LIMIT

1. CONCEPT OF LIMITS :

Suppose f(x) is a real-valued function c is a real number. The expression $\lim_{x\to c} f(x) = L$ means that f(x) can be as close to L as desired by making x sufficiently close to c. In such a case, we say that limit of f, as x approaches c, is L. Note that this statement is true even if $f(c) \neq L$. Indeed, the function f(x) need not even be defined at c. Two examples help illustrate this.

Consider $f(x) = x - 1$ as x approaches 2. In this case, f (x) is defined at 2, and it equals its limiting value 1.

f(1.9)	f(1.99)	f(1.999)	f(2)	f(2.001)	f(2.01)	f(2.1)
0.9	0.99	0.999	$\rightarrow 1 \leftarrow$	1.001	1.01	1.1

As x approaches 2, f(x) approaches 1 and hence we have $\lim_{x\to 2} f(x) = 1$.

$f(x) = \dfrac{x^2 - 4}{x - 2}$ in this case x approaches 2 the limiting value of f (x) is equal to 4 even if f (x) is not defined at x = 2.

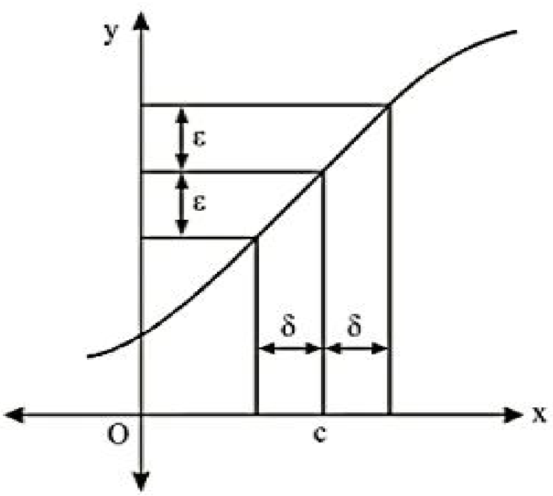
f(1.9)	f(1.99)	f(1.999)	f(2.0)	f(2.001)	f(2.01)	f(2.1)
3.9	3.99	3.999	$\Rightarrow$ undefined $\Leftarrow$	4.001	4.01	4.10

Thus, f (x) can be made arbitrarily close to the limit of 4 just by making x sufficiently close to 2.

Formal Definition of Limit :

Karl Weierstrass formally defined limit as follows :

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number.



$\lim_{x\to c} f(x) = L$ means that for each real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for all x with $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$ or, symbolically.

$\forall\ \varepsilon > 0,\ \exists\ \delta > 0,\ \forall\ x\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$

Neighbourhood (NBD) of a point :

Let 'a' be a real number and let be a positive real number. Then the set of all real numbers lying between $a-\delta$ and $a+\delta$ is called the neighbourhood of 'a' of radius 'δ' and is denoted by $N_\delta(a)$.
Thus, $N_\delta(a) = (a-\delta, a+\delta) = \{x \in R | a-\delta < x < a+\delta\}$
The set $(a-\delta, a)$ is called the left NBD of 'a' and the set $(a, a+\delta)$ is known as the right NBD of 'a'.

Left-and Right-Hand Limits :

Let $f(x)$ be a function with domain D and let 'a' be a point such that every NBD of 'a' contains infinitely many points of D. A real number l is called left limit of $f(x)$ at $x = a$ iff for every $\varepsilon > 0$ then exists a $\delta > 0$ such that $a-\delta < x < a \Rightarrow |f(x) - l| < \varepsilon$

In such a case, we write $\lim\limits_{x\to a^-} f(x) = l$.

Thus, $\lim\limits_{x\to a^-} f(x) = l$, if $f(x)$ tends to l as x tends to 'a' from the left-hand side.

Similarly, a real number l' is a right limit of f(x) at $x = a$ iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that $a < x < a+\delta \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim\limits_{x\to a^+} f(x) = l'$.

In order words, l' is a right limit of $f(x)$ at $x = a$ iff f(x) tends to l' as x tends to 'a' from the right-hand side.

Existence of Limit :

If follows from the discussions made in the previous two sections that $\lim\limits_{x\to a} f(x)$ exists if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exists and both are equal.

Thus, $\lim\limits_{x\to a} f(x)$ exists $\Leftrightarrow \lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$.

$f(a^-) = \lim\limits_{h\to 0} f(a-h)$

$f(a^+) = \lim\limits_{h\to 0} f(a+h)$

$\lim\limits_{x\to a} f(x)$ exists $\Leftrightarrow f(a^-) = f(a^+)$

One sided limit :

Let the function $f(x)$ is defined in $x \in [a, b]$. Sometime we need to calculate $\lim\limits_{x\to b} f(x)$ or $\lim\limits_{x\to a} f(x)$. In such $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a^+} f(x) = \text{RHL}$ at $x = a$, as their is no left neighbourhood of $x = a$.

Similarly $\lim\limits_{x\to b} f(x) = \lim\limits_{x\to b^-} f(x) = \text{LHL}$ at $x = b$. As their is no right neighbourhood of $x = b$.

For example, $f(x) = \cos^{-1} x$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1^-} \cos^{-1} x = 0$$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1^+} \cos^{-1} x = \pi$$

Difference between limit of function at x = a and f (a) :

Case	y = f(x)	Explanation
$\lim_{x\to a} f(x)$ exists but f(a) does not exist	$f(x)=\dfrac{x^2-a^2}{x-a}$	The value of function at x = a is of the form $\frac{0}{0}$ which is indeterminate, i.e., f (a) does not exist. But $\lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=2a$. Hence, $\lim_{x\to a} f(x)$ exists.
$\lim_{x\to a} f(x)$ does not exist but f (a) exists	$f(x)=[x]$ where $[\cdot]$ denotes greatest integer function.	The value of function at x = n ($n\in I$) is n i.e. f(n) = n. But $\lim_{x\to n^-}[x]=n-1$ and $\lim_{x\to n^+}[x]=n$ Hence $\lim_{x\to n}[x]=\text{DNE}$
$\lim_{x\to a} f(x)$ and f(a) both exist and are equal	$f(x)=\begin{cases}\sin x, & x<0\\ x, & x\ge 0\end{cases}$	The value of function at x = 0 is 0, i.e., 0. Also $\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\sin x=0$ and $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x=0$ $\Rightarrow \lim_{x\to 0} f(x)$ exists
$\lim_{x\to a} f(x)$ and f(a) both exist but are unequal	$f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & x\ne 3\\ 3, & x=3\end{cases}$	The value of function at x = 3 is 3. Also i.e. f(3) = 3. Also $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=\lim_{x\to 3}\dfrac{x^2-9}{x-3}$ $=\lim_{x\to 3}(x+3)=6$, i.e., $\lim_{x\to 3} f(x)$ exists. But $\lim_{x\to 3} f(x)\ne f(3)$

Thus, for limit to exist at x = a, it is not necessary that function is defined at that point.

Illustration :

Evaluate the following limits ([·] , {·} denotes greatest integer function and fractional part respectively)

(i) $\lim_{x\to n}[x]$, $\lim_{x\to n}\{x\}$, $n\in I$ (ii) $\lim_{x\to 0}\dfrac{|x|}{x}$ (iii) $\lim_{x\to 0}\tan^{-1}\dfrac{1}{x}$

(iv) $\lim_{x\to 0}\dfrac{1}{\ln|x|}$ (v) $\lim_{x\to 0}\cot^{-1}x^2$ (vi) $\lim_{x\to 0}[x]+\sqrt{\{x\}}$ (vii) $\lim_{x\to 1} x\,\mathrm{sgn}(x-1)$

(viii) $\lim_{x\to 1}\dfrac{x}{[x]}$ (ix) $\lim_{x\to 1}\dfrac{[x]}{x}$ (x) $\lim_{x\to 0}\sin^{-1}[\sec x]$

Sol.

(i) $\lim\limits_{x\to n^+}[x]=n,\ \lim\limits_{x\to n^-}[x]=n-1 \quad\Rightarrow\quad \lim\limits_{x\to n}[x]=DNE$ (Does not exists)

Similarly $\lim\limits_{x\to n^+}\{x\}=0,\ \lim\limits_{x\to n^-}\{x\}=1 \quad\Rightarrow\quad \lim\limits_{x\to n}\{x\}=DNE$ (Does not exists)

$\Rightarrow$ $f(x) = [x]$ and $\{x\}$ has no limit at all integers

(ii) $f(x) = \frac{|x|}{x}$ has no limit at $x = 0$ $\begin{cases} f(0^+)=1 \\ f(0^-)=-1 \end{cases}$

(iii) $\lim\limits_{x\to 0} \tan^{-1}\frac{1}{x}$ does not exist at $x = 0$ $\begin{cases} f(0^-)=\frac{-\pi}{2} \\ f(0^+)=\frac{\pi}{2} \end{cases}$ even if $f(0)$ is not defined.

(iv) $\lim\limits_{x\to 0}\frac{1}{\ln|x|}$ exists at $x = 0$ $f(0^-) = f(0^+) = 0$ even if $f(0)$ is not defined.

(v) $\lim\limits_{x\to 0}\cot^{-1}x^2=\frac{\pi}{2}$.

(vi) $f(x) = [x] + \sqrt{\{x\}}$ limit exists at $x = 0$ as $\lim\limits_{x\to 0^+} f(x)=0+0=0$, $\lim\limits_{x\to 0^-} f(x)=-1+\sqrt{1}=0$.

(vii) $\lim\limits_{x\to 1} x\,\mathrm{sgn}(x-1)$ does not exist $\begin{cases} f(1^+)\to 1 \\ f(1^-)\to -1 \end{cases}$ at $x = 1$.

(viii) $\lim\limits_{x\to 1}\frac{x}{[x]} = \lim\limits_{x\to 1^+}\frac{x}{[x]} = \lim\limits_{x\to 1^+}\frac{x}{1}=1,\ \lim\limits_{x\to 1^-}\frac{x}{[x]}$ is undefined

(ix) $\lim\limits_{x\to 1}\frac{[x]}{x}=$ D.N.E. as $\lim\limits_{x\to 1^+}\frac{[x]}{x}=1$, $\lim\limits_{x\to 1^-}\frac{[x]}{x}=0$

(x) $\lim\limits_{x\to 0}\sin^{-1}[\sec x]$ where [ ] denotes greatest integer function, exists and is equal to $\pi/2$.

Illustration :

Consider the function: $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x < -1 \\ 2, & \text{if } -1 \le x < 1 \\ 3, & \text{if } x = 1 \\ x+1, & \text{if } 1 < x \le 2 \\ \dfrac{-1}{(x-2)^2}, & \text{if } x > 2 \end{cases}$

(i) *Sketch the graph of f.*

(ii) *Determine the following limits.*

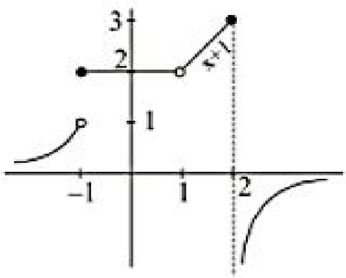
(a) $\lim\limits_{x\to -1^+} f(x)$ (b) $\lim\limits_{x\to -1^-} f(x)$ (c) $\lim\limits_{x\to -1} f(x)$ (d) $\lim\limits_{x\to 1^+} f(x)$

(e) $\lim\limits_{x\to 1^-} f(x)$ (f) $\lim\limits_{x\to 1} f(x)$ (g) $\lim\limits_{x\to 2^+} f(x)$ (h) $\lim\limits_{x\to 2^-} f(x)$

(i) $\lim\limits_{x\to 2} f(x)$ (j) $\lim\limits_{x\to -3} f(x)$ (k) $\lim\limits_{x\to 5} f(x)$ (l) $\lim\limits_{x\to 1.5} f(x)$

Sol.

(i)



(ii) (a) $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} 2 = 2$

(b) $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} \dfrac{1}{x^2} = \lim\limits_{x\to -1^-} \dfrac{1}{(-1)^2} = 1$

(c) $\lim\limits_{x\to -1} f(x) = DNE$

(d) $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (x+1) = 2$

(e) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} 2 = 2$

(f) $\lim\limits_{x\to 1} f(x) = DNE$

(g) $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} \dfrac{-2}{(x-2)^2} = DNE.$

(h) $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (x+1) = 3$

(i) $\lim\limits_{x\to 2} f(x) = DNE$

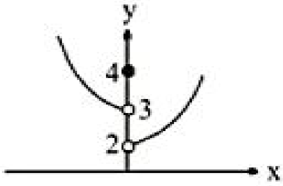
(j) $\lim\limits_{x\to -3} f(x) = \lim\limits_{x\to -3} \dfrac{1}{x^2} = \dfrac{1}{9}$

(k) $\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5} \dfrac{-1}{(x-2)^2} = \dfrac{-1}{9}$

(l) $\lim\limits_{x\to 1.5} f(x) = \lim\limits_{x\to 1.5} x+1 = 2.5.$

Illustration :

Refer the figure,



the value of λ *for which* $2\left(\lim_{x\to 0} f(x^3-x^2)\right) = \lambda\left(\lim_{x\to 0} f(2x^4-x^5)\right)$ *is*

(A) $\frac{4}{3}$ (B) 2 (C) 3 (D) 5

Sol. $\lim_{x\to 0} f\left(x^3-x^2\right) = \lim_{x\to 0^+} f\left(x^3-x^2\right) = \lim_{x\to 0^-} f\left(x^3-x^2\right) = f(0^-) = 3$

$\lim_{x\to 0} f\left(x^4-x^5\right) = \lim_{x\to 0^+} f\left(x^4-x^5\right) = \lim_{x\to 0^-} f\left(x^4-x^5\right) = f(0^+) = 2$

$2\lim_{x\to 0} f\left(x^3-x^2\right) = x\lim_{x\to 0} f\left(2x^4-x^5\right) \Rightarrow 2(3) = \lambda\,(2) \Rightarrow \lambda = 3.$ ***Ans.***

Illustration :

Evaluate the left and right-hand limits of the function $f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, x \neq 4 \\ 0, \quad x = 4 \end{cases}$.

Sol. *L.H.L of f(x) at x = 4*

$$= \lim_{x\to 4^-} f(x) = \lim_{h\to 0} f(4-h) = \lim_{h\to 0}\frac{|4-h-4|}{4-h-4} = \lim_{h\to 0}\frac{|-h|}{-h} = \lim_{h\to 0}\frac{h}{-h} = \lim_{h\to 0} -1 = -1$$

R.H.L. of f(x) at x = 4

$$= \lim_{x\to 4^+} f(x) = \lim_{h\to 0} f(4+h) = \lim_{h\to 0}\frac{|4+h-4|}{4+h-4} = \lim_{h\to 0}\frac{|h|}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1$$

Illustration :

Evaluate the left and the right-hand limits of the function defined by $f(x) = \begin{cases} 1+x^2, if\, 0 \le x < 1 \\ 2-x, \text{ if } x > 1 \end{cases}$

at x = 1. Also, show that $\lim_{x\to 1} f(x)$ *does not exist.*

Sol. *L.H.L. of f (x) at x = 1*

$$= \lim_{x\to 1^-} f(x) = \lim_{x\to 0} f(1-h) = \lim_{x\to 0} 1+(1-h)^2 = \lim_{h\to 0} 2-2h+h^2 = 2$$

R.H.L. of f (x) at x = 1.

$$= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{x\to 0}[2-(1+h)] = \lim_{h\to 0}(1-h) = 1$$

Clearly, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

So, $\lim_{x\to 1} f(x)$ *does not exist.*

Illustration :

Let $f(x) = \begin{cases} \cos x, \text{if } x \geq 0 \\ x + k, \text{if } x < 0 \end{cases}$. *Find the value of constant k, given that* $\lim\limits_{x \to 0} f(x)$ *exists.*

Sol. $\lim\limits_{x \to 0} f(x)$ *exists*

$$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \Rightarrow \lim_{x \to 0} x + k = \lim_{x \to 0} \cos x \Rightarrow 0 + k = \cos 0 \Rightarrow k = 1.$$

Practice Problem

Q.1 If $f(x) = \begin{cases} \dfrac{x - |x|}{2}, x \neq 0 \\ 2, \quad x = 0 \end{cases}$ show that $\lim\limits_{x \to 0} f(x)$ does not exist.

Q.2 Show that $\lim\limits_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ does not exist.

Q.3 Evaluate $\lim\limits_{x \to 0} \dfrac{3x + |x|}{7x - 5|x|}$.

Q.4 If $f(x) = \begin{cases} x, & x < 0 \\ 1, & x = 0 \\ x^2, & x > 0 \end{cases}$, then find $\lim\limits_{x \to 0} f(x)$ if exists.

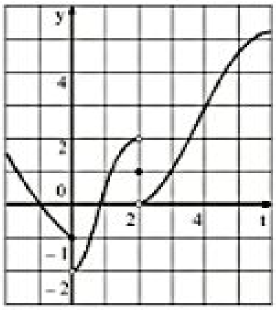
Q.5 For the function g whose graph is given, state the value of each quantity, if it exists.

(a) $\lim\limits_{t \to 0^-} g(t)$ (b) $\lim\limits_{t \to 0^+} g(t)$

(c) $\lim\limits_{t \to 0} g(t)$ (d) $\lim\limits_{t \to 2^-} g(t)$

(e) $\lim\limits_{t \to 2^+} g(t)$ (f) $\lim\limits_{t \to 2} g(t)$

(g) $g(2)$ (h) $\lim\limits_{t \to 4} g(t)$



Q.6 For the function R whose graph is shown, state the following

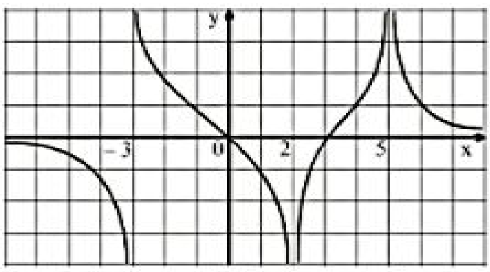
(a) $\lim\limits_{x \to 2} R(x)$

(b) $\lim\limits_{x \to 5} R(x)$

(c) $\lim\limits_{x \to 3^-} R(x)$

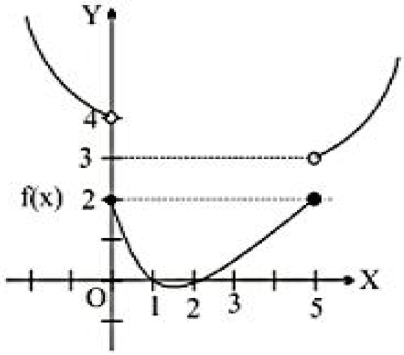
(d) $\lim\limits_{x \to 3^+} R(x)$

(e) The equations of the vertical asymptotes.



Q.7 Refer to the graph of $y = f(x)$
and $g(x) = (x-2)^2, x < 2$
$= 7 - x, x \geq 2$
then which of the following limits are non existent.

(a) $\lim\limits_{x \to 2} f(g(x))$ (b) $\lim\limits_{x \to 0} g(f(x))$ (c) $\lim\limits_{x \to 5} g(f(x))$



Answer key

Q.3 Does not exists **Q.4** **0**

Q.5 (a) – 1 ; (b) – 2 ; (c) does not exist ; (d) 2 ; (e) 0 ; (f) does not exist ; (g) 1 ; (h) 3

Q.6 (a) $-\infty$; (b) ∞ ; (c) 0 ; (d) 0 ; (e) $x = 5$; $x = 2$; $x + 3 = 0$ **Q.7** b, c

THE ALGEBRA OF LIMITS :

Let $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$. If l and m exist, then

1. $\lim\limits_{x \to a} (f \pm g)(x) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = l \pm m$

2. $\lim\limits_{x \to a} (fg)(x) = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x) = lm$

3. $\lim\limits_{x \to a} \left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

4. $\lim\limits_{x \to a} k\, f(x) = k. \lim\limits_{x \to a} f(x)$, where k is a constant

5. $\lim\limits_{x \to a} |f(x)| = \left|\lim\limits_{x \to a} f(x)\right| = |l|$

6. $\lim\limits_{x \to a} (f(x))^{g(x)} = \lim\limits_{x \to a} f(x)^{\lim\limits_{x \to a} g(x)} = l^m$

7. $\lim\limits_{x \to a} fog(x) = f\left(\lim\limits_{x \to a} g(x)\right) = f(m)$, only if f is continous at $g(x) = m$

In particular,

(a) $\lim\limits_{x \to a} \log f(x) = \log\left(\lim\limits_{x \to a} f(x)\right) = \log l$ (b) $\lim\limits_{x \to a} e^{f(x)} = e^{\lim\limits_{x \to a} f(x)} = e^l$

8. If $\lim\limits_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim\limits_{x \to a} \dfrac{1}{f(x)} = 0$.

9. If $f(x) \leq g(x)$ for every x in the NBD of a, then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$.

Points to Remember :

1. If $\lim\limits_{x \to c} f(x)\,g(x)$ exists, then we can have the following cases :

(a) Both $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ exist. Obviously, then $\lim\limits_{x \to c} f(x)\,g(x)$ exists.

(b) $\lim\limits_{x \to c} f(x)$ exists and $\lim\limits_{x \to c} g(x)$ does not exist.

Consider $f(x) = x$; $g(x) = \dfrac{1}{x}$, now $\lim\limits_{x \to 0} f(x) \cdot g(x)$ exists $= 1$. Also $\lim\limits_{x \to 0} f(x) = 0$ exists but $\lim\limits_{x \to 0} g(x)$ does not exist.

(c) Both $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ do not exist.

Let f be defined as $f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 2 & \text{if } x > 0 \end{cases}$. Let $g(x) = \begin{cases} 2 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$.

Then $f(x)\,g(x) = 2$, and so $\lim\limits_{x \to 0} f(x) \times g(x)$ exists, while $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 0} g(x)$ do not exist.

2. If $\lim\limits_{x \to c} [f(x) + g(x)]$ exists then we can have the following cases :

(a) If $\lim\limits_{x \to c} f(x)$ exists, then $\lim\limits_{x \to c} g(x)$ must exist.

Proof : This is true as $g = (f + g) - f$.

Therefore, by the limit theorem, $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} (f(x) + g(x)) - \lim\limits_{x \to 0} f(x)$ which exists.

(b) Both $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ do not exist.

Consider $\lim\limits_{x \to 1} [x]$ and $\lim\limits_{x \to 1} g[x]$, where $[\cdot]$ and $\{\cdot\}$ represent greatest integer and fractional part function, respectively. Here both the limits do not exist but $\operatorname{Lim}\limits_{x \to 1} [x] + \{x\} = \lim\limits_{x \to 1} x = 1$ exists.

For Example :

(a) $f(x) = \dfrac{1}{\sin x}$ and $g(x) = \dfrac{1}{\tan x}$ at $x = 0$ $\operatorname{Lim}\limits_{x \to 0} f(x) = \text{DNE}$, $\operatorname{Lim}\limits_{x \to 0} g(x) = \text{DNE}$

But $\operatorname{Lim}\limits_{x \to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{\tan x} \right) = 0 = \operatorname{Lim}\limits_{x \to 0} \dfrac{1 - \cos x}{\sin x}$

$$= \operatorname{Lim}_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} = \operatorname{Lim}_{x \to 0} \tan \frac{x}{2} = 0 \text{ (exist).}$$

(b) $f(x) = \text{sgn } x$ and $g(x) = [x]$ then $\lim\limits_{x \to 0} (\text{sgn } x + [x])$ does not exist

as $\lim\limits_{x \to 0^+} \sin x + [x] = 1$, $\lim\limits_{x \to 0^-} \sin x + [x] = -2$

while $\lim\limits_{x \to 0} f(x) = \text{DNE}$, $\lim\limits_{x \to 0} g(x) = \text{DNE}$.

(c) $f(x) = [x]$ and $g(x) = \{x\}$; $F(x) = [x]\{x\}$

$\lim\limits_{x \to 0} [x] \cdot \{x\}$ does not exist but $\lim\limits_{x \to 1} [x]\{x\}$ exist and is equal to zero.

(d) If $f(x) = e^{[x]}$; $g(x) = e^{\{x\}}$ then $\lim\limits_{x \to 0} e^{[x]} \cdot e^{\{x\}} = \lim\limits_{x \to 0} e^x$ which exist.

Illustration :

Let $f(x) = \begin{cases} x+1, & x > 0 \\ 2-x, & x \le 0 \end{cases}$ *and* $g(x) = \begin{cases} x+3, & x < 1 \\ x^2 - 2x - 2, & 1 \le x < 2 \\ x-5, & x \ge 2 \end{cases}$. *Find L.H.L. and R.H.L. of* $g(f(x))$ *at*

$x = 0$ *and hence find* $\lim\limits_{x \to 0} g(f(x))$

Sol. *As* $x \to 0^- \Rightarrow f(x) \to f(0^-) = 2^+ \Rightarrow \lim\limits_{x \to 0^-} g(f(x)) = g(2^+) = -3$

Also as $x \to 0^+ \Rightarrow f(x) \to f(0^+) = 1^+ \Rightarrow \lim\limits_{x \to 0^+} g(f(x)) = g(1^+) = -3$

Hence, $\lim\limits_{x \to 0} g(f(x))$ *exists and is equal to* $-3 \Rightarrow \lim\limits_{x \to 0} g(f(x)) = -3$

Indeterminate forms :

Indeterminant forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 1^∞ , 0^0 and ∞^0

EVALUATION OF ALGEBRAIC LIMITS :

(i) Direct Substitution Method :

Consider the following limits : (i) $\lim\limits_{x \to a} f(x)$ (ii) $\lim\limits_{x \to a} \frac{\Phi(x)}{\Psi(x)}$

If $f(a)$ and $\frac{\Phi(a)}{\Psi(a)}$ exist and are fixed real numbers and $\Psi(a) \ne 0$ then we say that $\lim\limits_{x \to a} f(x) = f(a)$ and

$\lim\limits_{x \to a} \frac{\Phi(x)}{\Psi(x)} = \frac{\Phi(a)}{\Psi(a)}$.

Illustration :

Evaluate

(i) $\lim\limits_{x \to 1} 3x^2 + 4x + 5$ (ii) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x + 3}$

Sol. (i) $\lim\limits_{x \to 1} 3x^2 + 4x + 5 = 3(1)^2 + 4(1) + 5 = 12$

(ii) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x + 3} = \dfrac{4 - 4}{2 + 3} = \dfrac{0}{5} = 0$

(ii) Fractorization Method :

Consider $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$

If by substituting $x = a$, $\dfrac{f(x)}{g(x)}$ reduces to the form $\dfrac{0}{0}$, then $(x - a)$ is a factor of both f(x) and g(x). So, we first factorize f(x) and g(x) and then cancel out the common factor to evaluate the limit.

Illustration :

Evaluate $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$.

Sol. $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4} = \lim\limits_{x \to 2} \dfrac{(x-2)(x-3)}{(x+2)(x-2)} = \lim\limits_{x \to 2} \dfrac{x-3}{x+2} = \dfrac{2-3}{2+2} = -\dfrac{1}{4}$

Illustration :

Evalute $\lim\limits_{x \to 1} \dfrac{x^2 + x\log_e x - \log_e x - 1}{(x^2 - 1)}$

Sol. $\lim\limits_{x \to 1} \dfrac{x^2 + x\log_e x - \log_e x - 1}{(x^2 - 1)} \left(\dfrac{0}{0} \text{ form}\right) = \lim\limits_{x \to 1} \dfrac{(x-1)(\log_e x + x + 1)}{(x+1)(x-1)} \left(\dfrac{0}{0} \text{ form}\right)$

$= \lim\limits_{x \to 1} \dfrac{\log_e x + x + 1}{(x+1)} = \dfrac{\log_e 1 + 1 + 1}{1 + 1} = \dfrac{0 + 2}{2} = 1$

Illustration :

Evaluate $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{1 - \sin 2x}{1 + \cos 4x}$

Sol. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{1 - \sin 2x}{1 + \cos 4x} \left(\dfrac{0}{0} \text{ form}\right) = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{(\sin x - \cos x)^2}{2\cos^2 2x} \left(\dfrac{0}{0} \text{ form}\right)$

$= \lim\limits_{x \to \frac{\pi}{4}} \dfrac{(\sin x - \cos x)^2}{2(\cos^2 x - \sin^2 x)^2} \left(\dfrac{0}{0} \text{ form}\right) = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{1}{2(\cos x + \sin x)^2} = \dfrac{1}{4}$

(iii) Rationalization Method :

This is particularly used when either the numerator or the denominator or both involve expression consists of squares roots and on substituting the value of x the rational expression takes the form $\frac{0}{0}, \frac{\infty}{\infty}$.
Following examples illustrate the procedure.

