DETERMINANT

HISTORICAL DEVELOPMENT :

Development of determinants took place while mathematicians were trying to solve a system of simultaneous linear equations.

e.g. $$\left.\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}\right] \Rightarrow x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

Mathematicians defined the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ as determinant of order 2 and the four numbers arranged in row and column were called its elements. Its value was taken as $a_1b_2 - a_2b_1$ which is the same as denominator.

This kind of definition helped then to state the solution of the simultaneous equation as

$$x = \frac{D_1}{D} \text{ and } y = \frac{D_2}{D} \text{ where } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}; D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}; D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Note : A determinant of order 1 is the number itself.

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order 3. Its value can be found as

$$D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad \text{or}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

In the way we can expand a determinant in 6 ways using elements of $R_1, R_2, R_3, C_1, C_2, C_3$.

COFACTOR AND MINORS OF AN ELEMENT :

Minors :

Minors of an element is defined as the minor determinant obtained by deleting a particular row or column in which that element lies. e.g. in the determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ minor of } a_{12} \text{ denoted as } M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ and so on}$$

Cofactor :

It has no separate identity and is related to the cofactor as

$C_{ij} = (-1)^{i+j} M_{ij}$ where 'i' denotes the row and 'j' denotes the column.

Hence the value of a determinant of order three in terms of 'Minor' and 'Cofactor' can be written as

$$D = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} \text{ or}$$
$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

Note : Determinant of order 3 will have 9 minors and each minor will be a determinant of order 2 and a determinant of order 4 will have 16 minors and each minor will be determinant of order 3.

Illustration :

The value of $\begin{vmatrix} a+1 & a-2 \\ a+2 & a-1 \end{vmatrix}$ *is*

(A) $2a^2$ *(B)* 0 *(C)* -3 *(D)* 3

Sol. $\begin{vmatrix} a+1 & a-2 \\ a+2 & a-1 \end{vmatrix}$

$= (a+1)(a-1) - (a+2)(a-2)$

$= (a^2-1) - (a^2-4) = 3$ *Ans. [D]*

Illustration :

The value of $\begin{vmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{vmatrix}$ *is*

(A) 2 *(B)* -1 *(C)* 0 *(D)* $\cos 2\theta$

Sol. $\begin{vmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{vmatrix}$

$= (1+\cos\theta)(1-\cos\theta) - (\sin\theta)(\sin\theta)$

$= 1 - \cos^2\theta - \sin^2\theta = 0$ *Ans. [C]*

Illustration :

The value of $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ *is*

(A) 213 *(B)* -231 *(C)* 231 *(D)* 39

Sol. $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = 1\begin{vmatrix} 3 & 6 \\ -7 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 3 \\ 2 & 7 \end{vmatrix}$

$= 1(3 \times 9 - 6(-7)) - 2(-4 \times 9 - 2 \times 6) + 3[(-4)(-7) - 3 \times 2]$

$= (27 + 42) - 2(-36 - 12) + 3(28 - 6) = 231$ *Ans. [C]*

PROPERTIES OF DETERMINANTS :

P-1: The value of a determinant remains unaltered, if the rows & columns are inter changed. e.g. if

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$$

D & D' are transpose of each other. If $D' = -D$ then it is **Skew symmetric** determinant but $D' = D \Rightarrow 2D = 0 \Rightarrow D = 0 \Rightarrow$ Skew symmetric determinant of third order has the value zero.

Remember: Without expanding prove that the value of the determinant

$$D = \begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$$

Note : The value of a skew symmetric determinant of odd order is zero.

P-2: If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

P-3: If a determinant has any two rows (or columns) identical , then its value is zero.

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then it can be verified that $D = 0$.

P-4: If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = KD$

P-5: If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants. e.g.

$$\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

P-6: The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 + na_1 & b_3 + nb_1 & c_3 + nc_1 \end{vmatrix}$. Then $D' = D$.

Note : that while applying this property **atleast one row (or column)** must remain unchanged.

P-7: If by putting $x = a$ the value of a determinant vanishes then $(x - a)$ is a factor of the determinant.

P-8: In a determinant the sum of the product's of the element's of any row (column) with their corresponding cofactor's is equal to the value of determinant.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let A_i, B_i, C_i be the cofactor's of the element's a_i, b_i, c_i $(i = 1, 2, 3)$

Then $a_1A_1 + b_1B_1 + c_1C_1 = D$

$a_2A_2 + b_2B_2 + c_2C_2 = D$

Similarly,

In a determinant the sum of the product's of the element's of any row(column) with the cofactor's of corresponding element's of any other row (column) is zero.

i.e. $a_1A_2 + b_1B_2 + c_1C_2 = 0$ or $a_2A_1 + b_2B_1 + c_2C_1 = 0.$

Remember:

Factorisation in respect the following determinants are very useful and should be remembered.

SOME IMPORTANT DETERMINANTS TO REMEMBER :

(1) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)$

Proof :

Let $D = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$\Rightarrow \quad D = \begin{vmatrix} 0 & x - y & x^2 - y^2 \\ 0 & y - z & y^2 - z^2 \\ 1 & z & z^2 \end{vmatrix}$

$D = (x - y)(y - z) \begin{vmatrix} 0 & 1 & x + y \\ 0 & 1 & y + z \\ 1 & z & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)$

$D = (x - y)(y - z)(z - x).$

Hence proved.

(2) $$\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

Proof :

Let $$D = \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$$

Apply $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$. Given

$$D = \begin{vmatrix} 0 & x-y & x^3-y^3 \\ 0 & y-z & y^3-z^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)\begin{vmatrix} 0 & 1 & x^2+xy+y^2 \\ 0 & 1 & y^2+yz+z^2 \\ 1 & z & z^3 \end{vmatrix}$$

$$D = (x-y)(y-z)[y^2+yz+z^2-x^2-xy-y^2]$$
$$D = (x-y)(y-z)[y(z-x)+z^2-x^2]$$
$$= (x-y)(y-z)(z-x)(x+y+z).$$

(3) $$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Proof :

Let $$D = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & y^4 \end{vmatrix}$$

Apply $R_1 \to xR_1 ; R_2 \to yR_2, R_3 \to zR_3$ divide by xyz balancing.

$$D = \frac{1}{xyz}\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$.

$$= \begin{vmatrix} 0 & x^2-y^2 & x^3-y^3 \\ 0 & y^2-z^2 & y^2-z^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(xy+yz+zy)$$

(4) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc) < 0$ if a, b, c are different and positive

Proof :

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\,[bc - a^2] - [b^2 - ac] + c(ab - c^2)$$

$$= 3abc - (a^3 + b^3 + c^3).$$

Illustration :

Prove that $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc$

Sol. $L.H.S. = \begin{vmatrix} 0 & c & b \\ -2a & c+a & a \\ -2a & a & a+b \end{vmatrix}$ $\quad [C_1 \to C_1 - (C_2 + C_3)]$

$$= -2a\begin{vmatrix} 0 & c & b \\ 1 & c+a & a \\ 1 & a & a+b \end{vmatrix} = -2a\begin{vmatrix} 0 & c & b \\ 0 & c & -b \\ 1 & a & a+b \end{vmatrix} \quad [R_2 \to R_2 - R_3]$$

$$= -2a\begin{vmatrix} c & b \\ c & -b \end{vmatrix} \quad \text{[expanding along } C_1]$$

$$= -(-2a)(-2bc) = 4abc = R.H.S.$$

Illustration :

Show that $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$

Sol. *We have*

$$L.H.S. = \begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} \quad [C_1 \to C_1 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & a+b & a \\ 1 & b+c & b \\ 1 & c+a & c \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a+b & a \\ 0 & c-a & b-a \\ 0 & c-b & c-a \end{vmatrix} \quad \begin{bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$= (a+b+c)\begin{vmatrix} c-a & b-a \\ c-b & c-a \end{vmatrix} \quad \text{[expending along } C_1]$$

$$= (a + b + c)\,[(c - a)^2 - (c - b)(b - a)]$$

$$= (a + b + c)\,[(c^2 + a^2 - 2ac)^2 - (cb - ca - b^2 + ab)]$$

$$= (a + b + c)\,[a^2 + b^2 + c^2 - ab - bc - ca]$$

$$= a^3 + b^3 + c^3 - 3abc = R.H.S.$$

Illustration :

$$\text{Show that } \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$$

Sol. $$L.H.S. = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \qquad [R_1 \to R_1 + R_2 + R_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix} \qquad \begin{bmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix}$$

$$= (a+b+c)\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix} \qquad [\text{expanding along } C_1]$$

$$= (a+b+c)(a+b+c)^2 = (a+b+c)^3 = R.H.S.$$

Illustration :

$$\text{Show that } \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$$

Sol. $$L.H.S. = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix} \qquad \begin{bmatrix} C_1 \to aC_1 \\ C_2 \to bC_2 \\ C_3 \to cC_3 \end{bmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} \qquad \begin{bmatrix} \text{taking } a,b,c \text{ common from} \\ C_1, C_2, C_3 \text{ respectively} \end{bmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} \qquad [C_1 \to C_1 + C_2 + C_3]$$

$$=(1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$=(1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$= (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad [\text{expanding along } C_1]$$

$$= 1 + a^2 + b^2 + c^2 = R.H.S.$$

Illustration :

If none of a, b, c is zero, show that $\Delta = \begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab + bc + ca)^3$.

Sol. *We have*

$$\Delta = \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & bc^2+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix} \qquad \begin{bmatrix} R_1 \to aR_1 \\ R_2 \to bR_2 \\ R_3 \to aR_3 \end{bmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} -bc & ab+bc & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix} \qquad \begin{bmatrix} \text{taking } a, b, c, \text{ common from} \\ C_1, C_2, C_3 \text{ respectively} \end{bmatrix}$$

$$= (ab + bc + ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix} \qquad [R_1 \to R_1 + R_2 + R_3]$$

$$= (ab + bc + ca)\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -(ab+bc+ca) & 0 \\ ac+bc & 0 & -(ab+bc+ca) \end{vmatrix}$$

$$[C_1 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$= (ab + bc + ca)\begin{vmatrix} -(ab+bc+ca) & 0 \\ 0 & -(ab+bc+ca) \end{vmatrix} \qquad [\text{expanding along } R_1]$$

$$= (ab + bc + ca)^3$$

Illustration :

Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

Sol. *Since, the answer contains abc, therefore, taking a, b, c common from R_1, R_2, R_3 respectively, we have*

$$\Delta = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} \quad [R_1 \to R_2 + R_2 + R_3]$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 1 \\ \frac{1}{b} & 1 & \frac{1}{b} \\ \frac{1}{c} & 0 & \frac{1}{c}+1 \end{vmatrix} \quad [C_2 \to C_2 - C_1]$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 \\ \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} \quad [\text{expanding along } C_2]$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Illustration :

Show that

$$\begin{vmatrix} ax-by-cz & ay+bx & cx+cz \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix} = (x^2+y^2+z^2)\,(a^2+b^2+c^2)\,(ax+by+cz).$$

Sol. *We have*

$$\Delta = \frac{1}{a}\begin{vmatrix} x(a^2+b^2+c^2) & ay+bx & cx+cz \\ y(a^2+b^2+c^2) & by-cz-ax & bz+cy \\ z(a^2+b^2+c^2) & bz+cy & cz-ax-by \end{vmatrix} \qquad [C_1 \to aC_1 + bC_2 + cC_3]$$

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} x & ay+bx & cx+cz \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$= \frac{1}{ax}(a^2+b^2+c^2)\begin{vmatrix} x^2+y^2+z^2 & b(x^2+y^2+z^2) & c(x^2+y^2+z^2) \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$[R_1 \to xR_1 + yR_2 + zR_3]$$

$$= \frac{1}{ax}(a^2+b^2+c^2)\,(x^2+y^2+z^2)\begin{vmatrix} 1 & b & c \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$= \frac{1}{ax}(a^2+b^2+c^2)\,(x^2+y^2+z^2)\begin{vmatrix} 1 & b & c \\ 0 & -cz-ax & bz \\ 0 & cy & -ax-by \end{vmatrix}$$

$$[R_2 \to R_2 - yR_1,\ R_3 \to R_3 - zR_1]$$

$$= \frac{1}{ax}(a^2+b^2+c^2)\,(x^2+y^2+z^2)\begin{vmatrix} 1 & b & c \\ 0 & -cz-ax & bz \\ 0 & cy & -ax-by \end{vmatrix}$$

$$= \frac{1}{ax}(a^2+b^2+c^2)\,(x^2+y^2+z^2)[(cz+ax)\,(ax+by) - bcyz]$$

$$= \frac{1}{ax}(a^2+b^2+c^2)\,(x^2+y^2+z^2)[acxz + a^2x^2 + bcyz + abxy - bcyz]$$

$$= (a^2+b^2+c^2)\,(x^2+y^2+z^2)(ax+by+cz).$$

Illustration :

If A, B, C are the angle of a triangle and $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0$

prove that ΔABC *must be isosceles*

Sol. *Let*

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B-\sin A & \sin C-\sin A \\ \sin A+\sin^2 A & (\sin B-\sin A)(\sin B+\sin A+1) & (\sin C-\sin A)(\sin C+\sin A+1) \end{vmatrix}$$

$$[C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1]$$

$= (\sin B - \sin A)\ (\sin C - \sin A)\ (\sin C - \sin B)$

Now, since Δ *is given to be zero, therefore we have*

$$(\sin B - \sin A)\ (\sin C - \sin A)\ (\sin C - \sin B) = 0$$

i.e.	$\sin B - \sin A = 0$	*or*	$\sin C - \sin A = 0$	*or*	$\sin C - \sin B = 0$
i.e.	$\sin B = \sin A$	*or*	$\sin C = \sin A$	*or*	$\sin C = \sin B$
i.e.	$B = A$	*or*	$C = A$	*or*	$C = B$

In all the three cases, the triangle will be isosceles.

Illustration :

Without expanding the determinant at any stage show that $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B.$

Sol. $L.H.S. = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$ $\quad [R_1 \to R_1 + R_3 - R_2]$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2x^2+2 & 3 & 3x+3 \\ x^2+4 & 0 & 2x-1 \end{vmatrix} \quad \begin{bmatrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - C_3 \end{bmatrix}$$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3x-3 \\ 4 & 0 & 2x-1 \end{vmatrix} \quad \begin{bmatrix} R_2 \to R_2 - \dfrac{x^2}{2} R_1 \\ R_3 \to R_3 - \dfrac{x^2}{4} R_1 \end{bmatrix}$$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3x \\ 4 & 0 & 2x \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix}$$

$$= x\begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3 \\ 4 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix} = xA + B = R.H.S.$$

Illustration :

If $f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix}$ *then show that* $f(x)$ *is linear in x.*

Hence, deduce that $f(0) = \dfrac{bg(a)-ag(b)}{(b-a)}$ *where* $g(x) = (c_1-x)(c_2-x)(c_3-x)$.

