COMPOUND ANGLE (Trigonometry Phase-I)

What is trigonometry?

The word trigonometry is derived from three greek words

Tri	Gon	Metron
↓	↓	↓
three	sides	Measure

In the ancient sense trigonometry defines relations between elements of a triangle. In a triangle there are six basic elements, three sides and three angles. Any three line segments will form a triangle iff they satisfy three triangular inequalities i.e. the sum of any two lines segment is greater than third side. In Euclidean geometry the sum of three angles of a triangle is 180º. These requirements impose limitations on the manner in which the relations between the elements are defined.

Basic definition of six trigonometric functions :

three triangular inequalities i.e. the sum of any two lines segment is greater than third side. In Euclidean geometry the sum of three angles of a triangle is 180º. These requirements impose limitations on the manner in which the relations between the elements are defined.

Basic definition of six trigonometric functions :

$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\text{cosec}\,\theta$	$\cot\theta$
$\frac{P}{H}$	$\frac{B}{H}$	$\frac{P}{B}$	$\frac{H}{B}$	$\frac{H}{P}$	$\frac{B}{P}$

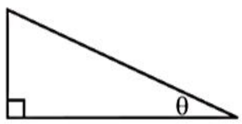

Illustration :

Which of the following reduces to unity for $0 < A < 90^o$?

(i) $\cos A \operatorname{cosec} A \sqrt{\sec^2 A - 1}$ (ii) $(1 + \tan^2 A)(1 - \sin^2 A)$

(iii) $\dfrac{1}{1+\sin^2 A} + \dfrac{1}{1+\operatorname{cosec}^2 A}$ (iv) $\dfrac{\cot^2 A \cos^2 A}{\cot^2 A - \cos^2 A}$

Sol. (i) $\cos A \dfrac{1}{\sin A} \tan A = \dfrac{\cos A}{\sin A} \times \dfrac{\sin A}{\cos A} = 1$

(ii) $(\sec^2 A)(\cos^2 A) = 1$

(iii) $\dfrac{1}{1+\sin^2 A} + \dfrac{\sin^2 A}{1+\sin^2 A} = 1$

(iv) $$\frac{\dfrac{\cos^2 A}{\sin^2 A} \times \cos^2 A}{\cos^2 A\left(\dfrac{1}{\sin^2 A} - 1\right)} = \frac{\dfrac{\cos^4 A}{\sin^2 A}}{\dfrac{\cos^4 A}{\sin^2 A}} = 1$$

Illustration :

Prove that $(\sec\theta + \operatorname{cosec}\theta)(\sin\theta + \cos\theta) = \sec\theta \cdot \operatorname{cosec}\theta + 2$

Sol. $\left(\dfrac{1}{\quad} + \dfrac{1}{\quad}\right)(\sin\theta + \cos\theta)$

Prove that $(\sec\theta + \operatorname{cosec}\theta)(\sin\theta + \cos\theta) = \sec\theta \cdot \operatorname{cosec}\theta + 2$

Sol. $\left(\dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta}\right)(\sin\theta + \cos\theta)$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} + 2$

$\Rightarrow \dfrac{1}{\sin\theta\cos\theta} + 2 = \sec\theta \operatorname{cosec}\theta + 2$

Illustration :

If $(1 + \sin A)(1 + \sin B)(1 + \sin C) = (1 - \sin A)(1 - \sin B)(1 - \sin C)$.

Prove that each side equal to $\cos A \cos B \cos C$.

Sol. *Multiply both side by $(1 - \sin A)(1 - \sin B)(1 - \sin C)$*

$= (1 - \sin^2 A)(1 - \sin^2 B)(1 - \sin^2 C) = (1 - \sin A)^2 (1 - \sin B)^2 (1 - \sin C)^2$

$= \cos^2 A \cos^2 B \cos^2 C = (1 - \sin A)^2 (1 - \sin B)^2 (1 - \sin C)^2$

$= \cos A \cos B \cos C = (1 - \sin A)(1 - \sin B)(1 - \sin C)$

Illustration :

Prove that $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1+\sin A}{\cos A}$

Sol. $= \dfrac{(\tan A + \sec A) - (\sec^2 A - \tan^2 A)}{(1 - \sec A + \tan A)} = \dfrac{(\tan A + \sec A)(1 - \sec A + \tan A)}{(1 - \sec A + \tan A)} = \dfrac{1+\sin A}{\cos A}$

Illustration :

$$\text{Prove that } \left(\frac{1}{sec^2 a - cos^2 a} + \frac{1}{cosec^2 - sec^2 a}\right) sin^2 a\, cos^2 a = \frac{1 - cos^2 a\, sin^2 a}{2 + cos^2 a\, sin^2 a}$$

Sol. $\left(\dfrac{cos^2 a}{1 - cos^4 a} + \dfrac{sin^2 a}{1 - sin^4 a}\right) sin^2 a\, cos^2 a$

$$= \left(\frac{cos^4 a \cdot sin^2 a}{1 - cos^4 a} + \frac{sin^4 a \cdot cos^4 a}{1 - sin^4 a}\right) = \frac{cos^4 a \cdot sin^2 a}{(1 + cos^2 a)(1 - cos^2 a)} + \frac{sin^4 a \cdot cos^2 a}{(1 - sin^2 a)(1 + sin^2 a)}$$

$$= \frac{cos^4 a + sin^4 a + cos^2 a\, sin^2 a\,(sin^2 a + cos^2 a)}{1 + sin^2 a + cos^2 a + sin^2 a\, cos^2 a}$$

$$= \frac{(cos^2 a + sin^2 a)^2 - 2 sin^2 a\, cos^2 a + cos^2 a\, sin^2 a\,(sin^2 a + cos^2 a)}{2 + sin^2 a\, cos^2 a} = \frac{1 - sin^2 a\, cos^2 a}{2 + sin^2 a\, cos^2 a}$$

Practice Problem

Q.1 Prove that $\dfrac{\cot\theta + \text{cosec }\theta - 1}{\cot\theta - \text{cosec }\theta} = (\cot\theta + \text{cosec}\theta)(\cot\theta + \text{cosec}\theta - 1)$

Q.2 If $\sin\theta + \sin^2\theta = 1$. Prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$

Q.3 Prove that $(1 + \cot A - \text{cosec } A)(1 + \tan A + \sec A) = 2$

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Q.4 Prove that $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\text{cosec}^2 A} - \dfrac{\text{cosec} A}{\sec^2 A}$

MEASUREMENT OF ANGLE AND SIGN CONVENTION :

Angle :

The measure of angle is the amount of rotation from the direction of one ray of the angle to the other. The initial and final position of the revolving ray are respectively called the initial side and terminal side.

Systems of Angle Measurement

- English system (Degree)
- French system (Grades)
- Circular system (Radians)

English System :

One right angle = 90° (degree)

1° = 60' (minutes)

1' = 60" (seconds)

Circular system :

If length of arc of a circle equal's to radius then angle impose by that arc on centre of circle is called one radian.

Otherwise $\ell = r \cdot \theta$

Note : Important Relation :-

(i) Radian and Degree's

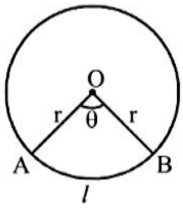
$\pi = 180^o$

(ii) Length of an arc of a circle

$\ell = r\,\theta$

(iii) Area of sector of a circle

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}r\ell$$



l = length of arc,
r = radius of circle
θ = angle in radian

REDUCTION FORMULAE :

(I) (90 + θ) Relation

Δ OPB and Δ OP'B' are congruent by ASA property one ∠θ, side r, ∠(90° – θ)

∴ In Δ OP'B', P'B' = x as side opposite to 90° – θ is x in Δ OPB

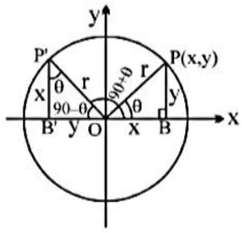
In Δ OP'B', OB' = y as side opposite to θ in Δ OPB is y.

In Δ OP'B'

$\sin(90° + \theta) = \frac{x}{r} = \cos\theta$; $\cos(90° + \theta) = \frac{-y}{r} = -\sin\theta$;

$\tan(90° + \theta) = -\cot\theta$; $\cot(90° + \theta) = -\tan\theta$;

$\sec(90° + \theta) = -\text{cosec}\,\theta$; $\text{cosec}(90° + \theta) = \sec\theta$



In all (90° + θ) relations

In all (90° + θ) relations

sin changes to cos
cos changes to sin
cosec changes to sec
tan changes to cot
cot changes to tan
and sec changes to cosec
with appropriate sign

∴ $\sin(120°) = \sin(90° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$

$\tan(135°) = \tan(90° + 45°) = -\cot 45° = -1$

$\cos(150°) = \cos(90° + 60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$

(II) Reduction (180° – θ)

Δ's OPB and OP'B' are congruent by A S A. ∠(90° – θ), side r, ∠θ

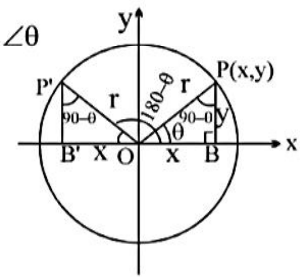
∴ side opposite to 90° – θ = x same as in ΔOPB

and side opposite to θ = y same as in ΔOPB

$\sin(180° - \theta) = \frac{y}{r} = \sin\theta$; $\cos(180° - \theta) = \frac{-x}{r} = -\cos\theta$;

$\tan(180° - \theta) = -\tan\theta$; $\cot(180° - \theta) = -\cot\theta$;

$\text{cosec}(180° - \theta) = \text{cosec}\theta$; $\sec(180° - \theta) = -\sec\theta$;



Sines of supplementary angles are equal supplementary angles are those whose sum is 180°.

Sum of the cosines, tangents, cotangents, secants of supplementary angles is zero.

since $\cos(180° - \theta) = -\cos\theta$

$\therefore \quad \cos(180° - \theta) + \cos\theta = 0$

same for tan, cot and sec

$$\therefore \quad \sin(120°) = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos(150°) = \cos(180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$$

For $(180^o - \theta)$

sin	remains sin
cos	remains cos
tan	remains tan
cot	remains cot
sec	remains sec
cosec	remains cosec with appropriate signs.

(III) Reduction $(180° + \theta)$

ΔOPB and $\Delta OP'B'$ are congruent by A S A in which $\angle(90° - \theta)$, side r, $\angle\theta$.

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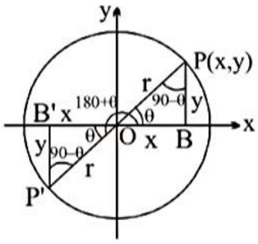
ΔOPB and $\Delta OP'B'$ are congruent by A S A in which $\angle(90° - \theta)$, side r, $\angle\theta$.



$$\therefore \quad \sin(180 + \theta) = \frac{-y}{r} = -\sin\theta;$$

$$\cos(180 + \theta) = \frac{-x}{r} = -\cos\theta;$$

$\tan(180 + \theta) = \tan\theta; \qquad \cot(180 + \theta) = \cot\theta;$

$\operatorname{cosec}(180 + \theta) = -\operatorname{cosec}\theta; \qquad \sec(180 + \theta) = -\sec\theta;$

$\sin(210°) = \sin(180° + 30°) = -\sin 30° = -1/2$

$\cos(240°) = \cos(180° + 60°) = -\cos 60° = -1/2$

$\tan(225°) = \tan(180° + 45°) = \tan 45° = 1$

In $(180 + \theta)$ relations

sin	remains sin
cos	remains cos
tan	remains tan
cot	remains cot
sec	remains sec
cosec	remains cosec with appropriate signs.

$\sin(270°) = \sin(180° + 90°) = -\sin 90° = -1$

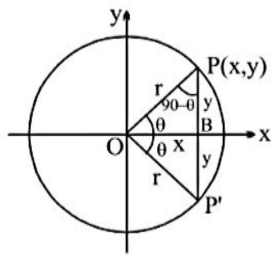
$\cos(270°) = \tan(180° + 90°) = -\cos 90° = 0$

(IV) Reduction $(360 - \theta)$ or $(2\pi - \theta)$

Any angle of the form $2\pi - \theta$ can be written as $-\theta$ because if we say $2\pi - \theta$ then it means we are moving clockwise from origin and by convention all angles measured clockwise are – ve.

