force of gravity is in all bodies universally, others have suspected or imagined; Newton was the first and only one who was able to demonstrate it [universal gravity] from phenomena and to make it a solid foundation for his brilliant theories.

I know indeed that some men, even of great reputation, unduly influenced by certain prejudices, have found it difficult to accept this new principle [of gravity] and have repeatedly preferred uncertainties to certainties. It is not my intention to carp at their reputation; rather, I wish to give you in brief, kind reader, the basis for making a fair judgment of the issue for yourself.

Therefore, to begin our discussion with what is simplest and nearest to us, let us briefly consider what the nature of gravity is in terrestrial bodies, so that when we come to consider celestial bodies, so very far removed from us, we may proceed more securely. It is now agreed among all philosophers that all bodies on or near the earth universally gravitate toward the earth. Manifold experience has long confirmed that there are no truly light bodies. What is called relative levity is not true levity, but only apparent, and arises from the more powerful gravity of contiguous bodies.

Furthermore, just as all bodies universally gravitate toward the earth, so the earth in turn gravitates equally toward the bodies; for the action of gravity is mutual and is equal in both directions. This is shown as follows. Let the whole body of the earth be divided into any two parts, whether equal or in any way unequal; now, if the weights of the parts toward each other were not equal, the lesser weight would yield to the greater, and the parts, joined together, would proceed to move straight on without limit in the direction toward which the greater weight tends, entirely contrary to experience. Therefore the necessary conclusion is that the weights of the parts are in equilibrium—that is, that the action of gravity is mutual and equal in both directions.

The weights of bodies equally distant from the center of the earth are as the quantities of matter in the bodies. This is gathered from the equal acceleration of all bodies falling from rest by the force of their weights; for the forces by which unequal bodies are equally accelerated must be proportional to the quantities of matter to be moved. Now, that all falling bodies universally are equally accelerated is evident from this, that in the vacuum produced by Boyle's air pump (that is, with the resistance of the air removed), they describe, in falling, equal spaces in equal times, and this is proved more exactly by experiments with pendulums.

The attractive forces of bodies, at equal distances, are as the quantities of matter in the bodies. For, since bodies gravitate toward the earth, and the earth in turn gravitates toward the bodies, with equal moments [i.e., strengths or powers], the weight of the earth toward each body, or the force by which the body attracts the earth, will be equal to the weight of the body toward the earth. But, as mentioned above, this weight is as the quantity of matter in the body, and so the force by which each body attracts the earth, or the absolute force of the body, will be as its quantity of matter.

Therefore the attractive force of entire bodies arises and is compounded from the attractive force of the parts, since (as has been shown), when the amount of matter is increased or diminished, its force is proportionally increased or diminished. Therefore the action of the earth must result from the combined actions of its parts; hence all terrestrial bodies must attract one another by absolute forces that are proportional to the attracting matter. This is the nature of gravity on earth; let us now see what it is in the heavens.

Every body perseveres in its state either of being at rest or of moving uniformly straight forward, except insofar as it is compelled by impressed forces to change that state: this is a law of nature accepted by all philosophers. It follows that bodies that move in curves, and so continually deviate from straight lines tangent to their orbits, are kept in a curvilinear path by some continually acting force. Therefore, for the planets to revolve in curved orbits, there will necessarily be some force by whose repeated actions they are unceasingly deflected from the tangents.

Now, it is reasonable to accept something that can be found by mathematics and proved with the greatest certainty: namely, that all bodies moving in some curved line described in a plane, which by a radius drawn to a point (either at rest or moving in any way) describe areas about that point proportional to the times, are urged by forces that tend toward that same point. Therefore, since it is agreed among astronomers that the primary planets describe areas around the sun proportional to the times, as do the secondary planets around their own primary planets, it follows that the force by which they are continually pulled away from rectilinear tangents and are compelled to revolve in curvilinear orbits is directed toward the bodies that are situated in the centers of the orbits. Therefore this force can, appropriately, be called centripetal with respect to the revolving body, and attractive with respect to the central body, from whatever cause it may in the end be imagined to arise.

The following rules must also be accepted and are mathematically demonstrated. If several bodies revolve with uniform motion in concentric circles, and if the squares of the periodic times are as the cubes of the distances from the common center, then the centripetal forces of the revolving bodies will be inversely as the squares of the distances. Again, if the bodies revolve in orbits that are very nearly circles, and if the apsides of the orbits are at rest, then the centripetal forces of the revolving bodies will be inversely as the squares of the distances. Astronomers agree that one or the other case holds for all the planets, [both primary and secondary]. Therefore the centripetal forces of all the planets are inversely as the squares of the distances from the centers of the orbits. If anyone objects that the apsides of the planets, especially the apsides of the moon, are not completely at rest but are carried progressively forward [or in consequentia] with a slow motion, it can be answered that even if we grant that this very slow motion arises from a slight deviation of the centripetal force from the proportion of the inverse square, this difference can be found by mathematical computation and is quite insensible. For the ratio of the moon's centripetal force itself, which should deviate most of all from the square, will indeed exceed the square by a very little, but it will be about sixty times closer to it than to the cube. But our answer to the objection will be truer if we say that this progression of the apsides does not arise from a deviation from the proportion of the [inverse] square but from another and entirely different cause, as is admirably shown in Newton's philosophy. As a result, the centripetal forces by which the primary planets tend toward the sun, and the secondary planets toward their primaries, must be exactly as the squares of the distances inversely.

From what has been said up to this point, it is clear that the planets are kept in their orbits by some force continually acting upon them, that this force is always directed toward the centers of the orbits, and that its efficacy is increased in approaching the center and decreased in receding from the center—actually increased in the same proportion in which the square of the distance is decreased, and decreased in the same proportion in which the square of the distance is increased. Let us now, by comparing the centripetal forces of the planets and the force of gravity, see whether or not they might be of the same kind. They will be of the same kind if the same laws and the same attributes are found in both. Let us first, therefore, consider the centripetal force of the moon, which is closest to us.

When bodies are let fall from rest, and are acted on by any forces whatever, the rectilinear spaces described in a given time at the very beginning of the motion are proportional to the forces themselves; this of course follows from mathematical reasoning. Therefore the centripetal force of the moon revolving in its orbit will be to the force of gravity on the earth's surface as the space that the moon would describe in a minimally small time in descending toward the earth by its centripetal force—supposing it to be deprived of all circular motion—is to the space that a heavy body describes in the same minimally small time in the vicinity of the earth, in falling by the force of its own gravity. The first of these spaces is equal to the versed sine of the arc described by the moon during the same time, inasmuch as this versed sine measures the departure of the moon from the tangent caused by centripetal force and thus can be calculated if the moon's periodic time and its distance from the center of the earth are both given. The second space is found by experiments with pendulums, as Huygens has shown. Therefore, the result of the calculation will be that the first space is to the second space, or the centripetal force of the moon revolving in its orbit is to the force of gravity on the surface of the earth, as the square of the semidiameter of the earth is to the square of the semidiameter of the orbit. By what is shown above, the same ratio holds for the centripetal force of the moon revolving in its orbit and the centripetal force of the moon if it were near the earth's surface. Therefore this centripetal force near the earth's surface is equal to the force of gravity. They are not, therefore, different forces, but one and the same; for if they were different, bodies acted on by both forces together would fall to the earth twice as fast as from the force of gravity alone. And therefore it is clear that this centripetal force by which the moon is continually either drawn or impelled from the tangent and is kept in its orbit is the very force of terrestrial gravity extending as far as the moon. And indeed it is reasonable for this force to extend itself to enormous distances, since one can observe no sensible diminution of it even on the highest peaks of mountains. Therefore the moon gravitates toward the earth. Further, by mutual action, the earth in turn gravitates equally toward the moon, a fact which is abundantly confirmed in this philosophy, when we deal with the tide of the sea and the precession of the equinoxes, both of which arise from the action of both the moon and the sun upon the earth. Hence finally we learn also by what law the force of gravity decreases at greater distances from the earth. For since gravity is not different from the moon's centripetal force, which is inversely proportional to the square of the distance, gravity will also be diminished in the same ratio.

Let us now proceed to the other planets. The revolutions of the primary planets about the sun and of the secondary planets about Jupiter and Saturn are phenomena of the same kind as the revolution of the moon about the earth; furthermore, it has been demonstrated that the centripetal forces of the primary planets are directed toward the center of the sun, and those of the secondary planets toward the centers of Jupiter and of Saturn, just as the moon's centripetal force is directed toward the center of the earth; and, additionally, all these forces are inversely as the squares of the distances from the centers, just as the force of the moon is inversely as the square of the distance from the earth. Therefore it must be concluded that all of these primary and secondary planets have the same nature. Hence, as the moon gravitates toward the earth, and the earth in turn gravitates toward the moon, so also all the secondary planets will gravitate toward their primaries, and the primaries in turn toward the secondaries, and also all the primary planets will gravitate toward the sun, and the sun in turn toward the primary planets.

Therefore the sun gravitates toward all the primary and secondary planets, and all these toward the sun. For the secondary planets, while accompanying their primaries, revolve with them around the sun. By the same argument, therefore, both kinds of planets gravitate toward the sun, and the sun toward them. Additionally, that the secondary planets gravitate toward the sun is also abundantly clear from the inequalities of the moon, concerning which a most exact theory is presented with marvelous sagacity in the third book of this work.

The motion of the comets shows very clearly that the attractive force of the sun is propagated in every direction to enormous distances and is diffused to every part of the surrounding space, since the comets, starting out from immense distances, come into the vicinity of the sun and sometimes approach so very close to it that in their perihelia they all seemingly touch its globe. Astronomers until now have tried in vain to find the theory of these comets; now at last, in our time, our most illustrious author has succeeded in finding the theory and has demonstrated it with the greatest certainty from observations. It is therefore evident that the comets move in conic sections having their foci in the center of the sun and by radii drawn to the sun describe areas proportional to the times. From these phenomena it is manifest and it is mathematically proved that the forces by which the comets are kept in their orbits are directed toward the sun and are inversely as the squares of their distances from its center. Thus the comets gravitate toward the sun; and so the attractive force of the sun reaches not only to the bodies of the planets, which are at fixed distances and in nearly the same plane, but also to the comets, which are in the most diverse regions of the heavens and at the most diverse distances. It is the nature of gravitating bodies, therefore, that they propagate their forces at all distances to all other gravitating bodies. From this it follows that all planets and comets universally attract one another and are heavy toward one another—which is also confirmed by the perturbation of Jupiter and Saturn, known to astronomers and arising from the actions of these planets upon each other; it is also confirmed by the very slow motion of the apsides that was mentioned above and that arises from an entirely similar cause.

We have at last reached the point where it must be acknowledged that the earth and the sun and all the celestial bodies that accompany the sun attract one another. Therefore every least particle of each of them will have its own attractive force in proportion to the quantity of matter, as was shown above for terrestrial bodies. And at different distances their forces will also be in the squared ratio of the distances inversely; for it is mathematically demonstrated that particles attracting by this law must constitute globes attracting by the same law.

The preceding conclusions are based upon an axiom which is accepted by every philosopher, namely, that effects of the same kind—that is, effects whose known properties are the same—have the same causes, and their properties which are not yet known are also the same. For if gravity is the cause of the fall of a stone in Europe, who can doubt that in America the cause of the fall is the same? If gravity is mutual between a stone and the earth in Europe, who will deny that it is mutual in America? If in Europe the attractive force of the stone and the earth is compounded of the attractive forces of the parts, who will deny that in America the force is similarly compounded? If in Europe the attraction of the earth is propagated to all kinds of bodies and to all distances, why should we not say that in America it is propagated in the same way? All philosophy is based on this rule, inasmuch as, if it is taken away, there is then nothing we can affirm about things universally. The constitution of individual things can be found by observations and experiments; and proceeding from there, it is only by this rule that we make judgments about the nature of things universally.

Now, since all terrestrial and celestial bodies on which we can make experiments or observations are heavy, it must be acknowledged without exception that gravity belongs to all bodies universally. And just as we must not conceive of bodies that are not extended, mobile, and impenetrable, so we should not conceive of any that are not heavy. The extension, mobility, and impenetrability of bodies are known only through experiments; it is in exactly the same way that the gravity of bodies is known. All bodies for which we have observations are extended and mobile and impenetrable; and from this we conclude that all bodies universally are extended and mobile and impenetrable, even those for which we do not have observations. Thus all bodies for which we have observations are heavy; and from this we conclude that all bodies universally are heavy, even those for which we do not have observations. If anyone were to say that the bodies of the fixed stars are not heavy, since their gravity has not yet been observed, then by the same argument one would be able to say that they are neither extended nor mobile nor impenetrable, since these properties of the fixed stars have not yet been observed. Need I go on? Among the primary qualities of all bodies universally, either gravity will have a place, or extension, mobility, and impenetrability will not. And the nature of things either will be correctly explained by the gravity of bodies or will not be correctly explained by the extension, mobility, and impenetrability of bodies.

I can hear some people disagreeing with this conclusion and muttering something or other about occult qualities. They are always prattling on and on to the effect that gravity is something occult, and that occult causes are to be banished completely from philosophy. But it is easy to answer them: occult causes are not those causes whose existence is very clearly demonstrated by observations, but only those whose existence is occult, imagined, and not yet proved. Therefore gravity is not an occult cause of celestial motions, since it has been shown from phenomena that this force really exists. Rather, occult causes are the refuge of those who assign the governing of these motions to some sort of vortices of a certain matter utterly fictitious and completely imperceptible to the senses.

But will gravity be called an occult cause and be cast out of natural philosophy on the grounds that the cause of gravity itself is occult and not yet found? Let those who so believe take care lest they believe in an absurdity that, in the end, may overthrow the foundations of all philosophy. For causes generally proceed in a continuous chain from compound to more simple; when you reach the simplest cause, you will not be able to proceed any further. Therefore no mechanical explanation can be given for the simplest cause; for if it could, the cause would not yet be the simplest. Will you accordingly call these simplest causes occult, and banish them? But at the same time the causes most immediately depending on them, and the causes that in turn depend on these causes, will also be banished, until philosophy is emptied and thoroughly purged of all causes.

Some say that gravity is preternatural and call it a perpetual miracle. Therefore they hold that it should be rejected, since preternatural causes have no place in physics. It is hardly worth spending time on demolishing this utterly absurd objection, which of itself undermines all of philosophy. For either they will say that gravity is not a property of all bodies—which cannot be maintained—or they will assert that gravity is preternatural on the grounds that it does not arise from other affections of bodies and thus not from mechanical causes. Certainly there are primary affections of bodies, and since they are primary, they do not depend on others. Therefore let them consider whether or not all these are equally preternatural, and so equally to be rejected, and let them consider what philosophy will then be like.

There are some who do not like all this celestial physics just because it seems to be in conflict with the doctrines of Descartes and seems scarcely capable of being reconciled with these doctrines. They are free to enjoy their own opinion, but they ought to act fairly and not deny to others the same liberty that they demand for themselves. Therefore, we should be allowed to adhere to the Newtonian philosophy, which we consider truer, and to prefer causes proved by phenomena to causes imagined and not yet proved. It is the province of true philosophy to derive the natures of things from causes that truly exist, and to seek those laws by which the supreme artificer willed to establish this most beautiful order of the world, not those laws by which he could have, had it so pleased him. For it is in accord with reason that the same effect can arise from several causes somewhat different from one another; but the true cause will be the one from which the effect truly and actually does arise, while the rest have no place in true philosophy. In mechanical clocks one and the same motion of the hour hand can arise from the action of a suspended weight or an internal spring. But if the clock under discussion is really activated by a weight, then anyone will be laughed at if he imagines a spring and on such a premature hypothesis undertakes to explain the motion of the hour hand; for he ought to have examined the internal workings of the machine more thoroughly, in order to ascertain the true principle of the motion in question. The same judgment or something like it should be passed on those philosophers who have held that the heavens are filled with a certain most subtle matter, which is endlessly moved in vortices. For even if these philosophers could account for the phenomena with the greatest exactness on the basis of their hypotheses, still they cannot be said to have given us a true philosophy and to have found the true causes of the celestial motions until they have demonstrated either that these causes really do exist or at least that others do not exist. Therefore if it can be shown that the attraction of all bodies universally has a true place in the nature of things, and if it further can be shown how all the celestial motions are solved by that attraction, then it would be an empty and ridiculous objection if anyone said that those motions should be explained by vortices, even if we gave our fullest assent to the possibility of such an explanation. But we do not give our assent; for the phenomena can by no means be explained by vortices, as our author fully proves with the clearest arguments. It follows that those who devote their fruitless labor to patching up a most absurd figment of their imagination and embroidering it further with new fabrications must be overly indulging their fantasies.

If the bodies of the planets and the comets are carried around the sun by vortices, the bodies carried around must move with the same velocity and in the same direction as the immediately surrounding parts of the vortices, and must have the same density or the same force of inertia in proportion to the bulk of the matter.

But it is certain that planets and comets, while they are in the same regions of the heavens, move with a variety of velocities and directions. Therefore it necessarily follows that those parts of the celestial fluid that are at the same distances from the sun revolve in the same time in different directions with different velocities; for there will be need of one direction and velocity to permit the planets to move through the heavens, and another for the comets. Since this cannot be accounted for, either it will have to be confessed that all the celestial bodies are not carried by the matter of a vortex, or it will have to be said that their motions are to be derived not from one and the same vortex, but from more than one, differing from one another and going through the same space surrounding the sun.

If it is supposed that several vortices are contained in the same space and penetrate one another and revolve with different motions, then—since these motions must conform to the motions of the bodies being carried around, motions highly regular in conic sections that are sometimes extremely eccentric and sometimes very nearly circular—it will be right to ask how it can happen that these same vortices keep their integrity without being in the least perturbed through so many centuries by the interactions of their matter. Surely, if these imaginary motions are more complex and more difficult to explain than the true motions of the planets and comets, I think it pointless to admit them into natural philosophy; for every cause must be simpler than its effect. Granted the freedom to invent any fiction, let someone assert that all the planets and comets are surrounded by atmospheres, as our earth is, a hypothesis that will certainly seem more reasonable than the hypothesis of vortices. Let him then assert that these atmospheres, of their own nature, move around the sun and describe conic sections, a motion that can surely be much more easily conceived than the similar motion of vortices penetrating one another. Finally, let him maintain that it must be believed that the planets themselves and the comets are carried around the sun by their atmospheres, and let him celebrate his triumph for having found the causes of the celestial motions. Anyone who thinks that this fiction should be rejected will also reject the other one; for the hypothesis of atmospheres and the hypothesis of vortices are as alike as two peas in a pod.

Galileo showed that when a stone is projected and moves in a parabola, its deflection from a rectilinear path arises from the gravity of the stone toward the earth, that is, from an occult quality. Nevertheless it can happen that some other philosopher, even more clever, may contrive another cause. He will accordingly imagine that a certain subtle matter, which is not perceived by sight or by touch or by any of the senses, is found in the regions that are most immediately contiguous to the surface of the earth. He will argue, moreover, that this matter is carried in different directions by various and—for the most part—contrary motions and that it describes parabolic curves. Finally he will beautifully show how the stone is deflected and will earn the applause of the crowd. The stone, says he, floats in that subtle fluid and, by following the course of that fluid, cannot but describe the same path. But the fluid moves in parabolic curves; therefore the stone must move in a parabola. Who will not now marvel at the most acute genius of this philosopher, brilliantly deducing the phenomena of nature from mechanical causes [i.e., matter and motion]—at a level comprehensible even to ordinary people! Who indeed will not jeer at that poor Galileo, who undertook by a great mathematical effort once more to bring back occult qualities, happily excluded from philosophy! But I am ashamed to waste any more time on such trifles.

It all finally comes down to this: the number of comets is huge; their motions are highly regular and observe the same laws as the motions of the planets. They move in conic orbits; these orbits are very, very eccentric. Comets go everywhere into all parts of the heavens and pass very freely through the regions of the planets, often contrary to the order of the signs. These phenomena are confirmed with the greatest certainty by astronomical observations and cannot be explained by vortices. Further, these phenomena are even inconsistent with planetary vortices. There will be no room at all for the motions of the comets unless that imaginary matter is completely removed from the heavens.

For if the planets are carried around the sun by vortices, those parts of the vortices that most immediately surround each planet will be of the same density as the planet, as has been said above. Therefore all the matter that is contiguous to the perimeter of the earth's orbit will have the same density as the earth, while all the matter that lies between the earth's orbit and the orbit of Saturn will have either an equal or a greater density. For, in order that the constitution of a vortex may be able to last, the less dense parts must occupy the center, and the more dense parts must be further away from the center. For since the periodic times of the planets are as the $\frac{3}{2}$ powers of the distances from the sun, the periods of the parts of the vortex should keep the same ratio. It follows that the centrifugal forces of these parts will be inversely as the squares of the distances. Therefore those parts that are at a greater distance from the center strive to recede from it by a smaller force; accordingly, if they should be less dense, it would be necessary for them to yield to the greater force by which the parts nearer to the center endeavor to ascend. Therefore the denser parts will ascend, the less dense will descend, and a mutual exchange of places will occur, until the fluid matter of the whole vortex has been arranged in such order that it can now rest in equilibrium [i.e., its parts are completely at rest with respect to one another or no longer have any motion of ascent or descent]. If two fluids of different density are contained in the same vessel, certainly it will happen that the fluid whose density is greater will go to the lowest place under the action of its greater force of gravity, and by similar reasoning it must be concluded that the denser parts of the vortex will go to the highest place under the action of their greater centrifugal force. Therefore the whole part of the vortex that lies outside the earth's orbit (much the greatest part) will have a density and so a force of inertia (proportional to the quantity of matter) that will not be smaller than the density and force of inertia of the earth. From this will arise a huge and very noticeable resistance to the comets as they pass through, not to say a resistance that rightly seems to be able to put a complete stop to their motion and absorb it entirely. It is however clear from the altogether regular motion of comets that they encounter no resistance that can be in the least perceived, and thus that they do not come upon any matter that has any force of resistance, or accordingly that has any density or force of inertia. For the resistance of mediums arises either from the inertia of fluid matter or from its friction.[a] That which arises from friction is extremely slight and indeed can scarcely be observed in commonly known fluids, unless they are very tenacious like oil and honey. The resistance that is encountered in air, water, quicksilver, and nontenacious fluids of this sort is almost wholly of the first kind and cannot be decreased in subtlety by any further degree, if the fluid's density or force of inertia—to which this resistance is always proportional—remains the same. This is most clearly demonstrated by our author in his brilliant theory of the resistance of fluids, which in this second edition is presented in a somewhat more accurate manner and is more fully confirmed by experiments with falling bodies.

As bodies move forward, they gradually communicate their motion to a surrounding fluid, and by communicating their motion lose it, and by losing it are retarded. Therefore the retardation is proportional to the motion so communicated, and the motion communicated (where the velocity of the moving body is given) is as the density of the fluid; therefore the retardation or resistance will also be as the density of the fluid and cannot be removed by any means unless the fluid, returning to the back of the body, restores the lost motion. But this cannot be the case unless the force of the fluid on the rear of the body is equal to the force the body exerts on the fluid in front, that is, unless the relative velocity with which the fluid pushes the body from behind is equal to the velocity with which the body pushes the fluid, that is, unless the absolute velocity of the returning fluid is twice as great as the absolute velocity of the fluid pushed forward, which cannot happen. Therefore there is no way in which the resistance of fluids that arises from their density and force of inertia can be taken away. And so it must be concluded that the celestial fluid has no force of inertia, since it has no force of resistance; it has

a. Literally, lack of lubricity or slipperiness.

no force by which motion may be communicated, since it has no force of inertia; it has no force by which any change may be introduced into one or more bodies, since it has no force by which motion may be communicated; it has no efficacy at all, since it has no faculty to introduce any change. Surely, therefore, this hypothesis, plainly lacking in any foundation and not even marginally useful to explain the nature of things, may well be called utterly absurd and wholly unworthy of a philosopher. Those who hold that the heavens are filled with fluid matter, but suppose this matter to have no inertia, are saying there is no vacuum but in fact are assuming there is one. For, since there is no way to distinguish a fluid matter of this sort from empty space, the whole argument comes down to the names of things and not their natures. But if anyone is so devoted to matter that he will in no way admit a space void of bodies, let us see where this will ultimately lead him.

For such people will say that this constitution of the universe as everywhere full, which is how they imagine it, has arisen from the will of God, so that a very subtle aether pervading and filling all things would be there to facilitate the operations of nature; this cannot be maintained, however, since it has already been shown from the phenomena of comets that this aether has no efficacy. Or they will say that this constitution has arisen from the will of God for some unknown purpose, which ought not to be said either, since a different constitution of the universe could equally well be established by the same argument. Or finally they will say that it has not arisen from the will of God but from some necessity of nature. And so at last they must sink to the lowest depths of degradation, where they have the fantasy that all things are governed by fate and not by providence, that matter has existed always and everywhere of its own necessity and is infinite and eternal. On this supposition, matter will also be uniform everywhere, for variety of forms is entirely inconsistent with necessity. Matter will also be without motion; for if by necessity matter moves in some definite direction with some definite velocity, by a like necessity it will move in a different direction with a different velocity; but it cannot move in different directions with different velocities; therefore it must be without motion. Surely, this world—so beautifully diversified in its forms and motions—could not have arisen except from the perfectly free will of God, who provides and governs all things.

From this source, then, have all the laws that are called laws of nature come, in which many traces of the highest wisdom and counsel certainly appear, but no traces of necessity. Accordingly we should not seek these laws by using untrustworthy conjectures, but learn them by observing and experimenting. He who is confident that he can truly find the principles of physics, and the laws of things, by relying only on the force of his mind and the internal light of his reason should maintain either that the world has existed from necessity and follows the said laws from the same necessity, or that although the order of nature was constituted by the will of God, nevertheless a creature as small and insignificant as he is has a clear understanding of the way things should be. All sound and true philosophy is based on phenomena, which may lead us—however unwilling and reluctant—to principles in which the best counsel and highest dominion of an all-wise and all-powerful being are most clearly discerned; these principles will not be rejected because certain men may perhaps not like them. These men may call the things that they dislike either miracles or occult qualities, but names maliciously given are not to be blamed on the things themselves, unless these men are willing to confess at last that philosophy should be based on atheism. Philosophy must not be overthrown for their sake, since the order of things refuses to be changed.

Therefore honest and fair judges will approve the best method of natural philosophy, which is based on experiments and observations. It need scarcely be said that this way of philosophizing has been illumined and dignified by our illustrious author's well-known book; his tremendous genius, enodating each of the most difficult problems and reaching out beyond the accepted limits of the human, is justly admired and esteemed by all who are more than superficially versed in these matters. Having unlocked the gates, therefore, he has opened our way to the most beautiful mysteries of nature. He has finally so clearly revealed a most elegant structure of the system of the world for our further scrutiny that even were King Alfonso himself to come to life again, he would not find it wanting either in simplicity or in grace of harmony. And hence it is now possible to have a closer view of the majesty of nature, to enjoy the sweetest contemplation, and to worship and venerate more zealously the maker and lord of all; and this is by far the greatest fruit of philosophy. He must be blind who does not at once see, from the best and wisest structures of things, the infinite wisdom and goodness of their almighty creator; and he must be mad who refuses to acknowledge them.

Therefore Newton's excellent treatise will stand as a mighty fortress against the attacks of atheists; nowhere else will you find more effective ammunition against that impious crowd. This was understood long ago, and was first splendidly demonstrated in learned discourses in English and in Latin, by a man of universal learning and at the same time an outstanding patron of the arts, Richard Bentley, a great ornament of his time and of our academy, the worthy and upright master of our Trinity College. I must confess that I am indebted to him on many grounds; you as well, kind reader, will not deny him due thanks. For, as a long-time intimate friend of our renowned author (he considers being celebrated by posterity for this friendship to be of no less value than becoming famous for his own writings, which are the delight of the learned world), he worked simultaneously for the public recognition of his friend and for the advancement of the sciences. Therefore, since the available copies of the first edition were extremely rare and very expensive, he tried with persistent demands to persuade Newton (who is distinguished as much by modesty as by the highest learning) and finally—almost scolding him—prevailed upon Newton to allow him to get out this new edition, under his auspices and at his own expense, perfected throughout and also enriched with significant additions. He authorized me to undertake the not unpleasant duty of seeing to it that all this was done as correctly as possible.

Cambridge, 12 May 1713

Roger Cotes,
Fellow of Trinity College,
Plumian Professor of Astronomy
and Experimental Philosophy

Author's Preface to the Third Edition

IN THIS THIRD EDITION, supervised by Henry Pemberton, M.D., a man greatly skilled in these matters, some things in the second book concerning the resistance of mediums are explained a little more fully than previously, and new experiments are added concerning the resistance of heavy bodies falling in air. In the third book, the argument proving that the moon is kept in its orbit by gravity is presented a little more fully; and new observations, made by Mr. Pound, on the proportion of the diameters of Jupiter to each other have been added. There are also added some observations of the comet that appeared in 1680, which were made in Germany during the month of November by Mr. Kirk, and which recently came into our hands; these observations make it clear how closely parabolic orbits correspond to the motions of comets. The orbit of that comet, by Halley's computations, is determined a little more accurately than heretofore, and in an ellipse. And it is shown that the comet traversed its course through nine signs of the heavens in this elliptical orbit just as exactly as the planets move in the elliptical orbits given by astronomy. There is also added the orbit of the comet that appeared in the year 1713, which was calculated by Mr. Bradley, professor of astronomy at Oxford.

London, Is. Newton
12 Jan. 1725/6.

[In the third edition, the final Author's Preface was followed by a two-page table of contents and a list of corrigenda.]

MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY

(PHILOSOPHIAE NATURALIS PRINCIPIA MATHEMATICA)

DEFINITIONS

Definition 1

[a]*Quantity of matter is a measure of matter that arises from its density and volume jointly.*[a]

[b]If the density of air is doubled in a space that is also doubled, there is four times as much air, and there is six times as much if the space is tripled.[b] The case is the same for snow and powders condensed by compression or liquefaction, and also for all bodies that are condensed in various ways by any causes whatsoever. For the present, I am not taking into account any medium, if there should be any, freely pervading the interstices between the parts of

aa. In translating def. 1, we have rendered Newton's "Quantitas materiae est mensura ejusdem . . ." as "Quantity of matter is a measure of matter . . ." rather than the customary ". . . is the measure . . ." The indefinite article is more in keeping with the Latin usage, with its absence of articles, and accords better with the sense in which we have interpreted this definition. See the Guide, §4.2. It should be noted that the indefinite article permits the possibility of the sense of either a definite or an indefinite article, whereas a definite article precludes the possibility of the sense of an indefinite article.

bb. Ed. 3 reads literally: "Air, if the density is doubled, in a space also doubled, becomes quadruple; in [a space] tripled, sextuple." The printer's manuscript for ed. 1 and the printed text of ed. 1 have: "Air twice as dense in twice the space is quadruple." Newton's interleaved copy of ed. 1 has: "Air twice as dense in twice the space is quadruple; in three times [the space], sextuple." Newton's annotated copy of ed. 1 has first: "Air twice as dense in twice the space becomes quadruple; in three times [the space], sextuple." This is then deleted and replaced with: "Air, by doubling the density, in the same container becomes double; in a container twice as large, quadruple; in one three times as large, sextuple; and by tripling the density, it becomes triple in the same container and sextuple in a container twice as large," but the last clause, "and by tripling . . . large," is then deleted.

The manuscript errata at the end of the annotated copy have: "For this quantity, if the density is given [or fixed], is as the volume and, if the volume is given, is as the density and therefore, if neither is given, is as the product of both. Thus indeed Air, if the density is doubled, in a space also doubled, becomes quadruple; in [a space] tripled, sextuple." The first sentence, "For this . . . product of both," and the following two words, "Thus indeed," are inserted over a caret preceding "Air."

An interleaf of the interleaved copy of ed. 1 and then the printed text of ed. 2 have exactly the same phrasing as ed. 3.

bodies. Furthermore, I mean this quantity whenever I use the term "body" or "mass" in the following pages. It can always be known from a body's weight, for—by making very accurate experiments with pendulums—I have found it to be proportional to the weight, as will be shown below.

Definition 2 *Quantity of motion is a measure of motion that arises from the velocity and the quantity of matter jointly.*

The motion of a whole is the sum of the motions of the individual parts, and thus if a body is twice as large as another and has equal velocity there is twice as much motion, and if it has twice the velocity there is four times as much motion.

Definition 3 *Inherent force of matter is the power of resisting by which every body, [a]so far as it is able,[a] perseveres in its state either of resting or of moving [b]uniformly straight forward.[b]*

This force is always proportional to the body and does not differ in any way from the inertia of the mass except in the manner in which it is conceived. Because of the inertia of matter, every body is only with difficulty put out of its state either of resting or of moving. Consequently, inherent force may also be called by the very significant name of force of inertia.[c] Moreover, a body exerts this force only during a change of its state, caused by another force impressed upon it, and this exercise of force is, depending on the viewpoint, both resistance and impetus: resistance insofar as the body, in order to maintain its state, strives against the impressed force, and impetus insofar as the same body, yielding only with difficulty to the force of a resisting obstacle, endeavors to change the state of that obstacle. Resistance is commonly attributed to resting bodies and impetus to moving bodies; but

aa. Newton's Latin clause is "quantum in se est," which here means "to the degree that it can of and by itself." See I. Bernard Cohen, " 'Quantum in se est': Newton's Concept of Inertia in Relation to Descartes and Lucretius," *Notes and Records of the Royal Society of London* 19 (1964): 131–155.

bb. Newton's "in directum" (used together with "uniformiter" ["uniformly"]) has the sense of moving straight on, of going continuously straight forward, and therefore in a straight line. In an earlier version, Newton had used the phrase "in linea recta" ("in a right line" or "in a straight line"), but by the time of the *Principia* he had rejected this expression in favor of "in directum." For details, see the Guide, §10.2. On Newton's "vis insita" and our rendition, see the Guide, §4.7.

c. Newton's interleaved copy of ed. 2 adds the following, which was never printed: "I do not mean Kepler's force of inertia, by which bodies tend toward rest, but a force of remaining in the same state either of resting or of moving."

motion and rest, in the popular sense of the terms, are distinguished from each other only by point of view, and bodies commonly regarded as being at rest are not always truly at rest.

Definition 4

Impressed force is the action exerted on a body to change its state either of resting or of moving uniformly straight forward.

This force consists solely in the action and does not remain in a body after the action has ceased. For a body perseveres in any new state solely by the force of inertia. Moreover, there are various sources of impressed force, such as percussion, pressure, or centripetal force.

Definition 5

Centripetal force is the force by which bodies are drawn from all sides, are impelled, or in any way tend, toward some point as to a center.

One force of this kind is gravity, by which bodies tend toward the center of the earth; another is magnetic force, by which iron seeks a lodestone; and yet another is that force, whatever it may be, by which the planets are continually drawn back from rectilinear motions and compelled to revolve in curved lines.

[a]A stone whirled in a sling endeavors to leave the hand that is whirling it, and by its endeavor it stretches the sling, doing so the more strongly the more swiftly it revolves; and as soon as it is released, it flies away. The force opposed to that endeavor, that is, the force by which the sling continually draws the stone back toward the hand and keeps it in an orbit, I call centripetal, since it is directed toward the hand as toward the center of an orbit. And the same applies to all bodies [b]that are made to move in orbits.[b] They all endeavor to recede from the centers of their orbits, and unless some force opposed to that endeavor is present, restraining them and keeping them in orbits and hence called by me centripetal, they will go off in straight lines with uniform motion. If a projectile were deprived of the force of gravity, it would not be deflected toward the earth but would go off in a straight line into the heavens and do so with uniform motion, provided that the resistance of the air were removed. The projectile, by its gravity, is drawn back from a rectilinear course and continually deflected toward the earth, and this is so

aa. Ed. 1 lacks this.

bb. See the Guide, §2.4.

to a greater or lesser degree in proportion to its gravity and its velocity of motion. The less its gravity in proportion to its quantity of matter, or the greater the velocity with which it is projected, the less it will deviate from a rectilinear course and the farther it will go. If a lead ball were projected with a given velocity along a horizontal line from the top of some mountain by the force of gunpowder and went in a curved line for a distance of two miles before falling to the earth, then the same ball projected with twice the velocity would go about twice as far and with ten times the velocity about ten times as far, provided that the resistance of the air were removed. And by increasing the velocity, the distance to which it would be projected could be increased at will and the curvature of the line that it would describe could be decreased, in such a way that it would finally fall at a distance of 10 or 30 or 90 degrees or even go around the whole earth or, lastly, go off into the heavens and continue indefinitely in this motion. And in the same way that a projectile could, by the force of gravity, be deflected into an orbit and go around the whole earth, so too the moon, whether by the force of gravity—if it has gravity—or by any other force by which it may be urged toward the earth, can always be drawn back toward the earth from a rectilinear course and deflected into its orbit; and without such a force the moon cannot be kept in its orbit. If this force were too small, it would not deflect the moon sufficiently from a rectilinear course; if it were too great, it would deflect the moon excessively and draw it down from its orbit toward the earth. In fact, it must be of just the right magnitude, and mathematicians have the task of finding the force by which a body can be kept exactly in any given orbit with a given velocity and, alternatively, to find the curvilinear path into which a body leaving any given place with a given velocity is deflected by a given force.[a]

The quantity of centripetal force is of three kinds: absolute, accelerative, and motive.

Definition 6 *The absolute quantity of centripetal force is the measure of this force that is greater or less in proportion to the efficacy of the cause propagating it from a center through the surrounding regions.*

An example is magnetic force, which is greater in one lodestone and less in another, in proportion to the bulk or potency of the lodestone.

Definition 7

The accelerative quantity of centripetal force is the measure of this force that is proportional to the velocity which it generates in a given time.

One example is the potency of a lodestone, which, for a given lodestone, is greater at a smaller distance and less at a greater distance. Another example is the force that produces gravity, which is greater in valleys and less on the peaks of high mountains and still less (as will be made clear below) at greater distances from the body of the earth, but which is everywhere the same at equal distances, because it equally accelerates all falling bodies (heavy or light, great or small), provided that the resistance of the air is removed.

Definition 8

The motive quantity of centripetal force is the measure of this force that is proportional to the motion which it generates in a given time.

An example is weight, which is greater in a larger body and less in a smaller body; and in one and the same body is greater near the earth and less out in the heavens. This quantity is the centripetency, or propensity toward a center, of the whole body, and (so to speak) its weight, and it may always be known from the force opposite and equal to it, which can prevent the body from falling.

These quantities of forces, for the sake of brevity, may be called motive, accelerative, and absolute forces, and, for the sake of differentiation, may be referred to bodies seeking a center, to the places of the bodies, and to the center of the forces: that is, motive force may be referred to a body as an endeavor of the whole directed toward a center and compounded of the endeavors of all the parts; accelerative force, to the place of the body as a certain efficacy diffused from the center through each of the surrounding places in order to move the bodies that are in those places; and absolute force, to the center as having some cause without which the motive forces are not propagated through the surrounding regions, whether this cause is some central body (such as a lodestone in the center of a magnetic force or the earth in the center of a force that produces gravity) or whether it is some other cause which is not apparent. This concept is purely mathematical, for I am not now considering the physical causes and sites of forces.

Therefore, accelerative force is to motive force as velocity to motion. For quantity of motion arises from velocity and quantity of matter jointly, and motive force from accelerative force and quantity of matter jointly. For the sum of the actions of the accelerative force on the individual particles of a body is the motive force of the whole body. As a consequence, near the surface of the earth, where the accelerative gravity, or the force that produces gravity, is the same in all bodies universally, the motive gravity, or weight, is as the body, but in an ascent to regions where the accelerative gravity becomes less, the weight will decrease proportionately and will always be as the body and the accelerative gravity jointly. Thus, in regions where the accelerative gravity is half as great, a body one-half or one-third as great will have a weight four or six times less.

Further, it is in this same sense that I call attractions and impulses accelerative and motive. Moreover, I use interchangeably and indiscriminately words signifying attraction, impulse, or any sort of propensity toward a center, considering these forces not from a physical but only from a mathematical point of view. Therefore, let the reader beware of thinking that by words of this kind I am anywhere defining a species or mode of action or a physical cause or reason, or that I am attributing forces in a true and physical sense to centers (which are mathematical points) if I happen to say that centers attract or that centers have forces.

Scholium Thus far it has seemed best to explain the senses in which less familiar words are to be taken in this treatise. Although time, space, place, and motion are very familiar to everyone, it must be noted that these quantities are popularly conceived solely with reference to the objects of sense perception. And this is the source of certain preconceptions; to eliminate them it is useful to distinguish these quantities into absolute and relative, true and apparent, mathematical and common.

1. Absolute, true, and mathematical time, in and of itself and of its own nature, without reference to anything external, flows uniformly and by another name is called duration. Relative, apparent, and common time is any sensible and external measure [a](precise or imprecise)[a] of duration by means of motion; such a measure—for example, an hour, a day, a month, a year—is commonly used instead of true time.

2. Absolute space, of its own nature without reference to anything external, always remains homogeneous and immovable. Relative space is any

aa. Newton uses the phrase "seu accurata seu inaequabilis"—literally, "exact or nonuniform."

movable measure or dimension of this absolute space; such a measure or dimension is determined by our senses from the situation of the space with respect to bodies and is popularly used for immovable space, as in the case of space under the earth or in the air or in the heavens, where the dimension is determined from the situation of the space with respect to the earth. Absolute and relative space are the same in species and in magnitude, but they do not always remain the same numerically. For example, if the earth moves, the space of our air, which in a relative sense and with respect to the earth always remains the same, will now be one part of the absolute space into which the air passes, now another part of it, and thus will be changing continually in an absolute sense.

3. Place is the part of space that a body occupies, and it is, depending on the space, either absolute or relative. I say the part of space, not the position of the body or its outer surface. For the places of equal solids are always equal, while their surfaces are for the most part unequal because of the dissimilarity of shapes; and positions, properly speaking, do not have quantity and are not so much places as attributes of places. The motion of a whole is the same as the sum of the motions of the parts; that is, the change in position of a whole from its place is the same as the sum of the changes in position of its parts from their places, and thus the place of a whole is the same as the sum of the places of the parts and therefore is internal and in the whole body.

4. Absolute motion is the change of position of a body from one absolute place to another; relative motion is change of position from one relative place to another. Thus, in a ship under sail, the relative place of a body is that region of the ship in which the body happens to be or that part of the whole interior of the ship which the body fills and which accordingly moves along with the ship, and relative rest is the continuance of the body in that same region of the ship or same part of its interior. But true rest is the continuance of a body in the same part of that unmoving space in which the ship itself, along with its interior and all its contents, is moving. Therefore, if the earth is truly at rest, a body that is relatively at rest on a ship will move truly and absolutely with the velocity with which the ship is moving on the earth. But if the earth is also moving, the true and absolute motion of the body will arise partly from the true motion of the earth in unmoving space and partly from the relative motion of the ship on the earth. Further, if the body is also moving relatively on the ship, its true motion will arise partly from

the true motion of the earth in unmoving space and partly from the relative motions both of the ship on the earth and of the body on the ship, and from these relative motions the relative motion of the body on the earth will arise. For example, if that part of the earth where the ship happens to be is truly moving eastward with a velocity of 10,010 units, and the ship is being borne westward by sails and wind with a velocity of 10 units, and a sailor is walking on the ship toward the east with a velocity of 1 unit, then the sailor will be moving truly and absolutely in unmoving space toward the east with a velocity of 10,001 units and relatively on the earth toward the west with a velocity of 9 units.

In astronomy, absolute time is distinguished from relative time by the equation of common time. For natural days, which are commonly considered equal for the purpose of measuring time, are actually unequal. Astronomers correct this inequality in order to measure celestial motions on the basis of a truer time. It is possible that there is no uniform motion by which time may have an exact measure. All motions can be accelerated and retarded, but the flow of absolute time cannot be changed. The duration or perseverance of the existence of things is the same, whether their motions are rapid or slow or null; accordingly, duration is rightly distinguished from its sensible measures and is gathered from them by means of an astronomical equation. Moreover, the need for using this equation in determining when phenomena occur is proved by experience with a pendulum clock and also by eclipses of the satellites of Jupiter.

Just as the order of the parts of time is unchangeable, so, too, is the order of the parts of space. Let the parts of space move from their places, and they will move (so to speak) from themselves. For times and spaces are, as it were, the places of themselves and of all things. All things are placed in time with reference to order of succession and in space with reference to order of position. It is of the essence of spaces to be places, and for primary places to move is absurd. They are therefore absolute places, and it is only changes of position from these places that are absolute motions.

But since these parts of space cannot be seen and cannot be distinguished from one another by our senses, we use sensible measures in their stead. For we define all places on the basis of the positions and distances of things from some body that we regard as immovable, and then we reckon all motions with respect to these places, insofar as we conceive of bodies as being changed in position with respect to them. Thus, instead of absolute places and motions we use relative ones, which is not inappropriate in ordinary human affairs, although in philosophy abstraction from the senses is required. For it is possible that there is no body truly at rest to which places and motions may be referred.

Moreover, absolute and relative rest and motion are distinguished from each other by their properties, causes, and effects. It is a property of rest that bodies truly at rest are at rest in relation to one another. And therefore, since it is possible that some body in the regions of the fixed stars or far beyond is absolutely at rest, and yet it cannot be known from the position of bodies in relation to one another in our regions whether or not any of these maintains a given position with relation to that distant body, true rest cannot be defined on the basis of the position of bodies in relation to one another.

It is a property of motion that parts which keep given positions in relation to wholes participate in the motions of such wholes. For all the parts of bodies revolving in orbit endeavor to recede from the axis of motion, and the impetus of bodies moving forward arises from the joint impetus of the individual parts. Therefore, when bodies containing others move, whatever is relatively at rest within them also moves. And thus true and absolute motion cannot be determined by means of change of position from the vicinity of bodies that are regarded as being at rest. For the exterior bodies ought to be regarded not only as being at rest but also as being truly at rest. Otherwise all contained bodies, besides being subject to change of position from the vicinity of the containing bodies, will participate in the true motions of the containing bodies and, if there is no such change of position, will not be truly at rest but only be regarded as being at rest. For containing bodies are to those inside them as the outer part of the whole to the inner part or as the shell to the kernel. And when the shell moves, the kernel also, without being changed in position from the vicinity of the shell, moves as a part of the whole.

A property akin to the preceding one is that when a place moves, whatever is placed in it moves along with it, and therefore a body moving away from a place that moves participates also in the motion of its place. Therefore, all motions away from places that move are only parts of whole and absolute motions, and every whole motion is compounded of the motion of a body away from its initial place, and the motion of this place away from its place, and so on, until an unmoving place is reached, as in the above-mentioned example of the sailor. Thus, whole and absolute motions can be determined only by means of unmoving places, and therefore in what has preceded I have referred such motions to unmoving places and relative motions to movable places. Moreover, the only places that are unmoving are those that all keep given positions in relation to one another from infinity to infinity and therefore always remain immovable and constitute the space that I call immovable.

The causes which distinguish true motions from relative motions are the forces impressed upon bodies to generate motion. True motion is neither generated nor changed except by forces impressed upon the moving body itself, but relative motion can be generated and changed without the impression of forces upon this body. For the impression of forces solely on other bodies with which a given body has a relation is enough, when the other bodies yield, to produce a change in that relation which constitutes the relative rest or motion of this body. Again, true motion is always changed by forces impressed upon a moving body, but relative motion is not necessarily changed by such forces. For if the same forces are impressed upon a moving body and also upon other bodies with which it has a relation, in such a way that the relative position is maintained, the relation that constitutes the relative motion will also be maintained. Therefore, every relative motion can be changed while the true motion is preserved, and can be preserved while the true one is changed, and thus true motion certainly does not consist in relations of this sort.

The effects distinguishing absolute motion from relative motion are the forces of receding from the axis of circular motion. For in purely relative circular motion these forces are null, while in true and absolute circular motion they are larger or smaller in proportion to the quantity of motion. If a bucket is hanging from a very long cord and is continually turned around until the cord becomes twisted tight, and if the bucket is thereupon filled with water and is at rest along with the water and then, by some sudden force, is made to turn around in the opposite direction and, as the cord unwinds, perseveres for a while in this motion; then the surface of the water will at first be level, just as it was before the vessel began to move. But after the vessel, by the force gradually impressed upon the water, has caused the water also to begin revolving perceptibly, the water will gradually recede from the middle and rise up the sides of the vessel, assuming a concave shape (as experience has shown me), and, with an ever faster motion, will rise further and further until, when it completes its revolutions in the same times as the vessel, it is relatively at rest in the vessel. The rise of the water reveals its endeavor to recede from the axis of motion, and from such an endeavor one can find out and measure the true and absolute circular motion of the water, which here is the direct opposite of its relative motion. In the beginning, when the relative motion of the water in the vessel was greatest, that motion was not giving rise to any endeavor to recede from the axis; the water did not seek the circumference by rising up the sides of the vessel but remained level, and therefore its true circular motion had not yet begun. But afterward, when the relative motion of the water decreased, its rise up the sides of the vessel revealed its endeavor to recede from the axis, and this endeavor showed the true circular motion of the water to be continually increasing and finally becoming greatest when the water was relatively at rest in the vessel. Therefore, that endeavor does not depend on the change of position of the water with respect to surrounding bodies, and thus true circular motion cannot be determined by means of such changes of position. The truly circular motion of each revolving body is unique, corresponding to a unique endeavor as its proper and sufficient effect, while relative motions are innumerable in accordance with their varied relations to external bodies and, like relations, are completely lacking in true effects except insofar as they participate in that true and unique motion. Thus, even in the system of those who hold that our heavens revolve below the heavens of the fixed stars and carry the planets around with them, the individual parts of the heavens, and the planets that are relatively at rest in the heavens to which they belong, are truly in motion. For they change their positions relative to one another (which is not the case with things that are truly at rest), and as they are carried around together with the heavens, they participate in the motions of the heavens and, being parts of revolving wholes, endeavor to recede from the axes of those wholes.

Relative quantities, therefore, are not the actual quantities whose names they bear but are those sensible measures of them (whether true or erroneous) that are commonly used instead of the quantities being measured. But if the meanings of words are to be defined by usage, then it is these sensible measures which should properly be understood by the terms "time," "space," "place," and "motion," and the manner of expression will be out of the ordinary and purely mathematical if the quantities being measured are understood here. Accordingly those who there interpret these words as referring to the quantities being measured do violence to the Scriptures. And they no less corrupt mathematics and philosophy who confuse true quantities with their relations and common measures.

It is certainly very difficult to find out the true motions of individual bodies and actually to differentiate them from apparent motions, because the parts of that immovable space in which the bodies truly move make no impression on the senses. Nevertheless, the case is not utterly hopeless. For it is possible to draw evidence partly from apparent motions, which are the differences between the true motions, and partly from the forces that are the causes and effects of the true motions. For example, if two balls, at a given distance from each other with a cord connecting them, were revolving about a common center of gravity, the endeavor of the balls to recede from the axis of motion could be known from the tension of the cord, and thus the quantity of circular motion could be computed. Then, if any equal forces were simultaneously impressed upon the alternate faces of the balls to increase or decrease their circular motion, the increase or decrease of the motion could be known from the increased or decreased tension of the cord, and thus, finally, it could be discovered which faces of the balls the forces would have to be impressed upon for a maximum increase in the motion, that is, which were the posterior faces, or the ones that are in the rear in a circular motion. Further, once the faces that follow and the opposite faces that precede were known, the direction of the motion would be known. In this way both the quantity and the direction of this circular motion could be found in any immense vacuum, where nothing external and sensible existed with which the balls could be compared. Now if some distant bodies were set in that space and maintained given positions with respect to one another, as the fixed stars do in the regions of the heavens, it could not, of course, be known from the relative change of position of the balls among the bodies whether the motion was to be attributed to the bodies or to the balls. But if the cord was examined and its tension was discovered to be the very one which the motion of the balls required, it would be valid to conclude that the motion belonged to the balls and that the bodies were at rest, and then, finally, from the change of position of the balls among the bodies, to determine the direction of this motion. But in what follows, a fuller explanation will be given of how to determine true motions from their causes, effects, and apparent differences, and, conversely, of how to determine from motions, whether true or apparent, their causes and effects. For this was the purpose for which I composed the following treatise.

AXIOMS, OR THE LAWS OF MOTION

Law 1 *Every body perseveres in its state of being at rest or of moving* [a]*uniformly straight forward,*[a] *except insofar as* [b]*it*[b] *is compelled to change* [c]*its*[c] *state by forces impressed.*

Projectiles persevere in their motions, except insofar as they are retarded by the resistance of the air and are impelled downward by the force of gravity. A spinning hoop,[d] which has parts that by their cohesion continually draw one another back from rectilinear motions, does not cease to rotate, except insofar as it is retarded by the air. And larger bodies—planets and comets—preserve for a longer time both their progressive and their circular motions, which take place in spaces having less resistance.

Law 2 *A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.*

If some force generates any motion, twice the force will generate twice the motion, and three times the force will generate three times the motion, whether the force is impressed all at once or successively by degrees. And if the body was previously moving, the new motion (since motion is always in the same direction as the generative force) is added to the original motion if that motion was in the same direction or is subtracted from the original motion if it was in the opposite direction or, if it was in an oblique direction,

aa. See note bb to def. 3.

bb. Ed. 1 and ed. 2 lack the pronoun "illud," which, by expressing the subject, renders it somewhat more emphatic than it is when conveyed only by the form of the verb ("is compelled") and which makes more explicit the reference to an antecedent noun ("body").

cc. Ed. 1 and ed. 2 have "that."

d. The Latin word is "trochus," i.e., a top or some kind of spinner.

is combined obliquely and compounded with it according to the directions of both motions.

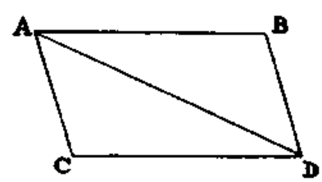
Law 3

To any action there is always an opposite and equal reaction; in other words, the actions of two bodies upon each other are always equal and always opposite in direction.

Whatever presses or draws something else is pressed or drawn just as much by it. If anyone presses a stone with a finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse will (so to speak) also be drawn back equally toward the stone, for the rope, stretched out at both ends, will urge the horse toward the stone and the stone toward the horse by one and the same endeavor to go slack and will impede the forward motion of the one as much as it promotes the forward motion of the other. If some body impinging upon another body changes the motion of that body in any way by its own force, then, by the force of the other body (because of the equality of their mutual pressure), it also will in turn undergo the same change in its own motion in the opposite direction. By means of these actions, equal changes occur in the motions, not in the velocities—that is, of course, if the bodies are not impeded by anything else.[a] For the changes in velocities that likewise occur in opposite directions are inversely proportional to the bodies because the motions are changed equally. This law is valid also for attractions, as will be proved in the next scholium.

Corollary 1

A body acted on by [two] forces acting jointly describes the diagonal of a parallelogram in the same time in which it would describe the sides if the forces were acting separately.



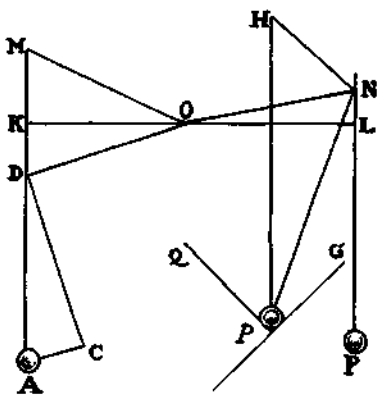
Let a body in a given time, by force M alone impressed in A, be carried with uniform motion from A to B, and, by force N alone impressed in the same place, be carried from A to C; then complete the parallelogram ABDC, and by both forces the body will be carried in the same time along the diagonal from A to D. For, since force N acts along the line AC parallel to

a. By "body" Newton means quantity of matter or mass (def. 1) and by "motion" he means quantity of motion (def. 2) or momentum.

BD, this force, by law 2, will make no change at all in the velocity toward the line BD which is generated by the other force. Therefore, the body will reach the line BD in the same time whether force N is impressed or not, and so at the end of that time will be found somewhere on the line BD. By the same argument, at the end of the same time it will be found somewhere on the line CD, and accordingly it is necessarily found at the intersection D of both lines. And, by law 1, it will go with [uniform] rectilinear motion from A to D.

Corollary 2 *And hence the composition of a direct force* AD *out of any oblique forces* AB *and* BD *is evident, and conversely the resolution of any direct force* AD *into any oblique forces* AB *and* BD. *And this kind of composition and resolution is indeed abundantly confirmed from mechanics.*

For example, let OM and ON be unequal spokes going out from the center O of any wheel, and let the spokes support the weights A and P by means of the cords MA and NP; it is required to find the forces of the weights to move the wheel. Draw the straight line KOL through the center O, so as to meet the cords perpendicularly at K and L; and with center O and radius OL, which is the greater of OK and OL, describe a circle meeting the cord MA at D; and draw the straight line OD, and let AC be drawn parallel to it and DC perpendicular to it. Since it makes no difference whether points K, L, and D of the cords are attached or not attached to the plane of the wheel, the weights will have the same effect whether they are suspended from the points K and L or from D and L. And if now the total force of the weight A is represented by line AD, it will be resolved into forces [i.e., components] AC and CD, of which AC, drawing spoke OD directly from the center, has no effect in moving the wheel, while the other force DC, drawing spoke DO perpendicularly, has the same effect as if it were drawing spoke OL (equal to OD) perpendicularly; that is, it has the same effect as the weight P, provided that the weight P is to the weight A as the force DC is to the force DA; that is (because triangles ADC and



DOK are similar), as OK to OD or OL. Therefore, the weights A and P, which are inversely as the spokes OK and OL (which are in a straight line), will be equipollent and thus will stand in equilibrium, which is a very well known property of the balance, the lever, and the wheel and axle. But if either weight is greater than in this ratio, its force to move the wheel will be so much the greater.

But if the weight p, equal to the weight P, is partly suspended by the cord Np and partly lies on the oblique plane pG, draw pH perpendicular to the plane of the horizon and NH perpendicular to the plane pG; then if the force of the weight p tending downward is represented by the line pH, it can be resolved into the forces [i.e., components] pN and HN. If there were some plane pQ perpendicular to the cord pN and cutting the other plane pG in a line parallel to the horizon, and the weight p were only lying on these planes pQ and pG, the weight p would press these planes perpendicularly with the forces pN and HN—plane pQ, that is, with force pN and plane pG with force HN. Therefore, if the plane pQ is removed, so that the weight stretches the cord, then—since the cord, in sustaining the weight, now takes the place of the plane which has been removed—the cord will be stretched by the same force pN with which the plane was formerly pressed. Thus the tension of this oblique cord will be to the tension of the other, and perpendicular, cord PN as pN to pH. Therefore, if the weight p is to the weight A in a ratio that is compounded of the inverse ratio of the least distances of their respective cords pN and AM from the center of the wheel and the direct ratio of pH to pN, the weights will have the same power to move the wheel and so will sustain each other, as anyone can test.

Now, the weight p, lying on those two oblique planes, has the role of a wedge between the inner surfaces of a body that has been split open; and hence the forces of a wedge and hammer can be determined, because the force with which the weight p presses the plane pQ is to the force with which weight p is impelled along the line pH toward the planes, whether by its own gravity or by the blow of a hammer, as pN is to pH, and because the force with which p presses plane pQ is to the force by which it presses the other plane pG as pN to NH. Furthermore, the force of a screw can also be determined by a similar resolution of forces, inasmuch as it is a wedge impelled by a lever. Therefore, this corollary can be used very extensively, and the variety of its applications clearly shows its truth, since the whole of mechanics—demonstrated in different ways by those who have written on this subject—depends on what has just now been said. For from this are easily derived the forces of machines, which are generally composed of wheels, drums, pulleys, levers, stretched strings, and weights, ascending directly or obliquely, and the other mechanical powers, as well as the forces of tendons to move the bones of animals.

Corollary 3 *The quantity of motion, which is determined by adding the motions made in one direction and subtracting the motions made in the opposite direction, is not changed by the action of bodies on one another.*

For an action and the reaction opposite to it are equal by law 3, and thus by law 2 the changes which they produce in motions are equal and in opposite directions. Therefore, if motions are in the same direction, whatever is added to the motion of the first body [*lit.* the fleeing body] will be subtracted from the motion of the second body [*lit.* the pursuing body] in such a way that the sum remains the same as before. But if the bodies meet head-on, the quantity subtracted from each of the motions will be the same, and thus the difference of the motions made in opposite directions will remain the same.

For example, suppose a spherical body A is three times as large as a spherical body B and has two parts of velocity, and let B follow A in the same straight line with ten parts of velocity; then the motion of A is to the motion of B as six to ten. Suppose that their motions are of six parts and ten parts respectively; the sum will be sixteen parts. When the bodies collide, therefore, if body A gains three or four or five parts of motion, body B will lose just as many parts of motion and thus after reflection body A will continue with nine or ten or eleven parts of motion and B with seven or six or five parts of motion, the sum being always, as originally, sixteen parts of motion. Suppose body A gains nine or ten or eleven or twelve parts of motion and so moves forward with fifteen or sixteen or seventeen or eighteen parts of motion after meeting body B; then body B, by losing as many parts of motion as A gains, will either move forward with one part, having lost nine parts of motion, or will be at rest, having lost its forward motion of ten parts, or will move backward with one part of motion, having lost its motion and (if I may say so) one part more, or will move backward with two parts of motion because a forward motion of twelve parts has been subtracted. And thus the sums, $15+1$ or $16+0$, of the motions in the same direction and the differences, 17−1 and 18−2, of the motions in opposite directions will always be sixteen parts of motion, just as before the bodies met and were reflected. And since the motions with which the bodies will continue to move after reflection are known, the velocity of each will be found, on the supposition that it is to the velocity before reflection as the motion after reflection is to the motion before reflection. For example, in the last case, where the motion of body A was six parts before reflection and eighteen parts afterward, and its velocity was two parts before reflection, its velocity will be found to be six parts after reflection on the basis of the following statement: as six parts of motion before reflection is to eighteen parts of motion afterward, so two parts of velocity before reflection is to six parts of velocity afterward.

But if bodies that either are not spherical or are moving in different straight lines strike against each other obliquely and it is required to find their motions after reflection, the position of the plane by which the colliding bodies are touched at the point of collision must be determined; then (by corol. 2) the motion of each body must be resolved into two motions, one perpendicular to this plane and the other parallel to it. Because the bodies act upon each other along a line perpendicular to this plane, the parallel motions [i.e., components] must be kept the same after reflection; and equal changes in opposite directions must be attributed to the perpendicular motions in such a way that the sum of the motions in the same direction and the difference of the motions in opposite directions remain the same as before the bodies came together. The circular motions of bodies about their own centers also generally arise from reflections of this sort. But I do not consider such cases in what follows, and it would be too tedious to demonstrate everything relating to this subject.

Corollary 4

The common center of gravity of two or more bodies does not change its state whether of motion or of rest as a result of the actions of the bodies upon one another; and therefore the common center of gravity of all bodies acting upon one another (excluding external actions and impediments) either is at rest or moves uniformly straight forward.

For if two points move forward with uniform motion in straight lines, and the distance between them is divided in a given ratio, the dividing point either is at rest or moves forward uniformly in a straight line. This is demonstrated below in lem. 23 and its corollary for the case in which the motions of the points take place in the same plane, and it can be demonstrated by the same reasoning for the case in which those motions do not take place in the same plane. Therefore, if any number of bodies move uniformly in straight lines, the common center of gravity of any two either is at rest or moves forward uniformly in a straight line, because any line joining these bodies through their centers—which move forward uniformly in straight lines—is divided by this common center in a given ratio. Similarly, the common center of gravity of these two bodies and any third body either is at rest or moves forward uniformly in a straight line, because the distance between the common center of the two bodies and the center of the third body is divided in a given ratio by the common center of the three. In the same way, the common center of these three and of any fourth body either is at rest or moves forward uniformly in a straight line, because that common center divides in a given ratio the distance between the common center of the three and the center of the fourth body, and so on without end. Therefore, in a system of bodies in which the bodies are entirely free of actions upon one another and of all other actions impressed upon them externally, and in which each body accordingly moves uniformly in its individual straight line, the common center of gravity of them all either is at rest or moves uniformly straight forward.

Further, in a system of two bodies acting on each other, since the distances of their centers from the common center of gravity are inversely as the bodies, the relative motions of these bodies, whether of approaching that center or of receding from it, will be equal. Accordingly, as a result of equal changes in opposite directions in the motions of these bodies, and consequently as a result of the actions of the bodies on each other, that center is neither accelerated nor retarded nor does it undergo any change in its state of motion or of rest. In a system of several bodies, the common center of gravity of any two acting upon each other does not in any way change its state as a result of that action, and the common center of gravity of the rest of the bodies (with which that action has nothing to do) is not affected by that action; the distance between these two centers is divided by the common center of gravity of all the bodies into parts inversely proportional to the total sums of the bodies whose centers they are, and (since those two centers maintain their state of moving or of being at rest) the common center of all maintains its state also—for all these reasons it is obvious that this common center of all never changes its state with respect to motion and rest as a result of the actions of two bodies upon each other. Moreover, in such a system all the actions of bodies upon one another either occur between two bodies or are compounded of such actions between two bodies and therefore never introduce any change in the state of motion or of rest of the common center of all. Thus, since that center either is at rest or moves forward uniformly in some straight line, when the bodies do not act upon one another, that center will, notwithstanding the actions of the bodies upon one another, continue either to be always at rest or to move always uniformly straight forward, unless it is driven from this state by forces impressed on the system from outside. Therefore, the law is the same for a system of several bodies as for a single body with respect to perseverance in a state of motion or of rest. For the progressive motion, whether of a single body or of a system of bodies, should always be reckoned by the motion of the center of gravity.

Corollary 5

When bodies are enclosed in a given space, their motions in relation to one another are the same whether the space is at rest or whether it is moving uniformly straight forward without circular motion.

For in either case the differences of the motions tending in the same direction and the sums of those tending in opposite directions are the same at the beginning (by hypothesis), and from these sums or differences there arise the collisions and impulses [*lit.* impetuses] with which the bodies strike one another. Therefore, by law 2, the effects of the collisions will be equal in both cases, and thus the motions with respect to one another in the one case will remain equal to the motions with respect to one another in the other case. This is proved clearly by experience: on a ship, all the motions are the same with respect to one another whether the ship is at rest or is moving uniformly straight forward.

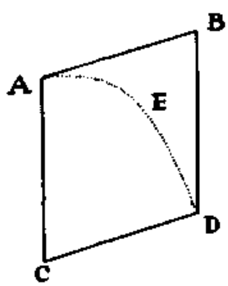
Corollary 6

If bodies are moving in any way whatsoever with respect to one another and are urged by equal accelerative forces along parallel lines, they will all continue to move with respect to one another in the same way as they would if they were not acted on by those forces.

For those forces, by acting equally (in proportion to the quantities of the bodies to be moved) and along parallel lines, will (by law 2) move all the bodies equally (with respect to velocity), and so will never change their positions and motions with respect to one another.

Scholium The principles I have set forth are accepted by mathematicians and confirmed by experiments of many kinds. By means of the first two laws and the first two corollaries Galileo found that the descent of heavy bodies is in the squared ratio of the time and that the motion of projectiles occurs in a parabola, as experiment confirms, except insofar as these motions are somewhat retarded by the resistance of the air. [a]When a body falls, uniform gravity, by acting equally in individual equal particles of time, impresses equal forces upon that body and generates equal velocities; and in the total time it impresses a total force and generates a total velocity proportional to the time. And the spaces described in proportional times are as the velocities and the times jointly, that is, in the squared ratio of the times. And when a body is projected upward, uniform gravity impresses forces and takes away velocities proportional to the times; and the times of ascending to the greatest heights are as the velocities to be taken away, and these heights are as the velocities and the times jointly, or as the squares of the velocities. And when a body is projected along any straight line, its motion arising from the projection is compounded with the motion arising from gravity.

For example, let body A by the motion of projection alone describe the straight line AB in a given time, and by the motion of falling alone describe the vertical distance AC in the same time; then complete the parallelogram ABDC, and by the compounded motion the body will be found in place D at the end of the time; and the curved line AED which the body will describe will be a parabola which the straight line AB touches at A and whose ordinate BD is as AB^2.[a]

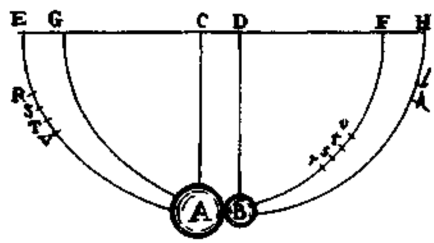


What has been demonstrated concerning the times of oscillating pendulums depends on the same first two laws and first two corollaries, and this is supported by daily experience with clocks. From the same laws and corollaries and law 3, Sir Christopher Wren, Dr. John Wallis, and Mr. Christiaan Huygens, easily the foremost geometers of the previous generation, independently found the rules of the collisions and reflections of hard bodies, and communicated them to the Royal Society at nearly the same time, entirely agreeing with one another (as to these rules); and Wallis was indeed the

aa. Ed. 1 and ed. 2 lack this.

first to publish what had been found, followed by Wren and Huygens. But Wren additionally proved the truth of these rules before the Royal Society by means of an experiment with pendulums, which the eminent Mariotte soon after thought worthy to be made the subject of a whole book.

However, if this experiment is to agree precisely with the theories, account must be taken of both the resistance of the air and the elastic force of the colliding bodies. Let the spherical bodies A and B be suspended from centers C and D by parallel and equal cords AC and BD. With these centers and with those distances as radii describe semicircles EAF and GBH bisected by radii CA and DB. Take away body B, and let body A be brought to any point R of the arc EAF and be let go from there, and let it return after one oscillation to point V. RV is the retardation arising from the resistance of the air. Let ST be a fourth of RV and be located in the middle so that RS and TV are equal and RS is to ST as 3 to 2. Then ST will closely approximate the retardation in the descent from S to A. Restore body B to its original place. Let body A fall from point S, and its velocity at the place of reflection A, without sensible error, will be as great as if it had fallen in a vacuum from place T. Therefore let this velocity be represented by the chord of the arc TA. For it is a proposition very well known to geometers that the velocity of a pendulum in its lowest point is as the chord of the arc that it has described in falling. After reflection let body A arrive at place *s*, and body B at place *k*. Take away body B and find place *v* such that if body A is let go from this place and after one oscillation returns to place *r*, *st* will be a fourth of *rv* and be located in the middle, so that *rs* and *tv* are equal; and let the chord of the arc *t*A represent the velocity that body A had in place A immediately after reflection. For *t* will be that true and correct place to which body A must have ascended if there had been no resistance of the air. By a similar method the place *k*, to which body B ascends, will have to be corrected, and the place *l*, to which that body must have ascended in a vacuum, will have to be found. In this manner it is possible to make all our experiments, just as if we were in a vacuum. Finally body A will have to be multiplied (so to speak) by the chord of the arc TA, which represents its velocity, in order



to get its motion in place A immediately before reflection, and then by the chord of the arc *t*A in order to get its motion in place A immediately after reflection. And thus body B will have to be multiplied by the chord of the arc B*l* in order to get its motion immediately after reflection. And by a similar method, when two bodies are let go simultaneously from different places, the motions of both will have to be found before as well as after reflection, and then finally the motions will have to be compared with each other in order to determine the effects of the reflection.

On making a test in this way with ten-foot pendulums, using unequal as well as equal bodies, and making the bodies come together from very large distances apart, say of eight or twelve or sixteen feet, I always found—within an error of less than three inches in the measurements—that when the bodies met each other directly, the changes of motions made in the bodies in opposite directions were equal, and consequently that the action and reaction were always equal. For example, if body A collided with body B, which was at rest, with nine parts of motion and, losing seven parts, proceeded after reflection with two, body B rebounded with those seven parts. If the bodies met head-on, A with twelve parts of motion and B with six, and A rebounded with two, B rebounded with eight, fourteen parts being subtracted from each. Subtract twelve parts from the motion of A and nothing will remain; subtract another two parts, and a motion of two parts in the opposite direction will be produced; and so, subtracting fourteen parts from the six parts of the motion of body B, eight parts will be produced in the opposite direction. But if the bodies moved in the same direction, A more quickly with fourteen parts and B more slowly with five parts, and after reflection A moved with five parts, then B moved with fourteen, nine parts having been transferred from A to B. And so in all other cases. As a result of the meeting and collision of bodies, the quantity of motion—determined by adding the motions in the same direction and subtracting the motions in opposite directions—was never changed. I would attribute the error of an inch or two in the measurements to the difficulty of doing everything with sufficient accuracy. It was difficult both to release the pendulums simultaneously in such a way that the bodies would impinge upon each other in the lowest place AB, and to note the places *s* and *k* to which the bodies ascended after colliding. But also, with respect to the pendulous bodies themselves, errors were introduced by the unequal density of the parts and by irregularities of texture arising from other causes.

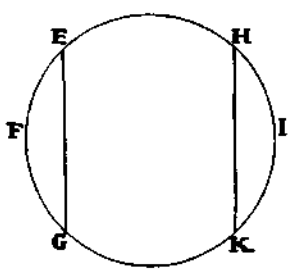
Further, lest anyone object that the rule which this experiment was designed to prove presupposes that bodies are either absolutely hard or at least perfectly elastic and thus of a kind which do not occur [b]naturally,[b] I add that the experiments just described work equally well with soft bodies and with hard ones, since surely they do not in any way depend on the condition of hardness. For if this rule is to be tested in bodies that are not perfectly hard, it will only be necessary to decrease the reflection in a fixed proportion to the quantity of elastic force. In the theory of Wren and Huygens, absolutely hard bodies rebound from each other with the velocity with which they have collided. This will be affirmed with more certainty of perfectly elastic bodies. In imperfectly elastic bodies the velocity of rebounding must be decreased together with the elastic force, because that force (except when the parts of the bodies are damaged as a result of collision, or experience some sort of extension such as would be caused by a hammer blow) is fixed and determinate (as far as I can tell) and makes the bodies rebound from each other with a relative velocity that is in a given ratio to the relative velocity with which they collide. I have tested this as follows with tightly wound balls of wool strongly compressed. First, releasing the pendulums and measuring their reflection, I found the quantity of their elastic force; then from this force I determined what the reflections would be in other cases of their collision, and the experiments which were made agreed with the computations. The balls always rebounded from each other with a relative velocity that was to the relative velocity of their colliding as 5 to 9, more or less. Steel balls rebounded with nearly the same velocity and cork balls with a slightly smaller velocity, while with glass balls the proportion was roughly 15 to 16. And in this manner the third law of motion—insofar as it relates to impacts and reflections—is proved by this theory, which plainly agrees with experiments.

I demonstrate the third law of motion for attractions briefly as follows. Suppose that between any two bodies A and B that attract each other any obstacle is interposed so as to impede their coming together. If one body A is more attracted toward the other body B than that other body B is attracted toward the first body A, then the obstacle will be more strongly pressed by body A than by body B and accordingly will not remain in equilibrium. The stronger pressure will prevail and will make the system of the two bodies and

bb. Evidently "in natural compositions" or "in natural bodies."

the obstacle move straight forward in the direction from A toward B and, in empty space, go on indefinitely with a motion that is always accelerated, which is absurd and contrary to the first law of motion. For according to the first law, the system will have to persevere in its state of resting or of moving uniformly straight forward, and accordingly the bodies will urge the obstacle equally and on that account will be equally attracted to each other. I have tested this with a lodestone and iron. If these are placed in separate vessels that touch each other and float side by side in still water, neither one will drive the other forward, but because of the equality of the attraction in both directions they will sustain their mutual endeavors toward each other, and at last, having attained equilibrium, they will be at rest.

[c]In the same way gravity is mutual between the earth and its parts. Let the earth FI be cut by any plane EG into two parts EGF and EGI; then their weights toward each other will be equal. For if the greater part EGI is cut into two parts EGKH and HKI by another plane HK parallel to the first plane EG, in such a way that HKI is equal to the part EFG that has been cut off earlier, it is manifest that the middle part EGKH will not preponderate toward either of the outer parts but will, so to speak, be suspended in equilibrium between both and will be at rest. Moreover, the outer part HKI will press upon the middle part with all its weight and will urge it toward the other outer part EGF, and therefore the force by which EGI, the sum of the parts HKI and EGKH, tends toward the third part EGF is equal to the weight of the part HKI, that is, equal to the weight of the third part EGF. And therefore the weights of the two parts EGI and EGF toward each other are equal, as I set out to demonstrate. And if these weights were not equal, the whole earth, floating in an aether free of resistance, would yield to the greater weight and in receding from it would go off indefinitely.[c]



As bodies are equipollent in collisions and reflections if their velocities are inversely as their inherent forces [i.e., forces of inertia], so in the motions of machines those agents [i.e., acting bodies] whose velocities (reckoned in the direction of their forces) are inversely as their inherent forces are equipol-

cc. Ed. 1 lacks this.

lent and sustain one another by their contrary endeavors. Thus weights are equipollent in moving the arms of a balance if during oscillation of the balance they are inversely as their velocities upward and downward; that is, weights which move straight up and down are equipollent if they are inversely as the distances between the axis of the balance and the points from which they are suspended; but if such weights are interfered with by oblique planes or other obstacles that are introduced and thus ascend or descend obliquely, they are equipollent if they are inversely as the ascents and descents insofar as these are reckoned with respect to a perpendicular, and this is so because the direction of gravity is downward. Similarly, in a pulley or combination of pulleys, the weight will be sustained by the force of the hand pulling the rope vertically, which is to the weight (ascending either straight up or obliquely) as the velocity of the perpendicular ascent to the velocity of the hand pulling the rope. In clocks and similar devices, which are constructed out of engaged gears, the contrary forces that promote and hinder the motion of the gears will sustain each other if they are inversely as the velocities of the parts of the gears upon which they are impressed. The force of a screw to press a body is to the force of a hand turning the handle as the circular velocity of the handle, in the part where it is urged by the hand, is to the progressive velocity of the screw toward the pressed body. The forces by which a wedge presses the two parts of the wood that it splits are to the force of the hammer upon the wedge as the progress of the wedge (in the direction of the force impressed upon it by the hammer) is to the velocity with which the parts of the wood yield to the wedge along lines perpendicular to the faces of the wedge. And the case is the same for all machines.

The effectiveness and usefulness of all machines or devices consist wholly in our being able to increase the force by decreasing the velocity, and vice versa; in this way the problem is solved in the case of any working machine or device: "To move a given weight by a given force" or to overcome any other given resistance by a given force. For if machines are constructed in such a way that the velocities of the agent [or acting body] and the resistant [or resisting body] are inversely as the forces, the agent will sustain the resistance and, if there is a greater disparity of velocities, will overcome that resistance. Of course the disparity of the velocities may be so great that it can also overcome all the resistance which generally arises from the friction of contiguous bodies sliding over one another, from the cohesion of continuous bodies that are to be separated from one another, or from the weights of bodies to be raised; and if all this resistance is overcome, the remaining force will produce an acceleration of motion proportional to itself, partly in the parts of the machine, partly in the resisting body.[d]

But my purpose here is not to write a treatise on mechanics. By these examples I wished only to show the wide range and the certainty of the third law of motion. For if the action of an agent is reckoned by its force and velocity jointly, and if, similarly, the reaction of a resistant is reckoned jointly by the velocities of its individual parts and the forces of resistance arising from their friction, cohesion, weight, and acceleration, the action and reaction will always be equal to each other in all examples of using devices or machines. And to the extent to which the action is propagated through the machine and ultimately impressed upon each resisting body, its ultimate direction will always be opposite to the direction of the reaction.

d. Newton writes of "instrumentorum" (literally, "equipment") and of "instrumentis mechanicis" (literally, "mechanical instruments"), as well as "machinae." See §5.7 of the Guide.

BOOK 1

THE MOTION OF BODIES

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SECTION 1

The method of first and ultimate ratios, for use in demonstrating what follows

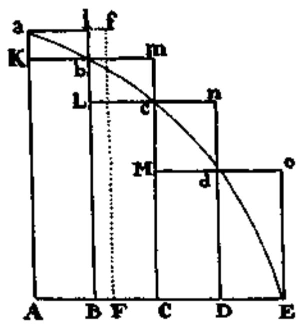
Lemma 1

Quantities, and also ratios of quantities, which in [a]*any finite time*[a] *constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal.*

If you deny this, [b]let them become ultimately unequal, and[b] let their ultimate difference be D. Then they cannot approach so close to equality that their difference is less than the given difference D, contrary to the hypothesis.

Lemma 2

If in any figure A*ac*E, *comprehended by the straight lines* A*a and* AE *and the curve ac*E, *any number of parallelograms* A*b*, B*c*, C*d*, . . . *are inscribed upon equal bases* AB, BC, CD, . . . *and have sides* B*b*, C*c*, D*d*, . . . *parallel to the side* A*a of the figure; and if the parallelograms a*K*bl*, *b*L*cm*, *c*M*dn*, . . . *are completed; if then the width of these parallelograms is diminished and their number increased indefinitely, I say that the ultimate ratios which the inscribed figure* AK*b*L*c*M*d*D, *the circumscribed figure* A*albmcndo*E, *and the curvilinear figure* A*abcd*E *have to one another are ratios of equality.*



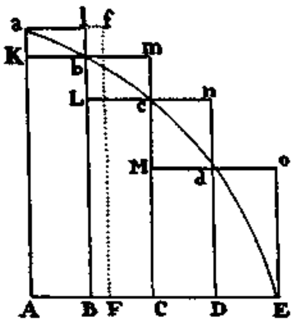
For the difference of the inscribed and circumscribed figures is the sum of the parallelograms K*l*, L*m*, M*n*, and D*o*, that is (because they all have equal bases), the rectangle having as base K*b* (the base of one of them) and as altitude A*a* (the sum of the altitudes), that is, the rectangle AB*la*. But this rectangle, because its width AB is diminished indefinitely, becomes less than any given rectangle. Therefore (by lem. 1) the inscribed figure and the circumscribed figure and, all the more, the intermediate curvilinear figure become ultimately equal. Q.E.D.

Lemma 3

The same ultimate ratios are also ratios of equality when the widths AB, BC, CD, . . . *of the parallelograms are unequal and are all diminished indefinitely.*

aa. Ed. 1 has "a given time."

bb. Ed. 1 lacks this.



For let AF be equal to the greatest width, and let the parallelogram FAaf be completed. This parallelogram will be greater than the difference of the inscribed and the circumscribed figures; but if its width AF is diminished indefinitely, it will become less than any given rectangle. Q.E.D.

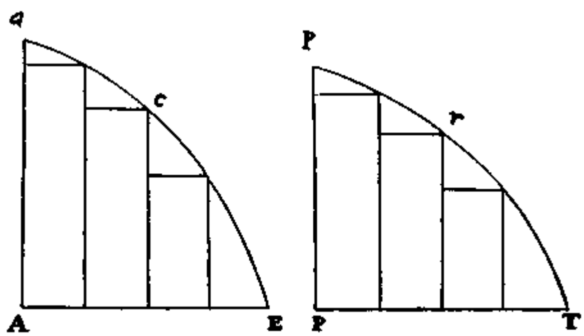
COROLLARY 1. Hence the ultimate sum of the vanishing parallelograms coincides with the curvilinear figure in its every part.

COROLLARY 2. And, all the more, the rectilinear figure that is comprehended by the chords of the vanishing arcs ab, bc, cd, . . . coincides ultimately with the curvilinear figure.

COROLLARY 3. And it is the same for the circumscribed rectilinear figure that is comprehended by the tangents of those same arcs.

COROLLARY 4. And therefore these ultimate figures (with respect to their perimeters acE) are not rectilinear, but curvilinear limits of rectilinear figures.

Lemma 4 *If in two figures* AacE *and* PprT *two series of parallelograms are inscribed (as above) and the number of parallelograms in both series is the same; and if, when their widths are diminished indefinitely, the ultimate ratios of the parallelograms in one figure to the corresponding parallelograms in the other are the same; then I say that the two figures* AacE *and* PprT *are to each other in that same ratio.*



For as the individual parallelograms in the one figure are to the corresponding individual parallelograms in the other, so (by composition [or componendo]) will the sum of all the parallelograms in the one become to the sum of all the parallelograms in the other, and so also the one figure to the other—the first figure, of course, being (by lem. 3) to the first sum, and the second figure to the second sum, in a ratio of equality. Q.E.D.

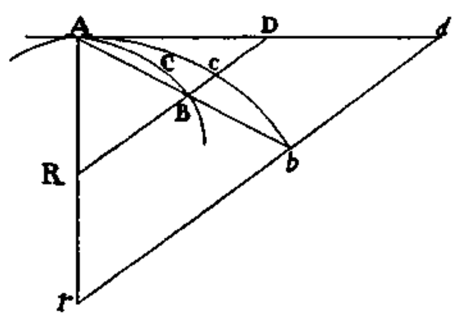
COROLLARY. Hence, if two quantities of any kind are divided in any way into the same number of parts, and these parts—when their number is increased and their size is diminished indefinitely—maintain a given ratio to one another, the first to the first, the second to the second, and so on in sequence, then the wholes will be to each other in the same given ratio. For if the parallelograms in the figures of this lemma are taken in the same proportion to one another as those parts, the sums of the parts will always be as the sums of the parallelograms; and therefore, when the number of parts and parallelograms is increased and their size diminished indefinitely, the sums of the parts will be in the ultimate ratio of a parallelogram in one figure to a corresponding parallelogram in the other, that is (by hypothesis), in the ultimate ratio of part to part.

Lemma 5

All the mutually corresponding sides—curvilinear as well as rectilinear—of similar figures are proportional, and the areas of such figures are as the squares of their sides.

Lemma 6

If any arc ACB, *given in position, is subtended by the chord* AB *and at some point* A, *in the middle of the continuous curvature, is touched by the straight line* AD, *produced in both directions, and if then points* A *and* B *approach each other and come together, I say that the angle* BAD *contained by the chord and the tangent will be indefinitely diminished and will ultimately vanish.*

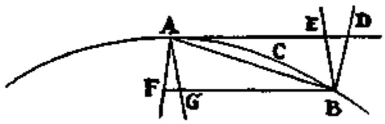


For [a]if that angle does not vanish, the angle contained by the arc ACB and the tangent AD will be equal to a rectilinear angle, and therefore the curvature at point A will not be continuous, contrary to the hypothesis.[a]

aa. Ed. 1 has "produce AB to *b* and AD to *d*; then, since points A and B come together and thus no part AB of A*b* still lies within the curve, it is obvious that this straight line A*b* will either coincide with the tangent A*d* or be drawn between the tangent and the curve. But the latter case is contrary to the nature of curvature; therefore, the former obtains. Q.E.D."

Lemma 7 *With the same suppositions, I say that the ultimate ratios of the arc, the chord, and the tangent to one another are ratios of equality.*

For while point B approaches point A, let AB and AD be understood always to be produced to the distant points *b* and *d*; and let *bd* be drawn parallel to secant BD. And let arc A*cb* be always similar to arc ACB. Then as points A and B come together, the angle *d*A*b* will vanish (by lem. 6), and thus the straight lines A*b* and A*d* (which are always finite) and the intermediate arc A*cb* will coincide and therefore will be equal. Hence, the straight lines AB and AD and the intermediate arc ACB (which are always proportional to the lines A*b* and A*d* and the arc A*cb* respectively) will also vanish and will have to one another an ultimate ratio of equality. Q.E.D.

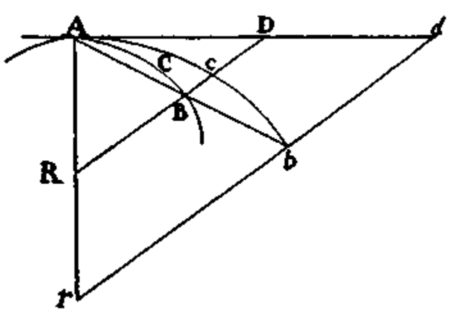


COROLLARY 1. Hence, if BF is drawn through B parallel to the tangent and always cutting at F any straight line AF passing through A, then BF will ultimately have a ratio of equality to the vanishing arc ACB, because, if parallelogram AFBD is completed, BF always has a ratio of equality to AD.

COROLLARY 2. And if through B and A additional straight lines BE, BD, AF, and AG are drawn cutting the tangent AD and its parallel BF, the ultimate ratios of all the abscissas AD, AE, BF, and BG and of the chord and arc AB to one another will be ratios of equality.

COROLLARY 3. And therefore all these lines can be used for one another interchangeably in any argumentation concerning ultimate ratios.

Lemma 8 *If the given straight lines* AR *and* BR, *together with the arc* ACB, *its chord* AB, *and the tangent* AD, *constitute three triangles* RAB, RACB, *and* RAD, *and if then points* A *and* B *approach each other, I say that the triangles as they vanish are similar in their ultimate form, and that their ultimate ratio is one of equality.*



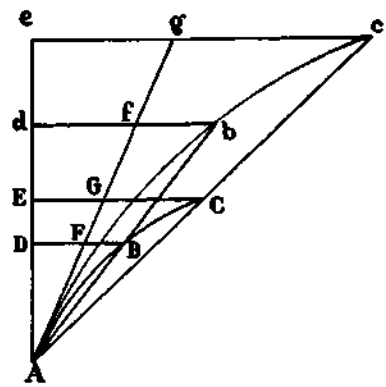
For while point B approaches point A, let AB, AD, and AR be understood always to be produced to the distant points *b*, *d*, and *r*, and *rbd* to be drawn parallel to RD; and let arc A*cb* be always similar to arc ACB. Then as points A and B come together, the angle *b*A*d* will vanish, and therefore the three triangles rAb, $rAcb$, and rAd, which are always finite, will coincide and on that account are similar and equal. Hence also RAB, RACB, and RAD, which will always be similar and proportional to these, will ultimately become similar and equal to one another. Q.E.D.

COROLLARY. And hence those triangles can be used for one another interchangeably in any argumentation concerning ultimate ratios.

Lemma 9

If the straight line AE *and the curve* ABC, *both given in position, intersect each other at a given angle* A, *and if* BD *and* CE *are drawn as ordinates to the straight line* AE *at another given angle and meet the curve in* B *and* C, *and if then points* B *and* C *simultaneously approach point* A, *I say that the areas of the triangles* ABD *and* ACE *will ultimately be to each other as the squares of the sides.*

For while points B and C approach point A, let AD be understood always to be produced to the distant points d and e, so that Ad and Ae are proportional to AD and AE; and erect ordinates db and ec parallel to ordinates DB and EC and meeting AB and AC, produced, at b and c. Understand the curve Abc to be drawn similar to ABC, and the straight line Ag to be drawn touching both curves at A and cutting the ordinates DB, EC, db, and ec at F, G, f, and g. Then, with the length Ae remaining the same, let points B and C come together with point A; and as the angle cAg vanishes, the curvilinear areas Abd and Ace will coincide with the rectilinear areas Afd and Age, and thus (by lem. 5) will be in the squared ratio of the sides Ad and Ae. But areas ABD and ACE are always proportional to these areas, and sides AD and AE to these sides. Therefore areas ABD and ACE also are ultimately in the squared ratio of the sides AD and AE. Q.E.D.



Lemma 10

The spaces which a body describes when urged by any [a]*finite*[a] *force,* [b]*whether that force is determinate and immutable or is continually increased or continually*

aa. Ed. 1 has "regular."
bb. Ed. 1 lacks this.

decreased,[b] *are at the very beginning of the motion in the squared ratio of the times.*

Let the times be represented by lines AD and AE, and the generated velocities by ordinates DB and EC; then the spaces described by these velocities will be as the areas ABD and ACE described by these ordinates, that is, at the very beginning of the motion these spaces will be (by lem. 9) in the squared ratio of the times AD and AE. Q.E.D.

COROLLARY 1. And hence it is easily concluded that when bodies describe similar parts of similar figures in proportional times, the errors that are generated by any equal forces similarly applied to the bodies, and that are measured by the distances of the bodies from those points on the similar figures at which the same bodies would arrive in the same proportional times without these forces, are very nearly as the squares of the times in which they are generated.

COROLLARY 2. But the errors that are generated by proportional forces similarly applied to similar parts of similar figures are as the forces and the squares of the times jointly.

[c]COROLLARY 3. The same is to be understood of any spaces which bodies describe when different forces urge them. These spaces are, at the very beginning of the motion, as the forces and the squares of the times jointly.

COROLLARY 4. And thus the forces are as the spaces described at the very beginning of the motion directly and as the squares of the times inversely.

COROLLARY 5. And the squares of the times are directly as the spaces described and inversely as the forces.

Scholium If indeterminate quantities of different kinds are compared with one another and any one of them is said to be directly or inversely as any other, the meaning is that the first one is increased or decreased in the same ratio as the second or as its reciprocal. And if any one of them is said to be as two or more others, directly or inversely, the meaning is that the first is increased or decreased in a ratio that is compounded of the ratios in which the others, or the reciprocals of the others, are increased or decreased. For example, if A is said to be as B directly and C directly and D inversely, the meaning is

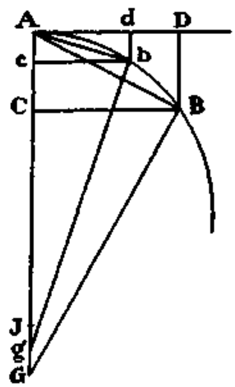
cc. Ed. 1 lacks corols. 3–5 and scholium.

that A is increased or decreased in the same ratio as $B \times C \times \frac{1}{D}$, that is, that A and $\frac{BC}{D}$ are to each other in a given ratio.[c]

Lemma 11

In all curves having a finite curvature at the point of contact, the vanishing subtense of the angle of contact is ultimately in the squared ratio of the subtense of the conterminous arc.

Case 1. Let AB be the arc, AD its tangent, BD the subtense of the angle of contact perpendicular to the tangent [angle BAD], and [the line] AB the subtense [i.e., the conterminous chord] of the arc [AB]. Erect BG and AG perpendicular to this subtense AB and tangent AD and meeting in G; then let points D, B, and G approach points d, b, and g, and let J be the intersection of lines BG and AG, which ultimately occurs when points D and B reach A. It is evident that the distance GJ can be less than any assigned distance. And (from the nature of the circles passing through points A, B, G and a, b, g) AB^2 is equal to $AG \times BD$, and Ab^2 is equal to $Ag \times bd$, and thus the ratio of AB^2 to Ab^2 is compounded of the ratios of AG to Ag and BD to bd. But since GJ can be taken as less than any assigned length, it can happen that the ratio of AG to Ag differs from the ratio of equality by less than any assigned difference, and thus that the ratio of AB^2 to Ab^2 differs from the ratio of BD to bd by less than any assigned difference. Therefore, by lem. 1, the ultimate ratio of AB^2 to Ab^2 is the same as the ultimate ratio of BD to bd. Q.E.D.



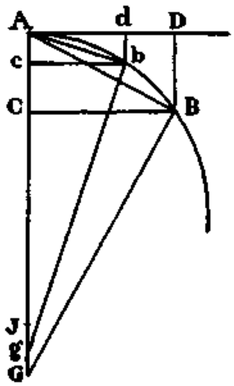
Case 2. Now let BD be inclined to AD at any given angle, and the ultimate ratio of BD to bd will always be the same as before and thus the same as AB^2 to Ab^2. Q.E.D.

Case 3. And even when angle D is not given, if the straight line BD converges to a given point or is drawn according to any other specification, still the angles D and d (constructed according to the specification common to both) will always tend to equality and will approach each other so closely that their difference will be less than any assigned quantity, and thus will ultimately be equal, by lem. 1; and therefore lines BD and bd are in the same ratio to each other as before. Q.E.D.

COROLLARY 1. Hence, since tangents AD and Ad, arcs AB and Ab, and their sines BC and bc become ultimately equal to chords AB and Ab, their squares will also be ultimately as the subtenses BD and bd.

[a]COROLLARY 2. The squares of these tangents, arcs, and sines are also ultimately as the sagittas of the arcs, which bisect the chords and converge to a given point. For these sagittas are as the subtenses BD and bd.

COROLLARY 3. And thus the sagitta is in the squared ratio of the time in which a body describes the arc with a given velocity.[a]



COROLLARY 4. The rectilinear triangles ADB and Adb are ultimately in the cubed ratio of the sides AD and Ad, and in the sesquialteral ratio [i.e., as the 3/2 power] of the sides DB and db, inasmuch as these triangles are in a ratio compounded of the ratios of AD and DB to Ad and db. So also the triangles ABC and Abc are ultimately in the cubed ratio of the sides BC and bc. [b]The ratio that I call sesquialteral is the halved of the tripled, namely, the one that is compounded of the simple and the halved.[b]

COROLLARY 5. And since DB and db are ultimately parallel and in the squared ratio of AD and Ad, the ultimate curvilinear areas ADB and Adb will be (from the nature of the parabola) two-thirds of the rectilinear triangles ADB and Adb; and the segments AB and Ab will be thirds of these same triangles. And hence these areas and segments will be in the cubed ratio of both of the tangents AD and Ad and of the chords AB and Ab and their arcs.

Scholium But we suppose throughout that the angle of contact is neither infinitely greater nor infinitely less than the angles of contact that circles contain with their tangents, that is, that the curvature at point A is neither infinitely small nor infinitely great—in other words, that the distance AJ is of a finite magnitude. For DB can be taken proportional to AD^3, in which case no circle can be drawn through point A between tangent AD and curve AB, and accordingly the angle of contact will be infinitely less than those of

aa. Ed. 1 lacks corols. 2 and 3.

bb. Ed. 1 lacks this sentence.

circles. And, similarly, if DB is made successively proportional to AD^4, AD^5, AD^6, AD^7, . . . , there will be a sequence of angles of contact going on to infinity, any succeeding one of which is infinitely less than the preceding one. And if DB is made successively proportional to AD^2, $AD^{3/2}$, $AD^{4/3}$, $AD^{5/4}$, $AD^{6/5}$, $AD^{7/6}$, . . . , there will be another infinite sequence of angles of contact, the first of which is of the same kind as those of circles, the second infinitely greater, and any succeeding one infinitely greater than the preceding one. Moreover, between any two of these angles a sequence of intermediate angles, going on to infinity in both directions, can be inserted, any succeeding one of which will be infinitely greater or smaller than the preceding one—as, for example, if between the terms AD^2 and AD^3 there were inserted the sequence $AD^{13/6}$, $AD^{11/5}$, $AD^{9/4}$, $AD^{7/3}$, $AD^{5/2}$, $AD^{8/3}$, $AD^{11/4}$, $AD^{14/5}$, $AD^{17/6}$, ... And again, between any two angles of this sequence a new sequence of intermediate angles can be inserted, differing from one another by infinite intervals. And nature knows no limit.

What has been demonstrated concerning curved lines and the [plane] surfaces comprehended by them is easily applied to curved surfaces and their solid contents. In any case, I have presented these lemmas before the propositions in order to avoid the tedium of working out [c]lengthy[c] proofs by *reductio ad absurdum* in the manner of the ancient geometers. Indeed, proofs are rendered more concise by the method of indivisibles. But since the hypothesis of indivisibles is [d]problematical[d] and this method is therefore accounted less geometrical, I have preferred to make the proofs of what follows depend on the ultimate sums and ratios of vanishing quantities and the first sums and ratios of nascent quantities, that is, on the limits of such sums and ratios, and therefore to present proofs of those limits beforehand as briefly as I could. For the same result is obtained by these as by the method of indivisibles, and we shall be on safer ground using principles that have been proved. Accordingly, whenever in what follows I consider quantities as consisting of particles or whenever I use curved line-elements [or minute curved lines] in

cc. For "lengthy" (Lat. "longas") ed. 1 and ed. 2 have "complicated" (Lat. "perplexas"), which Newton inserted with his own hand into the manuscript of ed. 1. Motte gives "perplexed," thus obviously using ed. 2, and Cajori has "involved," revealing that the Latin text was not consulted at this point. But in *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse* (Chicago and London: Open Court Publishing Co., 1919), Cajori notes on p. 5 that "in the third edition 'longas' takes the place of 'perplexas,' " and on p. 8 he uses Thorp's translation ("long").

dd. Newton uses the adjective "durior," which is traditionally translated by "rather harsh."

place of straight lines, I wish it always to be understood that I have in mind not indivisibles but evanescent divisibles, and not sums and ratios of definite parts but the limits of such sums and ratios, and that the force of such proofs always rests on the method of the preceding lemmas.

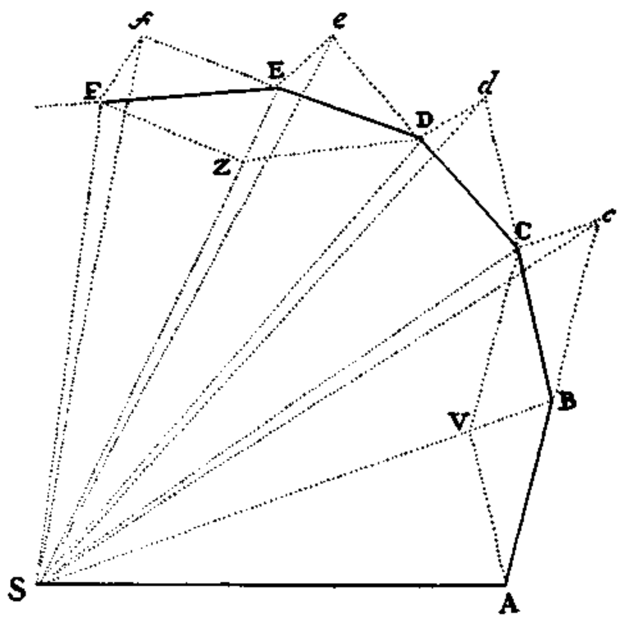
It may be objected that there is no such thing as an ultimate proportion of vanishing quantities, inasmuch as before vanishing the proportion is not ultimate, and after vanishing it does not exist at all. But by the same argument it could equally be contended that there is no ultimate velocity of a body reaching a certain place at which the motion ceases; for before the body arrives at this place, the velocity is not the ultimate velocity, and when it arrives there, there is no velocity at all. But the answer is easy: to understand the ultimate velocity as that with which a body is moving, neither before it arrives at its ultimate place and the motion ceases, nor after it has arrived there, but at the very instant when it arrives, that is, the very velocity with which the body arrives at its ultimate place and with which the motion ceases. And similarly the ultimate ratio of vanishing quantities is to be understood not as the ratio of quantities before they vanish or after they have vanished, but the ratio with which they vanish. Likewise, also, the first ratio of nascent quantities is the ratio with which they begin to exist [or come into being]. And the first and the ultimate sum is the sum with which they begin and cease to exist (or to be increased or decreased). There exists a limit which their velocity can attain at the end of the motion, but cannot exceed. This is their ultimate velocity. And it is the same for the limit of all quantities and proportions that come into being and cease existing. And since this limit is certain and definite, the determining of it is properly a geometrical problem. But everything that is geometrical is legitimately used in determining and demonstrating whatever else may be geometrical.

It can also be contended that if the ultimate ratios of vanishing quantities are given, their ultimate magnitudes will also be given; and thus every quantity will consist of indivisibles, contrary to what Euclid had proved concerning incommensurables in the tenth book of his *Elements*. But this objection is based on a false hypothesis. Those ultimate ratios with which quantities vanish are not actually ratios of ultimate quantities, but limits which the ratios of quantities decreasing without limit are continually approaching, and which they can approach so closely that their difference is less than any given quantity, but which they can never exceed and can never reach before the quantities are decreased indefinitely. This matter will be understood more clearly in the case of quantities that are indefinitely great. If two quantities whose difference is given are increased indefinitely, their ultimate ratio will be given, namely the ratio of equality, and yet the ultimate or maximal quantities of which this is the ratio will not on this account be given. Therefore, whenever, to make things easier to comprehend, I speak in what follows of quantities as minimally small or vanishing or ultimate, take care not to understand quantities that are determinate in magnitude, but always think of quantities that are to be decreased without limit.

SECTION 2

To find centripetal forces

Proposition 1[a] **Theorem 1** *The areas which bodies* [b]*made to move in orbits*[b] *describe by radii drawn to an unmoving center of forces lie in unmoving planes and are proportional to the times.*



Let the time be divided into equal parts, and in the first part of the time let a body by its inherent force describe the straight line AB. In the second part of the time, if nothing hindered it, this body would (by law 1) go straight on to *c*, describing line B*c* equal to AB, so that—when radii AS, BS, and *c*S were drawn to the center—the equal areas ASB and BS*c* would be described. But when the body comes to B, let a centripetal force act with a single but great impulse and make the body deviate from the straight line B*c* and proceed in the straight line BC. Let *c*C be drawn parallel to BS and meet BC at C; then, when the second part of the time has been completed, the body (by corol. 1 of the laws) will be found at C in the same plane as

a. For a gloss on this proposition see the Guide, §10.8.

bb. In the statement of prop. 1, Newton uses the phrase "in gyros acta"; see the Guide, §2.4.

triangle ASB. Join SC; and because SB and C*c* are parallel, triangle SBC will be equal to triangle SB*c* and thus also to triangle SAB. By a similar argument, if the centripetal force acts successively at C, D, E, . . . , making the body in each of the individual particles of time describe the individual straight lines CD, DE, EF, . . . , all these lines will lie in the same plane; and triangle SCD will be equal to triangle SBC, SDE to SCD, and SEF to SDE. Therefore, in equal times equal areas are described in an unmoving plane; and by composition [or componendo], any sums SADS and SAFS of the areas are to each other as the times of description. Now let the number of triangles be increased and their width decreased indefinitely, and their ultimate perimeter ADF will (by lem. 3, corol. 4) be a curved line; and thus the centripetal force by which the body is continually drawn back from the tangent of this curve will act uninterruptedly, while any areas described, SADS and SAFS, which are always proportional to the times of description, will be proportional to those times in this case. Q.E.D.

[c]COROLLARY 1. In nonresisting spaces, the velocity of a body attracted to an immobile center is inversely as the perpendicular dropped from that center to the straight line which is tangent to the orbit. For the velocities in those places A, B, C, D, and E are respectively as the bases of the equal triangles AB, BC, CD, DE, and EF, and these bases are inversely as the perpendiculars dropped to them.

COROLLARY 2. If chords AB and BC of two arcs successively described by the same body in equal times in nonresisting spaces are completed into the parallelogram ABCV, and diagonal BV (in the position that it ultimately has when those arcs are decreased indefinitely) is produced in both directions, it will pass through the center of forces.[c]

[d]COROLLARY 3. If chords AB, BC and DE, EF of arcs described in equal times in nonresisting spaces are completed into parallelograms ABCV and DEFZ, then the forces at B and E are to each other in the ultimate ratio of the diagonals BV and EZ when the arcs are decreased indefinitely. For the motions BC and EF of the body are (by corol. 1 of the laws) compounded of the motions B*c*, BV and E*f*, EZ; but in the proof of this

cc. In ed. 1, corols. 1 and 2 are earlier versions of prop. 2, corols. 1 and 2, and the corols. 1 and 2 of ed. 2 and ed. 3 are lacking.

dd. Ed. 1 lacks corols. 3–6.

proposition BV and EZ, equal to C*c* and F*f*, were generated by the impulses of the centripetal force at B and E, and thus are proportional to these impulses.

Corollary 4. The forces by which any bodies in nonresisting spaces are drawn back from rectilinear motions and are deflected into curved orbits are to one another as those sagittas of arcs described in equal times which converge to the center of forces and bisect the chords when the arcs are decreased indefinitely. For these sagittas are halves of the diagonals with which we dealt in corol. 3.

Corollary 5. And therefore these forces are to the force of gravity as these sagittas are to the sagittas, perpendicular to the horizon, of the parabolic arcs that projectiles describe in the same time.

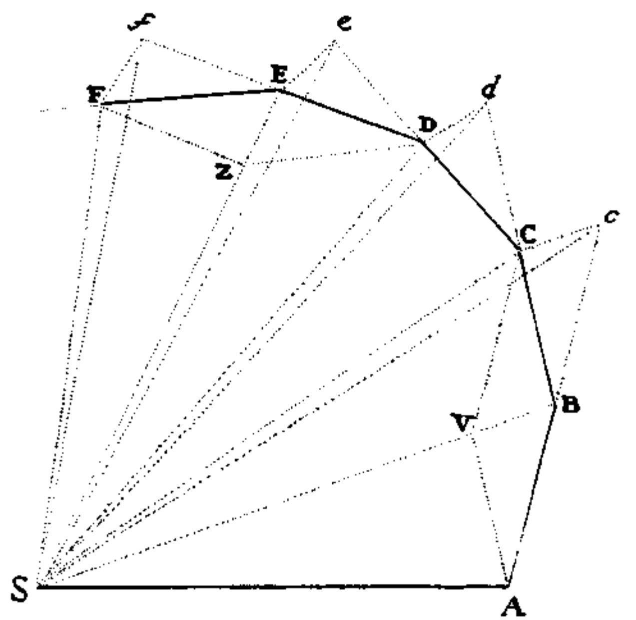
Corollary 6. All the same things hold, by corol. 5 of the laws, when the planes in which the bodies are moving, together with the centers of forces that are situated in those planes, are not at rest but move uniformly straight forward.[d]

Proposition 2
Theorem 2

Every body that moves in some curved line described in a plane and, by a radius drawn to a point, either unmoving or moving uniformly forward with a rectilinear motion, describes areas around that point proportional to the times, is urged by a centripetal force tending toward that same point.

Case 1. For every body that moves in a curved line is deflected from a rectilinear course by some force acting upon it (by law 1). And that force by which the body is deflected from a rectilinear course and in equal times is made to describe, about an immobile point S, the equal minimally small triangles SAB, SBC, SCD, . . . , acts in place B along a line parallel to *c*C (by book 1, prop. 40, of the *Elements*, and law 2), that is, along the line BS; and in place C, the force acts along a line parallel to *d*D, that is, along the line SC, ... Therefore it always acts along lines tending toward that unmoving point S. Q.E.D.

Case 2. And, by corol. 5 of the laws, it makes no difference whether the surface on which the body describes a curvilinear figure is at rest or whether it moves uniformly straight forward, together with the body, the figure described, and the point S.



[a]Corollary 1. [b]In nonresisting spaces or mediums, if the areas are not proportional to the times, the forces do not tend toward the point where the radii meet but deviate forward [or in consequentia] from it, that is, in the direction toward which the motion takes place, provided that the description of the areas is accelerated; but if it is retarded, they deviate backward [or in antecedentia, i.e., in a direction contrary to that in which the motion takes place].[b]

Corollary 2. [c]In resisting mediums also, if the description of areas is accelerated, the directions of the forces deviate from the point where the radii meet in the direction toward which the motion takes place.[a] [c]

Scholium

A body can be urged by a centripetal force compounded of several forces. In this case the meaning of the proposition is that the force which is compounded of all the forces tends toward point S. Further, if some force acts

aa. In ed. 1, prop. 2 has no corollaries. Corols. 1 and 2 of ed. 2 and ed. 3 are basically revised versions of corols. 1 and 2 to prop. 1 of ed. 1.

bb. Ed. 1 has (as prop. 1, corol. 1): "In nonresisting mediums, if the areas are not proportional to the times, the forces do not tend toward the point where the radii meet."

cc. Ed. 1 has (as prop. 1, corol. 2): "In all mediums, if the description of areas is accelerated, the forces do not tend toward the point where the radii meet but deviate forward [or in consequentia] from it."

continually along a line perpendicular to the surface described, it will cause the body to deviate from the plane of its motion, but it will neither increase nor decrease the quantity of the surface-area described and is therefore to be ignored in the compounding of forces.

Proposition 3
Theorem 3

[a]*Every body that, by a radius drawn to the center of a second body moving in any way whatever, describes about that center areas that are proportional to the times is urged by a force compounded of the centripetal force tending toward that second body and of the whole accelerative force by which that second body is urged.*

Let the first body be L, and the second body T; and (by corol. 6 of the laws) if each of the two bodies is urged along parallel lines by a new force that is equal and opposite to the force by which body T is urged, body L will continue to describe about body T the same areas as before; but the force by which body T was urged will now be annulled by an equal and opposite force, and therefore (by law 1) body T, now left to itself, either will be at rest or will move uniformly straight forward; and body L, since the difference of the forces [i.e., the remaining force] is urging it, will continue to describe areas proportional to the times about body T. Therefore, the difference of the forces tends (by theor. 2) toward the second body T as center. Q.E.D.

Corollary 1. Hence, if a body L, by a radius drawn to another body T, describes areas proportional to the times, and from the total force (whether simple or compounded of several forces according to corol. 2 of the laws) by which body L is urged there is subtracted (according to the same corol. 2 of the laws) the total accelerative force by which body T is urged, the whole remaining force by which body L is urged will tend toward body T as center.

Corollary 2. And if the areas are very nearly proportional to the times, the remaining force will tend toward body T very nearly.

aa. In both the statement and the demonstration of the proposition and also in the corollaries, ed. 1 lacks letters to designate the two bodies. In Newton's annotated copy of ed. 1, the letters L and T are added in all of these parts of the proposition. In Newton's interleaved copy of ed. 1, letters are added in all of these sections but are then deleted from the statement of the proposition, where the letters written in might have first been A and B and then been changed to L and T before being crossed out. In the first sentence of the demonstration in this interleaved copy, the first two letters added at the beginning of the sentence were originally A and B, which were then altered to L and T. It is these letters, L and T, that were added elsewhere and were kept in the demonstration and corollaries.

COROLLARY 3. And conversely, if the remaining force tends very nearly toward body T, the areas will be very nearly proportional to the times.

COROLLARY 4. If body L, by a radius drawn to another body T, describes areas which, compared with the times, are extremely unequal, and body T either is at rest or moves uniformly straight forward, either there is no centripetal force tending toward body T or the action of the centripetal force is mixed and compounded with the very powerful actions of other forces; and the total force compounded of all the forces, if there are several, is directed toward another center (whether fixed or moving). The same thing holds when the second body moves with any motion whatever, if the centripetal force is what remains after subtraction of the total force acting upon body T.[a]

Scholium

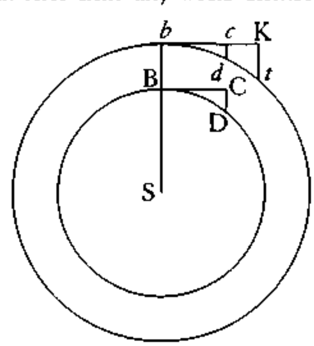
Since the uniform description of areas indicates the center toward which that force is directed by which a body is most affected and by which it is drawn away from rectilinear motion and kept in orbit, why should we not in what follows use uniform description of areas as a criterion for a center about which all orbital motion takes place in free spaces?

Proposition 4
Theorem 4

The centripetal forces of bodies that describe different circles with uniform motion tend toward the centers of those circles and are to one another as the squares of the arcs described in the same time divided by the radii of the circles.

[a]These forces tend toward the centers of the circles by prop. 2 and prop. 1, corol. 2, and are to another as the versed sines of the arcs described in

aa. Ed. 1 has: "Let bodies B and b, revolving in the circumferences of circles BD and bd, describe arcs BD and bd in the same time. Since by their inherent force alone they would describe tangents BC and bc equal to these arcs, it is obvious that centripetal forces are the ones which continually draw the bodies back from the tangents to the circumferences of the circles, and thus these forces are to each other in the first ratio of the nascent spaces CD and cd, and they tend toward the centers of the circles, by theor. 2, because the areas described by the radii are supposed proportional to the times. [Newton is using "first ratio" here in the special sense developed in sec. 1 above, where he introduces the concept of "first" and "ultimate" ratio.] Let figure tkb be similar to DCB and, by lem. 5, line-element CD will be to line-element kt as arc BD to arc bt, and also, by lem. 11, the nascent line-element tk will be to the nascent line-element dc as bt^2 to bd^2 and, from the equality of the ratios [or ex aequo], the nascent line-element DC will be to the nascent line-element dc as BD × bt to bd^2 or,



minimally small equal times, by prop. 1, corol. 4, that is, as the squares of those arcs divided by the diameters of the circles, by lem. 7; and therefore, since these arcs are as the arcs described in any equal times and the diameters are as their radii, the forces will be as the squares of any arcs described in the same time divided by the radii of the circles. Q.E.D.[a]

[b]COROLLARY 1. [c]Since those arcs are as the velocities of the bodies, the centripetal forces will be in a ratio compounded of the squared ratio of the velocities directly and the simple ratio of the radii inversely.[c]

COROLLARY 2. [d]And since the periodic times are in a ratio compounded of the ratio of the radii directly and the ratio of the velocities inversely, the centripetal forces are in a ratio compounded of the ratio of the radii directly and the squared ratio of the periodic times inversely.[d]

COROLLARY 3. Hence, if the periodic times are equal and therefore the velocities are as the radii, the centripetal forces also will be as the radii; and conversely.

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what comes to the same thing, as $\mathrm{BD} \times \frac{bt}{\mathrm{S}b}$ to $\frac{bd^2}{\mathrm{S}b}$ and thus (because the ratios $\frac{bt}{\mathrm{S}b}$ and $\frac{\mathrm{BD}}{\mathrm{SB}}$ are equal) as $\frac{\mathrm{BD}^2}{\mathrm{SB}}$ to $\frac{bd^2}{\mathrm{S}b}$. Q.E.D."

Here, as in the very similar earlier formulation of *De Motu* and in a later handwritten revision of ed. 1, the sentence specifying centrifugal forces has some ambiguity because the grammatical structure can indicate that Newton is redefining these forces whereas the context shows that he is giving one of their properties.

bb. Different versions of corols. 1, 2, 4, 5, and 6 exemplify interesting variations in basic mathematical terminology, as is indicated in the following notes.

cc. In ed. 1 this corollary reads: "Hence the centripetal forces are as the squares of the velocities divided by the radii of the circles." In manuscript revisions of ed. 1 "Hence" is deleted and the sentence begins with an additional clause: "Whence, since the arcs described in the same time are directly as the velocities and inversely as the periodic times." Ed. 2 reads: "Therefore, since those arcs are as the velocities of the bodies, the centripetal forces are as the squares of the velocities divided by the radii of the circles; that is, to express it as the geometers do, the forces are in a ratio compounded of the squared ratio of the velocities directly and the simple ratio of the radii inversely." And then, in ed. 3, Newton decides to eliminate the first formulation and express his result only "as the geometers do."

dd. In ed. 1 this corollary reads: "And inversely as the squares of the periodic times divided by the radii so are these forces to one another. That is (to express it as the geometers do), these forces are in a ratio compounded of the squared ratio of the velocities directly and the simple ratio of the radii inversely, and also in a ratio compounded of the simple ratio of the radii directly and the squared ratio of the periodic times inversely." The inversion in the first sentence suggests that originally it was not a full sentence but a continuation from corol. 1, as comparison with the earlier *De Motu* shows to be true. Ed. 2 reads: "And since the periodic times are in a ratio compounded of the ratio of the radii directly and the ratio of the velocities inversely, the centripetal forces are inversely as the squares of the periodic times divided by the radii of the circles: that is, in a ratio compounded of the ratio of the radii directly and the squared ratio of the periodic times inversely."

COROLLARY 4. [e]If both the periodic times and the velocities are as the square roots of the radii, the centripetal forces will be equal to one another; and conversely.[e]

COROLLARY 5. [f]If the periodic times are as the radii, and therefore the velocities are equal, the centripetal forces will be inversely as the radii; and conversely.[f]

COROLLARY 6. [g]If the periodic times are as the $\frac{3}{2}$ powers of the radii, and therefore the velocities are inversely as the square roots of the radii, the centripetal forces will be inversely as the squares of the radii; and conversely.[b] [g]

[h]COROLLARY 7. And universally, if the periodic time is as any power R^n of the radius R, and therefore the velocity is inversely as the power R^{n-1} of the radius, the centripetal force will be inversely as the power R^{2n-1} of the radius; and conversely.

COROLLARY 8. In cases in which bodies describe similar parts of any figures that are similar and have centers similarly placed in those figures, all the same proportions with respect to the times, velocities, and forces follow from applying the foregoing demonstrations to these cases. And the application is made by substituting the uniform description of areas for uniform motion, and by using the distances of bodies from the centers for the radii.

COROLLARY 9. From the same demonstration it follows also that the arc which a body, in revolving uniformly in a circle with a given centripetal force, describes in any time is a mean proportional between the diameter of the circle and the distance through which the body would fall under the action of the same given force and in the same time.[h]

ee. In ed. 1 this corollary reads: "If the squares of the periodic times are as the radii, the centripetal forces are equal, and the velocities are in the halved ratio of the radii, and vice versa."

ff. In ed. 1 this corollary reads: "If the squares of the periodic times are as the squares of the radii, the centripetal forces are inversely as the radii, and the velocities are equal, and vice versa." After "of the radii," handwritten revisions of ed. 1 add "that is, the times [are] as the radii."

gg. In ed. 1 this corollary reads: "If the squares of the periodic times are as the cubes of the radii, the centripetal forces are inversely as the squares of the radii, but the velocities are in the halved ratio of the radii, and vice versa."

hh. Ed. 1 lacks corols. 7 and 9, and in corol. 8, which is numbered 7, it lacks "in those figures" in the first sentence and all of the second sentence.

Scholium ⁱThe case of corol. 6 holds for the heavenly bodies (as our compatriots Wren, Hooke, and Halley have also found out independently). Accordingly, I have decided that in what follows I shall deal more fully with questions relating to the centripetal forces that decrease as the squares of the distances from centers [i.e., centripetal forces that vary inversely as the squares of the distances].

Further, with the help of the preceding proposition and its corollaries the proportion of a centripetal force to any known force, such as that of gravity, may also be determined. ʲFor if a body revolves by the force of its gravity in a circle concentric with the earth, this gravity is its centripetal force. Moreover, by prop. 4, corol. 9, both the time of one revolution and the arc described in any given time are given from the descent of heavy bodies.ʲ And by propositions of this sort Huygens in his excellent treatise *On the Pendulum Clock* compared the force of gravity with the centrifugal forces of revolving bodies.

This proposition can also be demonstrated in the following manner. In any circle, suppose that a polygon of any number of sides is described. And if a body moving with a given velocity along the sides of the polygon is reflected from the circle at each of the angles of the polygon, the force with which it impinges upon the circle at each reflection will be as its velocity; and therefore the sum of the forces in a given time will be as that velocity and the number of reflections jointly; that is (if the sides and angles of the polygon are specified), as the length described in that given time and increased or decreased in the ratio of the length to the radius of the above-mentioned circle, that is, as the square of that length divided by the radius. And, therefore,

ii. In the printer's manuscript of ed. 1 the scholium originally consisted of a single sentence, corresponding to the first sentence of ed. 3 but without the parenthesis containing the three proper names. A separate sheet in this manuscript and the printed text of ed. 1 contain the entire scholium, but in the addition to the manuscript the names are listed as Wren, Halley, and Hooke, whereas in ed. 1 they appear in the order retained in ed. 3. We cannot tell by whose authority Hooke's name was moved to a position before Halley's, but we can infer that the alteration was made in proof (and so presumably by Halley), since the handwritten addition to the manuscript as sent by Newton to Halley and by Halley to the printer is unaltered. It is very probable that Halley put Hooke's name ahead of his own because he did not want Hooke to be offended.

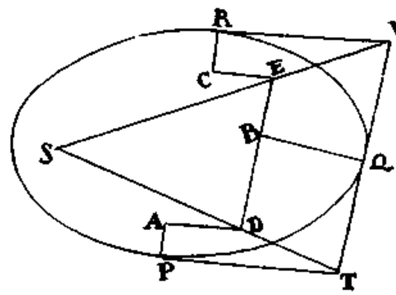
jj. Ed. 1 has: "For since the former force, in the time in which a body traverses arc BC, impels the body through space CD, which at the very beginning of the motion is equal to the square of that arc BD divided by the diameter of the circle, and since every body, by the same force continued always in the same direction, describes spaces that are in the squared ratio of the times, that force, in the time in which the revolving body describes any given arc, will cause the body as it advances directly forward to describe a space equal to the square of that arc divided by the diameter of the circle and thus is to the force of gravity as that space is to the space which a heavy body in falling describes in the same time."

if the sides are diminished indefinitely, the polygon will coincide with the circle, and the sum of the forces in a given time will be as the square of the arc described in the given time divided by the radius. This is the centrifugal force with which the body urges the circle; and the opposite force, with which the circle continually repels the body toward the center, is equal to this centrifugal force.[i]

Proposition 5
Problem 1

Given, in any places, the velocity with which a body describes a given curve when acted on by forces tending toward some common center, to find that center.

Let the curve so described be touched in three points P, Q, and R by three straight lines PT, TQV, and VR, meeting in T and V. Erect PA, QB, and RC perpendicular to the tangents and inversely proportional to the velocities of the body at the points P, Q, and R from which the perpendiculars are erected—that is, so that PA is to QB as the velocity at Q to the velocity at P, and QB to RC as the velocity at R to the velocity at Q. Through the ends A, B, and C of the perpendiculars draw AD, DBE, and EC at right angles to those perpendiculars, and let them meet in D and E; then TD and VE, when drawn and produced, will meet in the required center S.



For the perpendiculars dropped from center S to tangents PT and QT are (by prop. 1, corol. 1) inversely as the velocities of the body at points P and Q, and therefore, by the construction, as the perpendiculars AP and BQ directly, that is, as the perpendiculars dropped from point D to the tangents. Hence it is easily gathered that points S, D, and T are in one straight line. And, by a similar argument, the points S, E, and V are also in one straight line; and therefore the center S is at the point where the straight lines TD and VE meet. Q.E.D.

Proposition 6[a]
Theorem 5

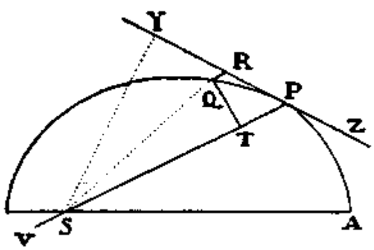
[b]*If in a nonresisting space a body revolves in any orbit about an immobile center and describes any just-nascent arc in a minimally small time, and if the sagitta of*

a. For a gloss on this proposition see the Guide, §10.8.

bb. In ed. 1 there is a different prop. 6, with its proof and single unnumbered corollary. In ed. 2 and ed. 3 the statement of this proposition becomes corol. 1 to the new prop. 6 and the single corollary *the arc is understood to be drawn so as to bisect the chord and, when produced, to pass through the center of forces, the centripetal force in the middle of the arc will be as the sagitta directly and as the time twice [i.e., as the square of the time] inversely.*

For the sagitta in a given time is as the force (by prop. 1, corol. 4), and if the time is increased in any ratio, then—because the arc is increased in the same ratio—the sagitta is increased in that ratio squared (by lem. 11, corols. 2 and 3) and therefore is as the force once and the time twice [i.e., as the force and the square of the time jointly]. Take away from both sides the squared ratio of the time, and the force will become as the sagitta directly and as the time twice [or as the square of the time] inversely. Q.E.D.

This proposition is also easily proved by lem. 10, corol. 4.



Corollary 1. If a body P, revolving about a center S, describes the curved line APQ, while the straight line ZPR touches the curve at any point P; and QR, parallel to distance SP, is drawn to the tangent from any other point Q of the curve, and QT is drawn perpendicular to that distance SP; then the centripetal force will be inversely as the solid $\frac{SP^2 \times QT^2}{QR}$, provided that the magnitude of that solid is always taken as that which it has ultimately when the points P and Q come together. For QR is equal to the sagitta of an arc that is twice the length of arc QP, with P being in the middle; and twice the triangle SQP (or $SP \times QT$) is proportional to the time in which twice that arc is described and therefore can stand for the time.

Corollary 2. By the same argument the centripetal force is inversely as the solid $\frac{SY^2 \times QP^2}{QR}$, provided that SY is a perpendicular dropped from the

becomes corol. 5. The proof in ed. 1 reads as follows: "For in the indefinitely small figure QRPT the nascent line-element QR, if the time is given, is as the centripetal force (by law 2) and, if the force is given, is as the square of the time (by lem. 10) and thus, if neither is given, is as the centripetal force and the square of the time jointly, and thus the centripetal force is as the line-element QR directly and the square of the time inversely. But the time is as the area SPQ, or its double $SP \times QT$, that is, as SP and QT jointly, and thus the centripetal force is as QR directly and SP^2 times QT^2 inversely, that is, as $\frac{SP^2 \times QT^2}{QR}$ inversely. Q.E.D." (The figure for prop. 6 in ed. 1 is the same as in eds. 2 and 3, except that the line PS does not extend below the line SA, so that there is no point V.)

center of forces to the tangent PR of the orbit. For the rectangles $SY \times QP$ and $SP \times QT$ are equal.

COROLLARY 3. If the orbit APQ either is a circle or touches a circle concentrically or cuts it concentrically—that is, if it makes with the circle an angle of contact or of section which is the least possible—and has the same curvature and the same radius of curvature at point P, and if the circle has a chord drawn from the body through the center of forces, then the centripetal force will be inversely as the solid $SY^2 \times PV$. For PV is equal to $\frac{QP^2}{QR}$.

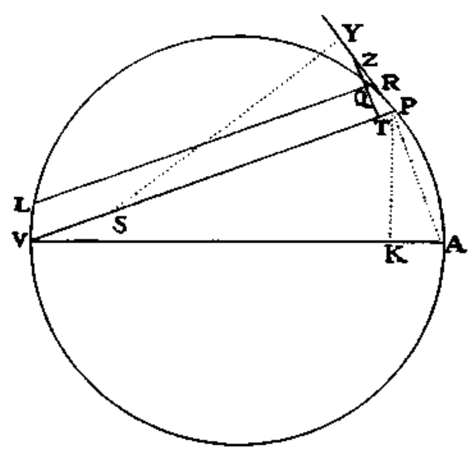
COROLLARY 4. Under the same conditions [as corol. 3], the centripetal force is directly as the square of the velocity and inversely as the chord. For, by prop. 1, corol. 1, the velocity is inversely as the perpendicular SY.

COROLLARY 5. Hence, if there is given any curvilinear figure APQ and on it there is given also point S, to which the centripetal force is continually directed, the law of the centripetal force can be found by which any body P, continually drawn away from a rectilinear course, will be kept in the perimeter of that figure and will describe it as an orbit. That is, the solid $\frac{SP^2 \times QT^2}{QR}$ or the solid $SY^2 \times PV$, inversely proportional to this force, is to be found by computation. We will give examples of this in the following problems.[b]

Proposition 7
Problem 2

Let a body revolve in the circumference of a circle; it is required to find the law of the centripetal force tending toward any given point.

Let VQPA be the circumference of the circle, S the given point toward which the force tends as to its center, P the body revolving in the circumference, Q the place to which it will move next, and PRZ the tangent of the circle at the previous place. Through point S draw chord PV; and when the diameter VA of the circle has been drawn, join AP; and to SP drop perpendicular

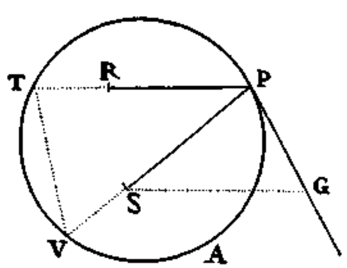


QT, which when produced meets the tangent PR at Z; and finally through point Q draw LR parallel to SP and meeting both the circle at L and the tangent PZ at R. Then because the triangles ZQR, ZTP, and VPA are similar, RP^2 (that is, $QR \times RL$) will be to QT^2 as AV^2 to PV^2. And therefore $\frac{QR \times RL \times PV^2}{AV^2}$ is equal to QT^2. Multiply these equals by $\frac{SP^2}{QR}$ and, the points P and Q coming together, write PV for RL. Thus $\frac{SP^2 \times PV^3}{AV^2}$ will become equal to $\frac{SP^2 \times QT^2}{QR}$. Therefore (by prop. 6, corols. 1 and 5), the centripetal force is inversely as $\frac{SP^2 \times PV^3}{AV^2}$, that is (because AV^2 is given), inversely as the square of the distance or altitude SP and the cube of the chord PV jointly. Q.E.I.

Another solution

Draw SY perpendicular to the tangent PR produced; then, because triangles SYP and VPA are similar, AV will be to PV as SP to SY, and thus $\frac{SP \times PV}{AV}$ will be equal to SY, and $\frac{SP^2 \times PV^3}{AV^2}$ will be equal to $SY^2 \times PV$. And therefore (by prop. 6, corols. 3 and 5), the centripetal force is inversely as $\frac{SP^2 \times PV^3}{AV^2}$, that is, because AV is given, inversely as $SP^2 \times PV^3$. Q.E.I.

Corollary 1. Hence, if the given point S to which the centripetal force always tends is located in the circumference of this circle, say at V, the centripetal force will be inversely as the fifth power of the altitude SP.



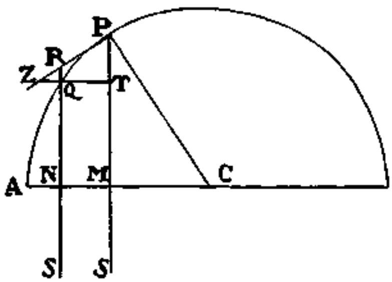
Corollary 2. The force by which body P revolves in the circle APTV around the center of forces S is to the force by which the same body P can revolve in the same circle and in the same periodic time around any other center of forces R as $RP^2 \times SP$ to the cube of the straight line SG, which is drawn from the first center of forces S to the tangent of the orbit PG and is parallel to the distance of the body from the second center of forces. For by the construction of this proposition the first force is

to the second force as $RP^2 \times PT^3$ to $SP^2 \times PV^3$, that is, as $SP \times RP^2$ to $\frac{SP^3 \times PV^3}{PT^3}$, or (because the triangles PSG and TPV are similar) to SG^3.

COROLLARY 3. The force by which body P revolves in any orbit around the center of forces S is to the force by which the same body P can revolve in the same orbit and in the same periodic time around any other center of forces R as the solid $SP \times RP^2$—contained under the distance of the body from the first center of forces S and the square of its distance from the second center of forces R—to the cube of the straight line SG, which is drawn from the first center of forces S to the tangent of the orbit PG and is parallel to the distance RP of the body from the second center of forces. For the forces in this orbit at any point of it P are the same as in a circle of the same curvature.

Proposition 8
Problem 3

Let a body move in the semicircle PQA; *it is required to find the law of the centripetal force for this effect, when the centripetal force tends toward a point* S *so distant that all the lines* PS *and* RS *drawn to it can be considered parallel.*



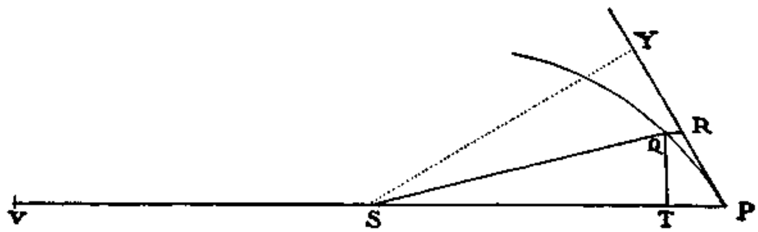
From the center C of the semicircle draw the semidiameter CA, intersecting those parallels perpendicularly at M and N, and join CP. Because triangles CPM, PZT, and RZQ are similar, CP^2 is to PM^2 as PR^2 to QT^2, and from the nature of a circle PR^2 is equal to the rectangle $QR \times (RN + QN)$, or, the points P and Q coming together, to the rectangle $QR \times 2PM$. Therefore, CP^2 is to PM^2 as $QR \times 2PM$ to QT^2, and thus $\frac{QT^2}{QR}$ is equal to $\frac{2PM^3}{CP^2}$, and $\frac{QT^2 \times SP^2}{QR}$ is equal to $\frac{2PM^3 \times SP^2}{CP^2}$. Therefore (by prop. 6, corols. 1 and 5), the centripetal force is inversely as $\frac{2PM^3 \times SP^2}{CP^2}$, that is $\left(\text{neglecting the determinate}^{a} \text{ ratio } \frac{2SP^2}{CP^2}\right)$, inversely as PM^3. Q.E.I.

The same is easily gathered also from the preceding proposition.

a. CP is the radius of the semicircle, and SP may be considered constant.

Scholium And by a not very different argument, a body will be found to move in an ellipse, or even in a hyperbola or a parabola, under the action of a centripetal force that is inversely as the cube of the ordinate tending toward an extremely distant center of forces.

Proposition 9
Problem 4 *Let a body revolve in a spiral* PQS *intersecting all its radii* SP, SQ, . . . , *at a given angle; it is required to find the law of the centripetal force tending toward the center of the spiral.*



Let the indefinitely small angle PSQ be given, and because all the angles are given, the species [i.e., the ratio of all the parts] of the figure SPRQT will be given. Therefore, the ratio $\frac{QT}{QR}$ is given, and $\frac{QT^2}{QR}$ is as QT, that is (because the species of the figure is given), as SP. Now change the angle PSQ in any way, and the straight line QR subtending the angle of contact QPR will be changed (by lem. 11) as the square of PR or QT. Therefore, $\frac{QT^2}{QR}$ will remain the same as before, that is, as SP. And therefore $\frac{QT^2 \times SP^2}{QR}$ is as SP^3, and thus (by prop. 6, corols. 1 and 5) the centripetal force is inversely as the cube of the distance SP. Q.E.I.

Another solution

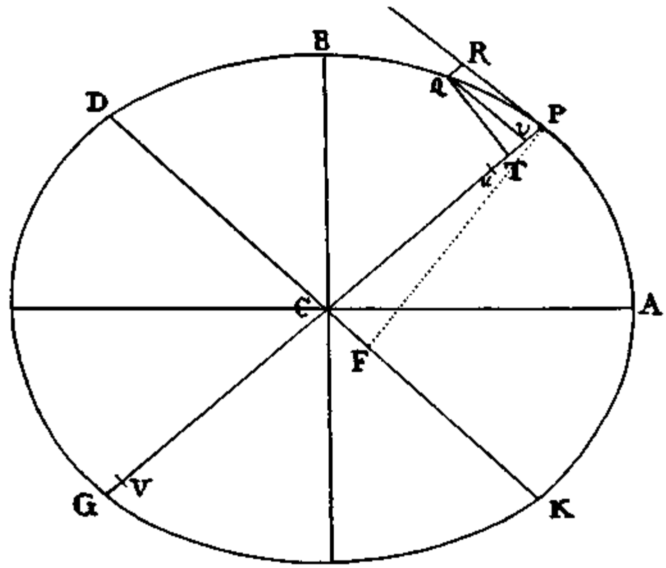
The perpendicular SY dropped to the tangent, and the chord PV of the circle cutting the spiral concentrically, are to the distance SP in given ratios; and thus SP^3 is as $SY^2 \times PV$, that is (by prop. 6, corols. 3 and 5), inversely as the centripetal force.

Lemma 12 *All the parallelograms described about any conjugate diameters of a given ellipse or hyperbola are equal to one another.*

This is evident from the *Conics*.

Proposition 10
Problem 5

Let a body revolve in an ellipse; it is required to find the law of the centripetal force tending toward the center of the ellipse.



Let CA and CB be the semiaxes of the ellipse, GP and DK other conjugate diameters, PF and QT perpendiculars to those diameters, Qv an ordinate to diameter GP; then, if parallelogram $QvPR$ is completed, the rectangle $Pv \times vG$ will (from the *Conics*[a]) be to Qv^2 as PC^2 to CD^2, and (because triangles QvT and PCF are similar) Qv^2 is to QT^2 as PC^2 to PF^2, and, when these ratios are combined, the rectangle $Pv \times vG$ is to QT^2 as PC^2 to CD^2 and PC^2 to PF^2; that is, vG is to $\frac{QT^2}{Pv}$ as PC^2 to $\frac{CD^2 \times PF^2}{PC^2}$. Write QR for Pv and (by lem. 12) $BC \times CA$ for $CD \times PF$, and also (points P and Q coming together) 2PC for vG, and, multiplying the extremes and means together, $\frac{QT^2 \times PC^2}{QR}$ will become equal to $\frac{2BC^2 \times CA^2}{PC}$. Therefore (by prop. 6, corol. 5), the centripetal force is as $\frac{2BC^2 \times CA^2}{PC}$ inversely, that is (because $2BC^2 \times CA^2$ is given), as $\frac{1}{PC}$ inversely, that is, as the distance PC directly. Q.E.I.

a. Concerning this reference to "the *Conics*," see the Guide, §10.10.

Another solution

On the straight line PG take a point u on the other side of point T, so that Tu is equal to Tv; then take uV such that it is to vG as DC^2 is to PC^2. And since (from the *Conics*) Qv^2 is to $Pv \times v$G as DC^2 to PC^2, Qv^2 will be equal to $Pv \times u$V. Add the rectangle uP × Pv to both sides, and the square of the chord of arc PQ will come out equal to the rectangle VP × Pv; and therefore a circle that touches the conic section at P and passes through point Q will also pass through point V. Let points P and Q come together, and the ratio of uV to vG, which is the same as the ratio of DC^2 to PC^2, will become the ratio of PV to PG or PV to 2PC; and therefore PV will be equal to $\frac{2DC^2}{PC}$. Accordingly, the force under the action of which body P revolves in the ellipse will (by prop. 6, corol. 3) be as $\frac{2DC^2}{PC} \times PF^2$ inversely, that is (because $2DC^2 \times PF^2$ is given), as PC directly. Q.E.I.

Corollary 1. Therefore, the force is as the distance of the body from the center of the ellipse; and, conversely, if the force is as the distance, the body will move in an ellipse having its center in the center of forces, or perhaps it will move in a circle, into which an ellipse can be changed.

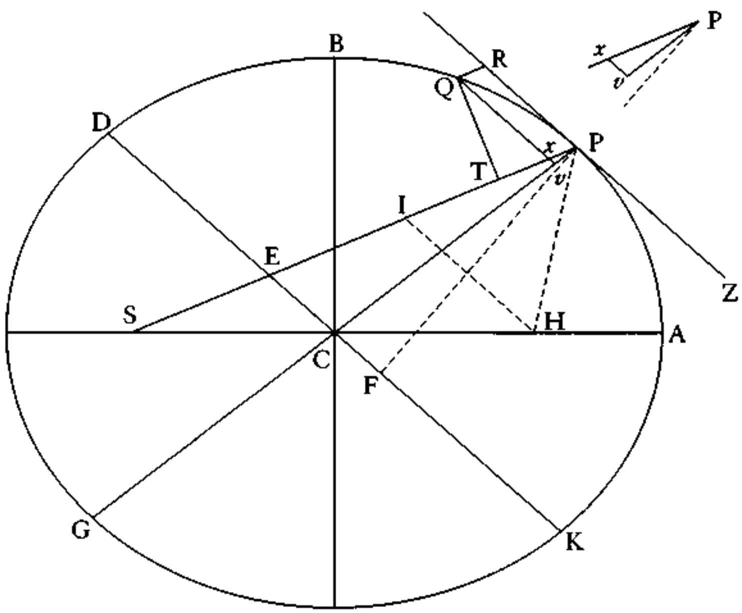
Corollary 2. And the periodic times of the revolutions made in all ellipses universally around the same center will be equal. For in similar ellipses those times are equal (by prop. 4, corols. 3 and 8), while in ellipses having a common major axis they are to one another as the total areas of the ellipses directly and the particles of the areas described in the same time inversely; that is, as the minor axes directly and the velocities of bodies in their principal vertices inversely; that is, as those minor axes directly and the ordinates to the same point of the common axis inversely; and therefore (because of the equality of the direct and inverse ratios) in the ratio of equality.

Scholium If the center of the ellipse goes off to infinity, so that the ellipse turns into a parabola, the body will move in this parabola, and the force, now tending toward an infinitely distant center, will prove to be uniform. This is Galileo's theorem. And if (by changing the inclination of the cutting plane to the cone) the parabolic section of the cone turns into a hyperbola, the body will move in the perimeter of the hyperbola, with the centripetal force turned into a centrifugal force. And just as in a circle or an ellipse, if the forces tend toward a figure's center located in the abscissa, and if the ordinates are increased or decreased in any given ratio or even if the angle of the inclination of the ordinates to the abscissa is changed, these forces are always increased or decreased in the ratio of the distances from the center, provided that the periodic times remain equal; so also in all figures universally, if the ordinates are increased or decreased in any given ratio or the angle of inclination of the ordinates is changed in any way while the periodic time remains the same, the forces tending toward any center located in the abscissa are, for each individual ordinate, increased or decreased in the ratio of the distances from the center.

SECTION 3

The motion of bodies in eccentric conic sections

Proposition 11[a] **Problem 6** *Let a body revolve in an ellipse; it is required to find the law of the centripetal force tending toward a focus of the ellipse.*



Let S be a focus of the ellipse. Draw SP cutting both the diameter DK of the ellipse in E and the ordinate Qv in x, and complete the parallelogram $QxPR$. It is evident that EP is equal to the semiaxis major AC because when line HI is drawn parallel to EC from the other focus H of the ellipse, ES and EI are equal because CS and CH are equal; so that EP is the half-sum of PS and PI, that is (because HI and PR are parallel and angles IPR and HPZ are equal), the half-sum of PS and PH (which taken together equal the whole axis 2AC). Drop QT perpendicular to SP, and if L denotes the principal latus rectum of the ellipse $\left(\text{or } \frac{2BC^2}{AC}\right)$, $L \times QR$ will be to $L \times Pv$ as QR to Pv, that is, as PE or AC to PC; and $L \times Pv$ will be to $Gv \times vP$ as

a. For a gloss on this proposition see the Guide, §10.9.

L to Gv; and[b] $Gv \times vP$ will be to Qv^2 as PC^2 to CD^2; and (by lem. 7, corol. 2) the ratio of Qv^2 to Qx^2, with the points Q and P coming together, is the ratio of equality; and Qx^2 or Qv^2 is to QT^2 as EP^2 to PF^2, that is, as CA^2 to PF^2 or (by lem. 12) as CD^2 to CB^2. And when all these ratios are combined, $L \times QR$ will be to QT^2 as $AC \times L \times PC^2 \times CD^2$, or as $2CB^2 \times PC^2 \times CD^2$ to $PC \times Gv \times CD^2 \times CB^2$, or as $2PC$ to Gv. But with the points Q and P coming together, $2PC$ and Gv are equal. Therefore, $L \times QR$ and QT^2, which are proportional to these, are also equal. Multiply these equals by $\frac{SP^2}{QR}$, and $L \times SP^2$ will become equal to $\frac{SP^2 \times QT^2}{QR}$. Therefore (by prop. 6, corols. 1 and 5) the centripetal force is inversely as $L \times SP^2$, that is, inversely as the square of the distance SP. Q.E.I.

Another solution

The force which tends toward the center of the ellipse, and by which body P can revolve in that ellipse, is (by prop. 10, corol. 1) as the distance CP of the body from the center C of the ellipse; hence, if CE is drawn parallel to the tangent PR of the ellipse and if CE and PS meet at E, then the force by which the same body P can revolve around any other point S of the ellipse will (by prop. 7, corol. 3) be as $\frac{PE^3}{SP^2}$; that is, if point S is a focus of the ellipse, and therefore PE is given, this force will be inversely as SP^2. Q.E.I.

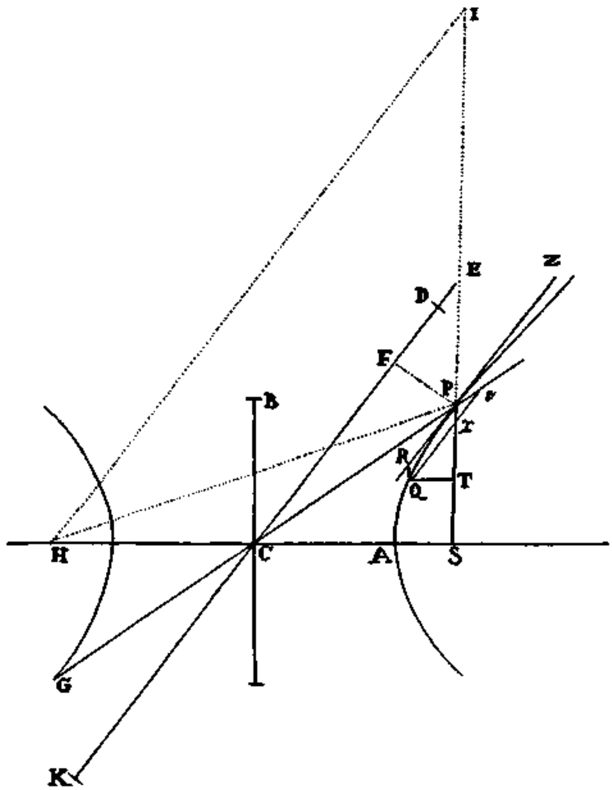
This solution could be extended to the parabola and the hyperbola as concisely as in prop. 10, but because of the importance of this problem and its use in what follows, it will not be too troublesome to confirm each of these other cases by a separate demonstration.

Proposition 12
Problem 7

Let a body move in a hyperbola; it is required to find the law of the centripetal force tending toward the focus of the figure.

Let CA and CB be the semiaxes of the hyperbola, PG and KD other conjugate diameters, PF a perpendicular to the diameter KD, and Qv an ordinate to the diameter GP. Draw SP cutting diameter DK in E and ordinate Qv in x, and complete the parallelogram $QRPx$. It is evident that EP is

b. This result is given in prop. 10 with reference to "the *Conics*"; see the Guide, §10.10.



equal to the transverse semiaxis AC, because when line HI is drawn parallel to EC from the other focus H of the hyperbola, ES and EI are equal because CS and CH are equal; so that EP is the half-difference of PS and PI, that is (because IH and PR are parallel and the angles IPR and HPZ are equal), of PS and PH, the difference of which equals the whole axis 2AC. Drop QT perpendicular to SP. Then, if L denotes the principal latus rectum of the hyperbola $\left(\text{or } \dfrac{2\text{BC}^2}{\text{AC}}\right)$, L × QR will be to L × Pv as QR to Pv, or Px to Pv, that is (because the triangles Pxv and PEC are similar), as PE to PC, or AC to PC. L × Pv will also be to Gv × Pv as L to Gv; and (from the nature of conics) the rectangle Gv × vP will be to Qv^2 as PC^2 to CD^2; and (by lem. 7, corol. 2) the ratio of Qv^2 to Qx^2, the points Q and P coming together, comes to be the ratio of equality; and Qx^2 or Qv^2 is to AT^2 as EP^2 to PF^2, that is, as CA^2 to PF^2, or (by lem. 12) as CD^2 to CB^2; and if all these ratios are combined, L × QR will be to QT^2 as AC × L × PC^2 × CD^2 or 2CB^2 × PC^2 × CD^2 to PC × Gv × CD^2 × CB^2, or as 2PC to Gv. But, the points

P and Q coming together, 2PC and Gv are equal. Therefore, $L \times QR$ and QT^2, which are proportional to these, are also equal. Multiply these equals by $\frac{SP^2}{QR}$, and $L \times SP^2$ will become equal to $\frac{SP^2 \times QT^2}{QR}$. Therefore (by prop. 6, corols. 1 and 5), the centripetal force is inversely as $L \times SP^2$, that is, inversely as the square of the distance SP. Q.E.I.