Sol. *We have*

$$f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix} \qquad \text{......(i)}$$

$$= \begin{vmatrix} x+c_1 & a-c_1 & 0 \\ x+b & c_2-b & a-c_2 \\ x+b & 0 & c_2-b \end{vmatrix} \qquad \begin{bmatrix} C_3 \to C_3 - C_2 \\ C_2 \to C_2 - C_1 \end{bmatrix}$$

$$= x\begin{vmatrix} 1 & a-c_1 & 0 \\ 1 & c_2-b & a-c_2 \\ 1 & 0 & c_3-b \end{vmatrix} + \begin{vmatrix} c_1 & a-c_1 & 0 \\ b & c_2-b & a-c_2 \\ b & 0 & c_3-b \end{vmatrix}$$

which proves that $f(x)$ *is linear.*

Let $\quad f(x) = Px + Q$

Then $\quad f(-a) = -aP + Q \qquad$ ......(ii)

$\quad f(-b) = -bP + Q \qquad$ ......(iii)

and $\quad f(0) = Q = \dfrac{bf(-a)-af(-b)}{(b-a)} \qquad$ *[from results (ii) and (iii)]* $\qquad$ ......(iv)

From equation (i), we have

$$f(-a) = \begin{vmatrix} c_1-a & 0 & 0 \\ b-a & c_2-a & 0 \\ b-a & b-a & c_3-a \end{vmatrix} = (c_1-a)(c_2-a)(c_3-a)$$

Similarly, $\quad f(-b) = (c_1-b)(c_2-b)(c_3-b)$

Since, $\quad g(x) = (c_1-x)(c_3-x)(c_2-x)$, *therefore, we can see that*

$g(a) = f(-a)$ *and* $g(b) = f(-b)$

Hence, from result (iv), we have

$$f(0) = \frac{bg(a)-ag(b)}{(b-a)}$$

Illustration :

$$\begin{vmatrix} a_1-b_1+x & a_1-b_2 & a_1-b_3 \\ a_2-b_1 & a_2-b_2+x & a_2-b_3 \\ a_3-b_1 & a_3-b_2 & a_3-b_3+x \end{vmatrix} = x^3+x^2\sum_{i=1}^{3}(a_i-b_i)+x\sum_{1\le i<j\le 3}(a_i-a_j)(b_i-b_j).$$

Sol. $\Delta = \begin{vmatrix} a_1-b_1+x & a_1-b_2 & a_1-b_3 \\ a_2-b_1 & a_2-b_2+x & a_2-b_3 \\ a_3-b_1 & a_3-b_2 & a_3-b_3+x \end{vmatrix} =$

$$\begin{vmatrix} a_1 & a_1-b_2 & a_1-b_3 \\ a_2 & a_2-b_2+x & a_2-b_3 \\ a_3 & a_3-b_2 & a_3-b_1+x \end{vmatrix} - \begin{vmatrix} b_1 & a_1-b_2 & a_1-b_3 \\ b_1 & a_2-b_2+x & a_2-b_3 \\ b_1 & a_3-b_2 & a_3-b_1+x \end{vmatrix} + \begin{vmatrix} x & a_1-b_2 & a_1-b_3 \\ 0 & a_2-b_2+x & a_2-b_3 \\ 0 & a_3-b_2 & a_3-b_3+x \end{vmatrix}$$

$= \Delta_1 - \Delta_2 + \Delta_3$ *(say)* ......(i)

Now, we have

$$\Delta_1 = \begin{vmatrix} a_1 & -b_2 & -b_3 \\ a_2 & -b_2+x & -b_3 \\ a_3 & -b_2 & -b_3+x \end{vmatrix} \qquad \begin{bmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix}$$

$$= \begin{vmatrix} a_1 & -b_2 & -b_3 \\ a_2-a_1 & x & 0 \\ a_3-a_1 & 0 & x \end{vmatrix} \qquad \begin{bmatrix} R_3 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$= xb_3(a_3 - a_1) + x^2a_1 + xb_2(a_2 - a_1)$ *[expanding along C_3]*

$$\Delta_2 = b_1\begin{vmatrix} 1 & a_1-b_2 & a_1-b_3 \\ 1 & a_2-b_2+x & a_2-b_3 \\ 1 & a_3-b_2 & a_3-b_3+x \end{vmatrix}$$

$$= b_1\begin{vmatrix} 1 & a_1 & a_1 \\ 1 & a_2+x & a_2 \\ 1 & a_3 & a_3+x \end{vmatrix} \qquad \begin{bmatrix} C_2 \to C_2 + b_2C_1 \\ C_3 \to C_3 + b_3C_1 \end{bmatrix}$$

$$= b_1\begin{vmatrix} 1 & a_1 & a_1 \\ 0 & a_2-a_1+x & a_2 \\ 0 & a_3-a_1 & a_3-a_1+x \end{vmatrix} \qquad \begin{bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$= b_1\begin{vmatrix} a_2-a_1+x & a_2-a_1 \\ a_3-a_1 & a_2-a_1+x \end{vmatrix} \qquad [\text{expanding along } C_1]$$

$$= b_1\begin{vmatrix} a_2-a_1+x & -x \\ a_3-a_1 & x \end{vmatrix} \qquad [C_2 \to C_2 - C_1]$$

$= b_1x(a_2 - a_1 + x + a_3 - a_1)$

$= x^2b_1 + xb_1(a_2 + a_3 + x - 2a_1)$

and $\Delta_3 = x[(a_2 - b_2 + x)(a_3 - b_3 + x) - (a_3 - b_2)(a_2 - b_3)]$ [expanding along C_1]

$$= x[x^2 + x(a_2 - b_2 + a_3 - b_3) + (a_2 - b_2)(a_3 - b_3) - (a_3 - b_2)(a_2 - b_3)]$$

$$= x^3 + x^2(a_2 - b_2 + a_3 - b_3) + x(a_2 - a_3)(b_2 - b_3)$$

Putting the values of $\Delta_1, \Delta_2, \Delta_3$ in equation (i), we have

$$\Delta = [xb_3(a_3 - a_1) + x^2 a_1 + xb_2(a_2 - a_1)] - [x^2 b_1 + xb_1(a_2 + a_3 - 2a_1)] + [x^3 + x^2(a_2 - b_2 + a_3 - b_3) + x(a_2 - a_3)(b_2 - b_3)]$$

$$= x[(a_1 - a_2)(b_1 - b_2) + (a_1 - a_3)(b_1 - b_3) + (a_2 - a_3)(b_2 - b_3)] + x^2(a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3)] + x^3$$

$$= x^3 + x^2 \sum_{i=1}^{3}(a_i - b_j) + x \sum_{1 \le i \le j \le 3}(a_i - a_j)(b_i - b_j)$$

Illustration :

If a, b, c are distinct, solve the equation $\begin{vmatrix} x^2-a^2 & x^2-b^2 & x^2-c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x+a)^3 & (x+b)^3 & (x+c)^3 \end{vmatrix} = 0.$

Sol. $\Delta = \begin{vmatrix} x^2-a^2 & x^2-b^2 & x^2-c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x+a)^3 & (x+b)^3 & (x+c)^3 \end{vmatrix}$ $[R_3 \to R_3 - R_2]$

$$= 2\begin{vmatrix} x^2-a^2 & x^2-b^2 & x^2-c^2 \\ x^3+3xa^2 & x^3+3xb^2 & x^3+3xc^3 \\ 3x^2a+a^3 & 3x^2b+b^3 & 3x^2c+c^3 \end{vmatrix} \quad [\text{taking 2 common from } R_1 \text{ and applying } R_2 \to R_2 + R_3]$$

$$= 2x\begin{vmatrix} x^2-a^2 & x^2-b^2 & x^2-c^2 \\ x^2+3a^2 & x^2+3b^2 & x^2+3c^2 \\ 3x^2a+a^3 & 3x^2b+b^3 & 3x^2c+c^3 \end{vmatrix} \quad [\text{taking } x \text{ common from } R_2]$$

$$= 2x\begin{vmatrix} x^2-a^2 & a^2-b^2 & a^2-c^2 \\ x^2+3a^2 & 3(b^2-a^2) & 3(c^2-a^2) \\ 3x^2a+a^3 & 3x^2(a-b)+b^3-a^3 & 3x^2(c-a)+c^3-a^3 \end{vmatrix} \quad [C_2 \to C_2 - C_1, C_3 \to C_3 - C_1]$$

$$= 2x(b-a)(c-a)\begin{vmatrix} x^2-a^2 & -(a+b) & -(a+c) \\ x^2+3a^2 & 3(b+a) & 3(a+c) \\ 3x^2a+a^3 & 3x^2+b^2+a^2+ba & 3x^2+c^2+a^2+ac \end{vmatrix}$$

$$= 2x(b-a)(c-a)\begin{vmatrix} x^2-a^2 & -(a+b) & -(a+c) \\ 4x^2 & 0 & 0 \\ 3x^2a+a^3 & 3x^2+b^2+a^2+ba & 3x^2+c^2+a^2+ac \end{vmatrix} \quad [R_2 \to R_2 + 3R_1]$$

$$= 8x^3(b-a)(c-a)\begin{vmatrix} a+b & a+c \\ 3x^2+b^2+a^2+ba & 3x^2+c^2+a^2+ac \end{vmatrix} \qquad [\text{expanding along } R_3]$$

$$= 8x^3(b-a)(c-a)\begin{vmatrix} a+b & c-b \\ 3x^2+b^2+a^2+ba & c^2-b^2+ca-ab \end{vmatrix} \qquad [C_2 \to C_2 - C_1]$$

$$= 8x^3(b-a)(c-b)(c-a)\begin{vmatrix} a+b & 1 \\ 3x^2+b^2+a^2+ba & a+b+c \end{vmatrix}$$

$= 8x^3 (b - a) (c - a)(c - b) [(a + b) (a + b + c) - 3x^2 - b^2 - a^2 - ba]$
$= 8x^3 (b - a) (c - a) (c - b) [(ab + bc + ca) - 3x^2]$
Now, since, Δ is given to be zero, therefore, we have
$8x^3 (b - a) (c - a) (c - b) [(ab + bc + ca) - 3x^2] = 0$

gives $x = 0, \pm\sqrt{\frac{1}{3}(ab+bc+ca)}$ $\quad [\because$ *a, b, c are distinct*, $\therefore (b-a)(c-a)(c-b) \neq 0]$

Illustration :

Find the coefficient of x in the determinant $\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix} = 0.$

Sol. *If $f(x)$ be a polynomial in x, then coefficient of x^n in* $f(x) = \dfrac{f^n(0)}{n!}$
(from differential calculus)

Let the given determinant be denoted by $f(x)$, then

$$f'(x) = \begin{vmatrix} a_1b_1(1+x)^{a_1b_1-1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ a_2b_1(1+x)^{a_2b_1-1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ a_3b_1(1+x)^{a_3b_1-1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix} + \begin{vmatrix} (1+x)^{a_1b_1} & a_1b_2(1+x)^{a_1b_2-1} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & a_2b_2(1+x)^{a_2b_2-1} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & a_3b_2(1+x)^{a_3b_2-1} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+ \begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & a_1b_3(1+x)^{a_1b_3-1} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & a_2b_3(1+x)^{a_2b_3-1} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & a_3b_3(1+x)^{a_3b_3-1} \end{vmatrix}$$

Thus, we have

$$f'(0) = \begin{vmatrix} a_1b_1 & 1 & 1 \\ a_2b_1 & 1 & 1 \\ a_3b_1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & a_1b_2 & 1 \\ 1 & a_2b_2 & 1 \\ 1 & a_3b_2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & a_1b_3 \\ 1 & 1 & a_2b_3 \\ 1 & 1 & a_3b_3 \end{vmatrix} = 0$$

Hence, we have

coeff. of x in $f(x) = \dfrac{f'(0)}{1!} = 0$

Practice Problem

Q.1 If x, y, z are postive and none of them is 1, then the value of the following determinant is

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} \text{ is}$$

(A) 1 (B) 0 (C) 2 (D) –2

Q.2 If $\Delta_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+1 & n^2+n \\ 2k-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{k=1}^{n} \Delta_k = 56$, then n is equal to

(A) 4 (B) 6 (C) 8 (D) 7

Q.3 If $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = ma^n b^n c^n$ then

(A) $m+n=6$ (B) $m+n=4$ (C) $m-n=0$ (D) $m-n=2$

Q.4 If $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^n$ (where $n \in N$), find the value of n.

Q.5 If $x \neq y \neq z$ and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ then, find the value of xyz.

Answer key

Q.1	B	Q.2	D	Q. 3	A, D	Q.4	3	Q.5	–1

MULTIPLICATION OF TWO DETERMINANTS :

1. Row by Row multiplication :

(i^{th} row of Δ_1) × (j^{th} row Δ_2)

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_3 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

2. Row by column Multiplication :

(i^{th} row of Δ_1) × (j^{th} column Δ_2)

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 & a_1\gamma_1 + b_1\gamma_2 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 & a_2\gamma_1 + b_2\gamma_2 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\alpha_2 + c_3\alpha_3 & a_3\beta_1 + b_3\beta_2 + c_3\beta_3 & a_3\gamma_1 + b_3\gamma_2 + c_3\gamma_3 \end{vmatrix}$$

3. Column by Row Multiplication :

(i^{th} column of Δ_1) × (j^{th} row of Δ_2)

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + a_2\beta_1 + a_3\gamma_1 & a_1\alpha_2 + a_2\beta_2 + a_3\gamma_2 & a_1\alpha_3 + a_2\beta_3 + a_3\gamma_3 \\ b_1\alpha_1 + b_2\beta_1 + b_3\gamma_1 & b_1\alpha_2 + b_2\beta_2 + b_2\gamma_2 & b_1\alpha_3 + b_2\beta_3 + b_3\gamma_3 \\ c_1\alpha_1 + c_2\beta_1 + c_3\gamma_1 & c_1\alpha_2 + c_2\beta_2 + c_3\gamma_2 & c_1\alpha_3 + c_2\beta_3 + c_3\gamma_3 \end{vmatrix}$$

4. Column by column Multiplication :

(i^{th} column of Δ_1) × (j^{th} column of Δ_2)

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 & a_1\beta_1 + a_2\beta_2 + a_3\beta_3 & a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 \\ b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 & b_1\beta_1 + b_2\beta_2 + b_3\beta_3 & b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3 \\ c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 & c_1\beta_1 + c_2\beta_2 + c_3\beta_3 & c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3 \end{vmatrix}$$

But we prefer row by column multiplication.

To express a determinants as a product of two determinants :

To express a determinant as product of two determinants one requires a lot of practice and this can be done only by inspection and trial. It can be understood by the following examples.

Illustration :

Let $\Delta = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$, *then* Δ *can be expressed as*

(A) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$ (B) $\begin{vmatrix} c & b & a \\ a & b & c \\ c & a & b \end{vmatrix}^2$ (C) $\begin{vmatrix} a & b & c \\ c & b & a \\ c & a & b \end{vmatrix}^2$ (D) None

Sol. $\Delta = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} \begin{vmatrix} c & a & b \\ b & c & a \\ -a & -b & -c \end{vmatrix}$

$$= \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} \qquad \left(\because \text{by properties} \begin{vmatrix} c & a & b \\ b & c & a \\ -a & -b & -c \end{vmatrix} = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} \right)$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

SYSTEM OF LINEAR EQUATIONS :

Definition-1 :

A system of linear equations in n unknowns $x_1, x_2, x_3, \ldots\ldots x_n$ is of the form :

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n \end{array} \right\} \quad \ldots\ldots (A)$$

If $b_1, b_2, \ldots\ldots, b_n$ are all zero, the system is called **homogeneous** and non-homogeneous if at least one b_i is non-zero.

Definition-2 :

The solution set of the system (A) is an n tupe $(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$ of real numbers (or complex numbers if the coefficients are complex) which satisfy each of the equations of the system.

Definition-3 :

A system of equations is called **consistent** if it has at least one solution; **inconsistent** if it does not have any solution; **determinate** if it has a unique solution; **indeterminate** if it has more than one solution.

(A) Non-homogeneous Equations in two unknowns :

Consider the system of equations

$$\begin{Bmatrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{Bmatrix} \qquad \ldots(i)$$

We consider the following cases.