$\therefore$ $\sin(2\pi - \theta) = \sin(-\theta)$
$\cos(2\pi - \theta) = \cos(-\theta)$
$\tan(2\pi - \theta) = \tan(-\theta)$

Again ΔOPB and $\Delta OP'B$ are congruent by ASA



$\sin(-\theta) = \dfrac{-y}{r} = -\sin\theta$; $\cos(-\theta) = \dfrac{x}{r} = \cos\theta$;
$\tan(-\theta) = -\tan\theta$; $\cot(-\theta) = -\cot\theta$;
$\mathrm{cosec}(-\theta) = -\mathrm{cosec}\,\theta$; $\sec(-\theta) = -\sec\theta$;

$$\cos(315°) = \cos(360° - 45°) = \cos(-45°) = \cos(45°) = \frac{1}{\sqrt{2}}$$

$$\tan(330°) = \tan(360° - 30°) = \tan(-30°) = -\tan 30° = -\frac{1}{\sqrt{3}}$$

In $(2\pi - \theta)$ relations

sin remains sin
cos remains cos
tan remains tan
cot remains cot
sec remains sec
cosec remains cosec with appropriate signs.

sec remains sec
cosec remains cosec with appropriate signs.

To remember the signs we use

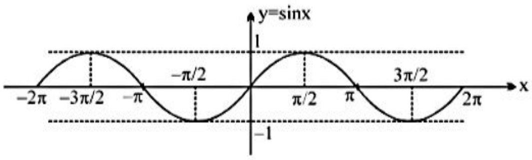
sin +ve Students	All All +ve
Take tan +ve	Coffee cos +ve

$\tan(-120°) = -\tan 120° = -\tan(180° - 60°) = \tan 60° = \sqrt{3}$
$\cos(180°) = \cos(90° + 90°) = \cos(180° - 0) = \cos(180 + 0°)$
$= -1$ $= -1$ $= -1$

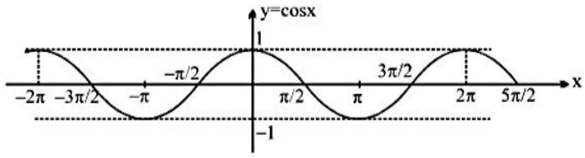
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degree	0	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
cot	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	ND

GRAPHS OF 6 TRIGONOMETRIC FUNCTIONS :

(1) $y = \sin x$, where $y \in [-1, 1]$, $x \in R$

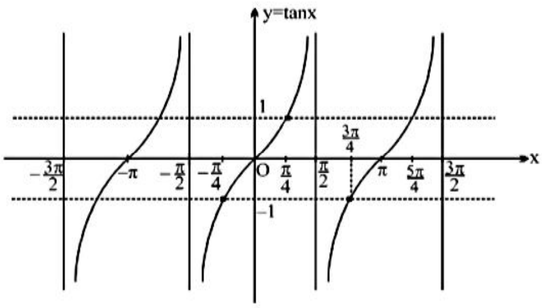


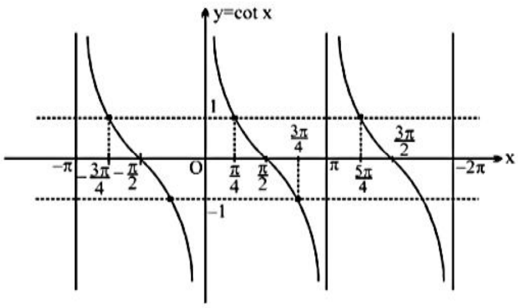
(2) $y = \cos x$, where $y \in [-1, 1]$, $x \in R$



(3) $y = \tan x$, $x \in R$, $y \in (-\infty, \infty)$, $x \neq (2n-1)\dfrac{\pi}{2}$ for $n \in I$

(4) $y = \cot x$, $x \in R$, $y \in (-\infty, \infty)$, $x \neq n\pi$ for $n \in I$



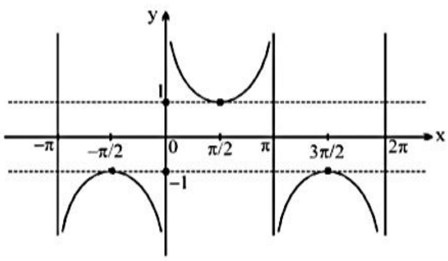


(5) $y = \operatorname{cosec} x$, $y \in (-\infty, -1] \cup [1, \infty)$, $x \in R - n\pi$, $x \neq n\pi$ for $n \in I$

(6) $y = \sec x$, $x \neq (2n+1)\dfrac{\pi}{2}$ for $n \in I$

(5) $y = \operatorname{cosec} x$, $y \in (-\infty, -1] \cup [1, \infty)$, $x \in R - n\pi$, $x \neq n\pi$ for $n \in I$

(6) $y = \sec x$, $x \neq (2n+1)\dfrac{\pi}{2}$ for $n \in I$



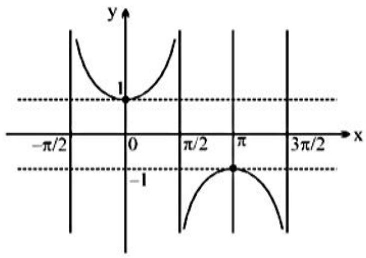


Illustration :

Consider a square of side 4cm. Now if a man runs at a distance of 1cm from the sides of the square. How much distance will he travel.

Sol. *Lienar distance = 16*

curved distance $= r\,\theta = 1.\dfrac{\pi}{2}$

Total curve part $4.\dfrac{\pi}{2} = 2\pi$

Total distance $16 + 2\pi$

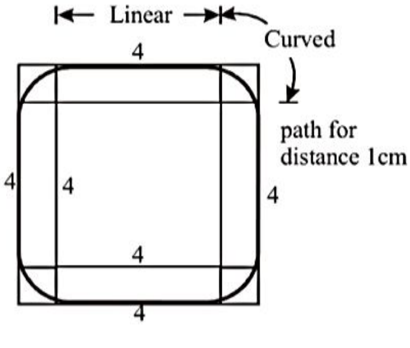
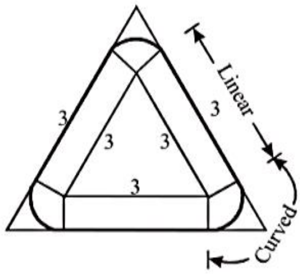


Illustration :

Consider an equilateral triangle with side 3cm. Now if a man runs around the triangle in such a ways that he is always at a distance of 1cm from the sides of the triangle then how much distance will he travel.



Sol. *Linear distance = 9*

$$\text{curved distance } \ell = 1 \times \frac{2\pi}{3}$$

$$3\ell = 2\pi$$

$$\text{Total distance} = 9 + 2\pi$$

Illustration :

Prove that

$$\cos 10° + \cos 20° + \cos 30° + ... + \cos 80° + \cos 100° + \cos 150° + \cos 160° + \cos 170° = 0.$$

Sol. $\cos 10° + \cos 20° + \cos 30° + ... + \cos 80° + \cos(180° - 80°) + \cos(180° - 30°)$
$+ \cos(180° - 20°) + \cos(180° - 10°) = 0$

$\cos 10° + \cos 20° + \cos 30° + ... + \cos 80° - \cos 80° - \cos 30° - \cos 20° - \cos 10° = 0$

Illustration :

$$\text{Prove that } \tan\frac{\pi}{11} + \tan\frac{2\pi}{11} + \tan\frac{4\pi}{11} + \tan\frac{7\pi}{11} + \tan\frac{9\pi}{11} + \tan\frac{10\pi}{11} = 0$$

Sol. *Sum of tangents of supplementary angles is zero.*

Illustration :

11 11 11 11 11 11

Sol. *Sum of tangents of supplementary angles is zero.*

Illustration :

Prove that $\sin 420° \cos 390° + \cos(-300°) \sin(-330°) = 1.$

Sol. $= \sin(360° + 60°) \cos(360° + 30°) + \cos(-360° + 60°) \sin(-360° + 30°)$

$= \sin 60° \cos 30° + \cos 60° \sin 30°$

$= \sin 90° = 1$

Illustration :

Prove that $\sin 240° \sin 510° - \sin 330° \cos 390° = 0$

Sol. $\sin(270° - 30°) \sin(540° - 30°) \cos(360° + 30°)$

$= -\cos 30° \sin 30° + \sin 30° \cos 30°$

$= 0$

Illustration :

What sign has $(\sin A + \cos A)$ *for the following vlaues of A?*

(a) $140°$ (b) $-1125°$

Sol. $\sin A + \cos A$

(a) $\sin 140° + \cos 140°$

$= \sin 40° - \cos 40°$

$= -ve$ (For $A < 45°$; $\cos 40° > \sin 40°$)

(b) $\sin(-1125°) + \cos(-1125°)$

$= \sin(-1080° - 45°) + \cos(-1080° - 45°)$

$= \sin(-45°) + \cos(-45°)$

$= -\sin 45° + \cos 45° = 0$

Illustration :

Prove that $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A) = 0$

Sol. $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A)$

$= \cos A - \cos A - (-\cos A) + (-\cos A)$

$= 0$

Practice Problem

Q.1 Prove that $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$

Q.2 What sign has $\sin A - \cos A$ for the following vlaues of A?
(a) $215°$ (b) $-457°$

Q.3 Prove that $\sec(270° - A)\sec(90° - A) - \tan(270° - A)\tan(90° + A) + 1 = 0$.

Q.4 Prove that $\sin^2 5° + \sin^2 10° + \sin^2 15° + ... + \sin^2 85° + \sin^2 90° = 9\frac{1}{2}$.

Q.5 Consider an triangle with sides 3, 6, 8. Now if a man run's around the triangle is such a way that he is always at a distance of 1cm from the sides of triangle then how much distance will he travel.

Answer key

Q.2 (a) Positive, (b) Negative Q.5 $17 + 2\pi$

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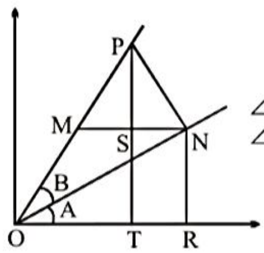
TRIGONOMETRY OF COMPOUND ANGLES :

Trigonometric ratios i.e., 'sin', 'cos', 'tan', cot, 'sec' and cosec are not distributed over addition and subtraction of 2 angles.

i.e., $\sin(A + B) \neq \sin A + \sin B$

Note that $A = 60°$, $B = 30°$

$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$



$\angle NOR = \angle PNM$

$\angle SPN = \angle A$

$$\sin(A + B) = \frac{PT}{OP} = \frac{PS + ST}{OP}$$

$$= \frac{PS}{OP} + \frac{ST}{OP}$$

$$= \frac{PS}{PN} \cdot \frac{PN}{OP} + \frac{NR}{ON} \cdot \frac{ON}{OP}$$

$$= \frac{PS}{PN} \cdot \frac{PN}{OP} + \frac{NR}{ON} \cdot \frac{ON}{OP}$$

$$= \cos A \cdot \sin B + \sin A \cdot \cos B$$

Hence we got $\boxed{\sin(A + B) = \sin A \cdot \cos B + \cos A \sin B}$ ....(i)

Now replace B by –B

$$\boxed{\sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B} \quad ...(ii)$$

In (i) replace A by $\frac{\pi}{2}+A$

$$\sin\left(\frac{\pi}{2}+A+B\right)=\sin\left(\frac{\pi}{2}+A\right)\cdot\cos B+\cos\left(\frac{\pi}{2}+A\right)\cdot\sin B$$

$$\boxed{\cos(A+B)=\cos A\cdot\cos B-\sin A\cdot\sin B} \qquad \text{...(iii)}$$

In (iii) relation replace B by –B

$$\boxed{\cos(A-B)=\cos A\cdot\cos B+\sin A\cdot\sin B} \qquad \text{...(iv)}$$

To deduce the value of tan (A + B) and cot (A + B) :

(1) $\boxed{\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\cdot\tan B}}$

(2) $\boxed{\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}}$ **Note :** $\boxed{\tan\left(\dfrac{\pi}{4}\pm A\right)=\dfrac{1\pm\tan A}{1\mp\tan A}}$

(3) $\boxed{\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}}$ or $\boxed{\cot A\cot B-1=\cot(A+B)(\cot B+\cot A)}$

(4) $\boxed{\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}}$

Important identities :

Important identities :

(a) $\boxed{\sin(A+B)\cdot\sin(A-B)=\sin^2A-\sin^2B}$

(b) $\boxed{\cos(A+B)\cdot\cos(A-B)=\cos^2A-\sin^2B}$

(c) $\boxed{\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}}$

Illustration :

Prove that $\dfrac{\cos 10^o+\sin 10^o}{\cos 10^o-\sin 10^o}=\tan 55^o$.