Illustration :

Evaluate $\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$

Sol. *When* $x = 0$, *the expression* $\frac{\sqrt{2+x}-\sqrt{2}}{x}$ *takes the form* $\frac{0}{0}$, *Rationalizing the numerator, we have*

$$\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$$

$$= \lim_{x \to 0} \frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x \to 0} \frac{2+x-2}{(\sqrt{2+x}+\sqrt{2})x} = \lim_{x \to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

Illustration :

Evalute $\lim_{x \to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

Sol. $\lim_{x \to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \left(form\ \frac{0}{0} \right)$

$$= \lim_{x \to a} \frac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{a+2x}+\sqrt{3x})}{(\sqrt{3a+x}-2\sqrt{x})(\sqrt{3a+x}+2\sqrt{2})} \frac{(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})} \left(form \frac{0}{0} \right)$$

$$= \lim_{x \to a} \frac{(a+2x-3x)}{(3a+x-4x)} \frac{(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})} = \lim_{x \to a} \frac{\sqrt{3a+x}+2\sqrt{x}}{3(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \frac{\sqrt{3a+a}+2\sqrt{a}}{3(\sqrt{a+2a}+\sqrt{3a})} = \frac{1}{3} \times \frac{4\sqrt{a}}{2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

(iv) Evalutation of Algebraic Limit Using Some Standard Limits :

Recall the binomial expansion for any rational power

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots\ldots$$

where $|x| < 1$
When x is infinitely small (approaching to zero) such that we can ignore higher powers of x, then we have $(1+x)^n = 1 + nx$ (approximately).

Following theorem will be used to evaluate some algebraic limits :

Theorem : If $n \in Q$, then $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Proof : We have $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a}$

$$= \lim_{x \to a^+} \frac{x^n - a^n}{x - a} = \lim_{h \to 0} \frac{(a+h)^n - a^n}{a+h-a} = \lim_{h \to 0} \frac{a^n\left\{\left(1+\frac{h}{a}\right)^n - 1\right\}}{h} \text{ [when } x \to 0, (1+x)^n \to 1+nx]$$

$$= a^n \lim_{h \to 0} \frac{\left\{1 + n\frac{h}{a}\right\} - 1}{h} = a^n \frac{n}{a} = na^{n-1}$$

Illustration :

Evaluate $\lim\limits_{x \to 2} \frac{x^{10} - 1024}{x^5 - 32}$

Sol. $$\lim_{x \to 2} \frac{x^{10} - 1024}{x^5 - 32} = \lim_{x \to 2} \frac{x^{10} - 2^{10}}{x^5 - 2^5} = \lim_{x \to 2} \frac{\frac{x^{10} - 2^{10}}{x - 2}}{\frac{x^5 - 2^5}{x - 2}} = \frac{\lim\limits_{x \to 2} \frac{x^{10} - 2^{10}}{x - 2}}{\lim\limits_{x \to 2} \frac{x^5 - 2^5}{x - 2}} = \frac{10 \times 2^{10-1}}{5 \times 2^{5-1}} = 64$$

Illustration :

Evaluate $\lim\limits_{x \to 2} \frac{\sqrt{(x+7)} - 3\sqrt{(2x-3)}}{\sqrt[3]{(x+6)} - 2\sqrt[3]{(3x-5)}}$

Sol. *We have* $L = \lim\limits_{x \to 2} \frac{\sqrt{(x+7)} - 3\sqrt{(2x-3)}}{\sqrt[3]{(x+6)} - 2\sqrt[3]{(3x-5)}} \left(\frac{0}{0} \text{ form}\right)$

Let $x - 2 = t$ *such that when* $x \to 2$, $t \to 0$

Then $L = \lim\limits_{t \to 0} \frac{(t+9)^{\frac{1}{2}} - 3(2t+1)^{\frac{1}{2}}}{(t+8)^{\frac{1}{3}} - 2(3t+1)^{\frac{1}{3}}} \left(\frac{0}{0} \text{ form}\right)$

$$= \frac{3}{2} \lim_{t \to 0} \frac{\left(1+\frac{t}{9}\right)^{\frac{1}{2}} - (2t+1)^{\frac{1}{2}}}{\left(1+\frac{t}{8}\right)^{\frac{1}{3}} - (3t+1)^{\frac{1}{3}}} \left(\frac{0}{0} \text{ form}\right) = \frac{3}{2} \lim_{t \to 0} \frac{\frac{1}{2}\frac{t}{9} - (2t)\frac{1}{2}}{\frac{t}{8}\frac{1}{3} - (3t)\frac{1}{3}} = \frac{3}{2} \frac{\left(\frac{1}{18} - 1\right)}{\left(\frac{1}{24} - 1\right)} = \frac{34}{23}$$

(v) Evaluation of Algebraic Limits at Infinity :

We know that $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$ and $\lim\limits_{x \to +\infty} \frac{1}{x^2} = 0$

$\therefore \quad \lim\limits_{x \to \infty} f(x) = \lim\limits_{y \to 0} f\left(\frac{1}{y}\right)$

Illustration :

Evaluate $\lim\limits_{x \to \infty} \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$.

Sol. *Herethe expression assumes the form* $\frac{\infty}{\infty}$. *We notice that the highest power of x in both the numberator and the denominator is 2. So we divide each term in both the numerator and denominator by* x^2.

$$\therefore \quad \lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} = \lim_{x \to \infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x} + \frac{f}{x^2}} = \frac{a + 0 + 0}{d + 0 + 0} = \frac{a}{d}$$

Illustration :

Eavalue $\lim\limits_{x \to \infty} \dfrac{\sqrt{3x^2 - 1} - \sqrt{2x^2 - 1}}{4x + 3}$

Sol. $$\lim_{x \to \infty} \frac{\sqrt{3x^2 - 1} - \sqrt{2x^2 - 1}}{4x + 3} = \mathop{Lim}_{x \to \infty} \frac{|x|\sqrt{3 - \frac{1}{x^2}} - |x|\sqrt{2 - \frac{1}{x^2}}}{4x + 3}$$

$$= \lim_{x \to \infty} \frac{\sqrt{3 - 1/x^2} - \sqrt{2 - 1/x^2}}{4 + 3/x} = \frac{\sqrt{3} - \sqrt{2}}{4}$$

Illustration :

Evaluate $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2 + 1} - \sqrt[3]{x^3 + 1}}{\sqrt[4]{x^4 + 1} - \sqrt[5]{x^4 + 1}}$

Sol. *We have* $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2 + 1} - \sqrt[3]{x^3 + 1}}{\sqrt[4]{x^4 + 1} - \sqrt[5]{x^4 + 1}}$

$$\lim_{x \to \infty} \frac{\sqrt{1 + \frac{1}{x^2}} - \sqrt[3]{1 + \frac{1}{x^3}}}{\sqrt[4]{1 + \frac{1}{x^4}} - \sqrt[5]{\frac{1}{x} + \frac{1}{x^5}}} = \frac{1 - 1}{1 - 0} = 0$$

Illustration :

Evaluate $\lim\limits_{x\to-\infty}\left(\sqrt{25x^2-3x}+5x\right)$

Sol. We have $\lim\limits_{x\to-\infty}\left(\sqrt{25x^2-3x}+5x\right)$ $\quad(\infty-\infty \text{ form})$

$$=\lim_{y\to\infty}\left(\sqrt{25y^2+3y}-5y\right), \text{ where } y=-x$$

$$=\lim_{y\to\infty}\frac{25y^2+3y-25y^2}{\sqrt{25y^2+3y}+5y}=\lim_{y\to\infty}\frac{3y}{\sqrt{25y^2+3y}+5y}$$

$$=\lim_{y\to\infty}\frac{3}{\sqrt{25+\frac{3}{y}}+5}=\frac{3}{5+5}=\frac{3}{10}$$

Illustration :

If $\lim\limits_{x\to 0}\dfrac{\sqrt{ax+b}-2}{x}=1$ find a and b.

Sol. $\lim\limits_{x\to 0}\dfrac{\sqrt{ax+b}-2}{x}=1$

For limit to exist, $\sqrt{0+b}-2=0\Rightarrow b=4$

$$\lim_{x\to 0}\frac{\sqrt{ax+4}-2}{x}=1\Rightarrow 2\lim_{x\to 0}\frac{\sqrt{1+\frac{ax}{4}}-1}{x}=1$$

$$\Rightarrow 2\lim_{x\to 0}\frac{1+\frac{ax}{2\cdot 4}-1}{x}=1\Rightarrow 2\cdot\frac{a}{8}=1\Rightarrow a=4.$$ ***Ans.***

Illustration :

(i) $\lim\limits_{x\to-2}\left(\dfrac{1}{x+2}-\dfrac{12}{x^3+8}\right)$ $(\infty-\infty)$ form $\qquad$ (ii) $\lim\limits_{x\to\infty}\left(\sqrt{4x^2+x}-\sqrt{\dfrac{4x^3}{x+2}}\right)$ $(\infty-\infty)$ form

Sol.

(i) $$\lim_{x\to-2}\left(\frac{1}{x+2}-\frac{12}{x^3+8}\right)=\lim_{x\to-2}\frac{x^2-2x+4-12}{x^3+8}$$

$$=\lim_{x\to-2}\frac{x^2-2x-8}{x^3+8}=\lim_{x\to-2}\frac{(x+2)(x-4)}{(x+2)(x^2-2x+4)}$$

$$=\lim_{x\to-2}\frac{x-4}{x^2-2x+4}=\frac{-2-4}{4+4+4}=\frac{-6}{12}=\frac{-1}{2}.$$ ***Ans.***

(ii) $$\lim_{x\to\infty}\sqrt{4x^2+x}-\sqrt{\frac{4x^3}{x+2}} = \lim_{y\to 0}\sqrt{\frac{4}{y^2}+\frac{1}{y}}-\sqrt{\frac{\frac{4}{y^3}}{\left(\frac{1}{y}+2\right)}} = \lim_{y\to 0}\frac{\sqrt{4+y}-2\sqrt{\frac{1}{1+2y}}}{y}$$

$$= 2\lim_{y\to 0}\frac{\sqrt{1+\frac{y}{4}}-(1+2y)^{\frac{-1}{2}}}{y} = 2\lim_{y\to 0}\frac{1+\frac{y}{8}-(1-y)}{y} = 2\left(\frac{9}{8}\right) = \frac{9}{4}.\ \textbf{Ans.}$$

Illustration :

$$\lim_{n\to\infty}\frac{1^2+2^2+3^2+\ldots.+n^2}{n^3}$$

Sol. $$\lim_{n\to\infty}\frac{1^2+2^2+3^2+\ldots\ldots+n^2}{n^3} = \lim_{n\to\infty}\frac{\frac{n}{6}(n+1)(2n+1)}{n^3}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6} = \frac{(1+0)(2+0)}{6} = \frac{1}{3}.\ \textbf{Ans.}$$

Illustration :

$$\lim_{x\to 0}\frac{(\cos x)^{1/3}-(\cos x)^{1/2}}{\sin^2 x}\ \text{equals}$$

(A) 1/12 (B) 1/6 (C) 1/3 (D) 1/2

Sol. Let $y = \cos x$

$$\lim_{x\to 0}\frac{(\cos x)^{\frac{1}{3}}-(\cos x)^{\frac{1}{2}}}{1-\cos^2 x} = \lim_{y\to 1}\frac{y^{\frac{1}{3}}-y^{\frac{1}{2}}}{1-y^2}$$

$$= \lim_{y\to 1}\frac{\left(\frac{y^{\frac{1}{3}}-1}{y-1}\right)-\left(\frac{y^{\frac{1}{2}}-1}{y-1}\right)}{\left(\frac{1-y^2}{1-y}\right)} = \frac{\frac{1}{3}(1)^{2/3}-\frac{1}{2}(1)^{-1/2}}{-2} = \frac{\frac{1}{3}-\frac{1}{2}}{-2} = \frac{1}{12}.\ \textbf{Ans.}$$

Illustration :

$$\lim_{x\to\infty}\left((x+a)(x+b)(x+c)\right)^{1/3}-x$$

Sol. $$\lim_{x\to\infty}\left((x+a)(x+b)(x+c)^{\frac{1}{3}}-x\right) \Rightarrow \lim_{y\to 0}\left(\frac{(1+ay)(1+by)(1+cy)^{\frac{1}{3}}-1}{y}\right)$$

$$\lim_{y\to 0}\frac{\left(\left(1+(a+b+c)+abcy^{3}\right)^{\frac{1}{3}}-1\right)}{y}$$

$$\lim_{y\to 0}\frac{1+\frac{1}{3}(a+b+c)\,y+\text{Higher terms}-1}{y}=\frac{(a+b+c)}{3}.\ \textbf{Ans.}$$

Illustration :

$\lim_{x\to 0}\dfrac{\ln(\sin 2x)}{\ln(\sin x)}$ *is equal to*

(A) 0 (B) 1 (C) 2 (D) non existent

Sol. $$\lim_{x\to 0}\frac{\ln(\sin 2x)}{\ln(\sin x)}=\lim_{x\to 0}\frac{\ln 2+\ln\sin x+\ln\cos x}{\ln\sin x}$$

$$=\lim_{x\to 0}\left(1+\frac{\ln 2}{\ln \sin x}+\frac{\ln\cos x}{\ln \sin x}\right)=1+0+0=1.\ \textbf{Ans.}$$

Illustration :

$$\lim_{x\to 0}\left(1^{\operatorname{cosec}^2 x}+2^{\operatorname{cosec}^2 x}+3^{\operatorname{cosec}^2 x}+\ldots\ldots+100^{\operatorname{cosec}^2 x}\right)^{\sin^2 x}\qquad [\infty^0]$$

Sol. $$\lim_{x\to 0}\left(1^{\operatorname{cosec}^2}+2^{\operatorname{cosec}^2 x}+3^{\operatorname{cosec}^2 x}+\ldots\ldots+100^{\operatorname{cosec}^2 x}\right)^{\sin^2 x}$$

$$100\lim_{x\to 0}\left(\left(\frac{1}{100}\right)^{\operatorname{cosec}^2 x}+\left(\frac{2}{100}\right)^{\operatorname{cosec}^2 x}+\ldots\ldots+\left(\frac{99}{100}\right)^{\operatorname{cosec}^2 x}+1\right)^{\sin^2 x}$$

$= 100\ (0 + 0 + \ldots\ldots\ldots + 0 + 1)^0 = 100.$ **Ans.**

Practice Problem

Q.1 Evaluate the following limits

(i) $\displaystyle \lim_{x\to\frac{\pi}{4}} \frac{-1+\cot^3 x}{-2+\cot x+\cot^3 x}$ (ii) $\displaystyle \lim_{x\to 2} \frac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}$ (iii) $\displaystyle \lim_{x\to 9} \frac{3-\sqrt{x}}{4-\sqrt{2x-2}}$

Q.2 Evaluate the following limits

(i) $\displaystyle \lim_{n\to\infty} \frac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^6+6n^5+2}-\sqrt[5]{n^7+3n^3+1}}$ (ii) $\displaystyle \lim_{x\to\pm\infty} \left(\sqrt{x^2-2x-1}-\sqrt{x^2-7x-3}\right)$

(iii) $\displaystyle \lim_{x\to\frac{\pi}{2}} \tan^2 x\left(\sqrt{2\sin^2 x+3\sin x+4}-\sqrt{\sin^2 x+6\sin x+2}\right)$

Q.3 Evaluate the following limits

(i) $\displaystyle \lim_{x\to\infty} \sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x}$ (ii) $\displaystyle \lim_{x\to\infty} ((x+1)(x+2)(x+3))^{\frac{1}{3}}-x$

(iii) $\displaystyle \lim_{x\to\infty} 100\left[[(x+1)(x+2)(x+3).......(x+100)]^{\frac{1}{100}}-x\right]$ (iv) $\displaystyle \lim_{x\to 1} \frac{x^n-1}{x^m-1}$ $(m, n \in N)$

Q.4 If $\displaystyle \lim_{x\to 2}\left(\frac{x^{n+1}-2^{n+1}}{x-2}\right)=80$ and $n \in N$, find n.

Q.5 Evaluate the following limits

(i) $\displaystyle \lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$ (ii) $\displaystyle \lim_{x\to 1} \frac{\left[\sum_{k=1}^{100} x^k\right]-100}{x-1}$

(iii) $\displaystyle \lim_{x\to\infty}\left[\sqrt{a^2x^2+ax+1}-\sqrt{a^2x^2+1}\right]$ (iv) $\displaystyle \lim_{x\to a} \frac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}$

(v) $\displaystyle \lim_{n\to\infty} \frac{(1^2-2^2+3^2-4^2+5^2+......n \text{ terms})}{n^2}$ (vi) $\displaystyle \lim_{h\to 0}\left[\frac{1}{h\sqrt[3]{8+h}}-\frac{1}{2h}\right]$

Answer key

Q.1 (i) $\frac{3}{4}$, (ii) 8, (iii) $\frac{2}{3}$

Q.2 (i) 1, (ii) $\pm\frac{5}{2}$, (iii) $\frac{1}{12}$

Q.3 (i) 2, (ii) 2, (iii) 5050, (iv) $\frac{n}{m}$

Q.4 $n = 4$

Q.5 (i) $\frac{-1}{10}$, (ii) 5050, (iii) $\frac{1}{2}$, (iv) $\frac{1}{\sqrt{2a}}$, (v) when n is even, limit $=\frac{-1}{2}$, when n is odd limit $=\frac{1}{2}$, (vi) $\frac{-1}{48}$

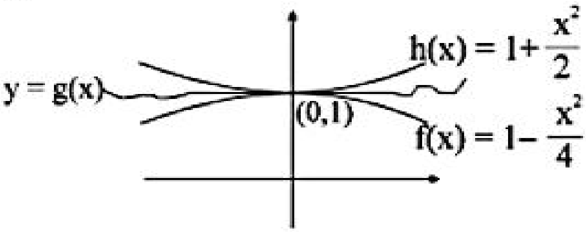
SANDWICH THEOREM OR SQUEEZE PLAY THEOREM FOR EVALUTATING LIMITS :

General: The squeeze principle is used on limit problems where the usual algebraic methods (factorisation or algebraic manipulation etc.) are not effective. However it requires to "squeeze" our problem in between two other simpler function whose limits can be easily computed and equal. Use of Squeeze principle requires accurate analysis, indepth algebra skills and careful use of inequalities.

Statement: If f, g and h are 3 functions such that $f(x) \le g(x) \le h(x)$ for all x in some interval containing the point x =c, and if

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L \Rightarrow \lim_{x\to c} g(x) = L$$

From the figure note that $\lim_{x\to 0} g(x) = 1$.



Note: **(i)** the quantity c may be a finite number, $+\infty$ or $-\infty$.
Similarly L may be finite number, $+\infty$ or $-\infty$.

Illustration :

Evaluate $\lim_{x\to\infty} \frac{x+7\sin x}{-2x+13}$ *using Sandwich theorem.*

Sol. *We know that* $-1 \le \sin x \le 1$ *for all x.*

$\Rightarrow \quad -7 \le 7\sin x \le 7$

$\Rightarrow \quad x - 7 \le x + 7\sin x \le x + 7$

Dividing throughout by $-2x + 13$, *we get*

$$\frac{x-7}{-2x+13} \ge \frac{x+7\sin x}{-2x+13} \ge \frac{x+7}{-2x+13} \text{ for all } x \text{ that are large.}$$

[Why did we switch the inequality signs?]

Now, $$\lim_{x\to\infty} \frac{x-7}{-2x+13} = \lim_{x\to\infty} \frac{1-\frac{7}{x}}{-2+\frac{13}{x}} = \frac{1-0}{-2+0} = -\frac{1}{2}$$

and $$\lim_{x\to\infty} \frac{x+7}{-2x+13} = \lim_{x\to\infty} \frac{1+\frac{7}{x}}{-2+\frac{13}{x}} = \frac{1+0}{-2+0} = -\frac{1}{2}$$

$$\therefore \quad \lim_{x\to\infty} \frac{x+7\sin x}{-2x+13} = \frac{-1}{2}.$$

Illustration :

If [·] denotes the greatest integer function, then find the value of $\lim_{x\to\infty}\frac{[x]+[2x]+......+[nx]}{n^2}$.

Sol. $nx-1<[nx]\le nx$. Putting $n = 1, 2, 3, ..., n$ and adding them, $x\Sigma n - n < \Sigma[nx] \le x\Sigma n$

$$\therefore \quad x\frac{\Sigma n}{n^2}-\frac{1}{n}<\frac{\Sigma[nx]}{n^2}\le x.\frac{\Sigma n}{n^2} \qquad ...(i)$$

Now, $\lim_{n\to\infty}\left\{x.\frac{\Sigma n}{n^2}-\frac{1}{n}\right\}=x.\lim_{n\to\infty}\frac{\Sigma n}{n^2}-\lim_{n\to\infty}\frac{1}{n}=\frac{x}{2}$

$$\lim_{n\to\infty}\left\{x.\frac{\Sigma n}{n^2}\right\}=x\lim_{n\to\infty}\frac{\Sigma n}{n^2}=\frac{x}{2}$$

As the two limits are equal by equation (i) $\lim_{n\to\infty}\frac{\Sigma[nx]}{n^2}=\frac{x}{2}$.

Illustration :

Evaluate $\lim_{n\to\infty}\frac{1}{1+n^2}+\frac{2}{2+n^2}+......+\frac{n}{n+n^2}$.

Sol. $P_n=\frac{1}{1+n^2}+\frac{2}{2+n^2}+......+\frac{n}{n+n^2}$

Now, $P_n<\frac{1}{1+n^2}+\frac{2}{1+n^2}+......+\frac{n}{1+n^2}=\frac{1+2+...+n}{(1+n^2)}=\frac{n(n+1)}{2(1+n^2)}$

Also, $P_n>\frac{1}{n+n^2}+\frac{2}{n+n^2}+\frac{3}{n+n^2}+......+\frac{n}{n+n^2}=\frac{1+2+.....+4}{(n+n^2)}=\frac{n(n+1)}{2(n+n^2)}$

Thus, $\frac{n(n+1)}{2(n+n^2)}<P_n<\frac{n(n+1)}{2(1+n^2)}$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{n(n+1)}{2(n+n^2)}<\lim_{n\to\infty}P_n<\lim_{n\to\infty}\frac{n(n+1)}{2(1+n^2)}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{1\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n}+1\right)}<\lim_{n\to\infty}P_n<\lim_{n\to\infty}\frac{1\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n^2}+1\right)}$$

$$\Rightarrow \quad \frac{1}{2}<\lim_{n\to\infty}P_n<\frac{1}{2} \quad \Rightarrow \quad \lim_{n\to\infty}P_n=\frac{1}{2}$$

Practice Problem

Evaluate following limits based on Sandwich theorem :

Q.1 If $4x - 9 \le f(x) \le x^2 - 4x + 7 \ \forall \ x \ge 0$ find $\lim\limits_{x \to 4} f(x)$

Q.2 $2x \le g(x) \le x^4 - x^2 + 2$ for all x then $\lim\limits_{x \to 1} g(x)$

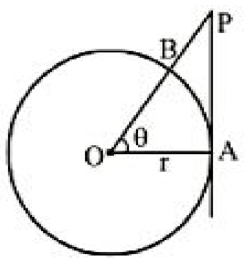
Answer key

Q.1 7 **Q.2** 2

EVALUATION OF TRIGONOMETRIC LIMITS :

(i) $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ (where θ is in radians)

Proof : Consider a circle of radius r. Let O be the centre of the circle such that $\angle AOB = \theta$ where θ is measured in radians and it is very small. Suppose the tangent at A meets OB produced at P. From figure, we have



Area of ΔOAB < Area of sector OAB < Area of ΔOAP

$\Rightarrow \quad \dfrac{1}{2} OA \times OB \sin\theta < \dfrac{1}{2}(OA)^2 \theta < \dfrac{1}{2} OA \times AP$

$\Rightarrow \quad \dfrac{1}{2} r^2 \sin\theta < \dfrac{1}{2} r^2 \theta < \dfrac{1}{2} r^2 \tan\theta$ [In ΔOAP, $AP = OA \tan\theta$]

$\Rightarrow \quad \sin\theta < \theta < \tan\theta \quad \Rightarrow \quad 1 > \dfrac{\sin\theta}{\theta} > \cos\theta$

$\Rightarrow \quad 1 > \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} > \lim\limits_{\theta \to 0} \cos\theta$ or, $\lim\limits_{\theta \to 0} \cos\theta < \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} < 1$

$\Rightarrow \quad 1 < \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} < 1 \quad \Rightarrow \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ (By Sandwich Theorem)

(ii) $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

We have $\lim\limits_{\theta \to 0} \dfrac{\tan\theta}{\theta} = \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} \dfrac{1}{\cos\theta} = \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} \cdot \lim\limits_{\theta \to 0} \dfrac{1}{\cos\theta} = 1$

(iii) $\lim\limits_{\theta\to a}\dfrac{\sin(\theta-a)}{\theta-a}=1$

We have $\lim\limits_{\theta\to a}\dfrac{\sin(\theta-a)}{\theta-a}=\lim\limits_{h\to 0}\dfrac{\sin(a+h-a)}{(a+h-a)}=\lim\limits_{h\to 0}\dfrac{\sin h}{h}=1$

(iv) $\lim\limits_{\theta\to a}\dfrac{\tan(\theta-a)}{\theta-a}=1$

(v) $\lim\limits_{\theta\to 0}\dfrac{\sin^{-1}x}{x}=1$

(vi) $\lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x}=1$

(vii) $\operatorname{Lim}\limits_{x\to\infty}\dfrac{1-\cos x}{x^2}=\dfrac{1}{2}$ (remember).

Note : Let [·] denotes greatest integer function

(i) $\operatorname{Lim}\limits_{x\to 0}\left[\dfrac{\sin x}{x}\right]=0$

(ii) $\left[\operatorname{Lim}\limits_{x\to 0}\dfrac{\sin x}{x}\right]=1$

(iii) $\operatorname{Lim}\limits_{x\to 0}\left[\dfrac{\tan x}{x}\right]=1$

(iv) $\operatorname{Lim}\limits_{x\to 0}\left[\dfrac{\sin^{-1}x}{x}\right]=1$

(v) $\left[\operatorname{Lim}\limits_{x\to 0}\dfrac{\sin^{-1}x}{x}\right]=0$

Illustration :

Evaluate the following limits

(i) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$ (ii) $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$ (iii) $\lim\limits_{x\to 1}\dfrac{\sin(\log x)}{\log x}$

Sol. (i) *We have* $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}=\lim\limits_{x\to 0}\left(3\dfrac{\sin 3x}{3x}\right)=3\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}=3(1)=3$ $\left[\because \lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}=1\right]$

(ii) *We have* $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}=\lim\limits_{x\to 0}\dfrac{\left(\dfrac{\sin ax}{ax}\right)ax}{\left(\dfrac{\sin bx}{bx}\right)ax}=\dfrac{a}{b}$

(iii) *Given* $L=\lim\limits_{x\to 1}\dfrac{\sin(\log x)}{\log x}$

Let $\log x = t$ *then*

$\Rightarrow\quad L=\lim\limits_{t\to 0}\dfrac{\sin t}{t}=1$

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{1}{x}\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$.

