RELATION & FUNCTION

1. ORDERED PAIRS :

An ordered pair consisting of two elements in a given fixed order.
Eg. (a, b).
An order pair is not a set consisting of two elements. The position of a point in two dimentional plane is
Eg. of an ordered pair (1, 2), (2, 2) ......

1.1 Equality of ordered pairs :

Two ordered pairs (a_1, b_1) and (a_2, b_2) are said to be equal if $a_1 = a_2$ & $b_1 = b_2$
Eg. Find the values of a and b if $(3a - 2, b + 3) = (2a - 1, 3)$
$b + 3 = 3,\ 3a - 2b = 2a - 1 \Rightarrow a = 1, b = 0$

1.2 Cartesian Product of two sets :

Let A and B be any two non empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called as cartesian product of sets A and B and is denotes by $A \times B$.
The cartesian product of two sets A, B is a non-void set of all ordered pairs (a, b).

$$A = \{1, 2, 3\};\quad B = \{p, q, r\}$$
$$A \times B = \{(a, b) / a \in A \text{ and } b \in B\}$$
$$= \{(1, p), (1, q), (1, r), (2, p), (2, q), (2, r), (3, p), (3, q), (3, r)\}$$

2. RELATION :

Every non-zero subset of $A \times B$ defined a relation from set A to set B.

Defination :
Relation is a linear operation which establishes relationship between the element's of two set's according to some definite rule of relationship.

$R : \{(a, b) \mid (a, b) \in A \times B \text{ and } a\,R\,b\}$

Eg:1 A is {2, 3, 5}
B is {1, 4, 9, 25, 30}
If $a\,R\,b \rightarrow$ b is square of a
Discreet element of relation are {(2, 4), (3, 9), (5, 25)}

Eg:2 A = {Jaipur, Patna, Kanpur, Lucknow}
B = {Rajasthan, Uttar Pradesh, Bihar}
$aRb \rightarrow$ a is capital of b,
$A \times B$ = {(Jaipur, Rajasthan), (Patna, Bihar), (Lucknow, Uttar Pradesh) }

2.1 Total number of Relation from A to B :

Let number of relations from A to B be x.
Let A contain 'm' elements and B contain 'n' element's.
Number of element's in $A \times B \rightarrow m \times n$
Number of non void subset's $= {}^{mn}C_1 + {}^{mn}C_2 + \ldots\ldots + {}^{mn}C_{mn} = 2^{mn} - 1$

2.2 Domain and Range of Relation:

If R be a relation from a set A to set B. Then set of all first component's or coordinates of ordered pairs is called the domain of R, while the set of all second component's or coordinates of the ordered pairs is called as range of relation.

Let $R : A \to B$ (R is a relation defined from set A to set B) then domain of this relation is

Domain : Set of all the first entries in R

$\{a \mid (a, b) \in R\}$

Range : Set of all the second entries in R

$\{b \mid (a, b) \in R\}$

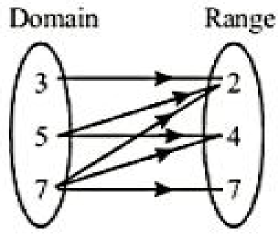
E.g. $A = \{1, 3, 5, 7\}$; $B = \{2, 4, 6, 8\}$

Relation is $aRb \Rightarrow a > b$; $a \in A, b \in B$

$R = \{(3, 2), (5, 2), (5, 3), (7, 2), (7, 4), (7, 6)\}$

Domain = $\{3, 5, 7\}$

Range = $\{2, 4, 6\}$



2.3 Inverse of a Relation:

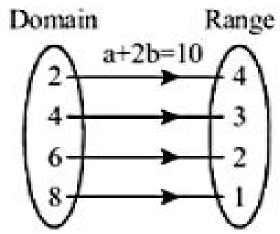
If R is a relation defined from $A \to B$ then R^{-1} is a relation defined from $B \to A$ as

$R^{-1} = \{(b, a) \mid (a, b) \in R\}$

i.e. domain is converted in to range element's and range is converted into domain elements.

i.e. Doman of R = Range of R^{-1}

Range of R = Domain of R^{-1}



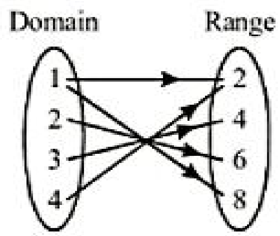
E.g. N is a set of first 10 natural nos.

$aRb \Rightarrow a + 2b = 10$

$N = \{1, 2, 3, ..., 10\}$ & $a, b \in N$

$R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

Inverse relation is $R^{-1} \to \{(1, 8), (2, 6), (3, 4), (4, 2)\}$



2.4 Types of Relation :

(i) Identity Relation

A relation defined on a set A is said to be an Identity relation if every element of A is related to itself and only to itself.

Eg.1 A relation defined on the set of natural number's as with rule $aRb \Leftrightarrow a = b$ is an identity relation

$R = \{(1, 1), (2, 2), (3, 3), ...\}$

Eg: 2 The relation $I_A = \{(1, 1), (2, 2), (3, 3), ...\}$ is the identity relation on set $A = \{1, 2, 3\}$ but $\{(1, 1), (2, 2), (1, 3)\}$ are not identity relation

Types of Relations

Reflexive | Symmetric | Transitive

(ii) Reflexive:

A relation defined on a set A is said to be an Identity relation if each & every element of A is related to itself.

i.e. if $(a, b) \in R$ then $(a, a) \in R$. However if there is a single ordered pair of $(a, b) \in R$ such $(a, a) \notin R$ then R is not reflexive.

Eg. 1 : Let $A = \{1, 2, 3\}$ be a set then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A.
$R_1 = \{(1, 1), (3, 3), (2, 1), (6, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \in R$.

E.g. 2: A relation defined on (set of natural numbers)
aRb $\Rightarrow$ 'a' divides 'b' $a, b \in N$

R would always contain (a, a) because every natural number divides itself and hence it is a reflexive relation.

Note :

Every identity relation is a reflexive relation but every reflexive relation need not be an identity.

(iii) Symmetric Relation:

A relation defined on a set is said to be symmetric
if $a R b \Rightarrow bRa$. If $(a, b) \in R$ then (b, a) must be necessarily there in the same relation.

Eg:

(i) $a R b \Rightarrow$ a is parallel to b
It is a symmetric relation because if a is parallel to b then the line b is parallel to a.

(ii) $L_1 R L_2$ ...... L_1 is perpendicular to L_2 is a symmetric relation.

(iii) $a R b \Rightarrow$ a is brother of b is not necessarily brother a.

(iv) $a R b \Rightarrow$ a is a cousin of b. This is a symmetric relation.

Note : If R is symmetric

(i) $R = R^{-1}$

(ii) Rangle of R = Domain of R

(iv) Transitive relation :

A relation on set A is said to be transitive if a R b and b R c implies a R c then it is transitive.

$(a, b) \in R$ & $(b, c) \in R \Rightarrow (a, c) \in R$ and (a, b, c) need not be distinct.

Eg. 1 : a R b $(a - b)$ is even
$(6, 4), (4, 20) \Rightarrow (6, 20) \in R$

Eg. 2 : On the set of natural numbers, the relation R defined by $x R y \Rightarrow x < y$ because for any
$x, y, z \in N$ $x < y, y < z \Rightarrow x < z$.

(v) Equivalence Relation:

If a relation is Reflexive, Symmetric and Transitive then it is said to be an equivalence relation.

Eg. 1 : A relation defined on N
xRy $\Rightarrow$ $x = y$
R is an equivalence relation.

Eg. 2 : A relation defined on a set of || lines in a plane
aRb $\Rightarrow$ a || b
It is an equivalence relation.

Eg. 3 : Relation defined on the set of integer (I)
xRy $\Rightarrow$ $(x - y)$ is even is an equivalence relation.

Illustration :

Check the following relations for being reflexive, symmetric, transitive and thus choose the equivalence relations if any.

(i) a R b if $|a| \le b$; $a, b \in$ *set of real numbers.*

(ii) $a\ R\ b$ *iff* $a < b$; $a, b \in N$.

(iii) $a\ R\ b$ *iff* $|a - b| > \frac{1}{2}$; $a, b \in R$.

(iv) $a\ R\ b$ *iff a divides b;* $a, b \in N$.

(v) $a\ R\ b$ *iff* $(a - b)$ *is divisible by n;* $a, b \in I$, *n is a fixed positive integer.*

Sol.

(i) *Not reflexive, not symmetric but transitive*

Let $a = -2$ *and* $b = 3$; $(-2, 3) \in R$. *Since* $|-2| \le 3$ *is true*

Since $|-2| = 2 \not\le -2$ *hence relation is not Reflexive*

Since $|3| \le -2$ *is wrong hence relation is not symmetric*

Now Let a, b, c be three real numbers such that $|a| \le b$ *and* $|b| \le c$

$|a| \le b \Rightarrow b \ge 0$, so $|b| \le c \Rightarrow b \le c$

Hence $|a| \le c$ *is true so the given relation is transitive.*

(ii) *Not reflexive, not symmetric but transitive.*
Since no natural number is less than itself hence not reflexive,
If $a < b$ *then* $b < a$ *is false. Hence not symmetric.*
If $a < b$ *then* $b < c$ *clearly* $a < c$. *Hence transitive*

(iii) *Not reflexive, symmetric, not transitive.*

$|a - a| = 0 \not> \frac{1}{2}$ *hence it is not reflexive.*

$\sqrt{x^2} = |x|$ *hence symmetric.*

Let $a = 1$, $b = -1$ *and* $c = \frac{3}{2}$, $|a - b| = 2 > \frac{1}{2}$ so $(a, b) \in R$; $|b - c| = \frac{5}{2} > \frac{1}{2}$ so $(b, c) \in R$

But $|a - c| = \left|1 - \frac{3}{2}\right| = \frac{1}{2} \not> \frac{1}{2}$ so $(a, c) \notin R$. *Hence R is not a transitive relation.*

(iv) *Reflexive, not symmetric, transitive*

Since $\frac{a}{a} = 1$ *i.e. every number divides itself, hence R is reflexive.*

If a divides b then b does not divide a (unless (a = b) hence the relation is not symmetric (but anti-symmetric).

If a divides b and b divides c then it is clear that a will divide c. Hence transitive.

(v) *Relfexive, symmetric as well as transitive, hence it is an equivalence relation.*

Since 0 is divisible by n $\left(\frac{0}{n} = 0\right)$ *so given relation is reflexive*

If $a - b$ *is divisible by n, then* $(b - a)$ *will also be divisible by n. Hence, symmetric.*

If $a - b = nI_1$ *and* $b - c = nI_2$, *where* I_1, I_2 *are integer.*

Then, $a - c = (a - b) + (b - c) = n(I_1 + I_2)$ *so* $a - c$ *is also divisible by n, hence transitive.*

Practice Problem

Q.1 If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is-

(A) 2^{mn} (B) $2^{mn} - 1$ (C) 2mn (D) m^n

Q.2 Let L denote the set of all straight lines in a plane. Let a relation R be defined by $\alpha R \beta \Leftrightarrow \alpha \perp \beta, \alpha, \beta \in L$. Then R is -

(A) Reflexive (B) Symmetric (C) Transitive (D) None of these

Q.3 Two points A and B in a plane are related if OA = OB, where O is a fixed point. This relation is -

(A) Reflexive but not symmetric (B) Symmetric but not transitive

(C) An equivalence relation (D) None of these

Answer key

Q.1 B Q.2 B Q.3 C

FUNCTION

3. INTRODUCTION :

A function is like a machine which gives unique output for each input that is fed into it. But every machine is designed for certain defined inputs for eg. a juicer is designed for fruits & not for wood. Similarly functions are defined for certain inputs which are called as its "domain and corresponding outputs are called "Range".

3.1 General Definition :

Definition-1 :

Let A and B be two sets and let there exist a rule or manner or correspondence 'f' which associates to each element of A to a unique element in B,then f is called a <u>function</u> or <u>mapping</u> from A to B. It is denoted by the symbol

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

which reads 'f' is a function from A to B' or 'f maps A to B,

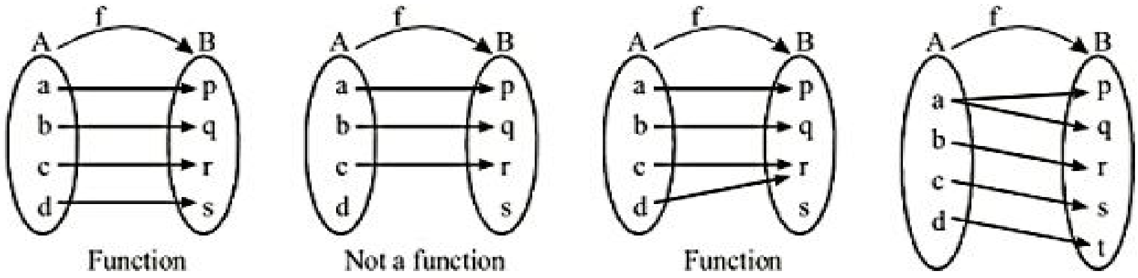
If an element $a \in A$ is associated with an element $b \in B$ then b is called 'the f image of a' or 'image of a under f' or 'the value of the function f at a'. Also a is called the pre-image of b or argument of b under the function f. We write it as

$$b = f(a) \text{ or } f : a \to b \text{ or } f : (a, b)$$

3.2 Function as a set of ordered pairs :

A function $f : A \to B$ can be expressed as a set of ordered pairs in which each ordered pair is such that its first element belongs to A and second element is the corresponding element of B.

