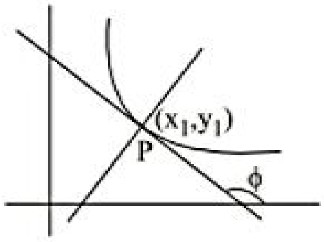
APPLICATION OF DERIVATIVE

TANGENT AND NORMAL

Define : $\tan\phi = \left.\frac{dy}{dx}\right|_P$



(1) Equation of a tangent at $P(x_1, y_1)$

$$y - y_1 = \left.\frac{dy}{dx}\right|_{x_1,y_1}(x - x_1)$$

(2) Equation of normal at (x_1, y_1)

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{x_1,y_1}}(x - x_1)\ ,\ \text{if}\ \left.\frac{dy}{dx}\right]_{x_1,y_1}\ \text{exists.}$$

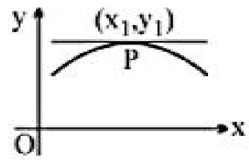
However in some cases $\frac{dy}{dx}$ fails to exist but still a tangent can be drawn e.g. case of vertical tangent.

Note that the point (x_1, y_1) must lie on the curve for the equation of tangent and normal.

Important notes to remember:

(a) If $\left.\frac{dy}{dx}\right|_{x_1,y_1} = 0 \Rightarrow$ tangent is parallel to x-axis and converse.

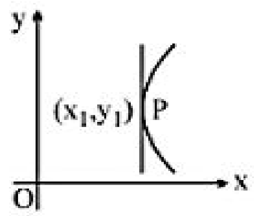
If tangent is parallel to $ax + by + c = 0 \Rightarrow \frac{dy}{dx} = -\frac{a}{b}$



(b) If $\left.\frac{dy}{dx}\right|_{x_1,y_1} \to \infty$ or $\left.\frac{dx}{dy}\right|_{x_1,y_1} = 0 \Rightarrow$ tangent is perpendicular to x-axis.

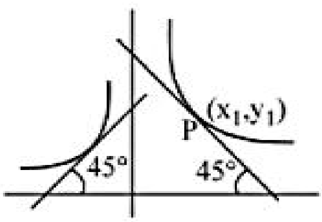
If tangent with a finite slope is perpendicular to $ax + by + c = 0$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x_1,y_1} \cdot \left(-\frac{a}{b}\right) = -1.$$



(3) If the tangent at $P(x_1, y_1)$ on the curve is equally inclined to the coordinate axes

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x_1,y_1} = \pm 1.$$



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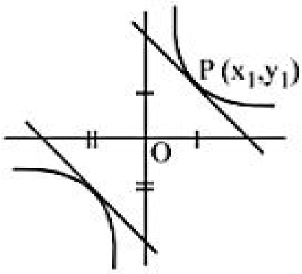
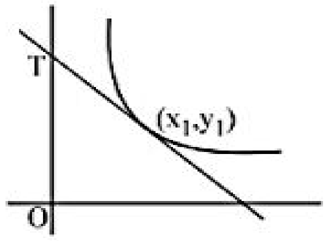
(4) If the tangent makes equal non zero intercept on the coordinate axes then $\left.\frac{dy}{dx}\right|_{x_1,y_1} = -1$



(5) If tangent cuts off from the coordinate axes equal distance from the origin $\Rightarrow \frac{dy}{dx} = \pm 1$.

(6) OT is called the initial ordinate of the tangent



$$Y - y = \frac{dy}{dx}(X - x)$$

put $X = 0$ to get

$\therefore \quad Y = OT = y - x\frac{dy}{dx}$ (It is the y intercept of a tangent at P)

(7) **Concept:** $F(x)=f(x)\cdot g(x)$ are such that $f(x)$ is continuous at $x = a$ and $g(x)$ is differentiable at $x = a$ with $g(a)=0$ then the product function $f(x)\cdot g(x)$ is diffenentiable at $x = a$.

Illustration :

Find the tangent and normal for $x^{2/3} + y^{2/3} = 2$ *at (1, 1).*

Sol. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2 \Rightarrow \frac{2}{3}\left(x^{\frac{-1}{3}} + y^{\frac{-1}{3}}y'\right) = 0$ or $y' = -\left(\frac{x}{y}\right)^{\frac{-1}{3}}$

At (1, 1) $\quad y' = -1$

Equation of tangent $\quad y - 1 = -1(x - 1) \quad \Rightarrow \quad x + y = 2$

Equation of normal $\quad y - 1 = 1(x - 1) \quad \Rightarrow \quad x - y = 0$

Illustration :

Find tangent to $x = a\sin^3 t$ *and* $y = a\cos^3 t$ *at* $t = \pi/2$.

Sol. $x = a\sin^3 t; \quad y = a\cos^3 t$

$$\frac{dy}{dx} = \frac{-3a\cos^2 t\sin t}{3a\sin^2 t\cos t} = -\cot t$$

At $\quad t = \frac{\pi}{2},\ \frac{dy}{dx} = 0 \quad$ *point is (a, 0)*

$\therefore \quad$ *Equation of tangent* $\Rightarrow y = 0$

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Illustration :

A curve in the plane is defined by the parametric eqautions $x = e^{2t} + 2e^{-t}$ *and* $y = e^{2t} + e^{t}$. *An equation for the line tangent to the curve at the point* $t = \ln 2$ *is*
(A) $5x - 6y = 7$ (B) $5x - 3y = 7$ (C) $10x - 7y = 8$ (D) $3x - 2y = 3$

Sol. $x = e^{2t} + 2e^{-t}, y = e^{2t} + e^{t}$

At $t = \ln 2$ $x = 4 + 1 = 5,\ y = 4 + 2 = 6$

$$\frac{dy}{dx} = \frac{2e^{2t} + e^{t}}{2e^{2t} - 2e^{-t}} = \frac{8+2}{8-1} = \frac{10}{7} \Rightarrow \text{equation of tangent is } y - 6 = \frac{10}{7}(x - 5)$$

$7y - 42 = 10x - 50$ or $10x - 7y = 8$

Illustration :

Equation of the normal to the curve $x^2 = 4y$ *which passes through (1, 2).*

Sol. $x^2 = 4y$ $2x = 4y'$

$$y' = \frac{x_1}{2} \quad \& \quad y_1 = \frac{x_1^2}{4}$$

Normal : $$y - y_1 = \frac{-2}{x_1}(x - x_1) \quad \text{or} \quad y - \frac{x_1^2}{4} = \frac{-2}{x_1}(x - x_1)$$

It passes through (1, 2)

$$2 - \frac{x_1^2}{4} = \frac{-2}{x_1}(1 - x_1) = -\frac{2}{x_1} + 2$$

$$x_1^3 = 8 \Rightarrow x_1 = 2 \quad \& \quad y_1 = \frac{x_1^2}{4} = 1$$

$$\therefore \text{ normal is } \quad y - 1 = \frac{-2}{2}(x - 2) = 2 - x$$

$$x + y = 3$$

Illustration :

Curve $C_1 : y = ex \ln x$ *and* $C_2 : y = \frac{\ln x}{ex}$ *intersect at point 'P' whose abscissa is less than 1. Find equation of normal to curve* C_1 *at point P.*

Sol. For point of intesection $ex \ln x = \frac{\ln x}{ex} \Rightarrow \ln x = 0$ or $e^2x^2 = 1$

$$\Rightarrow x = 1 \text{ or } x = \pm\frac{1}{e} \quad \text{but} \quad 0 < x < 1$$

$$\Rightarrow \text{Point P is } \left(\frac{1}{e}, -1\right)$$

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For curve C_1, $\frac{dy}{dx} = e(1 + \ln x)$ $\Rightarrow$ *Slope of tangent at point P is equal to* $e\left(1+\ln\frac{1}{e}\right)=0$

$\Rightarrow$ *Equation of normal is* $x = \frac{1}{e}$

Illustration :

A line is drawn touching the curve $y - \frac{2}{3-x} = 0$. *Find the line if its slope/gradient is 2.*

Sol. $y=\frac{2}{3-x}$ $\Rightarrow$ $y'=\frac{2}{(3-x)^2}$ *or* $\frac{2}{(3-x_1)^2}=2$

$\Rightarrow$ $(3-x_1)^2 = 1$

or $3 - x_1 = 1, -1$

or $x_1 = 2, 4$ $\quad\therefore$ *equation of line can be*

$\Rightarrow$ $y_1 = 2, -2$ $\quad y - 2 = 2(x - 2)$

or $y + 2 = 2(x - 4)$

SOME COMMON PARAMETRIC COORDINATES ON A CURVE:

(a) for $x^{2/3}+y^{2/3}=a^{2/3}$ take parametric coordinate $x=a\cos^3\theta$ & $y=a\sin^3\theta$.

(b) for $\sqrt{x}+\sqrt{y}=\sqrt{a}$ take $x=a\cos^4\theta$ & $y=a\sin^4\theta$.

(c) $\frac{x^n}{a^n}+\frac{y^n}{b^n}=1$ taken $x=a(\sin\theta)^{2/n}$ & $y=b(\sin\theta)^{2/n}$.

(d) for $c^2(x^2+y^2)=x^2y^2$ take $x=c\sec\theta$ and $y=c\,\mathrm{cosec}\theta$.

(e) for $y^2=x^3$, take $x=t^2$ and $y=t^3$.

Note: The tangent at P meeting the curve again at Q.

$$\Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{y_2-y_1}{x_2-x_1}$$

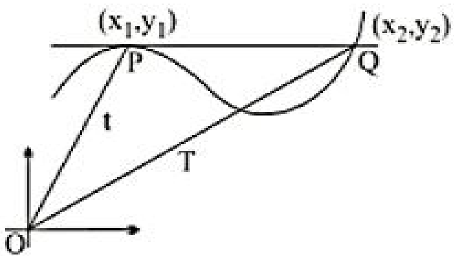

Illustration :

Tangent at point P on the curve $y^2 = x^3$ *meets the curve again at point Q. Find* $\frac{m_{OP}}{m_{OQ}}$, *where O is origin.*

Sol. *Take* $P(t^2, t^3)$ *and* $Q(T^2, T^3)$

$\frac{dy}{dx}=\frac{3x^2}{2y}$ *or* $\left(\frac{dy}{dx}\right)=\frac{3}{2}t$.

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Slope line joining P and Q is = $\dfrac{T^3-t^3}{T^2-t^2}=\dfrac{T^2+t^2+Tt}{T+t}$

$\Rightarrow \quad \dfrac{3}{2}t=\dfrac{T^2+t^2+Tt}{T+t}$ *or* $3tT+3t^2=2T^2+2t^2+2Tt \;\Rightarrow\; T=\dfrac{-t}{2} \;\Rightarrow\; \dfrac{m_{OP}}{m_{OQ}}=-2.$

Illustration :

The equation(s) of the straight lines which is (are) tangents at one point and normal at another point of the curve $x = 3t^2$, $y = 2t^3$ *is(are)*

(A) $\sqrt{3}x+y=2\sqrt{2}$ (B) $\sqrt{3}x-y=2\sqrt{2}$ (C) $\sqrt{2}x-y=2\sqrt{2}$ (D) $\sqrt{2}x+y=2\sqrt{2}$

Sol. *Parametric equation of the given curve is* $x = 3t^2$, $y = 2t^3$ ....(1)

$\therefore \quad \dfrac{dx}{dt}=6t$ ....(2)

and $\dfrac{dy}{dt}=6t^2$ ....(3)

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=t.$$

Let $A \equiv (3t_1^2, 2t_1^3)$, $B \equiv (3t_2^2, 2t_2^3)$

Let line AB be a tangent to curve (1) at A and normal to the curve at B, then

$t_1=\dfrac{-1}{t_2}$ *or* $t_1t_2=-1$ ....(4)

Also slope of AB = $\dfrac{2(t_1^3-t_2^3)}{3(t_1^2-t_2^2)}=\dfrac{2}{3}\,\dfrac{t_1^2+t_2^2+t_1t_2}{t_1+t_2}$

Since AB is tangent to the curve at A,

$\therefore \quad \dfrac{2}{3}\left(\dfrac{t_1^2+t_2^2-1}{t_1+t_2}\right)=t_1 \;\Rightarrow\; 2t_1^2+2t_2^2-2=3t_1^2-3$

$\Rightarrow \quad t_1^2-2t_2^2-1=0 \Rightarrow t_1^2-\dfrac{2}{t_1^2}-1=0$ $[\because t_1t_2=-1]$

$\Rightarrow \quad k^2-k-2=0$

$\therefore \quad k=2,-1,$ *where* $k=t_1^2$

$\Rightarrow \quad t_1^2=2 \Rightarrow t_1=\pm\sqrt{2}$

$\therefore \quad t_2=\mp\dfrac{1}{\sqrt{2}}$

Hence, $t_1=\sqrt{2},\ t_2=-\dfrac{1}{\sqrt{2}}$; $t_1=-\sqrt{2},\ t_2=\dfrac{1}{\sqrt{2}}$

$\therefore \quad A\equiv\left(6,4\sqrt{2}\right)$ *or* $A\equiv\left(6,-4\sqrt{2}\right).$

$\therefore$ *Equation of AB i.e., equation of tangent at A is*

$y-4\sqrt{2}=\sqrt{2}(x-6)$ *i.e.* $\sqrt{2}x-y-2\sqrt{2}=0$

or $y+4\sqrt{2}=-\sqrt{2}(x-6)$ *i.e.* $\sqrt{2}x+y-2\sqrt{2}=0.$

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Practice Problem

Q.1 Find the equation of tangent and normal to the curve $f(x) = \begin{cases} x-2 & \text{if } x<1 \\ x^2-x-1 & \text{if } x \ge 1 \end{cases}$ at $x=1$ if it exists.

Q.2 Find the points on the curve $y = x^2 - x^2 - x + 3$, where the tangent is parallel to x-axis.

Q.3 Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ is tangent to the curve, $y = be^{-x/a}$ at the point where the curve crosses the axis of y.

Q.4 For the curve $y = 4x^3 - 2x^5$, find all points at which the tangent passes through the origin.

Q.5 The tangent at any point on the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ meets the axes in P and Q. Prove that the locus of the mid point of PQ is a circle.

Answer key

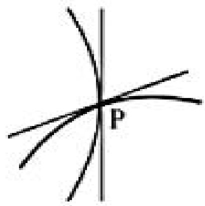
Q.1 T: $x - y - 2 = 0$; $x + y = 0$ Q.2 $(1, 2)$ and $\left(-\frac{1}{3}, 3\frac{5}{27}\right)$

Q.4 $(0, 0)$ and $(-1, -2)$

ANGLE OF INTERSECTION OF TWO CURVES :

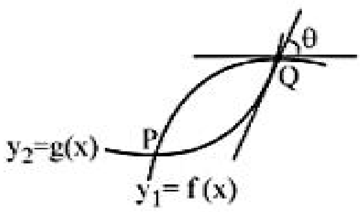
Definition :

The angle of intersection of two curves at a point P is defined as the angle between the two tangents to the curve at their point of intersection.



If the curves are orthogonal then

$\left(\frac{dy_1}{dx}\right)\left(\frac{dy_2}{dx}\right) = -1$ everywhere whereever they intersect.



If $\left(\frac{dy_1}{dx}\right)_P \left(\frac{dy_2}{dx}\right)_P = -1$ but $\left(\frac{dy_1}{dx}\right)_Q \left(\frac{dy_2}{dx}\right)_Q \ne -1$

then the two curves are orthogonal at P but not at Q hence they are not orthogonal.

e.g. $y^2 = 4ax$ & $y = e^{-x/2a}$; $xy = a^2$ & $x^2 - y^2 = b^2$ and $y = ax$ & $x^2 + y^2 = c^2$ are orthogonal but $y^2 = 4ax$ and $x^2 = 4by$ are not orthogonal.

Note : If the curves touch at $P(x_1, y_1)$ then $\theta = 0$ hence $f'(x_1) = g'(x_1)$

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Illustration :

Find the acute angle between the curve $y = \sin x$ & $y = \cos x$.

Sol. $y = \sin x$ & $y = \cos x$

Intersection point is $x = \frac{\pi}{4}$; $y = \frac{1}{\sqrt{2}}$

$y'_1 = \cos x = \frac{1}{\sqrt{2}}$ $\quad y'_2 = -\sin x = -\frac{1}{\sqrt{2}}$

$$\tan\theta = \left|\frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 - \frac{1}{2}}\right| = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(2\sqrt{2}\right)$$

Illustration :

If θ *is the angle between* $y = x^2$ *and* $6y = 7 - x^3$ *at* (a, a). *Find* θ.

Sol. $y = x^2$ and $6y = 7 - x^3$ Point (a, a) is $(1, 1)$

$y'_1 = 2x$ and $y'_2 = \frac{-x^2}{2}$

$y'_1 \times y'_2 = -1$

$\therefore \quad \theta = \frac{\pi}{2}$

Illustration :

Find the condition for the two concentric ellipses $a_1x^2 + b_1y^2 = 1$ *and* $a_2x^2 + b_2y^2 = 1$ *to intersect orthogonally.*

Sol. $a_1x^2 + b_1y^2 = 1$ $\quad$ Let curves intersect at (x_0, y_0) then

$a_2x^2 + b_2y^2 = 1$ $\quad (a_1 - a_2)x_0^2 = (b_2 - b_1)y_0^2$ ...(1)

For 1^{st} curve $2a_1x + 2b_1yy' = 0$, $\quad y' = \frac{-a_1x_0}{b_1y_0}$

Similarly for 2^{nd} curve $y' = -\frac{a_2}{b_2}\frac{x_0}{y_0}$

For orthogonal intersection $-\left(\frac{a_1}{b_1}\frac{a_2}{b_2}\frac{x_0^2}{y_0^2}\right) = -1$

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$$\frac{a_1a_2}{b_1b_2}\left(\frac{b_2-b_1}{a_1-a_2}\right)=-1 \qquad \text{[From relation (1)]}$$

$$\frac{b_2-b_1}{b_1b_2}=\frac{a_2-a_1}{a_1b_2}$$

$$\therefore \quad \frac{1}{b_1}-\frac{1}{b_2}=\frac{1}{a_1}-\frac{1}{a_2}$$

Illustration :

Match the following :

Column-I		Column-II	
(A)	If the parabola $y^2 = 4ax$, $a > 0$ cuts the hyperbola $xy = \sqrt{2}$ at right angles, then a =	(P)	$4\sqrt{2}$
(B)	If the curves $ay + x^2 = 7$, $a > 0$ and $x^3 = y$ cut orthogonally at (1, 1), then a =	(Q)	$2\sqrt{2}$
(C)	If the curves $y^2 = 4x$ and $xy = a$, $a > 0$ cut orthogonally, then a =	(R)	$\frac{1}{2}$
(D)	Curves $2x = y^2$ and $2xy = a$, $a > 0$ cut each other at right angles, then a =	(S)	6

[Ans. (A) R, (B) S, (C) P, (D) Q]

Sol.

(A) Given curves are, $y^2 = 4ax$ .....(1)

and $xy = \sqrt{2}$ .....(2)

From (1), $2y\frac{dy}{dx}=4a \quad \therefore \quad \frac{dy}{dx}=\frac{2a}{y}$ ......(3)

From (2), $y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x}$ ....(4)

Putting the value of y from (2) in (1), we get $\frac{2}{x^2} = 4ax \Rightarrow x^3 = \frac{1}{2a}$ .....(5)

For curves (1) and (2) to cut at right angles, $\left(\frac{2a}{y}\right)\left(\frac{y}{x}\right) = -1$

$\Rightarrow \quad 2a = x \qquad 8a^3 = x^3 = \frac{1}{2a}$ [From (5)]

$\Rightarrow \quad 16a^4 = 1 \Rightarrow a = \frac{1}{2}$ [$\because a > 0$]

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(B) Given curves are $ay + x^2 = 7$ ....(1)

and $y = x^3$ ....(2)

From (1), $\frac{dy}{dx} = \frac{-2x}{a}$ ....(3)

From (2), $\frac{dy}{dx} = 3x^2$ ....(4)

For curves (1) and (2) to cut each other orthogonally at (1, 1),

$$\left(-\frac{2}{a}\right) \cdot 3 = -1 \;\Rightarrow\; a = 6.$$

(C) Given curves are, $y^2 = 4x$ ....(1)

and $xy = a$ ....(2)

From (1), $2y\frac{dy}{dx} = 4$

$\therefore \quad \frac{dy}{dx} = \frac{2}{y}$ ......(3)

From (2), $1 \cdot y + x\frac{dy}{dx} = 0 \quad \therefore \quad \frac{dy}{dx} = -\frac{y}{x}$ ....(4)

Putting the value of y from (2) in (1), we get

$$\frac{a^2}{x^2} = 4x \;\Rightarrow\; a^2 = 4x^3$$ ....(5)

From (2), $y = \frac{a}{x} \Rightarrow \frac{dy}{dx} = \frac{-a}{x^2}$ $\quad [\because$ from (2), $y = \frac{a}{x}]$

$$\left(\frac{dy}{dx}\right)_{any\ curve\ (1)} \cdot \left(\frac{dy}{dx}\right)_{for\ curve\ (2)} = \frac{2}{y} \cdot \frac{-a}{x^2} = \frac{-2}{x}$$ ......(6)

For curves (1) and (2) to cut each other orthogonally,

$$\frac{-2}{x} = -1 \;\Rightarrow\; x = 2.$$ [From (6)]

$\therefore$ From (5), $a = 4\sqrt{2}$ $\quad [\because a > 1]$

(D) Given curves are, $y^2 = 2x$ ....(1)

and $xy = \frac{a}{2}$ ....(2)

From (1), $\frac{dy}{dx} = \frac{1}{y}$ ....(3)

From (2), $y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ ....(4)

$$\left(\frac{dy}{dx}\right)_{for\ curve\ (1)} \cdot \left(\frac{dy}{dx}\right)_{for\ curve\ (2)} = \frac{-1}{x}$$ ....(5)

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Putting the value of y from (2) in (1), we get

$$\frac{a^2}{4x^2} = 2x \Rightarrow 8x^3 = a^2 \qquad ....(6)$$

For the two curves to cut each other at ight angles,

$$-\frac{1}{x} = -1 \quad \Rightarrow x = 1$$

$\therefore$ From (6), $a^2 = 8 \quad \Rightarrow \quad a = 2\sqrt{2}$.