Another solution

Find the force that tends from the center C of the hyperbola. This will come out proportional to the distance CP. And hence (by prop. 7, corol. 3) the force tending toward the focus S will be as $\frac{PE^3}{SP^2}$, that is, because PE is given, inversely as SP^2. Q.E.I.

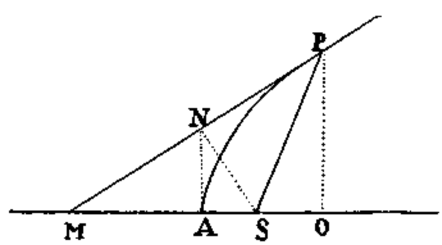
It is shown in the same way that if this centripetal force is turned into a centrifugal force, a body will move in the opposite branch of the hyperbola.

Lemma 13 *In a parabola the latus rectum belonging to any vertex is four times the distance of that vertex from the focus of the figure.*

This is evident from the *Conics*.

Lemma 14 *A perpendicular dropped from the focus of a parabola to its tangent is a mean proportional between the distance of the focus from the point of contact and its distance from the principal vertex of the figure.*

For let AP be the parabola, S its focus, A the principal vertex, P the point of contact, PO an ordinate to the principal diameter, PM a tangent meeting the principal diameter in M, and SN a perpendicular line from the focus to the tangent. Join AN, and because MS and SP, MN and NP, and MA and AO are equal, the straight lines AN and OP will be parallel; and hence triangle SAN will be right-angled at A and similar to the equal triangles SNM and SNP; therefore, PS is to SN as SN to SA. Q.E.D.

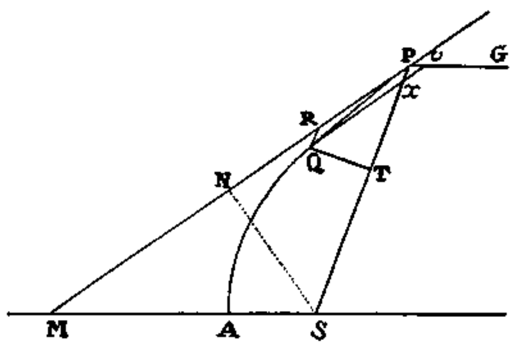


COROLLARY 1. PS^2 is to SN^2 as PS to SA.

COROLLARY 2. And because SA is given, SN^2 is as PS.

Corollary 3. And the point where any tangent PM meets the straight line SN, which is drawn from the focus perpendicular to that tangent, occurs in the straight line AN, which touches the parabola in the principal vertex.

Proposition 13
Problem 8
Let a body move in the perimeter of a parabola; it is required to find the law of the centripetal force tending toward a focus of the figure.



Let the construction be the same as in lem. 14, and let P be the body in the perimeter of the parabola; from the place Q into which the body moves next, draw QR parallel and QT perpendicular to SP and draw Qv parallel to the tangent and meeting both the diameter PG in v and the distance SP in x. Now, because triangles Pxv and SPM are similar and the sides SM and SP of the one are equal, the sides Px or QR and Pv of the other are equal. But from the *Conics* the square of the ordinate Qv is equal to the rectangle contained by the latus rectum and the segment Pv of the diameter, that is (by lem. 13), equal to the rectangle $4\text{PS} \times \text{P}v$, or $4\text{PS} \times \text{QR}$, and, the points P and Q coming together, the ratio of Qv to Qx (by lem. 7, corol. 2) becomes the ratio of equality. Therefore, in this case $\text{Q}x^2$ is equal to the rectangle $4\text{PS} \times \text{QR}$. Moreover (because triangles QxT and SPN are similar), $\text{Q}x^2$ is to QT^2 as PS^2 to SN^2, that is (by lem. 14, corol. 1), as PS to SA, that is, as $4\text{PS} \times \text{QR}$ to $4\text{SA} \times \text{QR}$, and hence (by Euclid's *Elements*, book 5, prop. 9) QT^2 and $4\text{SA} \times \text{QR}$ are equal. Multiply these equals by $\frac{\text{SP}^2}{\text{QR}}$, and $\frac{\text{SP}^2 \times \text{QT}^2}{\text{QR}}$ will become equal to $\text{SP}^2 \times 4\text{SA}$; and therefore (by prop. 6, corols. 1 and 5) the centripetal force is inversely as $\text{SP}^2 \times 4\text{SA}$, that is, because 4SA is given, inversely as the square of the distance SP. Q.E.I.

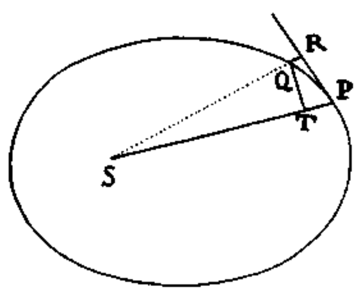
COROLLARY 1. From the last three propositions it follows that if any body P departs from the place P along any straight line PR with any velocity whatever and is at the same time acted upon by a centripetal force that is inversely proportional to the square of the distance of places from the center, this body will move in some one of the conics having a focus in the center of forces; and conversely. For if the focus and the point of contact and the position of the tangent are given, a conic can be described that will have a given curvature at that point. But the curvature is given from the given centripetal force and velocity of the body; and two different orbits touching each other cannot be described with the same centripetal force and the same velocity.

COROLLARY 2. If the velocity with which the body departs from its place P is such that the line-element PR can be described by it in some minimally small particle of time, and if the centripetal force is able to move the same body through space QR in that same time, this body will move in some conic whose principal latus rectum is the quantity $\frac{QT^2}{QR}$ which ultimately results when the line-elements PR and QR are diminished indefinitely. In these corollaries I include the circle along with the ellipse, but not for the case where the body descends straight down to a center.

Proposition 14
Theorem 6

If several bodies revolve about a common center and the centripetal force is inversely as the square of the distance of places from the center, I say that the principal latera recta of the orbits are as the squares of the areas which the bodies describe in the same time by radii drawn to the center.

For (by prop. 13, corol. 2) the latus rectum L is equal to the quantity $\frac{QT^2}{QR}$ that results ultimately when points P and Q come together. But the minimally small line QR is in a given time as the generating centripetal force, that is (by hypothesis), inversely as SP^2. Therefore, $\frac{QT^2}{QR}$ is as $QT^2 \times SP^2$, that is, the latus rectum L is as the square of the area $QT \times SP$. Q.E.D.



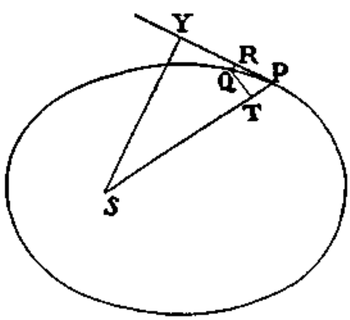
COROLLARY. Hence the total area of the ellipse and, proportional to it, the rectangle contained by the axes is as the square root of the latus rectum and as the periodic time. For the total area is as the area $QT \times SP$, which is described in a given time, multiplied by the periodic time.

Proposition 15
Theorem 7
Under the same suppositions as in prop. 14, I say that the squares of the periodic times in ellipses are as the cubes of the major axes.

For the minor axis is a mean proportional between the major axis and the latus rectum, and thus the rectangle contained by the axes is as the square root of the latus rectum and as the $^3\!/_2$ power of the major axis. But this rectangle (by prop. 14, corol.) is as the square root of the latus rectum and as the periodic time. Take away from both sides [i.e., divide through by] the square root of the latus rectum, and the result will be that the squares of the periodic times are as the cubes of the major axes. Q.E.D.

COROLLARY. Therefore the periodic times in ellipses are the same as in circles whose diameters are equal to the major axes of the ellipses.

Proposition 16
Theorem 8
Under the same suppositions as in prop. 15, if straight lines are drawn to the bodies in such a way as to touch the orbits in the places where the bodies are located, and if perpendiculars are dropped from the common focus to these tangents, I say that the velocities of the bodies are inversely as the perpendiculars and directly as the square roots of the principal latera recta.



From focus S to tangent PR drop perpendicular SY, and the velocity of body P will be inversely as the square root of $\frac{SY^2}{L}$. For this velocity is as the minimally small arc PQ described in a given particle of time, that is (by lem. 7), as the tangent PR, that is—because the proportion of PR to QT is as SP to SY—as $\frac{SP \times QT}{SY}$, or as SY inversely and $SP \times QT$ directly; and $SP \times QT$ is as the area described in the given time, that is (by prop. 14), as the square root of the latus rectum. Q.E.D.

COROLLARY 1. The principal latera recta are as the squares of the perpendiculars and as the squares of the velocities.

COROLLARY 2. The velocities of bodies at their greatest and least distances from the common focus are inversely as the distances and directly as the square roots of the principal latera recta. For the perpendiculars are now the distances themselves.

COROLLARY 3. And thus the velocity in a conic, at the greatest or least distance from the focus, is to the velocity with which the body would move in a circle, at the same distance from the center, as the square root of the principal latus rectum is to the square root of twice that distance.

COROLLARY 4. The velocities of bodies revolving in ellipses are, at their mean distances from the common focus, the same as those of bodies revolving in circles at the same distances, that is (by prop. 4, corol. 6), inversely as the square roots of the distances. For the perpendiculars now coincide with the semiaxes minor, and these are as mean proportionals between the distances and the latera recta. Compound this ratio [of the semiaxes] inversely with the square root of the ratio of the latera recta directly, and it will become the square root of the ratio of the distances inversely.

COROLLARY 5. In the same figure, or even in different figures whose principal latera recta are equal, the velocity of a body is inversely as the perpendicular dropped from the focus to the tangent.

COROLLARY 6. In a parabola the velocity is inversely as the square root of the distance of the body from the focus of the figure; in an ellipse the velocity varies in a ratio that is greater than this, and in a hyperbola in a ratio that is less. For (by lem. 14, corol. 2) the perpendicular dropped from the focus to the tangent of a parabola is as the square root of that distance. In a hyperbola the perpendicular is smaller, and in an ellipse greater, than in this ratio.

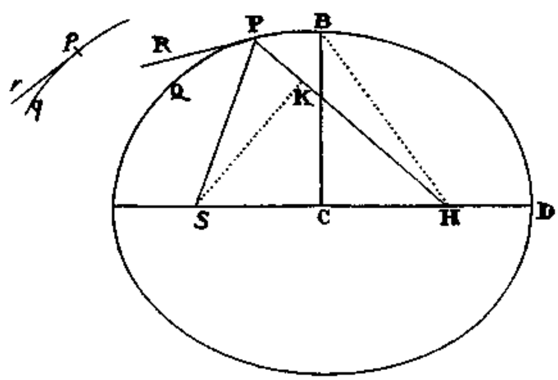
COROLLARY 7. In a parabola the velocity of a body at any distance from the focus is to the velocity of a body revolving in a circle at the same distance from the center as the square root of the ratio of 2 to 1; in an ellipse it is smaller and in a hyperbola greater than in this ratio. For by corol. 2 of this proposition the velocity in the vertex of a parabola is in this ratio, and—by corol. 6 of this proposition and by prop. 4, corol. 6—the same proportion is kept at all distances. Hence, also, in a parabola the velocity everywhere is equal to the velocity of a body revolving in a circle at half the distance; in an ellipse it is smaller and in a hyperbola greater.

COROLLARY 8. The velocity of a body revolving in any conic is to the velocity of a body revolving in a circle at a distance of half the principal latus rectum of the conic as that distance is to the perpendicular dropped from the focus to the tangent of the conic. This is evident by corol. 5.

COROLLARY 9. Hence, since (by prop. 4, corol. 6) the velocity of a body revolving in this circle is to the velocity of a body revolving in any other circle inversely in the ratio of the square roots of the distances, it follows from the equality of the ratios [or ex aequo] that the velocity of a body revolving in a conic will have the same ratio to the velocity of a body revolving in a circle at the same distance that a mean proportional between that common distance and half of the principal latus rectum of the conic has to the perpendicular dropped from the common focus to the tangent of the conic.

Proposition 17
Problem 9

Supposing that the centripetal force is inversely proportional to the square of the distance of places from the center and that the absolute quantity of this force is known, it is required to find the line which a body describes when going forth from a given place with a given velocity along a given straight line.



Let the centripetal force tending toward a point S be such that a body p revolves by its action in any given orbit pq, and let its velocity in the place p be found out. Let body P go forth from place P along line PR with a given velocity and thereupon be deflected from that line into a conic PQ under the compulsion of the centripetal force. Therefore the straight line PR will touch this conic at P. Let some straight line pr likewise touch the orbit pq at p, and if perpendiculars are understood to be dropped from S to these tangents, the principal latus rectum of the conic will (by prop. 16, corol. 1) be to the principal latus rectum of the orbit in a ratio compounded of the squares of the perpendiculars and the squares of the velocities and thus is given. Let L be the latus rectum of the conic. The focus S of the conic is also given. Let angle RPH be the complement of angle RPS to two right angles [i.e., the supplement of angle RPS]; then the line PH, on which the other focus H is located, will be given in position. Drop the perpendicular SK to PH and understand the conjugate semiaxis BC to be erected; then $SP^2 - 2KP \times PH + PH^2 = SH^2 = 4CH^2 = 4BH^2 - 4BC^2 = (SP + PH)^2 - L \times (SP + PH) = SP^2 + 2SP \times PH + PH^2 - L \times (SP + PH)$. Add to each side $2(KP \times PH) - SP^2 - PH^2 + L \times (SP + PH)$, and $L \times (SP + PH)$ will become $= 2(SP \times PH) + 2(KP \times PH)$, or SP + PH will be to PH as 2SP + 2KP to L. Hence PH is given in length as well as in position. Specifically, if the velocity of the body at P is such that the latus rectum L is less than 2SP + 2KP, PH will lie on the same side of the tangent PR as the line PS; and thus the figure will be an ellipse and will be given from the given foci S and H and the given principal axis SP + PH. But if the velocity of the body is so great that the latus rectum L is equal to 2SP + 2KP, the length PH will be infinite; and accordingly the figure will be a parabola having its axis SH parallel to the line PK, and hence will be given. But if the body goes forth from its place P with a still greater velocity, the length PH will have to be taken on the other side of the tangent; and thus, since the tangent goes between the foci, the figure will be a hyperbola having its principal axis equal to the difference of the lines SP and PH, and hence will be given. For if the body in these cases revolves in a conic thus found, it has been demonstrated in props. 11, 12, and 13 that the centripetal force will be inversely as the square of the distance of the body from the center of forces S; and thus the line PQ is correctly determined, which a body will describe under the action of such a force, when it goes forth from a given place P with a given velocity along a straight line PR given in position. Q.E.F.

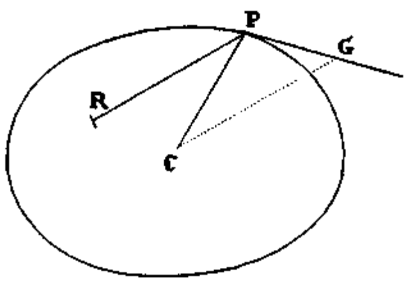
Corollary 1. Hence in every conic, given the principal vertex D, the latus rectum L, and a focus S, the other focus H is given when DH is taken to DS as the latus rectum is to the difference between the latus rectum and 4DS. For the proportion SP + PH to PH as 2SP + 2KP to L in the case of this corollary becomes DS + DH to DH as 4DS to L and, by separation [or dividendo], becomes DS to DH as 4DS − L to L.

Corollary 2. Hence, given the velocity of a body in the principal vertex D, the orbit will be found expeditiously, namely, by taking its latus rectum to twice the distance DS as the square of the ratio of this given velocity to the velocity of a body revolving in a circle at a distance DS (by prop. 16, corol. 3), and then taking DH to DS as the latus rectum to the difference between the latus rectum and 4DS.

Corollary 3. Hence also, if a body moves in any conic whatever and is forced out of its orbit by any impulse, the orbit in which it will afterward pursue its course can be found. For by compounding the body's own motion with that motion which the impulse alone would generate, there will be found the motion with which the body will go forth from the given place of impulse along a straight line given in position.

Corollary 4. And if the body is continually perturbed by some force impressed from outside, its trajectory can be determined very nearly, by noting the changes which the force introduces at certain points and estimating from the order of the sequence the continual changes at intermediate places.[a]

Scholium If a body P, under the action of a centripetal force tending toward any given point R, moves in the perimeter of any given conic whatever, whose center is C, and the law of the centripetal force is required, let CG be drawn parallel to the radius RP and meeting the tangent PG of the orbit at G; then the force (by prop. 10, corol. 1 and schol.; and prop. 7, corol. 3) will be as $\frac{CG^3}{RP^2}$.



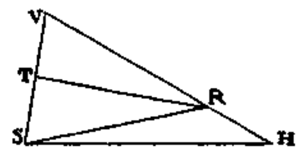
a. The sense of corol. 4 is that Newton can determine "the changes which the [impressed] force will make at certain points" and, by interpolation, estimate the changes continually made at intermediary points.

SECTION 4[a]

To find elliptical, parabolic, and hyperbolic orbits, given a focus

Lemma 15

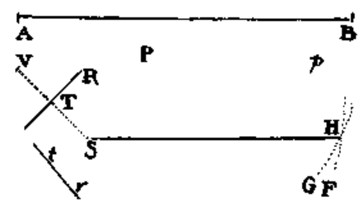
If from the two foci S *and* H *of any ellipse or hyperbola two straight lines* SV *and* HV *are inclined to any third point* V, *one of the lines* HV *being equal to the principal axis of the figure, that is, to the axis on which the foci lie, and the other line* SV *being bisected in* T *by* TR *perpendicular to it, then the perpendicular* TR *will touch the conic at some point; and conversely, if* TR *touches the conic,* HV *will be equal to the principal axis of the figure.*



For let the perpendicular TR cut the straight line HV (produced, if need be) in R; and join SR. Because TS and TV are equal, the straight lines SR and VR and the angles TRS and TRV will be equal. Hence the point R will be on the conic, and the perpendicular TR will touch that conic, and conversely. Q.E.D.

Proposition 18
Problem 10

Given a focus and the principal axes, to describe elliptical and hyperbolic trajectories that will pass through given points and will touch straight lines given in position.



Let S be the common focus of the figures, AB the length of the principal axis of any trajectory, P a point through which the trajectory ought to pass, and TR a straight line which it ought to touch. Describe the circle HG with P as center and AB − SP as radius if the orbit is an ellipse, or AB + SP if it is a hyperbola. Drop the perpendicular ST to the tangent TR and produce ST to V so that TV is equal to ST, and with center V and radius AB describe the circle FH. By this method, whether two points P and *p* are given, or two tangents TR and *tr*, or a point P and a tangent TR, two circles are to be described. Let H be their common intersection, and with foci S and H and the given axis, describe the trajectory. I say that the problem has been solved. For the trajectory described (because PH + SP in an ellipse, or PH − SP in a hyperbola,

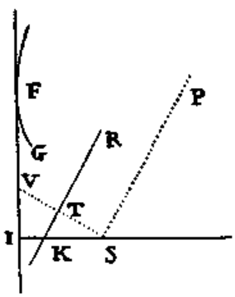
a. For Newton's statement of the reason for including secs. 4 and 5 in book 1, see book 3, prop. 41 (p. 901).

is equal to the axis) will pass through point P and (by lem. 15) will touch the straight line TR. And by the same argument, this trajectory will pass through the two points P and *p* or will touch the two straight lines TR and *tr*. Q.E.F.

Proposition 19
Problem 11

To describe about a given focus a parabolic trajectory that will pass through given points and will touch straight lines given in position.

Let S be the focus, P a given point, and TR a tangent of the trajectory to be described. With center P and radius PS describe the circle FG. Drop

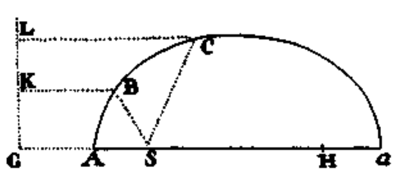


the perpendicular ST from the focus to the tangent and produce ST to V, so that TV is equal to ST. In the same manner, if a second point *p* is given, a second circle *fg* is to be described; or if a second tangent *tr* is given, or a second point *v* is to be found, then the straight line IF is to be drawn touching the two circles FG and *fg* if the two points P and *p* are given, or passing through the two points V and *v* if the two tangents TR and *tr* are given, or touching the circle FG and passing through the point V if the point P and tangent TR are given. To FI drop the perpendicular SI, and bisect it in K; and with axis SK and principal vertex K describe a parabola. I say that the problem has been solved. For, because SK and IK are equal, and SP and FP are equal, the parabola will pass through point P; and (by lem. 14, corol. 3) because ST and TV are equal and the angle STR is a right angle, the parabola will touch the straight line TR. Q.E.F.

Proposition 20
Problem 12

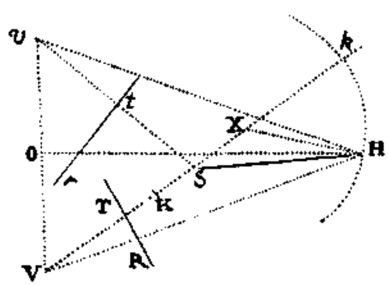
To describe about a given focus any trajectory, given in species [i.e., of given eccentricity], that will pass through given points and will touch straight lines given in position.

Case 1. Given a focus S, let it be required to describe a trajectory ABC through two points B and C. Since the trajectory is given in species, the

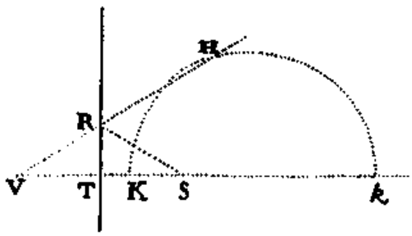


ratio of the principal axis to the distance between the foci will be given. Take KB to BS in this ratio and also LC to CS. With centers B and C and radii BK and CL, describe two circles, and drop the perpendicular SG to the straight line KL, which touches those circles in K and L, and cut SG in A and *a* so that GA is to AS, and G*a* to *a*S, as KB is to BS; and describe a trajectory with axis A*a* and vertices A and *a*. I say that the problem has been solved. For let H be the other focus of the figure described, and since GA is to AS as G*a* to *a*S, then by separation [or dividendo] G*a* − GA or A*a* to *a*S − AS or SH will be in the same ratio and thus in the ratio which the principal axis of the figure that was to be described has to the distance between its foci; and therefore the figure described is of the same species as the one that was to be described. And since KB to BS and LC to CS are in the same ratio, this figure will pass through the points B and C, as is manifest from the *Conics*.

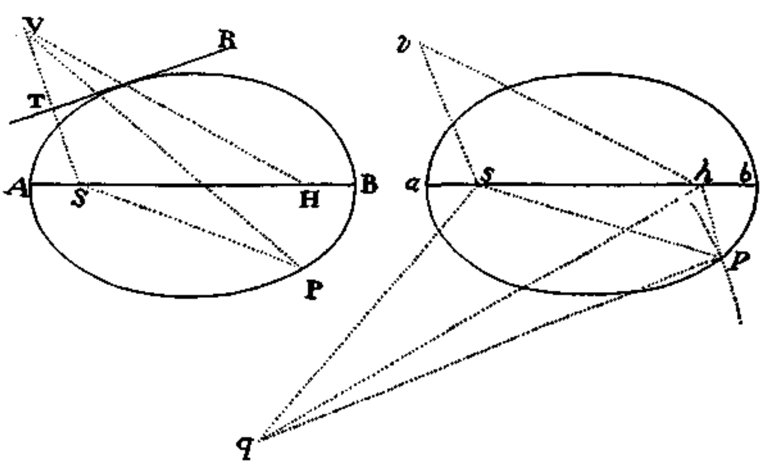
CASE 2. Given a focus S, let it be required to describe a trajectory which somewhere touches the two straight lines TR and *tr*. Drop the perpendiculars ST and S*t* from the focus to the tangents and produce ST and S*t* to V and *v*, so that TV and *tv* are equal to TS and *t*S. Bisect V*v* in O, and erect the indefinite perpendicular OH, and cut the straight line VS, indefinitely produced, in K and *k*, so that VK is to KS and V*k* to *k*S as the principal axis of the trajectory to be described is to the distance between the foci. On the diameter K*k* describe a circle cutting OH in H; and with foci S and H and a principal axis equal to VH, describe a trajectory. I say that the problem has been solved. For bisect K*k* in X, and draw HX, HS, HV, and H*v*. Since VK is to KS as V*k* to *k*S and, by composition [or componendo], as VK + V*k* to KS + *k*S and, by separation [or dividendo], as V*k* − VK to *k*S − KS, that is, as 2VX to 2KX and 2KX to 2SX and thus as VX to HX and HX to SX, the triangles VXH and HXS will be similar, and therefore VH will be to SH as VX to XH and thus as VK to KS. Therefore the principal axis VH of the trajectory which has been described has the same ratio to the distance SH between its foci as the principal axis of the trajectory to be described has to the distance between its foci and is therefore of the same species. Besides, since VH and *v*H are equal to the principal axis and since VS and *v*S are perpendicularly bisected by the straight lines TR and *tr*, it is clear (from lem. 15) that these straight lines touch the trajectory described. Q.E.F.



CASE 3. Given a focus S, let it be required to describe a trajectory which will touch the straight line TR in a given point R. Drop the perpendicular ST to the straight line TR and produce ST to V so that TV is equal to ST. Join VR and cut the straight line VS, indefinitely produced, in K and *k* so that VK is to SK and V*k* to S*k* as the principal axis of the ellipse to be described is to the distance between the foci; and after describing a circle on the diameter K*k*, cut the straight line VR, produced, in H, and with foci S and H and a principal axis equal to the straight line VH, describe a trajectory. I say that the problem has been solved. For, from what has been demonstrated in case 2, it is evident that VH is to SH as VK to SK and thus as the principal axis of the trajectory which was to be described to the distance between its foci, and therefore the trajectory which was described is of the same species as the one which was to be described, while it is evident from the *Conics* that the straight line TR by which the angle VRS is bisected touches the trajectory at point R. Q.E.F.



CASE 4. About a focus S let it be now required to describe a trajectory APB which touches the straight line TR and passes through any point P outside the given tangent and which is similar to the figure *apb* described with principal axis *ab* and foci *s* and *h*. Drop the perpendicular ST to the tangent TR and produce ST to V so that TV is equal to ST. Next make

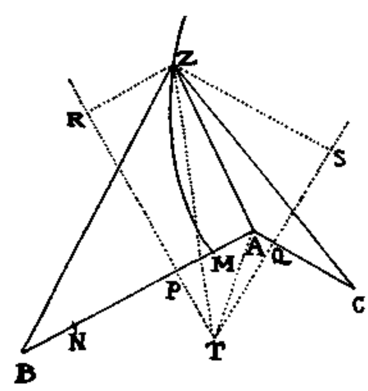


the angles *hsq* and *shq* equal to the angles VSP and SVP; and with center *q* and a radius that is to *ab* as SP to VS, describe a circle cutting the figure *apb* in *p*. Join *sp* and draw SH such that it is to *sh* as SP is to *sp* and makes the angle PSH equal to the angle *psh* and the angle VSH equal to the angle *psq*. Finally with foci S and H and with principal axis AB equaling the distance VH, describe a conic. I say that the problem has been solved. For if SV is drawn such that it is to *sp* as *sh* is to *sq*, and makes the angle *vsp* equal to the angle *hsq* and the angle *vsh* equal to the angle *psq*, the triangles *svh* and *spq* will be similar, and therefore *vh* will be to *pq* as *sh* is to *sq*, that is (because the triangles VSP and *hsq* are similar), as VS is to SP or *ab* to *pq*. Therefore *vh* and *ab* are equal. Furthermore, because the triangles VSH and *vsh* are similar, VH is to SH as *vh* to *sh*; that is, the axis of the conic just described is to the distance between its foci as the axis *ab* to the distance *sh* between the foci; and therefore the figure just described is similar to the figure *apb*. But because the triangle PSH is similar to the triangle *psh*, this figure passes through point P; and since VH is equal to the axis of this figure and VS is bisected perpendicularly by the straight line TR, the figure touches the straight line TR. Q.E.F.

Lemma 16

From three given points to draw three slanted straight lines to a fourth point, which is not given, when the differences between the lines either are given or are nil.

CASE 1. Let the given points be A, B, and C, and let the fourth point be Z, which it is required to find; because of the given difference of the lines AZ and BZ, point Z will be located in a hyperbola whose foci are A and B and whose principal axis is the given difference. Let the axis be MN. Take PM to MA as MN is to AB, and let PR be erected perpendicular to AB and let ZR be dropped perpendicular to PR; then, from the nature of this hyperbola, ZR will



be to AZ as MN is to AB. By a similar process, point Z will be located in another hyperbola, whose foci are A and C and whose principal axis is the difference between AZ and CZ; and QS can be drawn perpendicular to AC, whereupon, if the normal ZS is dropped to QS from any point Z of this hyperbola, ZS will be to AZ as the difference between AZ and CZ is to AC. Therefore the ratios of ZR and ZS to AZ are given, and consequently the ratio of ZR and ZS to each other is given; and thus if the straight lines RP and SQ meet in T, and TZ and TA are drawn, the figure TRZS will be given in species, and the straight line TZ, in which point Z is somewhere located, will be given in position. The straight line TA will also be given, as will also the angle ATZ; and because the ratios of AZ and TZ to ZS are given, their ratio to each other will be given; and hence the triangle ATZ, whose vertex is the point Z, will be given. Q.E.I.

Case 2. If two of the three lines, say AZ and BZ, are equal, draw the straight line TZ in such a way that it bisects the straight line AB; then find the triangle ATZ as above.

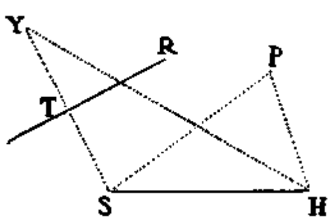
Case 3. If all three lines are equal, point Z will be located in the center of a circle passing through points A, B, and C. Q.E.I.

The problem dealt with in this lemma is also solved by means of Apollonius's book *On Tangencies*, restored by Viète.

Proposition 21
Problem 13

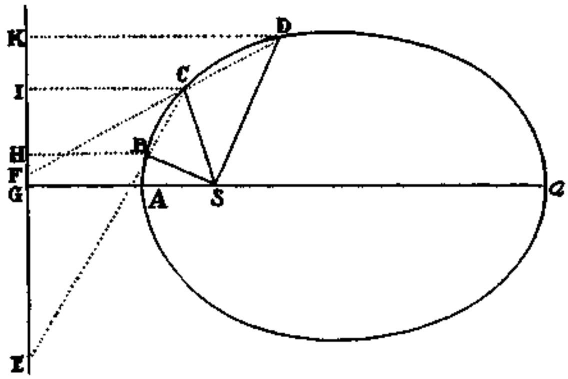
To describe about a given focus a trajectory that will pass through given points and will touch straight lines given in position.

Let a focus S, a point P, and a tangent TR be given; the second focus H is to be found. Drop the perpendicular ST to the tangent and produce ST to Y so that TY is equal to ST, and YH will be equal to the principal axis. Join SP and also HP, and SP will be the difference between HP and the principal axis. In this way, if more tangents TR or more points P are given, there will always be the same number of lines YH or PH, which can be drawn from the said points Y or P to the focus H, and which either are equal to the axes or differ from them by given lengths SP and so either are equal to one another or have given differences; and hence, by lem. 16, that second focus H is given. And once the foci are found, together with the length of the axis (which length is either YH, or PH + SP if the trajectory is an ellipse, but PH − SP if the trajectory is a hyperbola), the trajectory is found. Q.E.I.



Scholium

When the trajectory is a hyperbola, I do not include the opposite branch of the hyperbola as part of the trajectory. For a body going on with an uninterrupted motion cannot pass from one branch of a hyperbola into the other.



The case in which three points are given is solved more speedily as follows: Let the points B, C, and D be given. Join BC and also CD and produce them to E and F so that EB is to EC as SB to SC and FC is to FD as SC to SD. Draw EF, and drop the normals SG and BH to EF produced, and on GS indefinitely produced take GA to AS and G*a* to *a*S as HB is to BS; then A will be the vertex and A*a* the principal axis of the trajectory. According as GA is greater than, equal to, or less than AS, this trajectory will be an ellipse, a parabola, or a hyperbola, with point *a* in the first case falling on the same side of the line GF as point A, in the second case going off to infinity, in the third falling on the other side of the line GF. For if the perpendiculars CI and DK are dropped to GF, IC will be to HB as EC to EB, that is, as SC to SB; and by alternation [or alternando], IC will be to SC as HB to SB or as GA to SA. And by a similar argument it will be proved that KD is to SD in the same ratio. Therefore points B, C, and D lie in a conic described about the focus S in such a way that all the straight lines drawn from the focus S to the individual points of the conic are to the perpendiculars dropped from the same points to the straight line GF in that given ratio.

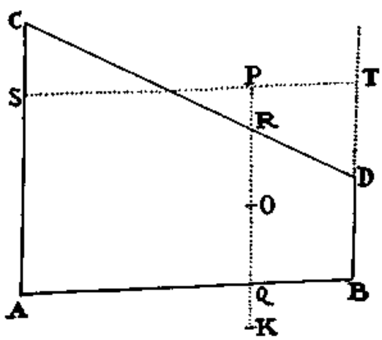
By a method that is not very different, the eminent geometer La Hire presents a solution of this problem in his *Conics*, book 8, prop. 25.

SECTION 5

To find orbits when neither focus is given

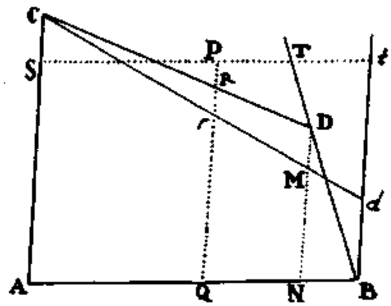
Lemma 17 *If four straight lines* PQ, PR, PS, *and* PT *are drawn at given angles from any point* P *of a given conic to the four indefinitely produced sides* AB, CD, AC, *and* DB *of some quadrilateral* ABDC *inscribed in the conic, one line being drawn to each side, the rectangle* PQ × PR *of the lines drawn to two opposite sides will be in a given ratio to the rectangle* PS × PT *of the lines drawn to the other two opposite sides.*

CASE 1. Let us suppose first that the lines drawn to opposite sides are parallel to either one of the other sides, say PQ and PR parallel to side AC, and PS and PT parallel to side AB. In addition, let two of the opposite sides, say AC and BD, be parallel to each other. Then the straight line which bisects those parallel sides will be one of the diameters of the conic and will bisect RQ also. Let O be the point in which RQ is bisected, and PO will be an ordinate to that diameter. Produce PO to K so that OK is equal to PO, and OK will be the ordinate on the opposite side of the diameter. Therefore, since points A, B, P, and K are on the conic and PK cuts AB at a given angle, the rectangle PQ × QK will be to the rectangle AQ × QB in a given ratio (by book 3, props. 17, 19, 21, and 23, of the *Conics* of Apollonius). But QK and PR are equal, inasmuch as they are differences of the equal lines OK and OP, and OQ and OR, and hence also the rectangles PQ × QK and PQ × PR are equal, and therefore the rectangle PQ × PR is to the rectangle AQ × QB, that is, to the rectangle PS × PT, in a given ratio. Q.E.D.

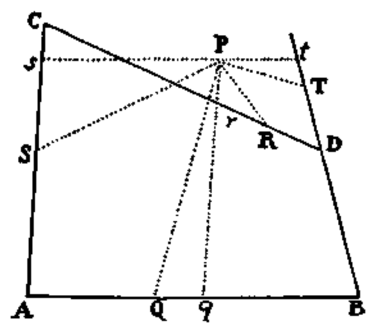


CASE 2. Let us suppose now that the opposite sides AC and BD of the quadrilateral are not parallel. Draw B*d* parallel to AC, meeting the straight line ST in *t* and the conic in *d*. Join C*d* cutting PQ in *r*; and draw DM parallel to PQ, cutting C*d* in M and AB in N. Now, because triangles BT*t* and DBN are similar, B*t* or PQ is to T*t* as DN to NB. So also R*r* is to AQ or PS as DM to AN. Therefore, multiplying the antecedents by the

antecedents and the consequents by the consequents, the rectangle $PQ \times Rr$ is to the rectangle $PS \times Tt$ as the rectangle $ND \times DM$ is to the rectangle $AN \times NB$, and (by case 1) as the rectangle $PQ \times Pr$ is to the rectangle $PS \times Pt$, and by separation [or dividendo] as the rectangle $PQ \times PR$ is to the rectangle $PS \times PT$. Q.E.D.

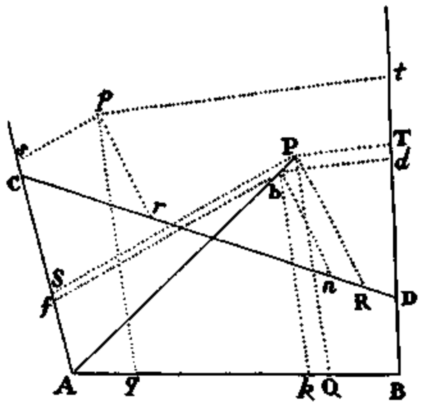


CASE 3. Let us suppose finally that the four lines PQ, PR, PS, and PT are not parallel to the sides AC and AB, but are inclined to them in any way whatever. In place of these lines draw Pq and Pr parallel to AC, and Ps and Pt parallel to AB; then because the angles of the triangles PQq, PRr, PSs, and PTt are given, the ratios of PQ to Pq, PR to Pr, PS to Ps, and PT to Pt will be given, and thus the compound ratios of $PQ \times PR$ to $Pq \times Pr$, and $PS \times PT$ to $Ps \times Pt$. But, by what has been demonstrated above, the ratio of $Pq \times Pr$ to $Ps \times Pt$ is given, and therefore also the ratio of $PQ \times PR$ to $PS \times PT$. Q.E.D.



With the same suppositions as in lem. 17, if the rectangle $PQ \times PR$ *of the lines drawn to two opposite sides of the quadrilateral is in a given ratio to the rectangle* $PS \times PT$ *of the lines drawn to the other two sides, the point* P *from which the lines are drawn will lie on a conic circumscribed about the quadrilateral.* **Lemma 18**

Suppose that a conic is described through points A, B, C, D, and some one of the infinite number of points P, say p; I say that point P always lies on this conic. If you deny it, join AP cutting this conic in some point other than P, if possible, say in b. Therefore, if from these points p and b the straight lines pq, pr, ps, pt and bk, bn, bf, bd are drawn at given angles to the sides of the quadrilateral, then $bk \times bn$ will be to $bf \times bd$ as (by lem. 17) $pq \times pr$ is to $ps \times pt$, and as (by hypothesis) $PQ \times PR$ is to $PS \times PT$. Also, because the quadrilaterals bkAf and PQAS are similar, bk is to bf as PQ to PS. And therefore, if the terms of the previous

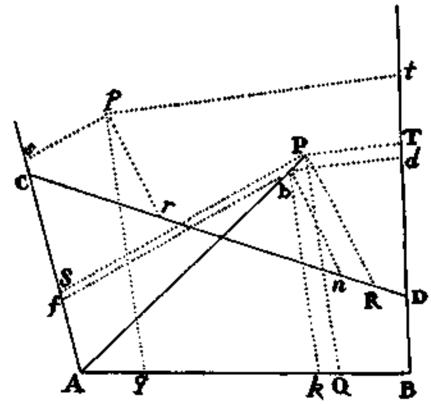


proportion are divided by the corresponding terms of this one, *bn* will be to *bd* as PR to PT. Therefore the angles of the quadrilateral D*nbd* are respectively equal to the angles of quadrilateral DRPT and the quadrilaterals are similar, and consequently their diagonals D*b* and DP coincide. And thus *b* falls upon the intersection of the straight lines AP and DP and accordingly coincides with point P. And therefore point P, wherever it is taken, falls on the assigned conic. Q.E.D.

Corollary. Hence if three straight lines PQ, PR, and PS are drawn at given angles from a common point P to three other straight lines given in position, AB, CD, and AC, one line being drawn to each of the other lines, and if the rectangle PQ $\times$ PR of two of the lines drawn is in a given ratio to the square of the third line PS, then the point P, from which the straight lines are drawn, will be located in a conic which touches lines AB and CD at A and C, and conversely. For let line BD coincide with line AC, while the position of the three lines AB, CD, and AC remains the same, and let line PT also coincide with line PS; then the rectangle PS $\times$ PT will come to be PS^2, and the straight lines AB and CD, which formerly cut the curve in points A and B, C and D, can no longer cut the curve in those points which now coincide, but will only touch it.

Scholium The term "conic" [or "conic section"] is used in this lemma in an extended sense, so as to include both a rectilinear section passing through the vertex of a cone and a circular section parallel to the base. For if point *p* falls on a straight line which joins points A and D or C and B, the conic section will turn into twin straight lines, one of which is the straight line on which point *p* falls and the other the straight line which joins the other two of the four points.

If two opposite angles of the quadrilateral, taken together, are equal to two right angles, and the four lines PQ, PR, PS, and PT are drawn to its sides either perpendicularly or at any equal angles, and the rectangle PQ × PR of two of the lines drawn is equal to the rectangle PS × PT of the other two, the conic will turn out to be a circle. The same will happen if the four lines are drawn at any angles and the rectangle PQ × PR of two of the lines drawn is to the rectangle PS × PT of the other two as the rectangle of the sines of the angles S and T, at which the last two lines PS and PT are drawn, is to the rectangle of the sines of the angles Q and R, at which the first two lines PQ and PR are drawn.

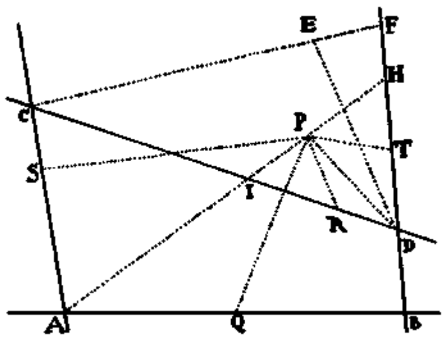


In the other cases the locus of point P will be some one of the three figures that are commonly called conic sections [or conics]. In place of the quadrilateral ABCD, however, there can be substituted a quadrilateral whose two opposite sides decussate each other as diagonals do. But also, one or two of the four points A, B, C, and D can go off to infinity, and in this way the sides of the figure which converge to these points can turn out to be parallel, in which case the conic will pass through the other points and will go off to infinity in the direction of the parallels.

Lemma 19

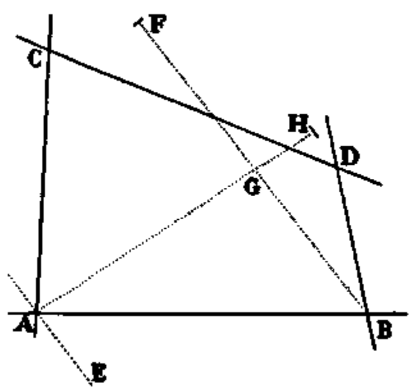
To find a point P *such that if four straight lines* PQ, PR, PS, *and* PT *are drawn from it at given angles to four other straight lines* AB, CD, AC, *and* BD *given in position, one line being drawn from the point* P *to each of the four other straight lines, the rectangle* PQ × PR *of two of the lines drawn will be in a given ratio to the rectangle* PS × PT *of the other two.*

Let lines AB and CD, to which the two straight lines PQ and PR containing one of the rectangles are drawn, meet the other two lines given in position in the points A, B, C, and D. From some one of them A draw any straight line AH, in which you wish point P to be found. Let this line AH cut the opposite lines BD and CD—that is, BD in H and CD in I—and because all the angles of the figure are given, the ratios of PQ to PA and PA to PS, and consequently the ratio



of PQ to PS, will be given. On eliminating this ratio of PQ to PS from the given ratio of PQ × PR to PS × PT, the ratio of PR to PT will be given; and when the given ratios of PI to PR and PT to PH are combined, the ratio of PI to PH, and thus the point P, will be given. Q.E.I.

Corollary 1. Hence also a tangent can be drawn to any point D of the locus of the infinite number of points P. For when points P and D come together—that is, when AH is drawn through the point D—the chord PD becomes a tangent. In this case the ultimate ratio of the vanishing lines IP and PH will be found as above. Therefore, draw CF parallel to AD and meeting BD in F and being cut in E in that ultimate ratio; then DE will be a tangent, because CF and the vanishing line IH are parallel and are similarly cut in E and P.



Corollary 2.[a] Hence also, the locus of all the points P can be determined. Through any one of the points A, B, C, D—say A—draw the tangent AE of the locus, and through any other point B draw BF parallel to the tangent and meeting the locus in F. The point F will be found by means of lem. 19. Bisect BF in G, and the indefinite line AG, when drawn, will be the position of the diameter to which BG and FG are ordinates. Let this line AG meet the locus in H, and AH will be the diameter or la-

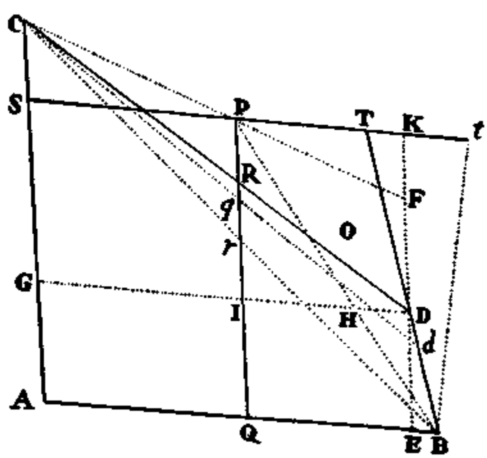
a. In the index prepared by Cotes for ed. 2 and retained in ed. 3, this corollary is keyed under "Problematis" ("of the problem") and characterized as follows: "Geometrical synthesis of the classical problem of four lines made famous by Pappus and attempted by Descartes through algebraic computation." As this description makes explicit, Newton's rejection of "an [analytical] computation" in favor of "a geometrical synthesis" is directed at Descartes, who reduced the four-line locus to a curve defined algebraically by an equation of the second degree. See *The Mathematical Papers of Isaac Newton*, ed. D. T. Whiteside (Cambridge: Cambridge University Press, 1967–1981), 6:252–254 n. 35, 4:291 n. 17, 4:274–282, esp. 274–276.

tus transversum [i.e., transverse diameter] to which the latus rectum will be as BG^2 to $AG \times GH$. If AG nowhere meets the locus, the line AH being indefinitely produced, the locus will be a parabola, and its latus rectum corresponding to the diameter AG will be $\frac{BG^2}{AG}$. But if AG does meet the locus somewhere, the locus will be a hyperbola when points A and H are situated on the same side of G, and an ellipse when G is between points A and H, unless angle AGB happens to be a right angle and additionally BG^2 is equal to the rectangle $AG \times GH$, in which case the locus will be a circle.

And thus there is exhibited in this corollary not an [analytical] computation but a geometrical synthesis, such as the ancients required, of the classical problem of four lines, which was begun by Euclid and carried on by Apollonius.

Lemma 20

If any parallelogram ASPQ *touches a conic in points* A *and* P *with two of its opposite angles* A *and* P, *and if the sides* AQ *and* AS, *indefinitely produced, of one of these angles meet the said conic in* B *and* C, *and if from the meeting points* B *and* C *two straight lines* BD *and* CD *are drawn to any fifth point* D *of the conic, meeting the other two indefinitely produced sides* PS *and* PQ *of the parallelogram in* T *and* R; *then* PR *and* PT, *the parts cut off from the sides, will always be to each other in a given ratio. And conversely, if the parts which are cut off are to each other in a given ratio, the point* D *will touch a conic passing through the four points* A, B, C, *and* P.



CASE 1. Join BP and also CP, and from point D draw two straight lines DG and DE, the first of which (DG) is parallel to AB and meets PB, PQ, and CA at H, I, and G, while the second (DE) is parallel to AC and meets PC, PS, and AB at F, K, and E; then (by lem. 17) the rectangle $DE \times DF$ will be to the rectangle $DG \times DH$ in a given ratio. But PQ is to DE (or IQ) as PB to

HB and thus as PT to DH; and by alternation [or alternando] PQ is to PT as DE to DH. Additionally, PR is to DF as RC to DC and hence as (IG or) PS to DG; and by alternation [or alternando] PR is to PS as DF to DG; and when the ratios are combined, the rectangle PQ × PR comes to be to the rectangle PS × PT as the rectangle DE × DF to the rectangle DG × DH, and hence in a given ratio. But PQ and PS are given, and therefore the ratio of PR to PT is given. Q.E.D.

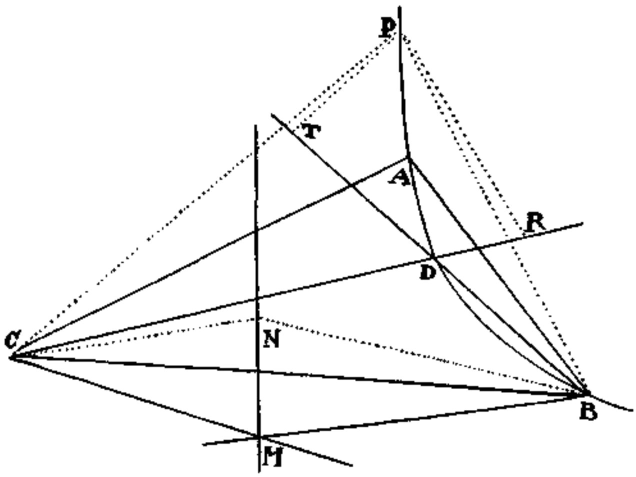
Case 2. But if PR and PT are supposed in a given ratio to each other, then on working backward with a similar argument, it will follow that the rectangle DE × DF is to the rectangle DG × DH in a given ratio and consequently that point D (by lem. 18) lies in a conic passing through points A, B, C, and P. Q.E.D.

Corollary 1. Hence, if BC is drawn cutting PQ in r, and if Pt is taken on PT in the ratio to Pr which PT has to PR, Bt will be a tangent of the conic at point B. For conceive of point D as coming together with point B in such a way that, as chord BD vanishes, BT becomes a tangent; then CD and BT will coincide with CB and Bt.

Corollary 2. And vice versa, if Bt is a tangent and BD and CD meet in any point D of the conic, PR will be to PT as Pr to Pt. And conversely, if PR is to PT as Pr to Pt, BD and CD will meet in some point D of the conic.

Corollary 3. One conic does not intersect another conic in more than four points. For, if it can be done, let two conics pass through five points A, B, C, P, and O, and let the straight line BD cut these conics in points D and d, and let the straight line Cd cut PQ in q. Then PR is to PT as Pq to PT; hence PR and Pq are equal to each other, contrary to the hypothesis.

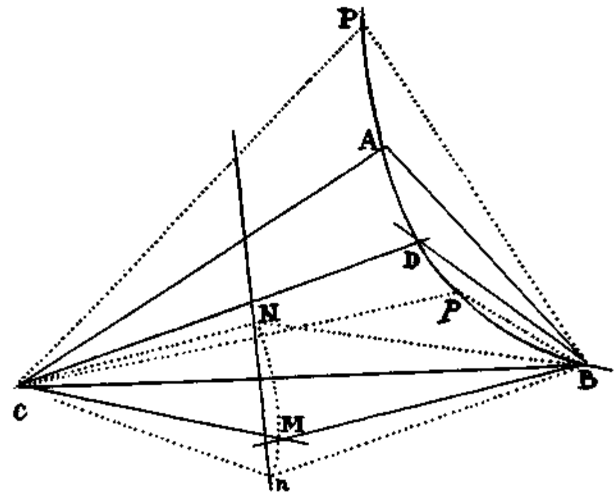
Lemma 21 *If two movable and infinite straight lines* BM *and* CM, *drawn through given points* B *and* C *as poles, describe by their meeting-point* M *a third straight line* MN *given in position, and if two other infinite straight lines* BD *and* CD *are drawn, making given angles* MBD *and* MCD *with the first two lines at those given points* B *and* C; *then I say that the point* D, *where these two lines* BD *and* CD *meet, will describe a conic passing through points* B *and* C. *And conversely, if the point* D, *where the straight lines* BD *and* CD *meet, describes a conic passing through the given points* B, C, *and* A, *and the angle* DBM *is always equal to the*



given angle ABC, *and the angle* DCM *is always equal to the given angle* ACB; *then point* M *will lie in a straight line given in position.*

For let point N be given in the straight line MN; and when the movable point M falls on the stationary point N, let the movable point D fall on the stationary point P. Draw CN, BN, CP, and BP, and from point P draw the straight lines PT and PR meeting BD and CD in T and R and forming an angle BPT equal to the given angle BNM, and an angle CPR equal to the given angle CNM. Since therefore (by hypothesis) angles MBD and NBP are equal, as are also angles MCD and NCP, take away the angles NBD and NCD that are common, and there will remain the equal angles NBM and PBT, NCM and PCR; and therefore triangles NBM and PBT are similar, as are also triangles NCM and PCR. And therefore PT is to NM as PB to NB, and PR is to NM as PC to NC. But the points B, C, N, and P are stationary. Therefore, PT and PR have a given ratio to NM and accordingly a given ratio to each other; and thus (by lem. 20) the point D, the perpetual meeting-point of the movable straight lines BT and CR, lies in a conic passing through points B, C, and P. Q.E.D.

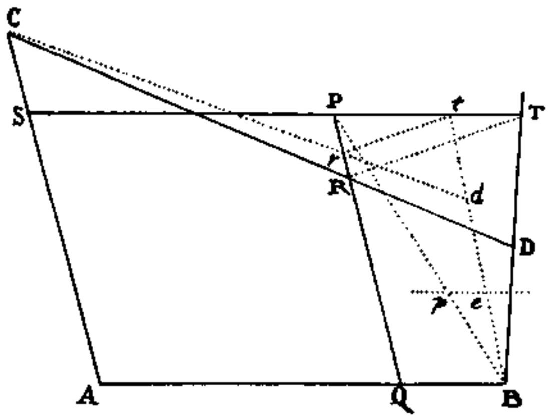
And conversely, if the movable point D lies in a conic passing through the given points B, C, and A; and if angle DBM is always equal to the given angle ABC, and the angle DCM is always equal to the given angle ACB; and if, when point D falls successively on any two stationary points *p* and P of the conic, the movable point M falls successively on the two stationary points *n* and N; then through these same points *n* and N draw the straight



line nN, and this will be the perpetual locus of the movable point M. For, if it can be done, let point M move in some curved line. Then the point D will lie in a conic passing through the five points B, C, A, p, and P when the point M perpetually lies in a curved line. But from what has already been demonstrated, point D will also lie in a conic passing through the same five points B, C, A, p, and P when point M perpetually lies in a straight line. Therefore, two conics will pass through the same five points, contrary to lem. 20, corol. 3. Therefore, it is absurd to suppose the point M to be moving in a curved line. Q.E.D.

Proposition 22 *To describe a trajectory through five given points.*

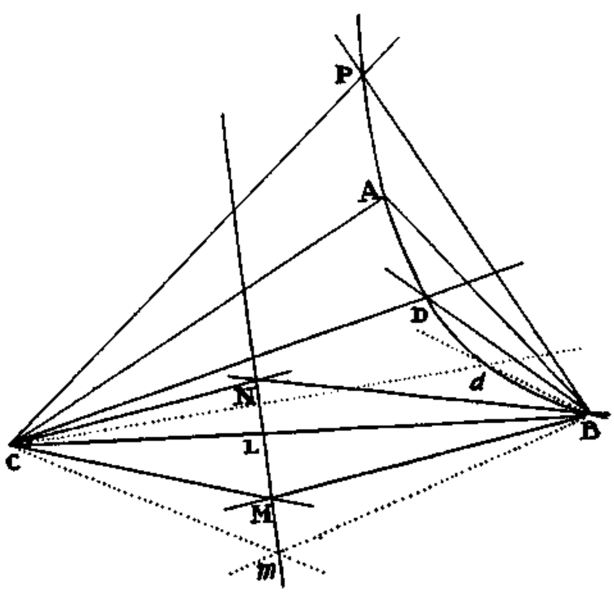
Problem 14 Let five points A, B, C, P, and D be given. From one of them A to any other two B and C (let B and C be called poles), draw the straight lines AB and AC, and parallel to these draw TPS and PRQ through the fourth point P. Then from the two poles B and C draw two indefinite lines BDT and CRD through the fifth point D, BDT meeting the line TPS (just drawn) in T, and CRD meeting PRQ in R. Finally, draw the straight line tr parallel to TR, and cut off from the straight lines PT and PR any straight lines Pt and Pr proportional to PT and PR; then, if through their ends t and r and poles B and C the lines Bt and Cr are drawn meeting in d, that point d will be located in the required trajectory. For that point d (by lem. 20) lies in a conic passing through the four points A, B, C, and P; and, the lines Rr and



T*t* vanishing, point *d* coincides with point D. Therefore, the conic section passes through the five points A, B, C, P, and D. Q.E.D.

Another solution

Join any three of the given points, A, B, and C; and, rotating the angles ABC and ACB, given in magnitude, around two of these points B and C as poles, apply the legs BA and CA first to point D and then to point P, and note the points M and N in which the other legs BL and CL cross in each

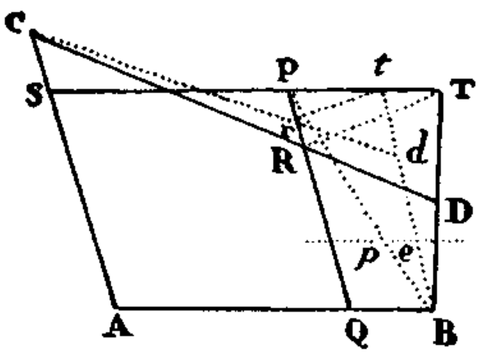


case. Draw the indefinite straight line MN, and rotate these movable angles around their poles B and C in such a way that the intersection of the legs BL and CL or BM and CM (which now let be *m*) always falls on that indefinite straight line MN; and the intersection of the legs BA and CA or BD and CD (which now let be *d*) will trace out the required trajectory PAD*d*B. For point *d* (by lem. 21) will lie in a conic passing through points B and C: and when point *m* approaches points L, M, and N, point *d* (by construction) will approach points A, D, and P. Therefore a conic will be described passing through the five points A, B, C, P, and D. Q.E.F.

COROLLARY 1. Hence a straight line can readily be drawn that will touch the required trajectory in any given point B. Let point *d* approach point B, and the straight line B*d* will come to be the required tangent.

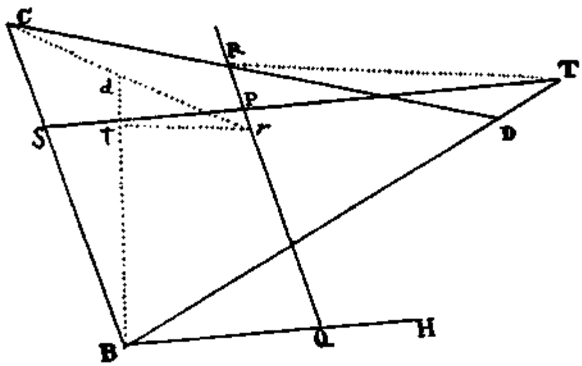
COROLLARY 2. Hence also the centers, diameters, and latera recta of the trajectories can be found, as in lem. 19, corol. 2.

Scholium The first of the constructions of prop. 22 will become a little simpler by joining BP, producing it if necessary, and in it taking B*p* to BP as PR is to PT, and then drawing through *p* the indefinite straight line *pe* parallel to SPT and in it always taking *pe* equal to P*r*, and then drawing the straight lines B*e* and C*r* meeting in *d*. For since the ratios P*r* to P*t*, PR to PT, *p*B to PB, and *pe* to P*t* are equal, *pe* and P*r* will always be equal. By this method the points of the trajectory are found most readily, unless you prefer to describe the curve mechanically, as in the second construction.



Proposition 23 Problem 15 *To describe a trajectory that will pass through four given points and touch a straight line given in position.*

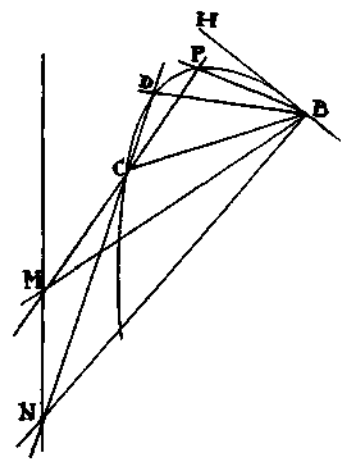
CASE 1. Let the tangent HB, the point of contact B, and three other points C, D, and P be given. Join BC, and by drawing PS parallel to the straight line BH, and PQ parallel to the straight line BC, complete the parallelogram BSPQ. Draw BD cutting SP in T, and CD cutting PQ in R. Finally, by drawing any line *tr* parallel to TR, cut off P*r* and P*t* from PQ



and PS in such a way that Pr and Pt are proportional respectively to PR and PT; then draw Cr and Bt, and their meeting-point d (by lem. 20) will always fall on the trajectory to be described.

Another solution

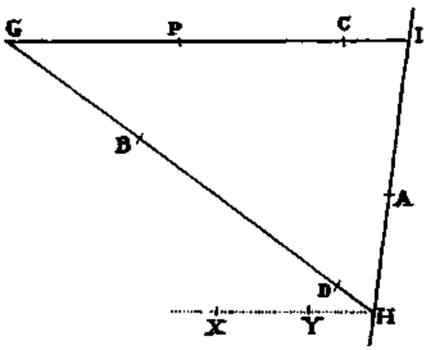
Revolve the angle CBH, given in magnitude, about the pole B, and revolve about the pole C any rectilinear radius DC, produced at both ends. Note the points M and N at which the leg BC of the angle cuts that radius when the other leg BH meets the same radius in points P and D. Then draw the indefinite line MN, and let that radius CP or CD and the leg BC of the angle meet perpetually in the line MN; and the meeting-point of the other leg BH with the radius will trace out the required trajectory.



For if, in the constructions of prop. 22, point A approaches point B, lines CA and CB will coincide, and line AB in its ultimate position will come to be the tangent BH; and therefore the constructions set forth in prop. 22 will come to be the same as the constructions described in this proposition. Therefore, the meeting-point of the leg BH with the radius will trace out a conic passing through points C, D, and P and touching the straight line BH in point B. Q.E.F.

Case 2. Let four points B, C, D, and P be given, situated outside the tangent HI. Join them in pairs by the lines BD and CP coming together in

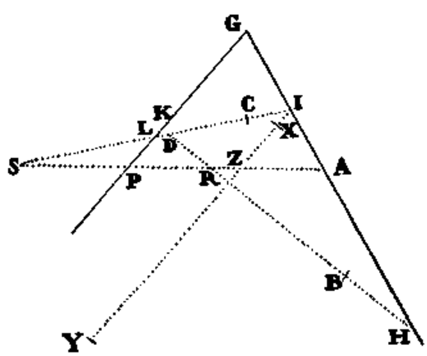
G and meeting the tangent in H and I. Cut the tangent in A in such a way that HA is to IA as the rectangle of the mean proportional between CG and GP and the mean proportional between BH and HD is to the rectangle of the mean proportional between DG and GB and the mean proportional between PI and IC, and A will be the point of contact. For if HX, parallel to the straight line PI, cuts the trajectory in any points X and Y, then (from the *Conics*) point A will have to be so placed that HA^2 is to AI^2 in a ratio compounded of the ratio of the rectangle $XH \times HY$ to the product $BH \times HD$, or of the rectangle $CG \times GP$ to the rectangle $DG \times GD$, and of the ratio of the rectangle $BH \times HD$ to the rectangle $PI \times IC$. And once the point of contact A has been found, the trajectory will be described as in the first case. Q.E.F.



But point A can be taken either between points H and I or outside them, and accordingly two trajectories can be described as solutions to the problem.

Proposition 24
Problem 16
To describe a trajectory that will pass through three given points and touch two straight lines given in position.

Let tangents HI and KL and points B, C, and D be given. Through any two of the points, B and D, draw an indefinite straight line BD meeting the tangents in points H and K. Then, likewise, through any two other points, C and D, draw the indefinite straight line CD meeting the tangents in points I and L. Cut BD in R and CD in S in such a way that HR will be to KR as the mean proportional between BH and HD is to the mean proportional between BK and KD and that IS will be to LS as the mean proportional between CI and ID is to the mean proportional between CL and LD. And cut these lines at will either between points K and H, and between I and L, or outside them; then draw RS cutting the tangents in A and P, and A and P will be



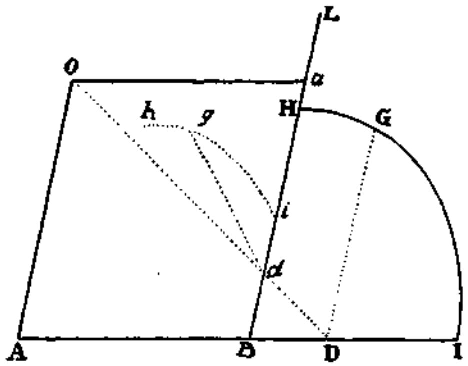
the points of contact. For if A and P are supposed to be the points of contact situated anywhere on the tangents, and if through any one of the points H, I, K, and L, say I, situated in either tangent HI, the straight line IY is drawn parallel to the other tangent KL and meeting the curve in X and Y; and if in this line, IZ is taken so as to be the mean proportional between IX and IY; then, from the *Conics*, the rectangle XI × IY or IZ^2 will be to LP^2 as the rectangle CI × ID to the rectangle CL × LD, that is (by construction), as SI^2 to SL^2, and thus IZ will be to LP as SI to SL. Therefore the points S, P, and Z lie in one straight line. Furthermore, since the tangents meet in G, the rectangle XI × IY or IZ^2 will be to IA^2 (from the *Conics*) as GP^2 to GA^2, and hence IZ will be to IA as GP to GA. Therefore, the points P, Z, and A lie in one straight line, and thus the points S, P, and A are in one straight line. And by the same argument it will be proved that the points R, P, and A are in one straight line. Therefore the points of contact A and P lie in the straight line RS. And once these points have been found, the trajectory will be described as in prop. 23, case 1. Q.E.F.

In this proposition and in prop. 23, case 2, the constructions are the same whether or not the straight line XY cuts the trajectory in X and Y, and they do not depend on this cut. But once the constructions have been demonstrated for the case in which the straight line does cut the trajectory, the constructions for the case in which it does not cut the trajectory also can be found; and for the sake of brevity I do not take the time to demonstrate them further.

Lemma 22

To change figures into other figures of the same class.

Let it be required to transmute any figure HGI. Draw at will two parallel straight lines AO and BL cutting in A and B any third line AB, given in position; and from any point G of the figure draw to the straight line AB any other straight line GD parallel to OA. Then from some point O, given in line OA, draw to the point D the straight line OD meeting BL at *d*, and from the meeting-point erect the straight line *dg* containing any given angle with the straight line BL and having the same ratio to O*d* that DG has to OD; and *g* will be the point in the new figure *hgi* corresponding to point G. By the same method, each of the points in the first figure will yield a corresponding point in the new figure. Therefore, suppose point G to be running through all the points in the first figure with a continual motion;



then point g—also with a continual motion—will run through all the points in the new figure and will describe that figure. For the sake of distinction let us call DG the first ordinate, dg the new ordinate, AD the first abscissa, ad the new abscissa, O the pole, OD the abscinding radius, OA the first ordinate radius, and Oa (which completes the parallelogram OAba) the new ordinate radius.

I say now that if point G traces a straight line given in position, point g will also trace a straight line given in position. If point G traces a conic, point g will also trace a conic. I here count a circle among the conic sections. Further, if point G traces a curved line of the third analytic order, point g will likewise trace a curved line of the third order; and so on with curves of higher orders, the two curved lines which points G and g trace will always be of the same analytic order. For as ad is to OA, so are Od to OD, dg to DG, and AB to AD; and hence AD is equal to $\frac{\mathrm{OA} \times \mathrm{AB}}{ad}$, and DG is equal to $\frac{\mathrm{OA} \times dg}{ad}$. Now, if point G traces a straight line and consequently, in any equation which gives the relation between the abscissa AD and the ordinate DG, the indeterminate lines AD and DG rise to only one dimension, and if in this equation $\frac{\mathrm{OA} \times \mathrm{AB}}{ad}$ is written for AD and $\frac{\mathrm{OA} \times dg}{ad}$ for DG, then the result will be a new equation in which the new abscissa ad and the new ordinate dg will rise to only one dimension and which therefore designates a straight line. But if AD and DG or either one of them rose to two dimensions in the first equation, then ad and dg will also rise to two dimensions in the second equation. And so on for three or more dimensions. The indeterminates ad and dg in the second equation, and AD and DG in the first, will always rise to the same number of dimensions, and therefore the lines which points G and g trace are of the same analytic order.

I say further that if some straight line touches a curved line in the first figure, this straight line—after being transferred into the new figure in the same manner as the curve—will touch that curved line in the new figure; and conversely. For if any two points of the curve approach each other and come together in the first figure, the same points—after being transferred—will approach each other and come together in the new figure; and thus the straight lines by which these points are joined will simultaneously come to be tangents of the curves in both figures.

The demonstrations of these assertions could have been composed in a more geometrical style. But I choose to be brief.

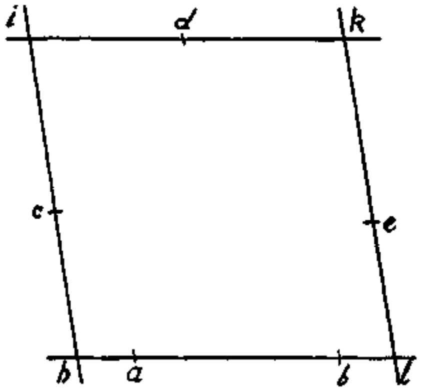
Therefore, if one rectilinear figure is to be transmuted into another, it is only necessary to transfer the intersections of the straight lines of which it is made up and to draw straight lines through them in the new figure. But if it is required to transmute a curvilinear figure, then it is necessary to transfer the points, tangents, and other straight lines which determine the curved line. Moreover, this lemma is useful for solving more difficult problems by transmuting the proposed figures into simpler ones. For any converging straight lines are transmuted into parallels by using for the first ordinate radius any straight line that passes through the meeting-point of the converging lines; and this is so because the meeting-point goes off this way to infinity, and lines that nowhere meet are parallel. Moreover, after the problem is solved in the new figure, if this figure is transmuted into the first figure by the reverse procedure, the required solution will be obtained.

This lemma is useful also for solving solid problems. For whenever two conics occur by whose intersection a problem can be solved, either one of them, if it is a hyperbola or parabola, can be transmuted into an ellipse; then the ellipse is easily changed into a circle. Likewise, in constructing plane problems, a straight line and a conic are turned into a straight line and a circle.

Proposition 25
Problem 17

To describe a trajectory that will pass through two given points and touch three straight lines given in position.

Through the meeting-point of any two tangents with each other and the meeting-point of a third tangent with the straight line that passes through two given points, draw an indefinite straight line; and using it as the first ordinate radius, transmute the figure, by lem. 22, into a new figure. In this figure the two tangents will come to be parallel to each other, and the third tangent will become parallel to the straight line passing through the two given points. Let hi and kl be the two parallel tangents,



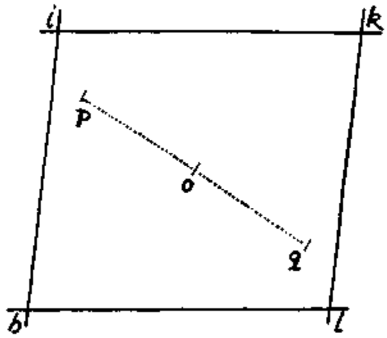
ik the third tangent, and hl the straight line parallel to it, passing through the points a and b through which the conic ought to pass in this new figure and completing the parallelogram $hikl$. Cut the straight lines hi, ik, and kl in c, d, and e, so that hc is to the square root of the rectangle $ah \times hb$, and ic is to id, and ke is to kd, as the sum of the straight lines hi and kl is to the sum of three lines, of which the first is the straight line ik and the other two are the square roots of the rectangles $ah \times hb$ and $al \times lb$; then c, d, and e will be the points of contact. For, from the *Conics*, hc^2 is to the rectangle $ah \times hb$ in the same ratio as ic^2 to id^2, and ke^2 to kd^2, and el^2 to the rectangle $al \times lb$; and therefore hc is to the square root of $ah \times hb$, and ic is to id, and ke is to kd, and el is to the square root of $al \times lb$, as the square root of that ratio and hence, by composition [or componendo], in the given ratio of all the antecedents hi and kl to all the consequents, which are the square root of the rectangle $ah \times hb$, the straight line ik, and the square root of the rectangle $al \times lb$ [i.e., in the given ratio of $hi + kl$ to $\sqrt{(ah \times hb)} + ik + \sqrt{(al \times lb)}$]. Therefore, the points of contact c, d, and e in the new figure are obtained from that given ratio. By the reverse procedure of lem. 22, transfer these points to the first figure, and there (by prop. 22) the trajectory will be described. Q.E.F.

But according as points a and b lie between points h and l or lie outside them, points c, d, and e must be taken either between points h, i, k, and l, or outside them. If either one of the points a and b falls between points h and l, and the other outside, the problem is impossible.

Proposition 26
Problem 18

To describe a trajectory that will pass through a given point and touch four straight lines given in position.

From the common intersection of any two of the tangents to the common intersection of the other two, draw an indefinite straight line; then, using this as the first ordinate radius, transmute the figure (by lem. 22) into

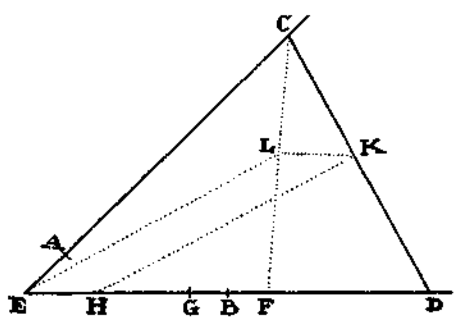


a new figure; then the tangents, which formerly met in the first ordinate radius, will now come to be parallel in pairs. Let those tangents be *hi* and *kl*, *ik* and *hl*, forming the parallelogram *hikl*. And let *p* be the point in this new figure corresponding to the given point in the first figure. Through the center O of the figure draw *pq*, and, on O*q* being equal to O*p*, *q* will be another point through which the conic must pass in this new figure. By the reverse procedure of lem. 22 transfer this point to the first figure, and in that figure two points will be obtained through which the trajectory is to be described. And that trajectory can be described through these same points by prop. 25.

Lemma 23

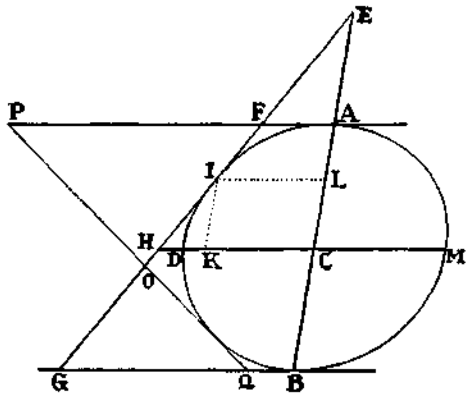
If two straight lines AC *and* BD, *given in position, terminate at the given points* A *and* B *and have a given ratio to each other; and if the straight line* CD, *by which the indeterminate points* C *and* D *are joined, is cut in* K *in a given ratio; I say that point* K *will be located in a straight line given in position.*

For let the straight lines AC and BD meet in E, and in BE take BG to AE as BD is to AC, and let FD always be equal to the given line EG; then, by construction, EC will be to GD, that is, to EF, as AC to BD, and thus in a given ratio, and therefore the species of the triangle EFC will be given. Cut CF in L so that CL is to CF in the ratio of CK to CD; then, because that ratio is given, the species of the triangle EFL will also be given, and accordingly point L will be located in the straight line EL given in position. Join LK, and the triangles CLK and CFD will be similar; and because FD and the ratio of LK to FD are given, LK will be given. Take EH equal to LK, and ELKH will always be a parallelogram. Therefore, point K is located in the side HK, given in position, of the parallelogram. Q.E.D.



Corollary. Because the species of the figure EFLC is given, the three straight lines EF, EL, and EC (that is, GD, HK, and EC) have given ratios to one another.

Lemma 24 *If three straight lines, two of which are parallel and given in position, touch any conic section, I say that the semidiameter of the section which is parallel to the two given parallel lines is a mean proportional between their segments that are intercepted between the points of contact and the third tangent.*



Let AF and GB be two parallel lines touching the conic ADB in A and B; and let EF be a third straight line touching the conic in I and meeting the first tangents in F and G; and let CD be the semidiameter of the figure parallel to the tangents; then I say that AF, CD, and BG are continually proportional.

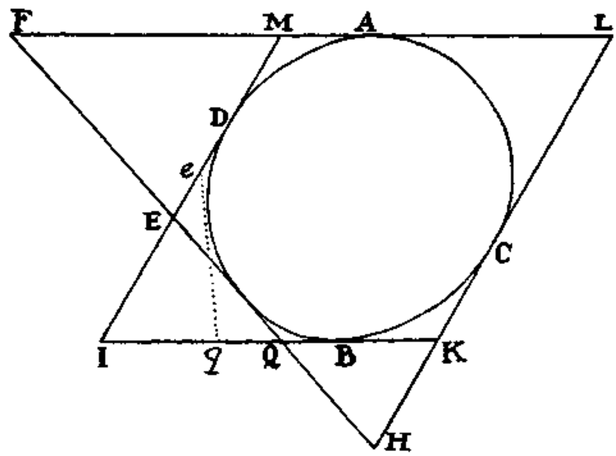
For if the conjugate diameters AB and DM meet the tangent FG in E and H and cut each other in C, and the parallelogram IKCL is completed, then, from the nature of conics, EC will be to CA as CA to CL, and by separation [or dividendo] as EC − CA to CA − CL, or EA to AL; and by composition [or componendo], EA will be to EA + AL or EL as EC to EC + CA or EB; and therefore, because the triangles EAF, ELI, ECH, and EBG are similar, AF will be to LI as CH to BG. And likewise, from the nature of conics, LI or CK is to CD as CD to CH and therefore from the equality of the ratios in inordinate proportion [or ex aequo perturbate] AF will be to CD as CD to BG. Q.E.D.

Corollary 1. Hence if two tangents FG and PQ meet the parallel tangents AF and BG in F and G, P and Q, and cut each other in O; then, from the equality of the ratios in inordinate proportion [or ex aequo perturbate] AF will be to BQ as AP to BG, and by separation [or dividendo] as FP to GQ, and thus as FO to OG.

Corollary 2. Hence also, two straight lines PG and FQ drawn through points P and G, F and Q, will meet in the straight line ACB that passes through the center of the figure and the points of contact A and B.

Lemma 25 *If the indefinitely produced four sides of a parallelogram touch any conic and are intercepted at any fifth tangent, and if the intercepts of any two conterminous sides are taken so as to be terminated at opposite corners of the parallelogram; I say that either intercept is to the side from which it is intercepted as the part of the other conterminous side between the point of contact and the third side is to the other intercept.*

Let the four sides ML, IK, KL, and MI of the parallelogram MLIK touch the conic section in A, B, C, and D, and let a fifth tangent FQ cut those sides in F, Q, H, and E; and take the intercepts ME and KQ of the sides MI and KI or the intercepts KH and MF of the sides KL and ML; I



say that ME is to MI as BK to KQ, and KH is to KL as AM to MF. For by lem. 24, corol. 1, ME is to EI as AM or BK to BQ, and by composition [or componendo] ME is to MI as BK to KQ. Q.E.D. Likewise, KH is to HL as BK or AM to AF, and by separation [or dividendo] KH is to KL as AM to MF. Q.E.D.

Corollary 1. Hence if the parallelogram IKLM is given, described about a given conic, the rectangle KQ × ME will be given, as will also the rectangle KH × MF equal to it. For those rectangles are equal because the triangles KQH and MFE are similar.

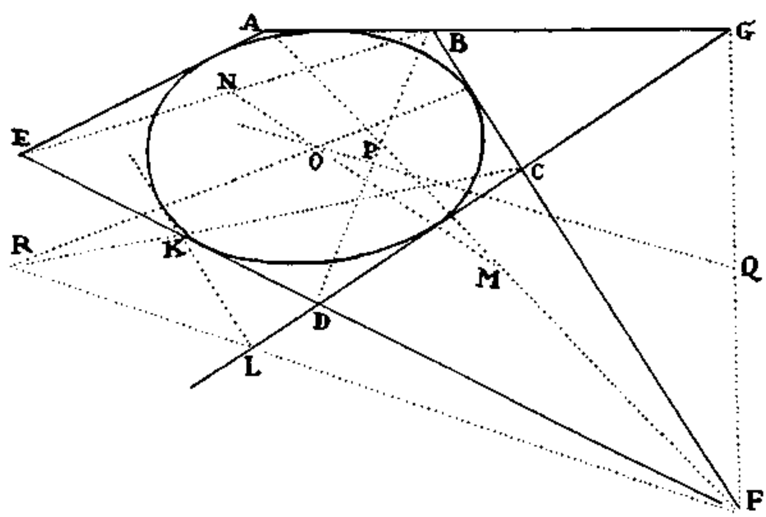
Corollary 2. And if a sixth tangent eq is drawn meeting the tangents KI and MI at q and e, the rectangle KQ × ME will be equal to the rectangle Kq × Me, and KQ will be to Me as Kq to ME, and by separation [or dividendo] as Qq to Ee.

Corollary 3. Hence also, if Eq and eQ are drawn and bisected and a straight line is drawn through the points of bisection, this line will pass through the center of the conic. For since Qq is to Ee as KQ to Me, the same straight line will (by lem. 23) pass through the middle of all the lines Eq, eQ, and MK, and the middle of the straight line MK is the center of the section.

Proposition 27
Problem 19

To describe a trajectory that will touch five straight lines given in position.

Let the tangents ABG, BCF, GCD, FDE, and EA be given in position. Bisect in M and N the diagonals AF and BE of the quadrilateral figure ABFE formed by any four of those tangents, and (by lem. 25, corol. 3) the straight line MN drawn through the points of bisection will pass through the center of the trajectory. Again, bisect in P and Q the diagonals (as I call them) BD and GF of the quadrilateral figure BGFD formed by any other four tangents; then the straight line PQ drawn through the points of

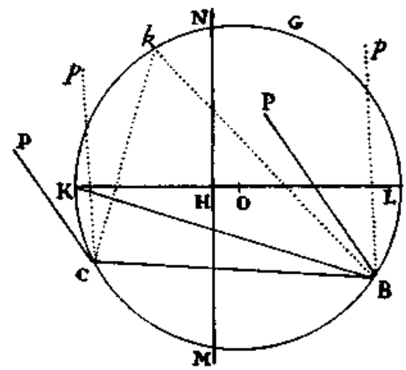


bisection will pass through the center of the trajectory. Therefore, the center will be given at the meeting-point of the bisecting lines. Let that center be O. Parallel to any tangent BC draw KL at such a distance that the center O is located midway between the parallels, and KL so drawn will touch the trajectory to be described. Let this line KL cut any other two tangents GCD and FDE in L and K. Through the meeting-points C and K, F and L, of these nonparallel tangents CL and FK with the parallels CF and KL, draw

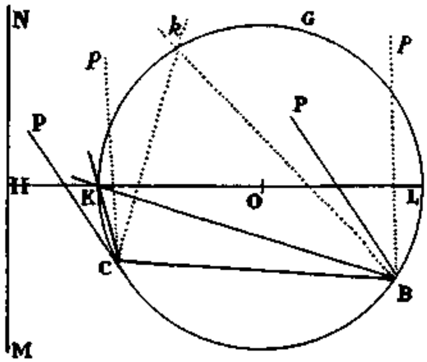
CK and FL meeting in R, and the straight line OR, drawn and produced, will cut the parallel tangents CF and KL in the points of contact. This is evident by lem. 24, corol. 2. By the same method other points of contact may be found, and then finally the trajectory may be described by the construction of prop. 22. Q.E.F.

Scholium

What has gone before includes problems in which either the centers or the asymptotes of trajectories are given. For when points and tangents are given together with the center, the same number of other points and tangents are given equally distant from the center on its other side. Moreover, an asymptote is to be regarded as a tangent, and its infinitely distant end-point (if it is permissible to speak of it in this way) as a point of contact. Imagine the point of contact of any tangent to go off to infinity, and the tangent will be turned into an asymptote, and the constructions of the preceding problems will be turned into constructions in which the asymptote is given.



After the trajectory has been described, its axes and foci may be found by the following method. In the construction and figure of lem. 21 make the legs BP and CP (by the meeting of which the trajectory was there described) of the mobile angles PBN and PCN be parallel to each other, and let them—while maintaining that position—revolve about their poles B and C in that figure. Meanwhile, let the circle BGKC be described by the point K or *k* in which the other legs CN and BN of those angles meet. Let the center of this circle be O. From this center to the ruler MN, at which those other legs CN and BN met while the trajectory was being described, drop the normal OH meeting the circle in K and L. And when those other legs CK and BK meet in K, the point that is nearer to the ruler, the first legs CP and BP will be parallel to the major axis and perpendicular to the minor axis; and the converse will occur if the same legs meet in the farther point L. Hence, if the center of a trajectory is given, the axes will be given. And when these are given, the foci are apparent.



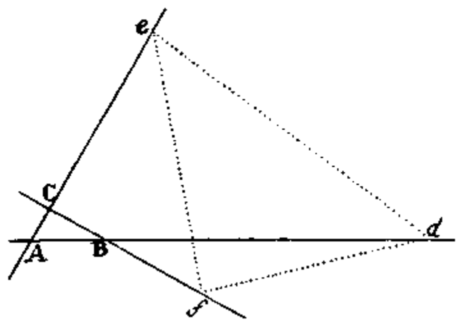
But the squares of the axes are to each other as KH to LH, and hence it is easy to describe a trajectory, given in species, through four given points. For if two of the given points constitute the poles C and B, a third will give the mobile angles PCK and PBK; and once these are given, the circle BGKC can be described. Then, because the species of the trajectory is given, the ratio of OH to OK, and thus OH itself, will be given. With center O and radius OH describe another circle, and the straight line that touches this circle and passes through the meeting-point of the legs CK and BK when the first legs CP and BP meet in the fourth given point will be that ruler MN by means of which the trajectory will be described. Hence, in turn, a quadrilateral given in species can (except in certain impossible cases) also be inscribed in any given conic.

There are also other lemmas by means of which trajectories given in species can be described if points and tangents are given. An example: if a straight line, drawn through any point given in position, intersects a given conic in two points, and the distance between the intersections is bisected, the point of bisection will lie on another conic that is of the same species as the first one and that has its axes parallel to the axes of the first. But I pass quickly to what is more useful.

Lemma 26 *To place the three corners of a triangle given in species and magnitude on three straight lines given in position and not all parallel, with one corner on each line.*

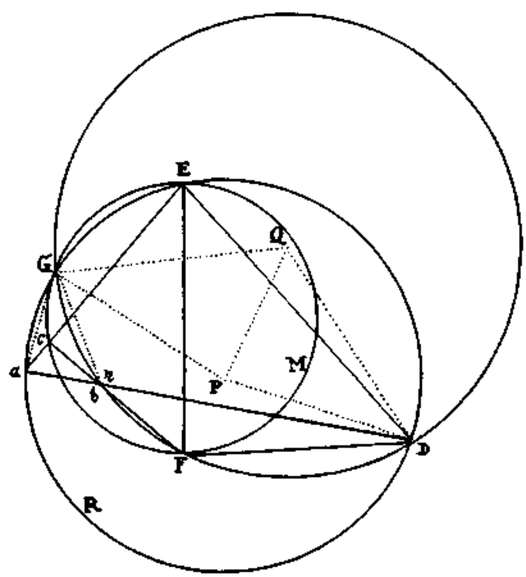
[a]Three indefinite straight lines, AB, AC, and BC, are given in position, and it is required to place triangle DEF in such a way that its corner D touches line AB, corner E line AC, and corner F line BC.[a] On DE, DF, and

aa. In all three editions, and in the preliminary manuscripts (see *The Mathematical Papers of Isaac Newton*, ed. D. T. Whiteside [Cambridge: Cambridge University Press, 1967–1981], 6:287), there is a minor discrepancy between the text and the accompanying diagram. The text refers (in the opening sentence) to "triangle DEF," but the corresponding diagram would indicate that this should rather be "triangle *def*," and similarly "corner [*lit.* vertex] D" and "corner F" should be respectively "corner *d*" and "corner *f*." At the end of the paragraph, however, and in the succeeding paragraph, Newton introduces lowercase letters *a*, *b*, *c* for the triangle *abc*.



EF describe three segments DRE, DGF, and EMF of circles, containing angles equal respectively to angles BAC, ABC, and ACB. And let these segments be described on those sides of the lines DE, DF, and EF that will make the letters DRED go round in the same order as the letters BACB, the letters DGFD in the same order as ABCA, and the letters EMFE in the same order as ACBA; then complete these segments into full circles. Let the first two circles cut each other in G, and let their centers be P and Q. Joining GP and also PQ, take G*a* to AB as GP is to PQ; and with center G and radius G*a* describe a circle that cuts the first circle DGE in *a*. Join *a*D cutting the second circle DFG in *b*, and *a*E cutting the third circle EMF in *c*. And now the figure ABC*def* may be constructed similar and equal to the figure *abc*DEF. This being done, the problem is solved.

For draw F*c* meeting *a*D in *n*, and join *a*G, *b*G, QG, QD, and PD. By construction, angle E*a*D is equal to angle CAB, and angle *ac*F is equal to angle ACB, and thus the angles of triangle *anc* are respectively equal to the angles of triangle ABC. Therefore angle *anc* or F*n*D is equal to angle



ABC, and hence equal to angle F*b*D; and therefore point *n* coincides with point *b*. Further, angle GPQ, which is half of angle GPD at the center, is equal to angle G*a*D at the circumference; and angle GQP, which is half of angle GQD at the center, is equal to the supplement of angle G*b*D at the circumference, and hence equal to angle G*ba*; and therefore triangles GPQ and G*ab* are similar, and G*a* is to *ab* as GP to PQ, that is (by construction), as G*a* to AB. And thus *ab* and AB are equal; and therefore triangles *abc* and ABC, which we have just proved to be similar, are also equal. Hence, since in addition the corners D, E, and F of the triangle DEF touch the sides *ab*, *ac*, and *bc* respectively of the triangle *abc*, the figure ABC*def* can be completed similar and equal to the figure *abc*DEF; and by its completion the problem will be solved. Q.E.F.

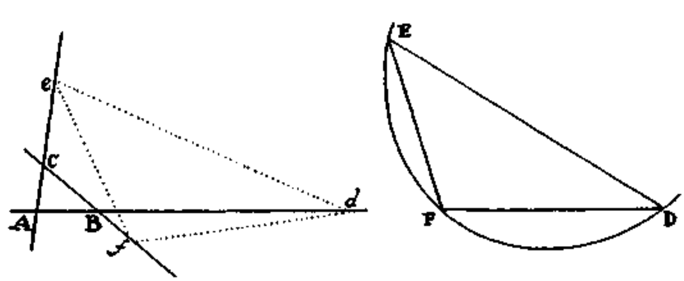
COROLLARY. Hence a straight line can be drawn whose parts given in length will lie between three straight lines given in position. Imagine that triangle DEF, with point D approaching side EF and sides DE and DF placed in a straight line, is changed into a straight line whose given part DE is to be placed between the straight lines AB and AC given in position and whose given part DF is to be placed between the straight lines AB and BC given in position; then, by applying the preceding construction to this case, the problem will be solved.

Proposition 28
Problem 20

To describe a trajectory given in species and magnitude, whose given parts will lie between three straight lines given in position.

Let it be required to describe a trajectory that is similar and equal to the curved line DEF and that will be cut by three straight lines AB, AC, and BC, given in position, into parts similar and equal to the given parts DE and EF of this curved line.

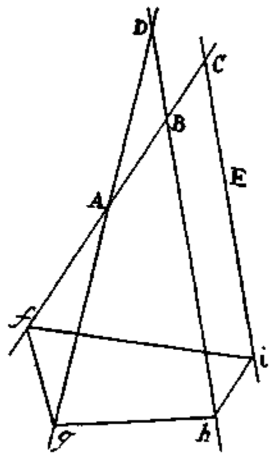
Draw the straight lines DE, EF, and DF, place one of the corners D, E, and F of this triangle DEF on each of those straight lines given in position



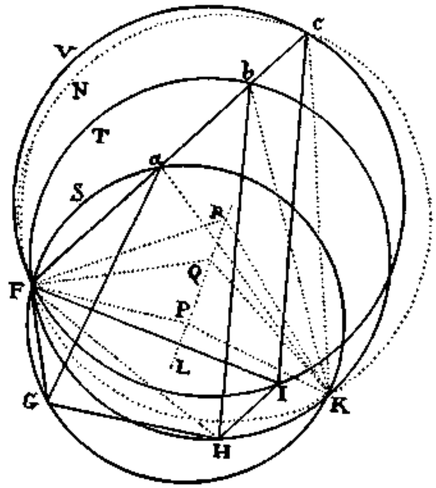
(by lem. 26); then about the triangle describe a trajectory similar and equal to the curve DEF. Q.E.F.

Lemma 27

To describe a quadrilateral given in species, whose corners will lie on four straight lines, given in position, which are not all parallel and do not all converge to a common point—each corner lying on a separate line.



Let four straight lines ABC, AD, BD, and CE be given in position, the first of which cuts the second in A, cuts the third in B, and cuts the fourth in C; let it be required to describe a quadrilateral *fghi* which is similar to the quadrilateral FGHI and whose corner *f*, equal to the given corner F, touches the straight line ABC, and whose other corners *g*, *h*, and *i*, equal to the other given corners G, H, and I, touch the other lines AD, BD, and CE respectively. Join FH, and on FG, FH, and FI describe three segments of circles, FSG, FTH, and FVI, of which the first (FSG) contains an angle equal to angle BAD, the second (FTH) contains an angle equal to angle CBD, and the third (FVI) contains an angle equal to angle ACE. The segments ought, moreover, to be described on those sides of the lines FG, FH, and FI that will make the circular order of the letters FSGF the same as that of the letters BADB, and will make the letters FTHF go round in the same order as CBDC, and the letters FVIF in the same order as ACEA.



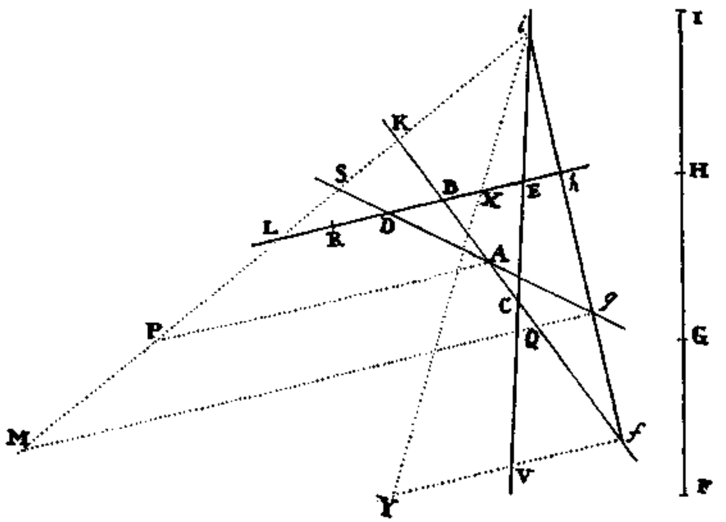
Complete the segments into whole circles, and let P be the center of the first circle FSG, and Q the center of the second circle FTH. Join PQ and produce it in both directions; and in it take QR in the ratio to PQ that BC has to AB. And take QR on the side of the point Q which makes the order of the letters P, Q, and R the same as that of the letters A, B, and C; and then with center R and radius RF describe a fourth circle FN*c* cutting the third circle FVI in *c*. Join F*c* cutting the first circle in *a* and the second in *b*. Draw *a*G, *b*H, and *c*I, and the figure ABC*fghi* can be constructed similar to the figure *abc*FGHI. When this is done, the quadrilateral *fghi* will be the very one which it was required to construct.

For let the first two circles FSG and FTH intersect each other in K. Join PK, QK, RK, *a*K, *b*K, and *c*K, and produce QP to L. The angles F*a*K, F*b*K, and F*c*K at the circumferences are halves of the angles FPK, FQK, and FRK at the centers, and hence are equal to the halves LPK, LQK, and LRK of these angles. Therefore, the angles of figure PQRK are respectively equal to the angles of figure *abc*K, and the figures are similar; and hence *ab* is to *bc* as PQ to QR, that is, as AB to BC. Besides, the angles *f*A*g*, *f*B*h*, and *f*C*i* are (by construction) equal to the angles F*a*G, F*b*H, and F*c*I. Therefore, ABC*fghi*, a figure similar to the figure *abc*FGHI, can be completed. When this is done, the quadrilateral *fghi* will be constructed similar to the quadrilateral FGHI with its corners *f*, *g*, *h*, and *i* touching the straight lines ABC, AD, BD, and CE. Q.E.F.

Corollary. Hence a straight line can be drawn whose parts, intercepted in a given order between four straight lines given in position, will have a given proportion to one another. Increase the angles FGH and GHI until the straight lines FG, GH, and HI lie in a single straight line; and by constructing the problem in this case, a straight line *fghi* will be drawn, whose parts *fg*, *gh*, and *hi*, intercepted between four straight lines given in position, AB and AD, AD and BD, BD and CE, will be to one another as the lines FG, GH, and HI, and will keep the same order with respect to one another. But the same thing is done more expeditiously as follows.

Produce AB to K and BD to L so that BK is to AB as HI to GH, and DL to BD as GI to FG; and join KL meeting the straight line CE in *i*. Produce *i*L to M, so that LM is to *i*L as GH to HI; and draw MQ parallel to LB and meeting the straight line AD in *g*, and draw *gi* cutting AB and BD in *f* and *h*. I declare it done.

For let M*g* cut the straight line AB in Q, and let AD cut the straight line KL in S, and draw AP parallel to BD and meeting *i*L in P; then *g*M will be to L*h* (*gi* to *hi*, M*i* to L*i*, GI to HI, AK to BK) and AP to BL in the same ratio. Cut DL in R so that DL is to RL in that same ratio; then, because *g*S to *g*M, AS to AP, and DS to DL are proportional, from the equality of



the ratios [or ex aequo] AS*a* will be to BL, and DS to RL, as *g*S to L*h*, and by a mixture of operations BL − RL will be to L*h* − BL as AS − DS to *g*S − AS. That is, BR will be to B*h* as AD to A*g* and thus as BD to *g*Q. And by alternation [or alternando] BR is to BD as B*h* to *g*Q or as *fh* to *fg*. But by construction the line BL was cut in D and R in the same ratio as the line FI in G and H; and therefore BR is to BD as FH to FG. As a result, *fh* is to *fg* as FH to FG. Therefore, since *gi* is also to *hi* as M*i* to L*i*, that is, as GI to HI, it is evident that the lines FI and *fi* are similarly cut in *g* and *h*, G and H. Q.E.F.

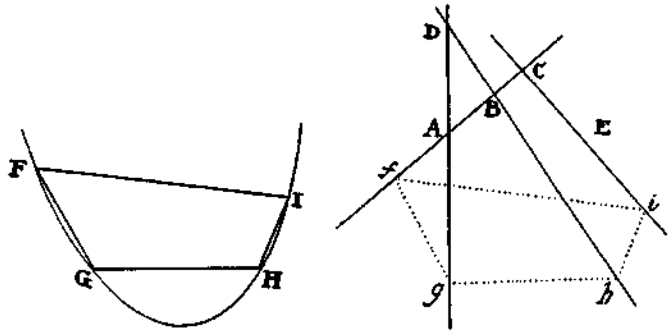
In the construction of this corollary, after LK is drawn cutting CE in *i*, it is possible to produce *i*E to V, so that EV is to E*i* as FH to HI, and then to draw V*f* parallel to BD. It comes to the same thing if with center *i* and radius IH a circle is described cutting BD in X, and if *i*X is produced to Y, so that *i*Y is equal to IF, and if Y*f* is drawn parallel to BD.

Other solutions of this problem were devised some time ago by Wren and Wallis.

Proposition 29
Problem 21

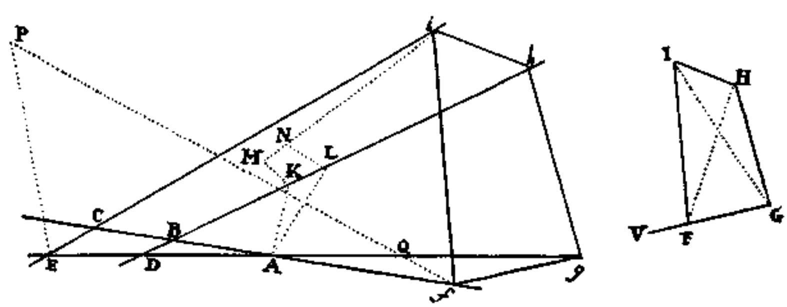
To describe a trajectory, given in species, which four straight lines given in position will cut into parts given in order, species, and proportion.

Let it be required to describe a trajectory that is similar to the curved line FGHI and whose parts, similar and proportional to the parts FG, GH, and HI of the curve, are intercepted between the straight lines AB and AD,



AD and BD, BD and CE given in position, the first part between the first two lines, the second between the second two lines, and the third between the third two lines. After drawing the straight lines FG, GH, HI, and FI, describe (by lem. 27) a quadrilateral *fghi* that is similar to the quadrilateral FGHI and whose corners *f*, *g*, *h*, and *i* touch the straight lines AB, AD, BD, and CE, given in position, each corner touching a separate line in the order stated. Then about this quadrilateral describe a trajectory exactly similar to the curved line FGHI.

Scholium This problem can also be constructed as follows. After joining FG, GH, HI, and FI, produce GF to V, join FH and IG, and make angles CAK and DAL equal to angles FGH and VFH. Let AK and AL meet the straight line BD in K and L, and from these points draw KM and LN, of which KM makes an angle AKM equal to angle GHI and is to AK as HI is to GH, and LN makes an angle ALN equal to angle FHI and is to AL as HI to FH. And draw AK, KM, AL, and LN on those sides of the lines AD, AK, and AL that will make the letters CAKMC, ALKA, and DALND go round in the same order as the letters FGHIF; and draw MN meeting the straight line CE in *i*. Make angle *i*EP equal to the angle IGF, and let PE



be to Ei as FG to GI; and through P draw PQf, which with the straight line ADE contains the angle PQE equal to the angle FIG and meets the straight line AB in f; and join fi. Now draw PE and PQ on those sides of the lines CE and PE that will make the circular order of the letters PEiP and PEQP the same as that of the letters FGHIF; and then, if on line fi a quadrilateral $fghi$ similar to the quadrilateral FGHI is constructed (with the same order of the letters), and a trajectory given in species is circumscribed about the quadrilateral, the problem will be solved.

So much for the finding of orbits. It remains to determine the motions of bodies in the orbits that have been found.

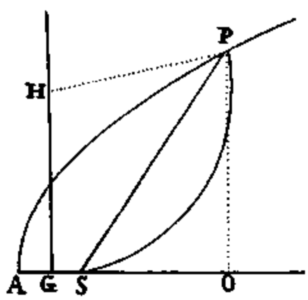
SECTION 6

To find motions in given orbits

Proposition 30
Problem 22

If a body moves in a given parabolic trajectory, to find its position at an assigned time.

Let S be the focus and A the principal vertex of the parabola, and let $4AS \times M$ be equal to the parabolic area APS to be cut off, which either was described by the radius SP after the body's departure from the vertex or is to be described by that radius before the body's arrival at the vertex. The quantity of that area to be cut off can be found from the time, which is proportional to it. Bisect AS in G, and erect the perpendicular GH equal to 3M, and a circle described with center H and radius HS will cut the parabola in the required place P. For,



when the perpendicular PO has been dropped to the axis and PH has been drawn, then $AG^2 + GH^2$ $(= HP^2 = (AO - AG)^2 + (PO - GH)^2) =$ $AO^2 + PO^2 - 2GA \times AO - 2GH \times PO + AG^2 + GH^2$. Hence $2GH \times PO$ $(= AO^2 + PO^2 - 2GA \times AO) = AO^2 + \frac{3}{4}PO^2$. For AO^2 write $\dfrac{AO \times PO^2}{4AS}$, and if all the terms are divided by 3PO and multiplied by 2AS, it will result that $\frac{4}{3}GH \times AS \left(= \frac{1}{6}AO \times PO + \frac{1}{2}AS \times PO = \dfrac{AO + 3AS}{6} \times PO = \dfrac{4AO - 3SO}{6} \times PO = \text{area } (APO - SPO)\right) = \text{area } APS$. But GH was 3M, and hence $\frac{4}{3}GH \times AS$ is $4AS \times M$. Therefore, the area APS that was cut off is equal to the area $4AS \times M$ that was to be cut off. Q.E.D.

Corollary 1. Hence GH is to AS as the time in which the body described the arc AP is to the time in which it described the arc between the vertex A and a perpendicular erected from the focus S to the axis.

Corollary 2. And if a circle ASP continually passes through the moving body P, the velocity of point H is to the velocity which the body had at the vertex A as 3 to 8, and thus the line GH is also in this ratio to the straight line which the body could describe in the time of its motion from A to P with the velocity which it had at the vertex A.

COROLLARY 3. Hence also, conversely, the time can be found in which the body described any assigned arc AP. Join AP and at its midpoint erect a perpendicular meeting the straight line GH in H.

Lemma 28

No oval figure exists whose area, cut off by straight lines at will, can in general be found by means of equations finite in the number of their terms and dimensions.

Within an oval let any point be given about which, as a pole, a straight line revolves continually with uniform motion, and meanwhile in that straight line let a mobile point go out from the pole and proceed always with the velocity that is as the square of that straight line within the oval. By this motion that point will describe a spiral with an infinite number of gyrations. Now, if the portion of the area of the oval cut off by that straight line can be found by means of a finite equation, there will also be found by the same equation the distance of the point from the pole, a distance that is proportional to this area, and thus all the points of the spiral can be found by means of a finite equation; and therefore the intersection of any straight line, given in position, with the spiral can also be found by means of a finite equation. But every infinitely produced straight line cuts a spiral in an infinite number of points; and the equation by which some intersection of two lines [i.e., curved lines] is found gives all their intersections by as many roots [as there are intersections] and therefore rises to as many dimensions as there are intersections. Since two circles cut each other in two points, one intersection will not be found except by an equation of two dimensions, by which the other intersection may also be found. Since two conics can have four intersections, one of these intersections cannot generally be found except by an equation of four dimensions, by means of which all four of the intersections may be found simultaneously. For if those intersections are sought separately, since they all have the same law and condition, the computation will be the same in each case, and therefore the conclusion will always be the same, which accordingly must comprehend all the intersections together and give them indiscriminately. Hence also the intersections of conics and of curves of the third power, because there can be six such intersections, are found simultaneously by equations of six dimensions; and intersections of two curves of the third power, since there can be nine of them, are found simultaneously by equations of nine dimensions. If this did not happen necessarily, all solid problems might be reduced to plane problems, and higher than solid to solid problems. I am speaking here of curves with a power that cannot be reduced. For if the equation by which the curve is defined can be reduced to a lower power, the curve will not be simple, but will be compounded of two or more curves whose intersections can be found separately by different computations. In the same way, the pairs of intersections of straight lines and conics are always found by equations of two dimensions; the trios of intersections of straight lines and of irreducible curves of the third power, by equations of three dimensions; the quartets of intersections of straight lines and of irreducible curves of the fourth power, by equations of four dimensions; and so on indefinitely. Therefore, the intersections of a straight line and of a spiral, which are infinite in number (since this curve is simple and cannot be reduced to more curves), require equations infinite in the number of their dimensions and roots, by which all the intersections can be given simultaneously. For they all have the same law and computation. For if a perpendicular is dropped from the pole to the intersecting straight line, and the perpendicular, together with the intersecting straight line, revolves about the pole, the intersections of the spiral will pass into one another, and the one that was the first or the nearest to the pole will be the second after one revolution, and after two revolutions will be third, and so on; nor will the equation change in the meantime except insofar as there is a change in the magnitude of the quantities by which the position of the intersecting line is determined. Hence, since the quantities return to their initial magnitudes after each revolution, the equation will return to its original form, and thus one and the same equation will give all the intersections and therefore will have an infinite number of roots by which all of the intersections can be given. Therefore, it is not possible for the intersection of a straight line and a spiral to be found universally by means of a finite equation, and on that account no oval exists whose area, cut off by prescribed straight lines, can universally be found by such an equation.

By the same argument, if the distance between the pole and the point by which the spiral is described is taken proportional to the intercepted part of the perimeter of the oval, it can be proved that the length of the perimeter cannot universally be found by a finite equation. [a]But here I am speaking of ovals that are not touched by conjugate figures extending out to infinity.[a]

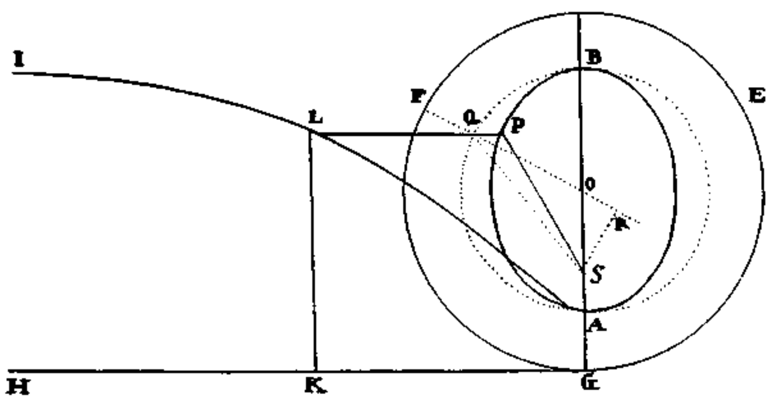
aa. This concluding sentence appeared for the first time in ed. 2.

Corollary. Hence the area of an ellipse that is described by a radius drawn from a focus to a moving body cannot be found, from a time that has been given, by means of a finite equation, and therefore cannot be determined by describing geometrically rational curves. I call curves "geometrically rational" when all of their points can be determined by lengths defined by equations, that is, by involved ratios of lengths, and I call the other curves (such as spirals, quadratrices, and cycloids) "geometrically irrational." For lengths that are or are not as integer to integer (as in book 10 of the *Elements*) are arithmetically rational or irrational. Therefore I cut off an area of an ellipse proportional to the time by a geometrically irrational curve as follows.

Proposition 31[a]
Problem 23

If a body moves in a given elliptical trajectory, to find its position at an assigned time.

Let A be the principal vertex of the ellipse APB, S a focus, and O the center, and let P be the position of the body. Produce OA to G so that OG is to OA as OA to OS. Erect the perpendicular GH, and with center O and radius OG describe the circle GEF; then, along the rule GH as a base let the wheel GEF move progressively forward, revolving about its own axis,



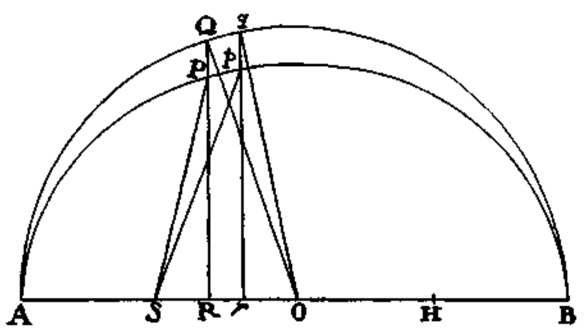
while the point A on the wheel describes the cycloid ALI. When this has been done, take GK so that it will have the same ratio to the perimeter GEFG of the wheel as the time in which the body, in moving forward from A, described the arc AP has to the time of one revolution in the ellipse.

a. In the index prepared by Cotes for ed. 2 and retained in ed. 3, this proposition is keyed under "Problematis" ("of the problem") and characterized as follows: "Solution of Kepler's problem by the cycloid and by approximations."

Erect the perpendicular KL meeting the cycloid in L; and when LP has been drawn parallel to KG, it will meet the ellipse in the required position P of the body.

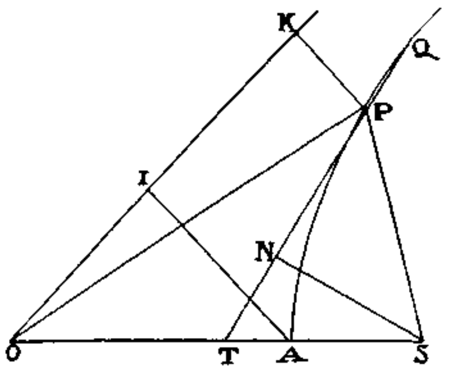
For with center O and radius OA describe the semicircle AQB, and let LP, produced if necessary, meet the arc AQ in Q, and join SQ and also OQ. Let OQ meet the arc EFG in F, and drop the perpendicular SR to OQ. Area APS is as area AQS, that is, as the difference between sector OQA and triangle OQS, or as the difference of the rectangles $\frac{1}{2}OQ \times AQ$ and $\frac{1}{2}OQ \times SR$, that is, because $\frac{1}{2}OQ$ is given, as the difference between the arc AQ and the straight line SR, and hence (because of the equality of the given ratios of SR to the sine of the arc AQ, OS to OA, OA to OG, AQ to GF, and so by separation [or dividendo] $AQ - SR$ to $GF -$ the sine of the arc AQ) as GK, the difference between the arc GF and the sine of the arc AQ. Q.E.D.

Scholium But the description of this curve is difficult; hence it is preferable to use a solution that is approximately true. Find a certain angle B that is to the angle of 57.29578° (which an arc equal to the radius subtends) as the distance SH between the foci is to the diameter AB of the ellipse; and also find a certain length L that is to the radius in the inverse of that ratio. Once these have been found, the problem can thereupon be solved by the following analysis.



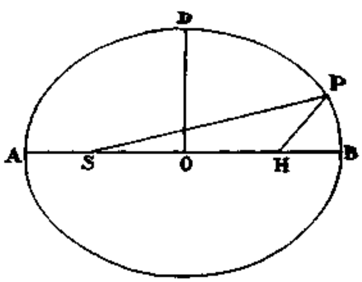
By any construction, or by making any kind of guess, find the body's position P very close to its true position p. Then, when the ordinate PR has been dropped to the axis of the ellipse, the ordinate RQ of the circumscribed circle AQB will be given from the proportion of the diameters of the ellipse, where the ordinate RQ is the sine of the angle AOQ (AO being the radius) and cuts the ellipse in P. It is sufficient to find this angle AOQ approximately by a rough numerical calculation. Also find the angle proportional to the time, that is, the angle that is to four right angles as the time in which the body described the arc Ap is to the time of one revolution in the ellipse. Let that angle be N. Then take an angle D that will be to angle B as the sine of angle AOQ is to the radius, and also take an angle E that will be to angle $N - AOQ + D$ as the length L is to this same length L minus the cosine of angle AOQ when that angle is less than a right angle, but plus that cosine when it is greater. Next take an angle F that will be to angle B as the sine of angle $AOQ + E$ is to the radius, and take an angle G that will be to angle $N - AOQ - E + F$ as the length L is to this same length minus the cosine of angle $AOQ + E$ when that angle is less than a right angle, and plus that cosine when it is greater. Thirdly, take an angle H that will be to angle B as the sine of angle $AOQ + E + G$ is to the radius, and take an angle I that will be to angle $N - AOQ - E - G + H$ as the length L is to this same length L minus the cosine of angle $AOQ + E + G$ when that angle is less than a right angle, but plus that cosine when it is greater. And so on indefinitely. Finally take angle AOq equal to angle $AOQ + E + G + I + \cdots$. And from its cosine Or and ordinate pr, which is to its sine qr as the minor axis of the ellipse to the major axis, the body's corrected place p will be found. If the angle $N - AOQ + D$ is negative, the $+$ sign of E must everywhere be changed to $-$, and the $-$ sign to $+$. The same is to be understood of the signs of G and I when the angles $N - AOQ - E + F$ and $N - AOQ - E - G + H$ come out negative. But the infinite series $AOQ + E + G + I + \cdots$ converges so very rapidly that it is scarcely ever necessary to proceed further than the second term E. And the computation is based on this theorem: that the area APS is as the difference between the arc AQ and the straight line dropped perpendicularly from the focus S to the radius OQ.

In the case of a hyperbola the problem is solved by a similar computation. Let O be its center, A a vertex, S a focus, and OK an asymptote. Find the quantity of the area to be cut off, which is proportional to the time. Let this quantity be A, and guess the position of the straight line SP that cuts off an approximately true area APS. Join OP, and from A and P to the asymptote OK draw AI and PK parallel to the second asymptote; then a table of logarithms will give the area AIKP and the equal area OPA,



which, on being subtracted from the triangle OPS, will leave the cut-off area APS. Divide 2APS − 2A or 2A − 2APS (twice the difference of the area A to be cut off and the cut-off area APS) by the line SN, which is perpendicular to the tangent TP from the focus S, so as to obtain the length of the chord PQ. Now, draw the chord PQ between A and P if the cut-off area APS is greater than the area A to be cut off, but otherwise draw PQ on the opposite side of point P, and then the point Q will be a more accurate position of the body. And by continually repeating the computation, a more and more accurate position will be obtained.

And by these computations a general analytical solution of the problem is achieved. But the particular computation that follows is more suitable for astronomical purposes. Let AO, OB, and OD be the semiaxes of the ellipse, and L its latus rectum, and D the difference between the semiaxis minor OD and half of the latus rectum ½L; find an angle Y, whose sine is to the radius as the rectangle of that difference D and the half-sum of the axes AO + OD is to the square of the major axis AB; and find also an angle Z, whose sine is to the radius as twice the rectangle of the distance SH between the foci and the difference D is to three times the square of the semiaxis major AO. Once these angles have been found, the position of the body will thereupon be determined as follows: Take an angle T proportional to the time in which arc BP was described, or equal to the mean motion (as it is called); and an angle V (the first equation of the mean motion) that shall be to angle Y (the greatest first equation) as the sine of twice angle T is to the radius; and an angle X (the second equation) that shall be to angle Z (the greatest second equation) as the cube of the sine of angle T is to the cube of the radius. Then take the angle BHP (the equated mean motion) equal either to the sum T + X + V of angles T, X, and V if angle T is less than a right angle, or to the difference T + X − V if angle



T is greater than a right angle and less than two right angles; and if HP meets the ellipse in P, SP (when drawn) will cut off the area BSP very nearly proportional to the time.

This technique seems expeditious enough because it is sufficient to find the first two or three figures of the extremely small angles V and X (reckoned in seconds, if it is agreeable). This technique is also accurate enough for the theory of the planets. For even in the orbit of Mars itself, whose greatest equation of the center is ten degrees, the error will hardly exceed one second. But when the angle BHP of equated mean motion has been found, the angle BSP of true motion and the distance SP are readily found by the very well known method.

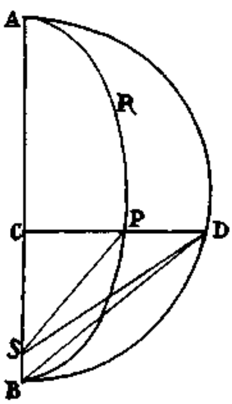
So much for the motion of bodies in curved lines. It can happen, however, that a moving body descends straight down or rises straight up; and I now go on to expound what relates to motions of this sort.

SECTION 7

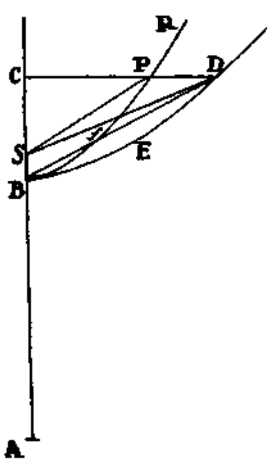
The rectilinear ascent and descent of bodies

Proposition 32[a] **Problem 24** *Given a centripetal force inversely proportional to the square of the distance of places from its center, to determine the spaces which a body in falling straight down describes in given times.*

Case 1. If the body does not fall perpendicularly, it will (by prop. 13, corol. 1) describe some conic having a focus coinciding with the center of forces. Let the conic be ARPB, and its focus S. And



first, if the figure is an ellipse, on its major axis AB describe the semicircle ADB, and let the straight line DPC pass through the falling body and be perpendicular to the axis; and when DS and PS have been drawn, area ASD will be proportional to area ASP and thus also to the time. Keeping the axis AB fixed, continually diminish the width of the ellipse, and area ASD will always remain proportional to the time. Diminish that width indefinitely; and, the orbit APB now coming to coincide with the axis AB, and the focus S with the terminus B of the axis, the body will descend in the straight line AC, and the area ABD will become proportional to the time. Therefore



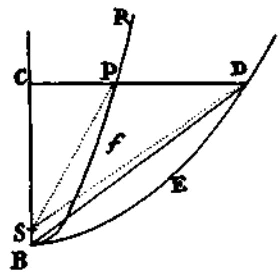
the space AC will be given, which the body in falling perpendicularly from place A describes in a given time, provided that area ABD is taken proportional to that time and the perpendicular DC is dropped from point D to the straight line AB. Q.E.I.

Case 2. If the figure RPB is a hyperbola, describe the rectangular hyperbola BED on the same principal diameter AB; and since the areas CSP, CBfP, and SPfB are respectively to the areas CSD, CBED, and SDEB in the given ratio of the distances CP and CD, and the area SPfB is

a. For a gloss on this proposition see the Guide, §10.11.

proportional to the time in which body P will move through the arc P*f*B, the area SDEB will also be proportional to that same time. Diminish the latus rectum of the hyperbola RPB indefinitely, keeping the principal diameter fixed, and the arc PB will coincide with the straight line CB, and the focus S with the vertex B, and the straight line SD with the straight line BD. Accordingly, the area BDEB will be proportional to the time in which body C, falling straight down, describes the line CB. Q.E.I.

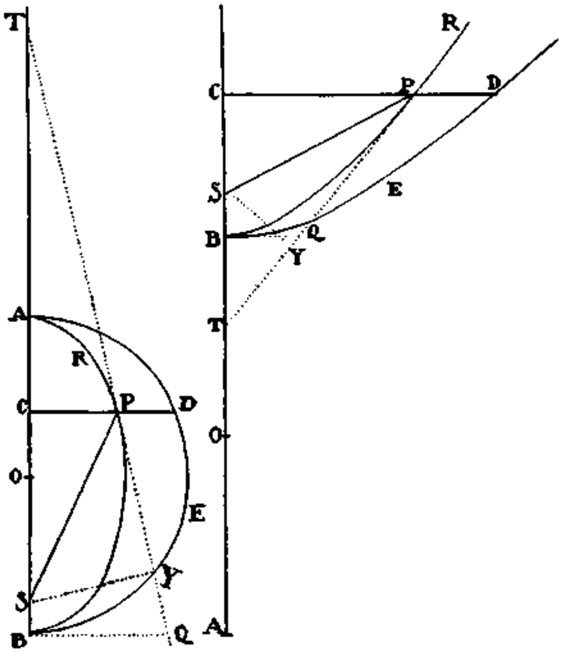
CASE 3. And by a similar argument, let the figure RPB be a parabola and let another parabola BED with the same principal vertex B be described and always remain given, while the latus rectum of the first parabola (in whose perimeter the body P moves) is diminished and reduced to nothing, so that this parabola comes to coincide with the line CB; then the parabolic segment BDEB will become proportional to the time in which the body P or C will descend to the center S or B. Q.E.I.



Proposition 33
Theorem 9

Supposing what has already been found, I say that the velocity of a falling body at any place C *is to the velocity of a body describing a circle with center* B *and radius* BC *as the square root of the ratio of* AC *(the distance of the body from the further vertex* A *of the circle or rectangular hyperbola) to* ½AB *(the principal semidiameter of the figure).*

Bisect AB, the common diameter of both figures RPB and DEB, in O; and draw the straight line PT touching the figure RPB in P and also cutting the common diameter AB (produced if necessary) in T, and let SY be perpendicular to this straight line and BQ be perpendicular to this diameter, and take the latus rectum of the figure RPB to be L. It is established by prop. 16, corol. 9, that at any place P the velocity of a body moving about the center S in the [curved] line RPB is to the velocity of a body describing a circle about the same center with the radius SP as the square root of the ratio of the rectangle $\frac{1}{2}\mathrm{L} \times \mathrm{SP}$ to SY^2. But from the *Conics*, $\mathrm{AC} \times \mathrm{CB}$ is to CP^2 as 2AO to L, and thus $\dfrac{2\mathrm{CP}^2 \times \mathrm{AO}}{\mathrm{AC} \times \mathrm{CB}}$ is equal to L. Therefore, the velocities are to each other as the square root of the ratio of $\dfrac{\mathrm{CP}^2 \times \mathrm{AO} \times \mathrm{SP}}{\mathrm{AC} \times \mathrm{CB}}$ to SY^2. Further, from the *Conics*, CO is to BO as BO to TO, and by composition [or componendo] or by separation [or dividendo], as CB to BT. Hence, either



by separation or by composition, BO $\mp$ CO becomes to BO as CT to BT, that is, AC to AO as CP to BQ; and hence $\dfrac{CP^2 \times AO \times SP}{AC \times CB}$ is equal to $\dfrac{BQ^2 \times AC \times SP}{AO \times BC}$. Now let the width CP of the figure RPB be diminished indefinitely, in such a way that point P comes to coincide with point C and point S with point B and the line SP with the line BC and the line SY with the line BQ; then the velocity of the body now descending straight down in the line CB will become to the velocity of a body describing a circle with center B and radius BC as the square root of the ratio of $\dfrac{BQ^2 \times AC \times SP}{AO \times BC}$ to SY^2, that is (neglecting the ratios of equality SP to BC and BQ^2 to SY^2), as the square root of the ratio of AC to AO or ½AB. Q.E.D.

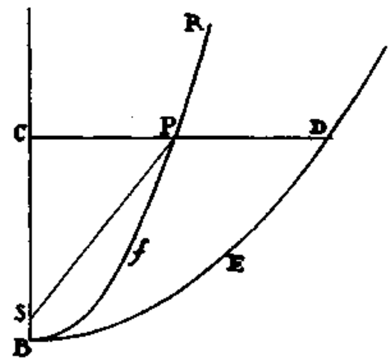
Corollary 1. When the points B and S come to coincide, TC becomes to TS as AC to AO.

Corollary 2. A body revolving in any circle at a given distance from the center will, when its motion is converted to an upward motion, ascend to twice that distance from the center.

Proposition 34
Theorem 10

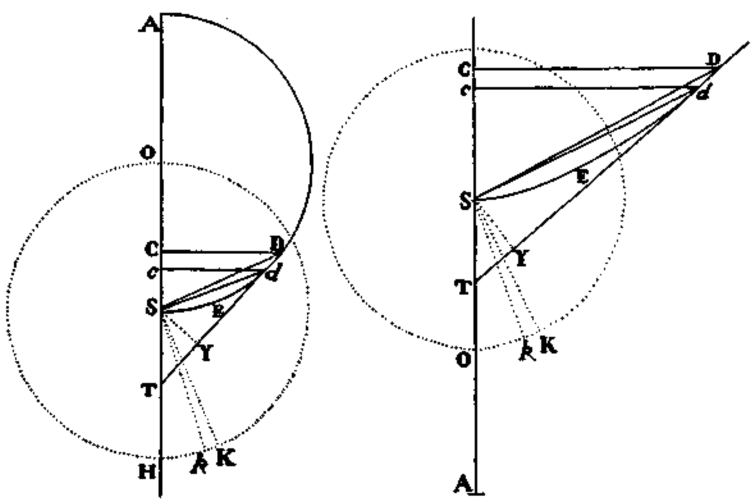
If the figure BED *is a parabola, I say that the velocity of a falling body at any place* C *is equal to the velocity with which a body can uniformly describe a circle with center* B *and a radius equal to one-half of* BC.

For at any place P the velocity of a body describing the parabola RPB about the center S is (by prop. 16, corol. 7) equal to the velocity of a body uniformly describing a circle about the same center S with a radius equal to half of the interval SP. Let the width CP of the parabola be diminished indefinitely, so that the parabolic arc P*f*B will come to coincide with the straight line CB, the center S with the vertex B, and the interval SP with the interval BC, and the proposition will be established. Q.E.D.



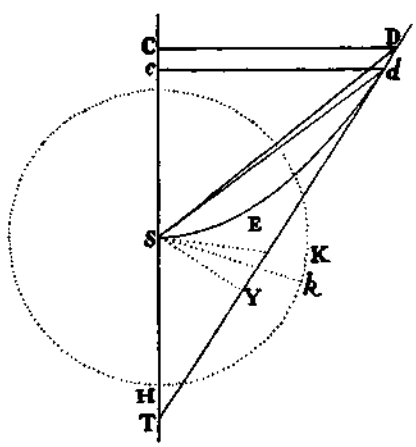
Proposition 35
Theorem 11

Making the same suppositions, I say that the area of the figure DES *described by the indefinite radius* SD *is equal to the area that a body revolving uniformly in orbit about the center* S *can describe in the same time by a radius equal to half of the latus rectum of the figure* DES.



For suppose that body C falling in a minimally small particle of time describes the line-element C*c* while another body K, revolving uniformly in the circular orbit OK*k* about the center S, describes the arc K*k*. Erect the perpendiculars CD and *cd* meeting the figure DES in D and *d*. Join SD, S*d*, SK, and S*k*, and draw D*d* meeting the axis AS in T, and drop the perpendicular SY to D*d*.

Case 1. Now, if the figure DES is a circle or a rectangular hyperbola, bisect its transverse diameter AS in O, and SO will be half of the latus rectum. And since TC is to TD as C*c* to D*d*, and TD to TS as CD to SY, from the equality of the ratios [or ex aequo] TC will be to TS as CD × C*c* to SY × D*d*. But (by prop. 33, corol. 1) TC is to TS as AC to AO, if, say, in the coming together of points D and *d* the ultimate ratios of the lines are taken. Therefore, AC is to AO or SK as CD × C*c* is to SY × D*d*. Further, the velocity of a descending body at C is to the velocity of a body describing a circle about the center S with radius SC as the square root of the ratio of AC to AO or SK (by prop. 33). And this velocity is to the velocity of a body describing the circle OK*k* as the square root of the ratio of SK to SC (by prop. 4, corol. 6), and from the equality of the ratios [or ex aequo] the first velocity is to the ultimate velocity, that is, the line-element C*c* is to the arc K*k*, as the square root of the ratio of AC to SC, that is, in the ratio of AC to CD. Therefore, CD × C*c* is equal to AC × K*k*, and thus AC is to SK as AC × K*k* to SY × D*d*, and hence SK × K*k* is equal to SY × D*d*, and $\frac{1}{2}$SK × K*k* is equal to $\frac{1}{2}$SY × D*d*, that is, the area KS*k* is equal to the area SD*d*. Therefore, in each particle of time, particles KS*k* and SD*d* of the two areas are generated such that, if their magnitude is diminished and their number increased indefinitely, they obtain the ratio of equality; and therefore (by lem. 4, corol.), the total areas generated in the same times are always equal. Q.E.D.



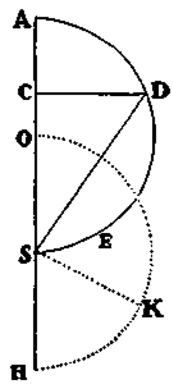
Case 2. But if the figure DES is a parabola, then it will be found that, as above, CD × C*c* is to SY × D*d* as TC to TS, that is, as 2 to 1, and thus $\frac{1}{4}$CD × C*c* will be equal to $\frac{1}{2}$SY × D*d*. But the velocity of the falling body at C is equal to the velocity with which a circle can be described uniformly with the radius $\frac{1}{2}$SC (by prop. 34). And this velocity is to the velocity with which a circle

can be described with the radius SK, that is, the line-element Cc is to the arc Kk (by prop. 4, corol. 6), as the square root of the ratio of SK to ½SC, that is, in the ratio of SK to ½CD. And therefore ½SK × Kk is equal to ¼CD × Cc and thus equal to ½SY × Dd; that is, the area KSk is equal to the area SDd, as above. Q.E.D.

Proposition 36
Problem 25

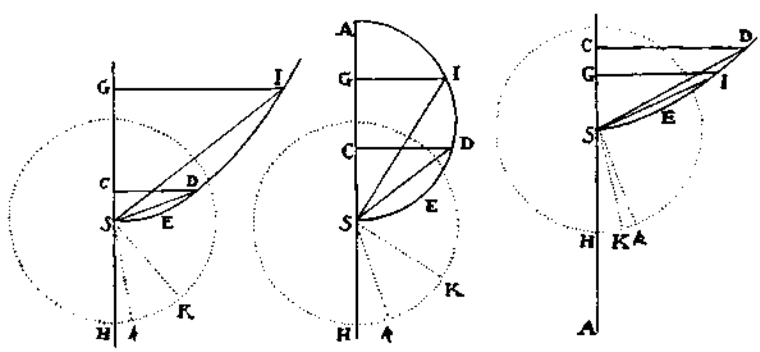
To determine the times of descent of a body falling from a given place A.

Describe a semicircle ADS with diameter AS (the distance of the body from the center at the beginning of the descent), and about the center S describe a semicircle OKH equal to ADS. From any place C of the body erect the ordinate CD. Join SD, and construct the sector OSK equal to the area ASD. It is evident by prop. 35 that the body in falling will describe the space AC in the same time in which another body, revolving uniformly in orbit about the center S, can describe the arc OK. Q.E.F.



Proposition 37
Problem 26

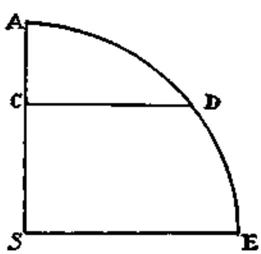
To define the times of the ascent or descent of a body projected upward or downward from a given place.



Let the body depart from the given place G along the line GS with any velocity whatever. Take GA to ½AS as the square of the ratio of this velocity to the uniform velocity in a circle with which the body could revolve about the center S at the given interval (or distance) SG. If that ratio is as 2 to 1, point A is infinitely distant, in which case a parabola is to be described with vertex S, axis SG, and any latus rectum, as is evident by prop. 34. But if that ratio is smaller or greater than the ratio of 2 to 1, then in the former case a circle, and in the latter case a rectangular hyperbola, must be described on the diameter SA, as is evident by prop. 33. Then, with center S and a radius equaling half of the latus rectum, describe the circle HkK, and to the place G of the descending or ascending body and to any other place C, erect the perpendiculars GI and CD meeting the conic or the circle in I and D. Then joining SI and SD, let the sectors HSK and HSk be made equal to the segments SEIS and SEDS, and by prop. 35 the body G will describe the space GC in the same time as the body K can describe the arc Kk. Q.E.F.

Proposition 38
Theorem 12

Supposing that the centripetal force is proportional to the height or distance of places from the center, I say that the times of falling bodies, their velocities, and the spaces described are proportional respectively to the arcs, the right sines, and the versed sines.



Let a body fall from any place A along the straight line AS; and with center of forces S and radius AS describe the quadrant AE of a circle, and let CD be the right sine of any arc AD; then the body A, in the time AD, will in falling describe the space AC and at place C will acquire the velocity CD.

This is demonstrated from prop. 10 in the same way that prop. 32 was demonstrated from prop. 11.

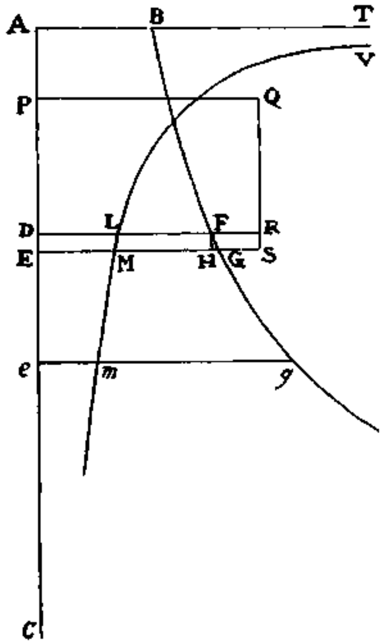
COROLLARY 1. Hence the time in which one body, falling from place A, arrives at the center S is equal to the time in which another body, revolving, describes the quadrantal arc ADE.

COROLLARY 2. Accordingly, all the times are equal in which bodies fall from any places whatever as far as to the center. For all the periodic times of revolving bodies are (by prop. 4, corol. 3) equal.

Proposition 39
Problem 27

Suppose a centripetal force of any kind, and grant the quadratures of curvilinear figures; it is required to find, for a body ascending straight up or descending straight down, the velocity in any of its positions and the time in which the body will reach any place; and conversely.

Let a body E fall from any place A whatever in the straight line ADEC, and let there be always erected from the body's place E the perpendicular



EG, proportional to the centripetal force in that place tending toward the center C; and let BFG be the curved line which the point G continually traces out. At the very beginning of the motion let EG coincide with the perpendicular AB; then the velocity of the body in any place E will be as the straight line whose square is equal to the curvilinear area ABGE. Q.E.I.

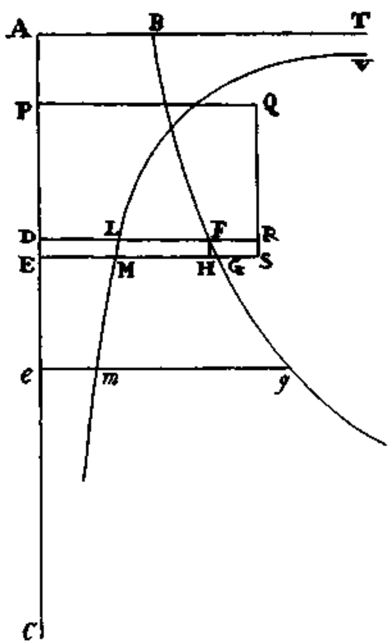
In EG take EM inversely proportional to the straight line whose square is equal to the area ABGE, and let VLM be a curved line which the point M continually traces out and whose asymptote is the straight line AB produced; then the time in which the body in falling describes the line AE will be as the curvilinear area ABTVME. Q.E.I.

For in the straight line AE take a minimally small line DE of a given length, and let DLF be the location of the line EMG when the body was at D; then, if the centripetal force is such that the straight line whose square is equal to the area ABGE is as the velocity of the descending body, the area itself will be as the square of the velocity, that is, if V and V + I are written for the velocities at D and E, the area ABFD will be as V^2, and the area ABGE as $V^2 + 2VI + I^2$, and by separation [or dividendo] the area DFGE will be as $2VI + I^2$, and thus $\frac{DFGE}{DE}$ will be as $\frac{2VI + I^2}{DE}$, that is, if the first ratios of nascent quantities are taken, the length DF will be as the quantity $\frac{2VI}{DE}$, and thus also as half of that quantity, or $\frac{I \times V}{DE}$. But the time in which the body in falling describes the line-element DE is as that line-element directly and the velocity V inversely, and the force is as the increment I of the velocity directly and the time inversely, and thus—if the first ratios of nascent quantities are taken—as $\frac{I \times V}{DE}$, that is, as the length DF. Therefore a force proportional to DF or EG makes the body descend with the velocity that is as the straight line whose square is equal to the area ABGE. Q.E.D.

Moreover, since the time in which any line-element DE of a given length is described is as the velocity inversely, and hence inversely as the straight line whose square is equal to the area ABFD, and since DL (and hence the nascent area DLME) is as the same straight line inversely, the time will be as the area DLME, and the sum of all the times will be as the sum of all the areas, that is (by lem. 4, corol.), the total time in which the line AE is described will be as the total area ATVME. Q.E.D.

COROLLARY 1. Let P be the place from which a body must fall so that, under the action of some known uniform centripetal force (such as gravity is commonly supposed to be), it will acquire at place D a velocity equal to the velocity that another body, falling under the action of any force whatever, acquired at the same place D. In the perpendicular DF take DR such that it is to DF as that uniform force is to the other force at the place D. Complete the rectangle PDRQ and cut off the area ABFD equal to it. Then A will be the place from which the other body fell.

For, when the rectangle DRSE has been completed, the area ABFD is to the area DFGE as V^2 to 2VI and hence as ½V to I, that is, as half of the total velocity to the increment of the velocity of the body falling under the action of the nonuniform force; and similarly, the area PQRD is to the area DRSE as half of the total velocity is to the increment of the velocity of



the body falling under the action of the uniform force, and those increments (because the nascent times are equal) are as the generative forces, that is, as the ordinates DF and DR, and thus as the nascent areas DFGE and DRSE. Therefore, the total areas ABFD and PQRD will then from the equality of the ratios [or ex aequo] be to each other as halves of the total velocities and therefore are equal because the velocities are equal.

COROLLARY 2. Hence if any body is projected with a given velocity either upward or downward from any place D and the law of centripetal force is given, the velocity of the body at any other place *e* will be found by erecting the ordinate *eg* and taking that velocity at place *e* to the velocity at place D as the straight line whose square is equal to the rectangle PQRD, either increased by the curvilinear area DF*ge* (if place *e* is lower than place D) or diminished by DF*ge* (if place *e* is higher), is to the straight line whose square is equal to the rectangle PQRD alone.

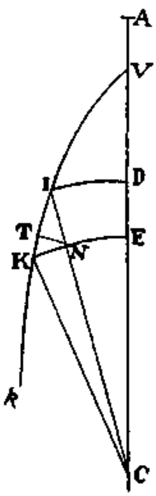
COROLLARY 3. The time, also, will be determined by erecting the ordinate *em* inversely proportional to the square root of PQRD ± DF*ge*, and by taking the time in which the body described the line D*e* to the time in which the other body fell under the action of a uniform force from P and (by so falling) reached D as the curvilinear area DL*me* to the rectangle 2PD × DL. For the time in which the body descending under the action of a uniform force described the line PD is to the time in which the same body described the line PE as the square root of the ratio of PD to PE, that is (the line-element DE being just now nascent), in the ratio of PD to PD + ½DE or 2PD to 2PD + DE and by separation [or dividendo] to the time in which the same body described the line-element DE as 2PD to DE, and thus as the rectangle 2PD × DL to the area DLME; and the time in which each of the two bodies described the line-element DE is to the time in which the second body with nonuniform motion described the line D*e* as the area DLME to the area DL*me*, and from the equality of the ratios [or ex aequo] the first time is to the ultimate time as the rectangle 2PD × DL to the area DL*me*.

SECTION 8

To find the orbits in which bodies revolve when acted upon by any centripetal forces

Proposition 40 Theorem 13 *If a body, under the action of any centripetal force, moves in any way whatever, and another body ascends straight up or descends straight down, and if their velocities are equal in some one instance in which their distances from the center are equal, their velocities will be equal at all equal distances from the center.*

Let some body descend from A through D and E to the center C, and let another body move from V in the curved line VIK*k*. With center C and



any radii describe the concentric circles DI and EK meeting the straight line AC in D and E and the curve VIK in I and K. Join IC meeting KE in N, and to IK drop the perpendicular NT, and let the interval DE or IN between the circumferences of the circles be minimally small, and let the bodies have equal velocities at D and I. Since the distances CD and CI are equal, the centripetal forces at D and I will be equal. Represent these forces by the equal line-elements DE and IN; then, if one of these forces IN is (by corol. 2 of the laws) resolved into two, NT and IT, the force NT, acting along the line NT perpendicular to the path ITK of the body, will in no way change the velocity of the body in that path but will only draw the body back from a rectilinear path and make it turn aside continually from the tangent of the orbit and move forward in the curvilinear path ITK*k*. That whole force will be spent in producing this effect, while the whole of the other force IT, acting along the body's path, will accelerate the body and in a given minimally small time will generate an acceleration proportional to itself. Accordingly, the accelerations of the bodies at D and I that are made in equal times (if the first ratios of the nascent lines DE, IN, IK, IT, and NT are taken) are as the lines DE and IT, but in unequal times they are as those lines and the times jointly. Now, the times in which DE and IK are described are as the described paths DE and IK (because the velocities are equal), and hence the accelerations in the path of the bodies along the lines DE and IK are jointly as DE and IT, DE and IK, that is, as DE^2 and the rectangle $IT \times IK$.

But the rectangle $IT \times IK$ is equal to IN^2, that is, equal to DE^2, and therefore the accelerations generated in the passing of the bodies from D and I to E and K are equal. Therefore the velocities of the bodies at E and K are equal, and by the same argument they will always be found equal at subsequent equal distances. Q.E.D.

But also by the same argument bodies that have equal velocities and are equally distant from the center will be equally retarded in ascending to equal distances. Q.E.D.

Corollary 1. Hence if a body either oscillates while hanging by a thread or is compelled by any very smooth and perfectly slippery impediment to move in a curved line, and another body ascends straight up or descends straight down, and their velocities are equal at any identical height, their velocities at any other equal heights will be equal. For the thread of the pendent body or the impediment of an absolutely slippery vessel produces the same effect as the transverse force NT. The body is neither retarded nor accelerated by these, but only compelled to depart from a rectilinear course.

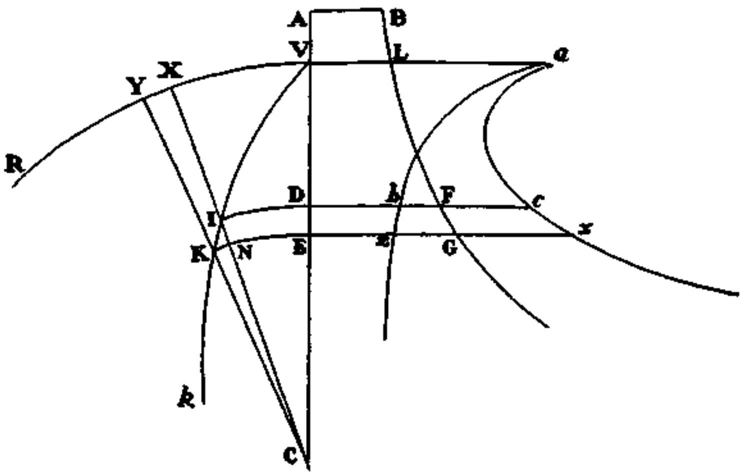
Corollary 2. Now let the quantity P be the greatest distance from the center to which a body, either oscillating or revolving in any trajectory whatever, can ascend when projected upward from any point of the trajectory with the velocity that it has at that point. Further, let the quantity A be the distance of the body from the center at any other point of the orbit. And let the centripetal force be always as any power A^{n-1} of A, the index $n-1$ being any number n diminished by unity. Then the velocity of the body at every height A [i.e., distance A] will be as $\sqrt{(P^n - A^n)}$ and therefore is given. For the velocity of a body ascending straight up and descending straight down is (by prop. 39) in this very ratio.

Proposition 41[a]
Problem 28

Supposing a centripetal force of any kind and granting the quadratures of curvilinear figures, it is required to find the trajectories in which bodies will move and also the times of their motions in the trajectories so found.

Let any force tend toward a center C; and let it be required to find the trajectory VIKk. Let the circle VR be given, described about the center C with any radius CV; and about the same center let there be described any other circles ID and KE cutting the trajectory in I and K and cutting the

a. For a gloss on this proposition see the Guide, §10.12.

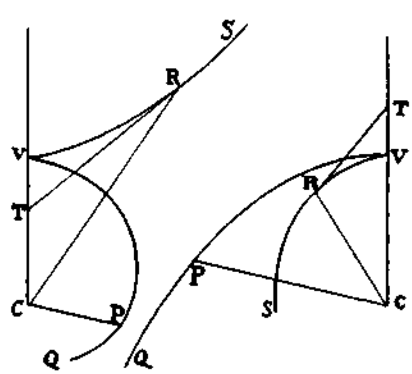


straight line CV in D and E. Then draw the straight line CNIX cutting the circles KE and VR in N and X, and also draw the straight line CKY meeting the circle VR in Y. Let the points I and K be very close indeed to each other, and let the body proceed from V through I and K to *k*; and let point A be the place from which another body must fall so as to acquire at place D a velocity equal to the velocity of the first body at I. And with everything remaining as it was in prop. 39, the line-element IK, described in a given minimally small time, will be as the velocity and hence as the straight line whose square equals the area ABFD, and the triangle ICK proportional to the time will be given; and therefore KN will be inversely as the height IC, that is, if some quantity Q is given and the height IC is called A, as $\frac{Q}{A}$. Let us denote this quantity $\frac{Q}{A}$ by Z, and let us suppose the magnitude of Q to be such that in some one case $\sqrt{ABFD}$ is to Z as IK is to KN, and in every case $\sqrt{ABFD}$ will be to Z as IK to KN and ABFD to Z^2 as IK^2 to KN^2, and by separation [or dividendo] $ABFD - Z^2$ will be to Z^2 as IN^2 to KN^2, and therefore $\sqrt{(ABFD - Z^2)}$ will be to Z, or $\frac{Q}{A}$, as IN to KN, and therefore $A \times KN$ will be equal to $\frac{Q \times IN}{\sqrt{(ABFD - Z^2)}}$. Hence, since $YX \times XC$ is to $A \times KN$ as CX^2 to A^2, the rectangle $XY \times XC$ will be equal to $\frac{Q \times IN \times CX^2}{A^2\sqrt{(ABFD - Z^2)}}$. Therefore, in the perpendicular DF take D*b* and D*c* always equal respectively to $\frac{Q}{2\sqrt{(ABFD - Z^2)}}$ and $\frac{Q \times CX^2}{2A^2\sqrt{(ABFD - Z^2)}}$, and describe the curved lines *ab* and *ac* which the points *b* and *c* continually trace out, and from point V erect V*a* perpendicular to the line AC so as to cut off the curvilinear areas VD*ba* and VD*ca*, and also erect the ordinates E*z* and E*x*. Then, since the rectangle D*b* × IN or D*bz*E is equal to half of the rectangle A × KN or is equal to the triangle ICK, and the rectangle D*c* × IN or D*cx*E is equal to half of the rectangle YX × XC or is equal to the triangle XCY—that is, since the nascent particles D*bz*E and ICK of the areas VD*ba* and VIC are always equal, and the nascent particles D*cx*E and XCY of the areas VD*ca* and VCX are always equal—the generated area VD*ba* will be equal to the generated area VIC and hence will be proportional to the time, and the generated area VD*ca* will be equal to the generated sector VCX. Therefore, given any time that has elapsed since the body set out from place V, the area VD*ba* proportional to it will be given and hence the body's height CD or CI will be given, and the area VD*ca* and, equal to that area, the sector VCX along with its angle VCI. And given the angle VCI and the height CI, the place I will be given, in which the body will be found at the end of that time. Q.E.I.

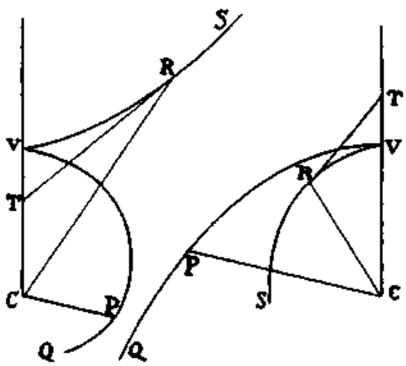
Corollary 1. Hence the greatest and least heights of bodies (that is, the apsides of their trajectories) can be found expeditiously. For the apsides are those points in which the straight line IC drawn through the center falls perpendicularly upon the trajectory VIK, which happens when the straight lines IK and NK are equal, and thus when the area ABFD is equal to Z^2.

