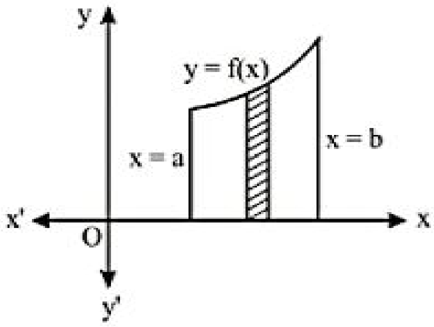
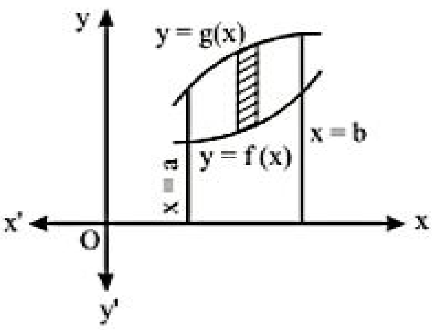
AREA UNDER THE CURVE

DIFFERENT CASES OF BOUNDED AREA :



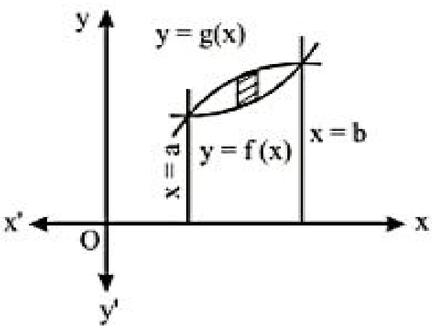
1. The area bounded by the continuous curve $y = f(x)$, the axis of x and the ordinates $x = a$ and $x = b$ (where $b > a$) is given by

$$A = \int_a^b f(x)dx = \int_a^b ydx$$



2. The area bounded by the straight line $x = a, x = b\ (a < b)$ and the curves $y = f(x)$ and $y = g(x)$, provided $f(x) < g(x)$ (where $a \le x \le b$), is given by

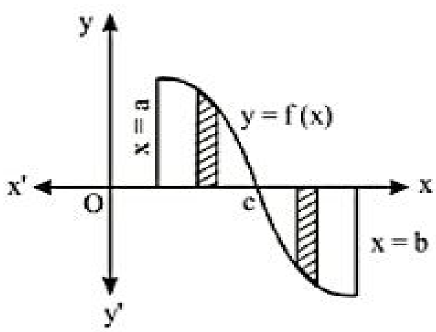
$$A = \int_a^b [g(x) - f(x)]dx$$



3. When two curves $y = f(x)$ and $y = g(x)$ intersect, the bounded area is

$$A = \int_a^b [g(x) - f(x)]dx \;;\; \text{where } a < b.$$

where a and b are the roots of the equation $f(x) = g(x)$.



4. If some part of a curve lies below the x-axis, then its area becomes negative but area cannot be negative. Therefore, we take its modulus.

If the curves crosses the x-axis at c, then the area bounded by the curve $y = f(x)$ and ordinates $x = a$ and $x = b$

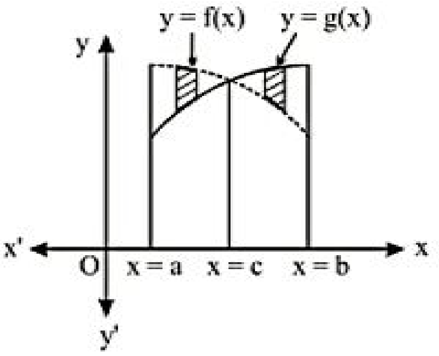
(where $b > a$) is given by $A = \left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|$

$$A = \int_a^c f(x)dx - \int_c^b f(x)dx$$

5. The area bounded by $y = f(x)$ and $y = g(x)$ (where $a \le x \le b$), when they intersect at $x = c \in (a, b)$ is given by

$$A = \int_a^b |f(x) - g(x)|\,dx$$

or $$\int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx$$



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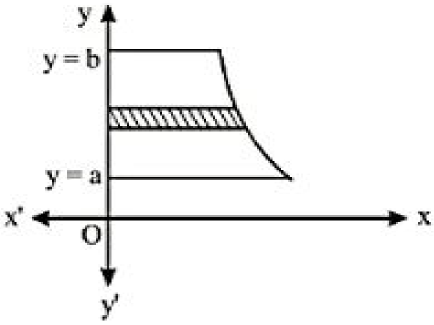

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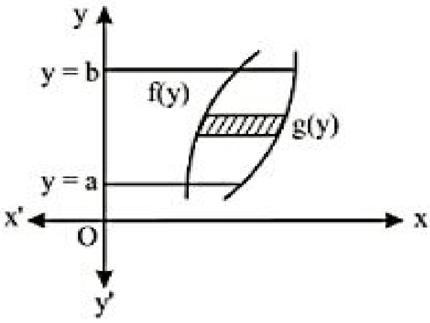

