DIFFERENTIAL EQUATION

1. INTRODUCTION :

1.0 An equation that involves independent and dependent variables and at least one derivative of the dependent variable w.r.t independent variable is called a differential equation.

For example : $x\frac{dy}{dx} + y\log x = xe^x\, x^{\frac{1}{2}\log x}, (x > 0)$; $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$

1.1 A differential equation is said to be **ordinary**, if the differential coefficients have reference to a single independent variable only and it is said to be **Partial** if there are two or more independent variables. We are concerned with ordinary differential equations only. While an ordinary differential equation containing two or more dependent variables with their differential coefficients w.r.t. to a single independent variable is called a **total differential equation.**

eg. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ is an ordinary diffrential equation

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$; $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y$ are partial differential equation.

1.2 Order and Degree of Diffrential Equation :

The order of a differential equation is the order of the highest differential coefficient occuring in it. The degree of a differential equation which is expressed or can be expressed as a polynomial in the derivatives is the degree of the highest order derivative occuring in it, after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation :

$$f(x, y)\left[\frac{d^my}{dx^m}\right]^p + \phi(x, y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + ... = 0 \text{ is order } m \text{ \& degree } p .$$

Illustration :

(i) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$ (ii) $\frac{dy}{dx} + y = \frac{1}{dy/dx}$ (iii) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$

(iv) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$ (v) $ln\left(\frac{dy}{dx}\right) = ax + by$

Sol.

(i) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4} \Rightarrow \left(\frac{d^2y}{dx^2}\right)^4 = \left[y + \left(\frac{dy}{dx}\right)^6\right]$

Hence order is 2 and degree is 4.

(ii) $\frac{dy}{dx} + y = \frac{1}{dy/dx} \Rightarrow \left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right) = 1$

Hence order is 1 and degree is 2.

(iii) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$

Clearly order is 3, but degree is not defined as it cannot be written as a polynomial equation in derivatives.

(iv) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y \Rightarrow \frac{dy}{dx} = \sin(x + y)$

Hence order is 1 and degree is 1.

(v) $ln\left(\frac{dy}{dx}\right) = ax + by \Rightarrow \frac{dy}{dx} = e^{ax + by}$

Hence order is one and degree is also 1.

Practice Problem

Q.1 Find the order and degree (if defined) of the following differential equations

(i) $\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^4\right\}^{5/3}$ (ii) $\frac{d^3y}{dx^3} = x\,ln\left(\frac{dy}{dx}\right)$

(iii) $\left(\frac{d^4y}{dx^4}\right)^3 + 3\left(\frac{d^2y}{dx^2}\right)^6 + \sin x = 2\cos x$ (iv) $\left(\frac{d^3y}{dx^3}\right)^{2/3} + 4 - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$

(v) $\rho = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$ (vi) $\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^3+6y=0$

(vii) $y(x) = 1 + \dfrac{dy}{dx}+\dfrac{1}{1.2}\left(\dfrac{dy}{dx}\right)^2+\dfrac{1}{1.2.3}\left(\dfrac{dy}{dx}\right)^3+......$ (viii) $\left(1+4\dfrac{dy}{dx}\right)^{2/3}=4\dfrac{d^2y}{dx^2}$

Answer key

Q.1	(i)	order = 2, degree = 3 ;	(ii)	order = 3, degree is not defined
	(iii)	order = 4, degree = 3 ;	(iv)	order = 3, degree = 2
	(v)	order = 2, degree = 2 ;	(vi)	order = 2, degree = 1
	(vii)	order = 1, degree = 1 ;	(viii)	order = 2, degree = 3

2. FORMATION OF DIFFERENTIAL EQUATIONS :

Consider a family of curves

$f(x, y, c_1, c_2, ...c_n) = 0$ ...(i)

where $c_1, c_2, ..., c_n$ are n independent parameters.

Equation (i) is known as an n parameter family of curves e.g. $y = mx$ is 1-parameter family of straight lines $x^2 + y^2 + ax + by = 0$ is a two-parameter family of circles.

If we differentiate equation (i) n times w.r.t x, we will get n more relations between $x, y, c_1, c_2, ... c_n$ and derivates of y with respect to x. By eliminating $c_1, c_2, ..., c_n$ from these n relations and equation (i), we get a differential equation.

Clearly order of this differential equation will be n, i.e., equal to the number of independent parameters in the family of curves.

Illustration :

From the differential equation of family of lines concurrent at the origin.

Sol. *Such lines are given by*

$y = mx$ ...(i)

$\Rightarrow \dfrac{dy}{dx} = m$

Putting the value of m is equation (i)

$$\Rightarrow \quad y = \frac{dy}{dx}x$$

$$\Rightarrow \quad xdy - ydx = 0$$

Note that the order is 1, same as number of constants.

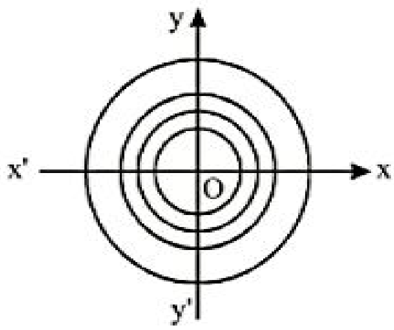
Illustration :

From the differential equation of all concentric circle at the origin.

Sol. *Such circles are given by*

$$x^2 + y^2 = r^2$$

Differentiating w.r.t. x,



$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 0$$

Illustration :

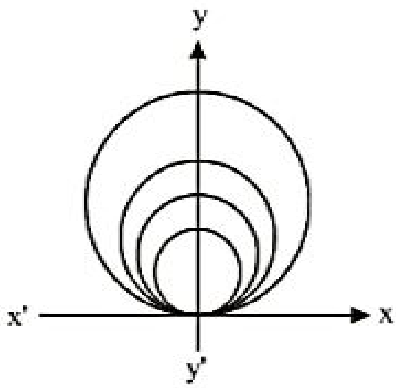
From the differential equation of all circles touching the x-axis at the origin and centre on y-axis.

Sol. *Such family of circles is given by*

$$x^2 + (y - a)^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ay = 0 \qquad ...(1)$$

Differentiating,



$$2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx}$$

$$\text{or} \quad x + y\frac{dy}{dx} = a\frac{dy}{dx}$$

substituting the value of a in equation (1)

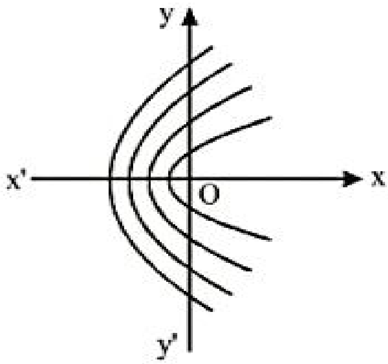
$$\Rightarrow \quad (x^2 - y^2)\frac{dy}{dx} = 2xy \qquad (\text{order} = 1 \text{ and degree} = 1)$$

From the differential equation of the family of parabolas with focus at the origin and axis of symmetry along the x-axis.

Sol. *Equation of such parabolas is* $y^2 = 4A(A + x)$

Differentiating w.r.t., we get

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4A \qquad \Rightarrow \quad y\frac{dy}{dx} = 2A$$



Eliminating A from equations (2) and (1)

$$y^2 = \left(y\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx}x \quad or \quad y^2 = y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

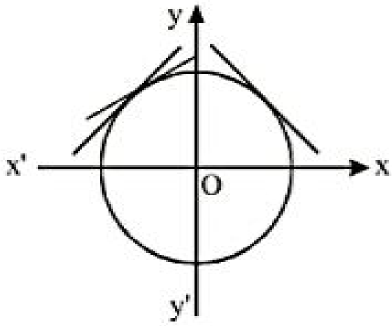
which has order 1 and degree 2.

Illustration :

Find the differential equation of family of lines situated at a constant distance p from the origin.

Sol. *All such lines are tangent to circle of radius p.*

$$y = mx + p\sqrt{1+m^2} \quad \Rightarrow \quad m = \frac{dy}{dx}$$



By eliminating m, we get

$$y = \frac{dy}{dx}x + p\sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \left(y - \frac{dy}{dx}x\right)^2 = p^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

which has order 1 and degree 2.

Practice Problem

Q.1 Find the differential equation of all the parabola having axis parallel to x-axis.

Q.2 Find the differential equation of all ellipse whose centre is at origin and axis are co-ordinate axis.

Q.3 Consider the equation $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ where a and b are specified constant and λ is an arbitrary parameter. Find a differential eqation satisfied by it.

Q.4 Find the degree of the differential equation satisfying the relation

$$\sqrt{1+x^2}+\sqrt{1+y^2}=\lambda\left(x\sqrt{1+y^2}-y\sqrt{1+x^2}\right)$$

Q.5 Find the differential equation of all non-vertical lines in a plane.