(1) **a_i, b_i, c_i (i = 1, 2) are all zero :**
Then any pair of numbers (x, y) is a solution of the system (i) since in this case equation reduces to an identity.
So, in this case equations are always **consistent and indeterminate.**

(2) **a_i, b_i (i = 1, 2)** are all zero, but at least one **c_1 and c_2** is non-zero. Then the system has solution i.e. the equation are **inconsistent.**

(3) **Al least one of $a_i b_i$ (i = 1, 2) is non-zero**
Suppose $b_2 \neq 0$. Then system (i), is equivalent to the system.

$$\begin{Bmatrix} a_1x + b_1y = c_1 \\ \dfrac{a_2}{b_2}x + y = \dfrac{c_2}{b_2} \end{Bmatrix} \qquad \ldots(ii)$$

i.e., if the pair (x_0, y_0) is a solution of system (i) then it is also a solution of system (ii), and vice-versa.
Multiplying the second equation of system (ii) by b_1 and subtracting from first, we get

$$\left(a_1 - \frac{a_2}{b_2}b_1\right)x = c_1 - \frac{c_2}{b_2}\cdot b_1 \qquad \ldots(iii)$$

Now replacing the first equation of system (ii) by equation (iii), we obtain the system

$$\begin{Bmatrix} \left(a_1 - \dfrac{a_2}{b_2}b_1\right)x = c_1 - \dfrac{c_2}{b_2}\cdot b_1 \\ \dfrac{a_2}{b_2}x + y = \dfrac{c_2}{b_2} \end{Bmatrix} \qquad \ldots(iv).$$

(a) If $a_1 - \dfrac{a_2}{b_2}\, b_1 \neq 0$ i.e., if $a_1b_2 - a_2b_1 \neq 0$.

then we find from the first equation of system (iv) that

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \qquad \ldots(v)$$

Substituting this value of x into the second equation of system (iv), we obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

For convenience, we write

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \qquad \text{...(vi)}$$

[Note that Δ_x and Δ_y are obtained by replacing the first and second columns in Δ by the column of c_1 and c_2 respectively].

Then (v) and (vi) can be written as

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta} \qquad \text{...(vii)}$$

This is known as **Cramer's rule**. If $a_1b_2 - a_2b_1 \neq 0$ then the system (iv) or system (i) has the unique solution given by (vii). Hence in this case, the equations are **consistent and determinate.**

(b) Now let $\Delta = a_1b_2 - a_2b_1 = 0$.

Then the system (iv) has the form

$$\begin{Bmatrix} 0.x = c_1b_2 - c_2b_1 \\ \dfrac{a_2}{b_2}x + y = \dfrac{c_2}{b_2} \end{Bmatrix} \qquad \text{...(ix)}$$

Obviously this system has no solution if

$$c_1b_2 - c_2b_1 = \Delta_x \neq 0$$

thus in this case, the equations are inconsistent.

But if $\Delta_x = 0$, then any pair of numbers (x, y),

where $y = \frac{c_2}{b_2} - \frac{a_2}{b_2}x$, $x \in R$, is a solution of system (9).

So in this case, the equations are consistent and indeterminate.

We summarize the whole discussion given in (A) as follows :

(i) If $\Delta \neq 0$, then the system is consistent and determinant and its solution is given by

$$x = \frac{\Delta_x}{\Delta}, \; y = \frac{\Delta_y}{\Delta} \qquad \text{(i.e., unique solution)}$$

(ii) If $\Delta = 0$, but at least one of the numbers Δ_x, Δ_y is non-zero, then the system is inconsistent i.e., it has no solution.

(iii) If $\Delta = 0$, and $\Delta_x = \Delta_y = 0$ but at least one of the numbers a_1, b_1, a_2, b_2 is non-zero, then the system has infinite number of solutions and hence it is consistent and indeterminante.

(iv) If $a_i = b_i = c_i = 0$ (i = 1, 2), then then system has infinite number of solutions and so it is consistent and indeterminate.

(B) Homogenous linear equations in two unknowns :

Consider the system of equations

$$\begin{Bmatrix} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{Bmatrix} \qquad \text{......(10)}$$

The system always has the solution x = 0, y = 0. If follows from the discussion in part (A) that if $\Delta \neq 0$, then the system (10) has the unique solution $x = 0, y = 0$.

And if $\Delta = 0$, and at least one of a_1, a_2, b_1, b_2 is non-zero then system (1) reduced to the single equation so that any pair of numbers (x, y) is a solution.
Thys system (10) is always consistent.

(C) Non-homogeneous linear equations in three unknowns :

Consider the system of equations

$$\begin{Bmatrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{Bmatrix} \qquad \text{.........(1)}$$

Let us introduce the following notations

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Without going into detials, we give the following rule for testing the consistency of the system (1).

(1) Let $a_i = b_i = c_i = d_i = 0, i = 1, 2, 3$
In this case any triplet (x, y, z) is a solution of the system.
Hence equations are consistent and indeterminate.

(2) If $a_i = b_i = c_i = 0, i = 1, 2, 3$ and at least one d_i $(i = 1, 2, 3)$ is non-zero, then the system has no solution, i.e., the equations in this case are inconsistent.

(3) Let $\Delta \neq 0$. In this case the system (1) has the unique solution

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta} \qquad \text{........(2)}$$

This is known as Crammer's rule. So equations in this case are consistent and determinate.

(4) If $\Delta = 0$, $\Delta_x \neq 0$ (or $\Delta_y \neq 0$ or $\Delta_z \neq 0$), then the system has no solution so the equations are inconsistent.

(5) If $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$ and at least one of the cofactors of Δ is non-zero, then the system will have an infinite number of solutions. In this case, any one of the variables can be given arbitrary value and other variables can be expressed in terms of that variable.

In such cases, the three equations reduce to two equations

If all the cofactors $\Delta, \Delta_x, \Delta_y, \Delta_z$ are zero but elements of Δ are not all zero, then in this case the system will reduced to single equation and any two variables can be given arbitrary values. So equations are consistent and indeterminate.

(D) Homogeneous linear equations :

If in (1), we take $d_i = 0$ $(i = 1, 2, 3)$ then the system is called the homogenous system of equations.
For such a system if $\Delta \neq 0$, then it has the unique solution $x = 0, y = 0, z = 0$. **(Trivial)**
So such system of equations is always consistent.

(1) Three equations in two unknowns :

Consider the equations

$$\left\{\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \\ a_3x + b_3y = c_2 \end{matrix}\right\} \qquad \text{.......(3)}$$

The system (3) will be consistent if the solutions set of any satisfies the third equations, i.e., if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Note : The factors of the following two determinants be remembered.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

(2) Gist of discussion in simple language :

(i) Consistent : Solution exists whether unique infinite number of solutions.
(ii) Inconsistent : Solution does not exist.
(iii) Homogeneous Equations : constant terms zero.
(iv) Trivial solution : All variables zero i.e., $x = 0, y = 0, z = 0$.
(v) Non-trivial solution : Infinite number of solutions.

Illustration :

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \Delta_1 \text{ or } \Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},$$

$$\Delta_2 \text{ or } \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

(3) **Case-I :** Intersecting lines

$2x + 3y = 10$ and $x + y = 4$

$\therefore$ $x = 2, y = 2$

$$\Delta = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, \Delta \neq 0.$$

(4) **Case II :** $2x + 3y = 10$

$4x + 6y = 20$

Here $\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0,$

but $\Delta_1 = \begin{vmatrix} 10 & 2 \\ 20 & 4 \end{vmatrix} = 0, \Delta_2 = 0$

As a matter of fact on division by 2 the second equation reduces to first. Thus we have got only one line $2x + 3y = 10$ on which lie infinite number of points. Thus there are infinite number of solutions and the system is consistent. $\left(k, \frac{10-3k}{2}\right)$ are infinite number of solutions by giving different values to k.

Case-III $\begin{matrix} 2x+3y=10 \\ 4x+16y=15 \end{matrix}$ or $\begin{matrix} 2x+3y=10 \\ 2x+3y=15/2 \end{matrix}$

i.e. parallel lines which we know do not intersect and hence no solution.
i.e. inconsistent. Here $\Delta = 0$ but $\Delta_1 \neq 0, \Delta_2 \neq 0$

***Summary :**

(i) $\Delta \neq 0$ Unique (Intersecting lines) Consistent

(ii) $\Delta = 0, \Delta_1 = 0, \Delta_2 = 0$ (Identical lines) Consistent, Infinite solution.

(iii) $\Delta = 0, \Delta_1 \neq 0$ (Parallel lines)
Inconsistent. No solution.

Homogeneous : $a_1x + b_1y = 0$
$a_2x + b_2y = 0$

$\Delta \neq 0$, Unique $x = 0, y = 0$, Trivial.

$\Delta = 0$, Identical line through origin, Non-trivial solution.

(5) Concurrent lines : Two variable, three equations :

$a_1x + b_1y = c, a_2x + b_2y = c_2, a_3x + b_3y = c_3$

The point of intersection of any two lines should satisfy the third.

$$\therefore \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

is the required condition.

Illustration :

For what value of λ the equations

$$2x + 3y = 8,\ 7x - 5y + 3 = 0 \text{ and } 4x - 6y + \lambda = 0$$

are consistent ? Also find the solution of the system of equations for the values of λ.

Sol. *Here the equations are linear. We have 3 equations in 2 unknowns.*

$$\therefore \quad \text{they are consistent if } \begin{vmatrix} 2 & 3 & -8 \\ 7 & -5 & 3 \\ 4 & -6 & \lambda \end{vmatrix} = 0$$

or $\quad 2(-5\lambda + 18) - 3(7y - 12) - 8(-42 + 20) = 0$

or $\quad -10\lambda + 36 - 21\lambda + 36 + 176 = 0$

or $\quad -31\lambda + 248 = 0\ ;\ \therefore \lambda = 8$

$\therefore$ *for $\lambda = 8$ the system has a solution which can be obtained by solving any two of the three equations.*

Solving $\quad 2x + 3y - 8 = 0$

$7x - 5y + 3 = 0$ *by Cramer's rule,*

$$\frac{x}{\begin{vmatrix} 3 & -8 \\ -5 & 3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & -8 \\ 7 & 3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & 3 \\ 7 & -5 \end{vmatrix}}$$

or $$\frac{x}{9-40} = \frac{-y}{6+56} = \frac{1}{-10-21}$$

or $$\frac{x}{-31} = \frac{-y}{62} = \frac{1}{-31}, \quad \therefore x = 1, y = 2$$

Illustration :

For what values of p and q the system of equations

$$2x + py + 6z = 8$$
$$x + 2y + qz = 5$$
$$x + y + 3z = 4$$

has (i) unique solution (ii) no solution (iii) infinite number of solutions?

Sol. *Here the system of linear equations in x, y, z are*

$$2x + py + 6z - 8 = 0$$
$$x + 2y + qz - 5 = 0$$
$$x + y + 3z - 4 = 0$$

$$\therefore \quad \Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & p-2 & 0 \\ 1 & 1 & q-3 \\ 1 & 0 & 0 \end{vmatrix}, \; C_2 \to C_2 - C_1, C_3 \to C_3 - 3 \times C_1$$

$$= \begin{vmatrix} p-2 & 0 \\ 1 & q-3 \end{vmatrix} = (p-2)(q-3)$$

$\therefore$ *If* $p \neq 2, q \neq 3$ *then* $D \neq 0$

and so the system will have unique solution, i.e., the system will be independent/solvable/consistent.

If $p = 2$ *or* $q = 3$ *then* $\Delta = 0$.

and so the system cannot have unique solution.

When $p = 2$,

$$\Delta_x = \begin{vmatrix} p & 6 & -8 \\ 2 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 6 & -8 \\ 2 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & -4 \\ 2 & 1 & -5 \\ 1 & 3 & -4 \end{vmatrix} = 0 \; (\because R_1 \equiv R_3)$$

$$\Delta_y = \begin{vmatrix} 2 & 6 & -8 \\ 1 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & -4 \\ 1 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = 0 \; (\because R_1 \equiv R_3)$$

$$\Delta_z = \begin{vmatrix} 2 & p & -8 \\ 1 & 2 & -5 \\ 1 & 1 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 2 & -8 \\ 1 & 2 & -5 \\ 1 & 1 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & -4 \\ 1 & 2 & -5 \\ 1 & 1 & -4 \end{vmatrix} = 0 \; (\because R_1 \equiv R_3)$$

$\therefore$ *when* $p = 2$, $\Delta = 0$, $\Delta_x = \Delta_y = \Delta_z$.

$\therefore$ *the system of equaitons will have infinite number of solutions (the system of equations will be dependent) for* $p = 2$ *and any real value of* q.

$$\text{When } q = 3,\; \Delta_x = \begin{vmatrix} p & 6 & -8 \\ 2 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} p & 6 & -8 \\ 2 & 3 & -5 \\ 1 & 3 & -4 \end{vmatrix} = 2\begin{vmatrix} p-2 & 0 & 0 \\ 2 & 3 & -5 \\ 1 & 3 & -4 \end{vmatrix}, \; R_1 \to R_1 - 2R_3$$

$$= (p-2)3$$

$\therefore p \neq 2$, $\Delta x \neq 0$ *and so the system of equaitons will have no solutions, i.e., the system is solvable/ inconsistent when* $q = 3$ *but* $p \neq 2$.

Thus we find that the system of equations will have

(i) *unique solution if* $p \neq 2$ *and* $q \neq 3$

(ii) *no solution if* $p \neq 2$ *and* $q = 3$

(iii) *infinite number of solutions if* $p = 2$.

Practice Problem

Q.1 If $\Sigma\cos^2\alpha_1 = \Sigma\cos^2\beta_1 = \Sigma\cos^2\gamma_1 = 1$;
$\Sigma\cos\alpha_1\cos\beta_1 = \Sigma\cos\beta_1\cos\gamma_1 = \Sigma\cos\gamma_1\cos\alpha_1 = 0$

Then the value of $\begin{vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \cos\gamma_1 & \cos\gamma_2 & \cos\gamma_3 \end{vmatrix}^2$

(A) 1 (B) –1 (C) 0 (D) None

Q.2 If $u = ax + by + cz$, $v = ay + bz + cx$, $w = az + bx + cy$, then the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$ is

(A) $u^2 + v^2 + w^2 - 2uvw$ (B) $u^3 + v^3 + w^3 - 3uvw$
(C) 0 (D) None of these

Q.3 If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(l'x + m'y + n')$

and $\Delta_1 = \begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix}$, then the product $\Delta_1\Delta_2$ is

(A) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ (B) $\begin{vmatrix} h & g & f \\ a & f & c \\ b & c & h \end{vmatrix} = 0$ (C) 1 (D) None

Q.4 The determinant $\Delta = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (x-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$ can be expressed as

(A) $2\begin{vmatrix} 1 & x & y \\ 1 & z & z^2 \\ 1 & x^2 & y^2 \end{vmatrix}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$ (B) $4\begin{vmatrix} 1 & x & y \\ 1 & z & y^2 \\ 1 & x^2 & x \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

(C) $2\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (D) None of these

Q.5 Find the values of c for which the following system of equations in x, y is consistent
$2x + 3y = 3$
$(c + 2)x + (c + 4)y = c + 6$
$(c + 2)^2x + (c + 4)^2y = (c + 6)^2$
(A) 0 (B) –4 (C) 10 (D) –10

Answer key

Q.1	A	Q.2	B	Q.3	A	Q.4	C	Q.5	A, D

MATRICES

Introduction :

Elementary matrix already has now becomes as integral part of the mathematical background necessary in field of electrical / computer engineering / chemistry.

A matrix is any rectangular array of numbers written within braekets. A matrix is usually represented by a capital letter and classified by its dimensions. The dimension of the matrices are the number of rows and columns.

A $m \times n$ matrix is usually written as

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

(where a_{ij} represents any number which lies i^{th} row (from top) & j^{th} column form left)

(i) The matrix is not a number. It has got no numerical value.

(ii) The determinant of matrix $A_{m\times m} = |A_{m\times m}| = \begin{vmatrix} a_{11}\cdots\cdots\cdots a_{1m} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}\cdots\cdots\cdots a_{mm} \end{vmatrix}$

Abbreviated as :

$A = [a_{ij}]$ $1 \le i \le m$; $1 \le j \le n$, i denotes the row and j denotes the column is called a matrix of order $m \times n$. The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real, the matrix is called real matrix.