Sol. *We know that* $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)=2\tan^{-1}x$, *for* $-1\le x\le 1$

$\Rightarrow\quad \lim\limits_{x\to 0}\dfrac{1}{x}\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)=\lim\limits_{x\to 0}\dfrac{2\tan^{-1}x}{x}=2$

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2}$

Sol. We know $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2}\left(\dfrac{0}{0}\text{ form}\right)$ $=\lim\limits_{x\to 2}\dfrac{2\sin^2 x}{x^2}=2\left(\lim\limits_{x\to 0}\dfrac{\sin x}{x}\right)^2=2$

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

Sol. We have $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}\left(\dfrac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 0}\left(\frac{\sin x-\sin x\cos x}{x^3\cos x}\right)$$

$$=\lim_{x\to 0}\left\{\frac{\sin x(1-\cos x)}{x^3\cos x}\right\}=\lim_{x\to 0}\left\{\frac{\sin x}{x}\,\frac{1-\cos x}{x^2}\,\frac{1}{\cos x}\right\}$$

$$=\left\{\lim_{x\to 0}\frac{\sin x}{x}\right\}\left\{\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2\times 4}\right\}\lim_{x\to 0}\frac{1}{\cos x}=\left\{\lim_{x\to 0}\frac{\sin x}{x}\right\}\frac{1}{2}\left\{\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\right\}\lim_{x\to 0}\frac{1}{\cos x}$$

$$=1\times\frac{1}{2}(1)^2\times\frac{1}{1}=\frac{1}{2}$$

Illustration :

Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1+\cos 2x}{(\pi-2x)^2}$

Sol. We have, $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1+\cos 2x}{(\pi-2x)^2}\left(\dfrac{0}{0}\text{ form}\right)$

$$=\lim_{h\to 0}\frac{1+\cos 2\left(\frac{\pi}{2}+h\right)}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2}=\lim_{h\to 0}\frac{1+\cos(\pi+2h)}{4h^2}=\lim_{h\to 0}\frac{1-\cos 2h}{4h^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 h}{4h^2}=\frac{2}{4}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)^2=\frac{1}{2}$$

Illustration :

Evaluate $\lim\limits_{x\to\infty} 2^{x-1}\tan\left(\frac{a}{2^x}\right)$.

Sol. We have $\lim\limits_{x\to\infty} 2^{x-1}\tan\left(\frac{a}{2^x}\right) = \lim\limits_{x\to\infty}\frac{a}{2}\frac{\tan\left(\frac{a}{2^x}\right)}{\left(\frac{a}{2^x}\right)}\left(\frac{0}{0}\text{ form}\right)$

$$=\frac{a}{2}\lim_{y\to 0}\frac{\tan y}{y},\ \text{where } y=\frac{a}{2^x}\ =\left(\frac{a}{2}\right)$$

Illustration :

Evaluate $\lim\limits_{x\to 2}\frac{x^2-x-2}{x^2-2x-\sin(x-2)}$

Sol. $\lim\limits_{x\to 2}\frac{x^2-x-2}{x^2-2x-\sin(x-2)}\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 2}\frac{(x-2)(x+1)}{x(x-2)-\sin(x-2)}\ =\lim_{x\to 2}\frac{(x+1)}{x-\frac{\sin(x-2)}{x-2}}=\frac{2+1}{2-1}=3$$

Illustration :

Evaluate $\lim\limits_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+4}-\frac{\pi}{4}\right)$.

Sol. We have $\lim\limits_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+4}-\frac{\pi}{4}\right)$

$$=\lim_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+4}-\tan^{-1}1\right)\ =\lim_{x\to\infty} x\tan^{-1}\left(\frac{\frac{x+1}{x+4}-1}{1+\frac{x+1}{x+4}}\right)$$

$$=\lim_{x\to\infty} x\tan^{-1}\left(\frac{-3}{2x+5}\right)=\lim_{x\to\infty}\left(\frac{\tan^{-1}\left(\frac{-3}{2x+5}\right)}{\frac{-3}{2x+5}}\right)\left(\frac{-3x}{2x+5}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\tan^{-1}\left(\frac{-3}{2x+5}\right)}{\frac{-3}{2x+5}}\right)\lim_{x\to\infty}\left(\frac{-3x}{2x+5}\right)=1\times\lim_{x\to\infty}\left(\frac{-3}{2+\frac{5}{x}}\right)=1\times\frac{-3}{2}=-\frac{3}{2}$$

Practice Problem

Q.1 Evaluate the following limits

(i) $\lim\limits_{x\to 0} \sin 8x \cot 3x$ (ii) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$ (iii) $\lim\limits_{x\to 0} \dfrac{1-\cos 5x}{3x^2}$

(iv) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$ (v) $\lim\limits_{x\to 1} \dfrac{x^3+x^2-2}{\sin(x-1)}$

Q.2 Evaluate the following limits

(i) $\lim\limits_{x\to 1} (1-x)\tan\dfrac{\pi x}{2}$ (ii) $\lim\limits_{x\to 0} \dfrac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$ (iii) $\lim\limits_{x\to 0} \dfrac{\cos 7x - \cos 9x}{\cos x - \cos 5x}$

(iv) $\lim\limits_{x\to 0} \dfrac{1-\cos x\sqrt{\cos 2x}}{\tan^2 x}$ (v) $\lim\limits_{x\to 0} \dfrac{1-\cos(1-\cos x)}{\sin^4 x}$

Q.3 Evaluate the following limits

(i) $\lim\limits_{x\to 0} \dfrac{1-\cos mx}{1-\cos nx}$ (ii) $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}$ (iii) $\lim\limits_{x\to 0} \dfrac{\cot 2x - \operatorname{cosec} 2x}{x}$

(iv) $\lim\limits_{x\to 0} \dfrac{\tan 2x - x}{3x - \sin x}$ (v) $\lim\limits_{n\to \infty} n\cos\left(\dfrac{\pi}{4n}\right)\sin\left(\dfrac{\pi}{4n}\right)$

Answer key

Q.1 (i) $\dfrac{8}{3}$, (ii) $\dfrac{1}{2}$, (iii) $\dfrac{25}{6}$, (iv) $\dfrac{1}{4}$, (v) 5

Q.2 (i) $\dfrac{2}{\pi}$, (ii) $\dfrac{1}{2}$, (iii) $\dfrac{4}{3}$, (iv) $\dfrac{3}{2}$, (v) $\dfrac{1}{8}$

Q.3 (i) $\dfrac{m^2}{n^2}$, (ii) $-\dfrac{1}{2}$, (iii) -1, (iv) $\dfrac{1}{2}$, (v) $\dfrac{\pi}{4}$

EVALUATION OF EXPONENTIAL AND LOGARITHMIC LIMITS :

In order to evaluate these type of limit, we use the following standard results.

1. $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log_e a$

Proof : $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x}$

$$= \lim_{x\to 0} \frac{\left(1+\dfrac{x(\log a)}{1!}+\dfrac{x^2(\log a)^2}{2!}+\ldots\ldots\right)-1}{x} = \lim_{x\to 0}\left(\frac{\log a}{1!}+\frac{x(\log a)^2}{2!}+\ldots\ldots\right) = \log_e a$$

2. $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$ (replace a by e in the above proof)

3. $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x}=1$

Proof : $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x}=\lim\limits_{x\to 0}\dfrac{x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots\ldots}{x}=\lim\limits_{x\to 0}\left(1-\dfrac{x}{2}+\dfrac{x^2}{3}\ldots\ldots\right)=1$

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{2^x-1}{\sqrt{1+x}-1}$

Sol. *We have* $\lim\limits_{x\to 0}\dfrac{2^x-1}{\sqrt{1-x}-1}\left(\dfrac{0}{0}\text{ form}\right)$

$=\lim\limits_{x\to 0}\dfrac{2^x-1}{\sqrt{1+x}-1}\dfrac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}=\lim\limits_{x\to 0}\dfrac{2^x-1}{x}\lim\limits_{x\to 0}\left(\sqrt{1+x}+1\right)$

$=(\log 2)\left(\sqrt{1+0}+1\right)=2\log 2.$

Illustration :

Evaluate $\lim\limits_{x\to 1}\dfrac{a^{x-1}-1}{\sin\pi x}.$

Sol. *We have* $\lim\limits_{x\to 1}\dfrac{a^{x-1}-1}{\sin\pi x}\left(\dfrac{0}{0}\text{ form}\right)$

$=\lim\limits_{h\to 0}\dfrac{a^{1+h-1}-1}{\sin\pi(1+h)}=\lim\limits_{h\to 0}\dfrac{a^h-1}{-\sin\pi h}=\dfrac{-1}{\pi}\lim\limits_{h\to 0}\left(\dfrac{a^h-1}{h}\right)\dfrac{\pi h}{\sin\pi h}=-\dfrac{1}{\pi}\log a$

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{10^x-2^x-5^x+1}{x\tan x}$

Sol. *We have* $\lim\limits_{x\to 0}\dfrac{10^x-2^x-5^x+1}{x\tan x}\left(\dfrac{0}{0}\text{ form}\right)$

$=\lim\limits_{x\to 0}\dfrac{5^x.2^x-2^x-5^x+1}{x\tan x}\quad=\lim\limits_{x\to 0}\dfrac{5^x-1}{x}\dfrac{2^x-1}{x}\dfrac{x}{\tan x}$

$=\lim\limits_{x\to 0}\dfrac{5^x-1}{x}\lim\limits_{x\to 0}\dfrac{2^x-1}{x}\lim\limits_{x\to 0}\dfrac{x}{\tan x}$

$=(\log 5)(\log 2)(1)=(\log 5)(\log 2)$

Illustration :

Evaluate $\lim\limits_{x \to a} \dfrac{\log x - \log a}{x - a}$

Sol. Let $x - a = h$, then if $x \to a$, $h \to 0 \Rightarrow \lim\limits_{x \to a} \dfrac{\log x - \log a}{x - a} \left(\dfrac{0}{0} \text{ form}\right)$

$$= \lim_{h \to 0} \frac{\log(a+h) - \log a}{h} \quad = \lim_{x \to 0} \frac{\log\left(1 + \frac{h}{a}\right)}{\frac{h}{a} a} = \frac{1}{a}$$

Illustration :

Evaluate $\lim\limits_{x \to 0} \dfrac{\log(5+x) - \log(5-x)}{c}$

Sol. We have $\lim\limits_{x \to 0} \dfrac{\log(5+x) - \log(5-x)}{c} \left(\dfrac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{\log\left\{5\left(1 + \frac{x}{5}\right)\right\} - \log\left\{5\left(1 - \frac{x}{5}\right)\right\}}{x} = \lim_{x \to 0} \frac{\left\{\log 5 \log\left(1 + \frac{x}{5}\right)\right\} - \left\{\log 5 + \log\left(1 - \frac{x}{5}\right)\right\}}{x}$$

$$= \lim_{x \to 0} \frac{\log\left(1 + \frac{x}{5}\right) - \log\left(1 - \frac{x}{5}\right)}{x}$$

$$= \lim_{x \to 0} \frac{1}{5} \frac{\log\left(1 + \frac{x}{5}\right)}{\frac{x}{5}} - \lim_{x \to 0} \frac{\log\left(1 - \frac{x}{5}\right)}{-\frac{x}{5}} \frac{1}{(-1)} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

Illustration :

Let $P_n = a^{P_{n-1}} - 1$, $\forall n = 2, 3 \ldots\ldots$ and let $P_1 = a^x - 1$ where $a \in R^+$, then evaluate $\lim\limits_{x \to 0} \dfrac{P_n}{x}$.

Sol. Clearly, if $P_k \to 0 \Rightarrow P_{k+1} \to 0$

Now, as $x \to 0 \Rightarrow P_1 \to 0 \Rightarrow P_2 P_3 P_4 \ldots\ldots P_n \to 0$

$$\Rightarrow \quad \lim_{x \to 0} \frac{P_n}{x} = \lim_{x \to 0} \frac{P_n}{P_{n-1}} \frac{P_{n-1}}{P_{n-2}} \ldots\ldots \frac{P_1}{x} = \lim_{x \to 0} \left(\frac{P_n}{P_{n-1}}\right) \cdot \lim_{x \to 0} \left(\frac{P_{n-1}}{P_{n-2}}\right) \ldots\ldots \lim_{x \to 0} \left(\frac{P_1}{x}\right)$$

Now, $\lim\limits_{x \to 0} \dfrac{P_k}{P_{k-1}} = \lim\limits_{x \to 0} \dfrac{a^{P_{k-1}-1}}{P_{k-1}} = \ln a$

$\Rightarrow$ Required limit $= (\ln a)^n$

Practice Problem

Q.1 Evaluate the following limits

(i) $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^{x}}{\tan x - x}$ (ii) $\lim\limits_{x\to 0}\dfrac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}, a>0$ (iii) $\lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos x}{x^2}$

(iv) $\lim\limits_{x\to 0}\dfrac{e^{\sin 2x}-e^{\sin x}}{x}$ (v) $\lim\limits_{x\to 0}\dfrac{\cos(xe^{x})-\cos(xe^{-x})}{x^3}$ (vi) $\lim\limits_{x\to \infty}\dfrac{e^{1/x^2}-1}{2\arctan x^2-\pi}$

Q.2 If $\lim\limits_{x\to 1}\dfrac{\frac{\pi}{4}-\tan^{-1}x}{e^{\sin(\ln x)}-x^{n}}$ exists and has the value equal to $\dfrac{1}{8}$, then find n.

Q.3 Evaluate the following limits :

(i) $\lim\limits_{x\to\infty}[x(a^{1/x}-1)], a>1$ (ii) $\lim\limits_{x\to 0}\dfrac{x2^{x}-x}{1-\cos x}$ (iii) $\lim\limits_{x\to 2}\dfrac{\sin(e^{x-2}-1)}{\log(x-1)}$

(iv) $\lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos x}{x^2}$ (v) $\lim\limits_{x\to 0}\dfrac{e^{x}+e^{-x}-2}{x^2}$ (vi) $\lim\limits_{x\to a}\dfrac{\log(x-a)}{\log(e^{x}-e^{a})}$

(vii) $\lim\limits_{x\to 0}\dfrac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}, a>0$ (viii) $\lim\limits_{x\to 0}\dfrac{(1-3^{x}-4^{x}+12^{x})}{\sqrt{(2\cos x+7)}-3}$

(ix) $\lim\limits_{x\to 0}\dfrac{(729)^{x}-(243)^{x}-(81)^{x}+9^{x}+3^{x}-1}{x^3}$

Answer key

Q.1 (i) 1, (ii) ln a, (iii) $\dfrac{3}{2}$, (iv) 1, (v) 2, (vi) $-\dfrac{1}{2}$ **Q.2** 5

Q.3 (i) $\ln a$, (ii) $\ln 4$, (iii) 1, (iv) $\dfrac{3}{2}$, (v) 1, (vi) 1, (vii) $\ln a$, (viii) $-12\ln 2\times\ln 3$, (ix) $6(\ln 3)^3$

LIMITS OF THE FORM $\lim\limits_{x\to a}(f(x))^{g(x)}$:

Form : $0^0, \infty^0$

Let $L = \lim\limits_{x\to a}(f(x))^{g(x)} \Rightarrow \log_e L = \log_e\left[\lim\limits_{x\to a}(f(x))^{g(x)}\right]$

$= \lim\limits_{x\to a} g(x)\log_e[f(x)]$

Form : 1^∞

1. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$ or $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Proof: $\lim\limits_{x\to\infty}(1+x)^{\frac{1}{x}}$

$$= \lim_{x\to 0}\left(1+\frac{1}{x}x+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3+......\right)$$

$$= \lim_{x\to 0}\left(1+1+\frac{1(1-x)}{2!}+\frac{1(1-x)(1-2x)}{3!}+......\right)$$

$$= \left(1+1+\frac{1}{2!}+\frac{1}{3!}+......\right) = e$$

2. $L = \lim\limits_{x\to a} f(x)^{g(x)}$ if $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$

Then $L = \lim\limits_{x\to a} f(x)^{g(x)}$

$$= \lim_{x\to a}(1+(f(x)-1))^{\frac{1}{f(x)-1}(f(x)-1)\times g(x)}$$

$$= \left[\lim_{x\to a}\left((1+(f(x)-1))^{\frac{1}{f(x)-1}}\right)\right]^{\lim\limits_{x\to a}(f(x)-1)\times g(x)} \qquad = e^{\lim\limits_{x\to a}(f(x)-1)\times g(x)}$$

Illustration :

Evaluate $\lim\limits_{x\to 0}(1+x)^{\operatorname{cosec} x}$.

Sol. $\lim\limits_{x\to 0}(1+x)^{\operatorname{cosec} x} = \lim\limits_{x\to 0}\left[(1+x)^{\frac{1}{x}}\right]^{\frac{x}{\sin x}} = \left[\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}\right]^{\lim\limits_{x\to 0}\frac{x}{\sin x}} = e^1 = e$

Illustration :

Evaluate $\lim\limits_{x\to 0}(\cos x)^{\cos 2x}$.

Sol. $\lim\limits_{x\to 0}(\cos x)^{\cos 2x}$

$$= \lim_{x\to 0}\left[(1+(\cos x-1))^{\frac{1}{\cos x-1}}\right]^{\frac{\cos x-1}{\tan x}}$$

$$= \left[\lim_{x\to 0}(1+(\cos x-1))^{\frac{1}{\cos x-1}}\right]^{\lim\limits_{x\to 0}\frac{\cos x-1}{\tan x}} = e^{\lim\limits_{x\to 0}\frac{\cos x-1}{\tan x}\cos x}$$

$$= e^{\lim\limits_{x\to 0}\frac{\cos x-1}{\sin^2 x}\cos x\sin x} = e^{\operatorname{Lim}\limits_{x\to 0}\left(\frac{\cos x-1}{1-\cos^2 x}\cdot\cos x\cdot\sin x\right)}$$

$$= e^{-\lim\limits_{x\to 0}\frac{\sin x\cos x}{1+\cos x}} = e^0 = 1$$

Illustration :

Evaluate $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{\sin x}{x-\sin x}\right)}$

Sol. Since $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$ and $\lim\limits_{x\to 0}\frac{\sin x}{x}=\lim\limits_{x\to 0}\frac{\sin x}{x-\sin x}=\lim\limits_{x\to 0}\frac{1}{\left(\frac{x}{\sin x}-1\right)}=\frac{1}{1-1}=\infty$

$$\Rightarrow \quad \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{\sin x}{x-\sin x}\right)} = e^{\lim\limits_{x\to 0}\left(\frac{\sin x}{x}-1\right)\left(\frac{\sin x}{x-\sin x}\right)} = e^{-\lim\limits_{x\to 0}\frac{\sin x}{x}} = e^{-1} = \frac{1}{e}$$

Illustration :

Evaluate $\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$; (a, b, c > 0).

Sol. We have $\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$

$$= e^{\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}-1\right)\frac{2}{x}} = e^{\frac{2}{3}\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)}$$

$$= e^{\frac{2}{3}\lim\limits_{x\to 0}\left(\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}\right)} = e^{\frac{2}{3}\left\{\lim\limits_{x\to 0}\frac{a^x-1}{x}+\lim\limits_{x\to 0}\frac{b^x-1}{x}+\lim\limits_{x\to 0}\frac{c^x-1}{x}\right\}}$$

$$= e^{\left(\frac{2}{3}\right)[\ln a+\ln b+\ln c]} = e^{\left(\frac{2}{3}\right)\ln(abc)} = e^{\ln(abc)^{\frac{2}{3}}} = (abc)^{\frac{2}{3}}$$

Practice Problem

Q.1 Evaluate the following limits

(i) $\lim\limits_{x\to\infty}\left(\dfrac{1^{1/x}+2^{1/x}+3^{1/x}+\dots\dots\dots+n^{1/x}}{n}\right)^{nx}$ $n\in N$ (ii) $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x+1}\right)^{x+3}$

(iii) $\operatorname{Lim}\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}$ (iv) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{a+bx}\right)^{c+dx}$ where a, b, c and d are positive.

(v) $\operatorname{Lim}\limits_{x\to 0}\left(\sin^2\dfrac{\pi}{2-ax}\right)^{\sec^2\frac{\pi}{2-bx}}$

Q.2 **Column-I** **Column-II**

	Column-I		Column-II
(A)	Let $f(x)=\begin{bmatrix}3-x & 2\le x\le 3\\ x-1 & 1\le x<2\end{bmatrix}$, then $\operatorname{Lim}\limits_{x\to 2}(f(x))^{\operatorname{cosec}\frac{\pi x}{2}}=$	(P)	1
(B)	$\operatorname{Lim}\limits_{x\to 1}(2-x)^{\tan\frac{\pi x}{2}}$	(Q)	e
(C)	$\operatorname{Lim}\limits_{x\to\infty}\left(\sin\dfrac{1}{x}+\cos\dfrac{1}{x}\right)^{x}$	(R)	e^2
(D)	$\operatorname{Lim}\limits_{x\to 0}\left[\tan\left(\dfrac{\pi}{4}+x\right)\right]^{\frac{1}{x}}$	(S)	$e^{\frac{2}{\pi}}$
(E)	$\operatorname{Lim}\limits_{x\to 0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$	(T)	$e^{\frac{-2}{\pi}}$
(F)	$\operatorname{Limit}\limits_{x\to 0}(\cos x)^{\cot^2 x}$	(U)	non existent
(G)	$\operatorname{Lim}\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^{x}$	(V)	$e^{-1/2}$
(H)	$\operatorname{Lim}\limits_{x\to\infty}\left(\dfrac{2\tan^{-1}x}{\pi}\right)^{x}$	(W)	e^{-1}
		(X)	$e^{1/2}$

Answer key

Q.1 (i) $n!$, (ii) e, (iii) e^5, (iv) $e^{d/b}$, (v) $e^{-\frac{a^2}{b^2}}$

Q.2 (A) U; (B) S; (C) Q; (D) R; (E) P ; (F) V; (G) W; (H) T

L' HOSPITAL'S RULE FOR EVALUATING LIMITS :

Rule : If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ takes $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ form, then, $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

Illustration :

Evaluate $\lim\limits_{x\to 0}\log_{\tan^2 x}(\tan^2 2x)$.

Sol. $L=\lim\limits_{x\to 0}\dfrac{\log(\tan^2 2x)}{\log(\tan^2 x)}\left(\dfrac{\infty}{\infty}\text{ form}\right)$

Using L' hospital Rule

$$\text{We have } L=\lim_{x\to 0}\frac{\left(\dfrac{1}{\tan^2 2x}2\tan 2x\sec^2 2x\right)\times 2}{\dfrac{1}{\tan^2 x}2\tan x\sec^2 x}$$

$$=\lim_{x\to 0}\frac{2\left(\dfrac{1}{\sin 2x\cos 2x}\right)}{\dfrac{1}{\sin x\cos x}}=\lim_{x\to 0}\frac{\left(\dfrac{1}{\sin 2x\cos 2x}\right)}{\left(\dfrac{1}{\sin 2x}\right)}=1\ \lim_{x\to 0}\frac{1}{\cos 2x}=1$$

Illustration :

Evaluate $\lim\limits_{x\to 0^+}x^m(\log x)^n,\ m,n\in N$.

Sol. $\lim\limits_{x\to 0^+}x^m(\log x^n)=\lim\limits_{x\to 0^+}\dfrac{(\log x)^n}{x^{-m}}\left(\dfrac{\infty}{\infty}\text{ form}\right)$

$$=\lim_{x\to 0^+}\frac{n(\log x)^{(n-1)}\dfrac{1}{x}}{-mx^{-m-1}} \qquad \textit{(using L' hospital Rule)}$$

$$=\lim_{x\to 0^+}\frac{n(\log x)^{n-1}}{-mx^{-m}}\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to 0^+}\frac{n(n-1)(\log x)^{(n-2)}\dfrac{1}{x}}{(-m)^2x^{-m-1}} \qquad \textit{(using L' hospital Rule)}$$

$$=\lim_{x\to 0^+}\frac{n(n-1)(\log x)^{(n-2)}}{m^2x^{-m}}\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to 0^+}\frac{n!}{(-m)^n x^{-m}}=0 \quad \textit{(differentiating } N^r \textit{ and } D^r\ n\ \textit{ times)}$$

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}$.

Sol. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}$

$$=\lim_{x\to 0}\frac{(1-x^2)-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}(1+x^2)} \qquad \textit{(Using L'hospital's Rule)}$$

$$=\lim_{x\to 0}\frac{(1+x^2)-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}(1+x^2)}$$

$$=\lim_{x\to 0}\frac{(1+x^2)^2-(1-x^2)}{3x^2\sqrt{1-x^2}(1+x^2)}\times\frac{1}{(1+x^2)+\sqrt{1-x^2}} \qquad \textit{(Rationalizing)}$$

$$=\lim_{x\to 0}\frac{x^4+3x^2}{3x^2\sqrt{1-x^2}(1+x^2)}\times\frac{1}{(1+x^2)+\sqrt{1-x^2}}$$

$$=\lim_{x\to 0}\frac{x^2+3}{3\sqrt{1-x^2}(1+x^2)}\times\frac{1}{(1+x^2)+\sqrt{1-x^2}}=\frac{1}{2}$$

LIMITS OF FUNCTIONS HAVING BUILT IN LIMIT WITH THEM :

EXAMPLES :

$$\lim_{n\to\infty}a^n=\begin{cases}0, & 0<a<1\\ 1, & a=1\\ \infty, & a>1\end{cases}, \quad \lim_{n\to\infty}a^n=\begin{cases}\infty, & 0<a<1\\ 1, & a=1\\ 0, & a>1\end{cases}.$$

Illustration :

$f(x)=\lim\limits_{n\to\infty}\dfrac{\tan\pi x^2+(x+1)^n\sin x}{x^2+(x+1)^n}$, find $\lim\limits_{x\to 0}f(x)$.

Sol. $f(x)=\begin{cases}\sin x, & x>0\\ \dfrac{\tan\pi x^2}{x^2}, & x<0\end{cases}$

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}\sin x=0$$

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\frac{\tan\pi x^2}{x^2}=\pi$$

$\Rightarrow$ $LHL\neq RHL\Rightarrow\lim\limits_{x\to 0}f(x)=DNE$.

Illustration :

$$f(x) = \lim_{n\to\infty} = \frac{\cos\pi x - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}},\ \text{find } \lim_{x\to 1} f(x)$$

Sol. $f(x) = \lim_{n\to\infty} \dfrac{\cos\pi x - x^{2n}\sin(x-1)}{1+x\cdot x^{2n}-x^{2n}}$

when $0 < x^2 < 1$, $\lim_{n\to\infty} x^{2n} \to 0$

$\therefore$ $f(x) = \cos\pi x$

when $x^2 = 1$ $\lim_{n\to\infty} x^2 \to 1$

$$\therefore\quad f(x) = \frac{\cos\pi x - \sin(x-1)}{1+x-1} = \frac{\cos\pi x - \sin(x-1)}{x}$$

when $x^2 > 1$

$\lim_{n\to\infty} x^{2n} \to \infty$

$$\therefore\quad \lim_{n\to\infty} \frac{1}{x^{2n}} \to 0$$

$$f(x) = \lim_{n\to\infty} \frac{\frac{\cos\pi x}{x^{2n}} - \sin(x-1)}{\frac{1}{x^{2n}}+x-1} = \frac{0-\sin(x-1)}{0+x-1} = \frac{-\sin(x-1)}{(x-1)}$$

$$f(x) = \begin{cases} \cos\pi x\ , & 0 < x^2 < 1 \\ \dfrac{\cos\pi x - \sin(x-1)}{x}, & x^2 = 1 \\ \dfrac{-\sin(x-1)}{(x-1)}, & x^2 > 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \cos\pi x = -1, \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{-\sin(x-1)}{(x-1)} = -1$$

$$\therefore\quad \lim_{x\to 1} f(x) = 1.$$

Illustration :

Let $f(x) = \min\left(1, x^{2n}, x^{2n+1}\right), n \in N$. The value of $\lim_{x\to 0} \dfrac{e^{\tan(f(x))} - e^{\sin(f(x))}}{\tan(f(x)) - \sin(f(x))}$, is equal to

(A) 0 (B) 1 (C) 2 (D) does not exist.