Illustration :

Find the angle of intersection of curves $y = [\,|\sin x| + |\cos x|\,]$ and $x^2 + y^2 = 5$, where [ ] denotes the greatest integer function.

Sol. Given curves are $y = [\,|\sin x| + |\cos x|\,]$ ......(1)

and $x^2 + y^2 = 5$ ......(2)

Let $|\sin x| = t$, then $|\cos x| = \sqrt{1-t^2}$ and $0 \le t \le 1$.

Now $|\sin x| + |\cos x| = t + \sqrt{1-t^2}$

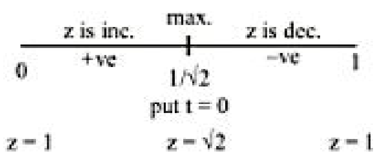
Let $z = t + \sqrt{1-t^2}$, $0 \le t \le 1$

Then $\dfrac{dz}{dt} = 1 - \dfrac{t}{\sqrt{1-t^2}} = \dfrac{\sqrt{1-t^2} - t}{\sqrt{1-t^2}}$

Since $\sqrt{1-t^2} > 0$, therefore sign scheme for $\dfrac{dz}{dt}$ will be same as that of $\sqrt{1-t^2} - t$.

Now $\sqrt{1-t^2} - t = 0 \Rightarrow 1 - t^2 = t^2 \Rightarrow t = \dfrac{1}{\sqrt{2}}$ $\qquad [\because t > 0]$

Sign scheme for $\left(\sqrt{1-t^2} - t\right)$ is



$\therefore$ Greatest value of $z = \sqrt{2}$

and least value of $z = 1$

$\therefore$ $1 \le z \le \sqrt{2}$ $\quad \therefore \quad [z] = 1$

$\therefore$ curve (1) becomes $y = 1$ ....(3)

Putting $y = 1$ in (2), we get $x^2 = 4$

$\therefore$ $x = \pm 2$

Hence points of intersection of curves (1) and (2) are $P(-2, 1)$ and $Q(2, 1)$

From (2), $\dfrac{dy}{dx} = -\dfrac{x}{y} = \begin{cases} 2, \text{ at } P(-2,1) \\ -2, \text{ at } Q(2,1) \end{cases}$

Clearly line (3) is parallel to x-axis. Its slope $m_1 = 0$

At $P(-2, 1)$, $m_1 = 0$ and $m_2 = 2$.

$\therefore$ Acute angle between the two curves at $P = \tan^{-1}\left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \tan^{-1} 2$.

Similarly acute angle between the two curves At $Q(-2, 0)$ is $\tan^{-1} 2$.

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Practice Problem

Q.1 Find the angle between the curve $2y^2 = x^3$ and $y^2 = 32x$

Q.2 Show that the curves $x^3 - 3xy^2 = a$ and $3x^2y - y^3 = b$ cut each other orthogonally, where a and b are constants.

Q.3 Find the acute angles between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection.

Q.4 Curves $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ cut each other at right angle then find the value of a^2.

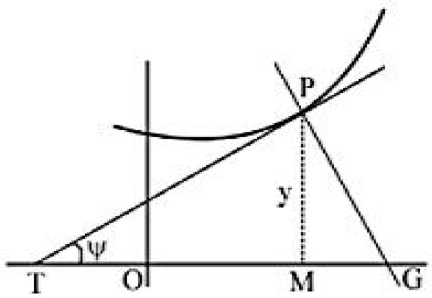
Answer key

Q.1 $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ Q.3 $\theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ Q.4 $a^2 = \frac{4}{3}$

LENGTH OF TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL:

(i) **Tangent :**

$$PT = MP \operatorname{cosec}\psi = y\sqrt{1+\cot^2\psi} = \left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}\right|$$



(ii) **Subtangent :** $TM = MP\cot\psi = \left|\frac{y}{(dy/dx)}\right|$

(iii) **Normal :** $GP = MP\sec\psi = y\sqrt{1+\tan^2\psi} = \left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$

(iv) **Subnormal :** $MG = MP\tan y = \left|y\left(\frac{dy}{dx}\right)\right|$

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Illustration :

Show that for the curve $by^2 = (x + a)^3$ the square of the subtangent varies as the subnormal.

Sol. $by^2 = (x + a)^3$ or $2byy' = 3(x + a)^2$

$$S.T. = \frac{y}{y'} = \frac{y}{3(x+a)^2}2by = \frac{2by^2}{3(x+a)^2} = \frac{2(x+a)}{3}$$

$$S.N. = yy' = y\frac{3(x+a)^2}{2by} = \frac{3(x+a)^2}{2b} \Rightarrow ST^2 \propto SN$$

Illustration :

Show that at any point on the hyperbola $xy = c^2$, the subtangent varies as the abscissa and the subnormal varies as the cube of the ordinate of the point of contact.

Sol. $xy = c^2 \Rightarrow xy' + y = 0$ or $y' = \frac{-y}{x}$

$$ST = \frac{y}{y'} = -x\,,\ SN = yy' = \frac{-y^2}{x} = \frac{-y^2}{c^2}y = -\frac{-y^3}{c}$$

APPROXIMATION AND DIFFERENTIALS :

For the figure it is clear that if Δx and Δy are sufficiently small quantities then

$$\frac{\Delta y}{\Delta x} = \tan\psi \cong \frac{dy}{dx} = f'(x)$$

Hence approximate change in the value of y, called its differential is given by

$$\Delta y = f'(x) \cdot \Delta x \qquad ....(1)$$

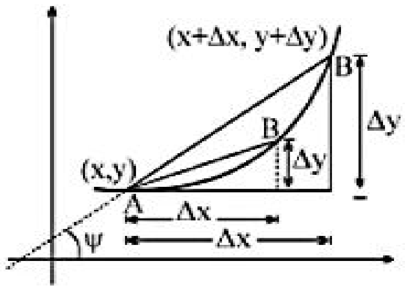

Illustration :

Use differential to approximate $\sqrt{101}$.

Sol. Let $f(x) = \sqrt{x}$ $\quad f(100) = 10$ & $\quad f'(x) = \frac{1}{2\sqrt{x}}$

$$\Delta y = \frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05 \Rightarrow f(101) = 10 + 0.05 = 10.05$$

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INTERPRETATION OF $\frac{dy}{dt}$ AS A RATE MEASURE :

Recall that by the derivative $\frac{ds}{dt}$, we mean the rate of change of distance s with respect to the time t. In a similar fashion, whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left(\frac{dy}{dx}\right)_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

Illustration :

Displacement s' of a particle at time t is expressed as $s = \frac{1}{2}t^3 - 6t$, find the acceleration at the time when the velocity vanishes (i.e., velocity tends to zero).

Sol. $s = \frac{1}{2}t^3 - 6t$

Thus velocity, $v = \frac{ds}{dt} = \left(\frac{3t^2}{2} - 6\right)$

and acceleration, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 3t$

velocity vanishes when $\frac{3t^2}{2} - 6 = 0$

$\Rightarrow \quad t^2 = 4 \qquad \Rightarrow \quad t = 2$

Thus, the acceleration when the velocity vanishes, is $a = 3t = 6$ units.

Illustration :

On the curve $x^3 = 12y$, find the interval of values of x for which the abscissa changes at a faster rate than the ordinate?

Sol. *Given $x^3 = 12y$, differentiating w.r.t. y*

$$3x^2 \frac{dx}{dy} = 12$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2}{4}$$

Now abscissa changes at a faster rate than the ordinate, then we must have $\left|\frac{dy}{dx}\right| < 1$

$\Rightarrow \quad |x^2| < 4, \; x \neq 0$

$\Rightarrow \quad -2 < x < 2, \; x \neq 0$

$\Rightarrow \quad x \in (-2, 2) - \{0\}$

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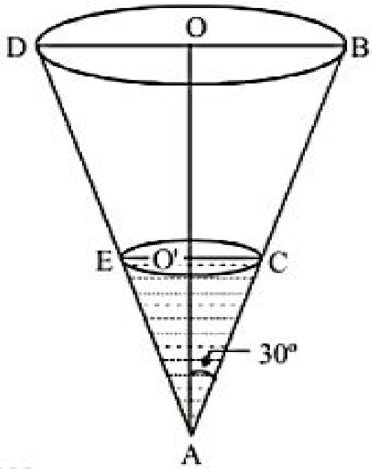
Illustration :

If water is poured into an inverted hollow cone whose semi-vertical angle is 30°, such that its depth (measured along axis) increases at the rate of 1 cm per sec, find the rate at which the volume of water increases when the depth is 24 cm.

Sol. Let A be the vertex and AO the axis of the cone.
Let O'A = h be the depth of water in the cone.

In $\Delta AO'C$, $\tan 30^o = \frac{O'C}{h}$

$\Rightarrow \quad O'C = \frac{h}{\sqrt{3}}$



V = volumeof water in the cone $= \frac{1}{3}\pi (O'C)^2 \times AO'$

$$= \frac{1}{3}\pi\left(\frac{h^2}{3}\right)\times h$$

$\Rightarrow \quad V = \frac{\pi}{9}h^3 \quad \Rightarrow \quad \frac{dV}{dt} = \frac{\pi}{3}h^2\frac{dh}{dt} \qquad ...(i)$

But given that depth of water increases at the rate of 1 cm/sec

$\Rightarrow \quad \frac{dh}{dt} = 1$ cm/s $\qquad ...(ii)$

From (i) and (ii), $\quad \frac{dV}{dt} = \frac{\pi h^2}{3}$

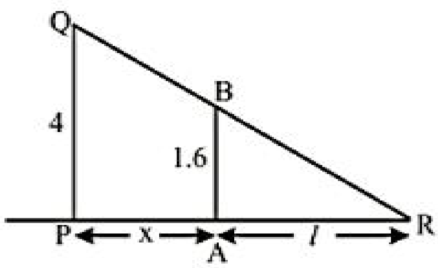
When h = 24cm, the rate of increase of volume $\frac{dV}{dt} = \frac{\pi(24)^2}{3} = 192$ cm³/s

Illustration :

A man 1.6 m high walks at the rate of 30 metre per minute away from a lamp which is 4m above ground. How fast does the man's shadow lengthen?

Sol. Let PQ = 4m be the height of pole and AB = 1.6 m be the height of the man.
Let the end of a shadow is R and it is at a distance of l from A when the man is at a distance x from PQ at some instant.

Since, ΔPQR and ΔABR are similar, $\frac{PQ}{AB} = \frac{PR}{AR}$



$\Rightarrow \quad \frac{4}{1.6} = \frac{x+l}{l}$

$\Rightarrow \quad 2x = 3l$

$\Rightarrow \quad 2\frac{dx}{dt} = 3\frac{dl}{dt} \qquad \left[\text{given } \frac{dx}{dt} = 30\ m/min\right]$

$\Rightarrow \quad \frac{dl}{dt} = \frac{2}{3}.30$ m/min = 20 m/min.

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Illustration :

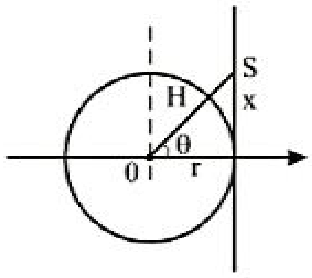
A horse runs along a circle with a speed of 20 km/hr. A lantern is at the centre of the circle. A fence is along the tangent to the circle at the point at which the horse starts. The speed with which the shadow of the horse moves along the fence at the moment when it covers 1/8 of the circle in km/hr is

(A) 20 *(B) 40* *(C) 30* *(D) 60*

Sol. $\tan\theta = \dfrac{x}{r} \quad \Rightarrow \quad x = r\tan\theta$

H is position of horse and S is its shadow on fence

$$\Rightarrow \quad \frac{dx}{dt} = r\sec^2\theta\left(\frac{d\theta}{dt}\right) = r\omega\sec^2\theta = v\sec^2\theta$$



$$\text{where } \theta = \frac{2\pi}{8}, \quad \frac{dx}{dt} = v\sec^2\left(\frac{\pi}{4}\right) = 2v = 40 \text{ km/hr},$$

Practice Problem

Q.1 A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x coordinate.

Q.2 An open Can of oil is accidently dropped into a lake; assume the oil spreads over the surface as a circular disc of uniform thickness whose radius increases steadily at the rate of 10 cm/sec. At the moment when the radius is 1 meter, the thickness of the oil slick is decreasing at the rate of 4 mm/sec, how fast is it decreasing when the radius is 2 meters.

Q.3 A circular ink blot grows at the rate of 2 cm^2 per second. Find the rate at which the radius is increasing after $2\dfrac{6}{11}$ seconds. Use $\pi = \dfrac{22}{7}$.

Q.4 If in a triangle ABC, the side 'c' and the angle 'C' remain constant, while the remaining elements are changed slightly, show that $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} = 0$.

Q.5 Using differentials, find the approximate value of $\sqrt{0.037}$, correct upto thre decimal places.

Answer key

Q.1	$(4, 11)$ & $(-4, -31/3)$	Q.2	0.05 cm/sec
Q.3	$\frac{1}{4}$ cm/sec.	Q.5	0.1924

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Solved Examples

Q.1 Using differentials, find the approximate value of $(82)^{1/4}$ upto 3 places of decimal.

Sol. Let $f(x) = x^{1/4}$

$\therefore \quad f'(x) = \frac{1}{4}(x)^{-\frac{3}{4}} = \frac{1}{4x^{3/4}}$

Also, $f(x + \delta x) = (x + \delta x)^{1/4}$

Now, $f(x + \delta x) = f(x) + \delta x\, f'(x)$ (approximately)

$\Rightarrow \quad (x + \delta x)^{1/4} = x^{1/4} + \delta x \cdot \frac{1}{4x^{3/4}}$

We have to find $(82)^{1/4}$ and we know the value of $(81)^{1/4}$ which is equal to 3

$\therefore$ Putting $x = 81$, $x + \delta x = 82$ so that $dx = 1$ in (4), we get

$$(82)^{1/4} = (81)^{1/4} + 1 \cdot \frac{1}{4(81)^{3/4}} = 3 + \frac{1}{4 \times 3^3} = 3 + \frac{1}{108} = 3.009.$$

Q.2 The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$ and cuts the y axis at a point Q where its slope is 3. Find a, b and c.

Sol. Since the curve $y = ax^3 + bx^2 + cx + 5$ touches x-axis at $P(-2, 0)$ then x-axis is the tangent at $(-2, 0)$. The curve meets y-axis in (0, 5). We have

given where is $y = ax^3 + bx^2 + cx + 5$ ....(1)

$\therefore \quad \left(\frac{dy}{dx}\right)_{at\ (0,5)} = 3 \Rightarrow 0 + 0 + c = 3$

$\therefore \quad c = 3$ ....(2)

Since x-axis touches the curve at $P(-2, 0)$

$\therefore \quad \left(\frac{dy}{dx}\right)_{at\ (-2,0)} = 0$

$\Rightarrow \quad 12a - 4b + c = 0$

$\Rightarrow \quad 12a - 4b + 3 = 0$ [From (2)] ....(3)

Also $(-2, 0)$ lies on curve (1)

$\therefore \quad 0 = -8a + 4b - 2c + 5 \Rightarrow 0 = -8a + 4b - 1$ [$\because c = 3$]

$\Rightarrow \quad 8a - 4b + 1 = 0$ ....(4)

From (3) and (4) we get $a = -\frac{1}{2}, b = -\frac{3}{4}$.

Hence $a = -\frac{1}{2}$ $b = \frac{3}{4}$ anc $c = 3$.

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Q.3 Show that angle between the tangents to the ellise $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b and the circle $x^2+y^2=ab$ at a point of intersection is $\tan^{-1}\frac{a-b}{\sqrt{ab}}$.

Sol. Given curves are $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ....(1)

and $x^2+y^2=ab$ ....(2)

Putting the value of y^2 from (2) in (1), we get

$$\frac{x^2}{a^2}+\frac{ab-x^2}{b^2}=1$$

or, $b^2x^2+a^3b-a^2x^2=a^2b^2$

or $(b^2-a^2)x^2=a^2b^2-a^3b=a^2b\,(b-a)$

or, $(b+a)x^2=a^2b$

$\therefore\quad x=\pm\sqrt{\frac{a^2b}{a+b}}=\pm a\sqrt{\frac{b}{a+b}}$

From (2), $y^2=ab-x^2=ab-\frac{a^2b}{a+b}=\frac{ab^2}{a+b}$

$\therefore\quad y=\pm b\sqrt{\frac{a}{a+b}}$

This points of intersection of curves (1) and (2) are

$P\left(a\sqrt{\frac{a}{a+b}},b\sqrt{\frac{a}{a+b}}\right)$, $Q\left(-a\sqrt{\frac{a}{a+b}},-b\sqrt{\frac{a}{a+b}}\right)$,

$R\left(-a\sqrt{\frac{b}{a+b}},b\sqrt{\frac{a}{a+b}}\right)$ and $S\left(a\sqrt{\frac{b}{a+b}},-b\sqrt{\frac{a}{a+b}}\right)$

Angle between the two curves at $P\left(a\sqrt{\frac{b}{a+b}},b\sqrt{\frac{a}{a+b}}\right)$:

From (1), $\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$; $\quad\therefore\quad \left(\frac{dy}{dx}\right)_{at\ P}=-\frac{b^2}{a^2}\cdot\frac{a}{b}\cdot\sqrt{\frac{b}{a}}=-\left(\frac{b}{a}\right)^{\frac{3}{2}}=m_1$ (say)

From (2), $\frac{dy}{dx}=-\frac{x}{y}$; $\quad\therefore\quad \left(\frac{dy}{dx}\right)_{at\ P}=-\frac{a}{b}\sqrt{\frac{b}{a}}=-\sqrt{\frac{a}{b}}=m_2$ (say)

If θ be the acute angle between curves (1) and (2) at P, then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\frac{-\left(\frac{b}{a}\right)^{\frac{3}{2}}+\sqrt{\frac{a}{b}}}{1+\frac{b}{a}}=\frac{a^2-b^2}{a^{\frac{3}{2}}\sqrt{b}}\ \frac{a}{a+b}=\frac{a-b}{\sqrt{ab}}$$

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$\therefore \quad \theta = \tan^{-1}\dfrac{a-b}{\sqrt{ab}}$

Acute angle between the two curves at Q, R, S will be each $\tan^{-1}\dfrac{a-b}{\sqrt{ab}}$.

Q.4 Tangent at a point P_1 (other than (0, 0) on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve again at P_3 and so on. Show that the abscissae of $P_1, P_2, ..., P_n$ form a G.P. Also find the ratio $\dfrac{\text{area}(\Delta P_1P_2P_3)}{\text{area}(\Delta P_2P_3P_4)}$.

Sol. Given curve is $y = x^3$ ....(1)

$\therefore \quad \dfrac{dy}{dx} = 3x^2$ ....(2)

Let P_1 (t_1, t_1^3) be a point on curve $y = x^3$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{\text{at}\,(t_1,\,t_1^3)} = 3t_1^2$

Equation of tangent at P_1 is $y - t_1^3 = 3t_1^2(x - t_1)$ .....(3)

Putting the value of y from (1) in (2), we get

$\Rightarrow \quad x^3 - t_1^3 = 3t_1^2(x - t_1)$

$\Rightarrow \quad (x - t_1)(x^2 + xt_1 + t_1^2) - 3t_1^2(x - t_1) = 0$

$\Rightarrow \quad (x - t_1)^2(x + 2t_1) = 0$

If P_2 (t_2, t_2^3), then $(t_2 - t_1)^2(t_2 + 2t_1) = 0$

$\therefore \quad t_2 = -2t_1 (t_2 \neq t_1)$

Similarly, the tangent at P_2 will meet the curve at the point $P_3(t_3, t_3^3)$, where $t_3 = -2t_2 = 4t_1$ and so on.

Thus the abscissae of $P_1, P_2, ..., P_n$ are

$t_1, -2t_1, 4t_1, ..., (-2)^{n-1}t_1$, which are in G.P.

$\therefore \quad \dfrac{t_2}{t_1} = \dfrac{t_3}{t_2} = \dfrac{t_4}{t_3} = ... = -2 (r \text{ say})$

$\therefore \quad t_2 = t_1 r, t_3 = t_2 r$ and $t_4 = t_3 r$

Now, area of $\Delta P_1P_2P_3 = \dfrac{1}{2}\begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix}$ and $\Delta P_2P_3P_4 = \dfrac{1}{2}\begin{vmatrix} rt_1 & r^3t_1^3 & 1 \\ rt_2 & r^3t_2^3 & 1 \\ rt_3 & r^3t_3^3 & 1 \end{vmatrix}$

$= r^4$ (Area of $(\Delta P_1P_2P_3)$)

$\therefore \quad \dfrac{\text{Area of }(\Delta P_1P_2P_3)}{\text{Area of }(\Delta P_2P_3P_4)} = \dfrac{1}{r^4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$.