Corollary 2. The angle KIN, in which the trajectory anywhere cuts the line IC, is also expeditiously found from the given height IC of the body, namely, by taking its sine to the radius as KN to IK, that is, as Z to the square root of the area ABFD.

Corollary 3. If with center C and principal vertex V any conic VRS is described, and from any point R of it the tangent RT is drawn so as to meet the axis CV, indefinitely produced, at point T; and, joining CR, there is drawn the straight line CP, which is equal to the abscissa CT and makes an angle VCP proportional to the sector VCR; then, if a centripetal



force inversely proportional to the cube of the distance of places from the center tends toward that center C, and the body leaves the place V with the proper velocity along a line perpendicular to the straight line CV, the body will move forward in the trajectory VPQ which point P continually traces out; and therefore, if the conic VRS is a hyperbola, the body will descend to the center. But if the conic is an ellipse, the body will ascend continually and will go off to infinity.

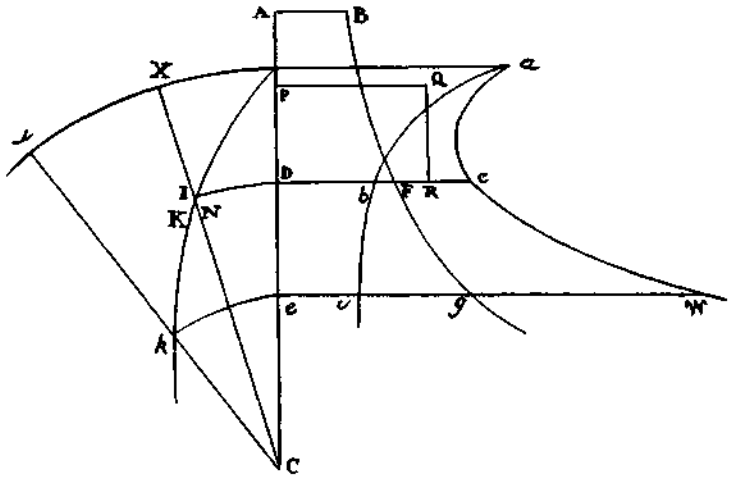


And conversely, if the body leaves the place V with any velocity and, depending on whether the body has begun either to descend obliquely to the center or to ascend obliquely from it, the figure VRS is either a hyperbola or an ellipse, the trajectory can be found by increasing or diminishing the angle VCP in some given ratio. But also, if the centripetal force is changed into a centrifugal force, the body will ascend obliquely in the trajectory VPQ, which is found by taking the angle VCP proportional to the elliptic sector VRC and by taking the length CP equal to the length CT, as above. All this follows from the foregoing (prop. 41), by means of the quadrature of a certain curve, the finding of which, as being easy enough, I omit for the sake of brevity.

Proposition 42
Problem 29

Let the law of centripetal force be given; it is required to find the motion of a body setting out from a given place with a given velocity along a given straight line.

With everything remaining as it was in the three preceding propositions, let the body go forth from the place I along the line-element IK, with the velocity which another body, falling from the place P under the action of some uniform centripetal force, could acquire at D; and let this uniform force be to the force with which the first body is urged at I as DR to DF. Let the body go on toward *k*; and with center C and radius C*k* describe the circle *ke* meeting the straight line PD at *e*, and erect the ordinates *eg*, *ev*, and *ew* of the curves BF*g*, *abv*, and *acw*. From the given rectangle PDRQ and the given law of the centripetal force acting on the first body, the curved line BF*g* is given by the construction of prop. 39 and its corol. 1. Then, from



the given angle CIK, the proportion of the nascent lines IK and KN is given, and hence, by the construction of prop. 41, the quantity Q is given, along with the curved lines *abv* and *acw*; and therefore, when any time D*bve* is completed, the body's height C*e* or C*k* is given and the area D*cwe* and the sector XC*y* equal to it and the angle IC*k* and the place *k* in which the body will then be. Q.E.I.

In these propositions we suppose that the centripetal force in receding from the center varies according to any law which can be imagined, but that at equal distances from the center it is everywhere the same. And so far we have considered the motion of bodies in nonmoving orbits. It remains for us to add a few things about the motion of bodies in orbits that revolve about a center of forces.

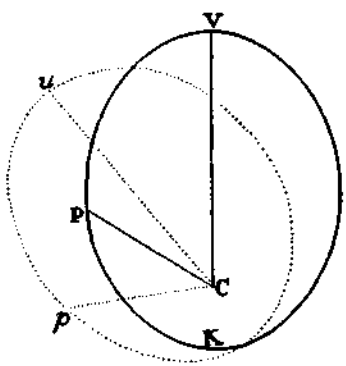
SECTION 9

The motion of bodies in mobile orbits, and the motion of the apsides

Proposition 43
Problem 30

[a]*It is required to find the force that makes a body capable of moving in any trajectory that is revolving about the center of forces in the same way as another body in that same trajectory at rest.*[a]

Let a body P revolve in the orbit VPK given in position, moving forward from V toward K. From center C continually draw Cp equal to CP and making the angle VCp which is proportional to the angle VCP; and the area that the line Cp describes will be to the area VCP that the line CP simultaneously describes as the velocity of the describing line Cp to the velocity of the describing line CP, that is, as the angle VCp to the angle VCP and thus in a given ratio and therefore proportional to the time. Since the area that line Cp describes in the immobile plane is proportional to the time, it is manifest that the body, under the action of a centripetal force of just the right quantity, can revolve along with point p in the curved line that the same point p, in the manner just explained, describes in an immobile plane. Let the angle VCu be made equal to the angle PCp, and the line Cu equal to the line CV, and the figure uCp equal to the figure VCP; then the body, being always at the point p, will move in the perimeter of the revolving figure uCp, and will describe its arc up in the same time in which another body P can describe the arc VP, similar and equal to up, in the figure VPK at rest. Determine, therefore, by prop. 6, corol. 5, the centripetal force by which a body can revolve in the curved line that point p describes in an immobile plane, and the problem will be solved. Q.E.F.

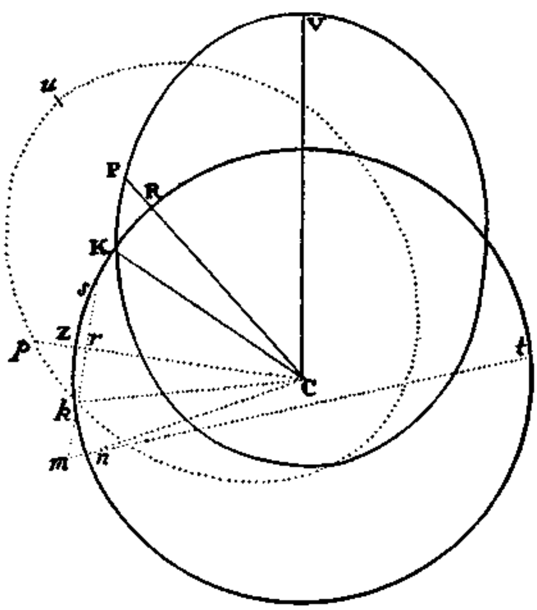


aa. Newton does not use the word "force" in the statement of prop. 43, but he does so in the conclusion of the demonstration. A literal translation of prop. 43 would read: "It is required to make it happen [*or*, It is to be effected] that a body may be able to move in any trajectory that is revolving about the center of forces exactly as another body moves in that same trajectory at rest." The force in question must be centripetal.

Proposition 44
Theorem 14

The difference between the forces under the action of which two bodies are able to move equally—one in an orbit that is at rest and the other in an identical orbit that is revolving—is inversely as the cube of their common height.

Let the parts *up* and *pk* of the revolving orbit be similar and equal to the parts VP and PK of the orbit at rest; and let it be understood that the distance between points P and K is minimally small. From point *k* drop the perpendicular *kr* to the straight line *p*C, and produce *kr* to *m* so that *mr* is to *kr* as the angle VC*p* to the angle VCP. Since the heights PC and *p*C, KC and *k*C, of the bodies are always equal, it is manifest that the increments and decrements of the lines PC and *p*C are always equal, and hence, if the motions of each of these bodies, when they are at places P and *p*, are resolved (by corol. 2 of the laws) into two components, of which one is directed toward the center, or along the line PC or *p*C, and the second is transverse to the first and has a direction along a line perpendicular to PC or *p*C, the components of motion toward the center will be equal, and the transverse component of motion of body *p* will be to the transverse component of motion of body P as the angular motion of line *p*C to the angular motion of line PC, that is, as the angle VC*p* to the angle VCP. Therefore, in the same time in which body P by the two components of its motion reaches point K, body *p* by its equal component of motion toward the center will move equally from *p*



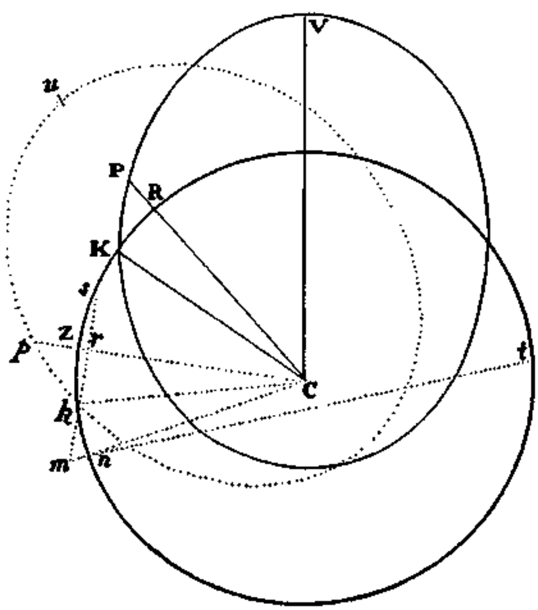
toward C and thus, when that time is completed, will be found somewhere on the line mkr (which is perpendicular to the line pC through point k) and by its transverse motion will reach a distance from the line pC that is to the distance from the line PC (which the other body P reaches) as the transverse motion of body p is to the transverse motion of the other body P. Therefore, since kr is equal to the distance from the line PC which body P reaches, and since mr is to kr as the angle VCp to the angle VCP, that is, as the transverse motion of body p to the transverse motion of body P, it is manifest that body p, at the completion of the time, will be found at the place m.

This will be the case when bodies p and P move equally along lines pC and PC and thus are urged along those lines by equal forces. But now, take the angle pCn to the angle pCk as the angle VCp is to the angle VCP, and let nC be equal to kC, and then body p—at the completion of the time—will actually be found at the place n; and thus body p is urged by a greater force than that by which body P is urged, provided that the angle nCp is greater than the angle kCp, that is, if the orbit upk either moves forward [or in consequentia] or moves backward [or in antecedentia] with a speed greater than twice that with which the line CP is carried forward [or in consequentia] and it is urged by a smaller force if the orbit moves backward [or in antecedentia] more slowly. And the difference between the forces is as the intervening distance mn through which the body p ought to be carried by the action of that difference in the given space of time.

Understand that a circle is described, with center C and radius Cn or Ck, cutting in s and t the lines mr and mn produced; then the rectangle $mn \times mt$ will be equal to the rectangle $mk \times ms$, and thus mn will be equal to $\dfrac{mk \times ms}{mt}$. But since the triangles pCk and pCn are, in a given time, given in magnitude, kr and mr and their difference mk and sum ms are inversely as the height pC, and thus the rectangle $mk \times ms$ is inversely as the square of the height pC. Also, mt is directly as $\frac{1}{2}mt$, that is, as the height pC. These are the first ratios of the nascent lines; and hence $\dfrac{mk \times ms}{mt}$ (that is, the nascent line-element mn and, proportional to it, the difference between the forces) becomes inversely as the cube of the height pC. Q.E.D.

COROLLARY 1. Hence the difference of the forces in the places P and p or K and k is to the force by which a body would be able to revolve with circular motion from R to K in the same time in which body P in an immobile orbit describes the arc PK as the nascent line-element mn is to the versed sine of the nascent arc RK, that is, as $\frac{mk \times ms}{mt}$ to $\frac{rk^2}{2kC}$ or as $mk \times ms$ to rk^2, that is, if the given quantities F and G are taken in the ratio to each other that the angle VCP has to the angle VCp, as $G^2 - F^2$ to F^2. And therefore, if with center C and any radius CP or Cp a circular sector is described equal to the total area VPC which the body P revolving in an immobile orbit has described in any time by a radius drawn to the center, the difference between the forces by which body P in an immobile orbit and body p in a mobile orbit revolve will be to the centripetal force by which some body, by a radius drawn to the center, would have been able to describe that sector uniformly in the same time in which the area VPC was described, as $G^2 - F^2$ to F^2. For that sector and the area pCk are to each other as the times in which they are described.

Corollary 2. If the orbit VPK is an ellipse having a focus C and an upper apsis V, and the mobile ellipse upk is supposed similar and equal to it, so that pC is always equal to PC and the angle VCp is to the angle VCP in the given ratio of G to F; and if A is written for the height PC or pC, and 2R is put for the latus rectum of the ellipse; then the force by which



a body can revolve in the mobile ellipse will be as $\frac{F^2}{A^2} + \frac{R(G^2 - F^2)}{A^3}$, and conversely. For let the force by which a body revolves in the unmoving ellipse be represented by the quantity $\frac{F^2}{A^2}$, and then the force at V will be $\frac{F^2}{CV^2}$. But the force by which a body could revolve in a circle at the distance CV with the velocity that a body revolving in an ellipse has at V is to the force by which a body revolving in an ellipse is urged at the apsis V as half of the latus rectum of the ellipse to the semidiameter CV of the circle, and thus has the value $\frac{R \times F^2}{CV^3}$; and the force that is to this as $G^2 - F^2$ to F^2 has the value $\frac{R(G^2 - F^2)}{CV^3}$; and this force (by corol. 1 of this prop.) is the difference between the forces at V by which body P revolves in the unmoving ellipse VPK and body *p* revolves in the mobile ellipse *upk*. Hence, since (by this proposition) that difference at any other height A is to itself at the height CV as $\frac{1}{A^3}$ to $\frac{1}{CV^3}$, the same difference at every height A will have the value $\frac{R(G^2 - F^2)}{A^3}$. Therefore, add the excess $\frac{R(G^2 - F^2)}{A^3}$ to the force $\frac{F^2}{A^2}$ by which a body can revolve in the immobile ellipse VPK, and the result will be the total force $\frac{F^2}{A^2} + \frac{R(G^2 - F^2)}{A^3}$ by which a body may be able to revolve in the same times in the mobile ellipse *upk*.

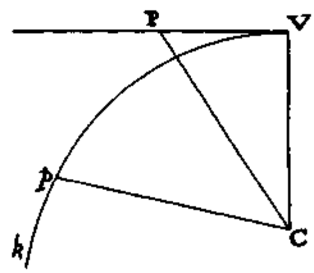
Corollary 3. In the same way it will be gathered that if the immobile orbit VPK is an ellipse having its center at the center C of forces, and a mobile ellipse *upk* is supposed similar, equal, and concentric with it; and if 2R is the principal latus rectum of this ellipse, and 2T the principal diameter or major axis, and the angle VC*p* is always to the angle VCP as G to F; then the forces by which bodies can revolve in equal times in the immobile ellipse and the mobile ellipse will be as $\frac{F^2A}{T^3}$ and $\frac{F^2A}{T^3} + \frac{R(G^2 - F^2)}{A^3}$ respectively.

Corollary 4. And universally, if the greatest height CV of a body is called T; and the radius of the curvature which the orbit VPK has at V (that is, the radius of a circle of equal curvature) is called R; and the centripetal force by which a body can revolve in any immobile trajectory VPK at place V is called $\frac{VF^2}{T^2}$ and at other places P is indefinitely styled X, while the height CP is called A; and if G is taken to F in the given ratio of the angle

VCp to the angle VCP; then the centripetal force by which the same body can complete the same motions in the same times in the same trajectory upk which is moving circularly will be as the sum of the forces $X + \frac{VR(G^2 - F^2)}{A^3}$.

COROLLARY 5. Therefore, given the motion of a body in any immobile orbit, its angular motion about the center of forces can be increased or diminished in a given ratio, and hence new immobile orbits can be found in which bodies may revolve by new centripetal forces.

COROLLARY 6. Therefore, if on the straight line CV, given in position, there is erected the perpendicular VP of indeterminate length, and CP is joined, and Cp is drawn equal to it making the angle VCp that is to the angle VCP in a given ratio; then the force by which a body can revolve in the curve Vpk which the point p continually traces out will be inversely as the cube of the height Cp. For body P, by its own force of inertia, and with no other force urging it, can move forward uniformly in the straight line VP.



Add the force toward the center C, inversely proportional to the cube of the height CP or Cp, and (by what has just been demonstrated) the rectilinear motion will be bent into the curved line Vpk. But this curve Vpk is the same as the curve VPQ found in prop. 41, corol. 3, and (as we said there) bodies attracted by forces of this kind ascend obliquely in this curve.

Proposition 45
Problem 31

It is required to find the motions of the apsides of orbits that differ very little from circles.

This problem is solved arithmetically by taking the orbit that is described in an immobile plane by a body revolving in a mobile ellipse (as in prop. 44, corol. 2 or 3) and making it approach the form of the orbit whose apsides are required, and by seeking the apsides of the orbit which that body describes in an immobile plane. Orbits will acquire the same shape if the centripetal forces with which those orbits are described, when compared with each other, are made proportional at equal heights. Let point V be the upper apsis, and write T for the greatest height CV, A for any other height CP or Cp, and X for the difference CV − CP of the heights; then the force by which a body moves in an ellipse revolving about its own focus C (as in corol. 2)—and which in corol. 2 was as $\dfrac{F^2}{A^2} + \dfrac{RG^2 - RF^2}{A^3}$, that is, as $\dfrac{F^2A + RG^2 - RF^2}{A^3}$— will, when $T - X$ is substituted for A, be as $\dfrac{RG^2 - RF^2 + TF^2 - F^2X}{A^3}$. Any other centripetal force is similarly to be reduced to a fraction whose denominator is A^3; and the numerators are to be made analogous [i.e., made proportional in the same degree] by bringing together homologous terms [i.e., corresponding terms, or terms of the same degree]. All of this will be clarified by the following examples.

EXAMPLE 1. Let us suppose the centripetal force to be uniform and thus as $\dfrac{A^3}{A^3}$, or (writing $T - X$ for A in the numerator) as

$$\frac{T^3 - 3T^2X + 3TX^2 - X^3}{A^3};$$

and by bringing together the corresponding [or homologous] terms of the numerators (namely, given ones with given ones, and ones not given with ones not given), $RG^2 - RF^2 + TF^2$ to T^3 will come to be as $-F^2X$ to $-3T^2X + 3TX^2 - X^3$ or as $-F^2$ to $-3T^2 + 3TX - X^2$. Now, since the orbit is supposed to differ very little from a circle, let the orbit come to coincide with a circle; and because R and T become equal and X is diminished indefinitely, the ultimate ratios will be RG^2 to T^3 as $-F^2$ to $-3T^2$, or G^2 to T^2 as F^2 to $3T^2$, and by alternation [or alternando] G^2 to F^2 as T^2 to $3T^2$, that is, as 1 to 3; and therefore G is to F, that is, the angle VCp is to the angle VCP, as 1 to $\sqrt{3}$. Therefore, since a body in an immobile ellipse, in descending from the upper apsis to the lower apsis, completes the angle VCP (so to speak) of 180 degrees, another body in the mobile ellipse (and hence in the immobile orbit with which we are dealing) will, in descending from the upper apsis to the lower apsis, complete the angle VCp of $\dfrac{180}{\sqrt{3}}$ degrees; this is so because of the similarity of this orbit, which the body describes under the action of a uniform centripetal force, to the orbit which a body completing its revolutions in a revolving ellipse describes in a plane at rest. By the above collation of terms, these orbits are made similar, not universally but at the time when they very nearly approach a circular form. Therefore a body revolving with uniform centripetal force in a very nearly circular orbit will always complete an angle of $\dfrac{180}{\sqrt{3}}$ degrees between the upper apsis and the lower apsis, or $103°55'23''$ at the center, arriving at the lower apsis from the upper apsis when it has completed this angle once, and returning from the lower to the upper apsis when it has completed the same angle again, and so on without end.

EXAMPLE 2. Let us suppose the centripetal force to be as the height A raised to any power, as A^{n-3} (that is, $\frac{A^n}{A^3}$), where $n - 3$ and n signify any indices of powers whatsoever—integral or fractional, rational or irrational, positive or negative. On reducing the numerator $A^n = (T - X)^n$ to an indeterminate series by our method of converging series, the result is $T^n - nXT^{n-1} + \frac{n^2 - n}{2}X^2T^{n-2} \cdots$. And by collating the terms of this with the terms of the other numerator $RG^2 - RF^2 + TF^2 - F^2X$, the result is that $RG^2 - RF^2 + TF^2$ is to T^n as $-F^2$ to $-nT^{n-1} + \frac{n^2 - n}{2}XT^{n-2} \cdots$. And after taking the ultimate ratios that result when the orbits approach circular form, RG^2 will be to T^n as $-F^2$ to $-nT^{n-1}$, or G^2 to T^{n-1} as F^2 to nT^{n-1}, and by alternation [or alternando] G^2 is to F^2 as T^{n-1} to nT^{n-1}, that is, as 1 to n; and therefore G is to F, that is, the angle VCp is to the angle VCP as 1 to $\sqrt{n}$. Therefore, since the angle VCP, completed in the descent of a body from the upper apsis to the lower apsis in an ellipse, is 180 degrees, the angle VCp, completed in the descent of a body from the upper apsis to the lower apsis in the very nearly circular orbit which any body describes under the action of a centripetal force proportional to A^{n-3}, will be equal to an angle of $\frac{180}{\sqrt{n}}$ degrees; and when this angle is repeated, the body will return from the lower apsis to the upper apsis, and so on without end.

For example, if the centripetal force is as the distance of the body from the center, that is, as A or $\frac{A^4}{A^3}$, n will be equal to 4 and $\sqrt{n}$ will be equal to 2; and therefore the angle between the upper apsis and the lower apsis will be equal to $\frac{180°}{2}$ or 90°. Therefore, at the completion of a quarter of a revolution the body will arrive at the lower apsis, and at the completion of another quarter the body will arrive at the upper apsis, and so on by turns without end. This is also manifest from prop. 10. For a body urged by this centripetal force will revolve in an immobile ellipse whose center is in the center of forces. But if the centripetal force is inversely as the distance, that is, directly as $\frac{1}{A}$ or $\frac{A^2}{A^3}$, n will be equal to 2, and thus the angle between the upper and the lower apsis will be $\frac{180}{\sqrt{2}}$ degrees, or $127°16'45''$, and therefore a body revolving under the action of such a force will—by the continual repetition of this angle—go alternately from the upper apsis to the lower and from the lower to the upper forever. Further, if the centripetal force is inversely as the fourth root of the eleventh power of the height, that is, inversely as $A^{11/4}$ and thus directly as $\frac{1}{A^{11/4}}$ or as $\frac{A^{1/4}}{A^3}$, n will be equal to $1/4$, and $\frac{180°}{\sqrt{n}}$ will be equal to 360°; and therefore a body, setting out from the upper apsis and continually descending from then on, will arrive at the lower apsis when it has completed an entire revolution, and then, completing another entire revolution by continually ascending, will return to the upper apsis; and so on by turns forever.

EXAMPLE 3. Let m and n be any indices of powers of the height, and let b and c be any given numbers, and let us suppose that the centripetal force is as $\frac{bA^m + cA^n}{A^3}$, that is, as $\frac{b(T-X)^m + c(T-X)^n}{A^3}$ or (again by our method of converging series) as

$$\frac{bT^m + cT^n - mbXT^{m-1} - ncXT^{n-1} + \frac{m^2-m}{2}bX^2T^{m-2} + \frac{n^2-n}{2}cX^2T^{n-2}\cdots}{A^3},$$

then, if the terms of the numerators are collated, the result will be $RG^2 - RF^2 + TF^2$ to $bT^m + cT^n$ as $-F^2$ to $-mbT^{m-1} - ncT^{n-1} + \frac{m^2-m}{2}bXT^{m-2} + \frac{n^2-n}{2}cXT^{n-2}\cdots$. And after taking the ultimate ratios that result when the orbits approach circular form, G^2 will be to $bT^{m-1} + cT^{n-1}$ as F^2 to $mbT^{m-1} + ncT^{n-1}$, and by alternation [or alternando] G^2 will be to F^2 as $bT^{m-1} + cT^{n-1}$ to $mbT^{n-1} + ncT^{n-1}$. This proportion, if the greatest height CV or T is expressed arithmetically by unity, becomes G^2 to F^2 as $b + c$ to $mb + nc$ and thus as 1 to $\frac{mb+nc}{b+c}$. Hence G is to F, that is, the angle VCp is to the angle VCP, as 1 to $\sqrt{\frac{mb+nc}{b+c}}$. And therefore, since the angle VCP between the upper apsis and the lower apsis in the immobile ellipse is 180 degrees, the angle VCp between the same apsides, in the orbit described by a body under the action of a centripetal force proportional to the quantity $\frac{bA^m + cA^n}{A^3}$, will be equal to an angle of $180\sqrt{\frac{b+c}{mb+nc}}$ degrees. And by the same argument, if the centripetal force is as $\frac{bA^m - cA^n}{A^3}$, the angle between the apsides will be found to be $180\sqrt{\frac{b-c}{mb-nc}}$ degrees. And the problem will be resolved in just the same way in more difficult cases. The quantity to which the centripetal force is proportional must always be resolved into converging series having the denominator A^3. Then the ratio of the given part of the numerator (resulting from that operation) to its other part, which is not given, is to be made the same as the ratio of the given part of this numerator $RG^2 - RF^2 + TF^2 - F^2X$ to its other part, which is not given; and when the superfluous quantities are taken away and unity is written for T, the proportion of G to F will be obtained.

COROLLARY 1. Hence, if the centripetal force is as some power of the height, that power can be found from the motion of the apsides, and conversely. That is, if the total angular motion with which the body returns to the same apsis is to the angular motion of one revolution, or 360 degrees, as some number m to another number n, and the height is called A, the force will be as the power of the height $A^{\frac{n^2}{m^2}-3}$ whose index is $\frac{n^2}{m^2} - 3$. This is manifest by the instances in ex. 2. Hence it is clear that the force, in receding from the center, cannot decrease in a ratio greater than that of the cube of the height; if a body revolving under the action of such a force and setting out from an apsis begins to descend, it will never reach the lower apsis or minimum height but will descend all the way to the center, describing that curved line which we treated in prop. 41, corol. 3. But if the body, on setting out from an apsis, begins to ascend even the least bit, it will ascend indefinitely and will never reach the upper apsis. For it will describe the curved line treated in the above-mentioned corol. 3 and in prop. 44, corol. 6. So also, when the force, in receding from the center, decreases in a ratio greater than that of the cube of the height, a body setting out from an apsis (depending on whether it begins to descend or to ascend) either will descend all the way to the center or will ascend indefinitely. But if the force, in receding from the center, either decreases in a ratio less than that of the cube of the height or increases in any ratio of the height whatever, the body will never descend all the way to the center, but will at some time reach a lower apsis; and conversely, if a body descending and ascending alternately from apsis to apsis never gets to the center, either the force in receding from the center will be increased or it will decrease in a ratio less than that of the cube of the height; and the more swiftly the body returns from apsis to apsis, the farther the ratio of the forces will recede from that of the cube.

For example, if by alternate descent and ascent a body returns from upper apsis to upper apsis in 8 or 4 or 2 or $1\frac{1}{2}$ revolutions, that is, if m is to n as 8 or 4 or 2 or $1\frac{1}{2}$ to 1, and therefore $\frac{n^2}{m^2} - 3$ has the value $\frac{1}{64} - 3$ or $\frac{1}{16} - 3$ or $\frac{1}{4} - 3$ or $\frac{4}{9} - 3$, the force will be as $A^{\frac{1}{64}-3}$ or $A^{\frac{1}{16}-3}$ or $A^{\frac{1}{4}-3}$ or $A^{\frac{4}{9}-3}$, that is, inversely as $A^{3-\frac{1}{64}}$ or $A^{3-\frac{1}{16}}$ or $A^{3-\frac{1}{4}}$ or $A^{3-\frac{4}{9}}$. If the body returns in each revolution to the same unmoving apsis, m will be to n as 1 to 1, and thus $A^{\frac{n^2}{m^2}-3}$ will be equal to A^{-2} or $\frac{1}{A^2}$; and therefore the decrease in force will be as the square of the height, as has been demonstrated in the preceding propositions. If the body returns to the same apsis in three-quarters or two-thirds or one-third or one-quarter of a single revolution, m will be to n as $\frac{3}{4}$ or $\frac{2}{3}$ or $\frac{1}{3}$ or $\frac{1}{4}$ to 1, and so $A^{\frac{n^2}{m^2}-3}$ will be equal to $A^{\frac{16}{9}-3}$ or $A^{\frac{9}{4}-3}$ or A^{9-3} or A^{16-3}; and therefore the force will be either inversely as $A^{\frac{11}{9}}$ or $A^{\frac{3}{4}}$, or directly as A^6 or A^{13}. Finally, if the body in proceeding from upper apsis to upper apsis completes an entire revolution and an additional three degrees (and therefore, during each revolution of the body, that apsis moves three degrees forward [or in consequentia]), m will be to n as 363° to 360° or as 121 to 120, and thus $A^{\frac{n^2}{m^2}-3}$ will be equal to $A^{-\frac{29{,}523}{14{,}641}}$, and therefore the centripetal force will be inversely as $A^{\frac{29{,}523}{14{,}641}}$ or inversely as $A^{2\frac{4}{243}}$ approximately. Therefore the centripetal force decreases in a ratio a little greater than that of the square, but $59\frac{3}{4}$ times closer to that of the square than to that of the cube.

COROLLARY 2. Hence also if a body, under the action of a centripetal force that is inversely as the square of the height, revolves in an ellipse having a focus in the center of forces, and any other extraneous force is added to or taken away from this centripetal force, the motion of the apsides that will arise from that extraneous force can be found out (by instances in ex. 3), and conversely. For example, if the force under the action of which the body revolves in the ellipse is as $\frac{1}{A^2}$ and the extraneous force which has been taken away is as cA, and hence the remaining force is as $\frac{A - cA^4}{A^3}$, then (as in ex. 3) b will be equal to 1, m will be equal to 1, and n will be equal to 4, and therefore the angle of the revolution between apsides will be equal to an angle of $180\sqrt{\frac{1-c}{1-4c}}$ degrees. Let us suppose the extraneous force to be 357.45 times less than the other force under the action of which the body revolves in the ellipse, that is, let us suppose c to be $\frac{100}{35{,}745}$, A or T being equal to 1, and then $180\sqrt{\frac{1-c}{1-4c}}$ will come to be $180\sqrt{\frac{35{,}645}{35{,}345}}$, or 180.7623, that is, 180°45′44″. Therefore a body, setting out from the upper apsis, will reach the lower apsis by an angular motion of 180°45′44″ and will return to the upper apsis if this angular motion is doubled; and thus in each revolution the upper apsis will move forward through 1°31′28″. [a]The [advance of the] apsis of the moon is about twice as swift.[a]

So much concerning the motion of bodies in orbits whose planes pass through the center of forces. It remains for us to determine additionally those motions which occur in planes that do not pass through the center of forces. For writers who deal with the motion of heavy bodies are wont to consider the oblique ascents and descents of weights in any given planes as well as perpendicular ascents and descents, and there is equal justification for considering here the motion of bodies that tend to centers under the action of any forces whatever and are supported by eccentric planes. We suppose, however, that these planes are highly polished and absolutely slippery, so as not to retard the bodies. Further, in these demonstrations, in place of the planes on which bodies rest and which they touch by resting on them, we even make use of planes parallel to them, in which the centers of the bodies move and by so moving describe orbits. And by the same principle we then determine the motions of bodies performed in curved surfaces.

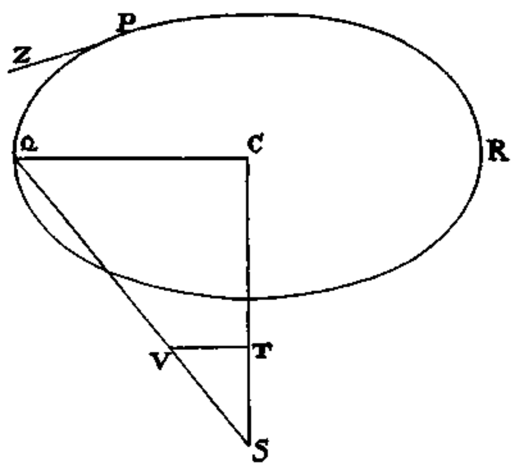
aa. Ed. 1 and ed. 2 lack this, but it appears both in the interleaved copy and in the annotated copy of ed. 2. The interleaved copy also has: "Query: Can this motion arise from twice the external force?" See further the Guide to the present translation, §6.10.

SECTION 10

The motion of bodies on given surfaces and the oscillating motion of [a]simple pendulums[a]

Proposition 46 Problem 32 *Suppose a centripetal force of any kind, and let there be given both the center of force and any plane in which a body revolves, and grant the quadratures of curvilinear figures; it is required to find the motion of a body setting out from a given place with a given velocity along a given straight line in that plane.*

Let S be the center of force, SC the least distance of this center from the given plane, P a body setting out from place P along the straight line PZ, Q the same body revolving in its trajectory, and PQR the required trajectory described in the given plane. Join CQ and also QS, and if SV is taken in



QS and is proportional to the centripetal force by which the body is drawn toward the center S, and VT is drawn parallel to CQ and meeting SC in T, then the force SV will be resolved (by corol. 2 of the laws) into the forces ST and TV, of which ST, by drawing the body along a line perpendicular to the plane, does not at all change the body's motion in this plane. But the other

aa. We use the term "simple pendulum" in its classical and technical sense. For example, according to Brougham and Routh, "A *simple pendulum* consists of a material particle suspended from a fixed point by an inflexible inextensible string without weight" (Henry Lord Brougham and E. J. Routh, *Analytical View of Sir Isaac Newton's "Principia"* [1855; reprint, with an introd. by I. Bernard Cohen, New York and London: Johnson Reprint Corp., 1972], pp. 240–241). See, further, §7.5 of the Guide.

force TV, by acting along the position of the plane, draws the body directly toward [i.e., along a line directed toward] the given point C in the plane and thus causes the body to move in this plane just as if the force ST were removed and as if the body revolved in free space about the center C under the action of the force TV alone. But, given the centripetal force TV under the action of which the body Q revolves in free space about the given center C, there are also given (by prop. 42) not only the trajectory PQR described by the body, but also the place Q in which the body will be at any given time, and finally the velocity of the body in that place Q; and conversely. Q.E.I.

Proposition 47
Theorem 15

Suppose that a centripetal force is proportional to the distance of a body from a center; then all bodies revolving in any planes whatever will describe ellipses and will make their revolutions in equal times; and bodies that move in straight lines, by oscillating to and fro, will complete in equal times their respective periods of going and returning.

For, under the same conditions as in prop. 46, the force SV, by which the body Q revolving in any plane PQR is drawn toward the center S, is as the distance SQ; and thus—because SV and SQ, TV and CQ are proportional—the force TV, by which the body is drawn toward the given point C in the plane of the orbit, is as the distance CQ. Therefore, the forces by which bodies that are in the plane PQR are drawn toward point C are, in proportion to the distances, equal to the forces by which bodies are drawn from all directions toward the center S; and thus in the same times the bodies will move in the same figures in any plane PQR about the point C as they would move in free spaces about the center S; and hence (by prop. 10, corol. 2, and prop. 38, corol. 2) in times which are always equal, they will either describe ellipses [i.e., complete a whole revolution in such ellipses] in that plane about the center C or will complete periods of oscillating to and fro in straight lines drawn through the center C in that plane. Q.E.D.

Scholium

The ascents and descents of bodies in curved surfaces are very closely related to the motions just discussed. Imagine that curved lines are described in a plane, that they then revolve around any given axes passing through the center of force and describe curved surfaces by this revolution, and then that bodies move in such a way that their centers are always found in these surfaces. If those bodies, in ascending and descending obliquely, oscillate to and fro, their motions will be made in planes passing through the axis and hence in curved lines by whose revolution those curved surfaces were generated. In these cases, therefore, it is sufficient to consider the motion in those curved lines.

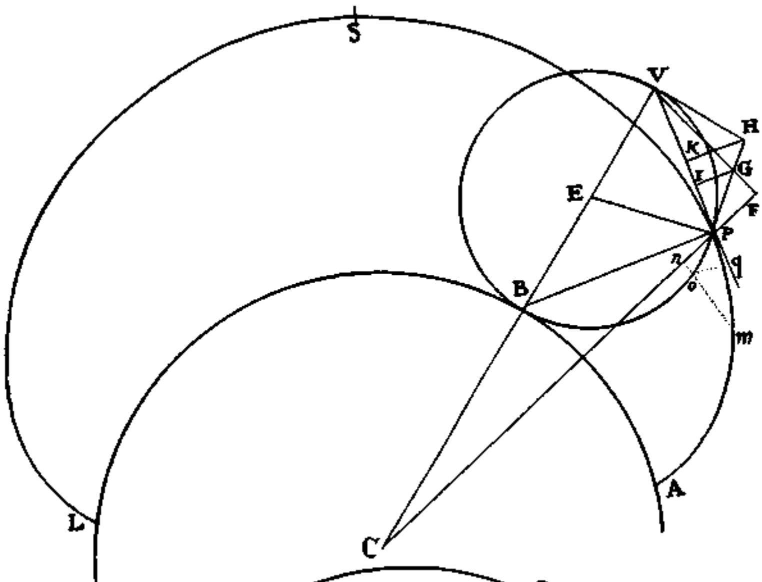
Proposition 48
Theorem 16

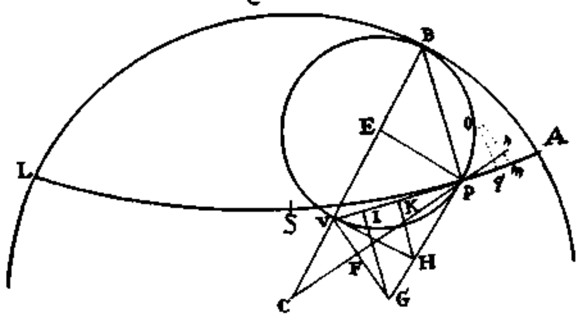
If a wheel stands upon the outer surface of a globe at right angles to that surface and, rolling as wheels do, moves forward in a great circle [in the globe's surface], the length of the curvilinear path traced out by any given point in the perimeter [or rim] of the wheel from the time when that point touched the globe (a curve which may be called a cycloid or epicycloid) will be to twice the versed sine of half the arc [of the rim of the wheel] which during the time of rolling has been in contact with the globe's surface as the sum of the diameters of the globe and wheel is to the semidiameter of the globe.

Proposition 49
Theorem 17

If a wheel stands upon the inner surface of a hollow globe at right angles to that surface and, rolling as wheels do, moves forward in a great circle [in the globe's surface], the length of the curvilinear path traced out by any given point in the perimeter [or rim] of the wheel from the time when that point touched the globe will be to twice the versed sine of half the arc [of the rim of the wheel] which during the time of rolling has been in contact with the globe's surface as the difference of the diameters of the globe and wheel is to the semidiameter of the globe.

Let ABL be the globe, C its center, BPV the wheel standing upon it, E the center of the wheel, B the point of contact, and P the given point in the perimeter of the wheel. Imagine that this wheel proceeds in the great circle ABL from A through B toward L and, while rolling, rotates in such a way that the arcs AB and PB are always equal to each other and that the given point P in the perimeter of the wheel is meanwhile describing the curvilinear path AP. Now, let AP be the whole curvilinear path described since the wheel was in contact with the globe at A, and the length AP of this path will be to twice the versed sine of the arc ½PB as 2CE to CB. For let the straight line CE (produced if need be) meet the wheel in V, and join CP, BP, EP, VP, and drop the normal VF to CP produced. Let PH and





VH, meeting in H, touch the circle in P and V, and let PH cut VF in G, and drop the normals GI and HK to VP. With the same center C and with any radius whatever describe the circle *nom* cutting the straight line CP in *n*, the wheel's perimeter BP in *o*, and the curvilinear path AP in *m*; and with center V and radius V*o* describe a circle cutting VP produced in *q*.

Since the wheel, in rolling, always revolves about the point of contact B, it is manifest that the straight line BP is perpendicular to the curved line AP described by the wheel's point P, and therefore that the straight line VP will touch this curve in point P. Let the radius of the circle *nom* be gradually increased or decreased, and so at last become equal to the distance CP; then, because the evanescent figure P*nomq* and the figure PFGVI are similar, the ultimate ratio of the evanescent line-elements P*m*, P*n*, P*o*, and P*q*, that is, the ratio of the instantaneous changes of the curve AP, the straight line CP, the circular arc BP, and the straight line VP, will be the same as that of the lines PV, PF, PG, and PI respectively. But since VF is perpendicular to CF, and VH is perpendicular to CV, and the angles HVG and VCF are therefore equal, and the angle VHG is equal to the angle CEP (because the angles of the quadrilateral HVEP are right angles at V and P), the triangles VHG and CEP will be similar; and hence it will come about that EP is to CE as HG to HV or HP and as KI to KP, and by composition [or componendo] or by separation [or dividendo] CB is to CE as PI to PK, and—by doubling of the consequents—CB is to 2CE as PI to PV and as P*q* to P*m*. Therefore the decrement of the line VP, that is, the increment of the line BV − VP, is to the increment of the curved line AP in the given ratio of CB to 2CE, and therefore (by lem. 4, corol.) the lengths BV − VP and AP, generated by those increments, are in the same ratio. But since BV is the radius, VP is the cosine of the angle BVP or ½BEP, and therefore BV − VP is the versed sine of the same angle; and therefore in this wheel, whose radius is ½BV, BV − VP will be twice the versed sine of the arc ½BP. And thus AP is to twice the versed sine of the arc ½BP as 2CE to CB. Q.E.D.

For the sake of distinction, we shall call the curved line AP in prop. 48 a *cycloid outside the globe*, and the curved line AP in prop. 49 a *cycloid inside the globe*.

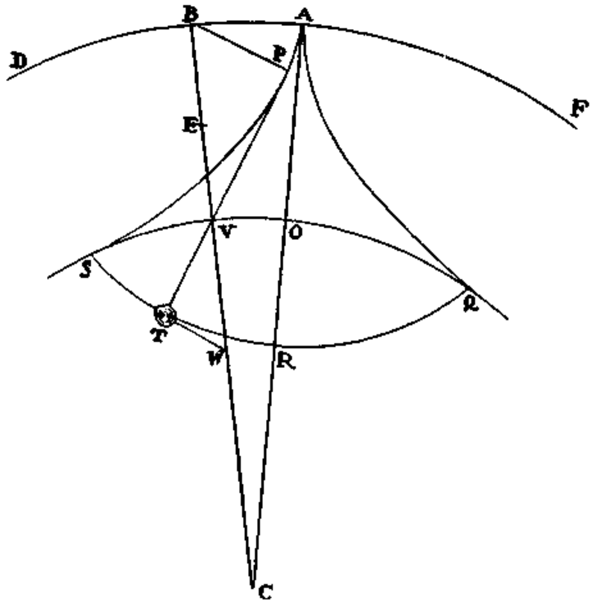
COROLLARY 1. Hence, if an entire cycloid ASL is described and is bisected in S, the length of the part PS will be to the length VP (which is twice the sine of the angle VBP, where EB is the radius) as 2CE to CB, and thus in a given ratio.

COROLLARY 2. And the length of the semiperimeter AS of the cycloid will be equal to a straight line that is to the diameter BV of the wheel as 2CE to CB.

Proposition 50
Problem 33

To make a pendulum bob oscillate in a given cycloid.

Within a globe QVS described with center C, let the cycloid QRS be given, bisected in R and with its end-points Q and S meeting the surface of the globe on the two sides. Draw CR bisecting the arc QS in O, and produce CR to A, so that CA is to CO as CO to CR. Describe an outer globe DAF with center C and radius CA; and inside this globe let two half-cycloids AQ and AS be described by means of a wheel whose diameter is AO, and let these two half-cycloids touch the inner globe at Q and S and meet the outer globe in A. Let a body T hang from the point A by a thread APT



equal to the length AR, and let this body T oscillate between the two half-cycloids AQ and AS in such a way that each time the pendulum departs from the perpendicular AR, the upper part AP of the thread comes into contact with that half-cycloid APS toward which the motion is directed, and is bent around it as an obstacle, while the other part PT of the thread, to which the half-cycloid is not yet exposed, stretches out in a straight line; then the weight T will oscillate in the given cycloid QRS. Q.E.F.

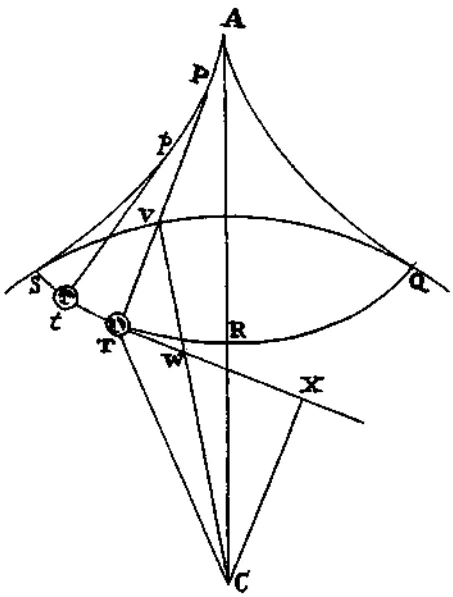
For let the thread PT meet the cycloid QRS in T and the circle QOS in V, and draw CV; and from the end-points P and T of the straight part PT of the thread, erect BP and TW perpendicular to PT, meeting the straight line CV in B and W. It is evident, from the construction and the generation of the similar figures AS and SR, that the perpendiculars PB and TW cut off from CV the lengths VB and VW equal respectively to OA and OR, the diameters of the wheels. Therefore, TP is to VP (which is twice the sine of the angle VBP, where ½BV is the radius) as BW to BV, or AO + OR to AO, that is (since CA is proportional to CO, CO to CR, and by separation [or dividendo] AO to OR), as CA + CO to CA, or, if BV is bisected in E, as 2CE to CB. Accordingly (by prop. 49, corol. 1), the length of the straight part PT of the thread is always equal to the arc PS of the cycloid, and the whole thread APT is always equal to the half-arc APS of the cycloid, that is (by prop. 49, corol. 2), to the length AR. And therefore, conversely, if the thread always remains equal to the length AR, point T will move in the given cycloid QRS. Q.E.D.

COROLLARY. The thread AR is equal to the half-cycloid AS and thus has the same ratio to the semidiameter AC of the outer globe that the half-cycloid SR, similar to it, has to the semidiameter CO of the inner globe.

Proposition 51
Theorem 18

If a centripetal force tending from all directions to the center C *of a globe is in each individual place as the distance of that place from the center; and if, under the action of this force alone, the body* T *oscillates (in the way just described) in the perimeter of the cycloid* QRS; *then I say that the times of the oscillations, however unequal the oscillations may be, will themselves be equal.*

For let the perpendicular CX fall to the indefinitely produced tangent TW of the cycloid and join CT. Now the centripetal force by which the body T is impelled toward C is as the distance CT, and CT may be resolved (by corol. 2 of the laws) into the components CX and TX, of which CX (by impelling the body directly from P) stretches the thread PT and is wholly nullified by the resistance of the thread and produces no other effect, while the other component TX (by urging the body transversely or toward X) directly accelerates the motion of the body in the cycloid; hence it is manifest that the body's acceleration, which is proportional to this accelerative force, is at each



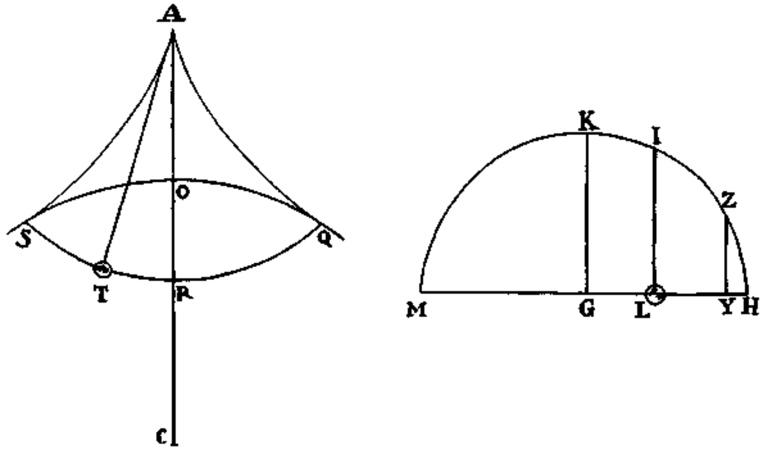
individual moment as the length TX, that is (because CV and WV—and TX and TW, proportional to them—are given), as the length TW, that is (by prop. 49, corol. 1), as the length of the arc of the cycloid TR. Therefore, if the two pendulums APT and A*pt* are drawn back unequally from the perpendicular [or vertical] AR and are let go simultaneously, their accelerations will always be as the respective arcs to be described TR and *t*R. But the parts of these arcs described at the beginning of the motion are as the accelerations, that is, as the whole arcs to be described at the beginning, and therefore the parts that remain to be described and the subsequent accelerations proportional to these parts are also as the whole arcs, and so on. Therefore the accelerations—and hence the velocities generated, the parts of the arcs described with these velocities, and the parts to be so described—are always as the whole arcs; and therefore the parts to be described, preserving a given ratio to one another, will vanish simultaneously, that is, the two oscillating bodies will arrive at the same time at the perpendicular [or vertical] AR. And since, conversely, the ascents of the pendulums, made from the lowest place R through the same cycloidal arcs with a reverse motion, are retarded in individual places by the same forces by which their descents were accelerated, it is evident that the velocities of the ascents and descents made through the same arcs are equal and hence occur in equal times; and therefore, since the two parts RS and RQ of the cycloid, each lying on a different side of the perpendicular [or vertical], are similar and equal, the two pendulums will always make their whole oscillations as well as their half-oscillations in the same times. Q.E.D.

COROLLARY. The force by which body T is accelerated or retarded in any place T of the cycloid is to the total weight of body T in the highest place S or Q as the arc TR of the cycloid to its arc SR or QR.

Proposition 52
Problem 34

To determine both the velocities of pendulums in individual places and the times in which complete oscillations, as well as the separate parts of oscillations, are completed.

With any center G and with a radius GH equal to the arc RS of the cycloid, describe the semicircle HKM bisected by the semidiameter GK. And if a centripetal force proportional to the distances of places from the center tends toward that center G, and if in the perimeter HIK that force is equal to the centripetal force in the perimeter of the globe QOS tending toward its center, and if, at the same time that the pendulum T is let go from its highest place S, some other body L falls from H to G; then, since the forces by which the bodies are urged are equal at the beginning of the motion, and are always proportional to the spaces TR and LG which are to be described, and are therefore equal in the places T and L if TR and LG are equal, it is evident that the two bodies describe the equal spaces ST and HL at the



beginning of the motion and thus will proceed thereafter to be equally urged and to describe equal spaces. Therefore (by prop. 38), the time in which the body describes the arc ST is to the time of one oscillation as the arc HI (the time in which the body H will reach L) to the semiperiphery HKM (the time in which the body H will reach M). And the velocity of the pendulum bob at the place T is to its velocity at the lowest place R (that is, the velocity of body H in the place L to its velocity in the place G, or the instantaneous increment of the line HL to the instantaneous increment of the line HG, where the arcs HI and HK increase with a uniform flow) as the ordinate LI to the radius GK, or as $\sqrt{(SR^2 - TR^2)}$ to SR. Hence, since in unequal oscillations arcs proportional to the total arcs of the oscillations are described in equal times, both the velocities and the arcs described in all oscillations universally can be found from the given times. As was first to be found.

Now let simple pendulums oscillate in different cycloids described within different globes, whose absolute forces are also different; and if the absolute force of any globe QOS is called V, the accelerative force by which the pendulum is urged in the circumference of this globe, when it begins to move directly toward its center, will be jointly as the distance of the pendulum bob from that center and the absolute force of the globe, that is, as $CO \times V$. Therefore the line-element HY (which is as this accelerative force $CO \times V$) will be described in a given time; and if the normal YZ is erected so as to meet the circumference in Z, the nascent arc HZ will denote that given time. But this nascent arc HZ is as the square root of the rectangle $GH \times HY$, and thus as $\sqrt{(GH \times CO \times V)}$. Hence the time of a complete oscillation in the cycloid QRS (since it is directly as the semiperiphery HKM, which denotes that complete oscillation, and inversely as the arc HZ, which similarly denotes the given time) will turn out to be as GH directly and $\sqrt{(GH \times CO \times V)}$ inversely, that is, because GH and SR are equal, as $\sqrt{\frac{SR}{CO \times V}}$, or (by prop. 50, corol.) as $\sqrt{\frac{AR}{AC \times V}}$. Therefore the oscillations in all globes and cycloids, made with any absolute forces whatever, are as the square root of the length of the thread directly and as the square root of the distance between the point of suspension and the center of the globe inversely and also as the square root of the absolute force of the globe inversely. Q.E.I.

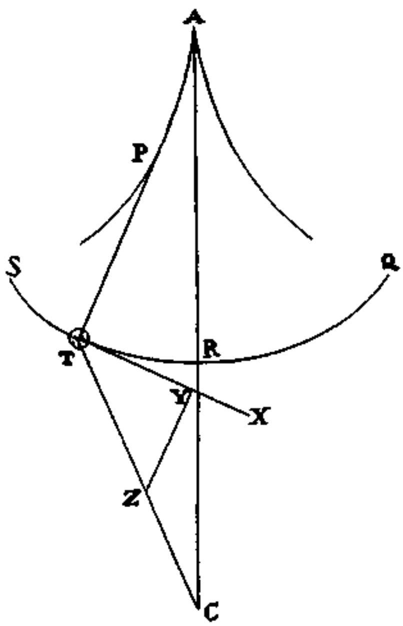
COROLLARY 1. Hence also the times of bodies oscillating, falling, and revolving can be compared one with another. For if the diameter of the wheel by which a cycloid is described within a globe is made equal to the semidiameter of the globe, the cycloid will turn out to be a straight line passing through the center of the globe, and the oscillation will now be a descent and subsequently an ascent in this straight line. Hence the time of the descent from any place to the center is given, as well as the time (equal to that time of descent) in which a body, by revolving uniformly about the center of the globe at any distance, describes a quadrantal arc. For this time (by the second case [that is, according to the second paragraph above]) is to the time of a half-oscillation in any cycloid QRS as 1 to $\sqrt{\frac{AR}{AC}}$.

COROLLARY 2. Hence also there follows what Wren and Huygens discovered about the common cycloid. For if the diameter of the globe is increased indefinitely, its spherical surface will be changed into a plane, and the centripetal force will act uniformly along lines perpendicular to this plane, and our cycloid will turn into a common cycloid. But in that case the length of the arc of the cycloid between that plane and the describing point will come out equal to four times the versed sine of half of the arc of the wheel between that same plane and the describing point, as Wren discovered; and a pendulum between two cycloids of this sort will oscillate in a similar and equal cycloid in equal times, as Huygens demonstrated. But also the descent of heavy bodies during the time of one oscillation will be the descent which Huygens indicated.

Moreover, the propositions that we have demonstrated fit the true constitution of the earth, insofar as wheels, moving in the earth's great circles, describe cycloids outside this globe by the motion of nails fastened in their perimeters; and pendulums suspended lower down in mines and caverns of the earth must oscillate in cycloids within globes in order that all their oscillations may be isochronous. For gravity (as will be shown in book 3) decreases in going upward from the surface of the earth as the square of the distance from the earth's center, and in going downward from the surface is as the distance from that center.

Proposition 53
Problem 35

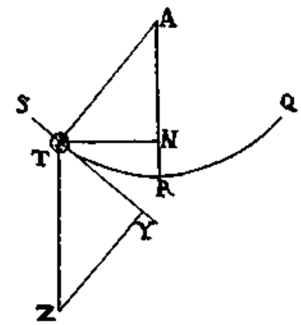
Granting the quadratures of curvilinear figures, it is required to find the forces by whose action bodies moving in given curved lines will make oscillations that are always isochronous.



Let a body T oscillate in any curved line STRQ whose axis is AR passing through the center of forces C. Draw TX touching that curve in any place T of the body, and on this tangent TX take TY equal to the arc TR. [This may be done] since the length of that arc can be known from the quadratures of figures by commonly used methods. From point Y draw the straight line YZ perpendicular to the tangent. Draw CT meeting the perpendicular in Z, and the centripetal force will be proportional to the straight line TZ. Q.E.I.

For if the force by which the body is drawn from T toward C is represented by the straight line TZ taken proportional to it, this will be resolved into the forces TY and YZ, of which YZ, by drawing the body along the length of the thread PT, does not change its motion at all, while the other force TY directly accelerates or directly retards its motion in the curve STRQ. Accordingly, since this force is as the projection TR to be described, the body's accelerations or retardations in describing proportional parts of two oscillations (a greater and a lesser oscillation) will always be as those parts, and will therefore cause those parts to be described simultaneously. And bodies that in the same time describe parts always proportional to the wholes will describe the wholes simultaneously. Q.E.D.

Corollary 1. Hence, if body T, hanging by a rectilinear thread AT from the center A, describes the circular arc STRQ and meanwhile is urged downward along parallel lines by some force that is to the uniform force of gravity as the arc TR to its sine TN, the times of any single oscillations will be equal. For, because TZ and AR are parallel, the triangles ATN and ZTY will be similar; and therefore TZ will be to AT as TY to TN; that is, if the uniform force of gravity is represented by the given length AT, the force TZ, by the action of which the oscillations will turn out to be isochronous, will be to the force of gravity AT as the arc TR (equal to TY) to the sine TN of that arc.

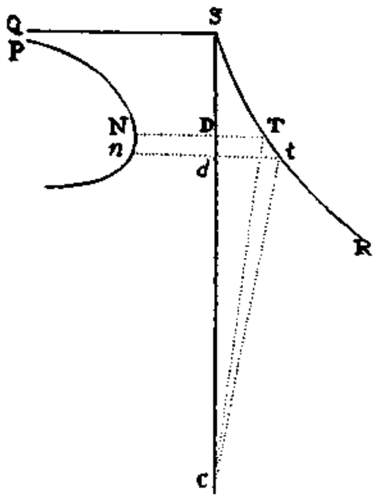


Corollary 2. And therefore in [pendulum] clocks, if the forces impressed by the mechanism upon the pendulum to maintain the motion can be compounded with the force of gravity in such a way that the total force downward is always as the line that arises from dividing the rectangle of the arc TR and the radius AR by the sine TN, all the oscillations will be isochronous.

Proposition 54
Problem 36

Granting the quadratures of curvilinear figures, to find the times in which bodies under the action of any centripetal force will descend and ascend in any curved lines described in a plane passing through the center of forces.

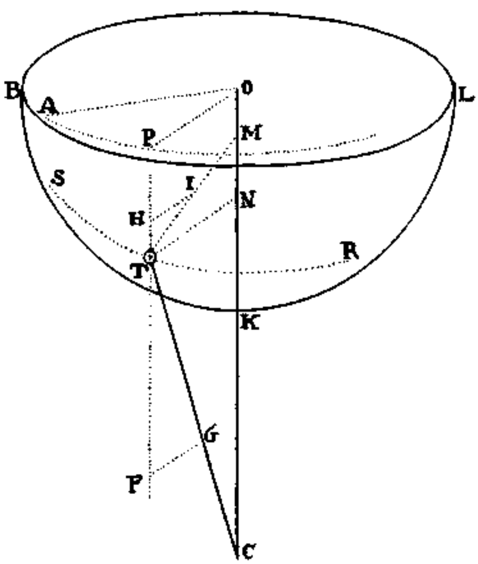
Let a body descend from any place S through any curved line ST*t*R given in a plane passing through the center of forces C. Join CS and divide it into innumerable equal parts, and let D*d* be some one of those parts. With center C and radii CD and C*d*, describe the circles DT and *dt*, meeting the curved line ST*t*R in T and *t*. Then, since both the law of centripetal force and the height CS from which the body has fallen are given, the velocity of the body at any other height CT will be given (by prop. 39). Moreover, the time in which the body describes the line-element T*t* is as the length of this line-element (that is, as the secant of the angle *t*TC) directly and as the velocity inversely. Let the ordinate DN be proportional to this time and perpendicular to the straight line CS through point D; then, because D*d* is



given, the rectangle $Dd \times DN$, that is, the area $DNnd$, will be proportional to that same time. Therefore if PNn is the curved line that point N continually traces out, [a]and its asymptote is the straight line SQ standing perpendicularly upon the straight line CS,[a] the area SQPND will be proportional to the time in which the body, by descending, has described the line ST; and accordingly, when that area has been found, the time will be given. Q.E.I.

Proposition 55
Theorem 19
If a body moves in any curved surface whose axis passes through a center of forces, and a perpendicular is dropped from the body to the axis, and a straight line parallel and equal to the perpendicular is drawn from any given point of the axis; I say that the parallel will describe an area proportional to the time.

Let BKL be the curved surface, T the body revolving in it, STR the trajectory which the body describes in it, S the beginning of the trajectory, OMK the axis of the curved surface, TN the perpendicular straight line dropped from the body to the axis; and let OP be the straight line parallel and equal to TN and drawn from a point O that is given in the axis, AP the path described by point P in the plane AOP of the revolving line OP, A the beginning of the projection (corresponding to point S); and let TC be a straight line drawn



aa. A clarification by Pemberton after he had called Newton's attention to the incorrect diagrams in eds. 1 and 2 (cf. *The Mathematical Papers of Isaac Newton*, ed. D. T. Whiteside [Cambridge: Cambridge University Press, 1967–1981], 6:409, nn. 308–309).

from the body to the center, TG the part of TC that is proportional to the centripetal force by which the body is urged toward the center C, TM a straight line perpendicular to the curved surface, TI the part of TM proportional to the force of pressure by which the body urges the surface and is in turn urged by the surface toward M; and let PTF be a straight line parallel to the axis and passing through the body, and GF and IH straight lines dropped perpendicularly from the points G and I to the parallel PHTF. I say now that the area AOP, described by the radius OP from the beginning of the motion, is proportional to the time. For the force TG (by corol. 2 of the laws) is resolved into the forces TF and FG, and the force TI into the forces TH and HI. But the forces TF and TH, by acting along the line PF perpendicular to the plane AOP, change the body's motion only insofar as it is perpendicular to this plane. And therefore the body's motion, insofar as it takes place in the position of the plane—that is, the motion of point P, by which the projection AP of the trajectory is described in this plane—is the same as if the forces TF and TH were taken away and the body were acted on by the forces FG and HI alone; that is, it is the same as if the body were to describe the curve AP in the plane AOP under the action of a centripetal force tending toward the center O and equal to the sum of the forces FG and HI. But by the action of such a force the area AOP is (by prop. 1) described proportional to the time.[a] Q.E.D.

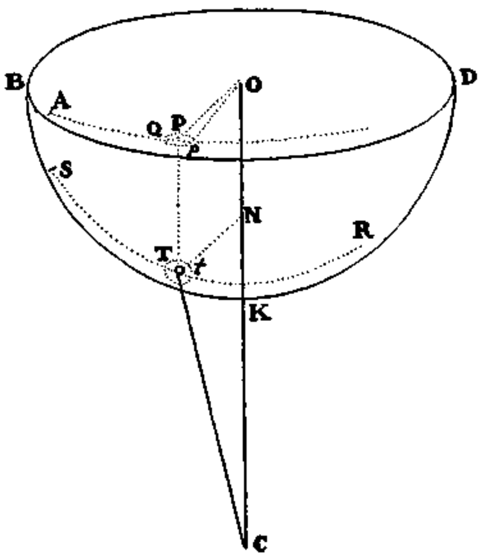
COROLLARY. By the same argument, if a body, acted on by forces tending toward two or more centers in any one given straight line CO, described any curved line ST in free space, the area AOP would always be proportional to the time.

Proposition 56
Problem 37

Granting the quadratures of curvilinear figures, and given both the law of centripetal force tending toward a given center and a curved surface whose axis passes through that center, it is required to find the trajectory that a body will describe in that same surface when it has set out from a given place with a given velocity, in a given direction in that surface.

Assuming the same constructions as in prop. 55, let body T go forth from the given place S, along a straight line given in position, in the required

a. In this proposition, Newton's "vestigium," literally, "a trace," has been translated as "projection," following D. T. Whiteside.



trajectory STR, and let the projection of this trajectory in the plane BLO be AP. And since the velocity of the body is given at the height SC, its velocity at any other height TC will be given. With this velocity, let the body in a given minimally small time describe the particle T*t* of its trajectory, and let P*p* be its projection described in the plane AOP. Join O*p*, and let the projection (in the plane AOP) of the little circle described with center T and radius T*t* in the curved surface be the ellipse *p*Q. Then, because the little circle T*t* is given in magnitude, and its distance TN or PO from the axis CO is given, the ellipse *p*Q will be given in species and in magnitude, as well as in its position with respect to the straight line PO. And since the area PO*p* is proportional to the time and therefore given because the time is given, the angle PO*p* will be given. And hence the common intersection *p* of the ellipse and the straight line O*p* will be given, along with the angle OP*p* in which the projection AP*p* of the trajectory cuts the line OP. And accordingly (by consulting prop. 41 with its corol. 2) the way of determining the curve AP*p* is readily apparent. Then, erecting perpendiculars to the plane AOP one by one, from the points P of the projection, so as to meet the curved surface in T, the points T of the trajectory will be given one by one. Q.E.I.

SECTION 11

The motion of bodies drawn to one another by centripetal forces

Up to this point, I have been setting forth the motions of bodies attracted toward an immovable center, such as, however, hardly exists in the natural world. For attractions are always directed toward bodies, and—by the third law—the actions of attracting and attracted bodies are always mutual and equal; so that if there are two bodies, neither the attracting nor the attracted body can be at rest, but both (by corol. 4 of the laws) revolve about a common center of gravity as if by a mutual attraction; and if there are more than two bodies that either are all attracted by and attract a single body or all attract one another, these bodies must move with respect to one another in such a way that the common center of gravity either is at rest or moves uniformly straight forward. For this reason I now go on to set forth the motion of bodies that attract one another, considering centripetal forces as attractions, although perhaps—if we speak in the language of physics—they might more truly be called impulses. For here we are concerned with mathematics; and therefore, putting aside any debates concerning physics, we are using familiar language so as to be more easily understood by mathematical readers.

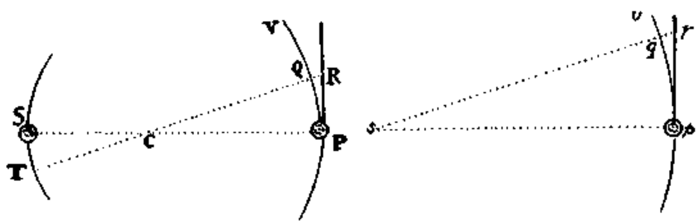
Proposition 57
Theorem 20

Two bodies that attract each other describe similar figures about their common center of gravity and also about each other.

For the distances of these bodies from their common center of gravity are inversely proportional to the masses of the bodies and therefore in a given ratio to each other and, by composition [or componendo], in a given ratio to the total distance between the bodies. These distances, moreover, rotate about their common end-point with an equal angular motion because, since they always lie in the same straight line, they do not change their inclination toward each other. And straight lines that are in a given ratio to each other and that rotate about their end-points with an equal angular motion describe entirely similar figures about the end-points in planes that, along with these end-points, either are at rest or move with any motion that is not angular. Accordingly, the figures described by the rotation of these distances are similar. Q.E.D.

Proposition 58
Theorem 21

If two bodies attract each other with any forces whatever and at the same time revolve about their common center of gravity, I say that by the action of the same forces there can be described around either body if unmoved a figure similar and equal to the figures that the bodies so moving describe around each other.



Let bodies S and P revolve about their common center of gravity C, going from S to T and from P to Q. From a given point *s* let *sp* and *sq* be drawn always equal and parallel to SP and TQ; then the curve *pqv*, which the point *p* describes by revolving around the motionless point *s*, will be similar and equal to the curves that bodies S and P describe around each other; and accordingly (by prop. 57) this curve *pqv* will be similar to the curves ST and PQV, which the same bodies describe around their common center of gravity C; and this is so because the proportions of the lines SC, CP, and SP or *sp* to one another are given.

Case 1. The common center of gravity C (by corol. 4 of the laws) either is at rest or moves uniformly straight forward. Let us suppose first that it is at rest, and at *s* and *p* let two bodies be placed, a motionless one at *s* and a moving one at *p*, similar and equal to bodies S and P. Then let the straight lines PR and *pr* touch the curves PQ and *pq* in P and *p*, and let CQ and *sq* be produced to R and *r*. Then, because the figures CPRQ and *sprq* are similar, RQ will be to *rq* as CP to *sp* and thus in a given ratio. Accordingly, if the force with which body P is attracted toward body S, and therefore toward the intermediate center C, were in that same given ratio to the force with which body *p* is attracted toward center *s*, then in equal times these forces would always attract the bodies from the tangents PR and *pr* to the arcs PQ and *pq* through the distances RQ and *rq* proportional to them; and therefore the latter force would cause body *p* to revolve in orbit in the curve *pqv*, which would be similar to the curve PQV, in which the former force causes body P to revolve in orbit, and the revolutions would be completed in the same times. But those forces are not to each other in the ratio CP to *sp* but are equal to each other (because bodies S and *s*, P and *p* are similar and equal, and distances SP and *sp* are equal); therefore, the bodies will in equal times be equally drawn away from the tangents; and therefore, for the second body *p* to be attracted through the greater distance *rq*, a greater time is required, which is as the square root of the distances, because (by lem. 10) the spaces described at the very beginning of the motion are as the squares of the times. Therefore, let the velocity of body *p* be supposed to be to the velocity of body P as the square root of the ratio of the distance *sp* to the distance CP, so that the arcs *pq* and PQ, which are in a simple ratio, are described in times which are as the square roots of the distances. Then bodies P and *p*, being always attracted by equal forces, will describe around the centers C and *s* at rest the similar figures PQV and *pqv*, of which *pqv* is similar and equal to the figure that body P describes around the moving body S. Q.E.D.

Case 2. Let us suppose now that the common center of gravity, along with the space in which the bodies are moving with respect to each other, is moving uniformly straight forward; then (by corol. 6 of the laws) all motions in this space will occur as in case 1. Hence the bodies will describe around each other figures which are the same as before and which therefore will be similar and equal to the figure *pqv*. Q.E.D.

Corollary 1. Hence (by prop. 10) two bodies, attracting each other with forces proportional to their distance, describe concentric ellipses, both around their common center of gravity and also around each other; and, conversely, if such figures are described, the forces are proportional to the distance.

Corollary 2. And (by props. 11, 12, and 13) two bodies, under the action of forces inversely proportional to the square of the distance, describe—around their common center of gravity and also around each other—conics having their focus in that center about which the figures are described. And, conversely, if such figures are described, the centripetal forces are inversely proportional to the square of the distance.

Corollary 3. Any two bodies revolving in orbit around a common center of gravity describe areas proportional to the times, by radii drawn to that center and also to each other.

Proposition 59
Theorem 22

The periodic time of two bodies S *and* P *revolving about their common center of gravity* C *is to the periodic time of one of the two bodies* P, *revolving in orbit* *about the other body* S *which is without motion, and describing a figure similar and equal to the figures that the bodies describe around each other, as the square root of the ratio of the mass of the second body* S *to the sum of the masses of the bodies* S + P.

For, from the proof of prop. 58, the times in which any similar arcs PQ and *pq* are described are as the square roots of the distances CP and SP or *sp*, that is, as the square root of the ratio of body S to the sum of the bodies S + P [or, as $\sqrt{S}$ to $\sqrt{(S + P)}$]. And by composition [or componendo] the sums of the times in which all the similar arcs PQ and *pq* are described, that is, the whole times in which the whole similar figures are described, are in that same ratio. Q.E.D.

Proposition 60
Theorem 23

If two bodies S *and* P, *attracting each other with forces inversely proportional to the square of the distance, revolve about a common center of gravity, I say that the principal axis of the ellipse which one of the bodies* P *describes by this motion about the other body* S *will be to the principal axis of the ellipse which the same body* P *would be able to describe in the same periodic time about the other body* S *at rest as the sum of the masses of the two bodies* S + P *is to the first of two mean proportionals between this sum and the mass of the other body* S.[a]

For if the ellipses so described were equal to each other, the periodic times would (by prop. 59) be as the square root of the mass of body S is to the square root of the sum of the masses of the bodies S + P. Let the periodic time in the second ellipse be decreased in this same ratio, and then the periodic times will become equal; but the principal axis of the second ellipse (by prop. 15) will be decreased as the $\frac{3}{2}$ power of the former ratio, that is, in the ratio of which the ratio S to S + P is the cube; and therefore the principal axis of the second ellipse will be to the principal axis of the first ellipse as the first of two mean proportionals between S + P and S to S + P. And inversely, the principal axis of the ellipse described about the body in motion will be to the principal axis of the ellipse described about the body not in motion as S + P to the first of two mean proportionals between S + P and S. Q.E.D.

Proposition 61
Theorem 24

If two bodies, attracting each other with any kind of forces and not otherwise acted on or impeded, move in any way whatever, their motions will be the same as if

a. That is, as (S + P) to the cube root of $S \times (S + P)^2$.

they were not attracting each other but were each being attracted with the same forces by a third body set in their common center of gravity. And the law of the attracting forces will be the same with respect to the distance of the bodies from that common center and with respect to the total distance between the bodies.

For the forces with which the bodies attract each other, in tending toward the bodies, tend toward a common center of gravity between them and therefore are the same as if they were emanating from a body between them. Q.E.D.

And since there is given the ratio of the distance of either of the two bodies from that common center to the distance between the bodies, there will also be given the ratio of any power of one such distance to the same power of the other distance, as well as the ratio that any quantity derived in any manner from one such distance together with given quantities has to another quantity derived in the same manner from the other distance together with the same number of given quantities having that given ratio of distances to the former ones. Accordingly, if the force with which one body is attracted by the other is directly or inversely as the distance of the bodies from each other or as any power of this distance or finally as any quantity derived in any manner from this distance and given quantities, the same force with which the same body is attracted to the common center of gravity will be likewise directly or inversely as the distance of the attracted body from that common center or as the same power of this distance or finally as a quantity derived in the same manner from this distance and analogous given quantities. That is, the law of the attracting force will be the same with respect to either of the distances. Q.E.D.

Proposition 62
Problem 38

To determine the motions of two bodies that attract each other with forces inversely proportional to the square of the distance and are let go from given places.

These bodies will (by prop. 61) move just as if they were being attracted by a third body set in their common center of gravity; and by hypothesis, that center will be at rest at the very beginning of the motion and therefore (by corol. 4 of the laws) will always be at rest. Accordingly, the motions of the bodies are (by prop. 36) to be determined just as if they were being urged by forces tending toward that center, and the motions of the bodies attracting each other will then be known. Q.E.I.

Proposition 63
Problem 39

To determine the motions of two bodies that attract each other with forces inversely proportional to the square of the distance and that set out from given places with given velocities along given straight lines.

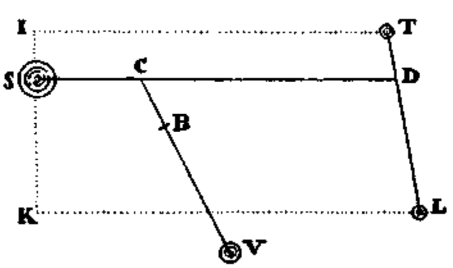
From the given motions of the bodies at the beginning the uniform motion of the common center of gravity is given, as well as the motion of the space that moves along with this center uniformly straight forward, and also the initial motions of the bodies with respect to this space. Now (by corol. 5 of the laws and prop. 61), the subsequent motions take place in this space just as if the space itself, along with that common center of gravity, were at rest, and as if the bodies were not attracting each other but were being attracted by a third body situated in that center. Therefore the motion of either body in this moving space, setting out from a given place with a given velocity along a given straight line and pulled by a centripetal force tending toward that center, is to be determined (by props. 17 and 37), and at the same time the motion of the other body about the same center will be known. This motion is to be compounded with that uniform progressive motion (found above) of the system of the space and bodies revolving in it, and the absolute motion of the bodies in an unmoving space will be known. Q.E.I.

Proposition 64
Problem 40

If the forces with which bodies attract one another increase in the simple ratio of the distances from the centers, it is required to find the motions of more than two bodies in relation to one another.

Suppose first that two bodies T and L have a common center of gravity D. These bodies will (by prop. 58, corol. 1) describe ellipses that have their centers at D and that have magnitudes which become known by prop. 10.

Now let a third body S attract the first two bodies T and L with accelerative forces ST and SL, and let it be attracted by those bodies in turn.



The force ST (by corol. 2 of the laws) is resolved into forces SD and DT, and the force SL into forces SD and DL. Moreover, the forces DT and DL, which are as their sum TL and therefore as the accelerative forces with which bodies T and L attract each other, when added respectively to those forces of bodies T and L, compose forces proportional to the distances DT and

DL, as before, but greater than those former forces, and therefore (by prop. 10, corol. 1, and prop. 4, corols. 1 and 8) they cause those bodies to describe ellipses as before, but with a swifter motion. The remaining accelerative forces, each of which is SD, by attracting those bodies T and L equally and along lines TI and LK (which are parallel to DS) with motive actions SD $\times$ T and SD $\times$ L (which are as the bodies), do not at all change the situations of those bodies in relation to one another, but make them equally approach line IK, which is to be conceived as drawn through the middle of body S, perpendicular to the line DS. That approach to line IK, however, will be impeded by causing the system of bodies T and L on one side and body S on the other to revolve in orbit with just the right velocities about a common center of gravity C. Body S describes an ellipse about that same point C with such a motion, because the sum of the motive forces SD $\times$ T and SD $\times$ L, which are proportional to the distance CS, tends toward the center C; and because CS and CD are proportional, point D will describe a similar ellipse directly opposite. But bodies T and L, being attracted respectively by motive forces SD $\times$ T and SD $\times$ L equally and along the parallel lines TI and LK (as has been said), will (by corols. 5 and 6 of the laws) proceed to describe their own ellipses about the moving center D, as before. Q.E.I.

Now let a fourth body V be added, and by a similar argument it will be concluded that this point and point C describe ellipses about B, the common center of gravity of all the bodies, while the motions of the former bodies T, L, and S about centers D and C remain the same as before, but accelerated. And by the same method it will be possible to add more bodies. Q.E.I.

These things are so, even if bodies T and L attract each other with accelerative forces that are greater or less than those by which they attract the rest of the bodies in proportion to the distance. Let the mutual accelerative attractions of all the bodies to one another be as the distances multiplied by the attracting bodies; then, from what has gone before, it will be easily deduced that all the bodies describe different ellipses in equal periodic times about B, the common center of gravity of them all, in a motionless plane. Q.E.I.

Proposition 65
Theorem 25

More than two bodies whose forces decrease as the squares of the distances from their centers are able to move with respect to one another in ellipses and, by radii drawn to the foci, are able to describe areas proportional to the times very nearly.

In prop. 64 the case was demonstrated in which the several motions occur exactly in ellipses. The more the law of force departs from the law there supposed, the more the bodies will perturb their mutual motions; nor can it happen that bodies will move exactly in ellipses while attracting one another according to the law here supposed, except by maintaining a fixed proportion of distances one from another. In the following cases, however, the orbits will not be very different from ellipses.

Case 1. Suppose that several lesser bodies revolve about some very much greater one at various distances from it, and that absolute forces proportional to these bodies [i.e., their masses] tend toward each and every one of them. Then, since the common center of gravity of them all (by corol. 4 of the laws) either is at rest or moves uniformly straight forward, let us imagine that the lesser bodies are so small that the greater body never is sensibly distant from this center. In this case, the greater body will—without any sensible error—either be at rest or move uniformly straight forward, while the lesser ones will revolve about this greater one in ellipses and by radii drawn to it will describe areas proportional to the times, except insofar as there are errors introduced either by a departure of the greater body from that common center of gravity or by the mutual actions of the lesser bodies on one another. The lesser bodies, however, can be diminished until that departure and the mutual actions are less than any assigned values, and therefore until the orbits square with ellipses and the areas correspond to the times without any error that is not less than any assigned value. Q.E.O.

Case 2. Let us now imagine a system of lesser bodies revolving in the way just described around a much greater one, or any other system of two bodies revolving around each other, to be moving uniformly straight forward and at the same time to be urged sideways by the force of another very much greater body, situated at a great distance. Then, since the equal accelerative forces by which the bodies are urged along parallel lines do not change the situations of the bodies in relation to one another, but cause the whole system to be transferred simultaneously, while the motions of the parts with respect to one another are maintained; it is manifest that no change whatsoever of the motion of the bodies attracted among themselves will result from their attractions toward the greater body, unless such a change comes either from the inequality of the accelerative attractions or from the inclination to one another of the lines along which the attractions take place. Suppose, therefore, that all the accelerative attractions toward the greater body are with respect to one another inversely as the squares of the distances; then by increasing the distance of the greater body until the differences (with respect to their length) among the straight lines drawn from this body to the other bodies and their inclinations with respect to one another are less than any assigned values, the motions of the parts of the system with respect to one another will persevere without any errors that are not less than any assigned values. And since, because of the slight distance of those parts from one another, the whole system is attracted as if it were one body, that system will be moved by this attraction as if it were one body; that is, by its center of gravity it will describe about the greater body some conic (namely, a hyperbola or parabola if the attraction is weak, an ellipse if the attraction is stronger) and by a radius drawn to the greater body will describe areas proportional to the times without any errors except the ones that may be produced by the distances between the parts, and these are admittedly slight and may be diminished at will. Q.E.O.

By a similar argument one can go on to more complex cases indefinitely.

Corollary 1. In case 2, the closer the greater body approaches to the system of two or more bodies, the more the motions of the parts of the system with respect to one another will be perturbed, because the inclinations to one another of the lines drawn from this great body to those parts are now greater, and the inequality of the proportion is likewise greater.

Corollary 2. But these perturbations will be greatest if the accelerative attractions of the parts of the system toward the greater body are not to one another inversely as the squares of the distances from that greater body, especially if the inequality of this proportion is greater than the inequality of the proportion of the distances from the greater body. For if the accelerative force, acting equally and along parallel lines, in no way perturbs the motions of the parts of the system with respect to one another, it will necessarily cause a perturbation to arise when there is an inequality in its action, and such perturbation will be greater or less according as this inequality is greater or less. The excess of the greater impulses acting on some bodies, but not acting on others, will necessarily change the situation of the bodies with respect to one another. And this perturbation, added to the perturbation that arises from the inclination and inequality of the lines, will make the total perturbation greater.

Corollary 3. Hence, if the parts of this system—without any significant perturbation—move in ellipses or circles, it is manifest that these parts either are not urged at all (except to a very slight degree indeed) by accelerative forces tending toward other bodies, or are all urged equally and very nearly along parallel lines.

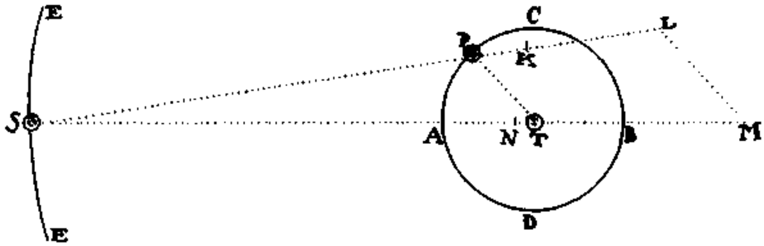
Proposition 66[a]
Theorem 26

Let three bodies—whose forces decrease as the squares of the distances—attract one another, and let the accelerative attractions of any two toward the third be to each other inversely as the squares of the distances, and let the two lesser ones revolve about the greatest. Then I say that if that greatest body is moved by these attractions, the inner body [of the two revolving bodies] will describe about the innermost and greatest body, by radii drawn to it, areas more nearly proportional to the times and a figure more closely approaching the shape of an ellipse (having its focus in the meeting point of the radii) than would be the case if that greatest body were not attracted by the smaller ones and were at rest, or if it were much less or much more attracted and were acted on either much less or much more.

This is sufficiently clear from the demonstration of the second corollary of prop. 65, but it is proved as follows by a more lucid and more generally convincing argument.

Case 1. Let the lesser bodies P and S revolve in the same plane about a greatest body T, and let P describe the inner orbit PAB, and S the outer orbit ESE. Let SK be the mean distance between bodies P and S, and let the accelerative attraction of body P toward S at that mean distance be represented by that same line SK. Let SL be taken to SK as SK^2 to SP^2, and SL will be the accelerative attraction of body P toward S at any distance SP. Join PT, and parallel to it draw LM meeting ST in M; then the attraction SL will be resolved (by corol. 2 of the laws) into attractions SM and LM. And

a. In ed. 1, Newton used a different system of letters. In imitation of the usual form of Copernican diagram, the central body was labeled S (for "Sol," the sun) and the encircling body was P (for "Planeta," or planet). The next or outer body continued the sequence from P to Q. In ed. 2, as in ed. 3, the central body is T (suggesting "Terra" for the earth), the encircling body is still P (but now secondary planet or planetary satellite), while the outermost or perturbing body is S (suggesting "Sol"). In this way, in ed. 2 and ed. 3, Newton quite properly alerts the reader to the fact that he is basically analyzing mathematically a form of the three-body problem, exemplified by the moon moving in orbit around the earth while being perturbed by the gravitational force of the distant sun. The corollaries will not only serve for the discussion of the moon's motion in book 3 but also be used in determining the mass of the moon in book 3, prop. 37, corol. 3.



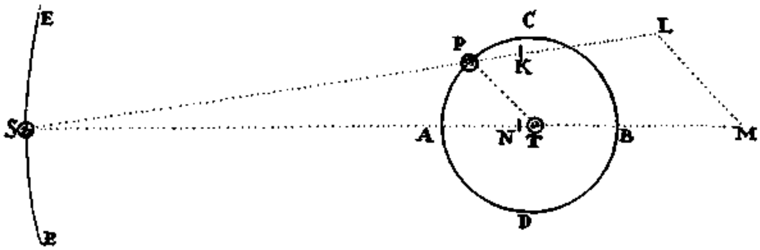
thus body P will be urged by a threefold accelerative force. One such force tends toward T and arises from the mutual attraction of bodies T and P. By this force alone (whether T is motionless or is moved by this attraction), body P must, by a radius PT, describe around body T areas proportional to the times and must also describe an ellipse whose focus is in the center of body T. This is clear from prop. 11 and prop. 58, corols. 2 and 3.

The second force is that of the attraction LM, which (since it tends from P to T) will, when added to the first of these forces, coincide with it and will thus cause areas to be described that are still proportional to the times, by prop. 58, corol. 3. But since this force is not inversely proportional to the square of the distance PT, it will, together with the first force, compose a force differing from this proportion—and the more so, the greater the proportion of this force is to that first force, other things being equal. Accordingly, since (by prop. 11 and by prop. 58, corol. 2) the force by which an ellipse is described about the focus T must tend toward that focus and be inversely proportional to the square of the distance PT, that composite force, by differing from this proportion, will cause the orbit PAB to deviate from the shape of an ellipse having its focus in T, and the more so the greater the difference from this proportion; and the difference from this proportion will be greater according as the proportion of the second force LM to the first force is greater, other things being equal.

But now the third force SM, by attracting body P along a line parallel to ST, will, together with the former forces, compose a force which is no longer directed from P to T and which deviates from this direction the more, the greater the proportion of this third force is to the former forces, other things being equal; and this compound force therefore will make body P describe, by a radius TP, areas no longer proportional to the times and will make the divergence from this proportionality be the greater, the greater the proportion of this third force is to the other forces. This third force will increase the deviation of the orbit PAB from the aforesaid elliptical shape for two reasons: not only is this force not directed from P to T, but also it is not inversely proportional to the square of the distance PT. Once these things have been understood, it is manifest that the areas will be most nearly proportional to the times when this third force is least, the other forces remaining the same as they were; and that the orbit PAB approaches closest to the aforesaid elliptical shape when both the second force and the third (but especially the third force) are least, the first force remaining the same as it was.

Let the accelerative attraction of body T toward S be represented by line SN; and if the accelerative attractions SM and SN were equal, they would, by attracting bodies T and P equally and along parallel lines, not at all change the situation of those two bodies with respect to each other. In this case, their motions with respect to each other would (by corol. 6 of the laws) be the same as it would be without these attractions. And for the same reason, if the attraction SN were smaller than the attraction SM, it would take away the part SN of the attraction SM, and only the part MN would remain, by which the proportionality of the times and areas and the elliptical shape of the orbit would be perturbed. And similarly, if the attraction SN were greater than the attraction SM, the perturbation of the proportionality and of the orbit would arise from the difference MN alone. Thus SM, the third attraction above, is always reduced by the attraction SN to the attraction MN, the first and second attractions remaining completely unchanged; and therefore the areas and times approach closest to proportionality, and the orbit PAB approaches closest to the aforesaid elliptical shape, when the attraction MN is either null or the least possible—that is, when the accelerative attractions of bodies P and T toward body S approach as nearly as possible to equality, in other words, when the attraction SN is neither null nor less than the least of all the attractions SM, but is a kind of mean between the maximum and minimum of all those attractions SM, that is, not much greater and not much smaller than the attraction SK. Q.E.D.

Case 2. Now let the lesser bodies P and S revolve about the greatest body T in different planes; then the force LM, acting along a line PT situated in the plane of orbit PAB, will have the same effect as before, and will not draw body P away from the plane of its orbit. But the second force NM, acting along a line that is parallel to ST (and therefore, when body S is outside the line of the nodes, is inclined to the plane of orbit PAB), besides



the perturbation of its motion in longitude, already set forth above, will introduce a perturbation of the motion in latitude, by attracting body P out of the plane of its orbit. And this perturbation, in any given situation of bodies P and T with respect to each other, will be as the generating force MN, and therefore becomes least when MN is least, that is (as I have already explained), when the attraction SN is not much greater and not much smaller than the attraction SK. Q.E.D.

Corollary 1. Hence it is easily gathered that if several lesser bodies P, S, R, . . . revolve about a greatest body T, the motion of the innermost body P will be least perturbed by the attractions of the outer bodies when the greatest body T is attracted and acted on as much by the other bodies (according to the ratio of the accelerative forces) as the other bodies are by one another.

Corollary 2. In a system of three bodies T, P, and S, if the accelerative attractions of any two toward the third are to each other inversely as the squares of the distances, body P will describe, by a radius PT, an area about body T more swiftly near their conjunction A and their opposition B than near the quadratures C and D. For every force by which body P is urged and body T is not, and which does not act along line PT, accelerates or retards the description of areas, according as its direction is forward and direct [or in consequentia] or retrograde [or in antecedentia]. Such is the force NM. In the passage of body P from C to A, this force is directed forward [or in consequentia] and accelerates the motion; afterward, as far as D, it is retrograde [or in antecedentia] and retards the motion; then forward up to B, and finally retrograde in passing from B to C.

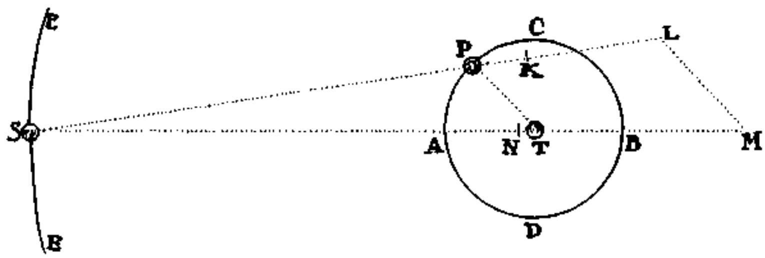
Corollary 3. And by the same argument it is evident that body P, other things being the same, moves more swiftly in conjunction and opposition than in the quadratures.

COROLLARY 4. The orbit of body P, other things being the same, is more curved in the quadratures than in conjunction and opposition. For swifter bodies are deflected less from a straight path. And besides, in conjunction and opposition the force KL, or NM, is opposite to the force with which body T attracts body P and therefore diminishes that force, while body P will be deflected less from a straight path when it is less urged toward body T.

COROLLARY 5. Accordingly, body P, other things being the same, will recede further from body T in the quadratures than in conjunction and opposition. These things are so if the motion of [i.e., change in] eccentricity is neglected. For if the orbit of body P is eccentric, its eccentricity (as will shortly be shown in corol. 9 of this proposition) will come out greatest when the apsides are in the syzygies; and thus it can happen that body P, arriving at the upper apsis, may be further away from body T in the syzygies than in the quadratures.

COROLLARY 6. Since the centripetal force of the central body T, which keeps body P in its orbit, is increased in the quadratures by the addition of the force LM and is diminished in the syzygies by the subtraction of the force KL and, because of the magnitude of the force KL [which is greater than LM], is more diminished than increased; and since that centripetal force (by prop. 4, corol. 2) is in a ratio compounded of the simple ratio of the radius TP directly and the squared ratio of the periodic time inversely [i.e., the force is directly as the radius and inversely as the square of the periodic time], it is evident that this compound ratio is diminished by the action of the force KL, and therefore that the periodic time (assuming the radius TP of the orbit to remain unchanged) is increased as the square root of the ratio in which that centripetal force is diminished. It is therefore further evident that, assuming this radius to be increased or diminished, the periodic time is increased more or diminished less than as the $\frac{3}{2}$ power of this radius, by prop. 4, corol. 6. If the force of the central body were gradually to weaken, body P, attracted always less and less, would continually recede further and further from the center T; and on the contrary, if the force were increased, body P would approach nearer and nearer. Therefore, if the action of the distant body S, whereby the force is diminished, is alternately increased and diminished, radius TP will at the same time also be alternately increased and diminished, and the periodic time will be increased and diminished in a ratio compounded of the $\frac{3}{2}$ power of the ratio of the radius and the square root of the ratio in which the centripetal force of the central body T is diminished or increased by the increase or decrease of the action of the distant body S.

COROLLARY 7. From what has gone before, it follows also that with respect to angular motion the axis of the ellipse described by body P, or the line of the apsides, advances and regresses alternately, but nevertheless advances more than it regresses and is carried forward [or in consequentia] by the excess of its direct forward motion. For the force whereby body P is



urged toward body T in the quadratures, when the force MN vanishes, is compounded of the force LM and the centripetal force with which body T attracts body P. If the distance PT is increased, the first force LM is increased in about the same ratio as this distance, and the latter force is decreased as the square of that ratio, and so the sum of these forces is decreased in a less than squared ratio of the distance PT, and therefore (by prop. 45, corol. 1) causes the auge, or upper apsis, to regress. But in conjunction and opposition the force whereby body P is urged toward body T is the difference between the force by which body T attracts body P and the force KL; and that difference, because the force KL is increased very nearly in the ratio of the distance PT, decreases in a ratio of the distance PT that is greater than the square of the distance PT, and so (by prop. 45, corol. 1) causes the upper apsis to advance. In places between the syzygies and quadratures the motion of the upper apsis depends on both of these causes jointly, so that according to the excess of the one or the other it advances or regresses. Accordingly, since the force KL in the syzygies is roughly twice as large as the force LM in the quadratures, the excess will have the same sense as the force KL and will carry the upper apsis forward [or in consequentia]. The truth of this corollary and its predecessor will be easily understood by supposing that a system of two bodies T and P is surrounded on all sides by more bodies S, S, S, . . . that are in an orbit ESE. For by the actions of these bodies, the action of T will be diminished on all sides and will decrease in a ratio greater than the square of the distance.

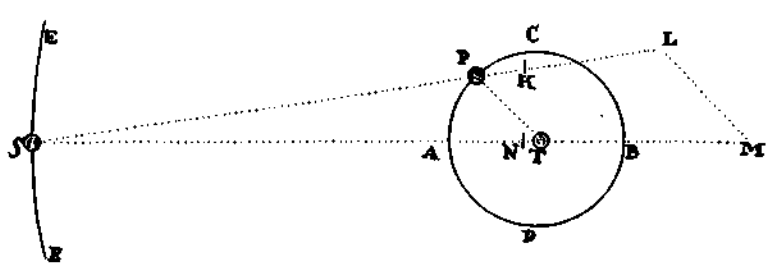
Corollary 8. Since, however, the advance or retrogression of the apsides depends on the decrease of the centripetal force, a decrease occurring in a ratio of the distance TP that is either greater or less than the square of the ratio of the distance TP, in the passage of the body from the lower to the upper apsis, and also depends on a similar increase in its return to the lower apsis, and therefore is greatest when the proportion of the force in the upper apsis to the force in the lower apsis differs most from the ratio of the inverse squares of the distances, it is manifest that KL or NM − LM, the force that subtracts, will cause the apsides to advance more swiftly in their syzygies and that LM, the force that adds, will cause them to recede more slowly in their quadratures. And because of the length of time in which the swiftness of the advance or slowness of the retrogression is continued, this inequality becomes by far the greatest.

Corollary 9. If a body, by the action of a force inversely proportional to the square of its distance from a center, were to revolve about this center in an ellipse, and if then, in its descent from the upper apsis or auge to the lower apsis, that force—because of the continual addition of a new force—were increased in a ratio that is greater than the square of the diminished distance, it is manifest that that body, being always impelled toward the center by the continual addition of that new force, would incline toward this center more than if it were urged only by a force increasing as the square of the diminished distance, and therefore would describe an orbit inside the elliptical orbit and in its lower apsis would approach nearer to the center than before. Therefore by the addition of this new force, the eccentricity of the orbit will be increased. Now if, during the receding of the body from the lower to the upper apsis, the force were to decrease by the same degrees by which it had previously increased, the body would return to its former distance; and so, if the force decreases in a greater ratio, the body, now attracted less, will ascend to a greater distance, and thus the eccentricity of its orbit will be increased still more. And therefore, if the ratio of the increase and decrease of the centripetal force is increased in each revolution, the eccentricity will always be increased; and contrariwise, the eccentricity will be diminished if that ratio decreases.

Now, in the system of bodies T, P, and S, when the apsides of the orbit PAB are in the quadratures, this ratio of the increase and decrease is least, and it becomes greatest when the apsides are in the syzygies. If the apsides are in the quadratures, the ratio near the apsides is smaller and near the syzygies is greater than the squared ratio of the distances, and from that greater ratio arises the forward or direct motion of the upper apsis, as has already been stated. But if one considers the ratio of the total increase or decrease in the forward motion between the apsides, this ratio is smaller than the squared ratio of the distances. The force in the lower apsis is to the force in the upper apsis in a ratio that is less than the squared ratio of the distance of the upper apsis from the focus of the ellipse to the distance of the lower apsis from that same focus; and conversely, when the apsides are in the syzygies, the force in the lower apsis is to the force in the upper apsis in a ratio greater than that of the squares of the distances.

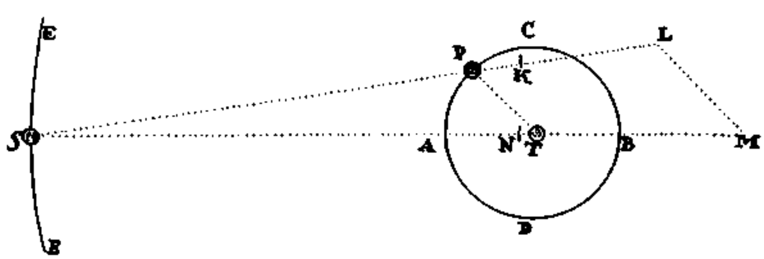
For the forces LM in the quadratures, added to the forces of body T, compose forces in a smaller ratio, and the forces KL in the syzygies, subtracted from the forces of body T, leave forces in a greater ratio. Therefore, the ratio of the total decrease and increase during the passage between apsides is least in the quadratures and greatest in the syzygies; and therefore, during the passage of the apsides from quadratures to syzygies, this ratio is continually increased and it increases the eccentricity of the ellipse; and in the passage from syzygies to quadratures, this ratio is continually diminished and it diminishes the eccentricity.

Corollary 10. To give an account of the errors in latitude, let us imagine that the plane of the orbit EST remains motionless; then from the cause of errors just expounded, it is manifest that of the forces NM and ML (which are the entire cause of these errors) the force ML, always acting in the plane of the orbit PAB, never perturbs the motions in latitude. It is likewise manifest that when the nodes are in the syzygies, the force NM, also acting in



the same plane of the orbit, does not perturb these motions; but when the nodes are in the quadratures, this force perturbs those motions to the greatest extent, and—by continually attracting body P away from the plane of its orbit—diminishes the inclination of the plane during the passage of the body from quadratures to syzygies and increases that inclination in turn during the passage from syzygies to quadratures. Hence it happens that when the body is in the syzygies the inclination turns out to be least of all, and it returns approximately to its former magnitude when the body comes to the next node. But if the nodes are situated in the octants after the quadratures, that is, between C and A, or D and B, it will be understood from what has just been explained that in the passage of body P from either node to a position 90 degrees from there, the inclination of the plane is continually diminished; then, in its passage through the next 45 degrees to the next quadrature, the inclination is increased; and afterward, in its next passage through another 45 degrees to the next node, it is diminished. Therefore, the inclination is diminished more than it is increased, and hence it is always less in each successive node than in the immediately preceding one. And by a similar reasoning, it follows that the inclination is increased more than it is diminished when the nodes are in the other octants between A and B, or B and C. Thus, when the nodes are in the syzygies, the inclination is greatest of all. In the passage of the nodes from syzygies to quadratures, the inclination is diminished in each appulse of the body to the nodes, and it becomes least of all when the nodes are in the quadratures and the body is in the syzygies; then it increases by the same degrees by which it had previously decreased, and at the appulse of the nodes to the nearest syzygies it returns to its original magnitude.

Corollary 11. When the nodes are in the quadratures, the body P is continually attracted away from the plane of its orbit in the direction toward S, during its passage from the node C through the conjunction A to the node D, and in the opposite direction in its passage from node D through



opposition B to node C; hence it is manifest that the body, in its motion from node C, continually recedes from the first plane CD of its orbit until it has reached the next node; and therefore at this node, being at the greatest distance from that first plane CD, it passes through EST, the plane of the orbit, not in the other node D of that plane but in a point that is closer to body S and which accordingly is a new place of the node, behind its former place. By a similar argument the nodes will continue to recede in the passage of the body from this node to the next node. Hence the nodes, when situated in the quadratures, continually recede; in the syzygies, when the motion in latitude is not at all perturbed, the nodes are at rest; in the intermediate places, since they share in both conditions, they recede more slowly; and therefore, since the nodes always either have a retrograde motion or are stationary, they are carried backward [or in antecedentia] in each revolution.

COROLLARY 12. All the errors described in these corollaries are slightly greater in the conjunction of bodies P and S than in their opposition; and this occurs because then the generating forces NM and ML are greater.

COROLLARY 13. And since the proportions in these corollaries do not depend on the magnitude of the body S, all the preceding statements are valid when the magnitude of body S is assumed to be so great that the system of two bodies T and P will revolve about it. And from this increase of body S, and consequently the increase of its centripetal force (from which the errors of body P arise), all those errors will—at equal distances—come out greater in this case than in the other, in which body S revolves around the system of bodies P and T.

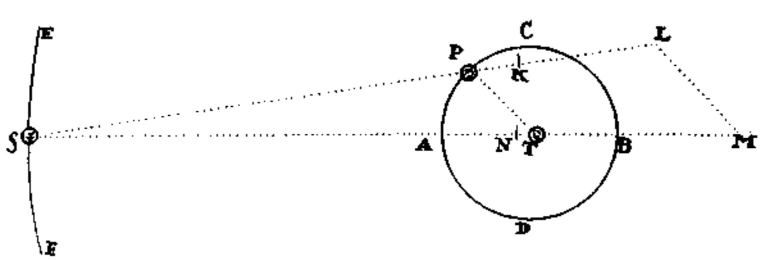
COROLLARY 14.[b] When body S is extremely far away, the forces NM and ML are very nearly as the force SK and the ratio of PT to ST jointly (that is, if both the distance PT and the absolute force of body S are given, as ST^3 inversely), and those forces NM and ML are the causes of all the errors and effects that have been dealt with in the preceding corollaries; hence it is manifest that all these effects—if the system of bodies T and P stays the same and only the distance ST and the absolute force of body S are changed—are very nearly in a ratio compounded of the direct ratio of the absolute force of body S and the inverse ratio of the cube of the distance ST. Accordingly, if the system of bodies T and P revolves about the distant body S, those forces NM and ML and their effects will (by prop. 4, corols. 2 and 6) be inversely

b. For a gloss on this corollary see the Guide, §10.16.

as the square of the periodic time. And hence also, if the magnitude of body S is proportional to its absolute force, those forces NM and ML and their effects will be directly as the cube of the apparent diameter of the distant body S when looked at from body T, and conversely. For these ratios are the same as the above-mentioned compounded ratio.

COROLLARY 15. If the magnitudes of the orbits ESE and PAB are changed, while their forms and their proportions and inclinations to each other remain the same, and if the forces of bodies S and T either remain the same or are changed in any given ratio, then these forces (that is, the force of body T, by whose action body P is compelled to deflect from a straight path into an orbit PAB; and the force of body S, by whose action that same body P is compelled to deviate from that orbit) will always act in the same way and in the same proportion; thus it will necessarily be the case that all the effects will be similar and proportional and that the times for these effects will be proportional as well—that is, all the linear errors will be as the diameters of the orbits, the angular errors will be the same as before, and the times of similar linear errors or of equal angular errors will be as the periodic times of the orbits.

COROLLARY 16. And hence, if the forms of the orbits and their inclination to each other are given, and the magnitudes, forces, and distances of the bodies are changed in any way, then from the given errors and given times of errors in one case there can be found the errors and times of errors in any other case very nearly. This may be done more briefly, however, by the following method. The forces NM and ML, other things remaining the same, are as the radius TP, and their periodic effects are (by lem. 10, corol. 2) jointly as the forces and the square of the periodic time of body P. These are the linear errors of body P, and hence the angular errors as seen from the center T (that is, the motions of the upper apsis and of the nodes, as well as all the apparent errors in longitude and latitude) are in any revolution of body P very nearly as the square of the time of revolution. Let these ratios



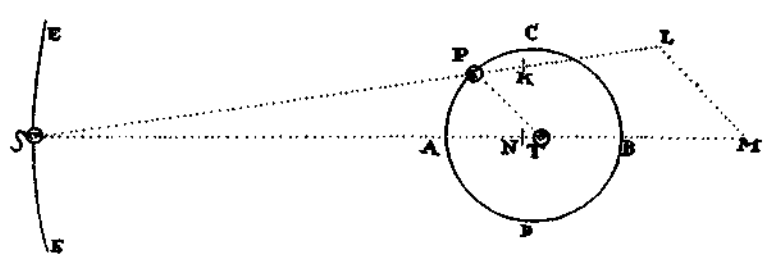
be compounded with the ratios of corol. 14; then in any system of bodies T, P, and S, in which P revolves around T which is near to it and T revolves around a distant S, the angular errors of body P, as seen from the center T, will—in each revolution of that body P—be as the square of the periodic time of body P directly and the square of the periodic time of body T inversely. And thus the mean motion of the upper apsis will be in a given ratio to the mean motion of the nodes, and each of the two motions will be as the periodic time of body P directly and the square of the periodic time of body T inversely. By increasing or decreasing the eccentricity and inclination of the orbit PAB, the motions of the upper apsis and of the nodes are not changed sensibly, except when the eccentricity and inclination are too great.

COROLLARY 17. Since, however, the line LM is sometimes greater and sometimes less than the radius PT, let the mean force LM be represented by that radius PT; then this force will be to the mean force SK or SN (which can be represented by ST) as the length PT to the length ST. But the mean force SN or ST by which body T is kept in its orbit around S is to the force by which body P is kept in its orbit around T in a ratio compounded of the ratio of the radius ST to the radius PT and the square of the ratio of the periodic time of body P around T to the periodic time of body T around S. And from the equality of the ratios [or ex aequo] the mean force LN is to the force by which a body P is kept in its orbit around T (or by which the same body P could revolve in the same periodic time around any immobile point T at a distance PT) in the same squared ratio of the periodic times. Therefore, if the periodic times are given, along with the distance PT, the mean force LM is also given; and if the force LM is given, the force MN is also given very nearly by the proportion of lines PT and MN.

COROLLARY 18. Let us imagine many fluid bodies to move around body T at equal distances from it according to the same laws by which body P revolves around the same body T; then let a ring—fluid, round, and concentric to body T—be produced by making these individual fluid bodies come into contact with one another; these individual parts of the ring, carrying out all their motions according to the law of body P, will approach closer to body T and will move more swiftly in the conjunction and opposition of themselves and body S than in the quadratures. The nodes of this ring, or its intersections with the plane of the orbit of body S or T, will be at rest in the syzygies, but outside the syzygies they will move backward [or in antecedentia], and do so most swiftly in the quadratures and more slowly in other places. The inclination of the ring will also vary, and its axis will oscillate in each revolution; and when a revolution has been completed, it will return to its original position except insofar as it is carried around by the precession of the nodes.

Corollary 19. Now imagine the globe T, which consists of nonfluid matter, to be so enlarged as to extend out to this ring, and to have a channel to contain water dug out around its whole circumference; and imagine this new globe to revolve uniformly about its axis with the same periodic motion. This water, being alternately accelerated and retarded (as in the previous corollary), will be swifter in the syzygies and slower in the quadratures than the surface of the globe itself, and thus will ebb and flow in the channel just as the sea does. If the attraction of body S is taken away, the water—now revolving about the quiescent center of the globe—will acquire no motion of ebb and flow. This is likewise the case for a globe advancing uniformly straight forward and meanwhile revolving about its own center (by corol. 5 of the laws), and also for a globe uniformly attracted away from a rectilinear path (by corol. 6 of the laws). But let body S now draw near, and by its nonuniform attraction of the water, the water will soon be disturbed. For its attraction of the nearer water will be greater and that of the more distant water will be smaller. Moreover, the force LM will attract the water downward in the quadratures and will make it descend as far as the syzygies, and the force KL will attract this same water upward in the syzygies and will prevent its further descent and will make it ascend as far as the quadratures, except insofar as the motion of ebb and flow is directed by the channel of water and is somewhat retarded by friction.

Corollary 20. If the ring now becomes hard and the globe is diminished, the motion of ebb and flow will cease; but the oscillatory motion of the inclination and the precession of the nodes will remain. Let the globe have the same axis as the ring and complete its revolutions in the same times, and let its surface touch the inside of the ring and adhere to it; then, with the



globe participating in the motion of the ring, the structure of the two will oscillate and the nodes will regress. For the globe, as will be shown presently, is susceptible to all impressions equally. The greatest angle of inclination of the ring alone, with the globe removed, occurs when the nodes are in the syzygies. From there in the forward motion of the nodes to the quadratures it endeavors to diminish its inclination and by that endeavor impresses a motion upon the whole globe. The globe keeps this impressed motion until the ring removes this motion by an opposite endeavor and impresses a new motion in the opposite direction; and in this way the greatest motion of the decreasing inclination occurs when the nodes are in the quadratures, and the least angle of inclination occurs in the octants after the quadratures; and the greatest motion of reclination occurs in the syzygies, and the greatest angle in the next octants. And this is likewise the case for a globe which has no such ring and which in the regions of the equator is either a little higher than near the poles or consists of matter a little denser. For that excess of matter in the regions of the equator takes the place of a ring. And although, by increasing the centripetal force of this globe in any way whatever, all its parts are supposed to tend downward, as the gravitating parts of the earth do, nevertheless the phenomena of this corollary and of corol. 19 will scarcely be changed on that account, except that the places of the greatest and least height of the water will be different. For the water is now sustained and remains in its orbit not by its own centrifugal force but by the channel in which it is flowing. And besides, the force LM attracts the water downward to the greatest degree in the quadratures, and the force KL or NM − LM attracts the same water upward to the greatest degree in the syzygies. And these forces conjoined cease to attract the water downward and begin to attract the water upward in the octants before the syzygies, and they cease to attract the water upward and begin to attract the water downward in the octants after the syzygies. As a result, the greatest height of the water can occur very nearly in the octants after the syzygies, and the least height can occur very nearly in the octants after the quadratures, except insofar as the motion of ascent or descent impressed on the water by these forces either perseveres a little longer because of the inherent force of the water or is stopped a little more swiftly because of the impediments of the channel.

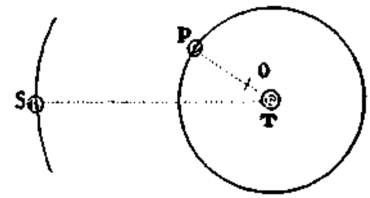
COROLLARY 21. In the same way that the excess matter of a globe near its equator makes the nodes regress (and thus the retrogression is increased by increase of equatorial matter and is diminished by its diminution and is removed by its removal), it follows that if more than the excess matter is removed, that is, if the globe near the equator is made either more depressed or more rare than near the poles, there will arise a motion of the nodes forward [or in consequentia].

Corollary 22. And thus, in turn, from the motion of the nodes the constitution of a globe can be found. That is to say, if a globe constantly preserves the same poles and there occurs a motion backward [or in antecedentia], there is an excess of matter near the equator; if there occurs a motion forward [or in consequentia], there is a deficiency. Suppose that a uniform and perfectly spherical globe is at first at rest in free space; then is propelled by any impetus whatever delivered obliquely upon its surface, from which it takes on a motion that is partly circular [i.e., rotational] and partly straight forward. Because the globe is indifferent to all axes passing through its center and does not have a greater tendency to turn around any one axis or an axis at any particular inclination, it is clear that the globe, by its own force alone, will never change its axis and the inclination of the axis. Now let the globe be impelled obliquely by any new impulse whatever, delivered to that same part of the surface as before; then, since the effect of an impulse is in no way changed by its being delivered sooner or later, it is manifest that the same motion will be produced by these two impulses being successively impressed as if they had been impressed simultaneously, that is, the resultant motion will be the same as if the globe had been impelled by a simple force compounded of these two (by corol. 2 of the laws), and hence will be a simple motion about an axis of a given inclination. This is likewise the case for a second impulse impressed in any other place on the equator of the first motion; and also for a first impulse impressed in any place on the equator of the motion which the second impulse would generate without the first, and hence for both impulses impressed in any places whatever. These two impulses will generate the same circular motion as if they had been impressed together and all at once in the place of intersection of the equators of the motions which each of them would generate separately. Therefore a homogeneous and perfect globe does not retain several distinct motions but compounds all the motions impressed on it and reduces them to one; and insofar as it can in and of itself, it always rotates with a simple and uniform motion about a single axis of a given and always invariable inclination. A centripetal force cannot change either this inclination of the axis or the velocity of rotation.

If a globe is thought of as divided into two hemispheres by any plane passing through the center of the globe and the center toward which a force is directed, that force will always urge both hemispheres equally and therefore will not cause the globe—as regards its motion of rotation—to incline in any direction. Let some new matter, heaped up in the shape of a mountain, be added to the globe anywhere between the pole and the equator; then this matter, by its continual endeavor to recede from the center of its motion, will disturb the motion of the globe and will make its poles wander over its surface and continually describe circles about themselves and the point opposite to them. And this tremendous wandering of the poles will not be corrected, save by placing the mountain either in one of the two poles, in which case (by corol. 21) the nodes of the equator will advance, or on the equator, in which case (by corol. 20) the nodes will regress, or finally by placing on the other side of the axis some additional matter by which the mountain is balanced in its motion, and in this way the nodes will either advance or regress, according as the mountain and this new matter are closer to a pole or to the equator.

Proposition 67
Theorem 27

With the same laws of attraction being supposed, I say that with respect to the common center of gravity O *of the inner bodies* P *and* T, *the outer body* S*—by radii drawn to that center—describes areas more nearly proportional to the times, and an orbit more closely approaching the shape of an ellipse having its focus in that same center, than it can describe about the innermost and greatest body* T *by radii drawn to that body.*

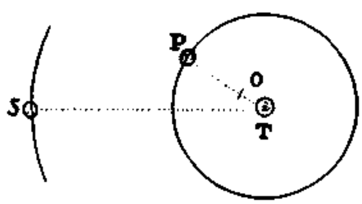


For the attractions of body S toward T and P compose its absolute attraction, which is directed more toward the common center of gravity O of bodies T and P than toward the greatest body T, and which is more nearly inversely proportional to the square of the distance SO than to the square of the distance ST, as will easily be seen by anyone carefully considering the matter.

Proposition 68
Theorem 28

With the same laws of attraction being supposed, I say that with respect to the common center of gravity O *of the inner bodies* P *and* T, *the outer body* S*—by* *radii drawn to that center—describes areas more nearly proportional to the times, and an orbit more closely approaching the shape of an ellipse having its focus in the same center, if the innermost and greatest body is acted on by these attractions just as the others are, than would be the case if it is either not attracted and is at rest or is much more or much less attracted or much more or much less moved.*

This is demonstrated in almost the same way as prop. 66, but the proof is more prolix and I therefore omit it. The following considerations should suffice.



From the demonstration of the last proposition it is apparent that the center toward which body S is urged by both forces combined is very near to the common center of gravity of the other bodies P and T. If this center were to coincide with the common center of those two bodies, and the common center of gravity of all three bodies were to be at rest, body S on the one hand and the common center of the other two bodies on the other would describe exact ellipses about the common center of them all which is at rest. This is clear from the second corollary of prop. 58 compared with what is demonstrated in props. 64 and 65. Such an exact elliptical motion is perturbed somewhat by the distance of the center of the two bodies from the center toward which the third body S is attracted. Let a motion be given, in addition, to the common center of the three, and the perturbation will be increased. Accordingly, the perturbation is least when the common center of the three is at rest, that is, when the innermost and greatest body T is attracted by the very same law as the others; and it always becomes greater when the common center of the three bodies, by a diminution of the motion of body T, begins to be moved and thereupon acted on more and more.

COROLLARY. And hence, if several lesser bodies revolve about a greatest one, it can be found that the orbits described will approach closer to elliptical orbits, and the descriptions of areas will become more uniform, if all the bodies attract and act on one another by accelerative forces that are directly as their absolute forces and inversely as the squares of the distances, and if the focus of each orbit is located in the common center of gravity of all the inner bodies (that is to say, with the focus of the first and innermost orbit in the center of gravity of the greatest and innermost body; the focus of the second orbit in the common center of gravity of the two innermost bodies; the focus of the third in the common center of gravity of the three inner bodies; and so on), than if the innermost body is at rest and is set at the common focus of all the orbits.

Proposition 69
Theorem 29

If, in a system of several bodies A, B, C, D, . . . , *some body* A *attracts all the others,* B, C, D, . . . , *by accelerative forces that are inversely as the squares of the distances from the attracting body; and if another body* B *also attracts the rest of the bodies* A, C, D, . . . , *by forces that are inversely as the squares of the distances from the attracting body; then the absolute forces of the attracting bodies* A *and* B *will be to each other in the same ratio as the bodies [i.e., the masses]* A *and* B *themselves to which those forces belong.*

For, at equal distances, the accelerative attractions of all the bodies B, C, D, . . . toward A are equal to one another by hypothesis; and similarly, at equal distances, the accelerative attractions of all the bodies toward B are equal to one another. Moreover, at equal distances, the absolute attractive force of body A is to the absolute attractive force of body B as the accelerative attraction of all the bodies toward A is to the accelerative attraction of all the bodies toward B at equal distances; and the accelerative attraction of body B toward A is also in the same proportion to the accelerative attraction of body A toward B. But the accelerative attraction of body B toward A is to the accelerative attraction of body A toward B as the mass of body A is to the mass of body B, because the motive forces—which (by defs. 2, 7, and 8) are as the accelerative forces and the attracted bodies jointly—are in this case (by the third law of motion) equal to each other. Therefore the absolute attractive force of body A is to the absolute attractive force of body B as the mass of body A is to the mass of body B. Q.E.D.

COROLLARY 1. Hence if each of the individual bodies of the system A, B, C, D, . . . , considered separately, attracts all the others by accelerative forces that are inversely as the squares of the distances from the attracting body, the absolute forces of all those bodies will be to one another in the ratios of the bodies [i.e., the masses] themselves.

COROLLARY 2. By the same argument, if each of the individual bodies of the system A, B, C, D, . . . , considered separately, attracts all the others by accelerative forces that are either inversely or directly as any powers whatever of the distances from the attracting body, or that are defined in terms of the distances from each one of the attracting bodies according to any law common to all these bodies; then it is evident that the absolute forces of those bodies are as the bodies [i.e., the masses].

COROLLARY 3. If, in a system of bodies whose forces decrease in the squared ratio of the distances [i.e., vary inversely as the squares of the distances], the lesser bodies revolve about the greatest one in ellipses as exact as they can be, having their common focus in the center of that greatest body, and—by radii drawn to the greatest body—describe areas as nearly as possible proportional to the times, then the absolute forces of those bodies will be to one another, either exactly or very nearly, as the bodies, and conversely. This is clear from the corollary of prop. 68 compared with corol. 1 of this proposition.

Scholium By these propositions we are directed to the analogy between centripetal forces and the central bodies toward which those forces tend. For it is reasonable that forces directed toward bodies depend on the nature and the quantity of matter of such bodies, as happens in the case of magnetic bodies. And whenever cases of this sort occur, the attractions of the bodies must be reckoned by assigning proper forces to their individual particles and then taking the sums of these forces.

I use the word "attraction" here in a general sense for any endeavor whatever of bodies to approach one another, whether that endeavor occurs as a result of the action of the bodies either drawn toward one another or acting on one another by means of spirits emitted or whether it arises from the action of aether or of air or of any medium whatsoever—whether corporeal or incorporeal—in any way impelling toward one another the bodies floating therein. I use the word "impulse" in the same general sense, considering in this treatise not the species of forces and their physical qualities but their quantities and mathematical proportions, as I have explained in the definitions.

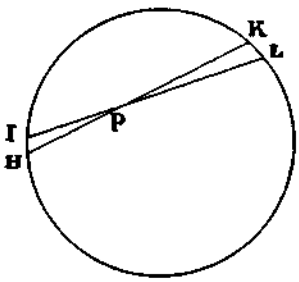
Mathematics requires an investigation of those quantities of forces and their proportions that follow from any conditions that may be supposed. Then, coming down to physics, these proportions must be compared with the phenomena, so that it may be found out which conditions [or laws] of forces apply to each kind of attracting bodies. And then, finally, it will be possible to argue more securely concerning the physical species, physical causes, and physical proportions of these forces. Let us see, therefore, what the forces are by which spherical bodies, consisting of particles that attract in the way already set forth, must act upon one another, and what sorts of motions result from such forces.

SECTION 12

The attractive forces of spherical bodies

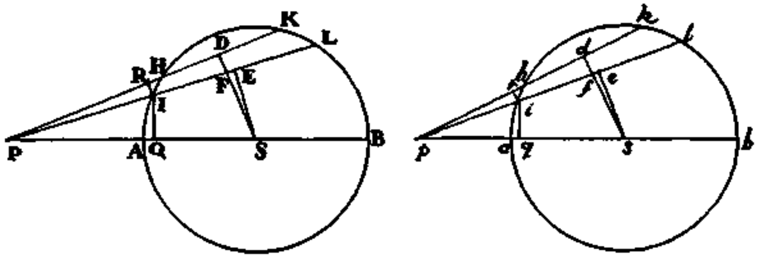
Proposition 70 Theorem 30 *If toward each of the separate points of a spherical surface there tend equal centripetal forces decreasing as the squares of the distances from the point, I say that a corpuscle placed inside the surface will not be attracted by these forces in any direction.*

Let HIKL be the spherical surface, and P the corpuscle placed inside. Through P draw to this surface the two lines HK and IL intercepting minimally small arcs HI and KL; and because triangles HPI and LPK are similar (by lem. 7, corol. 3), those arcs will be proportional to the distances HP and LP; and any particles of the spherical surface at HI and KL, terminated everywhere by straight lines passing through point P, will be in that proportion squared. Therefore the forces exerted on body P by these particles of surface are equal to one another. For they are as the particles directly and the squares of the distances inversely. And these two ratios, when compounded, give the ratio of equality. The attractions, therefore, being made equally in opposite directions, annul each other. And by a similar argument, all the attractions throughout the whole spherical surface are annulled by opposite attractions. Accordingly, body P is not impelled by these attractions in any direction. Q.E.D.



Proposition 71 Theorem 31 *With the same conditions being supposed as in prop. 70, I say that a corpuscle placed outside the spherical surface is attracted to the center of the sphere by a force inversely proportional to the square of its distance from that same center.*

Let AHKB and *ahkb* be two equal spherical surfaces, described about centers S and *s* with diameters AB and *ab*, and let P and *p* be corpuscles located outside those spheres in those diameters produced. From the corpuscles draw lines PHK, PIL, *phk*, and *pil*, so as to cut off from the great circles AHB and *ahb* the equal arcs HK and *hk*, and IL and *il*. And onto these lines drop perpendiculars SD and *sd*, SE and *se*, IR and *ir*, of which SD and *sd* cut PL and *pl* at F and *f*. Also drop perpendiculars IQ and



iq onto the diameters. Let angles DPE and dpe vanish; then, because DS and ds, ES and es are equal, lines PE, PF and pe, pf and the line-elements DF and df may be considered to be equal, inasmuch as their ultimate ratio, when angles DPE and dpe vanish simultaneously, is the ratio of equality.

On the basis of these things, therefore, PI will be to PF as RI to DF, and pf to pi as df or DF to ri, and from the equality of the ratios [or ex aequo] $\mathrm{PI} \times pf$ will be to $\mathrm{PF} \times pi$ as RI to ri, that is (by lem. 7, corol. 3), as the arc IH to the arc ih. Again, PI will be to PS as IQ to SE, and ps will be to pi as se or SE to iq; and from the equality of the ratios [or ex aequo] $\mathrm{PI} \times ps$ will be to $\mathrm{PS} \times pi$ as IQ to iq. And by compounding these ratios, $\mathrm{PI}^2 \times pf \times ps$ will be to $pi^2 \times \mathrm{PF} \times \mathrm{PS}$ as $\mathrm{IH} \times \mathrm{IQ}$ to $ih \times iq$; that is, as the circular surface that the arc IH will describe by the revolution of the semicircle AKB about the diameter AB to the circular surface that the arc ih will describe by the revolution of the semicircle akb about the diameter ab. And the forces by which these surfaces attract the corpuscles P and p (along lines tending to these same surfaces) are (by hypothesis) as these surfaces themselves directly and the squares of the distances of these surfaces from the bodies inversely, that is, as $pf \times ps$ to $\mathrm{PF} \times \mathrm{PS}$.

Now (once the resolution of the forces has been made according to corol. 2 of the laws), these forces are to their oblique parts, which tend along the lines PS and ps toward the centers, as PI to PQ and pi to pq; that is (because the triangles PIQ and PSF, piq and psf are similar), the forces are to their oblique parts as PS to PF and ps to pf. Hence, from the equality of the ratios [or ex aequo] the attraction of this corpuscle P toward S becomes to the attraction of the corpuscle p toward s as $\dfrac{\mathrm{PF} \times pf \times ps}{\mathrm{PS}}$ to $\dfrac{pf \times \mathrm{PF} \times \mathrm{PS}}{ps}$, that is, as ps^2 to PS^2. And by a similar argument, the forces by which the surface described by the revolution of the arcs KL and kl attract the corpuscles will be as ps^2 to PS^2. And the same ratio will hold for the forces of all the spherical surfaces into which each of the two spherical surfaces can be divided by taking *sd* always equal to SD and *se* equal to SE. And by composition [or componendo] the forces of the total spherical surfaces exercised upon the corpuscles will be in the same ratio. Q.E.D.

Proposition 72
Theorem 32

If toward each of the separate points of any sphere there tend equal centripetal forces, decreasing in the squared ratio of the distances from those points, and there are given both the density of the sphere and the ratio of the diameter of the sphere to the distance of the corpuscle from the center of the sphere, I say that the force by which the corpuscle is attracted will be proportional to the semidiameter of the sphere.

For imagine that two corpuscles are attracted separately by two spheres, one corpuscle by one sphere, and the other corpuscle by the other sphere, and that their distances from the centers of the spheres are respectively proportional to the diameters of the spheres, and that the two spheres are resolved into particles that are similar and similarly placed with respect to the corpuscles. Then the attractions of the first corpuscle, made toward each of the separate particles of the first sphere, will be to the attractions of the second toward as many analogous particles of the second sphere in a ratio compounded of the direct ratio of the particles and the inverse squared ratio of the distances [i.e., the attractions will be to one another as the particles directly and the squares of the distances inversely]. But the particles are as the spheres, that is, they are in the cubed ratio of the diameters, and the distances are as the diameters; and thus the first of these ratios directly combined with the second ratio taken twice inversely becomes the ratio of diameter to diameter. Q.E.D.

Corollary 1. Hence, if corpuscles revolve in circles about spheres consisting of equally attractive matter, and their distances from the centers of the spheres are proportional to the diameters of the spheres, the periodic times will be equal.

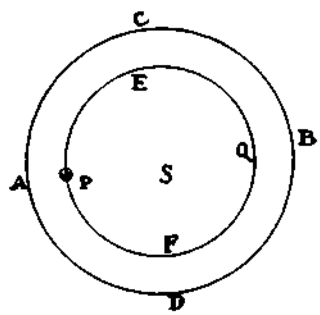
Corollary 2. And conversely, if the periodic times are equal, the distances will be proportional to the diameters. These two corollaries are evident from prop. 4, corol. 3.

Corollary 3. If toward each of the separate points of any two similar and equally dense solids there tend equal centripetal forces decreasing in the squared ratio of the distances from those points, the forces by which corpuscles will be attracted by those two solids, if they are similarly situated with regard to them, will be to each other as the diameters of the solids.

Proposition 73
Theorem 33

If toward each of the separate points of any given sphere there tend equal centripetal forces decreasing in the squared ratio of the distances from those points, I say that a corpuscle placed inside the sphere is attracted by a force proportional to the distance of the corpuscle from the center of the sphere.

Let a corpuscle P be placed inside the sphere ABCD, described about center S; and about the same center S with radius SP, suppose that an inner sphere PEQF is described. It is manifest (by prop. 70) that the concentric spherical surfaces of which the difference AEBF of the spheres is composed do not act at all upon body P, their attractions having been annulled by opposite attractions. There remains only the attraction of the inner sphere PEQF. And (by prop. 72) this is as the distance PS. Q.E.D.



Scholium

The surfaces of which the solids are composed are here not purely mathematical, but orbs [or spherical shells] so extremely thin that their thickness is as null: namely, evanescent orbs of which the sphere ultimately consists when the number of those orbs is increased and their thickness diminished indefinitely. Similarly, when lines, surfaces, and solids are said to be composed of points, such points are to be understood as equal particles of a magnitude so small that it can be ignored.

Proposition 74
Theorem 34

With the same things being supposed as in prop. 73, I say that a corpuscle placed outside a sphere is attracted by a force inversely proportional to the square of the distance of the corpuscle from the center of the sphere.

For let the sphere be divided into innumerable concentric spherical surfaces; then the attractions of the corpuscle that arise from each of the surfaces will be inversely proportional to the square of the distance of the corpuscle from the center (by prop. 71). And by composition [or componendo] the sum of the attractions (that is, the attraction of the corpuscle toward the total sphere) will come out in the same ratio. Q.E.D.

Corollary 1. Hence at equal distances from the centers of homogeneous spheres the attractions are as the spheres themselves. For (by prop. 72) if the distances are proportional to the diameters of the spheres, the forces will be as the diameters. Let the greater distance be diminished in that ratio; and, the distances having now become equal, the attraction will be increased in that ratio squared, and thus will be to the other attraction in that ratio cubed, that is, in the ratio of the spheres.

Corollary 2. At any distances the attractions are as the spheres divided by the squares of the distances.

Corollary 3. If a corpuscle placed outside a homogeneous sphere is attracted by a force inversely proportional to the square of the distance of the corpuscle from the center of the sphere, and the sphere consists of attracting particles, the force of each particle will decrease in the squared ratio of the distance from the particle.

Proposition 75
Theorem 35

If toward each of the points of a given sphere there tend equal centripetal forces decreasing in the squared ratio of the distances from the points, I say that this sphere will attract any other homogeneous sphere with a force inversely proportional to the square of the distance between the centers.[a]

For the attraction of any particle is inversely as the square of its distance from the center of the attracting sphere (by prop. 74), and therefore is the same as if the total attracting force emanated from one single corpuscle situated in the center of this sphere. Moreover, this attraction is as great as the attraction of the same corpuscle would be if, in turn, it were attracted by each of the individual particles of the attracted sphere with the same force by which it attracts them. And that attraction of the corpuscle (by prop. 74) would be inversely proportional to the square of its distance from the center of the sphere; and therefore the sphere's attraction, which is equal to the attraction of the corpuscle, is in the same ratio. Q.E.D.

Corollary 1. The attractions of spheres toward other homogeneous spheres are as the attracting spheres [i.e., as the masses of the attracting spheres] divided by the squares of the distances of their own centers from the centers of those that they attract.

a. Newton writes of a "sphaera quaevis alia similaris," literally, "any other like [or similar] sphere," but the context (see prop. 74, corols. 1 and 3) is that of a homogeneous sphere.

Corollary 2. The same is true when the attracted sphere also attracts. For its individual points will attract the individual points of the other with the same force by which they are in turn attracted by them; and thus, since in every attraction the attracting point is as much urged (by law 3) as the attracted point, the force of the mutual attraction will be duplicated, the proportions remaining the same.

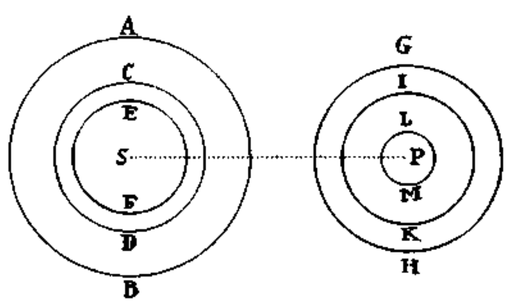
Corollary 3. Everything that has been demonstrated above concerning the motion of bodies about the focus of conics is valid when an attracting sphere is placed in the focus and the bodies move outside the sphere.

Corollary 4. And whatever concerns the motion of bodies around the center of conics applies when the motions are performed inside the sphere.

Proposition 76
Theorem 36

If spheres are in any way nonhomogeneous (as to the density of their matter and their attractive force) going from the center to the circumference, but are uniform throughout in every spherical shell at any given distance from the center, and the attractive force of each point decreases in the squared ratio of the distance of the attracted body, I say that the total force by which one sphere of this sort attracts another is inversely proportional to the square of the distance between their centers.

[a]Let there be any number of concentric homogeneous spheres [i.e., hollow spheres, or spherical shells or surfaces] AB, CD, EF, ...; and suppose that the addition of one or more inner ones to the outer one or ones forms a sphere composed of matter more dense, or the taking away leaves it less



dense, toward the center than at the circumference. Then these spheres will together (by prop. 75) attract any number of other concentric homogeneous spheres GH, IK, LM, ..., each sphere of one set attracting every one of

aa. The text of this proof has been translated somewhat freely, and in part expanded, for greater ease in comprehension.

the other set with forces inversely proportional to the square of the distance SP. And by adding up these forces (or by the reverse process when spheres are taken away) the sum of all those forces (or the excess of any one—or of some—of them above the others); that is, the force with which the whole sphere AB, composed of any concentric spheres (or the difference between some concentric spheres and others which have been taken away), attracts the whole sphere GH, composed of any concentric spheres (or the differences between some such concentric spheres and others)—will be in the same inverse ratio of the square of the distance SP. Let the number of concentric spheres be increased indefinitely, in such a way that the density of the matter, together with the force of attraction, may—on going from the circumference to the center—increase or decrease according to any law whatever; and by the addition of non-attracting matter, let the deficiencies in density be supplied wherever needed so that the spheres may acquire any desired form; then the force with which one of these spheres attracts the other will still be, by the former argument, in the same inverse ratio of the square of the distance.[a] Q.E.D.

Corollary 1. Hence, if many spheres of this sort, similar to one another in all respects, attract one another, the accelerative attraction of any one to any other of them, at any equal distances between the centers, will be as the attracting spheres.

Corollary 2. And at any unequal distances, as the attracting sphere divided by the square of the distances between the centers.

Corollary 3. And the motive attractions, or the weights of spheres toward other spheres, will—at equal distances from the centers—be as the attracting and the attracted spheres jointly, that is, as the products produced by multiplying the spheres by each other.

Corollary 4. And at unequal distances, as those products directly and the squares of the distances between the centers inversely.

Corollary 5. These results are valid when the attraction arises from each sphere's force of attraction being mutually exerted upon the other sphere. For the attraction is duplicated by both forces acting, the proportion remaining the same.

Corollary 6. If some spheres of this sort revolve about others at rest, one sphere revolving about each sphere at rest, and the distances between the centers of the revolving spheres and those at rest are proportional to the diameters of those at rest, the periodic times will be equal.

COROLLARY 7. And conversely, if the periodic times are equal, the distances will be proportional to those diameters.

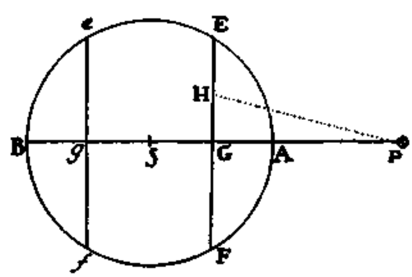
COROLLARY 8. Everything that has been demonstrated above about the motion of bodies around the foci of conics holds when the attracting sphere, of any form and condition that has already been described, is placed in the focus.

COROLLARY 9. As also when the bodies revolving in orbit are also attracting spheres of any condition that has already been described.

Proposition 77
Theorem 37

If toward each of the individual points of spheres there tend centripetal forces proportional to the distances of the points from attracted bodies, I say that the composite force by which two spheres will attract each other is as the distance between the centers of the spheres.

CASE 1. Let AEBF be a sphere, S its center, P an attracted exterior corpuscle, PASB that axis of the sphere which passes through the center of the corpuscle, EF and *ef* two planes by which the sphere is cut and which are perpendicular to this axis and equally distant on both sides from the center of the sphere, G and *g* the intersections of the planes and the axis, and H any point in the plane EF. The centripetal force of point H upon the corpuscle P,



exerted along the line PH, is as the distance PH; and (by corol. 2 of the laws) along the line PG, or toward the center S, as the length PG. Therefore the force of all the points in the plane EF (that is, of the total plane) by which the corpuscle P is attracted toward the center S is as the distance PG multiplied by the number of such points, that is, as the solid contained by that plane EF itself and the distance PG [i.e., as the product of the plane EF and the distance PG]. And similarly the force of the plane *ef*, by which the corpuscle P is attracted toward the center S, is as that plane multiplied by its distance P*g*, or as the plane EF equal thereto multiplied by that distance P*g*; and the sum of the forces of both planes is as the plane EF multiplied by the sum of the distances PG + P*g*; that is, as that plane multiplied by twice the distance

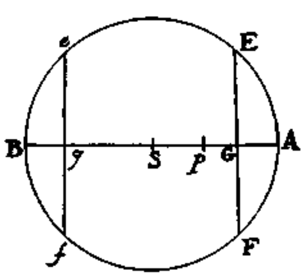
PS between the center S and the corpuscle P; that is, as twice the plane EF multiplied by the distance PS, or as the sum of the equal planes EF + *ef* multiplied by that same distance. And by a similar argument, the forces of all the planes in the whole sphere, equally distant on both sides from the center of the sphere, are as the sum of those planes multiplied by the distance PS, that is, as the whole sphere and the distance PS jointly. Q.E.D.

CASE 2. Now let the corpuscle P attract the sphere AEBF. Then by the same argument it can be proved that the force by which that sphere is attracted will be as the distance PS. Q.E.D.

CASE 3. Now let a second sphere be composed of innumerable corpuscles P; then, since the force by which any one corpuscle is attracted is as the distance of the corpuscle from the center of the first sphere and as that same sphere jointly, and thus is the same as if all the force came from one single corpuscle in the center of the sphere, the total force by which all the corpuscles in the second sphere are attracted (that is, by which that whole sphere is attracted) will be the same as if that sphere were attracted by a force coming from one single corpuscle in the center of the first sphere, and therefore is proportional to the distance between the centers of the spheres. Q.E.D.

CASE 4. Let the spheres attract each other mutually; then the force, now duplicated, will keep the former proportion. Q.E.D.

CASE 5. Now let a corpuscle *p* be placed inside the sphere AEBF. Then, since the force of the plane *ef* upon the corpuscle is as the solid contained by [or the product of] that plane and the distance *pg*; and the opposite force of the plane EF is as the solid contained by [or the product of] that plane and the distance *p*G; the force compounded of the two will be as the difference of the solids [or the products], that is, as the sum of the equal planes multiplied by half of the difference of the distances, that is, as that sum multiplied by *p*S, the distance of the corpuscle from the center of the sphere. And by a similar argument, the attraction of all the planes EF and *ef* in the whole sphere (that is, the attraction of the whole sphere) is jointly as the sum of all the planes (or the whole sphere) and as *p*S, the distance of the corpuscle from the center of the sphere. Q.E.D.



CASE 6. And if from innumerable corpuscles p a new sphere is composed, placed inside the former sphere AEBF, then it can be proved as above that the attraction, whether the simple attraction of one sphere toward the other, or a mutual attraction of both toward each other, will be as the distance pS between the centers. Q.E.D.

Proposition 78
Theorem 38

If spheres, on going from the center to the circumference, are in any way nonhomogeneous and nonuniform, but in every concentric spherical shell at any given distance from the center are homogeneous throughout; and the attracting force of each point is as the distance of the body attracted; then I say that the total force by which two spheres of this sort attract each other is proportional to the distance between the centers of the spheres.

This is demonstrated from prop. 77 in the same way that prop. 76 was demonstrated from prop. 75.

COROLLARY. Whatever was demonstrated above in props. 10 and 64 on the motion of bodies about the centers of conics is valid when all the attractions take place by the force of spherical bodies of the condition already described, and when the attracted bodies are spheres of the same condition.

Scholium

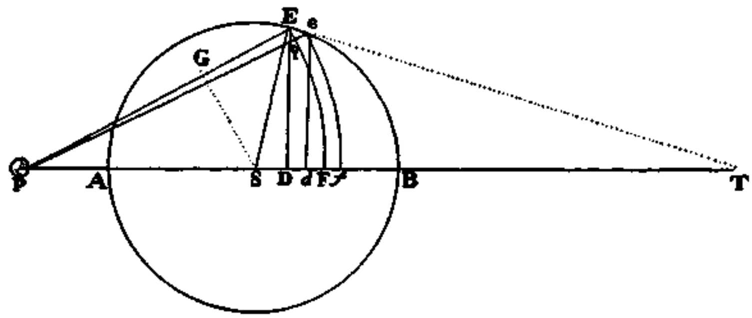
I have now given explanations of the two major cases of attractions, namely, when the centripetal forces decrease in the squared ratio of the distances or increase in the simple ratio of the distances, causing bodies in both cases to revolve in conics, and composing centripetal forces of spherical bodies that decrease or increase in proportion to the distance from the center according to the same law—which is worthy of note. It would be tedious to go one by one through the other cases which lead to less elegant conclusions. I prefer to comprehend and determine all the cases simultaneously under a general method as follows.

Lemma 29[a]

If any circle AEB *is described with center* S*; and then two circles* EF *and* ef *are described with center* P*, cutting the first circle in* E *and* e*, and cutting the line* PS *in* F *and* f*; and if the perpendiculars* ED *and* ed *are dropped to* PS*; then I say that if the distance between the arcs* EF *and* ef *is supposed to be diminished*

a. For a gloss on this lemma see the Guide, §10.13.

indefinitely, the ultimate ratio of the evanescent line Dd *to the evanescent line* Ff *is the same as that of line* PE *to line* PS.

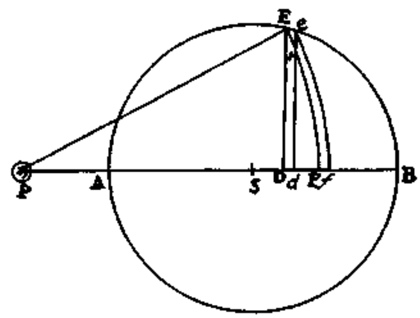


For if line Pe cuts arc EF in q, and the straight line Ee, which coincides with the evanescent arc Ee, when produced meets the straight line PS in T, and the normal SG is dropped from S to PE; then because the triangles DTE, dTe, and DES are similar, Dd will be to Ee as DT to TE, or DE to ES; and because the triangles Eeq and ESG (by sec. 1, lem. 8 and lem. 7, corol. 3) are similar, Ee will be to eq or Ff as ES to SG; and from the equality of the ratios [or ex aequo] Dd will be to Ff as DE to SG—that is (because the triangles PDE and PGS are similar), as PE to PS. Q.E.D.

Proposition 79
Theorem 39

If the surface EFfe, *just now vanishing because its width has been indefinitely diminished, describes by its revolution about the axis* PS *a concavo-convex spherical solid, toward each of whose individual equal particles there tend equal centripetal forces; then I say that the force by which that solid attracts an exterior corpuscle located in* P *is in a ratio compounded of the ratio of the solid [or product]* $DE^2 \times Ff$ *and the ratio of the force by which a given particle at the place* Ff *would attract the same corpuscle.*

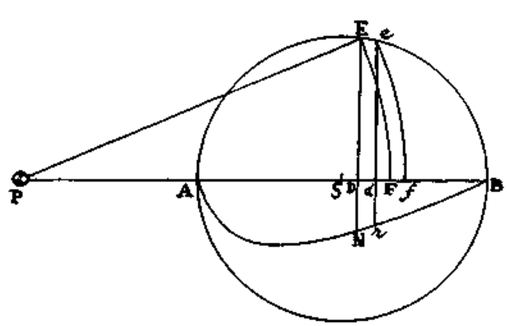
For if we first consider the force of the spherical surface FE, which is generated by the revolution of the arc FE and is cut anywhere by the line de in r, the annular part of this surface generated by the revolution of the arc rE will be as the line-element Dd, the radius PE of the sphere remaining the same (as Archimedes demonstrated in his book on the Sphere and Cylinder). And the force of that surface, exerted along lines PE or Pr, placed everywhere in the surface of a cone, will be as this annular part of the surface—that is, as the line-element Dd or, what comes to the same thing, as the rectangle of the given radius PE of the sphere and that line-element Dd;

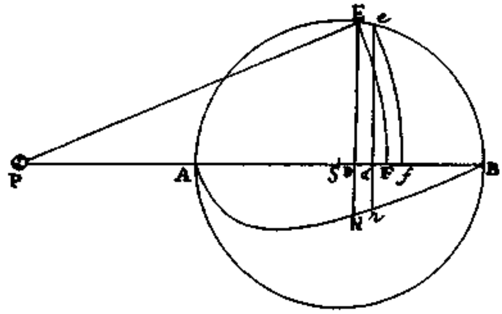


but along the line PS tending toward the center S, this force will be smaller in the ratio of PD to PE, and hence this force will be as PD × D*d*. Now suppose the line DF to be divided into innumerable equal particles, and let each of them be called D*d*; then the surface FE will be divided into the same number of equal rings, whose total forces will be as the sum of all the products PD × D*d*, that is, as $\frac{1}{2}PF^2 - \frac{1}{2}PD^2$, and thus as DE^2. Now multiply the surface FE by the altitude F*f*, and the force of the solid EF*fe* exerted upon the corpuscle P will become as $DE^2 \times Ff$, if there is given the force that some given particle F*f* exerts on the corpuscle P at the distance PF. But if that force is not given, the force of the solid EF*fe* will become as the solid $DE^2 \times Ff$ and that non-given force jointly. Q.E.D.

Proposition 80
Theorem 40

If equal centripetal forces tend toward each of the individual equal particles of some sphere ABE, *described about a center* S; *and if from each of the individual points* D *to the axis* AB *of the sphere, in which some corpuscle* P *is located, there are erected the perpendiculars* DE, *meeting the sphere in the points* E; *and if, on these perpendiculars, the lengths* DN *are taken, which are jointly as the quantity* $\dfrac{DE^2 \times PS}{PE}$ *and as the force that a particle of the sphere, located on the axis, exerts at the distance* PE *upon the corpuscle* P; *then I say that the total force with which the corpuscle* P *is attracted toward the sphere is as the area* ANB *comprehended by the axis* AB *of the sphere and the curved line* ANB, *which the point* N *traces out.*





For, keeping the same constructions as in lem. 29 and prop. 79, suppose the axis AB of the sphere to be divided into innumerable equal particles D*d*, and the whole sphere to be divided into as many spherical concavo-convex laminae EF*fe*; and erect the perpendicular *dn*. By prop. 79, the force with which the lamina EF*fe* attracts the corpuscle P is jointly as $DE^2 \times Ff$ and the force of one particle exerted at the distance PE or PF. But D*d* is to F*f* (by lem. 29) as PE to PS, and hence F*f* is equal to $\frac{PS \times Dd}{PE}$, and $DE^2 \times Ff$ is equal to $Dd\frac{DE^2 \times PS}{PE}$; and therefore the force of the lamina EF*fe* is jointly as $Dd\frac{DE^2 \times PS}{PE}$ and the force of a particle exerted at the distance PF; that is (by hypothesis) as $DN \times Dd$, or as the evanescent area DN*nd*. Therefore the forces upon body P exerted by all the laminae are as all the areas DN*nd*, that is, the total force of the sphere is as the total area ANB. Q.E.D.

Corollary 1. Hence, if the centripetal force tending toward each of the individual particles always remains the same at all distances, and DN is taken proportional to $\frac{DE^2 \times PS}{PE}$, the total force by which the corpuscle P is attracted by the sphere will be as the area ANB.

Corollary 2. If the centripetal force of the particles is inversely as the distance of the attracted corpuscle, and DN is taken proportional to $\frac{DE^2 \times PS}{PE^2}$, the force by which the corpuscle P is attracted by the whole sphere will be as the area ANB.

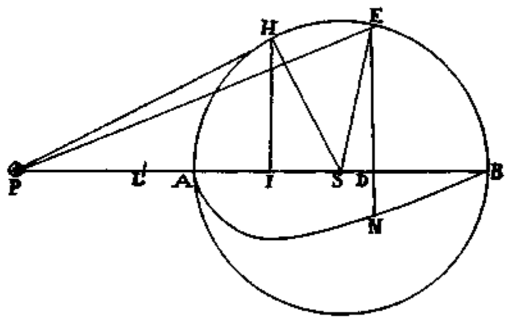
Corollary 3. If the centripetal force of the particles is inversely as the cube of the distance of the attracted corpuscle, and DN is taken proportional to $\frac{DE^2 \times PS}{PE^4}$, the force by which the corpuscle is attracted by the whole sphere will be as the area ANB.