Sol. $f(x) = \min.\ (1, x^{2n}, x^{2n+1}), n \in N$

$$= \begin{cases} x^{2n+1}, & x < 0 \\ x^{2n+1}, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases} = \begin{cases} x^{2n+1}, & x \le 1 \\ 1, & x \ge 1 \end{cases}$$

$$\lim_{x\to 0}\frac{e^{\tan(f(x))}-e^{\sin(f(x))}}{\tan(f(x))-\sin(f(x))}$$

$$=\lim_{x\to 0}\frac{e^{\sin(f(x))}\left(e^{\tan(f(x))-\sin(f(x))}-1\right)}{\tan(f(x))-\sin(f(x))}$$

$$=\lim_{x\to 0}\frac{\left(e^{\tan\left(x^{2n+1}\right)-\sin\left(x^{2n+1}\right)}-1\right)}{\tan\left(x^{2n+1}\right)-\sin\left(x^{2n+1}\right)}=\lim_{y\to 0}\frac{e^{y}-1}{y}=1$$

where $y = \tan(x^{2n+1}) - \sin(x^{2n+1})$ **Ans.**

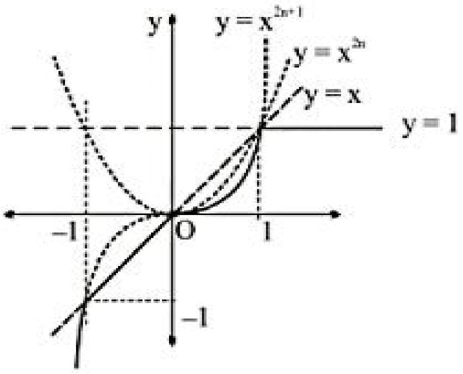


Illustration :

Let $f(x)=\lim_{n\to\infty}\frac{\sin\left(\pi x^{4}\right)+(x+2)^{n}\cdot\frac{\tan\pi x}{x+1}}{1+(x+2)^{n}-x^{4}}$, then $\lim_{x\to -1}f(x)$ is equal to

(A) π (B) 22/7 (C) 1 (D) non existent

Sol. Let $f(x)=\lim_{n\to\infty}\frac{\sin\left(\pi x^{4}\right)+(x+2)^{n}\frac{\tan\pi x}{(x+1)}}{1+(x+2)^{n}-x^{4}}$

$$f(x)=\begin{cases}\dfrac{\sin\left(\pi x^{4}\right)}{1-x^{4}}, & x<-1\\[2mm] \dfrac{\tan\pi x}{(x+1)}, & x>-1\end{cases}$$

$$\lim_{x\to -1}f(x)=\lim_{x\to -1^{+}}\frac{\tan\pi x}{(x+1)}=\lim_{x\to -1^{+}}\frac{(\tan\pi(1+x))}{(1+x)}=\pi$$

$$\lim_{x\to -1^{-}}f(x)=\lim_{x\to -1}\frac{\sin\left(\pi x^{4}\right)}{(1-x^{4})}=\lim_{x\to -1^{-}}\frac{\sin\left(\pi(1-x^{4})\right)}{(1-x^{4})}=\pi$$

$$\lim_{x\to -1^{-}}f(x)=\pi.$$

ONE SIDED LIMITS:

Illustration :

Evaluate $\lim_{x\to 0}\left(1+\tan^{2}\sqrt{x}\right)^{1/x}$

Sol. Let $\left(1+\tan^{2}\sqrt{x}\right)^{\frac{1}{x}}$, $l=\lim_{x\to 0^{+}}\frac{1}{x}\ln\left(1+\tan^{2}\sqrt{x}\right)$

$$l=e^{\lim_{x\to 0^{+}}\frac{1}{x}\left(\tan^{2}\sqrt{x}-\frac{\left(\tan^{2}\sqrt{x}\right)^{2}}{2}+\ldots\right)}=e^{1}=e$$

$$l = e^{\lim\limits_{x \to 0^-} \frac{1}{x} ln\left(1+tan^2\sqrt{x}\right)}.$$

So here left hand limit has so significance as $\sqrt{x}$ is not defined for x < 0.

Illustration :

$$\lim_{x \to tan^{-1}3} \frac{[tan^2 x] - 2[tan x] - 3}{[tan^2 x] - 4[tan x] + 3}$$ *(where [x] is the greatest integer function of x)*

(A) is 1/3 *(B) is 2* *(C) is 3* *(D) does not exist*

Sol. $$\lim_{x \to tan^{-1}3} \frac{\left[tan^2 x\right] - 2\left[tan x\right] - 3}{\left[tan^2 x\right] - 4\left[anx\right] + 3} = \lim_{x \to tan^{-1}3} \frac{8-4-3}{8-8+3} = \frac{1}{3}.$$ ***Ans.***

EXPANSION OF FUNCTION:

Expansion of function like Binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart & are given below :

(i) $a^x = 1 + \frac{x\,ln\,a}{1!} + \frac{x^2 ln^2 a}{2!} + \frac{x^3 ln^3 a}{3!} + ...a > 0$ **(ii)** $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \forall x \in R$

(iii) $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\text{for} -1 < x \le 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$ **(v)** $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ... \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **(vii)** $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

(viii) $\sin^{-1}x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + ...$ **(ix)** $\sec^{-1}x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + ...$

Illustration :

Evaluate the following limit :

(i) $\lim\limits_{x \to 0} \frac{e^x - 1 - x}{x^2}$ *(ii)* $\lim\limits_{x \to 0} \frac{x - \sin x}{x^3}$ *(iii)* $\lim\limits_{x \to 0} \frac{e^x - e^{-x} - 2x}{x^3}$ *(iv)* $\lim\limits_{x \to 0} \frac{x - \tan x}{x^3}$

Sol.

(i) $$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + ...\right) - 1 - x}{x^2} = \frac{1}{2!} = \frac{1}{2}.$$

(ii) $$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - .....\right)}{x^3} = \frac{1}{3!} = \frac{1}{6}.$$

(iii) $$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x^3} = \lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+....\right)-\left(1-x+\frac{x^2}{2!}...\right)-2x}{x^3}$$

$$= \lim_{x\to 0}\frac{\left(2x+\frac{2x^3}{3!}+....\right)-2x}{x^3} = \frac{2}{3!}=\frac{1}{3}.\ \textbf{Ans.}$$

(iv) $$\lim_{x\to 0}\frac{x-\tan x}{x^3} = \lim_{x\to 0}\frac{x-\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+......\right)}{x^3} = \frac{-1}{3}.\ \textbf{Ans.}$$

Illustration :

Evaluate $\lim_{x\to 0}\frac{e^{x^3}-1-x^3}{\sin^6 2x}$

Sol. $$\lim_{x\to 0}\frac{e^{x^3}-1-x^3}{\sin^6 2x} = \lim_{x\to 0}\frac{\left(1+x^3+\frac{x^6}{2!}+.....\right)-1-x^3}{\frac{\sin^6 2x}{(2x)^6}\cdot(2x)^6}$$

$$= \lim_{x\to 0}\frac{\frac{1}{2!}+\frac{x^9}{3!}+.....}{2^6}\ \frac{\frac{1}{2}}{2^6} = \frac{1}{128}.\ \textbf{Ans.}$$

Illustration :

Evaluate $\lim_{x\to\infty} x-x^2\ \ell n\left(1+\frac{1}{x}\right)$

Sol. $\lim_{x\to\infty} x-x^2\ \ln\left(1+\frac{1}{x}\right)$ put $x=\frac{1}{y}$

$$= \lim_{y\to 0}\frac{1}{y}-\frac{\ln(1+y)}{y^2}=\frac{y-\ln(1+y)}{y^2} = \lim_{y\to 0}\frac{y-\left(y-\frac{y^2}{2}+\frac{y^3}{3!}-...\right)}{y^2} = \frac{1}{2}.\ \textbf{Ans.}$$

Don't do it: $\lim_{y\to 0}\left(\frac{1}{y}-\frac{\ln(1+y)}{y}\cdot\frac{1}{y}\right)=\frac{1}{y}-\frac{1}{y}=0$ as $\lim_{y\to 0}\frac{\ln(1+y)}{y}=1$, *is not correct.*

Illustration :

$$\text{Evaluate } \lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}.$$

Sol. $$\lim_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = \lim_{x \to 0} \frac{e^{\frac{1}{x}\ln(1+x)} - e}{x} = \lim_{x \to 0} \frac{e\left(e^{\frac{\ln(1+x)-x}{x}} - 1\right)}{\left(\frac{\ln(1+x)-x}{x}\right)} \cdot \left(\frac{\ln(1+x)-x}{x^2}\right)$$

$$= e \lim_{x \to 0} \frac{e^{\frac{\ln(1+x)}{x}-1}}{\left(\frac{\ln(1+x)}{x} - 1\right)} \cdot \lim_{x \to 0} \left(\frac{\ln(1+x)-x}{x^2}\right) = e\,(1) \cdot \left(\frac{-1}{2}\right) = \frac{-e}{2}. \quad \textbf{Ans.}$$

Don't do it

$$\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} = l$$

Let $x = y/2$

$$\therefore l = \lim_{y \to 0} \frac{\left(1+\frac{y}{2}\right)^{2/y} - e}{y/2} = 2.\lim_{y \to 0} \frac{\left(1+\frac{y^2}{4}+y\right)^{1/y} - e}{y} = 2.\lim_{y \to 0} \frac{e^{\frac{1}{y}\left(1+\frac{y^2}{4}+y-1\right)} - e}{y} \,*$$

$$= 2.\lim_{y \to 0} \frac{e^{\left(\frac{y^2}{4}+y\right)\cdot\frac{1}{y}} - e}{y} = 2.\lim_{y \to 0} \frac{e^{\frac{y}{4}+1} - e}{y} = 2e.\lim_{y \to 0} \frac{e^{\frac{y}{4}} - 1}{y/4} \cdot \frac{1}{4} = \frac{2e}{4} = \frac{e}{2}$$

Note that mistake occured at *.

Illustration :

$$\text{Evaluate } \lim_{x \to 0} \frac{1}{(\sin^{-1} x)^2} - \frac{1}{x^2}.$$

Sol. Put $x = \sin\theta \Rightarrow \lim_{\theta \to 0} \frac{1}{\theta^2} - \frac{1}{\sin^2\theta} = \lim_{\theta \to 0} \frac{\sin^2\theta - \theta^2}{\theta^2 \sin^2\theta}$

$$= \lim_{\theta \to 0} \frac{(\sin\theta - \theta)(\sin\theta + \theta)}{\theta^4} = 2.\lim_{\theta \to 0} \frac{\sin\theta - \theta}{\theta^3} \qquad \textbf{Ans.} -\frac{1}{3}$$

Don't do it

$$\lim_{x \to 0} \frac{1}{(\sin^{-1} x)^2} - \frac{1}{x^2} = \lim_{x \to 0} \frac{1}{x^2} \cdot \frac{x^2}{(\sin^{-1} x)^2} - \cdot\frac{1}{x^2} = \lim_{x \to 0} \frac{1}{x^2} - \frac{1}{x^2} = 0, \text{ is wrong.}$$

Illustration :

If $\lim\limits_{x\to 0}\dfrac{A\cos x + Bx\sin x - 5}{x^4}$ exists & finite. Find A & B and also the limit.

Sol. Let $L = \lim\limits_{x\to 0}\dfrac{A\cos x + Bx\sin x - 5}{x^4}$

$$= \lim_{x\to 0}\frac{A\left(1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}.....\right)+B\,x\left(x-\dfrac{x^3}{3!}+....\right)-5}{x^4}$$

$$L = \lim_{x\to 0}\frac{(A-5)+\left(B-\dfrac{A}{2}\right)x^2+\left(\dfrac{A}{24}-\dfrac{B}{6}\right)x^4}{x^4} = \text{finite value}$$

$$\Rightarrow \quad A = 5,\ B = \frac{A}{2} = \frac{5}{2},\ L = \frac{A}{24}-\frac{B}{6} = \frac{-5}{24}.\ \textbf{Ans.}$$

Illustration :

Let $f(x) = \dfrac{4+\sin 2x + A\sin x + B\cos x}{x^2}$. If $\lim\limits_{x\to 0} f(x)$ exists and finite find A and B and the limit.

Sol. Let $L = \lim\limits_{x\to 0}\dfrac{4+\sin 2x + A\sin x + B\cos x}{x^2}$

$$L = \lim_{x\to 0}\frac{4+2x-\dfrac{(2x)^3}{3!}+.......+A\left(x-\dfrac{x^3}{3!}+.....\right)+B\left(1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+....\right)}{x^2}$$

$$L = \lim_{x\to 0}\frac{(4+B)+(A+2)x-\dfrac{B}{2}x^2+....}{x^2}$$

$$\Rightarrow \quad B = -4,\ A = -2,\ L = 2.\ \textbf{Ans.}$$

Illustration :

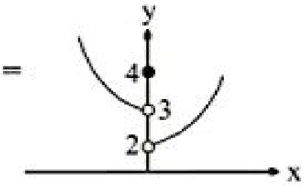
Evaluate $\lim\limits_{x\to 0}\dfrac{\tan^2 x - x^2}{x^2\tan^2 x}$

Sol. $\lim\limits_{x\to 0}\dfrac{\tan^2 x - x^2}{x^2\tan^2 x} = \lim\limits_{x\to 0}\dfrac{(\tan x - x)(\tan x + x)}{x^2\tan^2 x}$

$$= \lim_{x\to 0}\frac{\left(x+\dfrac{1}{3}x^3+......-x\right)(\tan x + x)}{x^4\left(\dfrac{\tan x}{x}\right)^2} = \frac{1}{3}\,\frac{\left(\dfrac{\tan x}{x}+1\right)}{\left(\dfrac{\tan x}{x}\right)^2} = \frac{1}{3}\left(\frac{1+1}{1}\right) = \frac{2}{3}.\ \textbf{Ans.}$$

Illustration :

Refer the figure, the value of $\lim\limits_{x\to 0^-}\left(\left[3f\left(\frac{x^3-\sin^3 x}{x^4}\right)\right]-f\left(\left[\frac{\sin x^3}{x}\right]\right)\right)=$



where [·] denote greatest integer function.

(A) 3 (B) 5 (C) 7 (D) 9

Sol. Let $L=\lim\limits_{x\to 0^-}\left(\left[3f\left(\frac{x^3-\sin^3 x}{x^4}\right)\right]\right)-f\left(\left[\frac{\sin x^3}{x}\right]\right)$

when $x\to 0^-$, $\frac{x^3-\sin^3 x}{x^4}=\frac{(x-\sin x)(x^2+\sin^2 x+x\sin x)}{x^4}$

$$=\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}.......\right)\left(1+\frac{\sin^2 x}{x^2}+\frac{\sin x}{x}\right)}{x^2}$$

$$=x\left(\frac{1}{3!}-\frac{x^2}{5!}+.......\right)\left(1+\frac{\sin^2 x}{x^2}+\frac{\sin x}{x}\right)\to 0^-$$

$f(0^-)=3^+$

$[3f(0^-)]=[9^+]=9$

$f\left(\left[\frac{\sin x^3}{x^3}\cdot x^2\right]\right)=f(0)=4$

$L=9-4=5.$ ***Ans.***

Illustration :

Evaluate the following limits

(a) $\lim\limits_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}$ (b) $\lim\limits_{x\to 0}\frac{e^{\sin 2x}-e^{\sin x}}{x}$

Sol. (a) $\lim\limits_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{1+x^2+\frac{(x^2)^2}{2!}+\frac{(x^2)^3}{3!}-\left(1-\frac{x^2}{2!}+.....\right)}{x^2}=\frac{3}{2}$

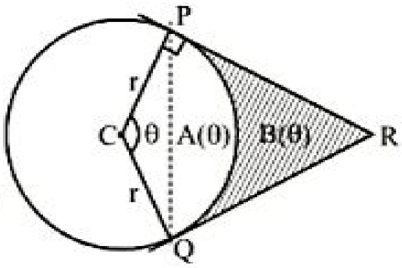
(b) $\lim\limits_{x\to 0}\frac{e^{\sin 2x}-e^{\sin x}}{x}=\lim\limits_{x\to 0}\frac{\left(1+\sin 2x+\frac{(\sin 2x)^2}{2!}+....\right)-(1+\sin x+....)}{x}$

$=\lim\limits_{x\to 0}\frac{\sin 2x-\sin x}{x}=2\ \ 1=1.$ ***Ans.***

Illustration :

An arc PQ of a circle subtends a central angle θ as shown. Let $A(\theta)$ be the area between the chord PQ and the arc PQ. Let $B(\theta)$ be the area between the tangent lines PR and QR and the arc PQ.

Find $\lim\limits_{\theta\to 0}\dfrac{A(\theta)}{B(\theta)}$



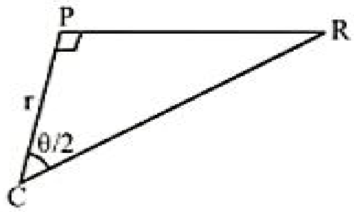
Sol. $A(\theta)$ = Area of sector PCQ – Area of ΔPCR

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$$

$B(\theta)$ = Area of quadrilateral PCQR – Area of sector PCQR

$$= 2\ (\text{Area of } \Delta CPR) - \text{Area of sector } PCQP$$

$$= 2\left(\frac{1}{2}\cdot r\cdot \tan\frac{\theta}{2}\right) - \frac{1}{2}r^2\theta = \frac{r^2}{2}\left(2\tan\frac{\theta}{2} - \theta\right)$$

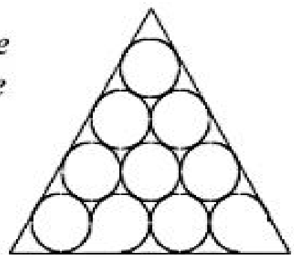


$$\lim_{\theta\to 0}\frac{A(\theta)}{B(\theta)} = -\lim_{\theta\to 0}\frac{\theta - \sin\theta}{2\tan\left(\frac{\theta}{2}\right) - \theta}$$

$$= \lim_{\theta\to 0}\frac{\theta - \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^4}{4!}.....\right)}{2\left(\frac{\theta}{2} + \frac{\left(\frac{\theta}{2}\right)^3}{3} + \frac{2\left(\frac{\theta}{2}\right)^5}{5!}.......\right) - \theta} = \frac{\frac{1}{3!}}{2\cdot\frac{1}{3\cdot 2^5}} = \frac{3\cdot 8}{2\cdot 6} = 2. \ \textit{Ans.}$$

Illustration :

Suppose that circle of equal diameter are packed tightly in n rows inside an equilateral triangle. (The figure illustrates the case n=4.) If A is the area of the triangle and A_n is the total area occupied by the circles in n rows then $\lim\limits_{n\to\infty}\dfrac{A_n}{A}$ *equals*



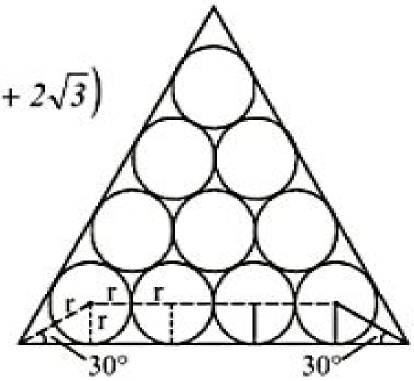
(A) $\dfrac{\pi}{\sqrt{3}}$ (B) $\dfrac{\pi\sqrt{3}}{6}$ (C) $\dfrac{\pi}{2\sqrt{3}}$ (D) $\dfrac{\pi}{6}$

Sol. Let radius of each circle = r and side of triangle = a

$$a = (n-2)\,2r + 2(r + 30°) = r\left(2n - 4 + 2 + 2\sqrt{3}\right) = r\left(2n - 2 + 2\sqrt{3}\right)$$

$$A = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}r^2\ 4\left(n - 1 + \sqrt{3}\right)^2$$

$$A = \sqrt{3}\,r^2\left(n + \sqrt{3} - 1\right)^2$$



$$A_n = \pi r^2 \quad (1 + 2 + \ldots\ldots + n) = \frac{n(n+1)}{2}\pi r^2$$

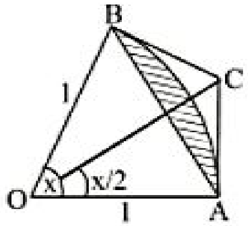
$$\lim_{n\to\infty}\frac{A_n}{A} = \lim_{n\to\infty}\frac{\frac{n(n+1)}{2}\pi r^2}{\sqrt{3}r^2\left(n+\sqrt{3}-1\right)^2}$$

$$= \lim_{n\to\infty}\frac{\pi}{2\sqrt{3}}\ \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{\sqrt{3}-1}{n}\right)^2} = \frac{\pi}{2\sqrt{3}}\ \frac{(1+0)}{(1+0)^2} = \frac{\pi}{2\sqrt{3}}.\ \textit{Ans.}$$

Illustration :

A circular arc of radius 1 subtends an angle of x radians, $0 < x < \frac{\pi}{2}$ as shown in the figure. The point C is the intersection of the two tangent lines at A & B. Let T(x) be the area of triangle ABC & let S(x) be the area of the shaded region. Compute:

(a) T(x) *(b) S(x)* & *(c) the limit of $\frac{T(x)}{S(x)}$ as $x \to 0$.*



Sol. $\tan\frac{x}{2} = \frac{CA}{1}$ so $CA = CB = \tan\frac{x}{2}$

$\angle ABC = \pi - x$; $\Delta ABC = \frac{1}{2}\left(\tan^2\frac{x}{2}\right)\sin x$

(a) so $T(x) = \frac{1}{2}\tan^2\frac{x}{2}\cdot\sin x = \frac{1}{2}\left(\sec^2\frac{x}{2} - 1\right)\sin x$

$$= \frac{1}{2}\sec^2\frac{x}{2}\sin x \quad \frac{\sin x}{2} - \frac{\sin x}{2} = \tan\left(\frac{x}{2}\right) - \frac{\sin x}{2}$$

$\Delta OAB = \frac{1}{2}\sin x$; sector $OAB = \frac{1}{2}x$.

(b) $S(x) = \frac{1}{2}x - \frac{1}{2}\sin x$.

(c) $$\lim_{x\to 0}\frac{T(x)}{S(x)} = \lim_{x\to 0}\frac{\tan\left(\frac{x}{2}\right) - \frac{\sin x}{2}}{\frac{x}{2} - \frac{\sin x}{2}} = \frac{\left(\frac{x}{2} + \frac{\left(\frac{x}{2}\right)^3}{3} + \ldots\ldots\right) - \left(\frac{x}{2} - \frac{x^3}{12} + \ldots\ldots\right)}{\frac{1}{2}\left(x - x + \frac{x^3}{3!} + \ldots\ldots\right)}$$

$$= \lim_{x\to 0}\frac{\frac{1}{24} + \frac{2}{24}}{\frac{1}{12}} = 12\left(\frac{3}{24}\right) = \frac{3}{2}.$$

CONTINUITY

1. GENERAL INTRODUCTION :

After conceiving the notion of limits the next element which is taken into consideration is the continuity of function. Qualitatively the graph of a function is said to be continuous at $x = a$ if while travelling along the graph of the function and in crossing over the point at $x = a$ either from Left to Right or from Right to Left one does not have to lift his pen. In case one has to lift his pen the graph of the function is said to have a break or discontinuous at $x = a$. Different type of situations which may come up at $x = a$ along the graph can be :

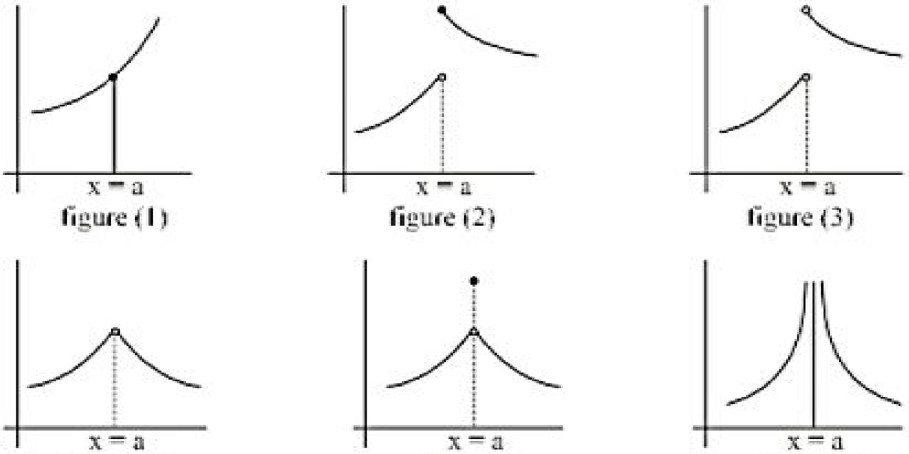


figure (1) figure (2) figure (3) figure (4) figure (5) figure (6)

Figure (2) – (6) is discontinuous at $x = a$ and in figure (1) f is continuous at $x = a$

2. DEFINITION OF CONTINUITY OF A FUNCTION :

A function f(x) is said to be continuous at $x = a$,

if $\quad \lim\limits_{x \to a} f(x) = f(a)$.

$\Rightarrow \quad \lim\limits_{h \to 0} f(a-h) = \lim\limits_{h \to 0} f(a+h) = f(a) =$ a finite quantity.

i.e. LHL at $x = a$ = RHL at $x = a$ = value of f (x) at $x = a$ = a finite quantity.

Note:

(i) Continuity at $x = a$ $\Rightarrow$ existence of limit at $x = a$, but not the converse

(ii) Continuity at $x = a$ $\Rightarrow$ f is well defined at $x = a$, but not the converse

(iii) Discontinuity at $x = a$ is meaningful to talk if in the immediate neighbourhood of $x = a$, i.e. the function has a graph in the immidiate neighbourhood of $x = a$, not necessarily at $x = a$.

(iv) Continuity is always talk in the domain of function and hence $f(x) = \dfrac{1}{x-1}, \dfrac{1}{x}$, tan x are all continuous functions but if you want to talk of discontinuity then we can say $\dfrac{1}{x-1}$ is discontinuous at $x = 1$, $\dfrac{1}{x}$ is discontinuous at $x = 0$.

Note that all rational functions are continuous. Because continuity is always talk in the domain of f (x).

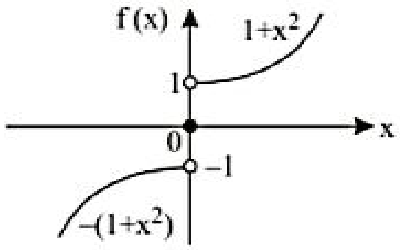
(v) Point function are continuous.