Q.5 If the sum of the squares of the intercepts on the axes cut off by tangent to the curve $x^{1/3} + y^{1/3} = a^{1/3}$, $a > 0$ at $\left(\dfrac{a}{8}, \dfrac{a}{8}\right)$ is 2, then a =

(A) 1 (B) 2 (C) 4 (D) 8

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Sol. Given, $x^{1/3} + y^{1/3} = a^{1/3}, a > 0$

$\therefore \quad \frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{2/3}$

At $\quad P\left(\frac{a}{8}, \frac{a}{8}\right), \frac{dy}{dx} = -1$

Equation of tangent at P is

$y - \frac{a}{8} = -1\left(x - \frac{a}{8}\right) \quad$ or, $\quad x + y = \frac{a}{4}$

$\therefore$ It intercepts on the axes are $\frac{a}{4}, \frac{a}{4}$.

Given, $\frac{a^2}{16} + \frac{a^2}{16} = 2 \Rightarrow a^2 = 16 \Rightarrow a = 4. \qquad (\because a > 0)$

Q.6 Tangents are drawn from the origin to the curve y = sin x, then their point of contact lie on the curve

(A) $\frac{1}{y^2} - \frac{1}{x^2} = 1$ (B) $\frac{1}{x^2} + \frac{1}{y^2} = 1$ (C) $x^2 - y^2 = 1$ (D) $x^2 + y^2 = 1$

Sol. Given curve is $y = \sin x$ ....(1)

Let the tangent to curve (1) at P (α, β) pass through (0, 0).

Equation of tangent at (α, β) is

$y - \beta = \cos\alpha(x - \alpha)$ ....(2)

Since (2) passes through (0, 0)

$\therefore \quad -\beta = -\alpha\cos\alpha$ or $\cos\alpha = \frac{\beta}{\alpha}$ ....(3)

Also, (α, β) lies on (1)

$\therefore \quad \sin\alpha = \beta$

From (3) and (4), $1 = \frac{\beta^2}{\alpha^2} + \beta^2$ ....(4)

or, $\alpha^2 - \beta^2 = \alpha^2\beta^2$ or, $\frac{\alpha^2 - \beta^2}{\alpha^2\beta^2} = 1$ or, $\frac{1}{\beta^2} - \frac{1}{\alpha^2} = 1$

$\therefore \quad (\alpha, \beta)$ lies on curve $\frac{1}{y^2} - \frac{1}{x^2} = 1$.

Q.7 The equation of the tangent(s) to the curve $y = \cos(x + y), -2\pi \le x \le 2\pi$ that is (are) parallel to the line $x + 2y = 0$ is (are)

(A) $2x + 4y + 3\pi = 0$ (B) $2x + 4y - \pi = 0$

(C) $x + 2y + \pi = 0$ (D) $2x + y + 3\pi = 0$

Sol. The given curve is $y = \cos(x + y)$

$\therefore \quad \frac{dy}{dx} = \sin(x + y)\left(1 + \frac{dy}{dx}\right)$ ....(1)

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or, $\dfrac{dy}{dx} = -\dfrac{\sin(x+y)}{1+\sin(x+y)}$ ....(2)

The given line is $x + 2y = 0$ ....(3)

Its slope $= -\dfrac{1}{2}$

Since tangent is parallel to given line

$\therefore \quad \dfrac{dy}{dx}$ = slope of the tangent $= -\dfrac{1}{2}$.

From (2) and (3), $-\dfrac{\sin(x+y)}{1+\sin(x+y)} = -\dfrac{1}{2}$

or, $2\sin(x+y) = 1 + \sin(x+y)$

or, $\sin(x+y) = 1 \Rightarrow \cos(x+y) = 0$ ....(4)

From (1) and (4)

$\therefore \quad y = 0$

$\therefore$ From (1), $\cos(x+0) = 0$ or $\cos x = 0 = \cos\dfrac{\pi}{2}$

$\therefore \quad x = 2n\pi \pm \dfrac{\pi}{2}$, where $n = 0, \pm 1, \pm 2, ...$

$\therefore$ Values of x such that $-2\pi \le x \le 2\pi$ are

$\pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}$ of which only $\dfrac{\pi}{2}$ and $-\dfrac{3\pi}{2}$ satisfy equation (4)

Hence possible points are, $\left(\dfrac{\pi}{2}, 0\right)$ and $\left(-\dfrac{3\pi}{2}, 0\right)$

Equation of tangent at $\left(\dfrac{\pi}{2}, 0\right)$ is $y - 0 = -\dfrac{1}{2}\left(x - \dfrac{\pi}{2}\right)$ or $2x + 4y - \pi = 0$

Equation of tangent at $\left(-\dfrac{3\pi}{2}, 0\right)$ is $y - 0 = -\dfrac{1}{2}\left(x + \dfrac{3\pi}{2}\right)$ or $2x + 4y - 3\pi = 0$.

Q.8 The coordinates of the feet of the normals drawn from the point (14, 7) to the curve $y^2 - 16x - 8y = 0$ are

(A) (0, 0) (B) (3, 2) (C) (3, – 4) (D) (8, 16)

Sol. Given curve is $y^2 - 16x - 8y = 0$ ....(1)

Let $P \equiv (14, 7)$

Equation (1) can be written as $y^2 - 8y = 16x$

or, $y^2 - 8y + 16 = 16x + 16$

or, $(y-4)^2 = 16(x+1)$ ....(2)

This is of the form $(y-\beta)^2 = 4a(x-\alpha)$, where $a = 4$, $\alpha = -1$, $\beta = 4$

Let $(-1 + 4t^2, 4 + 8t)$ be any point on curve (2).

From (2), $2(y-4)\dfrac{dy}{dx} = 16$

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$\therefore \quad \frac{dy}{dx} = \frac{8}{y-4}$

At $\quad (4t^2 - 1, 4 + 8t), \frac{dy}{dx} = \frac{8}{8t} = \frac{1}{t}$

$\therefore$ Equation of normal at $(-1 + 4t^2, 4 + 8t)$ is

$y - 4 - 8t = -t(x + 1 - 4t^2)$

or, $\quad tx + y - 4 - 8t + t - 4t^3 = 0 \quad$ ....(3)

If line (3) passes through the point P(14, 7), then

$14t + 7 - 4t^3 - 7t - 4 = 0$

or, $\quad 4t^3 - 7t - 3 = 0$

or, $\quad (t + 1)(4t^2 - 4t - 3) = 0$

$\therefore \quad t = -1, \frac{4 \pm 8}{8} = 1, \frac{3}{2}, -\frac{1}{2}$

when $t = -1$, foot of the normal is $(3, -4)$

when $t = \frac{3}{2}$, foot of the normal is $(8, 16)$

when $t = -\frac{1}{2}$, foot of the normal is $(0, 0)$.

Q.9 Find the equations of tangents to the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ which are parallel to the x-axis.

Sol. Equation of the curve is $x^2 + y^2 - 2x - 4y + 1 = 0$

$\therefore \quad 2x + 2y\frac{dy}{dx} - 2 - \frac{4dy}{dx} = 0 \quad \Rightarrow \quad (2y - 4)\frac{dy}{dx} = 2 - 2x$

$\therefore \quad \frac{dy}{dx} = \frac{2-2x}{2y-4} = \frac{1-x}{y-2}$

Since tangents are parallel to the x-axis, slope of each of the tangents = 0.

$\therefore \quad \frac{1-x}{y-2} = 0 \quad \Rightarrow \quad 1 - x = 0 \quad \Rightarrow \quad x = 1$

At $x = 1$, $1^2 + y^2 - 2(1) - 4y + 1 = 0$

$\Rightarrow \quad y^2 - 4y = 0 \quad \Rightarrow y(y - 4) = 0 \quad \Rightarrow \quad y = 0 \quad$ or $\quad y = 4$

$\therefore$ the points are $(1, 0)$ and $(1,4)$

$\therefore$ the equation of tangent through $(1, 0)$ and parallel to the x-axis is $y = 0$

and the equation of tangent through $(1, 4)$ and parallel to the x-axis is $y = 4$

$\therefore$ the equations of tangents are $y = 0$ and $y = 4$.

Q.10 Find the equation of tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at (x_0, y_0) and show that the sum of its intercepts on axes is constant.

Sol. The equation of the curve is $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$\therefore \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$

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At (x_0, y_0), $\dfrac{dy}{dx} = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}$

$\therefore$ slope of the tangent $= -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}$

$\therefore$ equation of the tangent is $y - y_0 = m(x - x_0)$

$\Rightarrow \quad y - y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \quad \Rightarrow \quad y\sqrt{x_0} - y_0\sqrt{x_0} = - - x\sqrt{y_0} + x_0\sqrt{y_0}$

$\Rightarrow \quad x\sqrt{y_0} + y\sqrt{x_0} = x_0\sqrt{y_0} + y_0\sqrt{x_0}$

$\Rightarrow \quad x\sqrt{y_0} + y\sqrt{x_0} = \sqrt{x_0 y_0}\left(\sqrt{x_0} + y_0\right)$ ......(i)

Since (x_0, y_0) is on the curve,

$$\sqrt{x_0} + \sqrt{y_0} = \sqrt{a} \qquad \text{......(ii)}$$

Putting this value in (i), we get the equation of tangent as

$$x\sqrt{y_0} + y\sqrt{x_0} = \sqrt{x_0 y_0}\sqrt{a} = \sqrt{a x_0 y_0}$$

$$y = 0 \quad \Rightarrow \quad x = \sqrt{a x_0} \quad \Rightarrow \quad \text{x-intercept} = \sqrt{a x_0}$$

$$x = 0 \quad \Rightarrow \quad y = \sqrt{a y_0} \quad \Rightarrow \quad \text{y-intercept} = \sqrt{a y_0}$$

$\therefore$ sum of intercepts $= \sqrt{a x_0} + \sqrt{a y_0} = \sqrt{a}\left(\sqrt{x_0} + \sqrt{y_0}\right)$

$= \sqrt{a} \cdot \sqrt{a}$ [from (ii)]

$= a =$ constant.

Q.11 Find the point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the line joining (3, 0) and (4, 1).

Sol. Given curve is $y = (x - 3)^2$ ...(i)

Let $A \equiv (3, 0)$ and $B \equiv (4, 1)$

Slope of AB $= \dfrac{1 - 0}{4 - 3} = 1$ ...(ii)

From (i), $\dfrac{dy}{dx} = 2(x - 3)$ ...(iii)

Since tangent is parallel to line AB

$\therefore \quad \dfrac{dy}{dx} = 1$

From (ii) and (iii), we have $2(x - 3) = 1 \Rightarrow x = \dfrac{7}{2}$ ...(iv)

From (i), when $x = \dfrac{7}{2}$, $y = \left(\dfrac{7}{2} - 3\right)^2 = \dfrac{1}{4}$

Hence the required point is $\left(\dfrac{7}{2}, \dfrac{1}{4}\right)$

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MONOTONICITY

GENERAL INTRODUCTION :

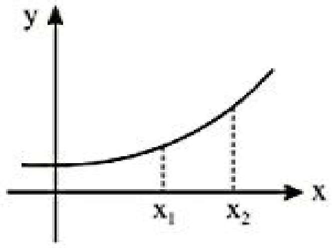
The most useful element taken into consideration amongst the total post mortuam activities of functions, is their monotonic behaviour.

(a) Monotonic function :

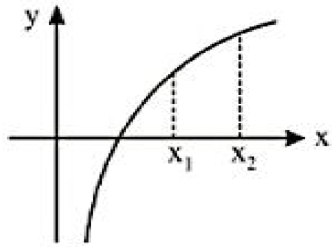
Functions are said to be monotonic if they are either increasing or decreasing in their entire domain *e.g.* $f(x) = e^x$; $f(x) = \ln x$ & $f(x) = 2x + 3$ are some of the examples of functions which are increasing whereas $f(x) = -x^3$; $f(x) = e^{-x}$ and $f(x) = \cot^{-1}(x)$ are some of the examples of the functions which are decreasing.

Increasing function

$f(x) = e^x$



$f(x) = \ln x$



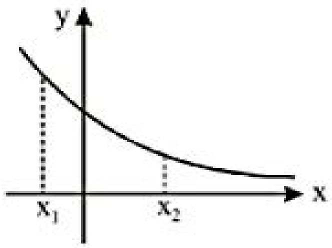
If $x_1 < x_2$ and $f(x_1) < f(x_2)$ then function is called increasing function or strictly increasing function.

Decreasing function

$f(x) = e^{-x}$

If $x_1 < x_2$

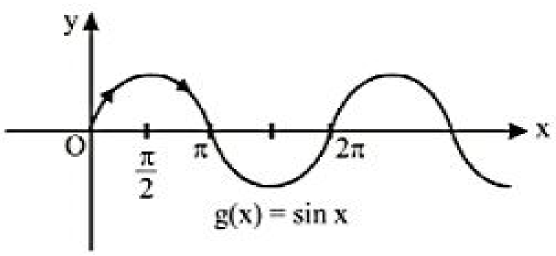
but $f(x_1) > f(x_2)$ in entire domain then fucntion is called decreasing function or strictly decreasing function.

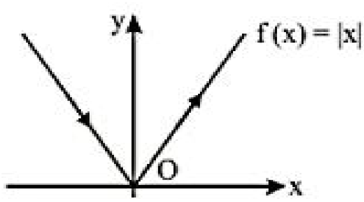


(b) Non Monotonic :

Functions which are increasing as well as decreasing in their domain are said to be non monotonic

e.g. $f(x) = \sin x$; $f(x) = ax^2 + bx + c$ and $f(x) = |x|$, however in the interval $\left[0, \frac{\pi}{2}\right]$, $f(x) = \sin x$ will be said to be increasing.





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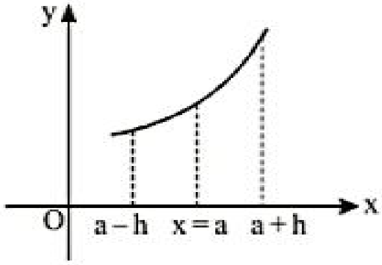
Monotonocity of a functon at a point :

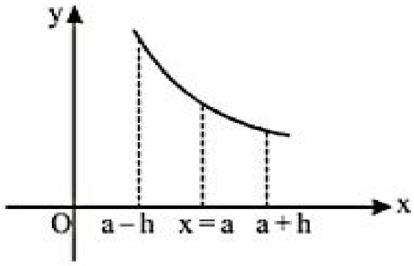
A function is said to be monotonically increasing $x = a$ if $f(x)$ satisfies

$$\left.\begin{array}{l} f(a+h) > f(a) \\ f(a-h) < f(a) \end{array}\right] \text{for a small Positive h.}$$

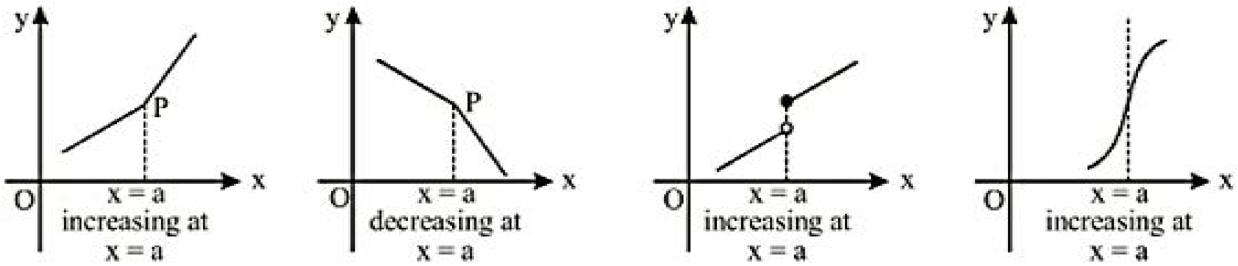


A function is said to be monotonically decreasing at $x = a$ if $f(x)$ satisfies

$$\left.\begin{array}{l} f(a+h) < f(a) \\ f(a-h) > f(a) \end{array}\right] \text{for a small Positive h.}$$



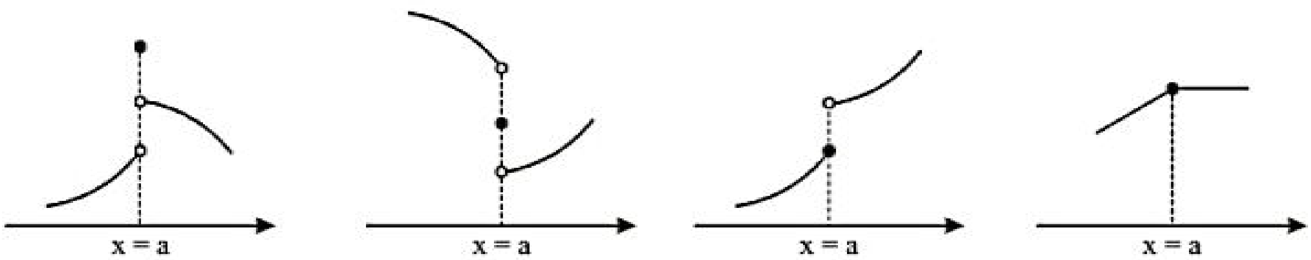
Note : It should be noted that we can talk of monotonocity of $f(x)$ at $x = a$ only if $x = a$ lies in the domain of $f(x)$, without any consideration of continuity or differentiability of $f(x)$ at $x = a$.



(a) (b) (c) (d)

Illustration :

For each of the following graph, comment whether f(x) is increasing or decreasing or neither incresing or decreasing at x = a.



(A) (B) (C) (D)

Sol. *(A)* *Neither monotonically increasing nor decreasing as* $f(a - h) < f(a)$ *and* $f(a + h) < f(a)$

(B) *Monotonically decreasing as* $f(a - h) < f(a) > f(a + h)$

(C) *Monotonically increasing as* $f(a - h) < f(a) < f(a + h)$

(D) *Neither monotonically increasing nor decreasing as* $f(a - h) < f(a)$ *but* $f(a + h) = f(a)$

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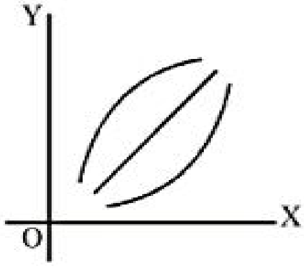
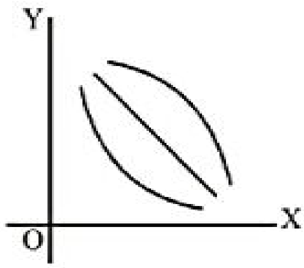
MONOTONICITY IN AN INTERVAL :

(a) For an increasing function in some interval,
If $\Delta x > 0 \Leftrightarrow \Delta y > 0$ or $\Delta x < 0 \Leftrightarrow \Delta y < 0$
then f is said to be monotonic (strictly) increasing in that interval. In other words if Δy and Δx have the same sign *i.e.* $\frac{dy}{dx} > 0$, for increasing function. Hence if $\frac{dy}{dx} > 0$ in some (interval) then y is said to be increasing function in that interval and conversely if $f(x)$ is increasing in some interval then $\frac{dy}{dx} > 0$ in that J.



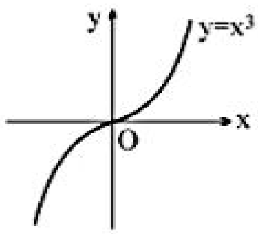
(b) Similarly if $\frac{dy}{dx} < 0$ in some interval then y is decreasing in that J and conversely.



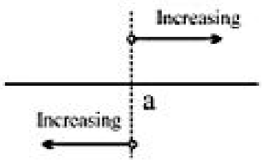
Note : Hence to find the intervals of monotonocity for a function $y = f(x)$ one has to find $\frac{dy}{dx}$ and solve the inequality, $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$. The solution of this inequality gives the interval of monotonocity.

It should however be noted that

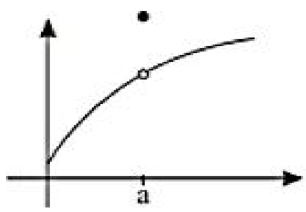
(a) $\frac{dy}{dx}$ at some point may be equal to zero but $f(x)$ may still be increasing at x = a. Consider $f(x) = x^3$ which is increasing at x = 0 although $f'(x)=0$. This is because $f(0 + h) > f(0)$ and $f(0 - h) < f(0)$. At all such points where $\frac{dy}{dx} = 0$ but y is still increasing or decreasing are known as **point of inflection**, which indicate the change of concavity of the curve.



(b) If f is increasing for x > a and f is also increasing for x < a then f is also increasing as x = a provided $f(x)$ is continuous at x = a.



(c) If f (x) is discontinuous at x = a then one can draw the graph as shown x = a is the point of maxima



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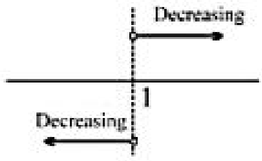
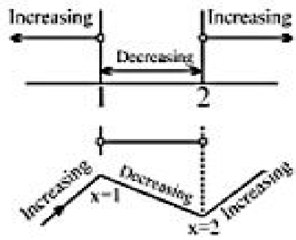
(d) Similarly if f is decreasing for $x > a$ and f is also decreasing for $x < a$ then f is also decreasing for $x = a$ provided $f(x)$ is continuous at $x = a$.



(e) However if $f(a)$ is not defined then monotonocity will not be indicated at $x = a$

e.g. $f(x) = \dfrac{1}{x-1}$ is decreasing for $x \in (-\infty, 1) \cup (1, \infty)$.

However if f is increasing and decreasing as shown then at $x = 1$ and $x = 2$, f will have extremum values, being maximum at $x = 1$ and minimum at $x = 2$.