COROLLARY 4. And universally, if the centripetal force tending toward each of the individual particles of a sphere is supposed to be inversely as the quantity V, and DN is taken proportional to $\frac{DE^2 \times PS}{PE \times V}$, the force by which a corpuscle is attracted by the whole sphere will be as the area ANB.

Proposition 81
Problem 41

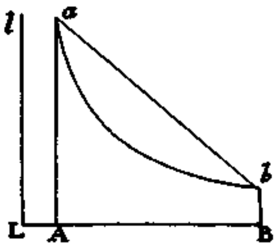
Under the same conditions as before, it is required to measure the area ANB.

From point P draw the straight line PH touching the sphere in H; and, having dropped the normal HI to the axis PAB, bisect PI in L; then (by book 2, prop. 12, of Euclid's *Elements*) PE^2 will be equal to $PS^2+SE^2+2(PS \times SD)$.



Moreover, SE^2 or SH^2 (because the triangles SPH and SHI are similar) is equal to the rectangle $PS \times SI$. Therefore PE^2 is equal to the rectangle of PS and $PS + SI + 2SD$, that is, of PS and $2LS + 2SD$, that is, of PS and 2LD. Further, DE^2 is equal to $SE^2 - SD^2$, or $SE^2 - LS^2 + 2(SL \times LD) - LD^2$, that is, $2(SL \times LD) - LD^2 - AL \times LB$. For $LS^2 - SE^2$ or $LS^2 - SA^2$ (by book 2, prop. 6, of the *Elements*) is equal to the rectangle $AL \times LB$. Write, therefore, $2(SL \times LD) - LD^2 - AL \times LB$ for DE^2; and the quantity $\frac{DE^2 \times PS}{PE \times V}$, which (according to corol. 4 of the preceding prop. 80) is as the length of the ordinate DN, will resolve itself into the three parts $\frac{2(SL \times LD \times PS)}{PE \times V} - \frac{LD^2 \times PS}{PE \times V} - \frac{AL \times LB \times PS}{PE \times V}$: where, if for V we write the inverse ratio of the centripetal force, and for PE the mean proportional between PS and 2LD, those three parts will become ordinates of as many curved lines, whose areas can be found by ordinary methods. Q.E.F.

EXAMPLE 1. If the centripetal force tending toward each of the individual particles of the sphere is inversely as the distance, write the distance PE in place of V, and then $2PS \times LD$ in place of PE^2, and DN will become as $SL - \frac{1}{2}LD - \frac{AL \times LB}{2LD}$. Suppose DN equal to its double $2SL - LD - \frac{AL \times LB}{LD}$; and the given part 2SL of that ordinate multiplied by the length AB will describe a rectangular area $2SL \times AB$, and the indefinite part LD multiplied perpendicularly by the same length AB in a continual motion (according to the rule that, while moving, either by increasing or decreasing, it is always equal to the length LD) will describe an area $\frac{LB^2 - LA^2}{2}$, that is, the area $SL \times AB$, which, subtracted from the first area $2SL \times AB$, leaves the area $SL \times AB$. Now the third part $\frac{AL \times LB}{LD}$, likewise multiplied perpendicularly by the same length AB in a local [i.e., continual] motion, will describe a hyperbolic area, which subtracted from the area $SL \times AB$ will leave the required area ANB. Hence, there arises the following construction of the problem.

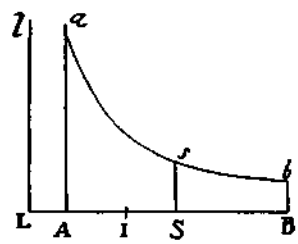


At points L, A, and B erect perpendiculars L*l*, A*a*, and B*b*, of which A*a* is equal to LB, and B*b* to LA. With asymptotes L*l* and LB, through points *a* and *b* describe the hyperbola *ab*. Then the chord *ba*, when drawn, will enclose the area *aba* equal to the required area ANB.

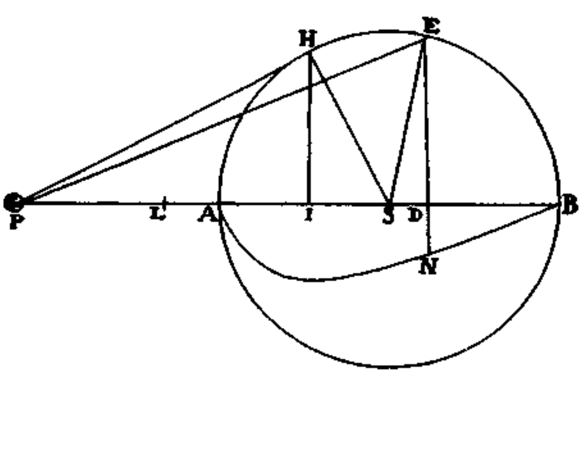
Example 2. If the centripetal force tending toward each of the individual particles of the sphere is inversely as the cube of the distance, or (which comes to the same thing) as that cube divided by any given plane, write $\frac{PE^3}{2AS^2}$ for V, and then $2PS \times LD$ for PE^2, and DN will become as $\frac{SL \times AS^2}{PS \times LD} - \frac{AS^2}{2PS} - \frac{AL \times LB \times AS^2}{2PS \times LD^2}$, that is (because PS, AS, and SI are continually proportional [or PS is to AS as AS to SI]), as $\frac{LS \times SI}{LD} - \frac{1}{2}SI - \frac{AL \times LB \times SI}{2LD^2}$. If the three parts of this quantity are multiplied by the length AB, the first, $\frac{LS \times SI}{LD}$, will generate a hyperbolic area; the second, $\frac{1}{2}SI$, will generate the area $\frac{1}{2}AB \times SI$; the third, $\frac{AL \times LB \times SI}{2LD^2}$, will generate the area $\frac{AL \times LB \times SI}{2LA} - \frac{AL \times LB \times SI}{2LB}$, that is, $\frac{1}{2}AB \times SI$.

From the first subtract the sum of the second and third, and the required area ANB will remain.

Hence there arises the following construction of the problem. At the points L, A, S, and B erect the perpendiculars L*l*, A*a*, S*s*, and B*b*, of which S*s* is equal to SI; and through the point *s*, with asymptotes L*l* and LB, describe the hyperbola *asb* meeting the perpendiculars A*a* and B*b* in *a* and *b*; then the rectangle $2\text{AS} \times \text{SI}$ subtracted from the hyperbolic area A*asb*B will leave the required area ANB.



EXAMPLE 3. If the centripetal force tending toward each of the individual particles of the sphere decreases as the fourth power of the distance from those particles, write $\frac{\text{PE}^4}{2\text{AS}^3}$ for V, and then $\sqrt{(2\text{PS} \times \text{LD})}$ for PE, and DN will become as $\frac{\text{SI}^2 \times \text{SL}}{\sqrt{(2\text{SI})}} \times \frac{1}{\sqrt{\text{LD}^3}} - \frac{\text{SI}^2}{2\sqrt{(2\text{SI})}} \times \frac{1}{\sqrt{\text{LD}}} - \frac{\text{SI}^2 \times \text{AL} \times \text{LB}}{2\sqrt{(2\text{SI})}} \times \frac{1}{\sqrt{\text{LD}^5}}$. Those three parts, multiplied by the length AB, produce three areas, namely $\frac{2\text{SI}^2 \times \text{SL}}{\sqrt{(2\text{SI})}}$ multiplied by $\left(\frac{1}{\sqrt{\text{LA}}} - \frac{1}{\sqrt{\text{LB}}}\right)$; $\frac{\text{SI}^2}{\sqrt{(2\text{SI})}}$ multiplied by $(\sqrt{\text{LB}} - \sqrt{\text{LA}})$; and $\frac{\text{SI}^2 \times \text{AL} \times \text{LB}}{3\sqrt{(2\text{SI})}}$ multiplied by $\left(\frac{1}{\sqrt{\text{LA}^3}} - \frac{1}{\sqrt{\text{LB}^3}}\right)$. And these, after the due reduction, become $\frac{2\text{SI}^2 \times \text{SL}}{\text{LI}}$, SI^2, and $\frac{\text{SI}^2 + 2\text{SI}^3}{3\text{LI}}$. And when the latter two quantities are subtracted from the first one, the result comes out to be $\frac{4\text{SI}^3}{3\text{LI}}$. Accordingly, the total force by which the cor-



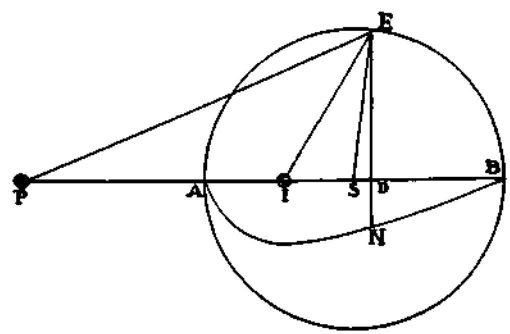
puscle P is attracted to the center of the sphere is as $\frac{SI^3}{PI}$, that is, inversely as $PS^3 \times PI$. Q.E.I.

The attraction of a corpuscle located inside a sphere can be determined by the same method, but more expeditiously by means of the following proposition.

Proposition 82
Theorem 41

If—in a sphere described about center S *with radius* SA—SI, SA, *and* SP *are taken continually proportional [i.e.,* SI *to* SA *as* SA *to* SP*], I say that the attraction of a corpuscle inside the sphere at any place* I *is to its attraction outside the sphere at place* P *in a ratio compounded of the square root of the ratio of the distances* IS *and* PS *from the center, and the square root of the ratio of the centripetal forces, tending at those places* P *and* I *toward the center.*

If, for example, the centripetal forces of the particles of the sphere are inversely as the distances of the corpuscle attracted by them, the force by which the corpuscle situated at I is attracted by the total sphere will be to the force by which it is attracted at P in a ratio compounded of the square



root of the ratio of the distance SI to the distance SP and the square root of the ratio of the centripetal force at place I arising from some particle in the center to the centripetal force at place P arising from the same particle in the center, which is the square root of the ratio of the distances SI and SP to each other inversely. Compounding these two square roots of ratios gives the ratio of equality, and therefore the attractions produced at I and P by the whole sphere are equal. By a similar computation, if the forces of the particles of the sphere are inversely in the squared ratio of the distances, it will be seen that the attraction at I is to the attraction at P as the distance

SP is to the semidiameter SA of the sphere. If those forces are inversely in the cubed ratio of the distances, the attractions at I and P will be to each other as SP^2 to SA^2; if as the inverse fourth power, as SP^3 to SA^3. Hence, since—in this last case [of the inverse fourth power, as in the final ex. 3 of prop. 81]—the attraction at P was found to be inversely as $PS^3 \times PI$, the attraction at I will be inversely as $SA^3 \times PI$, that is (because SA^3 is given), inversely as PI. And the progression goes on in the same way indefinitely. Moreover, the theorem is demonstrated as follows.

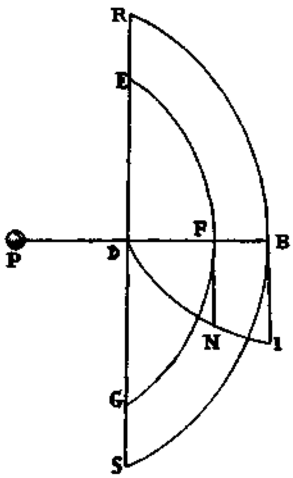
With the same construction and with the corpuscle being in any place P, the ordinate DN was found to be as $\frac{DE^2 \times PS}{PE \times V}$. Therefore, if IE is drawn, that ordinate for any other place I of the corpuscle will—*mutatis mutandis* [i.e., by substituting I for P in the considerations and arguments that have previously been applied to P]—come out as $\frac{DE^2 \times IS}{IE \times V}$. Suppose the centripetal forces emanating from any point E of the sphere to be to each other at the distances IE and PE as PE^n to IE^n (where let the number n designate the index of the powers of PE and IE); then those ordinates will become as $\frac{DE^2 \times PS}{PE \times PE^n}$ and $\frac{DE^2 \times IS}{IE \times IE^n}$, whose ratio to each other is as $PS \times IE \times IE^n$ to $IS \times PE \times PE^n$. Because SI, SE, and SP are continually proportional, the triangles SPE and SEI are similar, and hence IE becomes to PE as IS to SE or SA; for the ratio of IE to PE, write the ratio of IS to SA, and the ratio of the ordinates will come out $PS \times IE^n$ to $SA \times PE^n$. But PS to SA is the square root of the ratio of the distances PS and SI, and IE^n to PE^n (because IE is to PE as IS to SA) is the square root of the ratio of the forces at the distances PS and IS. Therefore the ordinates, and consequently the areas that the ordinates describe and the attractions proportional to them, are in a ratio compounded of the foregoing square-root ratios. Q.E.D.

Proposition 83
Problem 42

To find the force by which a corpuscle located in the center of a sphere is attracted toward any segment of it whatever.

Let P be the corpuscle in the center of the sphere, and RBSD a segment of the sphere contained by the plane RDS and the spherical surface RBS.

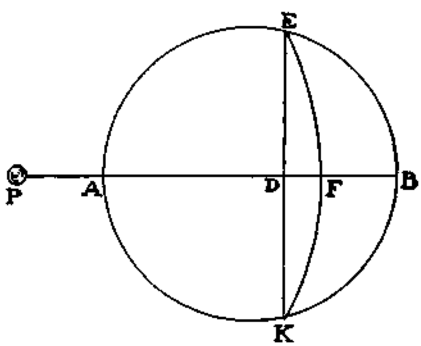
Let DB be cut at F by the spherical surface EFG described about the center P, and divide that segment into the parts BREFGS and FEDG. But let that surface be taken to be not purely mathematical, but physical, having a minimally small thickness. Call that thickness O, and this surface (by what Archimedes has demonstrated), will be as $PF \times DF \times O$. Let us suppose, additionally, the attractive forces of the particles of the sphere to be inversely as that power of the distances whose index is n; then the force by which the surface EFG attracts the body P will be (by prop. 79) as $\frac{DE^2 \times O}{PF^n}$, that is, as $\frac{2DF \times O}{PF^{n-1}} - \frac{DF^2 \times O}{PF^n}$.



Let the perpendicular FN drawn in [the thickness] O be proportional to this quantity; then the curvilinear area BDI, as described by the ordinate FN, drawn in a continual motion, applied to the length DB, will be as the whole force by which the whole segment RBSD attracts the corpuscle P. Q.E.I.

Proposition 84
Problem 43

To find the force with which a corpuscle is attracted by a segment of a sphere when it is located on the axis of the segment beyond the center of the sphere.



Let corpuscle P, located on the axis ADB of the segment EBK, be attracted by that segment. About center P and with radius PE describe the spherical surface EFK, which divides the segment into two parts EBKFE and EFKDE. Find the force of the first part by prop. 81 and the force of the second part by prop. 83, and the sum of these two forces will be the force of the whole segment EBKDE. Q.E.I.

Scholium Now that the attractions of spherical bodies have been explained, it would be possible to go on to the laws of the attractions of certain other bodies similarly consisting of attracting particles, but to treat these in particular cases is not essential to my design. It will be enough to subjoin certain more general propositions concerning the forces of bodies of this sort and the motions that arise from such forces, because these propositions are of some use in philosophical questions [i.e., questions of natural philosophy, or physical science].

SECTION 13

The attractive forces of nonspherical bodies

Proposition 85
Theorem 42

If the attraction of an attracted body is far stronger when it is contiguous to the attracting body than when the bodies are separated from each other by even a very small distance, then the forces of the particles of the attracting body decrease, as the attracted body recedes, in a more than squared ratio of the distances from the particles.

For if the forces decrease in the squared ratio of the distances from the particles, the attraction toward a spherical body will not be sensibly increased by contact, because (by prop. 74) it is inversely as the square of the distance of the attracted body from the center of the sphere; and still less will it be increased by contact, if the attraction decreases in a smaller ratio as the attracted body recedes. Therefore, this proposition is evident in the case of attracting spheres. It is the same for concave spherical orbs[a] attracting external bodies. And it is much more established in the case of orbs attracting bodies placed inside of them, since the attractions spreading out through the concavities of the orbs are annulled by opposite attractions (by prop. 70), and therefore the attracting forces are null, even in contact. But if any parts remote from the place of contact are taken away from these spheres and spherical orbs, and new parts are added anywhere away from the place of contact, the shapes of these attracting bodies can be changed at will; and yet the parts added or subtracted will not notably increase the excess of attraction that arises from contact, since they are remote from the place of contact. Therefore the proposition is established concerning bodies of all shapes. Q.E.D.

Proposition 86
Theorem 43

If the forces of the particles composing an attracting body decrease, as an attracted body recedes, in the cubed or more than cubed ratio of the distances from the particles, the attraction will be far stronger in contact than when the attracting body and attracted body are separated from each other by even a very small distance.

a. Here, as elsewhere in the *Principia*, Newton uses the word "orb" for what we would more precisely call a spherical shell.

For by the solution of prop. 81 given in exx. 2 and 3, it is established that the attraction is increased indefinitely in the approach of an attracted corpuscle to an attracting sphere of this sort. By the combination of those examples and prop. 82, the same result is easily inferred concerning the attractions of bodies toward concavo-convex orbs whether the attracted bodies are placed outside those orbs or in the cavities inside the orbs. But the proposition will also be established concerning all bodies universally by adding some attractive matter to these spheres and orbs, or taking some away from them, anywhere away from the place of contact, so that the attracting bodies take on any desired shape. Q.E.D.

Proposition 87
Theorem 44

If two bodies, similar to each other and consisting of equally attracting matter, separately attract corpuscles proportional to those bodies and similarly placed with respect to them, then the accelerative attractions of the corpuscles toward the whole bodies will be as the accelerative attractions of those corpuscles toward particles of those bodies proportional to the wholes and similarly situated in those whole bodies.

For if the bodies are divided into particles that are proportional to the whole bodies and similarly placed in those whole bodies, then the attraction toward an individual particle of the first body will be to the attraction toward the corresponding individual particle of the second body as the attractions toward any given particles of the first body are to the attractions toward the corresponding particles of the second body, and by compounding, the attraction toward the whole first body will be to the attraction toward the whole second body in that same ratio. Q.E.D.

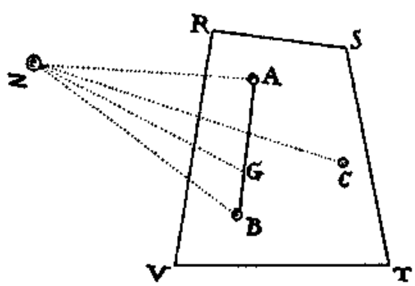
COROLLARY 1. Therefore, if the attracting forces of the particles, on increasing the distances of the attracted corpuscles, decrease in the ratio of any power of those distances, the accelerative attractions toward the whole bodies will be as the bodies directly and those powers of the distances inversely. For example, if the forces of the particles decrease in the squared ratio of the distances from the attracted corpuscles, and the bodies are as A^3 and B^3, and thus both the cube roots of the bodies and the distances of the attracted corpuscles from the bodies are as A and B, the accelerative attractions toward the bodies will be as $\frac{A^3}{A^2}$ and $\frac{B^3}{B^2}$, that is, as those cube roots A and B of the bodies. If the forces of the particles decrease in the cubed ratio of the distances from the attracted corpuscles, the accelerative attractions toward the whole bodies will be as $\frac{A^3}{A^3}$ and $\frac{B^3}{B^3}$, that is, will be equal. If the forces decrease in the fourth power of the distance, the attractions toward the bodies will be as $\frac{A^3}{A^4}$ and $\frac{B^3}{B^4}$, that is, inversely as the cube roots A and B. And so on.

COROLLARY 2. Hence, on the other hand, from the forces with which similar bodies attract corpuscles similarly placed with respect to such bodies, there can be gathered the ratio of the decrease of the forces of the attracting particles, as the attracted corpuscle recedes, so long as that decrease is directly or inversely in some ratio of the distances.

Proposition 88
Theorem 45

If the attracting forces of equal particles of any body are as the distances of places from the particles, the force of the whole body will tend toward its center of gravity, and will be the same as the force of a globe consisting of entirely similar and equal matter and having its center in that center of gravity.

Let the particles A and B of the body RSTV attract some corpuscle Z by forces which, if the particles are equal to each other, are as the distances AZ and BZ: but if the particles are supposed unequal, are as these particles and their distances AZ and BZ jointly, or (so to speak) as these particles multiplied respectively by their distances AZ and BZ. And let the forces be represented by those solids [or products] $A \times AZ$ and $B \times BZ$. Join AB, and let it be cut in G so that AG is to BG as the particle B to the particle A; then G will be the common center of gravity of the particles A and B. The force $A \times AZ$ (by corol. 2 of the laws) is resolved into the forces $A \times GZ$ and $A \times AG$, and the force $B \times BZ$ into the forces $B \times GZ$ and $B \times BG$. But the forces $A \times AG$ and $B \times BG$ are equal (because A is to B as BG to AG); and therefore, since they tend in opposite directions, they nullify each other. There remain the forces $A \times GZ$ and $B \times GZ$. These tend from Z toward the center G and compose the force $(A + B) \times GZ$—that is, the same force as if the attracting particles A and B were situated in their common center of gravity G and there composed a globe.



By the same argument, if a third particle C is added, and its force is compounded with the force $(A + B) \times GZ$ tending toward the center G, the force thence arising will tend toward the common center of gravity of the globe (at G) and the particle C (that is, toward the common center of gravity of the three particles A, B, and C), and will be the same as if the globe and the particle C were situated in their common center, there composing a greater globe. And so on indefinitely. Therefore the whole force of all the particles of any body RSTV is the same as if that body, while maintaining the same center of gravity, were to assume the shape of a globe. Q.E.D.

COROLLARY. Hence the motion of the attracted body Z will be the same as if the attracting body RSTV were spherical; and therefore, if that attracting body either is at rest or progresses uniformly straight forward, the attracted body will move in an ellipse having its center in the center of gravity of the attracting body.

Proposition 89
Theorem 46

If there are several bodies consisting of equal particles whose forces are as the distances of places from each individual particle, the force—compounded of the forces of all these particles—by which any corpuscle is attracted will tend toward the common center of gravity of the attracting bodies and will be the same as if those attracting bodies, while maintaining their common center of gravity, were united together and were formed into a globe.

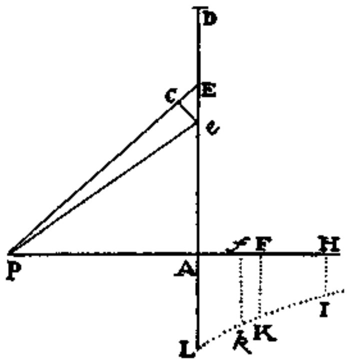
This is demonstrated in the same way as the preceding proposition.

COROLLARY. Therefore the motion of an attracted body will be the same as if the attracting bodies, while maintaining their common center of gravity, came together and were formed into a globe. And hence, if the common center of gravity of the attracting body either is at rest or progresses uniformly in a straight line, the attracted body will move in an ellipse having its center in the common center of gravity of the attracting bodies.

Proposition 90
Problem 44

If equal centripetal forces, increasing or decreasing in any ratio of the distances, tend toward each of the individual points of any circle, it is required to find the force by which a corpuscle is attracted when placed anywhere on the straight line that stands perpendicularly upon the plane of the circle at its center.

Suppose a circle to be described with center A and any radius AD in a plane to which the straight line AP is perpendicular; then it is required to find the force by which any corpuscle P is attracted toward the circle. From



any point E of the circle, draw the straight line PE to the attracted corpuscle P. In the straight line PA take PF equal to PE, and erect the normal FK so that it will be as the force by which the point E attracts the corpuscle P. And let IKL be the curved line that the point K traces out. Let that line meet the plane of the circle in L. In PA take PH equal to PD, and erect the perpendicular HI meeting the aforesaid curve at I, and the attraction of the corpuscle P toward the circle will be as the area AHIL multiplied by the altitude AP. Q.E.I.

For on AE take the minimally small line E*e*. Join P*e*, and in PE and PA take PC and P*f* equal to P*e*. And since the force by which any point E of the ring described with center A and radius AE in the aforesaid plane attracts body [i.e., corpuscle] P toward itself has been supposed to be as FK, and hence the force by which that point attracts body P toward A is as $\frac{AP \times FK}{PE}$; and the force by which the whole ring attracts body P toward A is as the ring and $\frac{AP \times FK}{PE}$ jointly; and that ring is as the rectangle of the radius AE and the width E*e*, and this rectangle (because PE is to AE as E*e* to CE) is equal to the rectangle PE × CE or PE × F*f*; it follows that the force by which that ring attracts body P toward A will be as PE × F*f* and $\frac{AP \times FK}{PE}$ jointly, that is, as the solid [or product] F*f* × FK × AP, or as the area FK*kf* multiplied by AP. And therefore the sum of the forces by which all the rings in the circle that is described with center A and radius AD attract body P toward A is as the whole area AHIKL multiplied by AP. Q.E.D.

Corollary 1. Hence, if the forces of the points decrease in the squared ratio of the distances, that is, if FK is as $\frac{1}{PF^2}$ $\left(\text{and thus the area AHIKL is as } \frac{1}{PA} - \frac{1}{PH}\right)$, the attraction of the corpuscle P toward the circle will be as $1 - \frac{PA}{PH}$, that is, as $\frac{AH}{PH}$.

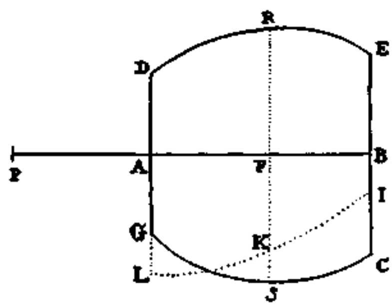
Corollary 2. And universally, if the forces of the points at the distances D are inversely as any power D^n of the distances $\left(\text{that is, if FK is as } \frac{1}{D^n}, \text{ and hence the area AHIKL is as } \frac{1}{PA^{n-1}} - \frac{1}{PH^{n-1}}\right)$, the attraction of the corpuscle P toward the circle will be as $\frac{1}{PA^{n-2}} - \frac{PA}{PH^{n-1}}$.

Corollary 3. And if the diameter of the circle is increased indefinitely and the number n is greater than unity, the attraction of the corpuscle P toward the whole indefinitely extended plane will be inversely as PA^{n-2}, because the other term, $\frac{PA}{PH^{n-1}}$, will vanish.

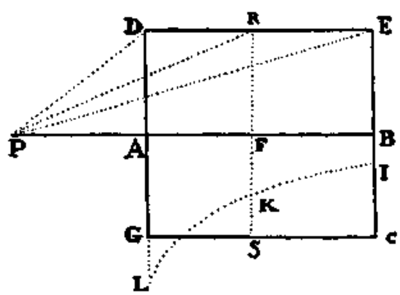
Proposition 91
Problem 45

To find the attraction of a corpuscle placed in the axis of a round solid, to each of whose individual points there tend equal centripetal forces decreasing in any ratio of the distances.

Let corpuscle P, placed in the axis AB of the solid DECG, be attracted toward that same solid. Let this solid be cut by any circle RFS perpendicular to this axis, and in its semidiameter FS, in a plane PALKB passing through the axis, take (according to prop. 90) the length FK proportional to the force by which the corpuscle P is attracted toward that circle. Let point K touch the curved line LKI meeting the planes of the outermost circles AL and BI at L and I, and the attraction of the corpuscle P toward the solid will be as the area LABI. Q.E.I.

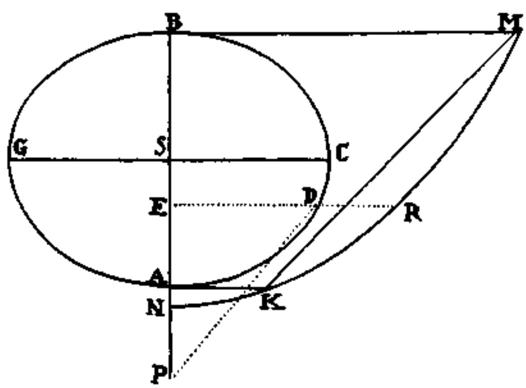


Corollary 1. Hence, if the solid is a cylinder described by parallelogram ADEB revolving about the axis AB, and the centripetal forces tending toward each of its individual points are inversely as the squares of the distances from the points, the attraction of the corpuscle P toward this cylinder will be as $AB - PE + PD$. For the ordinate FK (by prop. 90, corol. 1) will be as



$1 - \frac{PF}{PR}$. The unit part of this $\left[\text{or the quantity 1 in } 1 - \frac{PF}{PR}\right]$ multiplied by the length AB describes the area $1 \times AB$, and the other part $\frac{PF}{PR}$ multiplied by the length PB describes the area $1 \times (PE - AD)$, which can easily be shown from the quadrature of the curve LKI; and similarly the same part $\frac{PF}{PR}$ multiplied by the length PA describes the area $1 \times (PD - AD)$, and multiplied by the difference AB of PB and PA describes the difference of the areas $1 \times (PE - PD)$. From the first product $1 \times AB$ take away the last product $1 \times (PE - PD)$, and there will remain the area LABI equal to $1 \times (AB - PE + PD)$. Therefore the force proportional to this area is as $AB - PE + PD$.

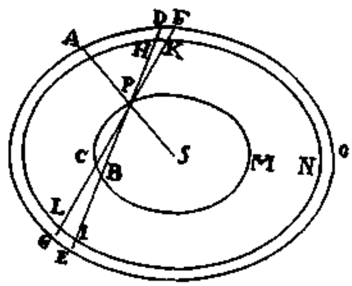
COROLLARY 2. Hence also the force becomes known by which a spheroid AGBC attracts any body P, situated outside the spheroid in its axis AB. Let NKRM be a conic whose ordinate ER, perpendicular to PE, is always



equal to the length of the line PD, which is drawn to the point D in which the ordinate cuts the spheroid. From the vertices A and B of the spheroid, erect AK and BM perpendicular to the axis AB of the spheroid and equal respectively to AP and BP, and therefore meeting the conic in K and M; and join KM cutting off the segment KMRK from the conic. Let the center of the spheroid be S, and its greatest semidiameter SC. Then the force by which the spheroid attracts the body P will be to the force by which a sphere described with diameter AB attracts the same body as $\frac{AS \times CS^2 - PS \times KMRK}{PS^2 + CS^2 - AS^2}$ to

$\frac{AS^3}{3PS^2}$. And by the same mode of computation it is possible to find the forces of the segments of the spheroid.

COROLLARY 3. But if the corpuscle is located inside the spheroid and in its axis, the attraction will be as its distance from the center. This is seen more easily by the following argument, whether the particle is in the axis or in any other given diameter. Let AGOF be the attracting spheroid, S its center, and P the attracted body. Through that body P draw both the semidiameter SPA and any two straight lines DE and FG meeting the spheroid in D and F on one side and in E and G on the other; and let PCM and HLN be the surfaces of two inner spheroids, similar to and concentric with the outer spheroid; and let the first of these pass through the body P and cut the straight lines DE and FG in B and C, and let the latter cut the same straight lines in H, I and K, L. Let all the spheroids have a common axis, and the parts of the straight lines intercepted on the two sides, DP and BE, FP and CG, DH and IE, FK and LG will be equal to one another, because the straight lines DE, PB, and HI are bisected in the same point, as are also the straight lines FG, PC, and KL. Now suppose that DPF and EPG designate opposite cones described with the infinitely small vertical angles DPF and EPG, and that the lines DH and EI also are infinitely small; then the particles of the cones—that is, the particles DHKF and GLIE—cut off by the surfaces of the spheroids will (because of the equality of the lines DH and EI) be to each other as the squares of their distances from the corpuscle P, and therefore will attract the corpuscle equally. And by a like reasoning, if the spaces DPF and EGCB are divided into particles by the surfaces of innumerable similar concentric spheroids, having a common axis, then all of these particles will attract the body P in opposite directions equally on both sides. Therefore the forces of the cone DPF and of the conical segment [or truncated cone] EGCB are equal, and—being opposite—annul each other. And it is the same with regard to the forces of all the matter outside the innermost spheroid PCBM. Therefore the body P is attracted only by the innermost spheroid PCBM, and accordingly (by prop. 72, corol. 3) its attraction is to the force by which



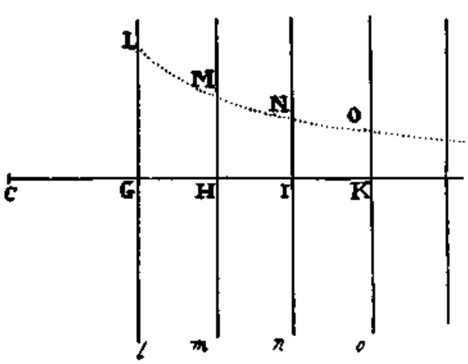
the body A is attracted by the whole spheroid AGOD as the distance PS to the distance AS. Q.E.D.

Proposition 92
Problem 46
Given an attracting body, it is required to find the ratio by which the centripetal forces tending toward each of its individual points decrease [i.e., decrease as a function of distance].

From the given body a sphere or cylinder or other regular figure is to be formed, whose law of attraction—corresponding to any ratio of decrease [in relation to distance]—can be found by props. 80, 81, and 91. Then, by making experiments, the force of attraction at different distances is to be found; and the law of attraction toward the whole that is thus revealed will give the ratio of the decrease of the forces of the individual parts, which was required to be found.

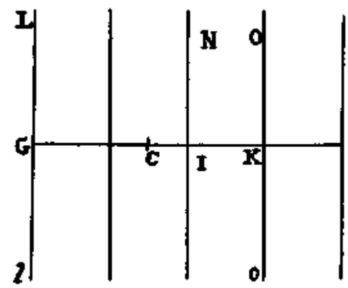
Proposition 93
Theorem 47
*If a solid, plane on one side but infinitely extended on the other sides, consists of equal and equally attracting particles, whose forces—in receding from the solid—decrease in the ratio of any power of the distances that is more than the square; and if a corpuscle set on either side of the plane is attracted by the force of the whole solid; then I say that that force of attraction of the solid in receding from its plane surface will decrease in the ratio of the distance of the corpuscle from the plane raised to a power whose index is less by 3 units than that of the power of the distances in the law of attractive force [*lit. *will decrease in the ratio of the power whose base is the distance of the corpuscle from the plane and whose index is less by 3 than the index of the power of the distances].*

Case 1. Let LG*l* be the plane by which the solid is terminated. Let the solid lie on the side of this plane toward I, and let it be resolved into innumerable planes *m*HM, *n*IN, *o*KO, . . . parallel to GL. And first let the attracted body C be placed outside the solid. Draw CGHI perpendicular to those innumerable planes, and let the forces of attraction of the points of the solid decrease in the ratio of a power of the distances whose index is the number *n* not



smaller than 3. Therefore (by prop. 90, corol. 3) the force by which any plane *m*HM attracts the point C is inversely as CH^{n-2}. In the plane *m*HM take the length HM inversely proportional to CH^{n-2}, and that force will be as HM. Similarly, on each of the individual planes *l*GL, *n*IN, *o*KO, . . . , take the lengths GL, IN, KO, . . . inversely proportional to CG^{n-2}, CI^{n-2}, CK^{n-2}, . . . ; then the forces of these same planes will be as the lengths taken, and thus the sum of the forces will be as the sum of the lengths; that is, the force of the whole solid will be as the area GLOK produced infinitely in the direction OK. But that area (by the well-known methods of quadratures) is inversely as CG^{n-3}, and therefore the force of the whole solid is inversely as CG^{n-3}. Q.E.D.

Case 2. Now let the corpuscle C be placed on the side of the plane *l*GL inside the solid, and take the distance CK equal to the distance CG. Then the part LG*lo*KO of this solid, terminated by the parallel planes *l*GL and *o*KO, will not attract the corpuscle C (situated in the middle) in any direction, the opposite actions of opposite points annulling each other because of their equality. Accordingly, corpuscle C is attracted only by the force of the solid situated beyond the plane OK. But this force (by case 1) is inversely as CK^{n-3}, that is (because CG and CK are equal), inversely as CG^{n-3}. Q.E.D.



Corollary 1. Hence, if the solid LGIN is terminated on both sides by two infinitely extended and parallel planes LG and IN, its force of attraction becomes known by subtracting from the force of attraction of the whole infinitely extended solid LGKO the force of attraction of the further part NIKO produced infinitely in the direction KO.

Corollary 2. If the more distant part of this infinitely extended solid is ignored, since its attraction compared with the attraction of the nearer part is of almost no moment, then the attraction of that nearer part, with an increase of the distance, will decrease very nearly in the ratio of the power CG^{n-3}.

Corollary 3. And hence, if any body that is finite and plane on one side attracts a corpuscle directly opposite the middle of that plane, and the distance between the corpuscle and the plane is exceedingly small compared with the dimensions of the attracting body, and the attracting body consists of homogeneous particles whose forces of attraction decrease in the ratio of any power of the distances that is more than the fourth; the force of attraction of the whole body will decrease very nearly in the ratio of a power of that exceedingly small distance, whose index is less by 3 than the index of the stated power. This assertion is not valid for a body consisting of particles whose forces of attraction decrease in the ratio of the third power of the distances, because in this case the attraction of the more distant part of the infinitely extended body in corol. 2 is always infinitely greater than the attraction of the nearer part.

Scholium If a body is attracted perpendicularly toward a given plane, and the motion of the body is required to be found from the given law of attraction, the problem will be solved by seeking (by prop. 39) the motion of the body descending directly to this plane and by compounding this motion (according to corol. 2 of the laws) with a uniform motion performed along lines parallel to the same plane. And conversely, if it is required to find the law of an attraction made toward the plane along perpendicular lines, under the condition that the attracted body moves in any given curved line whatever, the problem will be solved by the operations used in the third problem [i.e., prop. 8].

The procedure can be shortened by resolving the ordinates into converging series. For example, if B is the ordinate to the base A at any given angle, and is as any power $A^{\frac{m}{n}}$ of that base, and the force is required by which a body that is either attracted toward the base or repelled away from the base (according to the position of the ordinate) can move in a curved line that the upper end of the ordinate traces out; I suppose the base to be increased by a minimally small part O, and I resolve the ordinate $(A+O)^{\frac{m}{n}}$ into the infinite series

$$A^{\frac{m}{n}} + \frac{m}{n}OA^{\frac{m-n}{n}} + \frac{m^2-mn}{2n^2}O^2A^{\frac{m-2n}{n}} \ldots,$$

and I suppose the force to be proportional to the term of this series in which O is of two dimensions, that is, to the term $\frac{m^2-mn}{2n^2}O^2A^{\frac{m-2n}{n}}$. Therefore the required force is as $\frac{m^2-mn}{n^2}A^{\frac{m-2n}{n}}$, or, which is the same, as $\frac{m^2-mn}{n^2}B^{\frac{m-2n}{m}}$. For example, if the ordinate traces out a parabola, where $m = 2$ and $n = 1$, the force will become as the given quantity $2B^0$, and thus will be given. Therefore with a given [i.e., constant] force the body will move in a parabola, as Galileo demonstrated. But if the ordinate traces out a hyperbola, where $m = 0 - 1$ and $n = 1$, the force will become as $2A^{-3}$ or $2B^3$; and therefore with a force that is as the cube of the ordinate, the body will move in a hyperbola. But putting aside propositions of this sort, I go on to certain others on motion which I have not as yet considered.

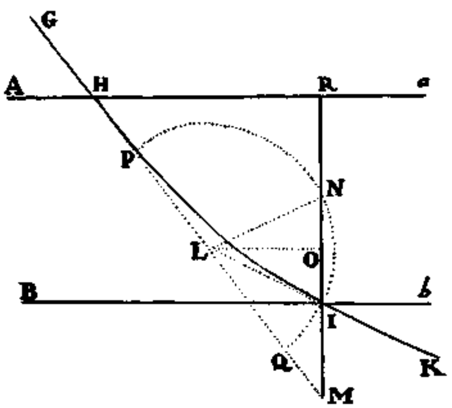
SECTION 14

The motion of minimally small bodies that are acted on by centripetal forces tending toward each of the individual parts of some great body

Proposition 94
Theorem 48

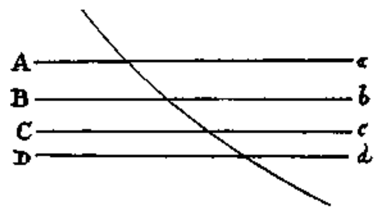
If two homogeneous mediums are separated from each other by a space terminated on the two sides by parallel planes, and a body passing through this space is attracted or impelled perpendicularly toward either medium and is not acted on or impeded by any other force, and the attraction at equal distances from each plane (taken on the same side of that plane) is the same everywhere; then I say that the sine of the angle of incidence onto either plane will be to the sine of the angle of emergence from the other plane in a given ratio.

CASE 1. Let A*a* and B*b* be the two parallel planes. Let the body be incident upon the first plane A*a* along line GH, and in all its passage through the intermediate space let it be attracted or impelled toward the medium of incidence, and by this action let it describe the curved line HI and emerge along the line IK. To the plane of emergence B*b* erect the perpendicular IM meeting the line of incidence GH produced in M and the plane of incidence A*a* in R; and let the line of emergence KI produced meet HM in L. With center L and radius LI describe a circle cutting HM in P and Q, as well as MI produced in N. Then first, if the attraction or impulse is supposed uniform, the curve HI (from what Galileo demonstrated) will be a parabola, of which this is a property: that the rectangle of its given latus rectum and the line IM is equal to HM squared; but also the line HM will be bisected in L. Hence, if the perpendicular LO is dropped to MI, MO and OR will be equal; and when the equals ON and OI have been added to these quantities, the totals MN and IR will become equal. Accordingly, since IR is given, MN is also given; and the rectangle NM × MI is to the rectangle of the latus rectum and IM (that is, to HM^2) in a given ratio. But the rectangle NM × MI is equal to the rectangle PM × MQ, that is, to the difference of



the squares ML^2 and PL^2 or LI^2; and HM^2 has a given ratio to its fourth part ML^2: therefore the ratio of $ML^2 - LI^2$ to ML^2 is given, and by conversion [or convertendo] the ratio LI^2 to ML^2 is given, and also the square root of that ratio, LI to ML. But in every triangle LMI, the sines of the angles are proportional to the opposite sides. Therefore the ratio of the sine of the angle of incidence LMR to the sine of the angle of emergence LIR is given. Q.E.D.

CASE 2. Now let the body pass successively through several spaces terminated by parallel planes, A*ab*B, B*bc*C, . . . , and be acted on by a force that is uniform in each of the individual spaces considered separately but is different in each of the different spaces. Then by what has just been demonstrated, the sine of the angle of incidence upon the first plane A*a* will be to the sine of the angle

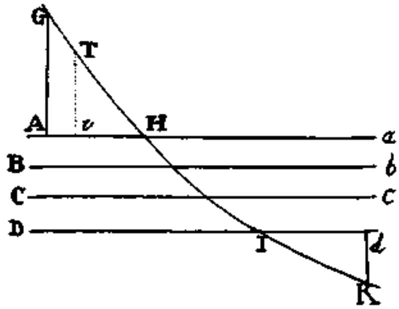


of emergence from the second plane B*b* in a given ratio; and this sine, which is the sine of the angle of incidence upon the second plane B*b*, will be to the sine of the angle of emergence from the third plane C*c* in a given ratio; and this sine will be in a given ratio to the sine of the angle of emergence from the fourth plane D*d*; and so on indefinitely. And from the equality of the ratios [or ex aequo] the sine of the angle of incidence upon the first plane will be in a given ratio to the sine of the angle of emergence from the last plane. Now let the distances between the planes be diminished and their number increased indefinitely, so that the action of attraction or of impulse, according to any assigned law whatever, becomes continuous; then the ratio of the sine of the angle of incidence upon the first plane to the sine of the angle of emergence from the last plane, being always given, will still be given now. Q.E.D.

Proposition 95
Theorem 49

With the same suppositions as in prop. 94, I say that the velocity of the body before incidence is to its velocity after emergence as the sine of the angle of emergence to the sine of the angle of incidence.

Let AH be taken equal to I*d*, and erect the perpendiculars AG and *d*K meeting the lines of incidence and emergence GH and IK in G and K. In GH take TH equal to IK, and drop T*v* perpendicular to the plane A*a*. And (by corol. 2 of the laws) resolve the motion of the body into two motions, one



perpendicular, the other parallel, to the planes Aa, Bb, Cc, ... The [component of the] force of attraction or of impulse acting along perpendicular lines does not at all change the motion in the direction of the parallels; and therefore the body, by this latter motion, will in equal times pass through equal distances along parallels between the line AG and the point H, and between the point I and the line dK, that is, it will describe the lines GH and IK in equal times. Accordingly, the velocity before incidence is to the velocity after emergence as GH to IK or TH; that is, as AH or Id to vH, that is (with respect to the radius TH or IK), as the sine of the angle of emergence to the sine of the angle of incidence. Q.E.D.

Proposition 96
Theorem 50

With the same suppositions, and supposing also that the motion before incidence is faster than afterward, I say that as a result of [a]changing the inclination[a] of the line of incidence, the body will at last be reflected, and the angle of reflection will become equal to the angle of incidence.

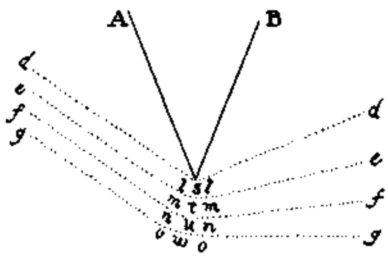
For suppose the body to describe parabolic arcs between the parallel planes Aa, Bb, Cc, ..., as before; and let those arcs be HP, PQ, QR, ...

And let the obliquity of the line of incidence GH to the first plane Aa be such that the sine of the angle of incidence is to the radius of the circle whose sine it is in the ratio which that same sine of the angle of incidence has to the sine of the angle of emergence from the plane Dd into the space DdeE; then, because the sine of the angle of emergence will now have become equal to the radius, the angle of emergence will be a right angle, and hence the line of emergence will coincide with the plane Dd. Let the body arrive at this plane at the point R; and since the line of emergence coincides with

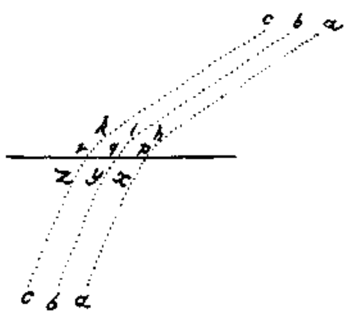
aa. The sense of Newton's "changing the inclination" is that of increasing the angle of incidence.

that same plane, it is obvious that the body cannot go any further toward the plane E*e*. But neither can it go on in the line of emergence R*d*, because it is continually attracted or impelled toward the medium of incidence. Therefore, this body will be turned back between the planes C*c* and D*d*, describing an arc of the parabola QR*q*, whose principal vertex (according to what Galileo demonstrated) is at R, and will cut the plane C*c* in the same angle at *q* as formerly at Q; and then, proceeding in the parabolic arcs *qp*, *ph*, . . . , similar and equal to the former arcs QP and PH, this body will cut the remaining planes in the same angles at *p*, *h*, . . . , as formerly at P, H, . . . , and will finally emerge at *h* with the same obliquity with which it was incident upon the plane at H. Now suppose the distances between the planes A*a*, B*b*, C*c*, D*d*, E*e*, . . . to be diminished and their number increased indefinitely, so that the action of attraction or impulse, according to any assigned law whatever, is made to be continuous; then the angle of emergence, being always equal to the angle of incidence, will still remain equal to it now. Q.E.D.

Scholium These attractions are very similar to the reflections and refractions of light made according to a given ratio of the secants, as Snel discovered, and consequently according to a given ratio of the sines, as Descartes set forth. For the fact that light is propagated successively [i.e., in time and not instantaneously] and comes from the sun to the earth in about seven or eight minutes is now established by means of the phenomena of the satellites of Jupiter, confirmed by the observations of various astronomers. Moreover, the rays of light that are in the air (as Grimaldi recently discovered, on admitting light into a dark room through a small hole—something I myself have also tried) in their passing near the edges of bodies, whether opaque or transparent (such as are the circular-rectangular edges of coins minted from gold, silver, and bronze, and the sharp edges of knives, stones, or broken glass), are inflected around the bodies, as if attracted toward them; and those of the rays that in such passing approach closer to the bodies are inflected the more, as if more attracted, as I myself have also diligently observed. And those that pass at greater distances are less inflected, and at still greater distances are inflected somewhat in the opposite direction and form three bands of colors.

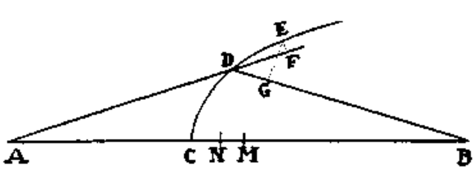


In the figure, *s* designates the sharp edge of a knife or of any wedge A*s*B, and *gowog*, *fnunf*, *emtme*, and *dlsld* are rays, inflected in the arcs *owo*, *nun*, *mtm*, and *lsl* toward the knife, more so or less so according to their distance from the knife. Moreover, since such an inflection of the rays takes place in the air outside the knife, the rays which are incident upon the knife must also be inflected in the air before they reach it. And the case is the same for those rays incident upon glass. Therefore refraction takes place not at the point of incidence, but gradually by a continual inflection of the rays, made partly in the air before the rays touch the glass, and partly (if I am not mistaken) within the glass after they have entered it, as has been delineated in the rays *ckzc*, *biyb*, and *ahxa* incident upon the glass at *r*, *q*, and *p*, and inflected between *k* and *z*, *i* and *y*, *h* and *x*. Therefore because of the analogy that exists between the propagation of rays of light and the motion of bodies, I have decided to subjoin the following propositions for optical uses, meanwhile not arguing at all about the nature of the rays (that is, whether they are bodies or not), but only determining the trajectories of bodies, which are very similar to the trajectories of rays.



Proposition 97
Problem 47

Supposing that the sine of the angle of incidence upon some surface is to the sine of the angle of emergence in a given ratio, and that the inflection of the paths of bodies in close proximity to that surface takes place in a very short space, which can be considered to be a point; it is required to determine the surface that may make all the corpuscles emanating successively from a given place converge to another given place.

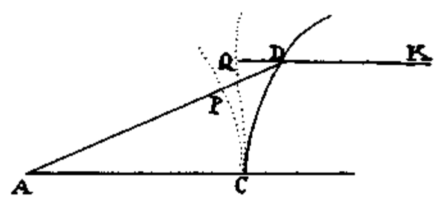


Let A be the place from which the corpuscles diverge, B the place to which they should converge, CDE the curved line that—by revolving about the axis

AB—describes the required surface, D and E any two points of that curve, and EF and EG perpendiculars dropped to the paths AD and DB of the body. Let point D approach point E; then the ultimate ratio of the line DF (by which AD is increased) to the line DG (by which DB is decreased) will be the same as that of the sine of the angle of incidence to the sine of the angle of emergence. Therefore the ratio of the increase of the line AD to the decrease of the line DB is given; and as a result, if a point C is taken anywhere on the axis AB, this being a point through which the curve CDE should pass, and the increase CM of AC is taken in that given ratio to the decrease CN of BC, and if two circles are described with centers A and B and radii AM and BN and cut each other at D, that point D will touch the required curve CDE, and by touching it anywhere whatever will determine that curve. Q.E.I.

COROLLARY 1. But by making point A or B in one case go off indefinitely, in another case move to the other side of point C, all the curves which Descartes exhibited with respect to refractions in his treatises on optics and geometry will be traced out. Since Descartes concealed the methods of finding these, I have decided to reveal them by this proposition.

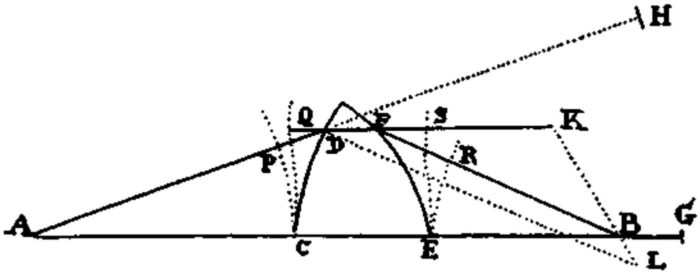
COROLLARY 2. If a body, incident upon any surface CD along the straight line AD drawn according to any law, emerges along any other straight line DK; and if from point C the curved lines CP and CQ, always perpendicular to AD and DK, are understood to be drawn; then the increments of the lines PD and QD, and hence the lines themselves PD and QD generated by those increments, will be as the sines of the angles of incidence and emergence to each other, and conversely.



Proposition 98
Problem 48

The same conditions being supposed as in prop. 97, and supposing that there is described about the axis AB *any attracting surface* CD, *regular or irregular, through which the bodies coming out from a given place* A *must pass; it is required to find a second attracting surface* EF *that will make the bodies converge to a given place* B.

Join AB and let it cut the first surface in C and the second in E, point D being taken in any way whatever. And supposing that the sine of the angle



of incidence upon the first surface is to the sine of the angle of emergence from that first surface, and that the sine of the angle of emergence from the second surface is to the sine of the angle of incidence upon the second surface, as some given quantity M is to another given quantity N; produce AB to G so that BG is to CE as M − N to N, and produce AD to H so that AH is equal to AG, and also produce DF to K so that DK is to DH as N to M. Join KB, and with center D and radius DH describe a circle meeting KB produced in L, and draw BF parallel to DL; then the point F will touch the line EF, which—on being revolved about the axis AB—will describe the required surface. Q.E.F.

Now suppose the lines CP and CQ to be everywhere perpendicular to AD and DF respectively, and the lines ER and ES to be similarly perpendicular to FB and FD, with the result that QS is always equal to CE; then (by prop. 97, corol. 2) PD will be to QD as M to N, and therefore as DL to DK or FB to FK; and by separation [or dividendo] as DL − FB or PH − PD − FB to FD or FQ − QD, and by composition [or componendo] as PH − FB to FQ, that is (because PH and CG, QS and CE are equal), as CE + BG − FR to CE − FS. But (because BG is proportional to CE and M − N is proportional to N) CE + BG is also to CE as M to N, and thus by separation [or dividendo] FR is to FS as M to N; and therefore (by prop. 97, corol. 2) the surface EF compels a body incident upon it along the line DF to go on in the line FR to the place B. Q.E.D.

Scholium It would be possible to use the same method for three surfaces or more. But for optical uses spherical shapes are most suitable. If the objective lenses of telescopes are made of two lenses that are spherically shaped and water is enclosed between them, it can happen that errors of the refractions that take place in the extreme surfaces of the lenses are accurately enough corrected by the refractions of the water. Such objective lenses are to be preferred to elliptical and hyperbolical lenses, not only because they can be formed more easily and more accurately but also because they more accurately refract the pencils of rays situated outside the axis of the glass. Nevertheless, the differing refrangibility of different rays [i.e., of rays of different colors] prevents optics from being perfected by spherical or any other shapes. Unless the errors arising from this source can be corrected, all labor spent in correcting the other errors will be of no avail.

BOOK 2

THE MOTION OF BODIES

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SECTION 1

The motion of bodies that are resisted in proportion to their velocity

Proposition 1
Theorem 1

If a body is resisted in proportion to its velocity, the motion lost as a result of the resistance is as the space described in moving.

For since the motion lost in each of the equal particles of time is as the velocity, that is, as a particle of the path described, then, by composition [or componendo], the motion lost in the whole time will be as the whole path. Q.E.D.

COROLLARY. Therefore, if a body, devoid of all gravity, moves in free spaces by its inherent force alone and if there are given both the whole motion at the beginning and also the remaining motion after some space has been described, the whole space that the body can describe in an infinite time will be given. For that space will be to the space already described as the whole motion at the beginning is to the lost part of that motion.

Lemma 1

Quantities proportional to their differences are continually proportional.

Let A be to A − B as B to B − C and C to C − D, . . . ; then, by conversion [or convertendo], A will be to B as B to C and C to D, ... Q.E.D.

Proposition 2
Theorem 2

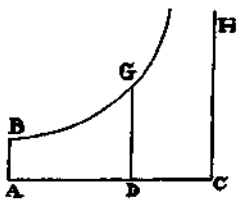
If a body is resisted in proportion to its velocity and moves through a homogeneous medium by its inherent force alone and if the times are taken as equal, the velocities at the beginnings of the individual times are in a geometric progression, and the spaces described in the individual times are as the velocities.

CASE 1. Divide the time into equal particles; and if, at the very beginnings of the particles, a force of resistance which is as the velocity acts with a single impulse, the decrease of the velocity in the individual particles of time will be as that velocity. The velocities are therefore proportional to their differences and thus (by book 2, lem. 1) are continually proportional. Accordingly, if any equal times are compounded of an equal number of particles, the velocities at the very beginnings of the times will be as the terms in a continual progression in which some have been skipped, omitting an equal number of intermediate terms in each interval. The ratios of these terms are indeed compounded of equally repeated equal ratios of the intermediate terms, and therefore these compound ratios are also equal to one another.

Therefore, since the velocities are proportional to these terms, they are in a geometric progression. Now let those equal particles of times be diminished, and their number increased indefinitely, so that the impulse of the resistance becomes continual; then the velocities at the beginnings of equal times, which are always continually proportional, will also be continually proportional in this case. Q.E.D.

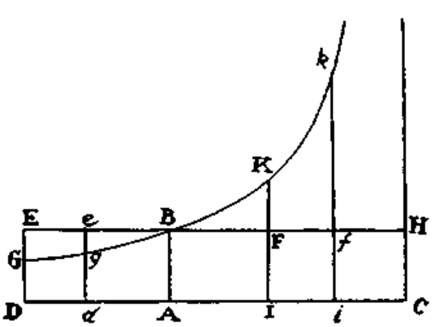
CASE 2. And by separation [or dividendo] the differences of the velocities (that is, the parts of them which are lost in the individual times) are as the wholes, while the spaces described in the individual times are as the lost parts of the velocities (by book 2, prop. 1) and are therefore also as the wholes. Q.E.D.

COROLLARY. Hence, if a hyperbola BG is described with respect to the rectangular asymptotes AC and CH and if AB and DG are perpendicular to asymptote AC and if both the velocity of the body and the resistance of the medium are represented, at the very beginning of the motion, by any given line AC, but after some time has elapsed, by the indefinite line DC, then the time can be represented by area ABGD, and the space described in that time can be represented by line AD. For if the area is increased uniformly by the motion of point D, in the same manner as the time, the straight line DC will decrease in a geometric ratio in the same way as the velocity, and the parts of the straight line AC described in equal times will decrease in the same ratio.



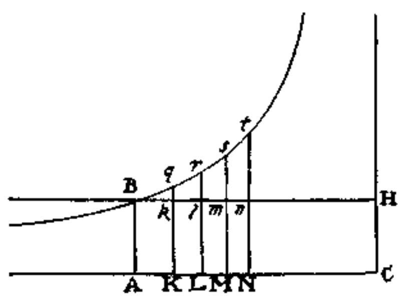
Proposition 3
Problem 1

To determine the motion of a body which, while moving straight up or down in a homogeneous medium, is resisted in proportion to the velocity, and which is acted on by uniform gravity.



When the body is moving up, represent the gravity by any given rectangle BACH, and the resistance of the medium at the beginning of the ascent by the rectangle BADE taken on the other side of the straight line AB. With respect to the rectangular asymptotes AC and CH, describe a hyperbola through point B, cutting perpendiculars DE and *de* in G and *g*; then the body, by ascending in the time DG*gd*, will describe the space EG*ge*; and in the time DGBA will describe the space of the total ascent EGB; and in the time ABKI will describe the space of descent BFK; and in the time IK*ki* will describe the space of descent KF*fk*; and the body's velocities (proportional to the resistance of the medium) in these periods of time will be ABED, AB*ed*, null, ABFI, and AB*fi* respectively; and the greatest velocity that the body can attain in descending will be BACH.

For resolve the rectangle BACH into innumerable rectangles A*k*, K*l*, L*m*, M*n*, . . . , which are as the increases of the velocities, occurring in the same number of equal times; then nil, A*k*, A*l*, A*m*, A*n*, . . . will be as the total velocities, and thus (by hypothesis) as the resistances of the medium at the beginning of each of the equal times. Make AC to AK, or ABHC to AB*k*K, as the force of gravity to the resistance at the beginning of the second time, and subtract the resistances from the force of gravity; then the remainders ABHC, K*k*HC, L*l*HC, M*m*HC, . . . will be as the absolute forces by which the body is urged at the beginning of each of the times, and thus (by the second law of motion) as the increments of the velocities, that is, as the rectangles A*k*, K*l*, L*m*, M*n*, . . . , and therefore (by book 2, lem. 1) in a geometric progression. Therefore, if the straight lines K*k*, L*l*, M*m*, N*n*, . . . , produced, meet the hyperbola in *q*, *r*, *s*, *t*, . . . , areas AB*q*K, K*qr*L, L*rs*M, M*st*N, . . . will be equal, and thus proportional both to the times and to the forces of gravity, which are always equal. But area AB*q*K (by book 1, lem. 7, corol. 3, and lem. 8) is to area B*kq* as K*q* to ½*kq* or AC to ½AK, that is, as the force of gravity to the resistance in the middle of the first time. And by a similar argument, areas *q*KL*r*, *r*LM*s*, *s*MN*t*, . . . are to areas *qklr*, *rlms*, *smnt*, . . . as the force of gravity to the resistance in the middle of the second time, of the third, of the fourth, ... Accordingly, since the equal areas BAK*q*, *q*KL*r*, *r*LM*s*, *s*MN*t*, . . . are proportional to the forces of gravity, areas B*kq*, *qklr*, *rlms*, *smnt*, . . . will be proportional to the resistance in the middle of each of the times, that is (by hypothesis), to the velocities, and thus to the spaces described. Take the sums of the pro-



portional quantities; then areas B*kq*, B*lr*, B*ms*, B*nt*, ... will be proportional to the total spaces described, and areas AB*q*K, AB*r*L, AB*s*M, AB*t*N, ... will be proportional to the times. Therefore the body, while descending in any time AB*r*L, describes the space B*lr*, and in the time L*rt*N describes the space *rlnt*. Q.E.D. And the proof is similar for an ascending motion. Q.E.D.

COROLLARY 1. Therefore the greatest velocity that a body can acquire in falling is to the velocity acquired in any given time as the given force of gravity by which the body is continually urged to [a]the force of the resistance by which it is impeded at the end of that time.[a]

COROLLARY 2. If the time is increased in an arithmetic progression, the sum of that greatest velocity and of the velocity in the ascent, and also their difference in the descent, decreases in a geometric progression.

COROLLARY 3. The differences of the spaces which are described in equal differences of the times decrease in the same geometric progression.

COROLLARY 4. The space described by a body is the difference of two spaces, of which one is as the time reckoned from the beginning of the descent, and the other is as the velocity; and these spaces are equal to each other at the very beginning of the descent.

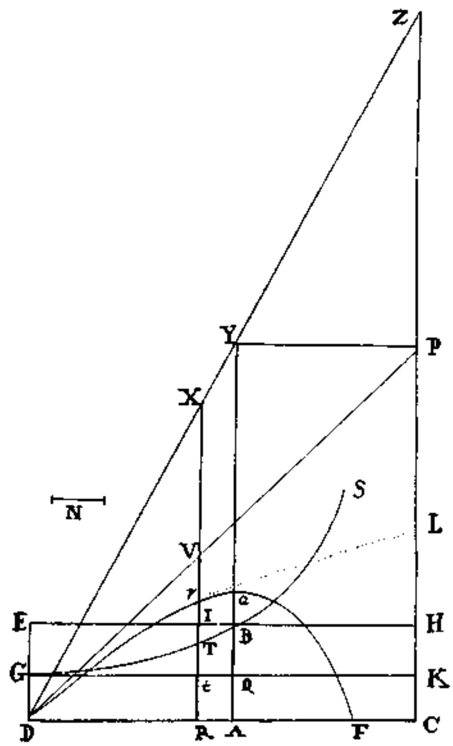
Proposition 4
Problem 2

Supposing that the force of gravity in some homogeneous medium is uniform and tends perpendicularly toward the plane of the horizon, it is required to determine the motion of a projectile in that medium, while it is resisted in proportion to the velocity.

From any place D let a projectile go forth along any straight line DP, and represent its velocity at the beginning of the motion by the length DP. Drop the perpendicular PC from point P to the horizontal line DC, and cut DC in A so that DA is to AC as the resistance of the medium arising from the upward motion at the beginning is to the force of gravity; or (which comes to the same thing) so that the rectangle of DA and DP is to the rectangle of AC and CP as the whole resistance at the beginning of the motion is to the force of gravity. Describe any hyperbola GTBS with asymptotes DC and CP which cuts the perpendiculars DG and AB in G and B; and complete the parallelogram DGKC, whose side GK cuts AB in Q. Take the line N

aa. Ed. 2 has "the excess of this force over the force by which it is resisted at the end of that time."

in the same ratio to QB as DC to CP, and at any point R of the straight line DC erect the perpendicular RT which meets the hyperbola in T and the straight lines EH, GK, and DP in I, *t*, and V, and then on RT take V*r* equal to $\frac{t\mathrm{GT}}{\mathrm{N}}$, or (which comes to the same thing) take R*r* equal to $\frac{\mathrm{GTIE}}{\mathrm{N}}$. Then in the time DRTG the projectile will arrive at point *r*, describing the curved line D*ra*F which point *r* traces out, reaching its greatest height *a* in the perpendicular AB, and afterward always approaching the asymptote PC. And its velocity at any point *r* is as the tangent *r*L of the curve. Q.E.I.

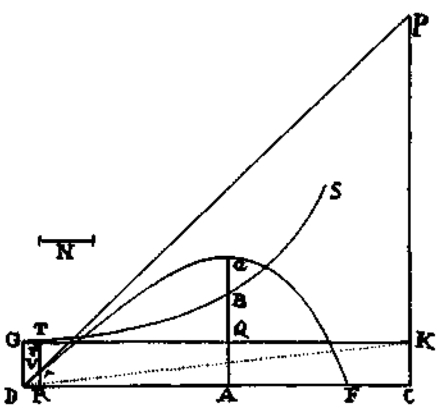


For N is to QB as DC to CP or DR to RV, and thus RV is equal to $\frac{\mathrm{DR} \times \mathrm{QB}}{\mathrm{N}}$, and R*r* $\left(\text{that is, RV} - \mathrm{V}r\text{, or } \frac{\mathrm{DR} \times \mathrm{QB} - t\mathrm{GT}}{\mathrm{N}}\right)$ is equal to $\frac{\mathrm{DR} \times \mathrm{AB} - \mathrm{RDGT}}{\mathrm{N}}$. Now represent the time by area RDGT, and (by corol. 2 of the laws) divide the motion of the body into two parts, one upward and the other lateral. Since the resistance is as the motion, it also will be divided into two parts proportional to and opposite to the parts of the motion; and thus the distance described by the lateral motion will be (by book 2, prop. 2) as line DR, and the distance described by the upward motion will be (by book 2, prop. 3) as the area $\mathrm{DR} \times \mathrm{AB} - \mathrm{RDGT}$, that is, as line R*r*. But at the very beginning of the motion the area RDGT is equal to the rectangle $\mathrm{DR} \times \mathrm{AQ}$, and thus that line R*r* $\left(\text{or } \frac{\mathrm{DR} \times \mathrm{AB} - \mathrm{DR} \times \mathrm{AQ}}{\mathrm{N}}\right)$ is then to DR as $\mathrm{AB} - \mathrm{AQ}$ or QB to N, that is, as CP to DC, and hence as the upward motion to the lateral motion at the beginning. Since, therefore, R*r* is always as the distance upward, and DR is always as the distance sideward, and R*r* is to DR at the beginning as the distance upward to the distance sideward, Rr must always be to DR as the distance upward to the distance sideward, and therefore the body must move in the line DraF, which the point r traces out. Q.E.D.

COROLLARY 1. Rr is therefore equal to $\frac{DR \times AB}{N} - \frac{RDGT}{N}$; and thus, if RT is produced to X so that RX is equal to $\frac{DR \times AB}{N}$, that is, if the parallelogram ACPY is completed, and DY is joined cutting CP in Z, and RT is produced until it meets DY in X, then Xr will be equal to $\frac{RDGT}{N}$, and therefore will be proportional to the time.

COROLLARY 2. Hence, if innumerable lines CR are taken (or, which comes to the same thing, innumerable lines ZX) in a geometric progression, then as many lines Xr will be in an arithmetic progression. And hence it is easy to draw curve DraF with the help of a table of logarithms.

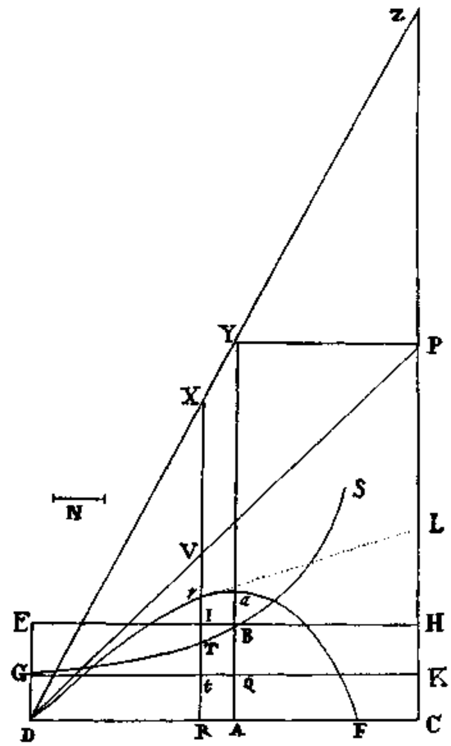
COROLLARY 3. If a parabola is constructed with vertex D and diameter DG (produced downward) and a latus rectum that is to 2DP as the whole



resistance at the very beginning of the motion is to the force of gravity, then the velocity with which a body must go forth from place D along the straight line DP in order to describe curve DraF in a uniform resisting medium will be the very one with which it must go forth from the same place D along the same straight line DP in order to describe the parabola in a nonresisting space. For the latus rectum of this parabola, at the very beginning of the motion, is $\frac{DV^2}{Vr}$; and Vr is $\frac{tGT}{N}$ or $\frac{DR \times Tt}{2N}$. But the straight line that, if it were drawn, would touch the hyperbola GTS in G is parallel to DK, and thus Tt is $\frac{CK \times DR}{DC}$, and N has been taken as $\frac{QB \times DC}{CP}$. Therefore Vr is $\frac{DR^2 \times CK \times CP}{2DC^2 \times QB}$, that is (because DR is to DC as DV is to DP),

$\frac{DV^2 \times CK \times CP}{2DP^2 \times QB}$; and the latus rectum $\frac{DV^2}{Vr}$ comes out $\frac{2DP^2 \times QB}{CK \times CP}$, that is (because QB is to CK as DA is to AC), $\frac{2DP^2 \times DA}{AC \times CP}$, and thus is to 2DP as DP × DA to CP × AC—that is, as the resistance to the gravity. Q.E.D.

COROLLARY 4. Hence, if a body is projected from any place D with a given velocity along any straight line DP given in position, and the resistance of the medium at the very beginning of the motion is given, the curve D*ra*F which the body will describe can be found. For from the given velocity the latus rectum of the parabola is given, as is well known. And if 2DP is taken to that latus rectum as the force of gravity to the force of resistance, DP is given. Then, if DC is cut in A so that CP × AC is to DP × DA in that same ratio of gravity to resistance, point A will be given. And hence curve D*ra*F is given.

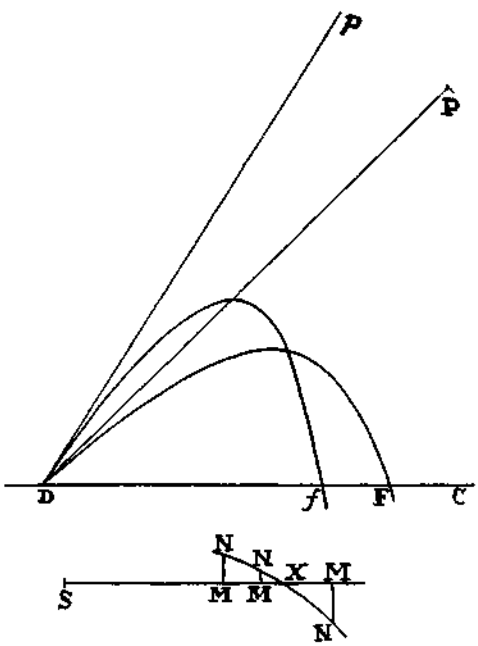


COROLLARY 5. And conversely, if curve D*ra*F is given, both the velocity of the body and the resistance of the medium in each of the places *r* will be given. For since the ratio of CP × AC to DP × DA is given, both the resistance of the medium at the beginning of the motion and the latus rectum of the parabola are also given; and hence the velocity at the beginning of the motion is also given. Then from the length of the tangent *r*L, both the velocity (which is proportional to it) and the resistance (which is proportional to the velocity) are given in any place *r*.

COROLLARY 6. The length 2DP is to the latus rectum of the parabola as the gravity to the resistance at D; and when the velocity is increased the resistance is increased in the same ratio, but the latus rectum of the parabola is increased in the square of that ratio; hence it is evident that the length

2DP is increased in the simple ratio and thus is always proportional to the velocity and is not increased or decreased when the angle CDP is changed unless the velocity is also changed.

COROLLARY 7. Hence the method is apparent for determining the curve D*ra*F from phenomena approximately and for obtaining thereby the resistance and the velocity with which the body is projected. Project two similar and equal bodies with the same velocity from place D along the different angles CDP and CD*p*, and let the places F and *f* where they fall upon the horizontal plane DC be known. Then, taking any length for DP or D*p*, suppose that the resistance at D is to the gravity in any ratio, and represent that ratio by any length SM. Then, by computation, find the lengths DF and D*f* from that assumed length DP, and from the ratio $\frac{Ff}{DF}$ (found by computation) take away the same ratio (found by experiment), and represent the difference by the perpendicular MN. Do the same thing a second and a third time, always taking a new ratio SM of resistance to gravity, and obtain a new difference MN. But draw the positive differences on one side of the straight line SM and the negative differences on the other, and through points N, N, N draw the regular curve NNN cutting the straight line SMMM in X, and then SX will be the true ratio of the resistance to the gravity, which it was required to find. From this ratio the length DF is to be obtained by calculation; then the length that is to the assumed length DP as the length DF (found out by experiment) to the length DF (just found by computation) will be the true length DP. When this is found, there will be known both the curved line D*ra*F that the body describes and the body's velocity and resistance in every place.



However, the hypothesis that the resistance encountered by bodies is in the ratio of the velocity belongs more to mathematics than to nature.[a] In mediums wholly lacking in rigidity, the resistances encountered by bodies are as the squares of the velocities. For by the action of a swifter body, a motion that is greater in proportion to that greater velocity is communicated to a given quantity of the medium in a smaller time; and thus in an equal time, because a greater quantity of the medium is disturbed, a greater motion is communicated in proportion to the square of the velocity, and (by the second and third laws of motion) the resistance is as the motion communicated. Let us see, therefore, what kinds of motions arise from this law of resistance. **Scholium**

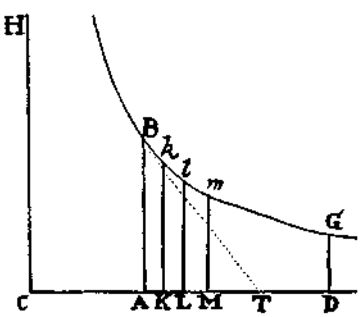
a. Ed. 1 and ed. 2 have an additional sentence: "This ratio obtains very nearly when bodies are moving very slowly in mediums having some rigidity." In Newton's annotated copy of ed. 2, "very nearly" is changed to "more closely."

SECTION 2

The motion of bodies that are resisted as the squares of the velocities

Proposition 5 Theorem 3 *If the resistance of a body is proportional to the square of the velocity and if the body moves through a homogeneous medium by its inherent force alone and if the times are taken in a geometric progression going from the smaller to the greater terms, I say that the velocities at the beginning of each of the times are inversely in that same geometric progression and that the spaces described in each of the times are equal.*

For since the resistance of the medium is proportional to the square of the velocity, and the decrement of the velocity is proportional to the resistance, if the time is divided into innumerable equal particles, the squares of the velocities at each of the beginnings of the times will be proportional to the differences of those same velocities. Let the particles of time be AK, KL, LM, . . . , taken in the straight line CD, and erect perpendiculars AB, Kk, Ll, Mm, . . . , meeting the hyperbola BklmG (described with center C and rectangular asymptotes CD and CH) in B, k, l, m, . . . ; then AB will be to Kk as CK to CA, and by separation [or dividendo] AB − Kk to Kk as AK to CA, and by alternation [or alternando] AB − Kk to AK as Kk to CA, and thus as AB × Kk to AB × CA. Hence, since AK and AB × CA are given, AB − Kk will be as AB × Kk; and ultimately, when AB and Kk come together, as AB^2. And by a similar argument Kk − Ll, Ll − Mm, . . . will be as $\text{K}k^2$, $\text{L}l^2$, ... The squares of lines AB, Kk, Ll, Mm, therefore, are as their differences; and on that account, since the squares of the velocities were also as their differences, the progression of both will be similar. It follows from what has been proved that the areas described by these lines are also in a progression entirely similar to that of the spaces described by the velocities. Therefore, if the velocity at the beginning of the first time AK is represented by line AB, and the velocity at the beginning of the second time KL by line Kk, and the length described in the first time is represented by area AKkB, then all the subsequent velocities will be represented by the subsequent lines Ll, Mm, . . . ,



and the lengths described will be represented by areas K*l*, L*m*, ... And by composition [or componendo], if the whole time is represented by the sum of its parts AM, the whole length described will be represented by the sum of its parts AM*m*B. Now imagine time AM to be divided into parts AK, KL, LM, . . . in such a way that CA, CK, CL, CM, . . . are in a geometric progression; then those parts will be in the same progression, and the velocities AB, K*k*, L*l*, M*m*, . . . will be in the same progression inverted, and the spaces described A*k*, K*l*, L*m*, . . . will be equal. Q.E.D.

COROLLARY 1. Therefore it is evident that if the time is represented by any part AD of the asymptote, and the velocity at the beginning of the time by ordinate AB, then the velocity at the end of the time will be represented by ordinate DG, and the whole space described will be represented by the adjacent hyperbolic area ABGD; and furthermore, the space that a body in a nonresisting medium could describe in the same time AD, with the first velocity AB, will be represented by the rectangle AB $\times$ AD.

COROLLARY 2. Hence the space described in a resisting medium is given by taking that space to be in the same proportion to the space which could be described simultaneously with a uniform velocity AB in a nonresisting medium as the hyperbolic area ABGD is to the rectangle AB $\times$ AD.

COROLLARY 3. The resistance of the medium is also given by setting it to be, at the very beginning of the motion, equal to the uniform centripetal force that in a nonresisting medium could generate the velocity AB in a falling body in the time AC. For if BT is drawn, touching the hyperbola in B and meeting the asymptote in T, the straight line AT will be equal to AC and will represent the time in which the first resistance uniformly continued could annul the whole velocity AB.

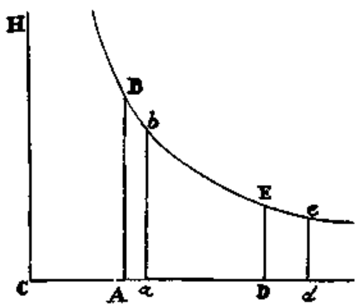
COROLLARY 4. And hence the proportion of this resistance to the force of gravity or to any other given centripetal force is also given.

COROLLARY 5. And conversely, if the proportion of the resistance to any given centripetal force is given, the time AC is given in which a centripetal force equal to the resistance could generate any velocity AB; and hence point B is given, through which the hyperbola with asymptotes CH and CD must be described, as is also the space ABGD which the body, beginning its motion with that velocity AB, can describe in any time AD in a homogeneous resisting medium.

Proposition 6
Theorem 4

Equal homogeneous spherical bodies that are resisted in proportion to the square of the velocity, and are carried forward by their inherent forces alone, will, in times that are inversely as the initial velocities, always describe equal spaces, and lose parts of their velocities proportional to the wholes.

Describe any hyperbola BbEe, with rectangular asymptotes CD and CH, which cuts perpendiculars AB, ab, DE, and de in B, b, E, and e; and represent the initial velocities by perpendiculars AB and DE and the times by lines Aa and Dd. Therefore Aa is to Dd as (by hypothesis) DE is to AB, and as (from the nature of the hyperbola) CA is to CD, and by composition [or componendo] as Ca is to Cd. Hence areas ABba and DEed, that is, the spaces described, are equal to each other, and the first velocities AB and DE are proportional to the ultimate velocities ab and de, and therefore, by separation [or dividendo], also to the lost parts of those velocities AB $-$ ab and DE $-$ de. Q.E.D.



Proposition 7
Theorem 5

Spherical bodies that are resisted in proportion to the squares of the velocities will, in times that are as the first motions directly and the first resistances inversely, lose parts of the motions proportional to the wholes and will describe spaces proportional to those times and the first velocities jointly.

For the lost parts of the motions are as the resistances and the times jointly. Therefore, for those parts to be proportional to the wholes, the resistance and time jointly must be as the motion. Accordingly, the time will be as the motion directly and the resistance inversely. Therefore, if the particles of times are taken in this ratio, the bodies will always lose particles of their motions proportional to the wholes and thus will retain velocities always proportional to their first velocities. And because the ratio of the velocities is given, they will always describe spaces that are as the first velocities and the times jointly. Q.E.D.

COROLLARY 1. Therefore, if equally swift bodies are resisted in proportion to the squares of their diameters, then homogeneous globes moving with any velocities will, in describing spaces proportional to their diameters, lose parts of their motions proportional to the wholes. For the motion of each globe will be as its velocity and mass jointly, that is, as its velocity and the cube of its diameter; the resistance (by hypothesis) will be as the square of the diameter and the square of the velocity jointly; and the time (by this proposition) is in the former ratio directly and the latter ratio inversely, that is, as the diameter directly and the velocity inversely; and thus the space, being proportional to the time and the velocity, is as the diameter.

COROLLARY 2. If equally swift bodies are resisted in proportion to the $\frac{3}{2}$ powers of the diameters, then homogeneous globes moving with any velocities will, in describing spaces that are as the $\frac{3}{2}$ powers of the diameters, lose parts of motions proportional to the wholes.

COROLLARY 3. And universally, if equally swift bodies are resisted in the ratio of any power of the diameters, the spaces in which homogeneous globes moving with any velocities will lose parts of their motions proportional to the wholes will be as the cubes of the diameters divided by that power. Let the diameters be D and E; and if the resistances, when the velocities are supposed equal, are as D^n and E^n, then the spaces in which the globes, moving with any velocities, will lose parts of their motions proportional to the wholes will be as D^{3-n} and E^{3-n}. And therefore homogeneous globes, in describing spaces proportional to D^{3-n} and E^{3-n}, will retain velocities in the same ratio to each other that they had at the beginning.

COROLLARY 4. But if the globes are not homogeneous, the space described by the denser globe must be augmented in proportion to the density. For the motion, with an equal velocity, is greater in proportion to the density, and the time (by this proposition) is increased in proportion to the motion directly, and the space described is increased in proportion to the time.

COROLLARY 5. And if the globes move in different mediums, the space in the medium that, other things being equal, resists more will have to be decreased in proportion to the greater resistance. For the time (by this proposition) will be decreased in proportion to the increase of the resistance, and the space will be decreased in proportion to the time.

Lemma 2[a] *The moment of a generated quantity is equal to the moments of each of the generating roots multiplied continually by the exponents of the powers of those roots and by their coefficients.*

I call "generated" every quantity that is, without addition or subtraction, generated from any roots or terms: in arithmetic by multiplication, division, or extraction of roots; in geometry by the finding either of products and roots or of extreme and mean proportionals. Quantities of this sort are products, quotients, roots, rectangles, squares, cubes, square roots, cube roots, and the like.[b] I here consider these quantities as indeterminate and variable, and increasing or decreasing as if by a continual motion or flux; and it is their

a. Newton's use of "terminus" and "latus" for "root" is of particular interest in lem. 2 and its cases, corollaries, and scholium. "Radix" appears only twice and is unchanged from edition to edition, but Newton tends to replace the "terminus" ("term," "root") of ed. 1 with the "latus" ("side," "root") of ed. 2 and ed. 3. In the statement of the lemma, for example, ed. 1 has "momentis Terminorum singulorum generantium" ("the moments of the individual generating terms," i.e., "the moments of each of the generating roots") and "eorundem laterum indices dignitatum" ("the exponents of the powers of the same sides," i.e., "the exponents of the powers of those roots"). Thus "terminus" and "latus" are obviously synonyms. In ed. 2 and ed. 3, however, "laterum" ("sides," "roots") is substituted for "Terminorum" ("terms," "roots"). In the first sentence of the explanation, where ed. 1 has "ex Terminis quibuscunque" ("from any terms," i.e., "from any roots"), ed. 2 and ed. 3 have "ex lateribus vel terminis quibuscunque" ("from any sides or terms," i.e., "from any roots or terms"). As the explanation proceeds, ed. 1 has, like ed. 2 and ed. 3, "extractionem radicum" ("extraction of roots"), "contentorum & laterum" ("of products and roots"), "Radices" ("roots"), and "latera quadrata, latera cubica" ("square roots, cube roots"), but ed. 1 has "Termini" and "Terminum" where ed. 2 and ed. 3 have "Lateris" and "latus" in the last sentence of the first paragraph: "And the coefficient of each generating root is the quantity that results from dividing the generated quantity by this root." In corol. 1, on the other hand, all the editions have "terminus" (with the ordinary sense of "term," not with the sense of "root"), while all have "latus" (with the sense of "root") in cases 1 and 2 and corols. 2 and 3. "Terminus" also occurs, in the phrase "in terminis surdis" ("in surd terms"), in the scholium of ed. 1 and ed. 2, which is, as will be seen below, very different from that of ed. 3, where, however, "quantitatibus surdis" ("surd quantities") is at least comparable, especially since "surdis" ("surd") appears nowhere else in all the editions of the *Principia*.

b. In the Latin text of this lemma, Newton referred to roots in two senses. The first occurs in the opening sentence, where he writes of "extraction of roots," using the Latin term "radix," or "root." The second occurs in the next sentence, where he writes of "products, quotients, roots, rectangles, squares, cubes, square roots, cube roots, and the like." Here both senses of "roots" appear in a single sentence, the first as "radices" (or "roots"), the second as "latera quadrata, latera cubica" (*lit.* "square sides" and "cubic sides"). In the geometric language of algebra, in which a "rectangle" of A and B indicates the product of two unequal quantities A and B as the area of a rectangle whose sides are A and B, the square root and cube root have similar geometric expression. Thus the square root of A is the side of a square whose area is A, while the cube root of A is the "side" (actually the edge) of a cube whose volume is A.

In his *Lexicon Technicum* (London, 1704), John Harris explained these two different mathematical senses of the word "root." An "Unknown Quantity in an Algebraick Equation," he wrote, "is often called the Root." This is the sense of the word as it appears in the first sentence of the lemma. But, as Harris explained, a root is also "whatever Quantity being multiplied by it self produces a Square" and when instantaneous increments or decrements that I mean by the word "moments," in such a way that increments are considered as added or positive moments, and decrements as subtracted or negative moments. But take care: do not understand them to be finite particles! [c]Finite particles are not moments, but the very quantities generated from the moments.[c] They must be understood to be the just-now nascent beginnings of finite magnitudes. For in this lemma the magnitude of moments is not regarded, but only their first proportion when nascent. It comes to the same thing if in place of moments there are used either the velocities of increments and decrements (which it is also possible to call motions, mutations, and fluxions of quantities) or any finite quantities proportional to these velocities. And the coefficient of each generating root is the quantity that results from dividing the generated quantity by this root.

Therefore, the meaning of this lemma is that if the moments of any quantities A, B, C, . . . increasing or decreasing by a continual motion, or the velocities of mutation which are proportional to these moments are called a, b, c, . . . , then the moment or mutation of the generated rectangle AB would be $a\mathrm{B} + b\mathrm{A}$, and the moment of the generated solid ABC would be $a\mathrm{BC} + b\mathrm{AC} + c\mathrm{AB}$, and the moments of the generated powers A^2, A^3, A^4, $\mathrm{A}^{1/2}$, $\mathrm{A}^{3/2}$, $\mathrm{A}^{1/3}$, $\mathrm{A}^{2/3}$, A^{-1}, A^{-2}, and $\mathrm{A}^{-1/2}$ would be $2a\mathrm{A}$, $3a\mathrm{A}^2$, $4a\mathrm{A}^3$, $\frac{1}{2}a\mathrm{A}^{-1/2}$, $\frac{3}{2}a\mathrm{A}^{1/2}$, $\frac{1}{3}a\mathrm{A}^{-2/3}$, $\frac{2}{3}a\mathrm{A}^{-1/3}$, $-a\mathrm{A}^{-2}$, $-2a\mathrm{A}^{-3}$, and $-\frac{1}{2}a\mathrm{A}^{-3/2}$ respectively. And generally, the moment of any power $\mathrm{A}^{\frac{n}{m}}$ would be $\frac{n}{m}a\mathrm{A}^{\frac{n-m}{m}}$. Likewise, the moment of the generated quantity $\mathrm{A}^2\mathrm{B}$ would be $2a\mathrm{AB} + b\mathrm{A}^2$, and the moment of the generated quantity $\mathrm{A}^3\mathrm{B}^4\mathrm{C}^2$ would be $3a\mathrm{A}^2\mathrm{B}^4\mathrm{C}^2 +$

once again "multiplied by that first Quantity produces a Cube, *&c.*" These, he said, are called "Square, Cube . . . *Root.*"

Even without any knowledge of the geometric sense of algebra, one might easily guess that Newton is referring to square and cube roots in the phrase "products, quotients, . . . squares, cubes, square sides, cube sides, and the like." Yet Andrew Motte, in his English translation (London, 1729), rendered these terms literally as "products, quotients, roots, rectangles, squares, cubes, square and cubic sides, and the like," which was carried over into the Motte-Cajori version. The marquise du Châtelet knew better and in her French translation (Paris, 1756) wrote, just as we would today, of "les produits, les quotiens, les racines, les rectangles, les quarrés, les cubes, les racines quarrées, & les racines cubes."

cc. Ed. 1 has: "Moments, as soon as they are of finite magnitude, cease to be moments. For being finite is somewhat incompatible with their continual increment or decrement." When one reads the "somewhat" ("aliquatenus": "to a certain extent," "in some respects") in the second of these sentences, one can understand why Newton decided to revise this portion of his explanation.

$4bA^3B^3C^2 + 2cA^3B^4C$, and the moment of the generated quantity $\frac{A^3}{B^2}$ or A^3B^{-2} would be $3aA^2B^{-2} - 2bA^3B^{-3}$, and so on. The lemma is proved as follows.

CASE 1. Any rectangle AB increased by continual motion, when the halves of the moments, $\frac{1}{2}a$ and $\frac{1}{2}b$, were lacking from the sides A and B, was $A - \frac{1}{2}a$ multiplied by $B - \frac{1}{2}b$, or $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$, and as soon as the sides A and B have been increased by the other halves of the moments, it comes out $A + \frac{1}{2}a$ multiplied by $B + \frac{1}{2}b$, or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. Subtract the former rectangle from this rectangle, and there will remain the excess $aB + bA$. Therefore by the total increments a and b of the sides there is generated the increment $aB + bA$ of the rectangle. Q.E.D.

CASE 2. Suppose that AB is always equal to G; then the moment of the solid ABC or GC (by case 1) will be $gC + cG$, that is (if AB and $aB + bA$ are written for G and g), $aBC + bAC + cAB$. And the same is true of the solid contained under any number of sides [or the product of any number of terms]. Q.E.D.

CASE 3. Suppose that the sides A, B, and C are always equal to one another; then the moment $aB + bA$ of A^2, that is, of the rectangle AB, will be $2aA$, while the moment $aBC + bAC + cAB$ of A^3, that is, of the solid ABC, will be $3aA^2$. And by the same argument, the moment of any power A^n is naA^{n-1}. Q.E.D.

CASE 4. Hence, since $\frac{1}{A}$ multiplied by A is 1, the moment of $\frac{1}{A}$ multiplied by A together with $\frac{1}{A}$ multiplied by a will be the moment of 1, that is, nil. Accordingly, the moment of $\frac{1}{A}$ or of A^{-1} is $-\frac{a}{A^2}$. And in general, since $\frac{1}{A^n}$ multiplied by A^n is 1, the moment of $\frac{1}{A^n}$ multiplied by A^n together with $\frac{1}{A^n}$ multiplied by naA^{n-1} will be nil. And therefore the moment of $\frac{1}{A^n}$ or A^{-n} will be $-\frac{na}{A^{n+1}}$. Q.E.D.

CASE 5. And since $A^{1/2}$ multiplied by $A^{1/2}$ is A, the moment of $A^{1/2}$ multiplied by $2A^{1/2}$ will be a, by case 3; and thus the moment of $A^{1/2}$ will be $\frac{a}{2A^{1/2}}$ or $\frac{1}{2}aA^{-1/2}$. And in general, if $A^{\frac{m}{n}}$ is supposed equal to B, A^m will be equal to B^n, and hence maA^{m-1} will be equal to nbB^{n-1}, and maA^{-1} will be equal to nbB^{-1} or $nbA^{\frac{-m}{n}}$, and thus $\frac{m}{n}aA^{\frac{m-n}{n}}$ equal to b, that is, equal to the moment of $A^{\frac{m}{n}}$. Q.E.D.

CASE 6. Therefore the moment of any generated quantity A^mB^n is the moment of A^m multiplied by B^n, together with the moment of B^n multiplied by A^m, that is, $maA^{m-1}B^n+nbB^{n-1}A^m$; and this is so whether the exponents m and n of the powers are whole numbers or fractions, whether positive or negative. And it is the same for a solid contained by more than two terms raised to powers. Q.E.D.

COROLLARY 1. Hence in continually proportional quantities, if one term is given, the moments of the remaining terms will be as those terms multiplied by the number of intervals between them and the given term. Let A, B, C, D, E, and F be continually proportional; then, if the term C is given, the moments of the remaining terms will be to one another as −2A, −B, D, 2E, and 3F.

COROLLARY 2. And if in four proportionals the two means are given, the moments of the extremes will be as those same extremes. The same is to be understood of the sides of any given rectangle.

COROLLARY 3. And if the sum or difference of two squares is given, the moments of the sides will be inversely as the sides.

Scholium

[d]In a certain letter written to our fellow Englishman Mr. J. Collins on 10 December 1672, when I had described a method of tangents that I suspected to be the same as Sluse's method, which at that time had not yet been made public, I added: "This is one particular, or rather a corollary, of a general

dd. In ed. 1 this scholium reads: "In correspondence which I carried on ten years ago with the very able geometer G. W. Leibniz, I indicated that I was in possession of a method of determining maxima and minima, drawing tangents, and performing similar operations, and that the method worked for surd as well as rational terms. I concealed this method under an anagram comprising this sentence: 'Given an equation involving any number of fluent quantities, to find the fluxions, and vice versa.' The distinguished gentleman wrote back that he too had come upon a method of this kind, and he communicated his method, which hardly differed from mine except in the forms of words and notations. The foundation of both methods is contained in this lemma." In ed. 2 the scholium is exactly the same except that "and the concept of the generation of quantities" is added at the end of the penultimate sentence.

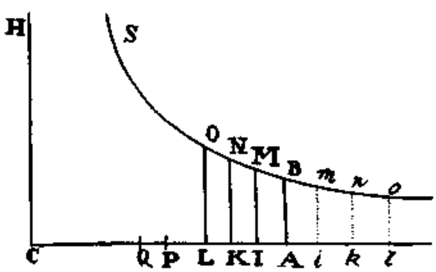
For the principal texts with interpretative comments on the Newton-Leibniz controversy over priority in the invention of the calculus, see *The Mathematical Papers of Isaac Newton*, ed. D. T. Whiteside (Cambridge: Cambridge University Press, 1967–1981), vol. 8, esp. pp. 469–697; also A. Rupert Hall, *Philosophers at War: The Quarrel between Newton and Leibniz* (Cambridge: Cambridge University Press, 1980).

method, which extends, without any troublesome calculation, not only to the drawing of tangents to all curve lines, whether geometric or mechanical or having respect in any way to straight lines or other curves, but also to resolving other more abstruse kinds of problems concerning curvatures, areas, lengths, centers of gravity of curves, . . . , and is not restricted (as Hudde's method of maxima and minima is) only to those equations which are free from surd quantities. I have interwoven this method with that other by which I find the roots of equations by reducing them to infinite series." So much for the letter. And these last words refer to the treatise that I had written on this topic in 1671. The foundation of this general method is contained in the preceding lemma.[d]

Proposition 8
Theorem 6

If a body, acted on by gravity uniformly, goes straight up or down in a uniform medium, and the total space described is divided into equal parts, and the absolute forces at the beginnings of each of the parts are found (adding the resistance of the medium to the force of gravity when the body is ascending, or subtracting it when the body is descending), I say that those absolute forces are in a geometric progression.

Represent the force of gravity by the given line AC; the resistance, by the indefinite line AK; the absolute force in the descent of the body, by the difference KC; the velocity of the body, by the line AP, which is the mean proportional between AK and AC, and thus is as the square root of the resistance; the increment of the resistance occurring in a given particle of time, by the line-element KL; and the simultaneous increment of the velocity, by the line-element PQ; then with center C and rectangular asymptotes CA and CH, describe any hyperbola BNS, meeting the erected perpendiculars AB, KN, and LO in B, N, and O. Since AK is as AP^2, the moment KL of AK will be as the moment $2AP \times PQ$ of AP^2, that is, as AP multiplied by KC, since the increment PQ of the velocity (by the second law of motion) is proportional to the generating force KC. Compound the ratio of KL with the ratio of KN, and the rectangle $KL \times KN$ will become as $AP \times KC \times KN$—that is, because the rectangle $KC \times KN$ is given, as



AP. But the ultimate ratio of the hyperbolic area KNOL to the rectangle KL × KN, when points K and L come together, is the ratio of equality. Therefore that evanescent hyperbolic area is as AP. Hence the total hyperbolic area ABOL is composed of the particles KNOL, which are always proportional to the velocity AP, and therefore this area is proportional to the space described with this velocity. Now divide that area into equal parts ABMI, IMNK, KNOL, . . . , and the absolute forces AC, IC, KC, LC, . . . will be in a geometric progression. Q.E.D.

And by a similar argument, if—in the ascent of the body—equal areas AB*mi*, *imnk*, *knol*, . . . are taken on the opposite side of point A, it will be manifest that the absolute forces AC, *i*C, *k*C, *l*C, . . . are continually proportional. And thus, if all the spaces in the ascent and descent are taken equal, all the absolute forces *l*C, *k*C, *i*C, AC, IC, KC, LC, . . . will be continually proportional. Q.E.D.

COROLLARY 1. Hence, if the space described is represented by the hyperbolic area ABNK, the force of gravity, the velocity of the body, and the resistance of the medium can be represented by lines AC, AP, and AK respectively, and vice versa.

COROLLARY 2. And line AC represents the greatest velocity that the body can ever acquire by descending infinitely.

COROLLARY 3. Therefore, if for a given velocity the resistance of the medium is known, the greatest velocity will be found by taking its ratio to the given velocity as the square root of the ratio of the force of gravity to that known resistance of the medium.[a]

Proposition 9
Theorem 7

Given what has already been proved, I say that if the tangents of the angles of a sector of a circle and of a hyperbola are taken proportional to the velocities, the radius being of the proper magnitude, the whole time [aa]*of ascending to the highest*

a. Ed. 1 has two additional corollaries as follows: "Corol. 4. But also the particle of time wherein the minimally small particle of space NKLO is described in descent is as the rectangle KN × PQ. For since the space NKLO is as the velocity multiplied by the particle of time, the particle of time will be as that space divided by the velocity, that is, as the minimally small rectangle KN × KL divided by AP. For KL, above, was as AP × PQ. Therefore the particle of time is as KN × PQ, or what comes to the same, as $\frac{\mathrm{PQ}}{\mathrm{CK}}$. Q.E.D."

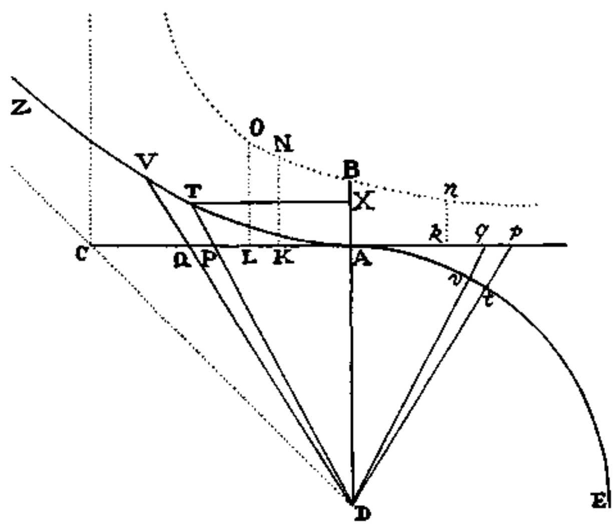
"Corollary 5. By the same argument the particle of time wherein the particle of space *nklo* is described in ascent is as $\frac{pq}{Ck}$."

aa. Ed. 1 and ed. 2 have "of the future ascent."

place[a] *will be as the sector of the circle, and the whole time* [b]*of descending from the highest place*[b] *will be as the sector of the hyperbola.*

Draw AD perpendicular and equal to the straight line AC, which represents the force of gravity. With center D and semidiameter AD describe the quadrant AtE of a circle and the rectangular hyperbola AVZ having axis AX, principal vertex A, and asymptote DC. Draw Dp and DP, and the sector AtD of the circle will be as [c]the whole time of ascending to the highest place,[c] and the sector ATD of the hyperbola will be as [d]the whole time of descending from the highest place,[d] provided that the tangents Ap and AP of the sectors are as the velocities.

CASE 1. Draw Dvq cutting off the moments or the minimally small particles tDv and qDp, described simultaneously, of the sector ADt and of the triangle ADp. Since those particles, because of the common angle D, are



as the squares of the sides, particle tDv will be as $\dfrac{q\mathrm{D}p \times t\mathrm{D}^2}{p\mathrm{D}^2}$, that is, because tD is given, as $\dfrac{q\mathrm{D}p}{p\mathrm{D}^2}$. But $p\mathrm{D}^2$ is $\mathrm{AD}^2 + \mathrm{A}p^2$, that is, $\mathrm{AD}^2 + \mathrm{AD} \times \mathrm{A}k$, or $\mathrm{AD} \times \mathrm{C}k$; and $q\mathrm{D}p$ is $\frac{1}{2}\mathrm{AD} \times pq$. Therefore particle tDv of the sector is as $\dfrac{pq}{\mathrm{C}k}$, that is, directly as the minimally small decrement pq of the velocity and

bb. Ed. 1 and ed. 2 have "of the past descent."

cc. Ed. 1 and ed. 2 have "the time of the whole future ascent."

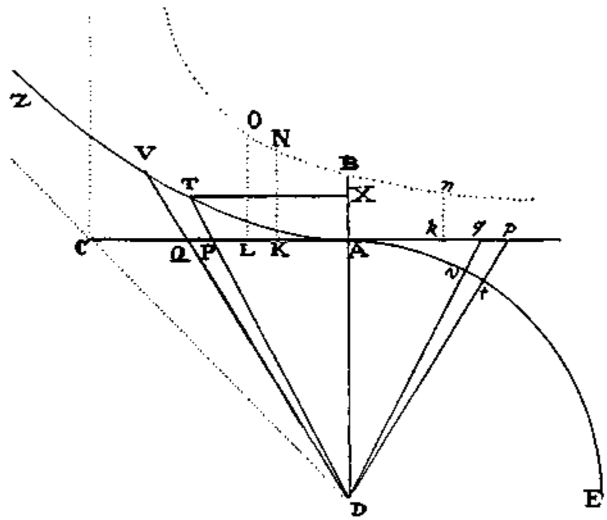
dd. Ed. 1 and ed. 2 have "the time of the whole past descent."

inversely as the force Ck that decreases the velocity, and thus as the particle of time corresponding to the decrement of the velocity. And by composition [or componendo] the sum of all the particles tDv in the sector ADt will be as the sum of the particles of time corresponding to each of the lost particles pq of the decreasing velocity Ap, until that velocity, decreased to nil, has vanished; that is, the whole sector ADt is as [e]the whole time of ascending to the highest place.[e] Q.E.D.

CASE 2. Draw DQV cutting off the minimally small particles TDV and PDQ of the sector DAV and of the triangle DAQ; and these particles will be to each other as DT^2 to DP^2, that is (if TX and AP are parallel), as DX^2 to DA^2 or TX^2 to AP^2, and by separation [or dividendo] as $DX^2 - TX^2$ to $DA^2 - AP^2$. But from the nature of the hyperbola, $DX^2 - TX^2$ is AD^2, and by hypothesis AP^2 is $AD \times AK$. Therefore the particles are to each other as AD^2 to $AD^2 - AD \times AK$, that is, as AD to AD − AK or AC to CK; and thus the particle TDV of the sector is $\frac{PDQ \times AC}{CK}$, and hence, because AC and AD are given, as $\frac{PQ}{CK}$, that is, directly as the increment of the velocity and inversely as the force generating the increment, and thus as the particle of time corresponding to the increment. And by composition [or componendo] the sum of the particles of time in which all the particles PQ of the velocity AP are generated will be as the sum of the particles of the sector ATD, that is, the whole time will be as the whole sector. Q.E.D.

COROLLARY 1. Hence, if AB is equal to a fourth of AC, the space that a body describes by falling in any time will be in the same ratio to the space that the body can describe by progressing uniformly in that same time with its greatest velocity AC as the ratio of area ABNK (which represents the space described in falling) to area ATD (which represents the time). For, since AC is to AP as AP to AK, it follows (by book 2, lem. 2, corol. 1) that LK will be to PQ as 2AK to AP, that is, as 2AP to AC, and hence LK will be to ½PQ as AP to ¼AC or AB; KN is also to AC or AD as AB to CK; and thus, from the equality of the ratios [or ex aequo], LKNO will be to DPQ as AP to CK. But DPQ was to DTV as CK to AC. Therefore, once again by the equality of the ratios [or ex aequo], LKNO is to DTV as AP to AC, that is, as the velocity of the falling body to the greatest velocity that the

ee. Ed. 1 and ed. 2 have "the time of the whole future ascent."



body can acquire in falling. Since, therefore, the moments LKNO and DTV of areas ABNK and ATD are as the velocities, all the parts of those areas generated simultaneously will be as the spaces described simultaneously, and thus the whole areas ABNK and ATD generated from the beginning will be as the whole spaces described from the beginning of the descent. Q.E.D.

COROLLARY 2. The same result follows for the space described in ascent: namely, the whole space is to the space described in the same time with a uniform velocity AC as area AB*nk* is to sector AD*t*.

COROLLARY 3. The velocity of a body falling in time ATD is to the velocity that it would acquire in the same time in a nonresisting space as the triangle APD to the hyperbolic sector ATD. For the velocity in a nonresisting medium would be as time ATD, and in a resisting medium is as AP, that is, as triangle APD. And the velocities at the beginning of the descent are equal to each other, as are those areas ATD and APD.

COROLLARY 4. By the same argument, the velocity in the ascent is to the velocity with which the body in the same time in a nonresisting space could lose its whole ascending motion as the triangle A*p*D is to the sector A*t*D of the circle, or as the straight line A*p* is to the arc A*t*.

COROLLARY 5. Therefore the time in which a body, by falling in a resisting medium, acquires the velocity AP is to the time in which it could acquire its greatest velocity AC, by falling in a nonresisting space, as sector ADT to triangle ADC; and the time in which it could lose the velocity A*p* by ascending in a resisting medium is to the time in which it could lose the same velocity by ascending in a nonresisting space as arc A*t* is to its tangent A*p*.

Corollary 6. Hence, from the given time, the space described by ascent or descent is given. For the greatest velocity of a body descending infinitely is given (by book 2, prop. 8, corols. 2 and 3), and hence the time is given in which a body could acquire that velocity by falling in a nonresisting space. And if sector ADT or AD*t* is taken to be to triangle ADC in the ratio of the given time to the time just found, there will be given both the velocity AP or A*p* and the area ABNK or AB*nk*, which is to the sector ADT or AD*t* as the required space is to the space that can be described uniformly in the given time with that greatest velocity which has already been found.

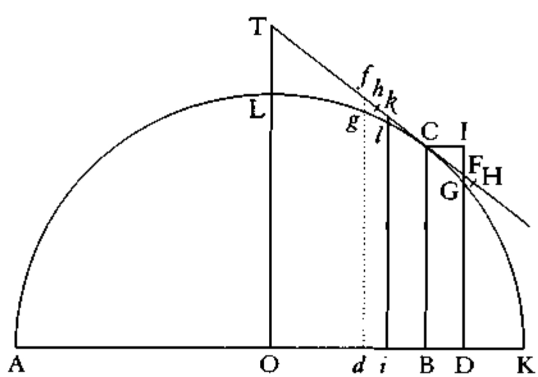
Corollary 7. And working backward, the time AD*t* or ADT will be given from the given space AB*nk* or ABNK of ascent or descent.

Proposition 10
Problem 3

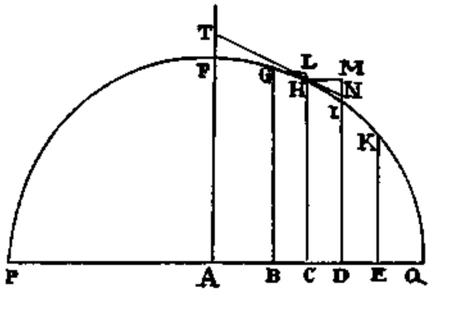
Let a uniform force of gravity tend straight toward the plane of the horizon, and let the resistance be as the density of the medium and the square of the velocity jointly; it is required to find, in each individual place, the density of the medium that makes the body move in any given curved line and also the velocity of the body and resistance of the medium.

[a]Let PQ be the plane of the horizon, perpendicular to the plane of the figure; PFHQ a curved line meeting this plane in points P and Q; G, H, I,

aa. Ed. 1 has: "Let AK be the plane of the horizon, perpendicular to the plane of the figure; ACK a curved line; C a body moving along the line; and FC*f* a straight line touching it in C. And suppose that body C now goes forward from A to K along the line ACK and now goes back along the same line and that in going forward it is impeded by the medium and in going back is equally assisted, so that in the same places the velocity of the body as it goes forward and back is always the same.



"And in equal times let the body as it goes forward describe the minimally small arc CG, and let the body as it goes back describe arc C*g*, and let CH and C*h* be equal rectilinear lengths which bodies and K four places of the body as it goes in the curve from F to Q; and GB, HC, ID, and KE four parallel ordinates dropped from these points to the horizon and standing upon the horizontal line PQ at points B, C, D, and E; and let BC, CD, and DE be distances between the ordinates equal to one another. From points G and H draw the straight lines GL and HN touching the curve in G and H, and meeting in L and N the ordinates CH and DI produced upward; and complete the parallelogram HCDM. Then the times in which the body describes arcs GH and HI will be as the square roots of the distances LH and NI which the body could describe in those times by falling from the tangents; and the velocities will be directly as GH and HI (the lengths described) and inversely as the times. Represent the times by T and *t*,



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moving away from place C would describe in these times without the actions of the medium and of gravity, and from points C, G, and g to the horizontal plane AK drop perpendiculars CB, GD, and gd, letting GD and gd meet the tangent in F and f. Through the resistance of the medium it comes about that the body as it goes forward describes, instead of length CH, only length CF, and through the force of gravity the body is transferred from F to G, and thus line-element HF and line-element FG are generated simultaneously, the first by the force of resistance and the second by the force of gravity. Accordingly (by book 1, lem. 10), line-element FG is as the force of gravity and the square of the time jointly and thus (since the gravity is given) as the square of the time, and line-element HF is as the resistance and the square of the time, that is, as the resistance and line-element FG. And hence the resistance comes to be as HF directly and FG inversely, or as $\frac{\mathrm{HF}}{\mathrm{FG}}$. This is so in the case of nascent line-elements. For in the case of line-elements of finite magnitude these ratios are not accurate.

"And by a similar argument fg is as the square of the time and thus, since the times are equal, is equal to FG, and the impulse by which the body going back is urged is as $\frac{hf}{fg}$. But the impulse upon the body as it goes back and the resistance to it as it goes forward are equal at the very beginning of the motion, and thus also $\frac{hf}{fg}$ and $\frac{\mathrm{HF}}{\mathrm{FG}}$, proportional to them, are equal, and therefore, because fg and FG are equal, hf and HF are also equal, and thus CF, CH (or Ch), and Cf are in arithmetic progression, and hence HF is half the difference between Cf and CF, and the resistance, which above was as $\frac{\mathrm{HF}}{\mathrm{FG}}$, is as $\frac{\mathrm{C}f-\mathrm{CF}}{\mathrm{FG}}$.

"But the resistance is as the density of the medium and the square of the velocity. And the velocity is as the described length CF directly and the time $\sqrt{\mathrm{FG}}$ inversely, that is, as $\frac{\mathrm{CF}}{\sqrt{\mathrm{FG}}}$, and thus the square of the velocity is as $\frac{\mathrm{CF}^2}{\mathrm{FG}}$. Therefore the resistance, proportional to $\frac{\mathrm{C}f-\mathrm{CF}}{\mathrm{FG}}$, is as the density of the medium and the velocities by $\frac{\mathrm{GH}}{\mathrm{T}}$ and $\frac{\mathrm{HI}}{t}$; and the decrement of the velocity occurring in time t will be represented by $\frac{\mathrm{GH}}{\mathrm{T}} - \frac{\mathrm{HI}}{t}$. This decrement arises from the resistance retarding the body and from the gravity accelerating the body. In a body falling and describing in its fall the space NI, gravity generates a velocity by which twice that space could have been described in the same time, as Galileo proved, that is, the velocity $\frac{2\mathrm{NI}}{t}$; but in a body describing arc HI, gravity increases the arc by only the length $\mathrm{HI} - \mathrm{HN}$ or $\frac{\mathrm{MI} \times \mathrm{NI}}{\mathrm{HI}}$, and thus generates only the velocity $\frac{2\mathrm{MI} \times \mathrm{NI}}{t \times \mathrm{HI}}$. Add this velocity to the above decrement, and the result is the decrement of the velocity arising from the resistance alone, namely $\frac{\mathrm{GH}}{\mathrm{T}} - \frac{\mathrm{HI}}{t} + \frac{2\mathrm{MI} \times \mathrm{NI}}{t \times \mathrm{HI}}$. And accordingly, since gravity generates the velocity $\frac{2\mathrm{NI}}{t}$ in the same time in a falling body, the resistance will be to the gravity as $\frac{\mathrm{GH}}{\mathrm{T}} - \frac{\mathrm{HI}}{t} + \frac{2\mathrm{MI} \times \mathrm{NI}}{t \times \mathrm{HI}}$ to $\frac{2\mathrm{NI}}{t}$, or as $\frac{t \times \mathrm{GH}}{\mathrm{T}} - \mathrm{HI} + \frac{2\mathrm{MI} \times \mathrm{NI}}{\mathrm{HI}}$ to 2NI.

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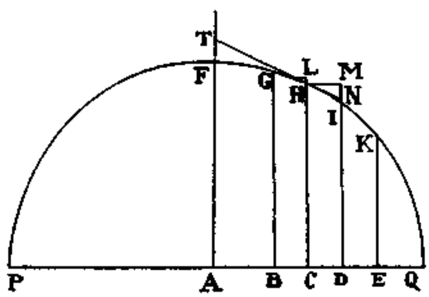
and $\frac{\mathrm{CF}^2}{\mathrm{FG}}$ jointly, and hence the density of the medium is as $\frac{\mathrm{C}f - \mathrm{CF}}{\mathrm{FG}}$ directly and $\frac{\mathrm{CF}^2}{\mathrm{FG}}$ inversely, that is, as $\frac{\mathrm{C}f - \mathrm{CF}}{\mathrm{CF}^2}$. Q.E.I.

"Corollary 1. And hence it is gathered that if $\mathrm{C}k$ on $\mathrm{C}f$ is taken as equal to CF and the perpendicular ki is dropped to the horizontal plane AK, cutting the curve ACK in l, the density of the medium will come to be as $\frac{\mathrm{FG} - kl}{\mathrm{CF} \times (\mathrm{FG} + kl)}$. For $f\mathrm{C}$ will be to $k\mathrm{C}$ as $\sqrt{fg}$ or $\sqrt{\mathrm{FG}}$ to $\sqrt{kl}$, and by separation [or dividendo] fk will be to $k\mathrm{C}$, that is, $\mathrm{C}f - \mathrm{CF}$ to CF, as $\sqrt{\mathrm{FG}} + \sqrt{kl}$ to $\sqrt{kl}$, that is, if both terms are multiplied by $\sqrt{\mathrm{FG}} + \sqrt{kl}$, as $\mathrm{FG} - kl$ to $kl + \sqrt{(\mathrm{FG} \times kl)}$, or to $\mathrm{FG} + kl$. For the first ratio of the nascent quantities $kl + \sqrt{(\mathrm{FG} \times kl)}$ and $\mathrm{FG} + kl$ is that of equality. And so let $\frac{\mathrm{FG} - kl}{\mathrm{FG} + kl}$ be written for $\frac{\mathrm{C}f - \mathrm{CF}}{\mathrm{CF}}$, and the density of the medium, which was as $\frac{\mathrm{C}f - \mathrm{CF}}{\mathrm{CF}^2}$, will turn out to be as $\frac{\mathrm{FG} - kl}{\mathrm{CF} \times (\mathrm{FG} + kl)}$.

"Corollary 2. Hence, since 2HF and $\mathrm{C}f - \mathrm{CF}$ are equal and FG and kl (because of the ratio of equality) compose 2FG, 2HF will be to CF as $\mathrm{FG} - kl$ to 2FG, and hence HF will be to FG, that is, the resistance will be to the gravity, as the rectangle $\mathrm{CF} \times (\mathrm{FG} - kl)$ to $4\mathrm{FG}^2$."

The demonstration in ed. 1 is incorrect, and the error was brought to Newton's attention only after the corresponding pages in ed. 2 had been printed off. For details see the Guide to the present translation, §7.3; also *The Mathematical Papers of Isaac Newton*, ed. D. T. Whiteside (Cambridge: Cambridge University Press, 1967–1981), 8:312–424; *The Correspondence of Isaac Newton*, vol. 5, ed. A. Rupert Hall and Laura Tilling (Cambridge: published for the Royal Society by Cambridge University Press, 1975); A. Rupert Hall, "Correcting the *Principia*," *Osiris* 13 (1958): 291–326; I. Bernard Cohen, *Introduction to Newton's "Principia"* (Cambridge, Mass.: Harvard University Press; Cambridge: Cambridge University Press, 1971), pp. 236–238.

Now for the abscissas CB, CD, and CE write $-o$, o, and $2o$. For the ordinate CH write P, and for MI write any series $Qo + Ro^2 + So^3 + \cdots$. And all the terms of the series after the first, namely $Ro^2 + So^3 + \cdots$, will be NI, and the ordinates DI, EK, and BG will be $P - Qo - Ro^2 - So^3 - \cdots$, $P - 2Qo - 4Ro^2 - 8So^3 - \cdots$, and $P + Qo - Ro^2 + So^3 - \cdots$ respectively. And by squaring the differences of the ordinates BG − CH and CH − DI and by adding to the resulting squares the squares of BC and CD, there will result the squares of the arcs GH and HI: $o^2 + Q^2o^2 - 2QRo^3 + \cdots$ and $o^2 + Q^2o^2 + 2QRo^3 + \cdots$. The roots of these, $o\sqrt{(1+Q^2)} - \dfrac{QRo^2}{\sqrt{(1+Q^2)}}$ and $o\sqrt{(1+Q^2)} + \dfrac{QRo^2}{\sqrt{(1+Q^2)}}$, are the arcs



GH and HI. Furthermore, if from ordinate CH half the sum of ordinates BG and DI is subtracted, and from ordinate DI half the sum of ordinates CH and EK is subtracted, the remainders will be the sagittas Ro^2 and $Ro^2 + 3So^3$ of arcs GI and HK. And these are proportional to the line-elements LH and NI, and thus as the squares of the infinitely small times T and t; and hence the ratio $\dfrac{t}{T}$ is $\sqrt{\dfrac{R + 3So}{R}}$ or $\dfrac{R + \frac{3}{2}So}{R}$; and if the values just found of $\dfrac{t}{T}$, GH, HI, MI, and NI are substituted in $\dfrac{t \times GH}{T} - HI + \dfrac{2MI \times NI}{HI}$, the result will be $\dfrac{3So^2}{2R}\sqrt{(1+Q^2)}$. And since 2NI is $2Ro^2$, the resistance will now be to the gravity as $\dfrac{3So^2}{2R}\sqrt{(1+Q^2)}$ to $2Ro^2$, that is, as $3S\sqrt{(1+Q^2)}$ to $4R^2$.

And the velocity is that with which a body going forth from any place H along tangent HN can then move in a vacuum in a parabola having a diameter HC and a latus rectum $\dfrac{HN^2}{NI}$ or $\dfrac{1+Q^2}{R}$.

And the resistance is as the density of the medium and the square of the velocity jointly, and therefore the density of the medium is as the resistance directly and the square of the velocity inversely, that is, as $\dfrac{3S\sqrt{(1+Q^2)}}{4R^2}$ directly and $\dfrac{1+Q^2}{R}$ inversely, that is, as $\dfrac{S}{R\sqrt{(1+Q^2)}}$. Q.E.I.

COROLLARY 1. If the tangent HN is produced in both directions until it meets any ordinate AF in T, $\frac{HT}{AC}$ will be equal to $\sqrt{(1+Q^2)}$ and thus can be written for $\sqrt{(1+Q^2)}$ above. And so the resistance will be to the gravity as $3S \times HT$ to $4R^2 \times AC$, the velocity will be as $\frac{HT}{AC\sqrt{R}}$, and the density of the medium will be as $\frac{S \times AC}{R \times HT}$.[a]

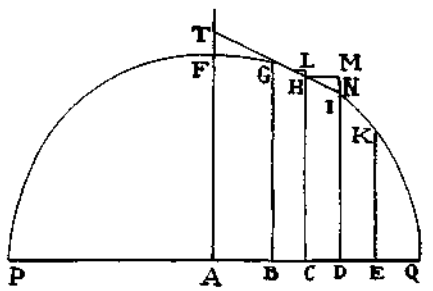
[b]COROLLARY 2. And hence, if the curved line PFHQ is defined by the relation between the base or abscissa AC and the ordinate CH, as is customary, and the value of the ordinate is resolved into a converging series, then the problem will be solved readily by means of the first terms of the series, as in the following examples.[b]

EXAMPLE 1. Let line PFHQ be a semicircle described on the diameter PQ, and let it be required to find the density of the medium that would make a projectile move in this semicircle.

Bisect diameter PQ in A; call AQ, n; AC, a; CH, e; and CD, o. Then DI^2 or $AQ^2 - AD^2$ will be $= n^2 - a^2 - 2ao - o^2$, or $e^2 - 2ao - o^2$, and when the root has been extracted by our method, DI will become $= e - \frac{ao}{e} - \frac{o^2}{2e} - \frac{a^2o^2}{2e^3} - \frac{ao^3}{2e^3} - \frac{a^3o^3}{2e^5} - \cdots$. Here write n^2 for $e^2 + a^2$, and DI will come out $= e - \frac{ao}{e} - \frac{n^2o^2}{2e^3} - \frac{an^2o^3}{2e^5} - \cdots$.

I divide series of this sort into successive terms in the following manner. What I call the first term is the term in which the infinitely small quantity o does not exist; the second, the term in which that quantity is of one dimension; the third, the term in which it is of two dimensions; the fourth, the term in which it is of three dimensions; and so on indefinitely. And the first term, which here is e, will always denote the length of the ordinate CH, standing at the beginning of the indefinite quantity o. The second term, which here is $\frac{ao}{e}$, will denote the difference between CH and DN, that is, the line-element MN, which is cut off by completing the parallelogram HCDM and thus always determines the position of the tangent HN; as, for example, in this case by taking MN to HM as $\frac{ao}{e}$ is to o, or a to e. The third term,

bb. In ed. 1 this is, with some variants, corol. 3.



which here is $\frac{n^2o^2}{2e^3}$, will designate the line-element IN, which lies between the tangent and the curve and thus determines the angle of contact IHN or the curvature that the curved line has in H. If that line-element IN is of a finite magnitude, it will be designated by the third term along with the terms following without limit. But if that line-element is diminished infinitely, the subsequent terms will come out infinitely smaller than the third and thus can be ignored. The fourth term determines the variation of the curvature, the fifth the variation of the variation, and so on. Hence, by the way, one can see clearly the not inconsiderable usefulness of these series in the solution of problems that depend on tangents and the curvature of curves.

[c]Now compare the series $e - \frac{ao}{e} - \frac{n^2o^2}{2e^3} - \frac{an^2o^3}{2e^5} - \cdots$ with the series $P - Qo - Ro^2 - So^3 - \cdots$, and in the same manner for P, Q, R, and S write e, $\frac{a}{e}$, $\frac{n^2}{2e^3}$, and $\frac{an^2}{2e^5}$, and for $\sqrt{(1+Q^2)}$ write $\sqrt{\left(1+\frac{a^2}{e^2}\right)}$ or $\frac{n}{e}$; then the density of the medium will come out[c] as $\frac{a}{ne}$, that is (because n is given), as $\frac{a}{e}$, or $\frac{AC}{CH}$, that is, as the tangent's length HT terminated at the semidiameter

cc. Ed. 1 has: "Besides, CF is the square root of CI^2 and IF^2, that is, of BD^2 and the square of the second term. And $FG + kl$ is equal to twice the third term, and $FG - kl$ is equal to twice the fourth. For the value of DG is converted into the value of il, and the value of FG into the value of kl, by writing Bi for BD, or $-o$ for $+o$. Accordingly, since FG is $-\frac{n^2o^2}{2e^3} - \frac{an^2o^3}{2e^5} \ldots$, kl will be $= -\frac{n^2o^2}{2e^3} + \frac{an^2o^3}{2e^5} \ldots$. And the sum of these is $-\frac{n^2o^2}{e^3}$; the difference, $-\frac{an^2o^3}{e^5}$. The fifth and following terms I ignore here as infinitely less than such as come under consideration in this problem. And so if the series is universally designated by the terms $\mp Qo - Ro^2 - So^3 \ldots$, CF will be equal to $\sqrt{(o^2 + Q^2o^2)}$, $FG + kl$ will be equal to $2Ro^2$, and $FG - kl$ will be equal to $2So^3$. For CF, $FG + kl$, and $FG - kl$, write these values of theirs, and the density of the medium, which was as $\frac{FG - kl}{CF \times (FG + kl)}$, will now be as $\frac{S}{R\sqrt{(1+Q^2)}}$. Therefore by reducing each problem to a converging series and here writing for Q, R, and S the terms of the series corresponding to these and then supposing the resistance of the medium in any place G to be to the gravity as $S\sqrt{(1+Q^2)}$ to $2R^2$, and the velocity to be the same as that with which a body, departing from place C along straight line CF, could subsequently move in a parabola having diameter CB and latus rectum $\frac{1+Q^2}{R}$, the problem will be solved.

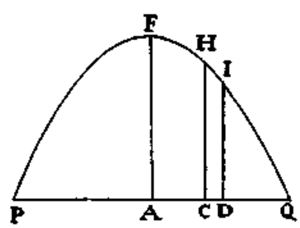
"Thus, in now solving the problem, if $\sqrt{\left(1+\frac{a^2}{e^2}\right)}$ or $\frac{n}{e}$ is written for $\sqrt{(1+Q^2)}$, $\frac{n^2}{2e^3}$ for R, and $\frac{an^2}{2e^5}$ for S, the density of the medium will come out."

AF, which stands perpendicularly upon PQ; and the resistance will be to the gravity as $3a$ to $2n$, that is, as 3AC to the diameter PQ of the circle, while the velocity will be as $\sqrt{\text{CH}}$. Therefore, if the body goes forth from place F with the proper velocity along a line parallel to PQ, and the density of the medium in each place H is as the length of the tangent HT, and the resistance, also in some place H, is to the force of gravity as 3AC to PQ, then that body will describe the quadrant FHQ of a circle. Q.E.I.

But if the same body were to go forth from place P along a line perpendicular to PQ and were to begin to move in an arc of the semicircle PFQ, AC or a would have to be taken on the opposite side of center A, and therefore its sign would have to be changed, and $-a$ would have to be written for $+a$. Thus the density of the medium would come out as $-\frac{a}{e}$. But nature does not admit of a negative density, that is, a density that accelerates the motions of bodies; and therefore it cannot naturally happen that a body by ascending from P should describe the quadrant PF of a circle. For this effect the body would have to be accelerated by an impelling medium, not impeded by a resisting medium.

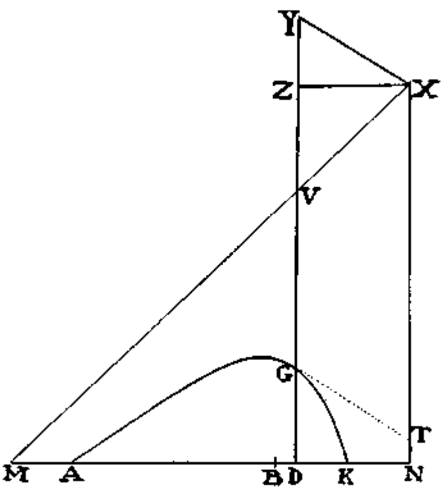
EXAMPLE 2. Let the line PFQ be a parabola having its axis AF perpendicular to the horizon PQ, and let it be required to find the density of the medium that would make a projectile move in that parabola.

From the nature of the parabola, the rectangle PD $\times$ DQ is equal to the rectangle of the ordinate DI and some given straight line. Let that straight line be called b; PC, a; PQ, c; CH, e; and CD, o. Then the rectangle $(a+o)\times(c-a-o)$, or $ac-a^2-2ao+co-o^2$, is equal to the rectangle $b\times$DI, and thus DI is equal to $\frac{ac-a^2}{b}+\frac{c-2a}{b}o-\frac{o^2}{b}$. Now the second term $\frac{c-2a}{b}o$ of this series should be written for Qo, the third term $\frac{o^2}{b}$ likewise for Ro^2. But since there are not more terms, the coefficient S of the fourth will have to vanish, and therefore the quantity $\frac{S}{R\sqrt{(1+Q^2)}}$, to which the density of the medium is proportional, will be nil. Therefore, if the density of the medium is null, a projectile will move in a parabola, as Galileo once proved. Q.E.I.



EXAMPLE 3. Let line AGK be a hyperbola having an asymptote NX perpendicular to the horizontal plane AK; and let it be required to find the density of the medium that would make a projectile move in this hyperbola.

Let MX be the other asymptote, meeting in V the ordinate DG produced; and from the nature of the hyperbola, the rectangle XV × VG will be given. Moreover, the ratio of DN to VX is given, and therefore the rectangle DN × VG is given also. Let this rectangle be b^2. And after completing the parallelogram DNXZ, call BN a; BD, o; NX, c; and suppose the given ratio of VZ to ZX or DN to be $\frac{m}{n}$. Then DN will be equal to $a - o$,



VG will be equal to $\frac{b^2}{a-o}$, VZ will be equal to $\frac{m}{n}(a - o)$, and GD or NX − VZ − VG will be equal to $c - \frac{m}{n}a + \frac{m}{n}o - \frac{b^2}{a-o}$. Resolve the term $\frac{b^2}{a-o}$ into the converging series $\frac{b^2}{a} + \frac{b^2}{a^2}o + \frac{b^2}{a^3}o^2 + \frac{b^2}{a^4}o^3 \ldots$, and GD will become equal to $c - \frac{m}{n}a - \frac{b^2}{a} + \frac{m}{n}o - \frac{b^2}{a^2}o - \frac{b^2}{a^3}o^2 - \frac{b^2}{a^4}o^3 \ldots$. The second term $\frac{m}{n}o - \frac{b^2}{a^2}o$ of this series is to be used for Qo, the third (with the sign changed) $\frac{b^2}{a^3}o^2$ for Ro^2, and the fourth (with the sign also changed) $\frac{b^2}{a^4}o^3$ for So^3, and their coefficients $\frac{m}{n} - \frac{b^2}{a^2}$, $\frac{b^2}{a^3}$, and $\frac{b^2}{a^4}$ are to be written in the above rule for Q, R, and S. When this is done, the density of the medium comes out as
$$\frac{\frac{b^2}{a^4}}{\frac{b^2}{a^3}\sqrt{\left(1 + \frac{m^2}{n^2} - \frac{2mb^2}{na^2} + \frac{b^4}{a^4}\right)}}$$
or
$$\frac{1}{\sqrt{\left(a^2 + \frac{m^2}{n^2}a^2 - \frac{2mb^2}{n} + \frac{b^4}{a^2}\right)}},$$
that is (if in VZ, VY is taken equal to VG),

as $\frac{1}{XY}$. For a^2 and $\frac{m^2}{n^2}a^2 - \frac{2mb^2}{n} + \frac{b^4}{a^2}$ are the squares of XZ and ZY. And the resistance is found to have the same ratio to gravity that 3XY has to 2YG; and the velocity is that with which the body would go in a parabola having vertex G, diameter DG, and latus rectum $\frac{XY^2}{VG}$. Therefore suppose that the densities of the medium in each of the individual places G are inversely as the distances XY and that the resistance in some place G is to gravity as 3XY to 2YG; then a body sent forth from place A with the proper velocity will describe that hyperbola AGK. Q.E.I.

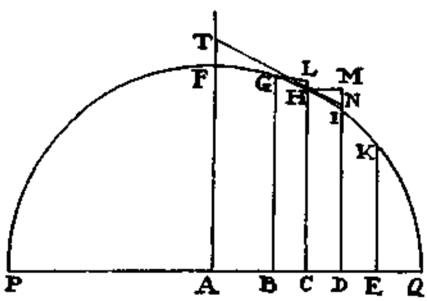
EXAMPLE 4. Suppose generally that line AGK is a hyperbola described with center X and asymptotes MX and NX with the condition that when the rectangle XZDN is described, whose side ZD cuts the hyperbola in G and its asymptote in V, VG would be inversely as some power DN^n (whose index is the number n) of ZX or DN; and let it be required to find the density of the medium in which a projectile would progress in this curve.

For BN, BD, and NX write A, O, and C respectively, and let VZ be to XZ or DN as d to e, and let VG be equal to $\frac{b^2}{DN^n}$; then DN will be equal to A − O, $VG = \frac{b^2}{(A-O)^n}$, $VZ = \frac{d}{e}(A - O)$, and GD or NX − VZ − VG will be equal to $C - \frac{d}{e}A + \frac{d}{e}O - \frac{b^2}{(A-O)^n}$. Resolve the term $\frac{b^2}{(A-O)^n}$ into the infinite series $\frac{b^2}{A^n} + \frac{nb^2}{A^{n+1}}O + \frac{n^2+n}{2A^{n+2}}b^2O^2 + \frac{n^3+3n^2+2n}{6A^{n+3}}b^2O^3 \ldots$, and GD will become equal to $C - \frac{d}{e}A - \frac{b^2}{A^n} + \frac{d}{e}O - \frac{nb^2}{A^{n+1}}O - \frac{+n^2+n}{2A^{n+2}}b^2O^2 - \frac{+n^3+3n^2+2n}{6A^{n+3}}b^2O^3 \ldots$. The second term of this series $\frac{d}{e}O - \frac{nb^2}{A^{n+1}}O$ is to be used for Qo, the third term $\frac{n^2+n}{2A^{n+2}}b^2O^2$ for Ro^2, the fourth term $\frac{n^3+3n^2+2n}{6A^{n+3}}b^2O^3$ for So^3. And hence the density of the medium, $\frac{S}{R\sqrt{(1+Q^2)}}$, in any place G, becomes

$$\frac{n+2}{3\sqrt{\left(A^2+\frac{d^2}{e^2}A^2-\frac{2dnb^2}{eA^n}A+\frac{n^2b^4}{A^{2n}}\right)}},$$

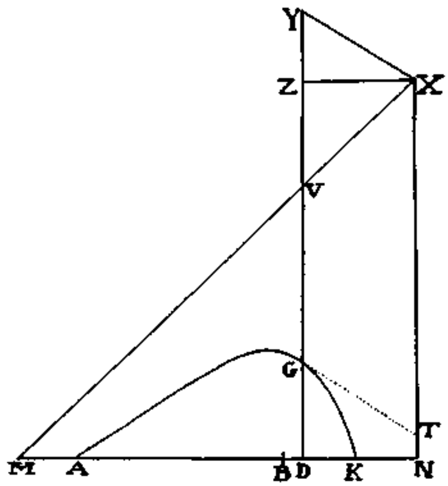
and thus if in VZ, VY is taken equal to $n \times$ VG, the density is inversely as XY. For A^2 and $\frac{d^2}{e^2}A^2 - \frac{2dnb^2}{eA^n}A + \frac{n^2b^4}{A^{2n}}$ are the squares of XZ and ZY. Moreover, the resistance in the same place G becomes to the gravity as $3S \times \frac{XY}{A}$ is to $4R^2$, that is, as XY to $\frac{2n^2+2n}{n+2}$VG. And the velocity in the same place is the very velocity with which a projected body would go in a parabola having vertex G, diameter GD, and latus rectum $\frac{1+Q^2}{R}$ or $\frac{2XY^2}{(n^2+n) \times VG}$. Q.E.I.

Scholium [d]In the same way in which the density of the medium turned out to be as $\frac{S \times AC}{R \times HT}$ in corol. 1, if the resistance is supposed to be as any power V^n of the velocity V, the density of the medium will turn out to be as $\frac{S}{R^{\frac{4-n}{2}}} \times \left(\frac{AC}{HT}\right)^{n-1}$. And therefore if a curve can be found under the condition that there would be given the ratio of $\frac{S}{R^{\frac{4-n}{2}}}$ to $\left(\frac{HT}{AC}\right)^{n-1}$, or $\frac{S^2}{R^{4-n}}$ to $(1+Q^2)^{n-1}$, a body will move in this curve in a uniform medium with a resistance that is as the power V^n of the velocity. But let us return to simpler curves.[d]



Since motion does not take place in a parabola except in a nonresisting medium, but does take place in the hyperbola here described if there is a continual resistance, it is obvious that the line which a projectile describes in a uniformly resisting medium approaches closer to these hyperbolas than to a parabola. At any rate, that line is of a hyperbolic kind, but about its vertex it is more distant from the asymptotes, and in

dd. Ed. 1 lacks this.



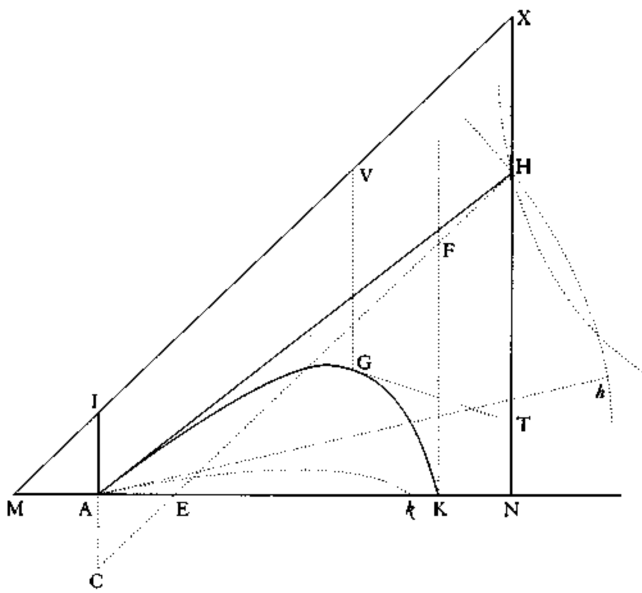
those parts that are further from the vertex it approaches the asymptotes more closely, than the hyperbolas which I have described here. But the difference between them is not so great that one cannot be conveniently used in place of the other in practice. And the hyperbolas which I have been describing will perhaps prove to be more useful than a hyperbola that is more exact and at the same time more compounded. And they will be brought into use as follows.

Complete the parallelogram XYGT, and the straight line GT will touch the hyperbola in G, and thus the density of the medium in G is inversely as the tangent GT, and the velocity in the same place is as $\sqrt{\frac{\mathrm{GT}^2}{\mathrm{GV}}}$, while the resistance is to the force of gravity as GT to $\frac{2n^2+2n}{n+2} \times \mathrm{GV}$.

Accordingly, if a body projected from place A along the straight line AH describes the hyperbola AGK and if AH produced meets the asymptote NX in H and if AI drawn parallel to NX meets the other asymptote MX in I, then the density of the medium in A will be inversely as AH, and the velocity of the body will be as $\sqrt{\frac{\mathrm{AH}^2}{\mathrm{AI}}}$, and the resistance in the same place will be to the gravity as AH to $\frac{2n^2+2n}{n+2} \times \mathrm{AI}$. Hence the following rules.

Rule 1. If both the density of the medium at A and the velocity with which the body is projected remain the same, and angle NAH is changed, lengths AH, AI, and HX will remain the same. And thus, if those lengths are found in some one case, the hyperbola can then be determined readily from any given angle NAH.

Rule 2. If both angle NAH and the density of the medium at A remain the same, and the velocity with which the body is projected is changed, the length AH will remain the same, and AI will be changed in the ratio of the inverse square of the velocity.



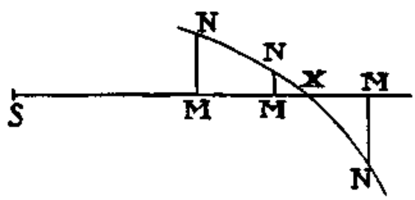
RULE 3. If angle NAH, the velocity of the body at A, and the accelerative gravity remain the same, and the proportion of the resistance at A to the motive gravity is increased in any ratio, the proportion of AH to AI will be increased in the same ratio, and the latus rectum of the above parabola as well as the length $\frac{\mathrm{AH}^2}{\mathrm{AI}}$ (proportional to it) will remain the same; and therefore AH will be decreased in the same ratio, and AI will be decreased as the square of that ratio. But the proportion of the resistance to the weight is increased when the specific gravity (the volume remaining constant) becomes smaller, or the density of the medium becomes greater, or the resistance (as a result of the decreased volume) is decreased in a smaller ratio than the weight.

RULE 4. The density of the medium near the vertex of the hyperbola is greater than at place A; hence, in order to have the mean density, the ratio of the least of the tangents GT to tangent AH must be found, and the density at A must be increased in a slightly greater ratio than that of half the sum of these tangents to the least of the tangents GT.

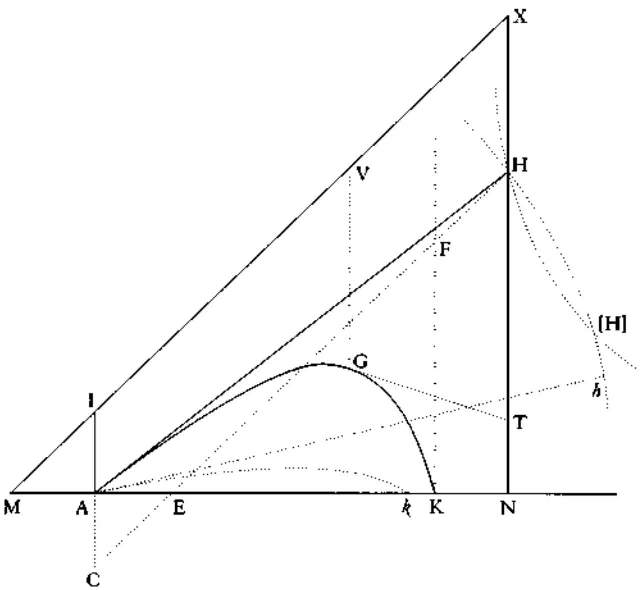
RULE 5. If lengths AH and AI are given, and it is required to describe the figure AGK, produce HN to X so that HX is to AI as $n + 1$ to 1, and with center X and asymptotes MX and NX, describe a hyperbola through point A in such a way that AI is to any VG as XV^n to XI^n.

RULE 6. The greater the number n, the more exact are these "hyperbolas" in the ascent of the body from A, and the less exact in its descent to K, and conversely. A conic hyperbola holds a mean ratio between them and is simpler than the others. Therefore, if the hyperbola is of this kind, and if it is required to find point K, where the projected body will fall upon any straight line AN passing through point A, let AN produced meet asymptotes MX and NX in M and N, and take NK equal to AM.

RULE 7. And hence a ready method of determining this kind of hyperbola from the phenomena is clear. Project two similar and equal bodies with the same velocity in different angles HAK and hAk, and let them fall upon the plane of the horizon in K and k, and note the proportion of AK to Ak (let this be d to e). Then, having erected a perpendicular AI of any length, assume length AH or Ah in any way and from this determine graphically lengths AK and Ak by rule 6. If the ratio of AK to Ak is the same as the ratio of d to e, length AH was correctly assumed. But if not, then on the indefinite straight line SM take a length SM equal to the assumed AH, and erect perpendicular MN equal to the difference of the ratios, $\frac{AK}{Ak} - \frac{d}{e}$, multiplied by any given straight line. From several assumed lengths AH find several points N by a similar method [e]and through them all draw a regular curved line NNXN cutting the straight line SMMM in X. Finally, assume AH equal to abscissa SX, and from this find length AK again; then the lengths that are to the assumed length AI and this last length AH as that length AK (found by experiment) is to the length AK (last found) will be those true lengths AI and AH which it was required to find. And these being given, the resistance of the medium in place A will also be given, inasmuch as it is to the force of gravity as AH to 2AI. The density of the medium, moreover, must be increased (by rule 4), and the resistance just found, if it is increased in the same ratio, will become more exact.[e]



ee. Ed. 1 has: "and then finally, if a regular curved line NN × N is drawn through them all, this will cut off SX equal to the required length AH. For mechanical purposes it suffices to keep the same lengths AH, AI in all angles HAK. But if the figure must be determined more exactly in order to find the resistance of the medium, these lengths must always be corrected (by rule 4)."



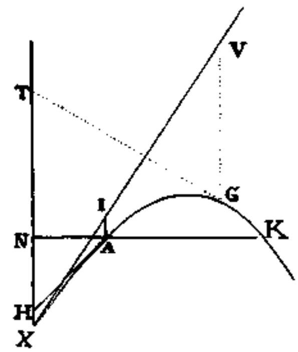
[In ed. 1, as well as in ed. 2, the same letter H is used for both the upper and the lower intersection of the two curves on the right side of the diagram, but in ed. 3 only the upper intersection is lettered. For the sake of clarity, we have introduced an [H] to designate the lower intersection and we have decreased the inclination of the tangent A*h* so that *h* is quite distinct from [H]. For further details, see the Guide, §7.4.]

RULE 8. If the lengths AH and HX have been found, and the position of the straight line AH is now desired along which a projectile sent forth with that given velocity falls upon any point K, erect at points A and K the straight lines AC and KF perpendicular to the horizon, of which AC tends downward and is equal to AI or ½HX. With asymptotes AK and KF describe a hyperbola whose conjugate passes through point C, and with center A and radius AH describe a circle cutting that hyperbola in point H; then a projectile sent forth along the straight line AH will fall upon point K. Q.E.I.

For point H, because length AH is given, is located somewhere in the circle described. Draw CH meeting AK and KF, the former in E, the latter in F; then, because CH and MX are parallel and AC and AI are equal, AE will be equal to AM, and therefore also equal to KN. But CE is to AE as FH to KN, and therefore CE and FH are equal. Point H therefore falls upon the hyperbola described with asymptotes AK and KF whose conjugate passes through point C, and thus H is found in the common intersection of this hyperbola and the circle described. Q.E.D.

It is to be noted, moreover, that this operation is the same whether the straight line AKN is parallel to the horizon or is inclined to the horizon at any angle, and that from the two intersections H and H two angles NAH and NAH arise, and that in a mechanical operation it is sufficient to describe a circle once, then to apply the indeterminate rule CH to point C in such a way that its part FH, placed between the circle and the straight line FK, is equal to its part CE situated between point C and the straight line AK.

What has been said about hyperbolas is easily applied to parabolas. For if XAGK designates a parabola that the straight line XV touches in vertex X and if ordinates IA and VG are as any powers XI^n and XV^n of abscissas XI and XV, draw XT, GT, and AH, of which XT is parallel to VG, and GT and AH touch the parabola in G and A; then a body projected with the proper velocity from any place A along the straight line AH (produced) will describe this parabola, provided that the density of the medium in each individual place G is inversely as tangent GT. The velocity in G, however, will be that with which a projectile would go, in a nonresisting space, in a conic parabola having vertex G, diameter VG produced downward, and latus rectum $\frac{2GT^2}{(n^2 - n) \times VG}$. And the resistance in G will be to the force of gravity as GT to $\frac{2n^2 - 2n}{n - 2}VG$. Hence, if NAK designates a horizontal line and if, while both the density of the medium in A and the velocity with which the body is projected remain the same, the angle NAH is changed in any way, then lengths AH, AI, and HX will remain the same; and hence vertex X of the parabola and the position of the straight line XI are given, and, by taking VG to IA as XV^n to XI^n, all the points G of the parabola, through which the projectile will pass, are given.



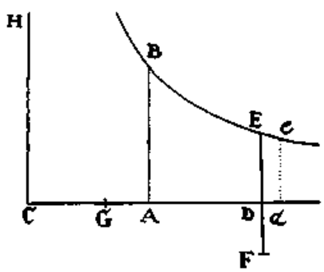
SECTION 3

The motion of bodies that are resisted partly in the ratio of the velocity and partly in the squared ratio of the velocity

Proposition 11
Theorem 8

If a body is resisted partly in the ratio of the velocity and partly in the squared ratio of the velocity and moves in a homogeneous medium by its inherent force alone, and if the times are taken in an arithmetic progression, then quantities inversely proportional to the velocities and increased by a certain given quantity will be in a geometric progression.

With center C and rectangular asymptotes CAD*d* and CH, describe a hyperbola BE*e*, and let AB, DE, and *de* be parallel to asymptote CH.



Let points A and G be given in asymptote CD. Then if the time is represented by the hyperbolic area ABED increasing uniformly, I say that the velocity can be represented by the length DF, whose reciprocal GD together with the given quantity CG composes the length CD increasing in a geometric progression.