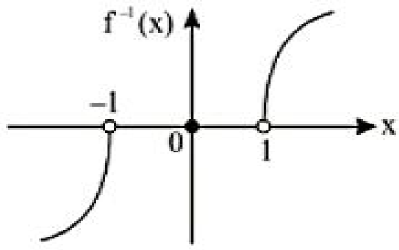
e.g. $\quad \sqrt{1-x} + \sqrt{x-1}, \ \sqrt{x} + \sqrt{-x}$

(vi) Inverse of a discontinuous function can be continuous.

e.g. $f(x)=\begin{cases} 1+x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \\ -(1+x^2) & \text{if } x \neq 0 \end{cases}$ is discontinuous at $x=0$ but its inverse function

$f^{-1}(x)=\begin{cases} \sqrt{x-1} & \text{if } x>1 \\ 0 & \text{if } x=0 \\ \sqrt{-(1+x)} & \text{if } x<-1 \end{cases}$ which is a continuous function and its graph is as shown.





3. CONTINUITY IN AN INTERVAL :

(a) A function f is said to be continuous in (a, b) if f is continuous at each & every point $\in (a, b)$.

(b) A function f is said to be continuous in a closed interval $[a, b]$ if :

(i) f is continuous in the open interval (a, b) &

(ii) f is right continuous at **'a'** i.e. $\lim\limits_{x \to a^+} f(x) = f(a) =$ a finite quantity.

(iii) f is left continuous at **'b'** i.e. $\lim\limits_{x \to b^-} f(x) = f(b) =$ a finite quantity.

4. REASONS OF DISCONTINUITY :

A function can be discontinuous due to the following reasons.

(i) $\lim\limits_{x \to a} f(x)$ does not exist (f (a) may or may not be defined)

i.e. $\lim\limits_{h \to 0} f(a+h) \neq \lim\limits_{h \to 0} f(a-h)$

e.g. $f(x)=[x]$ discontinuous at all integer points $f(x)=\operatorname{sgn} x$ discontinuous at $x=0$

$f(x)=\dfrac{x}{x-1}$ discontinuous at $x=1$.

(ii) $\lim\limits_{x \to a} f(x)$ exist but is not equal to $f(a)$ i.e. $\lim\limits_{h \to 0} f(a+h) = \lim\limits_{h \to 0} f(a-h) \neq f(a)$

$$f(x)=\begin{cases} (1-x)\tan\dfrac{\pi x}{2} & \text{if } x \neq 1 \\ \dfrac{\pi}{2} & \text{if } x=1 \end{cases}$$

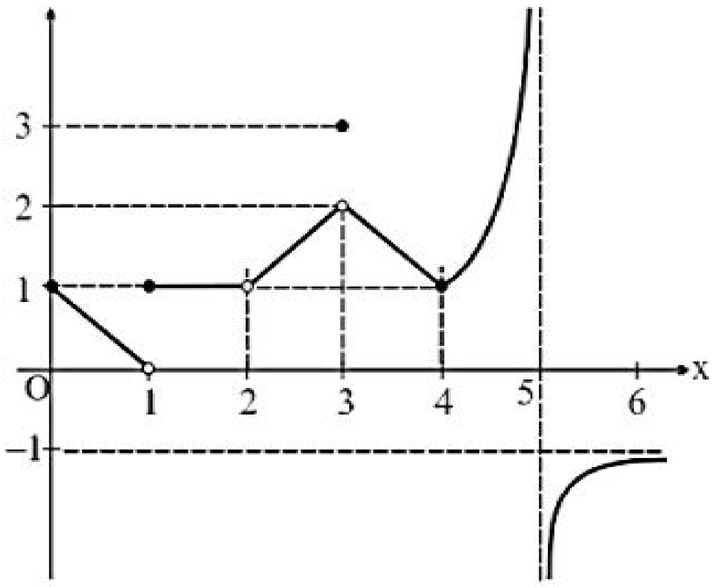
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \lim_{h \to 1} \frac{-h}{\tan\left(\frac{\pi h}{2}\right)} = \frac{2}{\pi}$$

$\Rightarrow$ $\lim f(x) \neq f(1) \Rightarrow f(x)$ is discontinuous at $x=1$.

(iii) f(a) is not defined

eg. $f(x) = \dfrac{1}{x-1}$

(iv) To understand explicitly the reasons of discontinuity. Consider the following graph of a function.



(a) f is continuous at x = 0 and x = 4

(b) f is discontinuous at x = 1 as limit does not exist

(c) f is discontinuous at x = 2 as f(2) is not defined although limit exist.

(d) f is discontinuous at x = 3 as $\lim\limits_{x\to 3} f(x) \neq f(3)$

(e) f is discontinuous at x = 5 as neither the limit exist nor f is defined at x = 5

Note:

(i) Every polynomial function is continous at every point of the real line.

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n \ \forall\ x \in R$$

(ii) Every rational function is continuous at every point where its denominator is different from zero.

(iii) Logarithmic functions, exponential functions, trigonometric functions, inverse circular functions, and modulus functions are continuous in their domain.

Illustration :

Find the points of discontinuity of the following functions.

(i) $f(x) = \dfrac{1}{2\sin x - 1}$; *(ii)* $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$; *(iii)* $f(x) = \dfrac{1}{x^4 + x^2 + 1}$; *(iv)* $f(x) = \dfrac{1}{1 - e^{\frac{x-1}{x-2}}}$

(v) $f(x) = [[x]] - [x-1]$, *where* $[\cdot]$ *represents the greatest integer function.*

Sol.

(i) $f(x) = \dfrac{1}{2\sin x - 1}$

f (x) is discontinuous when $2\sin x - 1 = 0$

$$\Rightarrow \ \sin x = \frac{1}{2} \ \Rightarrow \ x = 2n\pi + \frac{\pi}{6} \quad or \quad x = 2n\pi + \frac{5\pi}{6},\ n \in Z$$

(ii) $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

f (x) is discontinuous when $x^2 - 3|x| + 2 = 0$

$$\Rightarrow \quad |x|^2 - 3|x| + 2 = 0 \Rightarrow \quad (|x| - 1)(|x| - 2) = 0 \Rightarrow |x| = 1, 2 \Rightarrow x = \pm 1, \pm 2$$

(iii) $f(x)=\dfrac{1}{x^4+x^2+1}=\dfrac{1}{\left(x^2+\dfrac{1}{2}\right)^2+\dfrac{3}{4}}$

Now, $x^4+x^2+1=\left(x^2+\dfrac{1}{2}\right)^2+\dfrac{3}{4}\geq 1\ \forall x\in R$

$\Rightarrow$ $f(x)$ is continuous $\forall x\in R$

(iv) $f(x)=\dfrac{1}{1-e^{\frac{x+1}{x-2}}}$

$f(x)$ is discontinuous when $x-2=0$ also

when $1-e^{\frac{x+1}{x-2}}=0$

$\Rightarrow$ $x=2$ and $e^{\frac{x+1}{x-2}}=1$

$\Rightarrow$ $x=2$ and $\dfrac{x-1}{x-2}=0$

$\Rightarrow$ $x=2$ and $x=1$

(v) $f(x)=[[x]]-[x-1]=[x]-([x]-1)=1$

$\Rightarrow f(x)$ is continuous $\forall x\in R$.

Illustration :

(a) $f(x)=\begin{cases}(\cos x)^{\cot^2 x} & x\neq 0\\ e^{-1/2}\quad \text{if} & x=0\end{cases}$ find whether the $f(x)$ is continuous at $x=0$ or not.

(b) If $f(x)=\begin{cases}\dfrac{(e^x-1)^3\,cosec(ax)}{\ln(1+x^2)} & x\neq 0\\ b & x=0\end{cases}$ is continuous, find b.

Sol.

(a) $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}(\cos x)^{\cot^2 x}$

$=e^{\lim\limits_{x\to 0}(\cos x-1)\cot^2 x}=e^{\lim\limits_{x\to 0}\frac{-(1-\cos x)}{x^2}\frac{x^2}{\tan^2 x}}=e^{\frac{-1}{2}}=f(0)\Rightarrow f(x)$ is continuous at $x=0$.

(b) $\lim\limits_{x\to 0}\dfrac{(e^x-1)^3\,cosec\,(ax)}{\ln(1+x^2)}=\lim\limits_{x\to 0}\dfrac{\left(\dfrac{e^x-1}{x}\right)^3}{\dfrac{\ln(1+x^2)}{x^2}}\cdot\left(\dfrac{x}{\sin ax}\right)$

$b=\dfrac{1}{-}.$

Illustration :

Find the values of 'a' and 'b' so that the function $f(x)=\begin{cases} x+a\sqrt{2}\sin x & 0\le x<\frac{\pi}{4} \\ 2x\cot x+b & \frac{\pi}{4}\le x\le\frac{\pi}{2} \\ a\cos 2x-b\sin x & \frac{\pi}{2}<x\le\pi \end{cases}$

is continuous in $[0, \pi]$

Sol.

f(x) is continuous in the interval $0\le x<\frac{\pi}{4},\ \frac{\pi}{4}<x<\frac{\pi}{2},\ \frac{\pi}{2}<x\le\pi$.

We need to make the function continuous at $x=\frac{\pi}{4},\frac{\pi}{2}$

For continuity at $x=\frac{\pi}{4}$, $\lim\limits_{x\to\frac{\pi}{4}^-} f(x)=\lim\limits_{x\to\frac{\pi}{4}^+} f(x)=f\left(\frac{\pi}{4}\right)$

$$\lim_{x\to\left(\frac{\pi}{4}\right)^-}\left(x+a\sqrt{2}\sin x\right)=\lim_{x\to\left(\frac{\pi}{4}\right)^+}\left(2x\cot x+b\right)=f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\pi}{4}+a\sqrt{2}\sin\left(\frac{\pi}{4}\right)=2\cdot\frac{\pi}{4}\cdot\cot\left(\frac{\pi}{4}\right)+b=2\frac{\pi}{4}\cdot\cot\left(\frac{\pi}{4}\right)+b$$

$$\Rightarrow \quad \frac{\pi}{4}+a=\frac{\pi}{2}+b \Rightarrow a-b=\frac{\pi}{4} \qquad ....(1)$$

For continuity at $x=\frac{\pi}{2}$, $\lim\limits_{x\to\frac{\pi}{2}^-} f(x)=\lim\limits_{x\to\frac{\pi}{2}^+} f(x)=f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \quad \lim_{x\to\frac{\pi}{2}^-}\left(2x\cot x+b\right)=\lim_{x\to\left(\frac{\pi}{2}\right)^+}\left(a\cos 2x-b\sin x\right)=a\cos\pi-b\sin\pi$$

$$\Rightarrow \quad 0+b=-a-b \Rightarrow a+2b=0 \qquad ....(2)$$

From equation (1) and (2)

$$a=\frac{\pi}{6},\ b=\frac{-\pi}{12}.$$

Illustration :

$$f(x)=\begin{cases} \dfrac{1-\cos 4x}{x^2} & \text{if } x<0 \\ a & \text{if } x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & \text{if } x>0 \end{cases}$$

Determine 'a' if possible so that the function is continuous at x = 0.

Sol. $f(0^-) = \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{1-\cos 4x}{x^2} = 8$

$$f(0^+) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}\ -4} = \lim_{x\to 0^+} \frac{\sqrt{x}\cdot\left(\sqrt{16+\sqrt{x}}+4\right)}{16+\sqrt{x}\ -16}$$

$$= \lim_{x\to 0^+}\left(\sqrt{16+\sqrt{x}}+4\right) = 8$$

$f(0^-) = f(0^+) = 8 = f(0) \Rightarrow a = 8.$

Illustration :

$$\text{Let } f(x) = \begin{cases} (1+|\sin x|)^{\frac{a}{|\sin x|}} & \text{for } -\dfrac{\pi}{6}<x<0 \\ b & \text{for } x=0 \\ e^{\frac{\tan 2x}{\tan 3x}} & \text{for } 0<x<\dfrac{\pi}{6} \end{cases}$$

Find 'a' and 'b' if f is continuous at x = 0.

Sol. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0} (1+|\sin x|)^{\frac{a}{|\sin x|}} = e^a$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\frac{\tan 2x}{\tan 3x}} = e^{\lim\limits_{x\to 0^+} \frac{\tan 2x}{2x}\cdot\frac{3x}{\tan 3x}\cdot\frac{3}{2}} = e^{\frac{3}{2}}$$

$\therefore \quad e^a = b = e^{\frac{3}{2}} \Rightarrow a = \dfrac{3}{2},\ b = \ln\left(\dfrac{3}{2}\right).$

Illustration :

$$\text{Let } f(x) = \begin{cases} \dfrac{(e^{2x}+1)-(x+1)(e^x+e^{-x})}{x(e^x-1)} & \text{if } x\neq 0 \\ k & \text{if } x=0 \end{cases}$$

if f (x) is continuous at x = 0 then k is equal to

(A) 1/2 *(B) 1* *(C) 3/2* *(D) 2*

Sol. $k = \lim\limits_{x\to 0} \dfrac{\left(e^{2x}+1\right)-(x+1)(e^x+e^{-x})}{x(e^x-1)} = \lim\limits_{x\to 0} \dfrac{\left(e^{2x}+1\right)-(x+1)(e^x+e^{-x})}{x^2}$

$$= \lim_{x \to 0} \frac{2e^{2x} - (x+1)(e^x - e^{-x}) - (e^x + e^{-x})}{2x}$$

By L Hospital Rule

$$= \lim_{x \to 0} \frac{4e^{2x} - (x+1)\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) - \left(e^x - e^{-x}\right)}{2} = \frac{4-2-0-0}{2} = 1.$$

Illustration :

Let $f(x) = \dfrac{\sqrt{x^2 + kx + 1}}{x^2 - k}$. *The interval(s) of all possible values of k for which f is continuous for every* $x \in R$, *is*

(A) $(-\infty, -2]$ (B) $[-2, 0)$ (C) $R - (-2, 2)$ (D) $(-2, 2)$

Sol. $x^2 - k \neq 0 \ \forall x \in R$

$\Rightarrow \quad k < 0 \quad$ ....(1)

$\quad x^2 + x + 1 \geq 0 \ \forall x \in R$

$\Rightarrow \quad k^2 - 4 \leq 0 \Rightarrow -2 \leq k \leq 2 \quad$ ....(2)

$\therefore$ From (1) and (2)

$\quad k \in [-2, 0)$

Practice Problem

Q.1 What value must be assigned to k so that the function $f(x)$ is continuous at $x = 4$?

$$f(x) = \begin{cases} \dfrac{x^4 - 256}{x - 4}, & x \neq 4 \\ k, & x = 4 \end{cases}$$

Q.2 Let $f(x) = \begin{cases} \dfrac{\sin ax^2}{x^2}, & x \neq 0 \\ \dfrac{3}{4} + \dfrac{1}{4a}, & x = 0 \end{cases}$. For what values of a, $f(x)$ is continuous at $x = 0$.

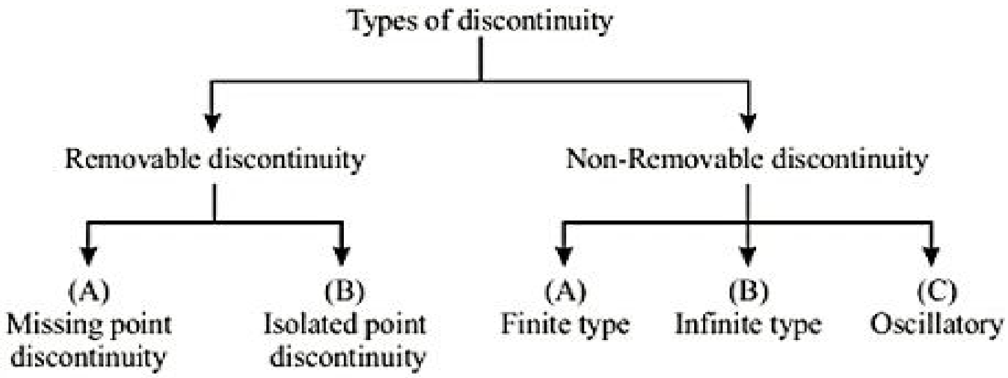
Q.3 Let $f(x) = \begin{cases} \dfrac{a + 3\cos x}{x^2}, & x \neq 0 \\ b\tan\left(\dfrac{\pi}{[x+3]}\right), & x = 0 \end{cases}$

If $f(x)$ is continuous at $x = 0$, then find a and b, where $[\cdot]$ denotes the greatest integer function.

Answer key

Q.1 $k = 256$ **Q.2** $a = -\dfrac{1}{4}, 1$ **Q.3** $a = -3, b = -\dfrac{\sqrt{3}}{2}$

5. TYPES OF DISCONTINUITY :



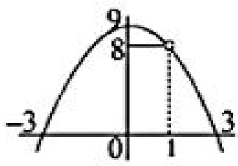
5.1 REMOVABLE DISCONTINUITY :

Here $\lim\limits_{x \to a} f(x)$ necessarily exists, but is either not equal to $f(a)$ or $f(a)$ is not defined. In this case, therefore it is possible to redefine the function in such a manner that $\mathop{\text{Lim}}\limits_{x \to a} f(x) = f(a)$ and thus making the function continuous. These discontinuities can be further classified as

(A) Missing point discontinuity :

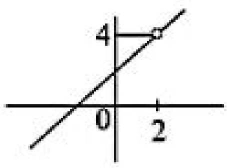
Here $\mathop{\text{Lim}}\limits_{x \to a} f(x)$ exists. But $f(a)$ is not defined.

(a) $f(x)=\dfrac{(x-1)(9-x^2)}{x-1}$ $x \neq 1$



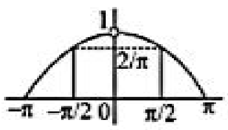
at $x = 1$, $f(1)$ is not defined. Hence $f(x)$ has missing point of discontinuity at $x = 1$.

(b) $f(x)=\dfrac{x^2-4}{x-2}$ $x \neq 2$



$f(2)$ is not defined. Hence, $f(x)$ has missing point of discontinuity at $x = 2$.

(c) $f(x)=\dfrac{\sin x}{x}, x \neq 0$

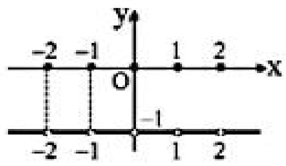


$f(0)$ is not defined. $f(x)$ has missing point of discontinuity at $x = 0$.

(B) Isolated point discontinuity :

Here $\lim_{x \to a} f(x)$ exists, also f(a) is defined but $\lim_{x \to a} f(x) \neq f(a)$

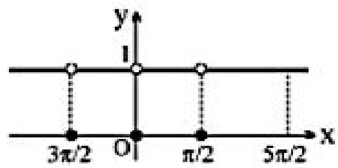
(a) $f(x) = [x] + [-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$



has isolated point of discontinuity at all integeral points.

(b) $f(x) = \text{sgn}(\cos 2x - 2\sin x + 3) = \text{sgn}\left(2(2+\sin x)(1-\sin x)\right) = \begin{cases} 0 & \text{if } x = 2n\pi + \dfrac{\pi}{2} \\ +1 & \text{if } x \neq 2n\pi + \dfrac{\pi}{2} \end{cases}$

has an isolated point at x = 0 discontinuity as $x = 2n\pi + \dfrac{\pi}{2}$



5.2 NON-REMOVABLE DISCONTINUITY :

Here $\lim_{x \to a} f(x)$ does not exists and therefore it is not possible to redefine the function in any manner to make it continuous. Such discontinuities can be further classified into 3 fold.

(a) Finite type (both limits finite and unequal)

(i) $\lim_{x \to 0} \tan^{-1}\left(\dfrac{1}{x}\right) \left\langle \begin{array}{l} f(0^+) = \dfrac{\pi}{2} \\ f(0^-) = -\dfrac{\pi}{2} \end{array} \right.$; jump $= \pi$

(ii) $\lim_{x \to 0} \dfrac{|\sin x|}{x} \left\langle \begin{array}{l} f(0^+) = 1 \\ f(0^-) = -1 \end{array} \right.$; jump $= 2$

(iii) $\lim_{x \to 2} \dfrac{[x]}{x} \left\langle \begin{array}{l} f(2^+) = 1 \\ f(2^-) = \dfrac{1}{2} \end{array} \right.$; jump $= \dfrac{1}{2}$

In this case non negative difference between the two limits is called the Jump of discontinuity. A function having a finite number of jumps in a given interval I is called a Piece Wise Continuous or Sectionally Continuous function in this interval.

(b) **Infinite type** (at least one of the two limit are infinity)

(i) $f(x)=\dfrac{x}{1-x}$ at $x=1$ $\begin{cases} f(1^+)=-\infty \\ f(1^-)=+\infty \end{cases}$

(ii) $f(x)=2^{\tan x}$ at $x=\dfrac{\pi}{2}$ $\begin{cases} f\left(\dfrac{\pi}{2}^+\right)=0 \\ f\left(\dfrac{\pi}{2}^-\right)=\infty \end{cases}$

(iii) $f(x)=\dfrac{1}{x^2}$ at $x=0$ $\begin{cases} f(0^+)=\infty \\ f(0^-)=\infty \end{cases}$

(c) **Oscillatory** (limits oscillate between two finite quantities)

(i) $\left.\begin{array}{l} f(x)=\sin\dfrac{1}{x} \\ \text{or} \\ f(x)=\cos\dfrac{1}{x} \end{array}\right]$ at $x=0$ oscillates between -1 & 1

(ii) $f(x)=\left[1+\dfrac{1}{3}\sin(ln|x|)\right]$ at $x=0$ oscillates between 0 & 1.

Illustration :

State the number of point of discontinuities and discuss the nature of discontinuity for the function $f(x)=\dfrac{1}{ln|x|}$ and also sketch its graph.

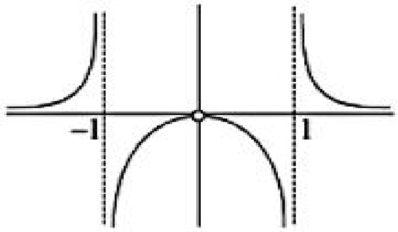
Sol. $f(x)=\begin{cases} \dfrac{1}{ln\,x} & \text{if } x>0,\ x\neq 1 \\ \dfrac{1}{ln(-x)} & \text{if } x<0,\ x\neq -1 \end{cases}$ *function is obviously discontinuous at x = 0, 1, –1. as it is not defined.*

$\left.\begin{array}{l} \lim\limits_{x\to 0^+} f(x)=0 \\ \lim\limits_{x\to 0^-} f(x)=0 \end{array}\right\}$ *Limit exists at x = 0. Hence removable discontinuity at x = 0. (Missing point discontinuity)*

$\left.\begin{array}{l} \lim\limits_{x\to 1^+} f(x)=\infty \\ \lim\limits_{x\to 1^-} f(x)=-\infty \end{array}\right\}$ *Limit DNE. Hence non removable discontinuity (infinite type) at x = 1*

$$\left.\begin{array}{l}\lim\limits_{x\to -1^+} f(x) = -\infty \\ \lim\limits_{x\to -1^-} f(x) = \infty\end{array}\right\}$$ *Limit DNE. Hence non removable discontinuity (infinite type) at x = 0*

Note that f (x) is even ⇒ symmetric about y axis.
The graph of f (x) is as follows.



Practice Problem

Q.1 The function $f : R - \{0\} \to R$ given by $f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}$, is continuous at $x = 0$, then find the value of $f(0)$.

Q.2 Let $f(x) = \begin{cases}(1+3x)^{1/x}, & x \neq 0 \\ e^3, & x = 0\end{cases}$. Discuss the continuity of $f(x)$ at (i) $x = 0$, (ii) $x = 1$.

Q.3 Which of the following functions is not continuous $\forall\, x \in R$?

(A) $\sqrt{2\sin x + 3}$ (B) $\frac{e^x + 1}{e^x + 3}$ (C) $\left(\frac{2^{2x}+1}{2^{3x}+5}\right)^{5/7}$ (D) $\sqrt{\operatorname{sgn} x + 1}$

Q.4 Discuss the continuity of $f(x) = \begin{cases}\frac{x^4 - 5x^2 + 4}{|(x-1)(x+2)|}, & x \neq 1, 2 \\ 6, & x = 1 \\ 12, & x = 2\end{cases}$

Q.5

	Column-I		Column-II
(A)	$f(x) = \frac{1}{x-1}$	(P)	Removable discontinuity
(B)	$f(x) = \frac{x^3 - x}{x^2 - 1}$	(Q)	Non-removable discontinuity
(C)	$f(x) = \frac{\lvert x-1\rvert}{x-1}$	(R)	Jump of discontinuity
(D)	$f(x) = \sin\left(\frac{1}{x-1}\right)$	(S)	Discontinuity due to vertical asymptote
		(T)	Missing point discontinuity
		(U)	Oscillating discontinuity

Answer key

Q.1 1

Q.2 Continuous at $x = 0$

Q.3 D

Q.4 $f(x)$ is discontinuous at $x = 1, 2$

Q.5 (A) $\to$ (S), (Q); (B) $\to$ (T), (P); (C) $\to$ (R), (Q); (D) $\to$ (U), (Q)

6. CONTINUITY OF FUNCTIONS DEFINED BY SOME FUNCTIONAL RULE :

Illustration :

If $f(x+y) = f(x) \cdot f(y)$ for all x & y & $f(x) = 1 + g(x)$. $G(x)$ where $\lim\limits_{x \to 0} g(x) = 0$ & $\lim\limits_{x \to 0} G(x)$ exist. Prove that $f(x)$ is continuous for all x.

Sol. $\lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a-h) = \lim\limits_{h \to 0} f(a)\, f(-h) = \lim\limits_{h \to 0} f(-h)$

$$= f(a)\left(1 + \lim_{h \to 0} g(-h)\, g(-h)\right) = f(a)$$

Similarly $\lim\limits_{x \to a^+} f(x) = f(a) = \lim\limits_{x \to a^-} f(x)$ *so continuous at $x = a$.*

7. THEOREMS ON CONTINUITY :

T-1 : Sum, difference, product and quotient of two continuous functions is always a continuous function.

However $h(x) = \dfrac{f(x)}{g(x)}$ is continuous at $x = a$ only if $g(a) \neq 0$.

8. FOLLOWING IMPORTANT NOTES SHOULD BE REMEMBERED :

(a) If $f(x)$ is continuous and $g(x)$ is discontinuous then prove that $f(x) + g(x)$ is a discontinuous function.

Proof : Let $f(x) + g(x)$ is a continuous function.

so, $\lim\limits_{x \to a} \left(f(x) + g(x)\right) = f(a) + g(a)$ ....(1)

Also, $f(x)$ is a continuous function $\lim\limits_{x \to a} f(x) + g(a)$ ....(2)

From (1) and (2)

$\lim\limits_{x \to a} g(x) = g(a) \Rightarrow g(x)$ is continuous at $x = a$.

But given $g(x)$ is discontinuous at $x = a$.

(b) If f(x) is continuous & g(x) is discontinuous at $x = a$ then the product function $\phi(x) = f(x) . g(x)$ is not necessarily be discontinuous at $x = a$. e.g.

(i) $f(x) = x$ & $g(x) = \begin{bmatrix} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{bmatrix}$.

Then $f(x)\cdot g(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at $x = 0$.

(ii) $f(x) = \cos\left(\frac{2x-1}{2}\right)\pi$ is continuous at $x = 1$ and $g(x) = [x]$ and $[\cdot]$ denotes the greatest integer functions is discontinuous at $x = 1$ but $f(x)\cdot g(x)$ is continuous at $x = 1$.

$$\lim_{x\to 1^+} \cos\left(\frac{2x-1}{2}\right)\pi\cdot[x] = \cos\frac{\pi}{2}\cdot(1) = 0$$

$$\lim_{x\to 1^-} \cos\left(\frac{2x-1}{2}\right)\pi\cdot[x] = \cos\left(\frac{\pi}{2}\right)\cdot(0) = 0,\ f(1) = 0 \Rightarrow \text{continuous at } x = 1.$$

(c) If f(x) and g(x) both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) . g(x)$ is not necessarily be discontinuous at $x = a$. e.g.