SPECIAL TYPE OF MATRICES :

(A) Row Matrix :

$A = [a_{11}, a_{12}, \ldots\ldots a_{1n}]$ having one row . $(1 \times n)$ matrix. (or row vectors)

(B) Column Matrix :

$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ having one column. $(m \times 1)$ matrix (or column vectors)

(C) Zero or Null Matrix :

$(A = O_{m \times n})$

An $m \times n$ matrix all whose entries are zero.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 null matrix & $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3×3 null matrix

(D) Horizontal Matrix :

A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

(E) Verical Matrix :

A matix of order $m \times n$ is a vertical matrix if $m > n$. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

Note: Every row matrix is also a Horizontal but not the converse.
|||ly every column matrix is also a vertical matrix but not the converse.

(F) Square Matrix : (Order n)

If number of rows = number of columns $\Rightarrow$ a square matrix. A real square matrix all whose elements are positive is called a positive matrices. Such matrices have application in mechanics and economics.

Note :

(i) In a square matrix the pair of elements a_{ij} & a_{ji} are called **Conjugate Elements.**

e.g. in the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, a_{21} and a_{12} are conjugate elements.

(ii) The elements a_{11}, a_{22}, a_{33}, ... a_{nn} are called **Diagonal Elements** . The line along which the diagonal elements lie is called **"Principal or Leading"** diagonal.
The quantity $\Sigma\, a_{ii}$ = trace of the matrix written as, $(t_r) A = t_r(A)$

Square Matrix

Triangular Matrix

Upper triangular matrix
if $a_{ij} = 0 \;\; \forall\, i > j$

$$A = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$$

;

Lower triangular matrix
if $a_{ij} = 0 \;\; \forall\, i < j$

$$A = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$$

Diagonal Matrix
atleast one, $a_{ii} \neq 0$ and $a_{ij} = 0$ if $i \neq j$

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

abbreviated as dia $(d_1, d_2, d_3 \,.....\, d_n)$

Scalar matrix
if $d_1 = d_2 = d_3 ..... = a \neq 0$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

Unit matrix
if $d_1 = d_2 = d_3 ..... = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Note:

(i) Minimum number of zeros in an upper or lower triangular matrix of order n

$$= 1 + 2 + 3 + ... + (n-1) = \frac{n(n-1)}{2}$$

(ii) Minimum number of cyphers in a diagonal/scalar/unit matrix of order $n = n(n-1)$ and maximum number of cyphers $= n^2 - 1$.

"It is to be noted that with every square matrix there is a corresponding determinant formed by the elements of A in the same order." If $|A| = 0$ then A is called a **singular matrix** and if $|A| \neq 0$ then a is called a **non singular matrix.**

Note: If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then det. A = 0 but not conversely.

ALGEBRA OF MATRICES :

ADDITION :

$A + B = [a_{ij} + b_{ij}]$ where A & B are of the same type . (same order)
If A and B are square matrices of the same type then, $t_r(A + B) = t_r(A) + t_2(B)$

(a) Addition of matrices is commutative :

i.e. $A + B = B + A$ where A and B must have the same order

(b) Addition of matrices is associative :

$(A + B) + C = A + (B + C)$ Provided A , B & C have the same order.

(c) Additive inverse :

If $A+B=\mathbf{O}=B+A \quad [A=m\times n]$
and both A and B have the same order then A and B are said to be the to be the additive inverse of each other where **O** is the null matrix of the same order as that of A and B. '**O**' is the additive identity element.

$$\left.\begin{array}{ll} & \text{If } A+B=A+C \Rightarrow B=C \\ \text{and} & \text{If } B+A=C+A \Rightarrow B=C \end{array}\right\} \text{ cancellation laws hold good.}$$

MULTIPLICATION OF A MATRIX BY A SCALAR :

If $A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$; $kA=\begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$ i.e. $k(A+B)=kA+kB$

Note:

(i) If A is a square matrix then $t_r(kA)=k[t_r(A)]$

(ii) $A=\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$ then $A+A+A=\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}+\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}+\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}=\begin{bmatrix} 3 & -6 \\ 6 & 9 \end{bmatrix}=3A$

Illustration :

Solve the equation, $[x \quad 2y \quad 3z]-2[y \quad z \quad -x]+3[-z \quad x \quad y]=[-12,\ 1,\ 17]$
on adding 3 matrices of LHS

$[x-2y-3z \quad 2y-2z+3x \quad 3z+2x+3y]=[-12 \quad 1 \quad 17]$

Sol. *Solving we get* $x=1,\ y=2,\ z=3$

MULTIPLICATION OF MATRICES :

(ROW BY COLUMN)

AB exists if, $\underset{2\times 3}{A=m\times n}$ & $\underset{3\times 3}{B=n\times p}$

A B is matrix of 2×3

Note that, AB exists, but BA does not $\Rightarrow AB\neq BA$

(number of columns in the pre multiplier = number of rows in post multiplier)

Note : In the product AB, $\begin{cases} A = \text{pre factor} \\ B = \text{post factor} \end{cases}$

$$\underset{1\times n}{A=(a_1, a_2, \ldots\ldots a_n)} \qquad \& \qquad \underset{n\times 1}{B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}$$

$AB=[a_1b_1+a_2b_2+\ldots\ldots a_nb_n]$

If $A=[a_{ij}]$ be an $m\times n$ matrix & $B=[b_{ij}]$ be an $n\times p$ matrix,

then $(AB)_{ij}=\sum_{r=1}^{n} a_{ir}.b_{rj}$ is a matrix of order $m\times p$.

PROPERTIES OF MATRIX MULTIPLICATION :

Matrix multiplication is not commutative

i.e. $AB \neq BA$ (in general)

e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

In fact if AB is defined it is possible that AB is not defined or may have different order

	A	B	
(1)	3×2	2×3	then AB is of order 3×3 and BA is of order 2×2
(2)	2×2	2×3	

Note :

(i) If $AB = 0 \not\Rightarrow$ that one of the matrices is zero however if any one of either A or B is null matrix then $AB = 0$ provided the product exist.

e.g. $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix}$

A and B are two square matrix of the same order such that $AB = 0$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and det. $A \neq 0$ then B must be a null matrix.

Verification: det. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot$ det. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

at least one of $|A|$ or $|B|$ must be if det. $(A) \neq 0 \quad \Rightarrow \quad$ det. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$$\begin{matrix} b = 0 \\ d = 0 \end{matrix}\left[\begin{matrix} a + 2b = 0 \\ b + 2d = 0 \\ 3a + 4c = 0 \\ 3b + 4d = 0 \end{matrix}\right] \Rightarrow a = 0 \Rightarrow c = 0$$

(ii) If $AB = AC \not\Rightarrow B = C$ but if $B = C \Rightarrow AB = AC$

(iii) In case $AB = BA$ is restrict of matrices A and B the two matrices are said to commute each other one if $AB = -BA$ then they are said to anticommute each other.

e.g. (i) $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ $\quad [AB = BA]$

Note that multiplication of diagonal matrices of the same order will be commutative.

(iv) For every square matrix A, there exist an identity matrix of the same order such that $IA = AI = A$ where I is the unit matrix of the same order.

(v) If $A = 0$ then det. $A = 0$, however if det. $A = 0 \not\Rightarrow A = 0$

MATRIX MULTIPLICATION IS ASSOCIATIVE :

If A, B & C are conformable for the product AB & BC, then

$$(A.B).C = A.(B.C)$$

$A = [a_{ij}]$ is $m \times n$; $B = [b_{ij}]$ is $n \times p$; $C = [c_{ij}]$ is $p \times q$

Note : $(A \cdot B) \cdot C$ & $A \cdot (B \cdot C)$ have the same order $\Rightarrow$ comparable.

$$[(AB).C]_{ij} = \sum_{r=1}^{p} (AB)_{ir}\, C_{rj}$$

$$= \sum_{r=1}^{p} \left(\sum_{s=1}^{n} a_{is}\, b_{sr} \right) c_{rj} = \sum_{r=1}^{p} \sum_{s=1}^{n} (a_{is}\, b_{sr})\, c_{rj}$$

$$= \sum_{r=1}^{p} \sum_{s=1}^{n} a_{is}\, (b_{sr} . c_{rj}) \qquad \text{(associativity in R)}$$

$$= \sum_{s=1}^{n} a_{is} \sum_{r=1}^{p} b_{sr}\, c_{rj} = \sum_{s=1}^{n} a_{is}\, (BC)_{sj}$$

$$= [A.(BC)]_{ij}$$

$$\therefore \quad [(A.B).C]_{ij} = [A.(B.C)]_{ij} \quad \Rightarrow \quad (AB)C = A.(BC)$$

DISTRIBUTIVITY :

$$\left.\begin{aligned} A(B+C) &= AB + AC \\ (A+B)C &= AC + BC \end{aligned}\right] \text{ Provided A, B \& C are conformable for respective products}$$

$A = m \times n$; $B = n \times p$; $c = n \times p$

$$[A.(B+C)]_{ij} = \sum_{r=1}^{n} a_{ir}\, (B+C)_{rj} = \sum_{r=1}^{n} a_{ir}\, (b_{rj} + c_{rj})$$

$$= \sum_{r=1}^{n} a_{ir}\, b_{rj} + \sum_{r=1}^{n} a_{ir}\, c_{rj}$$

$$= (AB)_{ij} + (AC)_{ij} = (AB + AC)_{ij}$$

POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX :

For a square matrix A, $A^2 A = (AA)A = A(AA) = A^3$.

Note that for a unit matrix I of any order, $I^m = I$ for all $m \in N$.
It can be easily seen that $A^m . A^n = A^{m+n}$ and $(A^m)^n = A^{mn}$.
In particular we define, $A^0 = I_n$, n being the order of A.

MATRIX POLYNOMIAL :

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots + a_nx^0$ then we define a matrix polynomial
$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots + a_nI_n$

where A is the given square matrix. If f (A) is the null matrix then A is called the zero or root of the matrix polynomial f (x).

Note that $(A)^0$ is not defined if A is a null matrix.

DEFINITIONS :

(A) Idempotent Matrix :

A square matrix is idempotent provide $A^2 = A$. For an idempotent matrix
A, $\mathbf{A^n = A}\ \forall\ \mathbf{n \geq 2, n \in N} \Rightarrow A^n = A,\ n \geq 2.$

For example if $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ then $A^2 = A$ i.e. A is idempotent.

(B) Nilpotent Matrix:

A square matrix is said to be nilpotent matrix of index p, $(p \in N)$, if $A^p = \mathbf{O}$, $A^{p-1} \neq \mathbf{O}$ i.e. if p is the least positive integer for which $A^p = \mathbf{O}$, then A is said to be nilpotent of index p.

e.g. (i) $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & 3 \end{bmatrix}$ Note that $A^3 = 0$ but $A^2 \neq 0 \Rightarrow$ index of nilpotency = 3

(ii) $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is a nilpotent matrix of index 2.

(iii) $A = \begin{bmatrix} a & -a^2 \\ 1 & -a \end{bmatrix} \begin{bmatrix} a & -a^2 \\ 1 & -a \end{bmatrix}$ nil potent

(C) Periodic Matrix :

A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer K then A is periodic with period K i.e. if K is the least positive integer for which $A^{K+1} = A$ then A is said to be periodic with period K. If K = 1 then A is called idempotent.

e.g. the matrix $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ has the period 1.

Note : (1) Period of a square null matrix is not defined.
(2) Period of an idempotent matrix is 1.

(D) Involutary Matrix :

If $A^2 = I$, the matrix is said to be an involutary matrix. An involutary matrix is its own inverse.

e.g. (i) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;

Illustration :

Prove that a square matrix A is involutary if and only if $(I - A)(I + A) = O$.

Sol. *L.H.S.* $= (I - A)(I + A) = I^2 + IA - AI - A^2 = I^2 - A^2$
above is null matrix if $A = I$.

Practice Problem

Q.1 If matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} = B + C$, where B is symmetric matrix and C is skew-symmetric matrix.

Then find matrix B and C.

Q.2 Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Q.3 If $k \in R_0$, then det $\{\text{adj }(kI)_n\}$ is equal to
(A) k^{n-1} (B) $k^{n(n-1)}$ (C) k^n (D) k

Q.4 If $A_1, A_3, \ldots\ldots, A_{2n-1}$ are n skew symmetric matrices of same order, then $B = \sum_{r=1}^{n}(2r-1)(A_{2r}-1)^{2r-1}$ will be
(A) symmetric (B) skew-symmetric
(C) neither symmetric nor skew-symmetric (D) data not adequate

Answer key

Q.1 $\frac{1}{2}\begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}$ Q.3 B Q.4 B

Illustration :

The matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ *be a zero divisor of the polynomial* $f(x) = x^2 - 4x - 5$. *Find the trace of matrix* A^3.

Sol. $A^3 = A \cdot A^2 = A(4A + 5I) = 4A^2 + 5A$

$$= 4(4A + 5I) + 5A.$$
$$= 21A + 20I$$
$$= \begin{bmatrix} 21 & 42 & 42 \\ 42 & 21 & 21 \\ 42 & 42 & 21 \end{bmatrix}$$

Trace $A^3 = 123$. ***Ans.***

Illustration :

Define $a = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ *find a vertical vector V such that* $(A^8 + A^6 + A^4 + A^2 + I)\, V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$.

Sol. $A^2 = 3I : (8I + 27 + 9 + 3 + 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$.

$$\begin{bmatrix} |21| \\ |21| \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

$a = 0;\ b = \dfrac{1}{11}$. ***Ans.***

Illustration :

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $\det(A^n - I) = 1 - \lambda^n \quad n \in N$ *then the value of* λ *is*

Sol. $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix};\ A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2^2 & 2^2 \\ 2^2 & 2^2 \end{bmatrix};\ A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$

$$A^n - I = \begin{bmatrix} 2^{n-1} - 1 & 2^{n-1} \\ 2^{n-1} & 2^{n-1} - 1 \end{bmatrix}$$

$\det(A^n - I) = (2^{n-1} - 1)^2 - (2^{n-1})^2$

$= 1 - 2^n \Rightarrow \lambda = 2$. ***Ans.***

Illustration :

The product of n matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdots\cdots \begin{bmatrix} 1 & n \\ 0 & n \end{bmatrix}$ *is equal to matrix* $\begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$. *Find n*

Sol. $\begin{bmatrix} 1 & \dfrac{n(n+1)}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$

$$\frac{n(n+1)}{2} = 378$$

$n = 27$ ***Ans.***

Illustration :

If $A^3 = B^3$ and $A^2 B = A^2B$ then prove that at least one of let $(A^2 + B^2)$ or det $(A - B)$ must be zero.

Sol. $A^3 - A^2B = B^3 - B^2A$

$A^2(A - B) = B^2 (B - A)$

$(A^2 + B^2) (A - B) = 0$

Let $\left((A^2+B^2)-(A-B)\right) = 0$

$\left(det\,(A^2+B^2)\,(det\ A-B)\right)$. **Ans.**

Illustration :

Let the matrices $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

then find $tr + tr\left(\frac{ABC}{2}\right) + tr\left(\frac{A(BC)^2}{2}\right) + tr\left(\frac{A(BC)^3}{8}\right)$ .....∞.

Sol. $tr\,(A) + tr\left(\frac{A}{2}\right) + tr\left(\frac{A}{2^2}\right) + ...$

$$= tr\,A\left(1+\frac{1}{2}+\frac{1}{2^2}.....\infty\right) = \frac{trA}{1-\frac{1}{2}} = 2tr(A) = 2(2 + 1) = 6.$$

Illustration :

If the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *comments find* $\frac{d-b}{a+c-b}$.

Sol. $AB = BA$

$$\begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$$

$3a + 4c = c + 3d$

$3(a + c) = 3d$

$a + 2c = a + 3b$

$b = \frac{2c}{3}$. **Ans.**

Illustration :

If A is involutary prove that $\frac{1}{2}(I+A)$ *and A is idempotent.*

Sol. $A^2 = I$, $A = \frac{I+A}{2}$; $A^2 = \frac{I^2 + A^2 + 2IA}{4} = \frac{1}{4}(I + 2A + I) = \frac{A+I}{2}$. ***Ans.***

Illustration :

If α *and* β *are roots of the equation*

$$[1\ \ 25]\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}^5 \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}^{10} \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}^5 \begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40]$$

then find the value of $(1-\alpha)(1-\beta)$.