Q.6 Form the differential equation

(i) $y=A+Bx+Ce^{-x}$ (ii) $y=e^{ax}\sin bx$ (iii) $y=ax\cos\left(\frac{1}{x}+b\right)$ (iv) $\sin^{-1}x+\sin^{-1}y=c$

Answer key

Q.1 $\frac{d^3y}{dx^3}=0$

Q.2 $x\left(yy''+(y')^2\right)=yy'$

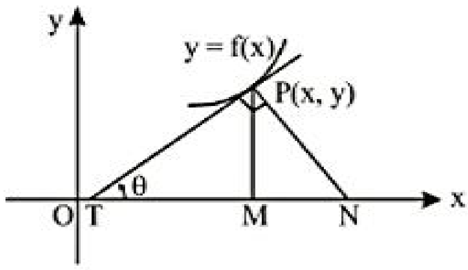
Q.3 $a^2-b^2=x^2-xy\frac{dx}{dy}-y^2+xy\frac{ydy}{dx}$

Q.4 degree 1

Q.5 $\frac{d^2y}{dx^2}=0$

Q.6 (i) $y'''+y''=0$, (ii) $y''-2ay'+(a^2+b^2)y=0$, (iii) $\sqrt{1-x^2}dy+\sqrt{1-y^2}dx=0$ (iv) $x^2y''+y=0$

3. LENGTH OF TANGENT, NORMAL SUB-TANGENT, SUB-NORMAL:



(i) Length of Tangent :

PT is defined as length of the tangent.

In ΔPMT, $PT=|y\ \text{cosec}\ \theta|$

$$=\left|y\sqrt{(1+\cot^2\theta)}\right| \quad \Rightarrow \quad =\left|y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\right|$$

$\Rightarrow$ Length of tangent $=\left|y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\right|$

(ii) Length of Normal :

PN is defined as length of the normal.

In ΔPMN, PN = $|y \operatorname{cosec}(90^\circ - \theta)|$

$$= |y \sec\theta| \quad \Rightarrow \quad = \left| y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} \right|$$

$$\Rightarrow \quad \text{Length of normal} = \left| y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} \right|$$

(iii) Length of Sub-tangent :

TM is defined as sub-tangent.

$$\text{In } \Delta PTM,\ TM = |y\cot\theta| = \left|\frac{y}{\tan\theta}\right| = \left|y\frac{dx}{dy}\right|$$

$$\Rightarrow \text{Length of sub-tangent} = \left|y\frac{dx}{dy}\right|$$

(iv) Length of Sub-normal :

MN is defined as sub-normal.

$$\text{In } \Delta PMN,\ MN = |y\cot(90^\circ - \theta)| = |y\tan\theta| = \left|y\frac{dy}{dx}\right|$$

$$\Rightarrow \text{Length of sub-normal} = \left|y\frac{dy}{dx}\right|$$

4. SOLUTION OF A DIFFERENTIAL EQUATION :

Elementary types of first order & first degree differential equations.

Elementry Type of first order and first degree

- Variable Separable
  - T-1 normal
  - T-2 $\frac{dy}{dx} = f(ax + by + z)$
- Homogeneous
  - Reducible to Homogeneous
- Linear
  - Reducible to linear (Bernouli's D.E.)

4.1 Variables Separable :

If the differential equation can be expressed as; f(x)dx + g(y)dy = 0 then this is said to be variable -separable type.

A general solution of this is given by $\int f(x)\,dx + \int g(y)\,dy = c$; where c is the arbitrary constant.

Illustration :

Solve the following differential equation

(i) $\ln\dfrac{dy}{dx} = 3x + 4y$ with $y(0) = 0$ (ii) $x(y^2 + 1)\,dx + y(x^2 + 1)\,dy = 0$

(iii) $y'\sin x = y\ln y$; $y\left(\dfrac{\pi}{2}\right) = e$ (iv) $(dy/dx) = e^{x-y} + x^2.\,e^{-y}$.

Sol.

(i) $\dfrac{dy}{dx} = e^{3x}\,e^{4y} \Rightarrow \int e^{-4y}dy = \int e^{3x}dx + C \Rightarrow \dfrac{-e^{4y}}{4} = \dfrac{e^{3x}}{3} + C \Rightarrow C = \dfrac{-1}{4} - \dfrac{1}{3} = \dfrac{-7}{12}$

Hence $4e^{3x} + 3e^{-4y} = 7$.

(ii) $x(y^2 + 1)\,dx + y(x^2 + 1)\,dy = 0$

$$\Rightarrow \int \frac{2x\,dx}{x^2+1} = -\int \frac{2y\,dy}{(y^2+1)} \Rightarrow \ln(x^2+1) = -\ln(y^2+1) + \ln C$$

Hence $(y^2 + 1)(x^2 + 1) =$ *where C is any constant.*

(iii) $\dfrac{dy}{dx}\sin x = y\log y \Rightarrow \int \dfrac{dy}{y\ln y} = \int \operatorname{cosec} x\,dx$

$\Rightarrow \ln(\ln y) = \ln(\operatorname{cosec} x - \cot x) + \ln C$

$\Rightarrow \ln y = C(\operatorname{cosec} x - \cot x)$

at $x = \dfrac{\pi}{2}, y = e$; so $\ln(e) = C(1 - 0) \Rightarrow C = 1$

$\Rightarrow y = e^{(\operatorname{cosec} x - \cot x)}$

(iv) $\dfrac{dy}{dx} = \left(e^x + x^2\right)e^{-y} \Rightarrow e^y\,dy = (e^x + x^2)\,dx \Rightarrow e^y = e^x + \dfrac{x^3}{3} + C$.

Illustration :

Find the foci of the conic passing through the point (1, 0) and satisfying the differential equation $(1+y^2)\,dx - xy\,dy = 0$. Find also the equation of a circle touching the conic at $(\sqrt{2}, 1)$ and passing through one of its foci.

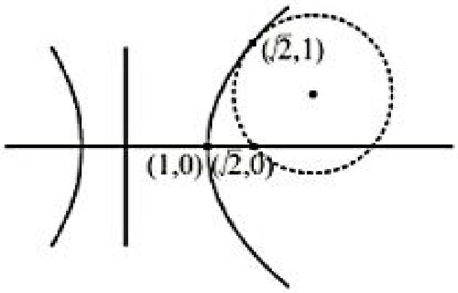
Sol. $(1+y^2)\,dx = xy\,dy \Rightarrow \int \frac{dx}{x} = \int \frac{2y}{2(1+y^2)}\,dy$

$$\Rightarrow \ln x = \frac{1}{2}\ln(1+y^2) + \frac{\ln C}{2} \Rightarrow x^2 = C(1+y^2)$$

at $\quad x = 1, y = 0 \Rightarrow 1 = C(1+0) \Rightarrow C = 1$ so $x^2 - y^2 = 1$

Equation of circle can be written as

$$(x-\sqrt{2})^2 + (y-1)^2 + \lambda L = 0$$



where L is tangent of hyperbola $x^2 - y^2 = 1$ at $(\sqrt{2}, 1)$

i.e., $L : \sqrt{2}\,x - y = 1$

so circle $(x-\sqrt{2})^2 + (y-1)^2 + \lambda(\sqrt{2}x - y - 1) = 0$

when this passes through focus $(\sqrt{2}, 0)$ so $0 + 1 + \lambda(1) = 0 \Rightarrow \lambda = -1$.

so equation of circle : $(x-\sqrt{2})^2 + (y-1)^2 - 1(\sqrt{2}x - y - 1) = 0$

when this passes through another focus $(-\sqrt{2}, 0)$.

so $\quad (-2\sqrt{2})^2 + (0-1)^2 + \lambda\left((\sqrt{2})(-\sqrt{2}) - 0 - 1\right) = 0$

$\quad 8 + 1 + \lambda(-3) = 0 \Rightarrow \lambda = 3$

Hence equation of circle, $(x-\sqrt{2})^2 + (y-1)^2 + 3(\sqrt{2}x - y - 1) = 0$.

Practice Problem

Q.1 Find the solution of the following differential equations

(i) $x^2\frac{dy}{dx} = 2$ (ii) $\frac{dy}{dx} = x\log x$ (iii) $\frac{dy}{dx} = e^{y+x} + e^{y-x}$

(iv) $\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} = 0$ (v) $\frac{dy}{dx}\tan y = \sin(x+y) + \sin(x-y)$

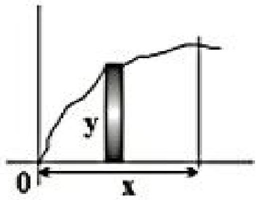
(vi) $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$ (vii) $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

Q.2 Solve $e^{\frac{dy}{dx}} = x + 1$, given that when $x = 0, y = 3$.

Q.3 Find the curve for which the segment of the tangent contained between the co-ordinate axes is bisected by the point. Curve passes through (2, 3).

Q.4 Find the $y = f(x)$ $(f(x) \geq 0$ and $f(0) = 0)$ bounding a curvilinear trapezoid with the base [0, x] if area bounded by curve, coordinate axes & var. ordinate.

Area $\propto (f(x))^{n+1}$ and $f(1) = 1$



[Hint : $\int_0^x y\,dx = K.(f(x))^{n+1}$; now differeniate both sides]

Q.5 Show that the curve passing through (1, 2) for which the segment of the tangent between P and T is bisected at its point of intersection with the y-axis is a parabola.