$f(x) = -g(x) = \begin{bmatrix} 1 & x \geq 0 \\ -1 & x < 0 \end{bmatrix}$

$\therefore$ $f(x)\,g(x) = 1\ \forall\ x \in R$ which is continuous function.

Illustration :

If $f(x) = \begin{cases} |x+1|; & x \leq 0 \\ x; & x > 0 \end{cases}$ *and* $g(x) = \begin{cases} |x+1|; & x \leq 0 \\ -|x+2|; & x > 0 \end{cases}$

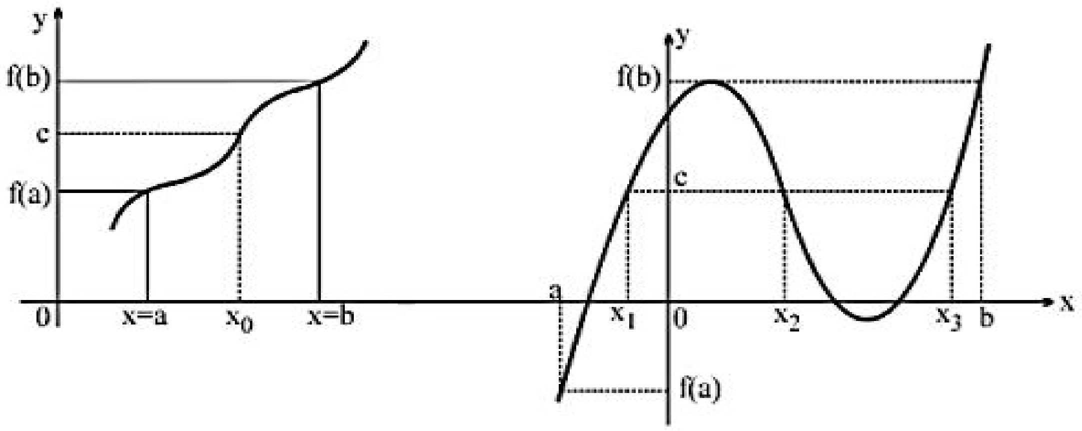
Draw its graph and discuss continuity of f (x) + g(x).

Sol. *Since f (x) is discontinuous at x = 0 and g(x) is continuous at x = 0, then f (x) + g(x) is discontinuous at x = 0.*

Since f (x) is continuous at x = 1 and g(x) is discontinuous at x = 1, then f (x) + g(x) discontinuous at x = 1.

T-2 : Intermediate value theorem :

If f is continuous on [a, b] and f (a) ≠ f (b) then for any value c ∈ $(f(a), f(b))$, there is at least one number x_0 in (a, b) for which f (x_0) = c



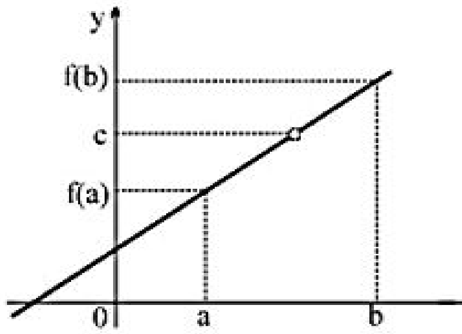


Figure-2

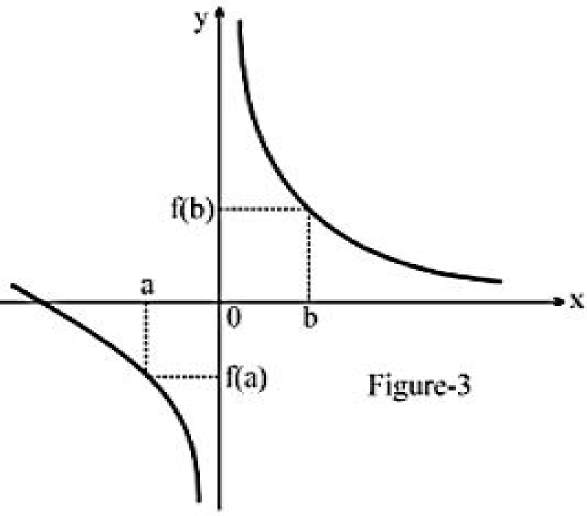


Figure-3

NOTE:

(1) Continuity through the interval [a, b] is essential for the validity of this theorem.

(2) in figure-3, f (a) and f (b) are of opposite sign but f(x) has no root in (a, b) as f is continuous.

Illustration :

Show that the function $f(x) = (x-a)^2 (x-b)^2 + x$ *takes the value* $\dfrac{a+b}{2}$ *for some value of* $x \in [a, b]$

Sol. $f(x) = (x-a)^2 (x-b)^2 + x$; *as* $f(x)$ *is continuous on* $[a, b]$ *and* $f(a) = a$ *and* $f(b) = b$, *then for any value* $c \in (a, b)$, *there is at least one number* x_0 *in* (a, b) *for which* $f(x_0) = c = \dfrac{a+b}{2}$.

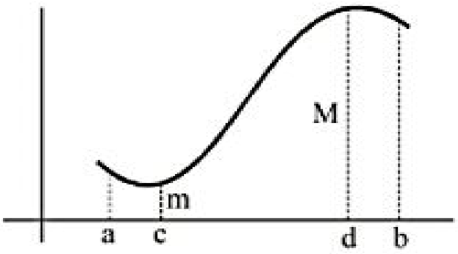
Illustration :

Suppose that $f(x)$ is continuous in $[0, 1]$ and $f(0) = 0$, $f(1) = 0$. Prove that $f(c)=1 - 2c^2$ for some $c \in (0, 1)$.

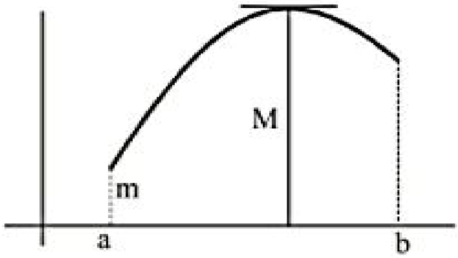
Sol. *Let $f(x) = f(x) + 2x^2 - 1$ is a continuous function in $(0, 1)$.*
$f(0) = f(0) - 1 = -1$ and $f(1) = F(1) + 1 = 1$ then there exists same $c \in (0, 1)$ such that $F(x) = 0$
$f(x) = 1 - 2c^2$.

T-3: Extreme Value Theorem :

If f is continuous on [a, b] then f takes on, a least value of m and a greatest value M on this interval.



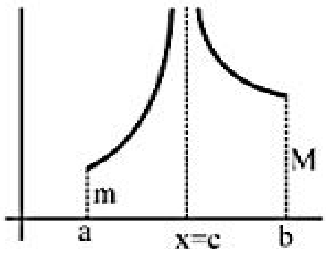
Minimum value 'm' occurs at x = c and maximum value M occurs at x = d. c, d $\in$ (a, b)



Minimum value 'm' occurs at the end point x = a and the maximum value M occurs inside the interval

☞ **Note :** To see that continuity is necessary for the extreme value theorem to be true refer the graph shown.

There is a discontinuity at x = c interval. The function has a minimum value at the left end point x = a and f has no maximum value.



9. PROPERTIES OF FUNCTION CONTINUOUS IN [a, b] :

(i) If a function f is continuous on a closed interval [a, b] then it is bounded.

(ii) A continuous function whose domain is some closed interval must have its range also in closed interval.

(iii) If f is continuous and onto on [a, b] and is onto then f^{-1}(from the range of f) is also continuous.

(iv) If f(a) and f(b) possess opposite signs then $\exists$ at least one solution of the equation f(x) = 0 in the open interval (a, b) provided f is continous in [a, b].

Illustration :

Let f be a continuous function defined onto on [0, 1] with range [0, 1]. Show that there is some c in [0,1] such that $f(c) = 1 - c$.

Sol. Consider $g(x) = f(x) - 1 + x$

$g(0) = f(0) - 1 \le 0$ [as $f(0) \le 1$]

$g(1) = f(1) \ge 0$ [as $f(1) \ge 0$]

Hence, g(0) and g(1) have valuse of opposite signs.

Hence, there exists at least one $c \in (0, 1)$ such that $g(c) = 0$.

$\therefore$ $g(c) = f(c) - 1 + c = 0$; $f(c) = 1 - c$.

Illustration :

Let f be continuous on the inteval [0, 1] to R such that f (0) = f (1). Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$.

Sol. Consider a continuous function $g(x) = f\left(x + \frac{1}{2}\right) - f(x)$ $\left(g \text{ is continous } \forall\, x \in \left[0, \frac{1}{2}\right]\right)$

$\Rightarrow$ $g(0) = f\left(\frac{1}{2}\right) - f(0) = f\left(\frac{1}{2}\right) - f(1)$ [as $f(0) = f(1)$]

and $g\left(\frac{1}{2}\right) = f(1) - f\left(\frac{1}{2}\right) = -\left[f\left(\frac{1}{2}\right) - f(1)\right]$

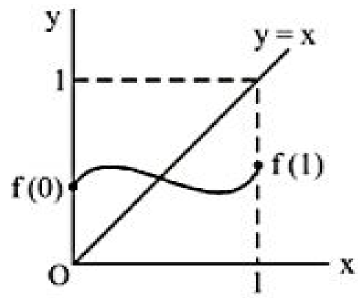
Since g is continuous and g(0) and $g\left(\frac{1}{2}\right)$ have opposite signs, hence the equation g(x) = 0 must have at one root in $\left[0, \frac{1}{2}\right]$.

Hence, for some $c \in \left[0, \frac{1}{2}\right]$, $g(c) = 0 \Rightarrow f\left(c + \frac{1}{2}\right) = f(c)$.

Illustration :

Let $f : [0, 1] \to [0, 1]$ be a continuous function. Then prove $f(x) = x$ for at least one $0 \le x \le 1$.

Sol. Clearly, $0 \le f(0) \le 1$ and $0 \le f(1) \le 1$. As f (x) is continuous, f (x) attains all the values between f (0) and f (1) and the graph will have no breaks. So, the graph will cut the line y = x at one point x at least where $0 \le x \le 1$. So, $f(x) = x$ at that point.

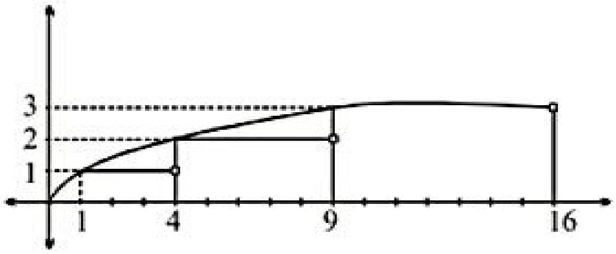


10. CONTINUITY OF SPECIAL TYPES OF FUNCTIONS :

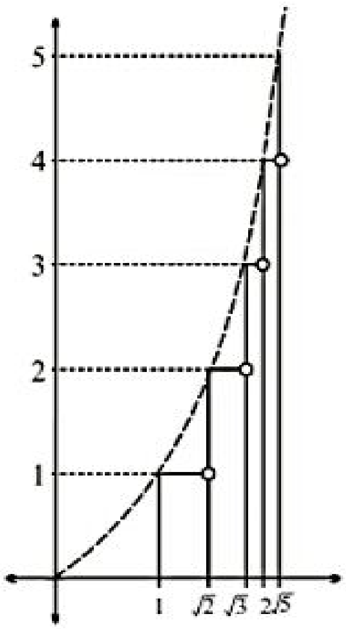
10.1 Continuity of functions in which greatest integer function is involved:

$f(x) = [x]$ is discontinuous when x is an integer.
Similarly, $f(x) = [g(x)]$ is discontinuous when g(x) is an integer, but this is true only when g(x) is monotonic (g(x) is strictly increasing or strictly decreasing).

For example, $f(x) = [\sqrt{x}\,]$ is discountinuous when $\sqrt{x}$ is an integer, as $\sqrt{x}$ is strictly increasing (monotonic function).



$f(x) = [x^2], x \geq 0$ is discountinuous when x^2 is an integer, as x^2 is strictly inreasing for $x \geq 0$.



Now consider, $f(x) = [\sin x], x \in [0, 2\pi]$. $g(x) = \sin x$ is not monotonic in $[0, 2\pi]$.

For this type of function, points of discontinuity can determined easily by graphical methods. We can note that at $x = \frac{3\pi}{2}$, sin x takes integral value –1, but at $x = \frac{3\pi}{2}$, $f(x) = [\sin x]$ is countinuous.

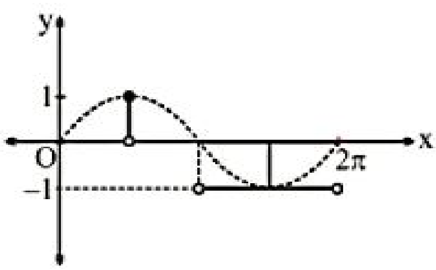


Illustration :

Discuss the continuity of following functions ([·] represents the greatest integer function.)

(a) $f(x) = [\log_e x]$ (b) $f(x) = [\sin^{-1} x]$ (c) $f(x) = \left[\dfrac{2}{1+x^2}\right], x \geq 0$

Sol.

(a) $\log_e x$ function is a montonically increasing function.

Hence $f(x) = [\log_e x]$ is discontinuous, where $\log_e x = k$ or $x = e^k$, $k \in Z$.

Thus $f(x)$ is discontinuous at $x = \ldots\ldots e^{-2}, e^{-1}, e^0, e^1, e^2, \ldots\ldots$

(b) $\sin^{-1} x$, is a monotonically increasing function.

Hence, $f(x) = [\sin^{-1} x]$ is discontinuous where $\sin^{-1} x$ is discontinuous where $\sin^{-1} x$ is an integer.

$\Rightarrow \quad \sin^{-1} x = -1, 0, 1$ or $x = -\sin 1, 0, \sin 1$

(c) $\dfrac{2}{1+x^2}, x \geq 0$, is a monotonically decreasing function.

Hence, $f(x) = \left[\dfrac{2}{1+x^2}\right], x \geq 0$ is discontinuous, when $\dfrac{2}{1+x^2}$ is an integer.

$\Rightarrow \quad \dfrac{2}{1+x^2} = 1,2$

$\Rightarrow \quad x = 1, 0$

Illustration :

Draw the graph and find the points of discontinuity for $f(x) = [2\cos x], x \in [0, 2\pi]$, ([·] represents the gratest integer function).

Sol. $f(x) = [2\cos x]$

Clearly from the graph given in figure

$f(x)$ is discontinuous at $x = 0$

and when $2\cos x = \pm 1$

or $\quad x = 0$ and when $2\cos x = \pm 1$

or $\quad x = 0$ and $\cos x = \pm\dfrac{1}{2}$

or $\quad x = 0$ and $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

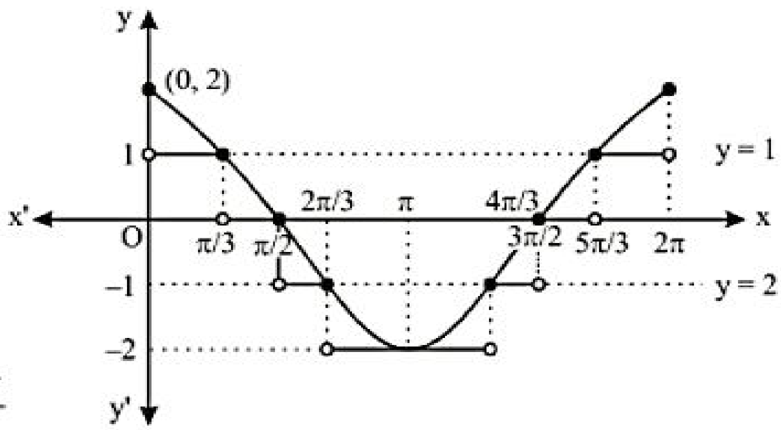


Illustration :

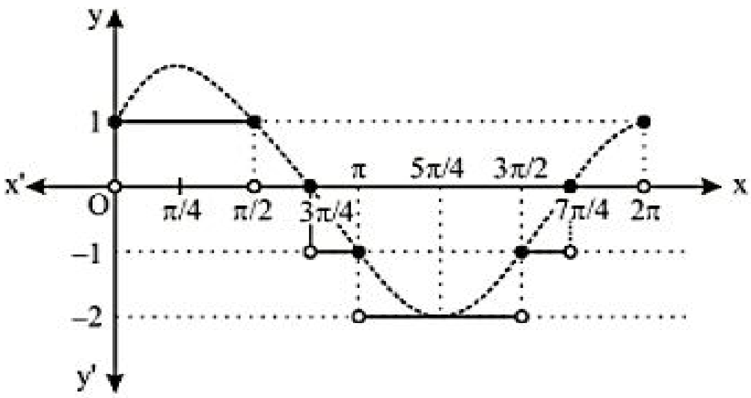
Draw the graph and discuss the continuity of $f(x) = [\sin x + \cos x]$, $x \in [0, 2\pi]$, where $[\cdot]$ represents the greatest integer function.

Sol. $f(x) = [\sin x + \cos x] = [g(x)]$ where $g(x) = \sin x + \cos x$

$g(0) = 1$, $g\left(\frac{\pi}{4}\right) = \sqrt{2}$, $g\left(\frac{\pi}{2}\right) = 1$

$g\left(\frac{3\pi}{4}\right) - 0$ or $g(\pi) = -1$ $g\left(\frac{5\pi}{2}\right) = \sqrt{2}$

$g\left(\frac{3\pi}{2}\right) = -1$, $g\left(\frac{7\pi}{2}\right) = 0$, $g(2\pi) = 1$



Clearly from the graph given in figure $f(x)$ is discontinuous at $x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\ \frac{3\pi}{2}, \frac{7\pi}{2}, 2\pi$.

Illustration :

If the function $f(x) = \left[\frac{(x-2)^3}{a}\right] \sin(x-2) + a\cos(x-2)$. *$[\cdot]$ denotes the greatest integer function which is continuous in $[4, 6]$, then find the value of a.*

Sol. *$\sin(x-2)$ and $\cos(x-2)$ are continuous for all x.*
Since $[x]$ is not continuous at integral point.

So, $f(x)$ is continuous in $[4, 6]$ if $\left[\frac{(x-2)^3}{a}\right] = 0\ \forall x \in [4, 6]$.

Now $(x-2)^3 \in [8, 64]$ for $x \in [4, 6]$.

$\Rightarrow \quad a > 64$ for $\left[\frac{(x-2)^3}{a}\right] = 0$

10.2 Continuity of functions is which signum function is involved :

We know that $f(x) = \text{sgn}(x)$ is discontinuous at $x = 0$.
In general, $f(x) = \text{sgn}(g(x))$ is discontinuous at $x = a$ if $g(a) = 0$.

Illustration :

Discuss the continuity of
(a) $f(x) = sgn(x^3 - x)$, *(b)* $f(x) = sgn(2\cos x - 1)$, *(c)* $f(x) = sgn(x^2 - 2x + 3)$.

Sol.

(a) $f(x) = sgn(x^3 - x)$
Here $x^3 - x = 0 \Rightarrow x = 0, -1, 1$
Here $f(x)$ *is discontinuous at* $x = 0, -1, 1$

(b) $f(x) = sgn(2\cos x - 1)$

Here, $2\cos x - 1 = 0 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = 2n\pi + \left(\frac{\pi}{3}\right)$

$n \in Z$, *where* $f(x)$ *is discontinuous.*

(c) $f(x) = sgn(x^2 - 2x + 3)$
Here, $x^2 - 2x + 3 > 0$ *for all x.*
Thus, $f(x) = 1$ *for all x, hence continuous for all x.*

Illustration :

If $f(x) = sgn(2\sin x + a)$ *is continuous for all x, then find the possible values of a.*

Sol. $f(x) = sgn(2\sin x + a)$ *is continuous for all x.*
Then $2\sin x + a \neq 0$ *for any real x.*

$$\Rightarrow \quad \sin x \neq -\frac{a}{2}$$

$$\Rightarrow \quad \left|\frac{a}{2}\right| > 1 \quad \Rightarrow \quad a < -2 \text{ or } a > 2$$

Illustration :

If $f(x) = \begin{cases} sgn(x-2) \times [\log_e x], & 1 \leq x \leq 3 \\ \{x^2\}, & 3 < x \leq 3.5 \end{cases}$ *where* $[\cdot]$ *denotes the greatest integer function and* $\{\cdot\}$ *represents the fractional part function. Find the point where the continuity of f(x) should be checked. Hence, find the points of discontinity.*

Sol.

(a) *Continuity should be checked at the endpoints of intervals of each definition, i.e.,* $x = 1, 3, 3.5$.
For $\{x^2\}$, *continuity should be checked when* $x^2 = 10$.

(b) $11, 12$ *or* $x = \sqrt{10}, \sqrt{11}, \sqrt{12}$. $[x^2]$ *discontinuous for those value of x where* x^2 *is an integer (note, here* x^2 *is monotonic for given domain).*

(c) *For* $sgn(x - 2)$, *continuity should be checked when* $x - 2 = 0$ *or* $x = 2$.

(d) *For* $[\log_e x]$, *continuity should be checked when* $\log_e x = 1$ *or* $x = e$ $(\in [1, 3])$.

Hence, the overall continuity must be checked at $x = 1, 2, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5$.

Checking continuity at $x = 1$

$$f(1) = 0 \text{ and } \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} sgn(x-2) \times [\log_e x] = 0.$$

Hence $f(x)$ *is continuous at* $x = 1$.

Checking continuity at $x = 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} sgn\,(x-2) \times [\log_e x] = (-1) \times 0 = 0$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} sgn\,(x-2) \times [\log_e x] = (1) \times 0 = 0$$

Hence, $f(x)$ is continuous at $x = 2$.

Checking continuity at $x = 3$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \{x^2\} = 0$$

Hence, $f(x)$ is discontinuous at $x = 3$.

Also $\{x^2\}$ and hence $f(x)$ is discontinuous at $x = \sqrt{10}, \sqrt{11}, \sqrt{12}$.

Checking continuity at $x = 3.5$

$$\lim_{x\to 3.5^-} f(x) = \lim_{x\to 3.5^-} \{x^2\} = 0.25 = f(3.5)$$

Hence, $f(x)$ is discontinuous at $x = 3, \sqrt{10}, \sqrt{11}, \sqrt{12}$.

10.3 Continuity of functions involving limit $\lim\limits_{n\to\infty} a^n$:

We know that $\lim\limits_{n\to\infty} a^n = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \end{cases}$

Illustration :

Discuss the continuity of $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n}-1}{x^{2n}+1}$.

Sol. $f(x) = \lim\limits_{n\to\infty} \dfrac{(x^2)^n - 1}{(x^2)^n + 1} = \lim\limits_{x\to\infty} \dfrac{1 - \dfrac{1}{(x^2)^n}}{1 + \dfrac{1}{(x^2)^n}} = \begin{cases} -1, & 0 \le x^2 < 1 \\ 0, & x^2 = 1 \\ 1, & x^2 > 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 0, & x = -1 \\ -1, & -1 < x < 1 \\ 0, & x = 1 \\ 1, & x > 1 \end{cases}$

Thus, $f(x)$ is discontinuous at $x = \pm 1$.

Illustration :

Discuss the continuity of $f(x) = \lim\limits_{n\to\infty} \cos^{2n} x$.

Sol. $f(x) = \lim\limits_{n\to\infty} (\cos^2 x)^n$

$= \begin{cases} 0, & 0 \le \cos^2 x < 1 \\ 1, & \cos^2 x = 1 \end{cases} = \begin{cases} 0, & x \ne n\pi,\ n \in I \\ 1, & x = n\pi,\ n \in I \end{cases}$

Hence, $f(x)$ is discontinuous when $x = n\pi$, $n \in I$.

10.4 Continuity of functions in which f (x) is defined differently for rational and irrational values of x :

Illustration :

Discuss the continuity of the following function : $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$.

Sol. *For any* $x = a$,

$L.H.L. = \lim_{x \to a^-} f(x) = \lim_{x \to a^-} f(a - h) = 0$ or 1 [as $\lim_{h \to 0} (a - h)$ can be rational or irrational]

Similarly, $R.H.L. = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h) = 0$ or 1.

Hence, $f(x)$ oscillates between 0 and 1 as for all values of a.

$\therefore$ L.H.L. and R.H.L. do not exist.

$\Rightarrow f(x)$ is discontinuous at a point $x = a$ for all values of a.

Illustration :

Find the value of x where $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$.

Sol. $f(x)$ is continuous at some $x = a$, where $x = 1 - x$ or $x = \frac{1}{2}$.

Hence, $f(x)$ is continuous at $x = \frac{1}{2}$

If $x \to \frac{1}{2^+}$ then x may be rational or irrational

$$\Rightarrow \quad f\left(\frac{1}{2^+}\right) = \frac{1}{2} \text{ or } 1 - \frac{1}{2} = \frac{1}{2}$$

If $x \to \frac{1}{2^-}$ then x may be rational or irrational

$$\Rightarrow \quad f\left(\frac{1}{2^-}\right) = \frac{1}{2} \text{ or } 1 - \frac{1}{2} = \frac{1}{2}$$

Hence $f(x)$ is continuous at $x = \frac{1}{2}$

For some other point, say, $x = 1$

$\Rightarrow \quad f(1) = 1$

If $x \to 1^+$ then x may be rational or irrational.

$\Rightarrow \quad f(1^+) = 1$ or $1 - 1 = 0$

Hence, $f(1^+)$ oscillates between 1 and 0, which causes discontinuity at $x = 1$.

Similarly, $f(x)$ osciallates between 0 and 1 for all $x \in R - \left(\frac{1}{2}\right)$.

11. CONTINUITY OF COMPOSITE FUNCTIONS :

If f is continuous at x = a & g is continuous at x = f(a) then the composite g[f(x)] is continuous at x = a. e.g. $f(x) = \dfrac{x \sin x}{x^2 + 2}$ & $g(x) = |x|$ are continuous at x = 0, hence the composite $(gof)(x) = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ will also be continuous at x = 0.

Illustration :

If $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{1}{x-2}$, then discuss the continuity of f (x), g(x) and fog (x).

Sol.

(a) $f(x) = \dfrac{x+1}{x-1}$

$\therefore$ f is not defined at x = 1. $\therefore$ f is discontinuous at x = 1.

(b) $g(x) = \dfrac{1}{x-2}$

g(x) is not defined at x = 2. $\therefore$ g is discontinuous at x = 2.

(c) *Now, fog will be discontinuous at*

x = 2 [point of discontinuity of g(x)]

g(x) = 1 [when g(x) = point of discontinuity of f (0)]

$$\text{if } g(x) = 1 \;\Rightarrow \frac{1}{x-2} = 1 \;\Rightarrow\; x = 3$$

$\therefore$ fog (x) is discontinuous at x = 2 and x = 3.

$$\text{Also,}\quad fog(x) = \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}$$

Here fog (2) is not defined.

$$\lim_{x\to 2} fog(x) = \lim_{x\to 2}\frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \lim_{x\to 2}\frac{1+x-2}{1-x+2} = 1$$

$\therefore$ fog (x) is discontinuous at x = 2 and it has a removable discontinuity at x = 2.

For continuity at x = 3.

$$\lim_{x\to 3^+} fog(x) = \lim_{x\to 3}\frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \infty$$

$$\lim_{x\to 3^-} fog(x) = \lim_{x\to 3}\frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = -\infty$$

$\therefore$ fog(x) is discontinuous at x = 3 and it is a non-removable discontinuity at x = 3.

Practice Problem

Q.1 Find the number of points in [1, 3] where the function is $[x^2 + 1]$ is discontinuous. ($[\cdot]$ represents the greatest integer function).

Q.2 Find the number of points of discontinuity for $f(x) = [6 \sin x], 0 \le x \le \pi$, ($[\cdot]$ represents the greatest integer function).