Increasing and decreasing functions :

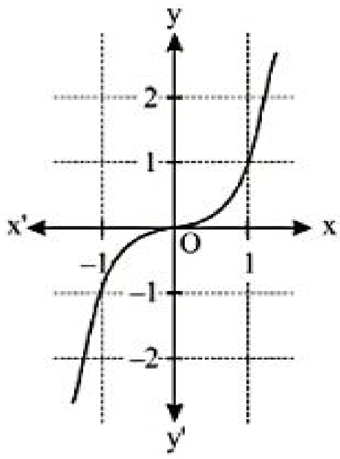
(i) A function f(x) is said to be monotonically increasing for all such interval (a, b) where $f'(x) \geq 0$ and equality may hold only for discrete values of x, i.e., $f'(x)$ does not identically become zero for $x \in (a, b)$ or any sub-interval.

(ii) f(x) is said to be monotonically decreasing for all such interval (a, b) where $f'(x) \leq 0$ and equility may hold only for discrete values of x.

Note : By discrete points, we mean those points where $f'(x) = 0$ does not form any interval.

Illustration :

Prove that $f(x) = x^3$ in an increasing function.

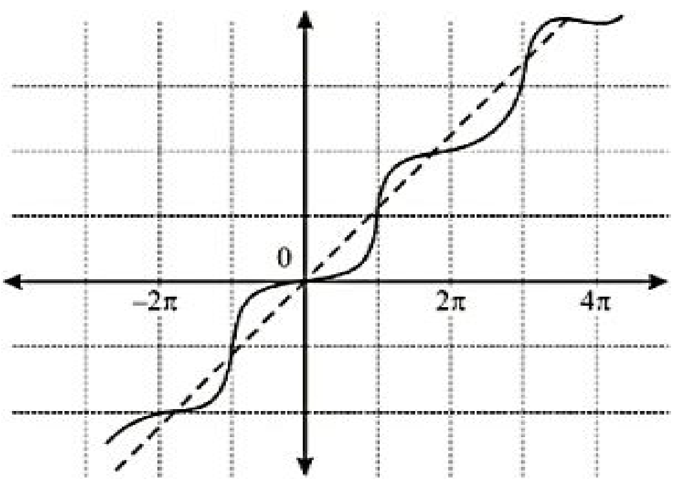


Sol. *Clearly $f'(x) = 3x^2 \geq 0$ in $(-\infty, \infty)$ and equility holds only at $x = 0$ and not in any interval, therefore $f(x) = x^3$ is an increasing fucnton in $(-\infty, \infty)$.*

Illustration :

Prove that $f(x) = x - \sin x$ is an increasing function.

Sol. *$f(x) = x - \sin x$*
$\Rightarrow$ $f'(x) = 1 - \cos x$
Now, $f'(x) > 0$ everywhere except at $x = 0, \pm 2\pi, \pm 4\pi$ etc. but all these points are discrete and do not form an interval hence we can conclude that f(x) is monotoically increasing for $x \in R$.
In fact, we can also see it graphically.



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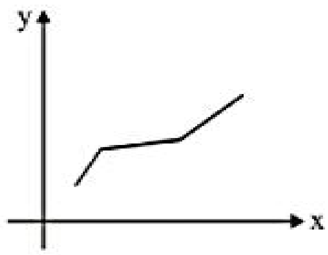
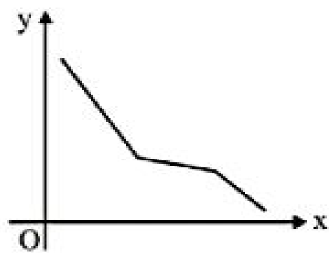
Non decreasing function :

f(x) is said to be non-decreasing in domain if for every $x_1, x_2 \in D_1$, $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$. It means that the value of f(x) would never decrease with an increase in the value of x (Figure).

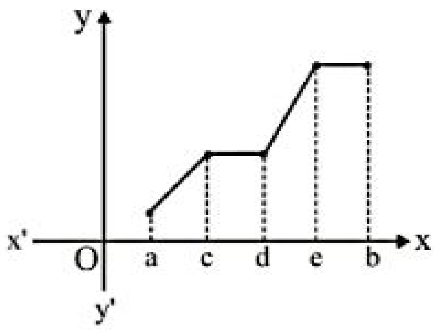


Non Increasing function :

f(x) is said to be non-increasing in domain if for every $x_1, x_2 \in D_1$, $x_1 > x_2$ $f(x_1) \leq f(x_2)$. It means that the value of f(x) would never increase with an increase in the value of x (Figure).



Let us consider another function whose graph is shown for $x \in (a, b)$.



Here also $f'(x) \geq 0$ for all $x \in (a, b)$ but note that in this case equality of $f'(x) = 0$ holds for all $x \in (c, d)$ and (e, b). Here f'(x) becomes identically zero and hence the given function cannot be assumed to be monotonically increasing for $x \in (a, b)$.

Note:

(i) If a function is monotonic at x = a it can not have extremum point at x = a and vice versa *i.e.* a point on the curve can not simultaneously be an extremum as well as monotonic point.

(ii) If f is increasing then nothing definite can be said about the function f'(x) w.r.t. its increasing or decreasing behaviour.

Illustration :

Find intervals of monotonicity of $f(x) = \dfrac{x}{\ln x}$

Sol. $f'(x) = \dfrac{\ln x - 1}{(\ln x)^2}$

sign of f'(x) – – – – – – – – – +++++ (points 0, 1, e)

strictly increasing in (e, ∞) *and strictly decreasing in* $(0, 1) \cup (1, e)$

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Illustration :

If the function $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ *is always decreasing* $\forall\, x \in R$, *find 'a'.*

Sol. $f'(x) = 3(a+2)\,x^2 - 6ax + 9a \le 0 \quad \forall\; x \in R$

$\Rightarrow\; 3(a+2) < 0 \qquad \& \qquad 36a^2 - 4 \cdot 3\,(a+2) \cdot 9a \le 0$

$\Rightarrow\; a < -2 \qquad \& \qquad a^2 - 3a(a+2) \le 0$

$\Rightarrow\; a < -2 \;\; ...(1) \qquad \& \qquad a^2 + 3a \ge 0 \quad \Rightarrow\; a \in (-\infty, -3] \cup [0, \infty) \;...\; (2)$

from (1) and (2)

$a \le -3$ *Ans.*

Illustration :

Find intervals of monotonicity of following functions :

(a) $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7$ (b) $f(x) = \dfrac{2x}{1+x^2}$

(c) $f(x) = \ln\,(x^2 - 2x)$ (d) $f(x) = \dfrac{|x-1|}{x^2}$

Sol.

(a) *We have*

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7,\;\; x \in R$

and $f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x-1)(x-2)(x-3)$

From the sign scheme for $f'(x)$, *we can see that* $f(x)$

From the sign scheme for $f'(x)$, *we can see that* $f(x)$

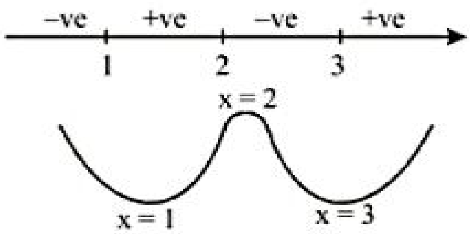
strictly decreases in $(-\infty, 1)$

strictly increases in $(1, 2)$

strictly decreases in $(2, 3)$

strictly increases in $(3, \infty)$.

The shape of the curve is drawn alongside.



(b) *We have* $f(x) = \dfrac{2x}{1+x^2},\; x \in R$

and $f'(x) = \dfrac{(1+x^2)2 - 2x(2x)}{(1+x^2)^2} = \dfrac{-2(x^2-1)}{(1+x^2)^2} = \dfrac{-2(x+1)(x-1)}{(1+x^2)^2}$

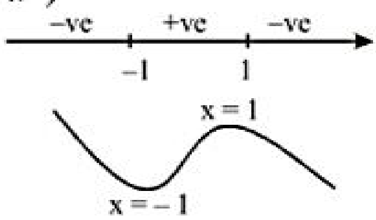
From the sign scheme for $f'(x)$, *we can see that* $f(x)$

strictly decreases in $(-\infty, 1)$

strictly increases in $(-1, 1)$

strictly decreases in $(1, \infty)$

The shape of the curve is shown alongside



(c) *We have* $f(x) = \ln\,(x^2 - 2x),\; x \in (-\infty, 0) \cup (2, \infty)$

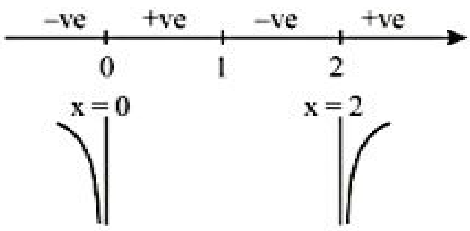
and $f'(x) = \dfrac{2x-2}{x^2-2x} = \dfrac{2(x-1)}{x(x-2)}$

From the sign scheme for $f'(x)$, *we can see that* $f(x)$

strictly decreases in $(-\infty, 0) \cup (1, 2)$

strictly increases in $(0, 1) \cup (2, \infty)$.

Also, we can see that $f(0^-) = -\infty$ *and* $f(2^+) = -\infty$.



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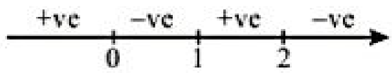
(d) *We have* $f(x) = -\dfrac{(x-1)}{x^2}$, $x < 1$ *and* $f(x) = \dfrac{x-1}{x^2}$, $x \ge 1$

and $f'(x) = \dfrac{-2}{x^3} + \dfrac{1}{x^2} = \dfrac{x-2}{x^3}$, $x < 1$ *and* $f'(x) = \dfrac{2-x}{x^3}$, $x > 1$

Now, from the sign scheme for $f'(x)$, *we have*



$\Rightarrow$ *$f(x)$ strictly increases in* $(-\infty, 0)$

strictly decreases in $(0, 1)$

strictly increases in $(1, 2)$

strictly decreases in $(2, \infty)$.

Illustration :

If $\phi(x) = f(x) + f(1-x)$ *and* $f''(x) < 0$ *in* $(-1, 1)$, *then show that* $\phi(x)$ *strictly increases in* $\left(0, \dfrac{1}{2}\right)$.

Sol. *We have* $\phi(x) = f(x) + f(1-x)$ *and* $\phi'(x) = f'(x) - f'(1-x)$

which vanishes at points given by $x = 1 - x$ *i.e.* $x = \dfrac{1}{2}$

$f''(x) < 0 \Rightarrow f'(x)$ *is decreasing for* $x \in \left(0, \dfrac{1}{2}\right)$ *i.e.* $1 - x > x \Rightarrow f'(1-x) < f'(x)$

hence $\phi'(x) > 0 \;\; \forall \; x \in \left(0, \dfrac{1}{2}\right)$

Hence, $\phi(x)$ *strictly increases in* $\left(0, \dfrac{1}{2}\right)$.

Practice Problem

Q.1 Compute the intervals of monotonicity for the following $f(x) = x^2 \cdot e^{-x}$

Q.2 Compute the intervals of monotonicity for the following $f(x) = x + \ln(1-4x)$

Q.3 Find value of a so that $f(x) = ax - \sin x$ is monotonic.

Q.4 Prove that $f(x) = \dfrac{2}{3}x^9 - x^6 + 2x^3 - 3x^2 + 6x - 1$ is always increasing.

Answer key

Q.1 $\uparrow$ in $(0, 2)$ and $\downarrow$ in $(-\infty, 0) \cup (2, \infty)$

Q.2 $\uparrow$ in $(-\infty, -3/4)$ and $\downarrow$ in $(-3/4, 1/4)$

Q.3 $\uparrow$ for $a \ge 1$ & $\downarrow$ for $a \le -1$

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GREATEST AND LEAST VALUE OF A FUNCTION :

Case-I :

If a continuous function $y = f(x)$ is strictly increasing in the closed interval [a, b] then $f(a)$ is the least value. (figure - 1) and f(b) is greatest value

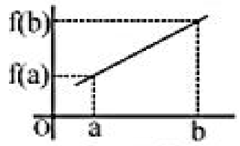

figure -1

Case-II :

If $f(x)$ is decreasing in [a, b] then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in [a, b].(figure - 2)

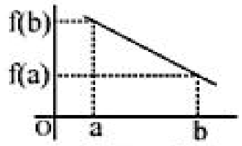

figure -2

Case-III :

However if $f(x)$ is non monotonic in [a, b] and is continuous then the greatest and least value of $f(x)$ in [a, b] are those where $f(x) = 0$ **or** $f'(x)$ does not exist or at the extreme values. (figure - 3)

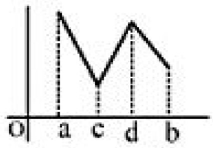

figure -3

Illustration :

Find least and greatest value of $f(x) = e^{x^2-4x+3}$ *in* $[-5, 5]$

Sol. $f(x) = e^{x^2-4x+3}$

For $f(x)$ *max* $\rightarrow x^2 - 4x + 3$ *be maximum in* $[-5, 5]$

$x^2 - 4x + 3$ *will be maximum at* $x = -5$ *in the given interval.*

i.e., $25 + 20 + 3 = 48$

$\therefore$ *Max* $f(x) = e^{48}$ *at* $x = -5$

$x^2 - 4x + 3$ *will be minimum at* $x = 2$ *i.e,* $4 - 8 + 3 = -1$

$\therefore$ *Min* $f(x) = e^{-1}$ *at* $x = 2$.

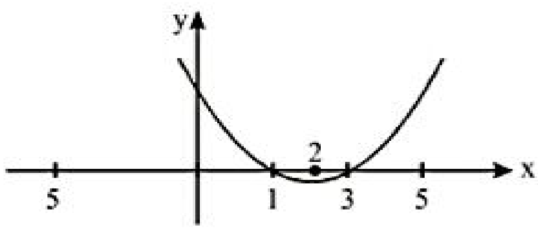

Illustration :

Find the image of interval $[-1, 3]$ *under the mapping specified by the function* $f(x) = 4x^3 - 12x$.

Sol. $f'(x) = 12x^2 - 12 = 12(x^2 - 1)$

$f'(x) = 0$ *at* $x = \pm 1$, $f(-1) = 8, f(1) = -8$

$f(3) = 72 \quad \Rightarrow$ *greatest value is 72 and least value is* -8.

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Illustration :

Find the range of the following functions $f(x) = \sqrt{x-3} + 2\sqrt{5-x}$.

Sol. *We have* $f(x) = \sqrt{x-3} + 2\sqrt{5-x}$

whose domain is $x \in [3, 5]$ *and its derivative is*

$$f'(x) = \frac{1}{2\sqrt{x-3}} - \frac{1}{\sqrt{5-x}} = \frac{\sqrt{5-x} - 2\sqrt{x-3}}{2\sqrt{x-3}\,\sqrt{5-x}}$$

Now, solving

$$\sqrt{5-x} > 2\sqrt{x-3} \quad \text{i.e.} \quad 5 - x > 4(x-3) \text{ given } x < \frac{17}{5}.$$

Hence, we have

$$f'(x) > 0 \ \forall x \in \left(3, \frac{17}{5}\right) \ \& \ f'(x) < 0 \ \forall x \in \left(\frac{17}{5}, 5\right)$$

$\Rightarrow$ $f(x)$ *strictly increases in* $\left(3, \frac{17}{5}\right)$ *and strictly decreases in* $\left(\frac{17}{5}, 5\right)$.

Now, we have

$$f(3) = 2\sqrt{2},\ f(5) = \sqrt{2} \quad \text{and} \quad f\left(\frac{17}{5}\right) = \sqrt{\frac{17}{5} - 3} + 2\sqrt{5 - \frac{17}{5}} = \sqrt{10}.$$

Hence, the range is $y \in \left[\sqrt{2}, \sqrt{10}\right]$

.

Illustration :

Find the range of the following functions $f(x) = \dfrac{x^4 - x^2 - 2x + 8}{x^4 - x^2 - 2x + 4}$

Sol. *We have*

$$f(x) = \frac{x^4 - x^2 - 2x + 8}{x^4 - x^2 - 2x + 4} = 1 + \frac{4}{x^4 - x^2 - 2x + 4} = 1 + \frac{4}{(x^2-1)^2 + (x-1)^2 + 2}.$$

Let $g(x) = (x^2 - 1)^2 + (x - 1)^2 + 2$, *whose least value* $= 2$

and *greatest value* $= \infty$

Thus, we have for $f(x)$ *greatest value* $= 1 + \frac{4}{2} = 3$ *and least value* $= 1 + \frac{4}{\infty} = 1$.

Also, $f(x)$ *is continuous and defined on R. Hence, the range of* $f(x)$, *is* $y \in (1, 3]$.

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ESTABLISHING INEQUALITIES :

Notion of monotonocity helps is establishing variety of inequalities involving algebraic and transcidental function with much greater ease.

If $f(x) \geq g(x)$ or $f(x) \leq g(x)$ is to be shown in some interval we create a new function $h(x) = f(x) - g(x)$ and using monotonicity check which $h(x) \geq 0$ or $h(x) \leq 0$ in the given interval.

Illustration :

Prove that $2 \sin x + \tan x \geq 3x \quad \left(0 \leq x < \frac{\pi}{2}\right)$

Sol. $f(x) = 2 \sin x + \tan x - 3x$

$f'(x) = 2 \cos x + \sec^2 x - 3$

$f''(x) = -2\sin x + 2\sec^2 x \tan x \qquad = 2\sin x\, [\sec^3 x - 1] \geq 0 \ \forall x \in \left[0, \frac{\pi}{2}\right)$

$\Rightarrow f'(x) \uparrow$ in $\left[0, \frac{\pi}{2}\right) \Rightarrow f'(x) \geq f'(0)$

or $f'(x) \geq 0 \Rightarrow f(x) \uparrow$ hence $f(x) \geq f(0)$

in $\left[0, \frac{\pi}{2}\right) \Rightarrow f(x) \geq 0 \Rightarrow 2\sin x + \tan x \geq 3x$

Illustration :

Find the set of values of x for which $\ln(1 + x) > \frac{x}{1+x}$ [Ans. $(-1, 0) \cup (0, \infty)$]

Sol. $f(x) = \ln(1 + x) - \frac{x}{1+x} = \ln(1 + x) + \frac{1}{1+x} - 1$

Domain : $x > -1$

$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2}$

$f'(x) \geq 0 \ \forall \ x \geq 0 \Rightarrow f(x)\uparrow$

& $f'(x) \leq 0 \ \forall \ x \leq 0 \Rightarrow f(x)\downarrow$

$f'(0) = 0$

$\therefore f(x) > f(0) \ \forall \ x \in D_f - \{0\}$

$\therefore f(x) > 0 \ \forall \ x \in (-1, 0) \cup (0, \infty)$

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Illustration :

Show that $\ln(1 + x) > x - \dfrac{x^2}{2}\ \forall\ x \in (0, \infty)$

Sol. Consider the function $f(x) = \ln(1 + x) - x + \dfrac{x^2}{2}, x \in (0, \infty)$

Then $f'(x) = \dfrac{1}{1+x} - 1 + x = \dfrac{x^2}{1+x} > 0\ \forall x \in (0, \infty)$

$\Rightarrow$ $f(x)$ strictly increases in $(0, \infty)$

$\Rightarrow$ $f(x) > f(0^+) = 0$ i.e. $\ln(1 + x) > x - \dfrac{x^2}{2}$ which is the desired result.

Illustration :

Show that the equation $x^5 - 3x - 1 = 0$ has a unique root in $[1, 2]$.

Sol. Consider the function

$f(x) = x^5 - 3x - 1,\ x \in [1, 2]$

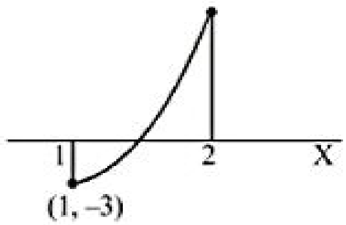
and $f'(x) = 5x^4 - 3 > 0\ \forall x \in [1, 2]$

$\Rightarrow$ $f(x)$ striclty increases in $[1, 2]$

Also, we have

$f(1) = 1 - 3 - 1 = -3$

and $f(2) = 32 - 6 - 1 = 25$



From the shape of the curve shown alongside, we can see that the curve $y = f(x)$ will cut the X-axis exactly once in $[1, 2]$

i.e. $f(x)$ will vanish exactly once in $[1, 2]$

Illustration :

Prove that $\dfrac{x}{1+x} < \ln(1 + x) < x\ \forall x > 0$

Sol. Consider the function $f(x) = \ln(1 + x) - \dfrac{x}{1+x}, x > 0.$

Then $f'(x) = \dfrac{x}{1+x} - \dfrac{x}{(1+x)^2} = \dfrac{x}{(1+x)^2} > 0\ \forall x > 0$

$\Rightarrow$ $f(x)$ strictly increases in $(0, \infty)$

$\Rightarrow$ $f(x) > f(0^+) = 0$ i.e. $\ln(1 + x) > \dfrac{x}{1+x}$ which proves the LHI.

Now, consider the function $g(x) = x - \ln(1 + x), x > 0$

Then $g'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0\ \forall x > 0$

$\Rightarrow$ $g(x)$ strictly increases in $(0, \infty)$ $\Rightarrow g(x) > g(0^+) = 0$

i.e. $x > \ln(1 + x)$ which proves the RHI.

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Practice Problem

Q.1 $f(x) = \cos 3x - 15 \cos x + 8$ in $\left[\frac{\pi}{3}, \frac{3\pi}{2}\right]$

Q.2 Use the function $f(x) = x^{\frac{1}{x}}$ $(x > 0)$ to ascertain whether π^e or e^π is greater.

Q.3 Prove that $e^x - e^{-x} - 2x > 0 \ \forall\ x > 0$. Hence, prove that $e^x + e^{-x} \geq x^2 + 2 \ \forall\ x \geq 0$.

