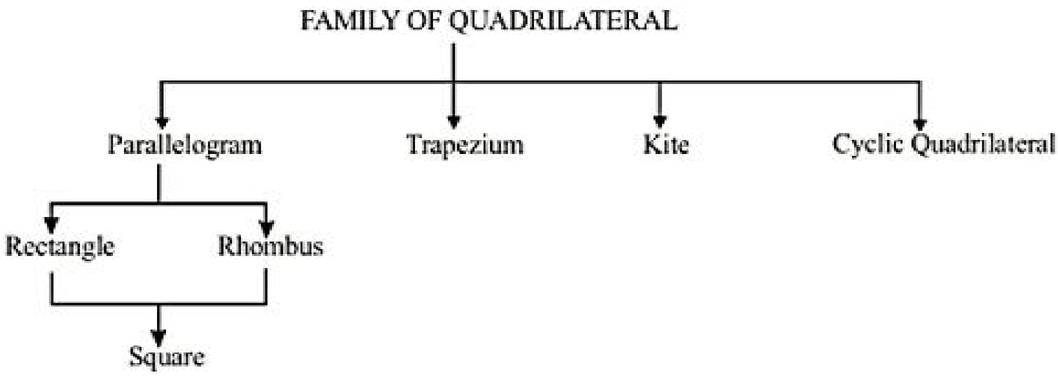
STRAIGHT LINE

1. INTRODUCTION :

We are familiar with 'Geometry' which is simply the study of the properties of figures and curves. The geometry usually studied upto the high school is known as 'Euclidean 'Geometry' as it is based upon the axioms laid by famous Greek mathematician Euclid in his first systematic treatise on geometry about 300 B.C. During this period and upto the seventeenth century geometric reasoning alone was employed in the study of geometry. This study of geometry is named as 'synthetic geometry'. There were problems whose solution were not available in the synthetic geometry. It was until about 17th century, A.D. that the geometry was linked with algebra, which is employed in solution of problems in synthetic geometry. By this mean the methods of algebra are applied in the study of geometry which is referred now as 'analytic geometry'. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician Rene' Descartes' (1596 – 1650), in his book 'La Geometrie', which was published in 1637. The book 'La Geometric is mainly concerned with the algebraic solutions of geometric problems and geometric interpretation of algebraic equations.

In order to relate algebra with geometry Descartes established a relationship between the basic geometric concept of 'point' with ordered pairs of real numbers. This association is named after Rene' Descartes' as Cartesian coordinate system which will be studied in this chapter.

2. TYPES OF QUADRILATERAL (EUCLIDIAN FIGURES) :



2.1 PARALLELOGRAM :

Definition :-

If opposite sides of quadrilateral are parallel and equal, then quadrilateral is called parallelorgam.

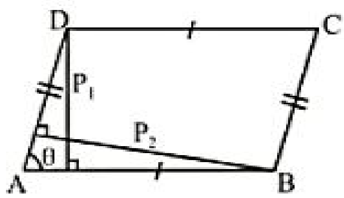
Four ways to prove that a quadrilateral is parallogram.

(i) Opposite sides are parallel i.e. AB || DC and AD || BC

(ii) Opposite sides are equal i.e. AB = DC and AD = BC

(iii) One pair of opposite sides are equal and parallel.

(iv) Diagonals bisect each other



Note: Area of parallelogram = $\dfrac{P_1P_2}{\sin\theta}$

Proof : Area = Base × Height (where P_1 and P_2 are distances between pair of parallel sides and θ is angle between two adjacent sides)

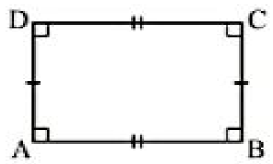
$= AB \times P_1$

$= \dfrac{P_1P_2}{\sin\theta}$ $\quad \{\because \sin\theta = \dfrac{P_2}{AB}\}$

(i) Rectangle :

Definition :

If all angles of parallelogram are equal then it is called rectangle.



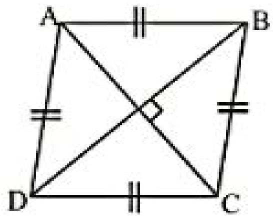
(i) Diagonal are equal and bisect each other.

(ii) Each diagonal divides the rectangle into two triangles of equal area.

(ii) Rhombus :

Definition :

If all sides of a parallelogram are equal then it is called Rhombus



(i) Diagonals are perpendicular

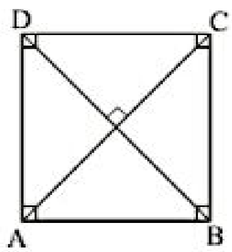
(ii) Area = $\dfrac{1}{2}\, d_1d_2$ where d_1 and d_2 are diagonal.

Note: If distance between pair of parallel sides are equal then it is a rhombus.

(iii) Square :

Definition :

If all the sides and all the interior angles of a parallelogram are equal then it is called a square.



(i) All sides are equal

$AB = BC = CD = DA$

(ii) Diagonals are equal and bisect each other at 90°.

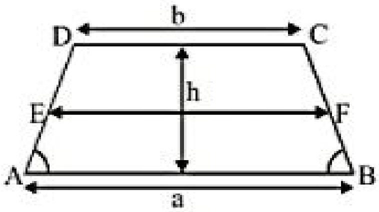
(iii) Area = $\dfrac{d^2}{2}$ (d = diagonal)

Note: Every square is a rectangle but not the converse.

2.2 TRAPEZIUM :

Definition :-

Trapezium is a quadrilateral which has exactly one pair of opposite sides parallel.



$\angle DAB$ & $\angle CBA$ = Base angle

(i) Area of trapezium = $\frac{1}{2}$ (sum of parallel sides) × distance between sides

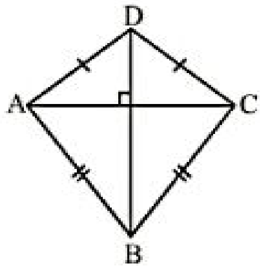
(iii) Median (EF) = $\frac{1}{2}(a+b)$

(iv) For equilateral /isosceles trapezium, non parallel sides are equal i.e. AD = BC

2.3 KITE :

Definition :

It is a quadrilateral in which two pairs of adjacent sides are equal.



(i) AD = DC and AB = BC

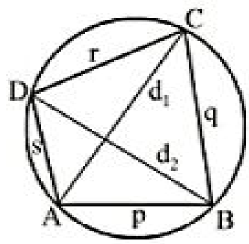
(ii) Diagonals are perpendicular but not bisect

(iii) Only one diagonal divide the figure into two congruent triangles.

2.4 CYCLIC QUADRILATERAL :

Definition :

If all vertices of quadrilateral lies on the circumference of a circle, then it is called cyclic quadrilateral.

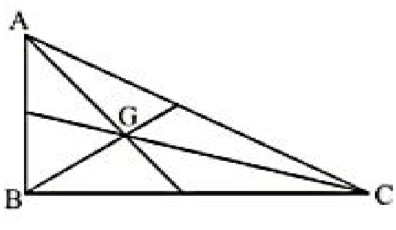


(i) Opposite angles are supplementary.

(ii) Sum of product of opposite sides are equal to product of diagonals

$pr + qs = d_1 d_2$

3. FIVE IMPORTANT POINTS WITH RESPECT TO A TRIANGLE :



3.1 CENTROID (G) :

Definition :

Centroid is a point of concurrency of medians.

⇒ Centroid always lies inside the triangle.

⇒ Centroid divides the median in the ratio 2 : 1, reckoning from the vertex.

⇒ The area of ΔGBC, ΔGCA, ΔGAB are equal (G is centroid)

⇒ Each median divides the triangle into the two triangles of equal areas.

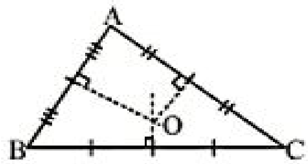
3.2 CIRCUMCENTRE (O) :

Definition :

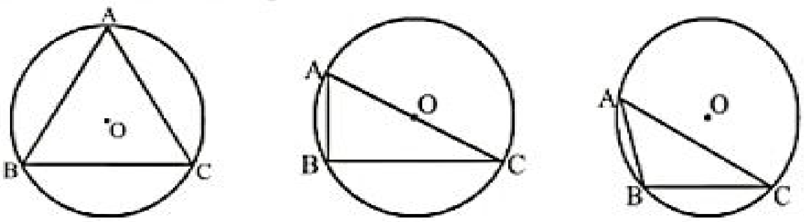
Circumcentre is a point of concurrency of perpendicular bisector of sides of triangle.

or

Circumcentre is a centre of circle circumscribing the triangle.



⇒ In case of acute angle triangle circumcentre lies inside the triangle
For right angle triangle it lies on mid-point of hypotenuse and in case of obtuse angle triangle it lies outside the triangle.



3.3 INCENTRE (I) :

Definition :

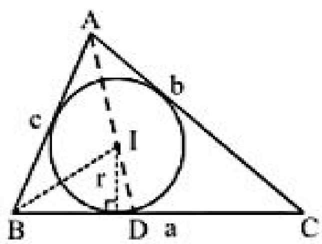
Incentre is a point of concurrency of internal angle bisector of triangle.

⇒ Incentre always lies inside the triangle.

⇒ Internal angle bisector AD divides the base BC in the ratio of the sides containing the angle i.e. BD : DC = c : b

⇒ Incentre I divides AD in the ratio AB : BD

⇒ AI : ID = (b + c) : a

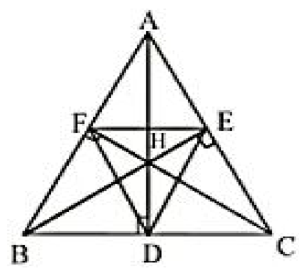


3.4 ORTHOCENTRE (H) :

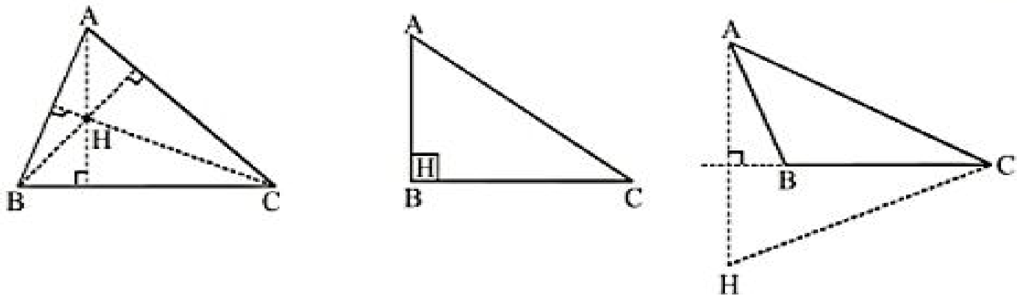
Definition :

Orthocentre is a point of concurrency of altitudes of triangle.

⇒ In case of acute angle triangle orthocentre lies inside the triangle.
For right angle triangle orthocentre lies at the vertex where it is right angled and in case of obtuse triangle orthocentre lies outside the triangle.



H- Orthocentre



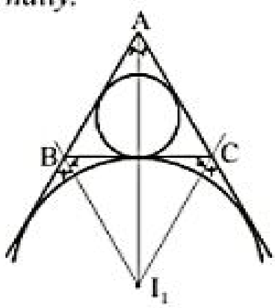
3.5 EXCENTRES (I_1, I_2, I_3) :

Definition :

Excentre is a point of concurrency of two external angle bisectors and one interior angle bisector.

or

Centre of a circle (excircle) which touches all the sides of the triangle externally.



$\Rightarrow$ There are three excentres with respect to a given triangle.

I_1 : Centre of the excircle opposite to vertex A (as shown in figure)

I_2 : Centre of the excircle opposite to vertex B

I_3 : Centre of the excircle opposite to vertex C

Imp. Points :

- For isosceles triangle centroid, circumcentre, orthocentre and incentre are collinear.
- For a triangle Orthocentre (O), Centroid (G), Circumcentre (C) are collinear and centroid divides orthocentre and circumcentre in the ratio 2 : 1 internally.
- For equilateral Δ, centroid, circumcentre, orthocentre and incentre coincide.

Illustration :

Find the distance between the orthocentre and circumcentre of a triangle whose vertices are

$P(3, 0), Q(0, 0)$ *and* $R\left(\frac{3}{2}, \frac{-3\sqrt{3}}{2}\right)$

Sol. $\therefore$ side $PQ = \sqrt{(3-0)^2+(0-0)^2} = 3$

$$QR = \sqrt{\left(\frac{3}{2}-0\right)^2+\left(\frac{-3\sqrt{3}}{2}-0\right)^2} = 3$$

$$PR = \sqrt{\left(3-\frac{3}{2}\right)^2+\left(0+\frac{3\sqrt{3}}{2}\right)^2} = 3$$

Hence $PQ = QR = PR$

Hence, the triangle is equilateral.

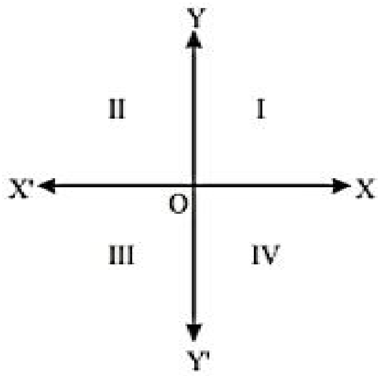
Now, since in an equilateral triangle orthocentre and circumcentre coincides therefore distance between them is zero.

4. CARTESIAN COORDINATE SYSTEM :

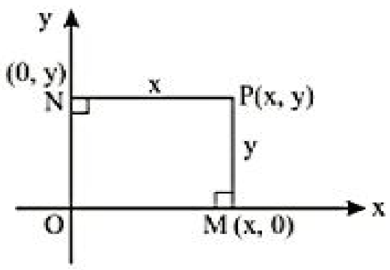
4.1 QUADRANT :

Two mutually perpendicular lines meeting at 'O' (origin) are called axes. Horizontal line X' OX is known as x-axis and vertical line Y'OY is called y-axis. These two perpendicular lines divide the plane into four quadrants, viz., as follows their names are given in anti-clockwise sense.

XOY = First quadrant, X'OY = Second quadrant
X'OY' = Third quadrant, Y'OX = fourth quadrant



4.2 COORDINATES OF A POINT :



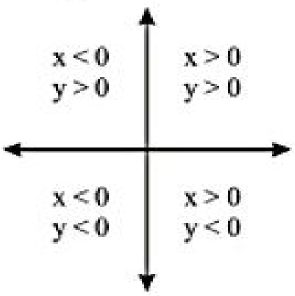
Co-ordinates of a point are given by ordered pair (x, y) whose first entry (x) denotes the x-coordinate or abscissa of the point and second entry (y) denotes the y-coordinate or ordinate of the point.
For x-coordinate (Abscissa) of the point, $|x|$ is the perpendicular distance of the point from y-axis.
For y-coordinate (ordinate) of the point, $|y|$ is the perpendicular distance of the point from x-axis.

Note :

(i) y-coordinate of any point lying on the x-axis is always zero.

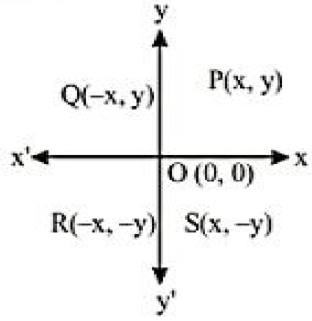
(ii) x-coordinate of any point lying on the y-axis is always zero.

In 1st quadrant $x > 0, y > 0$
IInd quadrant $x < 0, y > 0$
IIIrd quadrant $x < 0, y < 0$ and
IVth quadrant $x > 0, y < 0$



Quadrant wise sign of abscissa (x) and ordinate (y)

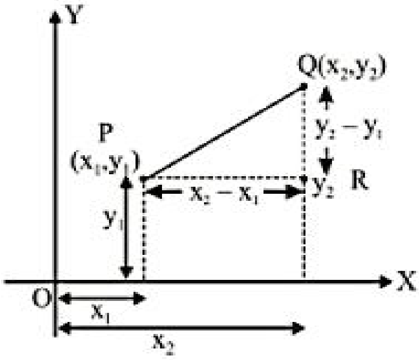
Let x and y are positve real numbers.



5. DISTANCE BETWEEN TWO POINTS :

Let $P(x_1, y_1)$ and $Q(x_2\ y_2)$ be two given points in the xy plane then distance between them is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Proof : For triangle PQR

$$PQ^2 = PR^2 + RQ^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note : distance of (x_1, y_1) from origin = $\sqrt{x_1^2 + y_1^2}$

Illustration :

Match the Column

	Column-I		***Column-II***
(A)	*The triangle with vertices A (7, 10), B (4, 5) and C (10, 15) is*	*(P)*	*Equilateral*
(B)	*The triangle with vertices P (2, 7), Q(4, –1) and R (–2, 6) is*	*(Q)*	*Isosceles*
(C)	*The triangle with verties L(3, 1), M (5, 6) and N(9, 16) is*	*(R)*	*Right angled*
(D)	*The triangle with vertices R (a, a), S(–a, –a) and T $\left(\sqrt{3}\,a, -\sqrt{3}\,a\right)$ is*	*(S)*	*Collinear*

Sol. *(A)* *In a ΔABC*

$\therefore$

$$AB = \sqrt{(4-7)^2 + (5-10)^2} = \sqrt{34}$$

$$BC = \sqrt{(10-4)^2 + (15-5)^2} = \sqrt{136}$$

$$AC = \sqrt{(10-7)^2 + (15-10)^2} = \sqrt{34}$$

Hence $AB = AC$

Hence ΔABC is isosceles triangle.

(B) $\therefore$ $PQ = \sqrt{(2-4)^2+(7+1)^2} = \sqrt{68}$

$QR = \sqrt{(4+2)^2+(-1-6)^2} = \sqrt{85}$

$PR = \sqrt{(-2-2)^2+(6-7)^2} = \sqrt{17}$

Hence $PQ^2 + PR^2 = QR^2$

Hence ΔABC is right angled.

(C) $\therefore$ $LM = \sqrt{(5-3)^2+(6-1)^2} = \sqrt{29}$

$MN = \sqrt{(9-5)^2+(16-6)^2} = \sqrt{116} = 2\sqrt{29}$

$LN = \sqrt{(9-3)^2+(16-1)^2} = \sqrt{261} = 3\sqrt{29}$

Hence $LM + MN = LN$

Hence points L, M, N are collinear.

(D) $\therefore$ side $RS = \sqrt{(a+a)^2+(a+a)^2} = 2\sqrt{2}\,a$

$ST = \sqrt{(\sqrt{3}a+a)^2+(-\sqrt{3}a+a)^2} = 2\sqrt{2}\,a$

$RT = \sqrt{(\sqrt{3}a-a)^2+(-\sqrt{3}a-a)^2} = 2\sqrt{2}\,a$

Hence $RS = ST = RT$

Hence ΔRST is equilateral.

Illustration :

If A(0, –1), B(6, 7), C(–2, 3) and D(λ, 3) forms a rectangle then find the value of λ.

Sol. $AB = CD$ and $AC = BD$.

$AB = \sqrt{6^2+8^2} = 10$

$CD = \sqrt{(\lambda+2)^2} = |\lambda+2|$

$\lambda + 2 = 10$ or $\lambda + 2 = -10$

$\lambda = 8$ or $\lambda = -12$ ... (1)

Now $AC = BD$

$4 + 16 = (6-\lambda)^2 + 4^2$

$4 = 36 + \lambda^2 - 12\lambda$

$\Rightarrow$ $\lambda^2 - 12\lambda + 32 = 0$

$\lambda = 4, 8$

Hence from (1) and (2)

$\lambda = 8$

Illustration :

Prove that the four points A(0, 0), B(2, 2), C(2 ($\sqrt{2}$ + 1), 2) and D($2\sqrt{2}$, 0) form a Rhombus but not a rectangle.

Sol. *Sides are*

$AB = 2\sqrt{2}$, $BC = 2\sqrt{2}$, $CD = 2\sqrt{2}$ $DA = 2\sqrt{2}$

Diagonals

$$AC = \sqrt{2^2(\sqrt{2}+1)^2+4}$$

$$BD = \sqrt{2^2(\sqrt{2}-1)^2+4}$$

Since $AC \neq BD$

and all sides are equal hence given points from a Rhombus but not a rectangle.

Illustration :

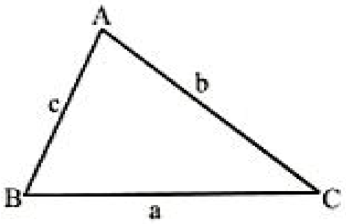
The vertices of a triangle are A(0, 0), B(2, 3) and C(0, 4). Find sinA.

Sol.

$a = BC = \sqrt{4+1} = \sqrt{5}$

$b = AC = 4$

$c = AB = \sqrt{4+9} = \sqrt{13}$



$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{16+13-5}{2.4.\sqrt{13}} = \frac{3}{\sqrt{13}}$$

$$\sin^2 A = 1 - \cos^2 A = 1 - \frac{9}{13} = \frac{4}{13}$$

$$\therefore \quad \sin A = \frac{2}{\sqrt{13}}$$

6. SECTION FORMULA :

6.1 FORMULA FOR INTERNAL DIVISION :

Coordinates of the point that divides the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio m : n are given by

$$x = \frac{mx_2+nx_1}{m+n}, \; y = \frac{my_2+ny_1}{m+n}$$

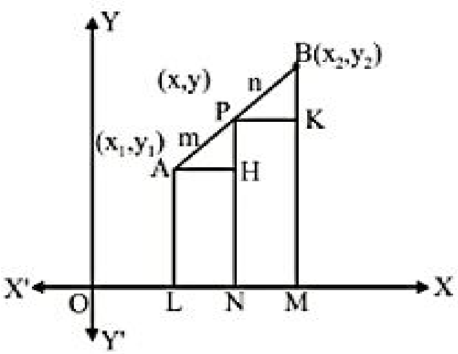
Proof :

From the figure,

Clearly, ΔAHP and ΔPKB are similar.

$$\Rightarrow \quad \frac{AP}{BP} = \frac{AH}{PK} = \frac{PH}{BK}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{x-x_1}{x_2-x} = \frac{y-y_1}{y_2-y}$$



Now, $\frac{m}{n} = \frac{x - x_1}{x_2 - x}$

$\Rightarrow \quad mx_2 - mx = nx - nx_1$

$\Rightarrow \quad x = \frac{mx_2 + nx_1}{m + n}$ and $\frac{m}{n} = \frac{y - y_1}{y_2 - y}$

$\Rightarrow \quad my_2 - my = ny - ny_1$

$\Rightarrow \quad y = \frac{my_2 + ny_1}{m + n}$

Thus, the coordinates of P are $\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right)$

6.2 FORMULA FOR EXTERNAL DIVISION :

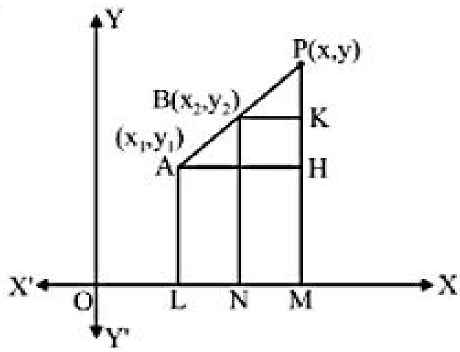
Coordinates of the point that divides the line segment joining the points (x_1, y_1) and (x_2, y_2) externally in the ratio m : n are given by

$$x = \frac{mx_2 - nx_1}{m - n}, \qquad y = \frac{my_2 - ny_1}{m - n}$$

Proof :

From the figure,

Clearly, trianlges PAH and PBK are similar. Therefore,



$$\frac{AP}{PB} = \frac{AH}{BK} = \frac{PH}{PK}$$

$\Rightarrow \quad \frac{m}{n} = \frac{x - x_1}{x - x_2} = \frac{y - y_1}{y - y_2}$

$\Rightarrow \quad mx - mx_2 = nx - nx_1$

$\Rightarrow \quad x = \frac{mx_2 - nx_1}{m - n}$

and $\quad \frac{m}{n} = \frac{y - y_1}{y - y_2}$

$\Rightarrow \quad my - my_2 = ny - ny_1$

$\Rightarrow \quad y = \frac{my_2 - ny_1}{m - n}$

Thus, the coordinates of P are $\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n} \right)$.