Q.3 Discuss the continuity of $f(x) = [\tan^{-1} x]$ ($[\cdot]$ represents the greatest integer function).

Q.4 Discuss the continuity of $f(x) = \{\cot^{-1} x\}$ ($\{\cdot\}$ represent the fraction part function).

Q.5 Discuss the continuity of f(x) in [0, 2], where $f(x) = \lim\limits_{n \to \infty} \left(\sin \dfrac{\pi x}{2}\right)^{2n}$.

Q.6 Discuss the continuity of $f(x) = \begin{cases} x^2, & x \text{ is rational} \\ -x^2, & x \text{ is irrational} \end{cases}$.

Q.7 If $y = \dfrac{1}{t^2 + t - 2}$ where $t = \dfrac{1}{x-1}$, then find the number of points where $f(x)$ is discontinuous.

Q.8 Prove that :

(i) $f(x) = \left[\begin{array}{l} x \text{ if } x \in Q \\ 0 \text{ if } x \notin Q \end{array}\right.$ is continuous only at $x = 0$.

(ii) $f(x) = \left[\begin{array}{ll} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{array}\right.$ is continuous only at $x = 0$.

(iii) $f(x) = \left[\begin{array}{ll} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{array}\right.$ is continuous only at $x = 1/2$.

(iv) $f(x) = \left[\begin{array}{ll} x^2 & \text{if } x \in Q \\ 1 & \text{if } x \notin Q \end{array}\right.$ is continuous only at $x = 1$ or -1.

Answer key

Q.1 9

Q.2 11

Q.3 Not continuous at $x = -\tan 1, 0, \tan 1$

Q.4 Not continuous at $x = \cot 1, \cot 2, \cot 3$

Q.5 Discontinuous at $x = 1$

Q.6 Continuous at $x = 0$

Q.7 Discontinuous at $x = 1, \dfrac{1}{2}, 2$

DIFFERENTIABILITY

1.0 DIFFERENTIABILITY / DERIVABILITY:

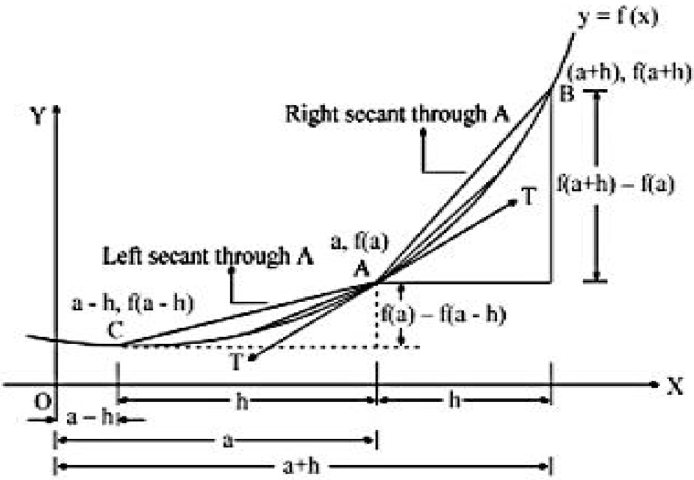
(Two fold meaning of derivability)

Geometrical meaning of derivative
Slope of the tangent drawn to the curve at $x = a$ if it exists

Physical meaning of derivative
(functions which are differentiable)
Instantaneous rate of change of function

Note : "Tangent at a point 'A' is the the limiting case of secant through A."

1.1 RIGHT AND LEFT HAND DERIVATIVES :



1.2 EXISTENCE OF DERIVATIVE :

(I) Right hand & Left hand Derivatives ; By definition : $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ if it exist

(a) The right hand derivative of f at $x = a$ denoted by $f'(a^+)$ is defined by : $f'(a^+) = \lim\limits_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h}$,

provided the limit exists & is finite.
when $h \to 0$, the point B moving along the curve tends to A, i.e., $B \to A$ then the chord AB approaches the tangent line AT at the point A.

$$\Rightarrow \qquad f'(a^+) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \tan\phi = \tan\psi$$

(b) The left hand derivative of f at x = a denoted by $f'(a^-)$ is defined by : $f'(a^-) = \lim\limits_{h \to 0^+} \dfrac{f(a-h)-f(a)}{-h}$,

Provided the limit exists & is finite.

When $h \to 0$, the point C moving along the curve tends to A, i.e., $C \to A$ then the chord CA approaches the tangent line AT at the point A then

$$\Rightarrow \quad f'(a^-) = \lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}$$

f is said to be derivable at x = a if $f'(a^+) = f'(a^-) =$ a finite quantity.

This geomtrically means that a unique tangent with finite slope can be drawn at x = a as shown in the figure.

(II) Theorem :

If a function f is derivable at x = a then f is continuous at x = a.

Proof :

Let f is derivable at x = a. Hence

For : $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$ exists .

Also $\quad f(a+h) - f(a) = \dfrac{f(a+h)-f(a)}{h} \cdot h \qquad (h \neq 0)$

Therefore : $\quad \lim\limits_{h \to 0} [(f(a+h) - f(a)] = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h} . h = f'(a).0 = 0$

Therefore $\lim\limits_{h \to 0} [(f(a+h) - f(a)] = 0 \Rightarrow \lim\limits_{h \to 0} f(a+h) = f(a) \Rightarrow f$ is continuous at x.

Note: If f(x) is derivable for every point of its domain of definition, then it is continuous in that domain.

The Converse of the above result is not true :

For a function f :

Differentiability $\Rightarrow$ Continuity ; $\qquad$ Continuity $\not\Rightarrow$ derivability ;

Non derivibality $\not\Rightarrow$ discontinuous $\qquad$ But discontinuity $\Rightarrow$ Non derivability

Note

(a) Let $f'(a^+) = p$ & $f'(a^-) = q$ where p & q are finite then :

(i) $p = q \Rightarrow f$ is derivable at x = a $\qquad \Rightarrow$ f is continuous at x = a .

(ii) $p \neq q \Rightarrow f$ is not derivable at x = a.

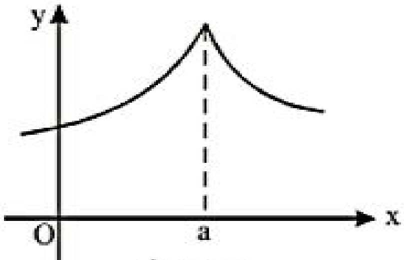
It is very important to note that f may be still continuous at x = a.

(b) If a function f is not differentiable but is continuous at x = a it geometrically implies a sharp corner at x = a.

(III) How can a function fail to be differntiable :

The function $f(x)$ is said to non-differentiable at $x = a$ if

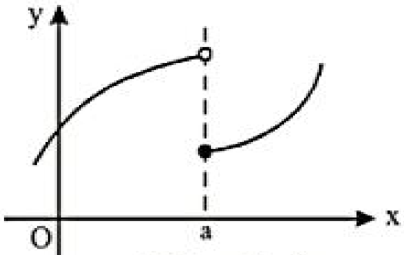
(a) Both left and right and hand derivative exists but are not equal



A corner

The function $y = |x|$ is not differentiable at 0 as its graph change direction abrupty when $x = 0$. In general, if the graph of a function has a 'corner' or 'kink' in it, then the graph of f has no tangent at this point and f is not differentiable there. (To compute $f'(a)$, we find that the left and right limits are different.)

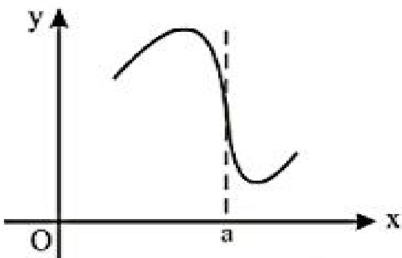
(b) Function is discontinuous at $x = a$



A discontinuity

If f is not continuous at a then f is not differentiable at a. So at any discontinuity (for instance, a jump of discontinuity) f fails to be differentiable.

(c) Either or both left and right hand derivative are not finite.



A vertical tangent

A third possibility is that the curve has a vertical tangent line when $x = a$, that is f is continuous at a and $\lim\limits_{x \to a} |f'(x)| = \infty$.

This means that the tangent lines becomes steeper and sleeper as $x \to a$.

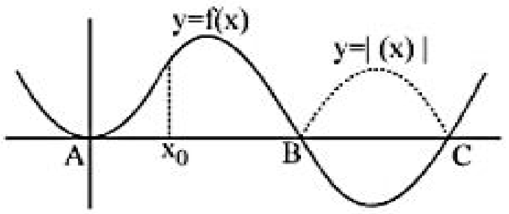
(IV) Derivability Over An Interval :

f (x) is said to be derivable over an open interval (a, b) if it is derivable at each & every point of the interval f(x) is said to be derivable over the closed interval [a, b] if :

(i) for the points a and b, $f'(a+)$ & $f'(b-)$ exist &

(ii) for any point c such that $a < c < b$, $f'(c+)$ & $f'(c-)$ exist & are equal.

Note : Consider the graph of a differentiable function



(1) If $f(x)$ is derivable at $x = x_0$ then $|f(x)|$ must be derivable at $x = x_0$ provided $f(x_0) \neq 0$. However if $f(x_0) = 0$ then $|f(x)|$ may or may not be derivable at $x = 0$

e.g. $f(x) = x^3$ is derivable at $x = 0$ and $|f(x)|$ is also derivable at $x = 0$.

$f(x) = x - 1$ is derivable at $x = 1$ but $|f(x)|$ is not derivable at $x = 1$

i.e. if $f'(x_0) = 0$ and $f(x_0) = 0$ then $|f(x)|$ will also derivable at $x = x_0$ and if $f'(x_0)$ is non zero finite then $|f(x)|$ is non derivable at $x = x_0$.

In figure $f(x) = 0$ at A, B, C and f(x) is derivable at A, B and C but $|f(x)|$ is non derivable at x = B and C but derivable at x = A.

Consider $f(x) = x^3$ at $x = 0$ when $f'(0) = 0$ and $f(0) = 0$ but from $|f(x)| = |x^3|$ is derivable but if $f(x) = x$, where $f'(x) \neq 0$,

hence $|f(x)|$ is not derivable at $x = 0$

Illustration :

Find the left and right hand derivative of the following function at given point

(a) $f(x) = |\ln x|$ at $x = 1$ (b) $f(x) = \ln^2 x$ at $x = 1$ (c) $f(x) = e^{-|x|}$ at $x = 0$

Sol.

(a) $f(x) = \begin{cases} \ln x, & x \geq 1 \\ -\ln x, & 0 < x \leq 1 \end{cases}$

$\Rightarrow$ $f(x)$ is continuous at $x = 1$

$$f'(1^+) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h \to 0} \frac{\ln(1+h)}{h} = 1$$

$$f'(1^-) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} \lim_{h \to 0} \frac{-\ln(1-h) - 0}{-h} = \lim_{h \to 0} \frac{\ln(1-h)}{h} = -1$$

(b) $f(x) = \ln^2 x$ is continuous and differentiable function.

$$f'(x) = 2\ln x\left(\frac{1}{x}\right)$$

$$f'(1) = 2\ln 1\left(\frac{1}{1}\right) = 0$$

(c) $f(x) = e^{-|x|} = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0 \end{cases}$

$\Rightarrow$ $f(x)$ is continuous at $x = 0$.

$$f'(0^+) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-h} - e^{0}}{h} = -1$$

$$f'(0^-) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{e^{-h} - e^{0}}{-h} = 1$$

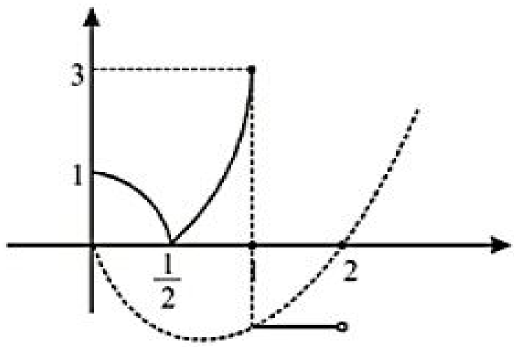
$\Rightarrow$ $f(x)$ is not differential at $x = 0$.

Illustration :

$f(x) = \begin{cases} |1 - 4x^2|, & 0 \leq x < 1 \\ [x^2 - 2x], & 1 \leq x \leq 2 \end{cases}$, *check differentiability in (0, 2), where [ ] denotes greatest integer function.*

Sol. $f(x) = \begin{cases} |1 - 4x^2|, & 0 \leq x < 1 \\ [x^2 - 2x], & 1 \leq x \leq 2 \end{cases}$

$$f(x) = \begin{cases} 1 - 4x^2, & 0 \leq x \leq \frac{1}{2} \\ 4x^2 - 1, & \frac{1}{2} \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 0, & x = 2 \end{cases}$$



$f(x)$ is not differentiable at $x = \frac{1}{2}$, 1 in $x \in (0, 2)$

Illustration :

$$If\ f(x) = \begin{cases} ax+b, & x \le -1 \\ ax^3+x+2b, & x > -1 \end{cases} \quad \text{is differentiable for all } x \in R \text{ find 'a' \& 'b'.}$$

Sol. $f(x) = \begin{cases} ax+b, & x \le -1 \\ ax^3+x+2b, & x > -1 \end{cases}$

For continuity at $x = -1$, $-a + b = a(-1)^3 + (-1) + 2b$

$\Rightarrow \quad b = 1$

For differentiability at $x = -1$

$$\frac{d}{dx}(ax+b)\Big|_{x=-1} = \frac{d}{dx}(ax^3+x+2b)\Big|_{x=-1}$$

$\Rightarrow \quad a = 3a(-1)^2 + 1 \quad \Rightarrow a = \dfrac{-1}{2}.$

Illustration :

Find derivative of

(i) $f(x) = \cos x + |\cos x|$ *at* $x = \dfrac{\pi}{2}$

(ii) $f(x) = \max.\{(1-x), (1+x), 2\}$ *number of points where f is not differentiable.*

(iii) *Find the number of points at which the function* $f(x) = \begin{cases} \max.(|x|, x^2) & \text{if } -\infty < x < 1 \\ \min.(2x-1, x^2) & \text{if } x \ge 1 \end{cases}$ *is not derivable.*

(iv) *Let f be differentiable at x = a and let* $f(a) \ne 0$. *Evaluate* $\lim\limits_{n\to\infty} \left\{\dfrac{f(a+1/n)}{f(a)}\right\}^n$.

Sol.

(i) $f(x) = \cos x + |\cos x| = \begin{cases} 2\cos x, & 0 \le x \le \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x < \pi \end{cases}$

$f(x)$ *is continuous at* $x = \dfrac{\pi}{2}$

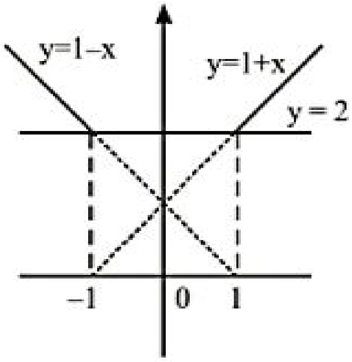
$$f'\left(\frac{\pi^-}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}-h\right) - f\left(\frac{\pi}{2}\right)}{-h} = \lim_{h\to 0} \frac{2\cos\left(\frac{\pi}{2}-h\right) - \cos\frac{\pi}{2}}{-h} = \lim_{h\to 0} \frac{2\sin h}{-h} = -2$$

$$f'\left(\frac{\pi^+}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\Rightarrow f(x)$ *is not differentiable at* $x = \dfrac{\pi}{}$

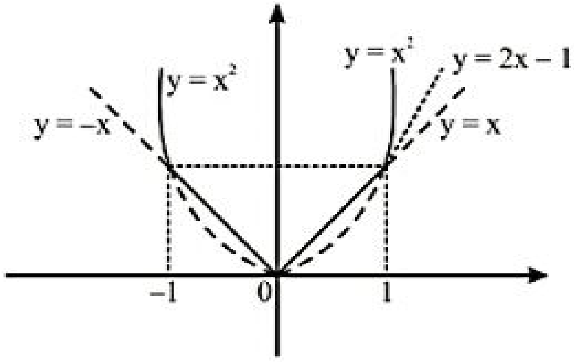
(ii) $f(x) = max\{(1-x), (1+x), 2\}$

$\Rightarrow$ $f(x)$ is not differentiable at $x = \pm 1$.



(iii) $f(x) = \begin{cases} max\{|x|, x^2\}, & x < 1 \\ min(2x-1, x^2) & x \geq 1 \end{cases}$

$\Rightarrow$ $f(x)$ is not differentiable at $x = \pm 1$.



(iv) $$\lim_{n\to\infty}\left(\frac{f\left(a+\frac{1}{n}\right)}{f(a)}\right)^n = e^{\lim_{n\to\infty}\left(\frac{f\left(a+\frac{1}{n}\right)}{f(a)}-1\right)n} = e^{\lim_{n\to\infty}\left(\frac{f\left(a+\frac{1}{n}\right)-f(a)}{\left(\frac{1}{n}\right)}\right)\cdot\frac{1}{f(a)}} = e^{\frac{f'(a)}{f(a)}}$$

Illustration :

If $f(x) = \left[\begin{array}{ll} x^m.\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array}\right.$ is continuous but not differentiable at $x = 0$ then find m.

Sol. $f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

For continuity at $x = 0$.

$$\lim_{h\to 0} f(0+h) = f(0) \Rightarrow \lim_{h\to 0} h^m \sin\left(\frac{1}{n}\right) = 0 \Rightarrow m > 0$$

For function to be not differentiable at $x = 0$

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = DNE$$

$$\lim_{h\to 0}\frac{h^m \sin\left(\frac{1}{h}\right)-0}{h} = \lim_{h\to 0} h^{m-1}\sin\left(\frac{1}{h}\right) = DNE$$

$\Rightarrow$ $m-1 \leq 0 \Rightarrow m \leq 1$.

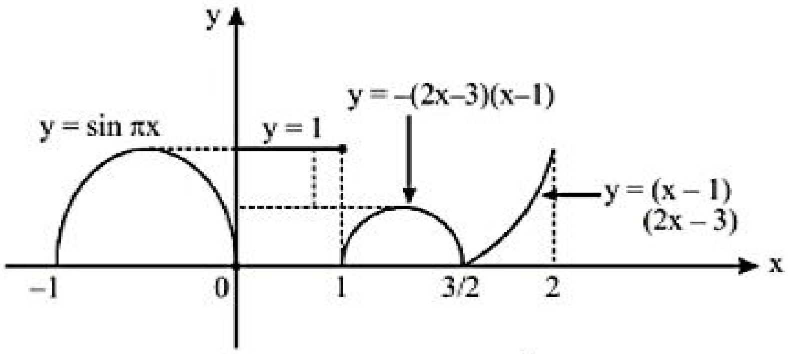
$\therefore$ $m \in [0, 1]$ for function $f(x)$ to be continuous and not differentiable at $x = 0$.

Illustration :

$$f(x) = \begin{cases} \sqrt{4x^2 - 12x + 9}.\{x\}; & x \geq 1 \\ \cos\left(\dfrac{\pi}{2}(|x| - \{x\})\right); & x < 1 \end{cases} \qquad \textit{check the differentiability in } x \in [-1, 2]$$

Sol. $f(x) = \begin{cases} \sqrt{4x^2 - 12x + 9} \cdot \{x\}, & x \geq 1 \\ \cos\left(\dfrac{\pi}{2}(|x|) - \{x\}\right), & x < 1 \end{cases} = \begin{cases} |2x - 3| \cdot \{x\}, & x \geq 1 \\ \cos\dfrac{\pi}{2}(|x| - \{x\}), & x < 1 \end{cases}$

$$= \begin{cases} -(2x-3)(x-1), & 1 \leq x \leq \dfrac{3}{2} \\ (2x-3)(x-1), & \dfrac{3}{2} \leq x \leq 2 \\ \cos\dfrac{\pi}{2}(-x-(x+1)), & -1 \leq x < 0 \\ \cos\dfrac{\pi}{2}(x-(x)), & 0 \leq x < 1 \end{cases} = \begin{cases} -\sin \pi x, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1 \\ -(2x-3)(x-1), & 1 \leq x < \dfrac{3}{2} \\ (2x-3)(x-1), & \dfrac{3}{2} \leq x \leq 2 \end{cases}$$



$\Rightarrow$ *f(x) is not differentiate at* $x = 0, 1, \dfrac{3}{2}$.

Illustration :

Let $f(x) = \operatorname{sgn} x$ *and* $g(x) = x(1 - x^2)$. *Investigate the composite functions* $f(g(x))$ *and* $g(f(x))$ *for continuity and differentiability.*

Sol. $f(x) = \operatorname{sgn}(x)$

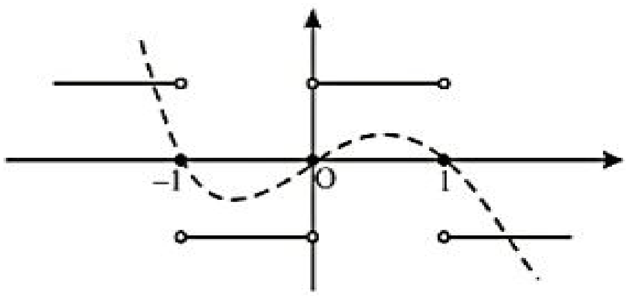
$g(x) = x(1 - x^2)$

$f(g(x)) = \operatorname{sgn}\left(x(1 - x^2)\right)$

$\Rightarrow$ *fog(x) is not differentiable at* $x = 0, \pm 1$.

$gof(x) = g(f(x)) = (\operatorname{sgn} x)\left(1 - (\operatorname{sgn} x)^2\right)$

as $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$



$\therefore$ $gof(x) = 0 \ \forall x \in R$

$\Rightarrow$ *gof (x) is always differentiable.*

f (g (x)) is discontinuous & non derivable at $-1, 0, 1$

Illustration :

$$f(x) = \begin{cases} \left(\ell n\left(e\left|[x]+[-x]\right|\right)^x\right)\cdot\left(\dfrac{2e^{\left(\frac{\{x\}+\{-x\}}{|x|}\right)}-5}{3+e^{\frac{1}{|x|}}}\right) & \text{for } x<0 \\ 0 & \text{for } x=0 \\ x\cdot\dfrac{1-e^{|x|+\{x\}}}{|x|+\{x\}} & \text{for } x>0 \end{cases}$$

Where [ ], { } represents integral and fractional part functions respectively. Compute the Right hand derivative and Left hand derivative at x = 0 and comment on the continuity and derivability at x = 0.

Sol. *As* $\{x\} + \{-x\} = x - [x] - x - [-x] = -\left([x]+[-x]\right) = -(-1+0)$ where $x < 0$.

$[x] + [-x] = -1 + 0 = -1$ when $x < 0$.

$$f(x) = \ln\left(e\left|(-1)\right|\right)^x\cdot\frac{\left(2e^{\frac{1}{|x|}}-5\right)}{\left(3+e^{\frac{1}{|x|}}\right)} \text{ for } x<0$$

$$= \frac{x\left(1-e^{x+x}\right)}{(x+x)} \text{ for } x>0 \qquad [\because |x| = x\ \forall x>0 \text{ and } \{x\} = x\ \forall 0<x<1]$$

$f(x) = 0$ at $x = 0$.

$$f(x) = x\frac{\left(2e^{\frac{-1}{x}}-5\right)}{\left(3+e^{\frac{-1}{x}}\right)} \text{ for } x<0$$

$$= x\frac{\left(1-e^{2x}\right)}{2x} \text{ for } x>0$$

$$\text{L.H.D.} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{(-h)}{(-h)}\frac{\left(2e^{\frac{1}{h}}-5\right)}{\left(3-e^{\frac{1}{h}}\right)} = \lim_{h\to 0}\frac{\left(2-5e^{\frac{-1}{h}}\right)}{\left(3e^{-\frac{1}{h}}+1\right)} = 2$$

$$\text{R.H.D.} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(1-e^{2h})}{2h} = -1$$

f (x) is continuous at x = 0 but not differentiable at x = 0.

Illustration :

$$f(x) = \begin{cases} x+a & \text{if } x<0 \\ |x-1| & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x+1 & \text{if } x<0 \\ (x-1)^2+b & \text{if } x\geq 0 \end{cases}$$

Where a and b are non negative real numbers. Determine the composite function gof. If (gof) (x) is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof differentiable at $x = 0$? Justify your answer.

Sol. $f(x) = \begin{Bmatrix} x+a & \text{if} & x<0 \\ |x-1| & \text{if} & x\geq 0 \end{Bmatrix} \Rightarrow f(x) = \begin{Bmatrix} x+a & \text{if} & x<0 & \\ (x-1) & & \text{if} & 0\leq x<1 \\ (x-1) & & \text{if} & 1\leq x<\infty \end{Bmatrix}$

$$g(x) = \begin{Bmatrix} x+1 & \text{if} & x<0 \\ (x-1)^2+b & \text{if} & x\geq 0 \end{Bmatrix}$$

$g(f(x)) = f(x) + 1$ if $f(x) < 0$

$\quad = (f(x)-1)^2 + b$ if $f(x) \geq 0$.

So $\quad g(f(x)) = x + a + 1$ if $\quad x + a < 0$ and $x < 0 \Rightarrow x < -a$

$\qquad = -x + 1 + 1$ if $-x + 1 < 0$ and $0 \leq x < 1 \Rightarrow$ Null set

$\qquad = x$ if $x - 1 < 0$ and $1 \leq x < \infty \Rightarrow$ Null set

$\qquad = (x + a - 1)^2 + b$ if $x + a \geq 0$ and $x < 0 \Rightarrow -a \leq x < 0$

$\qquad = (-x)^2 + b$ if $-x + 1 \geq 0$ and $0 \leq x < 1 \Rightarrow 0 \leq x < 1$

$\qquad = (x - 2)^2 + b$ if $x - 1 \geq 0$ and $1 \leq x < \infty \Rightarrow 1 \leq x \leq \infty$

$g(f(x)) = x + a + 1 \qquad$ if $\quad -\infty < x < -a \qquad$ ...(1)

$\qquad = (x + a - 1)^2 + b \qquad$ if $\quad -a \leq x < 0 \qquad$ ...(2)

$\qquad = x^2 + b \qquad$ if $\quad 0 \leq x < 1 \qquad$ ...(3)

$\qquad = (x - 2)^2 + b \qquad$ if $\quad 1 \leq x \leq \infty \qquad$ ...(4)

$g(f(x))$ *is continuous every where so it must be continuous at* $x = -a, 0, 1$.

from (1) and (2), checking continuity at $x = -a$

$\quad 1 = 1 + b \Rightarrow b = 0$

Similarly checking continuity at $x = 0$.

$(0 + a - 1)^2 + b = 0 + b \Rightarrow a = 1$

$g(f(x)) = x + 2 : \forall -\infty < x < -1$

$\quad = x^2$ if $-1 \leq x < 0$

$\quad = x^2$ if $0 \leq x < 1$

$\quad = (x - 2)^2$ if $1 \leq x < \infty$

so $\quad (gof)'(x) = 2x$ at $x = 0$ $(gof)'(0) = 0$

(V) Theorem :

If $f(x)$ and $g(x)$ both are derivable at $x = a$, then

(i) $f(x) \pm g(x)$ will be differentiable at $x = a$.

(ii) $f(x) \cdot g(x)$ will be differentiable at $x = a$.

(iii) $\frac{f(x)}{g(x)}$ will be differentiable at $x = a$ if $g(a) \neq 0$.