Answer key

Q.1 (i) $y = c - \frac{2}{x}$, (ii) $y = \frac{x^2}{2}\log x - \frac{x^2}{4} + c$, (iii) $e^{-y} = e^{-x} - e^{x} + c$, (iv) $y + \sin^{-1} x = c$

(v) $\sec y + 2\cos x = c$, (vi) $x\sqrt{1-y^2} + y\sqrt{1-x^2} = c$, (vii) $\tan y - \cot x = c$

Q.2 $y = (x + 1)\, ln\, (x + 1) - (x + 1) + 4$ Q.3 $xy = 6$ Q.4 $f(x) = x^{1/n}$ Q.5 $y^2 = 4x$

4.2 Differential Equation Reducible to the Separable Variable Type:

$$\frac{dy}{dx} = f(ax + by + c), \; a, b \neq 0$$

To solve this, substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Illustration :

Solve $\frac{dy}{dx} = (x + y)^2$.

Sol. $\frac{dy}{dx} = (x + y)^2$ ...(i)

Here the variable are not separable but by putting

$x + y = v$, *we have*

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$\Rightarrow$ *Equation (i) reduces to*

$$\frac{dv}{dx} = v^2 + 1 \quad or \quad \int \frac{dv}{v^2+1} = \int dx \qquad ...(ii)$$

in which variables are seperated.

Hence from equation (ii),

$\tan^{-1} v = x + c$ or $x + y = \tan(x + c)$, *which is a required solution.*

Illustration :

Solve $\dfrac{dy}{dx}\sqrt{1+x+y} = x + y - 1.$

Sol. *Putting* $\sqrt{1+x+y} = v$, *we have*

$\Rightarrow \quad x + y - 1 = v^2 - 2$

$\Rightarrow \quad 1 + \dfrac{dy}{dx} = 2v\dfrac{dv}{dx}$

Then the given equation transforms to

$$\left(2v\frac{dv}{dx} - 1\right)v = v^2 - 2$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{v^2+v-2}{2v^2} \qquad \Rightarrow \quad \int \frac{2v^2}{v^2+v-2}dv = \int dx$$

$$\Rightarrow \quad 2\int\left[1 + \frac{1}{3(v-1)} - \frac{4}{3(v+2)}\right]dv = \int dx \quad \Rightarrow \quad 2\left[v + \frac{1}{3}\log|v-1| - \frac{4}{3}\log|v+2|\right] = x + c$$

where $v = \sqrt{1+x+y}$

4.3 Differential Equation of the Form :

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad \text{where } b_1 + a_2 = 0$$

Following illustration will clarify the concept.

Illustration :

Solve the differential Equation $\frac{dy}{dx} = \frac{x-y}{x+y}$.

Sol. $\frac{dy}{dx} = \frac{x-y}{x+y}$

$x\,dy + y\,dy = x\,dx - y\,dx$

$x\,dy + y\,dx + y\,dy = x\,dx$

$\Rightarrow \quad d(xy) + y\,dy = x\,dx \quad \Rightarrow \quad \int d(xy) + \int y\,dy = \int x\,dx + \frac{C}{2}$.

$\Rightarrow \quad xy + \frac{y^2}{2} = \frac{x^2}{2} + \frac{C}{2} \quad \Rightarrow \quad 2xy + y^2 = x^2 + C.$

4.4 Polar Coordinates :

Sometimes transformation to the polar co–ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials. If $x = r\cos\theta$; $y = r\sin\theta$ where r and θ both are variable.

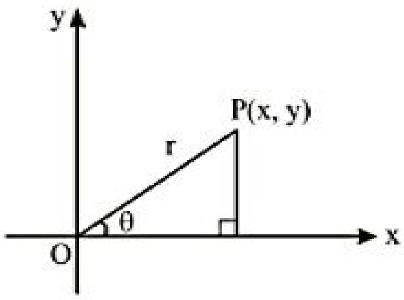
(a) (i) $x\,dx + y\,dy = r\,dr$ (ii) $x\,dy - y\,dx = r^2\,d\theta$

Proof : $x = r\cos\theta$; $y = r\sin\theta$ $\Rightarrow x^2 + y^2 = r^2$

$\Rightarrow x\,dx + y\,dy = rdr$

Also $\tan\theta = y/x$ $\Rightarrow$ $xdy - y\,dx = x^2\sec^2\theta\,d\theta$

$\Rightarrow$ $xdy - ydx = r^2\,d\theta$



(b) If $x = r\sec\theta$ & $y = r\tan\theta$ then

(i) $x\,dx - y\,dy = r\,dr$ and (ii) $x\,dy - y\,dx = r^2\sec\theta\,d\theta$.

Proof : $x = r\sec\theta$ and $y = r\tan\theta$

$\Rightarrow x^2 - y^2 = r^2 \Rightarrow xdx - ydy = rdr$

$y/x = \sin\theta \Rightarrow xdy - ydx = x^2\cos\theta\,d\theta = r^2\sec\theta\,d\theta$

Illustration :

Solve the following differential Equation

(i) $xdx + ydy = x(\,xdy - ydx\,)$ *(ii)* $\dfrac{x + y\frac{dy}{dx}}{x\frac{dy}{dx} - y} = \sqrt{\dfrac{1 - x^2 - y^2}{x^2 + y^2}}$

(iii) $\dfrac{xdx + ydy}{\sqrt{x^2 + y^2}} = \dfrac{ydx - xdy}{x^2}$

Sol.

(i) $x\,dx + y\,dy = x\,(x\,dy - y\,dx)$

Let $x = r\cos\theta,\ y = r\sin\theta$

$\therefore\quad r\,dr = r\cos\theta\,(r^2\,d\theta)$

$$\int\frac{dr}{r^2} = \int\cos\theta\,d\theta \Rightarrow \frac{-1}{r} = \sin\theta + c$$

$\Rightarrow 1 + r\sin\theta = -cr \Rightarrow (1 + r\sin\theta)^2 = c^2r^2$

$\Rightarrow (1 + y)^2 = c^2(x^2 + y^2)$

(ii) $$\frac{x + y\dfrac{dy}{dx}}{x\dfrac{dy}{dx} - y} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}} \quad\Rightarrow \frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1-(x^2+y^2)}{(x^2+y^2)}}$$

put $x = r\cos\theta,\ y = r\sin\theta$ so $x\,dx + y\,dy = r\,dr$; $x\,dy - y\,dy = r^2\,d\theta$

$$\Rightarrow\quad \frac{r\,dr}{r^2 d\theta} = \frac{\sqrt{1-r^2}}{r} \Rightarrow \int\frac{dr}{\sqrt{(1-r^2)}} = \int d\theta \Rightarrow \sin^{-1}(r) = \theta + c$$

$$\Rightarrow\quad r = \sin(\theta + c)\ ;\ \sqrt{(x^2+y^2)} = \sin(\theta + c) \text{ where } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

(iii) $$\frac{x\,dx + y\,dy}{\sqrt{(x^2+y^2)}} = \frac{y\,dx - x\,dy}{x^2}$$

put $x = r\cos\theta,\ y = r\sin\theta$

$x\,dx + y\,dy = r\,dr$; $x\,dy - y\,dx = r^2\,d\theta$

$$\frac{r\,dr}{r} = \frac{-r^2 d\theta}{r^2\cos^2\theta} \Rightarrow \int dr = -\int\sec^2\theta\,d\theta$$

$$r = -\tan\theta + c \Rightarrow \sqrt{(x^2+y^2)} = -\frac{y}{x} + c$$

so $$\sqrt{(x^2+y^2)} + \frac{y}{x} = c.$$

4.5 Homogeneous Equations :

The function f(x, y) is said to be a homogeneous function of degree n if for any real number t ($\neq 0$), we have $f(tx, ty) = t^n (x, y)$. For example, $f(x, y) = ax^{2/3} + hx^{1/3} \times y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3.

Homogeneous Differential Equation :

A differential equaiton of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, where f(x, y) and f(x, y) are homogeneous function of x and y, and of the same degree, is called Homogeneous. This equation may also be reduced to the form $\frac{dy}{dx} = g\left(\frac{x}{y}\right)$ and is solved by putting y = vx so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variable separable.

Illustration :

Solve $x^2\,dy + y(x + y)\,dx = 0$.

$$\frac{dy}{dx} = -\frac{y(x+y)}{x^2} \quad or \quad \frac{dy}{dx} = -\frac{y}{x} - \frac{y^2}{x^2}$$

Sol. *Putting y = vx*

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Given equation transforms to

$$v + x\frac{dv}{dx} = -v - v^2$$

$$\Rightarrow \int \frac{dv}{v^2 + 2v} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int \left[\frac{1}{v} - \frac{1}{v+2}\right] dv = -\int \frac{dx}{x}$$

$$\Rightarrow \log|v| - \log|v+2| = -2\log|x| + \log c \ (c > 0)$$

$$\Rightarrow \left|\frac{vx^2}{v+2}\right| = c$$

$$\Rightarrow \left|\frac{x^2 y}{2x+y}\right| = c \ (c > 0)$$

Solve $\left(x\sin\frac{y}{x}\right)dy=\left(y\sin\frac{y}{x}-x\right)dx$.