7. HARMONIC CONJUGATES :

If two points P and Q divides the line A B internally and externally in the same ratio m : n , then P and Q are said to be harmonic conjugate of each other with respect to A and B.

i.e. $\dfrac{AP}{PB} = \dfrac{AQ}{BQ} = \lambda$ .....(1)

Also, AP, AB and AQ are in H.P. ie. $\dfrac{2}{AB} = \dfrac{1}{AP} + \dfrac{1}{AQ}$

Proof : from (1), $\dfrac{AP}{AB - AP} = \dfrac{AQ}{AQ - AB}$

$$\frac{AB - AP}{AP} = \frac{AQ - AB}{AQ}$$

$$\frac{AB}{AP} - 1 = 1 - \frac{AB}{AQ}$$

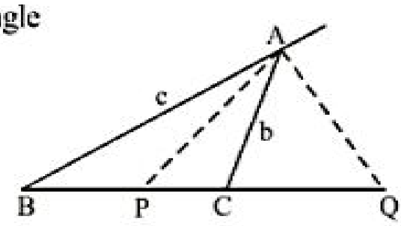
$$2 = \frac{AB}{AQ} + \frac{AB}{AP}$$

$$\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$$

Examples:

(i) Internal & exeternal angle bisector of an angle of a triangle divide the opposite base harmonically.

$$\frac{BP}{PC} = \frac{BQ}{CQ} = \frac{c}{b}$$



(ii) External and internal common tangents divide the line joining the centres of the two circles externally and internally in the ratio of the their radii.

$$\frac{C_1P}{PC_2} = \frac{C_1Q}{C_2Q} = \frac{r_1}{r_2}$$

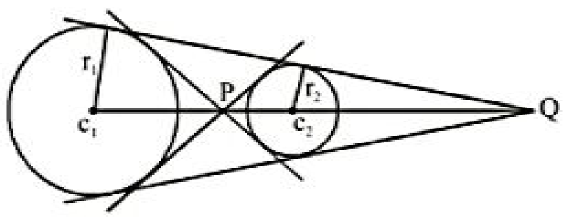


Illustration :

If mid points of the sides of a triangle are (1, 1), (2, 4) and (3, 5). Find the co-ordinates of vertices.

Sol. $h_1 + h_2 = 2, \quad h_1 + h_3 = 6, \quad h_3 + h_2 = 4$

$\Rightarrow \quad h_1 + h_2 + h_3 = 6$

$h_1 = 2, h_2 = 0, h_3 = 4$

$k_1 + k_2 = 2, \quad k_2 + k_3 = 8, \quad k_3 + k_1 = 10$

$k_1 + k_2 + k_3 = 10$

$k_3 = 8, k_2 = 0, k_1 = 2$

Points are A(2, 2), B(0, 0), C(4, 8).

Illustration :

Find the harmonic conjugate of point R(2, 4) with respect to the points P(2, 2) and Q(2, 5).

Sol. *Let R divides the PQ in ratio k : 1.*

$$\frac{5k+2}{k+1} = 4$$

$$5k + 2 = 4k + 4$$

$$k = 2$$

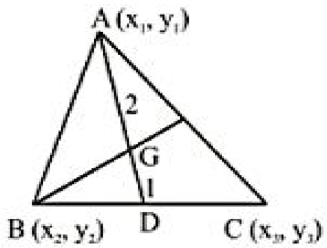
harmonic conjugate is $\left(\frac{2\times 2 - 1\times 2}{2-1}, \frac{2\times 5 - 1\times 2}{2-1}\right) = (2, 8)$

8. COORDINATES OF SPECIAL POINTS WITH RESPECT TO A TRIANGLE :

If co-ordinates of vertices of a triangle are given as $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ then the co-ordinates of its Centroid, Incentre, Excentre, Circumcentre and Orthocentre are as follows :

8.1 CENTROID :

As it is known that centroid is the point of corcurrency of medians.
From the figure D is the mid point of BC



co-ordinates of D are $\quad D \equiv \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$

We know that centroid G divides AD in the ratio 2 : 1

$\therefore$ co-ordinate of G are

x co-ordinate of G is $= \left(\frac{1(x_1) + 2\left(\frac{x_2 + x_3}{2}\right)}{1+2}\right) = \frac{x_1 + x_2 + x_3}{3}$

Similarly $\quad$ y-co-ordinate of G is $= \frac{y_1 + y_2 + y_3}{3}$

8.2 INCENTRE :

We know that point of concurrency of the internal bisector of the angles of a triangle is the incentre of the triangle.

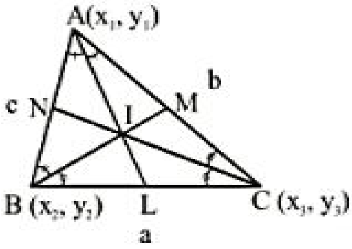
Angle bisector AL divides the base BC internally in the ratio c : b

$\therefore$ co-ordinates of L are

$$= \left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c} \right)$$

Now, I divides AL internally in the ratio b + c : a

$$\therefore \quad \text{x-co-ordinates} = \frac{ax_1 + (b+c)\left(\frac{bx_2 + cx_3}{b+c}\right)}{a+b+c}$$

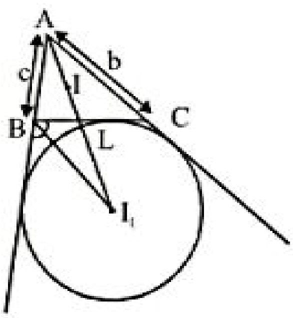


$$= \frac{ax_1 + bx_2 + cx_3}{a+b+c}$$

Hence, $$I \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$$

8.3 EX-CENTRES :

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC, and let a, b, c be the length of the sides BC, CA, AB, respectively. The circle which touches the side BC and the other two sides AB and AC produced is called the escribed angle B and C meet at a point I_1 which is the centre of the escribed circle opposite to the angle A.



$$\frac{BL}{LC} = \frac{c}{b}, \text{ also } \frac{AI_1}{I_1L} = -\frac{(b+c)}{a}$$

The coordinates of I_1 are given by

$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right)$$

Similarly, coordinates of I_2 and I_3 (centres of escribed circles opposite to the angles B and C, respectively) are given by

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c} \right)$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c} \right)$$

8.4 ORTHOCENTRE :

Important ratios which are useful to determine the coordinates of orthocentre are :

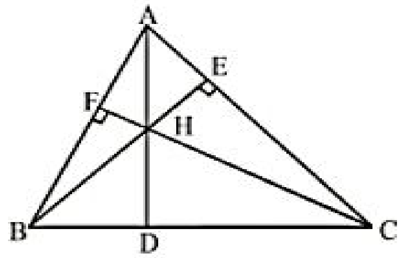
$BD : DC = c \cos B : b \cos C$

and $AH : HD = 2R \cos A : 2R \cos B \cos C$

where R is the circumradius of the triangle.

$\therefore$ Coordinates of orthocentre are

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$



8.5 CIRCUMCENTRE :

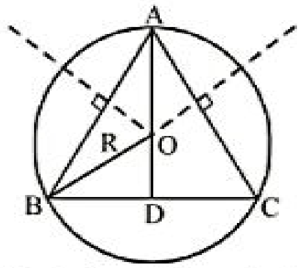
Important ratios which are useful to determine the coordinates of circumcentre are

$BD : DC = \sin 2C : \sin 2B$

and $AO : OD = \sin 2B + \sin 2C : \sin 2A$

$\therefore$ Coordinates of circumcentre are

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$



R = Radius of circumcircle

- Students are suggested to remeber the coordinates of circumcentre and orthocentre as it is, proof is not important.

Illustration :

If α, β and γ are the roots of equation $x^3 - 12x^2 + 44x - 48 = 0$. Find the centroid of the Δ whose co-ordinates are $A\left(\alpha, \frac{1}{\alpha}\right)$, $B\left(\beta, \frac{1}{\beta}\right)$ and $C\left(\gamma, \frac{1}{\gamma}\right)$.

Sol. $$\text{Centroid} = \left(\frac{\alpha+\beta+\gamma}{3}, \frac{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}{3}\right)$$

$$\frac{\alpha+\beta+\gamma}{3} = \frac{12}{3} = 4$$

$$\frac{1}{3}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right) = \frac{1}{3}\left(\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right) = \frac{1}{3}\left(\frac{44}{48}\right) = \frac{11}{36}$$

Centroid is $\left(4, \frac{11}{36}\right)$

Illustration :

Find the co-ordinates of circumcentre of the triangle whose vertices are (8, 6), (8, –2) and (2, –2).

Sol. Let $A(8, 6)$ and $B(8, -2)$ and $C(2, -2)$

$P(h, k)$ be the circumcentre

$$PA = PB = PC$$

$$\Rightarrow \quad PA^2 = PB^2$$

$$(h-8)^2 + (k-6)^2 = (h-8)^2 + (k+2)^2$$

$$16k = 32 \quad \Rightarrow \quad k = 2$$

$$PB^2 = PC^2$$

$$(h-8)^2 + (k+2)^2 = (h-2)^2 + (k+2)^2$$

$$12h = 60 \quad \Rightarrow \quad h = 5$$

Hence the co-ordinate of the circumcentre is (5, 2).

Illustration :

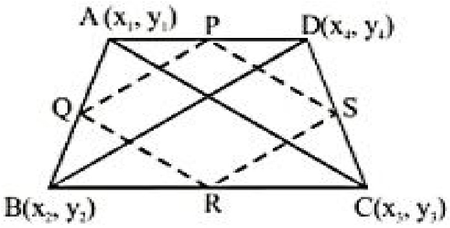
Prove the following results, analytically.

(i) Line joining the middle points of a quadrilateral forms a parallelogram.

(ii) Median to the hypotenuse in a right angled triangle is half as long as the hypotensue.

Sol. (i) P, Q, R & S are the mid point of sides AD, AB, CB & DC respectively.

$$\text{Mid point of } AD = P\left(\frac{x_1+x_4}{2}, \frac{y_1+y_4}{2}\right)$$



$$\text{Similarly } Q\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right),\ R\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right),$$

$$S\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}\right)$$

$$\text{Mid point of } \quad PR = \left(\frac{x_1+x_2+x_3+x_4}{2}, \frac{y_1+y_2+y_3+y_4}{2}\right)$$

$$\text{Mid point of } \quad QS = \left(\frac{x_1+x_2+x_3+x_4}{2}, \frac{y_1+y_2+y_3+y_4}{2}\right)$$

Since PR & QS bisects each other therefore PQRS is a parallelogram

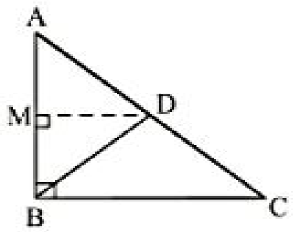
hence PQRS form a parallelogram

(ii) BD is a median to AC

where $\angle B = 90^o$

Let M is the mid point of AB

hence $DM \,||\, BC$



In ΔADM & ΔBDM

$AM = BM$

$DM = DM$ (common)

$\angle AMD = \angle BMD = 90^o$

$\therefore$ ΔADM & ΔBDM are congruent

hence $AD = BD = \frac{1}{2}AC$

Practice Problem

Single correct question

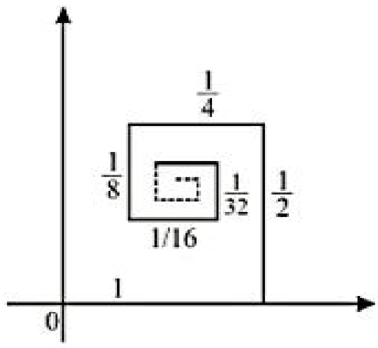
Q.1 The distance between the points $\left(t_1^3, \sqrt{2}\,t_1\right)$ and $\left(t_2^3, -\sqrt{2}\,t_2\right)$, if t_1 and t_2 are the roots of $x^2 - 2\sqrt{2}\,x + 1 = 0$ is

(A) 3 (B) $6\sqrt{6}$ (C) $5\sqrt{2}$ (D) 10

Q.2 Let ΔABC have vertices A (3, 4), B (4, 6) and C (–1, 6). The distance between orthocentre of ΔABC and the origin is :

(A) 0 (B) 3 (C) 8 (D) 5

Q.3 A particle begins at the origin and moves successively in the following manner as shown, 1 unit to the right, $\frac{1}{2}$ unit up, $\frac{1}{4}$ unit to the left, $\frac{1}{8}$ unit down, $\frac{1}{16}$ unit to the right, $\frac{1}{32}$ unit up etc. The length of each move is half the length of previous move and movement continues indefinitely. The co-ordinates of the point to which the movement converges is :



(A) $\left(\frac{4}{5}, \frac{2}{5}\right)$ (B) $\left(\frac{2}{5}, \frac{4}{5}\right)$ (C) $\left(\frac{4}{3}, \frac{2}{3}\right)$ (D) $\left(\frac{2}{3}, \frac{4}{3}\right)$

Q.4. Harmonic conjugate of the point C(5, 1) with respect to the point A(2, 10) and B(6, –2) is

(A) (8, –8) (B) (–1, 3) (C) (6, –4) (D) (2, 5)

Q.5 If in a triangle A ≡ (1, 10), circumcentre $\equiv \left(-\frac{1}{3}, \frac{2}{3}\right)$ and orthocentre $\equiv \left(\frac{11}{3}, \frac{4}{3}\right)$ then the co-ordinate of mid-point of side opposite to A is

(A) (1, 6) (B) (1, 5) (C) (1, –3) (D) $\left(1, -\frac{11}{3}\right)$

Multiple correct type question

Q.6 Coordinate of the vertices of ΔABC are (12, 8), (–2, 6) and (6, 0) then the correct statement (s) are :

(A) triangle is right but not isosceles

(B) triangle is right as well as isosceles

(C) triangle is obutse

(D) The product of abscissae of the centroid, orthocentre and circumcentre is 160

Paragraph type

Paragraph for question nos. 7 to 8

Consider the equation $x^3 - 3x^2 + 6x - 1 = 0$, let α, β, γ are the roots of the equation then

Q.7 The centroid of the triangle the coordinate of whose vertices are $\left(\alpha^2, \frac{1}{\alpha^2}\right), \left(\beta^2, \frac{1}{\beta^2}\right)$ and $\left(\gamma^2, \frac{1}{\gamma^2}\right)$ is

(A) (0, 3) (B) (–1, 10) (C) (2, 5) (D) (–3, 8)

Q.8 The ratio in which x-axis divides the join of centroid of ΔABC and the point (3, –4) is

(A) 1 : 3 externally (B) 2 : 5 externally

(C) 2 : 1 internally (D) 5 : 2 internally

Match the Column

Q.9

	Column-I		Column-II
(A)	The point (2, 0), (3, 3), (0, 2) and (–5, –5) taken in order are the vertices of	(P)	Rectangle
(B)	The points (2, –2), (8, 4), (5, 7) and (–1, 1) taken in order are the vertices of	(Q)	Trapezium
(C)	The point (–3, 4), (–1, 0), (1, 0) and (3, 4) taken in order are vertices of	(R)	Parallelogram
(D)	The points (3, –5), (–5, –4), (7, 10) and (15, 9) taken in order are the vertices of	(S)	Cyclic qudrilateral
		(T)	kite

Q.10 Prove the following result analytically

(i) Diagonals of an isosceles trapezium are equal

(ii) $l_1^2 + l_2^2 + l_3^2 = \frac{3}{4}(a^2 + b^2 + c^2)$ (where l_1, l_2, l_3 are the lengths of median of ΔABC)

(iii) Medians of the equal sides of an isosceles triangle are equal and converse.

Answer key

Q.1 B   Q.2 D   Q.3 A   Q.4. A   Q.5 D

Q.6 B,D   Q.7 B   Q.8 D

Q.9 (A) $\to$ T; (B) $\to$ P, R, S; (C) $\to$ Q, S; (D) $\to$ Q

9. DETERMINANT :

9.1 INTRODUCTION :

The quantity $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a second order determinant.

It consists of two rows and two columns, and it stands for the quantity $a_1b_2 - a_2b_1$. Thus

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order 3. Its value can be found as

$$D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad \text{or}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

In the way we can expand a determinant in 6 ways using elements of $R_1, R_2, R_3, C_1, C_2, C_3$.

9.2 MINORS AND COFACTORS OF A DETERMINANT :

Minors :

Minors (M_{ij}) of an element (a_{ij}) is defined as the value of the determinant obtained by deleting i^{th} row and j^{th} column in which that element lies. e.g. in the determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{minor of } a_{12} \text{ denoted as } M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad \text{and so on}$$

Cofactor :

It has no separate identity and is related to the minor as

$C_{ij} = (-1)^{i+j} M_{ij}$ where 'i' denotes the row and 'j' denotes the column.

Hence the value of a determinant of order three in terms of 'Minor' and 'Cofactor' can be written as

$$D = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} \text{ or}$$
$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

Illustration :

Expand $\begin{vmatrix} 2 & 3 & 0 \\ -2 & 1 & 2 \\ 6 & 5 & -1 \end{vmatrix}$.

Sol. $D = 2\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} - 3\begin{vmatrix} -2 & 2 \\ 6 & -1 \end{vmatrix} + 0\begin{vmatrix} -2 & 1 \\ 6 & 5 \end{vmatrix}$

$= 2(-1 - 10) - 3(2 - 12) + 0 = -22 + 30 = 8$

9.3 PROPERTIES OF DETERMINANTS :

P-1: The value of a determinant remains unaltered, if the rows & columns are inter changed. e.g. if

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$$

D & D' are transpose of each other. If $D' = -D$ then it is **Skew symmetric** determinant but $D' = D \Rightarrow 2D = 0 \Rightarrow D = 0 \Rightarrow$ Skew symmetric determinant of third order has the value zero.
The value of a skew symmetric determinant of odd order is zero.

P-2: If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ \& \ D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{Then } D' = -D.$$

P-3: If a determinant has any two rows (or columns) identical , then its value is zero.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

P-4: If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number.

$$\text{If } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \qquad D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{Then } D' = KD$$

P-5: If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants. We can say

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

P-6: The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

$$\text{e.g.} \quad \text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}. \text{ Then } D' = D.$$

Note that while applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

P-7: If by putting $x = a$ the value of a determinant vanishes then $(x - a)$ is a factor of the determinant.

Note : Factorisation in respect the following determinants are very useful and should be remembered.

(i) $$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

(ii) $$\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

(iii) $$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

(iv) $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) < 0$$ where a, b, c are different and positive

Proof : (i) $$D = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$D = 1\begin{vmatrix} y & y^2 \\ z & z^2 \end{vmatrix} - x\begin{vmatrix} 1 & y^2 \\ 1 & z^2 \end{vmatrix} + x^2\begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix}$$

$$= (yz^2 - y^2z) - x(z^2 - y^2) + x^2(z - y)$$
$$= yz(z - y) - x(z^2 - y^2) + x^2(z - y)$$
$$= (z - y)(yz - x(z + y) + x^2)$$
$$= (x - y)(y - z)(z - x)$$

Proof : (ii) $$D = \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$D = \begin{vmatrix} 0 & x-y & x^3-y^3 \\ 0 & y-z & y^3-z^3 \\ 1 & z & z^3 \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} 0 & 1 & x^2+xy+y^2 \\ 0 & 1 & y^2+yz+z^2 \\ 1 & z & z^3 \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} 1 & x^2+xy+y^2 \\ 1 & y^2+yz+z^2 \end{vmatrix}$$

$$= (x-y)\,(y-z)\,(y^2+yz+z^2-x^2-xy-y^2)$$

$$= (x-y)\,(y-z)\,(z-x)\,(x+y+z)$$

Proof : (iii) $D = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$$\begin{vmatrix} x-y & x^2-y^2 & z(y-z) \\ y-z & y^2-z^2 & x(z-y) \\ z & z^2 & xy \end{vmatrix} = (x-y)\,(y-z)\begin{vmatrix} 1 & x+y & -z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$= (x-y)\,(y-z)\begin{vmatrix} 0 & x-z & x-z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

$$= (x-y)\,(y-z)\,(x-z)\begin{vmatrix} 0 & 1 & 1 \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Applying $C_2 \to C_2 - C_3$

$$= (x-y)\,(y-z)\,(x-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & x+y+z & -x \\ z & z^2-xy & xy \end{vmatrix}$$

$$= (x-y)\,(y-z)\,(x-z)[z^2-xy-z\,(x+y+z)]$$

$$= (x-y)\,(y-z)\,(z-x)(xy+yz+zx)$$

Proof : (iv) $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & b+c+a & c+a+b \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix} = (a+b+c)\,[(b-c)(a-b)-(c-a)^2]$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = -(a^3+b^3+c^3-3abc)$$

Illustration :

If a, b,c are the roots of the equation $x^3 - 3x^2 + 2x - 1 = 0$ then find the value of $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$.

Sol. $D = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Taking a, b & c common from row 1, row 2 and row 3 respectively.

$$= abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc\left(\frac{abc+ab+bc+ca}{abc}\right)$$

$$= abc + ab + bc + ca = 1 + 2 = 3$$

Illustration :

If $a \neq p,\ b \neq q,\ c \neq r$ *and* $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, *then find the value of* $\frac{p}{p-r}+\frac{q}{q-b}+\frac{r}{r-c}$.