As such a function $f : A \to B$ can be considered as a set of ordered pairs $(a, f(a))$ where $a \in A$ and $f(a) \in B$ which is the f image of a. Hence f is a subset of $A \times B$.



Function   Not a function   Function   Not a function

As a particular type of relation, we can define a function as follows :

Definition-2 :

A relation R from a set A to a set B is called a function if

(i) each element of A is associated with some element of B.

(ii) each element of A has unique image in B.

Thus a function 'f' from a set A to a set B is a subset of $A \times B$ in which each 'a' belonging to A appears in one and only one ordered pair belonging to f. Hence a function f is a relation from A to B satisfying the following properties :

Every function from $A \to B$ satisfies the following conditions.

(i) $f \subset A \times B$ (ii) $\forall \ a \in A \Rightarrow (a, f(a)) \in f$ and

(iii) $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$.

Thus the ordered pairs of f must satisfy the property that each element of A appears in some ordered pair and no two ordered pairs have same first element.

Note : Every function is a relation but every relation is not necessarily a function.

3.3 Domain, Co-domain & Range of A Function :

Let $f: A \to B$, then the set A is known as the domain of f & the set B is known as co–domain of f. The set of all f images of elements of A is known as the range of f. Thus :

$$\text{Domain of } f = \{a \mid a \in A, (a, f(a)) \in f\}$$

$$\text{Range of } f = \{f(a) \mid a \in A, f(a) \in B, (a, f(a)) \in f\}$$

It should be noted that range is a subset of co–domain . If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

Let f and g be function with domain D_1 and D_2 then the functions

Note :

$f+g, f-g, fg, f/g$ are defined as

$(f+g)(x) = f(x) + g(x);$ Domain $D_1 \cap D_2$

$(f-g)(x) = f(x) - g(x);$ Domain $D_1 \cap D_2$

$(fg)(x) = f(x) \cdot g(x);$ Domain $D_1 \cap D_2$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)};$ Domain $= \{x \in D_1 \cap D_2 \mid g(x) \neq 0\}$

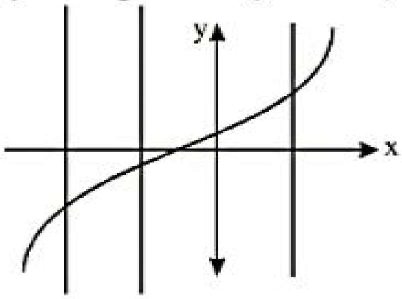
e.g. $f(x) = x^3 + 2x^2$ and $g(x) = 3x^2 - 1$. Find $f \pm g$, fg and f/g.

3.4 Graphical Representation of function :

Let f be a mapping with domain D such that $y = f(x)$ should assume angle value for each x,(i.e. the straight line drawn parallel to y-axis in its domain should cut at only one point.

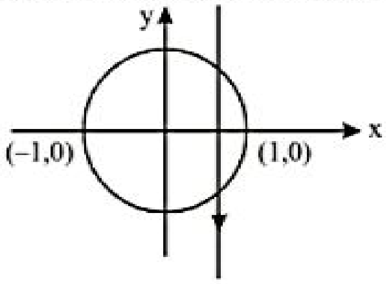
Eg. $y = x^3$

Here all the straight lines parallel to y-axis cut $y = x^3$ only at one print.



Eg. $x^2 + y^2 = 1^2$

Here line parallel to y-axis is intersecting the circle at two points hence it is not a function.



3.5 Domain :

Rule for finding Domain :

(i) Expression under even root (i.e. square root, fourth root etc) ≥ 0.

(ii) Denominator $\neq 0$

(iii) If domain of $y = f(x)$ & $y = g(x)$ are D_1 & D_2 respectively then the domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$

(iv) Domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

3.6 Classification of Functions:

(i) Polynomial Function:

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree *n*. A polynomial function is always continuous.

NOTE: **(A)** A polynomial of degree one with no constant term is called an odd linear function i.e. $f(x) = ax,\ a \neq 0$

(B) There are two polynomial functions, satisfying the relation ; $f(x).f(1/x) = f(x) + f(1/x)$. They are :

(a) $f(x) = x^n + 1$ & **(b)** $f(x) = 1 - x^n$, where n is a positive integer.

(C) A polynomial of degree odd has its range $(-\infty, \infty)$ put a polynomial of degree even has a range which is always subset of R.

(ii) Algebraic Function:

A function f is called an algebraic function if it can be constructed using algebraic operations such as addition, substraction, multiplication, division and taking roots, started with polynomials.

e.g. $f(x)=\sqrt{x^2+1}$; $g(x)=\dfrac{x^4-16x^2}{x+\sqrt{x}}+(x-2)\times\sqrt[3]{x+1}$

Note that all polynomial are algebraic but converse is not true. Functions which are not algebraic, are known as Transcidential function.

(iii) Fractional Rational Function:

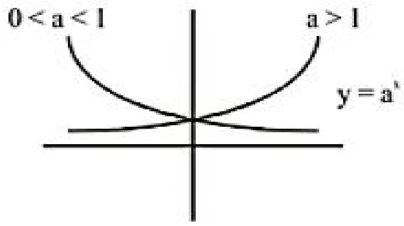
A rational function is a function of the form. $y=f(x)=\dfrac{g(x)}{h(x)}$, where g (x) & h (x) are polynomials & $h(x)\neq 0$. The domain of f (x) is set of realx such that $h(x)\neq 0$.

e.g. $f(x)=\dfrac{2x^4-x^2+1}{x^2-4}$; $D=\{x\mid x\neq \pm 2\}$

(iv) Exponential Function:

A function $f(x)=a^x=e^{x\,ln\,a}\ (a>0,\ a\neq 1,\ x\in R)$ is called an exponential function. $f(x)=a^x$ is called an exponential function because the variable x is the exponent. It should not be confused with power function. $g(x)=x^2$ in which variable x is the base.

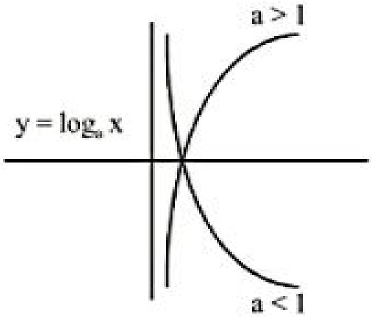
For $f(x)=e^x$ domain is R and range is R^+.



For $f(x)=e^{1/x}$ domain is $R-\{0\}$ and range is $R^+-\{1\}$. i.e. $(0, 1)\cup(1,\infty)$

$f(x)=\dfrac{1}{ln\,x}$ with domain $R^+-\{1\}$, range is $R-\{0\}$

(v) Logarithmic function: A function of the form $y=\log_a x,\ x>0,\ a>0,\ a\neq 1$, is called Logarithmic function.



(vi) Absolute Value Function:

A function $y=f(x)=|x|$ is called the absolute value function or Modulus function. It is defined as:

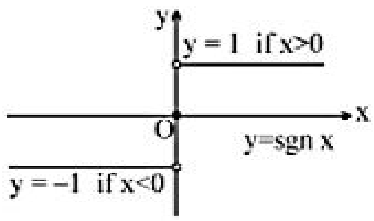
$$y=|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

For $f(x)=|x|$, domain is R and range is $R^+\cup\{0\}$.

For $f(x)=\dfrac{1}{|x|}$ or $\dfrac{|x|}{x^2}$, domain is $R-\{0\}$ and range is R^+.

(vii) Signum Function:

A function y= f (x) = Sgn (x) is defined as follows :



$$y = f(x) = \begin{bmatrix} 1 & \text{for} & x > 0 \\ 0 & \text{for} & x = 0 \\ -1 & \text{for} & x < 0 \end{bmatrix}$$

It is also written as Sgn x = |x|/ x or $\dfrac{x}{|x|}$

$x \neq 0$; $f(0) = 0$

Note that Sgn(Sgn x) = Sgn x;

$$y = \text{Sgn}(x^2 - 1) = \begin{cases} 1, & |x| > 1 \\ 0, & |x| = 1 \\ -1, & |x| < 1 \end{cases}$$

(viii) Greatest Integer Or Step Up Function :

The function y = f (x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x . Note that for :

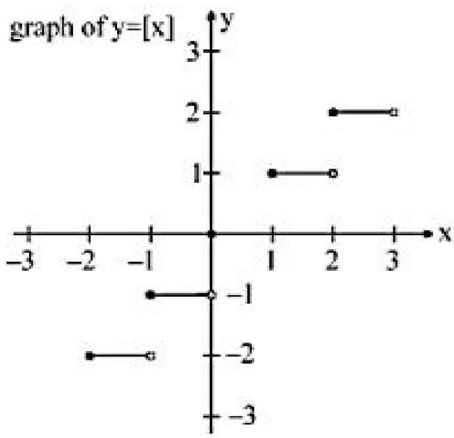
$-1 \le x < 0$; $[x] = -1$ $\quad 0 \le x < 1$; $[x] = 0$

$1 \le x < 2$; $[x] = 1$ $\quad 2 \le x < 3$; $[x] = 2$ and so on.

For f (x) = [x], domain is R and range is I.

For $f(x) = \dfrac{1}{[x]}$ domain is R – [0, 1) and range is $\left\{\dfrac{1}{n} \,\middle|\, n \in I - \{0\}\right\}$.

Properties of greatest integer function :



(a) $[x] \le x < [x] + 1$ and

$x - 1 < [x] \le x$, $0 \le x - [x] < 1$

(b) $[x + m] = [x] + m$, if m is an integer.

(c) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

(d) $[x] + [-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise.} \end{cases}$

(e) $[x] \ge n \Rightarrow x \in [n, \infty) \ \forall \ n \in I$

(f) $[x] > n \Rightarrow x \in [n + 1, \infty) \ \forall \ n \in I$

(g) $[x] \le n \Rightarrow x \in (-\infty, n + 1) \ \forall \ n \in I$

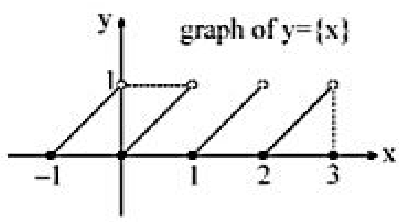
(h) $[x] < n \Rightarrow x \in (-\infty, n) \ \forall \ n \in I$

(ix) Fractional Part Function :

It is defined as :

g (x) = {x} = x – [x].

e.g. the fractional part of the number 2.1 is 2.1– 2 = 0.1 and the fractional part of – 3.7 is 0.3. The period of this function is 1 and graph of this function is as shown.



For f (x) = {x}, domain is R and range is [0, 1)

For $f(x) = \dfrac{1}{\{x\}}$, domain is R – I, range is $(1, \infty)$

Properties of fractional part :

(a) $0 \le \{x\} < 1$

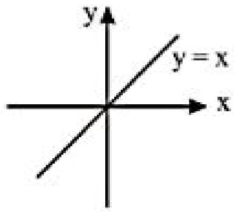
(b) $\{x+n\} = \{x\},\ n \in I$

(c) $\{x\} + \{-x\} = \begin{cases} 0, & x \in I \\ 1, & x \notin I \end{cases}$

(x) Identity function :

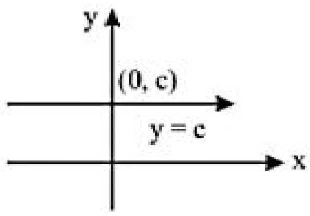
The function $f: A \to A$ defined by $f(x) = x\ \forall\ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function defined on R is a bijection.

$f: R \to R,\qquad f(x) = x$



(xi) Constant function :

A function $f: A \to B$ is said to be a constant function if every element of A has the same f image in B. Thus $f: A \to B$; $f(x) = c,\ \forall\ x \in A$, $c \in B$ is a constant function. Note that the range of a constant function is a singleton and a constant function may be one–one or many–one, onto or into.



e.g. $f(x) = [\{x\}]$; $g(x) = \sin^2 x + \cos^2 x$; $h(x) = \text{sgn}(x^2 - 3x + 4)$ etc, all are constant functions.

Illustration :

Find the domain of following function

(i) $f(x) = \sqrt{x^2 - 5x + 6}$

(ii) $f(x) = \sqrt{x^2 - 3x + 2} + \dfrac{1}{\sqrt{x^2 - 3x - 4}}$

(iii) $f(x) = \dfrac{2}{x^2 - 4} + \log_{10}(x^3 - x)$

(iv) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

(v) $f(x) = \dfrac{1}{\sqrt{[x] - x}}$

(vi) $f(x) = \sqrt{\log_{\frac{1}{2}}\left(\dfrac{5x - x^2}{4}\right)}$

(vii) $f(x) = \sqrt{\cos(\sin x)} + \sin^{-1}\left(\dfrac{1 + x^2}{2x}\right)$

(viii) $f(x) = \log_4 \log_2 \log_{1/2}(x)$

Sol.