Sol. $$[1\ \ 25]\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}^5 \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}^{10} \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}^5 \begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40].$$

Let $A = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$.

Here, $AB = BA = I$

$\therefore \quad A^5 B^{10} A^5 = I$

$$[1\ \ 25]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40] \Rightarrow [1\ \ 25]\begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40]$$

$x^2 - 5x + 20 + 25x + 50 = 40 \Rightarrow x^2 + 20x + 30 = 0 \left< \begin{matrix} \alpha \\ \beta \end{matrix} \right.$

$(1-\alpha)(1-\beta) = 1 - (\alpha + \beta) + \alpha\beta = 1 - (-20) + 30 + 51$. ***Ans.***

Practice Problem

Q.1 Given the matrices A of order m× n, B of order n × p and C of order r × q. Under what conditions on p, q and r would matrices be conformable for finding the product and what is the order of each
(a) ABC, (b) ACB, (c) A(B+C)

Q.2 The restriction on n, k and p so that PY + WY will be defined are
(A) $k = 3, p = n$ (B) k is arbitrary, $p = 2$
(C) p is arbitrary, $k = 3$ (D) $k = 2, p = 3$

Q.3 If $n = p$, then the order of the matrix $7X - 5Z$ is
(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

Q.4 If A, B, C are the given matries such that AB = O and BC = I then prove that $(A + B)^2 (A + C)^2 = I$ where I is an identity matrix.

Q.5 Find all matrices which commute with the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Q.6 Let $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}$. If $(A + B)^2 = A^2 + B^2$, find a and b.

Answer key

Q.1 (a) $p = r$, $m \times q$, (b) $r = n = q$, $m \times p$, (c) $r = n$, $p = q$, $m \times q$ Q.2 A Q.3 B

Q.5 $B = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$, where x, y are scalars; Let $B = \begin{pmatrix} x & y \\ a & b \end{pmatrix}$; now equate $AB = BA$ to get $a = 0$ and $b = x$.

Q.6 $a = 1$, $b = 4$

Illustration :

Paragraph for questions no. 1 & 2

Consider a square matrix A of order 2 which has it's element's as 0, 1, 2, 4. Let N denotes number of such matrices, all element's of which are distinct
Find

Q.1 *All possible matrices (distinct entries)*

Q.2 *Possible non negative values of det (A) and total number of such matrices.*

Sol.

(i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow 4 \cdot 3 \cdot 2 \cdot 1.$

(ii) $\begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix} = 2 \Rightarrow$ *4 matrices*

$$\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4 \quad \Rightarrow \quad 4 \text{ matrices}$$

$$\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = 8 \quad \Rightarrow \quad 4 \text{ matrices.}$$

Also 12 more matrices are possible whose determinant value can be {– 2, – 4, – 8}. **Ans. 12**

Illustration :

Let M be a 2 × 2 matric such that $M\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ *and* $M^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. *If* x_1 *and* x_2 $(x_1 > x_2)$ *are the two values of x for which det (M – xI) = 0, where I is an identity matrix of order 2 then find the value of* $(5x_1 + 2x_2)$.

Sol. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now, $M\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

So, $a - b = -1$ ....(1)

$c - d = 2$ ....(2)

Also, $M^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad M\left(M\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow M\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -a+2b \\ -c+2d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So, $-a + 2b = 1$ ....(3)

$-c + 2d = 0$ ....(4)

$\therefore$ From (1), (2), (3), (4), we get

$a = -1, b = 0, c = 4, d = 2$

Hence, $M = \begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix}$

Now, det $(M - x\,I) = 0$ (Given)

$$\Rightarrow \quad \begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix} - x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{vmatrix} -1-x & 0 \\ 4 & 2-x \end{vmatrix} = 0$$

$\Rightarrow \quad -(1 + x)(2 - x) = 0$

$(x + 1)\, x - 2) = 0$

$x = 2, -1$

$x_1 = 2, x_2 = -1$

$5x_1 + 2x_2 = 10 - 2 = 8.$ **Ans.**

Illustration :

Let $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -\alpha & 14\alpha & 7\alpha \\ 0 & 1 & 0 \\ \alpha & -4\alpha & -2\alpha \end{bmatrix}$. If $AB = I$, where I is an identity matrix of order 3 then trace B has value equal to

Sol. We have $AB = I$, so

$$\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -\alpha & 14\alpha & 7\alpha \\ 0 & 1 & 0 \\ \alpha & -4\alpha & -2\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 5\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10\alpha - 2 & 5\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (Given)}$$

$$\Rightarrow \quad 5a = 1 \Rightarrow \quad a = \frac{1}{5}.$$

Now, trace $(B) = (-\alpha) + 1 + (-2\alpha) = 1 - 3a = 1 - 3\left(\frac{1}{5}\right) = 1 - \frac{3}{5} = \frac{2}{5}$. ***Ans.***

Illustration :

Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and $(A + B)(A - B) = (A - B)(A + B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible value of k are

Sol. $(A + B)(A - B) = (A - B)(A + B) \Rightarrow AB = BA$

as A is symmetric and B is skew symmetric

$\Rightarrow \quad (AB)^t = -AB \Rightarrow \quad k = 1$ and $k = 3$.

$(AB)^T = (-1)^k AB$

$B^T \cdot A^T = (-1)^k AB$

$(-B)(A) = (-1)^k AB$

$(-1)^{k+1} AB = BA$

$(-1)^{k+1} = 1$

$k \Rightarrow$ odd.

Illustration :

Paragraph for questions no. 1 to 4

Let A be the 2 × 2 matrices given by $A = [a_{ij}]$ *where* $a_{ij} \in \{0, 1, 2, , 4\}$ *such that* $a_{11} + a_{12} + a_{21} + a_{22} = 4$.

Q.1 *The number of matrices A such that the trace of A is equal to 4 are*

(A) 3 (B) 4 (C) 5 (D) 6

[Note : The trace of a matrix is the sum of its diagonal entries.]

Q.2 *The number of matrices A such that A is invertible are*

(A) 20 (B) 18 (C) 15 (D) 12

Q.3 *The absolute value of the difference between maximum value and minimum value of det(A) is equal to*

(A) 0 (B) 4 (C) 6 (D) 8

Q.4 *The number of matrices A such that A is either symmetric or skew- symmetric or both and det (A) is divisible by 2 are*

(A) 5 (B) 3 (C) 7 (D) 9

Sol. *We have* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *where a, b, c, d* $\in \{0, 1, 2, 3, 4\}$ *and* $a + b + c + d = 4$

S.No.	Category	$a+b+c+d=4$	Cases
1	Type – I	4, 0, 0, 0	$\frac{4!}{3!} = \frac{24}{6} = 4$
2	Type – II	3, 1, 0, 0	$\frac{4!}{2!} = \frac{24}{2} = 12$
3	Type – III	2, 1, 1, 0	$\frac{4!}{2!} = \frac{24}{2} = 12$
4	Type – IV	2, 2, 0, 0	$\frac{4!}{2!\,2!} = \frac{24}{4} = 6$
5	Type – V	1, 1, 1, 1	$\frac{4!}{4!} = \frac{24}{24} = 1$

$\therefore$ *Total number of matrices A = 4 + 12 + 12 + 6 + 1 = 35.*

(i) $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ *are 5 matrices A where trace of A is equal to 4.*

(ii) *For matrix A to be invertible, det (A) ≠ 0*

Type-I has no determinant whose value is non-zero.

Type-II have 4 determinants whose value is non-zero.

Type-III have all its 12 determinants whose value is non-zero.

Type-IV have 2 determiants whose value is non-zero.

Type-V has no determinants whose value is non-zero.

∴ *Total number of matrices A such that A is invertible are = 0 + 4 + 12 + 2 + 0 = 18.* ***Ans.***

(iii) *Maximum value of det(A)* = $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$ *and minimum value of det(A)* = $\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -4$

∴ *Absolute value of difference* = $|4-(-4)| = 8.$

(iv) *There will not be any skew- symmetric matrix because no element is negative and sum of elements is 4. For symmetric matrix, pair of conjugate elements must be same.*

Type-I : $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$; *Type-II* : $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Type-III: $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$; *Type-IV* $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

Clearly, there are 5 symmetric matrices A such that det(A) is divisible by 2.

ORTHOGONAL MATRICES (NOT IN NCERT):

A square matrix is said to be orthogonal matrix, if $AA^T = I = A^T A$

Note that :

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

comparing,

$$\sum_{i=1}^{3} a_i^2 = \sum b_i^2 = \sum c_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{3} a_i b_i = \sum b_i c_i = \sum c_i a_i = 0$$

Note : All the 3 rows or 3 columns of an orthogonal matrix are pair wise orthogonal triad of 3 unit vectors.

Illustration :

Find a, b, c, x and y if the matrix A given by $A = \begin{bmatrix} a & 2/3 & 2/3 \\ 2/3 & 1/3 & b \\ c & x & y \end{bmatrix}$ *is orthogonal.*

Sol. (I) $a = \frac{1}{3}, b = -\frac{2}{3}, c = \frac{2}{3}, x = -\frac{2}{3}$ and $y = \frac{1}{3}$

(II) $a = \frac{1}{3}, b = -\frac{2}{3}, c = -\frac{2}{3}, x = \frac{2}{3}$ and $y = -\frac{1}{3}$

Illustration :

If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *and* $Q = PAP^T$ *and* $x = P^TQ^{2005}P$, *then x is equal to*

(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(C) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$

(D) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

Sol. $P^TP = I$

$Q = PAP^T$ so that

$x = P^TQ^{2005}P = P^T(PAP^T)^{2005}P$

$= P^TPAP^T(PAP^T)^{2004}P$

$= A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 21 \end{bmatrix}.$

ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is $= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$.

Then $(\text{adj } A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

Hence the transpose of the matrix of cofactors of elements of A in det. A is called the adj A.

Note:

(i) If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ then Adj A $= \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$ e.g. $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ the adj. $A = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

Hence adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing the signs of the off diagonal elements.

(ii) Adjoint of a scalar matrix is a also a scalar matrix, adjoint of a diagonal matrix and adjoint of a triangular matrix is a triangular matrix.

e.g., $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 4 \\ 2 & 6 & 7 \end{bmatrix}$ adj $A = \begin{bmatrix} -24 & 4 & 8 \\ -27 & 1 & 11 \\ 30 & -2 & -10 \end{bmatrix}$.

Illustration :

Which of the following statement(s) is(are) correct ?

(A) If A is square matrix of order 3, then the value of $\det\left\{(A - A^T)^{2011}\right\}$ *is equal to 0.*

(B) If A is a skew-symmetric matrix of order 3, then maxtrix A^4 *is symmetric.*

(C) If $3A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix}$ *and* $AA^T = I$*, then* $(x + y)$ *is equal to* -3.

(D) If α, β, γ *are the roots of the cubic* $x^3 + px^2 + q = 0$*, then the value of the determinant*

$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ *is equal to* $-p^3$

Sol.

(A) *If A is square matrix of order 3, then* $A - A^T$ *is skew symmetric of order 3.*

$\therefore \quad \left|(A - A^T)\right| = 0 \Rightarrow \left|(A - A^T)^{2011}\right| = \left|A - A^T\right|^{2011} = 0.$ *Ans.*

(B) *Given* $A^T = -A$

Let $C = A^4$

$C^T = (A^T)^4 = (-A)^4 = A^4 = C.$

Hence C is symmetric matrix $\Rightarrow$ *B is true.*

(C) *We have* $AA^T = \begin{pmatrix} 9 & 0 & x+4+2y \\ 0 & 0 & 2x+2-2 \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *(Given)*

$\Rightarrow \quad x = -2, y = -1$

Hence $(x + y) = (-2) + (-1) = -3.$ ***Ans.***

(D) *We have* $\alpha + \beta + \gamma = -p$, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$

Now, $2\alpha\beta\gamma - \alpha^3 - \beta^3 - \gamma^3 = -(\alpha + \beta + \gamma)(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= p(p^2) = p^3.$ ***Ans.***

PROPERTIES OF ADJOINT :

Theorem-1 :

$A(\text{adj } A) = (\text{adj } A) = A = |A| I_n$
whose A is any square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots a_{1n} \\ a_{21} & a_{22} & \dots\dots a_{2n} \\ a_{n1} & a_{n2} & \dots\dots a_{nn} \end{bmatrix}$$

$$(\text{adj } A) = \begin{bmatrix} A_{11} & A_{21} & \dots\dots A_{n1} \\ A_{12} & A_{22} & \dots\dots A_{n2} \\ \vdots & \vdots & \vdots \\ A_{1n} & A_{n2} & A_{nn} \end{bmatrix}$$

$(\text{adj } A) = \Sigma$ element's of ith row of A multiplied by corresponding element's jth column of adj A.

$= a_{ij}A_{ji} + a_{a2}A_{j2} + \dots\dots + a_{in}A_{in} = 0$

if $\quad i \neq j$ and $|A|$ when $i = j$.

Thus in product A (adj A) only diagonal element's exist all of them equal to $|A|$ while all other element's are zero.

$$A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0\dots\dots \\ 0 & |A| & 0\dots\dots \\ 0 & 0 & |A|\dots\dots \\ \vdots & \vdots & \vdots \\ \vdots & & |A| \end{bmatrix}$$

also for $(\text{adj } A) \cdot A = |A| I_n$.

Theorem-2 :

Let A be a non singular matrix of order $n \times n$. Then

$$|(\text{adj } A)| = |A|^{n-1}$$

Take det

$$|A(\text{adj } A)| = \begin{vmatrix} |A| & 0 & \dots\dots 0 \\ 0 & |A| & \dots\dots \\ \dots\dots & \dots\dots & |A| \end{vmatrix} = |A|^n$$

$|A||\text{adj } A| = |A|^n; \quad |\text{adj } A| = |A|^{n-1}$.

Also we can say that

$A(\text{adj } A) = |A| \cdot I_n$

$$A \cdot \frac{(\text{adj } A)}{|A|} = \frac{(\text{adj } A) \cdot A}{|A|} = I_n .$$

Theorem-3 :

If A and B are two $n \times n$ matrices, then adj (AB) = (adj B) · (adj A)
(A and B must be non-singular of same order)

Proof : We know that

$A \cdot (adj\ A) = |A|\ I$ mutliplying both sides by AB

L.H.S. $(AB) \cdot (adj\ AB) = |AB|\ I$ ....(1)

Now, $AB\ adj\ (AB) = AB\ (adj\ B) \cdot (adj\ A)$

R.H.S $(AB) \cdot (adj\ B) \cdot (adj\ A)$

$= A\ (B \cdot adj\ B) \cdot (adj\ A)$

$= A \cdot (|B|\ I) \cdot (adj\ A)$ $\because B \cdot adj\ B = |B|\ I$

$= A \cdot |B|\ I \cdot (adj\ A)$

$= A \cdot |B|\ (adj\ A)$ $\because I \cdot (adj\ A) = adj\ A$

$= |B|\ A \cdot (adj\ A)$

$= |B|\ |A|\ I$ $\because A(adj\ A) = |A|\ I$

$= |A|\ |B|\ I$

$= |AB|\ I$ $\because |A|\ |B| = |AB|$ ....(2)

In view of equation (1) and (2), we have

$(AB) \cdot (adj\ AB) = (AB) \cdot (adj\ B) \cdot (adj\ A) \Rightarrow adj\ (AB) = (adj\ B) \cdot (adj\ A)$

Generalisation : *The result can be generalised for square matrices A, B, C, D, ... each of order n as follows:*

$adj\ (ABCD) = ...(adj\ D) \cdot (adj\ C) \cdot (adj\ B) \cdot (adj\ A)$

Illustration :

If A is any square matrix of order $n \times n$, then $adj\ (adj\ A) = |A|^{n-2} A$.