Sol. $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

Applying $R_1 \to R_1 - R_2$ *and* $R_2 \to R_2 - R_3$

$$\begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

$(p-a)\,[(q-b)r - b\,(c-r)] + a[(b-q)\,(c-r)] = 0$

$(p-a)\,(q-b)r - (p-a)\,(c-r)b + a(b-q)\,(c-r) = 0$

Dividing the equation by $(p-a)\,(q-b)\,(r-c)$

$$\frac{r}{r-c}+\frac{b}{q-b}+\frac{a}{p-a} = 0$$

$$\frac{r}{r-c}+\frac{b-q+q}{q-b}+\frac{a-p+p}{p-a} = 0$$

$$\frac{r}{r-c}+\frac{q}{q-b}+\frac{p}{p-a} = 2$$

Practice Problem

Single correct question

Q.1 The value of $\begin{vmatrix} 1 & 1+ac & 1+bc \\ 1 & 1+ad & 1+bd \\ 1 & 1+ae & 1+be \end{vmatrix}$ equals

(A) 0 (B) 1 (C) 3 (D) $a+b+c$

Q.2 If $\begin{vmatrix} x+y & y & z \\ y+z & x & y \\ z+x & z & x \end{vmatrix} = k(x+y+z)(x-z)^2$ then k equals

(A) $x^2y^2z^2$ (B) xyz (C) 1 (D) 0

Q.3 The value of $\begin{vmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \\ \sin\alpha & \cos\alpha & \sin\beta \\ -\cos\alpha & \sin\alpha & \cos\beta \end{vmatrix}$ is dependent on

(A) α (B) β (C) both α and β (D) Neither α nor β

Multiple correct type question

Q.4 Let α, β, γ are the roots of the equation $\begin{vmatrix} 1 & -1 & x \\ x & 0 & 2 \\ -3 & x & -5 \end{vmatrix} = 0$ then, which of the following are correct?

(A) $(\alpha+\beta)^{-1} + (\beta+\gamma)^{-1} + (\gamma+\alpha)^{-1} = \frac{-7}{6}$ (B) $\alpha^3+\beta^3+\gamma^3 = 3\alpha\beta\gamma$

(C) $\alpha^2+\beta^2+\gamma^2 = 8$ (D) $\Sigma\alpha - \Sigma\alpha\beta = 4$

Q.5 The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$ if

(A) a, b, c are in A.P. (B) a, b, c are in G.P.

(C) α is a roots of $ax^2+bx+c=0$ (D) $(x-a)$ is a factor of $ax^2+2bx+c=0$

Answer key

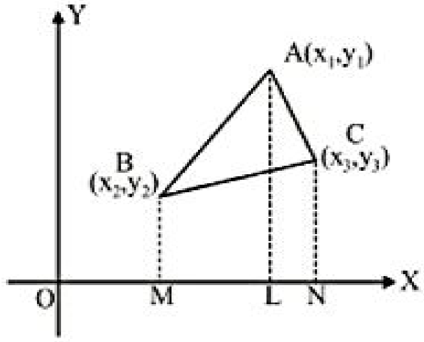
Q.1	A	Q.2	C	Q.3	D	Q.4	AB	Q.5	BD

10. AREA OF A TRIANGLE :

The area of a triangle, whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right| = \frac{1}{2}\left\| \begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix} \right\|$$

Proof:



Let ABC be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Draw AL, BM and CN as perpendicualr from A, B and C on the x-axis. Clearly, ABML, ALNC and BMNC all are trapziums.
We have
Area of ΔABC = Area of trapezium ABML + Area of trapedezium ALNC – Area of trapezium BMNC

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}(BM + AL)(ML) + \frac{1}{2}(AL + CN)(LN) - \frac{1}{2}(BM + CN)(MN)$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\left\| \begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix} \right\| \qquad \text{(Remember)}$$

Illustration :

If the area of the triangle formed by the points (1, 2), (2, 3) and (x, 4) is 40 sq. units then find x.

Sol. *Let A (1, 2), B (2, 3) & C(x, 4) be three points.*

$$Area\ (D) = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ x & 4 & 1 \end{vmatrix} = 40$$

$$= \left| 1(3-4) - 2(2-4) + x(2-3) \right| = 80$$

$$= \left| -1 + 4 - x \right| = 80$$

$$= \left| 3 - x \right| = 80$$

$3 - x = 80$ *or* $3 - x = -80$

$x = -77$ *or* $x = 83$

Illustration :

If $A(1, 1)$, $B(3, 4)$, $C(5, -2)$ and $D(4, -7)$ in order are the vertices of a quadrilateral. Find its area.

Sol. *To find the area of quadrilateral ABCD, we can calculate area of ΔABC & ΔADC*

$$Area\ of\ \Delta ABC = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \frac{18}{2}$$

$$Area\ of\ \Delta ADC = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 4 & -7 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \frac{23}{2}$$

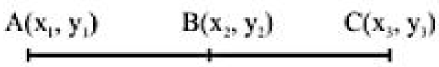
Hence Area of quadrilateral ABCD = $\Delta ABC + \Delta ADC$ $= \frac{18}{2} + \frac{23}{2} = \frac{41}{2}$

10.1 COLLINEARITY OF THREE POINTS :

Different conditions for three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ to be collinear are as follows

(i) $AB + BC = AC$, $AC - AB = BC$

(ii)



Slope of AB = Slope of BC

$$\Rightarrow \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

(iii) If the area of triangle ABC be zero then the three points will be collinear.

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Illustration :

Find the value of k for which points (2, 3), (3, 5) and (5, k) arecollinear.

Sol. *If the points are collinear then*

$$Area\ (\Delta) = 0$$

$$\frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 3 & 5 & 1 \\ 5 & k & 1 \end{vmatrix} = 0$$

$$|5(3 - k) - k\ (2 - 3) + 1\ (10 - 9)| = 0$$
$$|-10 + k + 1| = 0$$
$$|k - 9| = 0$$
$$k = 9$$

Illustration :

Show that points (b, c + a), (c, a + b) and (a, b + c) are always collinear where a, b, c $\in R$

Sol. *Area* $(\Delta) = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$$\alpha\Delta = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

Applying $c_1 \to c_1 + c_2$

$$= \begin{vmatrix} a+b+c & b+c & 1 \\ b+c+a & c+a & 1 \\ c+a+b & a+b & 1 \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = 0$$

Since $\Delta = 0$*, hence points are collinear.*

Practice Problem

Single correct question

Q.1 The area of quadrilateral whose vertices are A(1, 1), B(3, 4), C(5, –2) and D(4, –7) is :

(A) 12 sq. unit (B) $\frac{3\sqrt{3}}{2}$ sq. unit (C) 25 sq. unit (D) $\frac{41}{2}$ sq. unit

Q.2 If the points $(\sin\theta, \cos\theta)$, $(-2\sqrt{2}, 2)$ and $(-\sqrt{2}, 1)$ are collinear, then the number of values of $\theta \in [0, 2\pi]$ is :

(A) 0 (B) 1 (C) 2 (D) Infinite

Q.3 Let Δ_1 denotes the area of the triangle formed by the vertices $(a^3m_1^3, am_1)$, $(a^3m_2^3, am_2)$, $(a^3m_3^3, am_3)$ and Δ_2 denotes the area of the triangle formed by the vertices $(2am_1m_2, a^2(m_1^2+m_2^2))$, $(2am_2m_3, a^2(m_2^2+m_3^2))$ and $(2am_3m_1, a^2(m_3^2+m_1^2))$. Then $\frac{\Delta_1}{\Delta_2}$ $(a > 0)$ equals

(A) $\frac{a}{2}$ (B) 2a (C) $\frac{a^3}{8}$ (D) $8a^3$

Multiple correct type question

Q.4 If area of Hexagon whose vertices taken in order are (0, 0), (1, 1), (1, 3), (–1, 4), (–3, 2) and $\left(-2, \frac{1}{2}\right)$ can be written in the form $\frac{a}{b}$ (where $a, b \in N$ and a and b are in their lowest form) then

(A) a and b are co-prime (B) a + b is an even number
(C) a + b is an odd prime (D) $a^2 + b^2 = 1000$

Q.5 If the area of triangle formed by the points (5, 2), (0, 3) and (a, 4) is 8 square units. Then
(A) sum of all possible value(s) of 'a' is equal to 5
(B) sum of all possible value(s) of 'a' is equal to –10
(C) product of all possible value(s) of 'a' is equal to 125
(D) product of all possible value(s) of 'a' is equal to –210

Answer key

Q.1	D	Q.2	C	Q.3	A	Q.4	AC	Q.5	BD

11. LOCUS AND EQUATION TO A LOCUS :

11.1 LOCUS :

The curve described by a point which moves under given condition or conditions is called its locus. For example, suppose C is a point in the plane of the paper and P is a variable point in the plane of the paper such that its distance from C is always equal to a (say). It is clear that all the positions of the moving point P lie on the circumference of a circle whose centre is C and whose radius is a. The circumference of this circle is, therefore, the "Locus" of point P when it moves under the condition that its distance from the point C is always equal to constant a.

Let A and B be two fixed points in the plane of the paper, and P be a variable point in the plane of the paper which moves in such a way that its distance from A and B is always same. Thus, the "locus" of P is the perpendicular bisector of AB when it moves under the condition that its distance from A and B is always equal.

11.2 EQUATION TO LOCUS OF A POINT :

The equation to the locus of a point is the relation which is satisfied by the coordinates of every point on the locus of the point.

Note : Steps to find locus of a point.

Step I : Assume the coordinates of the point say (h, k) whose locus is to be determined.
Step II: Write the given condition in mathematical form involving h, k.
Step III : Eliminate the variable (s), if any.
Step IV : Replace h by x and k by y in the result obtained in step III.
The equation so obtained is the locus of the point which moves under some condition(s).

Illustration :

Find the locus of point P if P is equidistant from points

(i) $A(3, 4)$ & $B(5, -2)$

(ii) $A(a + b, a - b)$ & $B(a - b, a + b)$

Sol.

(i) If point $P(h, k)$ is equidistant from $A(3, 4)$ & $B(5, -2)$ then

$PA = PB$

$\Rightarrow \sqrt{(h-3)^2+(k-4)^2} = \sqrt{(h-5)^2+(k+2)^2}$

$\Rightarrow h^2 - 6h + 9 + k^2 - 8k + 16 = h^2 - 10h + 25 + k^2 + 4k + 4$

$\Rightarrow 4h - 12k = 4$

$\Rightarrow h - 3k = 1$

hence locus of P is $x - 3y = 1$

(ii) $PA = PB$

$\Rightarrow [h - (a + b)]^2 + [k - (a - b)]^2 = [h - (a - b)]^2 + [k - (a + b)]^2$

$\Rightarrow -2h(a + b) - 2k(a - b) = -2h(a - b) - 2k(a + b)$

$\Rightarrow 2h(2b) + 2k(2b) = 0$

$\Rightarrow h + k = 0$

hence locus of P is $x + y = 0$

Illustration :

Find the equation to the locus of a point which moves so that

(i) Its distance from the point $(a, 0)$ is always four times its distance from the axis of y.

(ii) Sum of the squares of its distances from the axes is equal to 3.

(iii) Its distance from x-axis is 3 times of its distance from y-axis.

Sol.

(i) Let the point be $P(h, k)$

Distance of P from axis of $y = |h|$

Distance of P from $(a, 0) = \sqrt{(h-a)^2+k^2}$

$\Rightarrow \sqrt{(h-a)^2+k^2} = 4|h|$

$\Rightarrow (h - a)^2 + k^2 = 16h^2$

$\Rightarrow h^2 - 2ah + a^2 + k^2 = 16h^2$

$\Rightarrow 15h^2 - k^2 + 2ah = a^2$

hence locus of P is

$15x^2 - y^2 + 2ax = a^2$

(ii) *Let the point be P(h, k)*

Distance of P from y-axis $= |h|$

Distance of P from x-axis $= |k|$

$$h^2 + k^2 = 3$$

hence Locus of P is

$$x^2 + y^2 = 3$$

(iii) *Let the point be P(h, k)*

Distance from x-axis $= |k|$

Distance from y-axis $= |h|$

$$|k| = 3|h|$$

$3h - k = 0$ *or* $3h + k = 0$

$3x - y = 0$ *or* $3x + y = 0$

Illustration :

A(0, 1) and B(0, – 1) are 2 points if a variable point P moves such that sum of its distance from A and B is 4. Then the locus of P is the equation of the form of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Find the value of* $(a^2 + b^2)$.

Sol. *Let the point P is (h, k)*

Given that PA + PB = 4 where A (0, 1) and B(0, –1)

$$\Rightarrow \quad \sqrt{(h-0)^2 + (k-1)^2} + \sqrt{(h-0)^2 + (k+1)^2} = 4$$

$$\Rightarrow \quad \sqrt{h^2 + k^2 - 2k + 1} = 4 - \sqrt{h^2 + k^2 + 2k + 1}$$

squaring both sides, we get

$$\Rightarrow \quad h^2 + (k-1)^2 = 16 + h^2 + (k+1)^2 - 8\sqrt{h^2 + (k+1)^2}$$

$$\Rightarrow \quad 8\sqrt{h^2 + (k+1)^2} = 16 + 4k$$

squaring again, we get

$$\Rightarrow \quad 4h^2 + 3k^2 = 12$$

$$\Rightarrow \quad \frac{h^2}{3} + \frac{k^2}{4} = 1$$

hence locus of P is $\frac{x^2}{3} + \frac{y^2}{4} = 1$

$$\Rightarrow \quad a^2 + b^2 = 3 + 4$$

$$= 7$$

Practice Problem

Single correct question

Q.1 The equation of locus of all points equidistant from the point $(-1, 2)$ and the origin is :

(A) $x - 3y - 7 = 0$ (B) $2x - 4y + 5 = 0$ (C) $4x + y - 3 = 0$ (D) $x + 2y + 1 = 0$

Q.2 Let A $(1, -3)$ and B$(-2, 5)$ be vertices of ΔABC. If the third vertex C of ΔABC move on the line $3x + y = 1$, then locus of centroid is :

(A) $x - y = 0$ (B) $x - y = 3$ (C) $x + y = 1$ (D) $3x + y = 0$

Q.3 If A $(\cos\alpha, \sin\alpha)$, B$(\sin\alpha, -\cos\alpha)$, C$(1, 2)$ are the vertices of a ΔABC, then as α varies, the locus of its centroid is

(A) $x^2 + y^2 - 2x - 4y + 1 = 0$ (B) $3(x^2 + y^2) - 2x - 4y + 1 = 0$

(C) $x^2 + y^2 - 2x - 4y + 3 = 0$ (D) None of these

Multiple correct type question

Q.4 A stick of length 10 units rests against the floor and a wall of a room . If the stick begins to slide on the floor then

(A) Locus of middle point is $x^2 + y^2 = 25$

(B) Locus of middle point is $x^2 + y^2 = 100$

(C) Locus of centroid of triangle formed by axes and stick is $x^2 + y^2 = 25/9$

(D) Locus of centroid of triangle formed by axes and stick is $x^2 + y^2 = 100/9$

Q.5 If the equation of locus of a point which moves so that its distance from the point $(ak, 0)$ is k $(k > 0, \neq 1)$ times the distance from the point $\left(\frac{a}{k}, 0\right)$ then

(A) Locus of the point depends on k (B) Locus of the point is independent of k

(C) Locus is $x^2 + y^2 = a^2$ (D) Locus is $(1 - k^2)x^2 + y^2 = k^2$

Answer key

Q.1	B	Q.2	D	Q.3	B	Q.4	AD	Q.5	BC

12. STRAIGHT LINE

12.1 DEFINITION :

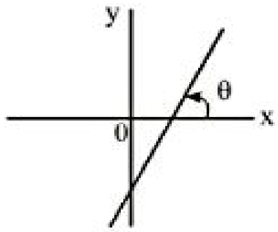
It is a locus of a point P (h, k) which moves in such a way that point P(h, k) is collinear with the two given points.

'or'

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

12.2 INCLINATION OF A LINE (θ) :

If a straight line intersects the x-axis, the inclination of the line is defined as the measure of the smallest non-negative angle which the line makes with the positive direction of the x-axis.

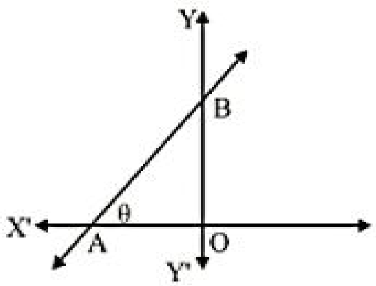


12.3 SLOPE (GRADIENT) OF A LINE :

If the inclination of a line (i.e. non vertical line) is θ and $\theta \neq \frac{\pi}{2}$, then the slope of a line is defined to be $\tan\theta$.

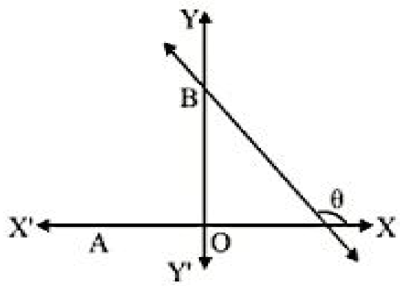
Imp.Point :-

(i) (a)



Slope = $\tan\theta$ = positive
$(0^\circ < \theta < 90^\circ)$

(b)



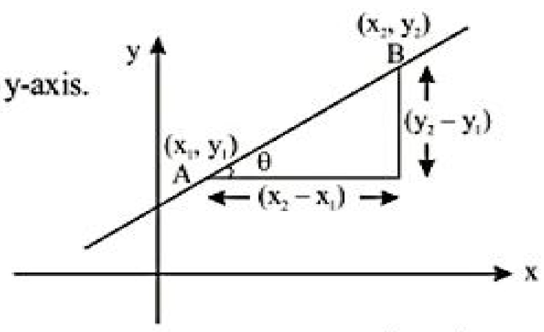
Slope = $\tan\theta$ = negative
$(90^\circ < \theta < 180^\circ)$

(ii) $0^\circ \leq \theta < 180^\circ$ $(\theta \neq 90^\circ)$

(iii) If $\theta = 0$ then line is parallel to x-axis
If $\theta = 90^\circ$ then line is perpendicular to x-axis or parallel to y-axis.

(iv) If $A(x_1, y_1)$ & $B(x_2, y_2)$, $x_1 \neq x_2$ are points on a straight line then the slope m of the line is given by

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$



slope = $\tan\theta = m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$

Example : Find the slopes if

(i) $\theta = \dfrac{\pi}{8}$ (ii) $\theta = \dfrac{3\pi}{8}$ (iii) $\theta = \dfrac{5\pi}{12}$

Ans. (i) $(\sqrt{2} - 1)$ (ii) $(\sqrt{2} + 1)$ (iii) $(2 + \sqrt{3})$

12.4 INTERCEPT :

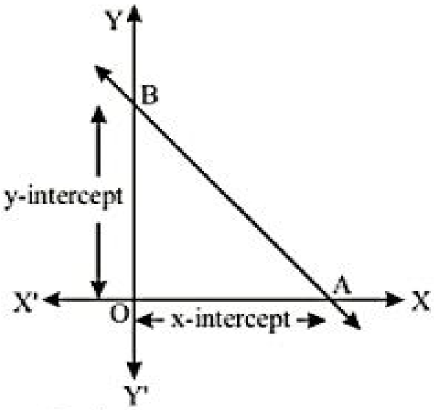
Definition :

The abscissa of the point where a line cuts the x-axis is called its x-intercept and ordinate of the point where it cuts the y-axis is called its y-intercept. If a line is parallel to x-axis its x-intercept is infinite, and if parallel to y-axis then y-intercept is not defined.

Intercepts of a line on the Axes :

If a straight line cuts x-axis at A and the y-axis at B then OA and OB are known as the intercepts of the line on x-axis and y-axis respectively.

The intercepts are positive or negative according as the line meets with positive or negative directions of the coordinate axes.



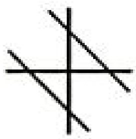
In figure OA = x-intercept, OB = y-intercept

OA is positive or negative according as A lies on OX or OX' respectively.

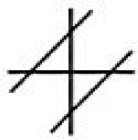
Similarly OB is positive or negative according as B lies on OY or OY' respectively.

Note:

(i) Line having equal interecept then $m = -1$.



(ii) Line having intercept equal in magnitude but opposite in sign then $m = +1$



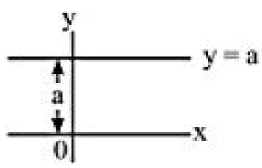
(iii) Line equally inclined with coordinate axes then $m = \pm 1$

(iv) Line cutting of equal non zero distance from origin then $m = \pm 1$.

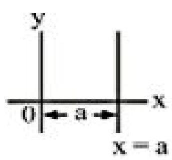
(v) $0.x + 0.y + c = 0$ $(c \neq 0)$ represents a straight line with x and y intercept both infinity

$\Rightarrow$ Straight line approaches at infinity.

(vi) Equations of lines parallel to x-axis are of form $y = a$.



(vii) Equations of lines parallel to y-axis are of form $x = a$.



13. DIFFERENT FORMS OF A LINE :

13.1 POINT-SLOPE FORM OF A LINE :

The equation of a line which passes through the point (x_1, y_1) and has the slope 'm' is

$$y - y_1 = m(x - x_1)$$

Proof :

Let $Q(x_1, y_1)$ be the point through which the line passes and let $P(x, y)$ be any point on the line. Then, the slope of the line is

$$\frac{y - y_1}{x - x_1}$$

But m is the slope of the line. Therefore,

$$m = \frac{y - y_1}{x - x_1} \quad \Rightarrow \quad y - y_1 = m(x - x_1)$$

Thus, $y - y_1 = m(x - x_1)$ is the required equation of the line.