For let the area-element DE*ed* be a minimally small given increment of time; then D*d* will be inversely as DE and thus directly as CD. And the decrement of $\frac{1}{\mathrm{GD}}$, which (by book 2, lem. 2) is $\frac{\mathrm{D}d}{\mathrm{GD}^2}$, will be as $\frac{\mathrm{CD}}{\mathrm{GD}^2}$ or $\frac{\mathrm{CG}+\mathrm{GD}}{\mathrm{GD}^2}$, that is, as $\frac{1}{\mathrm{GD}}+\frac{\mathrm{CG}}{\mathrm{GD}^2}$. Therefore, when the time ABED increases uniformly by the addition of the given particles ED*de*, $\frac{1}{\mathrm{GD}}$ decreases in the same ratio as the velocity. For the decrement of the velocity is as the resistance, that is (by hypothesis), as the sum of two quantities, of which one is as the velocity and the other is as the square of the velocity; and the decrement of $\frac{1}{\mathrm{GD}}$ is as the sum of the quantities $\frac{1}{\mathrm{GD}}$ and $\frac{\mathrm{CG}}{\mathrm{GD}^2}$, of which the former is $\frac{1}{\mathrm{GD}}$ itself and the latter $\frac{\mathrm{CG}}{\mathrm{GD}^2}$ is as $\frac{1}{\mathrm{GD}^2}$. Accordingly, because the decrements are analogous, $\frac{1}{\mathrm{GD}}$ is as the velocity. And if the quantity GD, which is inversely proportional to $\frac{1}{\mathrm{GD}}$, is increased by the given quantity

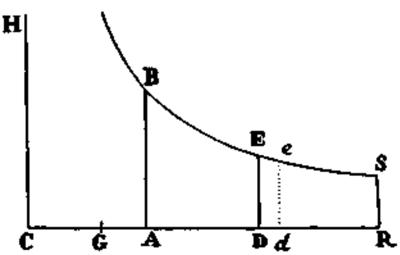
CG, then as the time ABED increases uniformly, the sum CD will increase in a geometric progression. Q.E.D.

COROLLARY 1. Therefore if, given the points A and G, the time is represented by the hyperbolic area ABED, the velocity can be represented by $\frac{1}{\mathrm{GD}}$, the reciprocal of GD.

COROLLARY 2. And by taking GA to GD as the reciprocal of the velocity at the beginning to the reciprocal of the velocity at the end of any time ABED, point G will be found. And when G has been found, then if any other time is given, the velocity can be found.

Proposition 12
Theorem 9

With the same suppositions, I say that if the spaces described are taken in an arithmetic progression, the velocities increased by a certain given quantity will be in a geometric progression.



Let point R be given in the asymptote CD, and after erecting perpendicular RS meeting the hyperbola in S, represent the described space by the hyperbolic area RSED; then the velocity will be as the length GD, which with the given quantity CG composes the length CD decreasing in a geometric progression while space RSED is increased in an arithmetic progression.

For, because the increment ED*de* of the space is given, the line-element D*d*, which is the decrement of GD, will be inversely as ED and thus directly as CD, that is, as the sum of GD and the given length CG. But the decrement of the velocity, in the time inversely proportional to it in which the given particle D*de*E of space is described, is as the resistance and the time jointly, that is, directly as the sum of two quantities (of which one is as the velocity and the other is as the square of the velocity) and inversely as the velocity; and thus is directly as the sum of two quantities, of which one is given and the other is as the velocity. Therefore the decrement of the velocity as well as of line GD is as a given quantity and a decreasing quantity jointly; and because the decrements are analogous, the decreasing quantities will always be analogous, namely, the velocity and the line GD. Q.E.D.

COROLLARY 1. If the velocity is represented by the length GD, the space described will be as the hyperbolic area DESR.

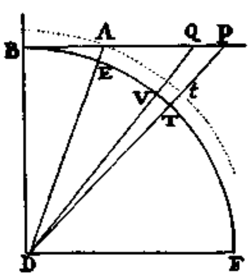
COROLLARY 2. And if point R is taken at will, point G will be found by taking GR to GD as the velocity at the beginning is to the velocity after any space RSED has been described. And when point G has been found, the space is given from the given velocity, and conversely.

COROLLARY 3. Hence, since (by prop. 11) the velocity is given from the given time, and by this prop. 12 the space is given from the given velocity, the space will be given from the given time, and conversely.

Proposition 13
Theorem 10

Supposing that a body attracted downward by uniform gravity ascends straight up or descends straight down and is resisted partly in the ratio of the velocity and partly in the squared ratio of the velocity, I say that if straight lines parallel to the diameters of a circle and a hyperbola are drawn through the ends of the conjugate diameters and if the velocities are as certain segments of the parallels, drawn from a given point, then the times will be as the sectors of areas cut off by straight lines drawn from the center to the ends of the segments, and conversely.

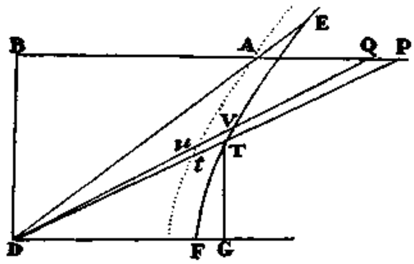
CASE 1. Let us suppose first that the body is ascending. With center D and any semidiameter DB describe the quadrant BETF of a circle, and



through the end B of semidiameter DB draw the indefinite line BAP parallel to semidiameter DF. Let point A be given in that line, and take segment AP proportional to the velocity. Since one part of the resistance is as the velocity and the other part is as the square of the velocity, let the whole resistance be as $AP^2 + 2BA \times AP$. Draw DA and DP cutting the circle in E and T, and represent the gravity by DA^2 in such a way that the gravity is to the resistance as DA^2 to $AP^2 + 2BA \times AP$; and the time of the whole ascent will be as sector EDT of the circle.

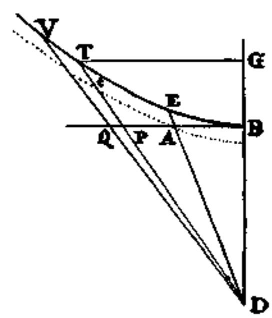
For draw DVQ cutting off both the moment PQ of velocity AP and the moment DTV (corresponding to a given moment of time) of sector DET; then that decrement PQ of the velocity will be as the sum of the forces of the gravity DA^2 and the resistance $AP^2 + 2BA \times AP$, that is (by book 2, prop. 12 of the *Elements*), as DP^2. Accordingly, the area DPQ, which is proportional to PQ, is as DP^2, and the area DTV, which is to the area DPQ as DT^2 to DP^2, is as the given quantity DT^2. The area EDT therefore decreases uniformly as the remaining time, by the subtraction of the given particles DTV, and therefore is proportional to the time of the whole ascent. Q.E.D.

Case 2. If the velocity in the ascent of the body is represented by the length AP as in case 1, and the resistance is supposed to be as $AP^2 + 2BA \times AP$, and if the force of gravity is less than what could be represented by DA^2, take BD of such a length that $AB^2 - BD^2$ is proportional to the gravity, and let DF be perpendicular and equal to DB, and through the vertex F describe the hyperbola FTVE, whose conjugate semidiameters are DB and DF and which cuts DA in E and cuts DP and DQ in T and V; then the time of the whole ascent will be as the sector TDE of the hyperbola.



For the decrement PQ of the velocity occurring in a given particle of time is as the sum of the resistance $AP^2 + 2BA \times AP$ and the gravity $AB^2 - BD^2$, that is, as $BP^2 - BD^2$. But area DTV is to area DPQ as DT^2 to DP^2 and thus, if a perpendicular GT is dropped to DF, is as GT^2 or $GD^2 - DF^2$ to BD^2, and as GD^2 to BP^2, and by separation [or dividendo] as DF^2 to $BP^2 - BD^2$. Therefore, since area DPQ is as PQ, that is, as $BP^2 - BD^2$, area DTV will be as DF^2, which is given. Area EDT therefore decreases uniformly in each equal particle of time, by the subtraction of the same number of given particles DTV, and therefore is proportional to the time. Q.E.D.

Case 3. Let AP be the velocity in the descent of the body, and $AP^2 + 2BA \times AP$ the resistance, and $BD^2 - AB^2$ the force of gravity, angle DBA being a right angle. And if with center D and principal vertex B the rectangular hyperbola BETV is described, cutting the produced lines DA, DP, and DQ in E, T, and V, then sector DET of this hyperbola will be as the whole time of descent.



For the increment PQ of the velocity, and the area DPQ proportional to it, is as the excess of the gravity over the resistance, that is, as $BD^2 - AB^2 - 2BA \times AP - AP^2$ or $BD^2 - BP^2$. And area DTV is to area DPQ as DT^2 to DP^2 and thus as GT^2 or $GD^2 - BD^2$ to BP^2, and as GD^2 to BD^2, and by separation [or dividendo] as BD^2 to $BD^2 - BP^2$. Therefore, since area DPQ is as $BD^2 - BP^2$, area DTV will be as BD^2, which is given. Therefore area EDT increases uniformly in each equal particle of time, by the addition of the same number of given particles DTV, and therefore is proportional to the time of descent. Q.E.D.

COROLLARY. If with center D and semidiameter DA, the arc A*t* similar to arc ET and similarly subtending angle ADT is drawn through vertex A, then the velocity AP will be to the velocity that the body in time EDT in a nonresisting space could lose by ascending, or acquire by descending, as the area of triangle DAP to the area of sector DA*t* and thus is given from the given time. For in a nonresisting medium the velocity is proportional to the time and thus proportional to this sector; in a resisting medium the velocity is as the triangle; and in either medium, when the velocity is minimally small, it approaches the ratio of equality just as the sector and the triangle do.

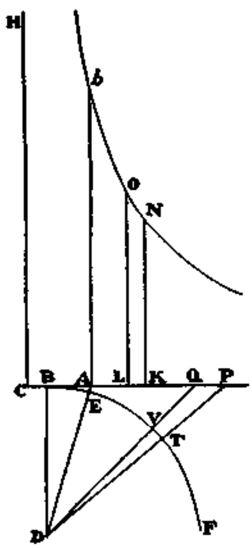
Scholium[a] The case could also be proved in the ascent of the body, where the force of gravity is less than what can be represented by DA^2 or $AB^2 + BD^2$ and greater than what can be represented by $AB^2 - BD^2$, and must be represented by AB^2. But I hasten to other topics.

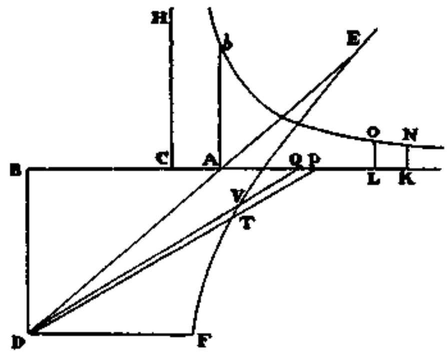
Proposition 14
Theorem 11

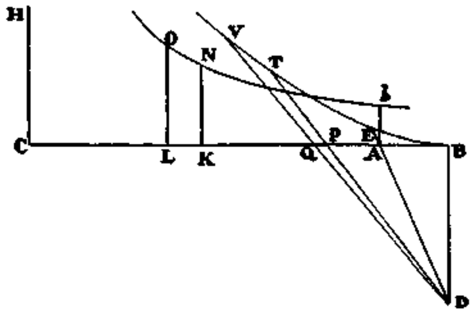
With the same suppositions, I say that the space described in the ascent or descent is as the difference between the area which represents the time and a certain other area that increases or decreases in an arithmetic progression, if the forces compounded of the resistance and the gravity are taken in a geometric progression.

Take AC (in the three figures) proportional to the gravity, and AK proportional to the resistance. And take them on the same side of point A if the body is descending, otherwise on opposite sides. Erect A*b*, which is to DB as DB^2 to $4BA \times AC$; and when the hyperbola *b*N has been described with respect to the rectangular asymptotes CK and CH, and KN has been erected perpendicular to CK, area A*b*NK will be increased or decreased in an arithmetic progression while the forces CK are taken in a geometric progression. I say therefore that the distance of the body from its greatest height is as the excess of area A*b*NK over area DET.

a. Ed. 1 and ed. 2 lack the scholium.







For since AK is as the resistance, that is, as $AP^2 + 2BA \times AP$, assume any given quantity Z, and suppose AK equal to $\frac{AP^2 + 2BA \times AP}{Z}$, and (by book 2, lem. 2) the moment KL of AK will be equal to $\frac{2AP \times PQ + 2BA \times PQ}{Z}$ or $\frac{2BP \times PQ}{Z}$, and the moment KLON of area AbNK will be equal to $\frac{2BP \times PQ \times LO}{Z}$ or $\frac{BP \times PQ \times BD^3}{2Z \times CK \times AB}$.

CASE 1. Now, if the body is ascending and the gravity is as $AB^2 + BD^2$, BET being a circle (in the first figure), then line AC, which is proportional to the gravity, will be $\frac{AB^2 + BD^2}{Z}$, and DP^2 or $AP^2 + 2BA \times AP + AB^2 + BD^2$ will be $AK \times Z + AC \times Z$ or $CK \times Z$; and thus area DTV will be to area DPQ as DT^2 or DB^2 or $CK \times Z$.

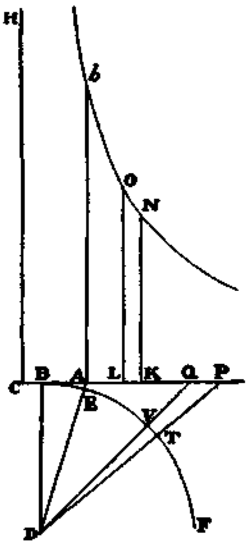
CASE 2. But if the body is ascending and the gravity is as $AB^2 - BD^2$, then line AC (in the second figure) will be $\frac{AB^2 - BD^2}{Z}$, and DT^2 will be to DP^2 as DF^2 or DB^2 to $BP^2 - BD^2$ or $AP^2 + 2BA \times AP + AB^2 - BD^2$, that is, to $AK \times Z + AC \times Z$ or $CK \times Z$. And thus area DTV will be to area DPQ as DB^2 to $CK \times Z$.

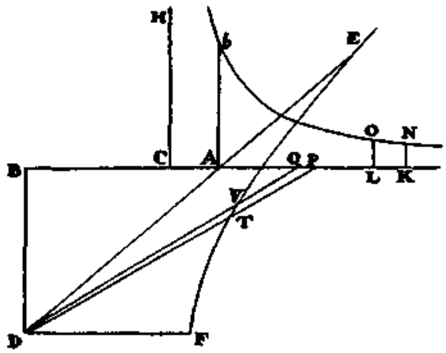
CASE 3. And by the same argument, if the body is descending and therefore the gravity is as $BD^2 - AB^2$, and line AC (in the third figure) is equal to $\frac{BD^2 - AB^2}{Z}$, then area DTV will be to area DPQ as DB^2 to $CK \times Z$, as above.

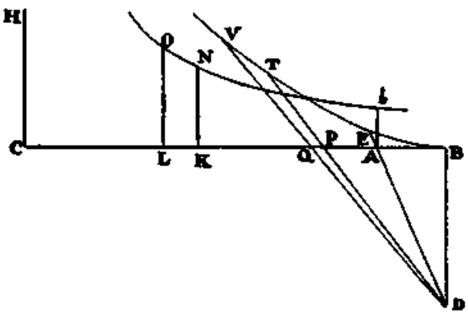
Since, therefore, those areas are always in this ratio, if for area DTV, which represents the moment of time always equal to it, any determinate rectangle is written, say $BD \times m$, then area DPQ, that is, $\frac{1}{2}BD \times PQ$, will be to $BD \times m$ as $CK \times Z$ to BD^2. And hence $PQ \times BD^3$ becomes equal to $2BD \times m \times CK \times Z$, and the moment KLON (found above) of area A*b*NK becomes $\frac{BP \times BD \times m}{AB}$. Take away the moment DTV or $BD \times m$ of area DET, and there will remain $\frac{AP \times BD \times m}{AB}$. Therefore the difference of the moments, that is, the moment of the difference of the areas, is equal to $\frac{AP \times BD \times m}{AB}$, and therefore (because $\frac{BD \times m}{AB}$ is given) is as the velocity AP, that is, as the moment of the space that the body describes in ascending or descending. And thus that space and the difference of the areas, increasing or decreasing by proportional moments and beginning simultaneously or vanishing simultaneously, are proportional. Q.E.D.

COROLLARY. If the length that results from dividing area DET by the line BD is called M, and another length V is taken in the ratio to length M that line DA has to line DE, then the space that a body describes in its whole ascent or descent in a resisting medium will be to the space that the body can describe in the same time in a nonresisting medium, by falling from a state of rest, as the difference of the above areas to $\frac{BD \times V^2}{AB}$, and thus is given from the given time. For the space in a nonresisting medium is in the squared ratio of the time, or as V^2, and, because BD and AB are given, as $\frac{BD \times V^2}{AB}$. [a]This area is equal to area $\frac{DA^2 \times BD \times M^2}{DE^2 \times AB}$, and the moment of

aa. Ed. 1 has: "But the time is as DET or $\frac{1}{2}BD \times ET$, and the moments of these areas are as $\frac{BD \times V}{2AB}$ multiplied by the moment of V and $\frac{1}{2}BD$ multiplied by the moment of ET, that is, as $\frac{BD \times V}{2AB} \times \frac{DA^2 \times 2m}{DE^2}$ and $\frac{1}{2}BD \times 2m$, or as $\frac{BD \times V \times DA^2 \times m}{AB \times DE^2}$ and $BD \times m$. And therefore the moment of area V^2 is to the moment of the difference of areas DET and AKN*b* as $\frac{BD \times V \times DA \times m}{AB \times DE}$







M is m; and therefore the moment of this area is $\frac{DA^2 \times BD \times 2M \times m}{DE^2 \times AB}$. But this moment is to the moment of the difference of the above areas DET and AbNK $\left(\text{that is, to } \frac{AP \times BD \times m}{AB}\right)$ as $\frac{DA^2 \times BD \times M}{DE^2}$ is to ½BD × AP, or as $\frac{DA^2}{DE^2} \times DET$ is to DAP; and thus, when areas DET and DAP are minimally small, in the ratio of equality. Therefore area $\frac{BD \times V^2}{AB}$ and the difference of areas DET and AbNK, when all these areas are minimally

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to $\frac{AP \times BD \times m}{AB}$ or as $\frac{V \times DA}{DE}$ to AP and thus, when V and AP are minimally small, in the ratio of equality. Therefore the minimally small area $\frac{BD \times V^2}{4AB}$ is equal to the minimally small difference of areas DET and AKNb. Hence, since the spaces described simultaneously in both mediums at the beginning of the descent or at the end of the ascent approach equality and thus are then to one another as area $\frac{BD \times V^2}{4AB}$ and the difference of areas DET and AKNb, it follows that, because of their analogous increments, in any equal times they must be to one another as the area $\frac{BD \times V^2}{4AB}$ and the difference of areas DET and AKNb. Q.E.D." In ed. 2 the passage is the same as in ed. 1 except that AKNb is AbNK and the first two sentences read: "The moment of this area or of its equivalent, $\frac{DA^2 \times BD \times M^2}{DE^2 \times AB}$, is to the moment of the difference of areas DET and AbNK as $\frac{DA^2 \times BD \times 2M \times m}{DE^2 \times AB}$ to $\frac{AP \times BD \times m}{AB}$, small, have equal moments and thus are equal. Hence, since the velocities, and therefore also the spaces described simultaneously in both mediums at the beginning of the descent or the end of the ascent, approach equality and thus are then to one another as area $\frac{BD \times V^2}{AB}$ and the difference of areas DET and A*b*NK; and furthermore since the space in a nonresisting medium is always as $\frac{BD \times V^2}{AB}$, and the space in a resisting medium is always as the difference of areas DET and A*b*NK; it follows that the spaces described in both mediums in any equal times must be to one another as the area $\frac{BD \times V^2}{AB}$ and the difference of areas DET and A*b*NK. Q.E.D.[a]

Scholium[b] The resistance encountered by spherical bodies in fluids arises partly from the tenacity, partly from the friction, and partly from the density of the medium. And we have said that the part of the resistance that arises from the density of the fluid is in the squared ratio of the velocity; the other part, which arises from the tenacity of the fluid, is uniform, or as the moment of the time; and thus it would now be possible to proceed to the motion of bodies that are resisted partly by a uniform force or in the ratio of the moments of the time and partly in the squared ratio of the velocity. But it is sufficient to have opened the way to the examination of this subject in the preceding props. 8 and 9 and their corollaries. In these propositions and corollaries, in place of the uniform resistance of the ascending body, which arises from its gravity, there can be substituted the uniform resistance that arises from the tenacity of the medium, when the body is moved by its inherent force alone; and when the body is ascending straight up, it is possible to add this uniform resistance to the force of gravity, and to subtract it when the body is descending straight down. It would also be possible to proceed to the motion of bodies that are resisted partly uniformly, partly in the ratio of the

that is, as $\frac{DA^2 \times BD \times M}{DE^2}$ to $\frac{1}{2}BD \times AP$, or as $\frac{DA^2}{DE^2} \times DET$ to DAP, and thus, when the areas DET and DAP are minimally small, in the ratio of equality." In both eds. 1 and 2 the fraction $\frac{BD \times V^2}{AB}$, which occurs just before this passage, is $\frac{BD \times V^2}{4AB}$.

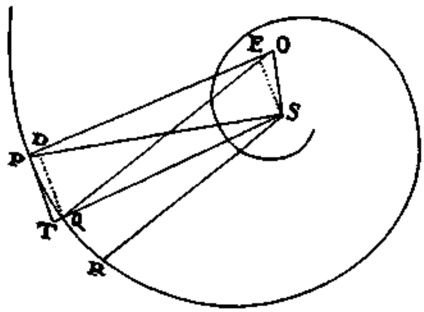
b. Ed. 1 and ed. 2 lack the scholium.

velocity, and partly in the squared ratio of the velocity. And I have opened the way in the preceding props. 13 and 14, in which the uniform resistance that arises from the tenacity of the medium can also be substituted for the force of gravity, or can be compounded with it as before. But I hasten to other topics.

SECTION 4

The revolving motion of bodies in resisting mediums

Lemma 3 *Let* PQR *be a spiral that cuts all the radii* SP, SQ, SR, . . . *in equal angles. Draw the straight line* PT *touching the spiral in any point* P *and cutting the radius* SQ *in* T; *erect* PO *and* QO *perpendicular to the spiral and meeting in* O, *and join* SO. *I say that if points* P *and* Q *approach each other and coincide, angle* PSO *will come out a right angle, and the ultimate ratio of rectangle* TQ × 2PS *to* PQ^2 *will be the ratio of equality.*

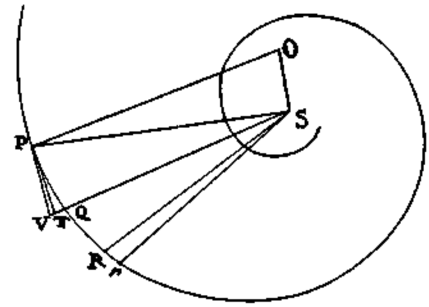


For, from the right angles OPQ and OQR subtract the equal angles SPQ and SQR, and the equal angles OPS and OQS will remain. Therefore a circle that passes through points O, S, and P will also pass through point Q. Let points P and Q come together, and this circle will touch the spiral in the place PQ where they coincide, and thus will cut the straight line OP perpendicularly. OP will therefore become a diameter of this circle, and OSP, an angle in a semicircle, will become a right angle. Q.E.D.

Drop perpendiculars QD and SE to OP, and the ultimate ratios of the lines will be as follows: TQ will be to PD as TS (or PS) to PE, or 2PO to 2PS; likewise, PD will be to PQ as PQ to 2PO; and from the equality of the ratios in inordinate proportion [or ex aequo perturbate] TQ will be to PQ as PQ to 2PS. Hence PQ^2 becomes equal to TQ × 2PS. Q.E.D.

Proposition 15
Theorem 12 *If the density of a medium in every place is inversely as the distance of places from a motionless center and if the centripetal force is in the squared ratio of the density, I say that a body can revolve in a spiral that intersects in a given angle all the radii drawn from that center.*

Let the same things be supposed as in lemma 3, and produce SQ to V, so that SV is equal to SP. In any time, in a resisting medium, let a body describe the minimally small arc PQ, and in twice the time, the minimally small arc PR; then the decrements of these arcs arising from the resistance, that is,



the differences between these arcs and the arcs that would be described in the same times in a nonresisting medium, will be to each other as the squares of the times in which they are generated. The decrement of arc PQ is therefore a fourth of the decrement of arc PR. Hence also, if area QSr is taken equal to area PSQ, the decrement of arc PQ will be equal to half of the line-element Rr; and thus the force of resistance and the centripetal force are to each other as the line-elements $\frac{1}{2}$Rr and TQ that they simultaneously generate. Since the centripetal force by which the body is urged in P is inversely as SP^2; and since (by book 1, lem. 10) the line-element TQ, which is generated by that force, is in a ratio compounded of the ratio of this force and the squared ratio of the time in which arc PQ is described (for I ignore the resistance in this case, as being infinitely smaller than the centripetal force); then it follows that TQ $\times$ SP^2, that is (by lem. 3), $\frac{1}{2}$PQ^2 $\times$ SP, will be in the squared ratio of the time, and thus the time is as PQ $\times \sqrt{\text{SP}}$; and the body's velocity with which arc PQ is described in that time will be as $\dfrac{\text{PQ}}{\text{PQ} \times \sqrt{\text{SP}}}$ or $\dfrac{1}{\sqrt{\text{SP}}}$, that is, as the square root of SP inversely. And by a similar argument, the velocity with which arc QR is described is as the square root of SQ inversely. But these arcs PQ and QR are as the velocities of description to each other, that is, as $\sqrt{\text{SQ}}$ to $\sqrt{\text{SP}}$, or as SQ to $\sqrt{(\text{SP} \times \text{SQ})}$; and because angles SPQ and SQr are equal and areas PSQ and QSr are equal, arc PQ is to arc Qr as SQ to SP. Take the differences of the proportional consequents, and arc PQ will become to arc Rr as SQ to SP $- \sqrt{(\text{SP} \times \text{SQ})}$, or $\frac{1}{2}$VQ. For, points P and Q coming together, the ultimate ratio of SP $- \sqrt{(\text{SP} \times \text{SQ})}$ to $\frac{1}{2}$VQ is the ratio of equality. [a]Since the decrement of arc PQ arising from the resistance, or its double Rr, is as the resistance and the square of the time jointly, the resistance will be as $\dfrac{\text{R}r}{\text{PQ}^2 \times \text{SP}}$.[a] But PQ was to R$r$ as SQ to $\frac{1}{2}$VQ, and

aa. Ed. 1 has: "In a nonresisting medium, equal areas PSQ, QSr would (by book 1, theor. 1) have to be described in equal times. From the resistance arises the difference RSr of the areas, and therefore the resistance is as decrement Rr of line-element Qr compared with the square of the time in which it is generated. For line-element Rr (by book 1, lem. 10) is as the square of the time. Therefore the resistance is as $\dfrac{\text{R}r}{\text{PQ}^2 \times \text{SP}}$."

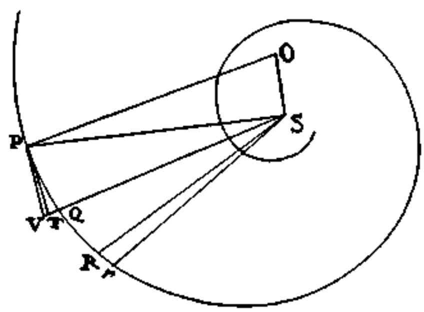
hence $\frac{Rr}{PQ^2 \times SP}$ becomes as $\frac{\frac{1}{2}VQ}{PQ \times SP \times SQ}$, or as $\frac{\frac{1}{2}OS}{OP \times SP^2}$. For, points P and Q coming together, SP and SQ coincide, and angle PVQ becomes a right angle; and because triangles PVQ and PSO are similar, PQ becomes to $\frac{1}{2}$VQ as OP to $\frac{1}{2}$OS. Therefore $\frac{OS}{OP \times SP^2}$ is as the resistance, that is, in the ratio of the density of the medium at P and the squared ratio of the velocity jointly. Take away the squared ratio of the velocity, namely the ratio $\frac{1}{SP}$, and the result will be that the density of the medium at P is as $\frac{OS}{OP \times SP}$. Let the spiral be given, and because the ratio of OS to OP is given, the density of the medium at P will be as $\frac{1}{SP}$. Therefore in a medium whose density is inversely as the distance SP from the center, a body can revolve in this spiral. Q.E.D.

Corollary 1. The velocity in any place P is always the velocity with which a body in a nonresisting medium, under the action of the same centripetal force, can revolve in a circle at the same distance SP from the center.

Corollary 2. The density of the medium, if the distance SP is given, is as $\frac{OS}{OP}$; but if that distance is not given, it is as $\frac{OS}{OP \times SP}$. And hence a spiral can be made to conform to any density of the medium.

Corollary 3. The force of resistance in any place P is to the centripetal force in the same place as $\frac{1}{2}$OS to OP. For those forces are to each other as $\frac{1}{2}Rr$ and TQ or as $\frac{\frac{1}{4}VQ \times PQ}{SQ}$ and $\frac{\frac{1}{2}PQ^2}{SP}$, that is, as $\frac{1}{2}$VQ and PQ, or $\frac{1}{2}$OS and OP. Given the spiral, therefore, the proportion of the resistance to the centripetal force is given; and conversely, from that given proportion the spiral is given.

Corollary 4. The body, therefore, cannot revolve in this spiral except when the force of resistance is less than half of the centripetal force. Let the resistance become equal to half of the centripetal force; then the spiral will coincide with the straight line PS, and the body will descend to the center in this straight line with a velocity that is (as we proved in book 1, prop. 34) to the velocity with which the body descends in a nonresisting medium in the case of a parabola in the ratio of 1 to $\sqrt{2}$. [b]And the times of descent will here be inversely as the velocities, and thus are given.[b]

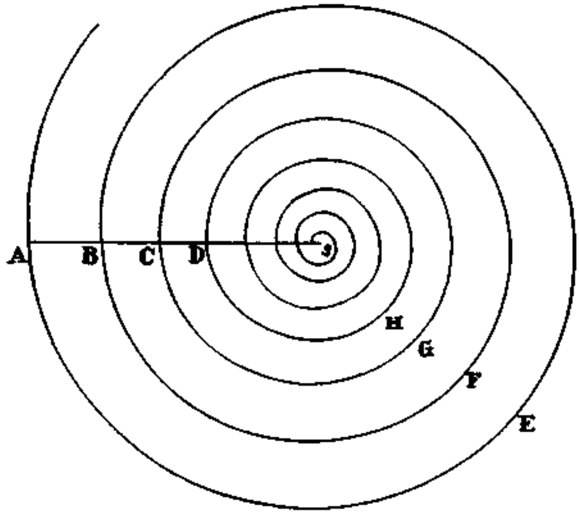


COROLLARY 5. And since at equal distances from the center the velocity is the same in the spiral PQR as in the straight line SP, and since the length of the spiral is in a given ratio to the length of the straight line PS, namely the ratio of OP to OS, the time of descent in the spiral will be to the time of descent in the straight line SP in that same given ratio, and accordingly is given.

COROLLARY 6. If, with center S and any two given radii, two circles are described, and if—these circles remaining the same—the angle that the spiral contains with radius PS is changed in any way, then the number of revolutions that the body can complete between the circumferences of the circles, by revolving in the spiral from one circumference to the other, is as $\frac{PS}{OS}$, or as the tangent of the angle that the spiral contains with radius PS. And the time of those revolutions is as $\frac{OP}{OS}$, that is, as the secant of that angle, or inversely as the density of the medium.

COROLLARY 7. If a body, in a medium whose density is inversely as the distance of places from the center, has made a revolution about that center in any curve AEB and has cut the first radius AS in the same angle in B as it did previously in A, with a velocity that was to its prior velocity in A inversely as the square roots of distances from the center—that is, as AS to a mean proportional between AS and BS—then that body will make innumerable entirely similar revolutions BFC, CGD, . . . , and by the intersections will divide the radius AS into the continually proportional parts AS, BS, CS, DS, ... And the times of revolution will be as the perimeters of the orbits AEB, BFC, CGD, . . . , directly, and the velocities in the beginnings A, B, C, inversely—that is, as $AS^{3/2}$, $BS^{3/2}$, $CS^{3/2}$. And the whole time in which the body will reach the center will be to the time of the first revolution as the

bb. Ed. 1 has: "Hence the times of descent will here be twice as great as those times and so are given."



sum of all the continually proportional quantities $AS^{3/2}$, $BS^{3/2}$, $CS^{3/2}$, going on indefinitely, is to the first term $AS^{3/2}$—that is, as that first term $AS^{3/2}$ is to the difference of the first two terms $AS^{3/2} - BS^{3/2}$, or very nearly as $\frac{2}{3}AS$ to AB. In this way the whole time is readily found.

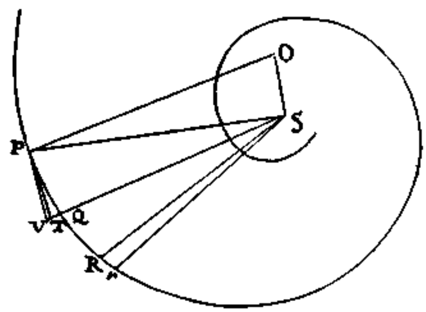
COROLLARY 8. From what has been presented, it is also possible to determine approximately the motions of bodies in mediums whose density either is uniform or accords with any other assigned law. With center S and radii SA, SB, SC, . . . which are continually proportional, describe any number of circles. And suppose that the time of the revolutions between the perimeters of any two of these circles in the medium treated in corol. 7 is to the time of revolutions between those perimeters in the proposed medium very nearly as the mean density of the proposed medium between those circles is to the mean density of the medium in corol. 7 between those same circles; and suppose additionally that the secant of the angle by which the spiral in corol. 7, in the medium treated in that corollary, cuts the radius AS is in the same ratio to the secant of the angle by which the new spiral cuts that same radius in the proposed medium; and also that the numbers of all the revolutions between those same two circles are very nearly as the tangents of those same angles. If this is done throughout between every pair of circles, the motion will be continued through all the circles. And thus we can imagine without difficulty in what ways and in what times bodies would have to revolve in any regular medium.

COROLLARY 9. And even if the motions are eccentric, being performed in spirals approaching an oval shape, nevertheless by conceiving that the single revolutions of those spirals are the same distance apart from one another and approach the center by the same degrees as the spiral described above, we shall also understand how the motions of bodies are performed in spirals of this sort.

Proposition 16
Theorem 13

If the density of the medium in every place is inversely as the distance of places from a motionless center and if the centripetal force is inversely as any power of that distance, I say that a body can revolve in a spiral that intersects in a given angle all the radii drawn from that center.

This is proved by the same method as prop. 15. For if the centripetal force in P is inversely as any power SP^{n+1} (whose index is $n+1$) of the distance SP, then it will be gathered, as above, that the time in which the body describes any arc PQ will be as $PQ \times PS^{\frac{1}{2}n}$, and the resistance in P will be as $\dfrac{Rr}{PQ^2 \times SP^n}$, or as $\dfrac{(1-\frac{1}{2}n) \times VQ}{PQ \times SP^n \times SQ}$, and thus as $\dfrac{(1-\frac{1}{2}n) \times OS}{OP \times SP^{n+1}}$, that is, because $\dfrac{(1-\frac{1}{2}n) \times OS}{OP}$ is given, inversely as SP^{n+1}. And therefore, since the velocity is inversely as $SP^{\frac{1}{2}n}$, the density in P will be inversely as SP.



COROLLARY 1. The resistance is to the centripetal force as $(1-\frac{1}{2}n) \times OS$ to OP.

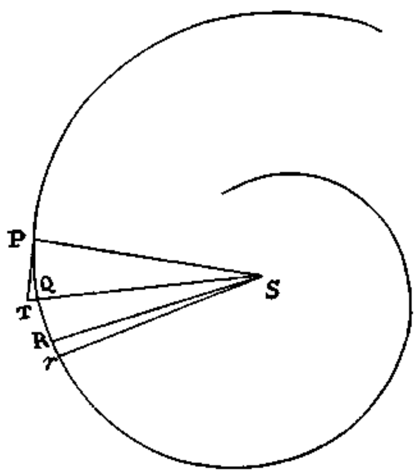
COROLLARY 2. If the centripetal force is inversely as SP^3, $1-\frac{1}{2}n$ will be $=0$, and thus the resistance and density of the medium will be null, as in book 1, prop. 9.

COROLLARY 3. If the centripetal force is inversely as some power of the radius SP whose index is greater than the number 3, positive resistance will be changed to negative.

Scholium

But this proposition and the previous ones, which relate to unequally dense mediums, are to be understood of the motion of bodies so small that no consideration need be taken of a greater density of the medium on one side of the body than on the other. I also suppose the resistance, other things being equal, to be proportional to the density. Hence, in mediums whose force of resisting is not as the density, the density ought to be increased or decreased to such an extent that either the excess of the resistance may be taken away or its defect supplied.

Proposition 17 Problem 4 *To find both the centripetal force and the resistance of the medium by means of which a body can revolve in a given spiral, if the law of the velocity is given.*



Let the spiral be PQR. The time will be given from the velocity with which the body traverses the minimally small arc PQ, and the force will be given from the height TQ, which is as the centripetal force and the square of the time. Then the retardation of the body will be given from the difference RS*r* of the areas PSQ and QSR traversed in equal particles of time, and the resistance and density of the medium will be found from the retardation.

Proposition 18 Problem 5 *Given the law of the centripetal force, it is required to find in every place the density of the medium with which a body will describe a given spiral.*

The velocity in every place is to be found from the centripetal force; then the density of the medium is to be sought from the retardation of the velocity, as in prop. 17.

I have presented the method of dealing with these problems in book 2, prop. 10 and lem. 2, and I do not wish to detain the reader any longer in complex inquiries of this sort. Some things must now be added on the forces of bodies in their forward motion, and on the density and resistance of the mediums in which the motions hitherto explained and motions related to these are performed.

SECTION 5

The density and compression of fluids, and hydrostatics

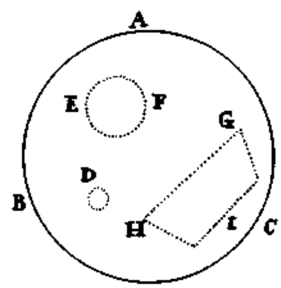
Definition of a Fluid

A fluid is any body whose parts yield to any force applied to it and yielding are moved easily with respect to one another.

Proposition 19 Theorem 14

All the parts of a homogeneous and motionless fluid that is enclosed in any motionless vessel and is compressed on all sides (apart from considerations of condensation, gravity, and all centripetal forces) are equally pressed on all sides and remain in their places without any motion arising from that pressure.

CASE 1. Let a fluid be enclosed in the spherical vessel ABC and be uniformly compressed on all sides; I say that no part of this fluid will move as a result of that pressure. For if some one part D moves, all the parts of this sort, standing on all sides at the same distance from the center, must move simultaneously with a similar motion; and this is so because the pressure on them all is similar and equal, and every motion is supposed excluded except that which arises from the pressure. But they cannot all approach closer to the center unless the fluid is condensed at the center, contrary to the hypothesis. They cannot recede farther from it unless the fluid is condensed at the circumference, also contrary to the hypothesis. They cannot move in any direction and keep their distance from the center, since by a like reasoning they will move in the opposite direction, and the same part cannot move in opposite directions at the same time. Therefore no part of the fluid will move from its place. Q.E.D.



CASE 2. I say additionally that all the spherical parts of this fluid are equally pressed on all sides. For let EF be a spherical part of the fluid; if this part is not pressed equally on all sides, let the lesser pressure be increased until this part is pressed equally on all sides; then its parts, by case 1, will remain in their places. But before the increase of the pressure they will remain in their places, also by case 1, and by the addition of new pressure they will be moved out of their places, by the definition of a fluid. These two results

are contradictory. Therefore it was false to say that the sphere EF was not pressed equally on all sides. Q.E.D.

Case 3. I say furthermore that there is equal pressure on different spherical parts. For contiguous spherical parts press one another equally in the point of contact, by the third law of motion. But by case 2, they are also pressed on all sides by the same force. Therefore any two noncontiguous spherical parts will be pressed by the same force, since an intermediate spherical part can touch both. Q.E.D.

Case 4. I say also that all the parts of the fluid are equally pressed on every side. For any two parts can be touched by spherical parts in any points, and there they press those spherical parts equally, by case 3, and in turn are equally pressed by them, by the third law of motion. Q.E.D.

Case 5. Since, therefore, any part GHI of the fluid is enclosed in the remaining fluid as if in a vessel and is pressed equally on all sides, while its parts press one another equally and are at rest with respect to one another, it is manifest that all the parts of any fluid GHI which is pressed equally on all sides press one another equally and are at rest with respect to one another. Q.E.D.

Case 6. Therefore, if that fluid is enclosed in a vessel that is not rigid and is not pressed equally on all sides, it will yield to a greater pressure, by the definition of a fluid.

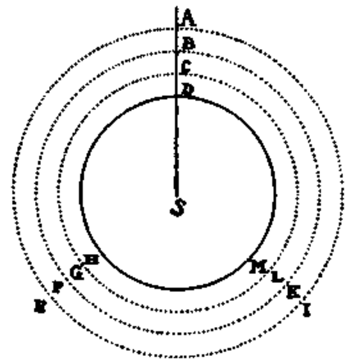
Case 7. And thus in a rigid vessel a fluid will not sustain a pressure that is greater on one side than on another, but will yield to it, and will do so in an instant of time, since the rigid side of the vessel does not follow the yielding liquid. And by yielding, it will press the opposite side, and thus the pressure will tend on all sides to equality. And since, as soon as the fluid endeavors to recede from the part that is pressed more, it is hindered by the resistance of the vessel on the opposite side, the pressure will be reduced on all sides to equality in an instant of time without local motion; and thereupon the parts of the fluid, by case 5, will press one another equally and will be at rest with respect to one another. Q.E.D.

Corollary. Hence the motions of the parts of the fluid with respect to one another cannot be changed by pressure applied to the fluid anywhere on the external surface, except insofar as either the shape of the surface is changed somewhere or all the parts of the fluid, by pressing one another more intensely or more remissly [i.e., by pressing one another more strongly or less strongly], flow among themselves with more or less difficulty.

Proposition 20
Theorem 15

If every part of a fluid that is spherical and homogeneous at equal distances from the center and rests upon a concentric spherical bottom gravitates toward the center of the whole, then the bottom will sustain the weight of a cylinder whose base is equal to the surface of the bottom and whose height is the same as that of the fluid resting upon it.

Let DHM be the surface of the bottom, and AEI the upper surface of the fluid. Divide the fluid into equally thick concentric spherical shells[a] by innumerable spherical surfaces BFK, CGL; and suppose the force of gravity to act only upon the upper surface of each spherical shell, and the actions upon equal parts of all the surfaces to be equal. The highest surface AE is pressed, therefore, by the simple force of its own gravity, by which also all the parts of the highest spherical shell, and the second surface BFK (by prop. 19), are equally pressed in accordance with their measure. The second surface BFK is pressed additionally by the force of its own gravity, which, added to the previous force, makes the pressure double. The third surface CGL is acted on by this pressure, in accordance with its measure, and additionally by the force of its gravity, that is, by a triple pressure. And similarly the fourth surface is urged by a quadruple pressure, the fifth by a quintuple, and so on. The pressure by which any one surface is urged is therefore not as the solid quantity of the fluid lying upon it, but as the number of spherical shells up to the top of the fluid, and is equal to the gravity of the lowest spherical shell multiplied by the number of shells; that is, it is equal to the gravity of a solid whose ultimate ratio to the cylinder specified above will become that of equality—provided that the number of shells is increased and their thickness decreased indefinitely, in such a way that the action of gravity is made continuous from the lowest surface to the highest. The lowest surface therefore sustains the weight of the cylinder specified above. Q.E.D. And



a. Here, as elsewhere in the *Principia*, Newton uses the noun "orbis" (orb) for a spherical shell.

by a similar argument this proposition is evident when the gravity decreases in any assigned ratio of the distance from the center, and also when the fluid is rarer upward and denser downward. Q.E.D.

Corollary 1. Therefore the bottom is not pressed by the whole weight of the incumbent fluid, but sustains only that part of the weight which is described in this proposition, the rest of the weight being sustained by the vaulted shape of the fluid.

Corollary 2. At equal distances from the center, moreover, the quantity of pressure is always the same, whether the pressed surface is parallel to the horizon or perpendicular or oblique, or whether the fluid—continued upward from the pressed surface—rises perpendicularly along a straight line or snakes obliquely through twisted cavities and channels, regular or extremely irregular, wide or very narrow. That the pressure is not at all changed by these circumstances is gathered by applying the proof of this theorem to the various cases of fluids.

Corollary 3. By the same proof it is also gathered (by prop. 19) that the parts of a heavy fluid acquire no motion with respect to one another as a result of the pressure of the incumbent weight, provided that the motion arising from condensation is excluded.

Corollary 4. And therefore, if another body, in which there is no condensation, of the same specific gravity is submerged in this fluid, it will acquire no motion as a result of the pressure of the incumbent weight; it will not descend, it will not ascend, and it will not be compelled to change its shape. If it is spherical, it will remain spherical despite the pressure; if it is square, it will remain square; and it will do so whether it is soft or very fluid, whether it floats freely in the fluid or lies on the bottom. For any internal part of a fluid is in the same situation as a submerged body, and the case is the same for all submerged bodies of the same size, shape, and specific gravity. If a submerged body, while keeping its weight, were to liquefy and assume the form of a fluid, then, if it were formerly ascending or descending or assuming a new shape as a result of pressure, it would also now ascend or descend or be compelled to assume a new shape, and would do so because its gravity and the other causes of motions remain fixed. But (by prop. 19, case 5) this body would now be at rest and would maintain its shape. Hence, this would also be the case under the earlier conditions.

Corollary 5. Accordingly, a body that is of a greater specific gravity than a fluid contiguous to it will sink, and a body that is of a lesser specific gravity will ascend, and will acquire as much motion and change of shape as that excess or deficiency of gravity can bring about. For that excess or deficiency acts like an impulse by which the body, otherwise in equilibrium with the parts of the fluid, is urged; and it can be compared with the excess or deficiency of weight in either of the scales of a balance.

Corollary 6. The gravity of bodies in fluids is therefore twofold: the one, true and absolute; the other, apparent, common, and relative. Absolute gravity is the whole force with which a body tends downward; relative or common gravity is the excess of gravity with which the body tends downward more than the surrounding fluid. By absolute gravity the parts of all fluids and bodies gravitate in their places, and thus the sum of the individual weights is the weight of the whole. For every whole is heavy, as can be tested in vessels full of liquids, and the weight of the whole is equal to the sum of the weights of all the parts, and thus is composed of them. By relative gravity bodies do not gravitate in their places; that is, compared with one another, one is not heavier than another, but each one opposes the endeavors of the others to descend, and they remain in their places just as if they had no gravity. Whatever is in the air and does not gravitate more than the air is not commonly considered to be heavy. Things that do gravitate more are commonly considered to be heavy, inasmuch as they are not sustained by the weight of the air. Weight as commonly conceived is nothing other than the excess of the true weight over the weight of the air. Bodies are commonly called light which are less heavy than the surrounding air and, by yielding to that air, which gravitates more, move upward. They are, however, only comparatively light and not truly so, since they descend in a vacuum. Similarly, bodies in water that descend or ascend because of their greater or smaller gravity are comparatively and apparently heavy or light, and their comparative and apparent heaviness or lightness is the excess or deficiency by which their true gravity either exceeds the gravity of the water or is exceeded by it. And bodies that neither descend by gravitating more nor ascend by yielding to water which gravitates more—even though they increase the weight of the whole by their own true weights—nevertheless, comparatively and as commonly understood, do not gravitate in water. For the demonstration of all these cases is similar.

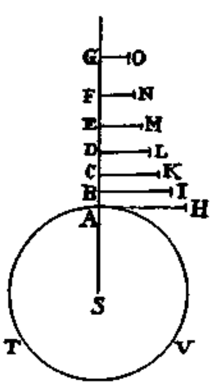
COROLLARY 7. What has been demonstrated concerning gravity is valid for any other centripetal forces.

COROLLARY 8. Accordingly, if the medium in which some body moves is urged either by its own gravity or by any other centripetal force, and the body is urged more strongly by the same force, then the difference between the forces is that motive force which we have considered to be the centripetal force in the preceding propositions. But if the body is urged more lightly by that force, the difference between the forces should be considered a centrifugal force.

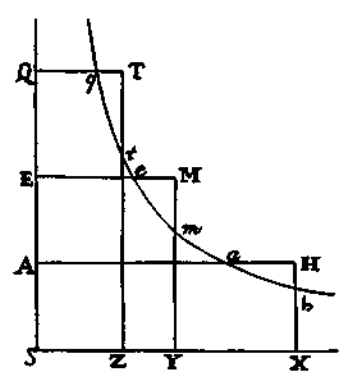
COROLLARY 9. Since fluids, moreover, do not change the external shapes of enclosed bodies that they press upon, it is evident in addition (by prop. 19, corol.) that fluids will not change the situation of the internal parts with respect to one another; and accordingly, if animals are immersed, and if all sensation arises from the motion of the parts, fluids will neither harm these immersed bodies nor excite any sensation, except insofar as these bodies can be condensed by compression. And the case is the same for any system of bodies that is surrounded by a compressing fluid. All the parts of the system will be moved with the same motions as if they were in a vacuum and retained only their relative gravity, except insofar as the fluid either resists their motions somewhat or is needed to make them cohere by compression.

Proposition 21
Theorem 16

Let the density of a certain fluid be proportional to the compression, and let its parts be drawn downward by a centripetal force inversely proportional to their distances from the center; I say that if the distances are taken continually proportional, the densities of the fluid at these distances will also be continually proportional.



Let ATV designate the spherical bottom on which the fluid lies, S the center, and SA, SB, SC, SD, SE, SF, . . . the continually proportional distances. Erect perpendiculars AH, BI, CK, DL, EM, FN, . . . , which are as the densities of the medium in places A, B, C, D, E, F; then the specific gravities in those places will be as $\frac{AH}{AS}$, $\frac{BI}{BS}$, $\frac{CK}{CS}$, . . . , or—which is the same—as $\frac{AH}{AB}$, $\frac{BI}{BC}$, $\frac{CK}{CD}$, ... Suppose first that these gravities continue uniformly, the first from A to B, the second from B to C, the third from C to D, . . . , the decrements thus occurring by degrees at points B, C, D, ... Then these specific gravities multiplied by the heights AB, BC, CD, . . . will give the pressures AH, BI, CK, . . . , by which the bottom ATV (according to prop. 20) is pressed. The particle A therefore sustains all the pressures AH, BI, CK, DL, going on indefinitely; and the particle B, all the pressures except the first, AH; and the particle C, all except the first two, AH and BI; and so on. And thus the density AH of the first particle A is to the density BI of the second particle B as the sum of all the AH + BI + CK + DL indefinitely, to the sum of all the BI + CK + DL . . . . And the density BI of the second particle B is to the density CK of the third particle C as the sum of all the BI + CK + DL . . . to the sum of all the CK + DL . . . . Those sums are therefore proportional to their differences AH, BI, CK, . . . , and thus are continually proportional (by book 2, lem. 1); and accordingly the differences AH, BI, CK, . . . , proportional to those sums, are also continually proportional. Therefore, since the densities in places A, B, C, . . . are as AH, BI, CK, . . . , these also will be continually proportional. Proceed now by jumps, and from the equality of the ratios [or ex aequo], at the continually proportional distances SA, SC, SE, the densities AH, CK, EM will be continually proportional. And by the same argument, at any continually proportional distances SA, SD, SG, the densities AH, DL, GO will be continually proportional. Now let points A, B, C, D, E, . . . come together so that the progression of the specific gravities is made continual from the bottom A to the top of the fluid; and at any continually proportional distances SA, SD, SG, the densities AH, DL, GO, being always continually proportional, will still remain continually proportional now. Q.E.D.

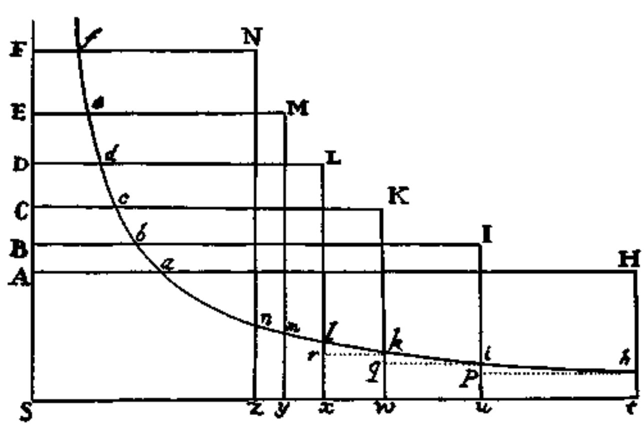


COROLLARY. Hence, if the density of a fluid is given in two places, say A and E, its density in any other place Q can be determined. With center S and rectangular asymptotes SQ and SX describe a hyperbola cutting perpendiculars AH, EM, and QT in *a*, *e*, and *q*, and also perpendiculars HX, MY, and TZ, dropped to asymptote SX, in *h*, *m*, and *t*. Make the area Y*mt*Z be to the given area Y*mh*X as the given area E*eq*Q is to the given area E*ea*A; and the line Z*t* produced will cut off the line QT proportional to the density. For if lines SA, SE, and SQ are continually proportional, areas E*eq*Q and E*ea*A will be equal, and hence the areas proportional to these, Y*mt*Z an X*hm*Y, will also be equal, and lines SX, SY, and SZ—that is, AH, EM, and QT—will be continually proportional, as they ought to be. And if lines SA, SE, and SQ obtain any other order in the series of continually proportional quantities, lines AH, EM, and QT, because the hyperbolic areas are proportional, will obtain the same order in another series of continually proportional quantities.

Proposition 22
Theorem 17

Let the density of a certain fluid be proportional to the compression, and let its parts be drawn downward by a gravity inversely proportional to the squares of their distances from the center; I say that if the distances are taken in a harmonic progression, the densities of the fluid at these distances will be in a geometric progression.

Let S designate the center, and SA, SB, SC, SD, and SE the distances in a geometric progression. Erect perpendiculars AH, BI, CK, . . . , which are as the densities of the fluid in places A, B, C, D, E, . . . ; then the specific gravities in those places will be $\frac{AH}{SA^2}, \frac{BI}{SB^2}, \frac{CK}{SC^2}, \ldots$. Imagine these specific gravities to be uniformly continued, the first from A to B, the second from B to C, the third from C to D, ... Then these, multiplied by the heights AB, BC,



CD, DE, . . .—or, which is the same, by the distances SA, SB, SC, . . . , proportional to those heights—will yield $\frac{AH}{SA}, \frac{BI}{SB}, \frac{CK}{SC}, \ldots$, which represent the pressures. Therefore, since the densities are as the sums of these pressures, the differences (AH − BI, BI − CK, . . .) of the densities will be as the differences $\left(\frac{AH}{SA}, \frac{BI}{SB}, \frac{CK}{SC}, \ldots\right)$ of the sums. With center S and asymptotes SA and Sx describe any hyperbola that cuts the perpendiculars AH, BI, CK, . . . in a, b, c, . . . and also cuts in h, i, and k the perpendiculars Ht, Iu, and Kw, dropped to asymptote Sx; then the differences tu, uw, . . . between the densities will be as $\frac{AH}{SA}, \frac{BI}{SB}, \ldots$. And the rectangles $tu \times th$, $uw \times ui$, . . . , or tp, uq, . . . , will be as $\frac{AH \times th}{SA}, \frac{BI \times ui}{SB}, \ldots$, that is, as A$a$, B$b$, ... For, from the nature of the hyperbola, SA is to AH or St as th to Aa, and thus $\frac{AH \times th}{SA}$ is equal to Aa. And by a similar argument, $\frac{BI \times ui}{SB}$ is equal to Bb, ... Moreover, Aa, Bb, Cc, . . . are continually proportional, and therefore proportional to their differences Aa − Bb, Bb − Cc, . . . ; and thus the rectangles tp, uq, . . . are proportional to these differences, and also the sums of the rectangles $tp + uq$ or $tp + uq + wr$ are proportional to the sums of the differences Aa − Cc or Aa − Dd. Let there be as many terms of this sort as you wish; then the sum of all the differences, say Aa − Ff, will be proportional to the sum of all the rectangles, say $zthn$. Increase the number of terms and decrease the distances of points A, B, C, . . . , indefinitely; then these rectangles will come out equal to the hyperbolic area $zthn$, and thus the difference Aa − Ff is proportional to this area. Now take any distances, say SA, SD, SF, in a harmonic progression, and the differences Aa − Dd and Dd − Ff will be equal; and therefore the areas $thlx$ and $xlnz$ which are proportional to these differences will be equal to each other, and the densities St, Sx, and Sz (that is, AH, DL, and FN) will be continually proportional. Q.E.D.

COROLLARY. Hence, if any two densities of a fluid are given, say AH and BI, the area $thiu$ corresponding to their difference tu will be given; and accordingly the density FN at any height SF will be found by taking the area *thnz* to be to that given area *thiu* as the difference $Aa - Ff$ is to the difference $Aa - Bb$.

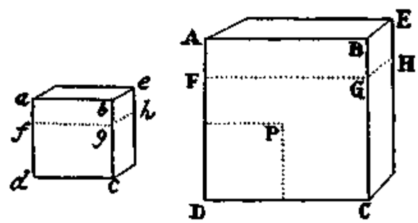
Scholium Similarly, it can be proved that if the gravity of the particles of a fluid is decreased as the cubes of the distances from the center, and if the reciprocals of the squares of the distances SA, SB, SC, ... $\left(\text{namely, } \frac{SA^3}{SA^2}, \frac{SA^3}{SB^2}, \frac{SA^3}{SC^2}\right)$ are taken in an arithmetic progression, then the densities AH, BI, CK, ... will be in a geometric progression. And if the gravity is decreased as the fourth power of the distances, and if the reciprocals of the cubes of the distances $\left(\text{say, } \frac{SA^4}{SA^3}, \frac{SA^4}{SB^3}, \frac{SA^4}{SC^3}, \ldots\right)$ are taken in an arithmetic progression, the densities AH, BI, CK, ... will be in a geometric progression. And so on indefinitely. Again, if the gravity of the particles of a fluid is the same at all distances, and if the distances are in an arithmetic progression, the densities will be in a geometric progression, as the distinguished gentleman Edmond Halley has found. If the gravity is as the distance, and if the squares of the distances are in an arithmetic progression, the densities will be in a geometric progression. And so on indefinitely.

These things are so when the density of a fluid condensed by compression is as the force of the compression or, which is the same, when the space occupied by the fluid is inversely as this force. Other laws of condensation can be imagined, as, for example, that the cube of the compressing force is as the fourth power of the density, or that the force ratio cubed is the same as the density ratio to the fourth power. In this case, if the gravity is inversely as the square of the distance from the center, the density will be inversely as the cube of the distance. Imagine that the cube of the compressing force is as the fifth power of the density; then, if the gravity is inversely as the square of the distance, the density will be inversely as the $3/2$ power of the distance. Imagine that the compressing force is as the square of the density, and that the gravity is inversely as the square of the distance; then the density will be inversely as the distance. It would be tedious to cover all cases. But it is established by experiments that the density of air is either exactly or at least very nearly as the compressing force; and therefore the density of the air in the earth's atmosphere is as the weight of the whole incumbent air, that is, as the height of the mercury in a barometer.

Proposition 23
Theorem 18

[a]*If the density of a fluid composed of particles that are repelled from one another is as the compression, the centrifugal forces [or forces of repulsion] of the particles are inversely proportional to the distances between their centers. And conversely, particles that are repelled from one another by forces that are inversely proportional to the distances between their centers constitute an elastic fluid whose density is proportional to the compression.*[a]

Suppose a fluid to be enclosed in the cubic space ACE, and then by compression to be reduced into the smaller cubic space *ace*; then the distances between particles maintaining similar positions with respect to one another in the two spaces will be as the edges AB and *ab* of the cubes; and the densities of the mediums will be inversely as the containing spaces AB^3 and ab^3. On the



plane side ABCD of the larger cube take the square DP equal to the plane side of the smaller cube *db*; then (by hypothesis) the pressure by which the square DP urges the enclosed fluid will be to the pressure by which the square *db* urges the enclosed fluid as the densities of the medium to each other, that is, as ab^3 to AB^3. But the pressure by which the square DB urges the enclosed fluid is to the pressure by which the square DP urges that same fluid as the square DB to the square DP, that is, as AB^2 to ab^2. Therefore, from the equality of the ratios [or ex aequo] the pressure by which the square DB urges the fluid is to the pressure by which the square *db* urges the fluid as *ab* to AB. Divide the fluid into two parts by planes FGH and *fgh* drawn through the middles of the cubes; then these parts will press each other with the same forces with which they are pressed by planes AC and *ac*, that is, in the proportion of *ab* to AB; and thus the centrifugal forces [or forces of repulsion] by which these pressures are sustained are in the same ratio. Because in both cubes the number of particles is the same and their situation similar, the forces that all the particles along planes FGH and *fgh* exert upon all the others are as the forces that each individual particle exerts upon every other particle. Therefore the forces that each particle exerts upon every other particle along the plane FGH in the larger cube are to the forces that individual particles exert on the particle next to them along the plane *fgh* in

aa. In ed. 1 the order of the two sentences is reversed.

the smaller cube as ab to AB, that is, inversely as the distances between the particles are to one another. Q.E.D.

And conversely, if the forces of the individual particles are inversely as the distances, that is, inversely as the edges AB and ab of the cubes, the sums of the forces will be in the same ratio, and the pressures of the sides DB and db will be as the sums of the forces; and the pressure of the square DP will be to the pressure of the side DB as ab^2 to AB^2. And from the equality of the ratios [or ex aequo] the pressure of the square DP will be to the pressure of the side db as ab^3 to AB^3; that is, the one force of compression will be to the other force of compression as the one density to the other density. Q.E.D.

Scholium By a similar argument, if the centrifugal forces [or forces of repulsion] of the particles are inversely as the squares of the distances between the centers, the cubes of the compressing forces will be as the fourth powers of the densities. If the centrifugal forces are inversely as the third or fourth powers of the distances, the cubes of the compressing forces will be as the fifth or sixth powers of the densities. And universally, if D is the distance, and E the density of the compressed fluid, and if the centrifugal forces are inversely as any power of the distance D^n, whose index is the number n, then the compressing forces will be as the cube roots of the powers E^{n+2}, whose index is the number $n + 2$; and conversely. In all of this, it is supposed that the centrifugal forces of particles are terminated in the particles which are next to them or do not extend far beyond them. We have an example of this in magnetic bodies. Their attractive virtue [or power] is almost terminated in bodies of their own kind which are next to them. The virtue of a magnet is lessened by an interposed plate of iron and is almost terminated in the plate. For bodies farther away are drawn not so much by the magnet as by the plate. In the same way, if particles repel other particles of their own kind that are next to them but do not exert any virtue upon more remote particles,[b] particles of this sort are the ones of which the fluids treated in this proposition will be composed. But if the virtue of each particle is propagated indefinitely, a greater force will be necessary for the equal condensation of a

b. Ed. 1 has in addition: "except perhaps through the increase of the intermediate particles by that virtue."

greater quantity of the fluid.[c] Whether elastic fluids consist of particles that repel one another is, however, a question for physics. We have mathematically demonstrated a property of fluids consisting of particles of this sort so as to provide natural philosophers with the means with which to treat that question.

c. Ed. 1 has in addition: "For example, if each particle by its own force, which is inversely as the distance of places from its center, repels all other particles indefinitely, the forces by which the fluid can be equally compressed and condensed in similar vessels will be as the squares of the diameters of the vessels, and thus the force by which the fluid is compressed in the same vessel will be inversely as the cube root of the fifth power of the density."

SECTION 6

Concerning the motion of [a]simple pendulums[a] and the resistance to them

Proposition 24
Theorem 19

In simple pendulums whose centers of oscillation are equally distant from the center of suspension, the quantities of matter are in a ratio compounded of the ratio of the weights and the squared ratio of the times of oscillation in a vacuum.

For the velocity that a given force can generate in a given time in a given quantity of matter is as the force and the time directly and the matter inversely. The greater the force, or the greater the time, or the less the matter, the greater the velocity that will be generated. This is manifest from the second law of motion. Now if the pendulums are of the same length, the motive forces in places equally distant from the perpendicular are as the weights; and thus if two oscillating bodies describe equal arcs and if the arcs are divided into equal parts, then, since the times in which the bodies describe single corresponding parts of the arcs are as the times of the whole oscillations, the velocities in corresponding parts of the oscillations will be to one another as the motive forces and the whole times of the oscillations directly and the quantities of matter inversely; and thus the quantities of matter will be as the forces and the times of the oscillations directly and the velocities inversely. But the velocities are inversely as the times, and thus the times are directly, and the velocities are inversely, as the squares of the times, and therefore the quantities of matter are as the motive forces and the squares of the times, that is, as the weights and the squares of the times. Q.E.D.

COROLLARY 1. And thus if the times are equal, the quantities of matter in the bodies will be as their weights.

COROLLARY 2. If the weights are equal, the quantities of matter will be as the squares of the times.

COROLLARY 3. If the quantities of matter are equal, the weights will be inversely as the squares of the times.

COROLLARY 4. Hence, since the squares of the times, other things being equal, are as the lengths of the pendulums, the weights will be as the lengths of the pendulums if both the times and the quantities of matter are equal.

aa. Newton uses the term "corpora funependula," literally "bodies hanging by a thread [or string]," which we have translated as "simple pendulums"; see the Guide, §7.5.

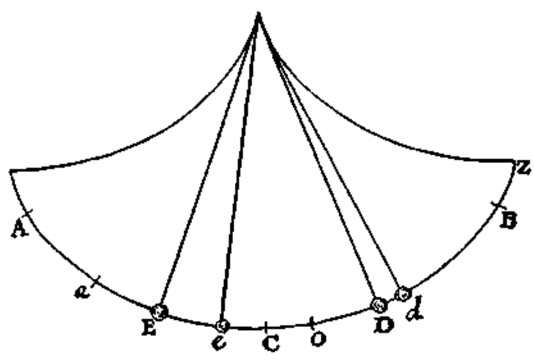
COROLLARY 5. And universally, the quantity of matter in a bob of a simple pendulum is as the weight and the square of the time directly and the length of the pendulum inversely.

COROLLARY 6. But in a nonresisting medium also, the quantity of matter in the bob of a simple pendulum is as the relative weight and the square of the time directly and the length of the pendulum inversely. For the relative weight is the motive force of a body in any heavy medium, as I have explained above, and thus fulfills the same function in such a nonresisting medium as absolute weight does in a vacuum.

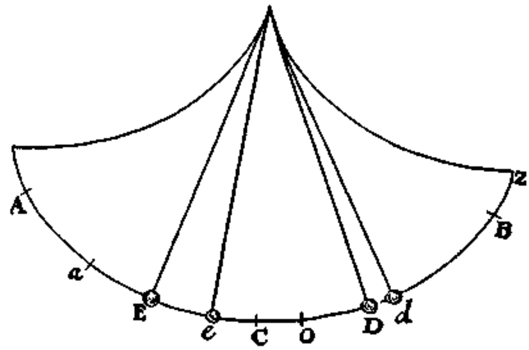
COROLLARY 7. And hence a method is apparent both for comparing bodies with one another with respect to the quantity of matter in each, and for comparing the weights of one and the same body in different places in order to find out the variation in its gravity. And by making experiments of the greatest possible accuracy, I have always found that the quantity of matter in individual bodies is proportional to the weight.

Proposition 25
Theorem 20

The bobs of simple pendulums that are resisted in any medium in the ratio of the moments of time, and those that move in a nonresisting medium of the same specific gravity, perform oscillations in a cycloid in the same time and describe proportional parts of arcs in the same time.



Let AB be the arc of a cycloid, which body D describes by oscillating in a nonresisting medium in any time. Bisect the arc AB in C so that C is its lowest point; then the accelerative force by which the body is urged in any place D or *d* or E will be as the length of arc CD or C*d* or CE. Represent that force by the appropriate arc [CD or C*d* or CE], and since



the resistance is as the moment of time, and thus is given, represent it by a given part CO of the arc of the cycloid, taking arc O*d* in the ratio to arc CD that arc OB has to arc CB; then the force by which the body at *d* is urged in the resisting medium (since it is the excess of the force C*d* over the resistance CO) will be represented by arc O*d*, and thus will be to the force by which body D is urged in a nonresisting medium in place D as arc O*d* to arc CD, and therefore also in place B as arc OB to arc CB. Accordingly, if two bodies D and *d* leave place B and are urged by these forces, then, since the forces at the beginning are as arcs CB and OB, the first velocities and the arcs first described will be in the same ratio. Let those arcs be DO and B*d*; then the remaining arcs CD and O*d* will be in the same ratio. Accordingly the forces, being proportional to CD and O*d*, will remain in the same ratio as at the beginning, and therefore the bodies will proceed simultaneously to describe arcs in the same ratio. Therefore the forces and the velocities and the remaining arcs CD and O*d* will always be as the whole arcs CB and OB, and therefore those remaining arcs will be described simultaneously. Therefore the two bodies D and *d* will arrive simultaneously at places C and O, the one in the nonresisting medium at place C, and the one in the resisting medium at place O. And since the velocities in C and O are as arcs CB and OB, the arcs that the bodies describe in the same time by going on further will be in the same ratio. Let those arcs be CE and O*e*. The force by which body D in the nonresisting medium is retarded in E is as CE, and the force by which body *d* in the resisting medium is retarded in *e* is as the sum of the force C*e* and the resistance CO, that is, as O*e*; and thus the forces by which the bodies are retarded are as arcs CB and OB, which are proportional to arcs CE and O*e*; and accordingly the velocities, which are retarded in that given ratio, remain in that same given ratio. The velocities, therefore, and the arcs described with those velocities are always to one another in the given ratio of arcs CB and OB; and therefore, if the whole arcs AB and *a*B are taken in the same ratio, bodies D and *d* will describe these arcs together and will simultaneously lose all motion in places A and *a*. The whole oscillations are therefore isochronal, and any parts of the arcs, BD and B*d* or BE and B*e*, that are described in the same time are proportional to the whole arcs BA and B*a*. Q.E.D.

COROLLARY. Therefore the swiftest motion in the resisting medium does not occur at the lowest point C, but is found in that point O by which *a*B, the whole arc described, is bisected. And the body, proceeding from that point to *a*, is retarded at the same rate by which it was previously accelerated in its descent from B to O.

Proposition 26
Theorem 21

If simple pendulums are resisted in the ratio of the velocities, their oscillations in a cycloid are isochronal.

For if two oscillating bodies equally distant from the centers of suspension describe unequal arcs and if the velocities in corresponding parts of the arcs are to one another as the whole arcs, then the resistances, being proportional to the velocities, will also be to one another as the same arcs. Accordingly, if these resistances are taken away from (or added to) the motive forces arising from gravity, which are as the same arcs, the differences (or sums) will be to one another in the same ratio of the arcs; and since the increments or decrements of the velocities are as these differences or sums, the velocities will always be as the whole arcs. Therefore, if in some one case the velocities are as the whole arcs, they will always remain in that ratio. But in the beginning of the motion, when the bodies begin to descend and to describe those arcs, the forces—since they are proportional to the arcs—will generate velocities proportional to the arcs. Therefore the velocities will always be as the whole arcs to be described, and therefore those arcs will be described in the same time. Q.E.D.

Proposition 27
Theorem 22

If simple pendulums are resisted as the squares of the velocities, the differences between the times of the oscillations in a resisting medium and the times of the oscillations in a nonresisting medium of the same specific gravity will be very nearly proportional to the arcs described during the oscillations.

For let the unequal arcs A and B be described by equal pendulums in a resisting medium; then the resistance to the body in arc A will be to the resistance to the body in the corresponding part of arc B very nearly in the squared ratio of the velocities, that is, as A^2 to B^2. If the resistance in arc B were to the resistance in arc A as AB to A^2, the times in arcs A and B would be equal, by the previous proposition. And thus the resistance A^2 in arc A, or AB in arc B, produces an excess of time in arc A over the time in a nonresisting medium; and the resistance B^2 produces an excess of time in arc B over the time in a nonresisting medium. And those excesses are very nearly as the forces AB and B^2 that produce them, that is, as arcs A and B. Q.E.D.

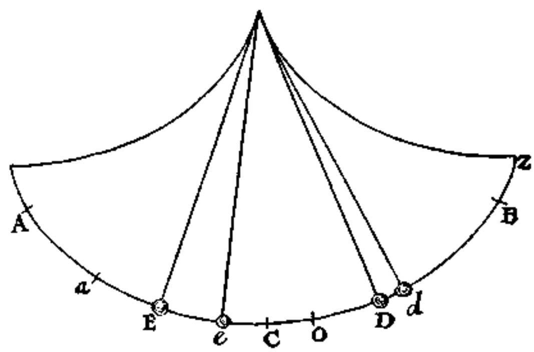
COROLLARY 1. Hence from the times of the oscillations made in a resisting medium in unequal arcs, the times of the oscillations in a nonresisting medium of the same specific gravity can be found. For the difference between these times will be to the excess of the time in the smaller arc over the time in the nonresisting medium as the difference between the arcs is to the smaller arc.

COROLLARY 2. Shorter oscillations are more isochronal, and the shortest are performed in very nearly the same times as in a nonresisting medium. In fact, the times of those that are made in greater arcs are a little greater, because the resistance in the descent of the body (by which the time is prolonged) is greater in proportion to the length described in the descent than the resistance in the subsequent ascent (by which the time is shortened). But also the time of short as well as long oscillations seems to be somewhat prolonged by the motion of the medium. For retarded bodies are resisted a little less in proportion to the velocity, and accelerated bodies a little more, than those that progress uniformly; and this is so because the medium, going in the same direction as the bodies with the motion that it has received from them, is in the first case more agitated, in the second less, and accordingly concurs to a greater or to a less degree with the moving bodies. The medium therefore resists the pendulums more in the descent, and less in the ascent, than in proportion to the velocity, and the time is prolonged as a result of both causes.

Proposition 28
Theorem 23

If a simple pendulum oscillating in a cycloid is resisted in the ratio of the moments of time, its resistance will be to the force of gravity as the excess of the arc described *in the whole descent over the arc described in the subsequent ascent is to twice the length of the pendulum.*

Let BC designate the arc described in the descent, C*a* the arc described in the ascent, and A*a* the difference between the arcs; then, with the same constructions and proofs as in prop. 25, the force by which the oscillating body is urged in any place D will be to the force of resistance as arc CD to arc CO, which is half of that difference A*a*. And thus the force by which the

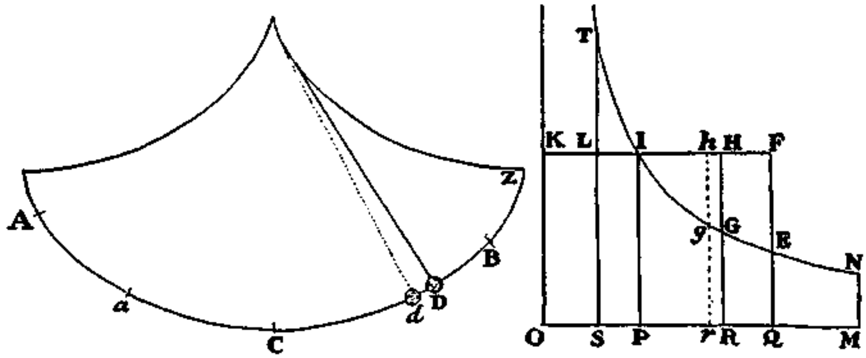


oscillating body is urged in the beginning (or highest point) of the cycloid—that is, the force of gravity—will be to the resistance as the arc of the cycloid between that highest point and the lowest point C is to arc CO, that is (if the arcs are doubled), as the arc of the whole cycloid, or twice the length of the pendulum, is to arc A*a*. Q.E.D.

Proposition 29
Problem 6

Supposing that a body oscillating in a cycloid is resisted as the square of the velocity, it is required to find the resistance in each of the individual places.

Let B*a* be the arc described in an entire oscillation, and let C be the lowest point of the cycloid, and let CZ be half of the arc of the whole cycloid and be equal to the length of the pendulum; and let it be required to find the resistance to the body in any place D. Cut the indefinite straight line OQ in points O, S, P, and Q, with the conditions that—if perpendiculars OK, ST, PI, and QE are erected; and if, with center O and asymptotes OK and OQ, hyperbola TIGE is described so as to cut perpendiculars ST, PI, and QE in T, I, and E; and if, through point I, KF is drawn parallel to asymptote OQ and meeting asymptote OK in K and perpendiculars ST and QE in L and F—the hyperbolic area PIEQ is to the hyperbolic area PITS as the arc BC described during the body's descent is to the arc C*a* described during the



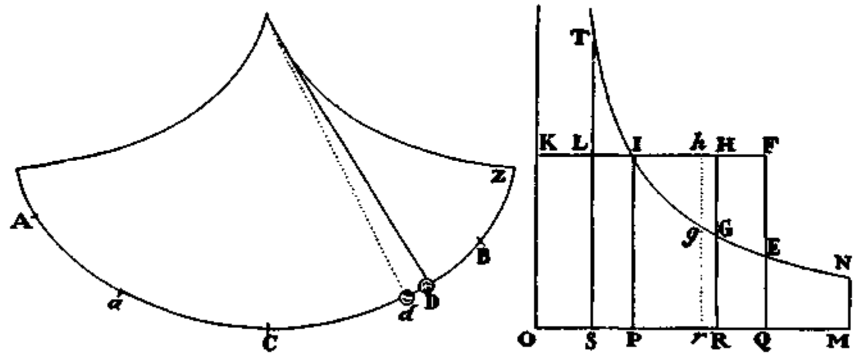
ascent, and area IEF is to area ILT as OQ to OS. Then let perpendicular MN cut off the hyperbolic area PINM, which is to the hyperbolic area PIEQ as arc CZ is to the arc BC described in the descent. And if perpendicular RG cuts off the hyperbolic area PIGR, which is to area PIEQ as any arc CD is to the arc BC described in the whole descent, then the resistance in place D will be to the force of gravity as the area $\frac{OR}{OQ}IEF - IGH$ to the area PINM.

For, since the forces which arise from gravity and by which the body is urged in places Z, B, D, and *a* are as arcs CZ, CB, CD, and C*a*, and those arcs are as areas PINM, PIEQ, PIGR, and PITS, let the arcs and the forces be represented by these areas respectively. In addition, let D*d* be the minimally small space described by the body while descending, and represent it by the minimally small area RG*gr* comprehended between the parallels RG and *rg*; and produce *rg* to *h*, so that GH*hg* and RG*gr* are decrements of areas IGH and PIGR made in the same time. And the increment $GHhg - \frac{Rr}{OQ}IEF$, or $Rr \times HG - \frac{Rr}{OQ}IEF$, of area $\frac{OR}{OQ}IEF - IGH$ will be to the decrement RG*gr*, or $Rr \times RG$, of area PIGR as $HG - \frac{IEF}{OQ}$ is to RG, and thus as $OR \times HG - \frac{OR}{OQ}IEF$ is to $OR \times GR$ or $OP \times PI$, that is (because $OR \times HG$, or $OR \times HR - OR \times GR$, ORHK − OPIK, PIHR, and PIGR + IGH are equal), as $PIGR + IGH - \frac{OR}{OQ}IEF$ is to OPIK. Therefore, if area $\frac{OR}{OQ}IEF - IGH$ is called Y, and if the decrement RG*gr* of area PIGR is given, then the increment of area Y will be as PIGR − Y.

But if V designates the force arising from gravity, by which the body is urged in D and which is proportional to the arc CD to be described, and if R represents the resistance, then V − R will be the whole force by which the body is urged in D. The increment of the velocity is therefore jointly as V − R and that particle of time in which the increment is made. But furthermore the velocity itself is directly as the increment of the space described in the same time and inversely as that same particle of time. Hence, since the resistance, by hypothesis, is as the square of the velocity, the increment of the resistance (by lem. 2) will be as the velocity and the increment of the velocity jointly, that is, as the moment of the space and V − R jointly; and thus, if the moment of the space is given, as V − R; that is, as PIGR − Z, if for the force V there is written PIGR (which represents it), and if the resistance R is represented by some other area Z.

Therefore, as area PIGR decreases uniformly by the subtraction of the given moments, area Y increases in the ratio of PIGR − Y, and area Z increases in the ratio of PIGR − Z. And therefore, if areas Y and Z begin simultaneously and are equal at the beginning, they will continue to be equal by the addition of equal moments and, thereafter decreasing by moments that are likewise equal, will vanish simultaneously. And conversely, if they begin simultaneously and vanish simultaneously, they will have equal moments and will always be equal; and this is so because, if the resistance Z is increased, the velocity will be decreased along with that arc C*a* which is described in the body's ascent, and as the point in which there is a cessation of all motion and resistance approaches closer to point C, the resistance will vanish more quickly than area Y. And the contrary will happen when the resistance is decreased.

Now area Z begins and ends where the resistance is nil, that is, in the beginning of the motion where arc CD is equal to arc CB and the straight line RG falls upon the straight line QE, and in the end of the motion where arc CD is equal to arc C*a* and RG falls upon the straight line ST. And area Y or $\frac{\mathrm{OR}}{\mathrm{OQ}}\mathrm{IEF} - \mathrm{IGH}$ begins and ends where the resistance is nil, and thus where $\frac{\mathrm{OR}}{\mathrm{OQ}}\mathrm{IEF}$ and IGH are equal; that is (by construction), where the straight line RG falls successively upon the straight lines QE and ST. And accordingly those areas begin simultaneously and vanish simultaneously and



therefore are always equal. Therefore area $\frac{OR}{OQ}$IEF − IGH is equal to area Z (which represents the resistance) and therefore is to area PINM (which represents the gravity) as the resistance is to the gravity. Q.E.D.

COROLLARY 1. The resistance in the lowest place C is, therefore, to the force of gravity as area $\frac{OP}{OQ}$IEF is to area PINM.

COROLLARY 2. And this resistance becomes greatest when area PIHR is to area IEF as OR is to OQ. For in that case its moment (namely, PIGR − Y) comes out nil.

COROLLARY 3. Hence also the velocity in each of the individual places can be known, inasmuch as it is as the square root of the resistance, and at the very beginning of the motion is equal to the velocity of the body oscillating without any resistance in the same cycloid.

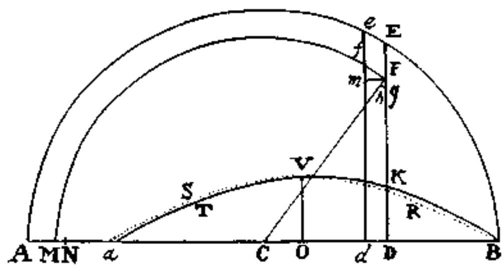
But because the computation by which the resistance and velocity are to be found by this proposition is difficult, it seemed appropriate to add the following proposition.[a]

Proposition 30
Theorem 24

If the straight line aB *is equal to a cycloidal arc that is described by an oscillating body, and if at each of its individual points* D *perpendiculars* DK *are erected that are to the length of the pendulum as the resistance encountered by the body in corresponding points of the arc is to the force of gravity, then I say that the difference between the arc described in the whole descent and the arc described in the whole subsequent ascent multiplied by half the sum of those same arcs will be equal to the area* BK*a occupied by all the perpendiculars* DK.

a. Ed. 1 and ed. 2 have in addition: "which is both more general and more than exact enough for use in natural philosophy."

Represent the cycloidal arc described in an entire oscillation by the straight line aB equal to it, and represent the arc that would be described in a vacuum by the length AB. Bisect AB in C, and point C will represent



the lowest point of the cycloid, and CD will be as the force arising from gravity (by which the body at D is urged along the tangent of the cycloid) and will have the ratio to the length of the pendulum that the force at D has to the force of gravity. Therefore represent that force by the length CD, and the force of gravity by the length of the pendulum; then, if DK is taken in DE in the ratio to the length of the pendulum that the resistance has to the gravity, DK will represent the resistance. With center C and radius CA or CB construct semicircle BEeA. And let the body describe space Dd in a minimally small time; then, when perpendiculars DE and de have been erected, meeting the circumference in E and e, they will be as the velocities that the body in a vacuum would acquire in places D and d by descending from point B. This is evident by book 1, prop. 52. Therefore represent these velocities by perpendiculars DE and de, and let DF be the velocity that the body acquires in D by falling from B in the resisting medium. And if with center C and radius CF circle FfM is described, meeting the straight lines de and AB in f and M, then M will be the place to which the body would then ascend if there were no further resistance, and df will be the velocity that it would acquire in d. Hence also, if Fg designates the moment of velocity that body D, in describing the minimally small space Dd, loses as a result of the resistance of the medium, and if CN is taken equal to Cg, then N will be the place to which the body would then ascend if there were no further resistance, and MN will be the decrement of the ascent arising from the loss of that velocity. Drop perpendicular Fm to df, and the decrement Fg (generated by the resistance DK) of the velocity DF will be to the increment fm (generated by the force CD) of that same velocity as the generating force DK is to the generating force CD. Furthermore, because triangles F*mf*, F*hg*, and FDC are similar, *fm* is to F*m* or D*d* as CD is to DF and from the equality of the ratios [or ex aequo] F*g* is to D*d* as DK is to DF. [a]Likewise F*h* is to F*g* as DF to CF, and from the equality of the ratios in inordinate proportion [or ex aequo perturbate] F*h* or MN is to D*d* as DK to CF or CM; and thus the sum of all the MN × CM will be equal to the sum of all the D*d* × DK. Suppose that a rectangular ordinate is always erected at the moving point M, equal to the indeterminate CM, which in its continual motion is multiplied by the whole length A*a*; then the quadrilateral described as a result of that motion—or the rectangle equal to it, A*a* × ½*a*B—will become equal to the sum of all the MN × CM, and thus equal to the sum of all the D*d* × DK, that is, equal to area BKVT*a*. Q.E.D.[a]

COROLLARY. Hence from the law of the resistance and the difference A*a* of arcs C*a* and CB, the proportion of the resistance to the gravity can be determined very nearly.

For if the resistance DK is uniform, the figure BKT*a* will be equal to the rectangle of B*a* and DK; and hence the rectangle of ½B*a* and A*a* will be equal to the rectangle of B*a* and DK, and DK will be equal to ½A*a*. Therefore, since DK represents the resistance, and the length of the pendulum represents the gravity, the resistance will be to the gravity as ½A*a* is to the length of the pendulum, exactly as was proved in prop. 28.

aa. Ed. 1 has: "Likewise F*g* is to F*h* as CF to DF, and from the equality of the ratios in inordinate proportion [or ex aequo perturbate] F*h* or MN is to D*d* as DK to CF. Take DR to ½*a*B as DK to CF, and MN will be to D*d* as DR to ½*a*B, and thus the sum of all the MN × ½*a*B, that is, A*a* × ½*a*B, will be equal to the sum of all the D*d* × DR, that is, to area BR*r*S*a*, which all the rectangles D*d* × DR or DR*rd* compose. Bisect A*a* and *a*B in P and O, and ½*a*B or OB will be equal to CP, and thus DR is to DK as CP to CF or CM, and by separation [or dividendo] KR will be to DR as PM to CP. And thus, since point M, when the body is in the midpoint O of the oscillation, falls approximately on point P and in the earlier part of the oscillation is between A and P and in the later part is between P and *a*, in both cases deviating equally from point P in opposite directions, it follows that point K, at about the midpoint of the oscillation, that is, over against point O, say in point V, will fall on point R and in the earlier part of the oscillation will lie between R and E and in the later part between R and D, in both cases deviating equally from point R in opposite directions. Accordingly, the area which line KR describes will in the earlier part of the oscillation lie outside area BRS*a* and in the later part within it and will do so within ranges nearly equal to each other on each of the two sides and therefore, when added to area BRS*a* in the first case and subtracted from it in the second, will result in area BKT*a* very nearly equal to area BRS*a*. Therefore the rectangle A*a* × ½*a*B, or A*a*O, will, since it is equal to area BRS*a*, also be very nearly equal to area BKT*a*. Q.E.D."

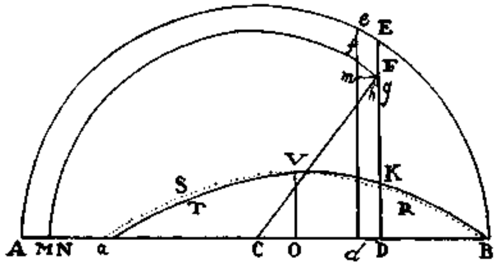
If the resistance is as the velocity, the figure BKTa will be very nearly an ellipse. For if a body in a nonresisting medium were to describe the length BA in a whole oscillation, the velocity in any place D would be as the ordinate DE of a circle described with diameter AB. Accordingly, since Ba in the resisting medium, and BA in a nonresisting medium, are described in roughly equal times, and the velocities in the individual points of Ba are thus very nearly to the velocities in the corresponding points of the length BA as Ba is to BA, the velocity in point D in the resisting medium will be as the ordinate of a circle or ellipse described upon diameter Ba; and thus the figure BKVTa will be very nearly an ellipse. Since the resistance is supposed proportional to the velocity, let OV represent the resistance in the midpoint O; then ellipse BRVSa, described with center O and semiaxes OB and OV, will be very nearly equal to the figure BKVTa and the rectangle equal to it, Aa × BO. Aa × BO is therefore to OV × BO as the area of this ellipse is to OV × BO; that is, Aa is to OV as the area of the semicircle is to the square of the radius, or as 11 to 7, roughly; and therefore $\frac{7}{11}$Aa is to the length of the pendulum as the resistance of the oscillating body in O is to its gravity.

But if the resistance DK is as the square of the velocity, the figure BKVTa will be almost a parabola having vertex V and axis OV, and thus will be very nearly equal to the rectangle contained by $\frac{2}{3}$Ba and OV. The rectangle contained by $\frac{1}{2}$Ba and Aa is therefore equal to the rectangle contained by $\frac{2}{3}$Ba and OV, and thus OV is equal to $\frac{3}{4}$Aa; and therefore the resistance on the oscillating body in O is to its gravity as $\frac{3}{4}$Aa is to the length of the pendulum.

And I judge that these conclusions are more than accurate enough for practical purposes. For, since the ellipse or parabola BRVSa and the figure BKVTa have the same midpoint V, if it is greater than that figure on either side BRV or VSa, it will be smaller than it on the other side, and thus will be very nearly equal to it.

Proposition 31
Theorem 25

If the resistance encountered by an oscillating body in each of the proportional parts of the arcs described is increased or decreased in a given ratio, the difference between the arc described in the descent and the arc described in the subsequent ascent will be increased or decreased in the same ratio.



For that difference arises from the retardation of the pendulum by the resistance of the medium, and thus is as the whole retardation and the retarding resistance, which is proportional to it. In the previous proposition the rectangle contained under the straight line ½*a*B and the difference A*a* of arcs CB and C*a* was equal to area BKT*a*. And that area, if the length *a*B remains the same, is increased or decreased in the ratio of the ordinates DK, that is, in the ratio of the resistance, and thus is as the length *a*B and the resistance jointly. And accordingly the rectangle contained by A*a* and ½*a*B is as *a*B and the resistance jointly, and therefore A*a* is as the resistance. Q.E.D.

COROLLARY 1. Hence, if the resistance is as the velocity, the difference of the arcs in the same medium will be as the whole arc described; and conversely.

COROLLARY 2. If the resistance is in the squared ratio of the velocity, that difference will be in the squared ratio of the whole arc; and conversely.

COROLLARY 3. And universally, if the resistance is in the cubed or any other ratio of the velocity, the difference will be in the same ratio of the whole arc; and conversely.

COROLLARY 4. And if the resistance is partly in the simple ratio of the velocity and partly in the squared ratio of the velocity, the difference will be partly in the simple ratio of the whole arc and partly in the squared ratio of it; and conversely. The law and ratio of the resistance in relation to the velocity will be the same as the law and ratio of that difference of the arcs in relation to the length of the arc itself.

COROLLARY 5. And thus if a pendulum successively describes unequal arcs and there can be found the ratio of the increment and decrement of this difference [i.e., the difference of the arcs] in relation to the length of the arc described, then there will also be had the ratio of the increment and decrement of the resistance in relation to a greater or smaller velocity.

General Scholium[a]

From these propositions it is possible to find the resistance of any mediums by means of pendulums oscillating in those mediums. In fact, I have investigated the resistance of air by the following experiments. I suspended a wooden ball by a fine thread from a sufficiently firm hook in such a way that the distance between the hook and the center of oscillation of the ball was $10\frac{1}{2}$ feet; the ball weighed $57\frac{7}{22}$ ounces avoirdupois and had a diameter of $6\frac{7}{8}$ London inches. I marked a point on the thread 10 feet and 1 inch distant from the center of suspension; and at a right angle at that point I placed a ruler divided into inches, by means of which I might note the lengths of the arcs described by the pendulum. Then I counted the oscillations during which the ball would lose an eighth of its motion. When the pendulum was drawn back from the perpendicular to a distance of 2 inches and was let go from there, so as to describe an arc of 2 inches in its whole descent and to describe an arc of about 4 inches in the first whole oscillation (composed of the descent and subsequent ascent), it then lost an eighth of its motion in 164 oscillations, so as to describe an arc of $1\frac{3}{4}$ inches in its final ascent. When it described an arc of 4 inches in its first descent, it lost an eighth of its motion in 121 oscillations, so as to describe an arc of $3\frac{1}{2}$ inches in its final ascent. When it described an arc of 8, 16, 32, or 64 inches in its first descent, it lost an eighth of its motion in 69, $35\frac{1}{2}$, $18\frac{1}{2}$, and $9\frac{2}{3}$ oscillations respectively. Therefore the difference between the arcs described in the first descent and the final ascent, in the first, second, third, fourth, fifth, and sixth cases, was $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, and 8 inches respectively. Divide these differences by the number of oscillations in each case, and in one mean oscillation—in which an arc of $3\frac{3}{4}$, $7\frac{1}{2}$, 15, 30, 60, and 120 inches was described—the difference between the arcs described in the descent and subsequent ascent will be $\frac{1}{656}$, $\frac{1}{242}$, $\frac{1}{69}$, $\frac{4}{71}$, $\frac{8}{37}$, and $\frac{24}{29}$ parts of an inch respectively. In the greater oscillations, moreover, these are very nearly in the squared ratio of the arcs described, while in the smaller oscillations they are a little greater than in that ratio; and therefore (by book 2, prop. 31, corol. 2) the resistance of the ball when it moves more swiftly is very nearly in the squared ratio of the velocity; when more slowly, a little greater than in that ratio.

a. In ed. 1 the general scholium appears at the end of book 2, sec. 7, with some variations, primarily in the numerical values.

Now let V designate the greatest velocity in any oscillation, and let A, B, and C be given quantities, and let us imagine the difference between the arcs to be $AV + BV^{3/2} + CV^2$. In a cycloid the greatest velocities are as halves of the arcs described in oscillating, but in a circle they are as the chords of halves of these arcs, and thus with equal arcs are greater in a cycloid than in a circle in the ratio of halves of the arcs to their chords, while the times in a circle are greater than in a cycloid in the inverse ratio of the velocity. Hence it is evident that the differences between the arcs (differences which are as the resistance and the square of the time jointly) would be very nearly the same in both curves. For those differences in the cycloid would have to be increased along with the resistance in roughly the squared ratio of the arc to the chord (because the velocity is increased in the simple ratio of the arc to the chord) and would have to be decreased along with the square of the time in that same squared ratio. Therefore, in order to reduce all of this to the cycloid, take the same differences between the arcs that were observed in the circle, while supposing the greatest velocities to correspond to the arcs, whether halved or entire, that is, to the numbers ½, 1, 2, 4, 8, and 16. In the second, fourth, and sixth cases, therefore, let us write the numbers 1, 4, and 16 for V; and the difference between the arcs will come out $\frac{1/2}{121} = A + B + C$ in the second case; $\frac{2}{35\frac{1}{2}} = 4A + 8B + 16C$ in the fourth case; and $\frac{8}{9\frac{2}{3}} = 16A + 64B + 256C$ in the sixth case. And by the proper analytic reduction of these equations taken together, A becomes $= 0.0000916$, $B = 0.0010847$, and $C = 0.0029558$. The difference between the arcs is therefore as $0.0000916V + 0.0010847V^{3/2} + 0.0029558V^2$; and therefore—since (by prop. 30, corol., applied to this case) the resistance of the ball in the middle of the arc described by oscillating, where the velocity is V, is to its weight as $\frac{7}{11}AV + \frac{7}{10}BV^{3/2} + \frac{3}{4}CV^2$ is to the length of the pendulum—if the numbers found are written for A, B, and C, the resistance of the ball will become to its weight as $0.0000583V + 0.0007593V^{3/2} + 0.0022169V^2$ is to the length of the pendulum between the center of suspension and the ruler, that is, to 121 inches. Hence, since V in the second case has the value 1, in the fourth 4, and in the sixth 16, the resistance will be to the weight of the ball in the second case as 0.0030345 to 121, in the fourth as 0.041748 to 121, and in the sixth as 0.61705 to 121.

The arc which in the sixth case was described by the point marked on the thread was $120 - \frac{8}{9\frac{2}{3}}$ or $119\frac{5}{29}$ inches. And therefore, since the radius was 121 inches, and the length of the pendulum between the point of suspension and the center of the ball was 126 inches, the arc that the center of the ball described was $124\frac{3}{31}$ inches. Since, because of the resistance of the air, the greatest velocity of an oscillating body does not occur at the lowest point of the arc described but is located near the midpoint of the whole arc, that velocity will be roughly the same as if the ball in its whole descent in a nonresisting medium described half that arc ($62\frac{3}{62}$ inches) and did so in a cycloid, to which we have above reduced the motion of the pendulum; and therefore that velocity will be equal to the velocity which the ball could acquire by falling perpendicularly and describing in its fall a space equal to the versed sine of that arc. But that versed sine in the cycloid is to that arc ($62\frac{3}{62}$) as that same arc is to twice the length of the pendulum (252) and thus is equal to 15.278 inches. Therefore the velocity is the very velocity that the body could acquire by falling and describing in its fall a space of 15.278 inches. With such a velocity, then, the ball encounters a resistance that is to its weight as 0.61705 to 121, or (if only that part of the resistance is considered which is in the squared ratio of the velocity) as 0.56752 to 121.

By a hydrostatic experiment, I found that the weight of this wooden ball was to the weight of a globe of water of the same size as 55 to 97; and therefore, since 121 is to 213.4 in the same ratio as 55 to 97, the resistance of a globe of water moving forward with the above velocity will be to its weight as 0.56752 to 213.4, that is, as 1 to $376\frac{1}{50}$. The weight of the globe of water, in the time during which the globe describes a length of 30.556 inches with a uniformly continued velocity, could generate all that velocity in the globe if it were falling; hence it is manifest that in the same time the force of resistance uniformly continued could take away a velocity smaller in the ratio of 1 to $376\frac{1}{50}$, that is, $\frac{1}{376\frac{1}{50}}$ of the whole velocity. And therefore in the same time in which the globe, with that velocity uniformly continued, could describe the length of its own semidiameter, or $3\frac{7}{16}$ inches, it would lose $\frac{1}{3{,}342}$ of its motion.

I also counted the oscillations in which the pendulum lost a fourth of its motion. In the following table the top numbers denote the length of the arc described in the first descent, expressed in inches and parts of an inch; the middle numbers signify the length of the arc described in the final ascent; and at the bottom stand the numbers of oscillations. I have described this experiment because it is more accurate than when only an eighth of the motion was lost. Let anyone who wishes test the computation.

First descent	2	4	8	16	32	64
Final ascent	1½	3	6	12	24	48
Number of oscillations	374	272	162½	83⅓	41⅔	22⅔

Later, using the same thread, I suspended a lead ball with a diameter of 2 inches and a weight of 26¼ ounces avoirdupois, in such a way that the distance between the center of the ball and the point of suspension was 10½ feet, and I counted the oscillations in which a given part of the motion was lost. The first of the following tables shows the number of oscillations in which an eighth of the whole motion was lost; the second shows the number of oscillations in which a fourth of it was lost.

First descent	1	2	4	8	16	32	64
Final ascent	⅞	7/4	3½	7	14	28	56
Number of oscillations	226	228	193	140	90½	53	30
First descent	1	2	4	8	16	32	64
Final ascent	¾	1½	3	6	12	24	48
Number of oscillations	510	518	420	318	204	121	70

Select the third, fifth, and seventh observations from the first table and represent the greatest velocities in these particular observations by the numbers 1, 4, and 16 respectively, and generally by the quantity V as above; then it will be the case that in the third observation $\frac{1/2}{193} = A+B+C$, in the fifth $\frac{2}{90\frac{1}{2}} = 4A+8B+16C$, in the seventh $\frac{8}{30} = 16A+64B+256C$. The reduction of these equations gives $A = 0.001414$, $B = 0.000297$, $C = 0.000879$. Hence the resistance of the ball moving with velocity V comes out to have the ratio to its own weight (26¼ ounces) that $0.0009V + 0.000208V^{3/2} + 0.000659V^2$ has to the pendulum's length (121 inches). And if we consider only that part of the resistance which is in the squared ratio of the velocity, it will be to the weight of the ball as $0.000659V^2$ is to 121 inches. But in the first experiment this part of the resistance was to the weight of the wooden ball (57 7/22 ounces) as $0.002217V^2$ to 121; and hence the resistance of the wooden ball becomes to the resistance of the lead ball (their velocities being equal) as $57\frac{7}{22} \times 0.002217$ to $26\frac{1}{4} \times 0.000659$, that is, as $7\frac{1}{3}$ to 1. The diameters of the two balls were $6\frac{7}{8}$ and 2 inches, and the squares of these are to each other as $47\frac{1}{4}$ and 4, or $11\frac{13}{16}$ and 1, very nearly. Therefore the resistances of equally swift balls were in a smaller ratio than the squared ratio of the diameters. But we have not yet considered the resistance of the thread, which certainly was very great and ought to be subtracted from the resistance of the pendulum that has been found. I could not determine this resistance of the thread accurately, but nevertheless I found it to be greater than a third of the whole resistance of the smaller pendulum; and I learned from this that the resistances of the balls, taking away the resistance of the thread, are very nearly in the squared ratio of the diameters. For the ratio of $7\frac{1}{3} - \frac{1}{3}$ to $1 - \frac{1}{3}$, or $10\frac{1}{2}$ to 1, is very close to the squared ratio of the diameters $11\frac{13}{16}$ to 1.

Since the resistance of the thread is of less significance in larger balls, I also tried the experiment in a ball whose diameter was $18\frac{3}{4}$ inches. The length of the pendulum between the point of suspension and the center of oscillation was $122\frac{1}{2}$ inches; between the point of suspension and a knot in the thread, $109\frac{1}{2}$ inches. The arc described by the knot in the first descent of the pendulum was 32 inches. The arc described by that same knot in the final ascent after five oscillations was 28 inches. The sum of the arcs, or the whole arc described in a mean oscillation, was 60 inches. The difference between the arcs was 4 inches. A tenth of it, or the difference between the descent and the ascent in a mean oscillation, was $\frac{2}{5}$ inch. The ratio of the radius $109\frac{1}{2}$ to the radius $122\frac{1}{2}$ is the same as the ratio of the whole arc of 60 inches described by the knot in a mean oscillation to the whole arc of $67\frac{1}{8}$ inches described by the center of the ball in a mean oscillation, and is equal to the ratio of the difference $\frac{2}{5}$ to the new difference 0.4475. If the length of the pendulum were to be increased in the ratio of 126 to $122\frac{1}{2}$ while the length of the arc described remained the same, the time of oscillation would be increased and the velocity of the pendulum would be decreased as the square root of that ratio, while the difference 0.4475 between the arcs described in a descent and subsequent ascent would remain the same. Then, if the arc described were to be increased in the ratio of $124\frac{3}{31}$ to $67\frac{1}{8}$, that difference 0.4475 would be increased as the square of that ratio, and thus would come out 1.5295. These things would be so on the hypothesis that the resistance of the pendulum was in the squared ratio of the velocity. Therefore, if the pendulum were to describe a whole arc of $124\frac{3}{31}$ inches, and its length between the point of suspension and the center of oscillation were 126 inches, the difference between the arcs described in a descent and subsequent ascent would be 1.5295 inches. And this difference multiplied by the weight of the ball of the pendulum, which was 208 ounces, yields the product 318.136. Again, when the above-mentioned pendulum (made with a wooden ball) described a whole arc of $124\frac{3}{31}$ inches by its center of oscillation (which was 126 inches distant from the point of suspension), the difference between the arcs described in the descent and ascent was $\frac{126}{121} \times \frac{8}{9\frac{2}{3}}$, which multiplied by the weight of the ball (which was $57\frac{7}{22}$ ounces) yields the product 49.396. And I multiplied these differences by the weights of the balls in order to find their resistances. For the differences arise from the resistances and are as the resistances directly and the weights inversely. The resistances are therefore as the numbers 318.136 and 49.396. But the part of the resistance of the smaller ball that is in the squared ratio of the velocity was to the whole resistance as 0.56752 to 0.61675, that is, as 45.453 to 49.396; and the similar part of the resistance of the larger ball is almost equal to its whole resistance; and thus those parts are very nearly as 318.136 and 45.453, that is, as 7 and 1. But the diameters of the balls are $18\frac{3}{4}$ and $6\frac{7}{8}$, and the squares of these diameters, $351\frac{9}{16}$ and $47\frac{17}{64}$, are as 7.438 and 1, that is, very nearly as the resistances 7 and 1 of the balls. The difference between the ratios is no greater than what could have arisen from the resistance of the thread. Therefore, those parts of the resistances that are (the balls being equal) as the squares of the velocities are also (the velocities being equal) as the squares of the diameters of the balls.

The largest ball that I used in these experiments, however, was not perfectly spherical, and therefore for the sake of brevity I have ignored certain minutiae in the above computation, being not at all worried about a computation being exact when the experiment itself was not sufficiently exact. Therefore, since the demonstration of a vacuum depends on such experiments, I wish that they could be tried with more, larger, and more exactly spherical balls. If the balls are taken in geometric proportion, say with diameters of 4, 8, 16, and 32 inches, it will be discovered from the progression of the experiments what ought to happen in the case of still larger balls.

To compare the resistances of different fluids with one another, I made the following experiments. I got a wooden box four feet long, one foot wide, and one foot deep. I took off its lid and filled it with fresh water, and I immersed pendulums in the water and made them oscillate. A lead ball weighing 166⅙ ounces, with a diameter of 3⅝ inches, moved as in the following table, that is, with the length of the pendulum from the point of suspension to a certain point marked on the thread being 126 inches, and to the center of oscillation being 134⅜ inches.

Arc described by the point marked on the thread in the first descent	64″	32″	16″	8″	4″	2″	1″	½″	¼″
Arc described in the final ascent	48″	24″	12″	6″	3″	1½″	¾″	⅜″	$\frac{3}{16}$″
Difference between the arcs, proportional to the motion lost	16″	8″	4″	2″	1″	½″	¼″	⅛″	$\frac{1}{16}$″
Number of oscillations in water			$\frac{29}{60}$	1⅕	3	7	11¼	12⅔	13⅓
Number of oscillations in air	85½	287	535						

In the experiment recorded in the fourth column, equal motions were lost in 535 oscillations in air, and 1⅕ in water. The oscillations were indeed a little quicker in air than in water. But if the oscillations in water were accelerated in such a ratio that the motions of the pendulums in both mediums would become equally swift, the number 1⅕ oscillations in water during which the same motion would be lost as before would remain the same because the resistance is increased and the square of the time simultaneously decreased in that same ratio squared. With equal velocities of the pendulums, therefore, equal motions were lost, in air in 535 oscillations and in water in 1⅕ oscillations; and thus the resistance of the pendulum in water is to its resistance in air as 535 to 1⅕. This is the proportion of the whole resistances in the case of the fourth column.

Now let $AV + CV^2$ designate the difference between the arcs described (in a descent and subsequent ascent) by the ball moving in air with the greatest velocity V; and since the greatest velocity in the case of the fourth column is to the greatest velocity in the case of the first column as 1 to 8, and since that difference between the arcs in the case of the fourth column is to the difference in the case of the first column as $\frac{2}{535}$ to $\frac{16}{85\frac{1}{2}}$, or as 85½ to 4,280, let us write 1 and 8 for the velocities in these cases and 85½ and 4,280 for the differences between the arcs; then $A + C$ will become $= 85\frac{1}{2}$

and $8A + 64C = 4,280$ or $A + 8C = 535$; and hence, by reduction of the equations, $7C$ will become $= 449\frac{1}{2}$ and $C = 64\frac{3}{14}$ and $A = 21\frac{2}{7}$; and thus the resistance, since it is as $\frac{7}{11}AV + \frac{3}{4}CV^2$, will be as $13\frac{6}{11}V + 48\frac{9}{56}V^2$. Therefore, in the case of the fourth column, where the velocity was 1, the whole resistance is to its part proportional to the square of the velocity as $13\frac{6}{11} + 48\frac{9}{56}$ or $61\frac{12}{17}$ to $48\frac{9}{56}$; and on that account the resistance of the pendulum in water is to that part of the resistance in air which is proportional to the square of the velocity (and which alone comes into consideration in swifter motions) as $61\frac{12}{17}$ to $48\frac{9}{56}$ and 535 to $1\frac{1}{5}$ jointly, that is, as 571 to 1. If the whole thread of the pendulum oscillating in water had been immersed, its resistance would have been still greater, to such an extent that the part of the resistance of the pendulum oscillating in water which is proportional to the square of the velocity (and which alone comes into consideration in swifter bodies) is to the resistance of that same whole pendulum oscillating in air, with the same velocity, as about 850 to 1, that is, very nearly as the density of water to the density of air.

In this computation also, that part of the resistance of the pendulum in water which would be as the square of the velocity ought to be taken into consideration, but (which may perhaps seem strange) the resistance in water was increased in more than the squared ratio of the velocity. In searching for the reason, I hit upon this: that the box was too narrow in proportion to the size of the ball of the pendulum, and because of its narrowness overly impeded the motion of the water as it yielded to the oscillation of the ball. For if a ball of a pendulum whose diameter was one inch was immersed, the resistance was increased in very nearly the squared ratio of the velocity. I tested this by constructing a pendulum out of two balls, so that the lower and smaller of them oscillated in the water, and the higher and larger one was fastened to the thread just above the water and, by oscillating in the air, aided the pendulum's motion and made it last longer. And the experiments made with this pendulum came out as in the following table.

Arc described in the first descent	16″	8″	4″	2″	1″	½″	¼″
Arc described in the final ascent	12″	6″	3″	1½″	¾″	⅜″	3/16″
Difference between the arcs, proportional to the motion lost	4″	2″	1″	½″	¼″	⅛″	1/16″
Number of oscillations	3⅜	6½	12 1/12	21⅕	34	53	62⅕

In comparing the resistances of the mediums with one another I also caused iron pendulums to oscillate in quicksilver. The length of the iron wire was about three feet, and the diameter of the ball of the pendulum was about 1/3 inch. And to the wire just above the mercury there was fastened another lead ball large enough to continue the motion of the pendulum for a longer time. Then I filled a small vessel (which held about three pounds of quicksilver) with quicksilver and common water successively, so that as the pendulum oscillated first in one and then in the other of the two fluids I might find the proportion of the resistances; and the resistance of the quicksilver came out to the resistance of the water as about 13 or 14 to 1, that is, as the density of quicksilver to the density of water. When I used [b]a slightly larger pendulum ball, say one whose diameter would be about 1/3 or 2/3 inch,[b] the resistance of the quicksilver came out in the ratio to the resistance of the water that the number 12 or 10 has to 1, roughly. But the former experiment is more trustworthy because in the latter the vessel was too narrow in proportion to the size of the immersed ball. With the ball enlarged, the vessel also would have to be enlarged. Indeed, I had determined to repeat experiments of this sort in larger vessels and in molten metals and certain other liquids, hot as well as cold; but there is not time to try them all, and from what has already been described it is clear enough that the resistance of bodies moving swiftly is very nearly proportional to the density of the fluids in which they move. I do not say exactly proportional. For the more viscous fluids, of an equal density, doubtless resist more than the more liquid fluids—as, for example, cold oil more than hot, hot oil more than rainwater, water more than spirit of wine. But in the liquids that are sufficiently fluid to the senses—as in air, in water (whether fresh or salt), in spirits of wine, of turpentine, and of salts, in oil freed of its dregs by distillation and then heated, and in oil of vitriol and in mercury, and in liquefied metals, and any others there may be which are so fluid that when agitated in vessels they conserve for some time a motion impressed upon them and when poured out are quite freely broken

bb. Here Newton makes a puzzling statement, namely, that the diameter of this ball, "about 1/3 or 2/3 inch," was larger than the one mentioned earlier, which was "about 1/3 inch." The source of this puzzling "about 1/3 or 2/3 inch" may be seen by comparing the various editions, as is done in our Latin edition. In the printer's manuscript and in ed. 1, the larger ball is said to have a diameter of "about 1/2 or 2/3 inch," which in ed. 2 was wrongly printed as "about 1/3 or 2/3 inch." In Newton's annotated copy of the *Principia*, it was noted that this should be corrected to "about 1/2 or 2/3 inch," but this was not done in ed. 3.

up into falling drops—in all these I have no doubt that the above rule holds exactly enough, especially if the experiments are made with pendulums that are larger and move more swiftly.

Finally, since [c]some people are of the opinion[c] that there exists a certain aethereal medium, by far the subtlest of all, which quite freely permeates all the pores and passages of all bodies, and that a resistance ought to arise from such a medium flowing through the pores of bodies, I devised the following experiment so that I might test whether the resistance that we experience in moving bodies is wholly on their external surface or whether the internal parts also encounter a perceptible resistance on their own surfaces. I suspended a round firwood box by a cord eleven feet long from a sufficiently strong steel hook, by means of a steel ring. The upper arc of the ring rested on the very sharp concave edge of the hook so that it might move very freely. And the cord was attached to the lower arc of the ring. I drew this pendulum away from the perpendicular to a distance of about six feet, and did so along the plane perpendicular to the edge of the hook, so that the ring, as the pendulum oscillated, would not slide to and fro on the edge of the hook. For the point of suspension, in which the ring touches the hook, ought to remain motionless. I marked the exact place to which I had drawn back the pendulum and then, letting the pendulum fall, marked another three places: those to which it returned at the end of the first, second, and third oscillations. I repeated this quite often, so that I might find those places as exactly as possible. Then I filled the box with lead and some of the other heavier metals that were at hand. But first I weighed the empty box along with the part of the cord that was wound around the box and half of the remaining part that was stretched between the hook and the suspended box. For a stretched cord always urges with half of its weight a pendulum drawn aside from the perpendicular. To this weight I added the weight of the air that the box contained. And the whole weight was about $\frac{1}{78}$ of the box full of metals. Then, since the box full of metals increased the length of the pendulum as a result of stretching the cord by its weight, I shortened the

cc. This reads literally: "the opinion of some is." Ed. 1 and ed. 2 have: "the most widely accepted opinion of the philosophers of this age is." The index prepared by Cotes for ed. 2 and retained for ed. 3 keys this opinion under "Materia" ("Matter") and specifies the "philosophers" (and hence the later "some") by thus describing the paragraph: "The subtle matter of the Cartesians is subjected to a certain examination."

cord so that the length of the pendulum now oscillating would be the same as before. Then, drawing the pendulum back to the first marked place and letting it fall, I counted about 77 oscillations until the box returned to the second marked place, and as many thereafter until the box returned to the third marked place, and again as many until the box on its return reached the fourth place. Hence I conclude that the whole resistance of the full box did not have a greater proportion to the resistance of the empty box than 78 to 77. For if the resistances of both were equal, the full box, because its inherent force was 78 times greater than the inherent force of the empty box, ought to conserve its oscillatory motion that much longer, and thus always return to those marked places at the completion of 78 oscillations. But it returned to them at the completion of 77 oscillations.

Let A therefore designate the resistance of the box on its external surface, and B the resistance of the empty box on its internal parts; then, if the resistances of equally swift bodies on their internal parts are as the matter, or the number of particles that are resisted, 78B will be the resistance of the full box on its internal parts; and thus the whole resistance $A + B$ of the empty box will be to the whole resistance $A + 78B$ of the full box as 77 to 78, and by separation [or dividendo] $A + B$ will be to $77B$ as 77 to 1, and hence $A + B$ will be to B as 77×77 to 1, and by separation [or dividendo] A will be to B as 5,928 to 1. The resistance encountered by the empty box on its internal parts is therefore more than 5,000 times smaller than the similar resistance on the external surface. This argument depends on the hypothesis that the greater resistance encountered by the full box does not arise from some other hidden cause but only from the action of some subtle fluid upon the enclosed metal.

I have reported this experiment from memory. For the paper on which I had once described it is lost. Hence I have been forced to omit certain fractions of numbers which have escaped my memory.

There is no time to try everything again. The first time, since I had used a weak hook, the full box was retarded more quickly. In seeking the cause, I found that the hook was so weak as to give way to the weight of the box and to be bent in this direction and that as it yielded to the oscillations of the pendulum. I got a strong hook, therefore, so that the point of suspension would remain motionless, and then everything came out as we have described it above.

SECTION 7[a]

The motion of fluids and the resistance encountered by projectiles

Proposition 32
Theorem 26

Let two similar systems of bodies consist of an equal number of particles, and let each of the particles in one system be similar and proportional to the corresponding particle in the other system, and let the particles be similarly situated with respect to one another in the two systems and have a given ratio of density to one another. And let them begin to move similarly with respect to one another in proportional times (the particles that are in the one system with respect to the particles in that system, and the particles in the other with respect to those in the other). Then, if the particles that are in the same system do not touch one another except in instants of reflection and do not attract or repel one another except by accelerative forces that are inversely as the diameters of corresponding particles and directly as the squares of the velocities, I say that the particles of the systems will continue to move similarly with respect to one another in proportional times.

I say that bodies which are similar and similarly situated move similarly with respect to one another in proportional times when their situations in relation to one another are always similar at the end of the times—for instance, if the particles of one system are compared with the corresponding particles of another. Hence the times in which similar and proportional parts of similar figures are described by corresponding particles will be proportional. Therefore, if there are two systems of this sort, the corresponding particles, because of the similarity of their motions at the beginning, will continue to move similarly until they meet one another. For if they are acted upon by no forces, they will, by the first law of motion, move forward uniformly in straight lines. If they act upon one another by some forces and if those forces are as the diameters of the corresponding particles inversely and the squares of the velocities directly, then, since the situations of the particles are similar and the forces proportional, the whole forces by which the corresponding particles are acted upon, compounded of the separate acting forces (by corol. 2 of the laws), will have similar directions, just as if they tended to centers similarly

a. In ed. 1, sec. 7 is very different. Props. 32–34 (32–35 in ed. 1) underwent partial alteration, including the suppression of the original prop. 34. The remainder of sec. 7 was completely rewritten for ed. 2 and essentially retained, with only minor changes, in ed. 3. For details, see the Guide to the present translation, §7.6.

placed among the particles, and those whole forces will be to one another as the separate component forces, that is, as the diameters of the corresponding particles inversely and the squares of the velocities directly, and therefore they will cause corresponding particles to continue describing similar figures. This will be so (by book 1, prop. 4, corols. 1 and 8) provided that the centers are at rest. But if they move, since their situations with respect to the particles of the systems remain similar (because the transferences are similar), similar changes will be introduced in the figures which the particles describe. The motions of corresponding similar particles will therefore be similar until they first meet, and therefore the collisions will be similar and the reflections similar, and then (by what has already been shown) the motions of the particles with respect to one another will be similar until they encounter one another again, and so on indefinitely. Q.E.D.

Corollary 1. Hence, if any two bodies that are similar and similarly situated (in relation to the corresponding particles of the systems) begin to move similarly with respect to the particles in proportional times, and if their volumes and densities are to each other as the volumes and densities of the corresponding particles, the bodies will continue to move similarly in proportional times. For the case is the same for the larger parts of both systems as for the particles.

Corollary 2. And if all the similar and similarly situated parts of the systems are at rest with respect to one another, and if two of them, which are larger than the others and correspond to each other in the two systems, begin to move in any way with a similar motion along lines similarly situated, they will cause similar motions in the remaining parts of the systems and will continue to move similarly with respect to them in proportional times and thus will continue to describe spaces proportional to their own diameters.

Proposition 33
Theorem 27

If the same suppositions are made, I say that the larger parts of the systems are resisted in a ratio compounded of the squared ratio of their velocities and the squared ratio of the diameters and the simple ratio of the density of the parts of the systems.

For the resistance arises partly from the centripetal or centrifugal forces with which the particles of the systems act upon one another and partly from the collisions and reflections of the particles and the larger parts. Resistances of the first kind, moreover, are to one another as the whole motive forces from which they arise, that is, as the whole accelerative forces and the quantities of matter in corresponding parts, that is (by hypothesis), as the squares of the velocities directly and the distances of the corresponding particles inversely and the quantities of matter in the corresponding parts directly. Thus, since the distances of the particles of the one system are to the corresponding distances of the particles of the other as the diameter of a particle or part in the first system to the diameter of the corresponding particle or part in the other, and since the quantities of matter are as the densities of the parts and the cubes of the diameters, the resistances are to one another as the squares of the velocities, the squares of the diameters, and the densities of the parts of the systems. Q.E.D.

Resistances of the second kind are as the numbers and forces of corresponding reflections jointly. The number of reflections in any one case, moreover, is to the number in any other as the velocities of the corresponding parts directly and the spaces between their reflections inversely. And the forces of the reflections are as the velocities and volumes and densities of the corresponding parts jointly, that is, as the velocities, the cubes of the diameters, and the densities of the parts. And if all these ratios are compounded, the resistances of the corresponding parts are to one another as the squares of the velocities, the squares of the diameters, and the densities of the parts, jointly. Q.E.D.

Corollary 1. Therefore, if the systems are two elastic fluids such as air and if their parts are at rest with respect to one another, and if two bodies which are similar and are proportional (with regard to volume and density) to the parts of the fluids and are similarly situated with respect to those parts are projected in any way along lines similarly situated, and if the accelerative forces with which the particles of the fluids act upon one another are as the diameters of the projected bodies inversely and the squares of the velocities directly, then the bodies will cause similar motions in the fluids in proportional times and will describe spaces that are similar and are proportional to their diameters.

Corollary 2. Accordingly, in the same fluid a swift projectile encounters a resistance that is very nearly in the squared ratio of the velocity. For if the forces with which distant particles act upon one another were increased in the squared ratio of the velocity, the resistance would be exactly in the squared ratio of the velocity; and thus, in a medium whose parts act upon one another with no forces because they are far apart, the resistance is exactly in the squared ratio of the velocity. Let A, B, and C, therefore, be three mediums consisting of parts that are similar and equal and regularly distributed at equal distances. Let the parts of mediums A and B recede from one another with forces that are to one another as T and V, and let the parts of medium C be entirely without forces of this sort. Then, let four equal bodies D, E, F, and G move in these mediums, the first two bodies D and E in the first two mediums A and B respectively, and the other two bodies F and G in the third medium C; and let the velocity of body D be to the velocity of body E, and let the velocity of body F be to the velocity of body G, as the square root of the ratio of the forces T to the forces V [i.e., as $\sqrt{T}$ to $\sqrt{V}$]; then the resistance of body D will be to the resistance of body E, and the resistance of body F to the resistance of body G, in the squared ratio of the velocities; and therefore the resistance of body D will be to the resistance of body F as the resistance of body E to the resistance of body G. Let bodies D and F have equal velocities, and also bodies E and G; then, if the velocities of bodies D and F are increased in any ratio and the forces of the particles of medium B are decreased in the same ratio squared, medium B will approach the form and condition of medium C as closely as is desired, and on that account the resistances of the equal and equally swift bodies E and G in these mediums will continually approach equality, in such a way that their difference finally comes out less than any given difference. Accordingly, since the resistances of bodies D and F are to each other as the resistances of bodies E and G, these also will similarly approach the ratio of equality. Therefore, the resistances of bodies D and F, when they move very swiftly, are very nearly equal, and therefore, since the resistance of body F is in the squared ratio of the velocity, the resistance of body D will be very nearly in the same ratio.

COROLLARY 3. The resistance of a body moving very swiftly in any elastic fluid is about the same as if the parts of the fluid lacked their centrifugal forces and did not recede from one another, provided that the elastic force of the fluid arises from the centrifugal forces of the particles and that the velocity is so great that the forces do not have enough time to act.

COROLLARY 4. Accordingly, since the resistances of similar and equally swift bodies, in a medium whose parts (being far apart) do not recede from one another, are as the squares of the diameters, the resistances of equally swift and very quickly moving bodies in an elastic fluid are also very nearly as the squares of the diameters.

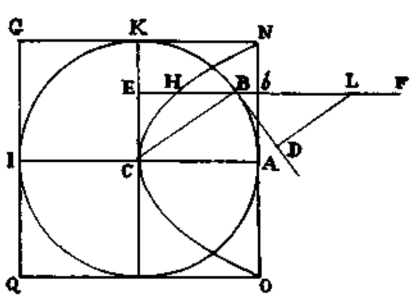
COROLLARY 5. And since similar, equal, and equally swift bodies, in mediums which have the same density and whose particles do not recede from one another, impinge upon an equal quantity of matter in equal times (whether the particles are more and smaller or fewer and larger) and impress upon it an equal quantity of motion and in turn (by the third law of motion) undergo an equal reaction from it (that is, are equally resisted), it is manifest also that in elastic fluids of the same density, when the bodies move very swiftly, the resistances they encounter are very nearly equal, whether those fluids consist of coarser particles or are made of the most subtle particles of all. The resistance to projectiles moving very quickly is not much diminished as a result of the subtlety of the medium.

COROLLARY 6. These statements all hold for fluids whose elastic force originates in the centrifugal forces [i.e., forces of repulsion] of the particles. But if that force arises from some other source, such as the expansion of the particles in the manner of wool or the branches of trees, or from any other cause which makes the particles move less freely with respect to one another, then the resistance will be greater than in the preceding corollaries because the medium is less fluid.

Proposition 34
Theorem 28

In a rare medium consisting of particles that are equal and arranged freely at equal distances from one another, let a sphere and a cylinder—described with equal diameters—move with equal velocity along the direction of the axis of the cylinder; then the resistance of the sphere will be half the resistance of the cylinder.

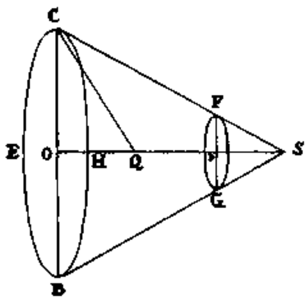
For since the action of a medium on a body is (by corol. 5 of the laws) the same whether the body moves in a medium at rest or the particles of the medium impinge with the same velocity on the body at rest, let us consider the body to be at rest and see with what force it will be urged by the moving medium. Let ABKI, therefore, designate a spherical body described with center C and semidiameter CA, and let the particles of the



medium strike the spherical body with a given velocity along straight lines parallel to AC; and let FB be such a straight line. On FB take LB equal to the semidiameter CB, and draw BD touching the sphere in B. To KC and BD drop the perpendiculars BE and LD; then the force with which a particle of the medium, obliquely incident along the straight line FB, strikes the sphere at B will be to the force with which the same particle would strike the cylinder ONGQ (described with axis ACI about the sphere) perpendicularly at *b* as LD to LB or BE to BC. Again, the efficacy of this force to move the sphere along the direction FB (or AC) of its incidence is to its efficacy to move the sphere along the direction of its determination—that is, along the direction of the straight line BC in which it urges the sphere directly [a direction through the center of the sphere]—as BE to BC. And, compounding the ratios, if a particle strikes the sphere obliquely along the straight line FB, its efficacy to move the sphere along the direction of its incidence is to the efficacy of the same particle to move the cylinder in the same direction, when striking the cylinder perpendicularly along the same straight line, as BE^2 to BC^2. Therefore, if in *b*E, which is perpendicular to the circular base NAO of the cylinder and equal to the radius AC, *b*H is taken equal to $\frac{BE^2}{CB}$, then *b*H will be to *b*E as the effect of a particle upon the sphere to the effect of the particle upon the cylinder. And therefore the solid that is composed of all the straight lines *b*H will be to the solid that is composed of all the straight lines *b*E as the effect of all the particles upon the sphere to the effect of all the particles upon the cylinder. But the first solid is a paraboloid described with vertex C, axis CA, and latus rectum CA, and the second solid is a cylinder circumscribed around the paraboloid; and it is known that a paraboloid is half of the circumscribed cylinder. Therefore the whole force of the medium upon the sphere is half of its whole force upon the cylinder. And therefore, if the particles of the medium were at rest and the cylinder and the sphere were moving with equal velocity, the resistance of the sphere would be half the resistance of the cylinder. Q.E.D.

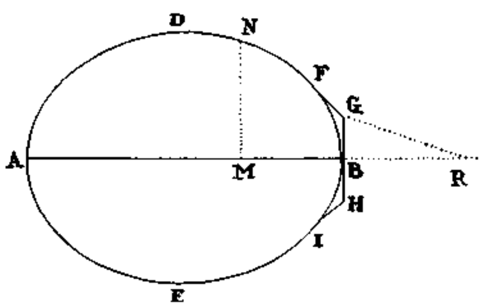
Scholium

By the same method other figures can be compared with one another with respect to resistance, and those that are more suitable for continuing their motions in resisting mediums can be found. For example, let it be required



to construct a frustum CBGF of a cone with the circular base CEBH (which is described with center O and radius OC) and with the height OD, which is resisted less than any other frustum constructed with the same base and height and moving forward along the direction of the axis toward D; bisect the height OD in Q, and produce OQ to S so that QS is equal to QC, and S will be the vertex of the cone whose frustum is required.

Note in passing that since the angle CSB is always acute, it follows that if the solid ADBE is generated by a revolution of the elliptical or oval figure ADBE about the axis AB, and if the generating figure is touched by the three straight lines FG, GH, and HI in points F, B, and I, in such a way that GH is perpendicular to the axis in the point of contact B, and FG and HI meet the said line GH at the angles FGB and BHI of 135 degrees, then the solid that is generated by the revolution of the figure ADFGHIE about the same axis AB is less resisted than the former solid, provided that each of the two moves forward along the direction of its axis AB, and the end B of each one is in front. Indeed, I think that this proposition will be of some use for the construction of ships.



But suppose the figure DNFG to be a curve of such a sort that if the perpendicular NM is dropped from any point N of that curve to the axis AB, and if from the given point G the straight line GR is drawn, which is parallel to a straight line touching the figure in N and cuts the axis (produced) in R, then MN would be to GR as GR^3 to $4BR \times GB^2$. Then, in this case, the solid that is described by a revolution of this figure about the axis AB will, in moving in the aforesaid rare medium from A toward B, be resisted less than any other solid of revolution described with the same length and width.

Proposition 35[a]
Problem 7

If a rare medium consists of minimally small equal particles that are at rest and arranged freely at equal distances from one another, it is required to find the resistance encountered by a sphere moving forward uniformly in this medium.

CASE 1. Let a cylinder described with the same diameter and height as before move forward with the same velocity along the length of its own axis in the same medium. And let us suppose that the particles of the medium upon which the sphere or cylinder impinges rebound with the greatest possible force of reflection. Then the resistance of the sphere (by prop. 34) is half the resistance of the cylinder, and the sphere is to the cylinder as 2 to 3, and the cylinder in impinging perpendicularly upon the particles and reflecting them as greatly as possible communicates twice its own velocity to them. Therefore, the cylinder, in the time in which it describes half the length of its axis by moving uniformly forward, will communicate to the particles a motion which is to the whole motion of the cylinder as the density of the medium is to the density of the cylinder; and the sphere, in the time in which it describes the whole length of its diameter by moving uniformly forward, will communicate the same motion to the particles, and in the time in which it describes ⅔ of its diameter it will communicate to the particles a motion which is to the whole motion of the sphere as the density of the medium to the density of the sphere. And therefore the sphere encounters a resistance that is to the force by which its whole motion could be either destroyed or generated, in the time in which it describes ⅔ of its diameter by moving uniformly forward, as the density of the medium is to the density of the sphere.

CASE 2. Let us suppose that the particles of the medium impinging upon the sphere or cylinder are not reflected; then the cylinder, in impinging perpendicularly upon the particles, will communicate its whole velocity to them and thus encounters half the resistance which it met in the former case, and the resistance encountered by the sphere will also be half of what it was before.

CASE 3. Let us suppose that the particles of the medium rebound from the sphere with a force of reflection that is neither the greatest nor nil but

a. A translation of the versions of book 2, props. 35–40, that appear in the first edition has been made by I. Bernard Cohen and Anne Whitman and will be published, together with a commentary by George Smith, in *Newton's Natural Philosophy*, ed. Jed Buchwald and I. Bernard Cohen (Cambridge: MIT Press, forthcoming).

some intermediate force; then the resistance encountered by the sphere will also be intermediate between the resistance in case 1 and the resistance in case 2. Q.E.I.

Corollary 1. Hence, if the sphere and the particles are infinitely hard without any elastic force and therefore also without any force of reflection, the resistance encountered by the sphere will be to the force by which its whole motion could be either destroyed or generated, in the time in which the sphere describes ⅔ of its diameter, as the density of the medium is to the density of the sphere.

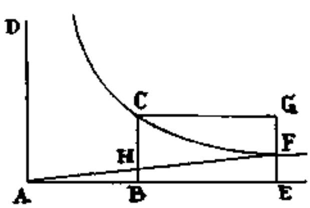
Corollary 2. The resistance encountered by the sphere, other things being equal, is in the squared ratio of the velocity.

Corollary 3. The resistance encountered by the sphere, other things being equal, is in the squared ratio of the diameter.

Corollary 4. The resistance encountered by the sphere, other things being equal, is as the density of the medium.

Corollary 5. The resistance encountered by the sphere is in a ratio that is compounded of the squared ratio of the velocity and the squared ratio of the diameter, and the simple ratio of the density of the medium.

Corollary 6. And the motion of the sphere with the resistance it encounters can be represented as follows. Let AB be the time in which the sphere can lose its whole motion when the resistance is continued uniformly. Erect AD and BC perpendicular to AB. And let BC be the whole motion, and through point C with asymptotes AD and AB describe the hyperbola CF. Produce AB to any point E. Erect the perpendicular EF meeting the hyperbola in F. Complete the parallelogram CBEG, and draw AF meeting BC in H. Then, if the sphere, in any time BE, when its first motion BC is continued uniformly, in a nonresisting medium, describes the space CBEG represented by the area of the parallelogram, it will in a resisting medium describe the space CBEF represented by the area of the hyperbola, and its motion at the end of that time will be represented by the ordinate EF of the hyperbola, with loss of part FG of its motion. And the resistance at the end of the same time will be represented by the length BH, with loss of part CH of the resistance. All of this is evident by book 2, prop. 5, corols. 1 and 3.



COROLLARY 7. Hence, if in time T, when the resistance R is continued uniformly, the sphere loses its whole motion M, then in time t in a resisting medium, when the resistance R decreases in the squared ratio of the velocity, the sphere will lose part $\frac{tM}{T+t}$ of its motion M without loss of part $\frac{TM}{T+t}$; and the sphere will describe a space that is to the space described by the uniform motion M, in the same time t, as the logarithm of the number $\frac{T+t}{T}$ multiplied by the number 2.302585092994 is to the number $\frac{t}{T}$, because the hyperbolic area BCFE is in this proportion to the rectangle BCGE.

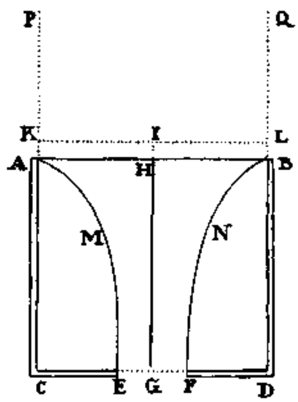
Scholium

In this proportion I have set forth the resistance and retardation encountered by spherical projectiles in noncontinuous mediums, and I have shown that this resistance is to the force by which the whole motion of a sphere could be either destroyed or generated in the time in which the sphere describes ⅔ of its diameter, with a velocity continued uniformly, as the density of the medium is to the density of the sphere, provided that the sphere and the particles of the medium are highly elastic and possess the greatest force of reflecting, and I have shown that this force is half as great when the sphere and the particles of the medium are infinitely hard and devoid of all force of reflecting. Moreover, in continuous mediums such as water, hot oil, and quicksilver, in which the sphere does not impinge directly upon all the particles of the fluid which generate resistance but presses only the nearest particles, and these press others and these still others, the resistance is half as great as in the second case. In extremely fluid mediums of this sort the sphere encounters a resistance that is to the force by which its whole motion could be either destroyed or generated, in the time in which it describes ⅛⁄₃ of its diameter with the motion continued uniformly, as the density of the medium is to the density of the sphere. We will try to show this in what follows.

Proposition 36
Problem 8

To determine the motion of water flowing out of a cylindrical vessel through a hole in the bottom.

Let ACDB be the cylindrical vessel, AB its upper opening, CD its bottom parallel to the horizon, EF a circular hole in the middle of the bottom, G the center of the hole, and GH the cylinder's axis perpendicular to the horizon.



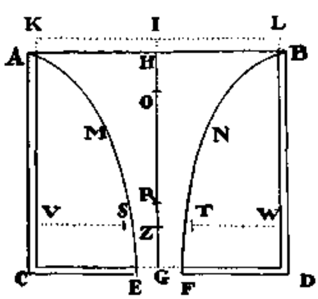
And imagine that a cylinder of ice APQB is of the same width as the interior of the vessel, has the same axis, and descends continually with a uniform motion. Imagine also that its parts liquefy as soon as they touch the surface AB, that when they have turned into water they flow down into the vessel as a result of their gravity, and that in falling these parts form a cataract or column of water ABNFEM and pass through the hole EF and fill it exactly. And let the uniform velocity of the descending ice, as well as that of the contiguous water in the circle AB, be the velocity which the water can acquire in falling and describing by its fall the space IH, and let IH and HG lie in a straight line, and through point I draw the straight line KL parallel to the horizon and meeting the sides of the ice in K and L. Then the velocity of the water flowing out through the hole EF will be that which the water can acquire in falling from I and describing by its fall the space IG. And thus, by Galileo's theorems, IG will be to IH as the square of the ratio of the velocity of the water flowing out through the hole to the velocity of the water in the circle AB, that is, as the square of the ratio of the circle AB to the circle EF, for these circles are inversely as the velocities of the water passing through them in the same time and with an equal quantity, filling them both exactly. Here it is the velocity of the water toward the horizon that is of concern. And the motion parallel to the horizon by which the parts of the falling water approach one another is not considered here, since it does not arise from gravity or change the motion perpendicular to the horizon that does arise from gravity. Indeed, we are supposing that the parts of the water cohere somewhat and that by their cohesion they approach one another with motions parallel to the horizon as they fall, so that they form only one single cataract and are not divided into several cataracts, but here we are not considering the motion parallel to the horizon arising from that cohesion.

Case 1. Now suppose that the interior of the vessel around the falling water ABNFEM is filled with ice, so that the water passes through the ice as if through a funnel. Then, if the water does not quite touch the ice, or (what comes to the same thing) if it touches it and, because of the great smoothness of the ice, slides through it with the greatest possible freedom and without any resistance, the water will flow down through the hole EF with the same velocity as before, and the whole weight of the column of water ABNFEM will be used in generating its downflow as before, and the bottom of the vessel will sustain the weight of the ice surrounding the column.

Now let the ice liquefy in the vessel; then the flow of the water will remain the same as before with respect to velocity. It will not be less, since the melted ice will endeavor to descend; and not greater, since the melted ice cannot descend without impeding an equal descent of the original water. The same force ought to generate the same velocity in the flowing water [i.e., since the force is the same, the velocity that it generates will also be the same].

But the hole in the bottom of the vessel, because of the oblique motions of the particles of the flowing water, ought to be a little larger than before. For now the particles of water do not all pass through the hole perpendicularly but, flowing together from all the sides of the vessel and converging into the hole, pass through with oblique motions and, turning their course downward, unite into a stream of water gushing out which is narrower a little below the hole than in the hole itself, its diameter being to the diameter of the hole as 5 to 6, or 5½ to 6½ very nearly, provided that I measured the diameters correctly. At any rate, I obtained a very thin flat plate perforated in the middle, the diameter of the circular hole being ⅝ inch. And so that the stream of water gushing out might not be accelerated in falling and made narrower by the acceleration, I fastened this plate not to the bottom but to the side of the vessel in such a way that the stream went out along a line parallel to the horizon. Then, when the vessel was full of water, I opened the hole so that the water might flow out, and the diameter of the stream, measured as accurately as possible at a distance of about ½ inch from the hole, came out 21/40 inch. The diameter of this circular hole, therefore, was to the diameter of the stream very nearly as 25 to 21. Therefore the water in passing through the hole converges from all directions, and after flowing out of the vessel the stream is made narrower by converging and is accelerated by narrowing until it has reached a distance of ½ inch from the hole and at that distance becomes narrower and swifter than it is in the hole itself in the ratio of 25 × 25 to 21 × 21 or very nearly 17 to 12, that is, roughly as the square root of the ratio of 2 to 1. And experiments prove that the quantity of water that flows out in a given time through a circular hole in the bottom of a vessel is the quantity that ought to flow out in the same time, with the velocity mentioned above, not through that hole but through a circular hole whose diameter is to the diameter of that hole as 21 to 25. And thus the flowing water has the downward velocity in the hole itself that a heavy body can acquire very nearly in falling and describing by its fall a space equal to half the height of the water standing in the vessel. But after the water has gone out of the vessel, it is accelerated by converging until it has reached a distance from the hole almost equal to the diameter of the hole and has acquired a velocity that is greater approximately as the square root of the ratio of 2 to 1, which is, as a matter of fact, very nearly the velocity that a heavy body can acquire in falling and describing by its fall a space equal to the whole height of the water standing in the vessel.

In what follows, therefore, let the diameter of the stream be designated by that smaller hole which we have called EF. And suppose that another higher plane VW is drawn parallel to the plane of the hole EF at a distance about equal to the diameter of the hole and pierced by a larger hole ST, and through this let a stream fall that exactly fills the lower hole EF and thus has a diameter which is to the diameter of this lower hole as about 25 to 21. For thus the stream will pass perpendicularly through the lower hole, and the quantity of the water flowing out, depending on the size of this hole, will be very nearly that which the solution of the problem demands. Now, the space which is enclosed by the two planes and the falling stream can be considered to be the bottom of the vessel. But so that the solution of the problem may be simpler and more mathematical, it is preferable to use only the lower plane for the bottom of the vessel and to imagine that the water which flowed down through the ice as if through a funnel and came out of the vessel through the hole EF in the lower plane keeps its motion continually and that the ice keeps its state of rest. In what follows, therefore, let ST be the diameter of a circular hole described with center Z, through which a cataract flows out of the vessel when all the water in the vessel is fluid. And let EF be the diameter of the hole which the cataract fills exactly when falling through it, whether the water comes out of the vessel through the upper hole ST or falls through the middle of the ice in the vessel as



if through a funnel. And let the diameter of the upper hole ST be to the diameter of the lower hole EF as about 25 to 21, and let the perpendicular distance between the planes of the holes be equal to the diameter of the smaller hole EF. Then the downward velocity of the water coming out of the vessel through the hole ST will in the hole itself be that which a body can acquire in falling from half of the height IZ; and the velocity of both falling cataracts will, in the hole EF, be that which a body will acquire in falling from the whole height IG.

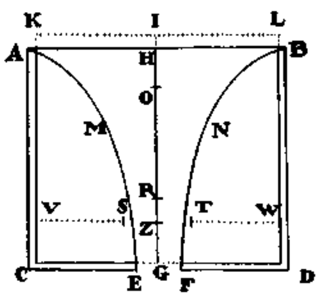
Case 2. If the hole EF is not in the middle of the bottom of the vessel, but the bottom is perforated elsewhere, the water will flow out with the same velocity as before, provided that the size of the hole is the same. For a heavy body does descend to the same depth in a greater time along an oblique line than along a perpendicular line, but in descending it acquires the same velocity in either case, as Galileo proved.

Case 3. The velocity of the water flowing out through a hole in the side of the vessel is the same. For if the hole is small, so that the distance between the surfaces AB and KL vanishes, so far as the senses can tell, and the stream of water gushing out horizontally forms a parabolic figure, it will be found from the latus rectum of this parabola that the velocity of the water flowing out is that which a body could have acquired by falling from the height HG or IG of the water standing in the vessel. Indeed, by making an experiment I found that when the height of the standing water above the hole was 20 inches and the height of the hole above a plane parallel to the horizon was also 20 inches, the stream of water gushing forth would fall upon the plane at a distance of about 37 inches, taken from a perpendicular that was dropped to the plane from the hole. For in the absence of resistance the stream would have had to fall upon the plane at a distance of 40 inches, the latus rectum of the parabolic stream being 80 inches.

Case 4. Further, if the water flowing out has an upward motion, it comes out with the same velocity. For a small stream of water gushing out ascends with a perpendicular motion to the height GH or GI of the water standing in the vessel, except insofar as its ascent is somewhat impeded by the resistance of the air; and accordingly it flows out with the velocity that it could have acquired in falling from that height. Any one particle of the standing water (by book 2, prop. 19) is pressed equally from all sides and, yielding to the pressure, goes with equal force in every direction, whether it descends through a hole in the bottom of the vessel or flows out horizontally through a hole in its side or comes out into a channel and ascends from there through a small hole in the upper part of the channel. And that the velocity with which the water flows out is that which we have designated in this proposition is not only found by reason but is also manifest from the well-known experiments already described.

CASE 5. The velocity of the water flowing out is the same whether the hole is circular or square or triangular or of any other shape equal in area to the circular one. For the velocity of the water flowing out does not depend on the shape of the hole but on the height of the water in relation to the plane KL.

CASE 6. If the lower part of the vessel ABDC is immersed in standing water, and the height of the standing water above the bottom of the vessel is GR, the velocity with which the water in the vessel will flow out through the hole EF into the standing water will be that which the water can acquire in falling and describing by its fall the space IR. For the weight of all the water in the vessel that is lower than the surface of the standing water will be sustained in equilibrium by the weight of the standing water and thus will not at all accelerate the motion of the descending water in the vessel. This case can also be shown by experiments, by measuring the times in which the water flows out.



COROLLARY 1. Hence, if the height CA of the water is produced to K, so that AK is to CK in the squared ratio of the area of a hole made in any part of the bottom to the area of the circle AB, the velocity of the water flowing out will be equal to the velocity that the water can acquire in falling and describing by its fall the space KC.

COROLLARY 2. And the force by which the whole motion of the water gushing out can be generated is equal to the weight of a cylindrical column of water whose base is the hole EF and whose height is 2GI or 2CK. For the gushing water, in the time in which it equals this column, can acquire in falling (by its weight) from the height GI the very velocity with which it gushes out.

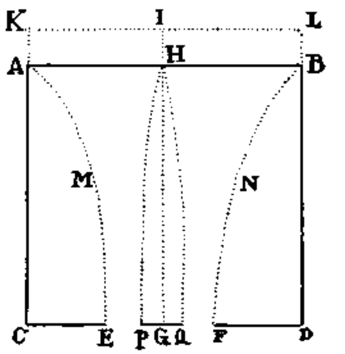
Corollary 3. The weight of all the water in the vessel ABDC is to the part of the weight that is used in making the water flow down as the sum of the circles AB and EF to twice the circle EF. For let IO be a mean proportional between IH and IG; then the water coming out through the hole EF, in the time in which a drop could describe a space equal to the height IG in falling from I, will be equal to a cylinder whose base is the circle EF and whose height is 2IG, that is, to a cylinder whose base is the circle AB and whose height is 2IO, for the circle EF is to the circle AB as the square root of the ratio of the height IH to the height IG, that is, in the simple ratio of the mean proportional IO to the height IG, and in the time in which a drop can describe a space equal to the height IH in falling from I, the water coming out will be equal to a cylinder whose base is the circle AB and whose height is 2IH, and in the time in which a drop describes a space equal to the difference HG between the heights in falling from I through H to G, the water coming out—that is, all the water in the solid ABNFEM—will be equal to the difference between the cylinders, that is, equal to a cylinder whose base is AB and whose height is 2HO. And therefore all the water in the vessel ABDC is to all the water falling in the solid ABNFEM as HG to 2HO, that is, as HO + OG to 2HO, or IH + IO to 2IH. But the weight of all the water in the solid ABNFEM is used in making the water flow down, and accordingly the weight of all the water in the vessel is to the part of the weight that is used in making the water flow down as IH + IO to 2IH and thus as the sum of the circles EF and AB to twice the circle EF.

Corollary 4. And hence the weight of all the water in the vessel ABDC is to the part of the weight sustained by the bottom of the vessel as the sum of the circles AB and EF is to the difference between these circles.

Corollary 5. And the part of the weight sustained by the bottom of the vessel is to the part of the weight used in making the water flow down as the difference between the circles AB and EF is to twice the smaller circle EF, or as the area of the bottom to twice the hole.

Corollary 6. And the part of the weight which alone presses upon the bottom is to the weight of all the water resting perpendicularly on the bottom as the circle AB is to the sum of the circles AB and EF, or as the circle AB is to the amount by which twice the circle AB exceeds the bottom. For the part of the weight which alone presses upon the bottom is to the weight of all the water in the vessel as the difference between the circles AB and EF is to the sum of these circles, by corol. 4; and the weight of all the water in the vessel is to the weight of all the water resting perpendicularly on the bottom as the circle AB is to the difference between the circles AB and EF. Therefore, from the equality of the ratios in inordinate proportion [or ex aequo perturbate], the part of the weight which alone presses upon the bottom is to the weight of all the water resting perpendicularly on the bottom as the circle AB is to the sum of the circles AB and EF, or to the amount by which twice the circle AB exceeds the bottom.

Corollary 7. If in the middle of the hole EF there is placed a little circle PQ described with center G and parallel to the horizon, the weight of the water which that little circle sustains is greater than the weight of ⅓ of a cylinder of water whose base is that little circle and whose height is GH. For let ABNFEM be a cataract or column of falling water, with axis GH as above, and suppose that there has been a freezing of all the water in the vessel (around the cataract as well as above the little circle) whose fluidity is not required for the very ready and very swift descent of the water. And let PHQ be the frozen column of water above the little circle, having vertex H and height GH. And imagine that this cataract falls with its whole weight and does not rest or press on PHQ but slides past freely and without friction, except perhaps at the very vertex of the ice, where at the very beginning of falling the cataract begins to be concave. And just as the frozen water (AMEC and BNFD) which is around the cataract is convex on the inner surface (AME and BNF) toward the falling cataract, so also this column PHQ will be convex toward the cataract, and therefore will be greater than a cone whose base is the little circle PQ and whose height is GH, that is, greater than ⅓ of a cylinder described with the same base and height. And the little circle sustains the weight of this column, that is, a weight that is greater than the weight of the cone or of ⅓ of the cylinder.