(i) $f(x) = \sqrt{x^2 - 5x + 6}$

$\Rightarrow\quad x^2 - 5x + 6 \ge 0$

$\Rightarrow\quad (x - 2)(x - 3) \ge 0$

$\Rightarrow\quad x \in (-\infty, 2] \cup [3, \infty)$

(ii) $f(x) = \sqrt{x^2 - 3x + 2} + \dfrac{1}{\sqrt{x^2 - 3x - 4}}$

$x^2 - 3x + 2 \geq 0$ and $x^2 - 3x - 4 > 0$

$x^2 - 3x + 2 \geq 0$

$\Rightarrow \quad (x - 2)(x - 1) \geq 0$

$\Rightarrow \quad x \in (-\infty, 1] \cup [2, \infty) \quad \ldots(i)$

and $\quad x^2 - 3x - 4 > 0$

$(x - 4)(x + 1) > 0$

$x \in (-\infty, -1) \cup (4, \infty) \quad \ldots(ii)$

Taking union of (i) & (ii)

$x \in (-\infty, -1) \cup (4, \infty)$

(iii) $f(x) = \dfrac{2}{x^2 - 4} + \log_{10}(x^3 - x)$

Following conditions should be followed

$x^2 - 4 \neq 0 \quad \& \quad x^3 - x > 0$

$x \neq \pm 2$

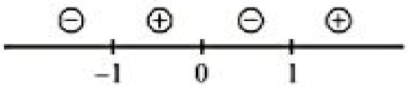
$x \in R - \{-2, 2\} \quad \ldots(i)$

$x^3 - x > 0$

$\Rightarrow \quad x(x^2 - 1) > 0$

$\Rightarrow \quad x(x - 1)(x + 1) > 0$

$x \in (-1, 0) \cup (1, \infty) \quad \ldots(ii)$



Taking union of (i) & (ii)

$x \in (-1, 0) \cup (1, 2) \cup (2, \infty)$

(iv) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

$|x| - x > 0$

$|x| > x$

This is possible only when x is negative i.e. $x < 0$, *hence*

$x \in (-\infty, 0)$

(v) $f(x) = \dfrac{1}{\sqrt{[x] - x}}$

$[x] - x > 0$

$[x] > x$

but we know that $[x] \leq x$

Hence domain is ϕ

(vi) $f(x) = \sqrt{\log_{\frac{1}{2}}\left(\frac{5x - x^2}{4}\right)}$

$\frac{5x - x^2}{4} > 0$

$\Rightarrow \quad x(5 - x) > 0 \qquad \Rightarrow \quad x(x - 5) < 0$

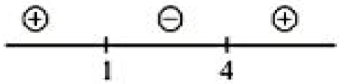
$x \in (0, 5) \qquad ...(i)$

Also $\log_{\frac{1}{2}}\left(\frac{5x - x^2}{4}\right) \geq 0$

$\Rightarrow \quad \frac{5x - x^2}{4} \leq \left(\frac{1}{2}\right)^0 \qquad \Rightarrow \quad 5x - x^2 \leq 4$

$\Rightarrow \quad x^2 - 5x + 4 \geq 0$

$x \in (-\infty, 1] \cup [4, \infty) \quad ...(ii)$



Using (i) and (ii)

$x \in (0, 1] \cup [4, 5)$

(vii) $f(x) = \sqrt{\cos(\sin x)} + \sin^{-1}\left(\frac{1 + x^2}{2x}\right)$

$\cos(\sin x) \geq 0$

$-1 \leq \sin x \leq 1 \ \forall x \in R$

$\therefore \cos\theta > 0$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

hence $x \in R \qquad ...(i)$

$\sin^{-1}\left(\frac{1 + x^2}{2x}\right)$

$\Rightarrow \quad \left|\frac{1 + x^2}{2x}\right| \leq 1 \qquad \Rightarrow |1 + x^2| \leq |2x|$

$\Rightarrow \quad |x|^2 - 2|x| + 1 \leq 0$

$\Rightarrow \quad (|x| - 1)^2 \leq 0$

Only at $x = \pm 1 \qquad ...(ii)$

Using (i) and (ii)

$x \in \{-1, 1\}$

(viii) $f(x) = \log_4 \log_2 \log_{1/2}(x)$

$\Rightarrow \log_2 \log_{1/2}(x) > 0 \qquad \Rightarrow \log_{1/2}(x) > (2^0)$

$\Rightarrow \log_{1/2} x > 1 \qquad \Rightarrow x < \left(\frac{1}{2}\right)^1$

Also $0 < x$

$\Rightarrow x \in \left(0, \frac{1}{2}\right)$

RANGE :

Range of y = f(x) is the collection of all outputs corresponding to each real number of the domain.

To find the range of function

(i) First of all find the domain of y = f(x).

(ii) If domain is a set having only finite number of points, then range is the set of corresponding f(x) values.

(iii) If domain of y = f(x) is R or R– {Some finite points}, then express x in terms of y. From this find y for x to be defined or real or form an equation in terms of x & apply the condition for real roots.

Illustration :

Find the range of the following function

(i) $f(x) = a \sin x + b.\ a > 0\ ,\ b \in R$ (ii) $f(x) = 4 \tan x \cos x$

(iii) $y = \dfrac{x^2}{1+x^2}$ (iv) $y = \log_e (3x^2 - 4x + 5)$

(v) $y = \dfrac{x^2 - x}{x^2 + 2x} = \dfrac{x(x-1)}{x(x+2)}$ (vi) $y = 3 - 2^x$

Sol.

(i) $f(x) = a \sin x + b.\ a > 0\ ,\ b \in R$

$f()x = a \sin x + b$

$\because -1 \le \sin x \le 1$

$\therefore -a + b \le f(x) \le a + b$

Range $\in [b - a,\ b + a]$

(ii) $f(x) = 4 \tan x \cos x$

$f(x) = 4 \sin x$ *for* $\cos x \neq 0$

$-1 \le \sin x \le 1$

but at $\sin x = \pm 1,\ \cos x = 0$

hence points with $\sin x = \pm 1$ *will not be included in range.*

Range $\in (-4, 4)$.

(iii) $y = \dfrac{x^2}{1+x^2}$

y is defined $\forall x \in R$*, domain is R*

from $y = \dfrac{x^2}{1-x^2} \quad \Rightarrow \quad x^2 = \dfrac{y}{1-y}$

$\Rightarrow x = \sqrt{\dfrac{y}{1-y}} \ge 0 \quad \Rightarrow \quad \dfrac{y}{1-y} \ge 0$

$\Rightarrow 0 \le y < 1$

Range $[0, 1)$

(iv) $y = \log_e(3x^2 - 4x + 5)$

y is defined if $3x^2 - 4x + 5 > 0$

$D < 0$ and coefficeint of $x^2 > 0$

hence domain is R and log is increaisng function.

Minimum value of $3x^2 - 4x + 5$ is $-\frac{D}{4a}$

$$\Rightarrow \quad = \frac{-(-44)}{4(3)} = \frac{11}{3} \quad \Rightarrow \quad y \geq \log_e\left(\frac{11}{3}\right)$$

Range $\in \left[\log_e\left(\frac{11}{3}\right), \infty\right)$

(v) $y = \frac{x^2 - x}{x^2 + 2x} = \frac{x(x-1)}{x(x+2)}$

Domain is $x \in R - \{-2, 0\}$

$$y = \frac{x(x-1)}{x(x+2)}$$

when $x \neq 0$, $y = \frac{x-1}{x+2} \quad \Rightarrow \quad x = \frac{1+2y}{1-y}$

$\therefore$ If x is real $y - 1 \neq 0 \Rightarrow y \neq 1$

Also for $x = \frac{1+2y}{1-y}$; $x \neq 0$

hence $y \neq -\frac{1}{2} \quad \Rightarrow \quad$ Hence range $y \in R - \left\{\frac{-1}{2}, 1\right\}$

(vi) $y = 3 - 2^x$

Domain is $x \in R$

$0 \leq 2^x < \infty$

Range $\in (-\infty, 3)$

Illustration :

Find the domain of the folloiwng function

(i) $f(x) = \ln(3x^2 - 4x + 5) + \sqrt{2\sin^2 x - 5\sin x + 2}$

(ii) $f(x) = \ln\{x\} + \sqrt{x - 2\{x\}}$

(iii) $f(x) = \sqrt{\log_{0.3}\left|\frac{x-2}{x}\right|}$

(iv) $f(x) = \frac{1}{[\lfloor x-1 \rfloor + \lceil |7-x| \rceil] - 6}$

(v) $f(x) = \log\sqrt{-\left(\cos x + \frac{1}{2}\right)}$

Sol.

(i) $f(x) = \ln(3x^2 - 4x + 5) + \sqrt{2\sin^2 x - 5\sin x + 2}$

$3x^2 - 4x + 5$

Coefficient of $x^2 = 3$

Discriminant $(D) = 16 - 4 \times 5 \times 3 = -44 < 0$

hence $3x^2 - 4x + 5 > 0,\ \forall x \in R$

$2\sin^2 x - 5\sin x + 2 \geq 0$

$$\Rightarrow 2\left(\sin^2 x - \frac{5}{2}\sin x\right) + 2 \geq 0 \quad \Rightarrow \quad 2\left[\left(\sin^2 x - \frac{5}{4}\right)^2 - \frac{25}{16}\right] + 2 \geq 0$$

$$\Rightarrow \left(\sin x - \frac{5}{4}\right)^2 \geq \frac{9}{16} \quad \Rightarrow \quad \sin x \geq 2 \ \text{ or } \ \sin x - \frac{5}{4} \leq \frac{-3}{4}$$

$$\sin x \geq 2 \qquad \sin x \leq \frac{1}{2} \quad \Rightarrow \quad x \in \left[2n\pi - \frac{7\pi}{6}, 2n\pi + \frac{\pi}{6}\right], n \in I$$

(ii) $f(x) = \ln\{x\} + \sqrt{x - 2\{x\}}$

for $\ln\{x\}$ to be defined

$\{x\} > 0$

$x \in R - I$ ... (i)

$x - 2\{x\} > 0$

$\Rightarrow [x] - \{x\} > 0 \Rightarrow [x] > \{x\}$

$x \geq 1$ ... (ii)

Using (i) and (ii)

$x \in (1, \infty) - I^+$

(iii) $f(x) = \sqrt{\log_{0.3}\left|\frac{x-2}{x}\right|}$

for $f(x)$ to be defined

$$0 < \left|\frac{x-2}{x}\right| \leq 1$$

$$\Rightarrow -1 \leq \frac{x-2}{x} \leq 1 \ \text{ and } \ \frac{x-2}{x} \neq 0 \quad \text{... (i)}$$

Solving LHS

$$\frac{x-2}{x} + 1 \geq 0$$

$\Rightarrow x < 0, x \geq 1$ ... (ii)

Solving RHS

$\frac{x-2}{x} \quad 1 \le 1$

$x > 0 \quad \ldots (iii)$

hence from (i), (ii) and (iii)

$x \in [1, \infty) - \{2\}$

(iv) $\quad f(x) = \dfrac{1}{[|x-1|]+[|7-x|] -6}$

$[|x-1|] + [|7-x|] - 6 \neq 0$

Case I :

$1 < x < 7$

$[x-1] + [7-x] - 6 \neq 0$

$[x] - 1 + [-x] + 7 \neq 0$

$[x] + [-x] \neq 0$

$x \notin I$

$x \in (1, 7) - \{2, 3, 4, 5, 6\}$

Case II :

$x \le 1$

$[1-x] + [7+x] - 6 \neq 0$

$2 + 2[-x] \neq 0$

$[-x] \neq -1$

$-x \notin -1$

$\Rightarrow x\ (0, 1] \Rightarrow x \in (-\infty, 0]$

Case III :

$[x-1] + [x-7] - 6 \neq 0$

$2[x] \neq 14$

$[x] \neq 7$

$x \notin [7, 8)$

using can I, II and III we get

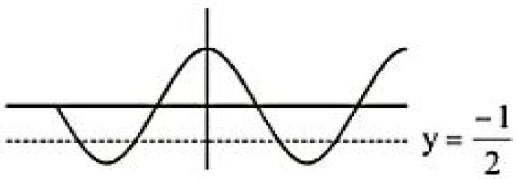
$x \in R - (0, 1] - [7, 8) - \{2, 3, 4, 5, 6)$

(v) $\quad f(x) = \log \sqrt{-\left(\cos x + \frac{1}{2}\right)}$

$-\left(\cos x + \frac{1}{2}\right) > 0$

$\Rightarrow \cos x < -\frac{1}{2}$

$x \in \left(2n\pi + \frac{2\pi}{3}, 2n\pi + \frac{4\pi}{3}\right)$



Practice Problem

Q.1 Find the domain of the following functions

(i) $f(x) = \dfrac{\sqrt{\cos x - \frac{1}{2}}}{\sqrt{6 + 35x - 6x^2}}$ (ii) $f(x) = \sqrt{(x^2 - 3x - 10)\ln^2(x-3)}$

(iii) $f(x) = \log_{2\{x\}-3}(x^2 - 5x + 13)$ (iv) $f(x) = \left(\log_{\frac{x-2}{x+3}} 2\right) + \sqrt{9 - x^2}$

Q.2 Find the range of the following functions

(i) $y = \dfrac{x^2 + x - 1}{x^2 - x + 2}$ (ii) $y = 3\sin\sqrt{\dfrac{9\pi^2}{16} - x^2}$

(iii) $y = 3\sin x + 4\sin\left(x + \dfrac{\pi}{3}\right) + 7$ (iv) $y = \dfrac{e^x - e^{-|x|}}{e^x + e^{|x|}}$

(v) $y = [x^2] - [x]^2$

4. EQUAL OR IDENTICAL FUNCTION :

Two functions f & g are said to be equal if

(i) The domain of f = the domain of g.

(ii) The range of f = the range of g and

(iii) $f(x) = g(x)$, for every x belonging to their common domain.

e.g. $f(x) = \dfrac{1}{x}$ & $g(x) = \dfrac{x}{x^2}$ are identical functions.