Illustration :

A line passes through the point (1, 0). Find the equation of line if

(a) *It is inclined at an angle of* $\frac{3\pi}{8}$ *with positive x-axis.*
(b) *It passes through (2, 1).*
(c) *It passes through the point of intersection of lines* $y = x$ *and* $y = 2x + 1$.
(d) *It has equal non zero intercepts.*
(e) *It has intercepts equal in magnitude but opposite in sign.*
(f) *It cuts an intercept of 4 units on x-axis.*
(g) *It cuts an intercept of –3 units on y-axis.*
(h) *It cuts equal non zero distances on co-ordinate axes from origin.*
(i) *It is equally inclined with co-ordinate axes.*
(j) *It has an angle of* 30^o *with positive y-axis.*

Sol. *Since equation passes through (1, 0)*

(a) $\theta = \frac{3\pi}{8}$

$m = \tan\theta = \sqrt{2} + 1$

$y - 0 = m(x - 1)$

$y = (\sqrt{2} + 1)(x - 1)$

(b) *If it passes through (2, 1) & (1, 0)*

$$slope = \frac{1-0}{2-1} = 1$$

Equation of line $\quad y - 0 = 1(x - 1)$

$$y = x - 1$$

(c) *Point of intersection of* $y = x$ *and* $y = 2x + 1$ *is* $x = -1, y = -1$

Line passing through $(-1, -1)$ & $(1, 0)$

$$\text{slope} \quad (m) = \frac{0-(-1)}{1-(-1)} = \frac{1}{2}$$

$$\text{Equation of line } (y - 0) = \frac{1}{2}(x - 1)$$

$$2y = x - 1$$

(d) *If it has equal non zero intercepts*

then slope $(m) = -1$

Equation of line $(y - 0) = -1\ (x - 1)$

$x + y = 1$

(e) *If it has intercepts equal in magnitude but opposite in sign then,* $m = 1$

$y - 0 = 1\ (x - 1)$

$y = x - 1$

(f) *It cuts an intercept of 4 units on x-axis then it passes through* $(4, 0)$

Slope of line through $(1, 0)$ *and* $(4, 0)$ *is*

$$m = \frac{0-0}{4-1} = 0$$

Equation of line $\quad y - 0 = 0(x - 1)$

$y = 0$

(g) *If cuts an intercept of* -3 *on y-axis, then it passes through* $(0, -3)$ & $(1, 0)$

$$\text{slope} \quad (m) = \frac{0-(-3)}{1-0} = 3$$

Equation of line $\quad (y - 0) = 3(x - 1) \quad \Rightarrow \quad 3x - y = 3$

(h) *If it cuts equal non zero distances then slope* $(m) = \pm 1$

Equation of lines are

$(y - 0) = 1\ (x - 1) \quad$ *or* $\quad (y - 0) = -1\ (x - 1)$

$x - y = 1 \quad$ *or* $\quad x + y = 1$

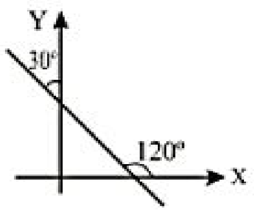
(i) *If it is equally inclined with co-roordinate axes then* $m = \pm 1$

Equation of lines are

$x - y = 1 \quad$ *or* $\quad x + y = 1$

(j) *If angle with y-axis is* 30^o *then angle with positive x-axis* $= 120^o$

slope $(m) = \tan 120^o = -\sqrt{3}$



Equation of line is

$y - 0 = -\sqrt{3}\ (x - 1)$

$\sqrt{3}\,x + y = \sqrt{3}$

13.2 TWO-POINT FORM OF A LINE :

The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Proof :

Let m be the slope of the line passing through (x_1, y_1) and (x_2, y_2) then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

So, the equation of the line is

$$y - y_1 = m(x - x_1) \text{ (Using point – slope form)}$$

Substituting the value of m, we obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This is the required equation of the line in two-point form.

Illustration :

Find the equation of a line passing through (2, 3) and (4, 5).

Sol. *Equation of line is given by*

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 3 = \left(\frac{5 - 3}{4 - 2}\right)(x - 2)$$

$$y = x + 1$$

Illustration :

Find number of straight lines passing through (2, 4) & forming a triangle of 16 sq. cm with the co-ordinate axes.

Sol. *Let the co-ordinates on x-axis be (h, 0)*

Equation of line becomes

$$y - 0 = \frac{4}{2 - h}(x - h)$$

$$y\text{-intercept} = \frac{4h}{h-2}$$

$$\text{area of triangle} = \frac{1}{2}\left|h \times \frac{4h}{h-2}\right| = 16$$

$$\left|\frac{h^2}{h-2}\right| = 8$$

$h^2 = 8h - 16$ or $h^2 = -8h + 16$

$h^2 - 8h + 16 = 0$ or $h^2 + 8h - 16 = 0$

$(h-4)^2 = 0$

Three values of h are possible hence three equations are possible.

13.3 SLOPE INTERCEPT FORM OF A LINE :

The equation of a line with slope m that makes an intercept c on y-axis is

$$y = mx + c$$

Proof :

Since y intercept = 'c'

Hence it passes through (0, c).

Equation of line with slope m and passing through (0, c) is given by

$$y - c = m(x - 0)$$

$$y = mx + c$$

Illustration :

Find the equation to the straight line cutting off an intercept 3 from the negative direction of the axis of y and inclined at 120° to the positive direction of x-axis.

Sol. *Slope of line (m)* $= \tan 120° = -\sqrt{3}$

y intercept (c) $= -3$

Equation of line is

$$y = mx + c$$

$$y = -\sqrt{3}\,x + (-3)$$

$$y + \sqrt{3}\,x + 3 = 0$$

13.4 INTERCEPT FORM OF A LINE :

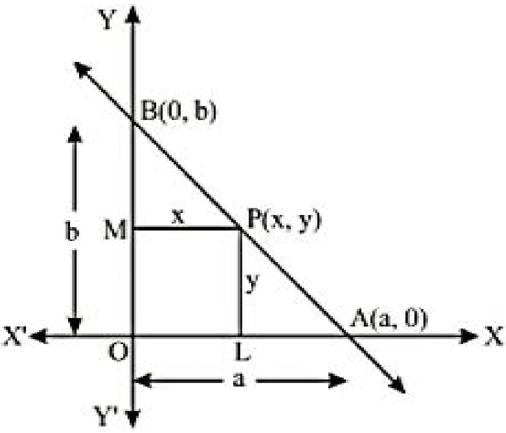
The equation of a line which cut-off intercepts a and b, respectively from the x and y-axes is

$$\frac{x}{a}+\frac{y}{b}=1$$

Proof :

Line cut-off intercepts a and b from the x and y-axes respectively,

Equation of line passes through the points (a, 0) and (0, b) is



$$y-0=\frac{-b}{a}(x-a)$$

$\Rightarrow \quad bx + ay = ab$

$\Rightarrow \quad \frac{x}{a}+\frac{y}{b}=1$

This is the equation of the line in the intercept form.

Illustration :

Find the equation to the straight line passing through the point (3, –4) and cutting off intercepts, equal but of opposite signs from the two axes.

Sol. *Let the intercepts cut off from the two axes are a & –a, then equation of straight line is given by*

$$\frac{x}{a}+\frac{y}{-a}=1$$

$$x-y=a$$

Since it passes through (3, –4)

hence $\quad 3-(-4)=a$

$$a=7$$

Required equation is

$$x-y=7$$

13.5 NORMAL FORM OR PERPENDICULAR FORM OF A LINE :

The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with positive direction of x-axis is

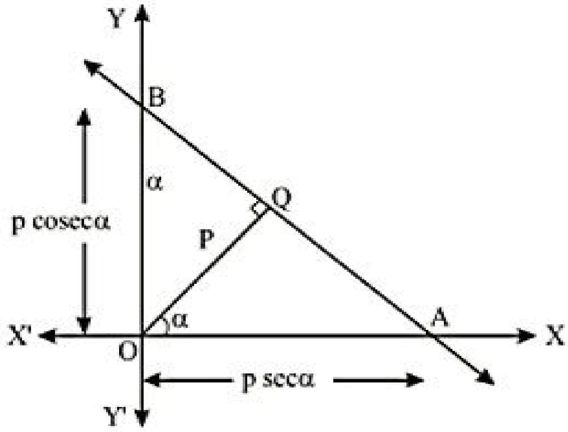
$$x \cos \alpha + y \sin \alpha = p$$

Proof :

Let the line AB be such that the length of the perpendicular OQ from the origin O to the line be p and $\angle XOQ = \alpha$.

From the diagram, using the intercept form, we get

Equation of line AB is



$$\frac{x}{p\sec\alpha} + \frac{y}{p\operatorname{cosec}\alpha} = 1$$

or $\quad x \cos \alpha + y \sin \alpha = p$

(i) $\quad 0 < \alpha < \frac{\pi}{2}$ $\qquad$ (ii) $\frac{\pi}{2} < \alpha < \pi$ $\qquad$ (iii) $\pi < \alpha < 3\pi$ $\qquad$ (iv) $\frac{3\pi}{2} < \alpha < 2\pi$

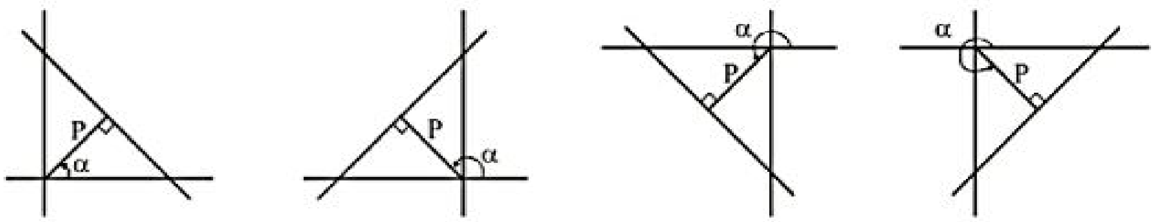


Illustration :

Find equation of a straight line on which length of perpendicular from the origin is four units and the line makes an anlge of 120° with the positive direction of x-axis.

Sol. $\alpha = 30^o$

$P = 4$

Equation line is given by

$$x \cos \alpha + y \sin \alpha = P$$

$$x \cos 30^o + y \sin 30^o = 4$$

$$x\cdot\frac{\sqrt{3}}{2} + \frac{y}{2} = 4$$

$$\sqrt{3}x + y = 8$$

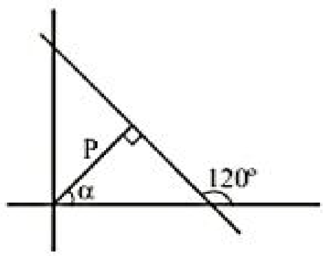


Illustration :

Match the Column

	Column-I		Column-II
(A)	Equation of line which cuts off an intercepts of 4 units on the x-axis & pass through $(2, -3)$	(P)	$5x - 3y = 0$
(B)	Equation of line which cuts off equal non-zero intercepts on co-ordinate axes and pass through $(2, 5)$	(Q)	$x + y = 1$
(C)	Equation of line passing through $(0, 0)$ & $(3, 5)$	(R)	$3x - 2y = 12$
(D)	Equation of line making an angle $135°$ with positive x-axis and pass through $(1, 0)$	(S)	$x + y = 7$
(E)	Equation of line passing through $(1, 0)$ and equally inclined with co-ordinate axes	(T)	$x - y = 1$

Sol. (A) *Line cutting off intercept 4 on the x-axis, then line passes through $(4, 0)$*
Equation of line passing through $(4, 0)$ & $(2, -3)$

$$y + 3 = \frac{3}{2}(x - 2)$$
$$2y + 6 = 3x - 6$$
$$3x - 2y = 12$$

(B) *Line cutting off equal intercepts = a*
Let the line be

$$\frac{x}{a} + \frac{y}{a} = 1$$
$$x + y = a$$

it passes through $(2, 5)$
$2 + 5 = a \quad \Rightarrow \quad a = 7$
Equation of line $\quad x + y = 7$

(C) *Equation of line passing through $(0, 0)$ & $(3, 5)$*

$$y - 0 = \frac{5}{3}(x - 0)$$
$$3y = 5x$$
$$5x - 3y = 0$$

(D) *Slope of line = $\tan 135° = -1$*
Equation of line through $(1, 0)$
$$y - 0 = -1\ (x - 1)$$
$$y + x = 1$$

(E) *If line is equally inclined then slope $(m) = \pm 1$*
Equation of lines are $y - 0 = \pm 1\ (x - 1)$
$$x + y = 1 \quad \text{or} \quad x - y = 1$$

Practice Problem

Single correct question

Q.1 The gradient of the line joining the points on the curve $y = x^2 - 2x + 3$ whose abscissa are -1 and 2 is

(A) 1 (B) -3 (C) -1 (D) $\frac{1}{2}$

Q.2 If the equation $x \cos\theta + y \sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$ then values of θ and p are

(A) $\frac{7\pi}{6}, 1$ (B) $\frac{5\pi}{3}, \frac{3}{2}$ (C) $\frac{2\pi}{3}, 1$ (D) $\frac{11\pi}{6}, 4$

Q.3 A variable straight line passes through a fixed point (a, b) intersecting the co-ordinates axes at A and B. If 'O' is the origin then the locus of the centroid of the triangle OAB is

(A) $bx + ay - 3xy = 0$ (B) $bx + ay - 2xy = 0$
(C) $ax + by - 3xy = 0$ (D) none

Q.4 The graph of function, $y = \cos x \cos(x + 2) - \cos^2(x + 1)$ is

(A) a straight line passing through $(0, -\sin^2 1)$ with slope 2
(B) a straight line passing through (0, 0)
(C) a parabola with vertex $(1, -\sin^2 1)$
(D) a straight line passing through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$ and parallel to the x-axis.

Multiple correct type question

Q.5 The equations of lines which cut off intercepts on the axes whose sum and product are 1 and -6 respectively are

(A) $5x + 3y + 2 = 0$ (B) $2x - 3y - 6 = 0$
(C) $3x - 4y + 1 = 0$ (D) $3x - 2y + 6 = 0$

Q.6 The equation to the straight lines each of which passes through the point (3, 2) and intersects the x-axis and y-axis in A, B respectively such that OA – OB = 2 are

(A) $x - y = 1$ (B) $8x - 5y - 2 = 0$
(C) $3x - y - 5 = 0$ (D) $2x + 3y = 12$

Answer key

Q.1 C	Q.2 A	Q.3 A	Q.4 D	Q.5 BD	Q.6 AD

14. ANGLE BETWEEN TWO STRAIGHT LINES WHEN THEIR EQUATIONS ARE GIVEN :

Let the equation of lines are

$$y_1 = m_1x + c_1 \quad \text{and} \quad y_2 = m_2x + c_2$$

then angle ϕ between lines L_1 & L_2 is given by

$$\boxed{\tan\phi = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|}$$

Proof :

By figure

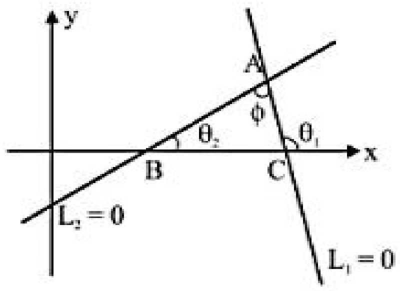
$$\text{slope of line } L_1 = \tan\theta_1 = m_1$$
$$\text{slope of line } L_2 = \tan\theta_2 = m_2$$

In triangle ABC

$$\theta_1 = \phi + \theta_2$$

or $\quad \phi = \theta_1 - \theta_2$

or $\quad \tan\phi = \dfrac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \cdot \tan\theta_1}$



$$\tan\phi = \frac{m_1 - m_2}{1 + m_1m_2}$$

& $\quad$ other angle of line $L_2 = 180^\circ - \phi$

$\therefore \quad \tan(180^\circ - \phi) = -\tan\phi = \dfrac{m_1 - m_2}{1 + m_1m_2}$

$\therefore \quad \tan\phi = \left|\dfrac{m_1 - m_2}{1 + m_1m_2}\right|$

Note :

If angle of any one line is 90° then slope of line is not define so to find angle between them.

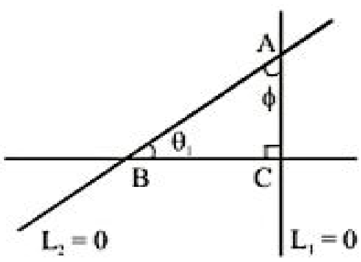
By figure, in triangle ABC

$$\theta_1 + \phi = 90^\circ$$

or $\quad \phi = 90^\circ - \theta_1$

or $\quad \tan\phi = \cot\theta_1$

or $\quad \tan\phi = \dfrac{1}{\tan\theta_1} = \dfrac{1}{m_1}$



and other angle of line is $180^\circ - \phi$ or is equal to $-\dfrac{1}{m_1}$.

Illustration :

Find the equation of a line passing through (1, 2) and making an angle of 45° with the line $2x + 3y = 10$.

Sol. Slope of $2x + 3y = 10$ is $\frac{-2}{3}$

Let the slope of the required line is m, then

$$\tan 45^o = \left|\frac{m - \left(-\frac{2}{3}\right)}{1 + m\left(-\frac{2}{3}\right)}\right|$$

$$1 = \left|\frac{3m+2}{3-2m}\right|$$

$3m + 2 = \pm(3 - 2m)$

$\Rightarrow$ $3m + 2 = 3 - 2m$ or $3m + 2 = -(3 - 2m)$

$m = \frac{1}{5}$ or $m = -5$

Equation of newly formed lines

$y - 2 = -5(x - 1)$ or $y - 2 = \frac{1}{5}(x - 1)$

$y + 5x = 7$ or $5y - x = 9$

Illustration :

Find the tangent of angle between pair of straight lines $x - y + 5 = 0$ and $x + 2y = 0$.

Sol. Slope of $x - y + 5 = 0$ is 1

Slope of $x + 2y = 0$ is $\frac{-1}{2}$

$m_1 = 1$, $m_2 = \frac{-1}{2}$

Angle between lines

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{1+\frac{1}{2}}{1+\left(\frac{-1}{2}\right)}\right| = 3$$

$\Rightarrow$ $\tan\theta = 3$

14.1 CONDITION FOR THE LINES TO BE PARALLEL :

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then

$$m_1 = m_2$$

$$\Rightarrow \quad -\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

$$\Rightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

14.2 CONDITION FOR THE LINES TO BE PERPENDICULAR :

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular, then

$$m_1\, m_2 = -1 \quad \Rightarrow \quad \left(-\frac{a_1}{b_1}\right)\times\left(-\frac{a_2}{b_2}\right) = -1$$

$$\Rightarrow \quad a_1a_2 + b_1b_2 = 0$$

If follows from the above discussion that the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(ii) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(iii) Interesecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(iv) Perpendicular, if $a_1a_2 + b_1b_2 = 0$

14.3 EQUATION OF A LINE PARALLEL TO A GIVEN LINE :

The equation of a line parallel to a given line $ax + by + c = 0$ is

$$ax + by + \lambda = 0$$

where λ is a constant and value of λ can be determined using another given condition.

Illustration :

Find the equation of straight line parallel to $3x + 2y + 4 = 0$ and passing through $(1, 1)$.

Sol. *Let the line | | to $3x + 2y + 4 = 0$ be*

$$3x + 2y + c = 0$$

Since it passes through $(1, 1)$

$$3(1) + 2(1) + c = 0$$

$$c = -5$$

Equation of line $3x + 2y = 5$

14.4 EQUATION OF A LINE PEREPENDICULAR TO A GIVEN LINE :

The equation of a line perpendicular to a given line $ax + by + c = 0$ is

$$bx - ay + \lambda = 0$$

where λ is a constant and value of λ can be determined using another given condition.

Illustration :

Find the equation of line perpendicular to $2x - y = 7$ and passing through point of intersections of line $3x + 4y = 8$ and y-axis.

Sol. *Point of intersection of $3x + 4y = 8$ & $x = 0$ is*

$$x = 0,\ y = 2$$

Let the equation of line $\perp$ to $2x - y = 7$ be

$$x + 2y = c$$

Passes through (0, 2)

$\Rightarrow \quad 0 + 4 = c$

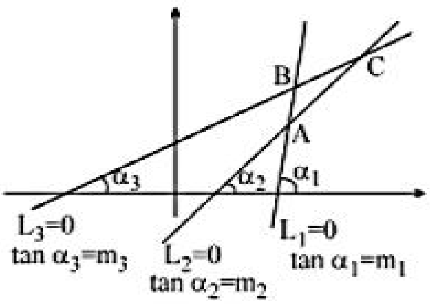
$\Rightarrow \quad c = 4$

hence equation is $x + 2y = 4$

14.5 THE TANGENTS OF THE INTERIOR ANGLES OF A TRIANGLE FORMED BY 3 GIVEN LINES :

Arrange the lines L_1, L_2 and L_3 in their descending order of slopes (as $m_1 > m_2 > m_3$) then tangents of interior angles of ΔABC can be written directly as

$$\tan A = \frac{m_1 - m_2}{1 + m_1 m_2}, \qquad \tan B = \frac{m_2 - m_3}{1 + m_2 m_3}, \tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$$



Explanation : $\quad A = \alpha_1 - \alpha_2$ (from the figure)

$$\tan A = \tan(\alpha_1 - \alpha_2) = \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$B = \alpha_2 - \alpha_3$$

$$\tan B = \tan(\alpha_2 - \alpha_3) = \frac{\tan\alpha_2 - \tan\alpha_3}{1 + \tan\alpha_2 \tan\alpha_3} = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$\pi - C = \alpha_1 - \alpha_3 \qquad \therefore\ C = \pi + \alpha_3 - \alpha_1$$

$$\tan C = \tan(\alpha_3 - \alpha_1) = \frac{\tan\alpha_3 - \tan\alpha_1}{1 + \tan\alpha_3 \tan\alpha_1} = \frac{m_3 - m_1}{1 + m_3 m_1}.$$

Illustration :

If a ΔABC is formed by the lines $2x + y - 3 = 0$; $x - y + 5 = 0$ and $3x - y + 1 = 0$, then obtain a cubic equation whose roots are the tangent of the interior angles of the triangle.

Sol. *Given lines are*

$$2x + y - 3 = 0 \qquad ...(i)$$
$$x - y + 5 = 0 \qquad ...(ii)$$
$$3x - y + 1 = 0 \qquad ...(iii)$$

Slope of line (i) = –2

Slope of line (ii) = 1

Slope of line (iii) = 3

Arranging the lines in descending order of their slopes

$\Rightarrow \quad m_1 = 3, \qquad m_2 = 1, \qquad m_3 = -2$

$$\Rightarrow \quad \tan A = \frac{m_1 - m_2}{1 + m_1 + m_2} = \frac{3-1}{1+(3)(1)} = \frac{1}{2}$$

$$\tan B = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{1-(-2)}{1+1(-2)} = -3$$

$$\tan C = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{(-2)-(3)}{1+(-2)3} = 1$$

Roots of cubic are $-3, \frac{1}{2}$ *and* 1.

$\Rightarrow \quad$ *Equation of cubic is* $(x + 3)\left(x - \frac{1}{2}\right)(x - 1) = 0$

$\Rightarrow \quad 2x^3 + 3x^2 - 8x + 3 = 0$

15. LENGTH OF THE PERPENDICULAR :

The length of the perpendicular from a point (x_1, y_1) to a line $ax + by + c = 0$ is

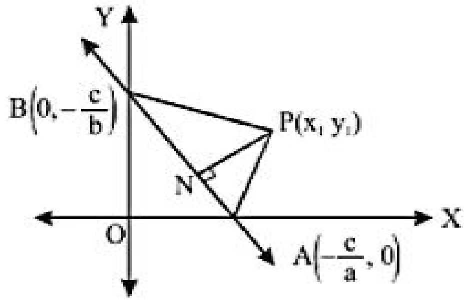
$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

Proof : The line $ax + by + c = 0$ meets x-axis at $A\left(\frac{-c}{a}, 0\right)$ and y-axis at $B\left(0, -\frac{c}{b}\right)$.

Let $P(x_1, y_1)$ be the point. Draw $PN \perp AB$.
Now, area of ΔPAB

$$= \frac{1}{2}\left|x_1\left(0+\frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0(y_1 - 0)\right|$$

$$= \frac{1}{2}\left|\frac{cx_1}{b} + \frac{cy_1}{a} + \frac{c^2}{ab}\right| = \left|(ax_1 + by_1 + c)\frac{c}{2ab}\right| \qquad \ldots(i)$$



Also, area of ΔPAB

$$= \frac{1}{2}\ AB \times PN$$

$$= \frac{1}{2}\sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} \times PN$$

$$= \frac{c}{2ab}\sqrt{a^2 + b^2} \times PN \qquad \ldots(ii)$$

From equation (i) and (ii), we get

$$\left|(ax_1 + by_1 + c)\frac{c}{2ab}\right| = \frac{c}{2ab}\sqrt{a^2 + b^2} \times PN$$

$$\Rightarrow \qquad PN = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Illustration :

Find the point on y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3

Sol. *Let the point on y-axis be $P(0, k)$*
Distance of $P(0, k)$ from $4x - 3y - 12 = 0$ is

$$\left|\frac{4(0) - 3(k) - 12}{\sqrt{4^2 + (-3)^2}}\right| = 3$$

$|3k + 12| = 15$

$|k + 4| = 5$

$k = 1, -9$

hence the points are $(0, 1)$ & $(0, -9)$

Illustration :

Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form three sides of two squares find the equations to the fourth sides of squares.