Corollary 8. The weight of the water sustained by the little circle PQ, when it is extremely small, appears to be less than the weight of ⅔ of a cylinder of water whose base is that little circle and whose height is HG. Keeping the same suppositions, imagine that half a spheroid is described,

whose base is the little circle and whose semiaxis or height is HG. Then this figure will be equal to ⅔ of that cylinder and will comprehend the column of frozen water PHQ whose weight that little circle sustains. In order that the motion of the water may be straight down, the outer surface of this column must meet the base PQ in a somewhat acute angle, because the water in falling is continually accelerated and the acceleration makes the column become narrower; and since that angle is less than a right angle, the lower parts of this column will lie within the half-spheroid. But higher up, the column will be acute or pointed, for otherwise the horizontal motion of the water at the vertex of the spheroid would be infinitely swifter than its motion toward the horizon. And the smaller the little circle PQ, the more acute the vertex of the column; and if the little circle is diminished indefinitely, the angle PHQ will be diminished indefinitely, and therefore the column will lie within the half-spheroid. That column is therefore less than the half-spheroid, or less than ⅔ of a cylinder whose base is that little circle and whose height is GH. Moreover, the little circle sustains the water's force equal to the weight of this column, since the weight of the surrounding water is used in making it flow down.

Corollary 9. The weight of the water sustained by the little circle PQ, when it is extremely small, is very nearly equal to the weight of a cylinder of water whose base is that little circle and whose height is ½GH. For this weight is an arithmetical mean between the weights of the cone and the said half-spheroid. If, however, the little circle is not extremely small but is increased until it equals the hole EF, it will sustain the weight of all the water resting perpendicularly on it, that is, the weight of a cylinder of water whose base is that little circle and whose height is GH.

Corollary 10. And (as far as I can tell) the weight that the little circle sustains always has the proportion to the weight of a cylinder of water whose base is that little circle and whose height is ½GH that EF^2 has to $EF^2 - \frac{1}{2}PQ^2$, or that the circle EF has to the excess of this circle over half of the little circle PQ, very nearly.

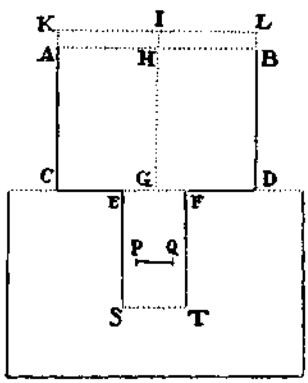
Lemma 4 *The resistance of a cylinder moving uniformly forward in the direction of its length is not changed by an increase or decrease in length and thus is the same as the resistance of a circle described with the same diameter and moving forward with the same velocity along a straight line perpendicular to its plane.*

For the sides of a cylinder offer no opposition to its motion, and a cylinder is turned into a circle if its length is decreased indefinitely.

Proposition 37
Theorem 29

If a cylinder moves uniformly forward in a compressed, infinite, and nonelastic fluid in the direction of its own length, its resistance arising from the magnitude of its transverse section is to the force by which its whole motion can be either destroyed or generated, while it is describing four times its length, very nearly as the density of the medium is to the density of the cylinder.

For if the bottom CD of the vessel ABDC touches the surface of stagnant water, and if water flows out of this vessel into the stagnant water through



the cylindrical channel EFTS perpendicular to the horizon, and if the little circle PQ is placed parallel to the horizon anywhere in the middle of the channel, and if CA is produced to K so that CK is to AK in the squared ratio of the circle AB to the amount by which the opening of the channel EF exceeds the little circle PQ, then it is obvious (by prop. 36, case 5, case 6, and corol. 1) that the velocity of the water passing through the annular space between the little circle and the sides of the vessel will be that which the water can acquire in falling and describing by its fall a space equal to the height KC or IG.

And (by prop. 36, corol. 10) if the width of the vessel is infinite, so that the line-element HI vanishes and the heights IG and HG are equal, then the force of the water flowing down into the little circle will be to the weight of a cylinder whose base is that little circle, and whose height is $\frac{1}{2}$IG, very nearly as EF^2 to $EF^2 - \frac{1}{2}PQ^2$. For the force of the water flowing down through the whole channel with uniform motion will be the same upon the little circle PQ in whatever part of the channel it is placed.

Now let the openings EF and ST of the channel be closed, and let the little circle ascend in the fluid compressed on all sides, and by its ascent let it make the upper water descend through the annular space between the little circle and the sides of the channel; then the velocity of the ascending little circle will be to the velocity of the descending water as the difference between the circles EF and PQ is to the circle PQ, and the velocity of the ascending little circle will be to the sum of the velocities (that is, to the relative velocity of the descending water, with which it flows past the ascending little circle) as the difference between the circles EF and PQ is to the circle EF, or as $EF^2 - PQ^2$ to EF^2. Let that relative velocity be equal to the velocity with which (as shown above) the water passes through the same annular space while the little circle remains unmoved, that is, to the velocity that the water can acquire in falling and describing by its fall a space equal to the height IG; then the force of the water upon the ascending little circle will be the same as before (by corol. 5 of the laws), that is, the resistance of the ascending little circle will be to the weight of a cylinder of water whose base is that little circle, and whose height is ½IG, very nearly as EF^2 to $EF^2 - \frac{1}{2}PQ^2$. And the velocity of the little circle will be to the velocity that the water acquires in falling, and describing by its fall a space equal to the height IG, as $EF^2 - PQ^2$ to EF^2.

Let the breadth of the channel be increased indefinitely; then those ratios between $EF^2 - PQ^2$ and EF^2 and between EF^2 and $EF^2 - \frac{1}{2}PQ^2$ will ultimately approach ratios of equality. And therefore the velocity of the little circle will now be that which the water can acquire in falling and describing by its fall a space equal to the height IG, and its resistance will come out equal to the weight of a cylinder whose base is that little circle and whose height is half of the height IG from which the cylinder must fall in order to acquire the velocity of the ascending little circle, and with this velocity the cylinder will, in the time of falling, describe four times its own length. And the resistance of the cylinder, moving forward with this velocity in the direction of its length, is the same as the resistance of the little circle (by lem. 4) and thus is very nearly equal to the force by which its motion can be generated while it is describing four times its length.

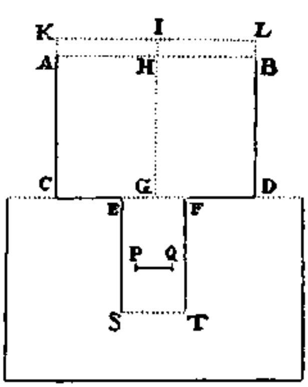
If the length of the cylinder is increased or decreased, its motion, and also the time in which it describes four times its length, will be increased or decreased in the same ratio; and thus that force by which the increased or decreased motion, in a time equally increased or decreased, could be generated or destroyed will not be changed and accordingly is under these circumstances still equal to the resistance of the cylinder; for this also remains unchanged, by lem. 4.

If the density of the cylinder is increased or decreased, its motion, and also the force by which the motion can be generated or destroyed in the same time, will be increased or decreased in the same ratio. The resistance, therefore, of any cylinder to the force by which its whole motion could be either generated or destroyed, while it is describing four times its length, will be very nearly as the density of the medium to the density of the cylinder. Q.E.D.

A fluid must be compressed in order to be continuous, and it must be continuous and nonelastic in order that every pressure arising from its compression may be propagated instantaneously and, by acting equally upon all parts of a moving body, not change the resistance. The pressure arising from the body's motion is of course used in generating the motion of the parts of the fluid and creates resistance. But the pressure arising from the compression of the fluid, however strong it may be, if it is propagated instantaneously, generates no motion in the parts of a continuous fluid, introduces no change of motion at all, and thus neither increases nor decreases the resistance. Certainly the action of a fluid that arises from its compression cannot be stronger upon the back of a moving body than upon the front and thus cannot decrease the resistance described in this proposition; and the action will not be stronger upon the front than upon the back provided that its propagation is infinitely swifter than the motion of the body pressed. And the action will be infinitely swifter and will be propagated instantaneously provided that the fluid is continuous and nonelastic.

COROLLARY 1. The resistances to cylinders that move uniformly forward in the direction of their lengths in infinite and continuous mediums are in a ratio compounded of the squared ratio of the velocities and the squared ratio of the diameters and the ratio of the density of the mediums.

COROLLARY 2. If the breadth of the channel is not increased indefinitely, but the cylinder moves forward in the direction of its own length in an enclosed medium at rest, and meanwhile its axis coincides with the axis of the channel, then the resistance to the cylinder will be to the force by which its whole motion could be either generated or destroyed, in the time in which it describes four times its length, in a ratio compounded of the simple ratio of EF^2 to $EF^2 - \frac{1}{2}PQ^2$ and the squared ratio of EF^2 to $EF^2 - PQ^2$ and the ratio of the density of the medium to the density of the cylinder.

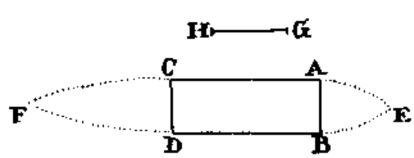


Corollary 3. With the same suppositions, let the length L be to four times the length of the cylinder in a ratio compounded of the simple ratio of $EF^2 - \frac{1}{2}PQ^2$ to EF^2 and the squared ratio of $EF^2 - PQ^2$ to EF^2; then the resistance of the cylinder will be to the force by which its whole motion could be either destroyed or generated, while it is describing the length L, as the density of the medium to the density of the cylinder.

Scholium

In this proposition we have investigated the resistance arising solely from the magnitude of the transverse section of a cylinder, without considering the part of the resistance that can arise from the obliquity of the motions. In prop. 36, case 1, the flow of the water through the hole EF was impeded by the obliquity of the motions with which the parts of the water in the vessel converged from all sides into the hole. Similarly, in this proposition, the obliquity of the motions with which the parts of the water pressed by the front end of the cylinder yield to the pressure and diverge on all sides has these effects: it retards the passage of those motions through the places around that front end toward the back of the cylinder, it makes the fluid move to a greater distance, and it increases the resistance in nearly the ratio with which it decreases the flow of the water from the vessel, that is, in the squared ratio of 25 to 21, roughly.

In case 1 of prop. 36 we made the parts of the water pass through the hole EF perpendicularly and in the greatest abundance by supposing that all the water in the vessel that had been frozen around the cataract, and whose motion was oblique and useless, remained without motion. Similarly, in this proposition, in order that the obliquity of the motions may be annulled, and the parts of the water, by yielding with the most direct and rapid motion, may provide the easiest passage to the cylinder, and in order that only the resistance may remain that arises



from the magnitude of the transverse section and that cannot be decreased except by decreasing the diameter of the cylinder, it must be understood that the parts of the fluid whose motions are oblique and useless and create resistance are at rest with respect to one another at both ends of the cylinder and cohere and are joined to the cylinder. Let ABDC be a rectangle, and let AE and BE be two parabolic arcs described with axis AB and with a latus rectum that is to the space HG, which is to be described by the falling cylinder while it is acquiring its velocity, as HG to $\frac{1}{2}$AB. Additionally, let CF and DF be two other parabolic arcs, described with axis CD and a latus rectum that is four times the former latus rectum; and by the revolution of the figure about its axis EF, let a solid be generated whose middle ABDC is the cylinder with which we are dealing, and whose extremities ABE and CDF contain the parts of the fluid which are at rest with respect to one another and solidified into two rigid bodies that adhere to the cylinder at the ends as head and tail. Then the resistance to the solid EACFDB moving forward in the direction of its axis FE from F toward E will be very nearly that which we have described in this proposition. That is, the density of the fluid is to the density of the cylinder very nearly in the ratio of this resistance to the force by which the whole motion of the cylinder could be either destroyed or generated, while the length 4AC is being described with that motion continued uniformly. And with this force the resistance cannot be less than in the ratio of 2 to 3, by prop. 36, corol. 7.

Lemma 5 *If a cylinder, a sphere, and a spheroid, whose widths are equal, are placed successively in the middle of a cylindrical channel in such a way that their axes coincide with the axis of the channel, these bodies will equally impede the flow of water through the channel.*

For the spaces through which the water passes between the channel and the cylinder, sphere, and spheroid are equal; and water passes equally through equal spaces.

This is so on the hypothesis that all the water is frozen which is above the cylinder, sphere, or spheroid, and whose fluidity is not required for the very swift passage of the water, as I have explained in prop. 36, corol. 7.

Lemma 6 *With the same suppositions, these bodies are equally urged by the water flowing through the channel.*

This is evident by lem. 5 and the third law of motion. Of course, the water and the bodies act equally upon one another.

Lemma 7 *If the water is at rest in the channel, and these bodies go through the channel with equal velocity in opposite directions, the resistances will be equal to one another.*

This is clear from lem. 6; for the relative motions remain the same with respect to one another.

Scholium

It is the same for all convex round bodies whose axes coincide with the axis of the channel. Some difference can arise from a greater or lesser friction; but in these lemmas we are supposing that the bodies are very smooth, that the tenacity and friction of the medium are nil, and that the parts of the fluid which by their oblique and superfluous motions can perturb, impede, and retard the flow of the water through the channel are at rest with respect to one another as if icebound and adhere to the front and back of the bodies, as I have explained in the scholium to prop. 37. For what follows deals with the least possible resistance of round bodies described with the greatest given transverse sections.

Bodies moving straight ahead in fluids make the fluid ascend in front of them and subside in back of them, especially if they are blunt in shape; and hence they encounter a slightly greater resistance than if they had pointed heads and tails. And bodies moving in elastic fluids, if they are blunt in front and in back, condense the fluid a little more at the front and make it a little less dense at the back; and hence they encounter a slightly greater resistance than if they had pointed heads and tails. But in these lemmas and propositions we are not dealing with elastic fluids but with nonelastic fluids, not with bodies floating on the surface of the fluid but with bodies deeply immersed. And once the resistance of bodies in nonelastic fluids is known, this resistance will have to be increased somewhat for elastic fluids such as air as well as for the surfaces of stagnant fluids such as seas and swamps.

Proposition 38
Theorem 30

The resistance to a sphere moving uniformly forward in an infinite and nonelastic compressed fluid is to the force by which its whole motion could either be destroyed or generated, in the time in which it describes 8/3 of its diameter, very nearly as the density of the fluid to the density of the sphere.

For a sphere is to the circumscribed cylinder as 2 to 3, and therefore the force that could take away all the motion of a cylinder, while the cylinder is describing a length of four diameters, will take away all the motion of the sphere while the sphere describes 2/3 of this length, that is, 8/3 of its own diameter. And the resistance of the cylinder is to this force very nearly as the density of the fluid to the density of the cylinder or sphere, by prop. 37, and the resistance of the sphere is equal to the resistance of the cylinder, by lems. 5, 6, and 7. Q.E.D.

COROLLARY 1. The resistances of spheres in infinite compressed mediums are in a ratio compounded of the squared ratio of the velocity and the squared ratio of the diameter and the ratio of the density of the mediums.

COROLLARY 2. The greatest velocity with which a sphere, by the force of its own relative weight, can descend in a resisting fluid is that which the same sphere with the same weight can acquire in falling without resistance and describing by its fall a space that is to 4/3 of its diameter as the density of the sphere to the density of the fluid. For the sphere in the time of its fall, with the velocity acquired in falling, will describe a space that will be to 8/3 of its diameter as the density of the sphere to the density of the fluid; and the force of its weight generating this motion will be to the force that could generate the same motion, in the time in which the sphere describes 8/3 of its diameter with the same velocity, as the density of the fluid to the density of the sphere; and thus, by this proposition, the force of its weight will be equal to the force of resistance and therefore cannot accelerate the sphere.

COROLLARY 3. Given both the density of the sphere and its velocity at the beginning of the motion, and also the density of the compressed fluid at rest in which the sphere moves, then by prop. 35, corol. 7, the velocity of the sphere, its resistance, and the space described by it are given for any time.

COROLLARY 4. A sphere moving in a compressed fluid at rest, having the same density as itself, will, by the same corol. 7, lose half of its motion before it has described the length of two of its diameters.

Proposition 39
Theorem 31

The resistance to a sphere moving uniformly forward through a fluid enclosed and compressed in a cylindrical channel is to the force by which its whole motion could be either generated or destroyed, while it describes 8/3 of its diameter, in a ratio compounded of three ratios, very nearly: the ratio of the opening of the channel to the excess of this opening over half of a great circle of the sphere, the squared ratio of the opening of the channel to the excess of this opening over a great circle of the sphere, and the ratio of the density of the fluid to the density of the sphere.

This is evident by prop. 37, corol. 2, and the proof proceeds as in prop. 38.

Scholium

In the last two propositions (as in lem. 5) I assume that all the water which is in front of the sphere, and whose fluidity increases the resistance to the sphere, is frozen. If all that water liquefies, the resistance will be somewhat increased. But in these propositions the increase will be small and can be ignored because the convex surface of the sphere has almost the same effect as ice.

Proposition 40
Problem 9

To find from phenomena the resistance of a sphere moving forward in a compressed, very fluid medium.

Let A be the weight of the sphere in a vacuum, B its weight in a resisting medium, D the diameter of the sphere, F a space that is to $\frac{4}{3}$D as the density of the sphere to the density of the medium (that is, as A to A − B), G the time in which the sphere in falling by its weight B without resistance describes the space F, and H the velocity that the sphere acquires by this fall. Then H will be the greatest velocity with which the sphere can descend by its weight B in the resisting medium, by prop. 38, corol. 2, and the resistance that the sphere encounters while descending with this velocity will be equal to its weight B; and the resistance that it encounters with any other velocity will be to the weight B as the square of the ratio of this velocity to the greatest velocity H, by prop. 38, corol. 1.

This is the resistance that arises from the inertia of matter of the fluid. And that which arises from the elasticity, tenacity, and friction of its parts can be investigated as follows.

Drop the sphere so that it descends in the fluid by its own weight B; and let P be the time of falling, in seconds if the time G is in seconds. Find the absolute number N that corresponds to the logarithm $0.4342944819\frac{2P}{G}$, and let L be the logarithm of the number $\frac{N+1}{N}$, then the velocity acquired in falling will be $\frac{N-1}{N+1}H$, and the space described will be $\frac{2PF}{G} - 1.3862943611F + 4.605170186LF$.

If the fluid is sufficiently deep, the term 4.605170186LF can be ignored, and $\frac{2PF}{G} - 1.3862943611F$ will be the space described, very nearly. These things are evident by book 2, prop. 9 and its corollaries, on the hypothesis that the sphere encounters no other resistance than that which arises from the inertia of matter. But if it encounters another resistance in addition, the descent will be slower, and the quantity of this resistance can be found from the retardation.

Times P	*Velocities of body falling in fluid*	*Spaces described by falling in fluid*	*Spaces described by greatest motion*	*Spaces described by falling in vacuum*
0.001G	99999 29/30	0.000001F	0.002F	0.000001F
0.01G	999967	0.0001F	0.02F	0.0001F
0.1G	9966799	0.0099834F	0.2F	0.01F
0.2G	19737532	0.0397361F	0.4F	0.04F
0.3G	29131261	0.0886815F	0.6F	0.09F
0.4G	37994896	0.1559070F	0.8F	0.16F
0.5G	46211716	0.2402290F	1.0F	0.25F
0.6G	53704957	0.3402706F	1.2F	0.36F
0.7G	60436778	0.4545405F	1.4F	0.49F
0.8G	66403677	0.5815071F	1.6F	0.64F
0.9G	71629787	0.7196609F	1.8F	0.81F
1G	76159416	0.8675617F	2F	1F
2G	96402758	2.6500055F	4F	4F
3G	99505475	4.6186570F	6F	9F
4G	99932930	6.6143765F	8F	16F
5G	99990920	8.6137964F	10F	25F
6G	99998771	10.6137179F	12F	36F
7G	99999834	12.6137073F	14F	49F
8G	99999980	14.6137059F	16F	64F
9G	99999997	16.6137057F	18F	81F
10G	99999999 3/5	18.6137056F	20F	100F

So that the velocity and descent of a body falling in a fluid may be found more easily, I have put together the accompanying table, in which the first column denotes the times of descent, the second shows the velocities acquired in falling (the greatest velocity being 100,000,000), the third shows the spaces described in falling in those times (2F being the space that the body describes in the time G with the greatest velocity), and the fourth shows the spaces described in the same times with the greatest velocity. The numbers in the fourth column are $\frac{2P}{G}$, and by subtracting the number 1.3862944 − 4.6051702L, the numbers in the third column are found, and these numbers must be multiplied by the space F in order to get the spaces described in falling. There has been added to these a fifth column, which contains the spaces described in the same times by a body falling in a vacuum by the force of its relative weight B.

Scholium In order to investigate the resistances of fluids by experiments, I got a square wooden vessel, with an internal length and width of 9 inches (of a London foot), and a depth of 9½ feet, and I filled it with rainwater; and making balls of wax with lead inside, I noted the times of descent of the balls, the space of the descent being 112 inches. A solid cubic London foot contains 76 Roman pounds [troy] of rainwater, and a solid inch of this foot contains 19/36 ounce of this pound or 253⅓ grains; and a sphere of water described with a diameter of 1 inch contains 132.645 grains in air, or 132.8 grains in a vacuum; and any other ball is as the excess of its weight in a vacuum over its weight in water.

Experiment 1. A ball which weighed 156¼ grains in air and 77 grains in water described the whole space of 112 inches [when dropped in water] in a time of 4 seconds. And when the experiment was repeated, the ball again fell in the same time of 4 seconds.

The weight of the ball in a vacuum is 156 13/38 grains, and the excess of this weight over the weight of the ball in water is 79 13/38 grains. And hence the diameter of the ball comes out 0.84224 inch. That excess is to the weight of the ball in a vacuum as the density of water to the density of the ball, and as 8/3 of the diameter of the ball (that is, 2.24597 inches) to the space 2F, which accordingly will be 4.4256 inches. In a time of 1 second the ball will fall in a vacuum by its whole weight of 156 13/38 grains through 193⅓ inches; and by a weight of 77 grains falling in water without resistance, it will in the same time describe 95.219 inches; and in the time G, which is to 1 second as the square root of the ratio of the space F or 2.2128 inches to 95.219 inches, it will describe 2.2128 inches and will attain the greatest velocity H with which it can descend in water. Therefore the time G is 0.15244 seconds. And in this time G, with that greatest velocity H, the ball will describe a space 2F of 4.4256 inches; and thus in the time of 4 seconds it will describe a space of 116.1245 inches. Subtract the space 1.3862944F or 3.0676 inches and there will remain a space of 113.0569 inches which the ball will describe in falling in water in a very wide vessel in the time of 4 seconds. This space, because of the narrowness of the wooden vessel, must be decreased in a ratio which is compounded of the square root of the ratio of the opening of the vessel to the excess of this opening over a great semicircle of the ball, and of the simple ratio of that same opening to its excess over a great circle of the ball, that is, in the ratio of 1 to 0.9914. When this has been done, the result will be a space of 112.08 inches which the ball should, according to the theory, have very nearly described in falling in water in this wooden vessel in the time of 4 seconds. And it described 112 inches in the experiment.

EXPERIMENT 2. Three equal balls, each of which weighed $76\frac{1}{3}$ grains in air and $5\frac{1}{16}$ grains in water, were dropped successively in water, and in a time of 15 seconds each one fell through 112 inches.

By computation the weight of a ball in a vacuum is $76\frac{5}{12}$ grains; the excess of this weight over the weight in water is $71\frac{17}{48}$ grains; the diameter of the ball is 0.81296 inch; $\frac{8}{3}$ of this diameter is 2.16789 inches; the space 2F is 2.3217 inches; the space that a ball describes in falling by a weight of $5\frac{1}{16}$ grains in the time of 1 second without resistance is 12.808 inches; and the time G is 0.301056 second. The ball, therefore, with the greatest velocity with which it can descend in water by the force of the weight of $5\frac{1}{16}$ grains, will describe in a time of 0.301056 second a space of 2.3217 inches, and in the time of 15 seconds a space of 115.678 inches. Subtract the space 1.3862944F or 1.609 inches, and there will remain a space of 114.069 inches which accordingly the ball ought to describe in falling in the same time in a very wide vessel. Because of the narrowness of our vessel a space of roughly 0.895 inch must be taken away. And thus there will remain a space of 113.174 inches which the ball, according to the theory, should have very nearly described in falling in this vessel in the time of 15 seconds. And it described 112 inches in the experiment. The difference is imperceptible.

EXPERIMENT 3. Three equal balls, each of which weighed 121 grains in air and 1 grain in water, were dropped successively in water, and in times of 46 seconds, 47 seconds, and 50 seconds, fell 112 inches.

According to the theory, these balls should have fallen in a time of roughly 40 seconds. I am uncertain whether their falling more slowly is to be attributed to the smaller proportion of the resistance that arises from the force of inertia in slow motions to the resistance that arises from other causes, or rather to some little bubbles adhering to the ball, or to the rarefaction of the wax from the heat either of the weather or of the hand dropping the ball, or even to imperceptible errors in weighing the balls in water. And thus the weight of the ball in water ought to be more than 1 grain, so that the experiment may be made certain and trustworthy.

EXPERIMENT 4. I began the experiments thus far described in order to investigate the resistances of fluids before formulating the theory set forth in the immediately preceding propositions. Afterward, in order to examine that theory, I obtained a wooden vessel with an internal width of $8\frac{2}{3}$ inches and a depth of $15\frac{1}{3}$ feet. Then I made four balls out of wax with lead inside, each one weighing $139\frac{1}{4}$ grains in air and $7\frac{1}{8}$ grains in water. And I let them fall in water in order to measure the times of falling, using a pendulum oscillating in half-seconds. When the balls were being weighed, and afterward when they were falling, they were cold and had remained cold for some time, because heat rarefies the wax and by the rarefaction diminishes the weight of the ball in water, and the rarefied wax is not immediately brought back to its original density by chilling. Before they fell, they were entirely immersed in water, so that their descent might not be accelerated at the beginning by the weight of some part projecting out of the water. And when totally immersed and at rest, they were let fall as carefully as possible, so as not to receive some impulse from the hand letting them fall. And they fell successively in the times of $47\frac{1}{2}$, $48\frac{1}{2}$, 50, and 51 oscillations, describing a space of 15 feet 2 inches. But the weather was now a little colder than when the balls were weighed, and so I repeated the experiment on another day, and the balls fell in the times of 49, $49\frac{1}{2}$, 50, and 53 oscillations, and on a third day in the times of $49\frac{1}{2}$, 50, 51, and 53 oscillations. The experiment was made quite often, and the balls for the most part fell in the times of $49\frac{1}{2}$ and 50 oscillations. When they fell more slowly, I suspect that they were retarded by hitting against the sides of the vessel.

Now by computation according to the theory, the weight of a ball in a vacuum is $139\frac{2}{5}$ grains; the excess of this weight over the weight of the ball in water is $132\frac{11}{40}$ grains; the diameter of the ball is 0.99868 inch; $\frac{8}{3}$ of the diameter is 2.66315 inches; the space 2F is 2.8066 inches; the space that a ball describes in falling with a weight of $7\frac{1}{8}$ grains in the time of 1 second without resistance is 9.88164 inches; and the time G is 0.376843 second. The ball, therefore, with the greatest velocity with which it can descend in water by a force of weight of $7\frac{1}{8}$ grains, describes in the time of 0.376843 second a space of 2.8066 inches; in the time of 1 second a space of 7.44766 inches; and in the time of 25 seconds, or 50 oscillations, a space of 186.1915 inches. Subtract the space 1.386294F, or 1.9454 inches, and there will remain the space of 184.2461 inches which the ball will describe in the same time in a very wide vessel. Because of the narrowness of our vessel, decrease this space in a ratio that is compounded of the square root of the ratio of the opening of the vessel to the excess of this opening over a great semicircle of the ball, and the simple ratio of that same opening to its excess over a great circle of the ball, and the result will be the space of 181.86 inches which the ball, according to the theory, should very nearly have described in this vessel in the time of 50 oscillations. And in the experiment it described a space of 182 inches in the time of 49½ or 50 oscillations.

EXPERIMENT 5. Four balls weighing 154⅜ grains in air and 21½ grains in water were dropped often and fell in the times of 28½, 29, 29½, and 30 oscillations, and sometimes 31, 32, and 33, describing a space of 15 feet 2 inches.

By the theory they ought to have fallen in the time of very nearly 29 oscillations.

EXPERIMENT 6. Five balls weighing 212⅜ grains in air and 79½ in water were dropped often and fell in the times of 15, 15½, 16, 17, and 18 oscillations, describing a space of 15 feet 2 inches.

By the theory they ought to have fallen in the time of very nearly 15 oscillations.

EXPERIMENT 7. Four balls weighing 293⅜ grains in air and 35⅞ grains in water were dropped often and fell in the times of 29½, 30, 30½, 31, 32, and 33 oscillations, describing a space of 15 feet 1½ inches.

By the theory they ought to have fallen in the time of very nearly 28 oscillations.

In investigating the reason why some of the balls which were of the same weight and size fell more quickly and others more slowly, I hit upon this: that when the balls were first dropped and were beginning to fall, the side which happened to be heavier descended first and generated an oscillatory motion, so that they oscillated around their centers. For by its oscillations a ball communicates a greater motion to the water than if it were descending without oscillations, and in the process loses part of its own motion with which it should descend; and it is retarded more or retarded less in proportion to the greatness or smallness of the oscillation. Further, the ball always recedes from that side which is descending in the oscillation and, by receding, approaches the sides of the vessel and sometimes strikes against the sides. In the case of heavier balls, this oscillation is stronger, and with larger balls, it agitates the water more. Therefore, in order to reduce the oscillation of the balls, I constructed new balls of wax and lead, fixing the lead into one side of the ball near its surface; and I dropped the ball in such a way that the heavier side, as far as possible, was lowest at the beginning of the descent. Thus the oscillations became much smaller than before, and the balls fell in less unequal times, as in the following experiments.

EXPERIMENT 8. Four balls, weighing 139 grains in air and $6\frac{1}{2}$ in water, were dropped often and fell in the times of not more than 52 oscillations, and not fewer than 50, and for the most part in the time of roughly 51 oscillations, describing a space of 182 inches.

By the theory they ought to have fallen in the time of roughly 52 oscillations.

EXPERIMENT 9. Four balls, weighing $273\frac{1}{4}$ grains in air and $140\frac{3}{4}$ in water, were dropped often and fell in the times of not fewer than 12 oscillations and not more than 13, describing a space of 182 inches.

And by the theory these balls ought to have fallen in the time of very nearly $11\frac{1}{3}$ oscillations.

EXPERIMENT 10. Four balls, weighing 384 grains in air and $119\frac{1}{2}$ in water, were dropped often and fell in the times of $17\frac{3}{4}$, 18, $18\frac{1}{2}$, and 19 oscillations, describing a space of $181\frac{1}{2}$ inches. And when they fell in the time of 19 oscillations, I sometimes heard them strike the sides of the vessel before they reached the bottom.

And by the theory they ought to have fallen in the time of very nearly $15\frac{5}{9}$ oscillations.

EXPERIMENT 11. Three equal balls, weighing 48 grains in air and $3\frac{29}{32}$ in water, were dropped often and fell in the times of $43\frac{1}{2}$, 44, $44\frac{1}{2}$, 45, and 46 oscillations, and for the most part 44 and 45, describing a space of very nearly $182\frac{1}{2}$ inches.

By the theory they ought to have fallen in the time of roughly $46\frac{5}{9}$ oscillations.

EXPERIMENT 12. Three equal balls, weighing 141 grains in air and $4\frac{3}{8}$ in water, were dropped several times and fell in the times of 61, 62, 63, 64, and 65 oscillations, describing a space of 182 inches.

And by the theory they ought to have fallen in the time of very nearly $64\frac{1}{2}$ oscillations.

From these experiments it is obvious that when the balls fell slowly (as in the second, fourth, fifth, eighth, eleventh, and twelfth experiments), the times of falling are shown correctly by the theory, but that when the balls fell more quickly (as in the sixth, ninth, and tenth experiments), the resistance was a little greater than in the squared ratio of the velocity. For the balls oscillate somewhat while falling, and this oscillation—in balls that are lighter and fall more slowly—ceases swiftly because the motion is weak, while in heavier and larger balls, because the motion is strong, the oscillation lasts longer and can be checked by the surrounding water only after more oscillations. Additionally, the swifter the balls, the less they are pressed by the fluid in back of them; and if the velocity is continually increased, they will at length leave an empty space behind, unless the compression of the fluid is simultaneously increased. The compression of the fluid, moreover, ought (by props. 32 and 33) to be increased in the squared ratio of the velocity in order for the resistance also to be in a squared ratio. Since this does not happen, the swifter balls are pressed a little less from behind, and because of this diminished pressure their resistance becomes a little greater than in the squared ratio of the velocity.

The theory therefore agrees with the phenomena of bodies falling in water; it remains for us to examine the phenomena of bodies falling in air.

Experiment 13. [a]From the top of St. Paul's Cathedral in London[a] in June 1710, glass balls were dropped simultaneously in pairs, one full of quicksilver, the other full of air; and in falling they described a space of 220 London feet. A wooden platform was suspended at one end by iron pivots, and at the other was supported by a wooden peg. The two balls were placed upon this platform and were let fall simultaneously by pulling out the peg by means of an iron wire extending to the ground, so that the platform, resting on the iron pivots alone, might swing downward upon the pivots and at the same moment a seconds pendulum, pulled by that iron wire, might be released and begin to oscillate. The diameters and weights of the balls and the times of falling are shown in the following table.

However, the observed times need to be corrected. For balls filled with mercury will (by Galileo's theory) describe 257 London feet in 4 seconds, and 220 feet in only 3 seconds 42 thirds. The wooden platform, when the

aa. In expt. 13, Newton writes of weights being dropped "a culmine ecclesiae Sancti Pauli, in urbe Londini." Newton is not referring to St. Paul's Church in Covent Garden, as is obvious from the fact that the distance through which the weights are let fall is 220 London feet. The only house of worship that tall (about twenty stories) was St. Paul's Cathedral. That these experiments were conducted in St. Paul's Cathedral is evident from the fact that in the cathedral there is a balcony, just below the cupola, at a height corresponding to Newton's 220 London feet. See, below, the note to expt. 14.

Balls full of mercury			*Balls full of air*		
Weights	Diameters	Times of falling	Weights	Diameters	Times of falling
grains	*inches*	*seconds*	*grains*	*inches*	*seconds*
908	0.8	4	510	5.1	8½
983	0.8	4−	642	5.2	8
866	0.8	4	599	5.1	8
747	0.75	4+	515	5.0	8¼
808	0.75	4	483	5.0	8½
784	0.75	4+	641	5.2	8

peg was withdrawn, swung downward more slowly than it should have [i.e., more slowly than in free fall] and as a result impeded the descent of the balls at the start. For the balls were lying upon the platform near its center, and were in fact a little closer to the pivots than to the peg. And hence the times of falling were prolonged by roughly 18 thirds and so need to be corrected by taking away those thirds, especially in the larger balls, which because of the magnitude of their diameters remained a little longer upon the platform as it swung downward. When this has been done, the times in which the six larger balls fell will come out 8 sec. 12 thirds, 7 sec. 42 thirds, 7 sec. 42 thirds, 7 sec. 57 thirds, 8 sec. 12 thirds, and 7 sec. 42 thirds.

Therefore the fifth of those balls filled with air, with a diameter of 5 inches and a weight of 483 grains, fell in the time of 8 sec. 12 thirds, describing the space of 220 feet. The weight of water equal to this ball is 16,600 grains; and the weight of air equal to it is $\frac{16,600}{860}$ grains, or 19³⁄₁₀ grains, and thus the weight of the ball in a vacuum is 502³⁄₁₀ grains, and this weight is to the weight of air equal to the ball as 502³⁄₁₀ to 19³⁄₁₀, as is the ratio of 2F to ⁸⁄₃ of the diameter of the ball (that is, 2F to 13⅓ inches). And hence 2F comes out 28 feet 11 inches. The ball in falling in a vacuum, with its whole weight of 502³⁄₁₀ grains, in the time of one second describes 193⅓ inches as above, and with a weight of 483 grains describes 185.905 inches, and with the same weight of 483 grains also in a vacuum describes the space F, or 14 feet 5½ inches, in the time of 57 thirds 58 fourths, and attains the greatest velocity with which it could descend in air. With this velocity the ball, in the time of 8 sec. 12 thirds, will describe a space of 245 feet 5⅓ inches. Take away 1.3863F, or 20 feet ½ inch, and there will remain 225 feet 5 inches. It is this space, therefore, that the ball should, by the theory, have described in falling in the time of 8 sec. 12 thirds. And it described a space of 220 feet in the experiment. The difference is negligible.

Applying similar computations also to the remaining balls filled with air, I constructed the following table.

Weights of the balls	*Diameters*	*Times of falling from a height of 220 feet*		*Spaces to be described by the theory*		*Excesses*	
grains	*inches*	*seconds*	*thirds*	*feet*	*inches*	*feet*	*inches*
510	5.1	8	12	226	11	6	11
642	5.2	7	42	230	9	10	9
599	5.1	7	42	227	10	7	10
515	5	7	57	224	5	4	5
483	5	8	12	225	5	5	5
641	5.2	7	42	230	7	10	7

EXPERIMENT 14. In July 1719, Dr. Desaguliers made experiments of this sort again, making hogs' bladders into a round shape by means of a concave wooden sphere, which the moist bladders, inflated with air, were forced to fill; after they were dried and taken out, they were dropped [b]from the lantern at the top of the cupola of the same cathedral, that is, from a height of 272 feet,[b] and at the same moment a lead ball was also dropped, whose weight was roughly two pounds troy. And meanwhile some people standing in the highest part of St. Paul's where the balls were released noted the whole times of falling, and others standing on the ground noted the difference between the times of fall of the lead ball and of the bladder. And the times were measured by half-second pendulums. And one of those who were standing on the ground had a clock with an oscillating spring, vibrating four times per second; someone else had another machine ingeniously constructed with a pendulum also vibrating four times per second. And one of those who were standing in the gallery of the cupola had a similar device. And these instruments were so constructed that their motions might begin or be stopped at will. The lead ball fell in a time of roughly 4¼ seconds. And by adding

bb. Newton here writes of weights dropped "ab altiore loco in templi ejusdem turri rotunda fornicata, nempe ab altitudine pedum 272," that is, "from a higher place in the round arched tower [i.e., from the lantern at the top of the cupola] of the same cathedral." This position corresponds to the height given by Newton, 272 feet.

this time to the aforesaid difference between the times, the whole time in which the bladder fell was determined. The times in which the five bladders continued to fall after the lead ball had completed its fall were 14¾ sec., 12¾ sec., 14⅝ sec., 17¾ sec., and 16⅞ sec. the first time, and 14½ sec., 14¼ sec., 14 sec., 19 sec., and 16¾ sec. the second time. Add 4¼ sec., the time in which the lead ball fell, and the whole times in which the five bladders fell were 19 sec., 17 sec., 18⅞ sec., 22 sec., and 21⅛ sec. the first time, and 18¾ sec., 18½ sec., 18¼ sec., 23¼ sec., and 21 sec. the second time. And the times noted from the cupola were 19⅜ sec., 17¼ sec., 18¾ sec., 22⅛ sec., and 21⅝ sec. the first time, and 19 sec., 18⅝ sec., 18⅜ sec., 24 sec., and 21¼ sec. the second time. But the bladders did not always fall straight down, but sometimes flew about and oscillated to and fro while falling. And the times of falling were prolonged and increased by these motions, sometimes by one-half of one second, sometimes by a whole second. The second and fourth bladders, moreover, fell straighter down the first time, as did the first and third the second time. The fifth bladder was wrinkled and was somewhat retarded by its wrinkles. I calculated the diameters of the bladders from their circumferences, measured by a very thin thread wound round them twice. And I compared the theory with the experiments in the following table, assuming the density of air to be to the density of rainwater as 1 to 860, and calculating the spaces that the balls should, by the theory, have described in falling.

Weights of bladders	*Diameters*	*Times of falling from a height of 272 feet*	*Spaces to be described in those same times, according to the theory*		*Difference between theory and experiments*	
grains	*inches*	*seconds*	*feet*	*inches*	*feet*	*inches*
128	5.28	19	271	11	− 0	1
156	5.19	17	272	0½	+ 0	0½
137½	5.3	18½	272	7	+ 0	7
97½	5.26	22	277	4	+ 5	4
99⅛	5	21⅛	282	0	+10	0

Therefore almost all the resistance encountered by balls moving in air as well as in water is correctly shown by our theory, and is proportional to the density of the fluids—the velocities and sizes of the balls being equal.

In the scholium at the end of sec. 6, we showed by experiments with pendulums that the resistances encountered by equal and equally swift balls moving in air, water, and quicksilver are as the densities of the fluids. We have shown the same thing here more accurately by experiments with bodies falling in air and water. For pendulums in each oscillation arouse in the fluid a motion always opposite to the motion of the pendulum when it returns; and the resistance arising from this motion, and also the resistance to the cord by which the pendulum was suspended, made the whole resistance to the pendulum greater than the resistance found by the experiments with falling bodies. For by the experiments with pendulums set forth in that scholium, a ball of the same density as water ought, in describing the length of its own semidiameter in air, to lose $\frac{1}{3,342}$ of its motion. But by the theory set forth in this seventh section and confirmed by experiments with falling bodies, that same ball ought, in describing that same length, to lose only $\frac{1}{4,586}$ of its motion, supposing that the density of water is to the density of air as 860 to 1. The resistances therefore were found to be greater by the experiments with pendulums (for the reasons already described) than by the experiments with falling balls, and in a ratio of roughly 4 to 3. But since the resistances to pendulums oscillating in air, water, and quicksilver are increased similarly by similar causes, the proportion of the resistances in these mediums will be shown correctly enough by the experiments with pendulums as well as by the experiments with falling bodies. And hence it can be concluded that the resistances encountered by bodies moving in any fluids that are very fluid, other things being equal, are as the densities of the fluids.

On the basis of what has been established, it is now possible to predict very nearly what part of the motion of any ball projected in any fluid will be lost in a given time. Let D be the diameter of the ball, and V its velocity at the beginning of the motion, and T the time in which the ball will—with velocity V in a vacuum—describe a space that is to the space $\frac{8}{3}$D as the density of the ball to the density of the fluid; then the ball projected in that fluid will, in any other time t, lose the part $\frac{tV}{T+t}$ of its velocity $\left(\text{the part } \frac{TV}{T+t} \text{ remaining}\right)$ and will describe a space that is to the space described in a vacuum in the same time with the uniform velocity V as the logarithm of the number $\frac{T+t}{T}$ multiplied by the number 2.302585093 is to the number t/T, by prop. 35, corol. 7. In slow motions the resistance can be a little less, because the shape of a ball is a little more suitable for motion than the shape of a cylinder described with the same diameter. In swift motions the resistance can be a little greater, because the elasticity and the compression of the fluid are not increased in the squared ratio of the velocity. But here I am not considering petty details of this sort.

And even if air, water, quicksilver, and similar fluids, by some infinite division of their parts, could be subtilized and become infinitely fluid mediums, they would not resist projected balls any the less. For the resistance which is the subject of the preceding propositions arises from the inertia of matter; and the inertia of matter is essential to bodies and is always proportional to the quantity of matter. By the division of the parts of a fluid, the resistance that arises from the tenacity and friction of the parts can indeed be diminished, but the quantity of matter is not diminished by the division of its parts; and since the quantity of matter remains the same, its force of inertia—to which the resistance discussed here is always proportional—remains the same. For this resistance to be diminished, the quantity of matter in the spaces through which bodies move must be diminished. And therefore the celestial spaces, through which the globes of the planets and comets move continually in all directions very freely and without any sensible diminution of motion, are devoid of any corporeal fluid, except perhaps the very rarest vapors and rays of light transmitted through those spaces.

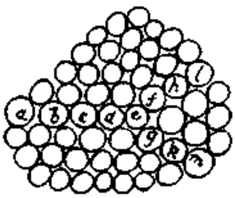
Projectiles, of course, arouse motion in fluids by going through them, and this motion arises from the excess of the pressure of the fluid on the front of the projectile over the pressure on the back, and cannot be less in infinitely fluid mediums than in air, water, and quicksilver in proportion to the density of matter in each. And this excess of pressure, in proportion to its quantity, not only arouses motion in the fluid but also acts upon the projectile to retard its motion; and therefore the resistance in every fluid is as the motion excited in the fluid by the projectile, and it cannot be less in the most subtle aether, in proportion to the density of the aether, than in air, water, and quicksilver, in proportion to the densities of these fluids.

SECTION 8

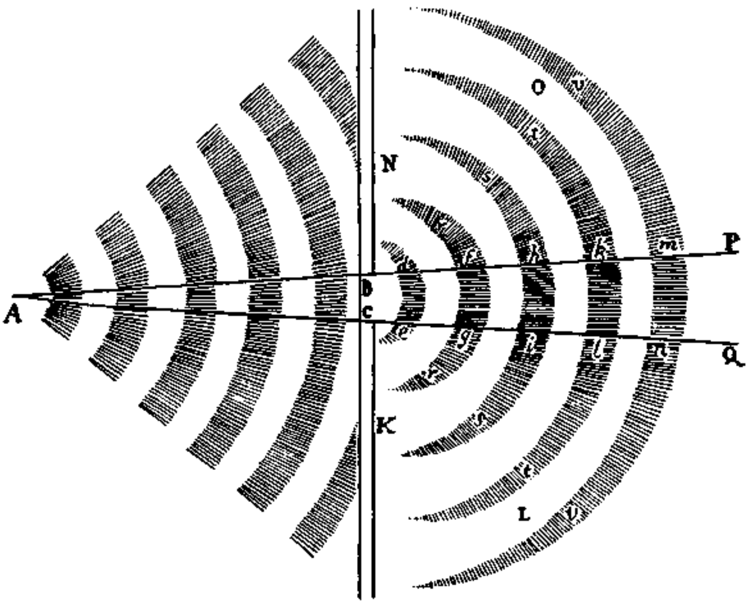
Motion propagated through fluids

Proposition 41 Theorem 32 *Pressure is not propagated through a fluid along straight lines, unless the particles of the fluid lie in a straight line.*

If the particles *a*, *b*, *c*, *d*, and *e* lie in a straight line, a pressure can indeed be propagated directly from *a* to *e*; but the particle *e* will urge the obliquely placed particles *f* and *g* obliquely, and those particles *f* and *g* will not sustain the pressure brought upon them unless they are supported by the further particles *h* and *k*; but to the extent that they are supported, they press the supporting particles, and these will not sustain the pressure unless they are supported by the further particles *l* and *m* and press them, and so on indefinitely. Therefore, as soon as a pressure is propagated to particles which do not lie in a straight line, it will begin to spread out and will be obliquely propagated indefinitely; and after the pressure begins to be propagated obliquely, if it should impinge upon further particles which do not lie in a straight line, it will spread out again, and will do so as often as it impinges upon particles not lying exactly in a straight line. Q.E.D.



COROLLARY. If some part of a pressure propagated through a fluid from a given point is intercepted by an obstacle, the remaining part (which is not intercepted) will spread out into the spaces behind the obstacle. This can be proved as follows. From point A let a pressure be propagated in any direction and, if possible, along straight lines; and by the obstacle NBCK, perforated in BC, let all the pressure be intercepted except the cone-shaped part APQ, which passes through the circular hole BC. By transverse planes *de*, *fg*, and *hi*, divide the cone APQ into frusta; then, while the cone ABC, by propagating the pressure, is urging the further conic frustum *degf* on the surface *de*, and this frustum is urging the next frustum *fgih* on the surface *fg*, and that frustum is urging a third frustum, and so on indefinitely, obviously (by the third law of motion) the first frustum *defg* will be as much urged and pressed on the surface *fg* by the reaction of the second frustum *fghi* as it urges and presses the second frustum. Therefore the frustum *degf* between the cone A*de* and the frustum *fhig* is compressed on both sides,

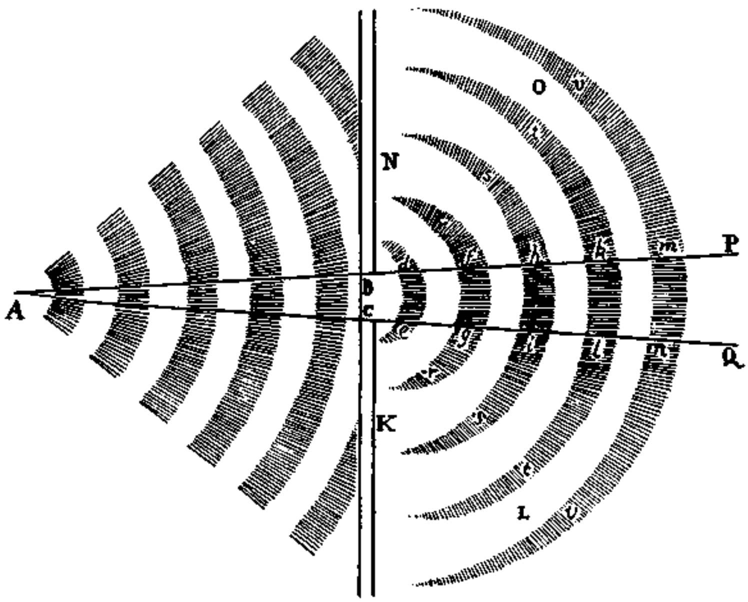


and therefore (by book 2, prop. 19, case 6) it cannot keep its figure unless it is compressed by the same force on all sides. With the same force, therefore, with which it is pressed on the surfaces *de* and *fg*, it will endeavor to yield at the sides *df* and *eg*; and there (since it is not rigid, but altogether fluid) it will run out and expand, unless a surrounding fluid is present to restrain that endeavor. Accordingly, by the endeavor to run out, it will press the surrounding fluid at the sides *df* and *eg*, as well as the frustum *fghi*, with the same force; and therefore the pressure will be no less propagated from the sides *df* and *eg* into the spaces NO on one side and KL on the other, than it is propagated from the surface *fg* toward PQ. Q.E.D.

Proposition 42
Theorem 33

All motion propagated through a fluid diverges from a straight path into the motionless spaces.

Case 1. Let a motion be propagated from point A through a hole BC, and let it proceed, if possible, in the conic space BCQP along straight lines diverging from point A. And let us suppose first that this motion is that of waves on the surface of stagnant water. And let *de*, *fg*, *hi*, *kl*, ... be the highest parts of the individual waves, separated from one another by the same number of intermediate troughs. Therefore, since the water is higher in the crests of the waves than in the motionless parts LK and NO of the



fluid, it will flow down from *e*, *g*, *i*, *l*, . . . , and *d*, *f*, *h*, *k*, . . . , the ends of the crests, toward KL on one side and NO on the other; and since it is lower in the troughs of the waves than in the motionless parts KL and NO of the fluid, it will flow down from those motionless parts into the troughs of the waves. In one case the crests of the waves, and in the other their troughs, are expanded and propagated toward KL on one side and NO on the other. And since the motion of the waves from A toward PQ takes place by the continual flowing down of the crests into the nearest troughs, and thus is not quicker than in proportion to the quickness of the descent, and since the descent of the water toward KL on one side and NO on the other ought to occur with the same velocity, the expansion of the waves will be propagated toward KL on one side and NO on the other with the same velocity with which the waves themselves progress directly from A toward PQ. And accordingly the whole space toward KL on one side and NO on the other will be occupied by the expanded waves *rfgr*, *shis*, *tklt*, *vmnv*, ... Q.E.D. Anyone can test this in stagnant water.

CASE 2. Now let us suppose that *de*, *fg*, *hi*, *kl*, and *mn* designate pulses successively propagated from point A through an elastic medium. Think of the pulses as propagated by successive condensations and rarefactions of the medium, in such a way that the densest part of each pulse occupies a spherical surface described about the center A, and that the spaces which come between successive pulses are equal. And let *de*, *fg*, *hi*, *kl*, . . . designate the densest parts of the pulses, parts which are propagated through the hole BC. And since the medium is denser there than in the spaces toward KL on one side and NO on the other, it will expand toward those spaces KL and NO situated on both sides as well as toward the rarer intervals between the pulses; and thus, always becoming rarer next to the intervals and denser next to the pulses, the medium will participate in their motion. And since the progressive motion of the pulses arises from the continual slackening of the denser parts toward the rarer intervals in front of them, and since the pulses ought to slacken with nearly the same speed into the medium's parts KL on one side and NO on the other, which are at rest, those pulses will expand on all sides into the motionless spaces KL and NO with nearly the same speed with which they are propagated straight forward from the center A, and thus will occupy the whole space KLON. Q.E.D. We find this by experience in the case of sounds, which are heard when there is a mountain in the way or which expand into all parts of a room when let in through a window and are heard in all corners, being not so much reflected from the opposite walls as propagated directly from the window, as far as the senses can tell.

CASE 3. Finally, let us suppose that a motion of any kind is propagated from A through the hole BC. That propagation does not occur except insofar as the parts of the medium that are nearer to the center A urge and move the further parts; and the parts that are urged are fluid and thus recede in every direction into regions where they are less pressed, and so will recede toward all the parts of the medium that are at rest, the parts KL and NO on the sides as well as the parts PQ in front. And therefore all the motion, as soon as it has passed through the hole BC, will begin to spread out and to be propagated directly from there into all parts as if from an origin and center. Q.E.D.

Proposition 43
Theorem 34

Every vibrating body in an elastic medium will propagate the motion of the pulses straight ahead in every direction, but in a nonelastic medium will produce a circular motion.

CASE 1. For the parts of a vibrating body, by going forward and returning alternately, will in their going urge and propel the parts of the medium that are nearest to them and by that urging will compress and condense them; then in their return they will allow the compressed parts to recede [i.e., to move apart from one another] and expand. Thus the parts of the medium that are nearest to the vibrating body will go and return alternately, like the parts of the vibrating body; and just as the parts of this body acted upon the parts of the medium, so the latter, acted upon by similar vibrations, will act upon the parts nearest to them, and these, similarly acted upon, will act upon further parts, and so on indefinitely. And just as the first parts of the medium condense in going and rarefy in returning, so the remaining parts will condense whenever they go and will expand [i.e., rarefy] whenever they return. And therefore they will not all go and return at the same time (for thus, by keeping determined distances from one another, they would not rarefy and condense alternately), but by approaching one another when they condense and moving apart when they rarefy, some of them will go while others return, and these conditions will alternate indefinitely. And the parts that are going and that condense in going (because of their forward motion with which they strike obstacles) are pulses; and therefore successive pulses will be propagated straight ahead from every vibrating body, and they will be so propagated at roughly equal distances from one another, because of the equal intervals of time in which the body produces each pulse by each of its vibrations. And even if the parts of the vibrating body go and return in some fixed and determined direction, nevertheless the pulses propagated from there through the medium will (by prop. 42) expand sideways and will be propagated in all directions from the vibrating body as if from a common center, in surfaces almost spherical and concentric. We have an example of this in waves, which, if they are produced by a wagging finger, not only will go to and fro according to the finger's motion but will immediately surround the finger like concentric circles and will be propagated in all directions. For the gravity of the waves takes the place of the elastic force.

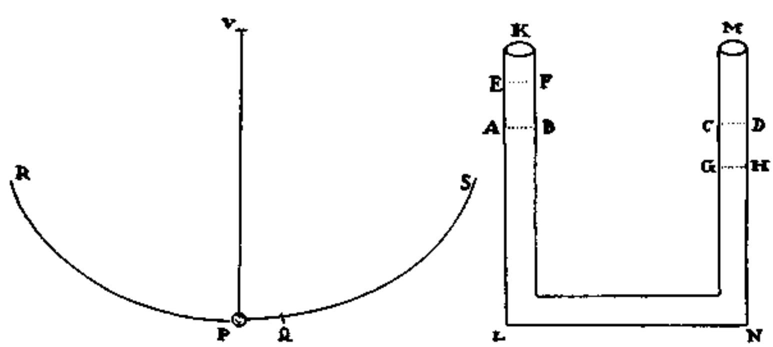
Case 2. But if the medium is not elastic, then, since its parts, pressed by the oscillating parts of the vibrating body, cannot be condensed, the motion will be propagated instantly to the parts where the medium yields most easily, that is, to the parts that the vibrating body would otherwise leave empty behind it. The case is the same as the case of a body projected in any medium. A medium, in yielding to projectiles, does not recede indefinitely, but goes with a circular motion to the spaces that the body leaves behind it. Therefore, whenever a vibrating body goes toward any place [or in any direction], the medium, in yielding, will go with a circular motion to the spaces that the body leaves; and whenever the body returns to its former place, the medium will be forced out and will return to its former place. And even though the vibrating body is not rigid but completely pliant, if it nevertheless remains of a fixed size, then, since it cannot urge the medium by its vibrations in any one place without simultaneously yielding to it in another, that body will make the medium, by receding from the parts where it is pressed, go always with a circular motion to the parts that yield to it. Q.E.D.

COROLLARY. Therefore it is a delusion to believe that the agitation of the parts of flame conduces to the propagation of a pressure along straight lines through a surrounding medium. A pressure of this sort must be derived not only from the agitation of the parts of the flame but from the dilation of the whole.

Proposition 44
Theorem 35

If water ascends and descends alternately in the vertical arms KL *and* MN *of a tube, and if a pendulum is constructed whose length between the point of suspension and the center of oscillation is equal to half of the length of the water in the tube, then I say that the water will ascend and descend in the same times in which the pendulum oscillates.*

I measure the length of the water along the axes of the tube and the arms and make it equal to the sum of these axes, and I do not here consider the resistance of the water that arises from the friction of the tube. Let AB and CD therefore designate the mean height of the water in the two arms, and when the water in the arm KL ascends to the height EF, the water in the arm MN will have descended to the height GH. Moreover, let P be a pendulum bob, VP the cord, V the point of suspension, RPQS the



cycloid described by the pendulum, P its lowest point, and PQ an arc equal to the height AE. The force by which the motion of the water is alternately accelerated and retarded is the amount by which the weight of the water in one of the two arms exceeds the weight in the other. And thus, when the water in the arm KL ascends to EF, and in the other arm descends to GH, that force is twice the weight of the water EABF and therefore is to the weight of all the water as AE or PQ to VP or PR. Furthermore, the force by which the weight P in any place Q is accelerated and retarded in the cycloid is (by book 1, prop. 51, corol.) to its whole weight as its distance PQ from the lowest place P to the length PR of the cycloid. Therefore the motive forces of the water and the pendulum, describing the equal spaces AE and PQ, are as the weights that are to be moved; and thus, if the water and the pendulum are at rest in the beginning, those forces will move them equally in equal times and will cause them to go and return synchronously with an alternating motion. Q.E.D.

COROLLARY 1. Therefore all the alternations of the ascending and descending water are isochronous, whether the motion is of greater intension or greater remission.[a]

COROLLARY 2. If the length of all the water in the tube is $6\frac{1}{9}$ Paris feet, the water will descend in the time of one second and will ascend in another second and will continue to alternate in this way indefinitely. For a pendulum $3\frac{1}{18}$ feet long oscillates in the time of one second.

COROLLARY 3. When the length of the water is increased or decreased, moreover, the time of alternation is increased or decreased as the square root of the length.

Proposition 45
Theorem 36

The velocity of waves is as the square roots of the lengths.

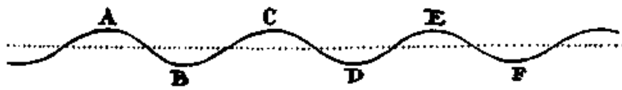
This follows from the construction of the following proposition.

Proposition 46
Problem 10

To find the velocity of waves.

Set up a pendulum whose length between the point of suspension and the center of oscillation is equal to the length of the waves; and in the same time in which the pendulum performs each of its oscillations, the waves as they move forward will traverse nearly their own lengths.

a. Newton evidently is referring to amplitude.



By length of a wave I mean the transverse distance either between bottoms of troughs or between tops of crests. Let ABCDEF designate the surface of stagnant water ascending and descending in successive waves; and let A, C, E, . . . be the crests of the waves, and B, D, F, . . . the troughs in between. Since the motion of the waves is caused by the successive ascent and descent of the water, in such a way that its parts, A, C, E, . . . , which now are highest, soon become lowest, and since the motive force by which the highest parts descend and the lowest ascend is the weight of the elevated water, the alternate ascent and descent will be analogous to the alternating motion of the water in the tube and will observe the same laws with respect to times; and therefore (by prop. 44), if the distances between the highest places A, C, and E of the waves and the lowest, B, D, and F, are equal to twice the length of a pendulum, the highest parts A, C, and E will in the time of one oscillation come to be the lowest, and in the time of a second oscillation will ascend once again. Therefore there will be a time of two oscillations between successive waves; that is, a wave will describe its own length in the time in which the pendulum oscillates twice; but in the same time a pendulum whose length is four times as great, and thus equals the length of the waves, will oscillate once. Q.E.I.

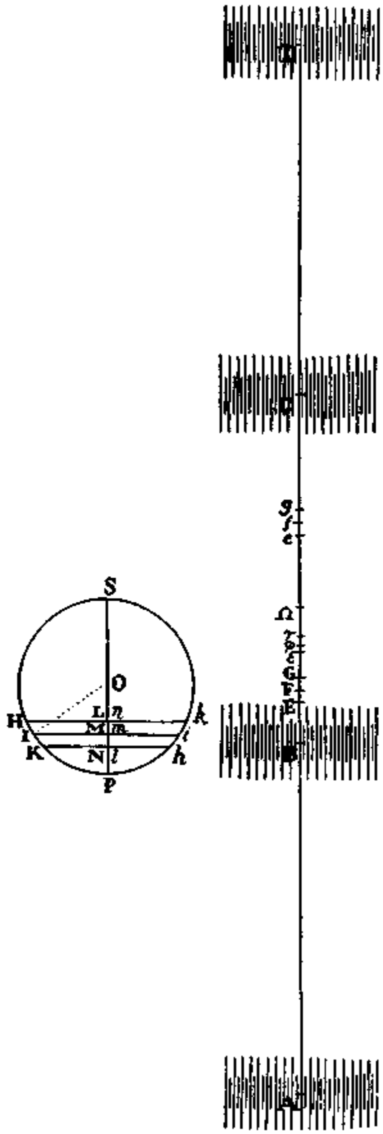
COROLLARY 1. Therefore waves with a length of 3$\frac{1}{18}$ Paris feet will move forward through their own length in the time of one second and thus in the time of one minute will traverse 183$\frac{1}{3}$ feet, and in the space of an hour very nearly 11,000 feet.

COROLLARY 2. And the velocity of waves of greater or smaller length will be increased or decreased as the square root of the length.

What has been said is premised on the hypothesis that the parts of the water go straight up or straight down; but this ascent and descent takes place more truly in a circle, and thus I admit that in this proposition the time has been determined only approximately.

Proposition 47
Theorem 37

If pulses are propagated through a fluid, the individual particles of the fluid, going and returning with a very short alternating motion, are always accelerated and retarded in accordance with the law of an oscillating pendulum.



Let AB, BC, CD, ... designate the equal distances between successive pulses; ABC the line of motion of the pulses, propagated from A toward B; E, F, and G three physical points in the medium at rest, situated at equal intervals along the straight line AC; Ee, Ff, and Gg very short equal spaces through which those points go and return in each vibration with an alternating motion; ε, φ, γ any intermediate positions of those same points; and EF and FG physical line-elements or linear parts of the medium, put between those points and successively transferred into the places $\varepsilon\varphi$, $\varphi\gamma$ and ef, fg. Draw the straight line PS equal to the straight line Ee. Bisect PS in O, and with center O and radius OP describe the circle SIPi.

Let the whole circumference of this circle with its parts represent the whole time of one vibration with its proportional parts, in such a way that when any time PH or PHSh is completed, if the perpendicular HL or hl is dropped to PS, and if Eε is taken equal to PL or Pl, then the physical point E is found in ε. By this law any point E, in going from E through ε to e and returning from there through ε to E, will perform each vibration with the same degrees of acceleration and retardation as the oscillating pendulum. It is to be proved that each of the physical points of the medium must move in such a way. Let us imagine, therefore, that there is such a motion in the medium, arising from any cause, and see what follows.

In the circumference PHSh take the equal arcs HI and IK or hi and ik, having the ratio to the whole circumference that the equal straight lines

EF and FG have to the whole interval BC between pulses. Drop the perpendiculars IM and KN and also *im* and *kn*. Then the points E, F, and G are successively agitated with similar motions and carry out their complete vibrations (consisting of a going and returning) while a pulse is transferred from B to C; accordingly, if PH or PHS*h* is the time from the beginning of the motion of point E, PI or PHS*i* will be the time from the beginning of the motion of point F, and PK or PHS*k* will be the time from the beginning of the motion of point G; and therefore $E\varepsilon$, $F\varphi$, and $G\gamma$ will be equal respectively to PL, PM, and PN in the going of the points, or to P*l*, P*m*, and P*n* in the returning of the points. Hence $\varepsilon\gamma$ or $EG + G\gamma - E\varepsilon$ will be equal to $EG - LN$ in the going of the points, and will be equal to $EG + ln$ in their returning. But $\varepsilon\gamma$ is the width or expansion of the part of the medium EG in the place $\varepsilon\gamma$; and therefore the expansion of that part in the going is to its mean expansion as $EG - LN$ to EG, and in the returning is as $EG + ln$ or $EG + LN$ to EG. Therefore, since LN is to KH as IM to the radius OP, and KH is to EG as the circumference PHS*h*P to BC, that is (if V is put for the radius of a circle having a circumference equal to the interval between the pulses BC), as OP to V, and since, from the equality of the ratios [or ex aequo], LN is to EG as IM to V, the expansion of the part EG or of the physical point F in the place $\varepsilon\gamma$ will be to the mean expansion which that part has in its own first place EG as $V - IM$ to V in the going, and as $V + im$ to V in the returning. Hence the elastic force of point F in the place $\varepsilon\gamma$ is to its mean elastic force in the place EG as $\frac{1}{V - IM}$ to $\frac{1}{V}$ in the going, and as $\frac{1}{V + im}$ to $\frac{1}{V}$ in the returning. And by the same argument the elastic forces of the physical points E and G in the going are as $\frac{1}{V - HL}$ and $\frac{1}{V - KN}$ to $\frac{1}{V}$; and the difference between the forces is to the mean elastic force of the medium as $\frac{HL - KN}{V^2 - V \times HL - V \times KN + HL \times KN}$ to $\frac{1}{V}$, that is, as $\frac{HL - KN}{V^2}$ to $\frac{1}{V}$, or as $HL - KN$ to V, provided that (because of the narrow limits of the vibrations) we suppose HL and KN to be indefinitely smaller than the quantity V. Therefore, since the quantity V is given, the difference between the forces is as $HL - KN$, that is, as OM (because $HL - KN$ is proportional to HK and OM to OI or OP; and HK and OP are given)—that is, if F*f* is bisected in Ω, as $\Omega\varphi$. And by the same argument the difference between the elastic forces of the physical points ε and γ, in the returning of the physical line-element $\varepsilon\gamma$, is as $\Omega\varphi$. But that difference (that is, the amount by which the elastic force of point ε exceeds the elastic force of point γ) is the force by which the intervening physical line-element $\varepsilon\gamma$ of the medium is accelerated in the going and retarded in the returning; and therefore the accelerative force of the physical line-element $\varepsilon\gamma$ is as its distance from the midpoint Ω of the vibration. Accordingly, the time (by book 1, prop. 38) is correctly represented by the arc PI, and the linear part $\varepsilon\gamma$ of the medium moves by the law previously mentioned, that is, by the law of an oscillating pendulum; and the same is true of all the linear parts of which the whole medium is composed. Q.E.D.

COROLLARY. Hence it is evident that the number of pulses propagated is the same as the number of vibrations of the vibrating body and does not increase as the pulses move forward. For as soon as the physical line-element $\varepsilon\gamma$ has returned to its first place, it will be at rest and will not move afterward unless it receives a new motion either by the impact of the vibrating body or by the impact of pulses that are propagated from the vibrating body. It will be at rest, therefore, as soon as the pulses cease to be propagated from the vibrating body.

Proposition 48
Theorem 38

The velocities of pulses propagated in an elastic fluid are as the square root of the elastic force directly and the square root of the density inversely, provided that the elastic force of the fluid is proportional to its condensation.

CASE 1. If the mediums are homogeneous and the distances between pulses in these mediums are equal to one another, but the motion in one medium is more intense, then the contractions and expansions of corresponding parts will be as the motions. In fact, this proportion is not exact. Even so, unless the contractions and expansions are extremely intense, the error will not be perceptible, and thus the proportion can be considered physically exact. But the motive elastic forces are as the contractions and expansions; and the velocities—generated in the same time—of equal parts are as the forces. And thus equal and corresponding parts of corresponding pulses will go and return together through spaces proportional to the contractions and expansions, with velocities that are as the spaces; and therefore the pulses, which advance through their own length in the time of one going and returning and which always succeed into the places of the immediately preceding pulses, will progress in both mediums with an equal velocity, because of the equality of the distances.

CASE 2. But if the distances between pulses, or their lengths, are greater in one medium than in the other, let us suppose that the corresponding parts by going and returning in each alternation describe spaces proportional to the lengths of the pulses; then their contractions and expansions will be equal. And thus if the mediums are homogeneous, those motive elastic forces by which they are agitated with an alternating motion will also be equal. But the matter to be moved by these forces is as the length of the pulses; and the space through which they must move by going and returning in each alternation is in the same ratio. And the time of going and returning is jointly proportional to the square root of the matter and the square root of the space and thus is as the space. But the pulses advance through their own lengths in the times of one going and returning, that is, traverse spaces proportional to the times, and therefore have equal velocities.

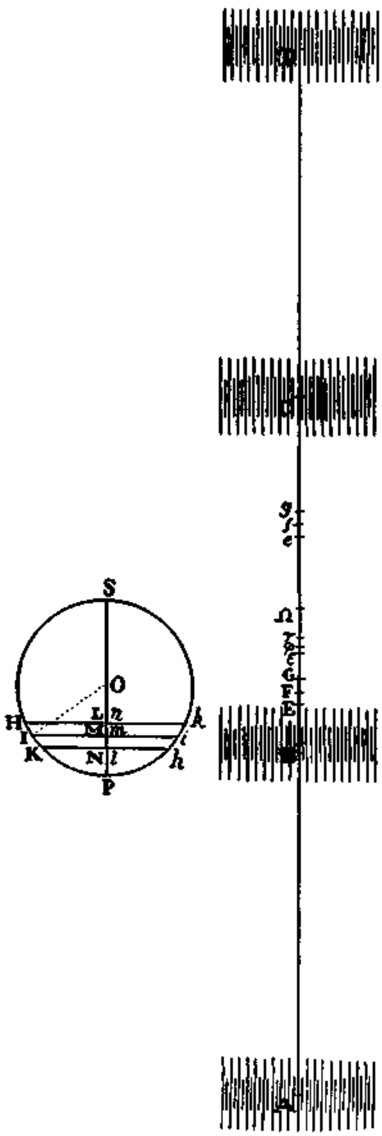
CASE 3. In mediums of the same density and elastic force, therefore, all pulses have equal velocities. But if either the density or the elastic force of the medium is intended [i.e., increased], then, since the motive force is increased in the ratio of the elastic force, and the matter to be moved is increased in the ratio of the density, the time in which the same motions as before can be performed will be increased as the square root of the density and will be decreased as the square root of the elastic force. And therefore the velocity of the pulses will be jointly proportional to the square root of the density of the medium inversely and the square root of the elastic force directly. Q.E.D.

This proposition will be clearer from the construction of the following proposition.

Proposition 49
Problem 11

Given the density and elastic force of a medium, it is required to find the velocity of the pulses.

Let us imagine the medium to be compressed, as our air is, by an incumbent weight and let A be the height of a homogeneous medium whose weight is equal to the incumbent weight and whose density is the same as the density of the compressed medium in which the pulses are propagated.



And suppose that a pendulum is set up, whose length between the point of suspension and the center of oscillation is A; then, in the same time in which that pendulum performs an entire oscillation composed of a going and a returning, a pulse will advance through a space equal to the circumference of a circle described with radius A.

For with the same constructions as in prop. 47, if any physical line EF, describing the space PS in each single vibration, is urged in the extremities P and S of each going and returning by an elastic force that is equal to its weight, it will perform each single vibration in the time in which it could oscillate in a cycloid whose whole perimeter is equal to the length PS; and this is so because equal forces will simultaneously impel equal corpuscles through equal spaces. Therefore, since the times of the oscillations are as the square root of the length of the pendulums, and since the length of the pendulum is equal to half the arc of the whole cycloid, the time of one vibration would be to the time of oscillation of a pendulum whose length is A as the square root of the length $\frac{1}{2}$PS or PO to the length A. But the elastic force by which the physical line-element EG is urged in its extremities P and S was (in the proof of prop. 47) to its whole elastic force as HL − KN to V, that is (since point K now falls upon P), as HK to V; and that whole force, that is, the incumbent weight by which the line-element EG is compressed, is to the weight of the line-element as the height A of the incumbent weight to the length EG of the line-element; and thus from the equality of the ratios [or ex aequo] the force by which the line-element EG is urged in its places P and S is to the weight of that line-element as HK $\times$ A to V $\times$ EG, or as PO $\times$ A to V^2 (for HK was to EG as PO to V). Therefore, since the times in which equal bodies are impelled through equal spaces are inversely as the square root of the forces, the time of one vibration under the action of that elastic force will be to the time of the vibration, under the action of the force of weight, as the square root of V^2 to PO $\times$ A, and thus will be to the time of oscillation of a pendulum having a length A as $\sqrt{\dfrac{V^2}{PO \times A}}$ and $\sqrt{\dfrac{PO}{A}}$ jointly, that is, as V to A. But in the time of one vibration, composed of a going and returning, a pulse advances through its own length BC. Therefore the time in which the pulse traverses the space BC is to the time of one oscillation (composed of a going and returning) as V to A, that is, as BC to the circumference of a circle whose radius is A. But the time in which the pulse will traverse the space BC is in the same ratio to the time in which it will traverse a length equal to this circumference; and thus in the time of such an oscillation the pulse will traverse a length equal to this circumference. Q.E.D.

COROLLARY 1. The velocity of the pulses is that which heavy bodies acquire in falling with a uniformly accelerated motion and describing by their fall half of the height A. For in the time of this fall, with the velocity acquired in falling, the pulse will traverse a space equal to the whole height A; and thus in the time of one oscillation (composed of a going and returning) it will traverse a space equal to the circumference of a circle described with radius A; for the time of fall is to the time of oscillation as the radius of the circle to its circumference.

COROLLARY 2. Hence, since that height A is as the elastic force of the fluid directly and its density inversely, the velocity of the pulses will be as the square root of the density inversely and the square root of the elastic force directly.

To find the distances between pulses. **Proposition 50**

Problem 12

In a given time, find the number of vibrations of the body by whose vibration the pulses are excited. Divide by that number the space that a pulse could traverse in the same time, and the part found will be the length of one pulse. Q.E.I.

Scholium The preceding propositions apply to the motion of light and of sounds. For since light is propagated along straight lines, it cannot consist in action alone (by props. 41 and 42). And because sounds arise from vibrating bodies, they are nothing other than propagated pulses of air (by prop. 43). This is confirmed from the vibrations that they excite in bodies exposed to them, provided that they are loud and deep, such as the sounds of drums. For swifter and shorter vibrations are excited with more difficulty. But it is also well known that any sounds impinging upon strings in unison with the sonorous bodies excite vibrations in them. It is confirmed also from the velocity of sounds. For since the specific weights of rainwater and quicksilver are to each other as roughly 1 to 13⅔, and since, when the mercury in a barometer reaches a height of 30 English inches, the specific weight of the air and that of rainwater are to each other as roughly 1 to 870, the specific weights of air and quicksilver will be as 1 to 11,890. Accordingly, since the height of the quicksilver is 30 inches, the height of uniform air whose weight could compress our air lying beneath it will be 356,700 inches, or 29,725 English feet. And this height is the very one that we called A in the construction of prop. 49. The circumference of a circle described with a radius of 29,725 feet is 186,768 feet. And since a pendulum 39⅕ inches long completes an oscillation composed of a going and returning in the time of 2 seconds, as is known, a pendulum 29,725 feet or 356,700 inches long must complete an entirely similar oscillation in the time of 190¾ seconds. In that time, therefore, sound will advance 186,768 feet, and thus in the time of one second, 979 feet.

[a]But in this computation no account is taken of the thickness of the solid particles of air, a thickness through which sound is of course propagated

aa. Ed. 1 has: "Mersenne writes in prop. 35 of his *Ballistics* that he found by making experiments that sound travels 1,150 French toises (that is, 6,900 French feet) in 5 seconds. Hence, since a French foot is to an English foot as 1,068 to 1,000, sound will have to travel 1,474 English feet in the time of 1 second. Mersenne also writes that the eminent geometer Roberval observed during the siege of Thionville that the noise of cannons was heard 13 or 14 seconds after the fire was seen, although he was scarcely half a league away from the cannons. A French league contains 2,500 toises, and thus, according to Roberval's observation, in the time of 13 or 14 seconds sound traveled 7,500 Paris feet, and in the time of 1 second 560 Paris feet, or about 600 English feet. These observations are very different from one another, and our computation falls in the middle. In the cloister of our college, which is 208 feet long, a sound excited at either end makes a fourfold echo in four returnings. And by making experiments I found that at each returning of the sound a pendulum of about 6 or 7 inches completed an oscillation, starting at the first returning of the sound and completing its oscillation at the second one. I was not able to determine the length of the pendulum exactly enough, but I judged that with a length of 4 inches the oscillations were instantaneously. Since the weight of air is to the weight of water as 1 to 870, and since salts are nearly twice as dense as water, if the particles of air are supposed to be of roughly the same density as the particles of either water or salts, and if the rarity of air arises from the distances between the particles, the diameter of a particle of air will be to the distance between the centers of the particles roughly as 1 to 9 or 10, and to the distance between the particles as 1 to 8 or 9. Accordingly, to the 979 feet which sound will travel in the time of 1 second according to the above calculation, $\frac{979}{9}$ feet or roughly 109 feet may be added, because of the density of the particles of air; and thus sound will travel roughly 1,088 feet in the time of 1 second.

Additionally, the vapors lying hidden in the air, since they are of another elasticity and another tone, participate scarcely or not at all in the motion of the true air by which sounds are propagated. And when these vapors are at rest, that motion will be propagated more swiftly through the true air alone

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too fast and that with a length of 9 inches they were too slow. Hence in going and returning the sound traveled 416 feet in a smaller time than that in which a pendulum of 9 inches oscillates and in a greater time than a pendulum of 4 inches, that is, in a smaller time than 28¾ thirds and a greater than 19⅙. And therefore in the time of 1 second the sound travels more than 866 English feet and fewer than 1,272 and thus is faster than according to Roberval's observation and slower than according to Mersenne's. Further, by more accurate observations made afterward, I determined that the length of the pendulum ought to be greater than 5½ inches and less than 8 inches and thus that sound in the time of 1 second traveled more than 920 English feet and fewer than 1,085. Therefore the motion of sounds, being between these limits according to the geometrical calculation given above, squares with the phenomena insofar as it has been possible to test it up to now. Accordingly, since this motion depends on the density of the whole air, it follows that sounds consist not in the motion of aether or of some more subtle air but in the agitation of the whole air.

"Certain experiments concerning sound propagated in vessels empty of air seem to contradict this, but vessels can scarcely be emptied of all air; and when they are sufficiently emptied, sounds are noticeably diminished. For example, if only a hundredth of the whole air remains in the vessel, a sound will have to be a hundred times weaker and thus should not be less audible than if someone, hearing the same sound excited in free air, immediately withdrew to ten times the distance from the sonorous body. Two equally sonorous bodies therefore must be compared, of which one is in an emptied vessel and the other is in free air and whose distances from the hearer are as the square roots of the densities of the air, and if the sound of the former body does not exceed the sound of the latter, the objection will cease.

"Once the velocity of sounds has been found, the intervals between the pulses can also be found. Mersenne writes (*Harmonics*, book 1, prop. 4) that (by making certain experiments which he describes in the same place) he found that a stretched musical string vibrates 104 times in the space of 1 second when it makes a unison with an open four-foot organ pipe or a stopped two-foot pipe, which organists call C fa ut. Accordingly, there are 104 pulses in a space of 968 feet, the distance which sound travels in the time of 1 second, and thus one pulse occupies a space of roughly 9¼ feet, that is, roughly twice the length of the pipe. Hence it is likely that the lengths of the pulses in the sounds of all open pipes are equal to twice the lengths of the pipes."

as the square root of the ratio of the total atmosphere of air and vapor to the matter of the particles of air alone. For example, if the atmosphere consists of 10 parts of true air and 1 part of vapors, the motion of sounds will be swifter as the square root of the ratio of 11 to 10, or in roughly the ratio of 21 to 20, than if it were propagated through 11 parts of true air; and thus the motion of sounds that was found above will have to be increased in this ratio. Thus in the time of 1 second, sound will travel 1,142 feet.

These things ought to be so in the springtime and autumn, when the air is rarefied by the temperate heat and its elastic force is somewhat intended [i.e., increased]. But in winter, when the air is condensed by the cold, and its elastic force is remitted [i.e., decreased], the motion of sounds should be slower as the square root of the density; and alternately, in summer it should be swifter.

It is established by experiments, moreover, that in the time of 1 second sounds advance through more or less 1,142 London feet, or 1,070 Paris feet.

Once the velocity of sounds has been found, the intervals between the pulses can also be found. Sauveur found by making experiments that an open pipe, whose length is more or less 5 Paris feet, produces a sound with the same pitch as the sound of a string that vibrates a hundred times in 1 second. Accordingly, there are more or less 100 pulses in the space of 1,070 Paris feet, the distance which sound travels in the time of 1 second, and thus 1 pulse occupies a space of about 10⁷⁄₁₀ Paris feet, that is, roughly twice the length of the pipe. Hence it is likely that the lengths of the pulses in the sounds of all open pipes are equal to twice the lengths of the pipes.[a]

Furthermore, it is evident from book 2, prop. 47, corol., why sounds immediately cease when the motion of the sonorous body ceases, and why they are not heard for a longer time when we are very far distant from the sonorous bodies than when we are very close. Why sounds are very much increased in megaphones is also manifest from the principles set forth. For every reciprocal motion is increased at each reflection by the generating cause. And the motion is lost more slowly and is reflected more strongly in tubes that impede the expansion of sounds, and therefore is more increased by the new motion impressed at each reflection. And these are the major phenomena of sounds.

SECTION 9

The circular motion of fluids

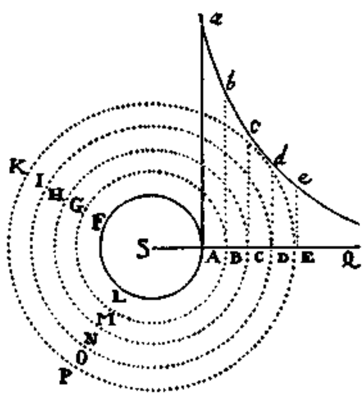
Hypothesis

*The resistance that arises from the friction [*lit. *lack of lubricity or slipperiness] of the parts of a fluid is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another.*

Proposition 51
Theorem 39

If an infinitely long solid cylinder revolves with a uniform motion in a uniform and infinite fluid about an axis given in position, and if the fluid is made to revolve by only the impulse of the cylinder, and if each part of the fluid perseveres uniformly in its motion, then I say that the periodic times of the parts of the fluid are as their distances from the axis of the cylinder.

Let AFL be the cylinder made to revolve uniformly about the axis S, and divide the fluid into innumerable concentric solid cylindrical orbs[a] of the same thickness by the concentric circles BGM, CHN, DIO, EKP, ... Then, since the fluid is homogeneous, the impressions that contiguous orbs make upon one another will (by hypothesis) be as their relative displacements and the contiguous surfaces on which the impressions are made. If the impression upon some orb is greater or less on its concave side than on its convex side, the stronger impression will prevail and will either accelerate or retard the motion of the orb, according as it is directed the same way as its motion or the opposite way. Consequently, so that each orb may persevere uniformly in its motion, the impressions on each of the two sides should be equal and be made in opposite directions. Hence,



a. In props. 51 and 52, Newton is using the word "orb" in two closely related senses. One is that of a series of nested hollow spheres or orbs, much as in the older Aristotelian universe, where the orbits of the planets were considered to be embedded in a set of nesting or concentric hollow spherical shells or orbs. In prop. 52, Newton writes of a set of "innumerable concentric orbs of the same thickness." In prop. 51, a similar concept is introduced for a cylinder, which Newton says is to be divided into "innumerable concentric solid cylindrical orbs of the same thickness." Today it would not be usual to call such cylindrical shells "orbs" as Newton did; nevertheless, we have kept Newton's "orbs" in prop. 51 so as to keep it in harmony with the language of prop. 52.

since the impressions are as the contiguous surfaces and their relative velocities, the relative velocities will be inversely as the surfaces, that is, inversely as the distances of the surfaces from the axis. And the differences between the angular motions about the axis are as these relative velocities divided by the distances, or as the relative velocities directly and the distances inversely—that is, if the ratios are compounded, as the squares of the distances inversely. Therefore, if the perpendiculars A*a*, B*b*, C*c*, D*d*, E*e*, . . . , inversely proportional to the squares of SA, SB, SC, SD, SE, . . . , are erected to each of the parts of the infinite straight line SABCDEQ and if a hyperbolic curve is understood to be drawn through the ends of the perpendiculars, then the sums of the differences, that is, the whole angular motions, will be as the corresponding sums of the lines A*a*, B*b*, C*c*, D*d*, E*e*; that is, if, in order to make the medium uniformly fluid, the number of orbs is increased and their width decreased indefinitely, as the hyperbolic areas A*a*Q, B*b*Q, C*c*Q, D*d*Q, E*e*Q, . . . , corresponding to these sums. And the times, which are inversely proportional to the angular motions, will also be inversely proportional to these areas. The periodic time of any particle D, therefore, is inversely as the area D*d*Q, that is (by the known quadratures of curves), directly as the distance SD. Q.E.D.

Corollary 1. Hence the angular motions of the particles of the fluid are inversely as the distances of the particles from the axis of the cylinder, and the absolute velocities are equal.

Corollary 2. If the fluid is contained in a cylindrical vessel of an infinite length and contains another inner cylinder, and if both cylinders revolve about a common axis, and the times of the revolutions are as the semidiameters of the cylinders, and each part of the fluid perseveres in its motion, then the periodic times of the individual parts will be as their distances from the axis of the cylinders.

Corollary 3. If any common angular motion is added to, or taken away from, the cylinder and the fluid moving in this way, then, since the mutual friction of the parts of the fluid is not changed by this new motion, the motions of the parts with respect to one another will not be changed. For the relative velocities of the parts depend upon the friction. Any part will persevere in that motion which is not more accelerated than retarded by the friction on opposite sides in opposite directions.

Corollary 4. Hence, if all the angular motion of the outer cylinder is taken away from the whole system of the cylinders and fluid, the result will be the motion of the fluid in the cylinder at rest.

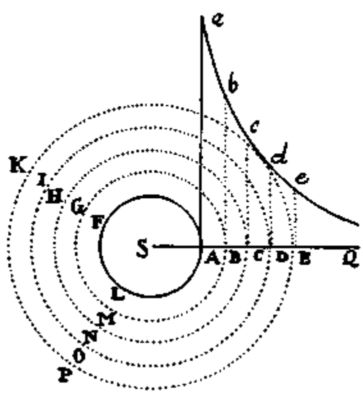
Corollary 5. Therefore, if, while the fluid and outer cylinder are at rest, the inner cylinder revolves uniformly, a circular motion will be communicated to the fluid and will be propagated little by little through the whole fluid, and it will not cease to be increased until the individual parts of the fluid acquire the motion defined in corol. 4.

Corollary 6. And since the fluid endeavors to propagate its own motion even further, its force will make the outer cylinder also revolve, unless that cylinder is forcibly held in place, and the motion of that cylinder will be accelerated until the periodic times of both cylinders become equal. But if the outer cylinder is forcibly held in place, it will endeavor to retard the motion of the fluid, and unless the inner cylinder conserves that motion by some force impressed from outside, the outer cylinder will cause the motion to cease little by little.

All of this can be tested in deep stagnant water.

Proposition 52
Theorem 40

If a solid sphere revolves with a uniform motion in a uniform and infinite fluid about an axis given in position, and if the fluid is made to revolve by only the impulse of this sphere, and if each part of the fluid perseveres uniformly in its motion, then I say that the periodic times of the parts of the fluid will be as the squares of the distances from the center of the sphere.



Case 1. Let AFL be a sphere made to revolve uniformly about the axis S, and divide the fluid into innumerable concentric orbs[a] of the same thickness by means of the concentric circles BGM, CHN, DIO, EKP, ... And imagine the orbs to be solid; then, since the fluid is homogeneous, the impressions that the contiguous orbs make upon one another will (by the hypothesis) be as their relative velocities and

a. On the use of "orbs" in prop. 52, and in the antecedent prop. 51, see the note to prop. 51.

the contiguous surfaces on which the impressions are made. If the impression upon some orb is greater or less on the concave side than on the convex side, the stronger impression will prevail and will either accelerate or retard the velocity of the orb, according as it is directed the same way as the motion of the orb or the opposite way. Consequently, so that each orb may persevere uniformly in its motion, the impressions on each of the two sides will have to be equal and to be made in opposite directions. Hence, since the impressions are as the contiguous surfaces and their relative velocities, the relative velocities will be inversely as the surfaces, that is, inversely as the squares of the distances of the surfaces from the center. But the differences in the angular motions about the axis are as these relative velocities divided by the distances, or as the relative velocities directly and the distances inversely—that is, if the ratios are compounded, as the cubes of the distances inversely. Therefore, if to each of the parts of the infinite straight line SABCDEQ there are erected the perpendiculars A*a*, B*b*, C*c*, D*d*, E*e*, . . . , inversely proportional to the cubes of SA, SB, SC, SD, SE, . . . , then the sums of the differences, that is, the whole angular motions, will be as the corresponding sums of the lines A*a*, B*b*, C*c*, D*d*, E*e*—that is (if, to make the medium uniformly fluid, the number of orbs is increased and their width decreased indefinitely), as the hyperbolic areas A*a*Q, B*b*Q, C*c*Q, D*d*Q, E*e*Q, . . . , corresponding to these sums. And the periodic times, inversely proportional to the angular motions, will also be inversely proportional to these areas. Therefore the periodic time of any orb DIO is inversely as the area D*d*Q, that is (by the known methods of quadratures of curves), directly as the square of the distance SD. And this is what I wanted to prove in the first place.

CASE 2. From the center of the sphere draw as many infinite straight lines as possible which with the axis contain given angles exceeding one another by equal differences, and imagine the orbs to be cut into innumerable rings by the revolution of these straight lines about the axis; then each ring will have four rings contiguous to it, one inside, another outside, and two at the sides. Each ring cannot be urged equally and in opposite directions by the friction of the inner ring and of the outer ring, except in a motion made according to the law of case 1. This is evident from the proof of case 1. And therefore any series of rings proceeding straight from the sphere indefinitely will be moved in accordance with the law of case 1, except insofar as it is impeded by the friction of the rings at the sides. But in motion made according to this law the friction of the rings at the sides is nil, and thus it will not impede the motion from being made according to this law. If rings equally distant from the center revolved either more quickly or more slowly near the poles than near the ecliptic, the slower rings would be accelerated and the swifter would be retarded by mutual friction, and thus the periodic times would always tend toward equality, in accordance with the law of case 1. This friction, therefore, does not prevent the motion from being made according to the law of case 1, and therefore that law will hold good; that is, the periodic time of each of the rings will be as the square of its distance from the center of the sphere. This is what I wanted to prove in the second place.

CASE 3. Now let each ring be divided by transverse sections into innumerable particles constituting an absolutely and uniformly fluid substance; then, since these sections have no relation to the law of circular motion but contribute only to the constitution of the fluid, the circular motion will continue as before. As a result of this sectioning, all the minimally small rings either will not change the unevenness and the force of their mutual friction or will change them equally. Furthermore, since the proportion of the causes remains the same, the proportion of the effects—that is, the proportion of the motions and periodic times—will also remain the same. Q.E.D.

But since the circular motion, along with the centrifugal force arising from it, is greater at the ecliptic than at the poles, there must be some cause by which each of the particles is kept in its circle; otherwise the matter at the ecliptic would always recede from the center and move on the outside of the vortex to the poles, and return from there along the axis to the ecliptic with a continual circulation.

COROLLARY 1. Hence the angular motions of the parts of the fluid about the axis of the sphere are inversely as the squares of the distances from the center of the sphere, and the absolute velocities are inversely as those same squares divided by the distances from the axis.

COROLLARY 2. If a sphere, in a homogeneous and infinite fluid at rest, revolves with a uniform motion about an axis given in position, it will communicate a motion to the fluid like that of a vortex, and this motion will be propagated little by little without limit, and this motion will not cease to be accelerated in each part of the fluid until the periodic time of each of the parts is as the squares of the distances from the center of the sphere.

Corollary 3. Since the inner parts of a vortex, because of their greater velocity, rub and push the outer parts and continually communicate motion to them by this action, and since those outer parts simultaneously transfer the same quantity of motion to others still further out and by this action conserve the quantity of their motion completely unaltered, it is evident that the motion is continually transferred from the center to the circumference of the vortex and is absorbed in that limitless circumference. The matter between any two spherical surfaces concentric with the vortex will never be accelerated, because all of the motion it receives from the inner matter is continually transferred to the outer matter.

Corollary 4. Accordingly, for a vortex to conserve the same state of motion constantly, some active principle is required from which the sphere may always receive the same quantity of motion that it impresses on the matter of the vortex. Without such a principle, it is necessary for the sphere and the inner parts of the vortex, always propagating their motion to outer parts and not receiving any new motion, to slow down little by little and cease to be carried around.

Corollary 5. If a second sphere were to be immersed in this vortex at a certain distance from the center, and meanwhile by some force were to revolve constantly about an axis given in inclination, then the fluid would be drawn into a vortex by the motion of this sphere; and first this new and tiny vortex would revolve along with the sphere about the center of the first vortex, and meanwhile its motion would spread more widely and little by little would be propagated without limit, in the same way as the first vortex. And for the same reason that the sphere of the new vortex was drawn into the motion of the first vortex, the sphere of the first vortex would also be drawn into the motion of this new vortex, in such a way that the two spheres would revolve about some intermediate point and because of that circular motion would recede from each other unless constrained by some force. Afterward, if the continually impressed forces by which the spheres persevere in their motions were to cease, and everything were left to the laws of mechanics, the motion of the spheres would weaken little by little (for the reason assigned in corols. 3 and 4), and the vortices would at last be completely at rest.

Corollary 6. If several spheres in given places revolved continually with certain velocities around axes given in position, the same number of vortices, going on without limit, would be made. For all of the spheres, for the same reason that any one of them propagates its motion without limit, will also propagate their motions without limit, in such a way that each part of the infinite fluid is agitated by that motion which results from the actions of all the spheres. Hence the vortices will not be limited by fixed bounds but will little by little run into one another, and the spheres will be continually moved from their places by the actions of the vortices upon one another, as was explained in corol. 5; nor will they keep any fixed position with respect to one another, unless constrained by some force. And when those forces, which conserve the motions by being continually impressed upon the spheres, cease, the matter—for the reason assigned in corols. 3 and 4—will little by little come to rest and will no longer be made to move in vortices.

Corollary 7. If a homogeneous fluid is enclosed in a spherical vessel and is made to revolve in a vortex by the uniform rotation of a sphere placed in the center, and if the sphere and the vessel revolve in the same direction about the same axis, and if their periodic times are as the squares of the semidiameters, then the parts of the fluid will not persevere in their motions without acceleration and retardation until their periodic times are as the squares of the distances from the center of the vortex. No other constitution of a vortex can be stable.

Corollary 8. If the vessel, the enclosed fluid, and the sphere conserve this motion and additionally revolve with a common angular motion about any given axis, then, since the friction of the parts of the fluid upon one another is not changed by this new motion, the motions of the parts with respect to one another will not be changed. For the relative velocities of the parts with respect to one another depend upon friction. Any part will persevere in that motion by which the friction on one side does not retard it more than the friction on the other accelerates it.

Corollary 9. Hence, if the vessel is at rest, and if the motion of the sphere is given, the motion of the fluid will be given. For imagine that a plane passes through the axis of the sphere and revolves with an opposite motion, and suppose that the sum of the time of the revolution of the plane and the revolution of the sphere is to the time of the revolution of the sphere as the square of the semidiameter of the vessel to the square of the semidiameter of the sphere; then the periodic times of the parts of the fluid with respect to the plane will be as the squares of their distances from the center of the sphere.

COROLLARY 10. Accordingly, if the vessel moves with any velocity either about the same axis as the sphere or about some different axis, the motion of the fluid will be given. For if the angular motion of the vessel is taken away from the whole system, all the motions with respect to one another will remain the same as before, by corol. 8. And these motions will be given by corol. 9.

COROLLARY 11. If the vessel and the fluid are at rest, and if the sphere revolves with a uniform motion, then the motion will be propagated little by little through the whole fluid to the vessel, and the vessel will be driven around unless forcibly constrained, and the fluid and vessel will not cease to be accelerated until their periodic times are equal to the periodic times of the sphere. But if the vessel is constrained by some force or revolves with any continual and uniform motion, the medium will little by little come to the state of the motion defined in corols. 8, 9, and 10, nor will it ever persevere in any other state. But then if, when those forces cease by which the vessel and the sphere were revolving with fixed motions, the whole system is left to the laws of mechanics, the vessel and the sphere will act upon each other by means of the intervening fluid and will not cease to propagate their motions to each other through the fluid until their periodic times are equal and the whole system revolves together like one solid body.

Scholium In the preceding propositions, I have been supposing the fluid to consist of matter which is uniform in density and fluidity. The fluid is such that a given sphere, set anywhere in it, would with a given motion in a given interval of time be able to propagate similar and equal motions, at distances always equal from itself. Indeed, matter endeavors by its circular motion to recede from the axis of a vortex and therefore presses all the further matter. From this pressure the friction of the parts becomes stronger and their separation from one another more difficult, and consequently the fluidity of the matter is decreased. Again, if there is any place where the parts of the fluid are thicker or larger, the fluidity will be less there, because the surfaces separating the parts from one another are fewer. In cases of this sort, I suppose the deficiency in fluidity to be supplied either by the slipperiness of the parts or by their pliancy or by some other condition. If this does not happen, the matter will cohere more and will be more sluggish where it is less fluid, and thus will receive motion more slowly and will propagate it further than according to the ratio assigned above. If the shape of the vessel is not spherical, the particles will move in paths which are not circular but correspond to the shape of the vessel, and the periodic times will be very nearly as the squares of the mean distances from the center. In the parts between the center and the circumference where the spaces are wider, the motions will be slower, and where the spaces are narrower the motions will be swifter, and yet the swifter particles will not seek the circumference. For they will describe less-curved arcs, and the endeavor to recede from the center will not be less decreased by the decrement of this curvature than it will be increased by the increment of the velocity. In going from the narrower spaces into the wider, they will recede a little further from the center, but they will be retarded by this receding, and afterward in approaching the narrower spaces from the wider ones they will be accelerated, and thus each of the particles will forever alternately be retarded and accelerated. All of this will be so in a rigid vessel. For the constitution of vortices in an infinite fluid can be found by corol. 6 of this proposition.

Moreover, in this proposition I have tried to investigate the properties of vortices in order to test whether the celestial phenomena could be explained in any way by vortices. For it is a phenomenon that the periodic times of the secondary planets that revolve about Jupiter are as the $\frac{3}{2}$ powers of the distances from the center of Jupiter; and the same rule applies to the planets that revolve about the sun. Moreover, these rules apply to both the primary and the secondary planets very exactly, as far as astronomical observations have shown up to now. And thus if those planets are carried along by vortices revolving about Jupiter and the sun, the vortices will also have to revolve according to the same law. But the periodic times of the parts of a vortex turned out to be in the squared ratio of the distances from the center of motion, and that ratio cannot be decreased and reduced to the $\frac{3}{2}$ power, unless either the matter of the vortex is the more fluid the further it is from the center, or the resistance arising from a deficiency in the slipperiness of the parts of the fluid (as a result of the increased velocity by which the parts of the fluid are separated from one another) is increased in a greater ratio than the ratio in which the velocity is increased. Yet neither of these seems reasonable. The thicker and less-fluid parts, if they are not heavy toward the center, will seek the circumference; and although—for the sake of the proofs—I proposed at the beginning of this section a hypothesis in which the resistance would be proportional to the velocity, it is nevertheless likely that the resistance is in a lesser ratio than that of the velocity. If this is conceded, then the periodic times of the parts of a vortex will be in a ratio greater than the squared ratio of the distances from its center. But if vortices (as is the opinion of some) move more quickly near the center, then more slowly up to a certain limit, then again more quickly near the circumference, certainly neither the $3/2$ power nor any other fixed and determinate ratio can hold. It is therefore up to philosophers to see how that phenomenon of the $3/2$ power can be explained by vortices.

Proposition 53 Theorem 41 *Bodies that are carried along in a vortex and return in the same orbit have the same density as the vortex and move according to the same law as the parts of the vortex with respect to velocity and direction.*

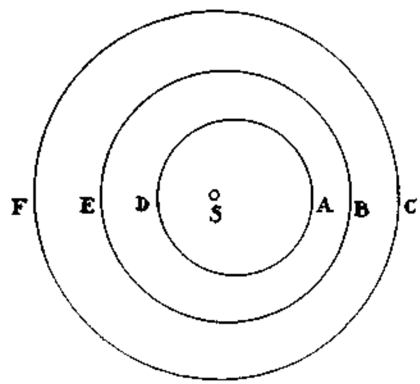
For if some tiny part of the vortex is composed of particles or physical points which preserve a given situation with respect to one another and is supposed to be frozen, then this part will move according to the same law as before, since it is not changed with respect to its density, or its inherent force or figure. And conversely, if a frozen and solid part of the vortex has the same density as the rest of the vortex and is resolved into a fluid, this part will move according to the same law as before, except insofar as its particles, which have now become fluid, move with respect to one another. Therefore, the motion of the particles with respect to one another may be ignored as having no relevance to the progressive motion of the whole, and the motion of the whole will be the same as before. But this motion will be the same as the motion of other parts of the vortex that are equally distant from the center, because the solid resolved into a fluid becomes a part of the vortex similar in every way to the other parts. Therefore, if a solid is of the same density as the matter of the vortex, it will move with the same motion as the parts of the vortex and will be relatively at rest in the immediately surrounding matter. But if the solid is denser, it will now endeavor to recede from the center of the vortex more than before; and thus, overcoming that force of the vortex by which it was formerly kept in its orbit as if set in equilibrium, it will recede from the center and in revolving will describe a spiral and will no longer return into the same orbit. And by the same argument, if the solid is rarer, it will approach the center. Therefore, the solid will not return into the same orbit unless it is of the same density as the fluid. And it has been shown that in this case the solid would revolve according to the same law as the parts of the fluid that are equally distant from the center of the vortex. Q.E.D.

Corollary 1. Therefore a solid that revolves in a vortex and always returns into the same orbit is relatively at rest in the fluid in which it is immersed.

Corollary 2. And if the vortex is of a uniform density, the same body can revolve at any distance from the center of the vortex.

Scholium

Hence it is clear that the planets are not carried along by corporeal vortices. For the planets, which—according to the Copernican hypothesis—move about the sun, revolve in ellipses having a focus in the sun, and by radii drawn to the sun describe areas proportional to the times. But the parts of a vortex cannot revolve with such a motion. Let AD, BE, and CF designate three orbits described about the sun S, of which let the outermost CF be a circle concentric with the sun, and let A and B be the aphelia of the two inner ones, and D and E their perihelia. Therefore, a body that revolves in the orbit CF, describing areas proportional to the times by a radius drawn to the sun, will move with a uniform motion. And a body that revolves in the orbit BE will, according to the laws of astronomy, move more slowly in the aphelion B and more swiftly in the perihelion E, although according to the laws of mechanics the matter of the vortex ought to move more swiftly in the narrower space between A and C than in the wider space between D and F, that is, more swiftly in the aphelion than in the perihelion. These two statements are contradictory. Thus in the beginning of the sign of Virgo, where the aphelion of Mars now is, the distance between the orbits of Mars and Venus is to the distance between these orbits in the beginning of the sign of Pisces as roughly 3 to 2, and therefore the matter of the vortex between these orbits in the beginning of Pisces must move more swiftly than in the beginning of Virgo in the ratio of 3 to 2. For the narrower the space through which a given quantity of matter passes in the given time of one revolution,



the greater the velocity with which it must pass. Therefore, if the earth, relatively at rest in this celestial matter, were carried by it and revolved along with it about the sun, its velocity in the beginning of Pisces would be to its velocity in the beginning of Virgo as 3 to 2. Hence the apparent daily motion of the sun in the beginning of Virgo would be greater than 70 minutes, and in the beginning of Pisces less than 48 minutes, although (as experience bears witness) the apparent motion of the sun is greater in the beginning of Pisces than in the beginning of Virgo, and thus the earth is swifter in the beginning of Virgo than in the beginning of Pisces. Therefore the hypothesis of vortices can in no way be reconciled with astronomical phenomena and serves less to clarify the celestial motions than to obscure them. But how those motions are performed in free spaces without vortices can be understood from book 1 and will now be shown more fully in book 3 on the system of the world.

BOOK 3

THE SYSTEM OF THE WORLD

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In the preceding books I have presented principles of philosophy[a] that are not, however, philosophical but strictly mathematical—that is, those on which the study of philosophy can be based. These principles are the laws and conditions of motions and of forces, which especially relate to philosophy. But in order to prevent these principles from seeming sterile, I have illustrated them with some philosophical scholiums [i.e., scholiums dealing with natural philosophy], treating topics that are general and that seem to be the most fundamental for philosophy, such as the density and resistance of bodies, spaces void of bodies, and the motion of light and sounds. It still remains for us to exhibit the system of the world from these same principles. On this subject I composed an earlier version of book 3 in popular form, so that it might be more widely read. But those who have not sufficiently grasped the principles set down here will certainly not perceive the force of the conclusions, nor will they lay aside the preconceptions to which they have become accustomed over many years; and therefore, to avoid lengthy disputations, I have translated the substance of the earlier version into propositions in a mathematical style, so that they may be read only by those who have first mastered the principles. But since in books 1 and 2 a great number of propositions occur which might be too time-consuming even for readers who are proficient in mathematics, I am unwilling to advise anyone to study every one of these propositions. It will be sufficient to read with care the Definitions, the Laws of Motion, and the first three sections of book 1, and then turn to this book 3 on the system of the world, consulting at will the other propositions of books 1 and 2 which are referred to here.

a. In this introduction to book 3, Newton uses "philosophy" and its adjective "philosophical" to refer to "natural philosophy." According to John Harris's *Lexicon Technicum* (London, 1704), natural philosophy is that "Science which contemplates the Powers of Nature, the Properties of Natural Bodies, and their mutual Action one upon another." The half title of the third edition of the *Principia* reads "Newtoni Principia Philosophiae" ("Newton's Principles of Philosophy"). The dedication page of the *Principia*, in all editions, refers to the Royal Society as founded "ad philosophiam promovendam" ("for the promotion of philosophy").

[a]RULES FOR THE STUDY OF NATURAL PHILOSOPHY

Rule 1 *No more causes of natural things should be admitted than are both true and sufficient to explain their phenomena.*

As the philosophers say: Nature does nothing in vain, and more causes are in vain when fewer suffice. For nature is simple and does not indulge in the luxury of superfluous causes.

aa. Ed. 1 has nine numbered "Hypotheses," most of which ed. 2 converts into two categories, now called "Rules for Natural Philosophy" and "Phenomena." Hyps. 1 and 2 become rules 1 and 2; hyp. 3 is discarded, to be replaced by rule 3; hyp. 4 becomes hyp. 1 and is transferred to a location between prop. 10 and prop. 11; hyps. 5–9 become phen. 1, 3–6, while phen. 2 is new in ed. 2. Ed. 3 further introduces rule 4. These changes may be tabulated as follows:

Ed. 1	*Ed. 2*	*Ed. 3*
hypothesis 1	rule 1	rule 1
hypothesis 2	rule 2	rule 2
hypothesis 3	—	—
—	rule 3	rule 3
—	—	rule 4
hypothesis 4	hypothesis 1*	hypothesis 1*
hypothesis 5	phenomenon 1	phenomenon 1
—	phenomenon 2	phenomenon 2
hypothesis 6	phenomenon 3	phenomenon 3
hypothesis 7	phenomenon 4	phenomenon 4
hypothesis 8	phenomenon 5	phenomenon 5
hypothesis 9	phenomenon 6	phenomenon 6

*between prop. 10 and prop. 11

Ed. 2 also has additions of explanatory phrases and sentences, alterations in wording, and, for the phenomena, revisions of numerical data and references to observers. Ed. 3 further expands or adds some explanatory sentences. For details see the Guide to the present translation, §8.2. Cf. also Alexandre Koyré, "Newton's 'Regulae Philosophandi,'" in his *Newtonian Studies* (Cambridge, Mass.: Harvard University Press, 1965), pp. 261–272; I. Bernard Cohen, "Hypotheses in Newton's Philosophy," *Physis: Rivista inter-*

Therefore, the causes assigned to natural effects of the same kind must be, so far as possible, the same. **Rule 2**

Examples are the cause of respiration in man and beast, or of the falling of stones in Europe and America, or of the light of a kitchen fire and the sun, or of the reflection of light on our earth and the planets.

Those qualities of bodies that cannot be intended and remitted [i.e., qualities that cannot be increased and diminished] and that belong to all bodies on which experiments can be made should be taken as qualities of all bodies universally. [b]**Rule 3**

For the qualities of bodies can be known only through experiments; and therefore qualities that square with experiments universally are to be regarded as universal qualities; and qualities that cannot be diminished cannot be taken away from bodies. Certainly idle fancies ought not to be fabricated recklessly against the evidence of experiments, nor should we depart from the analogy of nature, since nature is always simple and ever consonant with itself. The extension of bodies is known to us only through our senses, and yet there are bodies beyond the range of these senses; but because extension is found in all sensible bodies, it is ascribed to all bodies universally. We know by experience that some bodies are hard. Moreover, because the hardness of the whole arises from the hardness of its parts, we justly infer from this not only the hardness of the undivided particles of bodies that are accessible to our senses, but also of all other bodies. That all bodies are impenetrable we gather not by reason but by our senses. We find those bodies that we handle to be impenetrable, and hence we conclude that impenetrability is a property of all bodies universally. That all bodies are movable and persevere in motion or in rest by means of certain forces (which we call forces of inertia) we infer from finding these properties in the bodies that we have seen. The extension, hardness, impenetrability, mobility, and force of inertia of the whole arise from the extension, hardness, impenetrability, mobility, and force of inertia of each of the parts; and thus we conclude that every one of the least parts

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nazionale di storia della scienza 8 (1966): 163–184, reprinted in *Proceedings of the Boston Colloquium for the Philosophy of Science 1966/1968*, ed. Robert S. Cohen and Marx W. Wartofsky, Boston Studies in the Philosophy of Science, vol. 5 (Dordrecht: D. Reidel Publishing Co., 1969), pp. 304–326; I. Bernard Cohen, *Introduction to Newton's "Principia"* (Cambridge, Mass.: Harvard University Press; Cambridge: Cambridge University Press, 1971), pp. 23–26, 240–245.

bb. Ed. 1 has: "Hypothesis 3. Every body can be transformed into a body of any other kind and successively take on all the intermediate degrees of qualities." Cf. prop. 6, corol. 2, below.

of all bodies is extended, hard, impenetrable, movable, and endowed with a force of inertia. And this is the foundation of all natural philosophy. Further, from phenomena we know that the divided, contiguous parts of bodies can be separated from one another, and from mathematics it is certain that the undivided parts can be distinguished into smaller parts by our reason. But it is uncertain whether those parts which have been distinguished in this way and not yet divided can actually be divided and separated from one another by the forces of nature. But if it were established by even a single experiment that in the breaking of a hard and solid body, any undivided particle underwent division, we should conclude by the force of this third rule not only that divided parts are separable but also that undivided parts can be divided indefinitely.

Finally, if it is universally established by experiments and astronomical observations that all bodies on or near the earth gravitate [*lit.* are heavy] toward the earth, and do so in proportion to the quantity of matter in each body, and that the moon gravitates [is heavy] toward the earth in proportion to the quantity of its matter, and that our sea in turn gravitates [is heavy] toward the moon, and that all planets gravitate [are heavy] toward one another, and that there is a similar gravity [heaviness] of comets toward the sun, it will have to be concluded by this third rule that all bodies gravitate toward one another. Indeed, the argument from phenomena will be even stronger for universal gravity than for the impenetrability of bodies, for which, of course, we have not a single experiment, and not even an observation, in the case of the heavenly bodies. Yet I am by no means affirming that gravity is essential to bodies. By inherent force I mean only the force of inertia. This is immutable. Gravity is diminished as bodies recede from the earth.[b]

Rule 4 *In experimental philosophy, propositions gathered from phenomena by induction should be considered either exactly or very nearly true notwithstanding any contrary hypotheses, until yet other phenomena make such propositions either more exact or liable to exceptions.*

This rule should be followed so that arguments based on induction may not be nullified by hypotheses.

PHENOMENA

The circumjovial planets [or satellites of Jupiter], by radii drawn to the center of Jupiter, describe areas proportional to the times, and their periodic times—the fixed stars being at rest—are as the 3/2 powers of their distances from that center. **Phenomenon 1**

This is established from astronomical observations. The orbits of these planets do not differ sensibly from circles concentric with Jupiter, and their motions in these circles are found to be uniform. Astronomers agree that their periodic times are as the 3/2 power of the semidiameters of their orbits, and this is manifest from the following table.

Periodic times of the satellites of Jupiter			
$1^d 18^h 27^m 34^s$	$3^d 13^h 13^m 42^s$	$7^d 3^h 42^m 36^s$	$16^d 16^h 32^m 9^s$

Distances of the satellites from the center of Jupiter, in semidiameters of Jupiter				
	1	2	3	4
From the observations of				
Borelli	5⅔	8⅔	14	24⅔
Townly, by a micrometer	5.52	8.78	13.47	24.72
Cassini, by a telescope	5	8	13	23
Cassini, by eclipses of the satellites	5⅔	9	14 23/60	25 3/10
From the periodic times	5.667	9.017	14.384	25.299

Using the best micrometers, Mr. Pound has determined the elongations of the satellites of Jupiter and the diameter of Jupiter in the following way.

The greatest heliocentric elongation of the fourth satellite from the center of Jupiter was obtained with a micrometer in a telescope 15 feet long and came out roughly 8′16″ at the mean distance of Jupiter from the earth. That of the third satellite was obtained with a micrometer in a telescope 123 feet long and came out 4′42″ at the same distance of Jupiter from the earth. The greatest elongations of the other satellites, at the same distance of Jupiter from the earth, come out 2′56″47‴ and 1′51″6‴, on the basis of the periodic times.

The diameter of Jupiter was obtained a number of times with a micrometer in a telescope 123 feet long and, when reduced to the mean distance of Jupiter from the sun or the earth, always came out smaller than 40″, never smaller than 38″, and quite often 39″. In shorter telescopes this diameter is 40″ or 41″. For the light of Jupiter is somewhat dilated by its nonuniform refrangibility, and this dilation has a smaller ratio to the diameter of Jupiter in longer and more perfect telescopes than in shorter and less perfect ones. The times in which two satellites, the first and the third, crossed the disk of Jupiter, from the beginning of their entrance [i.e., from the moment of their beginning to cross the disk] to the beginning of their exit and from the completion of their entrance to the completion of their exit, were observed with the aid of the same longer telescope. And from the transit of the first satellite, the diameter of Jupiter at its mean distance from the earth came out 37⅛″ and, from the transit of the third satellite, 37⅜″. The time in which the shadow of the first satellite passed across the body of Jupiter was also observed, and from this observation the diameter of Jupiter at its mean distance from the earth came out roughly 37″. Let us assume that this diameter is very nearly 37¼″; then the greatest elongations of the first, second, third, and fourth satellites will be equal respectively to 5.965, 9.494, 15.141, and 26.63 semidiameters of Jupiter.

Phenomenon 2 *The circumsaturnian planets [or satellites of Saturn], by radii drawn to the center of Saturn, describe areas proportional to the times, and their periodic times—the fixed stars being at rest—are as the $^3/_2$ powers of their distances from that center.*

Cassini, in fact, from his own observations has established their distances from the center of Saturn and their periodic times as follows.

Periodic times of the satellites of Saturn				
$1^d21^h18^m27^s$	$2^d17^h41^m22^s$	$4^d12^h25^m12^s$	$15^d22^h41^m14^s$	$79^d7^h48^m00^s$

Distances of the satellites from the center of Saturn, in semidiameters of the ring					
From the observations	$1\frac{19}{20}$	$2\frac{1}{2}$	$3\frac{1}{2}$	8	24
From the periodic times	1.93	2.47	3.45	8	23.35

Observations yield a value of the greatest elongation of the fourth satellite from the center of Saturn that is very near eight semidiameters. But the greatest elongation of this satellite from the center of Saturn, as determined by an excellent micrometer in Huygens's 123-foot telescope, came out $8\frac{7}{10}$ semidiameters. And from this observation and the periodic times, the distances of the satellites from the center of Saturn are, in semidiameters of the ring, 2.1, 2.69, 3.75, 8.7, and 25.35. The diameter of Saturn in the same telescope was to the diameter of the ring as 3 to 7, and the diameter of the ring on the 28th and 29th day of May 1719 came out 43″. And from this the diameter of the ring at the mean distance of Saturn from the earth is 42″, and the diameter of Saturn is 18″. These are the results obtained with the longest and best telescopes, because the apparent magnitudes of heavenly bodies, as seen in longer telescopes, have a greater proportion to the dilation of light at the edges of these bodies than when seen in shorter telescopes. If all erratic light [i.e., dilated light] is disregarded, the diameter of Saturn will not be greater than 16″.

Phenomenon 3

The orbits of the five primary planets—Mercury, Venus, Mars, Jupiter, and Saturn—encircle the sun.

That Mercury and Venus revolve about the sun is proved by their exhibiting phases like the moon's. When these planets are shining with a full face, they are situated beyond the sun; when half full, to one side of the sun; when horned, on this side of the sun; and they sometimes pass across the sun's disk like spots. Because Mars also shows a full face when near conjunction with the sun, and appears gibbous in the quadratures, it is certain that Mars goes around the sun. The same thing is proved also with respect to Jupiter and Saturn from their phases being always full; and in these two planets, it is manifest from the shadows that their satellites project upon them that they shine with light borrowed from the sun.

Phenomenon 4 *The periodic times of the five primary planets and of either the sun about the earth or the earth about the sun—the fixed stars being at rest—are as the $3/2$ powers of their mean distances from the sun.*

This proportion, which was found by Kepler, is accepted by everyone. In fact, the periodic times are the same, and the dimensions of the orbits are the same, whether the sun revolves about the earth, or the earth about the sun. There is universal agreement among astronomers concerning the measure of the periodic times. But of all astronomers, Kepler and Boulliau have determined the magnitudes of the orbits from observations with the most diligence; and the mean distances that correspond to the periodic times as computed from the above proportion do not differ sensibly from the distances that these two astronomers found [from observations], and for the most part lie between their respective values, as may be seen in the following table.

Periodic times of the planets and of earth about the sun with respect to the fixed stars, in days and decimal parts of a day

♄	♃	♂	♁	♀	☿
10759.275	4332.514	686.9785	365.2565	224.6176	87.9692

Mean distances of the planets and of the earth from the sun

	♄	♃	♂	♁	♀	☿
According to Kepler	951000	519650	152350	100000	72400	38806
According to Boulliau	954198	522520	152350	100000	72398	38585
According to the periodic times	954006	520096	152369	100000	72333	38710

There is no ground for dispute about the distances of Mercury and Venus from the sun, since these distances are determined by the elongations of the planets from the sun. Furthermore, with respect to the distances of the superior planets from the sun, any ground for dispute is eliminated by the eclipses of the satellites of Jupiter. For by these eclipses the position of the shadow that Jupiter projects is determined, and this gives the heliocentric longitude of Jupiter. And from a comparison of the heliocentric and geocentric longitudes, the distance of Jupiter is determined.

Phenomenon 5

The primary planets, by radii drawn to the earth, describe areas in no way proportional to the times but, by radii drawn to the sun, traverse areas proportional to the times.

For with respect to the earth they sometimes have a progressive [direct or forward] motion, they sometimes are stationary, and sometimes they even have a retrograde motion; but with respect to the sun they move always forward, and they do so with a motion that is almost uniform—but, nevertheless, a little more swiftly in their perihelia and more slowly in their aphelia, in such a way that the description of areas is uniform. This is a proposition very well known to astronomers and is especially provable in the case of Jupiter by the eclipses of its satellites; by means of these eclipses we have said that the heliocentric longitudes of this planet and its distances from the sun are determined.

Phenomenon 6

The moon, by a radius drawn to the center of the earth, describes areas proportional to the times.

This is evident from a comparison of the apparent motion of the moon with its apparent diameter. Actually, the motion of the moon is somewhat perturbed by the force of the sun, but in these phenomena I pay no attention to minute errors that are negligible.[a]

PROPOSITIONS

Proposition 1
Theorem 1

The forces by which the circumjovial planets [or satellites of Jupiter] are continually drawn away from rectilinear motions and are maintained in their respective orbits are directed to the center of Jupiter and are inversely as the squares of the distances of their places from that center.

The first part of the proposition is evident from phen. 1 and from prop. 2 or prop. 3 of book 1, and the second part from phen. 1 and from corol. 6 to prop. 4 of book 1.

The same is to be understood for the planets that are Saturn's companions [or satellites] by phen. 2.

Proposition 2
Theorem 2

The forces by which the primary planets are continually drawn away from rectilinear motions and are maintained in their respective orbits are directed to the sun and are inversely as the squares of their distances from its center.

The first part of the proposition is evident from phen. 5 and from prop. 2 of book 1, and the latter part from phen. 4 and from prop. 4 of the same book. But this second part of the proposition is proved with the greatest exactness from the fact that the aphelia are at rest. For the slightest departure from the ratio of the square would (by book 1, prop. 45, corol. 1) necessarily result in a noticeable motion of the apsides in a single revolution and an immense such motion in many revolutions.

Proposition 3
Theorem 3

The force by which the moon is maintained in its orbit is directed toward the earth and is inversely as the square of the distance of its places from the center of the earth.

The first part of this statement is evident from phen. 6 and from prop. 2 or prop. 3 of book 1, and the second part from the very slow motion of the moon's apogee. For that motion, which in each revolution is only three degrees and three minutes forward [or in consequentia, i.e., in an easterly direction] can be ignored. For it is evident (by book 1, prop. 45, corol. 1) that if the distance of the moon from the center of the earth is to the semidiameter of the earth as D to 1, then the force from which such a motion may arise is inversely as $D^{2\frac{4}{243}}$, that is, inversely as that power of D of which the index is $2\frac{4}{243}$; that is, the proportion of the force to the distance is inversely as a little greater than the second power of the distance, but is $59\frac{3}{4}$ times closer to the square than to the cube. Now this motion of the apogee arises from the action of the sun (as will be pointed out below) and accordingly is to be ignored here. The action of the sun, insofar as it draws the moon away from the earth, is very nearly as the distance of the moon from the earth, and so (from what is said in book 1, prop. 45, corol. 2) is to the centripetal force of the moon as roughly 2 to 357.45, or 1 to $178\frac{29}{40}$. And if so small a force of the sun is ignored, the remaining force by which the moon is maintained in its orbit will be inversely as D^2. And this will be even more fully established by comparing this force with the force of gravity as is done in prop. 4 below.

COROLLARY. If the mean centripetal force by which the moon is maintained in its orbit is increased first in the ratio of $177\frac{29}{40}$ to $178\frac{29}{40}$, then also in the squared ratio of the semidiameter of the earth to the mean distance of the center of the moon from the center of the earth, the result will be the lunar centripetal force at the surface of the earth, supposing that that force, in descending to the surface of the earth, is continually increased in the ratio of the inverse square of the height.

Proposition 4
Theorem 4

The moon gravitates toward the earth and by the force of gravity is always drawn back from rectilinear motion and kept in its orbit.

The mean distance of the moon from the earth in the syzygies is, according to Ptolemy and most astronomers, 59 terrestrial semidiameters, 60 according to Vendelin and Huygens, $60\frac{1}{3}$ according to Copernicus, $60\frac{2}{5}$ according to Street, and $56\frac{1}{2}$ according to Tycho. But Tycho and all those who follow his tables of refractions, by making the refractions of the sun and moon (entirely contrary to the nature of light) be greater than those of the fixed stars—in fact greater by about four or five minutes—have increased the parallax of the moon by that many minutes, that is, by about a twelfth or fifteenth of the whole parallax. Let that error be corrected, and the distance will come to be roughly $60\frac{1}{2}$ terrestrial semidiameters, close to the value that has been assigned by others. Let us assume a mean distance of 60 semidiameters in the syzygies; and also let us assume that a revolution of the moon with respect to the fixed stars is completed in 27 days, 7 hours, 43 minutes, as has been established by astronomers; and that the circumference of the earth is 123,249,600 Paris feet, according to the measurements made by the French. If now the moon is imagined to be deprived of all its motion and to be let fall so that it will descend to the earth with all that force urging it by which (by prop. 3, corol.) it is [normally] kept in its orbit, then in the space of one minute, it will by falling describe $15\frac{1}{12}$ Paris feet. This is determined by a calculation carried out either by using prop. 36 of book 1 or (which comes to the same thing) by using corol. 9 to prop. 4 of book 1. For the versed sine of the arc which the moon would describe in one minute of time by its mean motion at a distance of 60 semidiameters of the earth is roughly $15\frac{1}{12}$ Paris feet, or more exactly 15 feet, 1 inch, and $1\frac{4}{9}$ lines [or twelfths of an inch]. Accordingly, since in approaching the earth that force is increased as the inverse square of the distance, and so at the surface of the earth is 60×60 times greater than at the moon, it follows that a body falling with that force, in our regions, ought in the space of one minute to describe $60 \times 60 \times 15\frac{1}{12}$ Paris feet, and in the space of one second $15\frac{1}{12}$ feet, or more exactly 15 feet, 1 inch, and $1\frac{4}{9}$ lines. And heavy bodies do actually descend to the earth with this very force. For a pendulum beating seconds in the latitude of Paris is 3 Paris feet and $8\frac{1}{2}$ lines in length, as Huygens observed. And the height that a heavy body describes by falling in the time of one second is to half the length of this pendulum as the square of the ratio of the circumference of a circle to its diameter (as Huygens also showed), and so is 15 Paris feet, 1 inch, $1\frac{7}{9}$ lines. And therefore that force by which the moon is kept in its orbit, in descending from the moon's orbit to the surface of the earth, comes out equal to the force of gravity here on earth, and so (by rules 1 and 2) is that very force which we generally call gravity. For if gravity were different from this force, then bodies making for the earth by both forces acting together would descend twice as fast, and in the space of one second would by falling describe $30\frac{1}{6}$ Paris feet, entirely contrary to experience.

This calculation is founded on the hypothesis that the earth is at rest. For if the earth and the moon move around the sun and in the meanwhile also revolve around their common center of gravity, then, the law of gravity remaining the same, the distance of the centers of the moon and earth from each other will be roughly 60½ terrestrial semidiameters, as will be evident to anyone who computes it. And the computation can be undertaken by book 1, prop. 60.

Scholium

The proof of the proposition can be treated more fully as follows. If several moons were to revolve around the earth, as happens in the system of Saturn or of Jupiter, their periodic times (by the argument of induction) would observe the law which Kepler discovered for the planets, and therefore their centripetal forces would be inversely as the squares of the distances from the center of the earth, by prop. 1 of this book 3. And if the lowest of them were small and nearly touched the tops of the highest mountains, its centripetal force, by which it would be kept in its orbit, would (by the preceding computation) be very nearly equal to the gravities of bodies on the tops of those mountains. And this centripetal force would cause this little moon, if it were deprived of all the motion with which it proceeds in its orbit, to descend to the earth—as a result of the absence of the centrifugal force with which it had remained in its orbit—and to do so with the same velocity with which heavy bodies fall on the tops of those mountains, because the forces with which they descend are equal. And if the force by which the lowest little moon descends were different from gravity and that little moon also were heavy toward the earth in the manner of bodies on the tops of mountains, this little moon would descend twice as fast by both forces acting together. Therefore, since both forces—namely, those of heavy bodies and those of the moons—are directed toward the center of the earth and are similar to each other and equal, they will (by rules 1 and 2) have the same cause. And therefore that force by which the moon is kept in its orbit is the very one that we generally call gravity. For if this were not so, the little moon at the top of a mountain must either be lacking in gravity or else fall twice as fast as heavy bodies generally do.

Proposition 5
Theorem 5

The circumjovial planets [or satellites of Jupiter] gravitate toward Jupiter, the circumsaturnian planets [or satellites of Saturn] gravitate toward Saturn, and the circumsolar [or primary] planets gravitate toward the sun, and by the force of their gravity they are always drawn back from rectilinear motions and kept in curvilinear orbits.

For the revolutions of the circumjovial planets about Jupiter, of the circumsaturnian planets about Saturn, and of Mercury and Venus and the other circumsolar planets about the sun are phenomena of the same kind as the revolution of the moon about the earth, and therefore (by rule 2) depend on causes of the same kind, especially since it has been proved that the forces on which those revolutions depend are directed toward the centers of Jupiter, Saturn, and the sun, and decrease according to the same ratio and law (in receding from Jupiter, Saturn, and the sun) as the force of gravity (in receding from the earth).

COROLLARY 1. Therefore, there is gravity toward all planets universally. For no one doubts that Venus, Mercury, and the rest [of the planets, primary and secondary,] are bodies of the same kind as Jupiter and Saturn. And since, by the third law of motion, every attraction is mutual, Jupiter will gravitate toward all its satellites, Saturn toward its satellites, and the earth will gravitate toward the moon, and the sun toward all the primary planets.

COROLLARY 2. The gravity that is directed toward every planet is inversely as the square of the distance of places from the center of the planet.

COROLLARY 3. All the planets are heavy toward one another by corols. 1 and 2. And hence Jupiter and Saturn near conjunction, by attracting each other, sensibly perturb each other's motions, the sun perturbs the lunar motions, and the sun and moon perturb our sea, as will be explained in what follows.

Scholium Hitherto we have called "centripetal" that force by which celestial bodies are kept in their orbits. It is now established that this force is gravity, and therefore we shall call it gravity from now on. For the cause of the centripetal force by which the moon is kept in its orbit ought to be extended to all the planets, by rules 1, 2, and 4.

Proposition 6
Theorem 6
All bodies gravitate toward each of the planets, and at any given distance from the center of any one planet the weight of any body whatever toward that planet is proportional to the quantity of matter which the body contains.

Others have long since observed that the falling of all heavy bodies toward the earth (at least on making an adjustment for the inequality of the retardation that arises from the very slight resistance of the air) takes place in equal times, and it is possible to discern that equality of the times, to a very high degree of accuracy, by using pendulums. I have tested this with gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I got two wooden boxes, round and equal. I filled one of them with wood, and I suspended the same weight of gold (as exactly as I could) in the center of oscillation of the other. The boxes, hanging by equal eleven-foot cords, made pendulums exactly like each other with respect to their weight, shape, and air resistance. Then, when placed close to each other [and set into vibration], they kept swinging back and forth together with equal oscillations for a very long time. Accordingly, the amount of matter in the gold (by book 2, prop. 24, corols. 1 and 6) was to the amount of matter in the wood as the action of the motive force upon all the gold to the action of the motive force upon all the [added] wood—that is, as the weight of one to the weight of the other. And it was so for the rest of the materials. In these experiments, in bodies of the same weight, a difference of matter that would be even less than a thousandth part of the whole could have been clearly noticed. Now, there is no doubt that the nature of gravity toward the planets is the same as toward the earth. For imagine our terrestrial bodies to be raised as far as the orbit of the moon and, together with the moon, deprived of all motion, to be released so as to fall to the earth simultaneously; and by what has already been shown, it is certain that in equal times these falling terrestrial bodies will describe the same spaces as the moon, and therefore that they are to the quantity of matter in the moon as their own weights are to its weight. Further, since the satellites of Jupiter revolve in times that are as the $\frac{3}{2}$ power of their distances from the center of Jupiter, their accelerative gravities toward Jupiter will be inversely as the squares of the distances from the center of Jupiter, and, therefore, at equal distances from Jupiter their accelerative gravities would come out equal. Accordingly, in equal times in falling from equal heights [toward Jupiter] they would describe equal spaces, just as happens with heavy bodies on this earth of ours. And by the same argument the circumsolar [or primary] planets, let fall from equal distances from the sun, would describe equal spaces in equal times in their descent to the sun. Moreover, the forces by which unequal bodies are equally accelerated are as the bodies; that is, the weights [of the primary planets toward the sun] are as the quantities of matter in the planets. Further, that the weights of Jupiter and its satellites toward the sun are proportional to the quantities of their matter is evident from the extremely regular motion of the satellites, according to book 1, prop. 65, corol. 3. For if some of these were more strongly attracted toward the sun in proportion to the quantity of their matter than the rest, the motions of the satellites (by book 1, prop. 65, corol. 2) would be perturbed by that inequality of attraction. If, at equal distances from the sun, some satellite were heavier [or gravitated more] toward the sun in proportion to the quantity of its matter than Jupiter in proportion to the quantity of its own matter, in any given ratio, say *d* to *e*, then the distance between the center of the sun and the center of the orbit of the satellite would always be greater than the distance between the center of the sun and the center of Jupiter and these distances would be to each other very nearly as the square root of *d* to the square root of *e*, as I found by making a certain calculation. And if the satellite were less heavy [or gravitated less] toward the sun in that ratio of *d* to *e*, the distance of the center of the orbit of the satellite from the sun would be less than the distance of the center of Jupiter from the sun in that same ratio of the square root of *d* to the square root of *e*. And so if, at equal distances from the sun, the accelerative gravity of any satellite toward the sun were greater or smaller than the accelerative gravity of Jupiter toward the sun, by only a thousandth of the whole gravity, the distance of the center of the orbit of the satellite from the sun would be greater or smaller than the distance of Jupiter from the sun by $\frac{1}{2{,}000}$ of the total distance, that is, by a fifth of the distance of the outermost satellite from the center of Jupiter; and this eccentricity of the orbit would be very sensible indeed. But the orbits of the satellites are concentric with Jupiter, and therefore the accelerative gravities of Jupiter and of the satellites toward the sun are equal to one another. And by the same argument the weights [or gravities] of Saturn and its companions toward the sun, at equal distances from the sun, are as the quantities of matter in them; and the weights of the moon and earth toward the sun are either nil or exactly proportional to their masses. But they do have some weight, according to prop. 5, corols. 1 and 3.

But further, the weights [or gravities] of the individual parts of each planet toward any other planet are to one another as the matter in the individual parts. For if some parts gravitated more, and others less, than in proportion to their quantity of matter, the whole planet, according to the kind of parts in which it most abounded, would gravitate more or gravitate less than in proportion to the quantity of matter of the whole. But it does not matter whether those parts are external or internal. For if, for example, it is imagined that bodies on our earth are raised to the orbit of the moon and compared with the body of the moon, then, if their weights were to the weights of the external parts of the moon as the quantities of matter in them, but were to the weights of the internal parts in a greater or lesser ratio, they would be to the weight of the whole moon in a greater or lesser ratio, contrary to what has been shown above.

COROLLARY 1. Hence, the weights of bodies do not depend on their forms and textures. For if the weights could be altered with the forms, they would be, in equal matter, greater or less according to the variety of forms, entirely contrary to experience.

COROLLARY 2. [a]All bodies universally that are on or near the earth are heavy [or gravitate] toward the earth, and the weights of all bodies that are equally distant from the center of the earth are as the quantities of matter in them. This is a quality of all bodies on which experiments can be performed and therefore by rule 3 is to be affirmed of all bodies universally. If the aether or any other body whatever either were entirely devoid of gravity or gravitated less in proportion to the quantity of its matter, then, since (according to the opinion of Aristotle, Descartes, and others) it does not differ from other bodies except in the form of its matter, it could by a change of its form be transmuted by degrees into a body of the same condition as those that gravitate the most in proportion to the quantity of their matter; and, on the other hand, the heaviest bodies, through taking on by degrees the form of the other body, could by degrees lose their gravity. And accordingly the weights would depend on the forms of bodies and could be altered with the forms, contrary to what has been proved in corol. 1.[a]

aa. Ed. 1 has: "Therefore all bodies universally that are on or near the earth are heavy [or gravitate] toward the earth, and the weights of all bodies that are equally distant from the center of the earth are as the quantities of matter in them. For if the aether or any other body whatever either were entirely devoid of gravity or gravitated less in proportion to the quantity of its matter, then, since it does not differ from other bodies except in the form of its matter, it could by a change of its form be changed by degrees into a body of the same condition as those that gravitate the most in proportion to the quantity of their matter (by hyp. 3), and, on the other hand, the heaviest bodies, through taking on by degrees the form of the other body, could by degrees lose their gravity. And accordingly the weights would depend on the forms of bodies and could be altered with the forms, contrary to what has been proved in corol. 1."

Some of the handwritten notes to Newton's copies of ed. 1 show various other alterations that never appeared in printed editions at this point. In one, for example, everything after the first sentence is replaced by "This is evident by hyp. 3, provided that this hypothesis holds here," while another has the substitution "This follows from the preceding proposition by hyp. 3, provided that this hypothesis holds here." See further the notes to the Rules and Phenomena above.

[b]COROLLARY 3. All spaces are not equally full. For if all spaces were equally full, the specific gravity of the fluid with which the region of the air would be filled, because of the extreme density of its matter, would not be less than the specific gravity of quicksilver or of gold or of any other body with the greatest density, and therefore neither gold nor any other body could descend in air. For bodies do not ever descend in fluids unless they have a greater specific gravity. But if the quantity of matter in a given space could be diminished by any rarefaction, why should it not be capable of being diminished indefinitely?

COROLLARY 4. If all the solid particles of all bodies have the same density and cannot be rarefied without pores, there must be a vacuum. I say particles have the same density when their respective forces of inertia [or masses] are as their sizes.[b]

COROLLARY 5. The force of gravity is of a different kind from the magnetic force. For magnetic attraction is not proportional to the [quantity of] matter attracted. Some bodies are attracted [by a magnet] more [than in proportion to their quantity of matter], and others less, while most bodies are not attracted [by a magnet at all]. And the magnetic force in one and the same body can be intended and remitted [i.e., increased and decreased] and is sometimes far greater in proportion to the quantity of matter than the force of gravity; and this force, in receding from the magnet, decreases not as the square but almost as the cube of the distance, as far as I have been able to tell from certain rough observations.

Proposition 7
Theorem 7

Gravity exists in all bodies universally and is proportional to the quantity of matter in each.

We have already proved that all planets are heavy [or gravitate] toward one another and also that the gravity toward any one planet, taken by itself, is inversely as the square of the distance of places from the center of the planet. And it follows (by book 1, prop. 69 and its corollaries) that the gravity toward all the planets is proportional to the matter in them.

Further, since all the parts of any planet A are heavy [or gravitate] toward any planet B, and since the gravity of each part is to the gravity of the whole

bb. In place of corols. 3 and 4, ed. 1 has a single corol. 3: "And thus a vacuum is necessary. For if all spaces were full, the specific gravity of the fluid with which the region of the air would be filled, because of the extreme density of its matter, would not be less than the specific gravity of quicksilver or of gold or of any other body with the greatest density, and therefore neither gold nor any other body could descend in air. For bodies do not ever descend in fluids unless they have a greater specific gravity."

as the matter of that part to the matter of the whole, and since to every action (by the third law of motion) there is an equal reaction, it follows that planet B will gravitate in turn toward all the parts of planet A, and its gravity toward any one part will be to its gravity toward the whole of the planet as the matter of that part to the matter of the whole. Q.E.D.

COROLLARY 1. Therefore the gravity toward the whole planet arises from and is compounded of the gravity toward the individual parts. We have examples of this in magnetic and electric attractions. For every attraction toward a whole arises from the attractions toward the individual parts. This will be understood in the case of gravity by thinking of several smaller planets coming together into one globe and composing a larger planet. For the force of the whole will have to arise from the forces of the component parts. If anyone objects that by this law all bodies on our earth would have to gravitate toward one another, even though gravity of this kind is by no means detected by our senses, my answer is that gravity toward these bodies is far smaller than what our senses could detect, since such gravity is to the gravity toward the whole earth as [the quantity of matter in each of] these bodies to the [quantity of matter in the] whole earth.

COROLLARY 2. The gravitation toward each of the individual equal particles of a body is inversely as the square of the distance of places from those particles. This is evident by book 1, prop. 74, corol. 3.

Proposition 8
Theorem 8

If two globes gravitate toward each other, and their matter is homogeneous on all sides in regions that are equally distant from their centers, then the weight of either globe toward the other will be inversely as the square of the distance between the centers.

After I had found that the gravity toward a whole planet arises from and is compounded of the gravities toward the parts and that toward each of the individual parts it is inversely proportional to the squares of the distances from the parts, I was still not certain whether that proportion of the inverse square obtained exactly in a total force compounded of a number of forces, or only nearly so. For it could happen that a proportion which holds exactly enough at very great distances might be markedly in error near the surface of the planet, because there the distances of the particles may be unequal and their situations dissimilar. But at length, by means of book 1, props. 75 and 76 and their corollaries, I discerned the truth of the proposition dealt with here.

[a]COROLLARY 1. Hence the weights of bodies toward different planets can be found and compared one with another. For the weights of equal bodies revolving in circles around planets are (by book 1, prop. 4, corol. 2) as the diameters of the circles directly and the squares of the periodic times inversely, and weights at the surfaces of the planets or at any other distances from the center are greater or smaller (by the same proposition) as the inverse squares of the distances. I compared the periodic times of Venus around the sun (224

aa. The text of the first part of corol. 1 as it appears in the later editions—that is, the first two sentences and part of the third sentence up to "of the moon around the earth (27 days, 7 hours, 43 minutes)"—is almost the same as in the version in ed. 1, except that the later editions have a more complete reference to prop. 4 (the addition of "corol. 2") and have more exact values for the periods of Venus (224 days and 16¾ hours) and of the outermost satellite of Jupiter (16 days and 16⁸⁄₁₅ hours). In the remainder of the text, however, the later versions are notably different from the earlier one. (For a gloss on this corollary, see the Guide, §8.10.) In ed. 1, corol. 1 reads as follows:

"Corollary 1. Hence the weights of bodies toward different planets can be found and compared one with another. For the weights of equal bodies [i.e., bodies with equal masses] revolving in circles around planets are (by book 1, prop. 4) as the diameters of the circles directly and the squares of the periodic times inversely, and weights at the surfaces of the planets or at any other distances from the center are greater or smaller (by the same proposition) inversely as the squared ratio of the distances. I compared the periodic times of Venus around the sun (224⅔ days), of the outermost circumjovial satellite around Jupiter (16¾ days), of Huygens's satellite around Saturn (15 days and 22⅔ hours), and of the moon around the earth (27 days, 7 hours, 43 minutes) respectively with the mean distance of Venus from the sun, with the greatest heliocentric elongation of the outermost circumjovial satellite, which (at the mean distance of Jupiter from the sun according to the observations of Flamsteed) is 8′13″, with the greatest heliocentric elongation of the satellite of Saturn (3′20″), and with the distance of the moon from the earth, on the hypothesis that the horizontal solar parallax or the semidiameter of the earth as seen from the sun is about 20″.

In this way I found by calculation that the weights of bodies which are equal and equally distant from the sun, Jupiter, Saturn, and the earth as directed toward the sun, Jupiter, Saturn, and the earth, were to one another as 1, $\frac{1}{1{,}100}$, $\frac{1}{2{,}360}$, and $\frac{1}{28{,}700}$. But the mean apparent semidiameter of the sun is about 16′6″. From the diameter of the shadow of Jupiter as found by eclipses of the satellites, Flamsteed determined that the mean apparent diameter of Jupiter as seen from the sun is to the elongation of the outermost satellite as 1 to 24.9, and since that elongation is 8′13″, the semidiameter of Jupiter as seen from the sun will be 19¾″. The diameter of Saturn is to the diameter of its ring as 4 to 9, and the diameter of the ring as seen from the sun (by Flamsteed's measurement) is 50″, and thus the semidiameter of Saturn as seen from the sun is 11″. I would prefer to say 10″ or 9″, because the globe of Saturn is somewhat dilated by a nonuniform refrangibility of light.

Thus, when the calculation is made, the true semidiameters of the sun, Jupiter, Saturn, and the earth to one another come out as 10,000, 1,063, 889, and 208. Whence, because the weights of bodies which are equal and equally distant from the centers of the sun, Jupiter, Saturn, and the earth are, respectively, toward the sun, Jupiter, Saturn, and the earth as 1, $\frac{1}{1{,}000}$, $\frac{1}{2{,}360}$, $\frac{1}{28{,}700}$, and because, when the distances are increased or decreased, the weights are decreased or increased in the squared ratio, [it follows that] the weights of the same equal bodies toward the sun, Jupiter, Saturn, and the earth at distances of 10,000, 1,063, 889, and 208 from their centers, and hence their weights at the surfaces, will be as 10,000, 804½, 536, and 805½ respectively. We shall show below that the weights of bodies on the surface of the moon are almost two times less than the weights of bodies on the surface of the earth."

days and 16¾ hours), of the outermost circumjovial satellite around Jupiter (16 days and 16⁸⁄₁₅ hours), of Huygens's satellite around Saturn (15 days and 22⅔ hours), and of the moon around the earth (27 days, 7 hours, 43 minutes) respectively with the mean distance of Venus from the sun, and with the greatest heliocentric elongations of the outermost circumjovial satellite from the center of Jupiter (8′16″), of Huygens's satellite from the center of Saturn (3′4″), and of the moon from the center of the earth (10′33″). In this way I found by computation that the weights of bodies which are equal and equally distant from the center of the sun, of Jupiter, of Saturn, and of the earth are respectively toward the sun, Jupiter, Saturn, and the earth as 1, $\frac{1}{1,067}$, $\frac{1}{3,021}$, and $\frac{1}{169,282}$. And when the distances are increased or decreased, the weights are decreased or increased as the squares of the distances. The weights of equal bodies toward the sun, Jupiter, Saturn, and the earth at distances of 10,000, 997, 791, and 109 respectively from their centers (and hence their weights on the surfaces) will be as 10,000, 943, 529, and 435. What the weights of bodies are on the surface of the moon will be shown below.[a]

[b]COROLLARY 2. The quantity of matter in the individual planets can also be found. For the quantities of matter in the planets are as their forces at equal distances from their centers; that is, in the sun, Jupiter, Saturn, and the earth, they are as 1, $\frac{1}{1,067}$, $\frac{1}{3,021}$, and $\frac{1}{169,282}$ respectively. If the parallax of the sun is taken as greater or less than 10″30‴, the quantity of matter in the earth will have to be increased or decreased in the cubed ratio.[b]

bb. In ed. 1, there was an additional corollary numbered 2, so that the corollaries numbered 2, 3, and 4 in the later editions were originally numbered 3, 4, and 5. (For a gloss on this corollary see the Guide, §8.10.) The corol. 2 of the first edition reads as follows:

"Corollary 2. Therefore the weights of equal bodies [i.e., bodies with equal masses], on the surfaces of the earth and of the planets, are almost proportional to the square roots of their apparent diameters as seen from the sun. With respect to the diameter of the earth as seen from the sun there is as yet no agreement. I have taken it to be 40″, because the observations of Kepler, Riccioli, and Vendelin do not permit it to be much greater; the observations of Horrocks and Flamsteed seem to make it a little smaller. And I have preferred to err on the side of excess. But if perhaps that diameter and the gravity on the surface of the earth are a mean among the diameters of the planets and the gravities on their surfaces, then, since the diameters of Saturn, Jupiter, Mars, Venus, and Mercury are about 18″, 39½″, 8″, 28″, 20″, the diameter of the earth will be about 24″ and therefore the parallax of the sun about 12″, as Horrocks and Flamsteed pretty nearly concluded. But a slightly larger diameter agrees better with the rule of this corollary." That is, the larger diameter of the earth as seen from the sun, and hence the larger solar parallax, agrees better with the rule about the weights of equal bodies on the surface of the earth and planets being "almost proportional to the square roots of their apparent diameters as seen from the sun."

COROLLARY 3. The densities of the planets can also be found. For the weights of equal and homogeneous bodies toward homogeneous spheres are, on the surfaces of the spheres, as the diameters of the spheres, by book 1, prop. 72; and therefore the densities of heterogeneous spheres are as those weights divided by the diameters of the spheres. Now, the true diameters of the sun, Jupiter, Saturn, and the earth were found to be to one another as 10,000, 997, 791, and 109, and the weights toward them are as 10,000, 943, 529, and 435 respectively, and therefore the densities are as 100, 94½, 67, and 400. The density of the earth that results from this computation does not depend on the parallax of the sun but is determined by the parallax of the moon and therefore is determined correctly here. Therefore the sun is a little denser than Jupiter, and Jupiter denser than Saturn, and the earth four times denser than the sun. For the sun is rarefied by its great heat. And the moon is denser than the earth, as will be evident from what follows [i.e., prop. 37, corol. 3].

[c]COROLLARY 4. Therefore, other things being equal, the planets that are smaller are denser. For thus the force of gravity on their surfaces approaches closer to equality. But, other things being equal, the planets that are nearer to the sun are also denser; for example, Jupiter is denser than Saturn, and the earth is denser than Jupiter. The planets, of course, had to be set at different distances from the sun so that each one might, according to the degree of its density, enjoy a greater or smaller amount of heat from the sun.[c] If the earth were located in the orbit of Saturn, our water would freeze; in the orbit of Mercury, it would immediately go off in a vapor. For the light of the sun, to which its heat is proportional, is seven times denser in the orbit of Mercury than on earth, and I have found with a thermometer that water boils at seven

cc. In place of this portion of corol. 4, ed. 1 has:

"Corollary 5. The densities of the planets, moreover, are to one another nearly in a ratio compounded of the ratio of the distance from the sun and the square roots of the diameters of the planets as seen from the sun. For the densities of Saturn, Jupiter, the earth, and the moon (60, 76, 387, and 700) are almost as the roots of the apparent diameters (18″, 39½″, 40″, and 11″) divided by the reciprocals of their distances from the sun $\left(\frac{1}{8,538}, \frac{1}{5,201}, \frac{1}{1,000}, \frac{1}{1,000}\right)$. We said, moreover [utique], in corol. 2, that the gravities at the surfaces of the planets are approximately as the square roots of their apparent diameters as seen from the sun; and in lem. 4 [i.e., corol. 4] that the densities are as the gravities divided by the true diameters; and so the densities are almost as the roots of the apparent diameters multiplied by the true diameters—that is, inversely as the roots of the apparent diameters divided by the distances of the planets from the sun. Therefore God placed the planets at different distances from the sun so that each one might, according to the degree of its density, enjoy a greater or smaller amount of heat from the sun."

times the heat of the summer sun. And there is no doubt that the matter of the planet Mercury is adapted to its heat and therefore is denser than this matter of our earth, since all denser matter requires a greater heat for the performance of the operations of nature.