Note : Functions are also equal if their graphs are same

Illustration :

Find the domain of x for which the function $f(x) = \ln x^2$ and $g(x) = 2\ln x$ are identical.

Sol. $f(x) = \ln x^2 = 2\ln|x|$

$g(x) = 2\ln x$

if $f(x) = g(x)$

$2\ln|x| = 2\ln x$

function are equal only if $x \in (0, \infty)$

Illustration :

Find out which of the following functions are identical.

(i) $f(x) = \text{cosec } x,\ g(x) = \dfrac{1}{\sin x}$ (ii) $f(x) = \tan x,\ g(x) = \dfrac{1}{\cot x}$

(iii) $f(x) = \ln e^x,\ g(x) = e^{\ln x}$ (iv) $f(x) = \sqrt{\dfrac{1-\cos 2x}{2}},\ g(x) = \sin x$

(v) $f(x) = \dfrac{1}{|x|},\ g(x) = \sqrt{x^{-2}}$

Sol.

(i) $f(x) = \text{cosec } x,\ g(x) = \dfrac{1}{\sin x}$

Domain of $f(x) \Rightarrow x \notin n\pi$

Domain of $g(x) \Rightarrow x \notin n\pi$

Since domain and range are same hence identical fucntion

(ii) $f(x) = \tan x,\ g(x) = \dfrac{1}{\cot x}$

$f(x) = \tan x$, $x = 0$ is domain of $f(x)$

$$g(x) = \frac{1}{\cot x}$$

$x = 0$ is not in the domain of $g(x)$

hence $f(x)$ and $g(x)$ are not identical.

(iii) $f(x) = \ln e^x,\ g(x) = e^{\ln x}$

$f(x) = \ln e^x$ Domain $= R$

$g(x) = e^{\ln x}$ Domain $= R^+$

hence not identical funciton

(iv) $f(x) = \sqrt{\dfrac{1-\cos 2x}{2}},\ g(x) = \sin x$

$f(x) = |\sin x|$ Range $[0, 1]$

$g(x) = \sin x$ Range $[0, 1]$

hence not identical.

(v) $f(x) = \dfrac{1}{|x|},\ g(x) = \sqrt{x^{-2}}$

$$f(x) = \frac{1}{|x|}$$

$$g(x) = \frac{1}{\sqrt{x^2}} = \frac{1}{|x|}$$

hence identical functions.

Practice Problem

Q.1 Identify the equal function

(i) $f(x) = \log_x e$; $g(x) = \dfrac{1}{\log_e x}$ (ii) $f(x) = \log_e x$; $g(x) = \dfrac{1}{\log_x e}$

(iii) $f(x) = \sqrt{x^2 - 1}$; $g(x) = \sqrt{x-1}\ \sqrt{x+1}$ (iv) $f(x) = \log(x+2) + \log(x-3)$; $g(x) = (x^2 - x - 6)$

(v) $f(x) = x\,|\,x\,|$; $g(x) = x^2 \operatorname{sgn} x$ (vi) $f(x) = \dfrac{1}{1+\dfrac{1}{x}}$; $g(x) = \dfrac{x}{1+x}$

(vii) $f(x) = [\{x\}]$; $g(x) = \{[x]\}$

5. CLASSIFICATION OF FUNCTIONS:

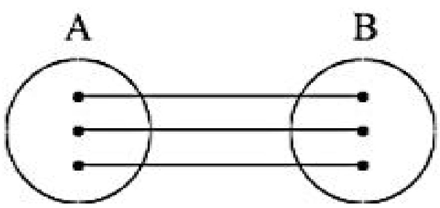
5.1 One-One Function (Injective mapping):

A function $f : A \to B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$,
$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

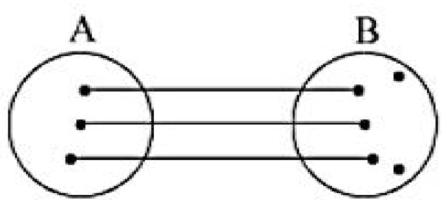
Examples: $R \to R$ $f(x) = x^3 + 1$; $f(x) = e^{-x}$; $f(x) = \ln x$

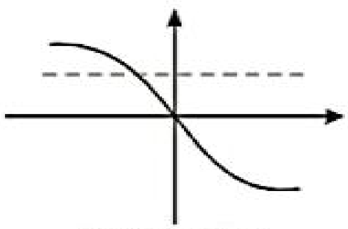
Remember that a linear function is always one-one.

Diagramatically an injective mapping can be shown as

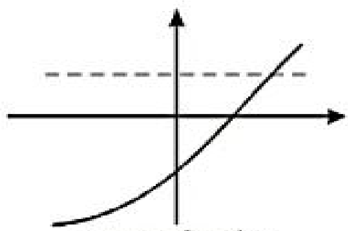


OR





one-one function



one-one function

Note:

(i) A continuous function which is always increasing or decreasing in whole domain, then f(x) is one–one.

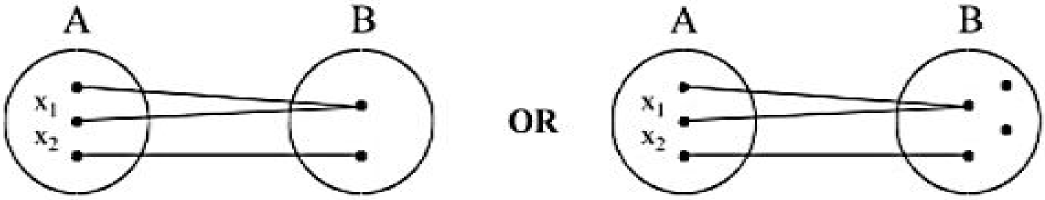
(ii) A function is one to one if and only if a horizontal line intersects its graph at most once.

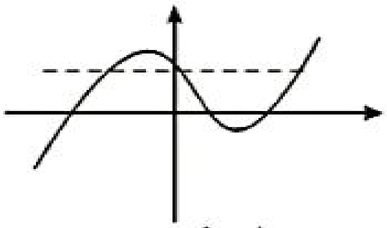
5.2 Many-one function (not injective) :

A function $f: A \to B$ is said to be a many one function if two or more elements of A have the same f image in B. Thus $f: A \to B$ is many one if for;
$x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

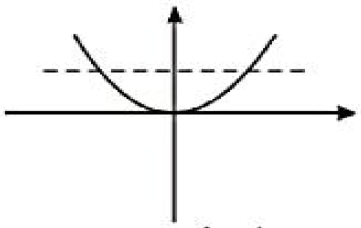
Examples : $R \to R$ $f(x) = [x]$; $f(x) = |x|$; $f(x) = ax^2 + bx + c$; $f(x) = \sin x$

Diagramatically a many one mapping can be shown as





many-one function



many-one function

Note:

(i) Any continuous function which has atleast one local maximum or local minimum in its domain, then f(x) is many–one. In other words, if a line parallel to x–axis cuts the graph of the function atleast at two points, then f is many–one.

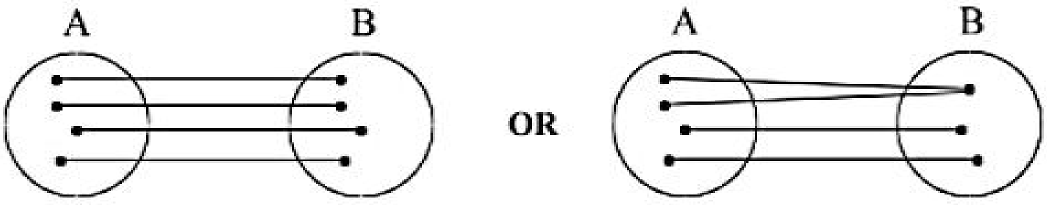
(ii) If a function is one–one, it cannot be many–one and vice versa.
One One + Many One = Total number of mappings.

5.3 Onto function (Surjective mapping) :

If the function $f: A \to B$ is such that each element in B (co–domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B. Thus $f: A \to B$ is surjective iff $\forall\, b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.

$f: R \to R$ $f(x) = 2x + 1$; $f: R \to R^+$ $f(x) = e^x$; $f: R^+ \to R$ $f(x) = \ln x$

Diagramatically surjective mapping can be shown as



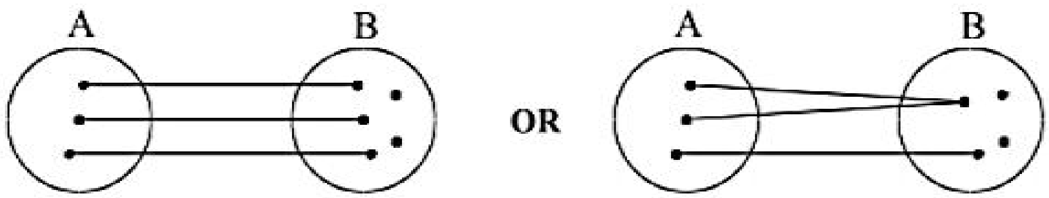
Note that: If range = co–domain, then f (x) is onto. Any polynomial of degree odd, $f: R \to R$ is onto.

5.4 Into function:

If $f : A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then $f(x)$ is into.

e.g. $\quad f : R \to R \quad f(x) = [x], |x|, \text{sgn } x, \; f(x) = ax^2 + bx + c$

Diagramatically into function can be shown as

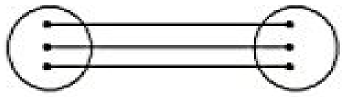


Note :

(i) If a function is onto, it cannot be into and vice versa . A polynomial of degree even define from $R \to R$ will always be into & a polynomial of degree odd defined from $R \to R$ will always be onto.

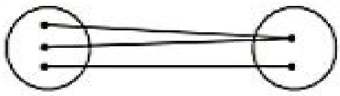
(ii) A function can be one of these four types :

(a) one–one onto (injective & surjective) $(I \cap S)$

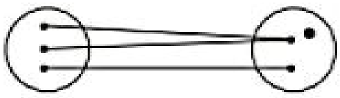


(b) one–one into (injective but not surjective) $(I \cap \overline{S})$

(c) many–one onto (surjective but not injective) $(S \cap \overline{I})$



(d) many–one into (neither surjective nor injective) $(\overline{I} \cap \overline{S})$



(iii) If f is both injective & surjective, then it is called a **Bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

Illustration :

Classify the following functions as many-one, one-one, onto or into functions.

(i) $f(x) = e^x + e^{-x}$ $\qquad$ *(ii)* $f(x) = x^3$

(iii) $f(x) = \sqrt{1+x^2}$ $\qquad$ *(iv)* $f : [-1, 1] \to [-1, 1], f(x) = \sin 2x$

Sol.

(i) $f(x) = e^x + e^{-x}$

Domain $\in R$

$$y = e^x + \frac{1}{e^x} \quad \Rightarrow \quad y = e^x + \frac{1}{e^x} \geq 2$$

Range $[2, \infty)$

also $f(x) = f(-x)$

hence function is many one into

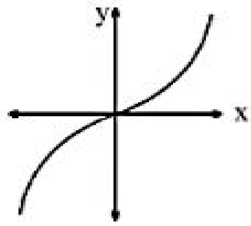
(ii) $f(x) = x^3$

Domain $\in R$

Range $\in R$

we know that ($y = x^3$) cubic, equation has a solution for all $x \in R$.

$f(x)$ is one-one onto $y = x^3$.



(iii) $f(x) = \sqrt{1+x^2}$

Domain $x \in R$

Range $y \in [1, \infty)$

$f(x) = f(-x)$

$f(x)$ is many one-into

(iv) $f : [-1, 1] \to [-1, 1], f(x) = \sin 2x$

From graph we can say that

$f(x)$ is many one onto.

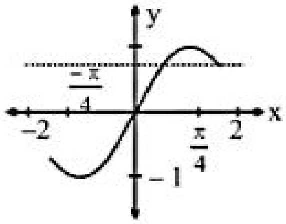


Illustration :

The function $f : [2, \infty) \to y$ defined by $f(x) = x^2 - 4x + 5$ is both one-one and onto if

(A) $y = R$ (B) $y = [1, \infty)$ (C) $y = [4, \infty]$ (D) $y = [5, \infty]$

Sol. $f(x) = x^2 - 4x + 5$

Minima at $x = 2$

at $x = 2, y = 4 - 8 + 5 = 1$

For function to be one-one it should be monotonic.

Hence, for $x \in [2, \infty)$, $f(x)$ is increasing.

at $x = 2, y = 1$. Hence $y \in [1, \infty)$

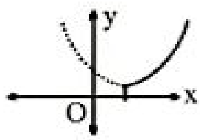


Illustration :

If $f(x) = x^2 + bx + 3$ is not injective for values of x in the interval, $0 \le x \le 1$. Find the interval in which b lies.

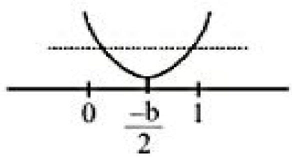
(A) $(-\infty, \infty)$ (B) $(-2, \infty)$ (C) $(-2, 0)$ (D) $(-\infty, 2)$

Sol. If $f(x)$ is not one-one then atleast one horizontal line should intersect it at two points

$$0 < \frac{-b}{2} < 1$$

$\Rightarrow$ $0 < -b < 2$

$\Rightarrow$ $b \in (-2, 0)$



5.6 Permutation and Combinations Problems :

Illustration :

A function $f : A \to B$, such that set "A" and "B" contain four elements each then find

(i) Total number of functions

(ii) Number of one-one functions

(iii) Number of many one functions

(iv) Number of onto functions

(v) Number of into functions

Sol.