Sol. *Distance between the lines $x + 2y + 3 = 0$ & $x + 2y - 7 = 0$ is*

$$d = \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$$

$$= \left|\frac{3-(-7)}{\sqrt{1+4}}\right| = \frac{10}{\sqrt{5}}$$

The fourth side is parallel to $2x - y - 4 = 0$
Let the fourth side be $2x - y + k = 0$

Distance between two sides $2x - y - 4 = 0$ and $2x - y + k = 0$ should be $\frac{10}{\sqrt{5}}$

$$\left|\frac{k+4}{\sqrt{4+1}}\right| = \frac{10}{\sqrt{5}}$$

$$|k + 4| = 10$$

$k = 6$ *or* $k = -14$

hence the 4th sides of squares are $2x - y + 6 = 0$ *or* $2x - y - 14 = 0$

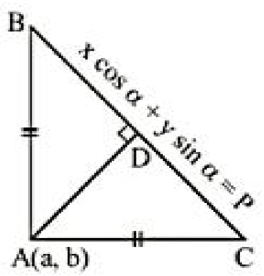
Illustration :

Two mutually perpendicular lines are drawn through the point (a, b) and enclose an isosceles triangle together with the line $x \cos \alpha + y \sin \alpha = P$. Find the area of triangle.

Sol. *ΔABC is right angled at A.*

$AB = AC$

AD perpendicular BC



Length of perpendicular from A (a, b) $= \left|\frac{a\cos\alpha + b\sin\alpha - P}{\sqrt{\cos^2\alpha + \sin^2\alpha}}\right|$

$= |a \cos \alpha + b \sin \alpha - P|$

Since ΔABC is isosceles $\therefore$ $AD = BD = DC$

$BC = 2\,(AD)$

$= 2\,|(a \cos \alpha + b \sin \alpha - P)|$

Area of $\Delta ABC = \frac{1}{2} \times |(a \cos \alpha + b \sin \alpha - P)| \cdot 2|(a \cos \alpha + b \sin \alpha - P)|$

$= (a \cos \alpha + b \sin \alpha - P)^2$

Illustration :

Find the condition so that lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + c_2 = 0$ and $a_2x + b_2y + c_1 = 0$, form a rhombus.

Sol. *If the given lines form a rhombus then perpendicular distances between opposite sides are equal*

Distance between $a_1x + b_1y + c_1 = 0$ & $a_1x + b_1y + c_2 = 0$

$$D_1 = \left|\frac{c_1 - c_2}{\sqrt{a_1^2 + b_1^2}}\right|$$

Distance between $a_2x + b_2y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

$$D_2 = \left|\frac{c_1 - c_2}{\sqrt{a_2^2 + b_2^2}}\right|$$

$D_1 = D_2$

$$\left|\frac{c_1 - c_2}{\sqrt{a_1^2 + b_1^2}}\right| = \left|\frac{c_1 - c_2}{\sqrt{a_2^2 + b_2^2}}\right|$$

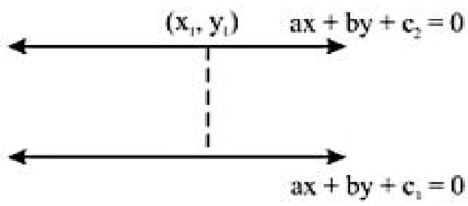
$$\Rightarrow \quad a_1^2 + b_1^2 = a_2^2 + b_2^2$$

15.1 DISTANCE BETWEEN TWO PARALLEL LINES :

Let (x_1, y_1) be any point on the line $ax + by + c_2 = 0$

Distance of point (x_1, y_1) from the line $ax + by + c_1 = 0$ is

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$



Now point (x_1, y_1) lies on $ax + by + c_2 = 0$ then

$$ax_1 + by_1 + c_2 = 0$$

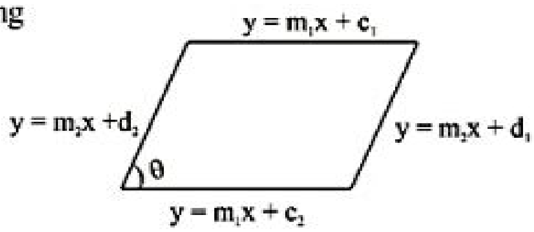
$$\Rightarrow \quad ax_1 + by_1 = -c_2$$

$$\Rightarrow \quad p = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

15.2 AREA OF THE PARALLELOGRAM :

Area of the | |gm whose 4 sides are as shown in the fig. using $A = p_1 p_2 \operatorname{cosec}\theta$ is given by

$$\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right| = \left(\frac{p_1 p_2}{\sin\theta}\right)$$



Practice Problem

Single correct question

Q.1 The product of the perpendiculars drawn from the two points $\left(\pm\sqrt{a^2-b^2},0\right)$ upon the straight line $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ is

(A) a^2 (B) b^2 (C) a^2+b^2 (D) 2ab

Q.2 Le the co-ordinates of the two points A and B be (1, 2) and (7, 5) respectively. The line AB is rotated through 45° in anti clockwise direction about the point of trisection of AB which is nearer to B. The equation of the line in new position is :

(A) $2x-y-6=0$ (B) $x-y-1=0$ (C) $3x-y-11=0$ (D) None of these

Q.3 The line L_1 given by $\frac{x}{5}+\frac{y}{b}=1$ passes through the point M (13, 32). The line L_2 is parallel to L_1 and has the equation $\frac{x}{c}+\frac{y}{3}=1$. Then the distance between L_1 and L_2 is

(A) $\sqrt{17}$ (B) $\frac{17}{\sqrt{15}}$ (C) $\frac{23}{\sqrt{15}}$ (D) $\frac{23}{\sqrt{15}}$

Multiple correct type question

Q.4 The equations of the straight lines which pass through the origin and are inclined at 75° to the straight line $x+y+\sqrt{3}(y-x)=a$ are

(A) $x=0$ (B) $y=0$ (C) $\sqrt{3}x+y=0$ (D) $x+\sqrt{3}y=0$

Q.5 The equation(s) of the straight lines drawn through the point (0, 1) on which the perpendiculars let fall from the point (2, 2) are each of length unity, is(are)

(A) $x-y=1$ (B) $x-1=0$ (C) $y-1=0$ (D) $4x-3y+3=0$

Answer key

Q.1	B	Q.2	C	Q.3	C	Q.4	AC	Q.5	CD

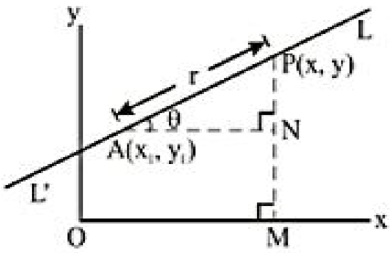
16. PARAMETRIC FORM OF A LINE :

The equation of straight line passing through a given point A (x_1, y_1) and making an angle θ from positive direction of x-axis in anticlockwise sence is –

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=\pm r$$

where, r is the distance of any point on the line from the given point $A(x_1, y_1)$.

Explanation :

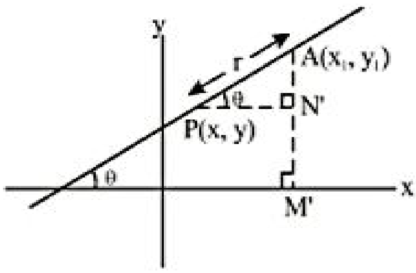


Let P(x, y) be taken on the line above the given point (x_1, y_1) then from the ΔPAN.

$$x-x_1=r\cos\theta$$
$$y-y_1=r\sin\theta$$

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \qquad \text{... (1)}$$

Again, if point is taken on the line below the given point $A(x_1, y_1)$ then from the ΔAPN'



$$x_1-x=r\cos\theta$$
$$y_1-y=r\sin\theta$$

$$\therefore \qquad \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=-r \qquad \text{... (2)}$$

Combining (1) & (2)

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=\pm r$$

Here, $x=x_1\pm r\cos\theta$, $y=y_1\pm r\sin\theta$ are the co-ordinates of the points situated on the line at a distance r from the given point $A(x_1, y_1)$.

Illustration :

A straight line is drawn through the point P(2, 3) and is inclined at an angle of 30° with the x-axis. Find the coordinates of two points on it at a distance 4 from P.

Sol. *Here $(x_1, y_1) = (2, 3)$, $\theta = 30^\circ$, the equation of the line is*

$$\frac{x-2}{\cos 30^\circ}=\frac{y-3}{\sin 30^\circ}$$

$$\Rightarrow \quad \frac{x-2}{\frac{\sqrt{3}}{2}}=\frac{y-3}{\frac{1}{2}}$$

$$\Rightarrow \quad x-2=\sqrt{3}\,(y-3)$$

$$\Rightarrow \quad x-\sqrt{3}\,y=2-3\sqrt{3}$$

Points on the line at a distance 4 from P(2, 3) are

$$(x_1\pm r\cos\theta,\ y_1\pm r\sin\theta)$$

$$\Rightarrow \quad (2\pm 4\cos 30^\circ,\ 3\pm 4\sin 30^\circ)$$

$$\Rightarrow \quad (2\pm 2\sqrt{3},\ 3\pm 2) \Rightarrow (2+2\sqrt{3},\ 5) \text{ or } (2-2\sqrt{3},\ 1)$$

Illustration :

Find the equation of the line passing through the point A(2, 3) and making an angle of 45° with the x-axis. Also determine the length of intercept on it between A and the line x + y + 1 = 0.

Sol. The equation of a line passing through A and making an angle of 45° with the x-axis is

$$\frac{x-2}{\cos 45^o} = \frac{y-3}{\sin 45^o}$$

$$\frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \qquad x - y + 1 = 0$$

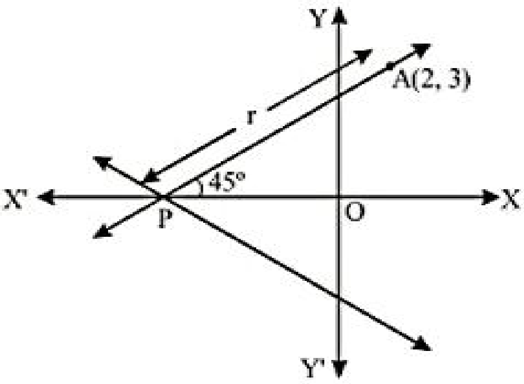
Suppose this line meets the line $x + y + 1 = 0$ at P such that $AP = r$.

Then, the coordinates of P are given by

$$\frac{x-2}{\cos 45^o} = \frac{y-3}{\sin 45^o} = r$$

$$\Rightarrow \qquad x = 2 + r\cos 45^o,\ y = 3 + r\sin 45^o$$

$$\Rightarrow \qquad x = 2 + \frac{r}{\sqrt{2}},\ y = 3 + \frac{r}{\sqrt{2}}$$



Thus, the coordinates of P are $\left(2+\frac{r}{\sqrt{2}}, 3+\frac{r}{\sqrt{2}}\right)$

Since P lies on $x + y + 1 = 0$. Therefore,

$$2+\frac{r}{\sqrt{2}}+3+\frac{r}{\sqrt{2}}+1=0$$

$$\Rightarrow \qquad \sqrt{2}r = -6 \qquad \Rightarrow \qquad r = -3\sqrt{2}$$

Therefore, $\quad$ length $AP = |r| = 3\sqrt{2}$

Illustration :

A (3, 2) and B(7, 4) are two vertices of a triangle. Find the third vertex C so that ABC is an equilateral triangle.

Sol. Let the point be C(h, k). If the triangle is equilateral then length of altitude is $\frac{\sqrt{3}}{2}$ times the length of side.

Length of side (AB) = $\sqrt{4^2+2^2} = \sqrt{20}$

Length of altitude = $\sqrt{15}$

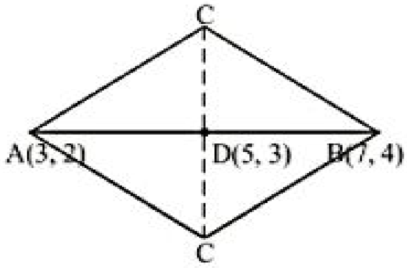
coordinates of point D = (5, 3)

slope of side (AB) = $\frac{1}{2}$

slope of altitude = –2

$\tan\theta = -2$

$\sin\theta = \frac{2}{\sqrt{5}}, \cos\theta = \frac{-1}{\sqrt{5}}$



Equation of line

$$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = \pm\sqrt{15}$$

$$\frac{x-5}{\frac{-1}{\sqrt{5}}} = \frac{y-3}{\frac{2}{\sqrt{5}}} = \pm\sqrt{15}$$

The points are

$\left(5-\sqrt{3},\ 3+2\sqrt{3}\right)$ and $\left(5+\sqrt{3},\ 3-2\sqrt{3}\right)$

Illustration :

A line through A(–5, –4) meets the line $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at B, C and D. If $(15/AB)^2 + (10/AC)^2 = (6/AD)^2$ then find the equation of the line.

Sol. Let the inclination of line be θ then equation of line through point A (–5, –4) is

$$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_i \qquad \text{where } i = 1, 2, 3$$

$r_1 = AB,\quad r_2 = AC,\quad \text{and } r_3 = AD$

$\therefore$ point B $(r_1\cos\theta - 5,\ r_1\sin\theta - 4)$

Points B satisfying the line $x + 3y + 2 = 0$

$\therefore \quad (r_1\cos\theta - 5) + 3(r_1\sin\theta - 4) + 2 = 0$

or $\quad r_1(\cos\theta + 3\sin\theta) - 5 - 12 + 2 = 0$ or $\quad \cos\theta + 3\sin\theta = \frac{15}{r_1} = \frac{15}{AB}$

Similarly

$$2\cos\theta + \sin\theta = \frac{10}{AC}$$

and $\quad \cos\theta - \sin\theta = \frac{6}{AD}$

Using the given relation

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

$\Rightarrow \quad (\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$

$\Rightarrow \quad (1 + 3\tan\theta)^2 + (2 + \tan\theta)^2 = (1 - \tan\theta)^2$

$\Rightarrow \quad 9\tan^2\theta + 12\tan\theta + 4 = 0$

$\Rightarrow \quad (3\tan\theta + 2)^2 = 0$

$\Rightarrow \quad \tan\theta = \dfrac{-2}{3}$

Equation of line is $y + 4 = \dfrac{-2}{3}(x + 5)$

$\Rightarrow \quad 3y + 2x + 22 = 0$

17. POSITION OF A POINT W.R.T. A LINE :

If the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lies on the same side of the line $Ax + Bx + C=0$ then the expressions $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will be of the same sign and if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are on the opposite side of the line then the expressions $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will be of opposite sign.

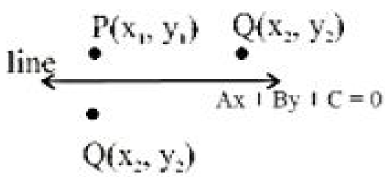


Illustration :

Are the points (3, 4) and (2, –6) on the same or opposite sides of the line $3x - 4y = 8$?

Sol. *Let* $L = 3x - 4y - 8$. *Then the value of L at (3, 4) is* $L_1 = -15$ *and the value of L at (2, –6) is* $L_2 = 22$. *Since* L_1 *and* L_2 *are of opposite signs, therefore the two points are on the opposite sides of the given line.*

Illustration :

If the point (a, a^2) *lies between the lines* $x + y - 2 = 0$ *and* $4x + 4y - 3 = 0$ *then find the range of values of a.*

Sol. *If* (a, a^2) *lies between the lines* $x + y - 2 = 0$ *and* $4x + 4y - 3 = 0$ *then sign of* $a + a^2 - 2$ *and* $4a + 4a^2 - 3$ *should be opposite hence*

$(a + a^2 - 2)(4a + 4a^2 - 3) < 0$

$\Rightarrow \quad (a - 1)(a + 2)(2a + 3)(2a - 1) < 0$

$\Rightarrow \quad a \in \left(-2, \dfrac{-3}{2}\right) \cup \left(\dfrac{1}{2}, 1\right)$

Practice Problem

Single correct question

Q.1 If points $(\sin\theta, \cos\theta)$ and $(3, 2)$ lies on the same side of the line $x + y = 1$, then θ lies between

(A) $\left(0, \frac{\pi}{4}\right)$ (B) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (C) $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ (D) $\left(\frac{3\pi}{4}, \pi\right)$

Q.2 A triangle ABC is formed by the lines $2x - 3y - 6 = 0$; $3x - y + 3 = 0$ and $3x + 4y - 12 = 0$. If the points $P(\alpha, 0)$ and $Q(0, \beta)$ always lie on or inside the ΔABC, then :

(A) $\alpha \in [-1, 2]$ and $\beta \in [-2, 3]$ (B) $\alpha \in [-1, 3]$ and $\beta \in [-2, 4]$

(C) $\alpha \in [-2, 4]$ and $\beta \in [-3, 4]$ (D) $\alpha \in [-1, 3]$ and $\beta \in [-2, 3]$

Q.3 A line $4x + y = 1$ passes through the point A $(2, -7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$ at the point B. The equation to the line AC so that AB = AC, is

(A) $52x + 89y - 519 = 0$ (B) $52x + 89y + 519 = 0$

(C) $89x + 52y + 519 = 0$ (D) $89x + 52y - 519 = 0$

Q.4 If the straight line through the point P(3, 4) makes an angle $\frac{\pi}{6}$ with the x-axis and meets the line $12x + 5y + 10 = 0$ at Q, then the length PQ is

(A) $\frac{132}{5\sqrt{3}+12}$ (B) $\frac{132}{5\sqrt{3}-12}$ (C) $\frac{132}{12\sqrt{3}+5}$ (D) $\frac{132}{12\sqrt{3}-5}$

Multiple correct type question

Q.5 If the slope of a line passing through the point A (3, 2) be 3/4, then the points on the line which are 5 units away from A, are

(A) (5, 5) (B) (7, 5) (C) (–1, –1) (D) (3, 4)

Q.6 The direction in which a straight line must be drawn through the point (1, 2), so that its point of intersection with the line $x + y = 4$ may be at a distance $\frac{1}{3}\sqrt{6}$ from this point.

(A) 15° (B) 30° (C) 60° (D) 75°

Answer key

Q.1 A	Q.2 D	Q.3 B	Q.4 C	Q.5 BC	Q.6 AD

18. CONCURRENCY OF THREE LINES :

Three lines are said to be concurrent if they pass through a common point, i.e. they meet at a point. Thus, if three line are concurrent the point of intersection of two lines lies on the third line. Let the three concurrent lines be

$$a_1x + b_1y + c_1 = 0 \qquad \ldots(i)$$
$$a_2x + b_2y + c_2 = 0 \qquad \ldots(ii)$$
$$a_3x + b_3y + c_3 = 0 \qquad \ldots(iii)$$

Then the point of intersection of equation (i) and (ii) must lie on the third.

The coordinates of the point of intersection of equation (i) and (ii) are

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

This point lies on the line (iii). Therefore, we get

$$\Rightarrow \quad a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

$$\Rightarrow \quad a_3 (b_1c_2 - b_2c_1) + b_3 (c_1a_2 - c_2a_1) + c_3 (a_1b_2 - a_2b_1) = 0$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of three lines.

Alternative Method :

Three lines $L_1 \equiv a_1x + b_1y + c_1 = 0$; $L_2 \equiv a_2x + b_2y + c_2 = 0$; $L_3 \equiv a_3x + b_3y + c_3 = 0$ are concurrent iff there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero at the same time so that $\lambda_1L_1 + \lambda_2L_2 + \lambda_3L_3 = 0$, i.e., $\lambda_1 (a_1x + b_1y + c_1) + \lambda_2 (a_2x + b_2y + c_2) + \lambda_3 (a_3x + b_3y + c_3) = 0$.

Illustration :

If lines $(\cos^2 A)x + (\cos A) \cdot y + 1 = 0$, $(\cos^2 B)x + (\cos B)y + 1 = 0$ and $(\cos^2 C)x + (\cos C)y + 1 = 0$ are concurrent, where A, B, C are angle of a triangle then prove that the triangle must be isosceles.

Sol. *If the given three lines are concurrent then*

$$\begin{vmatrix} \cos^2 A & \cos A & 1 \\ \cos^2 B & \cos B & 1 \\ \cos^2 C & \cos C & 1 \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$\begin{vmatrix} \cos^2 A - \cos^2 B & \cos A - \cos B & 0 \\ \cos^2 B - \cos^2 C & \cos B - \cos C & 0 \\ \cos^2 C & \cos C & 1 \end{vmatrix} = 0$$

$= (\cos A - \cos B)(\cos B - \cos C)(\cos C - \cos A) = 0$

hence $\cos A = \cos B$ *or* $\cos B = \cos C$ *or* $\cos C = \cos A$

we can say that triangle is isosceles.

Illustration :

If the lines $p_1x + q_1y + 1 = 0$, $p_2x + q_2y + 1 = 0$ *and* $p_3x + q_3y + 1 = 0$ *are concurrent then prove that the points* (p_1, q_1), (p_2, q_2) *and* (p_3, q_3) *are collinear.*

Sol. *If the given lines are concurrent then*

$$\begin{vmatrix} p_1 & q_1 & 1 \\ p_2 & q_2 & 1 \\ p_3 & q_3 & 1 \end{vmatrix} = 0$$

Above determinant also shows twice the area of triangle formed by points (p_1, q_1), (p_2, q_2) *and* (p_3, q_3)

Since area of triangle formed by these points is zero, hence the given points are collinear.

Illustration :

Let λ & $\alpha \in R$. *If lines* $\left.\begin{matrix} \lambda x + (\sin\alpha)y + \cos\alpha = 0 \\ x + (\cos\alpha)y + \sin\alpha = 0 \\ -x + (\sin\alpha)y - (\cos\alpha) = 0 \end{matrix}\right\}$; *are concurrent, then find the set of values of* λ.

Sol.