Sol. *Putting* $y = vx$, *we get* $dy = vdx + xdv$

$\Rightarrow x\sin v\,(vdx + xdv) = x(v\sin v - 1)\,dx$

$$\Rightarrow \sin v dv + \frac{dx}{x} = 0$$

Integrating, we get $\cos v = \ln x + c$

$$\Rightarrow \cos\frac{y}{x} = \ln x + c$$

4.6 Equations Reducible to the Homogenous Form :

Equation of the form $\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C}$ $(aB\neq Ab \text{ and } A+b\neq 0)$ can be reduced to a homogeneous form by changing the variable x, y, to X, Y by writing $x=X+h$ and $y=Y+k$; where h, k are constant to be chosen so as to make the given equation homogeneous. We have

$$\frac{dy}{dx}=\frac{d(Y+k)}{d(X+h)}=\frac{dY}{dX}$$

Hence the given equation becomes,

$$\frac{dY}{dX}=\frac{aX+bY+(ah+bk+c)}{Ah+Bk+(Ah+Bk+C)}$$

Let h and k be chosen to satisfy the relation $ah+bk+c=0$ and $Ah+Bk+C=0$.

Illustration :

Solve $x\frac{dy}{dx} = y + 2\sqrt{y^2-x^2}$.

Sol. *Putting,* $y = vx$ *and* $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$xv + x^2\frac{dv}{dx} = vx + 2x\sqrt{v^2-1} \Rightarrow \frac{dv}{2\sqrt{v^2-1}} = \frac{dx}{x}$, *integrating, we get*

$$\frac{1}{2}\ln\left(v+\sqrt{v^2-1}\right)=\ln(cx) \Rightarrow \frac{1}{2}\ln\left(\frac{y+\sqrt{y^2-x^2}}{x}\right)=\ln(cx)$$

$$\Rightarrow y+\sqrt{y^2-x^2}=c^2x^3$$

Illustration :

Solve x(dy/dx) = y(log y – log x + 1).

Sol. $x\left(\frac{dy}{dx}\right) = y\left(\log y - \log x + 1\right) \Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\log\left(\frac{y}{x}\right) + 1\right)$

Putting $y = vx$, *we get* $\frac{dy}{dx} = v + x\frac{dv}{dx}$

and the given equation transforms to $v + x\frac{dv}{dx} = v[\log v + 1] \Rightarrow x\frac{dv}{dx} = v \log v$

$$\Rightarrow \int \frac{dv}{v\log v} = \int \frac{dx}{x} \Rightarrow \log(\log v) = \log x + \log k,\ k > 0$$

$$\Rightarrow \log(v) = kx \Rightarrow \frac{y}{x} = e^{kx} \Rightarrow y = x\,e^{kx}\ ;\ \text{where } k > 0.$$

Illustration :

Solve (x + y sin (y/x))dx = x sin (y/x) dy.

Sol. $\frac{dy}{dx} = \frac{x + y\sin\left(\frac{y}{x}\right)}{x\sin\left(\frac{y}{x}\right)}$ *or* $\frac{dy}{dx} = \frac{1 + \left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)}$

Put $y = vx$ *then* $\frac{dy}{dx} = v + x\frac{dv}{dx}$ *and the given equation transforms to* $v + x\frac{dv}{dx} = \operatorname{cosec}(v) + v$

$\Rightarrow \sin v\,dv = \frac{dx}{x}$ *Integrating and replacing v by* $\frac{y}{x}$, *we get ;*

$$\cos\left(\frac{y}{x}\right) + \log|x| = c\ ,\ c \in R.$$

Illustration :

Solve (2x – y + 1) dx + (2y – x + 1) dy

Sol. $(2x - y + 1)\,dx = -(2y - x + 1)\,dy$

$\frac{dy}{dx} = \frac{-(2x - y + 1)}{(2y - x + 1)}$ *Let* $y = Y + k$ *and* $x = X + h$

then $\frac{dy}{dx} = \frac{dY}{dX} = \frac{-(2X - Y + 2h - k + 1)}{(2Y - X + 2k - h + 1)}$

Now, 'h' and 'k' can be chosen such that $2h - k + 1 = 0$ ......(1)

and $2k - h + 1 = 0$ ......(2)

By solving these equation $h = -1,\ k = -1$.

So $\frac{dy}{dx} = \frac{-(2X - Y)}{(2Y - X)}$. *Now homogenous equation so put* $Y = vX$; $\frac{dY}{dX} = v + x\frac{dv}{dX}$

$$v + X\frac{dv}{dX} = \frac{-(2-v)}{(2v-1)} \Rightarrow X\frac{dv}{dX} = -\left(\frac{2-v}{2v-1} + v\right)$$

$$X\frac{dv}{dX} = \left(\frac{2-v+2v^2-v}{2v-1}\right) \Rightarrow \int\frac{(2v-1)}{(2v^2-2v+2)}dv = -\int\frac{dX}{X}$$

$$\Rightarrow \frac{1}{2}\int\frac{(2v-1)}{(v^2-v+1)}dv = -\int\frac{dX}{X} \Rightarrow \frac{1}{2}\ln(v^2 - v + 1) = -\ln X + \ln k$$

$$\Rightarrow \ln\left(\sqrt{(v^2-v+1)}\,X\right) = \ln k$$

$$\Rightarrow X\sqrt{(v^2-v+1)} = k \Rightarrow \sqrt{(Y^2 - XY + X^2)} = k$$

$$\Rightarrow (y+1)^2 - (y+1)(x+1) + (x+1)^2 = k^2.$$

Practice Problem

Q.1 Solve the following equations

(i) $\frac{dy}{dx} = \frac{x - 2y + 5}{2x + 3y - 1}$ (ii) $\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$ (iii) $\frac{dy}{dx} = \cos(x+y) - \sin(x+y)$

Q.2 Solve the following equations

(i) $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ (ii) $\frac{dy}{dx} = \frac{x}{2y - x}$ (iii) $x + y\left(\frac{dy}{dx}\right) = 2y$

(iv) $y' = \frac{x - y}{x + y}$ (v) $(2x - y + 1)dx + (2y - x + 1)dy = 0$

(vi) $xdy - ydx = (\sqrt{x^2 + y^2})\,dx$ (vii) $(x + y)dx + xdy = 0$ (viii) $x(x - y)\frac{dy}{dx} = y(x + y)$

Answer key

Q.1 (i) $4xy + 3y^2 - 2y = x^2 + 10x + c$, (ii) $x + y = ce^{y-x}$ (iii) $\ln\left|1 - \tan\frac{x+y}{2}\right| + x + c = 0$

Q.2 (i) $cy^2 = e^{\frac{x^2}{y^2}}$, (ii) $(x - y)(x + 2y)^2 = c$, (iii) $\log(y - x) - \frac{x}{y-x} = c$,

(iv) $y^2 + 2xy - x^2 = c$, (v) $y^2 + x^2 - xy + x + y = c$ (vi) $y + \sqrt{x^2 + y^2} = cx^2$,

(vii) $x^2 + 2xy = c$, (viii) $\frac{x}{y} + \log xy = c$

4.7 Linear Differential Equations :

A differential equation is said to be linear if the dependent variable & all its differential coefficients occur in degree one only and are never multiplied together.

The nth order linear differential equation is of the form ;

$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_n(x) \, . \, y = \phi(x)$. Where $a_0(x)$, $a_1(x)$ ..... $a_n(x)$ are the coefficients of the differential equation.

Linear Differential Equations of First Order :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x (Independent variable).

For solving such equations we multiply both sides by

Integrating factor = I.F. = $e^{\int P dx}$

So we get $e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int P dx}$

$$\Rightarrow \frac{dy}{dx}e^{\int P dx} + yPe^{\int P dx} = Qe^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx} \quad \left[\text{Since } \frac{d}{dx}\left(e^{\int P dx}\right) = Pe^{\int P dx}\right]$$

$$\Rightarrow \int \frac{d}{dx}\left(ye^{\int P dx}\right)dx = \int Q\left(e^{\int P dx}\right)dx$$

$$\Rightarrow ye^{\int P\,dx} = \int Qe^{\int P\,dx} + C$$

which is the required solution of the given differential equation.

In some cases a linear differential equation may be of the form $\frac{dx}{dy} + P_1x = Q_1$, where P_1 and Q_1 are functions of y alone or constants. In such a case the integrating factor is $e^{\int P_1 dy}$, and solutions is given by

$$x\,e^{\int P_1 dy} = \int Q_1 e^{\int P_1 dy}\,dy + C$$

Illustration :

Solve $x^2\ (dy/dx) + y = 1$.