(i) I^{st} *element of A can have its image in 4 ways.*
Similarly II, III and IV can have 4 options for their image each.
hence number of functions $= 4^4$

(ii) *4 different elements can be matched in 4! ways*

(iii) *Number of many one functions*
$=$ *Total number of functions – number of one-one function*
$= 4^4 - 4!$

(iv) *Since 4 elements in B are given hence each should be image of atleast one.*
So number of onto function $= 4!$

(v) *Number of into functions* $= 4^4 - 4!$.

Illustration :

A function $f : A \to B$, such that set "A" contains five element and "B" contains four elements then find

(i) Total number of functions

(ii) Number of one-one functions

(iii) Number of onto function

(iv) Number of many one function

(v) Number of into functions

Sol.

(i) *Total number of functions*
Hence number of functions $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$

(ii) *Number of one-one functions*
Since A contains five elements hence one-one function is not possible.

(iii) *Number of onto function*
Divide 5 elements into 4 groups of size $= 1, 1, 1, 2$

Number of ways mapping 4 groups with four images $= \left(\dfrac{5!}{1!\,1!\,1!\,2!} \times \dfrac{1}{3!}\right) \times 4! = 240$

(iv) *Number of many one function*
All the possible functions are many-one.
$= 4^5 = 1024$

(v) *Number of into functions*
Number of into function = Total number of functions – number of onto functions
= 1024 – 240 =784

Illustration :

A function $f : A \to B$ such that set A contains 4 elements and set B contains 5 elements, then find the

(i) *Total number of functions*
(ii) *Number of injective (one-one) mapping.*
(iii) *Number of many-one functions*
(iv) *Number of onto function.*
(v) *Number of into functions*

Sol.

(i) *Total number of functions*
Every element in A has 5 options for image, hence
Total number of functions = 5^4 = 625.

(ii) *Number of injective (one-one) mapping.*
4 elements in A needs four images hence number of one one functions = ${}^5C_4 \times 4! = 120$.

(iii) *Number of many-one functions*
Number of many-one mapping
= Total number of mapping – number of one-one mapping
$= 5^4 - {}^5C_4 \times 4! = 505$

(iv) *Number of onto function = 0*

(v) *Number of into functions = 5^4 = 625*

Practice Problem

Q.1 Show that there are exactly two distinct linear function which map[–2, 0] onto [1, 3].

Q.2 Let f be a one-one function with domain ≡ {x, y, z} and range ≡ {1, 2, 3}. It is given that exactly one of the following statements is true & the remaining two are false. f(x) = 1, f(y) ≠ 1, f(z) ≠ 2. Find f(x), f(y) & f(z).

Q.3 If $f : R - \{3\} \to R - \{1\}$, where $f(x) = \dfrac{x-2}{x-3}$. Find out if f(x) is bijective or not.

Q.4 $f : R \to R$ is defined as $f(x) = \begin{cases} x^2 + 2mx - 1 & \text{for } x \le 0 \\ mx - 1 & \text{for } x > 0 \end{cases}$. If f(x) is one-one then m must lie in the interval

(A) $(-\infty, 0)$ (B) $(-\infty, 0]$ (C) $(0, \infty)$ (D) $[0, \infty)$

Answer key

Q.2 $f(x) = 2, f(y) = 1, f(z) = 3$ Q.3 Bijective Q.4 A

Some important points to remember :

If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\,ln\,x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n,\ n \in R$

(iii) $f(x+y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx} \Rightarrow f(x) = A^x, A > 0$.

(iv) $f(x+y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

(v) If P(x) is a polynomial function of degree n and $P(x) \cdot P\left(\frac{1}{x}\right) = P(x) + P\left(\frac{1}{x}\right)$ for $x \neq 0$. Then $P(x) = 1 + x^n$ or $1 - x^n$.

6. FUNCTIONAL EQUATIONS :

Illustration :

If $f(0) = 1, f(1) = 2$ & $f(x) = \frac{1}{2}[f(x+1) + f(x+2)]$, find the value of $f(5)$.

Sol. $f(x+2) = 2f(x) - f(x+1)$

thus $f(0+2) = f(2) = 2f(0) - f(1)$

$= 2(1) - 2 = 0$

$f(3) = 2f(1) - f(2) = 2(2) - 0 = 4$

$f(4) = 2f(2) - f(3) = 0 - 4 = -4$

$f(5) = 2f(3) - f(4) = 2(4) - (-4) = 12$

Illustration :

If $f(x) + 2f(1-x) = x^2 + 2 \ \forall\ x \in R$, find $f(x)$.

Sol. $f(x) + 2f(1-x) = x^2$ ...(i)

Replacing x by $1 - x$

$f(1-x) + 2f(x) = (1-x)^2$ ...(ii)

Solving (i) & (ii), we get

$3f(x) = 2x^2 - (1-x)^2$

$$f(x) = \frac{x^2 + 2x - 1}{3}$$

Illustration :

If $2f(x^2) + 3f\left(\frac{1}{x^2}\right) = x^2 - 1$ $(x \neq 0)$ then find $f(x^2)$.

Sol. $2f(x^2) + 3f\left(\frac{1}{x^2}\right) = x^2 - 1$ ...(i)

Replace x by $\frac{1}{x}$

$$2f\left(\frac{1}{x^2}\right) + 3f(x^2) = \frac{1}{x^2} - 1 \qquad ...(ii)$$

Solving (i) & (ii) we get

$$9f(x)^2 - 4f(x^2) = 3\left(\frac{1}{x^2} - 1\right) - 2(x^2 - 1)$$

$$5f(x^2) = \frac{3}{x^2} - 2x^2 - 1$$

$$f(x^2) = -\left(\frac{2x^4 + x^2 - 3}{5x^2}\right)$$

Illustration :

Let f(x) & g(x) be functions which take integers as arguments let f(x + y) = f(x) + g(y) + 8 for all integer x & y. Let f(x) = x for all negative integers x let g(8) = 17, find f(0).

Sol. *f(x) = x for integers less than zero*

$\therefore\ f(-8) = -8$

$f(x + y) = f(x) + g(y) + 8$

$f(-8 + 8) = f(-8) + g(8) + 8$

$f(0) = -8 + g(8) + 8$

$f(0) = 17$

Practice Problem

Q.1 Let $f(x) = ax^5 + bx^3 + cx - 5$, where a, b & c are constants. If $f(-7) = 7$, then find f(7).

Q.2 The function $f : R \to R$ satisfies the condition $mf(x-1) + nf(-x) = 2|x| + 1$. If $f(-2) = 5$ and $f(1) = 1$, then find $(m + n)$.

Q.3 If $f(x + y) = f(x)\,f(y)\ \forall\ x, y \in N$, $f(1) = 2$ and $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$. Find a.

Q.4 Solve the inequality $|f(x) - g(x)| < |f(x)| + |g(x)|$ where $f(x) = x - 3$ and $g(x) = 4 - x$.

Answer key

Q.1	-17	Q.2	$\frac{4}{3}$	Q.3	$a = 3$	Q.4	$x \in (3, 4)$

7. COMPOSITE FUNCTION :

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by

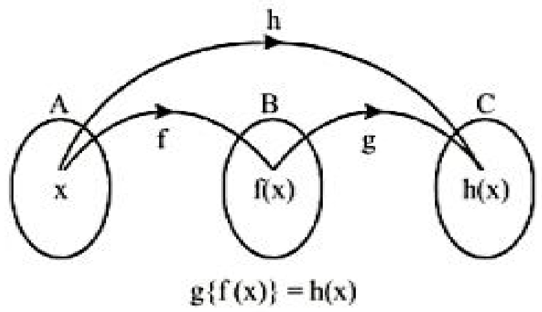
$$(gof)(x) = g(f(x))\ \forall\ x \in A$$

is called the composite of the two functions f & g. Diagramatically

$$\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x)).$$

Thus the image of every $x \in A$ under the function gof is the g–image of the f–image of x.

Note that gof is defined only if $\forall\ x \in A$, f(x) is an element of the domain of g so that we can take its g–image. Hence for gof of two functions f & g, the range of f must be a subset of the domain of g.



Note that gof in general not equal to fog.

Illustration :

$f : R \to R$ be the function defined by $f(x) = ax^2 - \sqrt{2}$ for some positive a. If $(fof)\sqrt{2} = -\sqrt{2}$, then find a.

Sol. $f(x) = ax^2 - \sqrt{2}$

$$(fof)(x) = a(ax^2 - \sqrt{2})^2 - \sqrt{2}$$

$$= 4a\left(a - \frac{1}{\sqrt{2}}\right)^2 = 0$$

$$= a = 0, \frac{1}{\sqrt{2}}$$

Illustration :

Let $f(x) = \sqrt{x}$; $g(x) = \sqrt{2-x}$ find the domain of

(i) (fog) (x) (ii) (gof) (x) (iii) (fof) (x) (iv) (gog) (x)

Sol.

(i) $(fog)(x) = f(g(x)) = f\left(\sqrt{2-x}\right) = \sqrt{\sqrt{2-x}}$

Domain $2 - x \geq 0$

$x \leq 2$

$\Rightarrow \quad x \in (-\infty, 2]$

(ii) $(gof)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{2-\sqrt{x}}$

$\therefore\ 2 - \sqrt{x} \geq 0 \quad \Rightarrow \quad 0 \leq \sqrt{x} \leq 2 \quad \Rightarrow \quad x \in [0, 4]$

(iii) $(fof)(x) = f(f(x)) = f\left(\sqrt{x}\right) = \sqrt{\sqrt{x}}$

Domain $\Rightarrow x \geq 0$

$x \in [0, \infty)$

(iv) $(gog)(x) = g(g(x)) = g\left(\sqrt{2-x}\right) = \sqrt{2-\sqrt{2-x}}$

$\therefore\ 0 \le \sqrt{2-x} \le 2 \Rightarrow 0 \le 2-x \le 4 \Rightarrow -2 \le x \le 4$

$x \in [-2, 2]$

Illustration :

Let $f(x) = x^x$ & $g(x) = x^{2x}$, *then find* $f(g(x))$.

Sol. $f(g(x)) = f\left(x^{2x}\right) = \left(x^{2x}\right)^{x^{2x}} = x^{2x.x^{2x}} = x^{2x^{2x+1}}$

7.1 Properties Of Composite Functions :

(i) The composite of functions is not commutative i.e. $gof \ne fog$.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that $fo(goh)$ & $(fog)oh$ are defined, then $fo(goh) = (fog)oh$.

Associativety: $f : (N) \to I_0 \quad f(x) = 2x$

$$g : I_0 \to Q \quad g(x) = \frac{1}{x}$$

$$h : Q \to R \quad h(x) = e^{\frac{1}{x}}$$

$$(hog)of = ho(gof) = e^{2x}$$

(iii) The composite of two bijections is a bijection i.e. if f and g are two bijections such that gof is defined, then gof is also a bijection.

Proof: Let $f : A \to B$ and $g : B \to C$ be two bijections. Then gof exists such that $gof : A \to C$

We have to prove that gof is one-one and onto.

One-one : Let $a_1, a_2 \in A$ such that $(gof)(a_1) = (gof)(a_2)$, then

$(gof)(a_1) = (gof)(a_2) \Rightarrow g[f(a_1)] = g[f(a_2)]$

$\Rightarrow f(a_1) = f(a_2)$ [$\because$ g is one-one]

$\Rightarrow a_1 = a_2$ [$\because$ f is one-one]

$\therefore$ gof is also one-one function.

Onto : Let $c \in C$, then

$c \in C \Rightarrow \exists\, b \in B$ s.t. $g(b) = c$ [$\because$ g is onto]

and $b \in B \Rightarrow \exists\, a \in A$ s.t. $f(a) = b$ [$\because$ f is onto]

Therefore, we see that

$c \in C \Rightarrow \{\exists\, a \in A \text{ s.t. } (gof)(a) = g[f(a)] = g(b) = c\}$

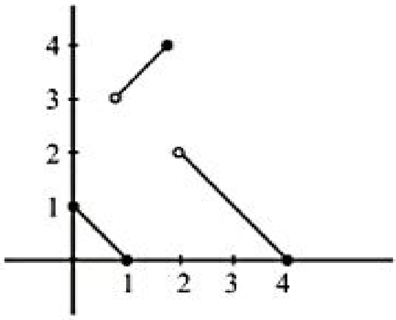
i.e. every element of C is the gof image of some element of A. As such gof is onto function. Hence gof befing one-one and onto, is a bijection.