Sol. We know that

$A \cdot (adj\ A) = |A|\ I$

$\therefore$ $adj\ \{A \cdot (adj\ A)\} = adj\ \{|A|\ I\}$

$\Rightarrow$ $\{adj\ (adj\ A)\} \cdot (adj\ A) = |A|^{n-1}$ | $\because adj\ \{k\ I_n\} = k^{n-1}\ I_n$

$\Rightarrow$ $\{adj\ (adj\ A)\} \cdot (adj\ A)\ A = |A|^{n-1}\ I \cdot A$ | Post-multiplying both sides by A

$\Rightarrow$ $\{adj\ (adj\ A)\} \cdot |A|\ I = |A|^{n-1}\ A$

$\Rightarrow$ $\{adj\ (adj\ A)\}\ |A| = |A|^{n-1}\ A$

$\Rightarrow$ $adj\ (adj\ A) = |A|^{n-2}\ A$

Illustration :

If A is a square matrix of order n and $|A| \neq 0$, then $|adj\ (adj\ A)| = |A|^{(n-1)^2}$

Sol. We know that

$|adj\ A| = |A|^{n-1}\ A$

$= \{|A|^{n-1}\}^{n-1}$

$= |A|^{(n-1)^2}$

Illustration :

Let matrix $A = \begin{bmatrix} x & y & -z \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ *where x, y, z* $\in N$. *If* $|adj(adj(adj(adj\ A)))| = 2^{32}.3^{16}$ *then number of such matrix A is* _____.

Sol. $|A| = x + y + z$

$adj\ (adj\ A) = |A|^{n-2} A = |A| A$

$|adj\ (adj\ (adj\ (adj\ A)))| = \left||A|^5 A\right| = |A|^{16}$

$|A|^{16} = 2^{32} \cdot 3^{16} = 12^{16}$

$|A| = 12$

$\Rightarrow x + y + z = 12$

Number of matrix = 55.

INVERSE OF A MATRIX (RECIPROCAL MATRIX) :

Definition:

A square matrix A said to be invertible (non singular) if there exists a matrix B such that,

$$AB = I = BA$$

B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA.$$

Note that for an involutary matrix $A^2 = I \Rightarrow A = A^{-1}$.

PROPERTIES OF INVERSE:

For a non singular matrix

Property :

$$A^{-1} = \frac{(\text{adj } A)}{|A|}$$

We have, $A.(\text{adj } A) = |A|\ I_n$

$A^{-1} A\ (\text{adj } A) = A^{-1} I_n\ |A|$

$I_n\ (\text{adj } A) = A^{-1}\ |A|\ I_n$

$$\therefore \quad A^{-1} = \frac{(\text{adj } A)}{|A|}$$

Remark: (i) Note that A^{-1} exists if A is non singular.

(ii) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|A| = 1$ then $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Note :

(i) The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

(ii) Inverse of a non singular diagonal matrix dia $(k_1\ k_2\ k_3 \ldots\ldots k_n)$ is the diagonal matrix diag $\left(k_1^{-1}, k_2^{-1}, k_3^{-1} \ldots\ldots, k_n^{-1}\right)$

Theorem-1 :

Every invertible matrix possesses a unique inverse.

Proof : Let A be an invertible matrix of order n. Let B and C be two inverse of A.

Then, $AB = BA = I_n$ ....(i)

and $AC = CA = I_n$ ....(ii)

Now, $AB = I_n$

$\Rightarrow$ $C(AB) = C\,I_n$ [pre-multiplying by C]

$\Rightarrow$ $(CA)B = C\,I_n$ [by associativity]

$\Rightarrow$ $I_n B = C\,I_n$ [$\because$ $CA = I_n$ from (ii)]

$\Rightarrow$ $B = C$ [$\because$ $I_n B = B$, $C\,I_n = C$]

Hence an invertible matrix possesses a unique inverse.

Theorem-2 :

If A is an invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

Proof : Since A is invertible matrix. Therefore,

$|A| \neq 0$

$\Rightarrow$ $|A^T| \neq 0$ [$\because$ $|A^T| = |A|$]

$\Rightarrow$ A^T is also invertible.

Now, $AA^{-1} = I_n = A^{-1}A$

$\Rightarrow$ $(AA^{-1})^T = (I_n)^T = (A^{-1}A)^T$

$\Rightarrow$ $(A^{-1})^T(A^T) = I_n = A^T(A^{-1})^T$ (as good as $AB = I = BA \Rightarrow A^{-1} = B$)

$\Rightarrow$ $(A^T)^{-1} = (A^{-1})^T$

Theorem-3 :

If A is a non-singular matrix, then prove that $|A^{-1}| = |A|^{-1}$ i.e. $|A^{-1}| = \dfrac{1}{|A|}$

Proof : Since $|A| \neq 0$, therefore A^{-1} exists such that $AA^{-1} = I = A^{-1}A$

$\Rightarrow$ $|AA^{-1}| = |I|$

$\Rightarrow$ $|A||A^{-1}| = 1$ [$\because$ $|AB| = |A||B|$ and $|I| = 1$]

$\Rightarrow$ $|A^{-1}| = \dfrac{1}{|A|}$ [$\because$ $|A| \neq 0$]

Theorem-4 :

If A & B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof : A & B are invertible. (reversal law of inverse)

$\Rightarrow$ $|A| \neq 0$ & $|B| \neq 0$ $\Rightarrow$ $|A|\,|B| \neq 0$ $\Rightarrow$ $|AB| \neq 0$

$\Rightarrow$ AB is invertible.

Now $(AB)(AB)^{-1} = I$

$A^{-1}(AB)(AB)^{-1} = A^{-1}$

$(IB)(AB)^{-1} = A^{-1}$

$B(AB)^{-1} = A^{-1}$

$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

Note :

(i) If A is invertible, **(a)** $(A^{-1})^{-1} = A$ **(b)** $(A^k)^{-1} = (A^{-1})^k = A^{-k}, \quad k \in N$

(ii) A square matrix is said to be orthogonal if, $A^{-1} = A^T$.

(iii) $(AB)^{-1}$ may be equal to $A^{-1}B^{-1}$.

EXPLANATION :

(i) As A is invertible hence $AA^{-1} = I = A^{-1}A$

Hence (A^{-1}) and A are inverse of each other

$\therefore \quad (A^{-1})^{-1} = A$

again $A^k = \underbrace{A\ A\ A.......A}_{k \text{ times}}$

$\therefore \quad (A^k)^{-1} = (AA.....A)^{-1} = A^{-1}.A^{-1}.A^{-1}$ .... k times $= (A^{-1})^k$

hence $(A^k)^{-1} = (A^{-1})^k k \in N$

(ii) again if A is a square matrix and is orthogonal

then $AA^T = A^TA = I$

hence A and A^T are inverse of each other

$\therefore \quad A^{-1} = A^T$

alternatively if $A^{-1} = A^T$

$$\left.\begin{array}{l} \therefore\ AA^{-1} = I = AA^T \\ \text{or } A^{-1}A = I = A^TA \end{array}\right] \Rightarrow A \text{ is orthogonal}$$

(iii) Let $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$AB = I = BA \Rightarrow A = B^{-1}$ and $B = A^{-1}$

also $(AB)^{-1} = I^{-1} = I$

SYSTEM OF EQUATION & CRITERIAN FOR CONSISTENCY :

GAUSS - JORDAN METHOD :

Q1. $x + y + z = 6$
$x - y + z = 2$
$2x + y - z = 1$

Q2. $x + 2y + 3z = 2$
$2x + 4y + 5z = 3$
$3x + 5y + 6z = 4$

$$\begin{pmatrix} x+y+z \\ x-y+z \\ 2x+y-z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}} \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{B}{\begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}}$$

$AX = B \quad \Rightarrow \quad A^{-1}AX = A^{-1}B$

$$X = A^{-1}B = \frac{(\text{Adj}\,A)B}{|A|}$$

Note:

(i) If $|A| \neq 0$, system is consistent having unique solution

(ii) If $|A| \neq 0$ & $(\text{adj } A) . B \neq$ Null matrix,
system is consistent having unique non – trivial solution.

(iii) If $|A| \neq 0$ & $(\text{adj } A) . B = 0$ (Null matrix),
system is consistent having trivial solution.

(iv) If $|A| = 0$, **matrix method fails**

If $(\text{adj } A) . B =$ null matrix $\downarrow$ Infinite solutions

If $(\text{adj } A) . B \neq 0$ $\downarrow$ Inconsistent (no solution)

EQUIVALENT MATRICES :

If a matrix B is obtained from a matrix A by one or more elementary transformations, then A and B are equivalent matrices and we write $A \sim B$. Let.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

Then $$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 \\ 3 & 1 & 2 & 4 \end{bmatrix} \qquad [\text{Applying } R_2 \to R_2 + (-1)R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & -1 & 1 & -2 \\ 3 & 1 & 2 & 2 \end{bmatrix} \qquad [\text{Applying } C_4 \to C_4 + (-1)C_3]$$

An elementary transformation is called a row transformation or a column transformation accordingly as it is appled to rows or columns.

Theorem-1 :

Every elementary row (column) transformation of an $m \times n$ matrix (not identity matrix) can be obtained by pre-multiplication (post-multiplication) with the corresponding elementary matrix obtained from the identity matrix $I_m (I_n)$ by subjecting it to the same elementary row (column) transformation.

Theorem-2 :

Let $C = AB$ be a product of two matrices. Any elementary row (column) transformation of AB can be obtained by subjecting the pre-factor A (post factor B) to the same elementary row (column) transformation.

Method of finding the inverse of a matrix by Elementary transformation :

Let A be a non sigular matrix of order n. Then A can be reduced to the identity matrix I_n by a finite sequence of elementary transformation only. As we have discussed every elementary row transformation of a matrix is equivalent to pre-multiplication by the corresponding elementary matrix. Therefore there exist elementary matrices $E_1, E_2, \ldots\ldots, E_4$ such that

$$(E_k\, E_{k-1} \ldots\ldots E_2E_1)\, A = I_n$$

$$\Rightarrow \quad (E_k\, E_{k-1} \ldots\ldots E_2E_1)AA^{-1} = I_nA^{-1} \qquad \text{(post multiplying by } A^{-1}\text{)}$$

$$\Rightarrow \quad (E_k\, E_{k-1} \ldots\ldots E_2E_1)\, I_n = A^{-1} \;\; (\because\; I_nA^{-1} = A^{-1} \text{ and } AA^{-1} = I_n)$$

$$\Rightarrow \quad A^{-1} = (E_k\, E_{k-1} \ldots\ldots E_2E_1)\, I_n$$

Algorithm for finding the inverse of a non singular matrix by elementary row transformations :

Let A be non-singular matrix of order n

Step-I : Write $A = I_nA$

Step-II : Perform a sequence of elementary row operations successively on A on the LHS and the pre factor I_n on the RHS till we obtain the result $I_n = BA$

Step-III : Write $A^{-1} = B$

The following steps will be helpful to find the inverse of a square matrix of order 3 by using elementary row transformations.

Step-I : Introduce unity at the intersection of first row and first column either by interchanging two rows or by adding a constant multiple of elements of some other row to first row.

Step-II : After introducting unity at (1, 1) place introduce zeros at all other places in first column.

Step-III : Introduce unity at the intersection 2nd row and 2nd column with the help of 2nd and 3rd row.

Step-IV : Introduce zeros at all other places in the second column except at the intersection of 2nd and 2nd column

Step-V : Introduce unity at the intersection of 3rd row and third column

Step-VI : Finally introduce zeros at all other places in the third column except at the intersection of third row and third column.

Illustration :

Using elementary transformation, find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{-a}\right) \end{bmatrix}$.

Sol. $A = \begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix}$

We write , $\begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \quad \Rightarrow \quad \begin{bmatrix} 1 & \dfrac{b}{a} \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} A \qquad \left(R_1 \to \dfrac{R_1}{a}\right)$

$$\text{or } \begin{bmatrix} 1 & \dfrac{b}{a} \\ 0 & \dfrac{1}{a} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ \dfrac{-c}{a} & 1 \end{bmatrix} A \quad (R_2 \to R_2 - cR_1) \qquad \text{or } \begin{bmatrix} 1 & \dfrac{b}{a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ -c & a \end{bmatrix} A \qquad (R_2 \to aR_2)$$

$$\text{or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1+bc}{a} & -b \\ -c & a \end{bmatrix} A \quad \left(R_1 \to R_1 - \dfrac{b}{a} R_2\right) \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} \dfrac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

Illustration :

Obtain the inverse of the following matrix using elementary operations $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol. *We, have* $A = IA$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (\text{Applying } R_1 \leftrightarrow R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \qquad (\text{Applying } R_3 \to R_3 - 3R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \qquad (\text{Applying } R_1 \to R_1 - 2R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \qquad (Applying\ R_3 \to R_3 + 5R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \qquad (Applying\ R_3 \to \frac{1}{2} R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \qquad (R_1 \to R_1 + R_3\ and\ R_2 \to R_2 \to -2R_3)$$

$$Hence,\ A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Practice Problem

Q.1 If $\begin{bmatrix} 1 & -2 & -3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ then find the product AB and BA.

Q.2 The matrix R(t) is defined by $R(t) = \begin{bmatrix} -\cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Show that $R(s)\,R(t) = R(s+t)$.

Q.3 If $A = \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $AB = I_3$, then x + y equals

(A) 0 (B) –1 (C) 2 (D) none of these

Q.4 Let $A+2B=\begin{bmatrix}1 & 2 & 0\\ 6 & -3 & 3\\ -5 & 3 & 1\end{bmatrix}$ and $2A-B=\begin{bmatrix}2 & -1 & 5\\ 2 & -1 & 6\\ 0 & 1 & 2\end{bmatrix}$

then $\mathrm{Tr}(A)-\mathrm{Tr}(B)$ has the value equal to

(A) 0 (B) 1 (C) 2 (D) none

Q.5 Consider a matrix $A=\begin{bmatrix}3 & 1\\ -6 & -2\end{bmatrix}$, then $(I+A)^{99}$ equals (where I is a unit matrix of order 2)

(A) $I+2^{98}A$ (B) $I+2^{99}A$ (C) $I+(2^{99}+1)A$ (D) $I+(2^{99}-1)A$

Q.6 Let the matrix A and B be defined as $A=\begin{bmatrix}3 & 2\\ 2 & \alpha\end{bmatrix}$ and $B=\begin{bmatrix}3 & 1\\ 7 & 3\end{bmatrix}$. If $\det(2A^9B^{-1})=-2$,

then the number of distinct possible real values of α equals

(A) 0 (B) 1 (C) 2 (D) 3

Answer key

Q.1 $\begin{bmatrix}-10 & 2 & 21\\ -16 & 2 & 37\\ -2 & -2 & 11\end{bmatrix}$ Q.3 A Q.4 C Q.5 D Q.6 B

SOLVED EXAMPLES

Q.1 If the system of equations

$$2x + 3y - z = 0$$
$$3x + 2y + kz = 0$$
$$4x + y + z = 0$$

have a set of non-zero integral solutions then, find the smallest positive value of z.

Sol. The system has a non-zero solution if $|A| = 0 \Rightarrow k = 0$.

Clearly, the solutions are $(2a, -3a, -5a)$.

So, the smallest positive integral value of $z = 5$. **Ans.**

Q.2 Given $a, b \in \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots, 9, 10\}$. Consider the system of equations

$$x + y + z = 4$$
$$2x + y + 3z = 6$$
$$x + 2y + az = b$$

Let **L :** denotes number of ordered pairs (a, b) so that the system of equations has unique solution,

M : denotes number of ordered pairs (a, b) so that the system of equations has no solution and

N : denotes number of ordered pairs (a, b) so that the system of equations has infinite solutions.

Find $(L + M - N)$.

Sol. Clearly, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & a \end{vmatrix} = 1\,(a - 6) - 1(2a - 3) + 1\,(4 - 1)$ (Expanding along R_1)

$\Rightarrow \quad \Delta = -a$

Case-I : If $a \neq 0$, then system of equations has unique solution.