Q.4 Prove that the function $f(x) = \frac{\ln x}{x}$, is strictly decreasing in (e, ∞). Hence, prove that $303^{202} < 202^{303}$.

Answer key

Q.1 Max. at $x = \pi = 22$ & Min. at $x = \frac{\pi}{3} = -\frac{1}{2}$ Q.2 $\pi^e < e^\pi$

ROLLE'S & MEAN VALUE THEOREM :

Rolle's Theorem :

Let f(x) be a function subject to the following conditions :

(i) f(x) is a continuous function in the closed interval of $a \leq x \leq b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$.

Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = 0$.

Alternative statement:

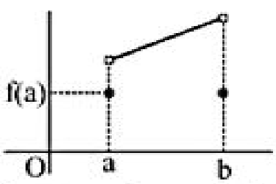
Rolles theorem states that between any two real zeroes of a differentiable real function f, lies at least one critical point of $f(x)$.

Remarks:

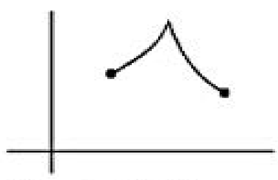
(i) Converse of Rolle's theorem is Not true i.e. $f'(x)$ may vanish at a point within (a, b) without satisfying all the three conditions of Rolle's Theorem.

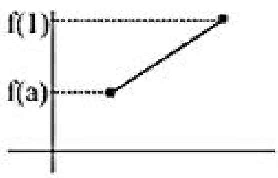
(ii) The three conditions are sufficient but not necessary for $f'(x) = 0$ for some x in (a, b)

(iii) If the function $y = f(x)$ defined over [a, b] does not satisfy even one of the 3 conditions then Rolle's Theorem fails i.e. there may or may not exist point in (a, b) where $f'(x) = 0$.



f is not continuous in [a, b] and Rolles theorem failing


f is not derivable in open (a, b)



$f(a) \neq f(b)$

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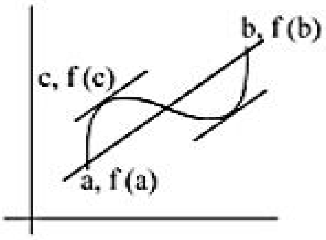
LMVT THEOREM (LAGRANGE'S MEAN VALUE THEOREM) :

Let f(x) be a function of x subject to the following conditions :



(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) \neq f(b)$.

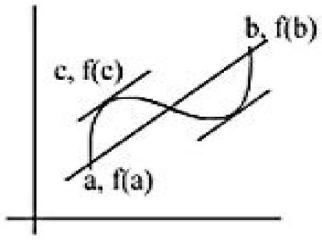
Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Geometrically, the slope of the secant line joining the curve at $x = a$ & $x = b$ is equal to the slope of the tangent line drawn to the curve at $x = c$. Note the following :

Note : Now $[f(b) - f(a)]$ is the change in the value of function f as x changes from a to b so that $[f(b) - f(a)] / (b - a)$ is the *average rate of change* of the function over the interval [a, b]. Also f'(c) is the instantaneous rate of change of the function at x = c. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval.
This interpretation of the theorem justifies the name "Mean Value" for the theorem.

Rolles theorem is a special case of LMVT since

$$f(a) = f(b) \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} = 0.$$



Alternative form of LMVT :

Another form of statement of Lagrange's Mean Value Theorem. If a function f is continuous in a closed interval [a, a + h] and derivable in the open interval] a, a + h [, then there exists at least one number 'θ' ∈ (0, 1) such that

$$f(a+h) = f(a) + h f'(a+\theta h).\ \ \theta \in (0, 1)$$

Proof:

We write $b - a = h$ so that h denotes the length of the interval [a, b] which may now be rewritten as [a, a + h]. The number, 'c' which lies between a and a + h, is greater than a and less than a + h so that we may write $c = a + \theta h$, where θ is some number between 0 and 1. Thus the equation (i) becomes

$$\frac{f(a+h) - f(a)}{h} = f'(a+\theta h) \quad \Rightarrow \quad f(a+h) = f(a) + h f'(a+\theta h)$$

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Illustration :

Verify Rolle's Theorem for

(a) $f(x) = x(x + 3)e^{-x/2}$ in $[-3, 0]$

(b) $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$ $\qquad [c = \frac{\pi}{4}]$

(c) $f(x) = 1 - x^{2/3}$ in $[-1, 1]$ $\qquad$ ($f'(0)$ non existent)

Sol.

(a) $f(x) = x(x + 3)e^{-x/2} \quad [-3, 0]$

$f(-3) = 0 \; f(0) = 0$

Clearly f(x) is continuous and diffentiable function

there exists $c \in (-3, 0)$ *s. t.* $f'(c) = 0$

$$f'(c) = (2c + 3)e^{-c/2} - \frac{1}{2}(c^2 + 3c)e^{-c/2} \quad \Rightarrow \quad e^{-c/2}\left[2c + 3 - \frac{(c^2 + 3c)}{2}\right] = 0$$

$4c + 6 - c^2 - 3c = 0$

$c^2 - c - 6 = 0$

$c = -2; c = 3 \notin (-3, 0)$

(b) $f(x) = \frac{\sin x}{e^x} \quad [0, \pi]$

$f(0) = f(\pi) = 0$

Continuity and differentiable function.

$f'(c) = e^{-c} \cos c - \sin c \, e^{-c} = 0$

$\tan c = 1 \Rightarrow c = \frac{\pi}{4} \in (0, \pi)$

(c) $f(x) = 1 - x^{2/3} \; [-1, 1]$

$f(-1) = f(1) = 0$

$f'(c) = \frac{-2}{3} c^{-1/3} \qquad$ *but for* $c = 0$

$f'(c)$ *is non-existent.*

$\therefore$ *f(x) is not differentiable in given interval.*

Hence Rolle's theorem not applicable.

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Illustration :

Find c of LMVT $f(x) = \sqrt{x-1}$ *in [1, 3]*

Sol. $f(x) = \sqrt{x-1}$ $[1, 3]$

continuous and diff. in given interval $f'(c) = \dfrac{f(3)-f(1)}{3-1}$

$$\frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2}}{2} \Rightarrow c = \frac{3}{2}$$

Illustration :

Using LMVT prove that $|\cos a - \cos b| \le |a - b|$

Sol. *Consider* $f(x) = \cos x$ *in [a, b]*

$$\left|\frac{\cos b - \cos a}{b-a}\right| = |-\sin c| \le 1 \quad \Rightarrow \quad |\cos b - \cos a| \le |b - a|$$

$|\cos a - \cos b| \le |a - b|$ *Hence proved.]*

Practice Problem

Q.1 Verify Rolles throrem for $f(x) = (x-a)^m (x-b)^n$ on [a, b] ; m, n being positive integer.

Q.2 Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, $0 < c < 1$ such that $f(c) = 0$.

Q.3 Let f be a twice differentiable function on $[0, 2]$ such that $f(0) = 0, f(1) = 2, f(2) = 4$, then prove that
(a) $f'(\alpha) = 2$ for some $\alpha \in (0, 1)$ (b) $f'(\beta) = 2$ for some $\beta \in (1, 2)$
(c) $f''(\gamma) = 0$ for some $\gamma \in (0, 2)$

Answer key

Q.1 $c = \dfrac{mb + na}{m + n}$ which lies between a & b

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Solved Examples

Q.1 Find the values of a, if the equation $x - \sin x = a$ has a unique root in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Sol. Consider the function

$$f(x) = x - \sin x - a, x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

Then $$f'(x) = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right) \geq 0 \ \forall \ x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$\Rightarrow$ f(x) increases in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Also, we have $f\left(\frac{-\pi}{2}\right) = \frac{-\pi}{2} + 1 - a$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 - a$

The curve $y = f(x)$ will cut the X-axis exactly once, if $f\left(\frac{-\pi}{2}\right)$ is negative or zero and $f\left(\frac{\pi}{2}\right)$ is positive or zero.

i.e. $\frac{-\pi}{2} + 1 - a \leq 0$ and $\frac{\pi}{2} - 1 - a \geq 0$ i.e. $a \geq \frac{-\pi}{2} + 1$ and $a \leq \frac{\pi}{2} - 1$

Hence, we have $a \in \left[1 - \frac{\pi}{2}, \frac{\pi}{2} - 1\right]$

Q.2 Find the intervals in which the given function increases or decreases $f(x) = \frac{x^2 + x + 1}{x^2 - x + 1}$.

Sol. We have $$f(x) = \frac{x^2 + x + 1}{x^2 - x + 1}, x \in R$$

and $$f'(x) = \frac{(x^2 - x + 1)(2x + 1) - (x^2 + x + 1)(2x - 1)}{(x^2 - x + 1)^2}$$

$$= \frac{(2x^3 - x^2 + x + 1) - (2x^3 + x^2 + x - 1)}{(x^2 - x + 1)^2} = \frac{-2(x + 1)(x - 1)}{(x^2 - x + 1)^2}$$

Now, from the sign scheme for f'(x), we have

–ve +ve –ve

–1 +1

$\Rightarrow$ f(x) strictly decreases in $(-\infty, -1)$
strictly increases in $(-1, 1)$
strictly decreases in $(1, \infty)$.

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Q.3 Prove the following inequalities $1+\cot x \le \cot\frac{x}{2}\ \forall\ x \in (0, \pi)$

Sol. Consider the function $f(x) = \cot\left(\frac{x}{2}\right) - 1 - \cot x,\ x \in (0, \pi)$.

Then $f'(x) = \frac{-1}{2}\csc^2\left(\frac{x}{2}\right) + \csc^2 x = \frac{1}{\sin^2 x} - \frac{1}{2\sin^2(x/2)}$

$$= \frac{1}{2\sin^2(x/2)}\left[\frac{1}{2\cos^2(x/2)} - 1\right] = \frac{-\cos x}{4\sin^2(x/2)\cos^2(x/2)}$$

$\Rightarrow$ $f'(x) = \frac{-\cos x}{\sin^2 x},\ f'(x) < 0\ \forall\ x \in \left(0, \frac{\pi}{2}\right),\ f'(x) > 0\ \forall\ x \in \left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow$ f(x) strictly decreases in $\left(0, \frac{\pi}{2}\right)$

strictly increases in $\left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow$ f(x) has least value at $x = \frac{\pi}{2}$

$\Rightarrow$ $f(x) \ge f\left(\frac{\pi}{2}\right) = 0$ i.e. $\cot\left(\frac{x}{2}\right) \ge 1 + \cot x$. which prove the desired result.

Q.4 Let $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ be an increasing function on the set R. Then find the condition on a and b.

Sol. $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ is increasing on R

$\Rightarrow$ $f'(x) > 0$ for $x \in R$

$\Rightarrow$ $3x^2 + 2ax + b + 5\sin 2x > 0$ for all $x \in R$

$\Rightarrow$ $3x^2 + 2ax + (b-5) > 0$ for all $x \in R$

$\Rightarrow$ $(2a)^2 - 4 \times 3 \times (b-5) < 0$

$\Rightarrow$ $a^2 - 3b + 15 < 0$

Q.5 Find the values of x where $f(x) = \sin(\ln x) - \cos(\ln x)$ is strictly increasing.

Sol. Since $f(x) = \sin(\ln x) - \cos(\ln x),\ x > 0$

$$= \sqrt{2}\sin\left(\ln x - \frac{\pi}{4}\right)$$

$$\therefore\ f'(x) = \frac{\sqrt{2}}{x}\cos\left(\ln x - \frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{2} + \ln x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{4} + \ln x\right) > 0 \qquad (\because\ x > 0)$$

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or $\quad \sin\left(\frac{\pi}{4}+\ln x\right)>0 \qquad$ or $\quad 2n\pi<\frac{\pi}{4}+\ln x<(2n+1)\pi,\ n\in I$

or $\quad 2n\pi-\frac{\pi}{4}<\ln x<2n\pi+\frac{3\pi}{4},\ n\in I \quad \Rightarrow \quad e^{2n\pi-\frac{\pi}{4}}<x<e^{2n\pi+\frac{3\pi}{4}},\ n\in I$

Therefore, f(x) is strictly increasing when $\quad x\in\left(e^{2n\pi-\frac{\pi}{4}},\ e^{2n\pi+\frac{3\pi}{4}}\right),\ n\in I$

Q.6 Prove that $\sin^2\theta<\theta\sin(\sin\theta)$ for $0<\theta<\frac{\pi}{2}$.

Sol. In this problem, first we have to select an appropriate function.

Now by observation, given inequality can be set as $\frac{\sin(\sin\theta)}{\sin\theta}>\frac{\sin\theta}{\theta}$. This clearly gives indication that one has to study the function $f(x)=\frac{\sin x}{x}$

$\Rightarrow \quad f'(x)=\frac{(x\cos x-\sin x)}{x^2}=\frac{\cos x(x-\tan x)}{x^2}<0 \qquad$ (as in first quadrant $x<\tan x$)

$\Rightarrow \quad f(x)$ is a decreasing function

Now, $\sin\theta<\theta$ for $0<\theta<\frac{\pi}{2}$

$\Rightarrow \quad f(\sin\theta)>f(\theta) \Rightarrow \frac{\sin(\sin\theta)}{\sin\theta}>\frac{\sin\theta}{\theta} \qquad$ [From (i)]

Hence, $\sin^2\theta<\theta\sin(\sin\theta)$ for $0<\theta<\frac{\pi}{2}$.

Q.7 Let $f:R\to R$ be a function such that $f(x)=ax+3\sin x+4\cos x$. Then f(x) is invertible if
(A) $a\in(-5,5)$ (B) $a\in(-\infty,5)$ (C) $a\in(-5,+\infty)$ (D) None of these

Sol. $f'(x)=a+3\cos x-4\sin x \;=a+5\cos(x+\alpha)$, where $\cos\alpha=\frac{3}{5}$

For invertible, f(x) must be monotonic

$\Rightarrow \quad f'(x)\ge 0\ \forall x$ or $f'(x)\le 0\ \forall x$

$\Rightarrow \quad a+5\cos(x+\alpha)\ge 0$ or $a+5\cos(x+\alpha)\le 0$

$\Rightarrow \quad a\ge -5\cos(x+\alpha)$ or $a\le -5\cos(x+\alpha)$

$\Rightarrow \quad a\ge 5$ or $a\le -5$

Q.8 $f(x)=(x-2)\,|x-3|$ is monotonically increasing in

(A) $\left(-\infty,\frac{5}{2}\right)\cup(3,\infty)$ (B) $\left(\frac{5}{2},\infty\right)$

(C) $(2,\infty)$ (D) $(-\infty,3)$

Sol. $f(x)=(x-2)\,|x-3|$

For $x\ge 3$, $\quad f(x)=(x-2)(x-3)=x^2-5x+6$

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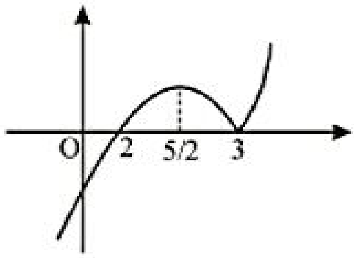
$$f'(x) = 2x - 5 = 0 \quad \Rightarrow \quad x = \frac{5}{2}$$

Now, the graph of $f(x) = (x-2)\,|x-3|$ is



Clearly from the graph, $f(x)$ increases in $\left(-\infty, \frac{5}{2}\right) \cup (3, \infty)$.

Q.9 A function $g(x)$ is defined as $g(x) = \frac{1}{4} f(2x^2 - 1) + \frac{1}{2} f(1 - x^2)$ and $f'(x)$ is an increasing function, then $g(x)$ is increasing in the interval

(A) $(-1, 1)$

(B) $\left(-\sqrt{\frac{2}{3}}, 0\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$

(C) $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

(D) None of these

Sol. $g'(x) = xf'(2x^2 - 1) - xf'(1 - x^2) = x\left(f'\left(2x^2 - 1\right) - f'\left(1 - x^2\right)\right)$,

$g'(x) > 0$

if $\quad x > 0,\ 2x^2 - 1 > 1 - x^2$ (as f' is an increasing function)

$$\Rightarrow \quad 3x^2 > 2 \ \Rightarrow \ x \in \left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right) \ \Rightarrow \ x \in \left(\sqrt{\frac{2}{3}}, \infty\right)$$

If $x < 0,\ 2x^2 - 1 < 1 - x^2$

$$\Rightarrow \quad 3x^2 < 2 \ \Rightarrow \ x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) \ \Rightarrow \ x \in \left(-\sqrt{\frac{2}{3}}, 0\right).$$

Q.10 Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$

Sol. $f(x) = 1 - x - x^3 \quad \Rightarrow \quad f'(x) = -1 - 3x^2$ which is – ve $\forall\ x \in R \Rightarrow f$ is decreasing

$f[f(x)] = 1 - f(x) - f^3(x)$

$\therefore \quad f[f(x)] > f(1 - 5x)]$ given

since, $f(x)$ is decreasing hence

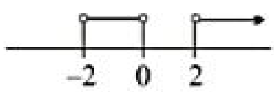
$f(x_1) > f(x_2) \quad \Rightarrow \quad x_1 < x_2$

$\therefore \quad f(x) < 1 - 5x$

$1 - x - x^3 < 1 - 5x$

$x^3 - 4x > 0$

$x(x^2 - 4) > 0$



$\therefore \quad x \in (-2, 0) \cup (2, \infty)$

Alternatively: $f'(x) = -(1 + 3x)^2 \quad \Rightarrow \quad f$ is decreasing

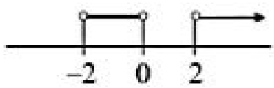
$1 - f(x) - f^3(x) > f(1 - 5x)$

$f[f(x)] > f(1 - 5x)$

but f is decreasing ; $\therefore$ $f(x) < 1 - 5x$; $1 - x - x^3 < 1 - 5x.$

$x^3 - 4x > 0$

$x(x^2 - 4) > 0$



$\therefore \quad x \in (-2, 0) \cup (2, \infty)$]

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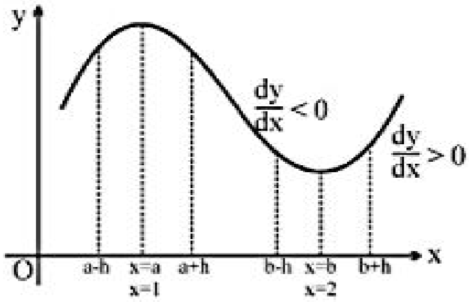
MAXIMA-MINIMA

(A) GENERAL INTRODUCTION :

The notion of optimising functions is one of the most useful application of calculus used in almost every sphere of life including geometry, business, trade, industries, economics, medicines and even at home. In this chapter we shall see how calculus defines the notion of maxima and minima and distinguishes it from the greatest and least value or global maxima and global minima of a function.

(B) DEFINITION MAXIMA & MINIMA :

A function $f(x)$ is said to have a maximum at $x = a$ if $f(a)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x = a$. Symbolically



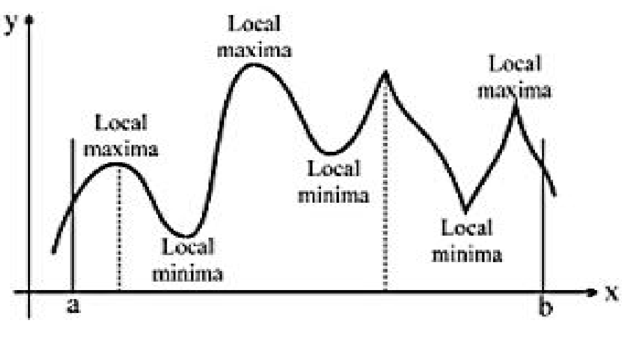
$$\left.\begin{matrix} f(a) > f(a+h) \\ f(a) > f(a-h) \end{matrix}\right] \Rightarrow x = a \text{ gives maxima}$$

for a sufficiently small positive h.

Similarly, a function $f(x)$ is said to have a minimum value at $x = b$ if $f(b)$ is least than every other value assumed by $f(x)$ in the immediate neighbourhood at $x = b$. Symbolically if

$$\left.\begin{matrix} f(b) < f(b+h) \\ f(b) < f(b-h) \end{matrix}\right] \Rightarrow x = b \text{ gives minima for a sufficiently small positive h.}$$

Note that :



(i) the maximum & minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) the term 'extremum' or (extremal) or 'turning value' is used both for maximum or a minimum value.

(iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) a function can have several maximum & minimum values & a minimum value may even be greater than a maximum value.

(v) maximum & minimum values of a continuous function occur alternately & between two consecutive maximum values there is a minimum value & vice versa.

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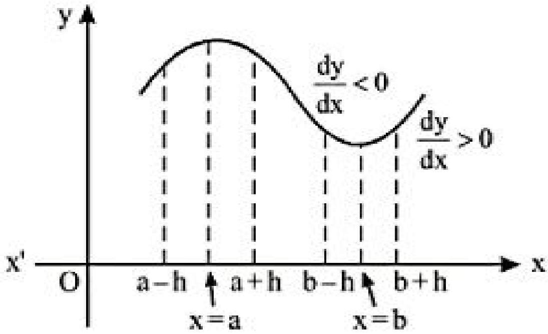
Tests for local maximum/minimum, when f(x) is differentiable :

(1) First-order derivative test in Ascertaining the maxima or minima :

Consider the interval $(a-h, a)$, we find $f(x)$ is increasing $\Rightarrow \frac{dy}{dx} > 0$. Similarly, for the interval $(a, a-h)$, we find $f(x)$ is decreasing $\Rightarrow \frac{dy}{dx} < 0$. Hence, at the point $x = a$ (maxima); $\frac{dy}{dx} = 0$.

Similarly, $\frac{dy}{dx} = 0$ at $x = b$ which is the point of minima.



Hence $\frac{dy}{dx} = 0$ is the necessary condition for maxima or minima.