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_3$

$$\begin{vmatrix} \lambda + 1 & 0 & 2\cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$\Rightarrow \quad (\lambda + 1)[-\cos^2\alpha - \sin^2\alpha] + 2\cos\alpha[\sin\alpha + \cos\alpha] = 0$

$\Rightarrow \quad -(\lambda + 1) + 2\cos\alpha[\sin\alpha + \cos\alpha]$

$\Rightarrow \quad \lambda + 1 = 2\cos\alpha[\sin\alpha + \cos\alpha]$

$\Rightarrow \quad \lambda = \sin 2\alpha + \cos 2\alpha$

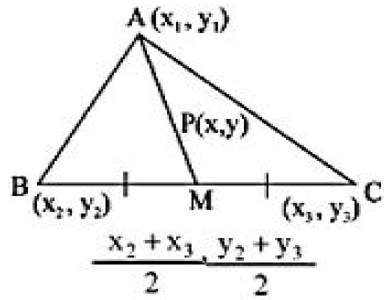
$\Rightarrow \quad \lambda \in \left[-\sqrt{2}, \sqrt{2}\right]$

19. EQUATION OF STRAIGHT LINE IN DETERMINANT FORM :

(i) Line passing through two points (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

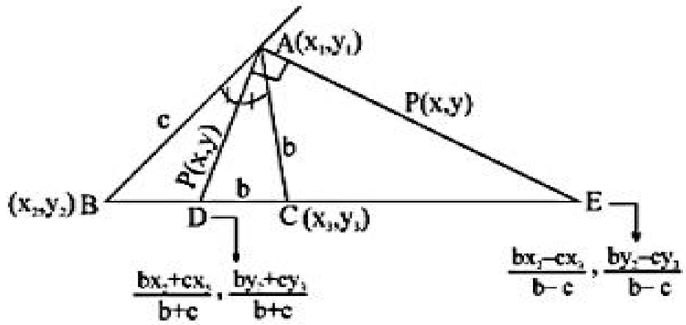
(ii) Equation of the median through A (x_1, y_1) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \frac{x_2+x_3}{2} & \frac{y_2+y_3}{2} & 1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$



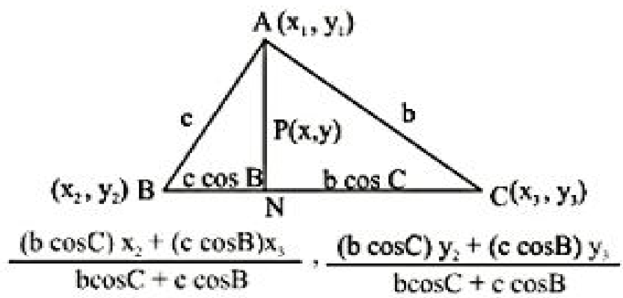
(iii) Equation of internal and external angle bisectors of A are

$$b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \pm c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$



(iv) Equation of the altitude through 'A' is

$$b\cos C\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\cos B\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$



(v) Equation of the line through A and parallel to the base BC is

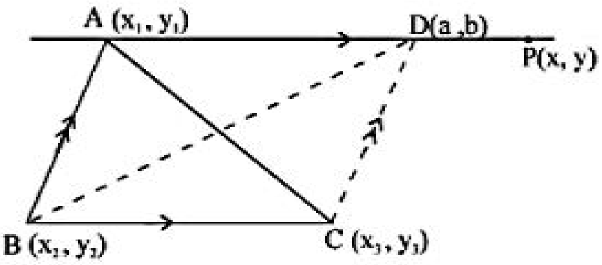
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ a & b & 1 \end{vmatrix} = 0 \quad \text{where (a, b) are assumed to be co-ordinates of D.}$$

Now, equating the middle point of BD and AC

$a + x_2 = x_1 + x_3 \Rightarrow a = x_1 - x_2 + x_3$

$b + y_2 = y_1 + y_3 \Rightarrow b = y_1 - y_2 + y_3$

Hence the equation of the line is



$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_1 + x_3 - x_2 & y_1 + y_3 - y_2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 - x_2 & y_3 - y_2 & 1-1 \end{vmatrix} = 0 \quad \Rightarrow \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Illustration :

Find the equation of median through A if three points of triangle are given by A(2, 1), B(3, 6), C(1, 0)

Sol. *Mid-point of BC = (2, 3)*

Equation of line passing through (2, 1) and (2, 3)

$$\begin{vmatrix} x & y & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$x(1-3) - y(2-2) + (6-2) = 0$$

$$-2x + 4 = 0$$

$$x = 2$$

Practice Problem

Single correct question

Q.1 Let (x_1, y_1); (x_2, y_2) and (x_3, y_3) are the vertices of a triangle ABC respectively. D is a point on BC such that BC = 3BD. The equation of the line through A and D, is

(A) $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + 2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(B) $3\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + 2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(C) $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + 3\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(D) $2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Q.2 Let (x_r, y_r) r = 1, 2, 3 are the coordinates of the vertices of a triangle ABC. If D is the point on BC dividing it in the ratio of 1 : 2 reckoning from the vertex B, then the equation of the line AE in the similar from where E is the harmonic conjugate of D w.r.t. the points B and C.

(A) $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - 2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ (B) $2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(C) $2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - 3\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ (D) $3\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - 2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Q.3 If the line $x + 2ay + a = 0$, $x + 3by + b = 0$ & $x + 4cy + c = 0$ are concurrent then
(A) a, b, c are in A.P. (B) a, b, c are in G.P. (C) a, b, c are in H.P. (D) None of these

Multiple correct type question

Q.4 If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of a triangle, then the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ represents}$$

(A) the median through A (B) the altitude through A
(C) the perpendicular bisector of BC (D) the line joining the centroid with a vertex A

Q.5 The line $x + y - 1 = 0$, $(m - 1)x + (m^2 - 7)y - 5 = 0$ and $(m - 2)x + (2m - 5)y = 0$ are
(A) concurrent for three values of m (B) concurrent for one value of m
(C) concurrent for no value of m (D) are parallel for m = 3

Answer key

Q.1	D	Q.2	B	Q.3	C	Q.4	AD	Q.5	CD

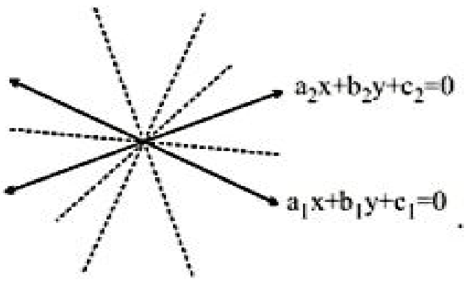
20. FAMILY OF STRAIGHT LINES :

Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$

Then, the general equation of any straight line passing through the point of intersection of lines L_1 and L_2 is given by $L_1 + \lambda L_2 = 0$, where $\lambda \in R$

These lines form a family of straight line

Also this general equation satisfies point of intersection of L_1 and L_2 for any value of λ.



Conversely, if a variable line is expressed in the form of $L_1 + \lambda L_2 = 0$ $(\lambda \in R)$ then it always passes through fixed point which is the point of intersection of $L_1 = 0$ and $L_2 = 0$.

Illustration :

Two lines are given by $L_1 : 3x - 4y + 6 = 0$ & $L_2 : x + y + 2 = 0$. Find the equation of line passing through the intersection by $L_1 = 0$ & $L_2 = 0$, if

(i) It passes through (1, 2)

(ii) It is parallel to $y = 2x + 3$.

(iii) It has intercepts equal in magnitude but opposite in sign.

(iv) It is parallel to x-axis

(v) It is at a distance of $\sqrt{2}$ units from origin.

Sol. Let equation of line through the intersection of L_1 & L_2 is

$$L_1 + \lambda L_2 = 0$$

$$3x - 4y + 6 + \lambda (x + y + 2) = 0$$

$$x (3 + \lambda) + y (\lambda - 4) + 6 + 2\lambda = 0$$

(i) If it passes through (1, 2)

$$3 + \lambda + 2\lambda - 8 + 6 + 2\lambda = 0$$

$$5\lambda = -1 \quad \Rightarrow \quad \lambda = -1/5$$

Equation of line becomes

$$x\left(3 - \frac{1}{5}\right) + y\left(\frac{-1}{5} - 4\right) + 6 + 2\left(\frac{-1}{5}\right) = 0$$

$$14x - 21y + 28 = 0 \Rightarrow 2x - 3y + 4 = 0$$

(ii) If it is parallel to $y = 2x + 3$

$$-\left(\frac{3+\lambda}{\lambda - 4}\right) = 2$$

$$-3 - \lambda = 2\lambda - 8$$

$$3\lambda = 5$$

$$\lambda = 5/3$$

Equation becomes $x\left(3 + \frac{5}{3}\right) + y\left(\frac{5}{3} - 4\right) + 6 + 2\left(\frac{5}{3}\right) = 0$

$$14x - 7y + 28 = 0 \Rightarrow 2x - y + 4 = 0$$

(iii) It has intercepts equal in magnitude but opposite in sign, then

$$\text{slope} = 1$$

$$\frac{-(3+\lambda)}{\lambda - 4} = 1$$

$$-3 - \lambda = \lambda - 4$$

$$2\lambda = 1$$

$$\lambda = 1/2$$

$$x - y + 2 = 0$$

(iv) *Its is parallel to x-axis*

slope = 0

$$\frac{-(3+\lambda)}{\lambda-4}=0$$

Equation becomes

$$x(3-3)+y(-3-4)+6+2(-3)=0$$

$$y=0$$

(v) *It is at a distance of* $\sqrt{2}$ *units from origin equation of line* $(3+\lambda)x+y(\lambda-4)+6+2\lambda=0$

Perpendicular distance from origin

$$\left|\frac{0(3+\lambda)+0(\lambda-4)+6+2\lambda}{\sqrt{(\lambda+3)^2+(\lambda-4)^2}}\right|=\sqrt{2}$$

$$\left|\frac{(2\lambda+6)}{\sqrt{(\lambda+3)^2+(\lambda-4)^2}}\right|=\sqrt{2}$$

$$(2\lambda+6)^2=2[2\lambda^2-2\lambda+25]$$

$$4\lambda^2+24\lambda+36=4\lambda^2-4\lambda+50$$

$$28\lambda=14$$

$$\lambda=1/2$$

$\therefore$ *Equation of the line is* $x-y+2=0$

Illustration :

It the family of straight lines $x(a+2b)+y(a+3b)=a+b$ *passes through a fixed point for all values of a and b. Find the point.*

Sol. $x(a+2b)+y(a+3b)=a+b$

$\Rightarrow$ $a(x+y-1)+b(2x+3y-1)=0$

This equation will always be satisfied for $x+y-1=0$ & $2x+3y-1=0$ *solving these equation we get*

$$x=2,\ y=-1$$

Illustration :

A variable line $ax+by+c=0$ *passes through a fixed point if a, b, c are in arithmetic progression. Find the fixed pont.*

Sol. *If a, b, c are in A.P. then*

$$2b=a+c$$

$$ax+by+c=0$$

$$ax+\left(\frac{a+c}{2}\right)y+c=0$$

$$a\left(x+\frac{y}{2}\right)+c\left(\frac{y}{2}+1\right)=0$$

Solving $x+\frac{y}{2}=0$ & $\frac{y}{2}+1=0$, we get

$y=-2,\qquad x=1$

Illustration :

If $a^2 + 9b^2 = 6ab + 4c^2$ and $ax + by + c = 0$ is a straight line that passes through one or the other of the two fixed points. Find the points.

Sol. *Given*

$$a^2 + 9b^2 = 6ab + 4c^2$$
$$a^2 - 6ab + 9b^2 - 4c^2 = 0$$
$$(a - 3b)^2 - (2c)^2 = 0$$
$$(a - 3b - 2c) \quad or \quad (a - 3b + 2c) = 0$$

Case-I

$$a = 3b + 3c$$
$$ax + by + c = 0$$
$$(3b + 2c)x + by + c = 0$$
$$b\,(3x + y) + c\,(2x + 1) = 0$$
$$3x + y = 0, \quad 2x + 1 = 0$$
$$x = \frac{-1}{2}, \quad y = \frac{3}{2}$$

Case II

$$If\ a = 3b - 2c$$
$$ax + by + c = 0$$
$$(3b - 2c)x + by + c = 0$$
$$b\,(3x + y) + c\,(-2x + 1) = 0$$
$$3x + y = 0, \quad -2x + 1 = 0$$
$$x = \frac{1}{2}, \quad y = \frac{-3}{2}$$

Hence it passes through one of the fixed points

$$\left(-\frac{1}{2}, \frac{3}{2}\right) \quad or \quad \left(\frac{1}{2}, \frac{-3}{2}\right)$$

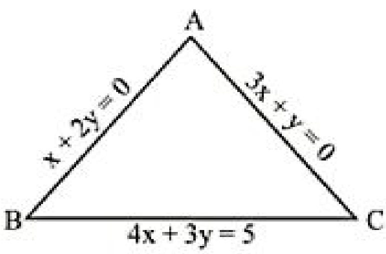
Illustration :

The equations of the sides of a triangle are $x + 2y = 0$, $4x + 3y = 5$ and $3x + y = 0$. Find the co-ordinate of the orthocentre of the triangle without finding the vertices of triangle.

Sol. *Equation of line passing through A can be given by*

$$x + 2y + \lambda\,(3x + y) = 0$$
$$x\,(1 + 3\lambda) + y\,(2 + \lambda) = 0$$

Altitude through A is $\perp$ to BC ($4x + 3y - 5 = 0$)



Slope of altitude through A is $= \dfrac{-1}{-\frac{4}{3}} = \dfrac{3}{4}$

$$\frac{-(3\lambda + 1)}{\lambda + 2} = \frac{3}{4}$$
$$-12\lambda - 4 = 3\lambda + 6$$
$$15\lambda = -10$$
$$\lambda = -\frac{2}{3}$$

Equation of altitude A is $x(-1) + y\left(\frac{4}{3}\right) = 0$

$-3x + 4y = 0 \qquad \ldots (1)$

Equation of line passing through B is

$x + 2y + k(4x + 3y - 5) = 0$

$x(1 + 4k) + y(2 + 3k) - 5k = 0$

line perpendicular to $3x + y = 0$, *has slope* $= \frac{1}{3}$

$$-\left(\frac{4k+1}{3k+2}\right) = \frac{1}{3}$$

$-(12k + 3) = 3k + 2$

$-15k = 5$

$$k = -\frac{1}{3}$$

Equation of altitude through B

$$x\left(1-\frac{4}{3}\right) + y(2-1) - 5\left(-\frac{1}{3}\right) = 0$$

$-x + 3y + 5 = 0$

$x - 3y - 5 = 0 \qquad \ldots (2)$

solving (1) and (2) we get orthocentre as $(-4, -3)$

Illustration :

Find equation of the diagonals of the parallelogram formed by the lines

$2x - y + 7 = 0,\ 2x - y - 5 = 0,\ 3x + 2y - 5 = 0$ and $3x + 2y + 4 = 0$

Sol. *Equation of diagonals can be formed by*

$\ell_1\ell_2 - \ell_3\ell_4 = 0 \quad$ or $\quad \ell_1\ell_4 - \ell_2\ell_3 = 0$

$\ell_1 : 2x - y + 7 = 0, \qquad \ell_2 : 3x + 2y - 5 = 0$

$\ell_3 : 2x - y - 5 = 0, \qquad \ell_4 : 3x + 2y + 4 = 0$

$(2x - y + 7)(3x + 2y - 5) - (2x - y - 5)(3x + 2y + 4) = 0$

$\Rightarrow \quad -5(2x - y) + 7(3x + 2y) - 35 - 4(2x - y) + 5(3x + 2y) + 20 = 0$

$\Rightarrow \quad -10x + 5y + 21x + 14y - 35 - 8x + 4y + 15x + 10 + 20 = 0$

$18x + 33y - 15 = 0$

$6x + 11y - 5 = 0$

Other diagonal

$(2x - y + 7)(3x + 2y + 4) - (2x - y - 5)(3x + 2y - 5) = 0$

$4(2x - y) + 7(3x + 2y) + 28 + (2x - y)5 + 5(3x + 2y) - 25$

$18x + 5y + 1 = 0$

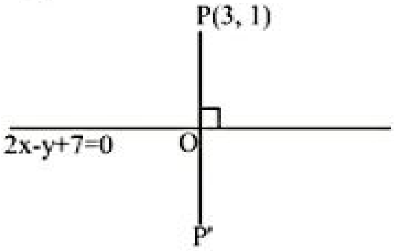
Optics Based Problem :

Illustration :

Find the image of (3, 1) across the line $y = 2x + 7$.

Sol. *Let the point P(3, 1) has image P′ across the line $2x - y + 7 = 0$*

Now PP′ is perpendicular to $2x - y + 7 = 0$



$$\text{Slope of } PP' = -\frac{1}{2}$$

Equation of PP′ is

$$y - 1 = -\frac{1}{2}(x - 3)$$
$$2y - 2 = -x + 3$$
$$x + 2y = 5$$

Point of intersection of lines $2x - y + 7 = 0$ and $x + 2y = 5$ is $O\left(\frac{-9}{5}, \frac{17}{5}\right)$

O is the mid point of PP′

Let P′ (h, k)

$$\frac{h+3}{2} = \frac{-9}{5}, \quad \frac{k+1}{2} = \frac{17}{5}$$

$$h = \frac{-33}{5}, k = \frac{29}{5} \Rightarrow \text{Image}\left(-\frac{33}{5}, \frac{29}{5}\right)$$

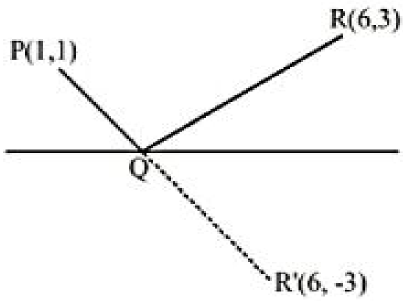
Illustration :

A ray starts from point (1, 1) and is reflected by x-axis and then it passes through the point (6, 3). Find the equation of

(i) Incident ray *(ii) Reflected ray*

Sol. *If the ray started from (1, 1) and after reflection it passes through (6, 3) then incident ray was supposed to pass from the image of (6, 3) across x-axis.*

Image of R(6, 3) in x-axis is R′(6, –3)



Equation of PQ $\Rightarrow y - 1 = \left(\frac{-3-1}{6-1}\right)(x - 1)$

$$5y - 5 = -4x + 4$$
$$4x + 5y - 9 = 0$$

to find equaiton of reflected ray we have to find Q.

Q is point of intersection of ray with x-axis $Q\left(\frac{9}{4}, 0\right)$

Equation of QR.

$$y - 3 = \left(\frac{3-0}{6-9/4}\right)(x - 6)$$

$$y - 3 = \frac{4}{5}(x - 6)$$

$$5y - 15 = 4x - 24$$
$$4x - 5y - 9 = 0$$

Practice Problem

Single correct question

Q.1 The equation of the line through the point of intersection of the lines $5x - 3y = 1$ and $2x + 3y = 23$ and perpendicular to the line $5x - 3y - 1 = 0$ is

(A) $33x - 47y + 21 = 0$ (B) $3x + 22y + 14 = 0$

(C) $63x + 105y - 781 = 0$ (D) $21x - 84y + 1023 = 0$

Q.2 The line $(k + 1)^2 x + ky - 2k^2 - 2 = 0$ passes through a point regardless of the value of k. Which of the following is the line with slope 2 passing through the point?

(A) $y = 2x - 8$ (B) $y = 2x - 5$ (C) $y = 2x - 4$ (D) $y = 2x + 8$

Q.3 Family of lines represented by the equation $(\cos\theta + \sin\theta)x + (\cos\theta - \sin\theta) - 3(3\cos\theta + \sin\theta) = 0$ passes through a fixed point M for all real values of θ. The reflection of M in the line $x - y = 0$, is

(A) (6,3) (B) (3, 6) (C) (–6, 3) (D) (3, –6)

Multiple correct type question

Q.4 If (–2, 6) is the reflection of the point (4, 2) with respect to line $L = 0$, If L can be written in the form $ax + by + c = 0$ (where a, b and c in their lowest form and $a, b, c \in N$) then

(A) $a + b = c$ (B) $a = b + c$ (C) $ab + bc + ac > 0$ (D) $a^2 + b^2 + c^2 > (a+b+c)^2$

Match the Column

Q.5 Set of family of lines are described in column-I and their mathematical equation are given in column-II. Match the entry of column-I with suitable entry of column-II. (*m* and *a* are parameters)

	Column-I		Column-II
(A)	having gradient 3	(P)	$mx - y + 3 - 2m = 0$
(B)	having y intercept three times the x-intercept	(Q)	$mx - y + 3m = 0$
(C)	having x intercept (–3)	(R)	$3x + y = 3a$
(D)	concurrent at (2, 3)	(S)	$3x - y + a = 0$

Answer key

Q.1 C Q.2 A Q.3 B Q.4 BD

Q.5 (A) $\to$ S; (B) $\to$ R; (C) $\to$ Q; (D) $\to$ P

21. SHIFTING OF ORIGIN :

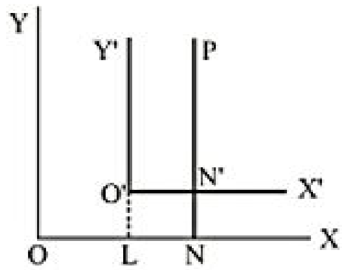
Let OX and OY be the original axes and let the new axes, parallel to the original, be

O'X' and O'Y'

Let the coordinates of the new origin O', referred to the original axes be h and k, so that, if O'L be perpendicular to OX, we have

$$OL = h \text{ and } LO' = k$$

Let P be any point in the plane of the paper, and let its coordinates, referred to the original axes, be x and y and referred to the new axes let them x' and y'.



Draw PN perpendicular to OX to meet O'X' in N'.

then, ON = x, NP = y, O'N' = x' and N'P = y'

we therefore, have

$$x = ON = OL + O'N' = h + x'$$

and $$y = NP = LO' + N'P = k + y'$$

The origin is therefore, transferred to the point (h, k) when we substitute for the coordinates x and y the quantities.

$$x' + h \text{ and } y' + k$$

The above article is true whether the axes be oblique or rectangular.

Illustration :

Find the new coordinates of point (3, –4) if the origin is shifted to (1, 2) by translation.

Sol. *Since origin is shifted to $x = 1$ and $y = 2$*

Hence $x - 1 = X$ and $y - 2 = Y$

Point (3, –4) is shifted to

$$X = 3 - 1, \qquad Y = -4 - 2$$

$$X = 2,\ Y = -6$$

Hence new coordinates are (2, –6)

Illustration :

Find the newly tranformed equation of the straight line $2x - 3y + 5 = 0$ if origin is shifted to (3, –1).

Sol. *If origin is shifted (3, –1)*

$$x - 3 = X,\ y - (-1) = Y$$

$$x = 3 + X,\ y = Y - 1$$

New equation is $2(X + 3) - 3(Y - 1) + 5 = 0$

$$2X - 3Y + 14 = 0$$

Illustration :

Find the point at which the origin be shifted so that the equation $x^2 + y^2 - 5x + 2y - 5 = 0$ has no first degree terms.