Sol. $\dfrac{\cos 10^o+\sin 10^o}{\cos 10^o-\sin 10^o}=\dfrac{1+\tan 10^o}{1-\tan 10^o}=\dfrac{\tan 45^o+\tan 10^o}{1-\tan 45^o\tan 10^o}=\tan(45^o+10^o)=\tan 55^o$

Illustration :

If $3\tan\theta\tan\varphi=1$, *prove that* $2\cos(\theta+\varphi)=\cos(\theta-\varphi)$

Sol. *Given* $3\tan\theta\tan\varphi=1$ *or* $\cot\theta\cot\varphi=3$

or $\dfrac{\sin\theta\cos\varphi}{\sin\theta\sin\varphi}=\dfrac{3}{1}$

By componendo and dividendo,

we have, $\dfrac{\cos\theta\cos\varphi+\sin\theta\sin\varphi}{\cos\theta\cos\varphi-\sin\theta\sin\varphi}\times\dfrac{3+1}{3-1}\quad\Rightarrow\quad\dfrac{\cos(\theta-\varphi)}{\cos(\theta+\varphi)}=2$

$\Rightarrow\quad 2\cos(\theta+\varphi)=\cos(\theta-\varphi)$

Illustration :

Show that $\cos^2\theta + \cos^2(\alpha+\theta) - 2\cos\alpha\cos\theta\cos(\alpha+\theta)$ *is independent of* θ.

Sol. $\cos^2\theta + \cos^2(\alpha+\theta) - 2\cos\alpha\cos\theta\cos(\alpha+\theta) = \cos^2\theta + \cos(\alpha+\theta)[\cos(\alpha+\theta) - 2\cos\alpha\cos\theta]$

$= \cos^2\theta + \cos(\alpha+\theta)\,[\cos\alpha\cos\theta - \sin\alpha\sin\theta - 2\cos^2\alpha\cos\theta]$

$= \cos^2\theta - \cos(\alpha+\theta)\,[\cos\alpha\cos\theta + \sin\alpha\sin\theta]$

$= \cos^2\theta - \cos(\alpha+\theta)\cos(\alpha-\theta)$

$= \cos^2\theta - [\cos^2\alpha - \sin^2\theta] = \cos^2\theta + \sin^2\theta - \cos^2\alpha$

$= 1 - \cos^2\alpha$, *which is independent of* θ.

Illustration :

To prove that $\cot 16^o \cdot \cot 44^o + \cot 44^o \cdot \cot 76^o - \cot 76^o \cdot \cot 16^o = 3$

or

$(\cot 16^o)(\cot 44^o) - 1 + (\cot 44^o \cdot \cot 76^o - 1) + (\cot 44^o \cdot \cot 76^o - 1) - (\cot 76^o \cdot \cot 16^o + 1) = 0$

Sol. $\cot A \cot B \mp 1 = \dfrac{\cos(A \pm B)}{\sin A \sin B}$

and $3 + 1 + 1 + 1$

$44° + 16° = 60°,\ 44° + 76° = 120°$

$76° - 16° = 60°$

$\therefore$ *L.H.S.*

$$= \frac{\cos 60°\sin 76° + \cos 120°\sin 16° - \cos 60°\sin 44°}{\sin 16°\sin 44°\sin 76°}$$

$$= \frac{1}{2}\left[\frac{\sin 76° + \sin 16° - \sin 44°}{\sin 16^o\sin 44^o\sin 76^o}\right] = \frac{1}{2}\frac{(2\sin 30°\cos 46° - \sin 44°)}{\sin 16^o\sin 44^o\sin 76^o}$$

$2[\ \sin 16^o \sin 44^o \sin 76^o\ ]\quad 2\quad \sin 16^o\sin 44^o\sin 76^o$

$$= \frac{1}{2\sin 16^o\sin 44^o\sin 76^o}(\cos 46° - \cos 46°), \text{ by comp. rule} = 0.$$

Practice Problem

Q.1 If $x - y = \dfrac{\pi}{4}$ and $\cot x + \cot y = 2$; Find smallest +ve angles x and y.

Q.2 If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, prove that $1 + \cot\alpha\tan\beta = 0$

Q.3 If $\sin\alpha = \dfrac{15}{17}$, $\cos\beta = \dfrac{-5}{13}$. Find $\sin(\alpha - \beta)$

Q.4 If $\dfrac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)} = \dfrac{\tan\gamma}{\tan\beta}$ then prove that $\sin(\beta-\gamma) = 0$ or $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

Q.5 If α and β are the solutions of the equation $a\tan\theta + b\sec\theta = c$, then show that

$$\tan(\alpha+\beta) = \frac{2ac}{(a^2 - c^2)}.$$

Answer key

Q.1 $x = 75^o,\ y = 30^o$

TRANSFORMATION FORMULAE:

Transform the product into sum or difference :

We know that

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$
$$\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$$
$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$
$$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$$

(1) $\boxed{2\sin A\cos B = \sin(A+B)+\sin(A-B)}$

(2) $\boxed{2\cos A\sin B = \sin(A+B)-\sin(A-B)}$

(3) $\boxed{2\cos A\cos B = \cos(A+B)+\cos(A-B)}$

(4) $\boxed{2\sin A\sin B = \cos(A-B)-\cos(A+B)}$

Transforming sum or differences into products :

Putting $A+B=C$ & $A-B=D$

$\Rightarrow$ $A=\dfrac{C+D}{2}$; $B=\dfrac{C-D}{2}$

in 1, 2, 3, 4

(5) $\boxed{\sin C+\sin D = 2\sin\frac{(C+D)}{2}\cos\frac{(C-D)}{2}}$

(5) $\boxed{\sin C+\sin D = 2\sin\frac{(C+D)}{2}\cos\frac{(C-D)}{2}}$

(6) $\boxed{\sin C-\sin D = 2\cos\frac{(C+D)}{2}\sin\frac{(C-D)}{2}}$

(7) $\boxed{\cos C+\cos D = 2\cos\frac{(C+D)}{2}\cos\frac{(C-D)}{2}}$

(8) $\boxed{\cos C-\cos D = -2\sin\frac{(C+D)}{2}\sin\frac{(C-D)}{2}}$ **(Imp.)**

VALUES OF TRIGONOMETRIC RATIO OF STANDARD ANGLES :

Finding values of 15° & 75°

(1) $\sin 15^\circ = \sin\dfrac{\pi}{12} = \sin(45^\circ-30^\circ) = \sin 45^\circ\cdot\cos 30^\circ - \cos 45^\circ\cdot\sin 30^\circ$

$$\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} = \cos 75^\circ = \cos\frac{5\pi}{12}$$

(2) $\sin 75^\circ = \sin\dfrac{5\pi}{12} = \sin(45^\circ+30^\circ) = \sin 45^\circ\cdot\cos 30^\circ + \cos 45^\circ\cdot\sin 30^\circ$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4} = \cos 15^\circ = \cos\frac{\pi}{12}$$

(3) $\tan 15^{o} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$ = **cot 75º**

(4) $\tan 75^{o} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$ = **cot 15º**

NOTE :- $\sin A \ \sin (60^{o}-A) \ \sin (60^{o}+A) = \dfrac{1}{4}\sin 3A$

$\cos A \ \cos (60^{o}-A) \ \cos (60^{o}+A) = \dfrac{1}{4}\cos 3A$

$\tan A \ \tan (60^{o}-A) \ \tan (60^{o}+A) = \tan 3A$

Illustration :

Prove that $\cos (36°-A) \cos (36°+A) + \cos (54°+A) \cos (54°-A) = \cos 2A$

Sol. $\cos (36°-A) \cos (36°+A) + \cos [90^{o}-(54°+A)] \sin [90^{o}-(54°-A)] = \cos 2A$

$= \cos (36^{o}-A) \cos (36^{o}+A) + \sin (36^{o}-A) \sin (36^{o}+A)$

$= \cos [(36^{o}+A)-(36^{o}-A)] = \cos 2A$

Illustration :

Prove that $\dfrac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta} = 2\cos\theta$

Sol. *In the* N^{r} *write 6 as 1 + 5 and 15 as 10 + 5*

$\cos 5\theta + 5\cos 3\theta + 10\cos\theta$

Sol. *In the* N^{r} *write 6 as 1 + 5 and 15 as 10 + 5*

$N^{r} = \cos 6\theta + \cos 4\theta + 5\cos 4\theta + 5\cos 2\theta + 10\cos 2\theta + 10$

$= 2\cos 5\theta\cos\theta + 5.2\cos\cos 3\theta\cos\theta + 10.2\cos 2\theta$

$= 2\cos\theta[\cos 5\theta + 5\cos 3\theta + 10\cos\theta] = 2\cos\theta[D^{r}]$

$\therefore \quad \dfrac{N'}{D'} = 2\cos\theta$

Illustration :

If $\dfrac{\tan(\theta+x)}{a} = \dfrac{\tan(\theta+y)}{b} = \dfrac{\tan(\theta+z)}{c}$

then show that $\dfrac{a+b}{a-b}\sin^{2}(x-y) + \dfrac{b+c}{b-c}\sin^{2}(y-z) + \dfrac{c+a}{c-a}\sin^{2}(z-x) = 0$

Sol. $\dfrac{a}{b} = \dfrac{\tan(\theta+x)}{\tan(\theta+y)}$. *Applying componendo and dividendo*

$\therefore \quad \dfrac{a+b}{a-b} = \dfrac{\sin(2\theta+x+y)}{\sin(x-y)}$

$\therefore \quad \dfrac{a+b}{a-b}\sin^{2}(x-y) = \sin(2\theta+x+y)\sin(x-y) = \dfrac{1}{2}[\cos(2\theta+2y) - \cos(2\theta+2x)]$

$\therefore \quad \sum\dfrac{a+b}{a-b}\sin^{2}(x-y) = 0$ *as the term in R.H.S will be cancelled.*

Illustration :

Prove that $\sin 20^o \sin 40^o \sin 80^o = \dfrac{\sqrt{3}}{8}$

Sol. $\dfrac{1}{2}(2\sin 20^o \sin 40^o)\sin 80^o = \dfrac{1}{2}(\cos 20^o - \cos 60^o)\sin 80^o$

$= \dfrac{1}{2}(\cos 20^o \sin 80^o - \dfrac{1}{2}\sin 80^o) = \dfrac{1}{4}(2\cos 20^o \sin 80^o - \sin 80^o)$

$= \dfrac{1}{4}(\sin 100^o + \sin 60^o - \sin 80^o) = \dfrac{1}{4}(2\cos 90^o \sin 10^o + \sin 60^o) = \dfrac{1}{4}\sin 60^o = \dfrac{\sqrt{3}}{8}$

Alternate method :-

$\sin 20^o \sin 40^o \sin 80^o = \sin 20^o \sin(60^o - 20^o)\sin(60^o + 20^o) = \dfrac{1}{4}\sin(3 \times 20^o) = \dfrac{\sqrt{3}}{8}$

Practice Problem

Q.1 Find the value of $\cos^2 73^o + \cos^2 47^o + \cos 73^o \cdot \cos 47^o$

Q.2 If α in first quadrant $= \dfrac{\pi}{19}$. Find value of $\dfrac{\sin 23\alpha - \sin 3\alpha}{\sin 16\alpha + \sin 4\alpha}$.