Sol. *The given diffential equation can be written as*

$$\frac{dy}{dx} + \frac{1}{x^2}y = \frac{1}{x^2}, \text{ which is linear}$$

Here $P = 1/x^2$ and $Q = \frac{1}{x^2}$

$$I.F. = e^{\int (1/x^2)dx} = e^{-1/x}$$

$$\Rightarrow \quad ye^{\frac{-1}{x}} = \int e^{\frac{-1}{x}} \cdot \frac{1}{x^2}\,dx = e^{\frac{-1}{x}} + C$$

$$\Rightarrow \quad y = 1 + Ce^{\frac{1}{x}}$$

Illustration :

Solve the following differential equations

(i) $\cos^2 x \frac{dy}{dx} + y = \tan x$ *(ii)* $x\ln x \frac{dy}{dx} + y = 2\ln x$

(iii) $x(x^2+1)\frac{dy}{dx} = y(1-x^2) + x^2 \ln x$ *(iv)* $\frac{dy}{dx} = \frac{1}{x\cos y + \sin 2y}$

Sol.

(i) $\cos^2 x \frac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \sec^2 x\, y = \tan x \sec^2 x$$

Integrating Factor $= e^{\int \sec^2 x\,dx} = e^{\tan x}$.

$$y\,e^{\tan x} = \int e^{\tan x} \tan x \sec^2 x\, dx \qquad \text{Let } t = \tan x \text{ so } dt = \sec^2 x\, dx$$

$$\int e^{\tan x} \tan x \sec^2 x dx = \int e^t\, t\, dt = (t-1)\, e^t$$

$$\text{so } y\, e^{\tan x} = (\tan x - 1)\, e^{\tan x} + k$$

$$y = (\tan x - 1) + k\, e^{-\tan x}$$

(ii) $x \ln x \dfrac{dy}{dx} + y = 2\ln x$

$$\frac{dy}{dx} + \frac{y}{x \ln x} = \frac{2}{x}$$

$$\text{Integrating factor} = e^{\int \frac{dx}{x \ln x}} = e^{\ln(\ln x)} = \ln x$$

$$\text{So, } y\,(\ln x) = \int \frac{2}{x} \ln x\, dx = (\ln x)^2 + c$$

(iii) $x\,(x^2+1)\dfrac{dy}{dx} = y\,(1-x^2) + x^2 \ln x$

$$\frac{dy}{dx} + \frac{(x^2-1)\,y}{x\,(x^2+1)} = \frac{x^2 \ln x}{x\,(x^2+1)}$$

$$\int \frac{(x^2-1)\,dx}{x(x^2+1)} = \int \frac{x}{(x^2+1)} dx - \int \frac{dx}{x\,(x^2+1)} = \frac{1}{2}\int \frac{2x}{(x^2+1)} dx - \int \left(\frac{1}{x} - \frac{2x}{2\,(x^2+1)}\right) dx$$

$$= \frac{1}{2} \ln (x^2+1) - \ln x + \frac{1}{2} \ln (x^2+1) = \ln\left(\frac{x^2+1}{x}\right)$$

$$\text{Integrating factor} = e^{\ln_e\left(\frac{x^2+1}{x}\right)} = \left(\frac{x^2+1}{x}\right)$$

$$\text{So, } y\left(\frac{x^2+1}{x}\right) = \int \left(\frac{x^2+1}{x}\right) \cdot \frac{x^2 \ln x}{x\,(x^2+1)} dx = \int \ln x\, dx$$

$$\Rightarrow y\left(\frac{x^2+1}{x}\right) = x\,(\ln x - 1) + C.$$

(iv) $\dfrac{dy}{dx} = \dfrac{1}{x\cos y + \sin 2y} \Rightarrow \dfrac{dx}{dy} = x \cos y + \sin 2y$

$$\frac{dy}{dx} - (\cos y)\, x = (\sin 2y)$$

$$\text{Integrating factor} = e^{-\int (\cos y)\,dy} = e^{-\sin y}$$

$$x\, e^{-\sin y} = \int e^{-\sin y}\, 2 \sin y \cos y\, dy \qquad \text{Put } \sin y = t,\ \cos y = \frac{dt}{dy}$$

$$\int e^{-\sin y}\, 2\sin y \cos y\, dy = \int e^{-t}\,(2t)\,dt = 2\,[-t\,e^{-t} - e^{-t}] = -2\,e^{-t}\,(t+1) = -2\,e^{-\sin y}\,(\sin y + 1)$$

$$\therefore\ x\, e^{-\sin y} = -2\, e^{-\sin y}\,(\sin y + 1) + C$$

$$x = -2\,(\sin y + 1) + c\, e^{\sin y}.$$

Practice Problem

Q.1 Find the intergrating factor for the following linear differential equations

(i) $\frac{dy}{dx} + y \tan x - \sec x = 0$ (ii) $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$

(iii) $(1-x^2)\frac{dy}{dx} - xy = 1$ (iv) $(x^2+1)\frac{dy}{dx} + 2xy = x^2 - 1$

Q.2 Solve the following D ifferential equation

(i) $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ (ii) $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

(iii) $\frac{dy}{dx} + \frac{xy}{(1-x^2)} = x\sqrt{y}$ (iv) $\frac{dy}{dx} = \frac{y}{2y\, \ln y + y - x}$

Q.3 Find the solution of the following differential equations

(i) $\frac{dy}{dx} + \frac{y}{x} = y^2$ (ii) $x \log x \frac{dy}{dx} + y = 2 \log x$

(iii) $\frac{dy}{dx} + 2y \cot x = 3x^2 \operatorname{cosec}^2 x$ (iv) $x\frac{dy}{dx} = y + x^2$

Answer key

Q.1 (i) $\sec x$, (ii) x, (iii) $\sqrt{1-x^2}$, (iv) x^2+1

Q.2 (i) $\frac{e^{-x}}{x} = \frac{x^{-2}}{2} + C$, (ii) $\tan y\, e^{x^2} = \frac{1}{2} e^{x^2}\left(x^2-1\right) + C$, (iii) $\frac{\sqrt{y}}{\left(1-x^2\right)^{\frac{1}{4}}} = \frac{-1}{3}\left(1-x^2\right)^{\frac{3}{4}} + C$

(iv) $xy = y^2 \ln y + c$

Q.3 (i) $xy \log_e\left(\frac{c}{x}\right) = 1$, (ii) $y \log x = (\log x)^2 + c$, (iii) $y \sin^2 x = x^3 + c$, (iv) $y = x^2 + cx$

4.8 Equations Reducible To Linear Form (Bernoulli's Equation):

The equation $\frac{dy}{dx} + py = Q \cdot y^n$ where P & Q functions of x, is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1} = Z$. Its solution can be obtained as in the normal case.

Illustration :

Solve the following Differential Equation.

(i) $\frac{dy}{dx} = xy + x^3y^2$

(ii) $\frac{dy}{dx} - 2y\tan x + y^2\tan^4 x = 0$

(iii) $y\sin x\frac{dy}{dx} = \cos x(\sin x - y^2)$

(iv) $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y$.

(v) $\frac{dy}{dx} - \frac{\tan y}{x+1} = (1+x)e^x\sec y$

Sol.

(i) $\frac{dy}{dx} = xy + x^3y^2 \Rightarrow \frac{dy}{dx} - yx = x^3y^2$

$\frac{1}{y^2}\frac{dy}{dx} - \frac{x}{y} = x^3$ put $t = \frac{-1}{y}$ so $dt = \frac{dy}{y^2}$

$\frac{dt}{dx} + tx = x^3 \Rightarrow$ Integrating factor $= e^{\int x\,dx} = e^{\frac{x^2}{2}}$

$t\,e^{\frac{x^2}{2}} = \int e^{\frac{x^2}{2}} x^3\,dx = \int e^{\frac{x^2}{2}} x\,x^2\,dx$ Put $z = \frac{x^2}{2}$; $dz = x\,dx$

$\int e^{\frac{x^2}{2}} x^3\,dx = \int e^z(2z)\,dz = 2(z-1)e^z = (x^2-2)\,e^{\frac{x^2}{2}}$

$t\,e^{\frac{x^2}{2}} = (x^2-2)\,e^{\frac{x^2}{2}} + C \Rightarrow \frac{-e^{\frac{x^2}{2}}}{y} = (x^2-2)\,e^{\frac{x^2}{2}} + C \Rightarrow y = \frac{-e^{\frac{x^2}{2}}}{(x^2-2)e^{\frac{x^2}{2}} + C}$

(ii) $\frac{dy}{dx} - 2y\tan x + y^2\tan^4 x = 0 \Rightarrow \frac{dy}{dx} - 2y\tan x = -y^2\tan^4 x$

$\frac{-1}{y^2}\frac{dy}{dx} + \frac{2\tan x}{y} = \tan^4 x$ put $t = \frac{1}{y}$; $dt = \frac{-1}{y^2}dy$