Proposition 9
Theorem 9

In going inward from the surfaces of the planets, gravity decreases very nearly in the ratio of the distances from the center.

If the matter of the planets were of uniform density, this proposition would hold true exactly, by book 1, prop. 73. Therefore the error is as great as can arise from the nonuniformity of the density.

Proposition 10
Theorem 10

The motions of the planets can continue in the heavens for a very long time.

In the scholium to prop. 40, book 2, it was shown that a globe of frozen water moving freely in our air would, as a result of the resistance of the air, lose $\frac{1}{4{,}586}$ of its motion in describing the length of its own semidiameter. And the same proportion obtains very nearly in any globes, however large they may be and however swift their motions. Now, I gather in the following way that the globe of our earth is denser than if it consisted totally of water. If this globe were wholly made of water, whatever things were rarer than water would, because of their smaller specific gravity, emerge from the water and float on the surface. And for this reason a globe made of earth that was covered completely by water would emerge somewhere, if it were rarer than water; and all the water flowing away from there would be gathered on the opposite side. And this is the case for our earth, which is in great part surrounded by seas. If the earth were not denser than the seas, it would emerge from those seas and, according to the degree of its lightness, a part of the earth would stand out from the water, while all those seas flowed to the opposite side. By the same argument the spots on the sun are lighter than the solar shining matter on top of which they float. And in whatever way the planets were formed, at the time when the mass was fluid, all heavier matter made for the center, away from the water. Accordingly, since the ordinary matter of our earth at its surface is about twice as heavy as water, and a little lower down, in mines, is found to be about three or four or even five times heavier than water, it is likely that the total amount of matter in the earth is about five or six times greater than it would be if the whole earth consisted of water, especially since it has already been shown above that the earth is about four times denser than Jupiter. Therefore, if Jupiter is a little denser than water, then in the space of thirty days (during which this planet describes a length of 459 of its semidiameters) it would, in a medium of the same density as our air, lose almost a tenth of its motion. But since the resistance of mediums decreases in the ratio of their weight and density (so that water, which is 13⅗ times lighter than quicksilver, resists 13⅗ times less; and air, which is 860 times lighter than water, resists 860 times less), it follows that up in the heavens, where the weight of the medium in which the planets move is diminished beyond measure, the resistance will nearly cease. We showed in the scholium to prop. 22, book 2, that at a height of two hundred miles above the earth, the air would be rarer than on the surface of the earth in a ratio of 30 to 0.0000000000003998, or 75,000,000,000,000 to 1, roughly. And hence the planet Jupiter, revolving in a medium with the same density as that upper air, would not, in the time of a million years, lose a millionth of its motion as a result of the resistance of the medium. In the spaces nearest to the earth, of course, nothing is found that creates resistance except air, exhalations, and vapors. If these are exhausted with very great care from a hollow cylindrical glass vessel, heavy bodies fall within the glass vessel very freely and without any sensible resistance; gold itself and the lightest feather, dropped simultaneously, fall with equal velocity and, in falling through a distance of four or six or eight feet, reach the bottom at the same time, as has been found by experiment. And therefore in the heavens, which are void of air and exhalations, the planets and comets, encountering no sensible resistance, will move through those spaces for a very long time.

Hypothesis 1 *The center of the system of the world is at rest.*

No one doubts this, although some argue that the earth, others that the sun, is at rest in the center of the system. Let us see what follows from this hypothesis.

Proposition 11
Theorem 11 *The common center of gravity of the earth, the sun, and all the planets is at rest.*

For that center (by corol. 4 of the Laws) either will be at rest or will move uniformly straight forward. But if that center always moves forward, the center of the universe will also move, contrary to the hypothesis.

Proposition 12
Theorem 12 *The sun is engaged in continual motion but never recedes far from the common center of gravity of all the planets.*

For since (by prop. 8, corol. 2) the matter in the sun is to the matter in Jupiter as 1,067 to 1, and the distance of Jupiter from the sun is to the semidiameter of the sun in a slightly greater ratio, the common center of gravity of Jupiter and the sun will fall upon a point a little outside the surface of the sun. By the same argument, since the matter in the sun is to the matter in Saturn as 3,021 to 1, and the distance of Saturn from the sun is to the semidiameter of the sun in a slightly smaller ratio, the common center of gravity of Saturn and the sun will fall upon a point a little within the surface of the sun. And continuing the same kind of calculation, if the earth and all the planets were to lie on one side of the sun, the distance of the common center of gravity of them all from the center of the sun would scarcely be a whole diameter of the sun. In other cases the distance between those two centers is always less. And therefore, since that center of gravity is continually at rest, the sun will move in one direction or another, according to the various configurations of the planets, but will never recede far from that center.

COROLLARY. Hence the common center of gravity of the earth, the sun, and all the planets is to be considered the center of the universe. For since the earth, sun, and all the planets gravitate toward one another and therefore, in proportion to the force of the gravity of each of them, are constantly put in motion according to the laws of motion, it is clear that their mobile centers cannot be considered the center of the universe, which is at rest. If that body toward which all bodies gravitate most had to be placed in the center (as is the commonly held opinion), that privilege would have to be conceded to the sun. But since the sun itself moves, an immobile point will have to be chosen for that center from which the center of the sun moves away as little as possible and from which it would move away still less, supposing that the sun were denser and larger, in which case it would move less.

Proposition 13
Theorem 13

The planets move in ellipses that have a focus in the center of the sun, and by radii drawn to that center they describe areas proportional to the times.

We have already discussed these motions from the phenomena. Now that the principles of motions have been found, we deduce the celestial motions from these principles a priori. Since the weights of the planets toward the sun are inversely as the squares of the distances from the center of the sun, it follows (from book 1, props. 1 and 11, and prop. 13, corol. 1) that if the sun were at rest and the remaining planets did not act upon one another, their orbits would be elliptical, having the sun in their common focus, and they would describe areas proportional to the times. The actions of the planets upon one another, however, are so very small that they can be ignored, and they perturb the motions of the planets in ellipses about the mobile sun less (by book 1, prop. 66) than if those motions were being performed about the sun at rest.

Yet the action of Jupiter upon Saturn is not to be ignored entirely. For the gravity toward Jupiter is to the gravity toward the sun (at equal distances) as 1 to 1,067; and so in the conjunction of Jupiter and Saturn, since the distance of Saturn from Jupiter is to the distance of Saturn from the sun almost as 4 to 9, the gravity of Saturn toward Jupiter will be to the gravity of Saturn toward the sun as 81 to $16 \times 1{,}067$, or roughly as 1 to 211. And hence arises a perturbation of the orbit of Saturn in every conjunction of this planet with Jupiter so sensible that astronomers have been at a loss concerning it. According to the different situations of the planet Saturn in these conjunctions, its eccentricity is sometimes increased and at other times diminished, the aphelion sometimes is moved forward and at other times perchance drawn back, and the mean motion is alternately accelerated and retarded. Nevertheless, all the error in its motion around the sun, an error arising from so great a force, can almost be avoided (except in the mean motion) by putting the lower focus of its orbit in the common center of gravity of Jupiter and the sun (by book 1, prop. 67); in which case, when that error is greatest, it hardly exceeds two minutes. And the greatest error in the mean motion hardly exceeds two minutes per year. But in the conjunction of Jupiter and Saturn the accelerative gravities of the sun toward Saturn, of Jupiter toward Saturn, and of Jupiter toward the sun are almost as 16, 81, and $\frac{16 \times 81 \times 3{,}021}{25}$, or 156,609, and so the difference of the gravities of the sun toward Saturn and of Jupiter toward Saturn is to the gravity of Jupiter toward the sun as 65 to 156,609, or 1 to 2,409. But the greatest power of Saturn to perturb the motion of Jupiter is proportional to this difference, and therefore the perturbation of the orbit of Jupiter is far less than that of Saturn's. The perturbations of the remaining orbits are still less by far, except that the orbit of the earth is sensibly perturbed by the moon. The common center of gravity of the earth and the moon traverses an ellipse about the sun, an ellipse in which the sun is located at a focus, and this center of gravity, by a radius drawn to the sun, describes areas (in that ellipse) proportional to the times; the earth, during this time, revolves around this common center with a monthly motion.

Proposition 14
Theorem 14

The aphelia and nodes of the [planetary] orbits are at rest.

The aphelia are at rest, by book 1, prop. 11, as are also the planes of the orbits, by prop. 1 of the same book; and if these planes are at rest, the nodes are also at rest. But yet from the actions of the revolving planets and comets upon one another some inequalities will arise, which, however, are so small that they can be ignored here.

COROLLARY 1. The fixed stars also are at rest, because they maintain given positions with respect to the aphelia and nodes.

COROLLARY 2. And so, since the fixed stars have no sensible parallax arising from the annual motion of the earth, their forces will produce no sensible effects in the region of our system, because of the immense distance of these bodies from us. Indeed, the fixed stars, being equally dispersed in all parts of the heavens, by their contrary attractions annul their mutual forces, by book 1, prop. 70.

Scholium

Since the planets nearer to the sun (namely, Mercury, Venus, the earth, and Mars) act but slightly upon one another because of the smallness of their bodies [i.e., because their masses are small], their aphelia and nodes will be at rest, except insofar as they are disturbed by the forces of Jupiter, Saturn, and any bodies further away. And by the theory of gravity it follows that their aphelia move slightly forward [or in consequentia] with respect to the fixed stars, and do this as the $\frac{3}{2}$ powers of the distances of these planets from the sun. For example, if in a hundred years the aphelion of Mars is carried forward [or in consequentia] $33'20''$ with respect to the fixed stars, then in a hundred years the aphelia of the earth, Venus, and Mercury will be carried forward $17'40''$, $10'53''$, and $4'16''$ respectively. And these motions are ignored in this proposition because they are so small.

Proposition 15
Problem 1

To find the principal diameters of the [planetary] orbits.

These diameters are to be taken as the $\frac{2}{3}$ powers of the periodic times by book 1, prop. 15; and then each one is to be increased in the ratio of the sum of the masses of the sun and each revolving planet to the first of two mean proportionals between that sum and the sun, by book 1, prop. 60.

Proposition 16
Problem 2

To find the eccentricities and aphelia of the [planetary] orbits.

The problem is solved by book 1, prop. 18.

Proposition 17
Theorem 15

The daily motions of the planets are uniform, and the libration of the moon arises from its daily motion.

This is clear from the first law of motion and book 1, prop. 66, corol. 22. With respect to the fixed stars Jupiter revolves in 9^h56^m, Mars in 24^h39^m, Venus in about 23 hours, the earth in 23^h56^m, the sun in 25½ days, and the moon in $27^d7^h43^m$. That these things are so is clear from phenomena. With respect to the earth, the spots on the body of the sun return to the same place on the sun's disc in about 27½ days; and therefore with respect to the fixed stars the sun revolves in about 25½ days. Now, since a lunar day (the moon revolving uniformly about its own axis) is a month long [i.e., is equal to a lunar month, the periodic time of the moon's revolution in its orbit], the same face of the moon will always very nearly look in the direction of the further focus of its orbit, and therefore will deviate from the earth on one side or the other according to the situation of that focus. This is the moon's libration in longitude; for the libration in latitude arises from the latitude of the moon and the inclination of its axis to the plane of the ecliptic. Mr. N. Mercator, in his book on astronomy, published in the beginning of the year 1676, set forth this theory of the moon's libration more fully on the basis of a letter from me.

The outermost satellite of Saturn seems to revolve about its own axis with a motion similar to our moon's, constantly presenting the same aspect toward Saturn. For in revolving about Saturn, whenever it approaches the eastern part of its own orbit, it is just barely seen and for the most part disappears from sight; and possibly this occurs because of certain spots in that part of its body which is then turned toward the earth, as Cassini noted. The outermost satellite of Jupiter also seems to revolve about its own axis with a similar motion, because in the part of its body turned away from Jupiter it has a spot which, whenever the satellite passes between Jupiter and our eyes, appears as if it were on the body of Jupiter.

Proposition 18
Theorem 16

The axes of the planets are smaller than the diameters that are drawn perpendicularly to those axes.

If it were not for the daily circular motion of the planets, then, because the gravity of their parts is equal on all sides, they would have to assume a spherical figure. Because of that circular motion it comes about that those parts, by receding from the axis, endeavor to ascend in the region of the equator. And therefore if the matter is fluid, it will increase the diameters at the equator by ascending, and will decrease the axis at the poles by descending. Thus the diameter of Jupiter is found by astronomical observations to be shorter between the poles than from east to west. By the same argument, if our earth were not a little higher around the equator than at the poles, the seas would subside at the poles and, by ascending in the region of the equator, would flood everything there.

Proposition 19[a]
Problem 3

To find the proportion of a planet's axis to the diameters perpendicular to that axis.

[bc]Our fellow countryman Norwood, in about the year 1635, measured a distance of 905,751 London feet between London and York and observed the

a. For a gloss on this proposition see the Guide, §10.14.

bb. Ed. 1 has: "The solution of this problem requires a complex computation, which is shown more easily by example than by precept. Through making the calculation, therefore, I find, by book 1, prop. 4, that the centrifugal force of the parts of the earth at the equator, arising from the daily motion, is to the force of gravity as 1 to 290⅘."

cc. Ed. 2 has: "Picard measured an arc of 1°22′55″ between Amiens and Malvoisine and found an arc of one degree to be 57,060 Paris toises. Hence the circumference of the earth is 123,249,600 Paris feet, as above. But since an error of four hundredths of an inch, either in the construction of the instruments or in their application to making observations, is imperceptible and, in the ten-foot sector by which the French observed the latitudes of places, corresponds to four seconds and, in single observations, can fall upon the center of the sector as well as on its circumference, and since errors in smaller arcs are of greater significance, therefore Cassini by the king's order undertook the measurement of the earth by means of greater intervals between the places (see the History of the Royal Academy of Sciences for the year 1700) and in the process, by using the distance between the Royal Paris Observatory and the village of Collioure in Roussillon and the difference of 6°18′ between the latitudes and supposing that the earth's shape is spherical, found one degree to be 57,292 toises, nearly as our fellow countryman Norwood had found earlier. For Norwood, in about the year 1635, measured a distance of 905,751 London feet between London and York and observed the difference of latitudes between those places to be 2°28′ and thereby found the measure of one degree to be 367,196 London feet, that is, 57,300 Paris toises. Because of the magnitude of the interval measured by Cassini, I shall use 57,292 toises for the measure of one degree in the middle of that interval, that is, between the latitudes of 45° and 46°. Hence, if the earth is spherical, its semidiameter will be 19,695,539 Paris feet.

"The length of a seconds pendulum oscillating in the latitude of Paris is three Paris feet and 8⁵⁄₉ lines. And the length which a heavy body describes by falling in the time of one second is to half the length of difference of latitudes between those places to be 2°28′ and thereby found the measure of one degree to be 367,196 London feet, that is, 57,300 Paris toises. Picard measured an arc of 1°22′55″ along the meridian between Amiens and Malvoisine and found an arc of one degree to be 57,060 Paris toises. The elder Cassini [Gian Domenico or Jean-Dominique] measured the distance along the meridian from the town of Collioure in Roussillon to the Paris observatory; and his son [Jacques] added the distance from the observatory to the tower of the city of Dunkerque. The total distance was 486,156½ toises, and the difference in latitudes between the town of Collioure and the city of Dunkerque was 8°31′11⅚″. Thus an arc of one degree comes out to be 57,061 Paris toises. And from these measures the circumference of the earth is found to be 123,249,600 Paris feet, and its semidiameter 19,615,800 feet, on the hypothesis that the earth is spherical.

At the latitude of Paris, a heavy body falling in the time of one second describes 15 Paris feet 1 inch 1 7/9 lines as has been mentioned above, that is, 2,173 7/9 lines. The weight of a body is diminished by the weight of the surrounding air. Let us suppose that the weight lost in this way is an eleven-thousandth part of the total weight; then such a heavy body falling in a vacuum will describe a space of 2,174 lines in the time of one second.[c]

A body revolving uniformly in a circle at a distance of 19,615,800 feet from the center, making a revolution in a single sidereal day of $23^h 56^m 4^s$, will describe an arc of 1,433.46 feet in the time of one second, an arc whose versed sine is 0.0523656 feet, or 7.54064 lines. And therefore the force by which heavy bodies descend at the latitude of Paris is to the [d]centrifugal[d] force of bodies on the equator (which arises from the daily motion of the earth) as 2,174 to 7.54064.

The centrifugal force of bodies on the earth's equator is to the centrifugal force by which bodies recede rectilinearly from the earth at the latitude of Paris (48°50′10″) as the square of the radius to the square of the cosine of that latitude, that is, as 7.54064 to 3.267. Let this force be added to the force by which heavy bodies descend at the latitude of Paris; then a body falling at that latitude with the total force of gravity will, in the time of one second, describe 2,177.267 lines, or 15 Paris feet 1 inch and 5.267 lines. And the total

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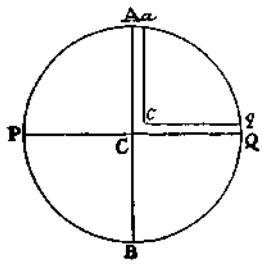
this pendulum or as the square of the ratio of the circumference of the circle to its diameter (as Huygens indicated) and thus is 15 Paris feet, 1 inch, 2 1/18 lines, or 2,174 1/18 lines."

dd. Ed. 2 has "centripetal."

force of gravity at that latitude will be to the [e]centrifugal[e] force of bodies on the earth's equator as 2,177.267 to 7.54064 or 289 to 1.[b]

Therefore, if APBQ represents the figure of the earth, which is now no longer spherical but generated by the rotation of an ellipse about its minor axis PQ; and if ACQqca is a channel full of water, going from the pole Qq to the center Cc and from that center out to the equator Aa; then the weight of the water in the leg ACca will have to be to the weight of the water in the other leg QCcq as 289 to 288, because the centrifugal force arising from the circular motion will sustain and take away one of the 289 parts of weight of the water in the leg ACca, and consequently the 288 parts of weight of the water in the leg QCcq will sustain the 288 parts remaining in the leg ACca. Further, on making the computation (according to book 1, prop. 91, corol. 2), I find that if the earth were composed of uniform matter and were deprived of all its motion, and its axis PQ were to its diameter AB as 100 to 101, then the gravity in place Q toward the earth would be to the gravity in the same place Q toward a sphere described about the center C with a radius PC or QC as 126 to 125. And by the same argument, the gravity in place A toward a spheroid generated by the rotation of the ellipse APBQ about the axis AB is to the gravity in the same place A toward a sphere described about a center C with a radius AC as 125 to 126. Moreover, the gravity in place A toward the earth is a mean proportional between the gravity toward the spheroid and the gravity toward the sphere, because the sphere, when its diameter PQ is diminished in the ratio of 101 to 100, is transformed into the figure of the earth; and this figure, when a third diameter (perpendicular to the two given diameters AB and PQ) is diminished in the same ratio, is transformed into the said spheroid; and the gravity in A, in either case, is diminished in very nearly the same ratio. Therefore the gravity in A toward a sphere described about the center C with a radius AC is to the gravity in A toward the earth as 126 to 125½; and the gravity in place Q, toward a sphere described about the center C with a radius QC, is to the gravity in place A, toward a sphere described about the center C with a radius AC, in the ratio of the diameters (by book 1, prop. 72),



ee. Ed. 2 has "centripetal."

that is, as 100 to 101. Now let these three ratios (126 to 125, 126 to 125½, and 100 to 101) be combined, and the gravity in place Q toward the earth will become to the gravity in place A toward the earth as $126 \times 126 \times 100$ to $125 \times 125\frac{1}{2} \times 101$, or as 501 to 500.

Now, since (by book 1, prop. 91, corol. 3) the gravity in either leg AC*ca* or QC*cq* of the channel is as the distance of places from the earth's center, if those legs are separated by transverse, equidistant surfaces into parts proportional to the wholes, the weights of any number of these individual parts in the leg AC*ca* will be to the weights of the same number of individual parts in the other leg as their magnitudes and accelerative gravities jointly, that is, as 101 to 100 and 500 to 501, which is as 505 to 501. And accordingly, if the centrifugal force of each part of the leg AC*ca* (which force arises from the daily motion) had been to the weight of the same part as 4 to 505, so that it would take away four parts from the weight of each part (supposing it to be divided into 505 parts), the weights would remain equal in each leg, and therefore the fluid would stay at rest in equilibrium. But the centrifugal force of each part is to the weight of the same part as 1 to 289; that is, the [f]centrifugal[f] force, which ought to have been $\frac{4}{505}$ of the weight, is only $\frac{1}{289}$ of it. And therefore I say, according to the golden rule [or rule of three], that if a centrifugal force of $\frac{4}{505}$ of the weight makes the height of the water in the leg AC*ca* exceed the height of the water in the leg QC*cq* by a hundredth of its total height, the centrifugal force of $\frac{1}{289}$ of the weight will make the excess of the height in the leg AC*ca* be only $\frac{1}{229}$ of the height of the water in the other leg QC*cq*. Therefore the diameter of the earth at the equator is to its diameter through the poles as 230 to 229. And thus, since the mean semidiameter of the earth, according to Picard's measurement, is 19,615,800 Paris feet, or 3,923.16 miles (supposing a mile to be 5,000 feet), the earth will be 85,472 feet or 17⅒ miles higher at the equator than at the poles. And its height at the equator will be about 19,658,600 feet, and at the poles will be about 19,573,000 feet.

If a planet is larger or smaller than the earth, while its density and periodic time of daily revolution remain the same, the ratio of centrifugal force

ff. Ed. 1 has "centripetal."

to gravity will remain the same, and therefore the ratio of the diameter between the poles to the diameter at the equator will also remain the same. But if the daily motion is accelerated or retarded in any ratio, the centrifugal force will be increased or decreased in that same ratio squared, and therefore the difference between the diameters will be increased or decreased very nearly in the same squared ratio. And if the density of a planet is increased or decreased in any ratio, the gravity tending toward the planet will also be increased or decreased in the same ratio, and the difference between the diameters in turn will be decreased in the ratio of the increase in the gravity or will be increased in the ratio of the decrease in the gravity. Accordingly, since the earth revolves [i.e., rotates] with respect to the fixed stars in 23^h56^m, and Jupiter in 9^h56^m, and the squares of their periodic times are as 29 to 5, and the densities of these revolving bodies are as 400 to 94½, the difference between the diameters of Jupiter will be to its smaller diameter as $\frac{29}{5} \times \frac{440}{94\frac{1}{2}} \times \frac{1}{229}$ to 1, or very nearly as 1 to 9⅓. Therefore Jupiter's diameter taken from east to west is to its diameter between the poles very nearly as 10⅓ to 9⅓. [g]Thus, since its larger diameter is 37″, its smaller diameter (which lies between the poles) will be 33″25‴. Because of the erratic light let about 3″ be added, and the apparent diameters of this planet will come out to be 40″ and 36″25‴, which are to each other nearly as 11⅙ to 10⅙. This argument has been based on the hypothesis that the body of Jupiter is uniformly dense. But if its body is denser toward the plane of the equator than toward the poles, its diameters can be to each other as 12 to 11, or 13 to 12, or even 14 to 13. As a matter of fact, Cassini observed in the year 1691 that the diameter of Jupiter extending from east to west would exceed its other diameter by about a fifteenth part of itself. Moreover, our fellow countryman Pound, with a 123-foot-long telescope and the best micrometer, measured the diameters of Jupiter in the year 1719 with the following results.

gg. Ed. 1 and ed. 2 have: "These things are so on the hypothesis that the matter of the planets is uniform. For if the matter is denser at the center than at the circumference, the diameter which is drawn from east to west will be still greater." In ed. 1 the proposition ends here, but in ed. 2 it continues: "Indeed, Cassini observed long ago that the diameter of Jupiter which lies between its poles is smaller than the other diameter, and it will be apparent from what is said in prop. 20 below that the diameter of the earth between the poles is smaller than the other diameter."

Times			*Largest diameter*	*Smallest diameter*	*Diameters to each other*
	days	*hours*	*parts*	*parts*	
January	28	6	13.40	12.28	as 12 to 11
March	6	7	13.12	12.20	13¾ to 12¾
March	9	7	13.12	12.08	12⅔ to 11⅔
April	9	9	12.32	11.48	14½ to 13½

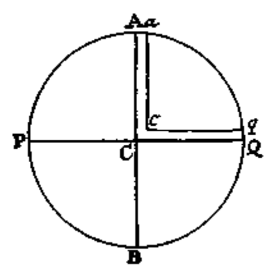
Therefore the theory agrees with the phenomena. Further, the planets are more exposed to the heat of sunlight toward their equators and as a result [h]are somewhat more thoroughly heated there[h] than toward the poles.

Even further, it will be apparent—from the experiments with pendulums reported in prop. 20 below—that gravity is decreased at the equator by the daily rotation of our earth, and therefore that the earth (supposing its matter to be uniformly dense) rises higher there than at the poles.[g]

Proposition 20
Problem 4

To find and compare with one another the weights of bodies in different regions of our earth.

Since the weights of the unequal legs of the water-channel ACQ*qca* are equal, and the weights of any parts that are proportional to the whole legs and similarly situated in those legs are to one another as the weights of the wholes, and thus are also equal to one another, the weights of parts that are equal and similarly situated in the legs will be inversely as the legs, that is, inversely as 230 to 229. This is likewise the case for any homogeneous equal bodies that are similarly situated in the legs of the channel. The weights of these bodies are inversely as the legs, that is, inversely as the distances of the bodies from the earth's center. Accordingly, if the bodies are in the topmost parts of the channels, or on the surface of the earth, their weights will be to one another inversely as their distances from the center. And by the same argument, weights that are in any other regions whatever, anywhere on the whole surface of the earth, are inversely as



hh. The Latin text reads "paulo magis ibi decoquuntur," employing the verb "decoquo," which could mean, literally, "boil down" or "cook" or "bake." The problems of interpretation are discussed in the Guide to the present translation, §8.11.

the distances of those places from the center; and therefore, on the hypothesis that the earth is a spheroid, the proportion of those weights is given.

From this the following theorem is deduced:[a] The increase of weight in going from the equator to the poles is very nearly as the versed sine of twice the latitude, or (which is the same) as the square of the sine of the latitude. [b]And the arcs of degrees of latitude on a meridian are increased in about the same ratio. Now, the latitude of Paris is 48°50′, the latitude of places on the equator 00°00′, and that of places at the poles 90°; the versed sines of twice those arcs of latitude are 11,334 and 00,000 and 20,000 (the radius being taken to be 10,000); the gravity at the pole is to the gravity at the equator as 230 to 229; and the excess of the gravity at the pole to the gravity at the equator is

a. See the Guide, §10.15.

bb. In place of the remaining part of this proposition, ed. 1 has: "For example, the latitude of Paris is 48°45′, that of the island of Gorée near Cape Verde 14°15′, that of Cayenne off the coast of Guiana about 50°, that of places at the pole 90°. If the arcs of latitude are doubled, they are 97.5°, 28.5°, 10°, and 180°. The versed sines are 11,305, 1,211, 152, and 2,000. Furthermore, since the gravity at the pole is to the gravity at the equator as 692 to 689 and the excess of gravity at the pole is to the gravity at the equator as 3 to 689, the excess of gravity at Paris, on the island of Gorée, and at Cayenne will be to the gravity at the equator as $\frac{3 \times 11{,}305}{2{,}000}$, $\frac{3 \times 1{,}211}{2{,}000}$, and $\frac{3 \times 152}{2{,}000}$ to 689, or as 33,915, 3,633, and 456 to 13,780,000, and therefore the total gravities in these places will be to one another as 13,813,915, 13,783,633, 13,780,456, and 13,780,000. Therefore, since the lengths of pendulums oscillating with equal periods are as the gravities, and the length of a seconds pendulum at Paris is 3 Paris feet and $^{17}/_{24}$ inches, the lengths of seconds pendulums on the island of Gorée, at Cayenne, and at the equator will be surpassed by the length of a Paris pendulum by excesses of $\frac{81}{1{,}000}$, $\frac{89}{1{,}000}$, and $\frac{90}{1{,}000}$ inches. All of these things will be so on the hypothesis that the earth consists of uniform matter. For if the matter at the center is a little denser than at the surface, those excesses will be a little greater. The reason is that if the superabundant matter at the center, by which the density there is rendered greater, is taken away and considered separately, the gravity toward the rest of the earth, which is uniformly dense, will be inversely as the distance of a weight from the center, but toward the superabundant matter inversely as the square of the distance from that matter very nearly. Therefore, gravity at the equator will be less toward that superabundant matter than as in the above computation, and therefore the earth there, on account of the deficiency of gravity, will rise a little higher than has been determined above. Indeed, the French by making experiments have already found that the length of seconds pendulums at Paris is greater than on the island of Gorée by $^1/_{10}$ of an inch and greater than at Cayenne by $^1/_8$. These differences are a little greater than the differences $\frac{81}{1{,}000}$ and $\frac{89}{1{,}000}$ which resulted from the above computation, and therefore (if one can have enough confidence in these rough observations) the earth will be somewhat higher at the equator than according to the above calculation and denser at the center than in mines near the surface. If the excess of gravity in these northern places over the gravity at the equator is finally determined exactly by experiments conducted with greater diligence, and then its excess is everywhere taken in the ratio of the versed sine of twice the latitude, then there will be determined a universal measure, an equalizing of the time of equal pendulums in the different places indicated, and also the proportion of the diameters of the earth and its density at the center, on the hypothesis that that density, as one goes to the circumference, decreases uniformly. And indeed this hypothesis, even though it is not exact, can be assumed for undertaking such a calculation."

as 1 to 229. Hence the excess of the gravity at the latitude of Paris will be to the gravity at the equator as $1 \times \frac{11{,}334}{20{,}000}$ to 229, or 5,667 to 2,290,000. And therefore the total gravities in these places will be to each other as 2,295,667 to 2,290,000. And thus, since the lengths of pendulums oscillating with equal periods are as the gravities, and at the latitude of Paris the length of a seconds pendulum is 3 Paris feet and 8½ lines (or rather, because of the weight of the air, 8⁵⁄₉ lines), the length of a pendulum at the equator will be shorter than the length of a pendulum with the same period at Paris in the amount of 1.087 lines. And a similar computation yields the following table.

Latitude of the place	*Length of the pendulum*		*Measure of one degree on the meridian*	*Latitude of the place*	*Length of the pendulum*		*Measure of one degree on the meridian*
degrees	*feet*	*lines*	*toises*	*degrees*	*feet*	*lines*	*toises*
0	3	7.468	56637	45	3	8.428	57010
5	3	7.482	56642	6	3	8.461	57022
10	3	7.526	56659	7	3	8.494	57035
15	3	7.596	56687	8	3	8.528	57048
20	3	7.692	56724	9	3	8.561	57061
25	3	7.812	56769	50	3	8.594	57074
30	3	7.948	56823	55	3	8.756	57137
35	3	8.099	56882	60	3	8.907	57196
40	3	8.261	56945	65	3	9.044	57250
1	3	8.294	56958	70	3	9.162	57295
2	3	8.327	56971	75	3	9.258	57332
3	3	8.361	56984	80	3	9.329	57360
4	3	8.394	56997	85	3	9.372	57377
				90	3	9.387	57382

[c] Moreover, it is established by this table that the inequality [in the length] of degrees [at different latitudes] is so small that in geographical matters the

cc. Ed. 2 has: "and that the inequality of the diameters of the earth can be ascertained more easily and more surely by experiments with pendulums or even by eclipses of the moon than by arcs measured geographically on the meridian.

"These things are so on the hypothesis that the earth consists of uniform matter. For if the matter at the center is a little denser than at the surface, the differences of pendulums and degrees on a meridian will be a little greater than according to the preceding table. The reason is that if superabundant matter at the center, by which the density there is rendered greater, is taken away and regarded separately, the gravity toward the rest of the earth, which is uniformly dense, will be inversely as the distance of a weight from the center; but toward the superabundant matter, it will be inversely as the square of the distance from that matter very nearly. Therefore gravity at the equator is less toward that superabundant matter than according to the above computation, and therefore the earth there, on account of the deficiency of shape of the earth can be considered to be spherical, especially if the earth is a little denser toward the plane of the equator than toward the poles.[c]

Now some astronomers, sent to distant regions to make astronomical observations, have observed that their pendulum clocks went more slowly near the equator than in our regions. And indeed M. Richer first observed this in the year 1672 on the island of Cayenne. For while he was observing the transit of the fixed stars across the meridian in the month of August, he found that his clock was going more slowly than in its proper proportion to the mean motion of the sun, the difference being $2^{m}28^{s}$ every day. Then by constructing a simple pendulum that would oscillate in seconds as measured by the best clock, he noted the length of the simple pendulum, and he did this frequently, every week for ten months. Then, when he had returned to France, he compared the length of this pendulum with the length of a seconds pendulum at Paris (which was 3 Paris feet and 8⅗ lines long) and found that it was shorter than the Paris pendulum, the difference being 1¼ lines.[d]

Afterward, our fellow countryman Halley, sailing in about the year 1677 to the island of St. Helena, found that his pendulum clock went more slowly there than in London, but he did not record the difference. He made the pendulum of his clock shorter by more than ⅛ of an inch, or 1½ lines. And to effect this, since the length of the threaded part at the lower end of the pendulum rod was not sufficient, he put a wooden ring between the nut (on the threaded part) and the weight at the end of the pendulum.

Then in the year 1682 M. Varin and M. Des Hayes found that the length of a seconds pendulum in the Royal Observatory at Paris was 3 feet 8⁵⁄₉ lines. And on the island of Gorée they found by the same method that the length of a pendulum with the same period was 3 feet 6⁵⁄₉ lines, the difference in lengths being 2 lines. And sailing in the same year to the islands of Guadeloupe and Martinique, they found that on these islands the length of a pendulum with the same period was 3 feet 6½ lines.

Afterward, in July 1697, M. Couplet the younger adjusted his pendulum clock to the mean motion of the sun in the Royal Observatory at Paris in such a way that for quite a long time the clock agreed with the motion of

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gravity, will rise a little higher, and the excesses of the lengths of pendulums and of the degrees at the poles will be a little greater than has been determined above."

d. Here ed. 2 has an additional sentence: "But from the slowness of the pendulum clock in Cayenne, the difference of the pendulums is gathered to be 1½ lines."

the sun. Then sailing to Lisbon, he found that by the next November the clock went more slowly than before, the difference being 2^m13^s in 24 hours. And sailing to Paraíba in the following March, he found that his clock went more slowly there than in Paris, the difference being 4^m12^s in 24 hours. And he declares that a seconds pendulum was 2½ lines shorter at Lisbon and 3⅔ lines shorter at Paraíba than at Paris. He might more correctly have put these differences as 1⅓ and 2⁵⁄₉; these are the differences that correspond to the differences in times of 2^m13^s and 4^m12^s. He is less trustworthy because of the crudity of his observations.

In the next years (1699 and 1700) M. Des Hayes, again sailing to America, determined that on the islands of Cayenne and Grenada the length of a seconds pendulum was a little less than 3 feet 6½ lines, that on the island of St. Kitts that length was 3 feet 6¾ lines, and that on the island of Santo Domingo it was 3 feet 7 lines.

And in the year 1704 Father Feuillée found that in Portobello in America the length of a seconds pendulum was 3 Paris feet and only 5⁷⁄₁₂ lines, that is, about 3 lines shorter than at Paris, but he made an error in his observation. For, sailing afterward to the island of Martinique, he found that the length of a pendulum with the same period was 3 Paris feet and 5¹⁰⁄₁₂ lines.

Moreover, the latitude of Paraíba is 6°38′ S, and that of Portobello is 9°33′ N; and the latitudes of the islands of Cayenne, Gorée, Guadeloupe, Martinique, Grenada, St. Kitts, and Santo Domingo are respectively 4°55′, 14°40′, 14°00′, 14°44′, 12°6′, 17°19′, and 19°48′ N. And the excesses of the length of the pendulum at Paris over the observed lengths of pendulums with the same period in these latitudes are a little greater than they would be according to the table of pendulum lengths computed above. And therefore the earth is somewhat higher at the equator than according to the above computation, and is denser toward the center than in mines near the surface, unless perhaps the heat in the torrid zone somewhat increased the length of the pendulums.

M. Picard, at any rate, observed that an iron rod, which in wintertime when the weather was freezing was 1 foot long, came to be 1 foot and ¼ of a line long when heated by a fire. Later M. La Hire observed that an iron rod, which in an exactly similar winter was 6 feet long, came to be 6 feet and ⅔ of a line long when it was exposed to the summer sun. The heat [i.e., temperature] was greater in the first example than in the second, and in the second it was greater than that of the external parts of the human body. For metals grow extremely hot in the summer sun. But the pendulum rod in a pendulum clock is ordinarily never exposed to the heat of the summer sun, and never acquires a heat equal to that of the external surface of the human body. And, therefore, although a 3-foot-long pendulum rod in a clock will indeed be a little longer in summertime than in wintertime, this increase will scarcely surpass ¼ of 1 line. Accordingly, all of the difference in the length of pendulums with the same period in different regions cannot be attributed to differences in heat. Nor can this difference be attributed to errors made by the astronomers sent from France. For although their observations do not agree perfectly with one another, the errors are so small that they can be ignored. And in this they all agree: that at the equator, pendulums are shorter than pendulums with the same period at the Royal Observatory in Paris, [e]the difference being neither less than 1¼ lines nor more than 2⅔ lines. By the observations of M. Richer made in Cayenne the difference was 1¼ lines. By those of M. Des Hayes that difference when corrected became 1½ or 1¾ lines. By the less accurate observations made by others, this difference came out as more or less 2 lines. And this discrepancy could have arisen partly from errors in observations, partly from the dissimilitude of the internal parts of the earth and from the height of mountains, and partly from the differences in heat [i.e., temperatures] of the air.

As far as I can tell, in England an iron rod 3 feet long is ⅙ of 1 line shorter in the wintertime than in the summertime. Let this quantity be sub-

ee. Ed. 2 has: "... the difference being about 2 lines or ⅙ of an inch. By the observations of M. Richer made in Cayenne, the difference was 1½ lines. An error of half a line is easily committed. And M. Des Hayes afterward, by his observations made on the same island, corrected the error, finding a difference of 2 1/18 lines. But also by observations made on the islands of Gorée, Guadeloupe, Martinique, Grenada, St. Kitts, and Santo Domingo and reduced to the equator, that difference came out to be scarcely smaller than 1 19/20 of a line and scarcely greater than 2½ lines. And the mean quantity between these limits is 2 9/40 lines. Because of the heat of places in the torrid zone, let us ignore 9/40 of a line, and a difference of 2 lines will remain.

"Therefore, since that difference, by the preceding table, on the hypothesis that the earth consists of uniformly dense matter, is only $1\frac{87}{1,000}$ of a line, the excess of the height of the earth at the equator over its height at the poles, which was 17⅙ miles, being now increased in the ratio of the differences, will become 31 7/12 miles. For the slowness of a pendulum at the equator proves the deficiency of the gravity; and the lighter the matter is, the greater its height must be in order that by its weight it may hold in equilibrium the matter at the poles.

"Hence the shape of the earth's shadow, which is to be determined by eclipses of the moon, will not be entirely circular, but its diameter drawn from east to west will be greater than its diameter drawn tracted (because of the heat at the equator) from the difference of $1\frac{1}{4}$ lines observed by Richer, and there will remain $1\frac{1}{12}$ lines, in excellent agreement with the $1\frac{87}{1,000}$ lines already found from the theory. Moreover, Richer repeated his observations in Cayenne every week during a ten-month period, and compared the lengths he found there for a pendulum consisting of an iron rod with its lengths similarly found in France [i.e., with its lengths adjusted in Paris so as to have the same period]. This diligence and caution seem to have been lacking in other observers. If his observations are to be trusted, the earth will be higher at the equator than at the poles by an excess of about seventeen miles, as came out above by the theory.[b e]

Proposition 21 Theorem 17 *The equinoctial points regress, and the earth's axis, by a nutation in every annual revolution, inclines twice toward the ecliptic and twice returns to its former position.*

This is clear by book 1, prop. 66, corol. 20. This motion of nutation, however, must be very small—either scarcely or not at all perceptible.

Proposition 22 Theorem 18 *All the motions of the moon and all the inequalities in its motions follow from the principles that have been set forth.*

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from south to north by an excess of about 55″. And the greatest longitudinal parallax of the moon will be a little greater than its greatest latitudinal parallax. And the greatest semidiameter of the earth will be 19,767,630 Paris feet, the least, 19,609,820 feet, and the mean, 19,688,725 feet, very nearly.

"Since one degree by Picard's measurement is 57,060 toises but by Cassini's measurement is 57,292 toises, some suspect that each degree, as one goes southward through France, is greater than the preceding degree by 72 toises more or less, or $\frac{1}{800}$ of one degree, the earth being an oblong spheroid whose parts are highest at the poles. Under this supposition, all bodies at the earth's poles would be lighter than at the equator, and the height of the earth at the poles would exceed its height at the equator by nearly 95 miles, and isochronous pendulums would be longer at the equator than in the Royal Observatory at Paris by an excess of about half an inch, as will be easily seen by anyone comparing the proportions set forth here with the proportions set forth in the preceding table. But also the diameter of the earth's shadow drawn from south to north would be greater than its diameter drawn from east to west by an excess of 2′46″, or $\frac{1}{12}$ of the moon's diameter. Experience contradicts all this. Certainly Cassini, in determining that one degree is 57,292 toises, took a mean between all his measurements, on the hypothesis of the equality of degrees. And although Picard on the northern border of France found a degree to be a little smaller, yet our compatriot Norwood in more northern reigons, by measuring a greater interval, found a degree to be a little greater than Cassini had found. And Cassini himself, when he attempted to determine the measurement of one degree by using a far greater interval, judged Picard's measurement to be insufficiently certain and exact because of the smallness of the interval measured. But the differences among the measurements of Cassini, Picard, and Norwood are nearly imperceptible and could easily have arisen from imperceptible errors in observations, not to mention the nutation of the earth's axis."

That the major planets, while they are being carried about the sun, can carry other or minor planets [or satellites], revolving around them, and that those minor planets must revolve in ellipses having their foci in the centers of the major planets, is evident from book 1, prop. 65. Moreover, their motions will be perturbed in many ways by the sun's action, and they will be influenced by those inequalities that are observed in our moon. Our moon, in any case (by book 1, prop. 66, corols. 2, 3, 4, and 5), moves more swiftly, and by a radius drawn to the earth describes an area greater for the time, and has a less curved orbit, and therefore approaches closer to the earth, in the syzygies than in the quadratures, except insofar as these effects are hindered by the motion of eccentricity. For the eccentricity is greatest (by book 1, prop. 66, corol. 9) when the moon's apogee is in the syzygies, and least when it stands in the quadratures; and thus the moon in its perigee is swifter and closer to us, while in its apogee it is slower and more remote, in the syzygies than in the quadratures. Additionally, the apogee advances and the nodes regress, but with a nonuniform motion. And indeed the apogee (by prop. 66, corols. 7 and 8) advances more swiftly in its syzygies, regresses more slowly in the quadratures, and by the excess of the advance over the regression is annually carried forward [or in consequentia, i.e., from east to west in the direction of the signs]. But the nodes (by prop. 66, corol. 2) are at rest in their syzygies and regress most swiftly in the quadratures. The moon's greatest latitude is also greater in its quadratures (by prop. 66, corol. 10) than in its syzygies, and (by prop. 66, corol. 6) the mean motion of the moon is slower in the earth's perihelion than in its aphelion. And these are the more significant inequalities [of the moon's motion] taken note of by astronomers.

There are also certain other inequalities not observed by previous astronomers, by which the lunar motions are so perturbed that until now these motions have not been reducible, by any law, to any definite rule. For the velocities or hourly motions of the moon's apogee and nodes, and their equations, and also the difference between the greatest eccentricity in the syzygies and the least in the quadratures, and that inequality which is called the variation, are increased and decreased annually (by prop. 66, corol. 14) as the cube of the sun's apparent diameter. And, additionally, the variation is increased or decreased very nearly as the square of the time between the quadratures (by book 1, lem. 10, corols. 1 and 2, and prop. 66, corol. 16), but in astronomical calculations this inequality is generally included under the moon's prosthaphaeresis [or equation of the center] and confounded with it.

Proposition 23
Problem 5

To derive the unequal motions [i.e., the inequalities in the motions] of the satellites of Jupiter and of Saturn from the motions of our moon.

From the motions of our moon the analogous motions of the moons or satellites of Jupiter are derived as follows. The mean motion of the nodes of Jupiter's outermost satellite is (by book 1, prop. 66, corol. 16) to the mean motion of the nodes of our moon in a ratio compounded of the square of the ratio of the earth's periodic time about the sun to Jupiter's periodic time about the sun, and of the simple ratio of the satellite's periodic time about Jupiter to the moon's periodic time about the earth, and so in one hundred years that node completes $8°24'$ backward [or in antecedentia, i.e., counter to the order of the signs]. The mean motions of the nodes of the inner satellites are (by the same corollary) to the motion of this outermost satellite as the periodic times of those inner satellites are to the periodic time of the outermost satellite and hence are given. Moreover (by the same corollary), the forward [or direct] motion of the upper apsis of each satellite [or its motion in consequentia] is to the backward [or retrograde] motion of its nodes [or the motion in antecedentia] as the motion of the apogee of our moon to the motion of its nodes, and hence is also given. However, the motion of the upper apsis found in this way must be decreased in the ratio of 5 to 9, or about 1 to 2, for a reason which would take too much time to explain here. The greatest equations of the nodes and upper apsis of each satellite are approximately to the greatest equations of the nodes and upper apsis of our moon respectively as the motions of the nodes and upper apsis of the satellites in the time of one revolution of the former equations are to the motions of the nodes and apogee of our moon in the time of one revolution of the latter equations. By the same corollary, the variation of a satellite as it would be observed from Jupiter is to the variation of our moon in the same proportion as the total motions of their nodes during the times in which respectively the satellite and our moon revolve as reckoned in relation to the sun; and therefore in the outermost satellite the variation does not exceed $5''12'''$.

Proposition 24
Theorem 19

The ebb and flow of the sea arise from the actions of the sun and moon.

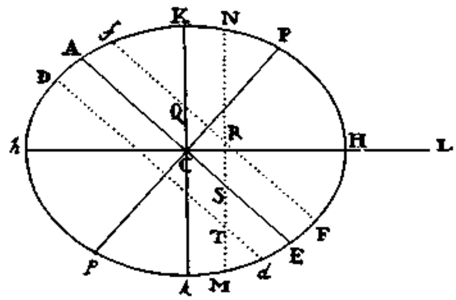
It is clear from book 1, prop. 66, corols. 19 and 20, that the sea should twice rise and twice fall in every day, lunar as well as solar, and also that the greatest height of the water, in deep and open seas, should occur less than six hours after the appulse of the luminaries to the meridian of a place, as happens in the whole eastern section of the Atlantic Ocean and the Ethiopic [or South Atlantic] Sea between France and the Cape of Good Hope, and also on the Chilean and Peruvian shore of the Pacific Ocean; on all these shores the tide comes in at about the second, third, or fourth hour, except in cases when the motion has been propagated from the deep ocean through shallow places and is delayed until the fifth, sixth, or seventh hour, or later. I number the hours from the appulse of either luminary to the meridian of a place, below the horizon as well as above, and by hours of a lunar day I mean twenty-fourths of that time in which the moon, by its apparent daily motion, returns to the meridian of the place. The force of the sun or moon to raise the sea is greatest in the very appulse of the luminary to the meridian of the place. But the force impressed upon the sea at that time remains for a while and is increased by a new force subsequently impressed, until the sea has ascended to its greatest height, which will happen in one or two hours, but more frequently at the shores in about three hours or even more if the sea is shallow.

Moreover, the two motions which the two luminaries excite will not be discerned separately but will cause what might be called a mixed motion. In the conjunction or the opposition of the luminaries their effects will be combined, and the result will be the greatest ebb and flow. In the quadratures the sun will raise the water while the moon depresses it and will depress the water while the moon raises it; and the lowest tide of all will arise from the difference between these two effects. And since, as experience shows, the effect of the moon is greater than that of the sun, the greatest height of the water will occur at about the third lunar hour. Outside of the syzygies and quadratures, the highest tide, which by the lunar force alone would always have to occur at the third lunar hour, and by the solar force alone at the third solar hour, will occur, as a result of the combining of the lunar and solar forces, at some intermediate time which is closer to the third lunar hour [than to the third solar hour]; and thus in the transit of the moon from the syzygies to the quadratures, when the third solar hour precedes the third lunar hour, the greatest height of the water will also precede the third lunar hour, and will do so by the greatest interval a little after the octants of the moon; and the highest tide will follow the third lunar hour with the same intervals in the transit of the moon from the quadratures to the syzygies. This is what happens in the open sea. For in the mouths of rivers the higher tides, other things being equal, will come to their peaks later.

Additionally, the effects of the luminaries depend on their distances from the earth. For at smaller distances their effects are greater, and at greater distances smaller, and this varies as the cubes of their apparent diameters. Therefore the sun in wintertime, when it is in its perigee, produces greater effects and makes the tides a little higher in the syzygies and a little lower (other things being equal) in the quadratures than in summertime; and the moon in its perigee every month produces higher tides than fifteen days before or after, when it is in its apogee. Accordingly, it happens that the two very highest tides do not follow each other in successive syzygies.

The effect of each luminary depends also on its declination, or distance from the equator. For if the luminary should be at one of the poles, it would constantly draw the individual parts of water, without intension and remission of action, and thus would produce no reciprocation of motion. Therefore the luminaries, in receding from the equator toward a pole, will lose their effects by degrees, and for this reason will produce lower tides in the solstitial syzygies than in the equinoctial syzygies. In the solstitial quadratures, however, they will produce higher tides than in the equinoctial quadratures, because the effect of the moon, which is now at the equator, most exceeds the effect of the sun. Therefore the highest tides occur at the syzygies of the luminaries, and the lowest at their quadratures, at about the times of either of the two equinoxes. And the highest tide in the syzygies is always acompanied by the lowest tide in the quadratures, as has been learned by experience. Moreover, as a result of the smaller distance of the sun from the earth in winter than in summer, it comes about that the highest and lowest tides more often precede the vernal equinox than follow it, and more often follow the autumnal equinox than precede it.

The effects of the luminaries depend also on the latitude of places. Let A*p*EP represent the earth covered everywhere with deep waters, C its center, P and *p* the poles, AE the equator, F any place not on the equator, F*f* the parallel of that place, D*d* the parallel corresponding to it on the other side of the equator, L the place that the moon was occupying three hours earlier, H the place on the earth situated perpendicularly beneath L, *h* the place opposite H, K and *k* places 90 degrees distant from H and *h*, CH and C*h* the greatest heights of the sea (measured from the center of the earth), and CK and C*k* the least heights. If an ellipse is described with axes H*h*, K*k*, and then if by the revolution of this ellipse about the major axis H*h* a spheroid HPK*hpk* is described, this spheroid will represent the figure of the sea very nearly, and CF, C*f*, CD, C*d* will be the heights of the sea at places F, *f*, D, *d*. Further, if in the aforesaid revolution of the ellipse any point N describes a circle NM which cuts parallels F*f*, D*d* in any places R, T, and cuts the equator AE in S, CN will be the height of the sea in all places R, S, T located on this circle. Hence, in the daily revolution of any place F, the greatest flood tide will be in F at the third hour after the appulse of the moon to the meridian above the horizon; afterward, the greatest ebb tide will be in Q at the third hour after the setting of the moon; then the greatest flood tide will be in *f* at the third hour after the appulse of the moon to the meridian below the horizon; finally, the greatest ebb tide will be in Q at the third hour after the rising of the moon; and the latter flood tide in *f* will be smaller than the former flood tide in F.



For the whole sea is divided into just two hemispherical flows [or flowing bodies of water], one in the hemisphere KH*k* verging to the north, the other in the opposite hemisphere K*hk*; and these may therefore be called the northern flow and the southern flow. These flowing bodies of waters, which are always opposite to each other, come by turns to the meridian of every single place, with an interval of twelve lunar hours between them. And since the northern regions partake more of the northern flow, and the southern regions more of the southern flow, higher and lower tides arise from them alternately, in every single place not on the equator in which the luminaries rise and set. Moreover, the higher tide, when the moon declines toward the vertex of the place, will occur at about the third hour after the appulse of the moon to the meridian above the horizon, and when the moon changes its declination[a], this higher tide will be turned into a lower one. And the greatest difference between these tides will occur at the times of the solstices, especially if the ascending node of the moon is in the first of Aries. Thus it has been found by experience that in winter, morning tides exceed evening tides and that in summer, evening tides exceed morning tides, at Plymouth by a height of about one foot, and at Bristol by a height of fifteen inches, according to the observations of Colepress and Sturmy.

Moreover, the motions hitherto described are changed somewhat by the force of reciprocation of the waters, by which a tide of the sea, even if the actions of the luminaries were to cease, would be able to persevere for a while. This conservation of impressed motion lessens the difference between alternate tides; and it makes the tides immediately after the syzygies higher and makes those immediately after the quadratures lower. Hence it happens that alternate tides at Plymouth and Bristol do not differ from each other by much more than a height of one foot or fifteen inches, and that the very highest tides in those same harbors are not the first tides after the syzygies but the third. All the motions are made slower also in their passing through shallows, to such an extent that the very highest tides, in certain straits and the mouths of rivers, are the fourth or even the fifth after the syzygies.

Further, it can happen that a tide is propagated from the ocean through different channels to the same harbor and passes more quickly through some channels than through others; in this case the same tide, divided into two or more tides arriving successively, can compose new motions of different kinds. Let us suppose that two equal tides come from different places to the same harbor and that the first precedes the second by a space of six hours and occurs at the third hour after the appulse of the moon to the meridian of the harbor. If the moon is on the equator at the time of this appulse to the meridian, then every six hours there will be equal flood tides coming upon corresponding equal ebb tides and causing those ebb tides to be balanced by the flood tides, and thus during the course of that day they will cause the water to stay quiet and still. If at that time the moon is declining from the equator, there will be alternately higher and lower tides in the ocean, as has been said; and from the ocean, two higher and two lower tides will each be alternately propagated toward this harbor. Moreover, the two greater flood

a. Motte adds, "to the other side of the equator."

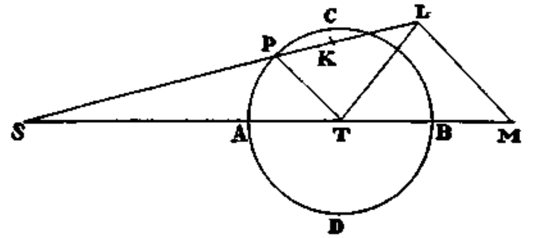
tides will produce the highest water in the middle time between them; the greater and lesser flood tides will make the water ascend to its mean height in the middle time between them; and between the two lesser flood tides the water will ascend to its least height. Thus in the space of twenty-four hours, the water will only once reach its greatest height, not twice as usually happens, and will only once reach its least height; and the greatest height, if the moon is declining toward the pole above the horizon of the place, will occur at either the sixth or the thirtieth hour after the appulse of the moon to the meridian; and when the moon changes its declination, this flood tide will be changed into an ebb tide. An example of all these things has been given by Halley, on the basis of sailors' observations, in Batsha harbor in the kingdom of Tonkin at a latitude of 20°50′ N. There the water stays still on the day following the transit of the moon over the equator; then, when the moon declines toward the north, the water begins to ebb and flow—not twice, as in other harbors, but only once every day; and the flood tide occurs at the setting of the moon, and the greatest ebb tide at its rising. This flood tide increases with the declination of the moon until the seventh or eighth day; then during the next seven days it decreases at the same rate at which it had previously increased. And when the moon changes its declination, the flood ceases and is then turned into an ebb. For thereafter the ebb tide occurs at the setting of the moon and the flood tide at its rising, until the moon again changes its declination. There are two different approaches from the ocean into this harbor and the neighboring channels, the one from the China Sea between the continent and the island of Leuconia, the other from the Indian Ocean between the continent and the island of Borneo. But whether there are tides coming through these channels in twelve hours from the Indian Ocean and in six hours from the China Sea, which thus occurring at the third and ninth lunar hours compound motions of this sort, or whether there is any other condition of those seas, I leave to be determined by observations of the neighboring shores.

Hitherto I have given the causes of the motions of the moon and seas. It is now proper to subjoin some things about the quantity of those motions.

Proposition 25
Problem 6

To find the forces of the sun that perturb the motions of the moon.

Let S designate the sun, T the earth, P the moon, CADB the orbit of the moon. On SP take SK equal to ST; and let SL be to SK as SK^2 to SP^2, and

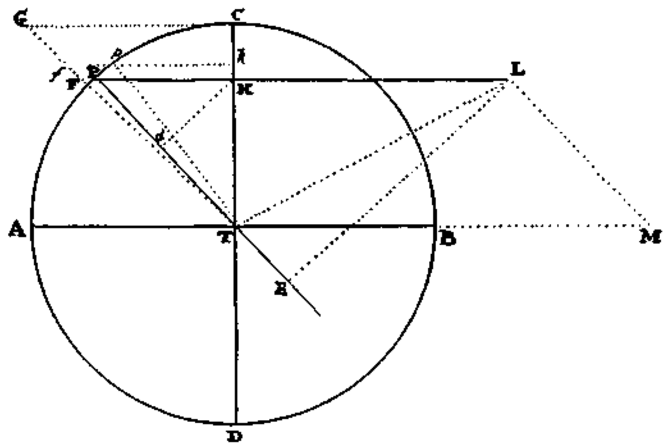


draw LM parallel to PT; and if the accelerative gravity of the earth toward the sun is represented by the distance ST or SK, SL will be the accelerative gravity of the moon toward the sun. This is compounded of the parts SM and LM, of which LM and the part TM of SM perturb the motion of the moon, as has been set forth in book 1, prop. 66 and its corollaries. Insofar as the earth and moon revolve around their common center of gravity, the motion of the earth about that center will also be perturbed by entirely similar forces; but it is possible to refer the sums of the forces and the sums of the motions to the moon, and to represent the sums of the forces by the lines TM and ML that correspond to them. The force ML, in its mean quantity, is to the centripetal force by which the moon could revolve in its orbit, about an earth at rest at a distance PT, as the square of the ratio of the periodic time of the moon about the earth to that of the earth about the sun (by book 1, prop. 66, corol. 17), that is, as the square of the ratio of $27^{d}7^{h}43^{m}$ to $365^{d}6^{h}9^{m}$, that is, as 1,000 to 178,725, or 1 to $178\frac{29}{40}$. But we found in prop. 4 of this book 3 that if the earth and moon revolve about their common center of gravity, their mean distance from each other will be very nearly $60\frac{1}{2}$ mean semidiameters of the earth. And the force by which the moon could revolve in orbit about the earth at rest at a distance PT of $60\frac{1}{2}$ terrestrial semidiameters is to the force by which it could revolve in the same time at a distance of 60 semidiameters as $60\frac{1}{2}$ to 60; and this force is to the force of gravity on the earth as 1 to 60×60 very nearly. And so the mean force ML is to the force of gravity on the surface of the earth as $1 \times 60\frac{1}{2}$ to $60 \times 60 \times 60 \times 178\frac{29}{40}$, or as 1 to 638,092.6. From this and from the proportion of the lines TM and ML, the force TM is also given; and these are the forces of the sun by which the motions of the moon are perturbed. Q.E.I.

Proposition 26
Problem 7

To find the hourly increase of the area that the moon, by a radius drawn to the earth, describes in a circular orbit.

We have said that the area which the moon describes by a radius drawn to the earth is proportional to the time, except insofar as the motion of the moon is disturbed by the action of the sun. We propose to investigate here the inequality of the moment, or of the hourly increase [under the foregoing condition of disturbance]. To make the computation easier, let us imagine



that the orbit of the moon is circular, and let us ignore all inequalities with the sole exception of the one under discussion here. Because of the enormous distance of the sun, let us suppose also that the lines SP and ST are parallel to each other. By this means the force LM will always be reduced to its mean quantity TP, and so will the force TM be reduced to its mean quantity 3PK. These forces (by corol. 2 of the laws of motion) compose the force TL; and if a perpendicular LE is dropped to the radius TP, this force is resolved into the forces TE and EL, of which TE, always acting along the radius TP, neither accelerates nor retards the description of the area TPC made by that radius TP; and EL, acting along the perpendicular to the radius, accelerates or retards the description of the area, as much as it accelerates or retards the moon. That acceleration of the moon, made in each individual moment of time, in the transit of the moon from the quadrature C to the conjunction A, is as the accelerating force itself EL, that is, as $\dfrac{3\text{PK} \times \text{TK}}{\text{TP}}$. Let the time be represented by the mean motion of the moon or (which comes to about the same thing) by the angle CTP or by the arc CP. On CT erect a normal CG (equal to CT). And when the quadrantal arc AC has been divided into innumerable equal particles $Pp, \ldots,$ by which the same innumerable quantity of equal particles of time can be represented, and when a perpendicular pk has been drawn to CT, draw TG meeting KP and kp (produced) in F and f; and FK will be equal to TK, and Kk will be to PK as Pp to Tp, that is, in a given ratio; and therefore FK × Kk, or the area FKkf, will be as $\frac{3\text{PK} \times \text{TK}}{\text{TP}}$, that is, as EL; and, by compounding, the total area GCKF will be as the sum of all the forces EL impressed on the moon in the total time CP, and so also as the velocity generated by this sum, that is, as the acceleration of the description of the area CTP, or the increase of its moment. The force by which the moon could revolve in its periodic time CADB of $27^{d}7^{h}43^{m}$ about the earth at rest, at the distance TP, would make a body, by falling in the time CT, describe the space ½CT, and at the same time acquire a velocity equal to the velocity with which the moon moves in its orbit. This is evident from book 1, prop. 4, corol. 9. However, since the perpendicular Kd dropped to TP is a third of EL, and is equal to a half of TP or ML in the octants, the force EL in the octants (where it is greatest) will exceed the force ML in the ratio of 3 to 2, and so will be to that force by which the moon could revolve in its periodic time about the earth at rest as 100 to ⅔ × 17,872½, or 11,915, and should in the time CT generate a velocity which would be $\frac{100}{11{,}915}$ of the moon's velocity; but in the time CPA this force would generate a greater velocity in the ratio of CA to CT or TP. Let the greatest force EL in the octants be represented by the area FK × Kk equal to the rectangle ½TP × Pp. And the velocity which that greatest force could generate in any time CP will be to the velocity which any other lesser force EL generates, in the same time, as the rectangle ½TP × CP to the area KCGF; but the velocities generated in the whole time CPA will be to each other as the rectangle ½TP × CA to the triangle TCG, or as the quadrantal arc CA to the radius TP. And so (by book 5, prop. 9 of the *Elements*) the latter velocity generated in the whole time will be $\frac{100}{11{,}915}$ of the velocity of the moon. Change this velocity of the moon, which corresponds to the mean moment of the area, by adding and subtracting half of the other velocity; and if the mean moment is represented by the number 11,915, the sum 11,915 + 50 (or 11,965) will represent the greatest moment of the area in the syzygy A, and the difference 11,915 − 50 (or 11,865) the least moment of the same area in the quadratures. Therefore the areas which are described in equal times in the syzygies and quadratures are to each other as 11,965 to 11,865. To the least moment 11,865 add the moment that is to the difference (100) of the two above-mentioned moments as the quadrilateral FKCG is to the triangle TCG or, which comes to the same thing, as the square of the sine PK to the square of the radius TP (that is, as P*d* to TP); then the sum will represent the moment of the area when the moon is in any intermediate place P.

All these things are so on the hypothesis that the sun and earth are at rest, and that the moon has a synodic period of revolution of $27^{d}7^{h}43^{m}$. But since the moon's synodic period is actually $29^{d}12^{h}44^{m}$, the increments of the moments should be increased in the ratio of the time, that is, in the ratio of 1,080,853 to 1,000,000. In this way the total increment, which was $\frac{100}{11,915}$ of the mean moment, will now become $\frac{100}{11,023}$ of it. And so the moment of the area in the quadrature of the moon will be to its moment in the syzygy as $11,023-50$ to $11,023+50$, or as 10,973 to 11,073; and to its moment, when the moon is in any other intermediate place P, as 10,973 to 10,973 + P*d*, taking TP to be equal to 100.

Therefore the area that the moon, by a radius drawn to the earth, describes in every equal particle of time is very nearly as the sum of the number 219.46 and of the versed sine of twice the distance of the moon from the nearest quadrature, with respect to a circle whose radius is unity. These things are so when the variation in the octants is at its mean magnitude. But if the variation there is greater or less, that versed sine should be increased or decreased in the same ratio.

Proposition 27
Problem 8

From the hourly motion of the moon, to find its distance from the earth.

The area that the moon, by a radius drawn to the earth, describes in every moment of time is as the hourly motion of the moon and the square of the distance of the moon from the earth jointly. And therefore the distance of the moon from the earth is directly proportional to the square root of the area and inversely proportional to the square root of the hourly motion. Q.E.I.

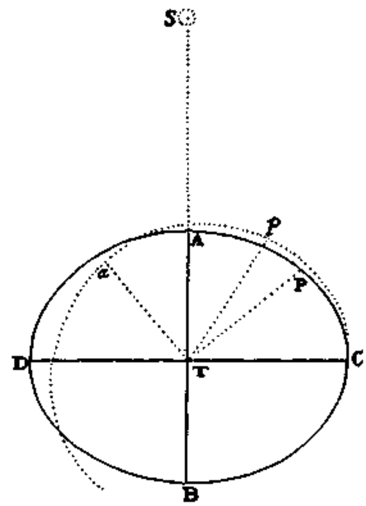
COROLLARY 1. Hence the apparent diameter of the moon is given, since it is inversely as the distance of the moon from the earth. Let astronomers test how accurately this rule agrees with the phenomena.

COROLLARY 2. Hence also the lunar orbit can be defined more exactly from the phenomena than could have been done before now.

Proposition 28
Problem 9
To find the diameters of the orbit in which the moon would have to move, if there were no eccentricity.

The curvature of the trajectory that a moving body describes, if it is attracted in a direction which is everywhere perpendicular to that trajectory, is as the attraction directly and the square of the velocity inversely. I reckon the curvatures of lines as being among themselves in the ultimate ratio of the sines or of the tangents of the angles of contact, with respect to equal radii, when those radii are diminished indefinitely. Now, the attraction of the moon toward the earth in the syzygies is the excess of its gravity toward the earth over the solar force 2PK (as in the figure to prop. 25), by which force the accelerative gravity of the moon toward the sun exceeds the accelerative gravity of the earth toward the sun or is exceeded by it. In the quadratures that attraction is the sum of the gravity of the moon toward the earth and the solar force KT (which draws the moon toward the earth). And these attractions, if $\frac{AT + CT}{2}$ is called N, are very nearly as $\frac{178{,}725}{AT^2} - \frac{2{,}000}{CT \times N}$ and $\frac{178{,}725}{CT^2} + \frac{1{,}000}{AT \times N}$, or as $178{,}725N \times CT^2 - 2{,}000AT^2 \times CT$ and $178{,}725N \times AT^2 + 1{,}000CT^2 \times AT$. For if the accelerative gravity of the moon toward the earth is represented by the number 178,725, then the mean force ML, which in the quadratures is PT or TK and draws the moon toward the earth, will be 1,000, and the mean force TM in the syzygies will be 3,000; if the mean force ML is subtracted from that, there will remain the force 2,000 by which the moon in the syzygies is drawn apart from the earth and which I have called 2PK above. Now, the velocity of the moon in the syzygies (A and B) is to its velocity in the quadratures (C and D) jointly as CT is to AT and as the moment of the area that the moon (by a radius drawn to the earth) describes in the syzygies is to the moment of that same area as described in the quadratures, that is, as 11,073CT to 10,973AT. Take this ratio squared inversely and the above ratio directly, and the curvature of the moon's orbit in the syzygies will become to its curvature in the quadratures as $120{,}406{,}729 \times 178{,}725AT^2 \times CT^2 \times N - 120{,}406{,}729 \times 2{,}000AT^4 \times CT$ to $122{,}611{,}329 \times 178{,}725AT^2 \times CT^2 \times N + 122{,}611{,}329 \times 1{,}000CT^4 \times AT$, that is, as $2,151,969AT \times CT \times N - 24,081AT^3$ to $2,191,371AT \times CT \times N + 12,261CT^3$.

Since the figure of the lunar orbit is unknown, in its place let us assume an ellipse DBCA, in whose center T the earth is placed, and let its major axis DC lie between the quadratures and its minor axis AB between the syzygies. And since the plane of this ellipse revolves about the earth with an angular motion, and since the trajectory whose curvature we are considering ought to be described in a plane that is entirely devoid of any angular motion, we must consider the figure that the moon, while revolving in that ellipse, describes in this place, that is, the figure C*pa*, whose individual points *p* are found by taking any point P on the ellipse to represent the place of the moon, and by drawing T*p* equal to TP in such a way that the angle PT*p* is equal to the apparent motion of the sun since the time of quadrature C, or (which comes to almost the same thing) in such a way that the angle CT*p* is to the angle CTP as the time of a synodic revolution of the moon is to the time of a periodic revolution, or as $29^d12^h44^m$ to $27^d7^h43^m$. Therefore, take the angle CT*a* in this same ratio to the right angle CTA, and let the length T*a* be equal to the length TA, then *a* will be the lower apsis and *c* the upper apsis of this orbit C*pa*. And by making calculations I find that the difference between the curvature of the orbit C*pa* at the vertex *a* and the curvature of the circle described with center T and radius TA has a ratio to the difference between the curvature of the ellipse at the vertex A and the curvature of that circle which is equal to the ratio of the square of the angle CTP to the square of the angle CT*p* and that the curvature of the ellipse at A is to the curvature of that circle in the ratio of TA^2 to TC^2; and the curvature of that circle is to the curvature of a circle described with center T and radius TC as TC to TA; but this curvature is to the curvature of the ellipse at C in the ratio of TA^2 to TC^2; and the difference between the curvature of the ellipse at the vertex C and the curvature of this last circle is to the difference between the curvature of the figure T*pa* at the vertex C and the curvature of the



same circle in the ratio of the square of the angle CTp to the square of the angle CTP. And these ratios are easily gathered from the sines of the angles of contact and of the differences of the angles. Moreover, by comparing these, the curvature of the figure Cpa at a comes out to its curvature at C as $AT^3 + \frac{16,824}{100,000}CT^2 \times AT$ to $CT^3 + \frac{16,824}{100,000}AT^2 \times CT$; where the factor $\frac{16,824}{100,000}$ represents the difference of the squares of the angles CTP and CTp divided by the square of the smaller angle CTP, or (which is the same) the difference of the squares of the times $27^d7^h43^m$ and $29^d12^h44^m$ divided by the square of the time $27^d7^h43^m$.

Therefore, since a designates the syzygy of the moon and C its quadrature, the proportion just found must be the same as the proportion of the curvature of the orbit of the moon in the syzygies to its curvature in the quadratures, which we found above. Accordingly, to find the proportion of CT to AT, I multiply the extremes by the means. And the resulting terms divided by AT × CT become $2,062.79CT^4 - 2,151,969N \times CT^3 + 368,676N \times AT \times CT^2 + 36,342AT^2 \times CT^2 - 362,047N \times AT^2 \times CT + 2,191,371N \times AT^3 + 4,051.4AT^4 = 0$. When I take the half-sum N of the terms AT and CT to be 1, and their half-difference to be x, there results $CT = 1 + x$ and $AT = 1 - x$; and when these values are put into the equation and the resulting equation is resolved, x is found equal to 0.00719, and hence the semidiameter CT comes out 1.00719 and the semidiameter AT 0.99281. These numbers are very nearly as $70\frac{1}{24}$ and $69\frac{1}{24}$. Therefore the distance of the moon from the earth in the syzygies is to its distance in the quadratures (setting aside, that is, any consideration of eccentricity) as $69\frac{1}{24}$ to $70\frac{1}{24}$, or in round numbers as 69 to 70.

Proposition 29 *To find the variation of the moon.*

Problem 10 This inequality arises partly from the elliptical form of the orbit of the moon and partly from the inequality of the moments of the area that the moon describes by a radius drawn to the earth. If the moon P moved in the ellipse DBCA about the earth at rest in the center of the ellipse and, by a radius TP drawn to the earth, described the area CTP proportional to the time, and if furthermore the greatest semidiameter CT of the ellipse were to the least semidiameter TA as 70 to 69, then the tangent of the angle CTP would be to the tangent of the angle of the mean motion (reckoned from the quadrature C) as the semidiameter TA of the ellipse to its semidiameter TC, or as 69 to 70. Moreover, the description of the area CTP ought, in the progress of the moon from quadrature to syzygy, to be accelerated in such a way that the moment of this area in the syzygy of the moon will be to its moment in its quadrature as 11,073 to 10,973, and in such a way that the excess of the moment in any intermediate place P over the moment in the quadrature will be as the square of the sine of the angle CTP. And this will occur exactly enough if the tangent of angle CTP is diminished in the ratio of $\sqrt{10{,}973}$ to $\sqrt{11{,}073}$, or in the ratio of 68.6877 to 69. In this way the tangent of angle CTP will now be to the tangent of the mean motion as 68.6877 to 70; and the angle CTP in the octants, where the mean motion is 45°, will be found to be 44°27′28″, which, when subtracted from the angle of the mean motion of 45°, leaves the greatest variation 32′32″. These things would be so if the moon, in going from quadrature to syzygy, described an angle CTA of only 90°. But because of the motion of the earth, by which the sun is transferred forward [or in consequentia] in its apparent motion, the moon, before it reaches the sun, describes an angle CT*a* greater than a right angle, in the ratio of the time of a synodic revolution of the moon to the time of its periodic revolution, that is, in the ratio of $29^{d}12^{h}44^{m}$ to $27^{d}7^{h}43^{m}$. And in this way all the angles about the center T are enlarged in the same ratio; and the greatest variation, which would otherwise be 32′32″, now increased in the same ratio, becomes 35′10″.

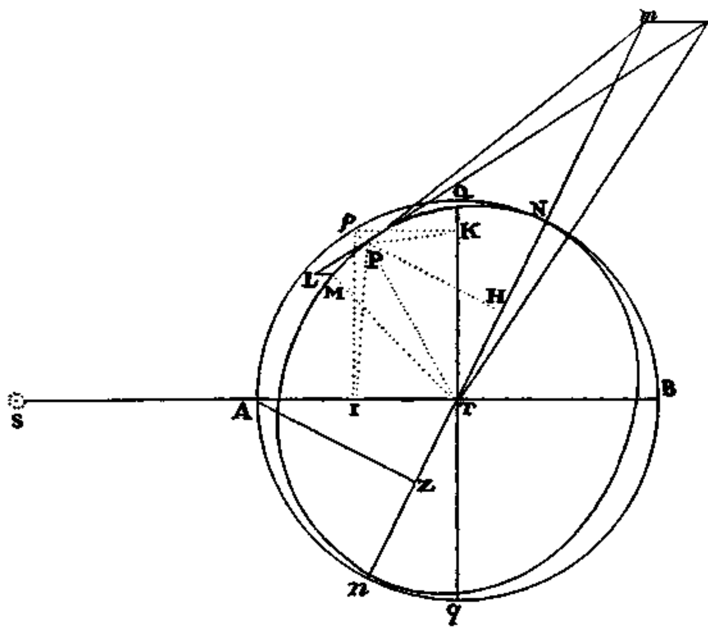
This is the magnitude of the greatest variation at the mean distance of the sun from the earth, ignoring the differences that can arise from the curvature of the earth's orbit and the greater action of the sun upon the sickle-shaped and the new moon than upon the gibbous and the full moon. At other distances of the sun from the earth, the greatest variation is directly as the square of the time of synodic revolution and inversely as the cube of the distance of the sun from the earth. And therefore in the apogee of the sun the greatest variation is 33′14″, and in its perigee 37′11″, provided that the eccentricity of the sun is to the transverse semidiameter of the great orbit [i.e., the earth's orbit] as $16\frac{15}{16}$ to 1,000.

Hitherto we have investigated the variation in a noneccentric orbit, in which the moon in its octants is always at its mean distance from the earth. If the moon, because of its eccentricity, is more distant or less distant from the earth than if it were placed in this orbit, the variation can be a little greater or a little less than according to the rule asserted here; but I leave the excess or deficiency for astronomers to determine from phenomena.

Proposition 30
Problem 11

To find the hourly motion of the nodes of the moon in a circular orbit.

Let S designate the sun, T the earth, P the moon, NP*n* the orbit of the moon, N*pn* the projection of the orbit in the plane of the ecliptic; N and *n* the nodes, *n*TN*m* the line of the nodes, indefinitely produced; PI and PK perpendiculars dropped to the lines ST and Q*q*; P*p* a perpendicular



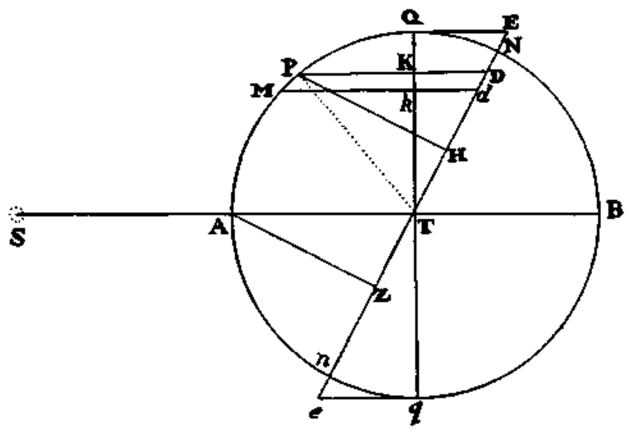
dropped to the plane of the ecliptic; A and B the syzygies of the moon in the plane of the ecliptic; AZ a perpendicular to the line of the nodes N*n*; Q and *q* the quadratures of the moon in the plane of the ecliptic; and *p*K a perpendicular to the line Q*q*, which lies between the quadratures. The force of the sun to perturb the motion of the moon has (by prop. 25) two components, one proportional to the line LM in the figure of that proposition, the other proportional to the line MT in that same figure. And the moon is attracted toward the earth by the first of these forces, and by the second it is attracted toward the sun along a line parallel to the straight line ST drawn from the earth to the sun. The first force LM acts in the plane of the moon's orbit and therefore can make no change in the position of that plane. Therefore this force is to be ignored. The second force MT, by which the plane of the lunar orbit is perturbed, is the same as the force 3PK or 3IT. And this force (by prop. 25) is to the force by which the moon could revolve uniformly in a circle in its periodic time about the earth at rest as 3IT to the radius of the circle multiplied by the number 178.725, or as IT to the radius multiplied by 59.575. But in this calculation and in what follows, I consider all lines drawn from the moon to the sun to be parallel to the line drawn from the earth to the sun, because the inclination diminishes all effects in some cases nearly as much as it increases them in others; and we are here seeking the mean motions of the nodes, ignoring those niceties of detail which would make the calculation too cumbersome.

Now let PM represent the arc that the moon describes in a minimally small given time, and ML the line-element one-half of which the moon could describe in the same time by the impulse of the above-mentioned force 3IT. Draw PL and MP, and produce them to *m* and *l*, and let them cut the plane of the ecliptic there, and upon T*m* drop the perpendicular PH. Since the straight line ML is parallel to the plane of the ecliptic and so cannot meet with the straight line *ml* (which lies in that plane) and yet these straight lines lie in a common plane LMP*ml*, these straight lines will be parallel, and therefore the triangles LMP and *lm*P will be similar. Now, since MP*m* is in the plane of the orbit in which the moon was moving while in place P, the point *m* will fall upon the line N*n* drawn through the nodes N and *n* of that orbit. The force by which half of the line-element LM is generated—if all of it were impressed all at once in place P—would generate that whole line and would cause the moon to move in an arc whose chord would be LP, and so would transfer the moon from the plane MP*m*T into the plane LP*l*T; therefore the angular motion of the nodes that is generated by that force will be equal to the angle *m*T*l*. Moreover, *ml* is to *m*P as ML is to MP, and so, since MP is given (because the time is given), *ml* is as the rectangle ML × *m*P, that is, as the rectangle IT × *m*P. And, provided that the angle T*ml* is a right angle, the angle *m*T*l* is as $\frac{ml}{Tm}$, and therefore as $\frac{IT \times Pm}{Tm}$, that is (because T*m* is to *m*P as TP is to PH), as $\frac{IT \times PH}{TP}$; and so, because TP is given, as IT × PH. But if the angle T*ml* or STN is oblique, the angle *m*T*l* will be still smaller, in the ratio of the sine of the angle STN to the radius, or of AZ to AT. Therefore the velocity of the nodes is as IT × PH × AZ, or as the solid contained by [or the product of] the sines of the three angles TPI, PTN, and STN.

If those angles are right angles, as happens when the nodes are in the quadratures and the moon is in the syzygy, the line-element *ml* will go off indefinitely and the angle *m*T*l* will become equal to the angle *m*P*l*. But in this case the angle *m*P*l* is to the angle PTM, which the moon describes about the earth in the same time by its apparent motion, as 1 to 59.575. For the angle *m*P*l* is equal to the angle LPM, that is, to the angle of the deflection of the moon from the straight-line path that the aforesaid solar force 3IT could generate by itself in that given time, if the gravity of the moon were then to cease; and the angle PTM is equal to the angle of the deflection of the moon from the straight-line path that the force by which the moon is kept in its orbit would generate in the same time, if the solar force 3IT were then to cease. And these forces, as we have said above, are to each other as 1 to 59.575. Therefore, since the mean hourly motion of the moon with respect to the fixed stars is $32'56''27'''12^{\text{iv}}\frac{1}{2}$, the hourly motion of the node in this case will be $33''10'''33^{\text{iv}}12^{\text{v}}$. But in other cases this hourly motion will be to $33''10'''33^{\text{iv}}12^{\text{v}}$ as the solid contained by [or the product of] the sines of the three angles TPI, PTN, and STN (or the distance of the moon from the quadrature, of the moon from the node, and of the node from the sun) to the cube of the radius. And whenever the sign of any of the angles is changed from positive to negative and from negative to positive, retrograde motion will have to be changed into progressive motion, and progressive into retrograde. Hence it happens that the nodes advance whenever the moon is between either of the quadratures and the node nearest to that quadrature. In other cases, the nodes are retrograde, and they are carried backward [or in antecedentia] each month by the excess of the retrograde motion over the progressive.

Corollary 1. Hence, if from the ends P and M of the minimally small given arc PM, the perpendiculars PK and M*k* are dropped to the line Q*q* that joins the quadratures, and these perpendiculars are produced until they cut the line of the nodes N*n* in D and *d*, then the hourly motion of the nodes will be as the area MPD*d* and the square of the line AZ jointly. For let PK, PH, and AZ be the above-mentioned three sines—namely, PK the sine of the distance of the moon from the quadrature, PH the sine of the distance of the moon from the node, and AZ the sine of the distance of the node from the sun—then the velocity of the node will be as the solid [or product] PK × PH × AZ. But PT is to PK as PM to K*k*, and so, because PT and PM are given, K*k* is proportional to PK. Also, AT is to PD as AZ to PH, and therefore PH is proportional to the rectangle PD × AZ; and, combining these ratios, PK × PH is as the solid K*k* × PD × AZ, and PK × PH × AZ is as K*k* × PD × AZ^2, that is, as the area PD*d*M and AZ^2 jointly. Q.E.D.

Corollary 2. In any given position of the nodes, the mean hourly motion is half of their hourly motion in the moon's syzygies, and thus is to $16''55'''16^{iv}36^{v}$ as the square of the sine of the distance of the nodes from the syzygies is to the square of the radius, or as AZ^2 to AT^2. For if the moon



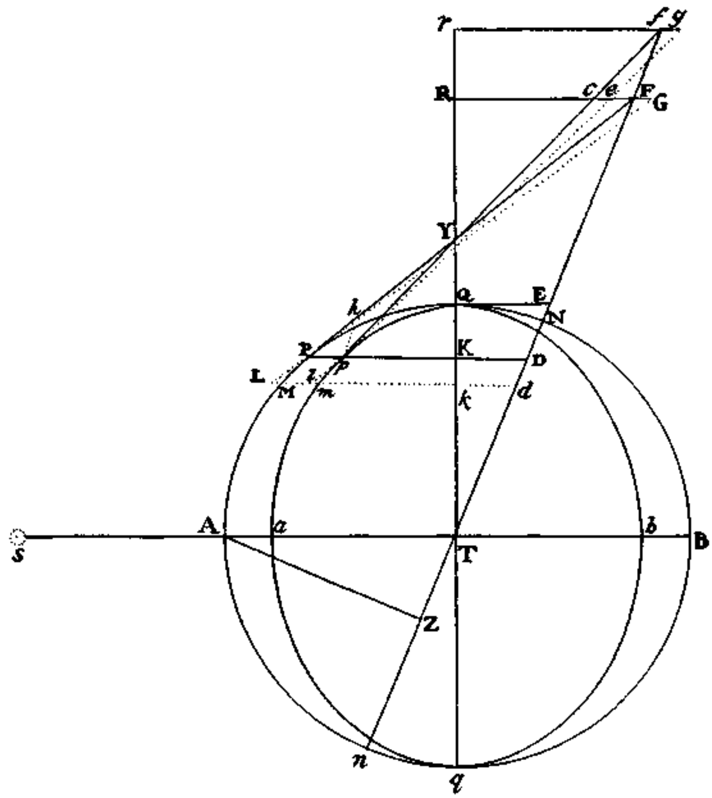
traverses the semicircle QA*q* with uniform motion, the sum of all the areas PD*d*M during the time in which the moon goes from Q to M will be the area QM*d*E, which is terminated at the tangent QE of the circle; and in the time in which the moon reaches point *n*, that sum will be the total area EQA*n*, which the line PD describes; then as the moon goes from *n* to *q*, the line PD will fall outside the circle and will describe the area *nqe* (which is terminated at the tangent *qe* of the circle)—which, since the nodes were previously retrograde but now are progressive, must be subtracted from the former area, and (since it is equal to the area QEN) will leave the semicircle NQA*n*. Therefore, during the time in which the moon describes a semicircle, the sum of all the areas PD*d*M is the area of that semicircle; and in the time in which the moon describes a circle, the sum of all those areas is the area of the whole circle. But the area PD*d*M, when the moon is in the syzygies, is the rectangle of the arc PM and the radius PT; and in the time in which the moon describes a circle, the sum of all the areas that are equal to this one is the rectangle of the whole circumference and the radius of the circle; and this rectangle, since it is equal to two circles, is twice as large as the former rectangle. Accordingly, if the nodes moved with the same velocity uniformly continued that they have in the lunar syzygies, they would describe a space twice as large as the space which they really describe; and therefore the mean motion—with which, if it were continued uniformly, they would describe the space that they really cover with their nonuniform motion—is one-half of the motion which they have in the moon's syzygies. Hence, since the greatest hourly motion of the nodes, if the nodes are in the quadratures, is $33''10'''33^{iv}12^{v}$, their mean hourly motion in this case will be $16''35'''16^{iv}36^{v}$. And since the hourly motion of the nodes is always as AZ^2 and the area PD*d*M jointly, and therefore the hourly motion of the nodes in the moon's syzygies is as AZ^2 and the area PD*d*M jointly, that is (because the area PD*d*M described in the syzygies is given), as AZ^2, the mean motion will also be as AZ^2; and hence this motion, when the nodes are outside the quadratures, will be to $16''35'''16^{iv}36^{v}$ as AZ^2 to AT^2. Q.E.D.

Proposition 31
Problem 12

To find the hourly motion of the nodes of the moon in an elliptical orbit.

Let Q*pmaq* represent an ellipse described with a major axis Q*q* and a minor axis *ab*, QA*q*B a circle circumscribed about this ellipse, T the earth in the common center of both, S the sun, *p* the moon moving in the ellipse, and *pm* the arc that the moon describes in a minimally small given particle of time, N and *n* the nodes joined by the line N*m*, *p*K and *mk* perpendiculars dropped to the axis Q*q* and produced on both sides until they meet the circle at P and M and the line of the nodes at D and *d*. And if the moon, by a radius drawn to the earth, describes an area proportional to the time, the hourly motion of the node in the ellipse will be as the area *p*D*dm* and AZ^2 jointly.

To demonstrate this, let PF touch the circle at P and, produced, meet TN at F; let *pf* touch the ellipse at *p* and, produced, meet the same TN at *f*; and let these tangents come together on the axis TQ at Y. And let ML designate the space that the moon, revolving in a circle, would describe by a transverse motion under the action and impulse of the aforesaid force 3IT or



3PK, while it describes the arc PM; and let *ml* designate the space that the moon, revolving in an ellipse, could describe in the same time, also under the action of the force 3IT or 3PK. Further, let L*p* and *lp* be produced until they meet the plane of the ecliptic at G and *g*; and let FG and *fg* be drawn, of which let FG produced cut *pf*, *pg*, and TQ at *c*, *e*, and R respectively; and let *fg* produced cut TQ at *r*. Then, since the force 3IT or 3PK in the circle is to the force 3IT or 3*p*K in the ellipse as PK is to *p*K, or as AT to *a*T, the space ML generated by the first force will be to the space *ml* generated by the second force as PK to *p*K, that is (because the figures PYK*p* and FYR*c* are similar), as FR to *c*R. Moreover, ML is to FG (because the triangles PLM and PGF are similar) as PL to PG, that is (because L*k*, PK, and GR are parallel), as *pl* to *pe*, that is (because the triangles *plm* and *cpe* are similar), as *lm* to *ce*; and thus LM is to *lm*, or FR is to *c*R, as FG is to *ce*. And therefore if *fg* were to *ce* as *f*Y to *c*Y, that is, as *fr* to *c*R (that is, as *fr* to FR and FR to *c*R jointly, that is, as *f*T to FT and FG to ce jointly), then, since the ratio FG to ce taken away from both sides leaves the ratios fg to FG and fT to FT, the ratio fg to FG would be as fT to FT, and so the angles that FG and fg would subtend at the earth T would be equal to each other. But these angles (by what we have set forth in the preceding prop. 30) are the motions of the nodes in the time in which the moon traverses the arc PM in the circle, and the arc pm in the ellipse; and therefore the motions of the nodes in the circle and in the ellipse would be equal to each other. These things would be so, if only fg were to ce as fY to cY, that is, if fg were equal to $\frac{ce \times f\mathrm{Y}}{c\mathrm{Y}}$. But because the triangles fgp and cep are similar, fg is to ce as fp to cp, and so fg is equal to $\frac{ce \times fp}{cp}$; and therefore the angle that fg really subtends is to the former angle that FC subtends (that is, the motion of the nodes in the ellipse is to the motion of the nodes in the circle) as this fg or $\frac{ce \times fp}{cp}$ to the former fg or $\frac{ce \times f\mathrm{Y}}{c\mathrm{Y}}$, that is, as $fp \times c\mathrm{Y}$ to $f\mathrm{Y} \times cp$, or as fp to fY and cY to cp; that is (if ph, parallel to TN, meets FP at h), as Fh to FY and FY to FP; that is, as Fh to FP or Dp to DP, and so as the area Dpmd to the area DPMd. And therefore, since (by prop. 30, corol. 1) the latter area and AZ^2 jointly are proportional to the hourly motion of the nodes in the circle, the former area and AZ^2 jointly will be proportional to the hourly motion of the nodes in the ellipse. Q.E.D.

COROLLARY. Therefore, since in any given position of the nodes, the sum of all the areas pDdm, in the time in which the moon goes from the quadrature to any place m, is the area mpQEd, which is terminated at the tangent QE of the ellipse, and the sum of all those areas in a complete revolution is the area of the whole ellipse, the mean motion of the nodes in the ellipse will be to the mean motion of the nodes in the circle as the ellipse to the circle, that is, as Ta to TA, or as 69 to 70. And therefore, since (by prop. 30, corol. 2) the mean hourly motion of the nodes in the circle is to $16''35'''16^{\mathrm{iv}}36^{\mathrm{v}}$ as AZ^2 to AT^2 if the angle $16''21'''3^{\mathrm{iv}}30^{\mathrm{v}}$ is taken to the angle $16''35'''16^{\mathrm{iv}}36^{\mathrm{v}}$ as 69 to 70, the mean hourly motion of the nodes in the ellipse will be to $16''21'''3^{\mathrm{iv}}30^{\mathrm{v}}$ as AZ^2 to AT^2, that is, as the square of the sine of the distance of the node from the sun to the square of the radius.

But the moon, by a radius drawn to the earth, describes an area more swiftly in the syzygies than in the quadratures, and on that account the time is shortened in the syzygies and lengthened in the quadratures, and along with the time the motion of the nodes is increased and decreased. Now, the moment of an area in the quadratures of the moon was to its moment in the syzygies as 10,973 to 11,073; and therefore the mean motion in the octants is to the excess in the syzygies and to the deficiency in the quadratures as the half-sum 11,023 of the numbers is to their half-difference 50. Accordingly, since the time of the moon in each equal particle of its orbit is inversely as its velocity, the mean time in the octants will be to the excess of time in the quadratures and its deficiency in the syzygies, arising from this cause, as 11,023 to 50 very nearly. With regard to positions of the moon between the quadratures and the syzygies, I find that the excess of the moments of the area in any one of these positions over the least moment in the quadratures is very nearly as the square of the sine of the distance of the moon from the quadratures; and therefore the difference between the moment in any position and the mean moment in the octants is as the difference between the square of the sine of the distance of the moon from the quadratures and the square of the sine of 45°, or half of the square of the radius; and the increase of the time in any one of the positions between the octants and the quadratures, and its decrease between the octants and the syzygies, is in the same ratio. But the motion of the nodes, in the time in which the moon traverses each equal particle of its orbit, is accelerated or retarded as the square of the time.

For that motion, while the moon traverses PM, is (other things being equal) as ML, and ML is in the squared ratio of the time. Therefore, the motion of the nodes in the syzygies, a motion completed in the time in which the moon traverses given particles of its orbit, is diminished as the square of the ratio of the number 11,073 to the number 11,023; and the decrement is to the remaining motion as 100 to 10,973 and to the total motion as 100 to 11,073 very nearly. But the decrement in positions between the octants and syzygies and the increment in positions between the octants and quadratures are to this decrement very nearly as [*i*] the total motion in those positions to the total motion in the syzygies and as [*ii*] the difference between the square of the sine of the distance of the moon from the quadrature and half of the square of the radius to half of the square of the radius, jointly. Hence, if the nodes are in the quadratures and two positions are taken equally distant from the octant, one on one side and one on the other, and another two are taken at the same distance from the syzygy and from the quadrature, and if from the decrements of the motions in the two positions between the syzygy and octant are subtracted the increments of the motions in the remaining two positions that are between the octant and quadrature, the remaining decrement will be equal to the decrement in the syzygy, as will be easily apparent upon examination. And accordingly the mean decrement, which must be subtracted from the mean motion of the nodes, is a fourth of the decrement in the syzygy. The total hourly motion of the nodes in the syzygies (when it was supposed the moon described, by a radius drawn to the earth, an area proportional to the time) was $32''42'''7^{\text{iv}}$. And according to what we have just said, the decrement of the motion of the nodes, in the time when the moon—now moving more swiftly—describes the same space, is to this motion as 100 to 11,073; and so the decrement is $17'''43^{\text{iv}}11^{\text{v}}$, of which a fourth ($4'''25^{\text{iv}}48^{\text{v}}$) subtracted from the mean hourly motion found above ($16''21'''3^{\text{iv}}30^{\text{v}}$) leaves $16''16'''37^{\text{iv}}42^{\text{v}}$, the corrected mean hourly motion.

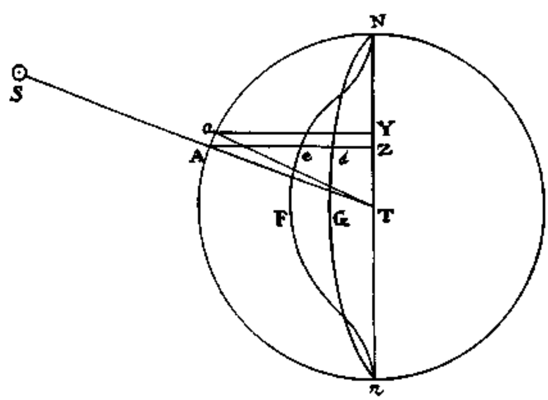
If the nodes are beyond the quadratures and two places equally distant from the syzygies are considered, one on one side and one on the other, the sum of the motions of the nodes when the moon is in these positions will be to the sum of their motions when the moon is in the same positions and the nodes are in the quadratures as AZ^2 to AT^2. And the decrements of the motions, arising from the causes just now set forth, will be to each other as the motions themselves, and therefore the remaining motions will be to each other as AZ^2 to AT^2, and the mean motions will be as the remaining motions. Therefore the corrected mean hourly motion, in any given situation of the nodes, is to $16''16'''37^{\text{iv}}42^{\text{v}}$ as AZ^2 to AT^2, that is, as the square of the sine of the distance of the nodes from the syzygies to the square of the radius.

Proposition 32
Problem 13

To find the mean motion of the nodes of the moon.

The mean annual motion is the sum of all the mean hourly motions in a year. Suppose that the node is in N and that as each hour is completed, it is drawn back into its former place so that, notwithstanding its own proper motion, it always maintains some given position with respect to the fixed stars.

And suppose that during this same time the sun S, as a result of the motion of the earth, advances from the node and completes its apparent annual course with a uniform apparent motion. Moreover, let A*a* be the minimally small given arc that the straight line TS, always drawn to the sun, describes in a minimally small given time by its intersection with the circle NA*n*; then (by what has already been shown) the mean hourly motion will be as AZ^2, that is (because AZ and ZY are proportional), as the rectangle of AZ and ZY, that is, as the area AZY*a*. And the sum of all the mean hourly motions from the beginning will be as the sum of all the areas *a*YZA, that is, as the area NAZ. Moreover, the greatest area AZY*a* is equal to the rectangle of the arc A*a* and the radius of the circle; and therefore the sum of all such rectangles in the whole circle will be to the sum of the same number of greatest rectangles as the area of the whole circle to the rectangle of the whole circumference and the radius, that is, as 1 to 2. Now, the hourly motion corresponding to the greatest rectangle was $16''16'''37^{\text{iv}}42^{\text{v}}$, and this motion, in a whole sidereal year of $365^{\text{d}}6^{\text{h}}9^{\text{m}}$, adds up to $39°38'7''50'''$. And so half of this, $19°49'3''55'''$, is the mean motion of the nodes that corresponds to the whole circle. And the motion of the nodes in the time during which the sun goes from N to A is to $19°49'3''55'''$ as the area NAZ is to the whole circle.



These things are so on the hypothesis that each hour the node is drawn back to its former place, in such a way that when a whole year is completed, the sun returns to the same node from which it had initially departed. But as a result of the motion of that node, it comes about that the sun returns to the node more quickly; and now this shortening of the time must be computed. Since in a total year the sun travels through 360°, and in the same time the node with its greatest motion would travel through 39°38′7″50‴, or 39.6355°, and the mean motion of the node in any place N is to its mean motion in its quadratures as AZ^2 to AT^2, the motion of the sun will be to the motion of the node in N as $360AT^2$ to $39.6355AZ^2$, that is, as $9.0827646AT^2$ to AZ^2. Hence, if the circumference N*An* of the whole circle is divided into equal particles A*a*, then the time in which the sun traverses the particle A*a* (the circle being at rest) will be to the time in which it traverses the same particle (if the circle revolves along with the nodes about the center T) inversely as $9.0827646AT^2$ to $9.0827646AT^2 + AZ^2$. For the time is inversely as the velocity with which the particle [of arc] is traversed, and this velocity is the sum of the velocities of the sun and of the node. Let the sector NTA represent the time in which the sun, without the motion of the node, would traverse the arc NA, and let the particle AT*a* of the sector represent the particle of time in which it would traverse the minimally small arc A*a*; furthermore, drop a perpendicular *a*Y to N*n* and on AZ take *d*Z of a length such that the rectangle of *d*Z and ZY is to the particle AT*a* of the sector as AZ^2 is to $9.0827646AT^2 + AZ^2$ (that is, such that *d*Z is to ½AZ as AT^2 is to $9.0827646AT^2 + AZ^2$); then the rectangle of *d*Z and ZY will designate the decrement of time arising from the motion of the node during the total time in which the arc A*a* is traversed. And if the point *d* touches the curve N*d*G*n*,[a] the curvilinear area N*d*Z will be the total decrement in the time in which the whole arc NA is traversed; and therefore the excess of the sector NAT over the area NAZ will be that total time. And since the motion of the node in a smaller time is smaller in proportion to the time, the area A*a*YZ also will have to be diminished in the same proportion. And this will happen if on AZ the length *e*Z is taken, which is to the length AZ as AZ^2 is to $9.0827646AT^2 + AZ^2$. For thus the rectangle of *e*Z and ZY will be to the area AZY*a* as the decrement of the time in which the arc A*a* is traversed is to the total time in which it would be traversed if the node were at rest; and therefore that rectangle will correspond to the decrement of the motion of the node. And if the point *e* touches the curve N*e*F*n*,[b] the total area N*e*Z, which is the sum of all the decrements of that motion, will correspond to the total decrement in the time during which the arc AN

a. That is, if the point *d* traces out the curve N*d*G*n*, or if N*d*G*n* is the curve which is the locus of the point *d*.

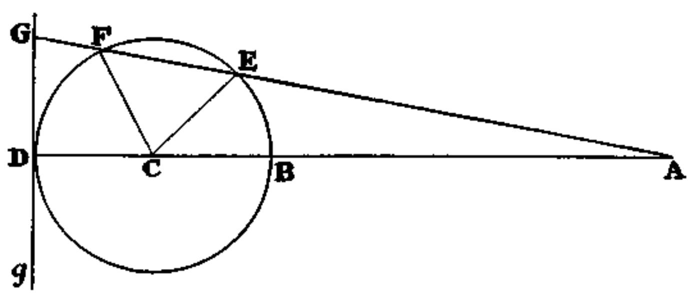
b. That is, if the curve N*e*F*n* is the locus of point *e*.

is traversed, and the remaining area NAe will correspond to the remaining motion, which is the true motion of the node in the time in which the total arc NA is traversed by the joint motions of the sun and the node. Now, the area of the semicircle is to the area of the figure NeFn, found by the method of infinite series, nearly as 793 to 60. And the motion that corresponds to the whole circle was 19°49′3″55‴, and therefore the motion that corresponds to double the figure NeFn is 1°29′58″2‴. Subtracting this from the former motion leaves 18°19′5″53‴, the total motion of the node with respect to the fixed stars between its successive conjunctions with the sun; and this motion, subtracted from the annual motion of the sun of 360°, leaves 341°40′54″7‴, the motion of the sun between the same conjunctions. And this motion is to the annual motion of 360° as the motion of the node just found (18°19′5″53‴) to its annual motion, which will therefore be 19°18′1″23‴. This is the mean motion of the nodes in a sidereal year. From the astronomical tables this is 19°21′21″50‴. The difference is less than $\frac{1}{300}$ of the total motion and seems to arise from the eccentricity of the moon's orbit and its inclination to the plane of the ecliptic. By the eccentricity of the orbit, the motion of the nodes is too much accelerated; and on the other hand, by its inclination it is retarded somewhat, and reduced to its correct velocity.

Proposition 33
Problem 14

To find the true motion of the nodes of the moon.

In the time which is as the area NTA − NdZ (in the preceding figure), that motion is as the area NAe, and hence is given. But because the calculation is too difficult, it is preferable to use the following construction of the problem. With center C and any interval CD as radius, describe a circle BEFD. Produce DC to A so that AB is to AC as the mean motion is to half



of the true mean motion when the nodes are in the quadratures (that is, as $19°18'1''23'''$ to $19°49'3''55'''$); and thus BC will be to AC as the difference of the motions ($0°31'2''32'''$) to the latter motion ($19°49'3''55'''$), that is, as 1 to $38\frac{3}{10}$. Next, through point D draw the indefinite line Gg, touching the circle in D; and let the angle BCE or BCF be taken equal to twice the distance of the sun from the place of the node, as found from the mean motion, and let AE or AF be drawn cutting the perpendicular DG in G. The true motion of the nodes will be found if now an angle is taken that is to the total motion of the node between its syzygies (that is, to $9°11'3''$) as the tangent DG is to the total circumference of the circle BED, and if that angle (for which the angle DAG can be used) is added to the mean motion of the nodes when the nodes are passing from quadratures to syzygies and is subtracted from the same mean motion when they are passing from syzygies to quadratures. For the true motion thus found will agree very nearly with the true motion which results from representing the time by the area NTA − N*d*Z and the motion of the node by the area NA*e*, as will be evident to anyone considering the matter and performing the computations. This is the semimonthly equation of the motion of the nodes. There is also a monthly equation, but it is not at all needed in order to find the latitude of the moon. For, since the variation of the inclination of the moon's orbit to the plane of the ecliptic is subject to a double inequality, one semimonthly and the other monthly, the monthly inequality of the variation and the monthly equation of the nodes so moderate and correct each other that both can be ignored in determining the latitude of the moon.

COROLLARY. From this and the preceding proposition it is clear that the nodes are stationary in their syzygies; in the quadratures, however, they regress by an hourly motion of $16''19'''26^{\text{iv}}$. It is also clear that the equation of the motion of nodes in the octants is $1°30'$. This all squares exactly with celestial phenomena.

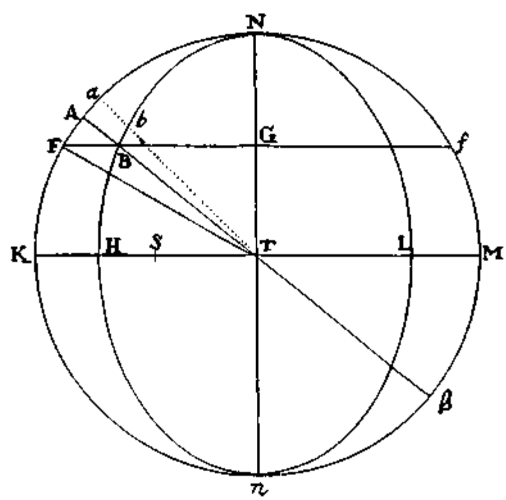
Scholium J. Machin, Gresham Professor of Astronomy, and Henry Pemberton, M.D., have independently found the motion of the nodes by yet another method. Some mention of the latter's method has been made elsewhere. And the papers (which I have seen) of both men contained two propositions, which agreed with each other. Here I shall present Mr. Machin's paper, since it was the first to come into my hands.

On the Motion of the Nodes of the Moon

Proposition 1

The mean motion of the sun from the node is defined by a mean geometrical proportional between the mean motion of the sun and that mean motion with which the sun recedes most swiftly from the node in the quadratures.

Let T be the place where the earth is, N*n* the line of the nodes of the moon at any given time, KTM a line drawn at right angles to this line, and TA a straight line revolving around the center with the angular velocity with which the sun and the node recede from each other, in such a way that the angle between the straight line N*n* (which is at rest) and TA (which is revolving) is always equal to the distance between the places of the sun and of the node. Now, if any straight line TK is divided into parts TS and SK, which are to each other as the hourly mean motion of the sun is to the hourly mean motion of the nodes in the quadratures, and if the straight line TH is taken so as to be a mean proportional between the part TS and the whole TK, this straight line among the rest will be proportional to the mean motion of the sun from the node.

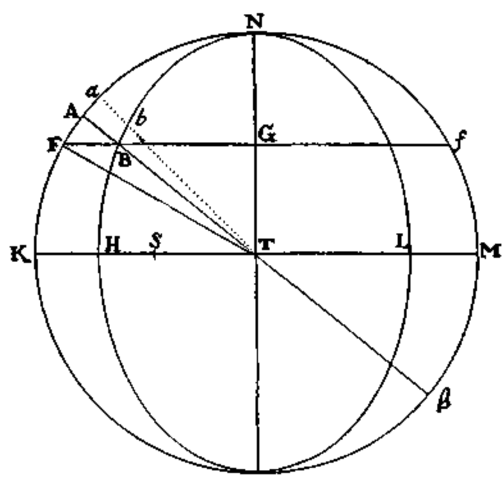


For describe a circle NK*n*M with center T and radius TK, and with the same center and the semiaxes TH and TN describe an ellipse NH*n*L, and in the time in which the sun recedes from the node through the arc N*a*, if the straight line T*ba* is drawn, the area of the sector NT*a* will represent the sum of the motions of the node and sun in that same time. Therefore let A*a* be the minimally small arc that the straight line T*ba*—revolving according to the aforesaid law—uniformly describes in a given particle of time, and the minimally small sector TA*a* will be as the sum of the velocities whereby the sun and the node are carried separately in that time. The velocity of the sun, however, is nearly uniform, since its small inequality introduces scarcely any variation in the mean motion of the nodes. The other part of this sum, namely the velocity of the node in its mean quantity, increases in receding from the syzygies as the square of the sine of its distance from the sun (by *Principia*, book 3, prop. 31, corol.) and, when it is greatest in the quadratures to the sun at K, has the same ratio to the velocity of the sun that SK has to TS; that is, it is as (the difference of the squares of TK and TH or) the rectangle KH $\times$ HM is to TH^2. But the ellipse NBH divides the sector AT*a*, which represents the sum of these two velocities, into two parts AB*ba* and BT*b*, which are proportional to the velocities. For, produce BT to the circle in β, and, from point B to the major axis, drop a perpendicular BG which, produced in both directions, meets the circle in points F and *f*. Then, since the space AB*ba* is to the sector TB*b* as the rectangle AB $\times$ Bβ to BT^2 (for that rectangle is equal to the difference of the squares of TA and TB because the straight line Aβ is cut equally at T and unequally at B), this ratio—when the space AB*ba* is greatest at K—will be the same as the ratio of the rectangle KH $\times$ HM to HT^2; but the greatest mean velocity of the node was [previously shown to be] in this ratio to the velocity of the sun. Therefore, in the quadratures, the sector AT*a* is divided into parts proportional to the velocities. And since the rectangle KH $\times$ HM is to HT^2 as FB $\times$ B*f* to BG^2 and since the rectangle AB $\times$ Bβ is equal to the rectangle FB $\times$ B*f*, it follows that the area-element AB*ba* when it is greatest will be to the remaining sector TB*b* as the rectangle AB $\times$ Bβ to BG^2. But the ratio of the area-elements was always as the rectangle AB $\times$ Bβ to BT^2; and therefore the area-element AB*ba* in the place A is smaller than the corresponding area-element in the quadratures, in the ratio of BG^2 to BT^2, that is, as the square of the sine of the distance of the sun from the node. And accordingly the sum of all the area-elements AB*ba* (namely, the space ABN) will be as the motion of the node in the time in which the sun departs from the node and passes through the arc NA. And the remaining space (namely, the elliptical sector NTB) will be as the mean motion of the sun in the same time. And, therefore, since the mean annual motion of the node is the motion that it makes in the time during which the sun has completed its period, the mean motion of the node from the sun will be to the mean motion of the sun itself as the area of the circle to the area of the ellipse, that is, as the straight line TK to the straight line TH (which is the mean proportional between TK and TS); or, which comes to the same thing, as the mean proportional TH to the straight line TS.

Proposition 2

Given the mean motion of the nodes of the moon, to find the true motion.

Let the angle A be the distance of the sun from the mean place of the node, or the mean motion of the sun from the node. Then if angle B is taken so that its tangent is to the tangent of angle A as TH to TK—that is, as the square root of the ratio of the mean



hourly motion of the sun to the mean hourly motion of the sun from the node when the node is in the quadratures—that same angle B will be the distance of the sun from the true place of the node. For draw FT, and (by the proof of the previous proposition) the angle FTN will be the distance of the sun from the mean place of the node, while the angle ATN will be the distance from the true place, and the tangents of these angles are to each other as TK to TH.

Corollary. Hence the angle FTA is the equation of the moon's nodes, and the sine of this angle, when it is greatest in the octants, is to the radius as KH to TK + TH. And the sine of this equation in any other place A is to the greatest sine as the sine of the sum of the angles FTN + ATN is to the radius—that is, nearly as the sine of 2FTN (that is, twice the distance of the sun from the mean place of the node) is to the radius.

Scholium

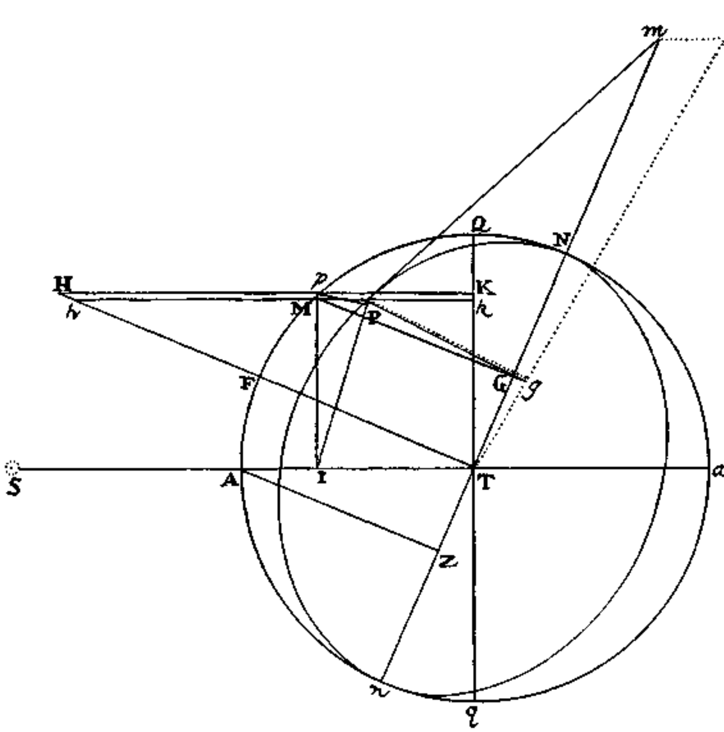
If the mean hourly motion of the nodes in the quadratures is $16''16'''37^{iv}42^{v}$ (that is, $39°38'7''50'''$ in a whole sidereal year), then TH will be to TK as the square root of the ratio of the number 9.0827646 to the number 10.0827646, that is, as 18.6524761 to 19.6524761. And therefore TH is to HK as 18.6524761 to 1, that is, as the motion of the sun in a sidereal year to the mean motion of the node, which is $19°18'1''23\frac{2}{3}'''$.

But if the mean motion of the nodes of the moon in twenty Julian years is $386°50'15''$, as is deduced from observations used in the theory of the moon, the mean motion of the nodes in a sidereal year will be $19°20'31''58'''$. And TH will be to HK as 360° to $19°20'31''58'''$, that is, as 18.61214 to 1. Hence the mean hourly motion of the nodes in the quadratures will come out $16''18'''48^{iv}$. And the greatest equation of the nodes in the octants will be $1°29'57''$.

Proposition 34
Problem 15

To find the hourly variation of the inclination of the lunar orbit to the plane of the ecliptic.

Let A and *a* represent the syzygies, Q and *q* the quadratures, N and *n* the nodes, P the place of the moon in its orbit, *p* the projection of that place on the plane of the ecliptic, and *m*T*l* the momentaneous motion of the nodes as above. Drop the perpendicular PG to the line T*m*, join *p*G and produce it until it meets T*l* in *g*, and also join P*g*; then the angle PG*p* will be the inclination of the moon's orbit to the plane of the ecliptic when the moon is in P, and the angle P*gp* will be the inclination of the same orbit after a moment of time has been completed; and thus the angle GP*g* will be the momentaneous variation of the inclination. But this angle GP*g* is to the angle GT*g* as TG to PG and P*p* to PG jointly. And therefore, if an hour is substituted for the moment of time, then—since the angle GT*g* (by prop. 30) is to the angle $33''10'''33^{iv}$ as IT × PG × AZ to AT^3—the angle GP*g* (or



the hourly variation of the inclination) will be to the angle $33''10'''33^{\text{iv}}$ as $\text{IT} \times \text{AZ} \times \text{TG} \times \dfrac{\text{P}p}{\text{PG}}$ to AT^3. Q.E.I.

These things are so on the hypothesis that the moon revolves uniformly in a circular orbit. But if that orbit is elliptical, the mean motion of the nodes will be diminished in the ratio of the minor axis to the major axis, as has been set forth above. And the variation of the inclination will also be diminished in the same ratio.

Corollary 1. If the perpendicular TF is erected on Nn, and pM is the hourly motion of the moon in the plane of the ecliptic, and if the perpendiculars pK and Mk are dropped to QT and produced in both directions to meet TF at H and h, then IT will be to AT as Kk to Mp, and TG to Hp as TZ to AT, and so $\text{IT} \times \text{TG}$ will be equal to $\dfrac{\text{K}k \times \text{H}p \times \text{TZ}}{\text{M}p}$, that is, equal to the area HpMh multiplied by the ratio $\dfrac{\text{TZ}}{\text{M}p}$; and therefore the hourly variation of the inclination will be to $33''10'''33^{\text{iv}}$ as HpMh multiplied by $\text{AZ} \times \dfrac{\text{TZ}}{\text{M}p} \times \dfrac{\text{P}p}{\text{PG}}$ is to AT^3.

COROLLARY 2. And so, if the earth and the nodes, as each hour is completed, were drawn back from their new places and were always restored instantly to their former places, so that their given position remained unchanged throughout an entire periodic month, the total variation of the inclination during the time of that month would be to $33''10'''33^{iv}$ as the sum of all the areas HpMh which are generated during a revolution of the point p (these areas being summed according to their proper signs + and −) multiplied by $AZ \times TZ \times \frac{Pp}{PG}$ is to $Mp \times AT^3$, that is, as the whole circle QAqa multiplied by $AZ \times TZ \times \frac{Pp}{PG}$ is to $Mp \times AT^3$, that is, as the circumference QAqa multiplied by $AZ \times TZ \times \frac{Pp}{PG}$ is to $2Mp \times AT^2$.

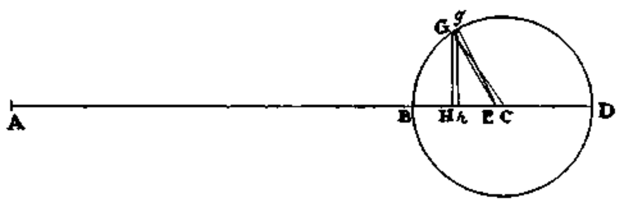
COROLLARY 3. Accordingly, in a given position of the nodes, the mean hourly variation, from which, continued uniformly for a month, that monthly variation could be generated, is to $33''10'''33^{iv}$ as $AZ \times TZ \times \frac{Pp}{PG}$ to $2AT^2$, or as $Pp \times \frac{AZ \times TZ}{\frac{1}{2}AT}$ to $PG \times 4AT$, that is (since Pp is to PG as the sine of the above-mentioned inclination to the radius, and $\frac{AZ \times TZ}{\frac{1}{2}AT}$ is to 4AT as the sine of twice the angle ATn to four times the radius), as the sine of that same inclination multiplied by the sine of twice the distance of the nodes from the sun to four times the square of the radius.

COROLLARY 4. Since the hourly variation of the inclination, when the nodes are in the quadratures, is (by this proposition) to the angle $33''10'''33^{iv}$ as $IT \times AZ \times TG \times \frac{Pp}{PG}$ to AT^3, that is, as $\frac{IT \times TG}{\frac{1}{2}AT} \times \frac{Pp}{PG}$ to 2AT, that is, as the sine of twice the distance of the moon from the quadratures multiplied by $\frac{Pp}{PG}$ is to twice the radius, it follows that the sum of all the hourly variations, in the time in which the moon in this situation of the nodes passes from quadrature to syzygy (that is, in the space of 177⅙ hours), will be to the sum of the same number of angles $33''10'''33^{iv}$, or $5{,}878''$, as the sum of all the sines of twice the distance of the moon from the quadratures multiplied by $\frac{Pp}{PG}$ is to the sum of the same number of diameters; that is, as the diameter multiplied by $\frac{Pp}{PG}$ is to the circumference; that is, if the inclination is $5°1'$, as $7 \times \frac{874}{10{,}000}$ to 22, or 278 to 10,000. And accordingly the total variation, composed of the sum of all the hourly variations in the aforesaid time, is 163″, or 2′43″.

Proposition 35
Problem 16

To find the inclination of the moon's orbit to the plane of the ecliptic at a given time.

Let AD be the sine of the greatest inclination, and AB the sine of the least inclination. Bisect BD in C, and with center C and radius BC describe a circle BGD. On AC take CE in the ratio to EB that EB has to 2BA. Now if, for the given time, the angle AEG is set equal to twice the distance of the nodes from the quadratures, and the perpendicular GH is dropped to AD, then AH will be the sine of the required inclination.



For GE^2 is equal to $\mathrm{GH}^2 + \mathrm{HE}^2 = \mathrm{BH} \times \mathrm{HD} + \mathrm{HE}^2 = \mathrm{HB} \times \mathrm{BD} + \mathrm{HE}^2 - \mathrm{BH}^2 = \mathrm{HB} \times \mathrm{BD} + \mathrm{BE}^2 - 2\mathrm{BH} \times \mathrm{BE} = \mathrm{BE}^2 + 2\mathrm{EC} \times \mathrm{BH} = 2\mathrm{EC} \times \mathrm{AB} + 2\mathrm{EC} \times \mathrm{BH} = 2\mathrm{EC} \times \mathrm{AH}$. And thus, since 2EC is given, GE^2 is as AH. Now let AEg represent twice the distance of the nodes from the quadratures after some given moment of time has been completed, and the arc Gg (because the angle GEg is given) will be as the distance GE. Moreover, Hh is to Gg as GH to GC, and therefore Hh is as the solid [or product] GH × Gg, or GH × GE; that is, as $\frac{\mathrm{GH}}{\mathrm{GE}} \times \mathrm{GE}^2$ or $\frac{\mathrm{GH}}{\mathrm{GE}} \times \mathrm{AH}$, that is, as AH and the sine of the angle AEG jointly. Therefore, if AH, in any given case, is the sine of the inclination, it will be increased by the same increments as the sine of the inclination, by corol. 3 of the preceding prop. 34, and therefore will always remain equal to that sine. But when the point G falls upon either point B or D, AH is equal to this sine and therefore remains always equal to it. Q.E.D.

In this demonstration, I have supposed that the angle BEG, which is twice the distance of the nodes from the quadratures, increases uniformly. For there is no time to consider all the minute details of inequalities. Now suppose that the angle BEG is a right angle and that in this case Gg is the hourly increment of twice the distance of the nodes and sun from each other; then (by corol. 3 of prop. 34) the hourly variation of the inclination in the same case will be to $33''10'''33^{iv}$ as the solid [or product] of the sine AH of the inclination and the sine of the right angle BEG (which is twice the distance of the nodes from the sun) is to four times the square of the radius; that is, as the sine AH of the mean inclination to four times the radius; that is (since that mean inclination is about $5°8\frac{1}{2}'$), as its sine (896) to four times the radius (40,000), or as 224 to 10,000. And the total variation, corresponding to BD, the difference of the sines, is to that hourly variation as the diameter BD to the arc G*g*; that is, as the diameter BD to the semicircumference BGD and the time of $2{,}079\frac{7}{10}$ hours (during which the node goes from the quadratures to the syzygies) to 1 hour jointly; that is, as 7 to 11 and $2{,}079\frac{7}{10}$ to 1. Therefore, if all the ratios are combined, the total variation BD will become to $33''10'''33^{iv}$ as $224 \times 7 \times 2{,}079\frac{7}{10}$ to 110,000, that is, as 29,645 to 1,000, and hence that variation BD will come out $16'23\frac{1}{2}''$.

This is the greatest variation of the inclination insofar as the place of the moon in its orbit is not considered. For if the nodes are in the syzygies, the inclination is not at all changed by the various positions of the moon. But if the nodes are in the quadratures, the inclination is less when the moon is in the syzygies than when it is in the quadratures, by a difference of $2'43''$, as we have indicated in corol. 4 of prop. 34. And the total mean variation BD, diminished when the moon is in its quadratures by $1'21\frac{1}{2}''$ (half of this excess), becomes $15'2''$; while in the syzygies it is increased by the same amount and becomes $17'45''$. Therefore, if the moon is in the syzygies, the total variation in the passage of the nodes from quadratures to syzygies will be $17'45''$; and so if the inclination, when the nodes are in the syzygies, is $5°17'20''$, it will be $4°59'35''$ when the nodes are in the quadratures and the moon in the syzygies. And that these things are so is confirmed by observations.

If now it is desired to find the inclination of the orbit when the moon is in the syzygies and the nodes are in any position whatever, let AB become to AD as the sine of $4°59'35''$ is to the sine of $5°17'20''$, and take the angle ABG equal to twice the distance of the nodes from the quadratures; then AH will be the sine of the required inclination. The inclination of the orbit is equal to this inclination when the moon is 90° distant from the nodes. In other positions of the moon, the monthly inequality that occurs in the variation of the inclination is compensated for in the calculation of the latitude of the moon (and, in a manner, canceled) by the monthly inequality in the motion of the nodes (as we have said above) and thus can be neglected in calculating that latitude.

Scholium

[a]I wished to show by these computations of the lunar motions that the lunar motions can be computed from their causes by the theory of gravity. By the same theory I found, in addition, that the annual equation of the mean motion of the moon arises from the varying dilatation [and contraction] of the orbit of the moon produced by the force of the sun, according to book 1, prop. 66, corol. 6. When the sun is in perigee, this force is greater and dilates the orbit of the moon; when the sun is in apogee, the force is smaller and permits the orbit to be contracted. The moon revolves more slowly in the dilated orbit, more swiftly in the contracted one; and the annual equation which compensates for this inequality vanishes in the apogee and perigee of the sun, rises to roughly 11′50″ in the mean distance of the sun from the earth, and in other places is proportional to the equation of the center of the sun; and it is added to the mean motion of the moon when the earth is going from its aphelion to its perihelion and is subtracted when the earth is in the opposite part of the orbit. Assuming the radius of the earth's orbit to be 1,000 and the eccentricity of the earth to be 16⅞, this equation, when it is greatest, came out 11′49″ by the theory of gravity. But the eccentricity of the earth seems to be a little greater; and if the eccentricity is increased, this equation should be increased in the same ratio. If the eccentricity is taken at 16 11/12, the greatest equation will be 11′51″.

aa. Ed. 1 has: "Up to now no consideration has been taken of the motions of the moon insofar as the eccentricity of the orbit is concerned. By similar computations, I found that the apogee, when it is in conjunction with or in opposition to the sun, moves forward 23′ each day with respect to the fixed stars but, when it is in the quadratures, regresses about 16⅓′ each day and that its mean annual motion is about 40°. By the astronomical tables which the distinguished Flamsteed adapted to the hypothesis of Horrocks, the apogee moves forward in its syzygies with a daily motion of 24′28″ but regresses in the quadratures with a daily motion of 20′12″ and is carried forward [or in consequentia] with a mean annual motion of 40°41′. The difference between the daily forward motion of the apogee in its syzygies and the daily regressive motion in its quadratures is 4′16″ by the tables but 6⅔′ by our computation, which we suspect ought to be attributed to a fault in the tables. But we do not think that our computation is exact enough either. For by means of a certain calculation the daily forward motion of the apogee in its syzygies and the daily regressive motion in its quadratures came out a little greater. But it seems preferable not to give the computations, since they are too complicated and encumbered by approximations and not exact enough."

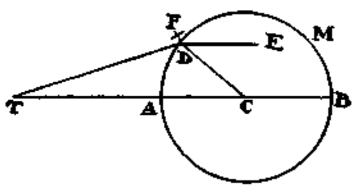
I found also that the apogee and nodes of the moon move more swiftly in the perihelion of the earth (because of the greater force of the sun) than in its aphelion, [and this] inversely as the cube of the distance of the earth from the sun. And from this there arise annual equations of these motions proportional to the equation of the sun's center. Now, the motion of the sun is inversely as the square of the distance of the earth from the sun, and the greatest equation of the center that this inequality generates is $1°56'20''$, corresponding to the above-mentioned eccentricity of the sun of $16^{11}/_{12}$. But if the motion of the sun were inversely as the cube of the distance, this inequality would generate a greatest equation of $2°54'30''$. And therefore the greatest equations that the inequalities of the motions of the apogee and nodes of the moon generate are to $2°54'30''$ as the daily mean motion of the apogee and the daily mean motion of the nodes of the moon are to the daily mean motion of the sun. Accordingly, the greatest equation of the mean motion of the apogee comes out $19'43''$, and the greatest equation of the mean motion of the nodes $9'24''$. And the first of these equations is added and the second subtracted when the earth is going from its perihelion to its aphelion, and the opposite happens in the opposite part of the orbit.

By the theory of gravity it was also established that the action of the sun upon the moon is a little greater when the transverse diameter of the moon's orbit is passing through the sun than when this diameter is at right angles to the line joining the earth and the sun; and therefore the moon's orbit is a little greater in the first case than in the second. And hence arises another equation of the moon's mean motion, one that depends on the position of the apogee of the moon with respect to the sun; this equation is greatest when the apogee of the moon is in an octant with the sun, and vanishes when the apogee reaches the quadratures or syzygies, and is added to the mean motion in the passage of the apogee of the moon from quadrature of the sun to syzygy, and is subtracted in the passage of the apogee from syzygy to quadrature. This equation, which I shall call semiannual, rises in the octants of the apogee (when it is greatest) to roughly $3'45''$, as far as I could gather from phenomena. This is its quantity at the mean distance of the sun from the earth. But it is increased and decreased inversely as the cube of the distance from the sun, and so at the greatest distance of the sun is $3'34''$ and at the least distance $3'56''$—very nearly; and when the apogee of the moon is situated outside the octants, it becomes less, and is to the greatest equation as the sine of twice the distance of the moon's apogee from the nearest syzygy or quadrature is to the radius.

By the same theory of gravity, the action of the sun upon the moon is a little greater when a straight line drawn through the nodes of the moon passes through the sun than when that line is at right angles to the straight line joining the sun and earth. And hence arises another equation of the moon's mean motion, which I shall call the second semiannual and which is greatest when the nodes are in the octants of the sun and vanishes when they are in the syzygies or quadratures, and in other positions of the nodes is proportional to the sine of twice the distance of either node from the next syzygy or quadrature; and it is added to the mean motion of the moon if the sun is ahead of [in antecedentia] the node nearest to it, and subtracted if beyond [in consequentia]; and in the octants, where it is greatest, it rises to 47″ at the mean distance of the sun from the earth, as I conclude from the theory of gravity. At other distances of the sun, this equation (which is greatest in the octants of the nodes) is inversely as the cube of the distance of the sun from the earth, and so in the perigee of the sun rises to about 49″ and in its apogee to about 45″.

By the same theory of gravity the apogee of the moon advances as much as possible when it is either in conjunction with the sun or in opposition, and regresses when it is in quadrature with the sun. And the eccentricity becomes greatest in the first case and least in the second, by book 1, prop. 66, corols. 7, 8, and 9. And these inequalities, by the same corollaries, are very great and generate the principal equation of the apogee, which I shall call the semiannual. And the greatest semiannual equation is roughly 12°18′, as far as I could gather from observations. Our fellow countryman Horrocks was the first to propose that the moon revolves in an ellipse around the earth, which is set in its lower focus. Halley placed the center of the ellipse in an epicycle, whose center revolves uniformly around the earth. And from the motion in this epicycle there arise the inequalities (mentioned above) in the advance and retrogression of the apogee and in the magnitude of the eccentricity. Suppose the mean distance of the moon from the earth to be divided into 100,000 parts, and let T represent the earth and TC the mean eccentricity of the moon, of 5,505 parts. Let TC be produced to B, so that CB is the sine of the greatest semiannual equation (12°18′) to the radius TC; then the circle BDA, described with center C and radius CB, will be that epicycle in which the center of the moon's orbit is located and which revolves according to the order of the letters BDA. Let the angle BCD be taken equal to twice the annual argument, or twice the distance of the true place of the sun from the moon's apogee equated one time [i.e., corrected by the equation applied once], and CTD will be the semiannual equation of the moon's apogee and TD the eccentricity of its orbit, tending to the apogee equated a second time. And once the moon's mean motion and apogee and eccentricity have been found, as well as the orbit's major axis of 200,000 parts, then from these data the true place of the moon in its orbit and its distance from the earth will be found by very well known methods.

In the perihelion of the earth, because of the greater force of the sun, the center of the moon's orbit moves more swiftly around the center C than in its aphelion, and does so inversely as the cube of the distance of the earth from the sun. Because the equation of the center of the sun is comprehended in the annual argument, the center of the moon's orbit moves more swiftly in the epicycle BDA inversely as the square of the distance of the earth from the sun.



In order for the center of the moon's orbit to move still more swiftly, inversely in the simple ratio of the distance, draw a straight line DE from the center D of the orbit toward the apogee of the moon, or parallel to the straight line TC, and take the angle EDF equal to the excess of the above-mentioned annual argument over the distance of the apogee of the moon from the perigee of the sun in a forward direction [or in consequentia]; or, which is the same, take the angle CDF equal to the complement of the true anomaly of the sun to 360°. And let DF be to DC jointly as twice the eccentricity of the earth's orbit is to the mean distance of the sun from the earth and as the daily mean motion of the sun from the apogee of the moon is to the daily mean motion of the sun from its own apogee, that is, as $33\frac{7}{8}$ to 1,000 and $52'27''16'''$ to $59'8''10'''$ jointly, or as 3 to 100. And suppose that the center of the moon's orbit is located in point F and revolves in an epicycle whose center is D and whose radius is DF, while the point D advances in the circumference of the circle DABD. For in this manner the velocity with which the center of the moon's orbit will move in a certain curved line described about the center C will be very nearly inversely as the cube of the distance of the sun from the earth, as it ought to be.

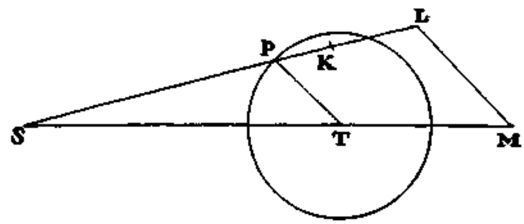
The computation of this motion is difficult, but it will be made easier by the following approximation. If the mean distance of the moon from the earth is 100,000 parts and the eccentricity TC is 5,505 parts as above, then the straight line CB or CD will be found to consist of 1,172¾ parts and the straight line DF of 35⅕ parts. And this straight line, at the distance TC, subtends at the earth the angle that the transfer of the center of the orbit from place D to place F generates in the motion of this center; and the same straight line DF doubled, in a position parallel to a line drawn from the earth to the upper focus of the moon's orbit, subtends the same angle, which of course that transfer generates in the motion of the focus; and at the distance of the moon from the earth it subtends the angle that the same transfer generates in the moon's motion and that therefore can be called the second equation of the center. And this equation, at the mean distance of the moon from the earth, is very nearly as the sine of the angle which that straight line DF contains with the straight line drawn from point F to the moon, and when it is greatest comes out $2'25''$. And the angle which the straight line DF contains with the straight line drawn from point F to the moon is found either by subtracting the angle EDF from the mean anomaly of the moon or by adding the distance of the moon from the sun to the distance of the apogee of the moon from the apogee of the sun. And as the radius is to the sine of the angle thus found, so $2'25''$ is to the second equation of the center, which should be added if that sum is less than a semicircle and subtracted if it is greater. In this way the longitude of the moon in the very syzygies of the luminaries will be found.

The atmosphere of the earth refracts the light of the sun up to a height of thirty-five or forty miles and, by refracting it, scatters it into the shadow of the earth, and by scattering the light at the edge of the shadow dilates the shadow. Hence, in lunar eclipses I add 1 minute, or 1⅓ minutes, to the diameter of the shadow as found from the parallax.

The theory of the moon, furthermore, should be examined and established by phenomena, first in the syzygies, then in the quadratures, and finally in the octants. And anyone who is going to undertake this task will not go wrong by using the following mean motions of the sun and moon at noon at the Royal Greenwich Observatory, on the last day of December 1700 (O.S.): namely, the mean motion of the sun ♑20°43′40″, and of its apogee ♋7°44′30″; and the mean motion of the moon ♒15°21′00″, and of its apogee ♓8°20′00″, and of its ascending node ♌27°24′20″; and the difference between the meridians of this observatory and the Royal Paris Observatory $0^h 9^m 20^s$; but the mean motions of the moon and of its apogee have not as yet been determined with sufficient exactness.[a]

Proposition 36
Problem 17

To find the force of the sun to move the sea.



The sun's force ML or PT to perturb the motions of the moon, in the moon's quadratures, was (by prop. 25 of this book 3) to the force of gravity here on earth as 1 to 638,092.6. And the force TM−LN or 2PK in the moon's syzygies is twice as great. Now these forces, in the descent to the surface of the earth, are diminished in the ratio of the distances from the center of the earth, that is, in the ratio 60½ to 1; and so the first force on the surface of the earth is to the force of gravity as 1 to 38,604,600. By this force the sea is depressed in places that are 90 degrees distant from the sun. By the other force, which is twice as great, the sea is elevated both in the region directly under the sun and in the region opposite to the sun. The sum of these forces is to the force of gravity as 1 to 12,868,200. And since the same force arouses the same motion, whether it depresses the water in the regions that are 90 degrees distant from the sun or elevates the water in regions under the sun and opposite to the sun, this sum will be the total force of the sun to agitate the sea, and it will have the same effect as if all of it elevated the sea in regions under the sun and opposite it and had no action at all in regions that are 90 degrees distant from the sun.

This is the force of the sun to put the sea in motion in any given place when the sun is in the Zenith of the place as well as at its mean distance from the earth. In other positions of the sun, the force for raising the sea is directly as the versed sine of twice the altitude of the sun above the horizon of the place and inversely as the cube of the distance of the sun from the earth.

Corollary. The centrifugal force of the parts of the earth, arising from the daily motion of the earth (a force that is to the force of gravity as 1 to 289), causes the height of the water under the equator to exceed its height under the poles by a measure of 85,472 Paris feet (as was seen above in prop. 19); therefore, the solar force with which we have been dealing (since it is to the force of gravity as 1 to 12,868,200 and so to that centrifugal force as 289 to 12,868,200 or 1 to 44,527) will cause the height of the water in regions directly under the sun and directly opposite to the sun to exceed its height in places that are 90 degrees distant from the sun by a measure of only 1 Paris foot and $11\frac{1}{30}$ inches. For this measure is to the measure of 85,472 feet as 1 to 44,527.

To find the force of the moon to move the sea. **Proposition 37 Problem 18**

[a]The force of the moon to move the sea is to be reckoned from its proportion to the force of the sun, and this proportion is to be determined from the proportion of the motions of the sea that arise from these forces. Before the mouth of the river Avon, at the third milestone below Bristol, in spring and autumn, the total ascent of the water in the conjunction and opposition of these two luminaries is (according to the observations of Samuel Sturmy) approximately 45 feet, but in the quadratures is only 25 feet. The first height arises from the sum of these two forces, the second from their difference. Therefore let the forces of the sun and the moon, when they are on the equator and at their mean distance from the earth, be S and L, and $L + S$ will be to $L - S$ as 45 to 25, or 9 to 5.

In Plymouth harbor, the tide of the sea (as observed by Samuel Colepress) is raised to approximately 16 feet in its mean height, and in spring and autumn the height of the tide in the syzygies can exceed its height in the quadratures by more than 7 or 8 feet. If the greatest difference of these heights is 9 feet, $L + S$ will be to $L - S$ as $20\frac{1}{2}$ to $11\frac{1}{2}$ or 41 to 23. And this proportion agrees well enough with the former one. Because of the magnitude of the tide in Bristol harbor, Sturmy's observations seem to be more trustworthy, and so, until something more certain is established, we shall use the proportion 9 to 5.

aa. In this proposition and its corollaries there are numerical differences in ed. 1 and sometimes also in ed. 2. The number 4.4815 at the end of the fifth paragraph, giving the ratio of the force of the sun to the force of the moon, was $6\frac{1}{3}$ in ed. 1.

But because of the reciprocating motions of the waters, the greatest tides do not occur at the syzygies of the luminaries but (as has been said earlier) are the third ones after the syzygies or follow next after the moon's third appulse to the meridian of the place after the syzygies, or rather (as is noted by Sturmy) are the third ones after the day of the new moon or full moon, or after approximately the twelfth hour from the new moon or full moon, and so occur at approximately the forty-third hour from the new moon or full moon. Now, in this harbor they occur at roughly the seventh hour from the appulse of the moon to the meridian of the place, and so they follow next after the appulse of the moon to the meridian, when the moon is approximately 18 or 19 degrees distant from the sun, or from the opposition of the sun, in a forward direction [or in consequentia]. The summer and winter reach their maximum, not in the solstices themselves, but when the sun has advanced through roughly a tenth of its whole circuit, or is approximately 36 or 37 degrees distant from the solstices. And similarly the greatest tide of the sea arises from the appulse of the moon to the meridian of the place, when the moon is distant from the sun by roughly a tenth part of its whole motion from one tide to the next. Let this distance be approximately 18½ degrees. Then the force of the sun at this distance of the moon from the syzygies and quadratures will be less effective to augment and to diminish that motion of the sea arising from the force of the moon than in the syzygies and quadratures themselves, in the ratio of the radius to the sine of the complement of twice this distance or the cosine of 37 degrees, that is, in the ratio of 10,000,000 to 7,986,355. And so in the above analogy, 0.7986355S ought to be written for S.

But additionally, the force of the moon must be diminished in the quadratures, because of the declination of the moon from the equator. For the moon in the quadratures, or rather at 18½ degrees beyond the quadratures, is in a declination of approximately $22°13'$. And the force of either luminary to move the sea is diminished when that luminary is declining from the equator, and diminished very nearly as the square of the cosine of the declination. And therefore the force of the moon in these quadratures is only 0.8570327L. Therefore L + 0.7986355S is to 0.8570327L − 0.7986355S as 9 to 5.

Besides, the diameters of the orbit in which the moon would have to move (supposing no eccentricity) are to each other as 69 to 70; and thus the distance of the moon from the earth in the syzygies is to its distance in the quadratures as 69 to 70, other things being equal. And its distances when 18½° beyond the syzygies (where the greatest tide is generated) and then 18½° beyond the quadratures (where the least tide is generated) are to its mean distance as 69.098747 and 69.897345 to 69½. But the forces of the moon to move the sea are as the cubes of the distances inversely; and thus the forces at the greatest and least of these distances are to the force at the mean distance as 0.9830427 and 1.017522 to 1. Hence 1.017522L + 0.7986355S will be to 0.9830427 × 0.8570327L − 0.7986355S as 9 to 5; and S will be to L as 1 to 4.4815. Therefore, since the force of the sun is to the force of gravity as 1 to 12,868,200, the force of the moon will be to the force of gravity as 1 to 2,871,400.

COROLLARY 1. Since the water acted on by the force of the sun ascends to a height of 1 foot and 11⅟₃₀ inches, by the force of the moon it will ascend to a height of 8 feet and 7⁵⁄₂₂ inches, and by both forces to a height of 10½ feet, and when the moon is in its perigee the water will ascend to a height of 12½ feet and beyond, especially when the tide is made greater by winds. And so great a force is more than sufficient to give rise to all the motions of the sea and corresponds exactly to the quantity of the motions. For in seas that extend widely from east to west, as in the Pacific Ocean and the parts of the Atlantic Ocean and the Ethiopic [or South Atlantic] Sea, which are outside the tropics, the water is generally raised to a height of 6, 9, 12, or 15 feet. And in the Pacific Ocean, which is deeper and wider, the tides are said to be greater than in the Atlantic Ocean and the Ethiopic Sea. For, to have the tide be full, the width of the sea from east to west should be no less than 90 degrees. In the Ethiopic Sea the ascent of the water within the tropics is less than in the temperate zones, because of the narrowness of the sea between Africa and the southern part of America. In the middle of the sea the water cannot rise unless it simultaneously falls on both shores, both the eastern and the western; nevertheless, in our narrow seas, the water ought to rise alternately on the two shores, that is, rise on one shore while it falls on the other. For this reason the ebb and flow are generally very small in islands that are farthest from the shores. In certain harbors, where the water is compelled to flow in and flow out with great impetus through shallow places, so as to fill and empty bays alternately, the ebb and flow must be greater than usual, as at Plymouth and Chepstow Bridge in England, at Mont-Saint-Michel and the city of Avranches in Normandy, at Cambay and Pegu [b]in the East Indies.[b] In these places the sea, coming in and going back out with great velocity, at times inundates the shores and at other times leaves them dry for many miles. And the impetus of flowing in or going back out cannot be broken before the water is raised or depressed to 30, 40, or 50 feet and more. And the same is true of oblong and shallow straits, such as the Straits of Magellan and that channel by which England is surrounded [presumably, the channel and seas, but not the ocean, bordering England]. The tide in harbors and straits of this sort is increased beyond measure by the impetus of running in and back. But on shores that face the deep and open sea with a steep descent, where the water can be raised and can fall without the impetus of flowing out and coming back, the magnitude of the tide corresponds to the forces of the sun and moon.

COROLLARY 2. Since the force of the moon to move the sea is to the force of gravity as 1 to 2,871,400, it is evident that this force is far smaller than what can be perceived in experiments with pendulums or in any statical or hydrostatical experiments. It is only in the tides of the sea that this force produces a sensible effect.

COROLLARY 3.[c] Since the force of the moon to move the sea is to the similar force of the sun as 4.4815 to 1, and since those forces (by book 1, prop. 66, corol. 14) are as the densities of the bodies of the moon and sun and the cubes of their apparent diameters jointly, the density of the moon will be to the density of the sun as 4.4815 to 1 directly and as the cube of the diameter of the moon to the cube of the diameter of the sun inversely, that is (since the apparent mean diameters of the moon and the sun are $31'16\frac{1}{2}''$ and $32'12''$), as 4,891 to 1,000. Now, the density of the sun was to the density of the earth as 1,000 to 4,000, and therefore the density of the moon is to the density of the earth as 4,891 to 4,000, or 11 to 9. Therefore the body of the moon is denser and more earthy than our earth.

bb. The Latin here is "in *India* orientali," *lit.* "in east India." In Newton's day the terms "East India" and "East Indies" were collective names applied to the whole area consisting of India, Indochina, Malaya, and the Malay Archipelago (see *Oxford English Dictionary*, s.vv. "East India" and "East Indies"; *Webster's New Geographical Dictionary*, s.v. "East Indies"). Although that usage is now obsolete, modern English provides no alternative collective name for that area, and so we have chosen the rendering "in the East Indies" used by Motte. In modern geographical terms, Cambay is in western India, and Pegu in Burma.

c. For a gloss on this corollary see the Guide, §10.16.

COROLLARY 4. And since the true diameter of the moon, from astronomical observations, is to the true diameter of the earth as 100 to 365, the mass of the moon will be to the mass of the earth as 1 to 39.788.

COROLLARY 5. And the accelerative gravity on the surface of the moon will be about three times smaller than the accelerative gravity on the surface of the earth.

[d]COROLLARY 6. And the distance of the center of the moon from the center of the earth will be to the distance of the center of the moon from the common center of gravity of the earth and the moon as 40.788 to 39.788.

COROLLARY 7. And the mean distance of the center of the moon from the center of the earth (in the octants of the moon) will be nearly $60\frac{2}{5}$ greatest semidiameters of the earth. For the greatest semidiameter of the earth was 19,658,600 Paris feet, and the mean distance between the centers of the earth and the moon, which consists of $60\frac{2}{5}$ such semidiameters, is equal to 1,187,379,440 feet. And this distance (by the preceding corollary) is to the distance of the center of the moon from the common center of gravity of the earth and the moon as 40.788 to 39.788; and hence the latter distance is 1,158,268,534 feet. And since the moon revolves with respect to the fixed stars in $27^{d}7^{h}43\frac{4}{9}^{m}$, the versed sine of the angle that the moon describes in the time of one minute is 12,752,341, the radius being 1,000,000,000,000,000. And the radius is to this versed sine as 1,158,268,534 feet to 14.7706353 feet. The moon, therefore, falling toward the earth under the action of that force with which it is kept in its orbit, will in the time of one minute describe 14.7706353 feet. And by increasing this force in the ratio of $178\frac{29}{40}$ to $177\frac{29}{40}$, the total force of gravity in the orbit of the moon will be found by prop. 3, corol. [of this book 3]. And falling toward the earth under the action of this force, the moon will describe 14.8538067 feet in the time of one minute. And at $\frac{1}{60}$ of the distance of the moon from the center of the earth, that is, at a distance of 197,896,573 feet from the center of the earth, a heavy body—falling in the time of one second—will likewise describe 14.8538067 feet. [e]And so, at a distance of 19,615,800 feet (which is the mean semidiameter of the earth), a heavy body in falling will describe—in the time of one second—15.11175 feet, or 15 feet 1 inch and $4\frac{1}{11}$ lines. This will be the descent of bodies at a latitude of 45

dd. Ed. 1 lacks this.

ee. This is considerably different in ed. 2.

degrees. And by the foregoing table, presented in prop. 20, the descent will be a little greater at the latitude of Paris by about $\frac{2}{3}$ of a line. Therefore, by this computation, heavy bodies falling in a vacuum at the latitude of Paris will—in the time of one second—describe approximately 15 Paris feet 1 inch and $4\frac{25}{33}$ lines. And if gravity is diminished by taking away the centrifugal force that arises from the daily motion of the earth at that latitude, heavy bodies falling there will—in the time of one second—describe 15 feet 1 inch and $1\frac{1}{2}$ lines. And that heavy bodies do fall with this velocity at the latitude of Paris has been shown above in props. 4 and 19 [of this book 3].[e]

COROLLARY 8. [f]The mean distance between the centers of the earth and the moon in the syzygies of the moon is 60 greatest semidiameters of the earth, taking away roughly $\frac{1}{30}$ of a semidiameter. And in the moon's quadratures, the mean distance between these centers is $60\frac{5}{6}$ semidiameters of the earth. For these two distances are to the mean distance of the moon in the octants as 69 and 70 to $69\frac{1}{2}$, by prop. 28.[f]

[g]COROLLARY 9. The mean distance between the centers of the earth and the moon in the syzygies of the moon is $60\frac{1}{10}$ mean semidiameters of the earth. And in the moon's quadratures, the mean distance of the same centers is 61 mean semidiameters of the earth, taking away $\frac{1}{30}$ of a semidiameter.