Q.3 If $x \sin\theta = y \sin\left(\theta + \dfrac{2\pi}{3}\right) = z \sin\left(\theta + \dfrac{4\pi}{3}\right)$ then. Prove that $\sum xy = 0$.

Q.4 If α, β are two values of θ satisfying equation $\dfrac{\cos\theta}{a} + \dfrac{\sin\theta}{b} = \dfrac{1}{c}$ then. Prove that $\cot\left(\dfrac{\alpha+\beta}{2}\right) = \dfrac{b}{a}$.

Q.5 Prove that $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5B}$.

Answer key

Q.1 3/4 Q.2 –1

Trigonometric Ratio's of Multiple and submultiple angles :

Multiple angles are 2θ and 3θ and sub multiple angles are $\dfrac{\theta}{2}, \dfrac{\theta}{4}, \dfrac{\theta}{8}$.

1. $\mathbf{\sin 2A = \sin(A + A) = 2\sin A \cos A}$

2. $\mathbf{\cos 2A}$

$$\boxed{\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A}$$

$$\boxed{\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)}$$

Formulaes in terms of $\tan^2 A$

3. $\tan 2A = \tan (A+A) = \dfrac{2\tan A}{1-\tan^2 A}$

4. $\sin 2A = \dfrac{(2\tan A)}{1+\tan^2 A}$

5. $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

Sine, cosines and tangent of 3A

6. $\sin 3A = 3\sin A - 4\sin^3 A$
7. $\cos 3A = 4\cos^3 A - 3\cos A$

8. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

9. $\tan (A_1 + A_2 + \ldots\ldots A_n) = \dfrac{S_1 - S_3 + S_5 - S_1 + \ldots\ldots}{1 - S_2 + S_4 - S_6 + \ldots\ldots}$,

where
$S_1 = \tan A_1 + \tan A_2 + \ldots\ldots + \tan A_n$ = Sum of the tangents of the separate angles,
$S_2 = \tan A_1 \tan A_2 + \tan A_1 + \tan A_3 + \ldots\ldots$ = Sum of the tangents taken two at a time,
$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots\ldots$ = Sum of the tangents taken three at a

$S_1 = \tan A_1 + \tan A_2 + \ldots\ldots + \tan A_n$ = Sum of the tangents of the separate angles,
$S_2 = \tan A_1 \tan A_2 + \tan A_1 + \tan A_3 + \ldots\ldots$ = Sum of the tangents taken two at a time,
$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots\ldots$ = Sum of the tangents taken three at a
time, and so on.

Continued product of cosine Series :

$$\cos A \cos 2A \cos 4A \cos 8A \ldots. \cos 2^{n-1} A = \frac{1}{2^n \sin A} \sin (2^n A)$$

Proof : Multiplying above and below by $2^n \sin A$

$\therefore$ LHS

$$= \frac{2^{n-1}}{2^n \sin A} [2\sin A \cos A \cos 2A \cos 4A \ldots. \cos 2^{n-1} A]$$

$$= \frac{2^{n-2}}{2^n \sin A} [2\sin 2A \cos 2A \cos 4A \ldots. \cos 2^{n-1} A]$$

$$= \frac{2^{n-3}}{2^n \sin A} [2\sin 4A \cos 4A \ldots. \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A} [2\sin 2^{n-1} A \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A} \sin (2 \cdot 2^{n-1} A) = \frac{\sin(2^n A)}{2^n \sin A}$$

Illustration :

Prove that $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{128}$

Sol. $\cos\frac{5\pi}{15}=\cos\frac{\pi}{3}=\frac{1}{2},\ \cos\frac{7\pi}{15}=-\cos\frac{8\pi}{15}$

$$L.H.S=\left[\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\left(-\cos\frac{8\pi}{15}\right)\right]\left(\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right)\cdot\frac{1}{2}$$

$$=-\frac{1}{2}\cdot\frac{\sin 2^4\left(\frac{\pi}{15}\right)}{2^4\sin\frac{\pi}{15}}\cdot\frac{\sin\left(2^2\frac{3\pi}{15}\right)}{2^2\sin\frac{3\pi}{15}}=\frac{1}{128},\text{ as }\sin(\pi+\theta)=-\sin\theta$$

Illustration :

Prove that $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}=\frac{1}{8}$

Sol. $\sin\frac{\pi}{14}=\sin\left(\frac{\pi}{2}-\frac{6\pi}{14}\right)=\cos\frac{6\pi}{14}=\cos\left(\pi-\frac{8\pi}{14}\right)=-\cos\frac{8\pi}{14}$

$\sin\frac{3\pi}{14}=\sin\left(\frac{\pi}{2}-\frac{4\pi}{14}\right)=\cos\frac{4\pi}{14}$

$\sin\frac{3\pi}{14}=\sin\left(\frac{\pi}{2}-\frac{4\pi}{14}\right)=\cos\frac{4\pi}{14}$

$\sin\frac{5\pi}{14}=\sin\left(\frac{\pi}{2}-\frac{2\pi}{14}\right)=\cos\frac{2\pi}{14}$

$$\therefore\ L.H.S.=-\cos\frac{2\pi}{14}\cos\frac{4\pi}{14}\cos\frac{8\pi}{14}=-\frac{1}{2^3\sin A}\sin(2^3A),\ A=\frac{2\pi}{14}$$

$$=-\frac{1}{8\sin\frac{\pi}{7}}\sin\frac{8\pi}{7}=-\frac{1}{8\sin\frac{\pi}{7}}\sin\left(\pi+\frac{\pi}{7}\right)=-\frac{1}{8}(1)=\frac{1}{8}$$

$\because\ \sin(\pi+\theta)=-\sin\theta$

Illustration :

Prove that $\frac{\cos 2\theta}{1+\sin 2\theta}=\tan\left(\frac{\pi}{4}-\theta\right)$

Sol. $L.H.S.=\frac{\cos 2\theta}{1+\sin 2\theta}$

$$=\frac{\sin\left(\frac{\pi}{2}-2\theta\right)}{1+\cos\left(\frac{\pi}{2}-2\theta\right)}\qquad\left[\because\cos A=\sin\left(\frac{\pi}{2}-A\right),\sin A=\cos\left(\frac{\pi}{2}-A\right)\right]$$

$$= \frac{2\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}{2\cos^2\left(\frac{\pi}{4}-\theta\right)} \qquad \left[\begin{array}{l} \because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \\ 1+\cos A = 2\cos^2\frac{A}{2} \end{array}\right]$$

$$= \tan\left(\frac{\pi}{4}-\theta\right) = R.H.S.$$

Illustration :

Prove that $\dfrac{\sin 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$

Sol. $L.H.S. = \dfrac{\sin 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\frac{1}{\cos 8\theta}-1}{\frac{1}{\cos 4\theta}-1} = \dfrac{1-\cos 8\theta}{\cos 8\theta}\cdot\dfrac{\cos 4\theta}{1-\cos 4\theta}$

$$= \frac{2^2\sin 4\theta}{\cos 8\theta}\times\frac{\cos 4\theta}{2\sin^2 2\theta} = \frac{(2\sin 4\theta\cos 4\theta)}{\cos 8\theta}\times\frac{\sin 4\theta}{2\sin^2 2\theta} \qquad \left[\begin{array}{l} \because 1-\cos 8\theta = 2\sin^2\frac{8\theta}{2} = 2\sin^2 4\theta \\ \text{and } 1-\cos 4\theta = 2\sin^2\frac{4\theta}{2} = 2\sin^2 2\theta \end{array}\right]$$

$$= \left(\frac{2\sin 4\theta\cos 4\theta}{\cos 8\theta}\right)\times\left(\frac{2\sin 2\theta\cos 2\theta}{2\sin^2 2\theta}\right)$$

$$= \left(\frac{\sin 2(4\theta)}{\cos 8\theta}\right)\times\left(\frac{\cos 2\theta}{2\sin^2 2\theta}\right) = \left(\frac{\sin 8\theta)}{\cos 8\theta}\right)\times\left(\frac{\cos 2\theta}{2\sin^2 2\theta}\right)$$

$$= \left(\frac{\sin 2(4\theta)}{\cos 8\theta}\right)\times\left(\frac{\cos 2\theta}{\sin 2\theta}\right) = \left(\frac{\sin 8\theta)}{\cos 8\theta}\right)\times\left(\frac{\cos 2\theta}{\sin 2\theta}\right)$$

$$= \tan 8\theta\cot 2\theta = \frac{\cos 8\theta}{\sin 2\theta} = R.H.S.$$

Illustration :

Prove that $\cos^3\theta + \cos^3\theta\,(120° + \theta) + \cos^3(240° + \theta) = \frac{3}{4}\cos 3\theta$

Sol. We know that $\cos 3A = 4\cos 3A - 3\cos A$

$\therefore \quad \cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$

Also, $\cos(2n\pi + \theta) = \cos\theta$

Applying the above we have

$$= \frac{1}{4}\,[(3\cos\theta + \cos 3\theta) + 3\cos(120° + \theta) + \cos(360° + 3\theta) + 3\cos(240° + \theta) + \cos(720° + 3\theta)]$$

$$= \frac{1}{4}[3\cos 3\theta] + \frac{3}{4}\,[\cos\theta + \cos(120° + \theta) + \cos(240° + \theta)]$$

$$= \frac{3}{4}\cos 3\theta + \frac{3}{4}[\cos\theta + 2\cos(180° + \theta)\cos 60°]$$

$$= \frac{3}{4}\cos 3\theta + \frac{3}{4}[\cos\theta + 2\cdot\frac{1}{2}(-\cos\theta)] = \frac{3}{4}\cos 3\theta$$

Illustration :

If A, B, C are the angles of triangle ABC and tan A, tan B, tan C are the roots of the equation $x^4 - 3x^3 + 3x^2 + 2x + 5 = 0$, then find the fourth root of the equation.

Sol. *Let the fourth root of the equation be tan D*

Now $A + B + C = \pi$

Consider $\tan(A + B + C + D) = \dfrac{\sum \tan A - \sum \tan A \tan B \tan C}{1 - \sum \tan A \tan B + \prod \tan A} = \dfrac{3-(-2)}{1-3+5} = \dfrac{5}{3}$

$\tan(\pi + D) = \tan D = \dfrac{5}{3}$ ***Ans.***

Practice Problem

Q.1 Prove that $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}$

Q.2 For a positive integer n, let $f_n(\theta) = \left(\tan\dfrac{\theta}{2}\right)(1+\sec\theta)(1+\sec^1\theta)(1+\sec^2\theta) \ldots (1+\sec 2^n\theta)$ then

Q.2 For a positive integer n, let $f_n(\theta) = \left(\tan\dfrac{\theta}{2}\right)(1+\sec\theta)(1+\sec^1\theta)(1+\sec^2\theta) \ldots (1+\sec 2^n\theta)$ then

(A) $f_2\left(\dfrac{\pi}{16}\right) = 1$ (B) $f_3\left(\dfrac{\pi}{32}\right) = 1$ (C) $f_4\left(\dfrac{\pi}{64}\right) = 1$ (D) $f_5\left(\dfrac{\pi}{128}\right) = 1$

Q.3 Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$.

Q.4 Prove that $\tan^2\dfrac{\pi}{16} + \tan^2\dfrac{2\pi}{16} + \tan^2\dfrac{3\pi}{16} + \ldots\ldots + \tan^2\dfrac{7\pi}{16} = 35$

Q.5 If $\sin x + \sin y = 3(\cos y - \cos x)$, prove that $\sin 3x + \sin 3y = 0$

Q.6 If $x + 270\left[\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7}\right] = 3746$, then find the value of x.

Q.7 Show that $\sin\dfrac{\pi}{14}\sin\dfrac{3\pi}{14}\sin\dfrac{5\pi}{14}\sin\dfrac{7\pi}{14}\sin\dfrac{9\pi}{14}\sin\dfrac{11\pi}{14}\sin\dfrac{13\pi}{14} = \dfrac{1}{64}$.