$\frac{dt}{dx} + (2\tan x)\,t = \tan^4 x$

$$Integrating\ factor = e^{2\int \tan x\, dx} = e^{2\log_e \sec x} = \sec^2 x$$

$$t \sec^2 x = \int \tan^4 x \sec^2 x\, dx = \frac{\tan^5 x}{5} + c$$

$$\frac{\sec^2 x}{y} = \frac{\tan^5 x}{5} + C$$

(iii) $$y \sin x \frac{dy}{dx} = \cos x (\sin x - y^2)$$

Let $z = y^2 \quad \therefore dz = 2y\, dy$

$$\frac{1}{2}\frac{dz}{dx} + \cot x\, z = \cos x \Rightarrow \frac{dz}{dx} + 2\cot x\, z = 2 \cos x$$

$$Integrating\ factor = e^{\int 2\cot x\, dx} = e^{2\ln(\sin x)} = \sin^2 x$$

$$z \cdot \sin^2 x = \int 2 \cos x \sin^2 x\, dx = \frac{2\sin^3 x}{3} + C$$

$$\therefore y^2 \sin^2 x = \frac{2\sin^3 x}{3} + C.$$

(iv) $$\frac{dy}{dx} + \frac{\sin 2y}{x} = x^3 \cos^2 y$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dx} + \frac{2\tan y}{x} = x^3$$

$$z = \tan y \Rightarrow dz = \sec^2 y\, dy \Rightarrow \frac{dz}{dx} + \frac{2z}{x} + x^3$$

$$Integrating\ factor = e^{\int \frac{2dx}{x}} = e^{2ln} = x^2$$

$$\Rightarrow \quad z \cdot x^2 = \int x^5 dx = \frac{x^6}{6} + C$$

$$\Rightarrow \quad x^2 \tan y = \frac{x^6}{6} + C.$$

(v) $$\frac{dy}{dx}-\frac{\tan y}{x+1}=(x+1)e^x \sec y$$

$$\cos y\frac{dy}{dx}-\frac{\sin y}{x+1}=(x+1)e^x$$

Let $z=\sin y \quad \therefore \quad dz=\cos y\, dy$

$$\frac{dz}{dx}-\frac{z}{x+1}=(x+1)e^x$$

$$\text{Integrating factor} = e^{\int\frac{-dx}{(x+1)}}=e^{-\ln(x+1)}=\frac{1}{x+1}$$

$$\therefore \quad \frac{z}{(x+1)}=\int e^x dx=e^x+c$$

$$z=(e^x+c)(x+1)$$

$$\therefore \quad \sin y=(e^x+c)(x+1).$$

Note : Following exact differentials must be remembered :

(i) $x\,dy+y\,dx=d(xy)$

(ii) $\dfrac{x\,dy-y\,dx}{x^2}=d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{y\,dx-x\,dy}{y^2}=d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{x\,dy+y\,dx}{xy}=d(\ln xy)$

(v) $\dfrac{dx+dy}{x+y}=d(\ln(x+y))$

(vi) $\dfrac{x\,dy-y\,dx}{xy}=d\left(\ln\dfrac{y}{x}\right)$

(vii) $\dfrac{y\,dx-x\,dy}{xy}=d\left(\ln\dfrac{x}{y}\right)$

(viii) $\dfrac{x\,dy-y\,dx}{x^2+y^2}=d\left(\tan^{-1}\dfrac{y}{x}\right)$

(ix) $\dfrac{y\,dx-x\,dy}{x^2+y^2}=d\left(\tan^{-1}\dfrac{x}{y}\right)$

(x) $\dfrac{x\,dx+y\,dy}{x^2+y^2}=d\left[\ln\sqrt{x^2+y^2}\right]$

(xi) $d\left(-\dfrac{1}{xy}\right)=\dfrac{x\,dy+y\,dx}{x^2y^2}$

(xii) $d\left(\dfrac{e^x}{y}\right)=\dfrac{ye^x\,dx-e^x\,dy}{y^2}$

(xiii) $d\left(\dfrac{e^y}{x}\right)=\dfrac{xe^y\,dy-e^y\,dx}{x^2}$

Illustration :

Solve $x\,dx + y\,dy = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$.

Sol. *The D.E. can be written as*

$$\frac{1}{2} d(x^2 + y^2) = d\{\tan^{-1}(y/x)\}$$

Integrating, we get

$$\frac{1}{2}(x^2 + y^2) = \tan^{-1}(y/x) + c$$

Illustration :

Solve $\{(x + 1)(y/x) + \sin y\}dx + (x + \ln x + x\cos y)\,dy = 0$.

Sol. *We can re-write the differential equation as*

$$(y\,dx + x\,dy) + \left(\frac{y}{x}dx + \ln x\,dy\right) + (\sin y\,dx + x\cos y\,dy) = 0$$

$\Rightarrow d(xy) + d(y\ln x) + d(x\sin y) = 0$

Integrating both sides we have

$xy + y\ln x + x\sin y = c$

Practice Problem

Q.1 Solve the following equations

(i) $y\,dx - x\,dy + 3x^2y^2\,e^{x^3}\,dx = 0$

(ii) $y\,dx + (x + x^2y)dy = 0$

(iii) $(xy^4 + y)\,dx - x\,dy = 0$

(iv) $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - 1 - 2y}$.

Q.2 The function $y(x)$ satisfies the equation $y(x) + 2x\displaystyle\int_0^x \frac{y(u)}{1+u^2}du = 3x^2 + 2x + 1$. Show that the substitution $z(x) = \displaystyle\int_0^x \frac{y(u)}{1+u^2}du$ converts the equation into a first order linear differential equation for z(x). Find its integrating factor.

Q.3 A differentiable function satisfies $f(x) = \displaystyle\int_0^x (f(t)\cos t - \cos(t - x))dt$. Which of the following hold good?

(A) $f(x)$ has a minimum value $1 - e$.

(B) $f(x)$ has a maximum value $1 - e^{-1}$.

(C) $f''\left(\dfrac{\pi}{2}\right) = e$

(D) $f'(0) = 1$

Answer key

Q.1 (i) $\frac{x}{y}+e^{x^3}=C$, (ii) $\frac{-1}{xy}+\log y=C$, (iii) $3x^4y^3+4x^3=cy^3$, (iv) $\frac{x^2}{y}=\frac{1}{y}-2\log y+C$

Q.2 $y(x)=\frac{(1+x^2)^2+2x}{(1+x^2)}$ Q.3 ABC

5. PHYSICAL APPLICATION OF DIFFERENTIAL EQUATION :

5.1 Mixture Problems :

A chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula.

$$\begin{matrix}\textit{Rate of change}\\ \textit{of amount}\\ \textit{in container}\end{matrix} = \begin{pmatrix}\textit{rate at which}\\ \textit{chemical}\\ \textit{arrives}\end{pmatrix} - \begin{pmatrix}\textit{rate at which}\\ \textit{chemical}\\ \textit{departs}\end{pmatrix} \qquad \text{....(1)}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\text{Departure rate} = \frac{y(t)}{V(t)}\cdot(\text{out flow rate})$$

$$= \begin{pmatrix}\textit{concentration in}\\ \textit{container at time t}\end{pmatrix}\cdot(\text{out flow rate})$$

Accordingly, Equation (1) becomes

$$\frac{dy}{dt} = (\text{chemical's given arrival rate}) - \frac{y(t)}{V(t)}\cdot(\text{out flow rate}) \qquad \text{....(2)}$$

If, say, y is measured in grams, V in liters, and t in minutes, then unit in equation (2) are

$$\frac{\textit{grams}}{\textit{minute}} = \frac{\textit{grams}}{\textit{minute}} - \frac{\textit{grams}}{\textit{litre}}\cdot\frac{\textit{litre}}{\textit{minute}}$$

Illustration :

A tank initially contains 100 litres of brine in which 50 gms of salt dissolved. A brine containing 2 gm/litre of salt runs into the tank at the rate of 5 litre/min. The mixture is kept stirring and flows out of the tank at the rate of 4 litres/min then

(a) At what rate (gms/min) does salt enter the tank at time t.

(b) What is the volume of the brine in the tank at time t.

(c) At what rate (gms/min) does salt leave the tank at time t.

(d) form the DE of the process and solve it to find an expression for the amount of salt present at time t.

Sol.