Sol. $x^2 + y^2 - 5x + 2y - 5 = 0$

$$\Rightarrow \quad \left(x^2 - 5x + \frac{25}{4}\right) - \frac{25}{4} + (y^2 + 2y + 1) - 1 - 5 = 0$$

$$\Rightarrow \quad \left(x - \frac{5}{2}\right)^2 + (y + 1)^2 - \frac{49}{4} = 0$$

$$\Rightarrow \quad X^2 + Y^2 = \frac{49}{4}$$

If new equation to be formed has no first degree terms then shift the origin to

$$x = \frac{5}{2}, y = -1$$

$$= \left(\frac{5}{2}, -1\right).$$

22. ROTATION OF AXES :

22.1 ROTATION OF AXES WITHOUT CHANGING THE ORIGIN :

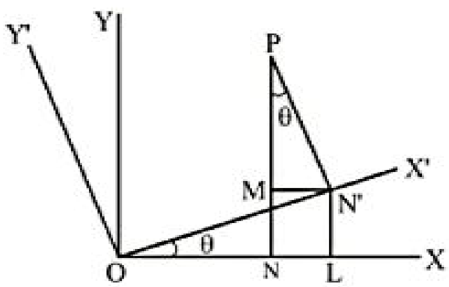
Let OX and OY be the original system of axes and OX' and OY' the new system, and let the angle, XOX', through which the axes are turned be called θ.
Take any point P in the plane of the paper.

Draw PN and PN' perpendicular to OX and OX', and also N'L and N'M perpendicular to OX and PN. If the coordinates of P, referred to the original axes, be x and y, and referred to the new axes, be x' and y', we have

$$ON = x, NP = y, ON' = x' \text{ and } N'P = y'$$

The angle,

$$MPN' = 90^\circ - \angle MN'P = \angle MN'O = \angle XOX' = \theta$$



We then have

$$x = ON = OL - MN' = ON' \cos\theta - N'P \sin\theta$$
$$= x' \cos\theta - y' \sin\theta$$

and

$$y = NP = LN' + MP = ON' \sin\theta + N'P \cos\theta$$
$$= x' \sin\theta + y' \cos\theta$$

If therefore, in any equation we wish to turn the axes, being rectangular, through an angle θ we must substitute

$$x = x' \cos\theta - y' \sin\theta \quad \text{and} \quad y = x' \sin\theta + y' \cos\theta$$

	x	y
x'	$\cos\theta$	$\sin\theta$
y'	$-\sin\theta$	$\cos\theta$

23. ANGLE BISECTOR :

23.1 EQUATION OF BISECTORS OF THE ANGLES BETWEEN TWO LINES :

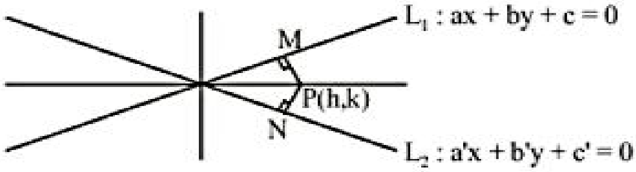
(i) Equations of the bisectors of angles between the lines $ax + by + c = 0$ & $a'x + b'y + c' = 0$ $(ab' \neq a'b)$ are : $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = \pm \dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

Explanation : As we know that angle bisector between two lines is the locus of the point which moves in a plane such that its perpendicular distance from both the lines are equal.

Let P(h, k) be a moving point

PM = PN

$$\left|\frac{ah+bk+c}{\sqrt{a^2+b^2}}\right| = \left|\frac{a'h+b'k+c'}{\sqrt{a'^2+b'^2}}\right|$$



$\therefore$ Locus of P(h, k) is $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = \pm \dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$.

(ii) **To discriminate between the acute angle bisector & the obtuse angle bisector**

If θ be the angle between one of the lines & one of the bisectors, find tan θ.

If $|\tan\theta| < 1$, then $2\theta < 90°$ so that this bisector is the acute angle bisector.

If $|\tan\theta| > 1$, then we get the bisector to be the obtuse angle bisector .

(iii) To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin . Rewrite the equations , $ax+by+c=0$ & $a'x+b'y+c'=0$ such that the constant terms c, c' are positive. Then ;

$\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = + \dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle containing the origin

& $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = - \dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle not containing the origin.

(iv) To discriminate between acute angle bisector & obtuse angle bisector proceed as follows Write $ax+by+c=0$ & $a'x+b'y+c'=0$ such that constant terms are positive.

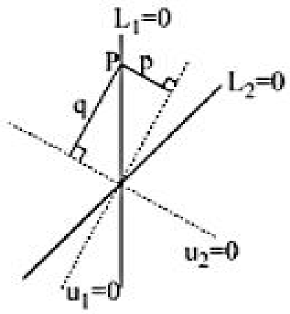
If $aa'+bb'<0$, then the angle between the lines that contains the origin is acute and the equation of the bisector of this acute angle is $\frac{ax+by+c}{\sqrt{a^2+b^2}}=+\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ therefore $\frac{ax+by+c}{\sqrt{a^2+b^2}}=-\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ is the equation of other bisector.

If, however, $aa'+bb'>0$, then the angle between the lines that contains the origin is obtuse & the equation of the bisector of this obtuse angle is:

$\frac{ax+by+c}{\sqrt{a^2+b^2}}=+\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$; therefore $\frac{ax+by+c}{\sqrt{a^2+b^2}}=-\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

is the equation of other bisector.

(v) Another way of identifying an acute and obtuse angle bisector is as follows :
Let $L_1=0$ & $L_2=0$ are the given lines & $u_1=0$ and $u_2=0$ are the bisectors between $L_1=0$ & $L_2=0$. Take a point P on any one of the lines $L_1=0$ or $L_2=0$ and drop perpendicular on $u_1=0$ & $u_2=0$ as shown. If,



$|p|<|q| \Rightarrow u_1$ is the acute angle bisector.

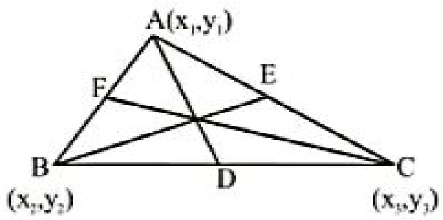
$|p|>|q| \Rightarrow u_1$ is the obtuse angle bisector.

$|p|=|q| \Rightarrow$ the lines L_1 & L_2 are perpendicular.

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ are those which are parallel to the bisectors between these two lines and passing through the point P.

(vi) **Bisector in case of triangle**

CASE-I : When vertices of triangle are known.



Method-1 :

1. Find the length of sides of triangle AB(c), BC(a) and CA(b) by distance formula.
2. Find incentre I of triangle where $I=\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$
3. With the help of incentre we can find all angle bisectors of triangle ABC.

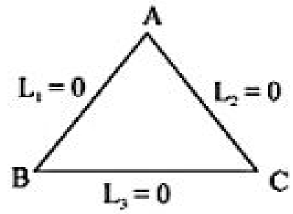
Method-2 :

1. Find the point D which divides the side BC in ratio c : b. and also find the points E and F similarly.
2. Find the equations of AD, BE and CF.

CASE-II : When the equation of sides are given.

Method-1 :

1. Compute tanA , tanB , tanC and arrange the lines in descending order of their slopes.
2. With the help of angles we can find all acute/obtuse angle bisectors.



Method-2 :

1. Plot the lines approximately and compute bisectors containing or not containing the origin.

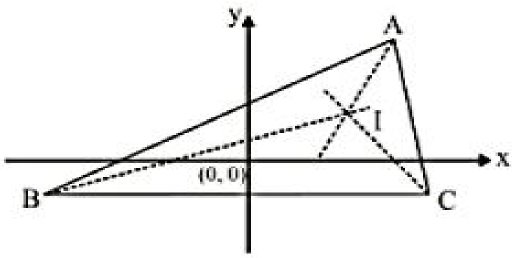


Illustration :

Find the angle bisectors between the lines $4x + 3y - 7 = 0$ and $24x + 7y - 31 = 0$. Also identify the acute angle bisector and bisector contianing origin.

Sol. *If the given equation are*

$$4x + 3y - 7 = 0 \text{ and } 24x + 7y - 31 = 0$$

These can be written as

$$-4x - 3y + 7 = 0 \text{ and } -24x - 7y + 31 = 0$$

Equation of bisector is given by

$$\left(\frac{-4x-3y+7}{\sqrt{4^2+3^2}}\right) = \pm\left(\frac{-24x-7y+31}{\sqrt{24^2+7^2}}\right)$$

$$5(-4x - 3y + 7) = \pm(-24x - 7y + 31)$$

Equation of bisector containing origin is

$$5(-4x - 3y + 7) = (-24x - 7y + 31)$$

$$x - 2y + 1 = 0$$

Equation of bisector not containing origin is

$$5(-4x - 3y + 7) = (-24x - 7y + 31)$$

$$2x + y - 3 = 0$$

Acute Angle Bisector

$$aa' + bb' = (-4)(-24) + (-3)(-7) > 0$$

Since $aa' + bb' > 0$ hence the angle bisector that contains origin is obtuse angle bisector i.e., $x - 2y + 1 = 0$ and equation not containing origin is acute angle bisector i.e., $2x + y - 3 = 0$.

Illustration :

Find the equation of bisectors between the lines $x + \sqrt{3}y = 6 + 2\sqrt{3}$ *and* $x - \sqrt{3}y = 6 - 2\sqrt{3}$.

Sol. *Slope of line* $x + \sqrt{3}y = 6 + 2\sqrt{3}$ *is* $m_1 = -\frac{1}{\sqrt{3}}$.

Slope of line $x - \sqrt{3}y = 6 - 2\sqrt{3}$ *is* $m_2 = \frac{1}{\sqrt{3}}$.

Now $m_1 + m_2 = 0$

Since sum of slopes is zero therefore x = h, and y = k are equations of angle bisector where (h, k) is point of intersection. Hence bisectors are x = 6 and y = 2.

Practice Problem

Single correct question

Q.1 The equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$ is

(A) $21x + 77y - 101 = 0$ (B) $11x - 3y + 9 = 0$

(C) $21x + 77y + 101 = 0$ (D) $11x - 3y - 9 = 0$

Q.2 The equation of the bisector of the lines $3x - 4y + 1 = 0$ and $5x + 12y - 11 = 0$ which do not contain origin is

(A) $7x - 56y + 34 = 0$ (B) $32x + 4y - 21 = 0$

(C) $3x + 7y + 11 = 0$ (D) $16x - 3y + 7 = 0$

Q.3 When the origin is shifted to (1, 1), the equation $xy - x - y + 1 = 0$ becomes

(A) $X^2 - Y^2 = 0$ (B) $XY = 0$

(C) $X^2 - Y^2 - 2X + 2Y = 0$ (D) $X^2 + Y^2 - 5X + 2Y - 5 = 0$

Multiple correct type question

Q.4 The bisectors of angle between the st. lines, $y - b = \frac{2m}{1-m^2}(x - a)$ and $y - b = \frac{2m'}{1-m'^2}(x - a)$ are

(A) $(y - b)(m + m') + (x - a)(1 - mm') = 0$

(B) $(y - b)(m + m') - (x - a)(1 - mm') = 0$

(C) $(y - b)(1 - mm') + (x - a)(m + m') = 0$

(D) $(y - b)(1 - mm') - (x - a)(m + m') = 0$

Q.5 Let (p, q) is the point to which the origin should be shifted so that the equation $y^2 - 6y - 4x + 13 = 0$ is transformed to the form $Y^2 + AX = 0$ then

(A) $p + q = 4$ (B) $p^2 + q^2 = 25$

(C) $p^2 + q^2 = 10$ (D) p and q are twin prime

Answer key

Q.1	A	Q.2	A	Q.3	B	Q.4	AD	Q.5	ACD

24. PAIR OF STRAIGHT LINES :

24.1 A PAIR OF STRAIGHT LINES THROUGH ORIGIN :

(i) A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through the origin & if:

(a) $h^2 > ab \Rightarrow$ lines are real & distinct.

(b) $h^2 = ab \Rightarrow$ lines are coincident.

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. $(0, 0)$.

Note: A homogeneous equation of degree n represents n straight lines through the origin.

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then; $m_1 + m_2 = -\frac{2h}{b}$ & $m_1 m_2 = \frac{a}{b}$.

Explanation :

$b(y - m_1x)(y - m_2x) = ax^2 + 2hxy + by^2$

Comparing co-efficients of x^2 and xy on both sides

$$-b(m_1 + m_2) = 2h \Rightarrow m_1 + m_2 = \frac{-2h}{b}$$

$$b\, m_1 m_2 = a \Rightarrow m_1 m_2 = \frac{a}{b}$$

24.2 ANGLE BETWEEN STRAIGHT LINES REPRESENTED BY THE EQUATION $ax^2 + 2hxy + by^2 = 0$:

Let the equation $ax^2 + 2hxy + by^2 = 0$ represents two lines which are $y - m_1x = 0$ and $y - m_2x = 0$

$\therefore$ Angle between these lines is

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right| = \left|\frac{\sqrt{\left(\frac{-2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + \frac{a}{b}}\right| = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$\therefore$ acute angle θ between the pair of straight lines represented by the equation,

$ax^2 + 2hxy + by^2 = 0$ is $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

The condition that these lines are :

(a) At right angles to each other if $a + b = 0$. i.e. co–efficient of x^2 + co–efficient of $y^2 = 0$.

(b) Coincident if $h^2 = ab$.

(c) Equally inclined to the axis of x if $h = 0$. i.e. coeff. of $xy = 0$.

24.3 BISECTORS OF ANGLE BETWEEN THE LINES REPRESENTED BY $ax^2 + 2hxy + by^2 = 0$:

Let the given equation represent the straight lines

$$y - m_1x = 0 \qquad \text{...(i)}$$
$$y - m_2x = 0 \qquad \text{...(ii)}$$

where $m_1 + m_2 = -2h/b$ and $m_1m_2 = a/b$ ...(iii)

The equation to the bisectors of the angles between the straight lines in equation (i) and (ii) are

$$\frac{y - m_1x}{\sqrt{1+m_1^2}} = \pm \frac{y - m_2x}{\sqrt{1+m_2^2}}$$

Therefore, the combined equation of the bisectors is

$$\left\{\frac{y - m_1x}{\sqrt{1+m_1^2}} - \frac{y - m_2x}{\sqrt{1+m_2^2}}\right\}\left\{\frac{y - m_1x}{\sqrt{1+m_1^2}} + \frac{y - m_2x}{\sqrt{1+m_2^2}}\right\} = 0$$

or $$\frac{(y - m_1x)^2}{1+m_1^2} - \frac{(y - m_2x)^2}{1+m_2^2} = 0$$

hence, by equation (iii), we get

$$\frac{-2h}{b}(x^2 - y^2) + 2\left(\frac{a}{b} - 1\right)xy = 0$$

or $$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

Important point

The product of the perpendiculars, dropped from (x_1, y_1) to the pair of lines represented by the equation, $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$.

24.4 GENERAL EQUATION OF SECOND DEGREE REPRESENTING A PAIR OF STRAIGHT LINES :

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0, \text{ i.e. if } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only.

Illustration :

Prove that the $x^2 - 4xy + y^2$ and $x + y = 1$ enclose an equilateral triangle. Find also its area.

Sol. Angle between the two lines $x^2 - 4xy + y^2 = 0$ is

$$\theta = tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= tan^{-1}\left|\frac{2\sqrt{4-1}}{1+1}\right|$$

$$= tan^{-1}\sqrt{3} = 60°$$

Again $x^2 - 4xy + y^2 = 0$

$\Rightarrow \quad x^2 - 4xy + 4y^2 - 3y^2 = 0$

$\Rightarrow \quad (x - 2y)^2 - (\sqrt{3}\,y)^2 = 0$

$\Rightarrow \quad (x - 2y + \sqrt{3}\,y)(x - 2y - \sqrt{3}\,y) = 0$

Equation of lines represented by $x^2 - 4xy + y^2 = 0$

are $\quad x - (2 - \sqrt{3})y = 0 \quad$ ... (i)

$\quad x - (2 + \sqrt{3})y = 0 \quad$ ... (ii)

Now angle between the line $x + y = 1$ and line (i) is

$$\theta = tan^{-1}\left|\frac{m_1 + m_2}{1 + m_1 m_2}\right|$$

$$= tan^{-1}\left|\frac{(-1) - \left(\frac{1}{2-\sqrt{3}}\right)}{1 + (-1)\left(\frac{1}{2-\sqrt{3}}\right)}\right|$$

$$= tan^{-1}\left|\frac{-1-(2+\sqrt{3})}{1-(2+\sqrt{3})}\right|$$

$$= tan^{-1}\left|\frac{-3-\sqrt{3}}{-1-\sqrt{3}}\right| = tan^{-1}|-\sqrt{3}| = tan^{-1}\sqrt{3} = 60°$$

Similarly, angle between the line $x + y = 1$ and line (ii) is $60°$.

$\therefore$ Angle between the lines taking any two of the given three lines be $60°$.
thus, the triangle formed by these lines is an equilateral triangle.

Again, point of intersection of lines $x^2 - 4xy + y^2 = 0$ is origin. So, distance of the line

$x + y - 1 = 0$ from the origin is $\quad h = \left|\frac{(0)+(0)-1}{\sqrt{1^2+1^2}}\right| = \frac{1}{\sqrt{2}}$

Let a is the side of the equilateral triangle
Hence, a sin 60° = h.

$$\Rightarrow \quad a = \frac{h}{\sin 60°} = \frac{\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \sqrt{\frac{2}{3}}$$

$$\text{Area of triangle} = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}\left(\frac{2}{3}\right) = \frac{\sqrt{3}}{6} \text{ sq. units.}$$

Illustration :

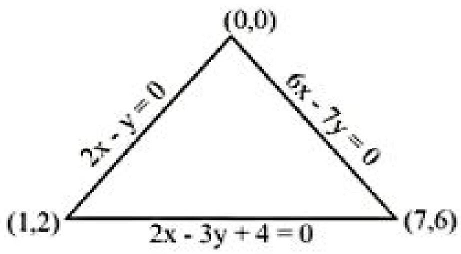
Find the centroid of the triangle the equation of whose sides $12x^2 - 20xy + 7y^2 = 0$ *and* $2x - 3y + 4 = 0$.

Sol. $12x^2 - 20xy + 7y^2 = 0$

$12x^2 - 14xy - 6xy + 7y^2 = 0$

$2x(6x - 7y) - y(6x - 6y) = 0$

$\Rightarrow (2x - y)(6x - 7y) = 0$



So three lines are $2x - y = 0$ .... (i)

$6x - 7y = 0$ .... (ii)

$2x - 3y + 4 = 0$ .... (iii)

Point of intersection of lines (i) and (ii) is (0, 0).
Point of intersection of lines (i) and (iii) is (1, 2).
and Point of intersection of lines (ii) and (iii) is (7, 6).

$$\text{Centroid is } \left(\frac{0+1+7}{3}, \frac{0+2+6}{3}\right) = \left(\frac{8}{3}, \frac{8}{3}\right).$$

Illustration :

Find the distance between the parallel lines $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$

Sol. *Given lines is* $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$

Again, $4x^2 + 4xy + y^2 = 0$

$(2x + y)^2 = 0$

$\Rightarrow \quad (2x + y + A)(2x + y + B) = 4x^2 + 4xy + y^2 - 6x - 3y - 4$

$\Rightarrow \quad 4x^2 + 4xy + y^2 + (2A + 2B)x + (A + B)y + AB = 4x^2 + 4xy + y^2 - 6x - 3y - 4$

Comparing both sides, we get

$A + B = -3$ ... (i)

$AB = -4$ ... (ii)

By solving (i) and (ii) we get

$A = -4, B = 1$

$\Rightarrow$ *Two parallel lines are* $2x - y - 4 = 0$ *and* $2x + y + 1 = 0$

$$\text{distance} = \left|\frac{-4-1}{\sqrt{2^2+1^2}}\right| = \sqrt{5}.$$

25. HOMOGENISATION :

The joint equation of a pair of straight lines joining origin to the points of intersection of the line given by

$$lx + my + n = 0 \quad \text{......(i)} \quad \&$$

the 2nd degree curve : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ......(ii)

is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx+my}{-n}\right) + 2fy\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$ ...... (iii)

(iii) is obtained by homogenizing (ii) with the help of (i), by writing (i) in the form : $\left(\frac{lx+my}{-n}\right) = 1.$

Important concept :

Any second degree curve through the four point of intersection of $f(x\,y) = 0$ & $xy = 0$ is given by $f(x\,y) + \lambda\, xy = 0$ where $f(xy) = 0$ is also a second degree curve.

Illustration :

Find the equation of the line pair joining origin and the point of intersections of the line $2x - y = 3$ and the curve $x^2 - y^2 - xy + 3x - 6y + 18 = 0$. Also find the angle between these two lines.

Sol. *Given*

curve : $x^2 + y^2 - xy + 3x - 6y + 18 = 0$

and line : $2x - y = 3$

covert the curve in homogeneous equation with the help of line

$$\Rightarrow x^2 - y^2 - xy + 3x\left(\frac{2x-y}{3}\right) - 6y\left(\frac{2x-y}{3}\right) + 18\left(\frac{2x-y}{3}\right)^2 = 0$$

$$\Rightarrow x^2 - y^2 - xy + x\,(2x - y) - 2y\,(2x - y) + 2(2x - y)^2 = 0$$

$$\Rightarrow x^2 - y^2 - xy + 2x^2 - xy - 4xy + 2y^2 + 8x^2 + 2y^2 - 8xy = 0$$

$$\Rightarrow 11x^2 + 3y^2 - 14xy = 0$$

Angle between them is

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} \quad \text{where } a = 11,\ b = 3 \text{ and } h = -7$$

$$= \frac{2\sqrt{49-33}}{14}$$

$$\text{or } \tan\theta = \frac{4}{7} \Rightarrow \quad \theta = \tan^{-1}\left(\frac{4}{7}\right)$$

Illustration :

Find the value of 'm' if the lines joining the origin to the points common to $x^2 + y^2 + x - 2y - m = 0$ & $x + y = 1$ are at right angles.

Sol. Given : $x^2 + y^2 + x - 2y - m = 0$

and a line : $x + y = 1$

Homogenize the curve with the help of line

$\Rightarrow \quad x^2 + y^2 + x(x + y) - 2y(x + y) - m(x + y)^2 = 0$

$\Rightarrow \quad x^2 + y^2 + x^2 + xy - 2xy - 2y^2 - mx^2 - my^2 - 2mxy = 0$

$\Rightarrow \quad x^2(2 - m) - y^2(1 + m) - xy(1 + 2m) = 0$

If lines are at right angle then

$$(2 - m) - (1 + m) = 0 \Rightarrow m = \frac{1}{2}$$

Illustration :

A line L passing through the point (2, 1) intersects the curve $4x^2 + y^2 - x + 4y - 2 = 0$ at the points A, B. If the lines joining origin and the points A, B are such that the coordinate axis are the bisectors between them then find the equation of line L.

Sol. Given :

Curve : $4x^2 + y^2 - x + 4y - 2 = 0$ and point (2, 1).