Answer key

Q.2 A, B, C, D Q.6 $x = 3881$

To find the trigonometrical funcitons of an angle of 18° :

Let θ stand for 18°, so that 2θ is 36° and 3θ is 54°.

Hence $2\theta = 90° - 3\theta$.

and therfore

$$\sin 2\theta = \sin(90° - 3\theta) = \cos 3\theta$$

$$\therefore \quad 2 \sin 6 \cos \theta = 4\cos^3\theta - 3\cos\theta$$

Hence, either $\cos\theta = 0$, which gives $\theta = 90°$, or

$$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta$$

$$\therefore \quad 4\sin^2\theta + 2\sin\theta = 1$$

By solving this quadratic equation, we have

(In our case sin θ is necessarily a positive quantity. Hence we take the upper sign, and have)

$$\sin\theta = \frac{\sqrt{5}-1}{4} = \sin 18°$$

Hence $\mathbf{\cos 18°} = \sqrt{1-\sin^2 18°} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{16}}$

The remaining trigonometrical ratio of 18° may be now found.

Since 72° is the completement of 18°, the vlaue of the ratios for 72° may be obtained.

To find the trigonometrical functions of an angle of 36°

Since $\cos 2\theta = 1 - 2\sin^2\theta$,

Since 72° is the completement of 18°, the vlaue of the ratios for 72° may be obtained.

To find the trigonometrical functions of an angle of 36°

Since $\cos 2\theta = 1 - 2\sin^2\theta$,

$$\therefore \cos 36° = 1 - 2\sin^2 18° = \frac{\sqrt{5}+1}{4}$$

The remaining trigonometrical functions of 36° may now be found.

Also, since 54° may be found.

Values of standard angles :

Angle → / ↓T. Ratio	$\left(\frac{\pi}{12}\right)$ 15°	$\left(\frac{\pi}{10}\right)$ 18°	$\left(\frac{\pi}{8}\right)$ 22.5°	$\left(\frac{\pi}{5}\right)$ 36°	$\left(\frac{3\pi}{8}\right)$ 67.5°	$\left(\frac{5\pi}{12}\right)$ 75°
sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
tan	$2-\sqrt{3}$	$\frac{1}{\sqrt{(5+2\sqrt{5})}}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{2}+1$	$2+\sqrt{3}$
cot	$2+\sqrt{3}$	$\sqrt{(5+2\sqrt{5})}$	$\sqrt{2}+1$	$\sqrt{\left(1+\frac{2}{\sqrt{2}}\right)}$	$\sqrt{2}-1$	$2-\sqrt{3}$

Illustration :

Prove that $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$

Sol. L.H.S. $= \sin 36^\circ \sin 72^\circ \sin 108^\circ \sin 144^\circ$

$= \sin 36^\circ \sin 72^\circ \sin(180^\circ - 72^\circ)\sin(108^\circ - 36^\circ)$

$= \sin 36^\circ \sin 72^\circ \sin 72^\circ \sin 36^\circ = \sin^2 36^\circ \sin^2 72^\circ$

$$= \frac{5-\sqrt{5}}{8} \times \frac{5+\sqrt{5}}{8} = \frac{25-5}{64} = \frac{20}{64} = \frac{5}{16}.$$

Illustration :

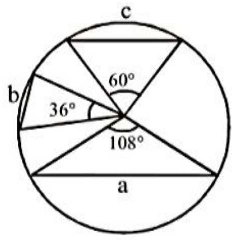
In any circle prove that the chord which subtends 108° at the centre is equal to the sum of the two chords which subtend angles of 36° and 60°.

Sol. *If r be the radius of the circle and a, b, c and chords which subtend at the centre angles = 108°, 36° and 60°, then*

$$a = 2r\sin\frac{108^o}{2} = r\sin 54^\circ$$

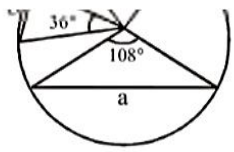
$$b = 2r\sin\frac{36^o}{2} = r\sin 18^\circ$$

$$c = 2r\sin\frac{60^o}{} = r\sin 30^\circ$$



$$b = 2r\sin\frac{36^o}{2} = r\sin 18^\circ$$

$$c = 2r\sin\frac{60^o}{2} = r\sin 30^\circ$$



$$= 2r\sin 54^\circ = 2r\sin 36^\circ$$

$$= 2r\frac{\sqrt{5}+1}{4}$$

or $$2r(\sin 18^\circ) + (2r\sin 30^\circ) = 2r\left(\frac{\sqrt{5}-1}{4} + \frac{1}{2}\right) = 2r\frac{\sqrt{5}+1}{4}$$

Hence the chords which subtends 108° at the centre is equal to the sum of the two chords which subtends angle of 36° and 60°.

Practice Problem

Q.1 If $\sec^2 60^o + k(\sin 54^o - \cos 72^o) = 10$, then find k.

Q.2 If $\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}$. Then find x.

Answer key

Q.1 k = 12 Q.2 x = 15°, 55°

APPLICATION OF TRIGONOMETRY IN MAXIMISING AND MINIMISING i.e. (Optimisation) :

1. Maximising and minimising by using the property of boundness of trigonometric functions.
   (a) Sine and cosine have bounded values between – 1 and 1.
   (b) Tangent and cotangent are unbounded functions.
   (c) Cosec and sec have values greater than 1 and less than – 1.
   (d) $0 \le \sin^2 x \le 1$, $0 \le \cos^2 x \le 1$, $\tan^2 x \ge 0$, $\sec^2 x \ge 1$.

Note : If maximum value of a function is **'b'**, and minimum value **'a'** then range is [a, b].

TYPE - I :

Illustration :

Find minimum and maximum values of $y = 2 + \cos x$.

Sol. $y = 2 + \cos x$

$\because -1 \le \cos x \le 1$

$\therefore y = 2 \pm 1$

$y_{max} = 3$ and $y_{min} = 1$

Illustration :

Find minimum and maximum values of $y = 1 + \cos x + \cos^2 x + \cos^3 x + \cos^4 x$.

Sol. for $x = 0°$ $\quad \cos x = 1$

for $x = \pi$ $\quad \cos x = -1$

Sol. for $x = 0°$ $\quad \cos x = 1$

for $x = \pi$ $\quad \cos x = -1$

for $x = \frac{\pi}{2}$ $\quad \cos x = 0$

for $\cos x = 1$, $\quad y_{max} = 5$

for $\cos x = -1$ or 0 $\quad y_{min} = 1$

Special Cases:

When argument of sine and cosine are same

General form $y = a \sin x + b \cos x + C$. Find min. and max. value of y.

[Ans. $y_{max.} = \sqrt{a^2 + b^2} + C$; $y_{min} = -\sqrt{a^2 + b^2} + C$, where $\theta = \cos^{-1}\frac{b}{\sqrt{a^2+b^2}}$]

Illustration :

Find minimum and maximum value of $7 \cos \theta + 24 \sin \theta$.

Sol. $y = 7 \cos \theta + 24 \sin \theta$.

$$-\sqrt{a^2 + b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2 + b^2}$$

$$\therefore -\sqrt{7^2 + 24^2} \le 7\cos\theta + 24\sin\theta \le \sqrt{7^2 + 24^2}$$

$-25 \le 7 \cos \theta + 24 \sin \theta \le 25$.

Illustration :

$$y = sin^2\left(\frac{15\pi}{8} - 4x\right) - sin^2\left(\frac{17\pi}{8} - 4x\right).\ \textit{Find range of y.}$$

Sol. $\Rightarrow\ sin\left(\frac{15\pi}{8} - 4x + \frac{17\pi}{8} - 4x\right) sin\left(\frac{15\pi}{8} - \frac{17\pi}{8}\right)$

$$\Rightarrow\ sin\,(4\pi - 8x)\ sin\left(-\frac{\pi}{4}\right)\ \Rightarrow - sin\,\frac{\pi}{4}\cdot sin\,8x \quad \Rightarrow \quad -\frac{1}{\sqrt{2}}\cdot sin\,8x$$

$$\Rightarrow\ \boxed{-\frac{1}{\sqrt{2}} \le y \le \frac{1}{\sqrt{2}}} \qquad\qquad \textbf{Ans. } y \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

Illustration :

$$y = log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right).\ \textit{Find range (minimum and maximum) value of y.}$$

Sol. $\therefore\quad -5 \le 3\ sin\ x - 4\ cos\ x \le 5$

$$1 \le \frac{3\sin x - 4\cos x + 15}{10} \le 2$$

$$log_2{}^1 \le log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right) \le log_2 2$$

10

$$log_2{}^1 \le log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right) \le log_2 2$$

$$0 \le log_2\left(-\frac{3\sin x - 4\cos x + 15}{10}\right) \le 1 \qquad \textbf{Ans. } y_{min} = log_2 1 = 0;\ y_{max} = 1;\ y \in [0, 1]$$

TYPE-II :

Argument of sine and cosine are different or a quadratic in sine / cos is given then we make a perfect square in sine / cosine and interpret.

Illustration :

$y = cos\ 2x + 3\ sin\ x$. *Find range of y.*

Sol. $\Rightarrow 1 - 2sin^2x + 3sinx$

$$\Rightarrow 1 - 2\left[\sin^2 x - \frac{3}{2}\sin x + \frac{9}{16} - \frac{9}{16}\right] \Rightarrow \qquad 1 - 2\left[\sin x - \frac{3}{4}\right]^2 + \frac{9}{8}$$

$$y = \frac{17}{8} - 2\left[\sin x - \frac{3}{4}\right]^2$$

$$y_{max} = \frac{17}{8} \qquad at\ sin\ x = \frac{3}{4}$$

$$y_{min} = -4 \qquad at\ sin\ x = -1 \qquad\qquad [Ans.\ y \in \left[-4, \frac{17}{8}\right];\ \ y_{max} = \frac{17}{8}\ ;\ y_{min} = -4]$$

TYPE-III:

Making use of reciprocal relationship between tan and cot, sin/cosec and cos/sec.

Illustration :

If $y = a^2\tan^2 x + b^2\cot^2 x$ $(a, b \geq 0)$. *Find minimum value of y.*

Sol. $y = a^2\tan^2 x + \dfrac{b^2}{\tan^2 x} - 2ab + 2ab \Rightarrow \left(a\tan x - \dfrac{b}{\tan x}\right)^2 + 2ab \geq 2ab$

$y_{min.} = 2ab$

Miscellaneous problems

Illustration :

If $x^2 + y^2 = 4$ *and* $a^2 + b^2 = 8$. *Find minimum and maximum value of* $(ax + by)$

Sol. *Let* $x = r_1 \cos\theta_1$, $y = r_1 \sin\theta$

and $a = r_2\cos\phi$; $b = r_2 \sin\phi$

$\therefore$ $r_1 = 2$, $r_2 = 2\sqrt{2}$

Then $(ax + by) = r_1 r_2 \cos(\theta - \phi)$

$-r_1 r_2 \leq (ax + by) \leq r_1 r_2$

$-4\sqrt{2} \leq (ax + by) \leq 4\sqrt{2}$ *Ans.* $y_{max} = 4\sqrt{2}$ *and* $y_{min} = -4\sqrt{2}$

Practice Problem

Practice Problem

Q.1 $y = 3\cos\left(x + \dfrac{\pi}{3}\right) + 5\cos x + 3$. Find minimum and maximum value of y.

Q.2 $y = \cos^2 x - 4\cos x + 13$. Find minimum and maximum value of y.

Q.3 $y = 8\sec^2 x + 18\cos^2 x$, find y_{min}.

Q.4 $y = \dfrac{\tan 3x}{\tan x}$, find range of y.

Q.5 The maximum possible value of $(xv - yu)^2$ over the surface given by the equations $x^2 + y^2 = 4$ and $u^2 + v^2 = 9$ is

(A) 13 (B) 26 (C) 36 (D) 40

Answer key

Q.1 $y_{max} = 10$ and $y_{min} = -4$ Q.2 $y_{max} = 18; y_{min} = 10$ Q.3 $y_{min} = 24$

Q.4 $\left(-\infty, \dfrac{1}{3}\right) \cup [3, \infty)$ Q.5 C

CONDITIONAL IDENTITIES AND INEQUALITIES :

(i) If A, B, C are the angles of a triangle, then

$A + B + C = 180°$

$\therefore \quad B + C = 180° - A, \quad C + A = 180° - B, \quad A + B = 180° - C,$

Hence $\sin(B + C) = \sin(180° - A) = \sin A$

Similarly $\sin(C + A) = \sin B$ and $\sin(A + B) = \sin C$.