COROLLARY 10. In the moon's syzygies, its mean horizontal parallax at latitudes of 0°, 30°, 38°, 45°, 52°, 60°, and 90° is 57′20″, 57′16″, 57′14″, 57′12″, 57′10″, 57′8″, and 57′4″ respectively.[g]

In these computations I have not considered the magnetic attraction of the earth, the magnitude of which is very small anyway and is unknown. But if this attraction can ever be determined—and if the measures of degrees on the meridian, and the lengths of isochronous pendulums at various parallels of latitude, and the laws of the motions of the sea, and the moon's parallax, together with the apparent diameters of the sun and moon, are ever determined more accurately from phenomena—it will then be possible to undertake all this calculation over again with a higher degree of accuracy.[a] [d]

Proposition 38
Problem 19

To find the figure of the body of the moon.

If the body of the moon were fluid like our sea, the force of the earth to elevate that fluid in both the nearest and farthest parts would be to the force

ff. This is very different in ed. 2.
gg. Ed. 2 lacks this.

of the moon by which our sea is raised in the regions both under the moon and opposite to the moon as the accelerative gravity of the moon toward the earth is to the accelerative gravity of the earth toward the moon and as the diameter of the moon is to the diameter of the earth, jointly—that is, as 39.788 to 1 and 100 to 365 jointly, or as 1,081 to 100. Hence, since our sea is raised by the force of the moon to 8⅗ feet, the lunar fluid would have to be raised by the force of the earth to 93 feet. And for this reason the figure of the moon would be a spheroid, the greatest diameter of which, produced, would pass through the center of the earth and would exceed by 186 feet the diameters perpendicular to that one. Therefore, it is just such a figure that the moon has and must have had from the beginning. Q.E.I.

COROLLARY. And hence it happens that the same face of the moon is always turned toward the earth. For in any other position, the moon cannot remain at rest, but by a motion of oscillation will always return to this position. But those oscillations would nevertheless be extremely slow because the forces producing them are small in magnitude; so that the face of the moon that should always look toward the earth can (for the reason given in prop. 17) be turned toward the other focus of the moon's orbit and not at once be drawn back from there and turned toward the earth.

[a]**Lemma 1**

Let APE*p represent the earth, uniformly dense, with a center* C *and poles* P *and* p *and equator* AE, *and suppose a sphere* Pape[b] *to be described with center* C *and radius* CP. *Let* QR *be the plane on which a straight line drawn from the center of the sun to the center of the earth stands perpendicularly. Then, if the individual particles of the whole exterior earth* PapAPepE, *which is higher than the sphere just described, endeavor to recede in both directions from the plane* QR, *and the endeavor of each particle is as its distance from the plane, I say, first of all, that the total force and efficacy of all the particles that lie in the circle of the equator* AE *(disposed uniformly outside the globe, in the manner of a ring completely encircling that globe) to rotate the earth around its center will be to the total force and efficacy of the same number of particles standing at point* A *of the equator (which is most distant from the plane* QR*) to move the earth with a similar circular*

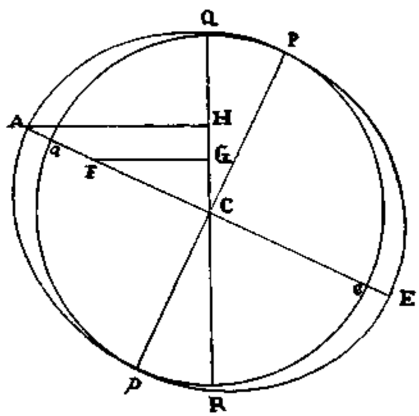
aa. In ed. 1, with certain variants, lems. 1–3 and hyp. 2 are simply three lemmas, of which the first contains the statement of lem. 1 followed by the demonstration (which is, however, greatly altered in ed. 2 and ed. 3) of lem. 2, the second corresponds to lem. 3, and the third corresponds to hyp. 1.

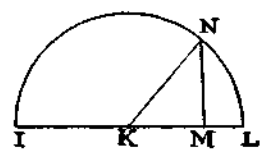
b. Strictly speaking, a sphere is defined as a solid body all of whose points are equidistant from the center, but the context and the diagram leave no doubt that in lem. 1, Newton's "sphere *Pape*" ("sphaera *Pape*") is not truly spherical but of an ellipsoidal shape.

motion around its center as 1 is to 2. And that circular motion will be performed around an axis lying in the common section of the equator and the plane QR.

For let a semicircle INLK be described with center K and diameter IL. Suppose the semicircumference INL to be divided into innumerable equal parts, and from the individual parts N to the diameter IL drop the sines NM. Then the sum of the squares of all the sines NM will be equal to the sum of the squares of the sines KM, and both sums will be equal to the sum of the squares of the same number of semidiameters KN; and so the sum of the squares of all the sines NM will be one-half of the sum of the squares of the same number of semidiameters KN.

Now let the perimeter of the circle AE be divided into the same number of equal particles, and from each one of them F to the plane QR drop a perpendicular FG, as well as a perpendicular AH from the point A. Then the



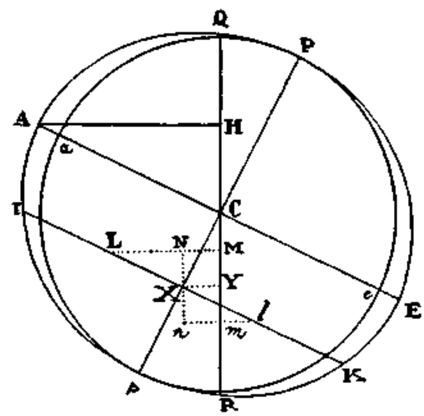


force by which the particle F recedes from the plane QR will (by hypothesis) be as that perpendicular FG, and this force multiplied by the distance CG will be the efficacy of the particle F to turn the earth around its center. And thus the efficacy of a particle in the place F will be to the efficacy of a particle in the place A as $FG \times GC$ to $AH \times HC$, that is, as FC^2 to AC^2; and therefore the total efficacy of all the particles in their places F will be to the efficacy of the same number of particles in place A as the sum of all the FC^2 to the sum of the same number of AC^2, that is (by what has already been demonstrated), as 1 to 2. Q.E.D.

And since the particles act by receding perpendicularly from the plane QR, and do so equally from each side of this plane, they will turn the circumference of the circle of the equator, together with the earth adhering to it, around an axis lying in the plane QR as well as in the plane of the equator.

Under the same conditions, I say, secondly, that the total force and efficacy of all the particles situated everywhere outside the globe to rotate the earth around the given axis is to the total force of the same number of particles, disposed uniformly throughout all of the circle of the equator AE *in the manner of a ring, to move the earth with a similar circular motion, as 2 is to 5.* **Lemma 2**

For let IK be any smaller circle parallel to the equator AE, and let L and *l* be any two equal particles situated in this circle outside the globe P*ape*.[c] And if perpendiculars LM and *lm* are dropped to the plane QR, which is perpendicular to a radius drawn to the sun, the total forces with which the particles recede from the plane QR will be proportional to the perpendiculars LM and *lm*. Now, let the straight line L*l* be parallel to the plane P*ape*; bisect L*l* at X; through the point X draw N*n* parallel to the plane QR and meeting the perpendiculars LM and *lm* at N and *n*; and drop a perpendicular XY to the plane QR. Then the contrary forces of the particles L and *l* to rotate the earth in opposite directions are as LM × MC and *lm* × *m*C, that is, as LN×MC+NM×MC and *ln*×*m*C−*nm*×*m*C, or LN×MC+NM×MC and LN × *m*C − NM × *m*C; and their difference LN × M*m* − NM × (MC + *m*C) is the force of both particles taken together to rotate the earth. The positive part of this difference, LN × M*m* or 2LN × NX, is to the force 2AH × HC of two particles of the same magnitude located at A as LX^2 to AC^2. And the negative part, NM × (MC + *m*C) or 2XY × CY is to the force 2AH × HC of the same particles located at A as CX^2 to AC^2. And accordingly the difference of the parts, that is, the force of the two particles L and *l* (taken together) to rotate the earth, is to the force of two particles equal to those and



c. In moving ahead from lem. 1 to lem. 2, Newton has shifted his vocabulary from "sphere" to "globe." He now writes of a circle "outside the globe *Pape*" ("extra globum *Pape*"), where again the context and the diagram leave no doubt that the "globe *Pape*" is ellipsoidal.

standing in the place A, likewise to rotate the earth, as $LX^2 - CX^2$ to AC^2. But if the circumference IK of the circle IK is divided into innumerable equal particles L, all the LX^2 will be to as many IX^2 as 1 to 2 (by lem. 1), and to this same number of AC^2 as IX^2 to $2AC^2$; and just as many CX^2 will be to the same number of AC^2 as $2CX^2$ to $2AC^2$. Therefore the combined forces of all the particles in the circumference of the circle IK are to the combined forces of as many particles in the place A as $IX^2 - 2CX^2$ to $2AC^2$, and therefore (by lem. 1) to the combined forces of as many particles in the circumference of the circle AE as $IX^2 - 2CX^2$ to AC^2.

Now, if the diameter Pp of the sphere[d] is divided into innumerable equal parts, on which the same number of circles IK stand, the matter in the perimeter of each circle IK will be as IX^2; and so the force of that matter to rotate the earth will be as IX^2 multiplied by $IX^2 - 2CX^2$. And the force of the same matter, if it stood in the perimeter of the circle AE, would be as IX^2 multiplied by AC^2. And therefore the force of all the particles of the total matter standing outside the globe in the perimeters of all the circles is to the force of as many particles standing in the perimeter of the greatest circle AE as all the IX^2 multiplied by $IX^2 - 2CX^2$ to as many IX^2 multiplied by AC^2, that is, as all the $AC^2 - CX^2$ multiplied by $AC^2 - 3CX^2$ to as many $AC^2 - CX^2$ multiplied by AC^2, that is, as all the $AC^4 - 4AC^2 \times CX^2 + 3CX^4$ to as many $AC^4 - AC^2 \times CX^2$, that is, as the total fluent quantity whose fluxion[e] is $AC^4 - 4AC^2 \times CX^2 + 3CX^4$ to the total fluent quantity whose fluxion is $AC^4 - AC^2 \times CX^2$; and accordingly, by the method of fluxions, as $AC^4 \times CX - \frac{4}{3}AC^2 \times CX^3 + \frac{3}{5}CX^5$ to $AC^4 \times CX - \frac{1}{3}AC^2 \times CX^3$, that is, if the whole of Cp or AC is written in place of CX, as $\frac{4}{15}AC^5$ to $\frac{2}{3}AC^5$, or as 2 to 5. Q.E.D.

Lemma 3 *Under the same conditions, I say, thirdly, that the motion of the whole earth around the axis described above, a motion that is composed of the motions of all the particles, will be to the motion of the above-mentioned ring around the same axis in a ratio that is compounded of the ratio of the matter in the earth to the matter in the ring and the ratio of three times the square of the quadrantal arc of any circle to two times the square of the diameter—that is, in the ratio of the matter to the matter and of the number 925,275 to the number 1,000,000.*

d. Newton has here changed his terminology, reverting to "sphere."

e. Note that here Newton makes explicit use of the "method" of fluxions, that is, the calculus.

For the motion of a cylinder revolving around its axis at rest is to the motion of an inscribed sphere revolving together with it as any four equal squares to three of the circles inscribed in them; and the motion of the cylinder is to the motion of a very thin ring surrounding the sphere and cylinder at their common contact as twice the matter in the cylinder to three times the matter in the ring; and this motion of the ring, continued uniformly around the axis of the cylinder, is to the uniform motion of the ring about its own diameter (in the same periodic time) as the circumference of the circle to twice its diameter.

Hypothesis 2

If the ring discussed above were to be carried alone in the orbit of the earth about the sun with an annual motion (supposing that all the rest of the earth were removed from it), and if this ring revolved at the same time with a daily motion about its axis, inclined to the plane of the ecliptic at an angle of 23½ degrees, then the motion of the equinoctial points would be the same whether that ring were fluid or consisted of rigid and solid matter.[a]

Proposition 39
Problem 20

To find the precession of the equinoxes.

The mean hourly motion of the nodes of the moon in a circular orbit was, for the nodes in the quadratures, $16''35'''16^{iv}36^{v}$, and half of this, $8''17'''38^{iv}18^{v}$, is (for the reasons explained above [at the end of corol. 2 to prop. 30]) the mean hourly motion of the nodes in such an orbit; and in a whole sidereal year the mean motion adds up to $20°11'46''$ [see beginning of prop. 32]. Therefore, since in a year the nodes of the moon would, in such an orbit, move backward [or in antecedentia] through $20°11'46''$; and since, if there were more moons, the motion of the nodes of each (by book 1, prop. 66, corol. 16) would be as the periodic times; it follows that if the moon revolved near the surface of the earth in the space of a sidereal day, the annual motion of the nodes would be to $20°11'46''$ as a sidereal day of $23^{h}56^{m}$ is to the periodic time of the moon, $27^{d}7^{h}43^{m}$—that is, as 1,436 to 39,343. And the same is true of the nodes of a ring of moons surrounding the earth, whether those moons do not touch one another, or whether they become liquid and take the form of a continuous ring, or finally whether that ring becomes rigid and inflexible.

Let us imagine therefore that this ring, as to its quantity of matter, is equal to all of the earth P*ap*AP*ep*E that lies outside of the globe P*ape*[a] (as in the figure to lem. 2). This globe is to the earth that lies outside of it as aC^2 to $AC^2 - aC^2$, that is (since the earth's smaller semidiameter PC or *a*C is to its greater semidiameter AC as 229 to 230), as 52,441 to 459. Hence, if this ring girded the earth along the equator and both together revolved about the diameter of the ring, the motion of the ring would be to the motion of the interior globe (by lem. 3 of this third book) as 459 to 52,441 and 1,000,000 to 925,275 jointly, that is, as 4,590 to 485,223; and so the motion of the ring would be to the sum of the motions of the ring and globe as 4,590 to 489,813. Hence, if the ring adheres to the globe and communicates to the globe its own motion with which its nodes or equinoctial points regress, the motion that will remain in the ring will be to its former motion as 4,590 to 489,813, and therefore the motion of the equinoctial points will be diminished in the same ratio. Therefore the annual motion of the equinoctial points of a body composed of the ring and the globe will be to the motion 20°11′46″ as 1,436 to 39,343 and 4,590 to 489,813 jointly, that is, as 100 to 292,369. But the forces by which the nodes of the moons [i.e., a ring of moons] regress (as I have explained above), and so by which the equinoctial points of the ring regress (that is, the forces 3IT in the figure to prop. 30), are—in the individual particles—as the distances of those particles from the plane QR, and it is with these forces that the particles recede from the plane; and therefore (by lem. 2), if the matter of the ring were scattered over the whole surface of the globe, as in the configuration P*ap*AP*ep*E, so as to constitute that exterior part of the earth, the total force and efficacy of all the particles to rotate the earth about any diameter of the equator, and thus to move the equinoctial points, would come out less than before in the ratio of 2 to 5. And hence the annual regression of the equinoxes would now be to 20°11′46″ as 10 to 73,092, and accordingly would become 9″56‴50^{iv}.

[b]But because of the inclination of the plane of the equator to the plane of the ecliptic, this motion must be diminished in the ratio of the sine 91,706

a. See the notes to lem. 1 and lem. 2.

bb. Ed. 1 has: "This is the precession of the equinoxes that arises from the force of the sun. Now the force of the moon to move the sea was to the force of the sun as 6⅓ to 1, and this force in proportion to its quantity will also increase the precession of the equinoxes. And therefore the precession arising from both causes will now become greater in the ratio of 7⅓ to 1 and thus will be 45″24‴15^{iv}. This is the motion of the equinoctial points arising from the actions of the sun and the moon on the parts of the (which is the sine of the complement of 23½ degrees [or the cosine of 23½ degrees]) to the radius 100,000. And thus this motion will now become $9''7'''20^{iv}$. This is the annual precession of the equinoxes that arises from the force of the sun.

Now, the force of the moon to move the sea was to the force of the sun as roughly 4.4815 to 1. And the force of the moon to move the equinoxes is to the force of the sun in this same proportion. And so the annual precession of the equinoxes that arises from the force of the moon comes out $40''52'''52^{iv}$, and the total annual precession arising from both forces will be $50''00'''12^{iv}$. And this motion of precession agrees with the phenomena. For, from astronomical observations, the precession of the equinoxes is more or less 50 seconds annually.

If the height of the earth at the equator exceeds its height at the poles by more than 17⅙ miles, its matter will be rarer at the circumference than

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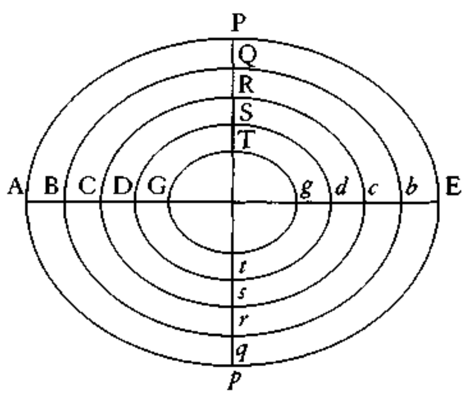
earth that lie on the globe *Pape*. For the earth cannot be inclined in any direction by those actions exerted upon the globe itself. [On Newton's use of "globe," see the note to lem. 2.]

"Now let APE*p* represent the body of the earth, possessing an elliptical shape and consisting of uniform matter. And if this is divided into innumerable elliptical, concentric, and similar figures APE*p*, BQ*bq*, CR*cr*, DS*ds*, . . . , whose diameters are in a geometric progression, then, since the figures are similar, the forces of the sun and the moon under the action of which the equinoctial points regress would cause those same equinoctial points of the remaining figures regarded separately to regress with the same velocity. And the case is the same for the motion of the single orbs AQE*q*, BR*br*, CS*cs*, . . . , which are the differences between those figures. [Newton here uses the word "orb" ("orbits") for the solid we would call an ellipsoid of revolution.] The equinoctial points of each orb, if it were alone, would have to regress with the same velocity. And it does not matter whether any orb



is denser or rarer, provided that it is made up of uniformly dense matter. Hence also if the orbs are denser at the center than at the circumference, the motion of the equinoxes of the whole earth will be the same as before, provided that each orb regarded separately consists of uniformly dense matter and that the shape of the orb is not changed. But if the shapes of the orbs are changed and if the earth now ascends higher than before at the equator AE because of the density of the matter at the center, the regression of the equinoxes will be increased as a result of the increase in the height, and will be increased in single separate orbs in the ratio of the greater height of the matter near the equator of that orb, and in the whole earth in the ratio of the greater height of the matter near the equator of an orb which is not the outermost AQE*q* and not the innermost G*g* but some mean orb CS*cs*. Moreover, we have implied above that the earth is denser at the center and therefore is higher at the equator than at the poles in a at the center; and the precession of the equinoxes has to be increased because of that excess in height, and diminished because of the greater rarity.[b]

We have now described the system of the sun, the earth, the moon, and the planets; something must still be said about comets.

Lemma 4 *The comets are higher than the moon and move in the planetary regions.*

Just as the lack of diurnal parallax requires that comets be located beyond the sublunar regions, so the fact that comets have an annual parallax is convincing evidence that they descend into the regions of the planets. For comets which move forward according to the order of the signs are all, toward the end of their visibility, either slower than normal or retrograde if the earth is between them and the sun, but swifter than they should be if the earth is approaching opposition. And conversely those comets that move

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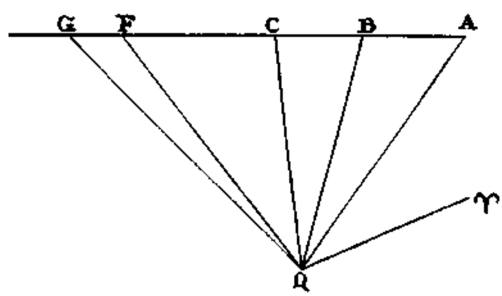
greater ratio than 692 to 689. And the ratio of the greater height can be gathered approximately from the greater diminution of gravity at the equator than that which ought to follow from a ratio of 692 to 689. The deviations by which the length of a seconds pendulum oscillating on Gorée Island and on Cayenne exceeded the length of a pendulum oscillating at Paris in the same time were found by the French to be $\frac{1}{10}$ and $\frac{1}{8}$ of an inch, which, however, from the proportion of 692 to 689, came out $\frac{81}{1,000}$ and $\frac{89}{1,000}$. Therefore the length of a pendulum on Cayenne is greater than it should be in the ratio of $\frac{1}{8}$ to $\frac{89}{1,000}$, or 1,000 to 712, and on Gorée Island, in the ratio of $\frac{1}{10}$ to $\frac{81}{1,000}$, or 1,000 to 810. If we take a mean ratio of 1,000 to 760, the gravity of the earth will have to be diminished at the equator, and its height increased in the same place, in the ratio of 1,000 to 760 very nearly. Hence the motion of the equinoxes (as was said above), if increased in the ratio of the height of the earth, not at the outermost orb, not at the innermost, but at some intermediate orb, that is, not in the greatest ratio of 1,000 to 760, not in the least ratio of 1,000 to 1,000, but in some mean ratio, say 10 to $8\frac{1}{3}$ or 6 to 5, will come out to be $54^{\text{ii}}29^{\text{iii}}6^{\text{iv}}$ annually.

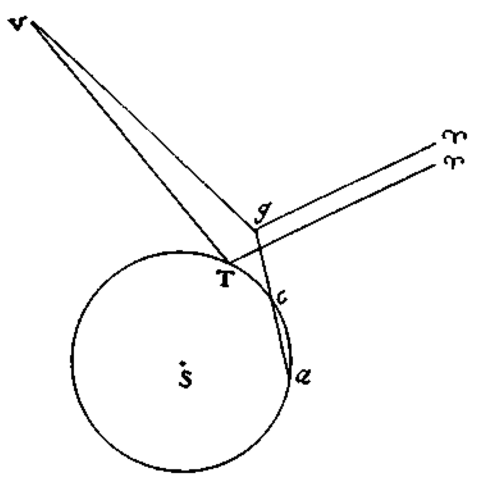
"Again, because of the inclination of the plane of the equator to the plane of the ecliptic, this motion must be diminished, and must be diminished in the ratio of the sine of the complement of the inclination to the radius. For the distance of each terrestrial particle from the plane QR, when the particle is farthest away from the plane of the ecliptic, being (so to speak) in its tropic, is diminished by the inclination of the planes of the ecliptic and the equator to each other, in the ratio of the sine of the complement of the inclination to the radius. And the force of the particle to move the equinoxes is also diminished in the ratio of that distance. The sum of the forces of that same particle is also diminished in the same ratio in places equally distant in both directions from the tropic, as could easily be shown from what has been demonstrated earlier, and therefore the whole force of that particle to move the equinoxes in an entire revolution, as well as the whole force of all the particles, and the motion of the equinoxes arising from that force, is diminished in the same ratio. Therefore since that inclination is $23\frac{1}{2}°$, the motion of $54^{\text{ii}}29^{\text{iii}}$ must be diminished in the ratio of the sine of 91,706 (which is the sine of the complement of $23\frac{1}{2}°$) to the radius 100,000. In this way that motion will now become $49^{\text{ii}}58^{\text{iii}}$. Therefore the points of the equinoxes regress with an annual motion (according to our calculation) of $49^{\text{ii}}58^{\text{iii}}$, nearly as the celestial phenomena require. For that annual regression, from the observations of astronomers, is 50^{ii}."

contrary to the order of the signs are swifter than they should be, at the end of their visibility when the earth is between them and the sun, and slower than they should be or retrograde if the earth is on the opposite side of the sun. This happens principally as a result of the motion of the earth in its different positions [with respect to the comets], just as is the case for the planets, which, according as the motion of the earth is either in the same direction or in an opposite one, are sometimes retrograde, and sometimes seem to advance more slowly and at other times more swiftly. If the earth goes in the same direction as the comet and by its angular motion is carried about the sun so much more swiftly that a straight line continually drawn through the earth and the comet converges toward the regions beyond the comet, then the comet as seen from the earth will appear to be retrograde because of its slower motion; but if the earth is going more slowly, the motion of the comet (taking away the motion of the earth) becomes at least slower. But if the earth goes in a direction opposite to the comet's motion, the motion of the comet will as a result appear speeded up. And from the acceleration or retardation or retrograde motion, the distance of the comet may be ascertained in the following way.

Let ♈QA, ♈QB, and ♈QC be three observed longitudes of a comet at the beginning of its [visible] motion, and let ♈QF be its last observed longitude, just as the comet ceases to be seen. Draw the straight line ABC,



whose parts AB and BC placed between the straight lines QA and QB, and between the straight lines QB and QC, are to each other as the times between the first three observations. Let AC be produced to G, so that AG is to AB as the time between the first and the last observation is to the time between the first and the second observation, and let QG be joined. Then, if the comet moved uniformly in a straight line and the earth were either at rest or also moved forward in a straight line with uniform motion, the angle ♈QG would be the longitude of the comet at the time of the last observation. Therefore, the angle FQG, which is the difference between the longitudes, arises from the inequality of the motions of the comet and of the earth. And this angle, if the earth and comet move in opposite directions, is added to the angle ♈QG, and thus makes the apparent motion of the comet swifter; but if the comet is going in the same direction as the earth, this angle is subtracted from that same angle ♈QG and makes the motion of the comet either slower or possibly retrograde, as I have just explained. Therefore this angle arises chiefly from the motion of the earth and on that account is rightly regarded as the parallax of the comet, ignoring, of course, any increase or decrease in it which could arise from the nonuniform motion of the comet in its own orbit. And the distance of the comet may be ascertained from this parallax in the following manner.



Let S represent the sun, *ac*T the earth's orbit, *a* the place of the earth in the first observation, *c* the place of the earth in the third observation, T the place of the earth in the last observation, and let T♈ be a straight line drawn toward the beginning of Aries. Let angle ♈TV be taken equal to angle ♈QF, that is, equal to the longitude of the comet when the earth is in T. Let *ac* be drawn and produced to *g*, so that *ag* is to *ac* as AG to AC; then *g* will be the place which the earth would reach at the time of the last observation, with its motion uniformly continued in the straight line *ac*. And so if *g*♈ is drawn parallel to T♈, and the angle ♈*g*V is taken equal to the angle ♈QG, this angle ♈*g*V will be equal to the longitude of the comet as seen from place *g*, and the angle TV*g* will be the parallax that arises from the transfer of the earth from place *g* to place T; and accordingly V will be the place of the comet in the plane of the ecliptic. And this place V is ordinarily lower than the orbit of Jupiter.

The same may be ascertained from the curvature of the path of comets. These bodies go almost in great circles as long as they move more swiftly, but at the end of their course, when that part of their apparent motion which arises from parallax has a greater proportion to the total apparent motion, they tend to deviate from such circles, and whenever the earth moves in one direction, they tend to go off in the opposite direction. Because this deviation corresponds to the motion of the earth, it arises chiefly from parallax, and its extraordinary quantity, according to my computation, has placed disappearing comets quite far below Jupiter. Hence it follows that when comets are closer to us, in their perigees and perihelions, they very often descend below the orbits of Mars and of the inferior planets.

The nearness of comets is confirmed also from the light of their heads. For the brightness of a heavenly body illuminated by the sun and going off into distant regions is diminished as the fourth power of the distance; that is, it is diminished as the square because of the increased distance of the body from the sun and diminished as the square again because of the diminished apparent diameter. Thus, if both the quantity of light [i.e., brightness] and the apparent diameter of the comet are given, its distance will be found by taking its distance to the distance of some planet directly in the ratio of diameter to diameter and inversely as the square root of the ratio of light to light. Thus, as observed by Flamsteed through a sixteen-foot telescope and measured with a micrometer, the least diameter of the coma[a] of the comet of the year 1682 equaled 2′0″, while the nucleus or star in the middle of the head occupied scarcely a tenth of this width and therefore was only 11″ or 12″ wide. But in the light and brilliance of its head it surpassed the head of the comet of the year 1680 and rivaled stars of the first or second magnitude. Let us suppose that Saturn with its ring was about four times brighter; then, because the light of the ring almost equals the light of the globe within it, and the apparent diameter of the globe is about 21″, so that the light of the globe and the ring together would equal the light of a globe whose diameter was 30″, it follows that the distance of the comet will be to the distance of Saturn as 1 to $\sqrt{4}$ inversely and 12″ to 30″ directly, that

a. The Latin word "coma" means "head of hair" and is used today to designate the nebulous envelope surrounding the nucleus or head of a comet. Another Latin word for "head of hair" is "capillitium." In book 3, lem. 4, Newton uses "capillitium" for the "head of hair" of a comet, but in prop. 41 he uses "coma." We have translated both as "coma," the term commonly used in English.

is, as 24 to 30 or as 4 to 5. Again, on the authority of Hevelius, the comet of April 1665 surpassed in its brilliance almost all the fixed stars, and even Saturn itself (that is, by reason of its far more vivid color). Indeed, this comet was brighter than the one which had appeared at the end of the preceding year and was comparable to stars of the first magnitude. The width of the comet's coma was about $6'$, but the nucleus, when compared with the planets by the aid of a telescope, was clearly smaller than Jupiter and was judged to be sometimes smaller than the central body of Saturn and sometimes equal to it. Further, since the diameter of the coma of comets rarely exceeds $8'$ or $12'$, and the diameter of the nucleus or central star is about a tenth or perhaps a fifteenth of the diameter of the coma, it is evident that such stars generally have the same apparent magnitude as the planets. Hence, since their light can often be compared to the light of Saturn and sometimes surpasses it, it is manifest that all the comets in their perihelions should be placed either below Saturn or not far above. Those who banish the comets almost to the region of the fixed stars are, therefore, entirely wrong; certainly in such a situation, they would not be illuminated by our sun any more than the planets in our solar system are illuminated by the fixed stars.

In treating these matters, we have not been considering the obscuring of comets by that very copious and thick smoke by which the head is surrounded, always gleaming dully as if through a cloud. For the darker the body is rendered by this smoke, the closer it must approach to the sun for the amount of light reflected from it to rival that of the planets. This makes it likely that the comets descend far below the sphere of Saturn, as we have proved from their parallax.

But this same result is, to the highest degree, confirmed from their tails. These arise either from reflection by the smoke scattered through the aether or from the light of the head. In the first case the distance of the comets must be diminished, since otherwise the smoke always arising from the head would be propagated through spaces far too great, with such a velocity and expansion as to be unbelievable. In the second case, all the light of both the tail and the coma must be ascribed to the nucleus of the head. Therefore, if we suppose that all this light is united and condensed within the disc of the nucleus, then certainly that nucleus, whenever it emits a very large and very bright tail, will far surpass in its brilliance even Jupiter itself. Therefore, if it has a smaller apparent diameter and is sending forth more light, it will be much more illuminated by the sun and thus will be much closer to the sun. By the same argument, furthermore, the heads ought to be located below the orbit of Venus, when they are hidden under the sun and emit tails both very great and very bright like fiery beams, as they do sometimes. For if all of that light were understood to be gathered together into a single star, it would sometimes surpass Venus itself, not to say several Venuses combined.

Finally, the same thing may be ascertained from the light of the heads, which increases as comets recede from the earth toward the sun and decreases as they recede from the sun toward the earth. Thus the latter comet of 1665 (according to the observations of Hevelius), from the time when it began to be seen, was always decreasing in its apparent motion and therefore had already passed its perigee; but the splendor of its head nevertheless increased from day to day until the comet, concealed by the sun's rays, ceased to be visible. The comet of 1683 (also according to the observations of Hevelius) at the end of July, when it was first sighted, was moving very slowly, advancing about 40′ or 45′ in its orbit each day. From that time its daily motion kept increasing continually until 4 September, when it came to about 5°. Therefore, in all this time the comet was approaching the earth. This is gathered also from the diameter of the head, as measured with a micrometer, since Hevelius found it to be on 6 August only 6′5″ including the coma, but on 2 September 9′7″. Therefore the head appeared far smaller at the beginning than at the end of the motion; yet at the beginning the head showed itself far brighter in the vicinity of the sun than toward the end of its motion, as Hevelius also reports. Accordingly, in all this time, because of its receding from the sun, it decreased with respect to its light, notwithstanding its approach to the earth.

The comet of 1618, about the middle of December, and that of 1680, about the end of the same month, were moving very swiftly and therefore were then in their perigees. Yet the greatest splendor of their heads occurred about two weeks earlier, when they had just emerged from the sun's rays, and the greatest splendor of their tails occurred a little before that, when they were even nearer to the sun. The head of the first of these comets, according to the observations of [Johann Baptist] Cysat, seemed on 1 December to be greater than stars of the first magnitude, and on 16 December (being now in its perigee) it had failed little in magnitude, but very much in the splendor or clarity of its light. On 7 January Kepler, being uncertain about its head, brought his observing to an end. On 12 December the head of the second of these comets was sighted, and was observed by Flamsteed at a distance of 9° from the sun, a thing which would scarcely have been possible in a star of the third magnitude. On 15 and 17 December it appeared as a star of the third magnitude, since it was diminished by the brightness of clouds near the setting sun. On 26 December, moving with the greatest speed and being almost in its perigee, it was less than the mouth of Pegasus, a star of the third magnitude. On 3 January it appeared as a star of the fourth magnitude, on 9 January as a star of the fifth magnitude, and on 13 January it disappeared from view, as a result of the splendor of the crescent moon. On 25 January it scarcely equaled stars of the seventh magnitude. If equal times are taken on both sides of the perigee (before and after), then the head, being placed at those times in distant regions, ought to have shone with equal brilliance because of its equal distances from the earth, but it appeared brightest in the region [on the side of the perigee] toward the sun and disappeared on the other side of the perigee. Therefore from the great difference of light in these two situations, it is concluded that there is a great nearness of the sun and the comet in the first of these situations. For the light of comets tends to be regular and be greatest when the heads move most swiftly, and accordingly are in their perigees, except insofar as this light becomes greater in the vicinity of the sun.

Corollary 1. Therefore comets shine by the sun's light reflected from them.

Corollary 2. From what has been said it will also be understood why comets appear so frequently in the region of the sun. If they were visible in the regions far beyond Saturn, they would have to appear more often in the parts of the sky opposite to the sun. For those that were in these parts would be nearer to the earth; and the sun, being in between, would obscure the others. Yet in running through the histories of comets, I have found that four or five times more have been detected in the hemisphere toward the sun than in the opposite hemisphere, besides without doubt not a few others which the sun's light hid from view. Certainly, in their descent to our regions comets neither emit tails nor are so brightly illuminated by the sun that they show themselves to the naked eye so as to be discovered before they are closer to us than Jupiter itself. But by far the greater part of the space described about the sun with so small a radius is situated on the side of the earth that faces the sun, and comets are generally more brightly illuminated in that greater part, since they are much closer to the sun.

COROLLARY 3. Hence also it is manifest that the heavens are lacking in resistance. For the comets, following paths that are oblique and sometimes contrary to the course of the planets, move in all directions very freely and preserve their motions for a very long time even when these are contrary to the course of the planets. Unless I am mistaken, comets are a kind of planet and revolve in their orbits with a continual motion. For there seems to be no foundation for the allegation of some writers, basing their argument on the continual changes of the heads, that comets are meteors. The heads of comets are encompassed with huge atmospheres, and the atmospheres must be denser as one goes lower. Therefore, it is in these clouds, and not in the very bodies of the comets, that those changes are seen. Thus, if the earth were viewed from the planets, it would doubtless shine with the light of its own clouds, and its solid body would be almost hidden beneath the clouds. Thus, the belts of Jupiter are formed in the clouds of that planet, since they change their situation relative to one another, and the solid body of Jupiter is seen with greater difficulty through those clouds. And the bodies of comets must be much more hidden beneath their atmospheres, which are both deeper and thicker.

Proposition 40
Theorem 20

Comets move in conics having their foci in the center of the sun, and by radii drawn to the sun, they describe areas proportional to the times.

This is evident by corol. 1 to prop. 13 of the first book compared with props. 8, 12, and 13 of the third book.

COROLLARY 1. Hence, if comets revolve in orbits, these orbits will be ellipses, and the periodic times will be to the periodic times of the planets as the $3/2$ powers of their principal axes. And therefore comets, for the most part being beyond the planets and on that account describing orbits with greater axes, will revolve more slowly. For example, if the axis of the orbit of a comet were four times greater than the axis of the orbit of Saturn, the time of a revolution of the comet would be to the time of a revolution of Saturn (that is, to 30 years) as $4\sqrt{4}$ (or 8) to 1, and accordingly would be 240 years.

COROLLARY 2. But these orbits will be so close to parabolas that parabolas can be substituted for them without sensible errors.

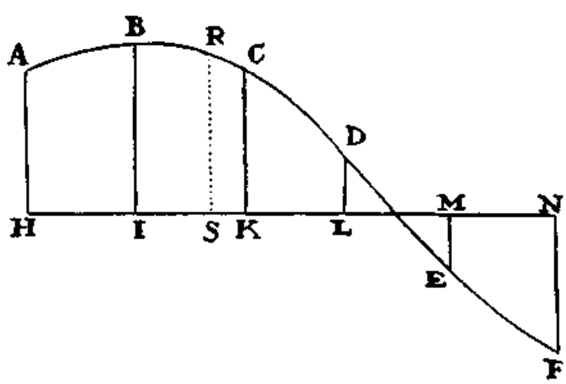
Corollary 3. And therefore (by book 1, prop. 16, corol. 7) the velocity of every comet will always be to the velocity of any planet, [considered to be] revolving in a circle about the sun, very nearly as the square root of twice the distance of the planet from the center of the sun to the distance of the comet from the center of the sun. Let us take the radius of the earth's orbit (or the greatest semidiameter of the ellipse in which the earth revolves) to be of 100,000,000 parts; then the earth will describe by its mean daily motion 1,720,212 of these parts, and by its hourly motion 71,675½ parts. And therefore the comet, at the same mean distance of the earth from the sun, and having a velocity that is to the velocity of the earth as $\sqrt{2}$ to 1, will describe by its daily motion 2,432,747 of these parts, and by its hourly motion 101,364½ parts. But at greater or smaller distances, both the daily and the hourly motion will be to this daily and hourly motion as the square root of the ratio of the distances inversely, and therefore is given.

Corollary 4. Hence, if the latus rectum of a parabola is four times greater than the radius of the earth's orbit, and if the square of that radius is taken to be 100,000,000 parts, the area that the comet describes each day by a radius drawn to the sun will be 1,216,373½ parts, and in each hour that area will be 50,682¼ parts. But if the latus rectum is greater or smaller in any ratio, then the daily and hourly area will be greater or smaller, as the square root of that ratio.

Lemma 5 *To find a parabolic curve that will pass through any number of given points.*

Let the points be A, B, C, D, E, F, . . . , and from them to any straight line HN, given in position, drop the perpendiculars AH, BI, CK, DL, EM, FN, ...

b *2b* *3b* *4b* *5b*
c *2c* *3c* *4c*
d *2d* *3d*
e *2e*
f



CASE 1. If the intervals HI, IK, KL, ... between the points H, I, K, L, M, N are equal, take the first differences b, b_2, b_3, b_4, b_5, ... of the perpendiculars AH, BI, CK, ...; the second differences c, c_2, c_3, c_4, ...; the third differences d, d_2, d_3, ...; that is, in such a way that $\text{AH} - \text{BI} = b$, $\text{BI} - \text{CK} = b_2$, $\text{CK} - \text{DL} = b_3$, $\text{DL} + \text{EM} = b_4$, $-\text{EM} + \text{FN} = b_5, \ldots,$ then $b - b_2 = c, \ldots,$ and go on in this way to the last difference, which here is f. Then, if any perpendicular RS is erected, which is to be an ordinate to the required curve, in order to find its length, suppose each of the intervals HI, IK, KL, LM, ... to be unity, and let AH be equal to a, $-\text{HS} = p$, $\frac{1}{2}p \times (-\text{IS}) = q$, $\frac{1}{3}q \times (+\text{SK}) = r$, $\frac{1}{4}r \times (+\text{SL}) = s$, $\frac{1}{5}s \times (+\text{SM}) = t$, proceeding, that is, up to the penultimate perpendicular ME, and prefixing negative signs to the terms HS, IS, ..., which lie on the same side of the point S as A, and positive signs to the terms SK, SL, ..., which lie on the other side of the point S. Then if the signs are observed exactly, RS will be $= a + bp + cq + dr + es + ft + \ldots$.

CASE 2. But if the intervals HI, IK, ... between the points H, I, K, L, ... are unequal, take b, b_2, b_3, b_4, b_5, ..., the first differences of the perpendiculars AH, BI, CK, ... divided by the intervals between the perpendiculars; take c, c_2, c_3, c_4, ..., the second differences divided by each two intervals; d, d_2, d_3, ..., the third differences divided by each three intervals; e, e_2, ..., the fourth differences divided by each four intervals, and so on—that is, in such a way that $b = \dfrac{\text{AH} - \text{BI}}{\text{HI}}$, $b_2 = \dfrac{\text{BI} - \text{CK}}{\text{IK}}$, $b_3 = \dfrac{\text{CK} - \text{DL}}{\text{KL}}, \ldots,$ and then $c = \dfrac{b - b_2}{\text{HK}}$, $c_2 = \dfrac{b_2 - b_3}{\text{IL}}$, $c_3 = \dfrac{b_3 - b_4}{\text{KM}}, \ldots,$ and afterward $d = \dfrac{c - c_2}{\text{HL}}$, $d_2 = \dfrac{c_2 - c_3}{\text{IM}} \ldots$. When these differences have been found, let AH be equal to a, $-\text{HS} = p$, $p \times (-\text{IS}) = q$, $q \times (+\text{SK}) = r$, $r \times (+\text{SL}) = s$, $s \times (+\text{SM}) = t$, proceeding, that is, up to the penultimate perpendicular ME; then the ordinate RS will be $= a + bp + cq + dr + es + ft + \ldots$.

COROLLARY. Hence the areas of all curves can be found very nearly. For if several points are found of any curve which is to be squared [i.e., any curve whose area is desired] and a parabola is understood to be drawn through them, the area of this parabola will be very nearly the same as the area of that curve which is to be squared. Moreover, a parabola can always be squared geometrically by methods which are very well known.

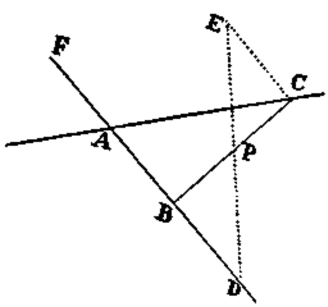
Lemma 6 *From several observed places of a comet, to find its place at any given intermediate time.*

Let HI, IK, KL, LM represent the times between the observations (in the figure to lem. 5), HA, IB, KC, LD, ME five observed longitudes of the comet, and HS the given time between the first observation and the required longitude. Then, if a regular curve ABCDE is understood to be drawn through the points A, B, C, D, E, and if the ordinate RS is found by the above lemma, RS will be the required longitude.

By the same method the latitude at a given time is found from five observed latitudes.

If the differences of the observed longitudes are small, say only 4 or 5 degrees, three or four observations would suffice for finding the new longitude and latitude. But if the differences are greater, say 10 or 20 degrees, five observations must be used.

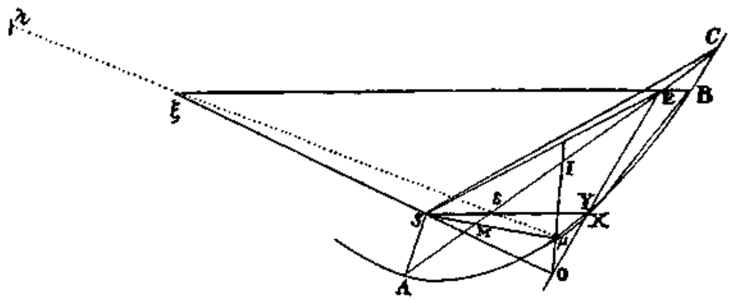
Lemma 7 *To draw a straight line* BC *through a given point* P, *so that the parts* PB *and* PC *of that line, cut off by two straight lines* AB *and* AC, *given in position, have a given ratio to each other.*



From that point P draw any straight line PD to either of the straight lines, say AB, and produce PD toward the other straight line AC as far as E, so that PE is to PD in the given ratio. Let EC be parallel to AD; and if CPB is drawn, PC will be to PB as PE to PD. Q.E.F.

Lemma 8 *Let* ABC *be a parabola with focus* S. *Let the segment* ABCI *be cut off by the chord* AC *(which is bisected at* I*), let its diameter be* Iμ, *and let its vertex be* μ. *On* Iμ *produced, take* μO *equal to half of* Iμ. *Join* OS *and produce it to* ξ, *so that* Sξ *is equal to* 2SO. *Then, if a comet* B *moves in the arc* CBA, *and if* ξB *is drawn cutting* AC *in* E, *I say that the point* E *will cut off from the chord* AC *the segment* AE *very nearly proportional to the time.*

For join EO, cutting the parabolic arc ABC in Y, and draw μX so as to touch the same arc in the vertex μ and meet EO in X; then the curvilinear area AEXμA will be to the curvilinear area ACYμA as AE to AC. And thus, since triangle ASE is in the same ratio to triangle ASC as the ratio of those curvilinear areas, the total area ASEXμA will be to the total area



ASCYμA as AE to AC. Moreover, since ξO is to SO as 3 to 1, and EO is in the same ratio to XO, SX will be parallel to EB; and therefore, if BX is joined, the triangle SEB will be equal to the triangle XEB. Thus, if the triangle EXB is added to the area ASEXμA and from that sum the triangle SEB is taken away, there will remain the area ASBXμA equal to the area ASEXμA, and thus it will be to the area ASCYμA as AE to AC. But the area ASBYμA is very nearly equal to the area ASBXμA, and the area ASBYμA is to the area ASCYμA as the time in which the arc AB is described to the time of describing the total arc AC. And thus AE is to AC very nearly in the ratio of the times. Q.E.D.

COROLLARY. When point B falls upon the vertex μ of the parabola, AE is to AC exactly in the ratio of the times.

Scholium

If $\mu\xi$ is joined, cutting AC at δ, and if ξn, which is to μB as 27MI to 16Mμ, is taken in this line, then when Bn is drawn it will cut the chord AC more nearly in the ratio of the lines than before. But the point n is to be taken so as to lie beyond point ξ if point B is more distant than point μ from the principal vertex of the parabola; and contrariwise if B is less distant from that vertex.

Lemma 9

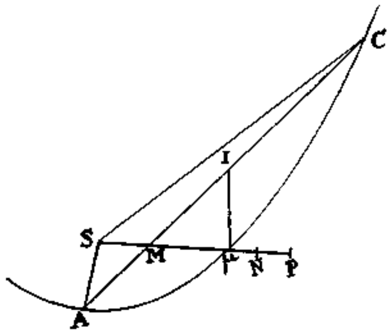
The straight lines Iμ *and* μM *and the length* $\frac{\text{AIC}}{4\text{S}\mu}$ *are equal to one another.*

For 4Sμ is the latus rectum of a parabola, extending to the vertex μ.

Lemma 10

Let Sμ *be produced to* N *and* P, *so that* μN *is one-third of* μI, *and so that* SP *is to* SN *as* SN *to* Sμ. *Then, in the time in which a comet describes the arc* AμC, *it would—if it moved forward always with the velocity that it has at a height equal to* SP—*describe a length equal to the chord* AC.

For if the comet were to move forward in the same time uniformly in the straight line that touches the parabola at μ, and with the velocity that it has in μ, then the area that it would describe by a radius drawn to point S would be equal to the parabolic area ASCμ. And hence the space determined by the length described along the tangent and the length Sμ would be to the space determined by the lengths AC and SM as the area ASCμ to the triangle ASC, that is, as SN to SM. Therefore, AC is to the length described along the tangent as Sμ to SN. But the velocity of the comet at the height SP is (by book 1, prop. 16, corol. 6) to its velocity at the height Sμ as the square root of the ratio of SP to Sμ inversely, that is, in the ratio of Sμ to SN; hence the length described in the same time with this velocity will be to the length described along the tangent as Sμ to SN. Therefore, since AC and the length described with this new velocity are in the same ratio to the length described along the tangent, they are equal to each other. Q.E.D.



COROLLARY. Therefore, in that same time, the comet, with the velocity that it has at the height S$\mu + \frac{2}{3}$Iμ, would describe the chord AC very nearly.

Lemma 11 *Suppose a comet, deprived of all motion, to be let fall from the height* SN *or* Sμ + $\frac{1}{3}$Iμ, *so as to fall toward the sun, and suppose this comet to be urged toward the sun always by that force, uniformly continued, by which it is urged at the beginning. Then in half of the time in which the comet describes the arc* AC *in its orbit, it would—in this descent toward the sun—describe a space equal to the length* Iμ.

For by lem. 10, in the same time in which the comet describes the parabolic arc AC, it will—with the velocity that it has at the height SP—describe the chord AC; and hence (by book 1, prop. 16, corol. 7), revolving by the force of its own gravity, it would—in that same time, in a circle whose semidiameter was SP—describe an arc whose length would be to the chord AC of the parabolic arc in the ratio of 1 to $\sqrt{2}$. And therefore, falling from the height SP toward the sun with the weight that it has toward the sun at that height, it would in half that time (by book 1, prop. 4, corol. 9) describe a space equal to the square of half of that chord, divided by four times the height SP, that is, the space $\frac{AI^2}{4SP}$. Thus, since the weight of the comet toward the sun at the height SN is to its weight toward the sun at the height SP as SP to $S\mu$, the comet—falling toward the sun with the weight that it has at the height SN—will in the same time describe the space $\frac{AI^2}{4S\mu}$, that is, a space equal to the length $I\mu$ or $M\mu$. Q.E.D.

Proposition 41
Problem 21

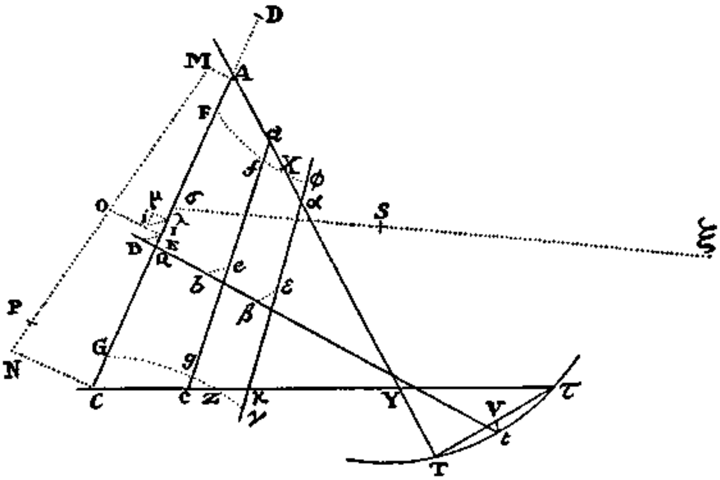
To determine the trajectory of a comet moving in a parabola, from three given observations.

Having tried many approaches to this exceedingly difficult problem, I devised certain problems [i.e., propositions] in book 1 which are intended for its solution. But later on, I conceived the following slightly simpler solution.

Let three observations be chosen, distant from one another by nearly equal intervals of time. But let that interval of time when the comet moves more slowly be a little greater than the other, that is, so that the difference of the times is to the sum of the times as the sum of the times to more or less six hundred days, or so that the point E (in the figure to lem. 8) falls very nearly on the point M and deviates from there toward I rather than toward A. If such observations are not at hand, a new place of the comet must be found by the method of lem. 6.

Let S represent the sun; T, t, and τ three places of the earth in its orbit; TA, tB, and τC three observed longitudes of the comet; V the time between the first observation and the second; W the time between the second and the third; X the length that the comet could describe in that total time [V + W] with the velocity that it has in the mean distance of the earth from the sun (which length is to be found by the method of book 3, prop. 40, corol. 3); and let tV be a perpendicular to the chord $T\tau$. In the mean observed longitude tB, let the point B be taken anywhere at all for the place of the comet in the plane of the ecliptic, and from there toward the sun S draw line BE so as to be to the sagitta tV as the content[a] of SB and St^2 to the cube of the hypotenuse of the right-angled triangle whose sides are SB and the tangent of the latitude of the comet in the second observation to the radius tB. And

a. Here "content" has the sense of Newton's "solid," that is, the result of multiplying SB by the square of St.



through point E (by lem. 7 of this third book) draw the straight line AEC so that its parts AE and EC, terminated in the straight lines TA and τC, are to each other as the times V and W. Then A and C will be the places of the comet in the plane of the ecliptic in the first and third observations very nearly, provided that B is its correctly assumed place in the second observation.

Upon AC, bisected in I, erect a perpendicular I*i*. Through point B let a line B*i* be imagined,[b] drawn parallel to AC. Let S*i* be a line imagined as cutting AC at λ, and complete the parallelogram $i\mathrm{I}\lambda\mu$. Take Iσ equal to 3Iλ, and through the sun S draw the dotted line $\sigma\xi$ equal to $3\mathrm{S}\sigma + 3i\lambda$.

b. In prop. 41, Newton refers to the line B*i* as "[lineam] occultam B*i*," directing that "Per punctum B age occultam B*i*," literally, "Through point B draw the occult line B*i*." In the next sentence, the same adjective, "occult," is applied to the line S*i*; in the following sentence, the line $\sigma\xi$ is said to be "occult." (In the first and second editions, there is another "occult" line OD.) In the final paragraph, there is a reference to the "occult" line AC.

In Newton's day, according to the *Oxford English Dictionary*, the adjective "occult" was used to denote "a line drawn in the construction of a figure, but not forming part of the finished figure," and also to denote a dotted line. The diagram for prop. 41 does not show any lines B*i*, S*i*, OD, or AC, but $\sigma\xi$ does appear as a dotted line. Accordingly, we have translated "occult" in the sense of "imagined" in the case of lines S*i*, B*i*, and AC, and as "dotted" in the case of line $\sigma\xi$.

Perhaps the reason why Newton has referred to all these lines (both invisible and dotted) as "occult" is that, in the original diagram that he drew for the cutter of the wood block, he did not show $\sigma\xi$ as a dotted line. In this case, all of these lines would have been "occult" or hidden from view, invisible and only imagined.

And after deleting the letters A, B, C, and I, let a new imagined line BE be drawn from the point B toward the point ξ so that it is to the former line BE as the square of the distance BS to the quantity $S\mu + \frac{1}{3}i\lambda$. And through point E again draw the straight line AEC according to the same rule as before, that is, so that its parts AE and EC are to each other as the times V and W between observations. Then A and C will be the places of the comet more exactly.

Upon AC, bisected in I, erect the perpendiculars AM, CN, and IO, so that, of these perpendiculars, AM and CN are the tangents of the latitudes[c] in the first and third observations (to the radii TA and τC). Join MN, cutting IO in O. Construct the rectangle $iI\lambda\mu$ as before. On IA produced, take ID equal to $S\mu + \frac{2}{3}i\lambda$. Then on MN, toward N, take MP so that it is to the length X found above as the square root of the ratio of the mean distance of the earth from the sun (or of the semidiameter of the earth's orbit) to the distance OD. If point P falls upon point N, then A, B, and C will be three places of the comet, through which its orbit is to be described in the plane of the ecliptic. But if point P does not fall upon point N, then on the straight line AC take CG equal to NP, in such a way that points G and P lie on the same side of the straight line NC.

Using the same method by which points E, A, C, and G were found from the assumed point B, find from other points b and β (assumed in any way whatever) the new points e, a, c, g, and ε, α, κ, γ. Then if the circumference of circle $Gg\gamma$ is drawn through G, g, and γ, cutting the straight line τC in Z, Z will be a place of the comet in the plane of the ecliptic. And if on AC, ac, and $\alpha\kappa$, there are taken AF, af, and $\alpha\varphi$, equal respectively to CG, cg, and $\kappa\gamma$, and if the circumference of a circle $Ff\varphi$ is drawn through points

c. Basically, here Newton is determining a comet's distance from its latitude and longitude as determined by a terrestrial observer. He guesses a position B of the comet in the plane of the ecliptic and then determines the altitude (or distance above the ecliptic), and so can construct a right triangle, of which one side is SB (a line drawn from the sun to the point B on the ecliptic) and other is tB times the tangent of the latitude of the comet (which Newton writes as: the tangent of the latitude of the comet in the second observation to the radius tB).

For an extensive gloss on prop. 41 and on Newton's theory of comets, see A. N. Kriloff, "On Sir Isaac Newton's Method of Determining the Parabolic Orbit of a Comet," *Monthly Notices of the Royal Astronomical Society* 85 (1925): 640–656; see also the enlightening discussion by S. Chandrasekhar, *Newton's "Principia" for the Common Reader* (Oxford: Clarendon Press, 1995), pp. 514–529, especially the diagram on p. 514.

F, f, and φ, cutting the straight line AT in X, then point X will be another place of the comet in the plane of the ecliptic. At the points X and Z, erect the tangents of the latitudes of the comet (to the radii TX and τZ), and two places of the comet in its orbit will be found. Finally (by book 1, prop. 19), let a parabola with focus S be described through those two places; this parabola will be the trajectory of the comet. Q.E.I.

The demonstration of this construction follows from the lemmas, since the straight line AC is cut in E in the ratio of the times, by lem. 7, as required by lem. 8; and since BE, by lem. 11, is that part of the straight line BS or Bξ which lies in the plane of the ecliptic between the arc ABC and the chord AEC; and since MP (by lem. 10, corol.) is the length of the chord of the arc that the comet must describe in its orbit between the first observation and the third, and therefore would be equal to MN, provided that B is a true place of the comet in the plane of the ecliptic.

But it is best not to choose the points B, b, and β any place whatever, but to take them as close to true as possible. If the angle AQt, at which the projection of the orbit described in the plane of the ecliptic cuts the straight line tB, is known approximately, imagine the straight line AC drawn at that angle so that it is to $\frac{4}{3}$Tτ as the square root of the ratio of SQ to St. And by drawing the straight line SEB, so that its part EB is equal to the length Vt, point B will be determined, which may be used the first time around. Then, after deleting the straight line AC and drawing AC anew according to the preceding construction, and after additionally finding the length MP, take point b on tB according to the rule that if TA and τC cut each other in Y, the distance Yb is to the distance YB in a ratio compounded of the ratio of MP to MN and the square root of the ratio of SB to Sb. And the third point β will have to be found by the same method, if it is desired to repeat the operation for the third time. But by this method two operations would, for the most part, be sufficient. For if the distance Bb happens to be very small, then after the points F, f and G, g have been found, the straight lines Ff and Gg (when drawn) will cut TA and τC in the required points X and Z.

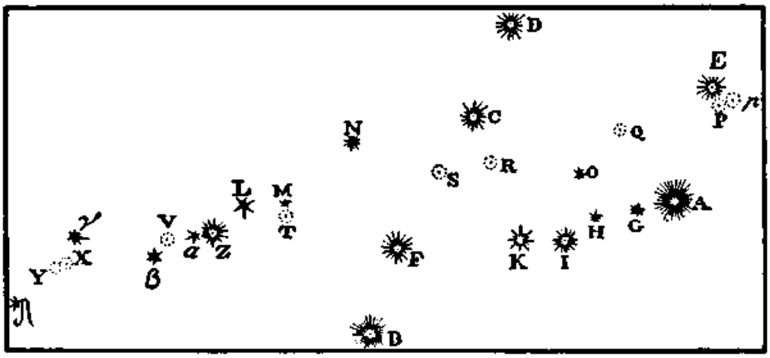
Example. Let the comet of 1680 be proposed as the example. The following table shows its motion as observed by Flamsteed and as calculated by him from these observations, and corrected by Halley on the basis of the same observations.

	Apparent time	*True time*	*Longitude of the sun*	*Longitude of the comet*	*North latitude of the comet*
	h m	h m s	° ′ ″	° ′ ″	° ′ ″
1680, Dec. 12	4 46	4 46 0	♑ 1 51 23	♑ 6 32 30	8 28 0
21	6 32½	6 36 59	11 6 44	♒ 5 8 12	21 42 13
24	6 12	6 17 52	14 9 26	18 49 23	25 23 5
26	5 14	5 20 44	16 9 22	28 24 13	27 0 52
29	7 55	8 3 2	19 19 43	♓ 13 10 41	28 9 58
30	8 2	8 10 26	20 21 9	17 38 20	28 11 53
1681, Jan. 5	5 51	6 1 38	26 22 18	♈ 8 48 53	26 15 7
9	6 49	7 0 53	♒ 0 29 2	18 44 4	24 11 56
10	5 54	6 6 10	1 27 43	20 40 50	23 43 52
13	6 56	7 8 55	4 33 20	25 59 48	22 17 28
25	7 44	7 58 42	16 45 36	♉ 9 35 0	17 56 30
30	8 7	8 21 53	21 49 58	13 19 51	16 42 18
Feb. 2	6 20	6 34 51	24 46 59	15 13 53	16 4 1
5	6 50	7 4 41	27 49 51	16 59 6	15 27 3

To these add certain observations of my own.

	Apparent time	*Longitude of the comet*	*North latitude of the comet*
	h m	° ′ ″	° ′ ″
1681, Feb. 25	8 30	♉ 26 18 35	12 46 46
27	8 15	27 4 30	12 36 12
Mar. 1	11 0	27 52 42	12 23 40
2	8 0	28 12 48	12 19 38
5	11 30	29 18 0	12 3 16
7	9 30	♊ 0 4 0	11 57 0
9	8 30	0 43 4	11 45 52

These observations were made with a seven-foot telescope, and a micrometer the threads of which were placed in the focus of the telescope; and with these instruments we determined both the positions of the fixed stars in relation to one another and the positions of the comet in relation to the fixed stars. Let A represent the star of the fourth magnitude in the left heel of Perseus (Bayer's o), B the following star of the third magnitude in the left foot (Bayer's ζ), C the star of the sixth magnitude in the heel of the same



foot (Bayer's n), and D, E, F, G, H, I, K, L, M, N, O, Z, α, β, γ, and δ other smaller stars in the same foot. And let p, P, Q, R, S, T, V, and X be the places of the comet in the observations described above; and, the distance AB being reckoned at $80\frac{7}{12}$ parts, AC was $52\frac{1}{4}$ parts, BC $58\frac{5}{6}$, AD $57\frac{5}{12}$, BD $82\frac{6}{11}$, CD $23\frac{2}{3}$, AE $29\frac{4}{7}$, CE $57\frac{1}{2}$, DE $49\frac{11}{12}$, AI $27\frac{7}{12}$, BI $52\frac{1}{6}$, CI $36\frac{7}{12}$, DI $53\frac{5}{11}$, AK $38\frac{2}{3}$, BK 43, CK $31\frac{5}{9}$, FK 29, FB 23, FC $36\frac{1}{4}$, AH $18\frac{6}{7}$, DH $50\frac{7}{8}$, BN $46\frac{5}{12}$, CN $31\frac{1}{3}$, BL $45\frac{5}{12}$, NL $31\frac{5}{7}$. HO was to HI as 7 to 6 and, when produced, passed between stars D and E in such a way that the distance of star D from this straight line was $\frac{1}{6}$CD. LM was to LN as 2 to 9 and, when produced, passed through star H. Thus the positions of the fixed stars in relation to one another were determined.

Finally our fellow countryman Pound again observed the positions of these fixed stars in relation to one another and recorded their longitudes and latitudes, as in the following table.

The fixed stars	*Longitudes*				*Latitudes north*			*The fixed stars*	*Longitudes*				*Latitudes north*		
		°	′	″	°	′	″			°	′	″	°	′	″
A	♉	26	41	50	12	8	36	L	♉	29	33	34	12	7	48
B		28	40	23	11	17	54	M		29	18	54	12	7	20
C		27	58	30	12	40	25	N		28	48	29	12	31	9
E		26	27	17	12	52	7	Z		29	44	48	11	57	13
F		28	28	37	11	52	22	α		29	52	3	11	55	48
G		26	56	8	12	4	58	β	♊	0	8	23	11	48	56
H		27	11	45	12	2	1	γ		0	40	10	11	55	18
I		27	25	2	11	53	11	δ		1	3	20	11	30	42
K		27	42	7	11	53	26								

I observed the positions of the comet in relation to these fixed stars as follows.

On Friday, 25 February (O.S.), at 8^h30^m P.M., the distance of the comet, which was at p, from star E was less than $\frac{3}{13}$AE, and greater than $\frac{1}{5}$AE, and thus was approximately $\frac{3}{14}$AE; and the angle ApE was somewhat obtuse, but almost a right angle. For if a perpendicular were dropped from A to pE, the distance of the comet from that perpendicular was $\frac{1}{5}p$E.

On the same night at 9^h30^m, the distance of the comet (which was at P) from star E was greater than $\frac{1}{4\frac{1}{2}}$AE and less than $\frac{1}{5\frac{1}{4}}$AE, and thus was very nearly $\frac{1}{4\frac{7}{8}}$AE, or $\frac{8}{39}$AE. And the distance of the comet from a perpendicular dropped from star A to the straight line PE was $\frac{4}{5}$PE.

On Sunday, 27 February, at 8^h15^m P.M., the distance of the comet (which was at Q) from star O equaled the distance between stars O and H; and the straight line QO, produced, passed between stars K and B. Because of intervening clouds, I could not determine the position of this straight line more exactly.

On Tuesday, 1 March, at 11^h P.M., the comet (which was at R) lay exactly between stars K and C; and the part CR of the straight line CRK was a little greater than $\frac{1}{3}$CK and a little smaller than $\frac{1}{3}$CK + $\frac{1}{8}$CR, and thus was equal to $\frac{1}{3}$CK + $\frac{1}{16}$CR, or $\frac{16}{45}$CK.

On Wednesday, 2 March, at 8^h P.M., the distance of the comet (which was at S) from star C was very close to $\frac{4}{9}$FC. The distance of star F from the straight line CS, produced, was $\frac{1}{24}$FC, and the distance of star B from that same straight line was five times greater than the distance of star F. Also, the straight line NS, produced, passed between stars H and I and was five or six times nearer to star H than to star I.

On Saturday, 5 March, at 11^h30^m P.M. (when the comet was at T), the straight line MT was equal to $\frac{1}{2}$ML, and the straight line LT, produced, passed between B and F four or five times closer to F than to B, cutting off from BF a fifth or sixth part of it toward F. And MT, produced, passed outside the space BF on the side of star B and was four times closer to star B than to star F. M was a very small star that could scarcely be seen through the telescope, and L was a greater star, of about the eighth magnitude.

On Monday, 7 March, at 9^h30^m P.M. (when the comet was at V), the straight line $V\alpha$, produced, passed between B and F, cutting off $\frac{1}{10}$BF from BF on the side of F, and was to the straight line $V\beta$ as 5 to 4. And the distance of the comet from the straight line $\alpha\beta$ was $\frac{1}{2}V\beta$.

On Wednesday, 9 March, at 8^h30^m P.M. (when the comet was at X), the straight line γX was equal to $\frac{1}{4}\gamma\delta$, and a perpendicular dropped from star δ to the straight line γX was $\frac{2}{5}\gamma\delta$.

On the same night at 12^h (when the comet was at Y), the straight line γY was equal to $\frac{1}{3}\gamma\delta$ or a little smaller, say $\frac{5}{16}\gamma\delta$, and a perpendicular dropped from star δ to the straight line γY was equal to about $\frac{1}{6}$ or $\frac{1}{7}\gamma\delta$. But the comet could scarcely be discerned because of its nearness to the horizon, nor could its place be determined so surely as in the preceding observations.

From observations of this sort, by constructions of diagrams, and by calculations, I found the longitudes and latitudes of the comet, and from the corrected places of the fixed stars our fellow countryman Pound corrected the places of the comet, and these corrected places are given above. I used a crudely made micrometer, but nevertheless the errors of longitudes and latitudes (insofar as they come from my observations) scarcely exceed one minute. Moreover, the comet (according to my observations) at the end of its motion began to decline noticeably toward the north from the parallel which it had occupied at the end of February.

Now, in order to determine the orbit of the comet, I selected—from the observations hitherto described—three that Flamsteed made, on 21 December, 5 January, and 25 January. From these observations I found St to be of 9,842.1 parts, and Vt to be of 455 parts (10,000 such parts being the semidiameter of the earth's orbit). Then for the first operation, assuming tB to be of 5,657 parts, I found SB to be of 9,747, BE the first time 412, $S\mu$ 9,503, $i\lambda$ 413; BE the second time 421, OD 10,186, X 8,528.4, MP 8,450, MN 8,475, NP 25. Hence for the second operation I reckoned the distance tb to be 5,640. And by this operation I found at last the distance TX to be 4,775 and the distance τZ to be 11,322. In determining the orbit from these distances, I found the descending node in ♋1°53′ and the ascending node in ♑1°53′, and the inclination of its plane to the plane of the ecliptic to be 61°20⅓′. I found that its vertex (or the perihelion of the comet) was 8°38′ distant from the node and was in ♐27°43′ with a latitude 7°34′ S; and that its latus rectum was 236.8, and that the area described each day by a radius drawn to the sun was 93,585, supposing the square of the semidiameter of the earth's orbit to be 100,000,000; and I found that the comet had advanced in this orbit in the order of the signs, and was on 8 December 0^h4^m A.M. in the vertex of the orbit or the perihelion. I made all these determinations graphically by a scale of equal parts and by chords of angles, taken from the table of natural sines, constructing a fairly large diagram, that is, one in which the semidiameter of the earth's orbit (of 10,000 parts) was equal to 16⅓ inches of an English foot.

Finally, in order to establish whether the comet moved truly in the orbit thus found, I calculated—partly by arithmetical and partly by graphical operations—the places of the comet in this orbit at the times of certain observations, as can be seen in the following table.

	Distance of the comet from the sun	*Calculated longitude*	*Calculated latitude*	*Observed longitude*	*Observed latitude*	*Difference in longitude*	*Difference in latitude*
		° ′	° ′	° ′	° ′		
Dec. 12	2,792	♑ 6 32	8 18½	♑ 6 31⅓	8 26	+1	− 7½
29	8,403	♓ 13 13⅔	28 0	♓ 13 11¾	28 10 1/12	+2	−10 1/12
Feb. 5	16,669	♉ 17 0	15 29⅔	♉ 16 59⅞	15 27⅖	+0	+ 2¼
Mar. 5	21,737	29 19¾	12 4	29 20 6/7	12 3½	−1	+ ½

[d]Later, our fellow countryman Halley determined the orbit more exactly by an arithmetical calculation than could be done graphically [*lit.* by the descriptions of lines]; and while he kept the place of the nodes in ♋1°53′ and in ♑1°53′ and the inclination of the plane of the orbit to the ecliptic 61°20⅓′, and also the time of the perihelion of the comet 8 December 0^h4^m, he found the distance of the perihelion from the ascending node (measured in the orbit of the comet) to be 9°20′, and the latus rectum of the parabola to be 2,430 parts, the mean distance of the sun from the earth being 100,000 parts. And by making the same kind of arithmetical calculation exactly (using these data), he calculated the places of the comet at the times of the observations, as follows.

dd. The next eight paragraphs (and the included tables) are not present in ed. 1. They were first published in ed. 2 and considerably revised and expanded in ed. 3.

True time	*Distance of the comet from the sun*	*Calculated longitude*	*Calculated latitude*	*Errors in longitude*	*Errors in latitude*
d h m		° ′ ″	° ′ ″	′ ″	′ ″
Dec. 12 4 46	28,028	♑ 6 29 25	8 26 0 N	−3 5	−2 0
21 6 37	61,076	♒ 5 6 30	21 43 20	−1 42	+1 7
24 6 18	70,008	18 48 20	25 22 40	−1 3	−0 25
26 5 21	75,576	28 22 45	27 1 36	−1 28	+0 44
29 8 3	84,021	♓ 13 12 40	28 10 10	+1 59	+0 12
30 8 10	86,661	17 40 5	28 11 20	+1 45	−0 33
Jan. 5 6 1½	101,440	♈ 8 49 49	26 15 15	+0 56	+0 8
9 7 0	110,959	18 44 36	24 12 54	+0 32	+0 58
10 6 6	113,162	20 41 0	23 44 10	+0 10	+0 18
13 7 9	120,000	26 0 21	22 17 30	+0 33	+0 2
25 7 59	145,370	♉ 9 33 40	17 57 55	−1 20	+1 25
30 8 22	155,303	13 17 41	16 42 7	−2 10	−0 11
Feb. 2 6 35	160,951	15 11 11	16 4 15	−2 42	+0 14
5 7 4½	166,686	16 58 25	15 29 13	−0 41	+2 10
25 8 41	202,570	26 15 46	12 48 0	−2 49	+1 14
Mar. 5 11 39	216,205	29 18 35	12 5 40	+0 35	+2 24

This comet also appeared in the preceding November and was observed by Mr. Gottfried Kirch at Coburg in Saxony on the fourth, sixth, and eleventh days of this month (O.S.); and from its positions with respect to the nearest fixed stars (observed with sufficient accuracy, sometimes through a two-foot telescope and sometimes through a ten-foot telescope), from the difference of the longitudes of Coburg and London, eleven degrees, and from the places of the fixed stars observed by our fellow countryman Pound, our own Halley has determined the places of the comet as follows.

On 3 November 17^h2^m, apparent time at London, the comet was in ♌29°51′ with latitude 1°17′45″ N.

On 5 November 15^h58^m, the comet was in ♍3°23′ with latitude 1°6′ N.

On 10 November 16^h31^m, the comet was equally distant from the stars σ and τ (Bayer) of Leo; it had not yet reached the straight line joining these stars, but was not far from it. In Flamsteed's catalog of stars, σ then was in ♍14°15′ with latitude about 1°41′ N, while τ was in ♍17°3½′ with latitude 0°34′ S. And the midpoint between these stars was ♍15°39¼′ with latitude 0°33½′ N. Let the distance of the comet from that straight line be about 10′ or 12′; then the difference of the longitudes of the comet and that midpoint will be 7′, and the difference of the latitudes roughly 7½′. And thus the comet was in ♍15°32′ with roughly latitude 26′ N.

The first observation of the position of the comet in relation to certain small fixed stars was more than exact enough. The second also was exact enough. In the third observation, which was less exact, there could have been an error of six or seven minutes, but hardly a greater one. And the longitude of the comet in the first observation, which was more exact than the others, being computed in the parabolic orbit mentioned above, was ♌29°30′22″, its latitude 1°25′7″ N, and its distance from the sun 115,546.

Further, Halley noted that a remarkable comet had appeared four times at intervals of 575 years—namely, in September after the murder of Julius Caesar; in A.D. 531 in the consulship of Lampadius and Orestes; in February A.D. 1106; and toward the end of 1680—and that this comet had a long and remarkable tail (except that in the year of Caesar's death the tail was less visible because of the inconvenient position of the earth); and he set out to find an elliptical orbit whose major axis would be 1,382,957 parts, the mean distance of the earth from the sun being 10,000 parts, that is, an orbit in which a comet might revolve in 575 years. Then he computed the motion of the comet in this elliptical orbit with the following conditions: the ascending node in ♋2°2′, the inclination of the plane of the orbit to the plane of the ecliptic 61°6′48″, the perihelion of the comet in this plane in ♐22°44′25″, the equated time of the perihelion 7 December $23^{h}9^{m}$, the distance of the perihelion from the ascending node in the plane of the ecliptic 9°17′35″, and the conjugate axis 18,481.2. The places of this comet, as deduced from observations as well as calculated for this orbit, are displayed in the following table [page 912].

The observations of this comet from beginning to end agree no less with the motion of a comet in the orbit just described than the motions of the planets generally agree with planetary theories, and this agreement provides proof that it was one and the same comet which appeared all this time and that its orbit has been correctly determined here.[d]

[e]In this table we have omitted the observations made on 16, 18, 20, and 23 November as being less exact. Yet the comet was observed at these times also.[e] In fact, [Giuseppe Dionigi] Ponteo and his associates, on 17 November

ee. These two sentences were added in ed. 3.

True time	*Observed longitude*	*Observed north latitude*	*Calculated longitude*	*Calculated latitude*	*Errors in longitude*	*Errors in latitude*
d h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	′ ″
Nov. 3 16 47	♌ 29 51 0	1 17 45	♌ 29 51 22	1 17 32 N	+0 22	−0 13
5 15 37	♍ 3 23 0	1 6 0	♍ 3 24 32	1 6 9	+1 32	+0 9
10 16 18	15 32 0	0 27 0	15 33 2	0 25 7	+1 2	−1 53
16 17 0			♎ 8 16 45	0 53 7 S		
18 21 34			18 52 15	1 26 54		
20 17 0			28 10 36	1 53 35		
23 17 5			♏ 13 22 42	2 29 0		
Dec. 12 4 46	♑ 6 32 30	8 28 0	♑ 6 31 20	8 29 6 N	−1 10	+1 6
21 6 37	♒ 5 8 12	21 42 13	♒ 5 6 14	21 44 42	−1 58	+2 29
24 6 18	18 49 23	25 23 5	18 47 30	25 23 35	−1 53	+0 30
26 5 21	28 24 13	27 0 52	28 21 42	27 2 1	−2 31	+1 9
29 8 3	♓ 13 10 41	28 9 58	♓ 13 11 14	28 10 38	+0 33	+0 40
30 8 10	17 38 0	28 11 53	17 38 27	28 11 37	+0 7	−0 16
Jan. 5 6 1½	♈ 8 48 53	26 15 7	♈ 8 48 51	26 14 57	−0 2	−0 10
9 7 1	18 44 4	24 11 56	18 43 51	24 12 17	−0 13	+0 21
10 6 6	20 40 50	23 43 32	20 40 23	23 43 25	−0 27	−0 7
13 7 9	25 59 48	22 17 28	26 0 8	22 16 32	+0 20	−0 56
25 7 59	♉ 9 35 0	17 56 30	♉ 9 34 11	17 56 6	−0 49	−0 24
30 8 22	13 19 51	16 42 18	13 18 28	16 40 5	−1 23	−2 13
Feb. 2 6 35	15 13 53	16 4 1	15 11 59	16 2 7	−1 54	−1 54
5 7 4½	16 59 6	15 27 3	16 59 17	15 27 0	+0 11	−0 3
25 8 41	26 18 35	12 46 46	26 16 59	12 45 22	−1 36	−1 24
Mar. 1 11 10	27 52 42	12 23 40	27 51 47	12 22 28	−0 55	−1 12
5 11 39	29 18 0	12 3 16	29 20 11	12 2 50	+2 11	−0 26
9 8 38	♊ 0 43 4	11 45 52	♊ 0 42 43	11 45 35	−0 21	−0 17

(O.S.) at 6^h A.M. in Rome, that is, at 5^h10^m London time, using threads applied to the fixed stars, observed the comet in ♎8°30′ with latitude 0°40′ S. Their observations may be found in the treatise which Ponteo published about this comet. [Marco Antonio] Cellio, who was present and sent his own observations in a letter to Mr. Cassini, saw the comet at the same hour in ♎8°30′ with latitude 0°30′ S. At the same hour Gallet in Avignon (that is, at 5^h42^m A.M. London time) saw the comet in ♎8° with null latitude; at which time, according to the theory, the comet was in ♎8°16′45″ with latitude 0°53′7″ S.

On 18 November at 6^h30^m A.M. in Rome (that is, at 5^h40^m London time) Ponteo saw the comet in ♎13°30′ with latitude 1°20′ S; Cellio saw it in

♎13°30′ with latitude 1°00′ S. Moreover, Gallet at 5^h30^m A.M. in Avignon saw the comet in ♎13°00′ with latitude 1°00′ S. And the Reverend Father Ango at the College of La Flèche in France, at 5^h A.M. (that is, at 5^h9^m London time), saw the comet midway between two small stars, of which one is the middle star of three in a straight line in the southern hand of Virgo, Bayer's ψ, and the other is the outermost star of the wing, Bayer's ϑ. Thus the comet was then in ♎12°46′ with latitude 50′ S. On the same day at Boston in New England, at a latitude of 42½°, at 5^h A.M. (that is, 9^h44^m London time), the comet was seen near ♎14° with latitude 1°30′ S, as I was informed by the distinguished Halley.

On 19 November at 4^h30^m A.M. in Cambridge, the comet (according to the observation of a certain young man) was about 2 degrees distant from Spica Virginis toward the northwest. And Spica was in ♎19°23′47″ with latitude 2°1′59″ S. On the same day at 5^h A.M. at Boston in New England, the comet was 1 degree distant from Spica Virginis, the difference of latitudes being 40 minutes. On the same day on the island of Jamaica, the comet was about 1 degree distant from Spica. On the same day Mr. Arthur Storer, at the Patuxent River, near Hunting Creek in Maryland, which borders on Virginia, at latitude 38½°, at 5^h A.M. (that is, 10^h London time), saw the comet above Spica Virginis and almost conjoined with Spica, the distance between them being about ¾ of a degree. And comparing these observations with one another, I gather that at 9^h44^m in London the comet was in ♎18°50′ with latitude roughly 1°25′ S. And by the theory the comet was then in ♎18°52′15″ with latitude 1°26′54″ S.[f]

On 20 November, Mr. Geminiano Montanari, professor of astronomy in Padua, at 6^h A.M. in Venice (that is, 5^h10^m London time), saw the comet in ♎23° with latitude 1°30′ S. On the same day at Boston the comet was distant from Spica by 4 degrees of longitude eastward and so was approximately in ♎23°24′.

On 21 November, Ponteo and his associates at 7^h15^m A.M. observed the comet in ♎27°50′ with latitude 1°16′ S, Cellio in ♎28°, Ango at 5^h A.M. in ♎27°45′, Montanari in ♎27°51′. On the same day on the island of Jamaica the comet was seen near the beginning of Scorpio and had roughly the same

f. Newton referred to the star α Virginis in the constellation Virgo as "spica ♍" and simply as "spica." We have rendered these as "Spica Virginis" and "Spica."

latitude as Spica Virginis, that is, 2°2′. On the same day at 5^h A.M. at Balasore in the East Indies (that is, at 11^h20^m the preceding night, London time) the comet was distant 7°35′ eastward from Spica Virginis. It was in a straight line between Spica and the scale [or pan of the Balance] and so was in ♎26°58′ with latitude roughly 1°11′ S, and after 5 hours and 40 minutes (that is, at 5^h A.M. London time) was in ♎28°12′ with latitutde 1°16′ S. And by the theory the comet was then in ♎28°10′36″ with latitude 1°53′35″ S.

On 22 November, the comet was seen by Montanari in ♏2°33′, while at Boston in New England it appeared in approximately ♏3°, with about the same latitude as before, that is, 1°30′. On the same day at 5^h A.M. at Balasore the comet was observed in ♏1°50′, and so at 5^h A.M. in London the comet was approximately in ♏3°5′. On the same day at London at 6^h30^m A.M. our fellow countryman Hooke saw the comet in approximately ♏3°30′, on a straight line that passes between Spica Virginis and the heart of Leo, not exactly indeed, but deviating a little from that line toward the north. Montanari likewise noted that a line drawn from the comet through Spica passed, on this day and the following days, through the southern side of the heart of Leo, there being a very small interval between the heart of Leo and this line. The straight line passing through the heart of Leo and Spica Virginis cut the ecliptic in ♍3°46′, at an angle of 2°51′. And if the comet had been located in this line in ♏3°, its latitude would have been 2°26′. But since the comet, by the agreement of Hooke and Montanari, was at some distance from this line toward the north, its latitude was a little less. On the 20th, according to the observation of Montanari, its latitude almost equaled the latitude of Spica Virginis and was roughly 1°30′; and by the agreement of Hooke, Montanari, and Ango, the latitude was continually increasing and so now (on the 22d) was sensibly greater than 1°30′. And the mean latitude between the limits now established, 2°26′ and 1°30′, will be roughly 1°58′. The tail of the comet, by the agreement of Hooke and Montanari, was directed toward Spica Virginis, declining somewhat from that star—southward according to Hooke, northward according to Montanari; and so that declination was hardly perceptible, and the tail, being almost parallel to the equator, was deflected somewhat northward from the opposition of the sun.

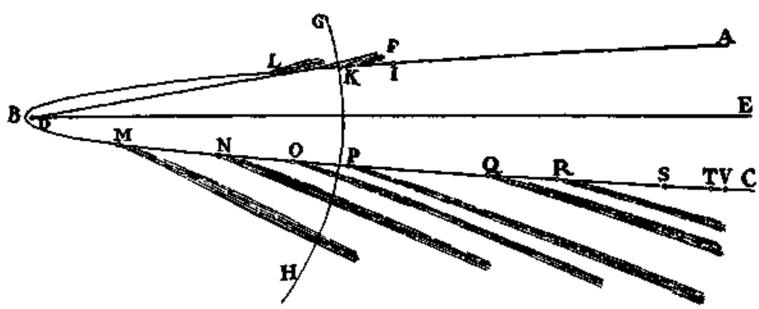
On 23 November (O.S.) at 5^h A.M. at Nuremberg (that is, at 4^h30^m London time) Mr. [Johann Jacob] Zimmermann saw the comet in ♏8°8′ with latitude 2°31′ S, determining its distances from the fixed stars.

On 24 November before sunrise the comet was seen by Montanari in ♏12°52′ on the northern side of a straight line drawn through the heart of Leo and Spica Virginis, and so had a latitude a little less than 2°38′. This latitude (as we have said), according to the observations of Montanari, Ango, and Hooke, was continually increasing, and so it was now (on the 24th) a little greater than 1°58′, and at its mean magnitude can be taken as 2°18′ without perceptible error. Ponteo and Gallet would have the latitude decreased now, and Cellio and the observer in New England would have it retained at about the same magnitude, namely 1 or 1½ degrees. The observations of Ponteo and Cellio are rather crude, especially those that were made by taking azimuths and altitudes, and so are those of Gallet; better are the ones that were made by means of the positions of the comet in relation to fixed stars by Montanari, Hooke, Ango, and the observer in New England, and sometimes by Ponteo and Cellio. On the same day at 5^h A.M. at Balasore, the comet was observed in ♏11°45′, and so at 5^h A.M. at London it was nearly in ♏13°. And by the theory the comet was at that time in ♏13°22′42″.

On 25 November before sunrise Montanari observed the comet approximately in ♏17¾°. And Cellio observed at the same time that the comet was in a straight line between the bright star in the right thigh of Virgo and the southern scale of Libra, and this straight line cuts the path of the comet in ♏18°36′. And by the theory the comet was at that time approximately in ♏18⅓°.

Therefore these observations agree with the theory insofar as they agree with one another, and by such agreement they prove that it was one and the same comet that appeared in the whole time from the 4th of November to the 9th of March. The trajectory of this comet cut the plane of the ecliptic twice and therefore was not rectilinear. It cut the ecliptic not in opposite parts of the heavens, but at the end of Virgo and at the beginning of Capricorn, at points separated by an interval of about 98 degrees; and thus the course of the comet greatly deviated from a great circle. For in November its course declined by at least 3 degrees from the ecliptic toward the south, and afterward in December verged from the ecliptic 29 degrees toward the north: the two parts of its orbit, in which the comet tended toward the sun and returned from the sun, declining from each other by an apparent angle of more than 30 degrees, as Montanari observed. This comet moved through nine signs, namely from the last degree of Leo to the beginning of Gemini, besides [that part of] the sign of Leo through which it moved before it began to be seen; and there is no other theory according to which a comet may travel over so great a part of the heaven with a motion according to some rule. Its motion was extremely nonuniform. For about the 20th of November it described approximately 5 degrees per day; then, with a retarded motion between 26 November and 12 December, that is, during 15½ days, it described only 40 degrees; and afterward, with its motion accelerated again, it described about 5 degrees per day until its motion began to be retarded again. And the theory that corresponds exactly to so nonuniform a motion through the greatest part of the heavens, and that observes the same laws as the theory of the planets, and that agrees exactly with exact astronomical observations cannot fail to be true.

Furthermore, it seemed appropriate to show the trajectory that the comet described and the actual tail that it projected in different positions, as in the accompanying figure, in the plane of the trajectory; in this figure, ABC denotes the trajectory of the comet, D the sun, DE the axis of the trajectory,



DF the line of nodes, GH the intersection of the sphere of the earth's orbit with the plane of the trajectory, I the place of the comet on 4 November 1680, K its place on 11 November, L its place on 19 November, M its place on 12 December, N its place on 21 December, O its place on 29 December, P its place on 5 January of the following year, Q its place on 25 January, R its place on 5 February, S its place on 25 February, T its place on 5 March, and V its place on 9 March. I used the following observations in determining the tail.

On 4 and 6 November the tail was not yet visible. On 11 November the tail, which had now begun to be seen, was observed through a ten-foot telescope to be no more than half a degree long. On 17 November the tail was observed by Ponteo to be more than 15 degrees long. On 18 November the tail was seen in New England to be 30 degrees long and directly opposite to the sun, and it was extended out to the star ♂ [i.e., the planet Mars], which was then in ♍9°54′. On 19 November, in Maryland, the tail was seen to be 15 or 20 degrees long. On 10 December the tail (according to the observations of Flamsteed) was passing through the middle of the distance between the tail of Serpens (the Serpent of Ophiuchus) and the star δ in the southern wing of Aquila and terminated near the stars A, ω, b in Bayer's tables. Therefore the end of the comet's tail was in ♑19½° with a latitude of about 34¼° N. On 11 December the tail was rising as far as the head of Sagitta (Bayer's α, β), terminating in ♑26°43′, with a latitude of 38°34′ N. On 12 December the tail was passing through the middle of Sagitta and did not extend very much further, terminating in ♒4°, with a latitude of about 42½° N.

These things are to be understood of the length of the brighter part of the tail. For when the light was fainter and the sky perhaps clearer, on 12 December at 5^h40^m in Rome, the tail was observed by Ponteo to extend to 10 degrees beyond the uropygium of Cygnus [i.e., the rump of the Swan], and its side toward the northwest terminated 45 minutes from this star. Moreover, in those days the tail was 3 degrees wide near its upper end, and so the middle of it was 2°15′ distant from that star toward the south, and its upper end was in ♓22° with a latitude of 61° N. And hence the tail was about 70 degrees long.

On 21 December the tail rose almost to Cassiopeia's Chair, being equally distant from β and Schedar [= α Cassiopeiae] and having a distance from each of them equal to their distance from each other, and so terminating in ♈24° with a latitude of 47½°. On 29 December the tail was touching Scheat, which was situated to the left of it, and exactly filled the space between the two stars in the northern foot of Andromeda; it was 54 degrees long; accordingly it terminated in ♉19° with a latitude of 35°. On 5 January the tail touched the star π in the breast of Andromeda on the right side and the star μ in the girdle on the left side, and (according to our observations) was 40 degrees long; but it was curved, and its convex side faced to the south. Near the head of the comet, the tail made an angle of 4 degrees with the circle passing through the sun and the head of the comet; but near the other end, it was inclined to that circle at an angle of 10 or 11 degrees, and the chord of the tail contained an angle of 8 degrees with that circle. On 13 January the tail was visible enough between Alamech and Algol [= β Persei], but it ended in a very faint light toward the star κ in Perseus's side. The distance of the end of the tail from the circle joining the sun and the comet was 3°50′, and the inclination of the chord of the tail to that circle was 8½ degrees. On 25 and 26 January the tail shone with a faint light to a length of 6 or 7 degrees; and, a night or so later, when the sky was extremely clear, it attained a length of 12 degrees and a little more, with a light that was very faint and scarcely to be perceived. But its axis was directed exactly toward the bright star in the eastern shoulder of Auriga, and accordingly declined from the opposition of the sun toward the north at an angle of 10 degrees. Finally on 10 February, my eyes armed [with a telescope], I saw the tail to be 2 degrees long. For the fainter light mentioned above was not visible through the glasses. But Ponteo writes that on 7 February he saw the tail with a length of 12 degrees. On 25 February and thereafter, the comet appeared without a tail.

Whoever considers the orbit just described and turns over in his mind the other phenomena of this comet will without difficulty agree that the bodies of comets are solid, compact, fixed, and durable, like the bodies of planets. For if comets were nothing other than vapors or exhalations of the earth, the sun, and the planets, this one ought to have been dissipated at once during its passage through the vicinity of the sun. For the heat of the sun is as the density of its rays, that is, inversely as the square of the distance of places from the sun. And thus, since the distance of the comet from the center of the sun on 8 December, when it was in its perihelion, was to the distance of the earth from the center of the sun as approximately 6 to 1,000, the heat of the sun on the comet at that time was to the heat of the summer sun here on earth as 1,000,000 to 36, or as 28,000 to 1. But the heat of boiling water is about three times greater than the heat that dry earth acquires in the summer sun, as I have found [by experiment]; and the heat of incandescent iron (if I conjecture correctly) is about three or four times greater than the heat of boiling water; and hence the heat that dry earth on the comet would have received from the sun's rays, when it was in its perihelion, would be about two thousand times greater than the heat of incandescent iron. But with so great a heat, vapors and exhalations, and all volatile matter, would have to have been consumed and dissipated at once.

Therefore the comet, in its perihelion, received an immense heat at [i.e., when near] the sun, and it can retain that heat for a very long time. For a globe of incandescent iron, one inch wide, standing in the air would scarcely lose all its heat in the space of one hour. But a larger globe would preserve its heat for a longer time in the ratio of its diameter, because its surface (which is the measure according to which it is cooled by contact with the surrounding air) is smaller in that ratio with respect to the quantity of hot matter it contains. And so a globe of incandescent iron equal to this earth of ours—that is, more or less 40,000,000 feet wide—would scarcely cool off in as many days, or about 50,000 years. Nevertheless, I suspect that the duration of heat is increased in a smaller ratio than that of the diameter because of some latent causes, and I wish that the true ratio might be investigated by experiments.

Further, it should be noted that in December, when the comet had just become hot at the sun, it was emitting a far larger and more splendid tail than it had done earlier in November, when it had not yet reached its perihelion. And, universally, the greatest and brightest tails all arise from comets immediately after their passage through the region of the sun. Therefore the heating up of the comet is conducive to a great size of its tail, and from this I believe it can be concluded that the tail is nothing other than extremely thin vapor that the head or nucleus of the comet emits by its heat.

There are indeed three opinions about the tails of comets: that the tails are the brightness of the sun's light propagated through the translucent heads of comets; that the tails arise from the refraction of light in its progress from the head of the comet to the earth; and finally that these tails are a cloud or vapor continually rising from the head of the comet and going off in a direction away from the sun. The first opinion is held by those who are not yet instructed in the science of optics. For beams of sunlight are not seen in a dark room except insofar as the light is reflected from particles of dust and smoke always flying about through the air, and for this reason in air darkened with thicker smoke the beams of sunlight appear brighter and strike the eye more strongly, while in clearer air these beams are fainter and are perceived with greater difficulty, but in the heavens, where there is no matter to reflect these beams of sunlight, they cannot be seen at all. Light is not seen insofar as it is in the beam, but only to the degree that it is reflected to our eyes; for vision results only from rays that impinge upon the eyes.

Therefore some reflecting matter must exist in the region of the tail, since otherwise the whole sky, illuminated by the light of the sun, would shine uniformly.

The second opinion is beset with many difficulties. The tails are never variegated in color, and yet colors are generally the inseparable concomitants of refractions. The light of the fixed stars and the planets which is transmitted to us is distinct [i.e., clearly defined]; this demonstrates that the celestial medium is empowered with no refractive force. It is said that the Egyptians sometimes saw the fixed stars surrounded by a head of hair, but this happens very rarely, and so it must be ascribed to some chance refraction by clouds. The radiation and scintillation of the fixed stars also should be referred to refractions both by the eyes and by the tremulous air, since they disappear when these stars are viewed through telescopes. By the tremor of the air and of the ascending vapors it happens that rays are easily turned aside alternately from the narrow space of the pupil of the eye but not at all from the wider aperture of the objective lens of a telescope. Thus it is that scintillation is generated in the former case while it ceases in the latter; and the cessation of scintillation in the latter case demonstrates the regular transmission of light through the heavens without any sensible refraction. And to counter the argument that tails are not generally seen in comets when their light is not strong enough, for the reason that the secondary rays do not then have enough force to affect the eyes, and that this is why the tails of the fixed stars are not seen, it should be pointed out that the light of the fixed stars can be increased more than a hundred times by means of telescopes, and yet no tails are seen. The planets also shine with more light, but they have no tails; and often comets have the greatest tails when the light of their heads is faint and exceedingly dull. For such was the case for the comet of 1680; in December, at a time when the light from its head scarcely equaled stars of the second magnitude, it was emitting a tail of notable splendor as great as 40, 50, 60, or 70 degrees in length and more. Afterward, on 27 and 28 January, the head appeared as a star of only the seventh magnitude, but the tail extended to 6 or 7 degrees in length with a very faint light that was sensible enough; and with a very dim light, which could scarcely be seen, it stretched out as far as 12 degrees or a little further, as was said above. But even on 9 and 10 February, when the head had ceased to be seen by the naked eye, the tail—when I viewed it through a telescope—was 2 degrees long. Further, if the tail arose from refraction by celestial matter, and if it deviated from the opposition of the sun in accordance with the form of the heavens, then, in the same regions of the heavens, that deviation ought always to take place in the same direction. But the comet of 1680, on 28 December at 8^h30^m P.M. London time, was in ♓8°41′ with a latitude of 28°6′ N, the sun being in ♑18°26′. And the comet of 1577, on 29 December, was in ♓8°41′ with a latitude of 28°40′ N, the sun again being in approximately ♑18°26′. In both cases the earth was in the same place and the comet appeared in the same part of the sky; yet in the former case the tail of the comet (according to my observations and those made by others) was declining by an angle of 4½ degrees from the opposition of the sun toward the north, but in the latter case (according to the observations of Tycho) the declination was 21 degrees toward the south. Therefore, since refraction by the heavens has been rejected, the remaining possibility is to derive the phenomena of comets' tails from some matter that reflects light.

Moreover, the laws which the tails of comets observe prove that these tails arise from the heads and ascend into regions turned away from the sun. For example, if the tails lie in planes of the comets' orbits which pass through the sun, they always deviate from being directly opposite the sun and point toward the region which the heads, advancing in those orbits, have left behind. Again, to a spectator placed in those planes, the tails appear in regions directly turned away from the sun; while for observers not in those planes, the deviation gradually begins to be perceived and appears greater from day to day. Furthermore, other things being equal, the deviation is less when the tail is more oblique to the orbit of the comet, and also when the head of the comet approaches closer to the sun, especially if the angle of deviation is taken near the head of the comet. And besides, the tails that do not deviate appear straight, while those that do deviate are curved. Again, this curvature is greater when the deviation is greater, and more sensible when the tail, other things being equal, is longer; for in shorter tails the curvature is scarcely noticed. Then, too, the angle of deviation is smaller near the head of the comet and larger near the other extremity of the tail; and thus the convex side of the tail faces the direction from which the deviation is made and which is along a straight line drawn from the sun through the head of the comet indefinitely. Finally, the tails that are more extended and wider and that shine with a more vigorous light are a little more resplendent on their convex sides and are terminated by a less indistinct limit than on their concave sides. For all these reasons, then, the phenomena of the tail depend on the motion of the head and not on the region of the sky in which the head is seen; and therefore these phenomena do not come about through refraction by the heavens, but arise from the head supplying the matter. For as in our air the smoke of any ignited body seeks to ascend and does so either perpendicularly (if the body is at rest) or obliquely (if the body is moving sideways), so in the heavens, where bodies gravitate toward the sun, smoke and vapors must ascend with respect to the sun (as has already been said) and move upward either directly, if the smoking body is at rest, or obliquely, if the body by advancing always leaves the places from which the higher parts of the vapor have previously ascended. And the swifter the ascent of the vapor, the less the obliquity, namely in the vicinity of the sun and near the smoking body. Moreover, as a result of this difference in obliquity, the column of vapor will be curved; and since the vapor on that side of the column in the direction of the comet's motion is a little more recent [i.e., more recently exhaled], so also the column will be somewhat more dense on that same side, and therefore will reflect light more abundantly and will be terminated by a less indistinct limit. I add nothing here concerning sudden and uncertain agitations of the tails, nor concerning their irregular shapes (which are sometimes described), because either these effects may arise from changes in our air and the motions of the clouds that may obscure those tails in one part or the other; or, perhaps, these effects may arise because some parts of the Milky Way may be confused with the tails as they pass by and may be considered as if they were parts of the tails.

Moreover, the rarity of our own air makes it understandable that vapors sufficient to fill such immense spaces can arise from the atmospheres of comets. For the air near the surface of the earth occupies a space about 850 times greater than water of the same weight, and thus a cylindrical column of air 850 feet high has the same weight as a foot-high column of water of the same width. Further, a column of air rising to the top of our atmosphere is equal in weight to a column of water about 33 feet high; and therefore if the lower part, 850 feet high, of the whole air column is taken away, the remaining upper part will be equal in weight to a column of water 32 feet high. And hence (by a rule confirmed by many experiments, that the compression of air is as the weight of the incumbent atmosphere and that gravity is inversely as the square of the distance of places from the center of the earth), by making a computation using the corollary of prop. 22, book 2, I found that air, at a height above the surface of the earth of one terrestrial semidiameter, is rarer than here on earth in a far greater ratio than that of all space below the orbit of Saturn to a globe described with a diameter of one inch. And thus a globe of our air one inch wide, with the rarity that it would have at the height of one terrestrial semidiameter, would fill all the regions of the planets as far out as the sphere of Saturn and far beyond. Accordingly, since still higher air becomes immensely rare and since the coma[g] or atmosphere of a comet is (as reckoned from the center) about ten times higher than the surface of the nucleus is, and the tail then ascends even higher, the tail will have to be exceedingly rare. And even if, because of the much thicker atmosphere of comets and the great gravitation of bodies toward the sun and the gravitation of the particles of air and vapors toward one another, it can happen that the air in the celestial spaces and in the tails of comets is not so greatly rarefied, it is nevertheless clear from this computation that a very slight quantity of air and vapors is abundantly sufficient to produce all those phenomena of the tails. For the extraordinary rarity of the tails is also evident from the fact that stars shine through them. The terrestrial atmosphere, shining with the light of the sun, by its thickness of only a few miles obscures and utterly extinguishes the light not only of all the stars but also of the moon itself; yet the smallest stars are known to shine, without any loss in their brightness, through the immense thickness of the tails, which are likewise illuminated by the light of the sun. Nor is the brightness of most cometary tails generally greater than that of our air reflecting the light of the sun in a beam, one or two inches wide, let into a dark room.

The space of time in which the vapor ascends from the head to the end of the tail can more or less be found by drawing a straight line from the end of the tail to the sun and noting the place where this straight line cuts the trajectory. For if the vapor has been ascending in a straight line away from the sun, then the vapor that is now in the end of the tail must have begun to ascend from the head at the time when the head was in that place of intersection. But the vapor does not ascend in a straight line away from the sun, but rather ascends obliquely, since the vapor retains the motion of

g. See note a on p. 891 above.

the comet which it had before its ascent and this motion is compounded with the motion of its own ascent. And therefore the solution of the problem will be nearer the true one if the straight line that cuts the orbit is drawn parallel to the length of the tail, or rather (because of the curvilinear motion of the comet) if it diverges from the line of the tail. In this way I found that the vapor that was in the end of the tail on 25 January had begun to ascend from the head before 11 December and thus had spent more than forty-five days in its total ascent. But all of the tail that appeared on 10 December had ascended in the space of those two days that had elapsed after the time of the perihelion of the comet. The vapor, therefore, rose most swiftly at the beginning of its ascent, in the vicinity of the sun, and afterward proceeded to ascend with a motion always retarded by the vapor's own gravity; and as the vapor ascended, it increased the length of the tail. The tail, however, as long as it was visible, consisted of almost all the vapor which had ascended from the comet's head since the time of the comet's perihelion; and that vapor which was the first to ascend, and which composed the end of the tail, did not disappear from view until its distance both from the sun which illuminated it and from our eyes became too great for it to be seen any longer. Hence it happens, also, that in other comets which have short tails, those tails do not rise up with a swift and continual motion from the heads of the comets and soon disappear, but are permanent columns of vapors and exhalations (propagated from the heads by a very slow motion that lasts many days) which, by sharing in the motion that the heads had at the beginning of the exhalations of the vapors, continue to move along through the heavens together with the heads. And hence again it may be concluded that the celestial spaces are lacking in any force of resisting, since in them not only the solid bodies of the planets and comets but also the rarest vapors of the tails move very freely and preserve their extremely swift motions for a very long time.

The ascent of the tails of comets from the atmospheres of the heads and the movement of the tails in directions away from the sun are ascribed by Kepler to the action of rays of light that carry the matter of the tail along with them. And it is not altogether unreasonable to suppose that in very free [or empty] spaces, the extremely thin upper air should yield to the action of the rays, despite the fact that gross substances in the very obstructed regions here on earth cannot be sensibly propelled by the rays of the sun. Someone else believes that there can be particles with the property of levity as well as gravity and that the matter of the tails levitates and through its levitation ascends away from the sun. But since the gravity of terrestrial bodies is as the quantity of matter in the bodies and thus, if the quantity of matter remains constant, cannot be intended and remitted [or increased and decreased], I suspect that this ascent arises rather from the rarefaction of the matter of the tails. Smoke ascends in a chimney by the impulse of the air in which it floats. This air, rarefied by heat, ascends because of its diminished specific gravity and carries along with it the entangled smoke. Why should the tail of a comet not ascend away from the sun in the same manner? For the sun's rays do not act on the mediums through which they pass except in reflection and refraction. The reflecting particles, warmed by this action, will warm the aethereal upper air in which they are entangled. This will become rarefied on account of the heat communicated to it; and because its specific gravity, with which it was formerly tending toward the sun, is diminished by this rarefaction, it will ascend and will carry with it the reflecting particles of which the tail is composed. This ascent of the vapors is also increased by the fact that they revolve about the sun and endeavor by this action to recede from the sun, while the atmosphere of the sun and the matter of the heavens are either completely at rest or revolve more slowly only by the motion that they have received from the rotation of the sun.

These are the causes of the ascent of tails of comets in the vicinity of the sun, where the orbits are more curved, and the comets are within the denser (and, on that account, heavier) atmosphere of the sun and soon emit extremely long tails. For the tails which arise at that point, by conserving their motion and meanwhile gravitating toward the sun, will move about the sun in ellipses as the heads of the comets do; and by that motion they will always accompany the heads and will very freely adhere to them. For the gravity of the vapors toward the sun will no more cause the tails to fall afterward from the heads toward the sun than the gravity of the heads can cause them to fall from the tails. By their common gravity they will either fall simultaneously and together toward the sun or will be simultaneously retarded in their ascent; and therefore this gravity does not hinder the tails and heads of comets from very easily acquiring (whether from the causes already described or any others whatsoever), and afterward very freely preserving, any position in relation to one another.

The tails that are formed when comets are in their perihelia will therefore go off into distant regions together with their heads, and either will return to us from there together with the heads after a long series of years or rather, having been rarefied there, will disappear by degrees. For afterward, in the descent of the heads toward the sun, new little tails should be propagated from the heads with a slow motion, and thereupon should be immeasurably increased in the perihelia of those comets which descend as far as the atmosphere of the sun. For vapor in those very free spaces becomes continually rarefied and dilated. For this reason it happens that every tail at its upper extremity is broader than near the head of the comet. Moreover, it seems reasonable that by this rarefaction the vapor—continually dilated—is finally diffused and scattered throughout the whole heavens, and then is by degrees attracted toward the planets by its gravity and mixed with their atmospheres. For just as the seas are absolutely necessary for the constitution of this earth, so that vapors may be abundantly enough aroused from them by the heat of the sun, which vapors either—being gathered into clouds—fall in rains and irrigate and nourish the whole earth for the propagation of vegetables, or—being condensed in the cold peaks of mountains (as some philosophize with good reason)—run down into springs and rivers; so for the conservation of the seas and fluids on the planets, comets seem to be required, so that from the condensation of their exhalations and vapors, there can be a continual supply and renewal of whatever liquid is consumed by vegetation and putrefaction and converted into dry earth. For all vegetables grow entirely from fluids and afterward, in great part, change into dry earth by putrefaction, and slime is continually deposited from putrefied liquids. Hence the bulk of dry earth is increased from day to day, and fluids—if they did not have an outside source of increase—would have to decrease continually and finally to fail. Further, I suspect that that spirit which is the smallest but most subtle and most excellent part of our air, and which is required for the life of all things, comes chiefly from comets.

In the descent of comets to the sun, their atmospheres are diminished by running out into tails and (certainly in that part which faces toward the sun) are made narrower; and, in turn, when comets are receding from the sun, and when they are now running out less into tails, they become enlarged, if Hevelius has correctly noted their phenomena. Moreover, these atmospheres appear smallest when the heads, after having been heated by the sun, have gone off into the largest and brightest tails, and the nuclei are surrounded in the lowest parts of their atmospheres by smoke possibly coarser and blacker. For all smoke produced by great heat is generally coarser and blacker. Thus, at equal distances from the sun and the earth, the head of the comet which we have been discussing appeared darker after its perihelion than before. For in December it was generally compared to stars of the third magnitude, but in November to stars of the first magnitude and the second magnitude. And those who saw both describe the earlier appearance as a greater comet. For a certain young man of Cambridge, who saw this comet on 19 November, found its light, however leaden and pale, to be equal to Spica Virginis and to shine more brightly than afterward. And on 20 November (O.S.) the comet appeared to Montanari greater than stars of the first magnitude, its tail being 2 degrees long. And Mr. Storer, in a letter that came into our hands, wrote that in December, at a time when the largest and brightest tail was being emitted, the head of the comet was small and in visible magnitude was far inferior to the comet which had appeared in November before sunrise. And he conjectured that the reason for this was that in the beginning the matter of the head was more copious and had been gradually consumed.

It seems to pertain to the same point that the heads of other comets that emitted very large and very bright tails appeared rather dull and very small. For on 5 March 1668 (N.S.) at 7^h P.M., the Reverend Father Valentin Stansel, in Brazil, saw a comet very close to the horizon toward the southwest with a very small head that was scarcely visible, but with a tail so shining beyond measure that those who were standing on the shore easily saw its appearance reflected from the sea. In fact it had the appearance of a brilliantly shining torch with a length of 23 degrees, verging from west to south and almost parallel to the horizon. But so great a splendor lasted only three days, decreasing noticeably immediately afterward; and meanwhile, as its splendor was decreasing, the tail was increasing in size. Thus in Portugal the tail is said to have occupied almost a quarter of the sky—that is, 45 degrees—stretched out from west to east with remarkable splendor, and yet not all of the tail was visible, since in those regions the head was always hidden below the horizon. From the increase of the size of the tail and the decrease of the splendor, it is manifest that the head was receding from the sun and had been nearest to the sun at the beginning of its visibility, as was the case for the comet of 1680. And in the *Anglo-Saxon Chronicle*, one reads about a similar comet of 1106, "of which the star was small and dim [h](as was that of 1680),[h] but the splendor that came out of it stretched out extremely bright and like a huge torch toward the northeast" as Hevelius also has it from Simeon the Monk of Durham. This comet appeared at the beginning of February, and thereafter was seen at about evening toward the southwest. And from this and from the position of the tail it is concluded that the head was near the sun. "Its distance from the sun," says Matthew of Paris, "was about one cubit, as from the third hour (more correctly, the sixth) until the ninth hour it emitted a long ray from itself." Such also was that fiery comet described by Aristotle (*Meteor.* 1.6), "whose head, on the first day, was not seen because it had set before the sun, or at least was hidden under the sun's rays; but on the following day, it was seen as much as it could be. For it was distant from the sun by the least possible distance, and soon set. Because of the excessive burning (of the tail, that is), the scattered fire of the head did not yet appear, but as time went on," says Aristotle, "since (the tail) was now flaming less, the comet's own face came back to (the head). And it extended its splendor as far as a third of the sky (that is, to 60 degrees). Moreover, it appeared in the winter (in the 4th year of the 101st Olympiad) and, ascending up to Orion's belt, vanished there."

The comet of 1618 which emerged out of the sun's rays with a very large tail seemed to equal stars of the first magnitude, or even to surpass them a little, but a number of greater comets have appeared which had shorter tails. Some of these are said to have equaled Jupiter, others Venus or even the moon.

We said that comets are a kind of planet revolving about the sun in very eccentric orbits. And just as among the primary planets (which have no tails) those which revolve in smaller orbits closer to the sun are generally smaller, so it seems reasonable also that the comets which approach closer to the sun in their perihelia are for the most part smaller, since otherwise they would act on the sun too much by their attraction. I leave the transverse diameters of the orbits and the periodic times of revolution of the comets to be determined by comparing comets that return in the same orbits after long

hh. The clause in parentheses was added by Newton, as were the following expressions within parentheses.

intervals of time. Meanwhile the following proposition may shed some light on this matter.

Proposition 42
Problem 22

To correct a comet's trajectory that has been found [by the method of prop. 41].

OPERATION 1. Assume the position of the plane of the trajectory, as found by prop. 41, and select three places of the comet which have been determined by very accurate observations and which are as greatly distant from one another as possible; let A be the time between the first and second observations, and B the time between the second and third. The comet should be in its perigee in one of these places, or at least not far from perigee. From these apparent places find, by trigonometric operations, three true places of the comet in that assumed plane of the trajectory. Then through those places thus found, describe a conic about the center of the sun as focus, by arithmetical operations made along the lines of prop. 21, book 1; and let D and E be areas of the conic which are bounded by radii drawn from the sun to those places—namely, D the area between the first and second observations, and E the area between the second and third. And let T be the total time in which the total area D + E should be described by the comet, with the velocity as found by prop. 16, book 1.

OPERATION 2. Let the longitude of the nodes of the plane of the trajectory be increased by adding 20 or 30 minutes (which can be called P) to that longitude; but keep constant the inclination of that plane to the plane of the ecliptic. Then from the three aforesaid observed places of the comet, let three true places of the comet be found in this new plane (as in oper. 1); and also the orbit passing through those places, two of its areas (which can be called d and e) described between observations, and the total time t in which the total area $d + e$ should be described.

OPERATION 3. Keep constant the longitude of the nodes in the first operation, and let the inclination of the plane of the trajectory to the plane of the ecliptic be increased by adding 20 or 30 minutes (which can be called Q) to that inclination. Then from the aforesaid three observed apparent places of the comet, let three true places be found in this new plane; and also the orbit passing through those places, two of its areas (which can be called δ and ϵ) described between observations, and the total time τ in which the total area $\delta + \epsilon$ should be described.

Now take C so as to be to 1 as A to B, and take G to 1 as D to E, and g to 1 as d to e, and γ to 1 as δ to ϵ, and let S be the true time between the first and third observations; and carefully observing the signs + and −, seek the numbers m and n, by the rule that $2G - 2C = mG - mg + nG - n\gamma$; and $2T - 2S = mT - mt + nT - n\tau$. And if, in the first operation, I designates the inclination of the plane of the trajectory to the plane of the ecliptic, and K the longitude of either node, $I + nQ$ will be the true inclination of the plane of the trajectory to the plane of the ecliptic, and $K + mP$ will be the true longitude of the node. And finally if in the first, second, and third operations, the quantities R, r, and ρ designate the latera recta of the trajectory, and the quantities $\frac{1}{L}, \frac{1}{l}, \frac{1}{\lambda}$ the transverse diameters [or latera transversa] respectively, $R + mr - mR + n\rho - nR$ will be the true latus rectum, and $\frac{1}{L + ml - mL + n\lambda - nL}$ will be the true transverse diameter of the trajectory that the comet describes. And given the transverse diameter, the periodic time of the comet is also given. Q.E.I.

But the periodic times of revolving comets, and the transverse diameters [latera transversa] of their orbits, will by no means be determined exactly enough except by the comparison with one another of comets that appear at diverse times. If several comets are found, after equal intervals of times, to have described the same orbit, it will have to be concluded that all these are one and the same comet revolving in the same orbit. And then finally from the times of their revolutions the transverse diameters of the orbits will be given, and from these diameters the elliptical orbits will be determined.

To this end, therefore, the trajectories of several comets should be calculated on the hypothesis that they are parabolic. For such trajectories will always agree very nearly with the phenomena. This is clear not only from the parabolic trajectory of the comet of 1680, which I compared above with the observations, but also from the trajectory of that remarkable comet which appeared in 1664 and 1665 and was observed by Hevelius. He calculated the longitudes and latitudes of this comet from his own observations, but not very accurately. From the same observations our own Halley calculated the places of this comet anew, and then finally he determined the trajectory of the comet from the places thus calculated. And he found its ascending node in ♊ $21°13'55''$, the inclination of its orbit to the plane of the ecliptic $21°18'40''$, the distance of its perihelion from the node in the orbit 49°27′30″. The perihelion in ♌8°40′30″ with heliocentric latitude 16°1′45″ S. The comet in its perihelion on 24 November, 11^h52^m P.M. mean time [*lit.* equated time] at London, or 13^h8^m (O.S.) at Gdansk, and the latus rectum of the parabola 410,286, the mean distance of the earth from the sun being 100,000. How exactly the calculated places of the comet in this orbit agree with the observations will be evident from the following table calculated by Halley [p. 932].

In February, in the beginning of 1665, the first star of Aries, which I shall from here on call γ, was in ♈28°30′15″ with latitude 7°8′58″ N. The second star of Aries was in ♈29°17′18″ with latitude 8°28′16″ N. And a certain other star of the seventh magnitude, which I shall call A, was in ♈28°24′45″ with latitude 8°28′33″ N. And on 7 February at 7^h30^m Paris time (that is, 7 February at 8^h30^m Gdansk time) (O.S.), the comet made a right triangle with those stars γ and A, with the right angle at γ. And the distance of the comet from the star γ was equal to the distance between the stars γ and A, that is, 1°19′46″ along a great circle, and therefore it was 1°20′26″ in the parallel of the latitude of the star γ. Therefore, if the longitude 1°20′26″ is taken away from the longitude of the star γ, there will remain the longitude of the comet ♈27°9′49″. Auzout, who had made this observation, put the comet in roughly ♈27°0′. And from the diagram with which Hooke delineated its motion, it was then in ♈26°59′24″. Taking the mean, I have put it in ♈27°4′46″. From the same observation, Auzout took the latitude of the comet at that time to be 7° and 4′ or 5′ toward the north. He would have put it more correctly at 7°3′29″, since the difference of the latitudes of the comet and of the star γ was equal to the difference of the longitudes of the stars γ and A.

On 22 February at 7^h30^m in London (that is, 22 February at 8^h46^m Gdansk time), the distance of the comet from the star A, according to Hooke's observation (which he himself delineated in a diagram) and also according to Auzout's observations (delineated in a diagram by Petit), was a fifth of the distance between the star A and the first star of Aries, or 15′57″. And the distance of the comet from the line joining the star A and the first star of Aries was a fourth of that same fifth part, that is, 4′. And hence the comet was in ♈28°29′46″ with latitude 8°12′36″ N.

On 1 March at 7^h0^m at London (that is, 1 March at 8^h16^m Gdansk time), the comet was observed near the second star of Aries, the distance between

Apparent time at Gdansk, O.S.	*Observed distances of the comet*		*Observed places*	*Calculated places in the orbit*
d h m		° ′ ″	° ′ ″	° ′ ″
December				
3 18 29½	from the heart of Leo	46 24 20	Long. ♎ 7 1 0	♎ 7 1 29
	from Spica Virginis	22 52 10	Lat. S. 21 39 0	21 38 50
4 18 1½	from the heart of Leo	46 2 45	Long. ♎16 15 0	♎ 6 16 5
	from Spica Virginis	23 52 40	Lat. S. 22 24 0	22 24 0
7 17 48	from the heart of Leo	44 48 0	Long. ♎ 3 6 0	♎ 3 7 33
	from Spica Virginis	27 56 40	Lat. S. 25 22 0	25 21 40
17 14 43	from the heart of Leo	53 15 15	Long. ♌ 2 56 0	♌ 2 56 0
	from the right shoulder of Orion	45 43 30	Lat. S. 49 25 0	49 25 0
19 9 25	from Procyon	35 13 50	Long. ♊28 40 30	♊28 43 0
	from the bright star in the jaw of Cetus	52 56 0	Lat. S. 45 48 0	45 46 0
20 9 53½	from Procyon	40 49 0	Long. ♊13 3 0	♊13 5 0
	from the bright star in the jaw of Cetus	40 4 0	Lat. S. 39 54 0	39 53 0
21 9 9½	from the right shoulder of Orion	26 21 25	Long. ♊ 2 16 0	♊ 2 18 30
	from the bright star in the jaw of Cetus	29 28 0	Lat. S. 33 41 0	33 39 40
22 9 0	from the right shoulder of Orion	29 47 0	Long. ♉24 24 0	♉24 27 0
	from the bright star in the jaw of Cetus	20 29 30	Lat. S. 27 45 0	27 46 0
26 7 58	from the bright star in Aries	23 20 0	Long. ♉ 9 0 0	♉ 9 2 28
	from Aldebaran	26 44 0	Lat. S. 12 36 0	12 34 13
27 6 45	from the bright star in Aries	20 45 0	Long. ♉ 7 5 40	♉ 7 8 45
	from Aldebaran	28 10 0	Lat. S. 10 23 0	10 23 13
28 7 39	from the bright star in Aries	18 29 0	Long. ♉ 5 24 45	♉ 5 27 52
	from the Hyades	29 37 0	Lat. S. 8 22 50	8 23 37
31 6 45	from the girdle of Andromeda	30 48 10	Long. ♉ 2 7 40	♉ 2 8 20
	from the Hyades	32 53 30	Lat. S. 4 13 0	4 16 25
Jan. 1665				
7 7 37½	from the girdle of Andromeda	25 11 0	Long. ♈28 24 47	♈28 24 0
	from the Hyades	37 12 25	Lat. N. 0 54 0	0 53 0
13 7 0	from the head of Andromeda	28 7 10	Long. ♈27 6 54	♈27 6 39
	from the Hyades	38 55 20	Lat. N. 3 6 50	3 7 40
24 7 29	from the girdle of Andromeda	20 32 15	Long. ♈26 29 15	♈26 28 50
	from the Hyades	40 5 0	Lat. N. 5 25 50	5 26 0
February				
7 8 37			Long. ♈27 4 46	♈27 24 55
			Lat. N. 7 3 29	7 3 15
22 8 46			Long. ♈28 29 46	♈28 29 58
			Lat. N. 8 12 36	8 10 25
March				
1 8 16			Long. ♈29 18 15	♈29 18 20
			Lat. N. 8 36 26	8 36 12
7 8 37			Long. ♉ 0 2 48	♉ 0 2 42
			Lat. N. 8 56 30	8 56 56

them being to the distance between the first and second stars of Aries, that is, to 1°33′, as 4 to 45 according to Hooke, or as 2 to 23 according to [Gilles François] Gottigniez. Accordingly, the distance of the comet from the second star of Aries was 8°16″ according to Hooke, or 8′5″ according to Gottigniez; or, taking the mean, was 8′10″. And according to Gottigniez the comet had now just gone beyond the second star of Aries by about a space of a fourth or a fifth of the course completed in one day, that is, roughly 1′35″ (and Auzout agrees well enough with this), or a little less according to Hooke, say 1′. Therefore, if 1′ is added to the longitude of the first star of Aries, and 8′10″ to its latitude, the longitude of the comet will be found to be ♈29°18′, and its latitude 8°36′26″ N.

On 7 March at 7^h30^m in Paris (that is, 7 March at 8^h37^m Gdansk time), the distance of the comet from the second star of Aries, according to Auzout's observations, was equal to the distance of the second star of Aries from the star A, that is, 52′29″. And the difference between the longitudes of the comet and of the second star of Aries was 45′ or 46′ or, taking the mean, 45′30″. And therefore the comet was in ♉0°2′48″. From the diagram of Auzout's observations that Petit constructed, Hevelius determined the latitude of the comet to be 8°54′. But the engraver curved the path of the comet irregularly toward the end of its motion, and Hevelius corrected the irregular curving in a diagram of Auzout's observations drawn by Hevelius himself, and thus made the latitude of the comet 8°55′30″. And by correcting the irregularity a little more, the latitude can come out to be 8°56′, or 8°57′.

This comet was also seen on 9 March and then must have been located in ♉0°18′ with latitude roughly 9°3½′ N.

This comet was visible for three months in all, during which time it passed through about six signs, completing about 20 degrees in each day. Its path deviated considerably from a great circle, being curved northward; and toward the end, its motion changed from retrograde to direct. And notwithstanding so unusual a path, the theory agrees with the observations from beginning to end no less exactly than theories of the planets tend to agree with observations of them, as will be clear upon examination of the table. Nevertheless, roughly 2 minutes must be subtracted when the comet was swiftest, and this will result by taking away 12 seconds from the angle between the ascending node and the perihelion, or by making that angle 49°27′18″. The annual parallax of each of the two comets (both this one and the previous one) was quite pronounced, and as a result it gave proof of the annual motion of the earth in its orbit.

The theory is confirmed also by the motion of the comet that appeared in 1683. It had a retrograde motion in an orbit whose plane contained almost a right angle with the plane of the ecliptic. Its ascending node (by Halley's calculation) was in ♍23°23′; the inclination of its orbit to the ecliptic 83°11′; its perihelion in ♊25°29′30″; its perihelial distance from the sun 56,020, the radius of the earth's orbit being taken at 100,000, and the time of its perihelion 2 July $3^{h}50^{m}$. And the places of the comet in this orbit, as calculated by Halley and compared with the places observed by Flamsteed, are displayed in the following table.

1683 Mean [lit. equated] time	*Place of the sun*	*Calculated longitude of the comet*	*Calculated latitude north*	*Observed longitude of the comet*	*Observed latitude north*	*Difference in longitude*	*Difference in latitude*
d h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	′ ″
July 13 12 55	♌ 1 2 30	♋13 5 42	29 28 13	♋13 6 42	29 28 20	+1 0	+0 7
15 11 15	2 53 12	11 37 48	29 34 0	11 39 43	29 34 50	+1 55	+0 50
17 10 20	4 45 45	10 7 6	29 33 30	10 8 40	29 34 0	+1 34	+0 30
23 13 40	10 38 21	5 10 27	28 51 42	5 11 30	28 50 28	+1 3	−1 14
25 14 5	12 35 28	3 27 53	24 24 47	3 27 0	28 23 40	−0 53	−1 7
31 9 42	18 9 22	♊27 55 3	26 22 52	♊27 54 24	26 22 25	−0 39	−0 27
31 14 55	18 21 53	27 41 7	26 16 57	27 41 8	26 14 50	+0 1	−2 7
Aug. 2 14 56	20 17 16	25 29 32	25 16 19	25 28 46	25 17 28	−0 46	+1 9
4 10 49	22 2 50	23 18 20	24 10 49	23 16 55	24 12 19	−1 25	+1 30
6 10 9	23 56 45	20 42 23	22 47 5	20 40 32	22 49 5	−1 51	+2 0
9 10 26	26 50 52	16 7 57	20 6 37	16 5 55	20 6 10	−2 2	−0 27
15 14 1	♍ 2 47 13	3 30 48	11 37 33	3 26 18	11 32 1	−4 30	−5 32
16 15 10	3 48 2	0 43 7	9 34 16	0 41 55	9 34 13	−1 12	−0 3
18 15 44	5 45 33	♉24 52 53	5 11 15	♉24 49 5	5 9 11	−3 48	−2 4
			South		South		
22 14 44	9 35 49	11 7 14	5 16 53	11 7 12	5 16 50	−0 2	−0 3
23 15 52	10 36 48	7 2 18	8 17 9	7 1 17	8 16 41	−1 1	−0 28
26 16 2	13 31 10	♈24 45 31	16 38 0	♈24 44 0	16 38 20	−1 31	+0 20

The theory is confirmed also by the motion of the retrograde comet that appeared in 1682. Its ascending node (by Halley's calculation) was in ♉21°16′30″. The inclination of the orbit to the plane of the ecliptic 17°56′0″. Its perihelion in ♒2°52′50″. Its perihelial distance from the sun 58,328, the radius of the earth's orbit being 100,000. And the perihelion 4 September

7^h39^m mean [*lit.* equated] time. And the places calculated from Flamsteed's observations and compared with the places calculated by the theory are shown in the following table.

1682 Apparent time	*Place of the sun*	*Calculated longitude of the comet*	*Calculated latitude north*	*Observed longitude of the comet*	*Observed latitude north*	*Difference in longitude*	*Difference in latitude*
d h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	′ ″
Aug. 19 16 38	♍ 7 0 7	♌ 18 14 28	25 50 7	♌ 18 14 40	25 49 55	−0 12	+0 12
20 15 38	7 55 52	24 46 23	26 14 42	24 46 22	26 12 52	+0 1	+1 50
21 8 21	8 36 14	29 37 15	26 20 3	29 38 2	26 17 37	−0 47	+2 26
22 8 8	9 33 55	♍ 6 29 53	26 8 42	♍ 6 30 3	26 7 12	−0 10	+1 30
29 8 20	16 22 40	♎ 12 37 54	18 37 47	♎ 12 37 49	18 34 5	+0 5	+3 42
30 7 45	17 19 41	15 36 1	17 26 43	15 35 18	17 27 17	+0 43	−0 34
Sept. 1 7 33	19 16 9	20 30 53	15 13 0	20 27 4	15 9 49	+3 49	+3 11
4 7 22	22 11 28	25 42 0	12 23 48	25 40 58	12 22 0	+1 2	+1 48
5 7 32	23 10 29	27 0 46	11 33 8	26 59 24	11 33 51	+1 22	−0 43
8 7 16	26 5 58	29 58 44	9 26 46	29 58 45	9 26 43	−0 1	+0 3
9 7 26	27 5 9	♏ 0 44 10	8 49 10	♏ 0 44 4	8 48 25	+0 6	+0 45

[a]The theory is confirmed also by the retrograde motion of the comet that appeared in 1723. Its ascending node (by the calculation of Mr. Bradley, Savilian professor of astronomy at Oxford) was in ♈14°16′. The inclination of the orbit to the plane of the ecliptic 49°59′. Its perihelion in ♉12°15′20″. Its perihelial distance from the sun 998,651, the radius of the earth's orbit being 1,000,000, and the mean [*lit.* equated] time of the perihelion being 16 September 16^h10^m. And the places of the comet in this orbit, as calculated by Bradley and compared with the places observed by himself and his uncle Mr. Pound, and by Mr. Halley, are shown in the following table.[a]

By these examples it is more than sufficiently evident that the motions of comets are no less exactly represented by the theory that we have set forth than the motions of planets are generally represented by planetary theories. And therefore the orbits of comets can be calculated by this theory, and the periodic time of a comet revolving in any orbit whatever can then be determined, and finally the transverse diameters [*lit.* latera transversa] of their elliptical orbits and their aphelian distances will become known.

The retrograde comet that appeared in 1607 described an orbit whose ascending node (according to Halley's calculation) was in ♉20°21′; the in-

aa. This paragraph and the accompanying table (on p. 936) appeared for the first time in ed. 3.

1723 Mean [lit. equated] time	*Observed longitude of the comet*	*Observed latitude north*	*Calculated longitude of the comet*	*Calculated latitude north*	*Difference in longitude*	*Difference in latitude*
d h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
Oct. 9 8 5	♒7 22 15	5 2 0	♒7 21 26	5 2 47	+49	−47
10 6 21	6 41 12	7 44 13	6 41 42	7 43 18	−50	+55
12 7 22	5 39 58	11 55 0	5 40 19	11 54 55	−21	+ 5
14 8 57	4 59 49	14 43 50	5 0 37	14 44 1	−48	−11
15 6 35	4 47 41	15 40 51	4 47 45	15 40 55	− 4	− 4
21 6 22	4 2 32	19 41 49	4 2 21	19 42 3	+11	−14
22 6 24	3 59 2	20 8 12	3 59 10	20 8 17	− 8	− 5
24 8 2	3 55 29	20 55 18	3 55 11	20 55 9	+18	+ 9
29 8 56	3 56 17	22 20 27	3 56 42	22 20 10	−25	+17
30 6 20	3 58 9	22 32 28	3 58 17	22 32 12	− 8	+16
Nov. 5 5 53	4 16 30	23 38 33	4 16 23	23 38 7	+ 7	+26
8 7 6	4 29 36	24 4 30	4 29 54	24 4 40	−18	−10
14 6 20	5 2 16	24 48 46	5 2 51	24 48 16	−35	+30
20 7 45	5 42 20	25 24 45	5 43 13	25 25 17	−53	−32
Dec. 7 6 45	8 4 13	26 54 18	8 3 55	26 53 42	+18	+36

clination of the plane of its orbit to the plane of the ecliptic was 17°2′; its perihelion was in ♒2°16′; and its perihelial distance from the sun was 58,680, the radius of the earth's orbit being 100,000. And the comet was in its perihelion on 16 October at 3^h50^m. This orbit agrees very closely with the orbit of the comet that was seen in 1682. If these two comets should be one and the same, this comet will revolve in a space of seventy-five years and the major axis of its orbit will be to the major axis of the earth's orbit as $\sqrt[3]{(75 \times 75)}$ to 1, or roughly 1,778 to 100. And the aphelial distance of this comet from the sun will be to the mean distance of the earth from the sun as roughly 35 to 1. And once these quantities are known, it will not be at all difficult to determine the elliptical orbit of this comet. What has just been said will be found to be true if the comet returns hereafter in this orbit in a space of seventy-five years. The other comets seem to revolve in a greater time and to ascend higher.

But because of the great number of comets, and the great distance of their aphelia from the sun, and the long time that they spend in their aphelia, they should be disturbed somewhat by their gravities toward one another, and hence their eccentricities and times of revolutions ought sometimes to be increased a little and sometimes decreased a little. Accordingly, it is not to be expected that the same comet will return exactly in the same orbit, and with the same periodic times. It is sufficient if no greater changes are found to occur than those that arise from the above-mentioned causes.

And hence a reason appears why comets are not restricted to the zodiac as planets are, but depart from there and are carried with various motions into all regions of the heavens—namely, for this purpose, that in their aphelia, when they move most slowly, they may be as far distant from one another as possible and may attract one another as little as possible. And this is the reason why comets that descend the lowest, and so move most slowly in their aphelia, should also ascend to the greatest heights.

The comet that appeared in 1680 was distant from the sun in its perihelion by less than a sixth of the sun's diameter; and because its velocity was greatest in that region and also because the atmosphere of the sun has some density, the comet must have encountered some resistance and must have been somewhat slowed down and must have approached closer to the sun; and by approaching closer to the sun in every revolution, it will at length fall into the body of the sun. But also, in its aphelion, when it moves most slowly, the comet can sometimes be slowed down by the attraction of other comets and as a result fall into the sun. So also fixed stars, which are exhausted bit by bit in the exhalation of light and vapors, can be renewed by comets falling into them and then, kindled by their new nourishment, can be taken for new stars. Of this sort are those fixed stars that appear all of a sudden, and that at first shine with maximum brilliance and subsequently disappear little by little. Of such sort was the star that Cornelius Gemma saw in Cassiopeia's Chair on 9 November 1572; it was shining brighter than all the fixed stars, scarcely inferior to Venus in its brilliance. But he did not see it at all on 8 November, when he was surveying that part of the sky on a clear night. Tycho Brahe saw this same star on the 11th of that month, when it shone with the greatest splendor; and he observed it decreasing little by little after that time, and he saw it disappearing after the space of sixteen months. In November, when it first appeared, it equaled Venus in brightness. In December, somewhat diminished, it equaled Jupiter. In January 1573 it was less than Jupiter and greater than Sirius, and it became equal to Sirius at the end of February and the beginning of March. In April and May it was equal to stars of the second magnitude; in June, July, and August, to stars of the third magnitude; in September, October, and November, to stars of the fourth magnitude; in December and in January 1574, to stars of the fifth magnitude; and in February, to stars of the sixth magnitude; and in March, it vanished from sight. Its color at the start was clear, whitish, and bright; afterward it became yellowish, and in March of 1573 reddish like Mars or the star Aldebaran, while in May it took on a livid whiteness such as we see in Saturn, and it maintained this color up to the end, yet all the while becoming fainter. Such also was the star in the right foot of Serpentarius, the beginning of whose visibility was observed by the pupils of Kepler in 1604, on 30 September (O.S.); they saw it exceeding Jupiter in its light, although it had not been visible at all on the preceding night. And from that time it decreased little by little and in the space of fifteen or sixteen months vanished from sight. It was when such a new star appeared shining beyond measure that Hipparchus is said to have been stimulated to observe the fixed stars and to put them into a catalog. But fixed stars that alternately appear and disappear, and increase little by little, and are hardly ever brighter than fixed stars of the third magnitude, seem to be of another kind and, in revolving, seem to show alternately a bright side and a dark side. And the vapors that arise from the sun and the fixed stars and the tails of comets can fall by their gravity into the atmospheres of the planets and there be condensed and converted into water and humid spirits, and then—by a slow heat—be transformed gradually into salts, sulphurs, tinctures, slime, mud, clay, sand, stones, corals, and other earthy substances.

GENERAL SCHOLIUM[a]

The hypothesis of vortices is beset with many difficulties. If, by a radius drawn to the sun, each and every planet is to describe areas proportional to the time, the periodic times of the parts of the vortex must be as the squares of the distances from the sun. If the periodic times of the planets are to be as the $\frac{3}{2}$ powers of the distances from the sun, the periodic times of the parts of the vortex must be as the $\frac{3}{2}$ powers of the distances. If the smaller vortices revolving about Saturn, Jupiter, and the other planets are to be preserved and are to float without agitation in the vortex of the sun, the periodic times of the parts of the solar vortex must be the same. The axial revolutions [i.e., rotations] of the sun and planets, [b]which would have to agree with the motions of their vortices,[b] differ from all these proportions. The motions of comets are extremely regular, observe the same laws as the motions of planets, and cannot be explained by vortices. Comets go with very eccentric motions into all parts of the heavens, which cannot happen unless vortices are eliminated.

The only resistance which projectiles encounter in our air is from the air. With the air removed, as it is in Boyle's vacuum, resistance ceases, since a tenuous feather and solid gold fall with equal velocity in such a vacuum. And the case is the same for the celestial spaces, which are above the atmosphere

a. Ed. 1 lacks the General Scholium, which includes Newton's famous discussions of God and of hypotheses. This scholium is first printed in ed. 2 but is documented further by its changing versions in five extant earlier holograph drafts and is treated also in contemporaneous correspondence between Newton and Roger Cotes, editor of ed. 2. For details see *Unpublished Scientific Papers of Isaac Newton*, ed. A. Rupert Hall and Marie Boas Hall (Cambridge: Cambridge University Press, 1962), pp. 348–364; I. Bernard Cohen, *Introduction to Newton's "Principia"* (Cambridge, Mass.: Harvard University Press; Cambridge: Cambridge University Press, 1971), pp. 240–245.

bb. Ed. 2 lacks this.

of the earth. All bodies must move very freely in these spaces, and therefore planets and comets must revolve continually in orbits given in kind and in position, according to the laws set forth above. They will indeed persevere in their orbits by the laws of gravity, but they certainly could not originally have acquired the regular position of the orbits by these laws.

The six primary planets revolve about the sun in circles concentric with the sun, with the same direction of motion, and very nearly in the same plane. Ten moons revolve about the earth, Jupiter, and Saturn in concentric circles, with the same direction of motion, very nearly in the planes of the orbits of the planets. And all these regular motions do not have their origin in mechanical causes, since comets go freely in very eccentric orbits and into all parts of the heavens. And with this kind of motion the comets pass very swiftly and very easily through the orbits of the planets; and in their aphelia, where they are slower and spend a longer time, they are at the greatest possible distance from one another, so as to attract one another as little as possible.

This most elegant system of the sun, planets, and comets could not have arisen without the design and dominion of an intelligent and powerful being. And if the fixed stars are the centers of similar systems, they will all be constructed according to a similar design and subject to the dominion of *One*, especially since the light of the fixed stars is of the same nature as the light of the sun, and all the systems send light into all the others. [c]And so that the systems of the fixed stars will not fall upon one another as a result of their gravity, he has placed them at immense distances from one another.[c]

He rules all things, not as the world soul but as the lord of all. And because of his dominion he is called Lord God *Pantokrator*[d]. For "god" is a relative word and has reference to servants, and godhood[e] is the lordship of God, not over his own body [f]as is supposed by those for whom God is the world soul[f], but over servants. The supreme God is an eternal, infinite, and absolutely perfect being; but a being, however perfect, without dominion is

cc. Ed. 2 lacks this.

d. Newton's note a: "That is, universal ruler."

e. Newton here uses the word "deitas," a nonclassical term which signifies the essential nature of the divinity or "god-ness." Although "Godhead" does fit, the term "godhood" (which is more abstract) may more accurately convey the sense of Newton's "deitas."

ff. Ed. 2 lacks this.

not the Lord God. For we do say my God, your God, the God of Israel, the God of Gods, and Lord of Lords, but we do not say my eternal one, your eternal one, the eternal one of Israel, the eternal one of the gods; we do not say my infinite one, or my perfect one. These designations [i.e., eternal, infinite, perfect] do not have reference to servants. The word "god" is used far and wide to mean "lord,"[g] but every lord is not a god. The lordship of a spiritual being constitutes a god, a true lordship constitutes a true god, a supreme lordship a supreme god, an imaginary lordship an imaginary god. And from true lordship it follows that the true God is living, intelligent, and powerful; from the other perfections, that he is supreme, or supremely perfect. He is eternal and infinite, omnipotent and omniscient, that is, he endures from eternity to eternity, and he is present from infinity to infinity; he rules all things, and he knows all things that happen or can happen. He is not eternity and infinity, but eternal and infinite; he is not duration and space, but he endures and is present. He endures always and is present everywhere, and by existing always and everywhere he constitutes duration and [h]space.[h] Since each and every particle of space is *always*, and each and every indivisible moment of duration is *everywhere*, certainly the maker and lord of all things will not be *never* or *nowhere*.

[i]Every sentient soul, at different times and in different organs of senses and motions, is the same indivisible person. There are parts that are successive in duration and coexistent in space, but neither of these exist in the person of man or in his thinking principle, and much less in the thinking substance of God. Every man, insofar as he is a thing that has senses, is one and the same man throughout his lifetime in each and every organ of his senses. God is one and the same God always and everywhere.[i] He is omnipresent not only *virtually* but also *substantially*; for action requires substance [*lit.* for active power [virtus] cannot subsist without substance]. In him all things are contained and move,[j] but he does not act on them nor they on him. God

g. Newton's note b, which ed. 2 lacks: "Our fellow countryman Pocock derives the word 'deus' from the Arabic word 'du' (and in the oblique case 'di'), which means lord. And in this sense princes are called gods, Psalms 82.6 and John 10.35. And Moses is called a god of his brother Aaron and a god of king Pharaoh (Exod. 4.16 and 7.1). And in the same sense the souls of dead princes were formerly called gods by the heathen, but wrongly because of their lack of dominion."

hh. Ed. 2 has "space, eternity, and infinity."

ii. Ed. 2 lacks this.

j. Newton's note c: "This opinion was held by the ancients: for example, by Pythagoras as cited in Cicero, *On the Nature of the Gods*, book 1; Thales; Anaxagoras; Virgil, *Georgics*, book 4, v. 221, and *Aeneid*, experiences nothing from the motions of bodies; the bodies feel no resistance from God's omnipresence.

It is agreed that the supreme God necessarily exists, and by the same necessity he is *always* and *everywhere*. It follows that all of him is like himself: he is all eye, all ear, all brain, all arm, all force of sensing, of understanding, and of acting, but in a way not at all human, in a way not at all corporeal, in a way utterly unknown to us. As a blind man has no idea of colors, so we have no idea of the ways in which the most wise God senses and understands all things. He totally lacks any body and corporeal shape, and so he cannot be seen or heard or touched, nor ought he to be worshiped in the form of something corporeal. We have ideas of his attributes, but we certainly do not know what is the substance of any thing. We see only the shapes and colors of bodies, we hear only their sounds, we touch only their external surfaces, we smell only their odors, and we taste their flavors. But there is no direct sense and there are no indirect reflected actions by which we know innermost substances; much less do we have an idea of the substance of God. We know him only by his properties and attributes and by the wisest and best construction of things and their final causes, [k]and we admire him because of his perfections;[k] but we venerate and worship him because of his dominion. [l]For we worship him as servants, and a god[l] without dominion, providence, and final causes is nothing other than fate and nature. [m]No variation in things arises from blind metaphysical necessity, which must be the same always and everywhere. All the diversity of created things, each in its place and time, could only have arisen from the ideas and the will of a necessarily existing being. But God is said allegorically to see, hear, speak, laugh, love, hate, desire, give, receive, rejoice, be angry, fight, build, form, construct. For all discourse about God is derived through a certain similitude

book 6, v. 726; Philo, *Allegorical Interpretation*, book 1, near the beginning; Aratus in the *Phenomena*, near the beginning. Also by the sacred writers: for example, Paul in Acts 17.27, 28; John in his Gospel 14.2; Moses in Deuteronomy 4.39 and 10.14; David, Psalms 139.7, 8, 9; Solomon, 1 Kings 8.27; Job 22.12, 13, 14; Jeremiah 23.23, 24. Moreover idolators imagined that the sun, moon, and stars, the souls of men, and other parts of the world were parts of the supreme god and so were to be worshiped, but they were mistaken." In ed. 2 this note reads: "This opinion was held by the ancients: Aratus in the *Phenomena*, near the beginning; Paul in Acts 7.27, 28; Moses, Deuteronomy 4.39 and 10.14; David, Psalms 139.7, 8; Solomon, Kings 8.27; Job 22.12; the prophet Jeremiah, 23.23, 24."

kk. Ed. 2 lacks this.

ll. Ed. 2 has: "For a god."

mm. Ed. 2 lacks this.

from things human, which while not perfect is nevertheless a similitude of some kind.[m] This concludes the discussion of God, and to treat of God from phenomena is certainly a part of [n]natural[n] philosophy.

Thus far I have explained the phenomena of the heavens and of our sea by the force of gravity, but I have not yet assigned a cause to gravity. Indeed, this force arises from some cause that penetrates as far as the centers of the sun and planets without any diminution of its power to act, and that acts not in proportion to the quantity of the *surfaces* of the particles on which it acts (as mechanical causes are wont to do) but in proportion to the quantity of *solid* matter, and whose action is extended everywhere to immense distances, always decreasing as the squares of the distances. Gravity toward the sun is compounded of the gravities toward the individual particles of the sun, and at increasing distances from the sun decreases exactly as the squares of the distances as far out as the orbit of Saturn, as is manifest from the fact that the aphelia of the planets are at rest, and even as far as the farthest aphelia of the comets, provided that those aphelia are at rest. I have not as yet been able to deduce from phenomena the reason for these properties of gravity, and I do not [o]feign[o] hypotheses. For whatever is not deduced from the phenomena must be called a hypothesis; and hypotheses, whether metaphysical or physical, or based on occult qualities, or mechanical, have no place in experimental philosophy. In this experimental philosophy, propositions are deduced from the phenomena and are made general by induction. The impenetrability, mobility, and impetus of bodies, and the laws of motion and the law of gravity have been found by this method. And it is enough that gravity really exists and acts according to the laws that we have set forth and is sufficient to explain all the motions of the heavenly bodies and of our sea.

[p]A few things could now be added concerning a certain very subtle spirit pervading gross bodies and lying hidden in them; by its force and actions, the particles of bodies attract one another at very small distances and cohere when

nn. Ed. 2 has "experimental."

oo. The word "fingo" in Newton's famous declaration, "Hypotheses non fingo," appears to be the Latin equivalent of the English word "feign." Andrew Motte translated "fingo" by "frame," a verb which at that time could have a pejorative sense. For details see the Guide, §9.1.

pp. The final paragraph of the General Scholium has attracted much scholarly attention, notably in an effort to discern what Newton intended (in the opening and closing sentences) by a "spirit" which may they become contiguous; and electrical [i.e., electrified] bodies act at greater distances, repelling as well as attracting neighboring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the limbs of animals move at command of the will, namely, by the vibrations of this spirit being propagated through the solid fibers of the nerves from the external organs of the senses to the brain and from the brain into the muscles. But these things cannot be explained in a few words; furthermore, there is not a sufficient number of experiments to determine and demonstrate accurately the laws governing the actions of this spirit.[p]

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be operative in various types of phenomena. It might even appear that Newton was here introducing a speculation—we dare not call it a hypothesis—although Newton's actual language indicates that for him there was no question about whether this spirit "really" exists, only about the laws according to which this spirit acts.

A puzzle relating to the interpretation of this "spirit" is the appearance of the qualifying adjectives "electric and elastic," introduced in the original Motte translation and retained in the Motte-Cajori version. Although these words are not found in either the second or the third Latin editions, they have a Newtonian provenience since they occur in Newton's personal interleaved copy of the second edition as one of the proposed emendations. Furthermore, thanks to the research of A. Rupert Hall and Marie Boas Hall, we know that the spirit in question is indeed "electrical." In particular, as Newton worked toward the second edition of the *Principia*, he composed various drafts of proposed conclusions which, together with other manuscripts, provide evidence for the importance of electrical phenomena in his thinking about gravity during the years 1711–1713. For details see the Guide to the present translation, §9.3.

One possible reason why Newton decided not to insert the qualifying phrase "electric and elastic" into the text of the third edition (1726) is that in his interleaved copy of the second edition he has finally drawn a line through the whole paragraph, showing his intention of deleting it in a third edition. The reason for this decision seems to be that some time after 1713 Newton lost his enthusiasm for electricity as a possible agent in gravitation.

We may readily understand why Newton omitted to carry out either the revision or the proposed cancellation of the final paragraph. By the time that the third edition was fully printed, in about February 1726, Newton and Pemberton had spent several years revising the text and reading the proofs and Newton was within a little more than a year of his death. When Newton reached the last paragraph he was probably so weary that he overlooked his proposed alteration of the conclusion.

The third edition concludes with an "Index Rerum Alphabeticus" (pp. 531–536) and an advertisement of books sold by William and John Innys (pp. 537–538).

Notes Added in Second Printing

1. In order to understand why Newton used the term "vis inertiae" rather than simply "inertia," we must keep in mind that Newton, as he says explicitly in Def. 3, was giving a new and better name to the then-current concept of "vis insita." Thus he merely changed one qualifier ("insita") to another ("inertiae").

2. On Newton's mathematical methods in the *Principia* and the history of the ways in which his successors read and revised Newton's rational mechanics, see Niccolò Guicciardini's masterful analysis, *Reading the "Principia": The Debate on Newton's Mathematical Methods for Natural Philosophy from 1687 to 1736* (Cambridge: Cambridge University Press, 1999).

3. On Newton's concept of limit and on the methods of book 1, sec. 1, of the *Principia,* see Bruce Pourciau, "Newton and the Notion of Limit," *Historia Mathematica* 28 (2001): 18–30, and "The Preliminary Mathematical Lemmas of Newton's *Principia,*" *Archive for History of Exact Sciences* 52 (1998): 279–295.

4. On Book 1, Lemma 28, see Bruce Pourciau, "The Integrability of Ovals," *Archive for History of Exact Sciences* 55 (2001): 478–499.

5. On the solid of least resistance and Newton's thoughts concerning the design of ships, see A. Rupert Hall, *Ballistics in the Seventeenth Century* (Cambridge: Cambridge University Press, 1952), pp. 141–145.