(a) *Inflow rate of brine solution = (2 gm/litre) · (5 litre/min) = 10 gm/min.*

(b) *Volume of the brine in the tank at time t = initial volume + (inflow – outflow) · t*

$$= 100 + (5 - 4)\,t = (100 + t) \text{ litres.}$$

(c) *Let y(t) is the amount of salt at time t then outflow rate of salt =* $\dfrac{4y}{(100+t)}$

(d) *Rate of change of salt in container = (rate at which salt arrivrs) – rate of which salt leaves*

$$\frac{dy}{dt} = 10 - \frac{4y}{(100+t)} \Rightarrow \frac{dy}{dx} + \frac{4y}{(100+t)} = 10$$

Integrating factor $= e^{\int \frac{4dt}{(100+t)}} = e^{4\ln(100+t)} = (100+t)^4$

$$y\,(100 + t)^4 = 10\int (100+t)^4 dt = 2\,(100 + t)^5 + C \text{ at } t = 0,\quad y = 50$$

$$\Rightarrow \quad 50 \cdot (100)^4 = 2\,(100)^5 + C \Rightarrow C = -150\,(100)^4$$

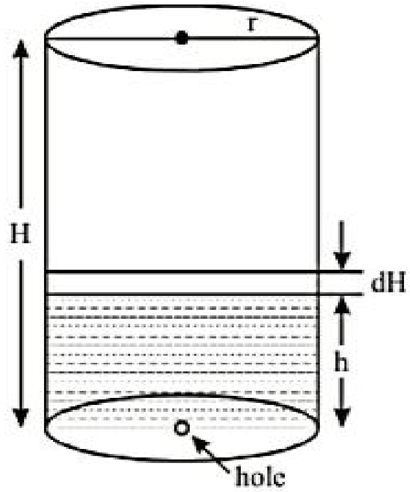
$$\Rightarrow \quad y\,(100 + t)^4 = 2(100 + t)^5 - 150\,(100)^4 \Rightarrow y = 2\,(100 + t) - \frac{150}{\left(1 + \frac{t}{100}\right)^4} \quad \textit{Ans.}$$

Illustration :

Find the time required for a cylindrical tank of radius r and height H to empty through a round hole of area 'a' at the bottom. The flow through the hole is according to the law $v(t) = k\sqrt{2gh(t)}$ *where v(t) and h(t) are respectively the velocity of flow through the hole and the height of the water level above the hole at time t and g is the acceleration due to gravity.*

Sol. *Let at time t the depth of water is h and radius of water surface is r.*
If in time dt the decrease of water level is dh then

$$-\pi r^2 dh = ak\sqrt{(2gh)}\, dt$$



$$\Rightarrow \quad \frac{-\pi r^2}{ak\sqrt{2g}\sqrt{h}} dh = dt \quad \Rightarrow \quad -\frac{\pi r^2}{ak\sqrt{2g}} \frac{dh}{\sqrt{h}} = dt$$

Now when $t = 0,\ h = H$ *and when* $t = t,\ h = 0$

$$\text{then,} \quad -\frac{\pi r^2}{ak\sqrt{2g}} \int_H^0 \frac{dh}{\sqrt{h}} = \int_0^t dt$$

$$\Rightarrow \quad -\frac{\pi r^2}{ak\sqrt{2g}} \left\{2\sqrt{h}\right\}_H^0 = t$$

$$\Rightarrow \quad t = \frac{\pi r^2 2\sqrt{H}}{ak\sqrt{2g}} = \frac{\pi r^2}{ak}\sqrt{\left(\frac{2H}{g}\right)}$$

Illustration :

Suppose that a mothball loses volume by evaporation at a rate proportional to its instantaneous area. If the diameter of the ball decreases from 2 cm to 1cm in 3 months, how long will it take until the ball has practically gone?

Sol. *Let at any instance (t), radius of moth ball be 'r' and 'v' be its volume*

$$\Rightarrow \quad v = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Thus as per the information

$$4\pi r^2 \frac{dr}{dt} = -k\, 4\pi r^2, \text{ where } k \in R^+$$

$$\Rightarrow \quad \frac{dr}{dt} = -k \quad \text{or} \quad r = -kt + c \quad \text{at} \quad t = 0,\ r = 2\text{cm};\ t = 3 \text{ month},\ r = 1 \text{ cm}$$

$$\Rightarrow \quad c = 2,\ k = \frac{1}{3} \quad \Rightarrow \quad r = -\frac{1}{3}t + 2$$

now for $r \to 0,\ t \to 6$
Hence, it will take six months until the ball is practically gone.

Illustration :

A body at a temperature of 50F is placed outdoors where the temperature is 100 F. If the rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium. If after 5 min. the temperature of the body is 60F, find

(a) how long it will take the body to reach a temperature of 75F and

(b) the temperature of the body after 20 min.

Sol. *Let T be the temperature of the body at time t and* $T_m = 100$ *(the temperature of the surrounding medium)*

$$\frac{dT}{dt} = -k(T - T_m) \quad \text{or} \quad \frac{dT}{dt} + kT = kT_m,$$ *where k is constant of proportionally.*

$$\Rightarrow \quad \frac{dT}{dt} + kT = 100k$$

This differential equation whose solution is

$$T = ce^{-kt} + 100 \qquad \text{...(i)}$$

Since $T = 50$ *when* $t = 0$*, then from equation (i)* $50 = ce^{-k(0)} + 100$, *or* $c = -50$.

Substituting this value in equation (i), we obtain

$$T = -50e^{-kt} + 100 \qquad \text{...(ii)}$$

At $t = 5$*, we are given that* $T = 60$*; hence, from equation (ii),* $60 = -50e^{-5k} + 100$.

Solving for k, we obtain $-40 = -50\,e^{-5k}$ *or* $k = -\frac{1}{5}\ln\frac{40}{50}$

Substituting this value in equation (ii), we obtain the temperature of the body at any time t as

$$T = -50e^{\frac{1}{5}\ln\frac{4}{5}t} + 100 \qquad \text{...(iii)}$$

(a) We require t when $T = 75$*, Substituting* $T = 75$ *in equation (iii), we have*

$$75 = -50e^{\frac{1}{5}\ln\frac{4}{5}t} + 100,$$ *from which we get t*

(b) We require T when $t = 20$*. Substituting* $t = 20$ *in equation (iii) and then solving for T, we find*

$$T = -50e^{\frac{1}{5}\ln\frac{4}{5}(20)} + 100$$

5.2 Statistical Applications of Differential Equation:

Illustration :

The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled and after three years the population is 20,000, estimate the number of people initially living in the country.

Sol. *Let N denote the number of people living in the country at any time t, and let N_0 denote the number of people initially living in the country.*

Then, from equation $\frac{dN}{dt} - kN = 0$

which has the solution $N = ce^{kt}$ ...(i)

At $t = 0$, $N = N_0$; *hence equation (i) states that* $N_0 = ce^{k(0)}$, *or that* $c = N_0$

Thus, $N = N_0e^{kt}$ ...(ii)

At $t = 2$, $N = 2N_0$

Substituting these values into equation (ii), we have

$$2N_0 = N_0e^{2k} \text{ from which } k = \frac{1}{2}\ln 2$$

Substituting this value into equation (i) gives

$$N = N_0\, e^{t/2 \ln 2} \qquad \text{...(iii)}$$

At $t = 3$, $N = 20{,}000$

Substituting these values into equation (iii), we obtain

$$20{,}000 = N_0e^{3/2 \ln 2} \quad \Rightarrow \quad N_0 = \frac{20000}{2\sqrt{2}} \approx 7071.$$

Illustration :

What constant interest rate is required if an initial deposit placed into an account the accrues interest compounded continuously is to double its value in six years? ($\ln|2| = 0.6930$)

Sol. *The balance N(t) in the account at any time t.*

$$\frac{dN}{dt} - kN = 0, \text{ its solution is } N(t) = ce^{kt} \qquad \text{...(i)}$$

Let initial deposit be N_0.

At $t = 0$, $N(0) = N_0$, *which when substituted into equation (i) yields*

$$N_0 = ce^{k(0)} = c$$

and equation (i) becomes $N(t) = N_0e^{kt}$ .....(ii)

We seek the value of k for which $N = 2N_0$ *when* $t = 6$, *Substituting these values into equation (ii) and solving for k we find* $2N_0 = N_0e^{k(6)} \Rightarrow e^{6k} = 2 \Rightarrow k = \frac{1}{6}\ln|2| = 0.1155$

An interest rate of 11.55 percent is required.

5.3 Geometrical Applications of Differential Equation :

We also use differential equations for finding the family of curves for which some conditiong involving the derivatives are given. For this we proceed in the following way

Equation of the tangent at a point (x, y) to the curve $y = f(x)$ is given by $Y - y = \frac{dy}{dx}(X - x)$.

At the X axis, $Y = 0$, and $X = x - \frac{y}{dy/dx}$ (intercept on X-axis)

At the Y axis, $X = 0$, and $Y = y - \frac{dy}{dx}$ (intercept on Y-axis)

Similar infomation can be obtained for normals by writing equations as $(Y - y)\frac{dy}{dx} + (X - x) = 0$.

Illustration :

Find the equation of the curve passing through (2, 1) which has constant sub-tangent.

Sol. *We are given that*

$$\text{sub-tangent} = \frac{y}{\frac{dy}{dx}} = \text{(constant)} \qquad = k \text{ (say)}$$

$$\Rightarrow \quad k\frac{dy}{y} = dx$$

Integrating we get, $k \ln y = x + c$

Given that curve passes through $(2, 1) \Rightarrow c = -2$

Hence the equation of such curve is $k \ln y = x - 2$.

Illustration :

Find the curve such that the intercept on the x-axis cut off between the origin and the tangent at a point is twice the abscissa and which passes through the point (1, 2).