These points, where $\frac{dy}{dx}$ vanishes, are known as stationary points as instantaneous rate of change of function momentarily ceases at this point.

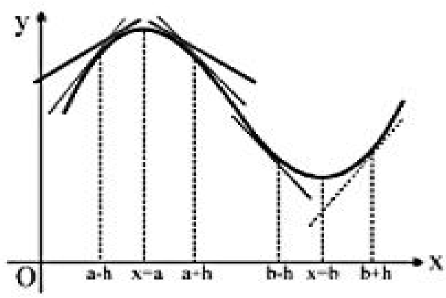
Hence, if $\left.\begin{matrix} f'(a-h) > 0 \\ f'(a+h) < 0 \end{matrix}\right] \Rightarrow x = a$ is a point of local maxima, where $f'(a) = 0$. It means that $f'(x)$ should change its sign from positive to negative.

Similarly, $\left.\begin{matrix} f'(b-h) < 0 \\ f'(b+h) < 0 \end{matrix}\right] \Rightarrow x = b$ is a point of local minima, where $f'(b) = 0$. It means that $f'(x)$ should change its sign from negative to positive.

However, if $f'(x)$ does not change sign, i.e., has the same sign in a certain complete neighbourhood of c, then $f(x)$ is either increasing or decreasing throughout this neighbourhood implying that $f(c)$ is not an extreme value of f, e.g., $f(x) = x^3$ at $x = 0$.

(2) Use of second order derivative in ascertaining the Maxima or Minima for a differentiable function:

As shown in the figure it is clear that as x increases from $a-h$ to $a+h$, the function $\frac{dy}{dx}$ continuously decreases, i.e. (+) ve for $x < a$, zero at $x = a$ and (–) ve for $x > a$. Hence $\frac{dy}{dx}$ itself is a decreasing function. Therefore $\frac{d^2y}{dx^2} < 0$ in $(a-h, a+h)$.



Hence at local maxima, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

$f'(a) = 0$ and $f''(a) < 0$

Similarly at local minima, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

i.e. $f'(b) = 0$ and $f''(b) > 0$

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Hence if

(a) f (a) is a maximum value of the function f then $f'(a) = 0$ & $f''(a) < 0$.

(b) f (b) is a minimum value of the function f, if $f'(b) = 0$ & $f''(b) > 0$.
However, if $f''(c) = 0$ then the test fails. In this case f can still have a maxima or minima or point of inflection (neither maxima nor minima). In this case revert back to the first order derivative check for ascertaning the maxima or minima.

(3) n th Derivative Test :

It is nothing but the general version of the second derivative test. It says that if $f'(a) = f''(a) = f'''(a) = \ldots\ldots f^n(a) = 0$ and $f^{n+1}(a) \neq 0$ (all derivatives of the function up to order n vanish and (n + 1)th order derivative does not vanish at x = a), then f (x) would have a local maximum or minimum at x = a iff n is odd natural number and that x = a would be a point of local maxima if $f^{n+1}(a) < 0$ and would be a point of local minima if $f^{n+1}(a) > 0$. However, if n is even, then f has neither a maxima nor a minima at x = a.

Illustration :

If $f(x) = \begin{cases} x^2, & x \le 0 \\ 2\sin x, & x > 0 \end{cases}$, *investigate the function at x = 0 for maxima/minima.*

Sol. *Analyzing the graph of f(x), we get x = 0 is a point of minima.*

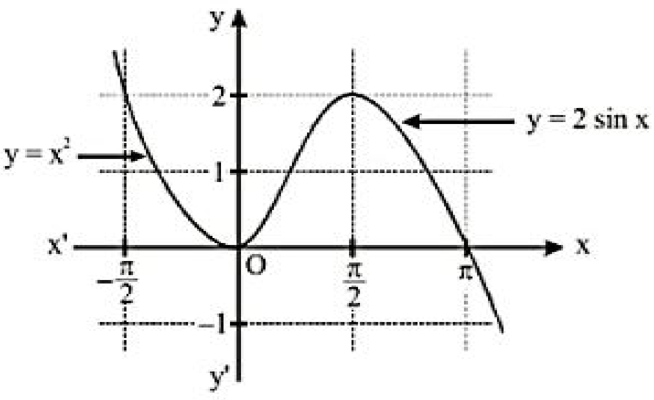

Illustration :

The function $y = \dfrac{ax+b}{(x-1)(x-4)}$ *has turning point at P(2, 1). Then find the value of a and b.*

Sol. $y = \dfrac{ax+b}{(x-1)(x-4)} = \dfrac{ax+b}{x^2-5x+4}$ *has turning point at P(2, –1)*

$\Rightarrow$ *P(2, –1) lies on the curve* $\Rightarrow$ $2a + b = 2$ ...(i)

Also $\dfrac{dy}{dx} = 0$ *at P(2, –1)*

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Now $\dfrac{dy}{dx} = \dfrac{a(x^2-5x+4)-(2x-5)(ax+b)}{(x^2-5x+4)}$

At $P(2, -1)$, $\dfrac{dy}{dx} = \dfrac{-2a+2a+b}{4} = 0$

$\Rightarrow$ $b = 0$ $\Rightarrow$ $a = 1$ [from equation (i)]

Illustration :

Find the points of maxima and minima of the function

$f(x) = 12x^5 - 45x^4 + 40\,x^3 + 40.$

Check whether second derivative can be used to find the point of extrema.

Sol. *We have*

$f(x) = 12x^5 - 45x^4 + 40x^3 + 40$

$f'(x) = 60x^4 - 180x^3 + 120x^2 = 60x^2\,(x-1)\,(x-2)$

and $f''(x) = 60\,(4x^3 - 9x^2 + 4x)$

The critical points of $f(x)$ are

$x = 0, 1, 2$

At the critical points, we have

$f''(x) = 0 \Rightarrow$ *more investigation required*

$f''(1) = 60\,(4-9+4) < 0 \Rightarrow$ *maxima at* $x = 1$

and $f''(2) = 60\,(32-36+8) > 0 \Rightarrow$ *minima at* $x = 2$.

Hence, we can see that the nature of the critical points cannot be predicted by the use of second derivative.

Since $f'(x)$ does not change sign as x passes through 0, hence $x = 0$ is not an extrema.

Illustration :

Discuss the extremum of $f(x) = x^2 + \dfrac{1}{x^2}$.

Sol. $f(x) = x^2 + \dfrac{1}{x^2}$

$f'(x) = 2x - \dfrac{2}{x^3}$

Let $f'(x) = 0$ $\Rightarrow$ $x^4 = 1$ $\Rightarrow$ $x = \pm 1$

Also, $f''(x) = 2 + \dfrac{6}{x^4} > 0$ for all $x \neq 0$

$\Rightarrow$ *Both the points* $x = 1$ *and* $x = -1$ *are the points of minima.*

Illustration :

The function $f(x) = (4\sin^2 x - 1)^n\,(x^2 - x + 1)$, $n \in N$, *has a local minimum at* $x = \dfrac{\pi}{6}$, *then*

(A) *n is any even number* (B) *n is an odd number*

(C) *n is odd prime number* (D) *n is any natural number*

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Sol. $f(x) = (4\sin^2 x - 1)^n (x^2 - x + 1)$

$x^2 - x + 1 > 0 \ \forall x \in R$

$$f\left(\frac{\pi}{6}\right) = 0 \quad \Rightarrow \quad f\left(\frac{\pi^+}{6}\right) = \lim_{x \to \frac{\pi^+}{6}} (4\sin^2 x - 1)^n (x^2 - x + 1) = \to 0^+$$

$$f\left(\frac{\pi^-}{6}\right) = \lim_{x \to \frac{\pi^-}{6}} (4\sin^2 x - 1)^n (x^2 - x + 1) \to (0^-)^n \ \text{(a positive value)}$$

$$f\left(\frac{\pi^-}{6}\right) > 0 \text{ if } n \text{ is an even number.}$$

When F(x) is not differentiable at x = a :

Case-I :

When f(x) is continuous at $x = a$ and $f'(a-h)$ and $f'(a+h)$ exist and are non-zero, then f(x) has a local maximum or minimum at $x = a$ if $f'(a-h)$ and $f'(a+h)$ are of opposite signs.

If $f'(a-h) > 0$ and $f'(a+h) < 0$, then $x = a$ will be a point of local maximum.

If $f'(a-h) < 0$ and $f'(a+h) > 0$, then $x = a$ will be a point of local minimum.

Case-II :

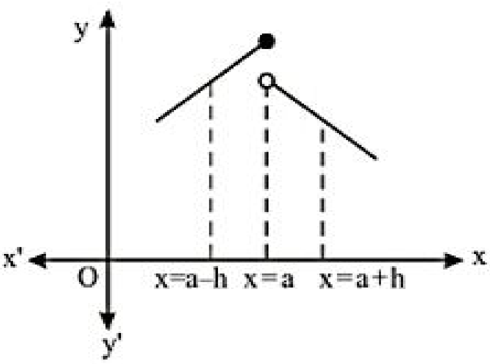
When f(x) is continuous and $f'(a-h)$ and $f'(a+h)$ exist but one of them is zero, we should infer the information about the existence of local maxima/minima from the basic definition of local maxima/minima.

Case-III :

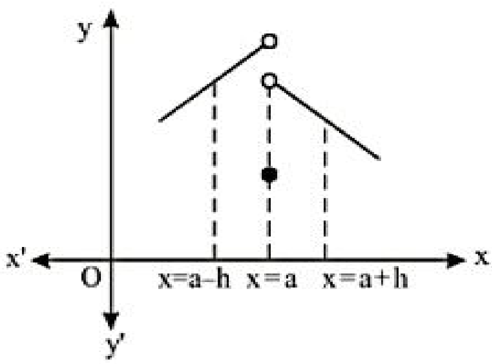
If $f(x)$ is not continuous at $x = a$ and $f'(a-h)$ and/or $f'(a+h)$ are not finite, then compare the values of $f(x)$ at the neighbouring points of $x = a$.

It is advisable to draw the graph of the function in the vicinity of the point $x = a$, because the graph would give us the clear picture about the existence of local maxima/minima at $x = a$.

Consider the following cases :



$x = a$ is the point of maxima as
$f(a) > f(a-h)$ and
$f(a) > f(a+h)$



$x = a$ is the point of minima as
$f(a) < f(a-h)$ and
$f(a) < f(a+h)$

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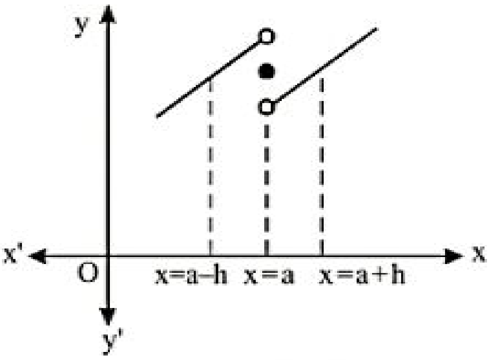
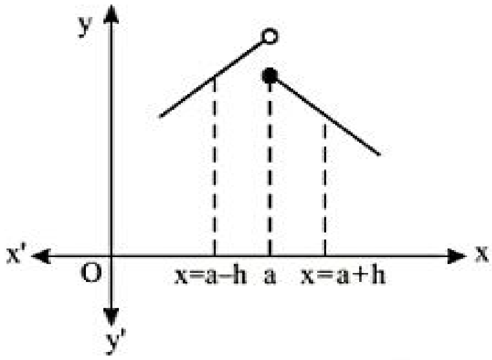


$x = a$ is not the point of extremum as
$f(a) < f(a-h)$ and
$f(a) > f(a+h)$



$x = a$ is not the point of extremum as
$f(a) < f(a-h)$ and
$f(a) > f(a+h)$

Practice Problem

Q.1 Consider the function $f(x) = \sin^3 x + \lambda \sin^2, \ x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Find the possible values of λ such that f(x) has exactly one minima and one maxima.

Q.2 $f(x) = \begin{cases} \cos\frac{\pi x}{2}, & x > 0 \\ x + a, & x \le 0 \end{cases}$. Find the values of a if $x = 0$ is a point of maxima.

Q.3 Investigate for maxima & minima for the function, $f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]\,dt$

Q.4 Let $f(x)$ be a cubic polynomial which has local maximum at $x = -1$ and $f'(x)$ has a local minimum at $x = 1$. If $f(-1) = 10$ and $f(3) = -22$, then find the distance between its two horizontal tangents.

Q.5 If $f(x) = x^5 - 5x^4 + 5x^3 - 10$ has local maximum and minimum at $x = p$ and $x = q$, respectively, then $(p, q) =$

(A) (0, 1) (B) (1, 3) (C) (1, 0) (D) None of these

Answer key

Q.1 $\frac{-3}{2} < \lambda < \frac{3}{2}$ and $\lambda \ne 0$.

Q.2 $a \ge 1$

Q.3 Max. at $x = 1$; $f(1) = 0$, Min. at $x = 7/5$; $f(7/5) = -108/3125$

Q.4 32

Q.5 B

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Concept of global maximum/minimum :

Let $y = f(x)$ be a given function with domain D. Let $[a, b] \subseteq D$. Global maximum / minimum of $f(x)$ in [a, b].

Global maximum and minimum in [a, b] would occur at critical point f(x) within [a, b] or at the endpoints of the interval.

Global maximum/minimum in [a, b] :

In order to find the global maximum and minimum of f(x) in [a, b], find the critical points of f(x) in (a, b).
Let $c_1, c_2, \ldots\ldots, c_n$ be the different critical points. Find the value of the function at these critical points.
Let $f(c_1), f(c_2), \ldots\ldots, f(c_n)$ be the values of the function at critical points.
Say, $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots\ldots, f(c_n), f(b)\}$
and $M_2 = \min\{f(a), f(c_1), f(c_2), \ldots\ldots, f(c_n), f(b)\}$
Then M_1 is the greatest value of f(x) in [a, b] and M_2 is the least value of f(x) in [a, b].

Illustration :

Find the maximum value of $f(x) = \left(\frac{1}{x}\right)^x$

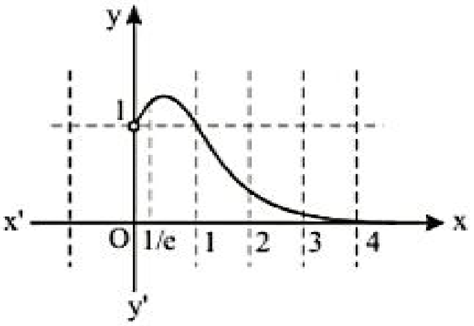
Sol. $f(x) = \left(\frac{1}{x}\right)^x \Rightarrow f'(x) = \left(\frac{1}{x}\right)^x \left(\ln\frac{1}{x} - 1\right)$

$f'(x) = 0 \Rightarrow \ln\frac{1}{x} = 1 \Rightarrow \frac{1}{x} = e \Rightarrow x = \frac{1}{e}$

Also for $x < \frac{1}{e}$, $f'(x)$ *is positive and for* $x > \frac{1}{e}$, $f'(x)$ *is negative.*

Hence, $x = \frac{1}{e}$ *is point of maxima.*

Therefore, the maximum value of function is $e^{1/e}$.



Also $\lim_{x\to 0}\left(\frac{1}{x}\right)^x = e^{\lim_{x\to 0} x\ln\left(\frac{1}{x}\right)} = e^{-\lim_{x\to 0} x\ln x} = e^0 = 1$

$\lim_{x\to 0}\left(\frac{1}{x}\right)^x = 1.$

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Illustration :

Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. *Discuss the global maxima and minima of* $f(x)$ *in* $[0, 2]$ *and* $(1, 3)$.

Sol. $f(x) = 2x^3 - 9x^2 + 12x + 6$

$\Rightarrow \quad f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$

Clearly the critical point of $f(x)$ *in* $[0, 2]$ *is* $x = 1$.

Now $f(0) = 6,\ f(1) = 11,\ f(2) = 10$

Thus $x = 0$ *is the point of minimum of* $f(x)$ *in* $[0, 2]$ *and* $x = 1$ *is the point of global maximum.*

For $x \in (1, 3)$

Clearly $x = 2$ *is the only critical point in* $(1, 3)$

$$f(2) = 10. \quad \lim_{x \to 1^+} f(x) = 11 \text{ and } \lim_{x \to 3^-} f(x) = 15$$

Thus $x = 2$ *is the point of global minimum in* $(1, 3)$ *and the global maximum in* $(1, 3)$ *does not exist.*

Illustration :

Find the greatest and least values of function $f(x) = 3x^4 - 8x^3 - 18x^2 + 1$.

Sol. *We have*

$$f(x) = 3x^4 - 8x^3 - 18x^2 + 1$$

and $f'(x) = 12x^3 - 24x^2 - 36x = 12x(x+1)(x-3)$.

The points at which $f(x)$ *may have extreme values, are the critical points* $x = -1, 0, 3$

and the end points $x = \pm\infty$.

Now, $f(-1) = -6, f(0) = 1, f(3) = -134$, *and* $f(\pm\infty) \to +\infty$.

Hence, the least value of the function is -134 *whereas the greatest value does not exist.*

Illustration :

Find the greatest and least values of function $f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 2-(x-1)^2, & 0 \le x \le 2 \end{cases}$.

Sol.

We have

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 2-(x-1)^2, & 0 \le x \le 2 \end{cases} \quad \text{and} \quad f'(x) = \begin{cases} -1, & -1 < x < 0 \\ -2x(x-1), & 0 < x < 2 \end{cases}$$

Thus, the points at which $f(x)$ *may have extreme values, are the critical points* $x = 0, 1$

$[f'(1) = 0$ *and* $f'(0) = DNE]$

and the end points $x = -1, 2$

Now, $f(-1) = 1, f(1) = 2$ *and* $f(2) = 1$.

Since f *is discontinuous at* $x = 0$, *we also need to find the limiting values of* $f(x)$ *as* $x \to 0$.

We have

$$f(0^-) \to 0, f(0^+) \to 1 \text{ and } f(0) = 1$$

the largest and the smallest among the above six values are 2 and 0 respectively.

Hence, the greatest value is 2 but the least value does not exist since the function approaches 0 but is never equal to 0.

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Illustration :

Find greatest and least values of $f(x) = \dfrac{a^2}{x} + \dfrac{b^2}{1-x}$, $x \in (0, 1)$ $(a, b > 0)$.

Sol. *We have*

$$f(x) = \frac{a^2}{x} + \frac{b^2}{1-x},\ x \in (0, 1)$$

and $$f'(x) = \frac{-a^2}{x^2} + \frac{b^2}{(1-x)^2}$$

which exists everywhere in $(0, 1)$ *and vanishes at points, given by*

$$\frac{b^2}{(1-x^2)} = \frac{a^2}{x^2}$$

$$a^2(1-x)^2 = b^2x^2$$

i.e. $a(1-x) = bx$ *i.e.* $x = \dfrac{a}{a+b}$

To find the greatest and least value, we need to check the values of $f(x)$ *at* $x = 0^+, 1^-, \dfrac{a}{a+b}$.

We have $f(0^+) \to +\infty, f(1^-) \to +\infty$ *and* $f\left(\dfrac{a}{a+b}\right) = (a+b)^2$

Hence, we have

least value $= (a+b)^2$

and greatest value does not exist. ***Ans.***

Illustration :

Find greatest and least values of $f(x) = \dfrac{(a+x)(b+x)}{(c+x)}$, $x > -c$.

Sol. *We have* $f(x) = \dfrac{(a+x)(b+x)}{(c+x)}$, $x \in (-c, \infty)$

and $$f'(x) = \frac{(c+x)(2x+a+b) - [x^2+(a+b)x+ab]}{(c+x)^2}$$

$$= \frac{x^2+2cx+ac+bc-ab}{(c+x)^2},\ x \in (-c, \infty)$$

which vanishes at points given by

$$x^2 + 2cx + ac + bc - ab = 0$$

i.e. $x = -c \pm \sqrt{c^2-(ac+bc-ab)} = -c \pm \sqrt{(a-c)(b-c)}$

Thus, the expression for $f'(x)$ *can be written as*

$$f'(x) = \frac{(x-\alpha)(x-\beta)}{(c+x)^2}$$

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choosing $\alpha = -c - \sqrt{(a-c)(b-c)}$ *and* $\beta = -c + \sqrt{(a-c)(b-c)}$

The critical point $x = \alpha$ *is of no interest since it does lie in the interval* $(-c, \infty)$.

Now, we have

$$f(-c^+) \to \infty, f(\infty) \to \infty$$

and $$f(\beta) = \frac{(a-c+\sqrt{(a-c)(b-c)})(b-c+\sqrt{(a-c)(b-c)})}{c-c+\sqrt{(a-c)(b-c)}}$$

$$= \frac{(a-c)(b-c)+(a+b-2c)\sqrt{(a-c)(b-c)}+(a-c)(b-c)}{\sqrt{(a-c)(b-c)}}$$

$$= 2\sqrt{(a-c)(b-c)} + a + b - 2c$$

$$= a - c + b - c + 2\sqrt{(a-c)(b-c)}$$

$$= \left(\sqrt{(a-c)} + \sqrt{(b-c)}\right)^2$$

Hence, we have

Least value $= \left(\sqrt{(a-c)} + \sqrt{(b-c)}\right)^2$ *and greatest value does not exist.*

Practice Problem

Q.1 Find greatest and least values of $f(x) = x - \sin 2x + \frac{1}{3}\sin 3x, x \in [0, \pi]$.

Q.2 Discuss the global maxima and global minima of $f(x) = \tan^{-1} x - \log_e x$ in $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$.