Illustration :

Evaluate $f(f(x))$, *where*

$$f(x) = (1-x), \quad 0 \le x \le 1$$
$$= (x+2), \quad 1 < x \le 2$$
$$= (4-x), \quad 2 < x \le 4$$

Sol. $f(x)=\begin{cases} 1-x, & 0\le x\le 1 \\ x+2, & 1<x\le 2 \\ 4-x, & 2<x\le 4 \end{cases}$



graph of $f(x)$

$$f(f(x))=\begin{cases} 1-f(x), & 0\le f(x)\le 1 \\ f(x)+2, & 1<f(x)\le 2 \\ 4-f(x), & 2<f(x)\le 4 \end{cases}$$

from the graph we can see that

$0 \le f(x) \le 1$ when $x \in [0, 1] \cup [3, 4]$

$1 < f(x) \le 2$ when $x \in (2, 3]$

$2 < f(x) \le 4$ when $x \in [1, 2]$

$$f(f(x))=\begin{cases} 1-(1-x), & f(x)=1-x, \quad 0\le x\le 1 \\ 1-(4-x), & f(x)=4-x, \quad 3\le x\le 4 \\ (4-x)+2, & f(x)=4-x, \quad 2<x<3 \\ 4-(x+2), & f(x)=x+2, \quad 1<x\le 2 \end{cases}$$

$$f(f(x))=\begin{cases} x, & 0\le x\le 1 \\ x-3, & 3\le x\le 4 \\ 6-x, & 2<x<3 \\ 2-x, & 1<x\le 2 \end{cases}$$

Alternative method :

We have

$$f\{f(x)\} = 1-f \qquad 0\le f\le 1 \qquad \ldots(i)$$
$$= f+2 \qquad 1<f\le 2 \qquad \ldots(ii)$$
$$= 4-f \qquad 2<f\le 4 \qquad \ldots(iii)$$

Putting the values of $f(x)$ *in (i)*

$$1-f = 1-(1-x), \quad 0\le f\le 1, \quad 0\le x\le 1$$
$$= 1-(x+2), \quad 0\le x+2\le 1, \quad 1<x\le 2$$
$$= 1-(4-x), \quad 0\le 4-x\le 1, \quad 2<x\le 4$$

On solving $0\le 1-x\le 1$ & $0<x\le 1$ *we get* $0\le x\le 1$

Solving $0\le x+2\le 1$ & $2<x\le 4 \Rightarrow$ *null set*

Solving $0\le 4-x\le 1$ & $2<x\le 4 \Rightarrow 3<x\le 4$

thus $\Rightarrow$ $1-f=\begin{cases} x, & 0\le x\le 1 \\ x-3, & 3<x\le 4 \end{cases}$ $\qquad \ldots(iv)$

$$f+2 = (1-x)+2, \quad 1<(1-x)\le 2, \quad 0\le x\le 1$$
$$= (x+2)+2, \quad 1<(x+2)\le 2, \quad 1<x\le 2$$
$$= (4-x)+2, \quad 1<(4-x)\le 2, \quad 2<x\le 4$$

Solving $1 < (1-x) \le 2$, *we have* $-1 \le x \le 0$ *& its intersection with* $0 \le x \le 1$ *gives null set*
Solving $1 < (x+2) \le 2$, *we get* $-1 \le x \le 0$ *& intersection with* $1 < x \le 2$ *gives null set*
Solving $1 < (4-x) \le 2$, *we get* $2 \le x < 3$ *& its intersection with* $2 < x \le 4$ *gives* $2 < x < 3$
thus $f + 2 = 6 - x, \quad 2 < x < 3$ ...(v)
Putting the values of $f(x)$ *in (iii), we have*

$$\begin{aligned} 4 - f &= 4-(1-x), && 2 < 1-x \le 4, && 0 \le x \le 1 \\ &= 4-(x+2), && 2 < x+2 \le 4, && 1 < x \le 2 \\ &= 4-(4-x), && 2 < 4-x \le 4, && 2 < x \le 4 \end{aligned}$$

Solving $2 \le (1-x) \le 4$, *we get* $-3 \le x \le -1$ *& its intersection with* $0 \le x \le 1$ *gives null set*
Solving $2 < (x+2) \le 4$, *we get* $0 < x \le 2$ *& intersection with* $1 < x \le 2$ *gives* $1 < x \le 2$
Solving $2 < (4-x) \le 4$, *we get* $0 \le x < 3$ *& its intersection with* $2 < x \le 4$ *gives null set*
thus $4 - f = 2 - x, \quad 1 < x \le 2$ ...(vi)
Using (iv),(v) & (vi)

$$f(f(x)) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 6-x, & 2 < x < 3 \\ x-3, & 3 \le x \le 4 \end{cases}$$

Practice Problem

Q.1 If $f(x) = \begin{cases} 1-x & \text{if} \quad 0 \le x \le 1 \\ x+2 & \text{if} \quad 1 < x < 2 \\ 4-x & \text{if} \quad 2 \le x \le 4 \end{cases}$. Find fof(x)

Q.2 If $g(x) = 2x+1$ & $h(x) = 4x^2 + 4x + 7$, find a function f such that fog = h.

Q.3 If $f(x) = \dfrac{2x-7}{x+3}$, find a function g such that $g[f(x)] = x$ for all x in the domain of f and find its domain & range.

Q.4 Evaluate $g\{f(x)\}$, where

$$\begin{aligned} f(x) &= 1 + x^3, && x < 0 \\ &= x^2 - 1, && x \ge 0 \\ g(x) &= (x-1)^{1/3}, && x < 0 \\ &= (x+1)^{1/2}, && x \ge 0 \end{aligned}$$

Answer key

Q.1 $f(f(x)) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 6-x, & 2 < x < 3 \\ x-3, & 3 \le x \le 4 \end{cases}$

Q.2 $f(x) = x^2 + 6$

Q.3 Domain $= R - \{-3\}$, Range $= R - \{2\}$

Q.4 $g(f(x)) = \begin{cases} x, & x < -1 \\ (x^3+2)^{\frac{1}{2}}, & -1 \le x < 0 \\ (x^2-2)^{\frac{1}{3}}, & 0 \le x < 1 \\ x, & x \ge 1 \end{cases}$

8.1 Homogeneous Functions :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2 + 3y^2 - xy$ is homogeneous in x & y.
$f(x, y)$ is a homogeneous function iff
$f(tx, ty) = t^n f(x, y)$

or $f(x, y) = x^n g\left(\frac{y}{x}\right) = y^n h\left(\frac{x}{y}\right)$, where n is the degree of homogenity

$f(x, y) = \frac{x - y\cos x}{y\sin x + x}$ is not a homogeneous function and

$f(x, y) = \frac{x}{y} ln\frac{y}{x} + \frac{y}{x} ln\frac{x}{y}$; $\sqrt{x^2 - y^2} + x$; $x + y\cos\frac{y}{x}$ are homogeneous functions of degree one.

8.2 Bounded Function :

A function is said to be bounded if $|f(x)| \le M$, where M is a finite quantity.
e.g. $f(x) = \sin x$ is bounded in $[-1, 1]$

8.3 Implicit & Explicit Function :

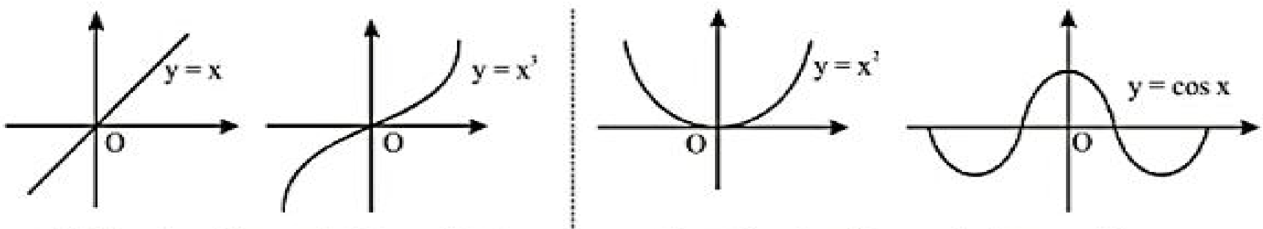
A function defined by an equation not solved for the dependent variable is called an IMPLICIT FUNCTION. For eg. the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an EXPLICIT FUNCTION.

8.4 Odd & Even Functions :

A function $f(x)$ defined on the symmetric interval $(-a, a)$

If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.
e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$.

If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.
e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.



Odd functions (Symmetric about origin) | Even functions (Symmetric about y-axis)

NOTE :

(a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

(b) A function may neither be odd nor even.

(c) Inverse of an even function is not defined and an even function can not be strictly monotonic

(d) Every even function is symmetric about the y–axis & every odd function is symmetric about the origin.

(e) Every function can be expressed as the sum of an even & an odd function.

e.g. $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$ $\qquad$ $2^x = \underbrace{\frac{2^x+2^{-x}}{2}}_{\text{EVEN}} + \underbrace{\frac{2^x-2^{-x}}{2}}_{\text{ODD}}$

(f) The only function which is defined on the entire number line & is even and odd at the same time is $f(x)=0$. Any non zero constant is even.

(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd then f.g will be odd.

f(x)	g(x)	f(x) + g(x)	f(x) - g(x)	f(x).g(x)	f(x) / g(x)	(gof)(x)	(fo g)(x)
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Illustration :

Identity the functions, as even, odd or neither nor odd.

(i) $f(x) = \left(ln\ x + \sqrt{1+x^2}\right)$ $\qquad$ *(ii)* $f(x) = x \cdot \left(\frac{2^x+1}{2^x-1}\right)$

(iii) $f(x) = 2x^3 - x + 1$ $\qquad$ *(iv)* $f(x) = 3$

(v) $f(x) = x^2 - |x|$

Sol.

(i) $f(x) = \left(ln + \sqrt{1+x^2}\right)$

$$f(-x) = ln\left(-x + \sqrt{1+x^2}\right) = ln\left(\sqrt{1+x^2} - x\right)$$

$$= ln\left(\frac{\left(\sqrt{1+x^2}-x\right)\left(\sqrt{1+x^2}+x\right)}{\left(\sqrt{1+x^2}+x\right)}\right) = ln\left(\sqrt{1+x^2}\right) = -f(x)$$

Hence odd function.

(ii) $f(x) = x\left(\frac{2^x+1}{2^x-1}\right)$

$f(-x) = (-x)\left(\frac{2^{-x}+1}{2^{-x}-1}\right)$

$= (-x)\left(\frac{1+2^x}{1-2^x}\right) = x\left(\frac{2^x+1}{2^x-1}\right) = f(x)$

hence even function

(iii) $f(x) = 2x^3 - x + 1$
$f(-x) = -2x^3 + x + 1 \neq f(x)$ or $-f(x)$
Hence neither even nor odd function

(iv) $f(x) = 3$
$f(-x) = 3 = f(x)$
Hence even function

(v) $f(x) = x^2 - |x|$
$f(-x) = x^2 - |-x| = f(x)$
even function

Practice Problem

Q.1 Let $f(x) = [x], x \geq 0$
$= g(x), x < 0$
Find $g(x)$ if $f(x)$ is even

Q.2 Let $f : [-2, 2] \to R$, where $f(x) = x^3 + \sin x + \left[\frac{x^2+1}{a}\right]$ be an odd function. Then find the values of the parameter a.

Q.3 Identify whether the given function is even odd or neither even nor odd where

$$f(x) = \begin{cases} x\,|\,x\,|, & x \leq -1 \\ [1+x]+[1-x], & -1 < x < 1 \\ -x\,|\,x\,|, & x \geq 1 \end{cases}$$

where $|\,|$ & $[\cdot]$ represents modulus and greatest integral function

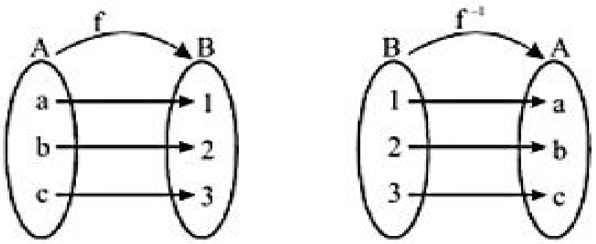
Answer key

Q.1 $g(x) = -[x]$ Q.2 $a > 5$ Q.3 $f(x)$ is even

9. INVERSE OF A FUNCTION :

Let $f : A \to B$ be a one–one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A\ \&\ y \in B$. Then g is said to be inverse of f. Thus $g = f^{-1} : B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Consider a one-one onto function with domain A= {a, b, c} & range B = {1, 2, 3}



Domain of $f = \{a, b, c\}$ = Range of f^{-1}
Range of $f = \{1, 2, 3\}$ = Domain of f^{-1}

Note: (a) Only one-one onto functions (i.e., Bijections) are invertible.

(b) To find the inverse
Step-1: write $y = f(x)$
Step-2: solve this equation for x in terms of y (if possible)
Step-3: To express f^{-1} as a function of x, interchange x and y.

Illustration :

Find the inverse of the following bijective function

(i) $f : R \to R^+$, $f(x) = 10^{x+1}$ (ii) $f(x) = 3x - 5$

(iii) $f : [1, \infty) \to [2, \infty), f(x) = x + \dfrac{1}{x}$ (iv) $f : R \to (0, 1),\ f(x) = \dfrac{2^x}{1+2^x}$

Sol.

(i) $y = 10^{x+1}$
$x + 1 = \log_{10} y$
$x = -1 + \log_{10} y$
$\Rightarrow f^{-1} = y = -1 + \log_{10} x, \quad f^{-1} : R^+ \to R$

(ii) $f(x) = 3x - 5$
$y = 3x - 5$

$$x = \frac{y+5}{3}$$

$$\Rightarrow f^{-1}(x) = y = \frac{x+5}{3}$$

(iii) $f : [1, \infty) \to [2, \infty)$

$$y = f(x) = x + \frac{1}{x}$$

$$\Rightarrow \quad x^2 - xy + 1 = 0$$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{x \pm \sqrt{x^2 - 4}}{2}$$

Since range is [1, ∞), hence

$$f^{-1}(x) = \frac{x + \sqrt{x^2 - y}}{2}$$

(iv) $f : R \to (0, 1),\ f(x) = \dfrac{2^x}{1 + 2^x}$

$$y = \frac{2^x}{1+2^x} \quad \Rightarrow \quad y + 2^x y = 2^x$$

$$\Rightarrow \quad 2^x = \frac{y}{1-y} \quad \Rightarrow \quad x = \log_2\left(\frac{y}{1-y}\right)$$

$$\Rightarrow \quad f^{-1}(x) = y = \log_2\left(\frac{x}{1-x}\right)$$

9.1 Properties of inverse of a function :

(i) The inverse of Bijection is unique.