Sol. *The equation of the tangent at any point P(x, y) is*

$$Y - y = \frac{dy}{dx}(X - x)$$

Given that intercept on X-axis (putting $Y = 0$*)* $= 2$*(x-coordinate of P)*

$$\Rightarrow \quad x - y\frac{dx}{dy} = 2x$$

$$\Rightarrow \quad -\frac{dy}{y} = -\frac{dx}{x}$$

Integrating we get $xy = c$

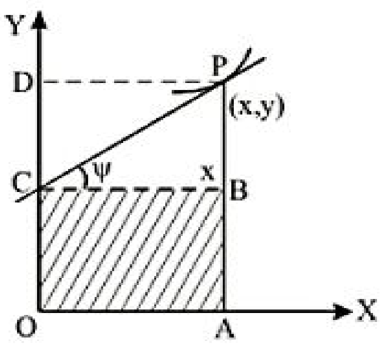
Since the curve passes through (1, 2), $c = 2$

Hence, the equation of the required curve is $xy = 2$

Illustration :

Find the equation of the curve which is such that the area of the rectangle constructed on the abscissa of any point and the intercept of the tangent at this point on y-axis is equal to 4.

Sol. *Equation of tangent P(x, y) is $Y - y = \frac{dy}{dx}(X - x)$*



$$\therefore \text{ Y-intercept} = y - x\frac{dy}{dx}$$

$$\therefore \text{ area of } OABC = \left| x\left(y - x\frac{dy}{dx}\right)\right| = 4$$

$$\Rightarrow xy - x^2\frac{dy}{dx} = \pm 4$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x}y = \pm\frac{4}{x^2}$$

$$\therefore \text{ I.F.} = e^{-\int\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\therefore \text{ the solution is } \left(\frac{y}{x}\right) = \pm 4\int\frac{1}{x^3}dx + c$$

$$\Rightarrow \frac{y}{x} = \pm\frac{2}{x^2} + c$$

Illustration :

Find the equation of the curve passing through the origin if the middle point of the segment of its normal from any point of the curve to the x-axis lies on the parabola $2y^2 = x$.

Sol. *Equation of normal at any point P(x, y) is*

$$\frac{dy}{dx}(Y - y) + (X - x) = 0$$

This meets the x-axis at $A\left(x + y\frac{dy}{dx}, 0\right)$

Mid point of AP is $\left(x + \frac{1}{2}y\frac{dy}{dx}, \frac{y}{2}\right)$ which lies on the parabola $2y^2 = x$.

$$\therefore \quad 2 \times \frac{y^2}{4} = x + \frac{1}{2} y \frac{dy}{dx} \text{ or } y^2 = 2x + y\frac{dy}{dx}$$

Putting $y^2 = t$, *so that* $2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$,

we get $\dfrac{dt}{dx} - 2t = -4x$ *(linear)*

$I.F. = e^{-2\int dx} = e^{-2x}$

$\therefore$ *solution is*

$$t\, e^{-2x} = -4\int xe^{-2x}\, dx + c = -4\left[\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\, dx\right] + c$$

$\Rightarrow \quad y^2 e^{-2x} = 2xe^{-2x} + e^{-2x} + C$

or $\quad y^2 = 2x + 1 - e^{2x}$ *is the equation of the required curve.*

5.4 Trajectories :

Suppose we are given the family of plane curves $\phi(x, y, a) = 0$, depending on a single parameter a.

A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called an ***isogonal trajectory*** of that family ; if in particular $\alpha = \pi/2$, then it is called an ***orthogonal trajectory***.

To find Orthogonal trajectories :

We set up the differential equation of the given family of curves. Let it be of the form $F(x, y, y') = 0$

The differential equation of the orthogonal trajectories is of the form $F\left(x,\, y,\, -\dfrac{1}{y'}\right) = 0$

The general integral of this equation $\Phi_1(x, y, C) = 0$, gives the family of orthogonal trajectories.

Illustration :

Find the orthogonal trajectory of the following curves

(i) $\quad x^2 + y^2 = a^2$

(ii) $\quad x^2 + y^2 - 2ay = 0$

Sol.

(i) $\quad x^2 + y^2 = a^2 \quad \Rightarrow \quad 2x + 2yy' = 0 \quad \Rightarrow \quad x + yy' = 0$

For orthogonal trajectory, replace y' *by* $\dfrac{-1}{y'}$

$$x - \frac{y}{y'} = 0 \quad \Rightarrow \quad xy' - y = 0$$

$$\Rightarrow \quad x\,dy - y\,dx = 0 \qquad \Rightarrow \quad \frac{dy}{y} - \frac{dx}{x} = 0$$

$$\Rightarrow \quad \ln y - \ln x = \ln c \qquad \Rightarrow \quad y = cx$$

(ii) $\quad x^2 + y^2 - 2ay = 0 \quad \Rightarrow \quad 2x + 2yy' - 2ay' = 0 \quad \Rightarrow \quad 2a = \left(\frac{2x + 2yy'}{y'}\right)$

$$\therefore \quad x^2 + y^2 - \left(\frac{2x + 2yy'}{y'}\right)y = 0$$

For orthogonal trajectory, replace y' by $\frac{-1}{y'}$

$$x^2 + y^2 - \frac{\left(2x - \frac{2y}{y'}\right)y}{\left(\frac{-1}{y'}\right)} = 0 \quad \Rightarrow \quad x^2 + y^2 + (2xy' - 2y)\,y = 0 \quad \Rightarrow \quad x^2 + y^2 + 2xy\,\frac{dy}{dx} - 2y^2 = 0$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ *(a homogeneous equation)*

Put $y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{-(v^2 + 1)}{2v} \quad \Rightarrow \quad \frac{2v}{v^2 + 1}dv = \frac{-dx}{x}$$

$$\Rightarrow \quad \ln(v^2 + 1) = -\ln x \quad + \ln c \quad \Rightarrow \quad x^2 + y^2 = cx$$

Illustration :

Find the orthogonal trajector of $y^2 = 4ax$ (a being the parameter).

Sol. $\quad y^2 = 4ax$

$$2y\,\frac{dy}{dx} = 4a$$

Eliminating a from equaiton (1) and (2)

$$y^2 = 2y\,\frac{dy}{dx}\,x$$

Replacing $\frac{dy}{dx}$ *by* $-x\,\frac{dx}{dy}$, *we get*

$$y = 2\left(-\frac{dx}{dy}\right)x$$

$2\,x\,dx + y\,dy = 0$

Integrating each term,

$$x^2 + \frac{y^2}{2} = c$$

$2x^2 + y^2 = 2c$

which is required orthogonal trajectory.

Illustration :

Find the orthogonal trajectories of $x\,y = c$.

Sol. $x\,y = c$

Differentiating w.r.t. x, we get $x\,\frac{dy}{dx} + y = 0.$

Replace $\frac{dy}{dx}$ *by* $-\frac{dx}{dy}$ *to get* $x\frac{dx}{dy} - y = 0$

Integrating $x\,dx - y\,dy = 0$

$\Rightarrow x^2 - y^2 = c$

This is the family of required orthogonal trajectories.

Practice Problem

Q.1 A storage tank contains 2000 litres of gasoline which initially has 100 gms of an additive dissolved in it. Gasoline containing 2 gms of additive per litre is pumped into the tank at a rate of 40 litre/min. The well mixed solution is pumped out at a rate of 45 litre/min. Form the DE and express the amount of additive in gasoline as a function of t.

Q.2 A tank contains 200 litres of brine in which 20 gms of salt dissolved. Brine containing 1/4 gm of salt/lit. runs into the tank at the rate of 2 litre/min. The mixture is kept stirring runs out at the same rate. Express the concentration of solution in tank in grams as a function of time t. What is the limiting value approached by the amount of salt as $t \to \infty$. Whenever was the amount of salt in solution be 20 gm.

Q.3 Find the time required for a cylindrical tank of radius 2.5m and height 3 m to empty through a round hole of 2.5 cm with a velocity 2.5 $\sqrt{h}\ ms^{-1}$, h being the depth of the water in the tank.

Q.4 If the population of country doubles in 50 years, in how many years will it triple under the assumption that the rate of increse in proportional to the number of inhabitants.

Q.5 The rate at which a substance cools in moving air is proportional to the difference between the temperatures of the substance and that of the air. If the temperature of the air is 290 K and the substance cools from 370 K to 330 K in 10 min.,when will the temperature be 295 K.

Q.6 Find the orthogonal trajectory of the following curves

(i) $x^2 - \frac{1}{3}y^2 = a^2$ (ii) $y = \tan x + c$ (iii) $\cos y = ae^{-x}$ (iv) $y^2 = 4(x-a)$

Answer key

Q.1 $y = 10(400-t) - 3900\left(1-\frac{t}{400}\right)^9$

Q.2 (i) $y = 50 - 30 \cdot e^{-\frac{t}{100}}$, (ii) y when $t \to \infty = 50$ gms, (iii) $y > 20$ for all $t > 0$

Q.3 $8000\sqrt{3}$ s

Q.4 $50 \log_2 3$

Q.5 40 min

Q.6 (i) $xy^3 = c$, (ii) $2x + 4y + \sin 2x = k$, (iii) $\sin y = c\,e^{-x}$, (iv) $y = c\,e^{-x/2}$