Q.3 The maximum value of $x^4 e^{-x^2}$ is

(A) e^2 (B) e^{-2} (C) $12e^{-2}$ (D) $4e^{-2}$

Q.4 Find the greatest & least value for the function ;

(i) $y = x + \sin 2x, \ 0 \le x \le 2\pi$ (ii) $y = 2\cos 2x - \cos 4x, 0 \le x \le \pi$

Answer key

Q.1 Least value $= -\frac{3\sqrt{3}-(\pi+2)}{6}$ and greatest value $= \frac{5\pi+2+3\sqrt{3}}{6}$

Q.2 $\frac{\pi}{3} - \ln\sqrt{3}, \ \frac{\pi}{6} - \ln\frac{1}{\sqrt{3}}$ Q.3 D

Q.4 (i) Max at $x = 2\pi$, Max value $= 2\pi$, Min. at $x = 0$, Min value $= 0$

(ii) Max at $x = \pi/6$ & also at $x = 5\pi/6$ and Max value $= 3/2$, Min at $x = \pi/2$, Min value $= -3$

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PROBLEMS BASED ON MENSURATION AND GEOMETRY :

Summary-Working Rule :

(1) When possible, draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

(2) Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

(3) If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

(4) Test each values of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither. The usual tests are :

(a) If d^2y/dx^2 is positive when $dy/dx = 0 \Rightarrow$ y is minimum.
If d^2y/dx^2 is negative when $dy/dx = 0 \Rightarrow$ y is maximum.
If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

(b) If $\frac{dy}{dx}$ is $\left.\begin{array}{lll} \text{positive} & \text{for} & x < x_0 \\ \text{zero} & \text{for} & x = x_0 \\ \text{negative} & \text{for} & x > x_0 \end{array}\right] \Rightarrow$ a maximum occurs at $x = x_0$.

But if dy/dx changes sign from negative to zero to positive as x advances through x_0 there is a minimum. If dy/dx does not change sign, neither a maximum nor a minimum. Such points are called INFLECTION POINTS.

(5) If the function $y = f(x)$ is defined for only a limited range of values $a \le x \le b$ then examine $x = a$ & $x = b$ for possible extreme values.

(6) If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Useful formulae of Mensuration to remember :

- ☞ Volume of a cuboid = lbh.
- ☞ Surface area of a cuboid = $2(lb + bh + hl)$.
- ☞ Volume of a prism = area of the base x height.
- ☞ Lateral surface of a prism = perimeter of the base x height.
- ☞ Total surface of a prism = lateral surface + 2 area of the base
  (Note that lateral surfaces of a prism are all rectangles).
- ☞ Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).
- ☞ Curved surface of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.
  (Note that slant surfaces of a pyramid are triangles).

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- Volume of a cone $= \frac{1}{3}\pi r^2 h$.
- Curved surface of a cylinder $= 2\pi r h$.
- Total surface of a cylinder $= 2\pi r h + 2\pi r^2$.
- Volume of a sphere $= \frac{4}{3}\pi r^3$.
- Surface area of a sphere $= 4\pi r^2$.
- Area of a circular sector $= \frac{1}{2} r^2 \theta$, when θ is in radians.

Illustration :

Find two positive numbers x and y such that $x + y = 60$ and $x^3 y$ is maximum.

Sol. $x + y = 60$

$\Rightarrow y = 60 - x$

$\Rightarrow x^3 y = (60 - x)x^3$

Let $f(x) = (60 - x)\, x^3$; $x \in (0, 60)$

For maximizing $f(x)$*, let us find critical points*

$f'(x) = 3x^2 (60 - x) - x^3 = 0$

$f'(x) = x^2 (180 - 4x) = 0$

$\Rightarrow x = 45$ ($\because\ x \neq 0$)

$f'(45^+) < 0$ *and* $f'(45) > 0$

Hence local maxima at $x = 45$.

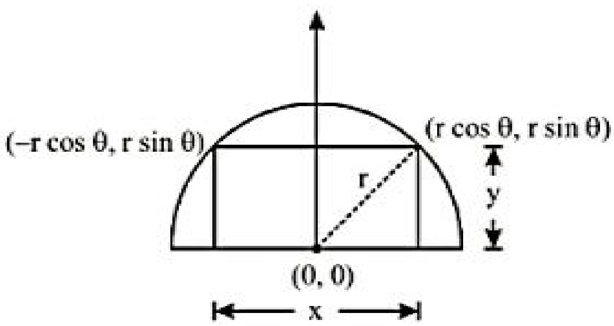
So $x = 45$ *and* $y = 15$

Illustration :

Rectangles are inscribe inside a semi-circle of radius r. Find the rectangle with maximum area.

Sol. *Let us choose co-ordinate system with origin as centre of circle*

Area, $A = xy$



$\Rightarrow \quad A = 2(r\cos\theta)(r\sin\theta),\ \theta \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \quad A = r^2 \sin 2\theta$

A is maximum when $\sin 2\theta = 1 \ \Rightarrow\ 2\theta = \frac{\pi}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

$\Rightarrow$ *Sides of the rectangle are* $2r\cos\left(\frac{\pi}{4}\right) = \sqrt{2}\, r$ *and* $r\sin\left(\frac{\pi}{4}\right) = \frac{r}{\sqrt{2}}$.

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Illustration :

The tangent to the parabola $y = x^2$ has been drawn so that the abscissa x_0 of the point of tangency belong to the interval (1, 2). Find x_0 for which the triangle is to be bounded by the tangent, the axis of ordinates, and the straight line $y = x_0^2$ has the greatest area.

Sol. $y = x^2,\ \dfrac{dy}{dx} = 2x$

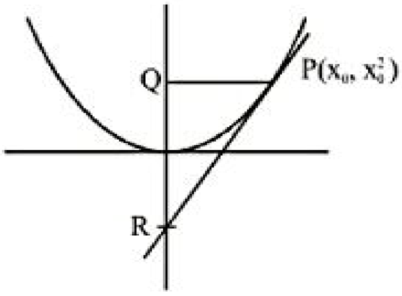
$\Rightarrow$ Equation of the tangent at (x_0, x_0^2) is $y - x_0^2 = 2x_0(x - x_0)$. It meets y-axis in $R(0, -x_0^2)$.

Q is $(0, x_0^2)$

$\Rightarrow$ Z = area of the triangle PQR

$$= \frac{1}{2} 2x_0^2 x_0 = x_0^3,\ 1 \le x_0 \le 2$$

$$\frac{dZ}{dx_0} = 3x_0^2 > 0 \text{ in } 1 \le x_0 \le 2$$



$\Rightarrow$ Z is an increasing function in [1, 2]

Hence, Z, i.e., the area of ΔPQR is greatest at $x_0 = 2$.

Illustration :

A sheet of area 40 m^2 is used to make an open tank with square base. Find the dimensions of the base such that volume of this tank is maximum.

Sol. Let the length of base be x m and height be y m

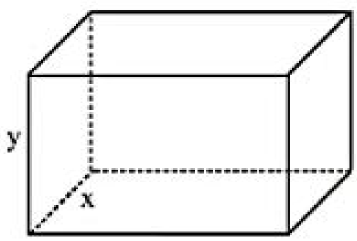
Volume $V = x^2y$

Again x and y are related to the surface area of this tank which is equal to 40 m^2.

$\Rightarrow$ $x^2 + 4xy = 40$

$$y = \frac{40 - x^2}{4x},\quad x \in \left(0, \sqrt{40}\right)$$

$$\Rightarrow \quad V(x) = x^2\left(\frac{40 - x^2}{4x}\right) = \frac{40x - x^3}{4}$$



Maximizing volume,

$$V'(x) = \frac{40 - 3x^2}{4} = 0 \Rightarrow x = \sqrt{\frac{40}{3}}\, m$$

and $$V''(x) = \frac{-3x}{2} \quad \Rightarrow V''\left(\sqrt{\frac{40}{3}}\right) < 0$$

$\Rightarrow$ volume is maximum at $x = \sqrt{\dfrac{40}{3}}$ m.

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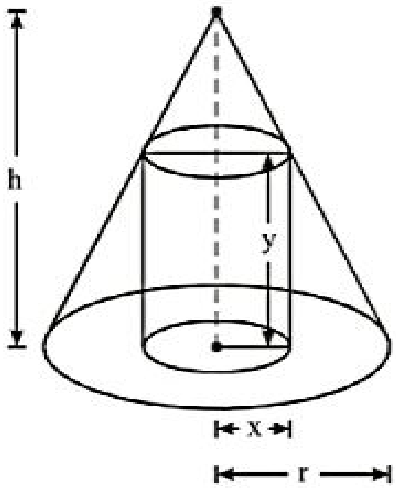
Illustration :

If a right-circular cylinder is inscribed in a given cone. Find the dimensions of the cylinder such that its volume is maximum.

Sol. *Let x be the radius of cylinder and y be its height*

Volume $V = \pi x^2 y$

x, y can be related by using similar triangles



$$\frac{y}{r-x} = \frac{h}{r}$$

$$\Rightarrow \quad y = \frac{h}{r}(r-x)$$

$$\Rightarrow \quad V(x) = \pi x^2 \frac{h}{r}(r-x),\ x \in (0, r)$$

$$\Rightarrow \quad V(x) = \frac{\pi h}{r}(rx^2 - x^3)$$

$$\Rightarrow \quad V'(x) = \frac{\pi h}{r} x(2r - 3x)$$

$$V'(x) = 0 \quad \Rightarrow \quad x = \frac{2r}{3}$$

Also $\quad V''(x) = \frac{\pi h}{r}(2r - 6x) \quad \Rightarrow \quad V''\left(\frac{2r}{3}\right) < 0$

This volume is maximum when, $x = \frac{2r}{3}$ *and* $y = \frac{h}{3}$.

Practice Problem

Q.1 For a right circular cone of given total surface area (including the base) and maximum volume, find the value of the semi-vertical angle.

Q.2 If the sum of the lengths of the hypotenuse and another side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between these sides is $\pi/3$.

Q.3 Find the point (α, β) on the ellipse $4x^2 + 3y^2 = 12$, in the first quadrant, so that the area enclosed by the lines $y = x$, $y = \beta$, $x = \alpha$ and the x-axis is maximum.

Q.4 A rectangle is inscribed in an equilateral trianle of side length 2a units. The maximum are of this rectangle can be

(A) $\sqrt{3}\,a^2$ (B) $\frac{\sqrt{3}\,a^2}{4}$ (C) a^2 (D) $\frac{\sqrt{3}\,a^2}{2}$

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Q.5 Tangents are drawn to $x^2 + y^2 = 16$ from the point P(0, h). These tangents meet the x-axis at A and B. If the area of triangle PAB is minimum, then

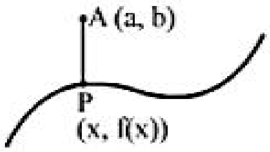
(A) $h = 12\sqrt{2}$ (B) $h = 6\sqrt{2}$ (C) $h = 8\sqrt{2}$ (D) $h = 4\sqrt{2}$

Answer key

Q.1 $\theta = \sin^{-1}(1/3)$ Q.3 $\left(\frac{3}{2}, 1\right)$ Q.4 D Q.5 D

GENERAL CONCEPT :

Given a fixed point A (a, b) and a moving point P (x, f (x)) on the curve y = f (x). Then AP will be maximum or minimum if it is normal to the curve at P.



Significance of the Sign of 2nd order Derivative and Points of Inflection :

A point where the graph of function is continuous and has a tangent line and where the concavity changes is called point of inflection.

- At the point of inflection either $y'' = 0$ and changes sign or y'' fails to exist.
- At the point of inflection, the curve crosses its tangent at that point.
- A function can not have point of inflection and extrema at same point.

Note: If $\frac{d^2y}{dx^2} > 0$ then y is concave up and if $\frac{d^2y}{dx^2} < 0$ the y is concave down.

Illustration :

$f(x) = x^{1/5}$ at $x = 0$ has inflection point.
y'' D.N.E. at $x = 0$
Note that $f(x)$ has a vertical tangent and the curve crosses its tangent line.

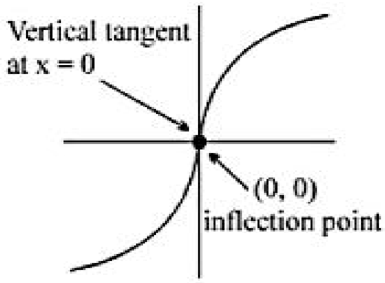


Illustration :

$f(x) = x^3$ at $x = 0$ has inflection point
$y'' = 0$ at $x = 0$ and changes sign

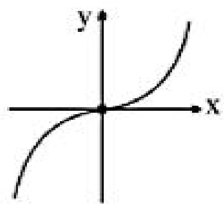


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Illustration :

$f(x) = |x^2 - 1|$
has no inflection point in its domain.
as no tangent can be drawn at these points.

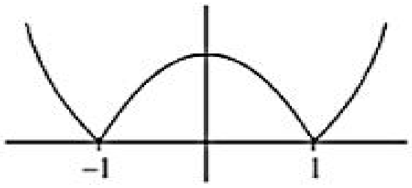


Illustration :

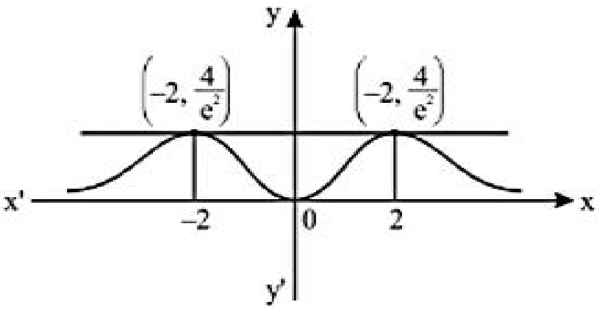
Number of points of inflection for $f(x) = x^2 e^{-|x|}$ is
(A) 1 *(B) 2* *(C) 3* *(D) 4*

Sol. *We have* $f(x) = x^2 e^{-|x|} = \begin{cases} x^2 e^{-x}, & x \geq 0 \\ x^2 e^{x}, & x < 0 \end{cases}$

$\Rightarrow f'(x) = \begin{cases} e^{-x}(2x - x^2), & x \geq 0 \\ e^{x}(x^2 + 2x), & x < 0 \end{cases}$

$f(x)$ increases in $(-\infty, -2) \cup (0, 2)$
and $f(x)$ decreases in $(-2, 0) \cup (2, \infty)$

$\Rightarrow f''(x) = \begin{cases} e^{-x}(x^2 - 4x + 2), & x \geq 0 \\ e^{x}(x^2 + 4x + 2), & x < 0 \end{cases}$

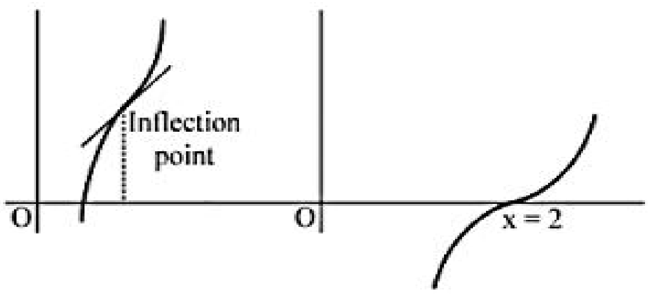


DIFFERENT GRAPHS OF THE CUBIC:

$y = ax^3 + bx^2 + cx + d$

(1) One real & two imaginary roots. (always monotonic) $\forall \; x \in R$

Condition :
$f'(x) \geq 0$ or $f'(x) \leq 0$ together with either $f'(x) = 0$ has no root (i.e. $D < 0$) or $f'(x) = 0$ has a root $x = \alpha$ then $f(\alpha) = 0$.



(i) either $f'(x) = 0$ has no real root
or (ii) if $f'(x) = 0$ has a root $x = \alpha$ then $f(\alpha) = 0$

e.g. $y = x^3 - 2x^2 + 5x + 4$ $\qquad$ $y = (x-2)^3$
$y' = 3x^2 - 4x + 5 \quad (D < 0)$ $\qquad$ $y' = 3(x-2)^2 = 0 \Rightarrow x = 2$, also $f(2) = 0$
gives $x = 2, \quad y(2) = 0$

Note: In this case if $f'(x) = 0$ has a root $x = \alpha$ and $f(\alpha) = 0$ this would mean $f(x) = 0$ has repeated roots which is dealt separately.

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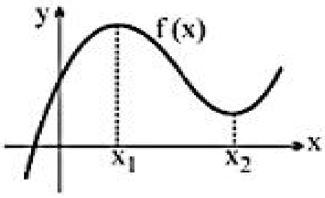
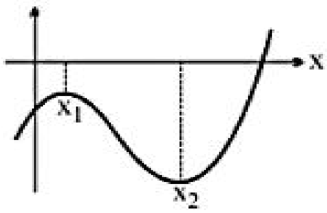
(2) Exactly one root and non monotonic.

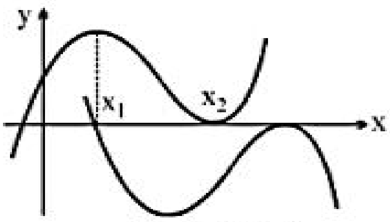




$f(x_1) \cdot f(x_2) > 0$

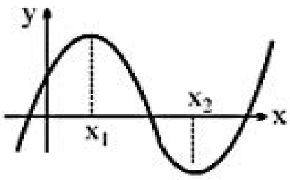
where x_1 & x_2 are the roots of $f'(x) = 0$

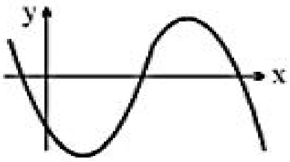
(3) Three roots < two concident / One different



$f(x_1) \cdot f(x_2)=0$

(4) All three distinct real roots

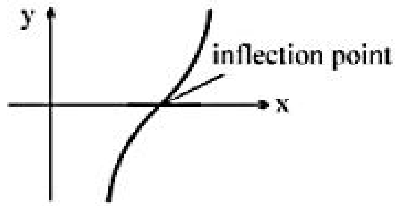




$f(x_1) \cdot f(x_2) < 0$

where x_1 & x_2 are the roots of $f'(x) = 0$

(5) All three roots concident



$f'(x) \geq 0$ or $f'(x) \leq 0$ & $f(\alpha) = 0$

where α is a root of $f'(x) = 0$

e.g. $y = (x - 1)^3$

Note :

(i) Graph of every cubic polynomial must have exactly one point of *inflection.*

(ii) In case (4) if f(a), f(b), f(c) and f(d) alternatively change sign.

Illustration :

Find the value of a if $x^3 - 3x + a = 0$ has three real distinct roots.

Sol. Let $f(x) = x^3 - 3x + a$

Let $f'(x) = 0$

$\Rightarrow \ 3x^2 - 3 = 0 \ \Rightarrow \ x = \pm 1$

For three distinct roots, $f(1)\, f(-1) < 0$

$\Rightarrow \ (1 - 3 + a)(-1 + 3 + a) < 0$

$\Rightarrow \ (a + 2)\,(a - 2) < 0$

$\Rightarrow \ -2 < a < 2$

Illustration :

Prove that three exist exactly two non-similar isosceles triangle ABC such that tan A+tan B+tan C=100.

Sol. Let $A = B$, then $2A + C = 180^o$ and $2 \tan A + \tan C = 100$

Now $\ 2A + C = 180^o \ \Rightarrow \ \tan 2A = -\tan C$ ...(i)

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Also $2 \tan A + \tan C = 100$

$\Rightarrow \quad 2 \tan A - 100 = -\tan C \qquad \ldots(ii)$

From (i) and (ii), $2 \tan A - 100 = \dfrac{2 \tan A}{1 - \tan^2 A}$

Let $\tan A = x$, then $\dfrac{2x}{1-x^2} = 2x - 100$

$\Rightarrow \quad x^3 - 50x^2 + 50 = 0$

Let $f(x) = x^3 - 50x^2 + 50$. Then $f'(x) = 3x^2 - 100x$. Thus $f'(x) = 0$ has roots $0, \dfrac{100}{3}$. Also

$f(0)\, f\left(\dfrac{100}{3}\right) < 0$. Thus $f(x) = 0$ has exactly three distinct real roots. Therefore, $\tan A$ and hence A has three distinct values but one of them will be obtuse angle. Hence, there exist exactly two non similar isosceles triangles.

Illustration :

Find the set of value of m for the cubic $x^3 - \dfrac{3}{2}x^2 + \dfrac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

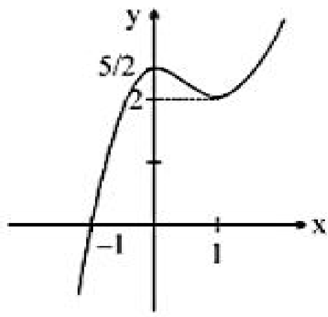
Sol. Consider $y = x^3 - \dfrac{3}{2}x^2 + \dfrac{5}{2}$

$$\frac{dy}{dx} = 3x^2 - 3x = 3x(x-1) = 0 \quad \Rightarrow \quad x = 0 \text{ or } 1$$

$$\frac{d^2y}{dx^2} = 6x - 3 \;;\; \left.\frac{d^2y}{dx^2}\right|_{x=0} = -3 \text{ i.e. } < 0 \Rightarrow \text{maximum at } x = 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \text{is } 3 \text{ i.e.} > 0 \Rightarrow \text{mimimum}$$

Hence the graph of the cubic is now for 3 distinct roots



$$2 < \log_{1/4}(m) < \frac{5}{2}$$

$$2 < -\log_4(m) < \frac{5}{2}$$

$$-\frac{5}{2} < \log_4(m) < -2$$

$$\frac{1}{32} < m < \frac{1}{16}$$

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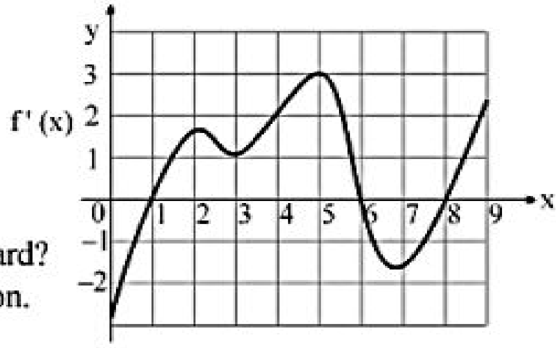
Practice Problem

Q.1 The graph of the derivative f' of a continuous function f is shown with $f(0) = 0$



(i) On what intervals is f increasing or decreasing?