Thus in a triangle, sine of any angle is equal to the sine of the sum of the remaining angles.

(ii) Again $\cos(B + C) = \cos(180° - A) = -\cos A$

In the same manner

$\cos(C + A) = -\cos B$ and $\cos(A + B) = -\cos C$.

Thus in a triangle, the cosine of any one angle is equal to minus times the cosine of the remaining two angles.

(iii) If $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ then $\frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

$$\sin\left(\frac{A}{2} + \frac{B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\frac{C}{2}$$

Note : Some standard identities in triangle ($A + B + C = \pi$) :-

(1) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(2) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(3) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(2) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(3) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(4) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

(5) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(6) $\tan\frac{A}{2} \tan\frac{B}{2} + \tan\frac{C}{2} \tan\frac{B}{2} + \tan\frac{C}{2} \tan\frac{A}{2} = 1$

Illustration :

Prove that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

Sol. $\sin 2A + \sin 2B + \sin 2C = 2\sin(A + B)\cos(A - B) + 2 \sin C \cos C$

Since $A + B + C = 180°$

We have $A + B = 180° - C,$

and therfore $\sin(A + B) = \sin C,$

and $\cos(A + B) = -\cos C$

Hence the expression

$= 2\sin C \cos(A - B) + 2 \sin C \cos C$

$= 2\sin C [\cos(A - B) + \cos C]$

$= 2\sin C [\cos(A - B) - \cos(A + B)]$

$= 2\sin C, 2 \sin A \sin B$

$= 4\sin A \sin B \sin C.$

Note: Remember this identity: $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

Illustration :

Prove that $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

Sol. $(\cos A + \cos B) + \cos C$

$$= 2 \cos\frac{A+B}{2} \cos \frac{A-B}{2} + \cos C - 1$$

$$= 2 \cos\left(\frac{\pi}{2}-\frac{C}{2}\right) \cos \frac{A-B}{2} + \cos C - 1$$

$$= 2\sin\frac{C}{2} \cos \frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} - 1$$

$$= 2 \sin\frac{C}{2} \cos \frac{A-B}{2} - 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)\right]$$

$$= 2 \sin\frac{C}{2}\ 2 \sin \frac{A}{2} \sin \frac{B}{2} = 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$$

Illustration :

If A, B, C are the angles of a triangle, show that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Sol. *We have*

Illustration :

If A, B, C are the angles of a triangle, show that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Sol. *We have*

$$\tan (A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1-(\tan B \tan C + \tan C \tan A + \tan A \tan B)}$$

But $\tan (A + B + C) = \tan 180° = 0$

Hence $0 = \tan A + \tan B + \tan C - \tan A \tan B \tan C$

i.e, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

This may also be proved independently. For

$\tan (A + B) = \tan (180° - C) = -\tan C.$

$$\therefore \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$\tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Illustration :

In a $\Delta\, ABC$ *show that* $\cos A \cos B \cos C \le \frac{1}{8}$

Sol. *Let* $y = \cos A \cos B \cos C$

$$= \frac{1}{2}\cos A\left[\cos\frac{(B+C)}{2} + \cos\frac{(B-C)}{2}\right]$$

$$= \frac{1}{2}\cos A\, [-\cos A + \cos (B - C)]$$

$= \frac{1}{2} \cos A \, [\cos (B - C) - \cos A]$

$\leq \frac{\cos A}{2} \, (1 - \cos A)$

$\therefore \cos (B - C)_{max} = 1)$

$\leq -\frac{1}{2} \, (\cos^2 A - \cos A)$

$\leq -\frac{1}{2} \left[\left(\cos A - \frac{1}{2} \right)^2 - \frac{1}{4} \right]$

$\leq \frac{1}{8} - \frac{1}{2} \left(\cos A - \frac{1}{2} \right)^2$

$y \leq \frac{1}{8} - \frac{1}{2} \left(\cos A - \frac{1}{2} \right)^2$

max vlaues of y occurs when $\cos A = \frac{1}{2}$

$A = 60°.$

max vlaues of y occurs when $\cos A = \frac{1}{2}$

$A = 60°.$

$y_{max} = \frac{1}{8}$ $\quad \boxed{y \leq \frac{1}{8}}$

Practice Problem

Q.1 In ΔABC, prove that $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

Q.2 In ΔABC, prove that $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

Q.3 In ΔABC, prove that $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.

Q.4 If $\alpha + \beta = \gamma$, prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 + 2 \cos \alpha \cdot \cos \beta \cdot \cos \gamma$.

Q.5 In ΔABC, prove that $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \cdot \tan \frac{C}{2} + \tan \frac{C}{2} \cdot \tan \frac{A}{2} = 1$.

Q.6 In a ΔABC show that $1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$.

Q.7 For all real θ prove that $\cos (\sin \theta) > \sin (\cos \theta)$

SUMMATION OF TRIGONOMETRIC SERIES :

Type-I

Sum of the sin and cosine series when the angles are in A.P.

(1) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+\overline{n-1}\,\beta) = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \times \sin\left[\alpha + (n-1)\frac{\beta}{2}\right]$

Proof :

Let $S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+\overline{n-1}\,\beta)$

Here angle are in A.P. and common difference of angles $= \beta$

$\therefore$ multiplying both sides by $2\sin\frac{\beta}{2}$, we get

$$2S\sin\frac{\beta}{2} = 2\sin\alpha\sin\frac{\beta}{2} + 2\sin(\alpha+\beta)\sin\frac{\beta}{2} + \ldots\ldots + 2\sin(\alpha+\overline{n-1}\,\beta)\sin\frac{\beta}{2} \qquad \ldots(i)$$

Now, $2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)$

$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)$

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$2\sin(\alpha+\overline{n-1}\,\beta)\sin\frac{\beta}{2} = \cos\left[\alpha + (2n-3)\frac{\beta}{2}\right] - \cos\left[a(2n-1)\frac{\beta}{2}\right]$

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adding, we get R.H.S. of equation (i) $= \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left[\alpha + (2n-1)\frac{\beta}{2}\right]$

hence, $$S = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left[\alpha + (n-1)\frac{\beta}{2}\right]$$

(2) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos(\alpha+\overline{n-1}\,\beta) = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \times \cos\left[\alpha + (n-1)\frac{\beta}{2}\right]$

Illustration :

Find the sum of series $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$.

Sol. *Here the angles are in A.P. of 5 terms*

$n = 5,\ \alpha = \frac{\pi}{11},\ \beta = \frac{2\pi}{11}\ \therefore\ \frac{\beta}{2} = \frac{\pi}{11}$

$$S = \frac{\sin \cdot \frac{5\pi}{11}}{\sin \frac{\pi}{11}} \cdot \cos \frac{1}{2}\left[\frac{\pi}{11} + \frac{9\pi}{11}\right]$$

$$S = \frac{2\sin \cdot \frac{5\pi}{11} \cos \cdot \frac{5\pi}{11}}{2\sin \frac{\pi}{11}} = \frac{\sin \cdot \frac{10\pi}{11}}{2\sin \frac{\pi}{11}} = \frac{\sin\left(\pi - \frac{\pi}{11}\right)}{2\sin \frac{\pi}{11}} = \frac{1}{2}$$

Type - II

For n sided regular polygon
Sum of all exterior angles = 2π

(i) **The value of one exterior angle =** $\frac{2\pi}{n}$

(ii) **The value of one interior angle =** $\left(\pi - \frac{2\pi}{n}\right) = \pi\frac{(n-2)}{n}$

(iii) **Sum of interior angles =** $\pi (n-2)$

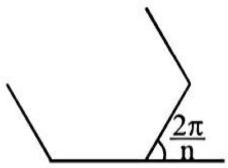


Illustration :

In a regular polygon of n-sides with $A_1, A_2, \ldots\ldots A_n$ vertices prove that
$(A_1A_2)^2 + (A_1A_3)^2 + (A_1A_4)^2 + \ldots\ldots\ldots + (A_1A_n)^2 = 2nR^2$

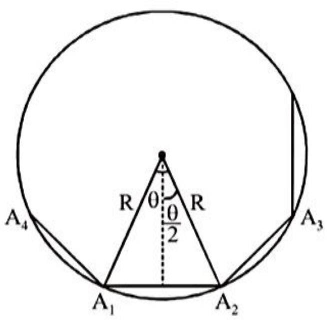
Illustration :

In a regular polygon of n-sides with $A_1, A_2, \ldots\ldots A_n$ vertices prove that
$$(A_1A_2)^2 + (A_1A_3)^2 + (A_1A_4)^2 + \ldots\ldots\ldots + (A_1A_n)^2 = 2nR^2$$
Where R is the radius of circumcircle circumscribing it.

Sol. $(A_1A_2)^2 + (A_1A_3)^2 + \ldots. (A_1A_n)^2 = 2nR^2$

$\theta = \frac{2\pi}{n}$

By trigonometry



$(A_1A_2)^2 = 4R^2 \cdot \sin^2 \frac{\pi}{n}$

$(A_1A_3)^2 = 4R^2 \cdot \sin^2 \frac{2\pi}{n}$

$(A_1A_4)^2 = 4R^2 \cdot \sin^2 \frac{3\pi}{n}$

...

$(A_1A_n)^2 = 4R^2 \cdot \sin^2 \frac{(n-1)\pi}{n}$

$\therefore (A_1A_2)^2 + (A_1A_3)^2 + \ldots. + (A_1A_n)^2$

$$4R^2\left[\sin^2\frac{\pi}{n} + \sin^2 x\frac{2\pi}{n} + \sin^2\frac{3\pi}{n} + \ldots.. \sin^2\frac{(n-1)\pi}{n}\right]$$

$$\frac{1}{2}4R^2\left[1 - \cos\frac{2\pi}{n} + 1 - \cos\frac{4\pi}{n} + 1 - \cos\frac{6\pi}{n} + \ldots.. 1 - \cos\frac{2(n-1)\pi}{n}\right]$$

$$2R^2(n-1) - 2R^2\left[\frac{\sin\frac{(n-1)\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}\cdot\cos\left(\frac{\frac{2\pi}{n}+\frac{2(n-1)\pi}{n}}{2}\right)\right]$$

$\Rightarrow \quad 2R^2(n-1) - 2R^2(-1) \qquad \Rightarrow \quad 2R^2n - 2R^2 + 2R^2$

$\Rightarrow \quad 2nR^2$

Type-III of summation of sine/cosine series

Splitting the sum series as difference of 2 terms.

Illustration :

$S = cosec\,x + cosec\,2x + cosec\,4x + \ldots\ldots + cosec\,2^n x = \cot\frac{x}{2} - \cot 2^n x.$

Sol. Let $cosec\,x$

$$= \frac{\sin\left(\frac{x}{2}\right)}{\sin\frac{x}{2}\cdot\sin x} = \frac{\sin\left(x-\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)\cdot\sin x} = \frac{\sin\cdot\cos\frac{x}{2} - \cos x\cdot\sin\frac{x}{2}}{\sin\frac{x}{2}\cdot\sin x} = \cot\frac{x}{2} - \cot x$$

$$S = \left(\cot\frac{x}{2} - \cot x\right) + (\cot x - \cot 2x) + (\cot 2x - \cot 2^2 x) + \ldots (\cot 2^{n-1}x - \cot 2^n x)$$

$$\Rightarrow S = \left(\cot\frac{x}{2} - \cot 2^n\cdot x\right)$$

$$S = \left(\cot\frac{x}{2} - \cot x\right) + (\cot x - \cot 2x) + (\cot 2x - \cot 2^2 x) + \ldots (\cot 2^{n-1}x - \cot 2^n x)$$

$$\Rightarrow S = \left(\cot\frac{x}{2} - \cot 2^n\cdot x\right)$$

Illustration :

Given $\theta \in (0, 2\pi)$ and if $\sum_{r=1}^{r=3}\sec\left(\frac{r\pi}{6}+\theta\right)\sec\left(\frac{(r-1)\pi}{6}+\theta\right) = 4$, then which of the following alternative(s) is/are correct?