Case-II: If $a = 0$, put $z = k$, we get $x + y = 4 - k$ and $2x + y = 6 - 3k$

$\therefore$ On solving, we get

$x = 2 - 2k,\ y = 2 + k$

Now, substituting these values of x, y and z in equation $x + 2y + a \cdot z = b$, we get

$(2 - 2k) + 2\,(2 + k) + a \cdot k = bn \Rightarrow 6 + 0k = b$ i.e, $b = 6$

Thus for $b \neq 6$, there is no solution and for $b = 6$, there are infinite solution.

Hence, for unique solution $a \neq 0,\ b \in R \quad \Rightarrow \quad L = 10 \times 11 = 110$

for no solution we must have $a = 0, b \neq 6 \Rightarrow \quad M = 1 \times 10 = 10$

for infinite solution $a = 0$ and $b = 6 \Rightarrow \quad N = 1 \times 1 = 1$

$\Rightarrow \quad L + M - N = 110 + 10 - 1 = 119$ **Ans.**

Alternatively:

$$x+y+z=4 \quad .....(1)$$
$$2x+y+3z=6 \quad .....(2)$$
$$x+2y+az=b \quad .....(3)$$

Solving (1) and (2) $\Rightarrow$ $x=2-2z$ and $y=2+z$

Put in equation (3), we get

$$az=b-6$$

Hence, for unique solution $a\neq 0,\ b\in R$ $\Rightarrow$ $L=10\times 11=110$

for no solution we must have $a=0, b\neq 6 \Rightarrow$ $M=1\times 10=10$

for infinite solution $a=0$ and $b=6$ $\Rightarrow$ $N=1\times 1=1$

$\Rightarrow$ $L+M-N=110+10-1=119$ **Ans.**

Q.3 If $f(x)=\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$, then $f(100)$ is equal to

(A) 0 (B) 1 (C) 100 (D) –100

Sol. We have

$$f(x)=x\,(x+1)\,(x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix}$$

$$=x\,(x+1)\,(x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix} \quad [C_1\to C_1-C_3 \text{ and } C_2\to C_2-C_3]\ =0$$

Hence, $f(100)=0$.

Q.4 If α,β,γ are the roots of $x^3-3x+2=0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is equal to

(A) –3 (B) 2 (C) 1 (D) None of these

Sol. We have

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}=\begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix} \quad [C_1\to C_1+C_2+C_3]$$

$$=0 \quad [\because \alpha+\beta+\gamma=0 \text{ from the equation } x^3-3x+2=0]$$

Q.5 If $ax^4 + bx^3 + cx^2 + dx + e = \begin{vmatrix} 2x & x-1 & x+1 \\ x+1 & x^2-x & x-1 \\ x-1 & x+1 & 3x \end{vmatrix}$, then the value of e, is

(A) 0 (B) –2 (C) 3 (D) –2

Sol. Putting $x = 0$, we have

$$\begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 0 \end{vmatrix} \qquad [C_2 \to C_2 + C_3]$$

$= 1 - 1 = 0.$

Q.6 If the value of the determinant $\begin{vmatrix} x+1 & \alpha & \beta \\ \alpha & x+\beta & 1 \\ \beta & 1 & x+\alpha \end{vmatrix}$ is equal to -8, then the value of x, is

(A) ±2 (B) –2 (C) 0 (D) 1

Sol. We have

$\alpha = \omega$ and $\beta = \omega^2$

Thus, we have

$$\Delta = \begin{vmatrix} x & \alpha & \beta \\ x & x+\beta & 1 \\ x & 1 & x+\alpha \end{vmatrix} \qquad [C_1 \to C_1 + C_2 + C_3 \text{ and using } 1 + \omega + \omega^2 = 0]$$

$$= \begin{vmatrix} 1 & \alpha & \beta \\ 0 & x+\beta-\alpha & 1-\beta \\ 0 & 1-\alpha & x+\alpha-\beta \end{vmatrix} \qquad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$= x\,[x^2 - (\alpha - \beta)^2 - (1-\alpha)(1-\beta)]$

$= x\,[x^2 - \alpha^2 - \beta^2 + 2\alpha\beta - 1 + \alpha + \beta - \alpha\beta]$

$= x\,[x^2 - \omega^2 - \omega + 2 - 1 + \omega + \omega^2 - 1]$ [using $\omega^3 = 1$]

$= x^3$

According to the given condition, we have

$x^3 = -8$

gives $x = -2.$

Q.7 If $\begin{vmatrix} \sin 2x & \cos^2 x & \cos 4x \\ \cos^2 x & \cos 2x & \sin^2 x \\ \cos 4x & \sin^2 x & \sin 2x \end{vmatrix} = a_0 x + a_1(\cos x) + a_2(\cos^2 x) + \ldots\ldots + a_n(\cos^n x)$, then the value of a_0, is

(A) –1 (B) 1 (C) 0 (D) 2

Sol. We can see that

$a_0 = \Delta'(0)$

Now, we have

$$\Delta'(x) = \begin{vmatrix} 2\cos 2x & -2\cos x \sin x & -4\sin 4x \\ \cos^2 x & \cos 2x & \sin^2 x \\ \cos 4x & \sin^2 x & \sin 2x \end{vmatrix} + \begin{vmatrix} \sin 2x & \cos^2 x & \cos 4x \\ -2\cos x \sin x & -2\sin 2x & 2\sin x \cos x \\ \cos 4x & \sin^2 x & \sin 2x \end{vmatrix}$$

$$+ \begin{vmatrix} \sin 2x & \cos^2 x & \cos 4x \\ \cos^2 x & \cos 2x & \sin^2 x \\ -4\sin 4x & 2\sin x \cos x & 2\cos 2x \end{vmatrix}$$

Hence, we have

$$a_0 = \Delta'(0) = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= 0 + 0 + (-1) = -1.$

Q.8 Let α, β be the roots of $ax^2 + bx + c = 0$. Let $S_n = \alpha^n + \beta^n, n \geq 1$ and $\Delta = \begin{vmatrix} 1+S_0 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

If α, β are distinct and real, then

(A) $\Delta \leq 0$ (B) $\Delta > 0$ (C) $\Delta < 0$ (D) $\Delta = 0$

Sol. We have

$$\Delta = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha-1 & \beta-1 \\ 1 & \alpha^2-1 & \beta^2-1 \end{vmatrix}^2 \qquad [C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$= [(\alpha-1)(\beta^2-1) - (\alpha^2-1)(\beta-1)]^2$

$= [\alpha\beta(\beta-\alpha) + (\beta-\alpha) + (\alpha^2-\beta^2)]^2$

$= [\alpha\beta + 1 - (\alpha+\beta)]^2 [(\alpha+\beta)^2 - 4\alpha\beta]$

$$= \left(\frac{c}{a} + \frac{b}{a} + 1\right)^2 \left(\frac{b^2}{a^2} - \frac{4c}{a}\right)$$

$$= \frac{(a+b+c)^2(b^2-4ac)}{a^4} > 0 \text{ if roots are real and distinct.}$$

Q.9 Let $\Delta(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$. The value of $x\left(0 \le x \le \frac{\pi}{2}\right)$ for which $\Delta(x)$ is maximum, is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

Sol. We have

$$\Delta(x) = (1 + \sin^2 x + \cos^2 x + 4\sin 2x)\begin{vmatrix} 1 & \cos^2 x & 4\sin 2x \\ 1 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= (2 + 4\sin 2x)\begin{vmatrix} 1 & \cos^2 x & 4\sin 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= (2 + 4\sin 2x)$$

which attains maxima at $x = \frac{\pi}{4}$.

Q.10 If $D_k = \begin{vmatrix} 3^k & \frac{1}{(k+1)(k+2)} & \cos(k+1)d\theta \\ \frac{3^n - 1}{2} & \frac{n}{(n+1)} & \frac{\sin\frac{n\theta}{2}\cos\frac{(n-1)\theta}{2}}{\sin\frac{\theta}{2}} \\ a & b & c \end{vmatrix}$ then $\sum_{k=0}^{n-1} D_k$ is

(A) independent of n (B) independent of a, b, c

(C) a + b + c (D) none of these

Sol. We have

$$\sum_{k=0}^{n-1} 3^k = 1 + 3 + 3^2 + \ldots\ldots + 3^{n-1} = \frac{3^n - 1}{3-1} = \frac{3^n - 1}{2}$$

$$\sum_{k=0}^{n-1} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n-1}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

and $\sum_{k=0}^{n-1}\cos(k+1)\theta = 1+\cos\theta+\cos 2\theta+\cos 3\theta+\ldots\ldots+\cos(n-1)\theta$

$= \text{Re}\,[1+e^{i\theta}+e^{i2\theta}+\ldots\ldots+e^{i(n-1)\theta}]$

$$= \text{Re}\left(\frac{1-e^{in\theta}}{1-e^{i\theta}}\right) = \text{Re}\left(\frac{(1-e^{in\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}\right)$$

$$= \text{Re}\left(\frac{1-e^{-i\theta}-e^{in\theta}+e^{i(n-1)\theta}}{2-(e^{i\theta}+e^{-i\theta})}\right) = \frac{1-\cos\theta-\cos n\theta+\cos(n-1)\theta}{2(1-\cos\theta)}$$

$$= \frac{2\sin^2\frac{\theta}{2}+2\sin\left(n-\frac{1}{2}\right)\theta\sin\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}+\sin\left(n-\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} = \frac{\sin\frac{n\theta}{2}\cos\frac{(n-1)\theta}{2}}{\sin\frac{\theta}{2}}.$$

Q.11 The roots of the equation $\begin{vmatrix} \frac{1}{x}+x^2+\frac{1}{x} & \frac{1}{x}+ax+\frac{1}{a} & \frac{1}{x^2}+x+\frac{1}{b} \\ \frac{1}{a}+ax+\frac{1}{x} & \frac{1}{a}+a^2+\frac{1}{a} & \frac{1}{a}+ab+\frac{1}{b} \\ \frac{1}{b}+bx+\frac{1}{x} & \frac{1}{b}+ab+\frac{1}{a} & \frac{1}{b}+b^2+\frac{1}{b} \end{vmatrix} = 0$ are

(A) a, b (B) –a, –b (C) a + b, a – b (D) None of these

Sol. We have

$$\Delta = \begin{vmatrix} \frac{1}{x} & x & 1 \\ \frac{1}{a} & a & 1 \\ \frac{1}{b} & b & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & \frac{1}{x} \\ 1 & a & \frac{1}{a} \\ 1 & b & \frac{1}{b} \end{vmatrix} \text{ [row by row] } = (-1)\begin{vmatrix} \frac{1}{x} & x & 1 \\ \frac{1}{a} & a & 1 \\ \frac{1}{b} & b & 1 \end{vmatrix}^2$$

Thus, we have

$$\begin{vmatrix} \frac{1}{x} & x & 1 \\ \frac{1}{a} & a & 1 \\ \frac{1}{b} & b & 1 \end{vmatrix} = 0$$

i.e. $\frac{1}{abx}\begin{vmatrix} 1 & x^2 & x \\ 1 & a^2 & a \\ 1 & b^2 & b \end{vmatrix} = 0$ i.e. $\begin{vmatrix} 1 & x^2 & x \\ 1 & a^2-x^2 & a-x \\ 1 & b^2-x^2 & b-x \end{vmatrix} = 0$ [$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

i.e. $(a^2-x^2)(b-x)-(b^2-x^2)(x-a) = 0$ [expanding along C_1]

i.e. $(a-x)(b-x)[(a+x)-(b+x)] = 0$

gives $x = a, b$

Q.12 If α, β are the roots of the equation $ax^2 + bx + c = 0$, then the value of the determinant

$$\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos\alpha \\ \cos(\alpha-\beta) & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 1 \end{vmatrix} \text{ is equal to}$$

(A) a + b (B) 0 (C) a – b (D) a + b + c

Sol. We have

$$\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos\alpha \\ \cos(\alpha-\beta) & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 1 \end{vmatrix} \quad [\text{expanding along } R_1]$$

$= (1 - \cos^2\beta) + \cos(\alpha - \beta)[\cos\alpha \cos\beta - \cos(\alpha - \beta)] + \cos\alpha[\cos(\alpha - \beta)\cos\beta - \cos\alpha]$

$= \sin^2\beta + \cos(\alpha - \beta)[2\cos\alpha\cos\beta - \cos(\alpha - \beta)]$

$= \sin^2 + \cos(\alpha - \beta)\cos(\alpha + \beta) - \cos^2\alpha$

$= \sin^2\beta + (\cos^2\alpha - \sin^2\beta) - \cos^2\alpha = 0.$

Q.13 If p + q + r = 0, prove that $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Sol. We have

L.H.S. $= pa(qra^2 - p^2bc) - qb(q^2ca - prb^2) + rc(pqc^2 - r^2ab)$

$= pqra^3 - abcp^3 - abcq^3 + pqrb^3 + pqrc^3 - abcr^3$

$= pqr(a^3 + b^3 + c^3) - abc(p^3 + q^3 + r^3)$

$= pqr(a^3 + b^3 + c^3 - 3abc) - abc(p^3 + q^3 + r^3 - 3pqr)$

$= pqr(a^3 + b^3 + c^3 - 3abc) - 0$

$[p^3 + q^3 + r^3 - 3pqr = (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp)$

$= 0 \quad \therefore\ p + q + r = 0 \text{ (given)}]$

$= pqr(a^3 + b^3 + c^3 - 3abc)$

and R.H.S. $= pqr(a + b + c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$

$= pqr(a + b + c)\begin{vmatrix} 0 & b-c & c-a \\ 0 & a-c & b-a \\ 1 & c & a \end{vmatrix} \quad \begin{bmatrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{bmatrix}$

$= pqr(a + b + c)[(b - c)(b - a) + (c - a)^2]$

$= pqr(a + b + c)[a^2 + b^2 + c^2 - ab - bc - ca]$

$= pqr(a^2 + b^2 + c^3 - 3abc) =$ L.H.S.

Q.14 If A is non-singular, prove that the eigen values of A^{-1} are the reciprocals of the eigen value of A.

Sol. Let A be an eigen value of A and X be a corresponding eigenvector. Then

$$AX = \lambda X$$

$\Rightarrow$ $X = A^{-1}(\lambda X) = \lambda(A^{-1}X)$

$\Rightarrow$ $\frac{1}{\lambda}X = A^{-1}X \quad [\because \text{ A is non-singular} \Rightarrow \lambda = 0]$

$\Rightarrow$ $A^{-1}X = \frac{1}{\lambda}X$

Therefore, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and X is a corresponding eigenvector.

Q.15 If α is a characteristic root of a non-singular matrix, then prove that $\left[\frac{A}{\alpha}\right]$ is a characteristic root of adj A.

Sol. Since a is a characteristic root of a non-singular matrix, therefore $a \neq 0$. Also a is a characteristic root of A implies that there exists a non-zero vector X such that

$$AX = \alpha X$$

$\Rightarrow$ $(\text{adj } A)(AX) = (\text{adj } A)(\alpha X) \quad \Rightarrow \quad [(\text{adj } A) A] X = \alpha(\text{adj } A) X$

$\Rightarrow$ $|A|\ IX = \alpha(\text{adj } A)X \quad [\because (\text{adj}) A] A = |A| I]$

$\Rightarrow$ $|A| X = \alpha(\text{adj } A)X \quad \Rightarrow \quad \frac{|A|}{\alpha}X = (\text{adj } A)X$

$\Rightarrow$ $(\text{adj } A)X = \frac{|A|}{\alpha}X$

Since X is a non-zero vector, therefore $\left[\frac{A}{\alpha}\right]$ is a characteristic root of the matrix adj A.

Q.16 Solve the following system of equations, using matrix method : $x + 2y + z = 7,\ x + 3z = 11,\ 2x - 3y = 1$.