(ii) The inverse of Bijection is also bijection.

(iii) If $f : A \to B$ is Bijection & $g : B \to A$ is inverse of f, then $fog = I_B$ & $gof = I_A$, where I_A, I_B are the identical function on the set A and B respectively

(iv) If $f : A \to B$ and $g : B \to C$ are two bijections, then $gof : A \to C$ is bijections and $(gof)^{-1} = f^{-1}og^{-1}$.

(v) In general $fog \neq gof$ but if $fog = gof$ then either $f^{-1} = g$ or $g^{-1} = f$ also $(fog)(x) = (gof)(x) = x$.

(vi) The graphs of f & g are the mirror images of each other in the line y = x. As shown in the figure given below a point (x', y') corresponding to $y = x^2 (x \geq 0)$ changes to (y', x') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

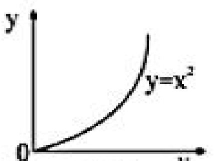


fig. 1

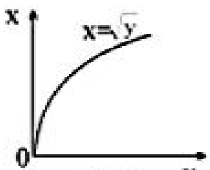


fig. 2

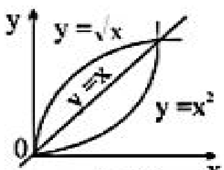


fig. 3

Illustration :

If $f : R \to R$ *is defined by* $f(x) = x^3 + 1$, *then find value of* $f^{-1}(28)$.

Sol. $f^{-1}(28) = x \quad \Rightarrow f(x) = 28 \quad \Rightarrow x^3 + 1 = 28 \quad \Rightarrow x = 3$

Illustration :

If the function f & g be defined as $f(x) = e^x$ *and* $g(x) = 3x - 2$ *where* $f : R \to R$ *and* $g : R \to R$ *then find the function fog and gof. Also find the domain of* $(fog)^{-1}$ *and* $(gof)^{-1}$.

Sol. $(fog)(x) = f\{g(x)\}$

$f\{g(x)\} = f(3x - 2)$

$= e^{3x-2}$

$(gof)(x) = g\{f(x)\} = g(e^x) = 3e^x - 2$

To find $(fog)^{-1}$ & $(gof)^{-1}$

$(fog)(x) = y = e^{3x-2}$

$\Rightarrow \quad 3x - 2 = \log y$

$$\Rightarrow \quad x = \frac{\log y + 2}{3} \Rightarrow \quad (fog)^{-1} x = \frac{\log x + 2}{3}$$

Domain of $(fog)^{-1}$ is $x > 0$ i.e. $x \in (0, \infty)$

Again $gof(x) = y = 3e^x - 2$

$$\Rightarrow \quad e^x = \frac{y+2}{3} \quad \Rightarrow \quad x = \log\left(\frac{y+2}{3}\right) \quad \Rightarrow \quad (gof)^{-1} x = \log\left(\frac{x+2}{3}\right)$$

$$\text{Domain of } (gof)^{-1} \text{ is } \frac{x+2}{3} > 0$$

$x > -2$

$\Rightarrow \quad x \in (-2, \infty)$

Illustration :

$$\text{If } f: [0, \infty) \to [1, \infty),\ f(x) = \frac{e^x + e^{-x}}{2}.\ \text{Find } f^{-1}(x).$$

Sol. $$f(x) = \frac{e^x + e^{-x}}{2}$$

$$\Rightarrow \quad 2y = e^x + \frac{1}{e^x} \qquad \Rightarrow \quad e^{2x} - 2e^x y + 1 = 0$$

$$\Rightarrow \quad e^{2x} - 2e^x y + y^2 = y^2 - 1 \qquad \Rightarrow \quad (e^x - y)^2 = y^2 - 1$$

$$\Rightarrow \quad e^x = y \pm \sqrt{y^2 - 1} \qquad \Rightarrow \quad x = \log\left(y \pm \sqrt{y^2 - 1}\right)$$

$$\Rightarrow \quad f^{-1}(x) = y = \log\left(x \pm \sqrt{x^2 - 1}\right)$$

Since range is $[0, \infty)$ hence

$$\Rightarrow \quad f^{-1}(x) = y = \log\left(x + \sqrt{x^2 - 1}\right)$$

Illustration :

Find the inverse of the function $f : N \to N,\ f(x) = x + (-1)^{x-1}$.

Sol. $f(x) = x + (-1)^{x-1},\ x \in N$

Then we have $f(1) = 1 + 1 = 2,\quad f(2) = 1$

$f(3) = 4,\quad f(4) = 3$

$f(5) = 6,\quad f(6) = 5$

The points on graph are (1, 2), (2, 1), (3, 4) (4, 3), (5, 6), (6, 5) .... etc. Thus if (a, b) is a point on the graph then (b, a) is also a point on the graph. Hence f is the inverse of itself.

i.e. $f^{-1}(x) = x + (-1)^{x-1},\quad x \in N$

Practice Problem

Q.1 If $y = ax + b$ and the equation $f(x) = f^{-1}(x)$ is satisfied by every real value of x then

(A) $a = 2, b = -1$ (B) $a = -1, b \in R$

(C) $a = 1, b \in R$ (D) $a = 1, b = -1$

Q.2 Find the inverse of following functions

(i) $f(x) = 5^{\log_e x},\ x > 0$ (ii) $f(x) = \begin{cases} x & , x < 1 \\ x^2 & , 1 \le x \le 4 \\ 8\sqrt{x} & , x > 4 \end{cases}$

(ii) $f(x) = \log_e(x^2 + 3x + 1),\ x \in [1, 3]$

Answer key

Q.1 B **Q.2** (i) $f^{-1}(x) = e^{\log_5 x}$ (ii) $f^{-1}(x) = \begin{cases} x, & -\infty < x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ \log_2 x, & 16 < x < \infty \end{cases}$ (ii) $f^{-1}(x) = \dfrac{\sqrt{5 + 4e^x} - 3}{2}$

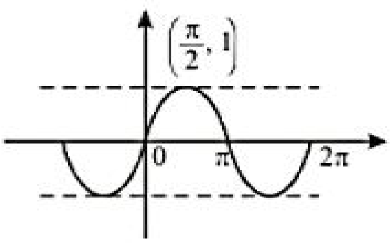
10. PERIODIC FUNCTION:

A function $f(x)$ is called periodic if there exists a positive number $T(T > 0)$ called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of x.

e.g. The function $\sin x$ & $\cos x$ both are periodic over 2π & $\tan x$ is periodic over π.

Graphically :

If the graph repeats at fixed interval then function is said to be periodic and its period is the width of that interval. For example graph of sin x repeats itself at an interval of 2π



f (x) = sin x

10.1 Properties of periodic function :

(i) $f(T) = f(0) = f(-T)$, where 'T' is the period.
(ii) Inverse of a periodic function does not exist.
(iii) Every constant function is always periodic, with no fundamental period.
(iv) If f(x) has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.
(v) if f(x) has a period T then f(ax + b) has a period $\frac{T}{|a|}$.
(vi) If f(x) has a period T & g(x) also has a period T then it does not mean that f(x) + g(x) must have a period T. e.g. $f(x) = |\sin x| + |\cos x|$; $\sin^4 x + \cos^4 x$ has fundamental period equal to $\frac{\pi}{2}$.
(vii) If f(x) and g(x) are periodic then f(x) + g(x) need not be periodic.
e.g. $f(x) = \cos x$ and $g(x) = \{x\}$

Illustration :

Find the period of the following functions.

(i) $f(x) = \cos\frac{2x}{3} - \sin\frac{4x}{5}$;
(ii) $f(x) = \cos(\sin x)$
(iii) $f(x) = \sin(\cos x)$;
(iv) $f(x) = [x] + [2x] + [3x] + \ldots\ldots + [nx] - \frac{n(n+1)}{2}x$

where $n \in N$ & [ ] denotes greatest integer function

Sol.(i) $f(x) = \cos\left(\frac{2x}{3}\right) - \sin\left(\frac{4x}{5}\right)$

Period of $\cos\left(\frac{2x}{3}\right) = \frac{2\pi(3)}{2} = 3\pi$

Period of $\sin\left(\frac{4x}{5}\right) = \frac{2\pi}{4} \times 5 = \frac{5}{2}\pi$

L.C.M. of 3π & $\frac{5}{2}\pi = 15\pi$

(ii) $f(x) = \cos(\sin x)$
Since cos is even functions $f(\pi + x) = \cos(\sin(\pi + x)) = \cos(-\sin x) = \cos(\sin x) = f(x)$
Hence π is period.

(iii) $f(x) = \sin(\cos x)$
Period is 2π

(iv) $f(x) = [x] + [2x] + [3x] + \ldots\ldots + [nx] - \frac{n(n+1)}{2}x$

$= -\{x\} - \{2x\} - \ldots\ldots - \{nx\}$

Period of $\{x\} = 1$

period of $\{2x\} = \frac{1}{2}$

period of $\{3x\} = \frac{1}{3}$

………………………

………………………

L.C.M. of $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}\right) = 1$

Illustration :

If $f(x) = \dfrac{\sin nx}{\sin \frac{x}{n}}$ *has its period as* 4π, *then find the integral values of* n.

Sol. Period of $\sin nx = \frac{2\pi}{n}$

Period of $\sin \frac{x}{n} = 2n\pi$

L.C.M. of $\left(\frac{2\pi}{n}, 2n\pi\right) = 2n\pi$

$2n\pi = 4\pi$

$n = 2, -2$

Illustration :

Find the period of $f(x) = |\sin x| + |\cos x|$.

Sol. $|\sin x|$ has period π

$|\cos x|$ has period π

$f(x)$ is an even function & $\sin x$, $\cos x$ are complementary then period of

$f(x) = \frac{1}{2}\{\text{LCM of } \pi \ \& \ \pi\} = \frac{\pi}{2}$

Illustration :

Prove that if $f(x) = \sin x + \cos ax$ *is a periodic function then a must be rational.*

Sol. $f(x) = \sin x + \cos ax$

Period of $\sin x = 2\pi$

Period of $\cos ax = \frac{2\pi}{a}$

LCM of 2π & $\frac{2\pi}{a}$ is possible only when a is rational, hence a must be rational.

Practice Problem

Q.1 Find the period of the function $f(x)=\frac{1}{2}\left(\frac{|\sin x|}{\cos x}+\frac{\sin x}{|\cos x|}\right)$

Q.2 Find the period of following functions
(i) $f(x)=\cos 2\pi x$ (ii) $f(x)=2\sin(3x-\pi)$

Q.3 Let a function satisfying $f(x+4)+f(x-4)=f(x)$ for all real x is periodic, then period p for them is
(A) 8 (B) 12 (C) 16 (D) 24

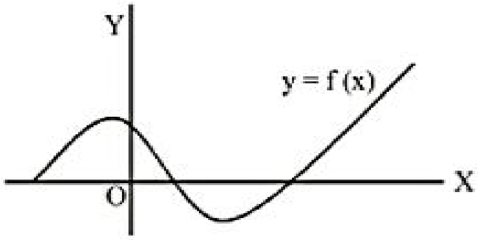
Q.4 If $f(x)=(a+3)x+5a,\ x\in R$ is periodic then find the value of a.

Answer key

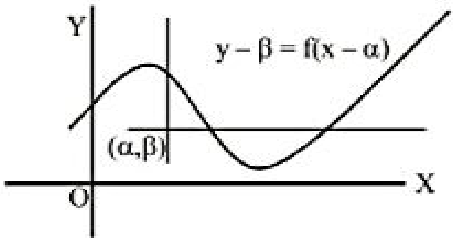
Q.1 2π Q.2 (i) 1, (ii) $\frac{2\pi}{3}$ Q.3 D Q.4 $a=-3$

11. SOME GRAPHICAL TRANSFORMATION :

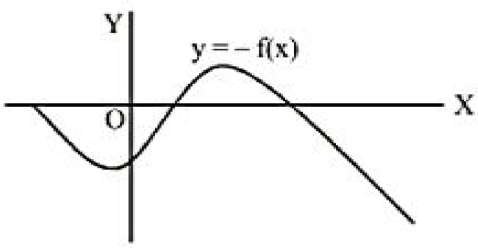
Consider the graph $y=f(x)$ shown alongside.



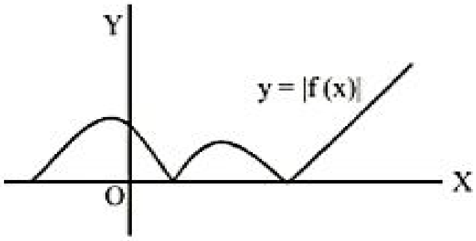
(i) Graph of $y-\beta=f(x-\alpha)$ is drawn by shifting the origin to (α,β) & then translating the graph of $y=f(x)$ w.r.t. new axes



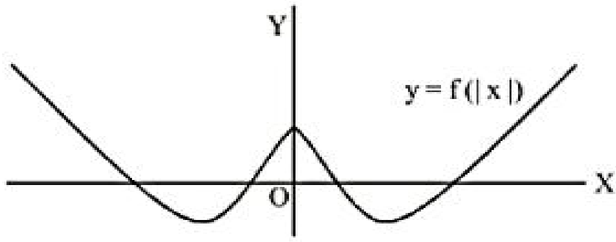
(ii) The graph of $y=-f(x)$ is the mirror image of f(x) in X-axis.