(A) $\theta = \frac{3\pi}{4}$ (B) $\theta = \frac{\pi}{8}$ (C) $\theta = \frac{7\pi}{4}$ (D) $\theta = \frac{3\pi}{2}$

Sol.
$$\frac{1}{\sin\frac{\pi}{6}}\left[\frac{1}{\cos\left(\frac{r\pi}{6}+\theta\right)}\cdot\frac{\sin\frac{\pi}{6}}{\cos\left(\frac{(r-1)\pi}{6}+\theta\right)}\right] = \frac{1}{\sin\frac{\pi}{6}}\left[\frac{\sin\left(\left(\frac{\pi}{6}+\theta\right)-\left(\frac{(r-1)\pi}{6}+\theta\right)\right)}{\cos\left(\frac{r\pi}{6}+\theta\right)\cos\left(\frac{(r-1)\pi}{6}+\theta\right)}\right]$$

$$= \frac{1}{\sin\frac{\pi}{6}}\left[\frac{\sin A\cdot\cos B - \cos A\cdot\sin B}{\cos A\cdot\cos B}\right] \qquad \text{Let} \quad \left(\frac{\pi}{6}+\theta\right) = A,\ \left(\frac{(r-1)\pi}{6}+\theta\right) = B$$

$$= \frac{1}{\sin\frac{\pi}{6}}[\tan A - \tan B] = \frac{1}{\sin\frac{\pi}{6}}\left[\tan\left(\frac{r\pi}{6}+\theta\right) - \tan\left(\frac{(r-1)\pi}{6}+\theta\right)\right]$$

$$\sum_{r=1}^{3} S = -2\,[\cot\theta + \tan\theta] \quad \Rightarrow \quad \frac{-4}{\sin 2\theta} = +4$$

$$\text{or} \quad \sin 2\theta = -1 \quad \therefore \theta = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

Practice Problem

Q.1 Prove that $\sum_{r=1}^{n-1} \sin^2 \frac{r\pi}{n} = \frac{n}{2}$.

Q.2 Prove that $\tan\frac{x}{2}\sec x + \tan\frac{x}{2^2}\sec\frac{x}{2} + \tan\frac{x}{2^3}\sec\frac{x}{2^2} + \ldots$ up to n terms $= \tan x - \tan\frac{x}{2^n}$

Q.3 Find the sum to n terms of the series $\frac{\sin x}{\cos x + \cos 2x} + \frac{\sin 2x}{\cos x + \cos 4x} + \frac{\sin 3x}{\cos x + \cos 6x} + \ldots\ldots$

Answer key

Q.3 $\frac{1}{4}\operatorname{cosec}\frac{x}{2}\left[\sec(2n+1)\frac{x}{2} - \sec\frac{x}{2}\right]$

ELIMINATION :

ELIMINATION :

Between any two equations involving one unknown quantity we can, in theory, always eliminate that quantity. In practice, a considerable amount of artifice and ingenuity is often required in seemingly simple cases. So, between any three equations involving two unknown quantities, we can theoretically eliminate both of the unknown quantities.

Eliminate θ from the equations

Illustration :

Eliminate θ from the equations $a\cos\theta + b\sin\theta = c$ *and* $b\cos\theta - a\sin\theta = d$.

Sol. $a\cos\theta + b\sin\theta = c$ ... (i) ; $b\cos\theta - a\sin\theta = d$. ... (ii)

Square (i) and (ii) and adding,

$a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = c^2 + d^2$

or $a^2 + b^2 = c^2 + d^2$.

Illustration :

Eliminate θ from the equations

$a\sin\alpha - b\cos\alpha = 2b\sin\theta$, *and* $a\sin 2\alpha - b\cos 2\theta = a$,

Sol. $a\sin\alpha - b\cos\alpha = 2b\sin\theta$ ... (i)

$a\sin 2\alpha - b\cos 2\theta = a$ ... (ii)

From (i) $\sin\theta = \frac{a\sin\alpha - b\cos\alpha}{2b}$

From (ii) $a \sin 2\alpha - b\,(1 - 2\sin^2\theta) = a$

or $\quad a \sin 2\alpha - b\left\{1 - 2\left(\dfrac{a\sin\alpha - b\cos\alpha}{2b}\right)^2\right\} = a$

or $\quad a^2\sin^2\alpha + b^2\cos^2\alpha + 4ab\sin\alpha\cos\alpha - 2ab\sin\alpha\cos\alpha = 2ab + 2b^2$

or $\quad a\sin\alpha + b\cos\alpha = \sqrt{\{2b(a+b)\}}$

Illustration :

Eliminate θ from the equations

$$x\sin\theta - y\cos\theta = \sqrt{(x^2+y^2)},\ \text{and}\ \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} = \frac{1}{x^2+y^2}.$$

Sol. $\quad x\sin\theta - y\sin\theta = \sqrt{x^2+y^2} \qquad \ldots (1)$

and $\quad \dfrac{\sin^2\theta}{a^2} + \dfrac{\cos^2\theta}{b^2} = \dfrac{1}{x^2+y^2} \qquad \ldots (2)$

Squaring (1) we have

$x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta = x^2 + y^2$

or $\quad x\cos\theta + y\sin\theta = 0$

or $\quad \tan\theta = \dfrac{-x}{y}$; $\sin^2\theta = \left\{\dfrac{x}{\sqrt{(x^2+y^2)}}\right\}^2 = \dfrac{x^2}{x^2+y^2}$

and $\quad \cos^2\theta = \dfrac{y^2}{\ 2 \ . \ 2}$

and $\quad \cos^2\theta = \dfrac{y^2}{x^2+y^2}$

Substituting value in (2) we have

or $\quad \dfrac{x^2}{a^2(x^2+y^2)} + \dfrac{y^2}{b^2(x^2+y^2)} = \dfrac{1}{x^2+y^2} \quad \Rightarrow \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Practice Problem

Q.1 Eliminate θ from the following

$\sin\theta - \cos\theta = p$, and $\operatorname{cosec}\theta - \sin\theta = q$.

Q.2 Eliminate θ from the following

$a\cos 2\theta = b\sin\theta$, and $c\sin 2\theta = d\cos\theta$.

Q.3 Eliminate x, y from the following

$\cos x + \cos y = a$

$\cos 2x + \cos 2y = b$

$\cos 3x + \cos 3y = c$.

Answer key

Q.1 $(p^2+1)^2 + 2q(p^2+1) = 4(p+q)^2$ Q.2 $a(2c^2 - d^2) = bcd$

Q.3 $2a^3 + c = 3a(b+1)$

Solved Examples

Q.1 Compute the square of the value of the expression $\frac{4+\sec 20°}{\text{cosec } 20°}$.

Sol. $\frac{4+\sec 20°}{\text{cosec} 20°}$

$$=\left(\frac{4\cos 20°+1}{\cos 20°}\right)\sin 20° = \frac{4\sin 20° \cos 20° + \sin 20°}{\cos 20°}$$

$$= \frac{2\sin 40° + \sin 20°}{\cos 20°} = \frac{\sin 40° + 2\sin 30° \cos 10°}{\cos 20°}$$

$$= \frac{\sin 40° + \cos 10°}{\cos 20°} = \frac{\sin 40° + \sin 80°}{\cos 20°} = \frac{2\sin 60° \cos 20°}{\cos 20°} = \sqrt{3}$$

$= \left(\sqrt{3}\right)^2 = 3$ **Ans.**

Q.2 If $\alpha + \beta + \gamma = \pi$ and $\prod \tan\left(\frac{\alpha+\beta-\gamma}{4}\right) = 1$, then find $\sum \sin\alpha + \prod \sin\alpha$.

Sol. $\Pi \tan\left(\frac{\alpha+\beta-\gamma}{4}\right) = 1$

$\left(\pi - 2\gamma\right)$

Sol. $\Pi \tan\left(\frac{\alpha+\beta-\gamma}{4}\right) = 1$

$\Rightarrow \Pi \tan\left(\frac{\pi-2\gamma}{4}\right) = 1$ (since $\alpha + \beta + \gamma = \pi$)

$\Rightarrow \Pi \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = 1$ $\Rightarrow$ $\Pi\left(\frac{\cos\frac{\gamma}{2} - \sin\frac{\gamma}{2}}{\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}}\right) = 1$

$\Rightarrow \Pi\left(\cos\frac{\gamma}{2} - \sin\frac{\gamma}{2}\right) = \Pi\left(\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\right)$

After squaring both sides

$\Rightarrow \Pi(1 - \sin\gamma) = \Pi(1 + \sin\gamma)$

$\Rightarrow 1 - \sum\sin\gamma + \sum\sin\gamma\sin\alpha - \Pi\sin\alpha = 1 + \sum\sin\gamma + \sum\sin\gamma\sin\alpha + \Pi\sin\alpha$

$\Rightarrow \sum\sin\gamma + \Pi\sin\gamma = 0$ **Ans.**

Q.3 Let $f(x) = \frac{1+\cos 2x + 8\sin^2 x}{\sin 2x}$, $x \in (0, \pi/2)$. Find the minimum value of $f(x)$.

Sol. $f(x) = \frac{1+\cos 2x + 8\sin^2 x}{\sin 2x}$

$$f(x) = \frac{2\cos^2 x + 8\sin^2 x}{2\sin x \cos x}$$

Divide by $2\cos^2 x$, we get

$$f(x)=\frac{1+4\tan^2 x}{\tan x} \quad \Rightarrow \quad f(x)=\frac{1}{\tan x}+4\tan x$$

Apply A.M. $\geq$ G.M.

$$\therefore \quad \frac{\frac{1}{\tan x}+4\tan x}{2}\geq\sqrt{4} \qquad \therefore \quad \frac{1}{\tan x}+4\tan x\geq 4$$

Q.4 α and β are the positive acute angles and satisfying simultaneously the equations

$5\sin 2\beta=3\sin 2\alpha$ and $\tan\beta=3\tan\alpha$,

find the value of $\tan\alpha+\tan\beta$.

Sol. $\dfrac{\sin 2\alpha}{\sin 2\beta}=\dfrac{5}{3}$

$$\Rightarrow \frac{(2\tan\alpha)}{(1+\tan^2\alpha)}\frac{(1+\tan^2\beta)}{(2\tan\beta)}=\frac{5}{3} \qquad \Rightarrow \qquad \frac{(2\tan\alpha)}{(1+\tan^2\alpha)}\frac{(1+9\tan^2\alpha)}{(6\tan\alpha)}=\frac{5}{3} \quad (\tan\beta=3\tan\alpha)$$

$\Rightarrow 1+9\tan^2\alpha=5+5\tan^2\alpha \qquad \Rightarrow \qquad 4\tan^2\alpha=4$

$\Rightarrow \boxed{\tan\alpha=+1}$

$\Rightarrow \tan\beta=3$

$\therefore \tan\alpha+\tan\beta=1+3=4$ **Ans.**

Q.5 If $\dfrac{\cos 42°+\sin 42°}{\cos 42°-\sin 42°}=(\tan\alpha+\sec\alpha)$ where $0<\alpha<90°$, then find the value of α in degree.

Sol. RHS $=\tan\alpha+\sec\alpha$

$$=\frac{1+\sin\alpha}{\cos\alpha}=\frac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}{\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)}=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$

$\therefore \dfrac{\alpha}{2}=42° \Rightarrow \alpha=84°$ **Ans.**

Q.6 Let $x=\dfrac{\sum_{n=1}^{44}\cos n°}{\sum_{n=1}^{44}\sin n°}$ find the greatest integer that does not exceed $100x$.