Sol. The given system of equation is

$$x + 2y + z = 7, \quad x + 0y + 3z = 11, \quad 2x - 3y + 0z = 1$$

or $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$ or $AX = B$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$

Now, $|A| = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} = 18$

So, the given system of equation has a unique solution given by $X = A^{-1}B$

$\therefore$ $\text{adj } A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & 2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{8}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$

Now $X = A^{-1}B$

$$\Rightarrow X = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63-33+6 \\ 42-22-2 \\ -21+77-2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow \; x = 2, y = 1, \; z = 3$

Q.17 If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then find the value of $|A| \; |\text{adj } A|$.

Sol. $|A| \; |\text{adj } A| = |A \text{ adj } A| = | \; |A| \; I| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3 = (a^3)^3 = a^9$

Q.18 For the matrix $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, then x and y so that $A^2 + xI = yA$. Hence, find A^{-1}.

Sol. We have

$$A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9+7 & 3+5 \\ 21+35 & 7+25 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

Now, $A^2 + xI = yA$

$$\Rightarrow \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 16+x & 8+0 \\ 56+0 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$\Rightarrow \quad 16 + x = 3y, \;\; y = 8, \;\; 7y = 56, \;\; 5y = 32 + x$

Putting $y = 8$ in $16 + x = 3y$, we get $x = 24 - 16 = 8$. Clearly, $x = 8$ and $y = 8$ also satisfy $7y = 56$ and $5y = 32 + x$. Hence, $x = 8$ and $y = 8$. We have

$$|A| = \begin{vmatrix} 3 & 1 \\ 7 & 5 \end{vmatrix} = 8 \neq 0$$

So, A is invertible.

Putting $x = 8, y = 8$ in $A^2 = xI = yA$, we get

$A^2 + 8I = 8A$

$\Rightarrow \quad A^{-1}(A^2 + 8I) = 8A^{-1}A$ [re-multiplying througout by A^{-1}]

$\Rightarrow \quad A^{-1}A^2 + 8A^{-1}I = 8A^{-1}A$

$\Rightarrow \quad A + 8A^{-1} = 8I$ [$\because \; A^{-1}A^2 = (A^{-1}A)A = IA = A, \; A^{-1}I = A^{-1}$ and $A^{-1}A = I$]

$\Rightarrow \quad 8A^{-1} = 8I - A$

$$\Rightarrow \quad A^{-1} = \frac{1}{8}(8I - A) = \frac{1}{8}\left\{\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right\}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{8}\begin{bmatrix} 8-3 & 0-1 \\ 0-7 & 8-5 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5/8 & -1/8 \\ -7/8 & 3/8 \end{bmatrix}$$

Q.19 By the method of matrix inversion, solve the system.

$$\begin{bmatrix}1 & 1 & 1\\ 2 & 5 & 7\\ 2 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ x_3 & y_3\end{bmatrix}=\begin{bmatrix}9 & 2\\ 52 & 15\\ 0 & 1\end{bmatrix}$$

Sol. $$\begin{bmatrix}1 & 1 & 1\\ 2 & 5 & 7\\ 2 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ x_3 & y_3\end{bmatrix}=\begin{bmatrix}9 & 2\\ 52 & 15\\ 0 & 1\end{bmatrix}$$

$\Rightarrow \quad AX = B \qquad \ldots(i)$

Clearly $|A| = -4 \neq 0$. Therefore

$$\text{adj } A = \begin{bmatrix}-12 & 16 & -8\\ 2 & -3 & 1\\ 2 & 1 & 3\end{bmatrix}^T = \begin{bmatrix}-12 & 2 & 2\\ 16 & -3 & -5\\ -8 & 1 & 3\end{bmatrix} \qquad \therefore \; A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{-1}{4}\begin{bmatrix}-12 & 2 & 2\\ 16 & -3 & -5\\ -8 & 1 & 3\end{bmatrix}$$

Now, $$A^{-1}B = \frac{-1}{4}\begin{bmatrix}-12 & 2 & 2\\ 16 & -3 & -5\\ -8 & 1 & 3\end{bmatrix}\begin{bmatrix}9 & 2\\ 52 & 15\\ 0 & -1\end{bmatrix} = \frac{-1}{4}\begin{bmatrix}-4 & 4\\ -12 & -8\\ -20 & -4\end{bmatrix} = \begin{bmatrix}1 & -1\\ 3 & 2\\ 5 & 1\end{bmatrix}$$

From equation (i),

$$X = A^{-1}B \;\Rightarrow\; \begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ x_3 & y_3\end{bmatrix} = \begin{bmatrix}1 & -1\\ 3 & 2\\ 5 & 1\end{bmatrix}$$

$\Rightarrow \quad x_1 = 1,\; x_2 = 3,\; x_3 = 5 \quad \text{or} \quad y_1 = -1,\; y_2 = 2,\; y_3 = 1$

Q.20 If A, B and C are $n \times n$ matrix and $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 5$, then find the value of the $\det(A^2BC^{-1})$.

Sol. Given that $|A| = 2$, $|B| = 3$, $|C| = 5$. Now

$$\det(A^2BC^{-1}) = |A^2BC^{-1}| = \frac{|A|^2|B|}{|C|} = \frac{4\times 3}{5} = \frac{12}{5}$$

Q.21 Matrices A and B satisfy $AB = B^{-1}$, where $B = \begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix}$. Find

(i) without finding B^{-1}, the value of K for which $KA = 2B^{-1} + I = 0$

(ii) without finding A^{-1}, the matrix X satsifying $A^{-1}XA = B$.

Sol.

(i) $AB = B^{-1} \quad \Rightarrow \quad AB^2 = I$

Now,

$KA - 2B^{-1} + I = O \quad \Rightarrow \quad KAB - 2B^{-1}B + IB = O \quad \Rightarrow \quad KAB - 2I + B = O$

$\Rightarrow \quad KAB^2 - 2B + B^2 = O \quad \Rightarrow \quad KI - 2B + B^2 = O$

$$\Rightarrow \quad K\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - 2\begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix} + \begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix}\begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} K-2 & 0 \\ 0 & K-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad K = 2$

(ii) $A^{-1}XA = B$

$\Rightarrow \quad AA^{-1}XA = AB \quad \Rightarrow \quad IXA = AB \quad \Rightarrow \quad XAB = AB^2$

$\Rightarrow \quad XAB = I \quad \Rightarrow \quad XAB^2 = B \quad \Rightarrow \quad XI = B$

$\Rightarrow \quad X = B$

Paragraph for question nos. 22 to 24

For $x > 0$, let $A = \begin{bmatrix} x+\frac{1}{x} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 16 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{5x}{x^2+1} & 0 & 0 \\ 0 & \frac{3}{x} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ be two matrices.

Three other matrices X, Y and Z are defined as

$X = (AB)^{-1} + (AB)^{-2} + (AB)^{-3} + \ldots\ldots\ldots\ldots\ldots (AB)^{-n}$, $\quad Y = \lim\limits_{n \to \infty} X$ and $Z = Y^{-1} - 2I$,

where I is identity matrix of order 3.

[**Note:** tr (P) denotes the trace of matrix P.]

Q.22 The value of $\det.\left(\text{adj}\,(\sqrt{5}\,Y^{-1})\right)$ is equal to

(A) $(5!)^2$ (B) $5^3\,(5!)^2$ (C) $5\,(5!)^2$ (D) $5^2\,(5!)^2$

Q.23 Least positive integral value of tr (AY) is equal to

(A) 8 (B) 7 (C) 6 (D) 5

Q.24 If $\text{tr}\,(Z + Z^2 + Z^3 + \ldots\ldots + Z^{10}) = 2^a + b$ where $a, b \in N$, then least value of $(a + b)$ is equal to

(A) 11 (B) 12 (C) 18 (D) 19

Sol. $AB = \begin{bmatrix} \frac{x^2+1}{x} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} \frac{5x}{x^2+1} & 0 & 0 \\ 0 & \frac{3}{x} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$(AB)^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$, $(AB)^{-2} = \begin{bmatrix} \frac{1}{5^2} & 0 & 0 \\ 0 & \frac{1}{3^2} & 0 \\ 0 & 0 & \frac{1}{4^2} \end{bmatrix}$ and so on

$$\therefore \quad X=\begin{bmatrix}\frac{1}{5}+\frac{1}{5^2}+......\frac{1}{5^n} & 0 & 0\\ 0 & \frac{1}{3}+\frac{1}{3^2}+......\frac{1}{3^n} & 0\\ 0 & 0 & \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+........+\frac{1}{4^n}\end{bmatrix}$$

$$Y=\lim_{n\to\infty}X=\begin{bmatrix}\frac{\frac{1}{5}}{1-\frac{1}{5}} & 0 & 0\\ 0 & \frac{\frac{1}{3}}{1-\frac{1}{3}} & 0\\ 0 & 0 & \frac{\frac{1}{4}}{1-\frac{1}{4}}\end{bmatrix}\Rightarrow Y=\begin{bmatrix}\frac{1}{4} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix}$$

$$\therefore \quad Y^{-1}=\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$$

(i) $\left|\operatorname{adj}\left(\sqrt{5}\,Y^{-1}\right)\right| = \left|\left(\sqrt{5}\right)^2\operatorname{adj}\left(Y^{-1}\right)\right| = 5^3\left|\operatorname{adj}\left(Y^{-1}\right)\right| = 5^3\cdot\left|Y^{-1}\right|^2 = 5^3\cdot(4\cdot 2\cdot 3)^2 = 5\cdot(5\,!)^2$

(ii) $$AY=\begin{bmatrix}x+\frac{1}{x} & 0 & 0\\ 0 & \frac{1}{x} & 0\\ 0 & 0 & 16\end{bmatrix}\begin{bmatrix}\frac{1}{4} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix}=\begin{bmatrix}\frac{1}{4}\left(x+\frac{1}{x}\right) & 0 & 0\\ 0 & \frac{1}{2x} & 0\\ 0 & 0 & \frac{16}{3}\end{bmatrix}$$

$$\therefore \quad t_r(AY)=\frac{1}{4}x+\frac{1}{4x}+\frac{1}{2x}+\frac{16}{3}$$

$$=\frac{1}{4}x+\frac{3}{4x}+\frac{16}{3}\geq 2\cdot\sqrt{\frac{1}{4}\cdot x\cdot\frac{3}{4x}}+\frac{16}{3}\geq\frac{\sqrt{3}}{2}+\frac{16}{3}\geq 0.866+5.333\geq 6.199$$

$\therefore$ Least integral value of $t_r(AY)=7$.

(iii) $$Z=Y^{-1}-2I=\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}-2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\therefore \quad t_r(Z+Z^2+Z^3+............Z^{10}) = 2+2^2+2^3+...........+2^{10}+10$$

$$= 2\cdot\left(\frac{2^{10}-1}{2-1}\right)+10 = 2^{11}+8 = 2^a+b$$

$\therefore \quad a+b=11+8=19$ Ans.

Paragraph for question nos. 25 to 27

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ satisfy $A^n = A^{n-2} + A^2 - I \qquad \forall\, n \geq 3$

Further consider a matrix $\underset{3\times 3}{U}$ with its column as U_1, U_2, U_3 such that

$$A^{50}U_1 = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}; \qquad A^{50}U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; A^{50}U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then answer the following

Q.25 Trace of A^{50} equals

(A) 0 (B) 1 (C) 2 (D) 3

Q.26 $|\text{Adj Adj Adj Adj Adj } A^{50}| =$

(A) 2^n (B) 2^5 (C) 2^{25} (D) 2^{50}

Q.27 Find sum of all entries of $UA^{50}U^{-1}$

(A) 50 (B) 51 (C) 52 (D) 53

Sol. $A^n - A^{n-2} = A^2 - I$

$A^{50} - A^{48} = A^2 - I$

$A^{48} - A^{46} = A^2 - I$

.. .. ..

.. .. ..

$A^4 - A^2 = A^2 - I$

$A^{50} - A^2 = 24A^2 - 24I$

$$A^{50} = 25A^2 - 24I \Rightarrow A^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

$|A^{50}| = 1 \qquad ; \qquad \text{tr } A^{50} = 3$

$$A^{50}U_1 = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}$$

$$\Rightarrow \quad U_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ similarly } \quad U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ So, } \underset{3\times 3}{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ **Ans.**

Q.28 Consider $I_{n,m} = \int_0^1 \frac{x^n}{x^m - 1} dx$ and $J_{n,m} = \int_0^1 \frac{x^n}{x^m + 1} dx \ \forall \ n > m$ and $n, m \in N$.

(a) Consider a matrix $A = [a_{ij}]_{3 \times 3}$, where $a_{ij} = \begin{cases} I_{6+i,3} - I_{i+3,3}, & i = j \\ 0, & i \neq j \end{cases}$. Then find trace (A^{-1}).

[**Note:** Trace of a square matrix is sum of the diagonal elements.]

(b) Let $A = \begin{bmatrix} J_{6,5} & 72 & J_{11,5} \\ J_{7,5} & 63 & J_{12,5} \\ J_{8,5} & 56 & J_{13,5} \end{bmatrix}$ and $B = \begin{bmatrix} I_{6,5} & 72 & I_{11,5} \\ I_{7,5} & 63 & I_{12,5} \\ I_{8,5} & 56 & I_{13,5} \end{bmatrix}$,

then find the value of $\det(A) - \det(B)$.

[Ans. (a) 18; (b) 0]

Sol.

(a) Given, $I_{n,m} = \int_0^1 \frac{x^n}{x^m - 1} dx$ and $A = [a_{ij}]_{3 \times 3}$, where $a_{ij} = \begin{cases} I_{6+i,3} - I_{i+3,3}, & i = j \\ 0, & i \neq j \end{cases}$.

Hence, $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

Now, $a_{11} = I_{7,3} - I_{4,3} = \int_0^1 x^4 \frac{(x^3-1)}{(x^3-1)} dx = \frac{1}{5}$

$|||^{ly}$, $a_{22} = I_{8,3} - I_{5,3} = \int_0^1 x^5 \frac{(x^3-1)}{(x^3-1)} dx = \frac{1}{6}$

and $a_{33} = I_{9,3} - I_{6,3} = \int_0^1 x^6 \frac{(x^3-1)}{(x^3-1)} dx = \frac{1}{7}$

Hence, $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$

$\therefore \ A^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix} \Rightarrow$ trace $(A^{-1}) = 5 + 6 + 7 = 18$. **Ans.**

(b) Clearly, $\det.(A) = \begin{vmatrix} J_{6,5} & 72 & J_{11,5} \\ J_{7,5} & 63 & J_{12,5} \\ J_{8,5} & 56 & J_{13,5} \end{vmatrix} = \begin{vmatrix} \frac{1}{7} & 72 & J_{11,5} \\ \frac{1}{8} & 63 & J_{12,5} \\ \frac{1}{9} & 56 & J_{13,5} \end{vmatrix}$

Applying $C_1 \to C_1 + C_3$ and using equation (2), we get

$$= \frac{1}{7 \times 8 \times 9} \begin{vmatrix} 72 & 72 & J_{11,5} \\ 63 & 63 & J_{12,5} \\ 56 & 56 & J_{13,5} \end{vmatrix} = 0. \qquad \text{(As } C_1 \text{ and } C_2 \text{ are identical.)}$$

Similarly, $\det.(B) = \begin{vmatrix} I_{6,5} & 72 & I_{11,5} \\ I_{7,5} & 63 & I_{12,5} \\ I_{8,5} & 56 & I_{13,5} \end{vmatrix} = \begin{vmatrix} I_{6,5} & 72 & \frac{1}{7} \\ I_{7,5} & 63 & \frac{1}{8} \\ I_{8,5} & 56 & \frac{1}{9} \end{vmatrix}$

Applying $C_3 \to C_3 - C_1$ and using equation (1), we get

$$= \frac{1}{7 \times 8 \times 9} \begin{vmatrix} I_{6,5} & 72 & 72 \\ I_{7,5} & 63 & 63 \\ I_{8,5} & 56 & 56 \end{vmatrix} = 0 \qquad \text{(As, } C_2 \text{ and } C_3 \text{ are identical.)}$$