Let the slope of a line is m, then the equation of the line passing through a point (2, 1) is

$(y - 1) = m(x - 2)$

$\Rightarrow \quad y - 1 = mx - 2m \Rightarrow y - mx + 2m - 1 = 0$

Make the curve homogeneous with the help of line then

$$\Rightarrow \quad 4x^2 + y^2 - x\left(\frac{y - mx}{1 - 2m}\right) + 4y\left(\frac{y - mx}{1 - 2m}\right) - 2\left(\frac{y - mx}{1 - 2m}\right)^2 = 0$$

$\Rightarrow \quad 4x^2(1 - 2m)^2 + y^2(1 - 2m)^2 - x(y - mx)(1 - 2m) + 4y(y - mx)(1 - 2m) - 2(y - mx)^2 = 0$

$\therefore$ If the co-ordinate axis are the bisectors between them the coefficient of xy should be zero.

$(2m - 1) + 4m(2m - 1) + 4m = 0$

$8m^2 + 2m - 1 = 0$

$8m^2 + 4m - 2m - 1 = 0$

$4m(2m + 1) - 1(2m + 1) = 0$

$(4m - 1)(2m + 1) = 0$

$$m = \frac{1}{4} \quad \text{or} \quad -\frac{1}{2}$$

$\therefore$ lines are

$$(y - 1) = \frac{1}{4}(x - 2) \quad \text{or} \quad (y - 1) = -\frac{1}{2}(x - 2)$$

$$x - 4y + 2 = 0 \quad \text{or} \quad x + 2y - 4 = 0$$

Practice Problem

Single correct question

Q.1 The combined equation of straight lines which pass through (1, 2) and perpendicular to the line pair $3x^2 - 8xy + 5y^2 = 0$ is

(A) $5x^2 - 8xy + 3y^2 + 13x - 15y + 7 = 0$ (B) $5x^2 + 8xy + 3y^2 - 26x - 20y + 33 = 0$

(C) $3x^2 + 4xy + 3y^2 - 15x + 25y + 12 = 0$ (D) $3x^2 + 4xy - 3y^2 - 46x + 8y + 33 = 0$

Q.2 If the pair of lines $ax^2 + 2(a + b)xy + by^2 = 0$ lie along diameters of a circle and divide the circle into four sectors such that the area of one of the sectors is thrice the area of another sector, then

(A) $3a^2 + 2ab + 3b^2 = 0$ (B) $3a^2 + 10ab + 3b^2 = 0$

(C) $3a^2 - 2ab + 3b^2 = 0$ (D) $3a^2 - 10ab + 3b^2 = 0$

Q.3 If $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ bisect angles between each other, then

(A) $p + q = 1$ (B) $pq = 1$ (C) $pq + 1 = 0$ (D) $p^2 + pq + q^2 = 0$

Q.4 If the straight lines joining the origin and the points of intersection of the curve $5x^2 + 12xy - 6y^2 + 4x - 2y + 3 = 0$ and $x + ky - 1 = 0$ are equally inclined to the co-ordinate axes then the value of k :

(A) is equal to 1 (B) is equal to -1

(C) is equal to 2 (D) does not exist in the set of real numbers

Multiple correct type question

Q.5 The lines L_1 and L_2 denoted by $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$ intersect at the point P and have gradients m_1 and m_2 respectively. The acute angle between them is θ. Which of the following relations hold good?

(A) $m_1 + m_2 = 5/4$

(B) $m_1 m_2 = 3/5$

(C) acute angle between L_1 and L_2 is $\sin^{-1}\left(\frac{2}{5\sqrt{5}}\right)$

(D) sum of the abscissa and ordinate of the point P is -1.

Q.6 If angle between the lines joining origin and the point of intersections of the line $x - y = 2$ and the curve $x^2 - 4xy + 2y^2 - 2x + y + k = 0$ is 45° then

(A) Sum of all possible value (s) of k is equal to 3

(B) Sum of all possible value (s) of k is equal to -2

(C) Product of all possible value (s) of k is equal to -16

(D) Product of all possible value (s) of k is equal to 12

Answer key

Q.1	B	Q.2	A	Q.3	C
Q.4	B	Q.5	BCD	Q.6	BC

SOLVED EXAMPLES

Q.1 If A (1, 2) and B (3, 8) be two given points, find a point P such that |PA| = |PB| and ΔPAB = 10.

Sol. Let $P = (\alpha, \beta)$, then $|PA| = |PB| \quad \Rightarrow \quad PA^2 = PB^2$

$\Rightarrow \quad (\alpha - 1)^2 + (\beta - 1)^2 = (\alpha - 3)^2 + (\beta - 8)^2$

$\Rightarrow \quad \alpha^2 - 2\alpha + 1 + \beta^2 + 4 - 4\beta = \alpha^2 + 9 - 6\alpha + \beta^2 + 64 - 16\beta$

$\Rightarrow \quad 4\alpha + 12\beta = 68 \quad \Rightarrow \quad \alpha + 3\beta = 17$ ...(i)

Also, area of ΔPAB = 10

$$\Rightarrow \quad \frac{1}{2}|2\alpha - \beta + 8 - 6 + 3\beta - 8\alpha| = 10$$

$\Rightarrow \quad -6\alpha + 2\beta + 2 = \pm 20$

$\Rightarrow \quad -3\alpha + \beta + 1 = \pm 10$

$\Rightarrow \quad -3\alpha + \beta = -1 \pm 10$

$\Rightarrow \quad -3\alpha + \beta = -11$ ...(ii)

or $\quad -3\alpha + \beta = 9$ ...(iii)

Solving (i) and (ii), we obtain $\alpha = 5, \beta = 4$, solving (i) and (iii), we obtain $\alpha = -1, \beta = 6$

Hence, the point P is either (5, 4) or (–1, 6)

Q.2 Factorize $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$.

Sol. Applying $C_1 \to C_1 + C_2 - 2C_3$

$$D = \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$= (a^2+b^2+c^2)\begin{vmatrix} 0 & a^2-b^2 & c(b-a) \\ 0 & b^2-c^2 & a(c-b) \\ 1 & c^2 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} 0 & a+b & -c \\ 0 & b+c & -a \\ 1 & c^2 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} a+b & -c \\ b+c & -a \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$$= (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} a+b+c & -c \\ b+c+a & -a \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a-b)(b-c)(a+b+c)\begin{vmatrix} 1 & -c \\ 1 & -a \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a+b+c)(a-b)(b-c)(c-a)$$

Q.3 If $\left(\frac{a^3}{a-1}, \frac{a^2-3}{a-1}\right)$, $\left(\frac{b^3}{b-1}, \frac{b^2-3}{b-1}\right)$ and $\left(\frac{c^3}{c-1}, \frac{c^2-3}{c-1}\right)$ are collinear and α abc + β (a + b + c) = γ (ab + bc + ca), $\alpha, \beta, \gamma \in N$, then find the least value of $\alpha + \beta + \gamma$.

Sol. Let the equation of line on which these three points lie be

$$lx + my + n = 0$$

and the point $\left(\frac{t^3}{t-1}, \frac{t^2-3}{t-1}\right)$ lie on the line where t = a, b, c

$$l\left(\frac{t^3}{t-1}\right) + m\left(\frac{t^2-3}{t-1}\right) + n = 0$$

$$l\,t^3 + m\,(t^2 - 3) + n\,(t - 1) = 0$$

$$t^3\,l + t^2\,m + t\,n - 3m - n = 0$$

If a, b, c are the roots of given equation then

$$a + b + c = \frac{-m}{l} \qquad \ldots(i)$$

$$ab + bc + ca = \frac{n}{l} \qquad \ldots(ii)$$

$$abc = \frac{3m}{l} + \frac{n}{l} \qquad \ldots(iii)$$

using (i), (ii) & (iii) we get

$$abc = -3\,(a + b + c) + ab + bc + ca$$

$$abc + 3\,(a + b + c) = ab + bc + ca$$

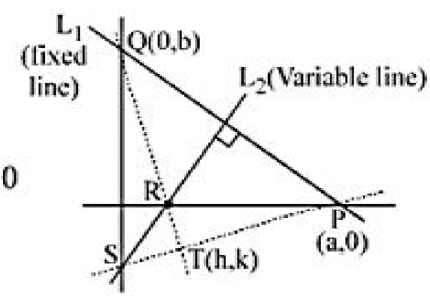
$\Rightarrow \quad \alpha + \beta + \gamma = 1 + 3 + 1 = 5.$

Q.4 A fixed line PQ : $\frac{x}{a} + \frac{y}{b} = 1$; cut the x and y axis at P & Q. and a variable line perpendicular to it cut the x-axis at R and y axis at S. Find the locus of the point of intersection of QR and PS.

Sol. Line PQ cut the x-axis and y-axis at P & Q respectively.

So, P = (a, 0) and Q = (0, b)

$$\text{Slope of PQ} = \frac{-b}{a}$$



Slope of line perpendicular to PQ = $\frac{a}{b}$

Equation of line perpendicular to PQ is $ax - by + \lambda = 0$

It intersect x-axis at R and y-axis at S

Hence, $R = \left(\frac{-\lambda}{a}, 0\right)$, $\quad S = \left(0, \frac{\lambda}{b}\right)$

$$\text{Slope of QR} = \frac{b-0}{0-\left(\frac{-\lambda}{a}\right)} = \frac{ab}{\lambda}$$

$$\text{Slope of PS} = \frac{\frac{\lambda}{b}-0}{0-a} = \frac{-\lambda}{ab}$$

$\Rightarrow$ QR $\perp$ PS

Let the point of intersection of QR and PS is T(h, k)

$\Rightarrow$ PT $\perp$ QT

$\Rightarrow$ $\left(\frac{k-0}{h-a}\right)\left(\frac{k-b}{h-0}\right) = -1$

$\Rightarrow$ $h(h-a) + k(k-b) = 0$

$\Rightarrow$ $h^2 + k^2 - ah - bk = 0$

So, Locus of T is $x^2 + y^2 - ax - by = 0$

Q.5 A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1 : n. Find the equation of the line, $(n \neq -1)$

Sol. The given points are A (1, 0) and B (2, 3). The required line divides the segment [AB] in the ratio 1 : n $(n \neq -1)$.

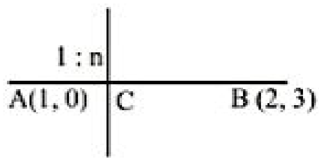
$\therefore$ Dividing point C is $\left(\frac{1\times 2 + n\times 1}{1+n}, \frac{1\times 3 + n\times 0}{n+1}\right) = \left(\frac{2+n}{1+n}, \frac{3}{n+1}\right)$

$\because$ Slope of line $AB = \frac{3-0}{2-1} = 3$

$\therefore$ Slope of the required line $(\perp AB) = -1/3$

As the line passes through C, therefore, its equation is $y - \frac{3}{n+1} = -\frac{1}{3}\left(x - \frac{n+2}{n+1}\right)$

Point slope form



or $\frac{1}{3}x + y - \frac{3}{n+1} - \frac{n+2}{3(n+1)} = 0$

or $\frac{1}{3}x + y - \left(\frac{9+n+2}{3(n+1)}\right) = 0$

or $(n+1)x + 3(n+1)y - (n+11) = 0$

Q.6 The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 liters of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many liters could he sell weekly at Rs 17/litre?

Sol. Let the milk store owner sell L litre of milk at Rs p per litre and let the linear relation ship between L and p be

$$L = mp + b \qquad \ldots(i)$$

when $p = 14, L = 980$

$\therefore \quad 980 = 14m + b \qquad \ldots(ii)$

when $p = 16, L = 1220$

$\therefore \quad 1220 = 16m + b \qquad \ldots(iii)$

Substituting (ii) from (iii), we get $240 = 2m$

$$\Rightarrow \quad m = \frac{240}{2} = 120$$

Substituting this value of m in (ii), we get

$$980 = 14 \times 120 + b \quad \Rightarrow \quad 980 - 1680 = b \quad \Rightarrow \quad b = -700$$

Substituting $m = 120$ and $b = -700$ in (i), we obtain $L = 120p - 700$.

when $p = 17$, then $L = 120 \times 17 - 700 = 2040 - 700 = 1340$

Hence, the man can sell 1340 litres of milk at Rs. 17 per litre.

Q.7 If p and q are the lengths of perpendiculars from the origin to the lines $x \cos\theta - y \sin\theta = 3 \cos 2\theta$ and $x \sec\theta + y \operatorname{cosec}\theta = 3$, respectively, then find the value of $p^2 + 4q^2$.

Sol. Here, $p = \dfrac{|0\cos\theta - 0\sin\theta - 3\cos 2\theta|}{\sqrt{(\cos\theta)^2 + (-\sin\theta)^2}}$

or $\quad p = \dfrac{|3\cos 2\theta|}{1} \quad$ or $\quad p = |3 \cos 2\theta| \qquad \ldots(i)$

and $\quad q = \dfrac{|0\sec\theta + 0\operatorname{cosec}\theta - 3|}{\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}}$

$$= \frac{|3|}{\sqrt{\dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta}}} = |3\cos\theta\sin\theta| = \frac{|3\sin 2\theta|}{2} \qquad (\because \sin 2\theta = 2\sin\theta\cos\theta)$$

$\Rightarrow \quad 2q = |3 \sin 2\theta| \qquad \ldots(ii)$

Squaring (i) and (ii) and adding

or $\quad p^2 + 4q^2 = (3\cos 2\theta)^2 + (3\sin 2\theta)^2$

or $\quad p^2 + 4q^2 = 9(\cos^2 2\theta + \sin^2 2\theta)$

or $\quad p^2 + 4q^2 = 9.$

Q.8 A variable line through origin meets two fixed lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ at P and Q. On it is taken a point R. If $\frac{2}{OR} = \frac{1}{OP} + \frac{1}{OQ}$ then prove that locus of R is also a st. line.

Sol. $a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0$

Let the point R be (h, k)

Equation of line through origin with inclination θ is

$$\frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta} = r_i \qquad \text{where } i = 1, 2, 3$$

$$r_1 = OP, \qquad r_2 = OQ \quad \text{and} \quad r_3 = OR$$

P lies on $a_1x + b_1y + c_1 = 0$

$$a_1r_1 \cos\theta + b_1r_1 \sin\theta + c_1 = 0$$

$$r_1 = \frac{-c_1}{a_1\cos\theta + b_1\sin\theta}$$

Similarly $r_2 = \dfrac{-c_2}{a_2\cos\theta + b_2\sin\theta}$

$$R = (r_3\cos\theta, r_3\sin\theta)$$

$$h = r_3\cos\theta, \quad k = r_3\sin\theta$$

If $\dfrac{2}{OR} = \dfrac{1}{OP} + \dfrac{1}{OQ}$

$$\frac{2}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}$$

$$\frac{2}{r_3} = -\left(\frac{a_1\cos\theta + b_1\sin\theta}{c_1} + \frac{a_2\cos\theta + b_2\sin\theta}{c_2}\right)$$

$$2 = -\left[\frac{a_1r_3\cos\theta + b_1r_3\sin\theta}{c_1}\right] - \left[\frac{a_2r_3\cos\theta + b_2r_3\sin\theta}{c_2}\right]$$

$$2 = -\left[\frac{a_1h + b_1k}{c_1}\right] - \left[\frac{a_2h + b_2k}{c_2}\right]$$

$$-2 = h\left(\frac{a_1}{c_1} + \frac{a_2}{c_2}\right) + k\left(\frac{b_1}{c_1} + \frac{b_2}{c_2}\right)$$

$Ax + By + 2 = 0$ which is the equation of line.

Q.9 If $bc \neq ad$ and the lines $\left.\begin{matrix}(\sin 3\theta)x + ay + b = 0 \\ (\cos 2\theta)x + cy + d = 0 \\ 2x + (a+2c)y + (b+2d) = 0\end{matrix}\right\}$ are concurrent then find the number of values of θ in $[0, 2\pi]$.

Sol. If the lines are concurrent then

$$\begin{vmatrix} \sin 3\theta & a & b \\ \cos 2\theta & c & d \\ 2 & (a+2c) & (b+2d) \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - R_1 - 2R_2$

$$\begin{vmatrix} \sin 3\theta & a & b \\ \cos 2\theta & c & d \\ 2 - \sin 3\theta - 2\cos 2\theta & 0 & 0 \end{vmatrix} = 0$$

$$(2 - \sin\theta - 2\cos 2\theta)(ad - bc) = 0$$

since $bc \neq ad$, hence

$$2 - \sin 3\theta - 2\cos 2\theta = 0$$
$$2 - (3\sin\theta - 4\sin^3\theta) - 2(1 - 2\sin^2\theta) = 0$$
$$-3\sin\theta + 4\sin^3\theta + 4\sin^2\theta = 0$$
$$\sin\theta\,[4\sin^2\theta + 4\sin\theta - 3] = 0$$
$$\sin\theta\,(2\sin\theta - 1)(2\sin\theta + 3) = 0$$
$$\sin\theta = 0,\ \sin\theta = \frac{1}{2},\ \sin\theta = \frac{-3}{2} \text{ (rejected)}$$

$\Rightarrow \quad \theta = 0, \pi, 2\pi \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore$ number of values of θ in $[0, 2\pi]$ is 5.

Q.10 Let ABC be a given isosceles triangle with AB = AC. Sides AB and AC are extended up to E and F, respectively, such that $BE \times CF = AB^2$. Prove that the line EF always passes through a fixed point.

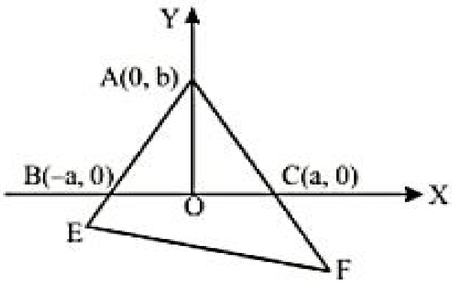
Sol. Let ABC be the triangle having vertices $(-a, 0)$, $(0, b)$ and $(a, 0)$. Now,

$$BE \times CF = AB^2 \quad \Rightarrow \quad \frac{BE}{AB} = \frac{AB}{CF} = \lambda \text{ (let)}$$

$$\Rightarrow \quad \frac{BE}{AB} = \frac{AB}{CF} = \lambda$$

Hence, the coordinates of E and F are $(-a(\lambda + 1), -\lambda b)$ and $(a(1 + 1/\lambda) - b/\lambda)$. Equation of line EF is

$$y + \lambda b = \frac{-\lambda b + \frac{b}{\lambda}}{-a(\lambda+1) - \frac{a(\lambda+1)}{\lambda}}\,[x + a(\lambda + 1)]$$



or $\quad y+\lambda b=\dfrac{\frac{b}{\lambda}(1-\lambda^2)}{-\frac{a(\lambda+1)}{\lambda}[1+\lambda]}\,[x+a(\lambda+1)]$

or $\quad y+\lambda b=\dfrac{b(\lambda-1)}{a(\lambda+1)}\,[x+a(\lambda+1)]$

or $\quad a(\lambda+1)y+ab\,\lambda\,(\lambda+1)=b\,(\lambda-1)\,x+ab\,(\lambda^2-1)$

or $\quad (bx+ay+ab)-\lambda\,(bx-ay-ab)=0$

which is the equation of a family of lines passing through the point of intersection of the lines $bx+ay+ab=0$ and $bx-ay-ab=0$, the point of intersection being $(0,-b)$. Hence, the line EF passes through a fixed point.

Q.11 The straight line $2x+3y+1=0$ bisects the angle between two straight lines, one of which is $3x+2y+4=0$. Find the equation of the other straight line.

Sol. The given line is $\quad 3x+2y+4=0 \quad$ ...(i)

and bisector is $\quad 2x+3y+1=0 \quad$ ...(ii)

Since the other line passes through the intersection of (i) and (ii), its equation is

$$2x+3y+1+k\,(3x+2y+4)=0. \qquad \text{...(iii)}$$

Since, P lies on one bisector

$\therefore$ length of perpendicular from P on (i) = length of perpendicular from P on (iii)

$$\Rightarrow \quad \left|\frac{3.1+2.(-1)+4}{\sqrt{9+4}}\right|=\frac{|(2+3k).1+(3+2k)(-1)+(1+4k)|}{\sqrt{(2+3k)^2+(3+2k)^2}}$$

$$\Rightarrow \quad \frac{5}{\sqrt{15}}=\frac{5|k|}{\sqrt{13k^2+24k+13}}$$

$$\Rightarrow \quad 13k^2+24k+13=13k^2$$

$$\Rightarrow \quad 24k+13=0 \quad \Rightarrow \quad k=-\frac{13}{24}$$

Substituting this value of k in (iii), we get

$$\left(2-\frac{13}{8}\right)x+\left(3-\frac{13}{12}\right)y+\left(1-\frac{13}{6}\right)=0 \Rightarrow \frac{3}{8}x+\frac{23}{12}y-\frac{7}{6}=0$$

Q.12 Show that all chords of the curve $3x^2-y^2-2x+4y=0$ subtending right angles at the origin pass through a fixed point.find also the coordinates of the fixed point.

Sol. $3x^2-y^2-2x+4y=0$

Let the point through which it passes is (h, k)

Equation of line through (h, k)

$$y-k=m\,(x-h)$$

$$y-mx=k-mh$$

Homogenising the curve with the line, then

$$3x^2 - y^2 - 2x\left(\frac{y-mx}{k-mh}\right) + 4y\left(\frac{y-mx}{k-mh}\right) = 0$$

$$x^2\left(3 + \frac{2m}{k-mh}\right) + y^2\left(\frac{4}{k-mh} - 1\right) + xy\left(\frac{-2}{k-mh}\frac{-4m}{k-mh}\right) = 0$$

If they subtend a right angle at origin then, coeffi. of x^2 + coeffi. of $y^2 = 0$

$$3 + \frac{2m}{k-mh} + \frac{4}{k-mh} - 1 = 0$$

$$2 + \frac{2m}{k-mh} + \frac{4}{k-mh} = 0$$

$k - mh + m + 2 = 0$

$k + 2 + m(1 - h) = 0$

$k = -2$ and $h = 1$

Hence the fixed point is $(1, -2)$

Q.13 A straight line is drawn from the point (1, 0) to intersect the curve $x^2 + y^2 + 6x - 10y + 1 = 0$ such that the intercept made by it on the curve subtend a right angle at the origin. Find the equation of the line.

Sol. Let the equation of line is $y = m(x - 1)$

Equation of curve $x^2 + y^2 + 6x - 10y + 1 = 0$

Homogenising the curve with the line, then

$$x^2 + y^2 + 6x\left(\frac{mx-y}{m}\right) - 10y\left(\frac{mx-y}{m}\right) + \left(\frac{mx-y}{m}\right)^2 = 0$$

coeffi. of $x^2 = 1 + 6 + 1 = 8$

coeffi. of $y^2 = 1 + \frac{10}{m} + \frac{1}{m^2}$

If it subtends a right angle at origin the

$$8 + 1 + \frac{10}{m} + \frac{1}{m^2} = 0$$

$$\frac{1}{m^2} + \frac{10}{m} + 9 = 0$$

$9m^2 + 10m + 1 = 0$

$9m^2 + 9m + m + 1 = 0$

$(m + 1)(9m + 1) = 0$

$m = -1$ or $m = \frac{-1}{9}$

Equation of lines are

$y = -1(x - 1)$ or $y = \frac{-1}{9}(x - 1)$

$x + y = 1$ or $x + 9y = 1$