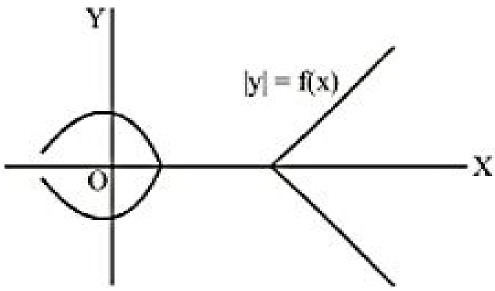
(iii) $y = |f(x)|$ is mirror image of negative portion of $y = f(x)$ in X-axis.



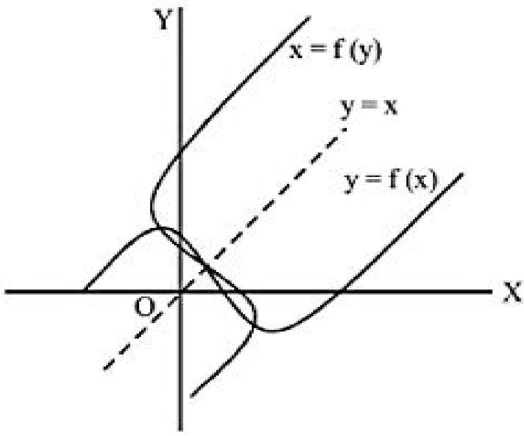
(iv) $y = f(|x|)$ is drawn by taking the mirror image of positive x-axis graph in y-axis.



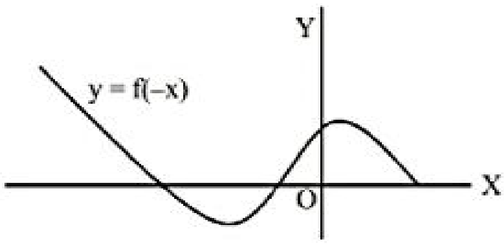
(v) The graph of $|y| = f(x)$ is drawn by deleting those portions of the graph $y = f(x)$ which lie below the X-axis and then taking the mirror image of the remaining portion in the X-axis, as shown alongside.



(vi) $x = f(y)$ is drawn by taking mirror image of $y = f(x)$ in the line $y = x$.



(vii) $y = f(-x)$ is drawn by taking the mirror image of $y = f(x)$ in Y-axis.



Solved Examples

Q.1 Evaluate $g\{f(x)\}$, where $f(x)=\begin{cases}\cos x, & x \neq n\pi \\ 3, & \text{otherwise}\end{cases}$ and $g(x)=\begin{cases}x^2+1, & x\neq 0,3 \\ 3, & x=0 \\ 5, & x=3\end{cases}$.

Sol. We have $g\{f(x)\}=\begin{cases}f^2+1, & f\neq 0,3 & \text{.......(1)} \\ 3, & f=0 & \text{.......(2)} \\ 5, & f=3 & \text{.......(3)}\end{cases}$

Putting values of f(x) in (1)

$$f^2+1=\begin{cases}\cos^2 x+1, & \cos x \neq 0,3;\ x\neq n\pi \\ 3^2+1, & 3\neq 0,\ 3\neq 3,\ x=n\pi\end{cases}$$

i.e., $f^2+1=\cos^2 x+1,\ x\neq(2n+1)\frac{\pi}{2},\ x\neq n\pi$ ....(4)

Putting the values of f(x) in (2),

$g\{f(x)\}=3,\quad \cos x=0,\ x\neq n\pi$

$\quad\quad =3,\quad 3=0,\ x=n\pi$

On solving above equation $g\{f(x)\}=3,\ x=(2n+1)\frac{\pi}{2}$ ....(5)

Putting the values of f(x) in (3)

$g\{f(x)\}=5,\quad \cos x=3,\ x\neq n\pi$

$\quad\quad =3,\quad 3=3,\ x=n\pi$

On solving above equation $g\{f(x)\}=n\pi$ ....(6)

Using (4), (5) and (6), we get

$$g\{f(x)\}=\begin{cases}1+\cos^2 x, & x\neq(2n+1)\frac{\pi}{2},\ x\neq n\pi \\ 3, & x=(2n+1)\frac{\pi}{2} \\ 5 & x=n\pi\end{cases}.$$

Q.2 Consider the function $f(x)=\begin{cases}x|x|, & 0\le x<1 \\ 2x, & x\ge 1\end{cases}$. Find the extension of the function $\forall\ x\in R$ if

(i) f(x) is even (ii) f(x) is odd.

Sol.

(i) We have $f(x)=\begin{cases}x^2, & 0\le x<1 \\ 2x, & x\ge 1\end{cases}$

If f is even $\forall\ x\in R$, then $f(-x)=f(x)$

Hence $f(-x)=\begin{cases}x^2, & -1<x\le 0 \\ -2x, & x\le 1\end{cases}$; $f(x)=\begin{cases}-2x, & x\le 1 \\ x^2, & -1<x<1 \\ 2x, & 1\le x\end{cases}$

(ii) For f(x) to be odd function

$-f(-x) = -x^2, -1 < x \le 0$

$= 2x, \quad x \le -1$

$$f(x) = \begin{cases} 2x, & x \le -1 \\ -x^2, & -1 < x \le 0 \\ x^2, & 0 \le x < 1 \\ 2x, & 1 \le x \end{cases} \qquad \text{or}$$

$$f(x) = \begin{cases} x\,|x|, & |x| < 1 \\ 2x, & |x| \ge 1 \end{cases}.$$

Q.3 Let $f(x) = \dfrac{x-1}{x+1}$, $f^2(x) = f\{f(x)\}$, $f^3(x) = f\{f^2(x)\}$ .... $f^{k+1}(x) = f\{f^k(x)\}$, for k = 1, 2, 3, ...
Find $f^{1998}(x)$.

Sol. $f(x) = \dfrac{x-1}{x+1}$

$f^2(x) = f\{f(x)\} = \dfrac{f-1}{f+1} = \dfrac{-1}{x}$

$f^3(x) = \dfrac{x+1}{1-x}$

$f^4(x) = x$

$f^5(x) = f\{f^4(x)\} = f(x)$

$f^{1998}(x) = f^2(x) = \dfrac{-1}{x}$.

Q.4 Find the domain of the function

(i) $f(x) = \log_3 \log_{(1/3)}(x^2 + 10x + 25) + \dfrac{1}{[x]+5}$ (where [·] denotes greatest integer function.)

(ii) $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$

Sol.

(i) $f(x) = \log_3 \log_{(1/3)}(x^2 + 10x + 25) + \dfrac{1}{[x]+5}$

$x^2 + 10x + 25 > 0 \Rightarrow (x+5)^2 > 0 \Rightarrow x \ne -5$ ....(1)

Also $\log_{(1/3)}(x^2 + 10x + 25) > 0$

$x^2 + 10x + 25 < 1$

$(x+6)(x+4) < 0$

$\Rightarrow x \in (-6, -4)$ ....(2)

From (1) and (2) $x \in (-6, -5) \cup (-5, -4)$

Also $[x] + 5 \ne 0$; $[x] \ne -5 \Rightarrow x \notin [-5, -4)$

Hence, domain of $f(x) \in (-6, -5)$.

(ii) $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$

$[x]^2 - [x] - 6 > 0 \Rightarrow ([x] - 3)([x] + 2) > 0 \Rightarrow [x] \in (3, \infty) \cup (-\infty, -2)$

$x \in [4, \infty)$ also $x \in (-\infty, -2)$

$x \in (-\infty, -2) \in [4, \infty)$.

Q.5 Find the range of following functions

(i) $f(x) = \log_3\{\log_{1/2}(x^2 + 4x + 4)\}$ (ii) $f(x) = \sin^2 x - 5\sin x - 6$.

Sol.

(i) $f(x) = \log_3\{\log_{1/2}(x^2 + 4x + 4)\}$

firstly finding the domain

$\log_{1/2}(x^2 + 4x + 4) > 0$

$x^2 + 4x + 4 < 1 \Rightarrow x^2 + 4x + 3 < 0 \Rightarrow (x + 1)(x + 3) < 0 \Rightarrow -3 < x < -1$

Also $x^2 + 4x + 4 > 0$

$(x + 2)^2 > 0 \Rightarrow x \neq -2$

Hence, $x \in (-3, -1) - \{2\}$

Since $0 < \log_{1/2}(x^2 + 4x + 4) < \infty \ \forall x \in$ Domain thus

Range $\in R$

(ii) $f(x) = \sin^2 x - 5\sin x - 6 = \sin^2 x - 2\left(\dfrac{5}{2}\right)\sin x + \dfrac{25}{4} - 6 - \dfrac{25}{4}$

$= \left(\sin x - \dfrac{5}{2}\right)^2 - \dfrac{49}{4}$

where $\dfrac{9}{4} \le \left(\sin x - \dfrac{5}{2}\right)^2 \le \dfrac{49}{4}$

Hence, $f(x) \in [-10, 0]$. **Ans.**

Q.6 Find the period of f(x) satisfying the condition

(i) $f(x-1) + f(x + 3) = f(x + 1) + f(x + 5)$

(ii) $f(x) = \{x\} + \cos \pi x$

where $\{\cdot\}$ denotes fraction part.

Sol.

(i) $f(x - 1) + f(x + 3) = f(x + 1) + f(x + 5)$ ....(1)

Replacing x by x + 2

$f(x + 1) + f(x + 5) = f(x + 3) + f(x + 7)$ ....(2)

Adding (1) and (2), we get

$f(x - 1) = f(x + 7)$ i.e. $f(x) = f(x + 8)$

Hence, period is 8.

(ii) $f(x) = \{x\} + \cos \pi x$

Period of $\{x\} = 1$

Period of $\cos \pi x = \dfrac{2\pi}{\pi} = 2$

Hence period of $f(x) = 2$.

Q.7 Let $f : R \to R$, $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find the value of α for f(x) to be onto.

Sol. $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

$\Rightarrow (\alpha + 8y)\, x^2 + 6\,(1 - y)\, x - (\alpha y + 8) = 0$

According to condition, y takes all real values for all real x,

i.e. $D \geq 0 \;\forall\; y \in R$.

$\Rightarrow 36\,(1 - y)^2 + 4\,(\alpha y + 8)\,(\alpha + 8y) \geq 0 \;\forall\; y \in R$

$\Rightarrow (9 + 8\alpha)y^2 + (\alpha^2 + 46)y + (9 + 8\alpha) \geq 0 \;\forall\; y \in R$

i.e. $D \leq 0$ and coefficient of $y^2 > 0$

$\Rightarrow (\alpha^2 + 46)^2 \leq 4\,(9 + 8\alpha)^2$ and $9 + 8\alpha > 0$

$\Rightarrow \alpha^2 - 16\alpha + 28 \leq 0$ and $\alpha > \dfrac{-9}{8}$

$\Rightarrow 2 \leq \alpha \leq 14$ and $\alpha > \dfrac{-9}{8}$

Hence, $\alpha \in [2, 14]$. **Ans.**

Q.8 If $f(x) = ax^7 + bx^3 + cx - 5$, where a, b and c are constants. If $f(-7) = 7$, then find f(7).

Sol. $f(-7) = a\,(-7)^7 + b(-7)^3 + c(-7) - 5$

$7 = -(a \cdot 7^7 + b \cdot 7^3 + c \cdot 7) - 5$

$a \cdot 7^7 + b \cdot 7^3 + c \cdot 7 = -12$

$f(7) = a \cdot 7^7 + b \cdot 7^3 + c \cdot 7 - 5 = -12 - 5 = -17$. **Ans.**

Q.9 Let f be a real valued function of real and positive argument such that $f(x) + 3x\, f\left(\dfrac{1}{x}\right) = 2\,(x + 1) \;\forall\; x > 0$, find the value of f(10099).

Sol. $f(x) + 3x\, f\left(\dfrac{1}{x}\right) = 2\,(x + 1)$ ......(1)

replacing x by $\dfrac{1}{x}$

$f\left(\dfrac{1}{x}\right) + \dfrac{3}{x} f(x) = 2\left(\dfrac{1}{x} + 1\right)$

$x\, f\left(\dfrac{1}{x}\right) + 3f(x) = 2\,(1 + x)$ ......(2)

On solving (1) and (2)

$8f(x) = 4\,(1 + x) \Rightarrow f(x) = \dfrac{x + 1}{2}$

$f(10099) = \dfrac{10100}{2} = 5050$. **Ans.**

Q.10 Find the inverse of the function $f(x) = \log_a\left(x + \sqrt{x^2+1}\right), a > 0, a \neq 1.$

Sol. Since $\sqrt{x^2+1} > \sqrt{x^2} \quad \forall\, x \in R$

Hence $x + \sqrt{x^2+1} > 0 \quad \forall\, x \in R$

f(x) is one-one onto hence invertible

$$y = \log_a\left(x + \sqrt{x^2+1}\right)$$

$$a^y = x + \sqrt{x^2+1} \qquad \ldots(i)$$

$$a^{-y} = \frac{1}{x+\sqrt{x^2+1}} = -x + \sqrt{x^2+1} \qquad \ldots(ii)$$

(i) − (ii)

$$a^y - a^{-y} = 2x \quad \Rightarrow \quad x = \frac{1}{2}\left(a^y - a^{-y}\right)$$

Hence $f^{-1}(x) = \frac{1}{2}\left(a^x - a^{-x}\right)$