Sol. $x=\dfrac{\sum_{n=1}^{44}\cos n°}{\sum_{n=1}^{44}\sin n°}=\dfrac{\cos 1°+\cos 2°+....\cos 44°}{\sin 1°+\sin 2°+....\sin 44°}$

$$\frac{\dfrac{\sin\left(\dfrac{44}{2}\right)^\circ}{\sin\left(\dfrac{1}{2}\right)^\circ}\cos\left(\dfrac{1^\circ+44^\circ}{2}\right)}{\dfrac{\sin\left(\dfrac{44}{2}\right)^\circ}{\sin\left(\dfrac{1}{2}\right)^\circ}\sin\left(\dfrac{1^\circ+44^\circ}{2}\right)} \Rightarrow \cot\left(\frac{45^\circ}{2}\right) \Rightarrow \cot\left(22\frac{1}{2}^\circ\right)$$

$\Rightarrow x = \left(\sqrt{2}+1\right) = 2.414$

$\Rightarrow [100\,x] = 241$ **Ans.**

Q.7 For $A \in \left(0, \frac{\pi}{4}\right)$, if $\left(1+\frac{\cos 3A}{\cos A}\right)+\left(1+\frac{\cos 6A}{\cos 2A}\right)+\left(1+\frac{\cos 9A}{\cos 3A}\right)+\left(1+\frac{\cos 12A}{\cos 4A}\right) = 0$, then find the value of $(\operatorname{cosec} A - \sec 2A)$.

Sol. $\left(1+\frac{\cos 3A}{\cos A}\right)+\left(1+\frac{\cos 6A}{\cos 2A}\right)+\left(1+\frac{\cos 9A}{\cos 3A}\right)+\left(1+\frac{\cos 12A}{\cos 4A}\right) = 0$

$\Rightarrow \quad \left(\frac{\cos A+\cos 3A}{\cos A}\right)+\left(\frac{\cos 2A+\cos 6A}{\cos 2A}\right)+\left(\frac{\cos 3+\cos 9A}{\cos 3A}\right)+\left(\frac{\cos 4A+\cos 12A}{\cos 4A}\right) = 0$

Sol. $\left(1+\frac{\cos 3A}{\cos A}\right)+\left(1+\frac{\cos 6A}{\cos 2A}\right)+\left(1+\frac{\cos 9A}{\cos 3A}\right)+\left(1+\frac{\cos 12A}{\cos 4A}\right) = 0$

$\Rightarrow \quad \left(\frac{\cos A+\cos 3A}{\cos A}\right)+\left(\frac{\cos 2A+\cos 6A}{\cos 2A}\right)+\left(\frac{\cos 3+\cos 9A}{\cos 3A}\right)+\left(\frac{\cos 4A+\cos 12A}{\cos 4A}\right) = 0$

$\Rightarrow \quad \left(\frac{2\cos 2A\cdot\cos A}{\cos A}\right)+\left(\frac{2\cos 4A\cdot\cos 2A}{\cos 2A}\right)+\left(\frac{\cos 6A\cdot\cos 3A}{\cos 3A}\right)+\left(\frac{\cos 8A\cdot\cos 4A}{\cos 4A}\right) = 0$

$\Rightarrow \quad 2[\cos 2A+\cos 4A+\cos 6A+\cos 8A = 0$

$\Rightarrow \quad 2\left[\frac{\sin 4A}{\sin A}\cos(5A)\right] = 0$

$\Rightarrow \quad \frac{4\sin 2A\cos 2A\cos 5A}{\sin A} = 0$

$\cos 5A = 0 \quad$ or $\quad \cos 2A = 0 \quad$ or $\quad \cos A = 0$

$\therefore \quad 5A = \frac{\pi}{2} \quad$ or $\quad 2A = \frac{\pi}{2} \quad$ or $\quad A = \frac{\pi}{2}$

$A = \frac{\pi}{10} \quad$ or $\quad A = \frac{\pi}{4}$

$A \in \left(0, \frac{\pi}{4}\right) \Rightarrow A = \frac{\pi}{10}$

$\therefore \quad (\operatorname{cosec} A - \sec 2A)$

$= \left(\frac{1}{\sin 18^\circ}-\frac{1}{\cos 36^\circ}\right) = \left(\frac{4}{\sqrt{5}-1}-\frac{4}{\sqrt{5}+1}\right) = 2$ **Ans.**

Q.8 Prove that $\dfrac{2\cos 2^n\theta+1}{2\cos\theta+1} = (2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1).......(2\cos 2^{n-1}\theta - 1)$

Sol. We have to prove $\dfrac{2\cos 2^n\theta+1}{2\cos\theta+1} = (2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1).......(2\cos 2^{n-1}\theta - 1)$

or $2\cos 2^n\theta + 1 = [(2\cos\theta + 1)(2\cos\theta - 1)](2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ ....\ (2\cos 2^{n-1}\theta - 1)$

Now $[(2\cos\theta + 1)(2\cos\theta - 1)](2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ ....\ (2\cos 2^{n-1}\theta - 1)$

$= (4\cos^2\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ ....\ (2\cos 2^{n-1}\theta - 1)$

$= (2\cos 2\theta + 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)....\ (2\cos 2^{n-1}\theta - 1)$ [using $\cos 2\theta = 2\cos^2\theta - 1$]

$= (4\cos^2\theta - 1)(2\cos 2^2\theta - 1)\ ....\ (2\cos 2^{n-1}\theta - 1)$

$= (2\cos^2\theta + 1)(2\cos 2^2\theta - 1)\ ....\ (2\cos 2^{n-1}\theta - 1)$

$= (4\cos^2 2^2\theta - 1)(2\cos 2^3\theta - 1)\ ....\ (2\cos 2^{n-1}\theta - 1)$

$\vdots$

$= (2\cos 2^{n-1}\theta + 1)(2\cos 2^{n-1}\theta - 1)$

$= 4\cos 2\, 2^{n-1}\theta - 1$

$= 2\cos 2^n\theta + 1$ **Hence proved.**

Q.9 Find the sum

$$\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\cdot\tan\frac{\theta}{2^2} + \frac{1}{2^3}\cdot\tan\frac{\theta}{2^3} + \ldots\ldots\infty$$

Sol. Let t_n denote the n^{th} term of the given series.

Now, first term $= \tan\theta$

$\sin\theta \quad \sin\theta$

Sol. Let t_n denote the n^{th} term of the given series.

Now, first term $= \tan\theta$

$$= \frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta - (\cos^2\theta - \sin^2\theta)}{\sin\theta\cos\theta}$$

$= \cot\theta - 2\cot 2\theta$

i.e., $t_1 = \cot\theta - 2\cot 2\theta$

similarly, $t_2 = \dfrac{1}{2}\cot\dfrac{\theta}{2} - \cot\theta$

$t_3 = \dfrac{1}{2^2}\cdot\cot\dfrac{\theta}{2^2} - \dfrac{1}{2}\cot\dfrac{\theta}{2}$

....

....

$t_n = \dfrac{1}{2^{n-1}}\cot\dfrac{\theta}{2^{n-1}} - \dfrac{1}{2^{n-2}}\cdot\cot\dfrac{\theta}{2^{n-2}}$

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adding we get, $S_n = \dfrac{1}{2^{n-1}}\cdot\cot\dfrac{\theta}{2^{n-1}} - 2\cot 2\theta$

$$\text{Required sum} \quad S = \underset{n\to\infty}{\text{Lt}}\, S_n$$

$$= \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - 2\cot 2\theta\right)$$

$$= \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{2^{n-1}}\cdot\frac{1}{\tan\frac{\theta}{2^{n-1}}} - 2\cot 2\theta\right)$$

$$= \underset{n\to\infty}{\text{Lt}}\left[\frac{1}{\theta}\cdot\frac{\theta}{2^{n-1}}\cdot\frac{1}{\tan\frac{\theta}{2^{n-1}}} - 2\cot 2\theta\right]$$

$$= \frac{1}{\theta} - 2\cot 2\theta \ \textbf{Ans.}$$

Q.10 If $A + B + C = \pi$, then prove : $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

Sol. $\quad$ L.H.S. $= \dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$

Sol. $\quad$ L.H.S. $= \dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$

$$= \frac{2\sin(A+B)\cos(A-B) + 2\sin C\cos C}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$= \frac{2\sin C[\cos(A-B) - \cos(A+B)]}{2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right]}$$

$$= \frac{2\sin C\cdot 2\sin A\sin B}{2\cos\frac{C}{2}\ 2\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{\sin A\sin B\sin C}{\cos\frac{A}{2}\ \cos\frac{B}{2}\ \cos\frac{C}{2}}$$

$$= 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \text{R.HS. } \textbf{Hence proved.}$$

Q.11 If $A + B + C = 180^\circ$, prove $\tan A + \tan B + \tan C \le 3\sqrt{3}$, where A, B, C are acute angles.

Sol. $\quad \tan(A + B) = \tan(180^\circ - C)$

or, $\quad \dfrac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C$

or, $\quad \tan A + \tan B + \tan C = \tan A\tan B\tan C$

$$\frac{\tan A + \tan B + \tan C}{3} \ge \sqrt[3]{\tan A\tan B\tan C} \qquad [\text{since} \quad \text{A.M.} \ge \text{G.M.}]$$

or, $\quad \dfrac{\tan A \tan B \tan C}{3} \geq \sqrt[3]{\tan A \tan B \tan C}$

or, $\quad \tan^2 A \tan^2 B \tan^2 C \geq 27 \qquad$ [cubing both sides]

or $\quad \tan A \tan B \tan C \geq 3\sqrt{3} \;\Rightarrow\; \tan A + \tan B + \tan C \geq 3\sqrt{3}$. **Hence proved.**

Q.12 Let A, B, C be three angles such that $A = \dfrac{\pi}{4}$ and $\tan B \tan C = p$. Find all possible values of p such that A, B, C are the angles of a triangle.

Sol. $A + B + C = \pi$

$$\Rightarrow \quad B + C = \frac{3\pi}{4} \quad \Rightarrow \quad 0 < B, C < \frac{3\pi}{4}$$

Also $\tan B \tan C = p$

$$\Rightarrow \quad \frac{\sin B \sin C}{\cos B \cos C} = \frac{p}{1} \quad \Rightarrow \quad \frac{\cos B \cos C - \sin B \sin C}{\cos B \cos C + \sin B \sin C} = \frac{1-p}{1+p}$$

$$\Rightarrow \quad \frac{\cos(B+C)}{\cos(B-C)} = \frac{1-p}{1+p} \quad \Rightarrow \quad \frac{1+p}{\sqrt{2}(p-1)} = \cos(B-C) \qquad \ldots\text{(i)}$$

Since B or C can vary from 0 to $\dfrac{3\pi}{4}$,

$$0 \leq B - C < \frac{3\pi}{4} \quad \Rightarrow \quad -\frac{1}{\sqrt{2}} < \cos(B-C) \leq 1.$$

$$\Rightarrow \quad \frac{\cos(B+C)}{\cos(B-C)} = \frac{1-p}{1+p} \quad \Rightarrow \quad \frac{1+p}{\sqrt{2}(p-1)} = \cos(B-C) \qquad \ldots\text{(i)}$$

Since B or C can vary from 0 to $\dfrac{3\pi}{4}$,

$$0 \leq B - C < \frac{3\pi}{4} \quad \Rightarrow \quad -\frac{1}{\sqrt{2}} < \cos(B-C) \leq 1.$$

Equation (i) will now lead to

$$-\frac{1}{\sqrt{2}} < \frac{p+1}{\sqrt{2}(p-1)} \leq 1 \;\Rightarrow\; \quad 0 < 1 + \frac{p+1}{p-1}$$

$$\Rightarrow \quad \frac{2p}{(p-1)} > 0 \quad \Rightarrow \quad p < 0 \text{ or } p > 1 \qquad \ldots\text{(ii)}$$

Also $\quad \dfrac{p+1-\sqrt{2}(p-1)}{\sqrt{2}(p-1)} \leq 0 \;\Rightarrow\; \dfrac{\left(p - \left(\sqrt{2+1}\right)^2\right)}{(p-1)} \geq 0$

$$\Rightarrow \quad p < 1 \text{ or } p \leq \left(\sqrt{2+1}\right)^2 \qquad \ldots\text{(iii)}$$

Combining (ii) and (iii), we get $p < 0$ or $p \leq \left(\sqrt{2+1}\right)^2$