Note that :

(1) If $f(x)$ and $g(x)$ are both derivable at $x = a$, $f(x) \pm g(x)$; $g(x) \cdot f(x)$ and $\frac{f(x)}{g(x)}$ will also be derivable at $x = a$. (only if $g(a) \neq 0$)

(2) If $f(x)$ is derivable at $x = a$ and $g(x)$ is not derivable at $x = a$ then the $f(x) + g(x)$ or $f(x) - g(x)$ will not be derivable at $x = a$.

e.g. $f(x) = \cos|x|$ is derivable at $x = 0$ and $g(x) = |x|$ is not derivable at $x = 0$,
then $\cos|x| + |x|$ or $\cos|x| - |x|$ will not be derivable at $x = 0$.
However nothing can be said about the product function in this case.

$f(x) = x$ derivable at $x = 0$

$g(x) = |x|$ not derivable at $x = 0$

$$x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

(3) If both $f(x)$ and $g(x)$ are non derivable then nothing definite can be said about the sum / difference / product function.

e.g. $f(x) = \sin|x|$ not derivable at $x = 0$

$g(x) = |x|$ not derivable at $x = 0$

then the function

$F(x) = \sin|x| - |x|$ is derivable at $x = 0$

$G(x) = \sin|x| + |x|$ is not derivable at $x = 0$.

(4) If $f(x)$ is derivable at $x = a$ and $f(a) = 0$ and $g(x)$ is continuous at $x = a$ then the product function
$F(x) = f(x) \cdot g(x)$ will be derivable at $x = a$

$$F'(a^+) = \lim_{h \to 0} \frac{f(a+h)g(a+h) - 0}{h} = f'(a) \cdot g(a)$$

$$F'(a^-) = \lim_{h \to 0} \frac{f(a-h)\,g(a-h) - 0}{-h} = f'(a) \cdot g(a)$$

(5) Derivative of a continuous function need not be a continuous function

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases};\quad \begin{matrix} f'(0^+)=0 \\ f'(0^-)=0 \end{matrix}$$

$$f'(x)=\begin{cases} \sin\frac{1}{x}\cdot 2x - x^2\cos\frac{1}{x}\left(\frac{-1}{x^2}\right) & x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$f'(x)$ is not continuous at $x=0$.

Illustration :

Let f be defined as follows :

$$f(x)=\begin{cases} \sin x & \text{if } x<\pi \\ mx+n & \text{if } x\geq\pi \end{cases}$$

where m and n are constants. Determine m and n such that f is derivable on set of real numbers.

Sol. $f(x)=\begin{cases} \sin x, & \text{if } x<\pi \\ mx+n, & \text{if } x\geq\pi \end{cases}.$

Since f(x) is continuous at $x=\pi$

$0=m\pi+n$ ....(1)

Now f(x) is differentiable at $x=\pi$

$\cos x\big|_{x=\pi}=m \;\Rightarrow\; m=-1 \;\Rightarrow\; n=\pi$

Illustration :

Check the differentiability of function at x = e.

$$f(x)=\begin{cases} (x-e)2^{-2(1/(e-x))}, & x\neq e \\ 0, & x=e \end{cases}$$

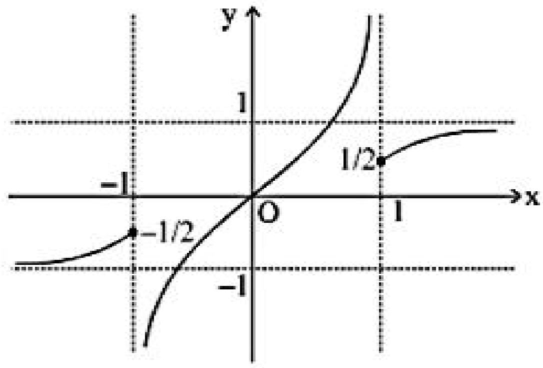
Sol. $f(x)=\begin{cases} (x-e)\,2^{-2\frac{1}{(e-x)}}, & x\neq e \\ 0, & x=e \end{cases}$

$$R.H.D.=\lim_{h\to 0}\frac{f(e+h)-f(e)}{h}=\lim_{h\to 0}\frac{(-h)\,2^{\frac{-2}{h}}}{h}=\infty$$

$$L.H.D.=\lim_{h\to 0}\frac{f(e-h)-f(e)}{-h}=\lim_{h\to 0}\frac{(-h)\,2^{\frac{-2}{h}}-0}{(-h)}=0$$

$\Rightarrow$ *f (x) is not differentiable at x = 0.*

Illustration :



Find the domain of $f'(x)$ *if* $f(x) = \begin{cases} \dfrac{x}{1+|x|} & \text{if } |x| \geq 1 \\ \dfrac{x}{1-|x|} & \text{if } |x| < 1 \end{cases}$.

Sol. $f(x) = \begin{cases} \dfrac{x}{1-x}; & -\infty < x < -1 \\ \dfrac{x}{1+x}; & -1 < x < 0 \\ \dfrac{x}{1-x}; & 0 < x < 1 \\ \dfrac{x}{1+x}; & 1 < x < \infty \end{cases}$

so domain of $f'(x)$ *will be* $R \sim \{-1, 1\}$

Illustration :

If $f(x)$ *is differentiable at* $x = a$ *and* $f'(a) = \dfrac{1}{4}$. *Find* $\lim\limits_{h \to 0} \dfrac{f(a+2h^2) - f(a-2h^2)}{h^2}$

Sol. $\text{Limit} = \lim\limits_{h \to 0} \dfrac{f(a+2h^2) - f(a-2h^2)}{h^2}$ *put* $t = h^2$

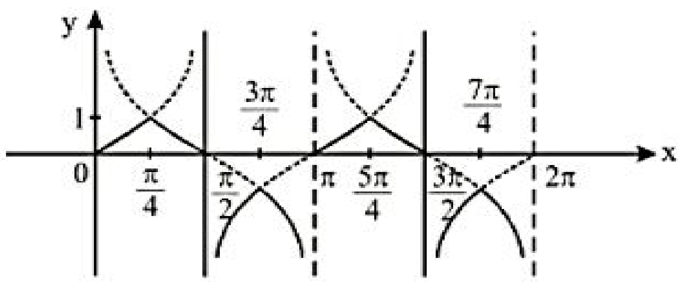
$= \lim\limits_{h \to 0} \dfrac{f(a+2t) - f(a-2t)}{t}$ *differentiating numerator and denominator*

$= \lim\limits_{t \to 0} \dfrac{2f'(a+2t) - 2f'(a-2t)}{1} = 2f'(a) + 2f'(a) = 4f'(a) = 1.$

Illustration :

Consider the function $f(x) = \text{Min}(\tan x, \cot x)$ *in* $(0, 2\pi)$. *Number of points where f is either fails to be derivable is 'm' and number of points where it is discontinuous in 'n'. Find (m, n)*

Sol. *Number of points where* f *is non derivable* $= 7$.



Number of points where $f(x)$ *is discontinuous* $= 3$

$(m, n) = (7, 3)$.

(VI) Determination of function which are differentiable and satisfying the given functional rule :

BASIC STEPS :

(i) Write down the expression for $f'(x)$ as $f'(x) = \frac{f(x+h)-f(x)}{h}$

(ii) Manipulate $f(x+h) - f(x)$ in such a way that the given functional rule is applicable. Now apply the functional rule and simplify the RHS to get $f'(x)$ as a function of x along with constants if any.

(iii) Integrate $f'(x)$ get $f(x)$ as a function of x and a constant of integration. In some cases a Differential Equation in formed which can be solved to get $f(x)$.

(iv) Apply the boundary value conditions to determine the value of this constant.

Illustration :

Let f be a differentiable function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)$ *for all x, y > 0. If* $f'(1) = 1$ *then find f (x).*

Sol. $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{1}{h} f\left(\frac{x+h}{x}\right) = \lim\limits_{h\to 0} \frac{1}{h} f\left(1+\frac{h}{x}\right) = \lim\limits_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)-f(1)}{h}$

as $f(1) = 0$

So, $f'(x) = \frac{f'(1)}{x} = \frac{1}{x}$

integrating w.r.t. to x

$f(x) = \ln x + k$ *as* $f(1) = 0$ *so* $k = 0$

$f(x) = \ln x$

Illustration :

Suppose f is a derivable function that satisfies the equation

$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose that $\lim\limits_{x\to 0} \frac{f(x)}{x} = 1$*, find*

(a) $f(0)$ *(b)* $f'(0)$ *(c)* $f'(x)$ *(d)* $f(3)$

Sol. $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{f(h)+x^2h+xh^2}{h}$

$f'(x) = \lim\limits_{h\to 0} \frac{f(h)}{h} + x^2 = 1 + x^2$

$f(x) = x + \frac{x^3}{3} + c$ *as* $f(0) = 0$ *so* $c = 0$

$f(x) = x + \frac{x^3}{3}$

Illustration :

A differentiable function satisfies the relation

$$f(x+y) = f(x) + f(y) + 2xy - 1 \quad \forall x, y \in R$$

If $f'(0) = \sqrt{3+a-a^2}$ *find* $f(x)$ *and prove that* $f(x) > 0 \quad \forall x \in R$

Sol. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)+2xh-1}{h}$

$f(0) = 2f(0) - 1 \Rightarrow f(0) = 1$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} + 2x = f'(0) + 2x$

$f'(x) = 2x + \sqrt{(3+a-a^2)}$

$f(x) = x^2 + \sqrt{(3+a-a^2)}\,x + c$ $\qquad$ as $f(0) = 1$ so $c = 1$

$f(x) = x^2 + \sqrt{(3+a-a^2)}\,x + 1$

$D = 3 + a - a^2 - 4 = a - a^2 - 1 = -(a^2 - a + 1) < 0$

as $a^2 - a + 1 > 0 \ \forall a$

So, $f(x) > 0 \quad \forall x \in R$

Illustration :

(i) If $f(x+y) = f(x) \cdot f(y)$, $\forall\ x,\ y \in R$ and $f(x)$ is a differentiable function everywhere. Find $f(x)$.

(ii) If $f(x+y) = f(x) + f(y)$, $\forall\ x,\ y \in R$ then prove that $f(kx) = k f(x)$ for $\forall k, x \in R$.

Sol.

(i) $f(x+y) = f(x) f(y) \Rightarrow f(0) = f(0)^2 \Rightarrow f(0) = 0$ or $f(0) = 1$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)f(h)-f(x)}{h} = f(x) f'(0)$ assume $f'(0) = k_2$

$\displaystyle\int \frac{f'(x)}{f(x)} = \int k_2 dx \quad \Rightarrow \quad f(x) = k_1 e^{k_2 x}$

(ii) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)+f(h)-f(x)}{h} = f'(0) = k$

$\dfrac{d}{dx}(f(x)) = k \Rightarrow f(x) = kx + c$ as $f(0) = 0$

$f(x) = kx \Rightarrow f(f(x)) = k f(x)$.

Illustration :

Discuss the differentiability of $f(x)=\begin{cases}\dfrac{\sin x^2}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ *at* $x=0$

Sol. *For continuity,* $\lim\limits_{x\to 0} f(x)=\lim\limits_{h\to 0}\dfrac{\sin h^2}{h}$

$$=\lim_{h\to 0} h\frac{\sin h^2}{h^2}=0$$

Hence, f (x) is continuous at x = 0.

Also, $f'(0^+)=\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{\sin h^2}{h^2}=1$

and $f'(0)=\lim\limits_{h\to 0}\dfrac{f(-h)-f(0)}{-h}=\lim\limits_{h\to 0}\dfrac{\sin h^2}{h^2}=1$

Thus, f (x) is differentiable at x =0.

DIFFERENTIABILITY USING THEOREMS ON DIFFERENTIABILITY :

Illustration :

Discuss the differentiability of $f(x)=|x|+|x-1|$.

Sol. $f(x)=|x|+|x-1|$

f (x) is continuous everywhere at |x| and |x – 1| are continuous for all x.

Also |x| and |x – 1| are non-differentiable at x = 0 and x = 1, respectively.

Hence, f (x) is non-differentiable at x = 0 and x = 1.

Illustration :

Discuss the differentiability of $f(x)=\text{maximum }\{2\sin x, 1-\cos x\}\ \forall x\in(0,\pi)$.

Sol. $f(x)=\max.\{2\sin x, 1-\cos x\}$ *can be plotted as*

Thus, $f(x)=\text{maximum }\{2\sin x, 1-\cos x\}$ *is not differentiable, when* $2\sin x = 1-\cos x$

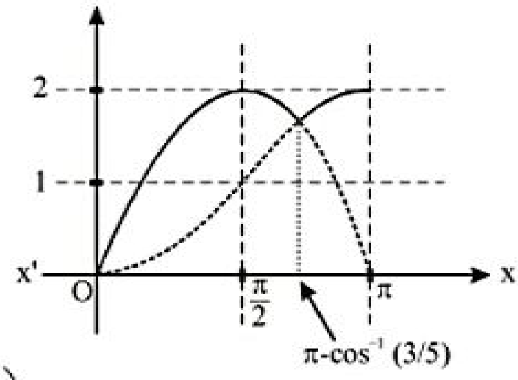
$\Rightarrow\quad 4\sin^2 x=(1-\cos x^2)$

$\Rightarrow\quad 4(1+\cos x)=(1-\cos x)$

$\Rightarrow\quad 4+4\cos x=1-\cos x$

$\Rightarrow\quad \cos x=-\dfrac{3}{5}$

$\Rightarrow\quad x=\cos^{-1}\left(-\dfrac{3}{5}\right)$



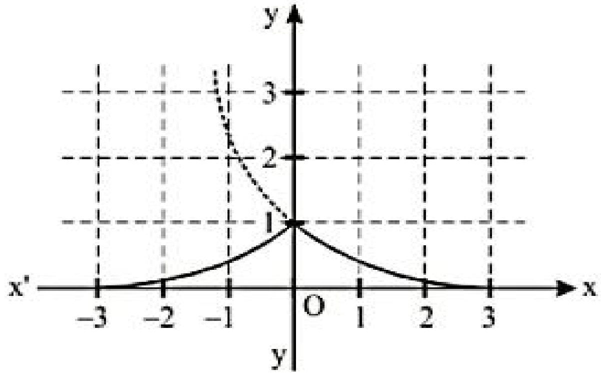
$\Rightarrow\quad f(x)$ *is not differentiable at* $x=\pi-\cos^{-1}\left(\dfrac{3}{5}\right),\ \forall x\in(0,\pi)$.

Illustration :

Discuss the differentiability of $f(x) = e^{-|x|}$.

Sol. *We have* $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0 \end{cases}$

Clearly from the graph,
$f(x)$ *is non-differentiable at* $x = 0$.



DIFFERENTIABILITY BY DIFFERENTIATION :

Illustration :

If $f(x) = \begin{cases} x, & x \leq 1 \\ x^2 + bx + c, & x > 1 \end{cases}$, *then find the values of b and c if* $f(x)$ *differentiable at* $x = 1$.

Sol. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2 + bx + c, & x > 1 \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} 1, & x < 1 \\ 2x + b, & x > 1 \end{cases}$

$f(x)$ *is differentiable at* $x = 1$
Then, it must be continuous at $x = 1$

for which $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x)$

$\Rightarrow \quad 1 + b + c = 1$

$\Rightarrow \quad b + c = 0 \qquad \qquad ...(i)$

Also $f'(0) = f'(0^-)$

$\Rightarrow \quad \lim\limits_{x \to 1^+} f'(x) = \lim\limits_{x \to 1^-} f'(x) \quad \Rightarrow \quad 2 + b = 1 \quad \Rightarrow \quad b = -1$

$\Rightarrow \quad c = 1 \qquad \qquad$ *[from equation (i)]*

Illustration :

Find the values of a and b if $f(x) = \begin{cases} a + \sin^{-1}(x + b), & x \geq 1 \\ x, & x < 1 \end{cases}$ *is differentiable at* $x = 1$.

Sol. $f(x) = \begin{cases} a + \sin^{-1}(x + b), & x \geq 1 \\ x, & x < 1 \end{cases} \Rightarrow f'(x) = \begin{cases} \dfrac{1}{\sqrt{1 - (x + b)^2}}, & x > 1 \\ 1, & x < 1 \end{cases}$

For $f(x)$ *to be continuous at* $x = 1$.

$f(1^+) = f(1^-) \Rightarrow \quad a + \sin^{-1}(1 + b) = 1 \qquad \qquad ...(i)$

Also $\quad f'(1^+) = f'(1) \Rightarrow \quad \dfrac{1}{\sqrt{1 - (1 + b)^2}} = 1 \quad \Rightarrow \quad b = -1.$

Practice Problem

Q.1 Discuss the continuity and differentiability of $f(x) = |x+1| + |x| + |x-1|, \forall x \in R$; also draw the graph of $f(x)$.

Q.2 Find x where $f(x) = \text{maximum}\left\{\sqrt{x(2-x)}, 2-x\right\}$ is non-differentiable.

Q.3 Discuss the differentiability of function $f(x) = |[x]\,x|$ in $-1 < x \le 2$, where $[\cdot]$ represents gratest integer function.

Q.4 Discuss the differentiability of $f(x) = \max\{\tan^{-1} x, \cot^{-1} x\}$.

Q.5 Find the value of a and b if $f(x) = \begin{cases} ax^2 + 1, & x \le 1 \\ x^2 + ax + b, & x > 1 \end{cases}$ is differentiable at $x = 1$.

Q.6 Discuss the differentiability of $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Answer key

Q.1 Continuous $\forall x \in R$, differentiable for $x \in R - \{-1, 0, 1\}$

Q.2 $x = 1$

Q.3 $x = 0, 1, 2$

Q.4 $x = 0$

Q.5 $a = 2, b = 0$

Q.6 $a = 2, b = 0$

Solved Examples

Q.1 The value of $\lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$ is

(A) $\dfrac{1}{2\sqrt{2}}$ (B) $\dfrac{1}{8\sqrt{2}}$ (C) $\dfrac{1}{4\sqrt{2}}$ (D) $-\dfrac{1}{4\sqrt{2}}$

Sol. We have $\lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

$$= \lim_{x\to 0}\frac{2-(1+\cos x)}{\sin^2 x}\times\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$

$$= \lim_{x\to 0}\frac{1-\cos x}{(1+\cos x)(1-\cos x)}\times\lim_{x\to 0}\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$

$$= \lim_{x\to 0}\frac{1}{(1+\cos x)}\times\frac{1}{2\sqrt{2}}=\frac{1}{4\sqrt{2}}$$

Q.2 If $a_1 = 1$ and $a_{n+1} = \dfrac{4+3a_n}{3+2a_n}$, $n \ge 1$ and if $\lim\limits_{n\to\infty} a_n = n$ then the value of a is

(A) $\sqrt{2}$ (B) $-\sqrt{2}$ (C) 2 (D) None of these

Sol. We have $a_{n+1} = \dfrac{4+3a_n}{3+2a_n}$

$$\Rightarrow \quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\frac{4+3a_n}{3+2a_n}$$

$$\Rightarrow \quad a = \frac{4+3a}{3+2a} \quad \Rightarrow \quad 2a^2 = 4 \quad \Rightarrow a = \sqrt{2} \quad (\text{where } \lim_{n\to\infty} a_n = 0)$$

($a \ne -\sqrt{2}$ because each $a_n > 0$. therefore $\lim a_n = a > 0$)

Q.3 Evaluate $\lim\limits_{n\to\infty}\cos\left(\pi\sqrt{n^2+n}\right)$. when n is an integer.

(A) 1 (B) – 1 (C) 0 (D) None

Sol. $L = \lim\limits_{n\to\infty}\cos\left(\pi\sqrt{n^2+n}\right) = \lim\limits_{x\to\infty}\pm\cos\left(n\pi-\pi\sqrt{n^2+n}\right) = \lim\limits_{x\to\infty}\pm\cos\left(\pi\left(n-\sqrt{n^2+n}\right)\right)$

$$= \pm\lim_{x\to\infty}\cos\left(\frac{-n\pi}{n+\sqrt{n^2+n}}\right) = \pm\lim_{x\to\infty}\cos\left(\frac{n\pi}{n+n\sqrt{1+\frac{1}{n}}}\right) = \pm\lim_{x\to\infty}\cos\left(\frac{\pi}{1+\sqrt{1+\frac{1}{n}}}\right) = \pm\cos\frac{\pi}{2}\to 0$$

Q.4 Evaluate $\lim_{n\to\infty}\prod_{r=3}^{n}\frac{r^3-8}{r^3+8}$, where $\prod$ represents product of function.

(A) $\frac{2}{3}$ (B) $\frac{2}{5}$ (C) $\frac{1}{3}$ (D) $\frac{2}{7}$

Sol. Let $P=\lim_{n\to\infty}\prod_{r=3}^{n}\frac{r^3-8}{r^3+8}=\lim_{n\to\infty}\prod_{r=3}^{n}\left(\frac{r-2}{r+2}\right)\left(\frac{r^2+2r+4}{r^2-2r+4}\right)=\lim_{n\to\infty}\prod_{r=3}^{n}\left(\frac{r-2}{r+2}\right)\prod_{r=3}^{n}\left(\frac{r^2+2r+4}{r^2-2r+4}\right)$

$$=\lim_{n\to\infty}\left\{\frac{1}{5}\frac{2}{6}\frac{3}{7}\frac{4}{8}\frac{5}{9}\cdots\cdots\frac{(n-5)}{(n-1)}\frac{(n-4)}{(n)}\frac{(n-3)}{(n+1)}\frac{(n-2)}{(n+2)}\right\}$$

$$\times\left\{\frac{19}{7}\times\frac{28}{12}\times\frac{39}{19}\cdots\cdots\frac{(n^2-2n+4)}{(n^2-6n+12)}\times\frac{(n^2+3)}{(n^2-4n+7)}\times\frac{(n^2+2n+4)}{(n^2-2n+4)}\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{1\cdot2\cdot3\cdot4}{(n-1)n(n+1)(n+2)}\times\frac{(n^2+3)(n^2+2n+4)}{7\times12}\right\}=\frac{2}{7}\lim_{n\to\infty}\left\{\frac{(n^2+3)(n^2+2n+4)}{(n-1)n(n+1)(n-2)}\right\}$$

$$=\frac{2}{7}\lim_{n\to\infty}\left\{\frac{\left(1+\frac{3}{n^2}\right)\left(1+\frac{2}{n}+\frac{4}{n^2}\right)}{\left(1-\frac{1}{n}\right)1\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}\right\}=\frac{2}{7}\frac{(1+0)(1+0+0)}{(1-0)1(1+0)(1+0)}=\frac{2}{7}$$

Hence $P=\frac{2}{7}$

Q.5 If [x] denotes the greatest integer $\le x$, then evaluate $\lim_{x\to\infty}\frac{1}{n^3}\{[1^2x]+[2^2x]+[3^2x]+\ldots\ldots+[n^2x]\}$

(A) $\frac{x}{2}$ (B) $\frac{x}{3}$ (C) $\frac{x}{4}$ (D) x

Sol. $\lim_{x\to\infty}\frac{1}{n^3}\{[1^2x]+[2^2x]+[3^2x]+\ldots\ldots+[n^2x]\}$

$$=\lim_{n\to\infty}\left\{\frac{\sum_{r=1}^{n}[r^2x]}{n^3}\right\}=\lim_{n\to\infty}\left(\frac{\sum_{r=1}^{n}r^2x-\{r^2x\}}{n^3}\right)=\lim_{n\to\infty}\left(\frac{x\,\frac{n(n+1)(2n+1)}{6}}{n^3}-\sum_{r=1}^{n}\frac{\{r^2x\}}{n^3}\right)$$

$$=x\frac{(1)(1)(2)}{6}-0=\frac{x}{3}$$

Q.6 Let $f(x)=\begin{cases}\dfrac{(x^2+3x-1)\tan x}{x^2+2x} & \text{if } x\neq 0\\ k & \text{if } x=0\end{cases}$, then find the value of k.

Sol. $f(x)=\begin{cases}\dfrac{(x^2+3x-1)\tan x}{x^2+2x} & \text{if } x\neq 0\\ k & \text{if } x=0\end{cases}$; $k=\lim\limits_{x\to 0}\dfrac{(x^2+3x-1)}{(x+2)x}\tan x$

so $k=\dfrac{-1}{2}$.

Q.7 $f(x)=\begin{cases}\dfrac{\sin x+\sin 5x}{\cos x+\cos 5x} & \text{if } x\neq -\dfrac{\pi}{4}\\ k & \text{if } x=-\dfrac{\pi}{4}\end{cases}$ Find k if f is continuous at $x=-\dfrac{\pi}{4}$

(A) 1 (B) -1 (C) 0 (D) $-1/\sqrt{2}$

Sol. $f(x)=\begin{cases}\dfrac{\sin x+\sin 5x}{\cos x+\cos 5x} & \text{if } x\neq -\dfrac{\pi}{4}\\ k & \text{if } x=-\dfrac{\pi}{4}\end{cases}$

$$k=\lim_{x\to\frac{-\pi}{4}}\frac{\sin x+\sin 5x}{\cos x+\cos 5x}$$

$$k=\lim_{x\to\frac{-\pi}{4}}\frac{2\sin 3x\cos 2x}{2\cos 3x\cos 2x}=1.$$

Q.8 If $f(x)=\begin{cases}x+1 & x\leq 1\\ 3-ax^2 & x>1\end{cases}$ is continuous at $x=1$, then find the value of a.

Sol. $f(x)=\begin{cases}x+1, & x\leq 1\\ 3-ax^2, & x>1\end{cases}$; $\lim\limits_{x\to 1^-}f(x)=\lim\limits_{x\to 1^+}f(x)=\lim\limits_{x\to 1^+}f(x)$

$2=3-a\Rightarrow a=1.$

Q.9 What kind of discontinuity the function $\dfrac{\cos x}{x}$ has at $x=0$.

Sol. $f(x)=\dfrac{\cos x}{x}$ $\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^+}\dfrac{\cos x}{x}=\infty$

$\lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^-}\dfrac{\cos x}{x}=-\infty$

Non removable infinite type

Q.10 If $f(x)=\begin{cases} \dfrac{8^x-4^x-2^x+1^2}{x^2}, & x>0 \\ e^x\sin x+4x+k\,ln4, & x\le 0\end{cases}$ is continuous at $x=0$, then find the value of k.

Sol. $f(x)=\begin{cases} \dfrac{8^x-4^x-2^x+1^2}{x^2}, & x>0 \\ e^x\sin x+4x+k\,ln4, & x\le 0\end{cases}$

$$\lim_{x\to 0^+}\frac{8^x-4^x-2^x+1}{x^2}=\lim_{x\to 0^-}\frac{e^{x\,ln_e 8}-e^{x\,ln_e 4}-e^{x\,ln_e 2}+1}{x^2}$$

$$=\lim_{x\to 0^+}\frac{\left(1+x\,ln_e 8+\dfrac{(x\,ln_e 8)^2}{2!}+....\right)-\left(1+x\,ln_e 4+\dfrac{(x\,ln_e 4)^2}{2!}+....\right)-\left(1+x\,ln_e 2+\dfrac{(x\,ln_e 2)^2}{2!}+.....\right)+1}{x^2}$$

$$=\lim_{x\to 0^+}\frac{x\left(ln_e 8-ln_e 4-ln_e 2\right)+x^2\left(\dfrac{(ln_e 8)^2}{2}-\dfrac{(ln_e 4)^2}{2}-\dfrac{(ln_e 2)^2}{2}\right)}{x^2}$$

$$=\lim_{x\to 0^+}\frac{1}{2}\left((ln_e 8)^2-(ln_e 4)^{2x^2}-(ln_e 2)^2\right)=\frac{1}{2}\left((ln_e 32)(ln_e 2)-(ln_e 2)^2\right)^2$$

$$=\frac{1}{2}(ln_e 2)(ln_e 16)-(ln_e 2)(ln_e 4)$$

$$\lim_{x\to 0^-} f(x)=0+0+k\,ln\,4=(ln_e 2)\,(ln_e 4)\Rightarrow k=ln_e 2.$$