(ii) At what values of x does f have a local maximum or minimum?

(iii) On what intervals is f concave upward or downward?

(iv) State the x-coordinate(s) of the point(s) of inflection.

(v) Assuming that $f(0) = 0$, sketch a graph of f.

Q.2 The set of value(s) of a for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ possesses a negative point inflection is

(A) $(-\infty, -2)$ (B) $\left\{-\frac{4}{5}\right\}$ (C) $(-2, 0)$ (D) empty set

Q.3 The number of values of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots lying in the interval (0, 1) is

(A) three (B) two (C) infinitely many (D) zero

Q.4 For the function $f(x) = x^4(12 \log_e x - 7)$

(A) there is no point of inflection (B) $x = e^{1/3}$ is the point of minima

(C) the graph is concave upwards in (0, 1) (D) the graph is concave downwards in $(1, \infty)$

Answer key

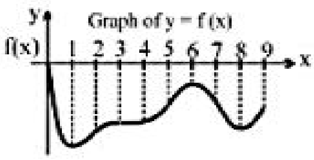
Q.1 (i) I in $(1, 6) \cup (8, 9)$ and D in $(0, 1) \cup (6, 8)$;

(ii) L.Min. at $x = 1$ and $x = 8$; L.Max. $x = 6$

(iii) CU in $(0, 2) \cup (3, 5) \cup (7, 9)$ and CD in $(2, 3) \cup (5, 7)$;

(iv) $x = 2, 3, 5, 7$

(v) One of the possible graph is



Q.2 A Q.3 D Q.4 B

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Solved Examples

Q.1 Find the shortest distance between the curves $y^2 = x^3$ and $9x^2 + 9y^2 - 30y + 16 = 0$.

Sol. We have

$$9x^2 + 9y^2 - 30y + 16 = 0$$

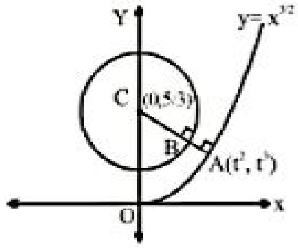
which is a circle having

$$\text{centre} \equiv \left(0, \frac{5}{3}\right) \text{ and radius} = \sqrt{\left(\frac{5}{3}\right)^2 - \frac{16}{9}} = 1.$$

Let us choose any point on the curve $y^2 = x^3$ as $A(t^2, t^3)$. If B is the point on the circle and nearest to A(see figure), then

$$AB = AC - \text{radius}$$

$$= \sqrt{(t^2 - 0)^2 + \left(t^3 - \frac{5}{3}\right)^2} - 1$$



Let $f(t) = t^4 + \left(t^3 - \frac{5}{3}\right)^2$

and $f'(t) = 4t^3 + 2\left(t^3 - \frac{5}{3}\right) \cdot 3t^2 = 2t^2\left[2t + 3\left(t^3 - \frac{5}{3}\right)\right]$

$$= 2t^2(3t^3 + 2t - 5) = 2t^2(t-1)(3t^2 + 3t + 5).$$

The value of 't' at which AB attains minima is given by the equation

$$f'(t) = 0$$

gives $t = 0, 1$

But $t = 0$ is not a point of extrema, since $f'(t)$ does not change sign in the neighbourhood of $t = 0$. However, $t = 1$ is a point of minima, since $f'(1^-) < 0$ and $f'(1^+) > 0$. Hence, the minimum value of AB is

$$\sqrt{1 + \left(1 - \frac{5}{3}\right)^2} - 1 \simeq 0.2. \text{ **Ans.**}$$

Q.2 Consider the function :

$$f(x) = \begin{cases} -x^3 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$$

Find all possible real values of b such that $f(x)$ has the least value at $x = 1$.

Sol. We have

$$f(x) = -x^3 + g(b), \qquad 0 \le x < 1$$
$$= 2x - 3, \qquad 1 \le x \le 3$$

where $g(b) = \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}$ (b is a constant)

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and $f'(x) = -3x^2, \quad 0 < x < 1$
$\qquad = 2, \qquad 1 < x < 3$

$\Rightarrow$ f(x) strictly decreases in (0, 1)
strictly increases in (1, 3).

Now, we have for minima at $x = 1$

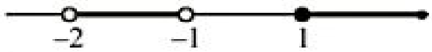
$f(1) \le f(1^-)$

i.e. $2 - 3 \le -1 + g(b)$

i.e. $g(b) \ge 0$

i.e. $\dfrac{(b^2+1)(b-1)}{(b+1)(b+2)} \ge 0$

The number line shown alongside, gives



$b \in (-2, -1) \cup [1, \infty)$.

Q.3 If the function

$$f(x) = (a + 3)x^3 + (a - 3)x^2 + 4(a - 4)x + 5$$

has maxima at some $x \in R^-$ and a minima at some $x \in R^+$, find the possible values of a.

Sol. We have

$$f(x) = (a + 3)\, x^3 + (a - 3)\, x^2 + 4\,(a - 4)\, x + 5$$

and $f'(x) = 3\,(a + 3)\, x^2 + 2\,(a - 3)\, x + 4\,(a - 4)$

According to the given condition, $f'(x)$ must vanish at two real and distinct points say α, β such that $\alpha < 0$ and $\beta > 0$. Thus, we have $f'(x) = 3\,(a + 3)\,(x - \alpha)\,(x - \beta)$

and $f''(x) = 3\,(a + 3)\,(x - \alpha + x - \beta)$.

According to the given condition, α is to be a maxima

i.e. $f''(\alpha) < 0$

i.e. $3\,(a + 3)\,(\alpha - \beta) < 0$

i.e. $a + 3 > 0 \qquad [\alpha - \beta < 0]$

i.e. $a > -3$

and β is to be a minima

i.e. $f''(\beta) > 0$

i.e. $3\,(a + 3)\,(\beta - \alpha) > 0$

i.e. $a + 3 > 0 \qquad [\beta - \alpha < 0]$

i.e. $a > -3$

Thus, taking together all the above results $(a + 3 > 0)$, the graph of

$$y = 3\,(a + 3)\, x^2 + 2\,(a - 3)\, x + 4\,(a - 4)$$

must look as shown alongside. For the curve to look like this, the neccessary and sufficient condition is

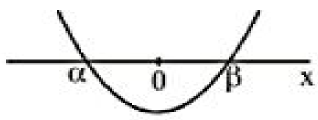
$y(0) < 0$

i.e. $4(a - 4) < 0$

i.e. $a < 4$

Hence, the possible values of a, are

$a \in (-3, 4)$.



[$a + 3 > 0 \Rightarrow$ concavity along the +ve Y-axis]

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Q.4 Find a point M on the curve $y = \frac{3}{\sqrt{2}}\ x\ ln\,x, x \in (e^{-1.5}, \infty)$ such that the segment of the tangent at M intercepted between M and the Y-axis is shortest.

Sol. We have

$$y = \frac{3}{\sqrt{2}}\ x\ ln\,x, x \in (e^{-1.5}, \infty)$$

and $$y' = \frac{3}{\sqrt{2}}(1 + ln\,x), x \in (e^{-1.5}, \infty) \qquad ....(1)$$

Any point on the curve can be chosen as $M \equiv \left(h, \frac{3}{\sqrt{2}} h\ ln\ h\right)$

Slope of the tangent at $M = \frac{3}{\sqrt{2}}\ (1 + ln\,h)$ [using eq. (1)]

Equation of the tangent at M, is given by

$$y - \frac{3}{\sqrt{2}} h\ ln\ h = \frac{3}{\sqrt{2}}\ (1 + ln\,h)\,(x - h) \qquad ......(2)$$

The tangent cuts the Y-axis at

$$N \equiv \left(0, \frac{-3h}{\sqrt{2}}\right) \qquad \text{[putting } x = 0 \text{ in eq. (2)]}$$

Length l of the tangent segment MN, is given by

$$l^2 = h^2 + \left\{\frac{3}{\sqrt{2}} h(1 + ln\ h)\right\}^2 = h^2 + \frac{9}{2}h^2\ (1 + ln\,h)^2$$

The value of h at which l^2 attains minima, is given by the equation

$$\frac{d}{dh}(l^2) = 0$$

i.e. $$2h + 9h\,(1 + ln\,h)^2 + 9h^2\,(1 + ln\,h)\left(\frac{1}{h}\right) = 0$$

i.e. $$h\left[(1 + ln\ h)^2 + (1 + ln\ h) + \frac{2}{9}\right] = 0$$

i.e. $$1 + ln\,h = \frac{-1 \pm \sqrt{1 - \frac{8}{9}}}{2} = \frac{-2}{3}, \frac{-1}{3} \qquad [h = 0 \text{ does not lie in } (e^{-1.5}, \infty)]$$

gives $h = e^{-5/3}, e^{-4/3}$.

Only $h = e^{-4/3}$ is acceptable since $e^{-5/3}$ does not lie in $(e^{-1.5}, \infty)$. Hence, the required point, is

$$M \equiv \left(e^{-4/3}, -2\sqrt{2}e^{-4/3}\right).$$

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Q.5 Assuming that the petrol burnt per unit time in driving a motor boat, varies as the cube of its velocity. Find the most economical speed of the motor boat when moving against a current whose speed is cm/sec.

Sol. Let s m/sec be the speed of the motor boat w.r.t stream. Then, the rate at which petrol is burnt, is given by

$$\frac{dp}{dt} = ks^3 \qquad \text{........(1)}$$

where k is a constant of proportionality.

Now that the boat is moving against a current of c m/sec, the absolute speed of the motor boat is (s – c) m/sec. Then, the rate at which distance is covered by the motor boat, is given by

$$\frac{dx}{dt} = s - c \qquad \text{........(2)}$$

Dividing equation (1) by equation (2), gives

$$\frac{dp}{dx} = \frac{ks^3}{s-c}$$

which is the amount of petrol consumed per unit distance travelled by the boat,.

Let $\quad f(s) = \dfrac{ks^3}{s-c}$

The value of s at which f(s) attains minima, is given by

$$\frac{df}{ds} = 0$$

i.e. $\quad \dfrac{(s-c)\,3s^2 - s^3}{(s-c)^2} = 0$

i.e. $\quad 2s - 3c = 0$

gives $\quad s = \dfrac{3}{2}c$

At this value of s, the petrol consumption per unit distance, is minimum and the absolute speed of the motor boat is

$$\frac{3}{2}c - c = \frac{3c}{2} \text{ m/sec.}$$

Q.6 Find the point on the curve $y = \dfrac{x}{1+x^2}$ where the tangent to the curve has the greatest slope.

Sol. We have

$$y = \frac{x}{1+x^2}$$

The slope of the tangent at any point on the curve, is given by

$$\frac{dy}{dx} = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

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The values of x at which the slope becomes maximum or minimum, is given by

$$\frac{d^2y}{dx^2}=0$$

i.e. $$\frac{d}{dx}\left[\frac{1-x^2}{(1+x^2)^2}\right]=0$$

i.e. $$\frac{(1+x^2)^2(-2x)-(1-x^2)\,2(1+x^2)\cdot 2x}{(1+x^2)^4}$$

i.e. $$\frac{2x\,(1+x^2)\,(x^2-3)}{(1+x^2)^4}=0$$

i.e. $x=0,\pm\sqrt{3}$.

Now, from the sign scheme for $\frac{d^2y}{dx^2}$, we have

$$\underset{-\sqrt{3}}{\overset{-ve\quad +ve}{\longleftarrow\!\!+\!\!}}\;\underset{0}{+}\;\overset{-ve\quad +ve}{\underset{\sqrt{3}}{+\!\!\longrightarrow}}$$

$\frac{d^2y}{dx^2}$ changes sign from + ve to – ve at $x=0$

$\Rightarrow$ $\frac{dy}{dx}$ is maximum at $x=0$

Hence, the point on the curve where slope of the tangent is maximum, is (0, 0).

Q.7 Discuss the extremum of $f(x)=a\sec x+b\operatorname{cosec} x,\ 0<a<b$.

Sol. $f(x)=a\sec x+b\operatorname{cosec} x,\ 0<a<b.$
$f'(x)=a\sec x\tan x-b\operatorname{cosec} x\cot x$

Let $f'(x)=0 \Rightarrow a\frac{\sin x}{\cos^2 x}-b\frac{\cos x}{\sin^2 x}=0$

$\Rightarrow$ $\tan^3 x=\frac{b}{a}$ $\Rightarrow$ $x=\tan^{-1}\left(\frac{b}{a}\right)^{1/3}$; $a, b>0$ $\Rightarrow$ $x=\tan^{-1}\left(\frac{b}{a}\right)^{1/3}>0$

$\Rightarrow$ x lies in either the first or third quadrant for extremum.

Case-I : $0<x<\frac{\pi}{2}$

$$\lim_{x\to 0}(a\sec x+b\operatorname{cosec} x)\to\infty$$

$$\lim_{x\to\frac{\pi}{2}}(a\sec x+b\operatorname{cosec} x)\to\infty$$

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Also f(x) is +ve for this value of x.
Hence, only one point of extremum is the point of minima.

and $\tan x = \left(\frac{b}{a}\right)^{1/3}$

$$\Rightarrow \quad \cos x = \frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}, \sin x = \frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$$

$$\Rightarrow \quad \text{Minimum value of } f = \frac{a\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + \frac{b\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}} = (a^{2/3}+b^{2/3})^{3/2}$$

Case-II : $\pi < x < \frac{3\pi}{2}$

$$\lim_{x\to\pi} (a \sec x + b \operatorname{cosec} x) \to -\infty$$

$$\lim_{x\to\frac{3\pi}{2}} (a \sec x + b \operatorname{cosec} x) \to -\infty$$

Also f(x) is –ve for this values of x.
Hence, only one point of extremum is the point of maxima.
$\Rightarrow$ Maximum value $f_{max} = -(a^{2/3}+b^{2/3})^{3/2}$

Q.8 The function $f(x) = |ax - b| + c\ |x|\ \forall\ x \in (-\infty, \infty)$, where $a > 0, b > 0, c > 0$. Find the condition if f(x) attains the minimum value only at one point.

Sol. $$f(x) = \begin{cases} b-(a+c)x, & x<0 \\ b+(c-a)x, & 0 \le x < \frac{b}{a} \\ (a+c)x+b, & x \ge \frac{b}{a} \end{cases}$$

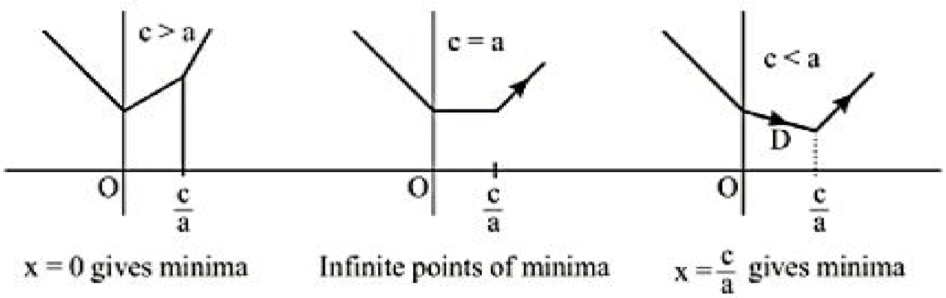

Figure clearly indicates that for exactly one point of minima, $a \ne c$.

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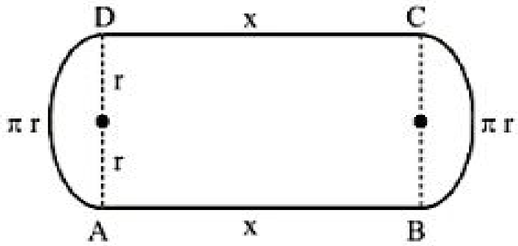
Q.9 A running track of 440 ft is to be laid out enclosing a football field, the shape of which is a rectangle with a semi-circle at each end. If the area of the rectangular portion is to be maximum, then find the lengths of its sides.

Sol. Perimeter = 440 ft.

$\Rightarrow \quad 2x + \pi r + \pi r = 440 \quad$ or $\quad 2x + 2\pi r = 440$

A = Area of the rectangular portion = x 2r

$$\Rightarrow A = x\frac{(440-2x)}{\pi} = \frac{1}{\pi}(440x - 2x^2)$$



Let $\quad \dfrac{dA}{dx} = \dfrac{1}{\pi}(440 - 4x) = 0$

$\Rightarrow \quad x = 110$ for which $\dfrac{d^2A}{dx^2} < 0$

$\Rightarrow \quad$ A is maximum when $x = 110$

$$\Rightarrow \quad 2r = \frac{440-2x}{\pi} = \frac{440-220}{22/7} = 70$$

$\Rightarrow \quad r = 35$ ft and $x = 110$ ft

Q.10 Find the points on the curves $5x^2 - 8xy + 5y^2 = 4$ whose distance from the origin is maximum or minimum.

Sol. Let (r, θ) be the polar coordinates of any point P on the curve where r is the distance of the point from the origin.

$\Rightarrow \quad r^2[5(\cos^2\theta + \sin^2\theta) - 8\sin\theta\cos\theta] = 4$

$$\Rightarrow \quad r^2 = \frac{4}{5 - 4\sin 2\theta}$$

r^2 is maximum when $5 - 4\sin 2\theta$ is minimum $= 5 - 4 = 1$ (when $\sin 2\theta = 1$)

$\Rightarrow \quad 2\theta = 90^\circ \Rightarrow \theta = 45^\circ \Rightarrow r = \pm 2,\ \theta = 45^\circ$ ......(i)

Again r^2 is minimum when $5 - 4\sin 2\theta$ is maximum

$$= 5 + 4 = 9 \text{ when } \sin 2\theta = -1 \Rightarrow 2\theta = \frac{3\pi}{2} \Rightarrow \theta = \frac{3\pi}{4}$$

$$\Rightarrow \quad r = \pm\frac{2}{3},\ \theta = \frac{3\pi}{4}$$

Hence, the points are $(r\cos\theta, r\sin\theta)$ where r and θ are given by equations (i) and (ii).

Thus, we get four points $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$, $\left(\dfrac{\sqrt{2}}{3}, -\dfrac{\sqrt{2}}{3}\right)$ and $\left(-\dfrac{\sqrt{2}}{3}, \dfrac{\sqrt{2}}{3}\right)$.

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Q.11 Discuss the monotonocity of $Q(x)$, where $Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)\ \forall\ x \in R$. It is given that $f''(x) > 0\ \forall\ x \in R$. Find also the points of maxima and minima of $Q(x)$.

Sol. Given $Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$$\therefore \quad Q'(x) = 2xf'\left(\frac{x^2}{2}\right) - 2xf'(6 - x^2) = 2x\left\{f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right\}$$

But given that $f''(x) > 0 \Rightarrow f'(x)$ is increasing for all $x \in R$.

Case-I : Let $\frac{x^2}{2} > (6 - x^2) \Rightarrow x^2 > 4$

$\therefore \quad x \in (-\infty, -2) \cup (2, \infty)$

$\Rightarrow \quad f\left(\frac{x^2}{2}\right) > f(6 - x^2)$

$\Rightarrow \quad f\left(\frac{x^2}{2}\right) - f(6 - x^2) > 0$

If $x > 0$, then $Q'(x) > 0 \quad \Rightarrow \quad x \in (2, \infty)$
and if $x < 0$, then $Q'(x) < 0 \quad \Rightarrow \quad x \in (-\infty, -2)$

Case-II : Let $\frac{x^2}{2} < (6 - x^2) \Rightarrow x^2 < 4 \Rightarrow x \in (-2, 2)$

$\Rightarrow \quad f'\left(\frac{x^2}{2}\right) < f'(6 - x^2) \Rightarrow f'\left(\frac{x^2}{2}\right) - f'(6 - x^2) < 0$

If $x > 0$, then $Q'(x) < 0 \quad \Rightarrow \quad x \in (0, 2)$
and If $x < 0$, then $Q'(x) > 0 \quad \Rightarrow \quad x \in (-2, 0)$

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Q.12 The largest term in the sequence $a_n = \dfrac{n^2}{n^3+200}$ is given by

(A) $\dfrac{529}{49}$ (B) $\dfrac{8}{89}$ (C) $\dfrac{49}{543}$ (D) None of these

Sol. Conisder the function $f(x) = \dfrac{x^2}{(x^3+200)}$ ......(i)

$$f'(x) = x\frac{(400-x^3)}{(x^3+200)^2} = 0$$

When $x = (400)^{1/3}$ $(\because\ x \neq 0)$

$x = (400)^{1/3} - h \Rightarrow f'(x) > 0$

$x = (400)^{1/3} + h \Rightarrow f'(x) < 0$

$\therefore$ $f(x)$ has maxima at $x = (400)^{1/3}$

Since $7 < (400)^{1/3} < 8$, either a_7 or a_8 is the greatest term of the sequence.

$\because$ $a_7 = \dfrac{49}{543}$ and $a_8 = \dfrac{8}{89}$ and $\dfrac{49}{543} > \dfrac{8}{89}$

$\Rightarrow$ $a_7 = \dfrac{49}{543}$ is the greatest term.

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