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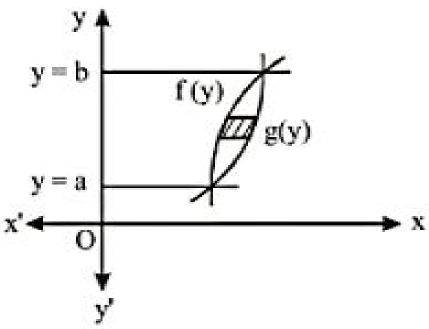

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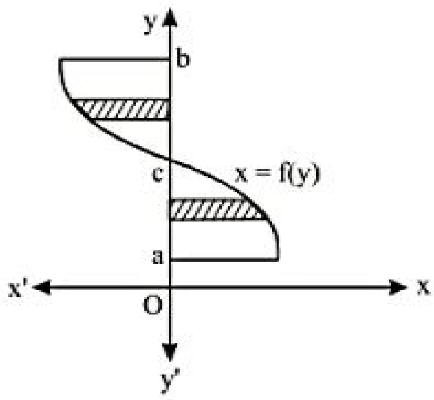

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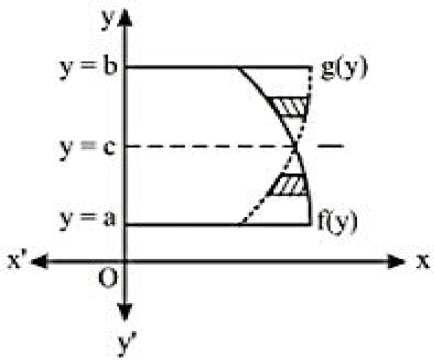


Illustration :

Find the area bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$.

Sol. *The two curves meet at ponts where $3 - x = x^2 + 1$ i.e., $x^2 + x - 2 = 0$*

$\Rightarrow (x + 2)(x - 1) = 0 \Rightarrow x = -2, 1$

$$\therefore \text{ required area } = \int_{-2}^{1}\left[(3-x)-(x^2+1)\right]dx$$

$$= \int_{-2}^{1}\left(2-x-x^2\right)dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right)$$

$$= \frac{9}{2} \text{ sq. units.}$$

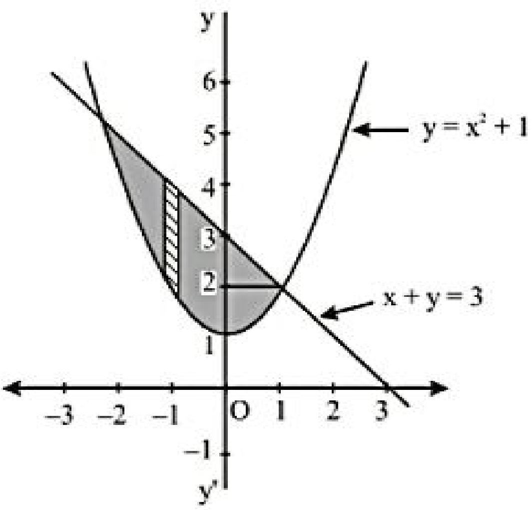
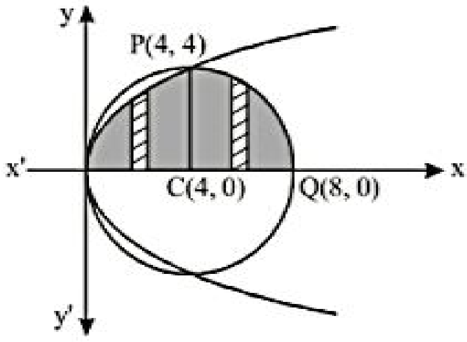


Illustration :

Find the area, lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

Sol. Solving the curves, we get $x^2 + 4x = 8x \Rightarrow x = 0, 4$.

$$\text{Required area} = \int_0^4 y_{parabola}\, dx + \int_4^8 y_{circle}\, dx$$



Circle is $(x - 4)^2 + y^2 = 4^2$.

$$\text{Area of circle in } 1^{st} \text{ quadrant} = \frac{1}{4}\pi 4^2 = 4\pi$$

$$A = 2\int_0^4 \sqrt{x}\, dx + 4\pi$$

$$= \frac{4}{3}\left[x^{3/2}\right]_0^4 + 4\pi = \frac{2}{3} \times 4\sqrt{4} + 4\pi \text{ sq. units}$$

$$= \frac{32}{3} + 4\pi \text{ sq. units}$$

Illustration :

Find the area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ lying between the ordinates $x = 0$ and $x = 3$.

Sol. $y = (x - 1)(x - 2)(x - 3)$

The curves will intersect the x-axis, when $y = 0$.

$\Rightarrow \quad (x - 1)(x - 2)(x - 3) = 0$

$\Rightarrow \quad x = 1, 2, 3$

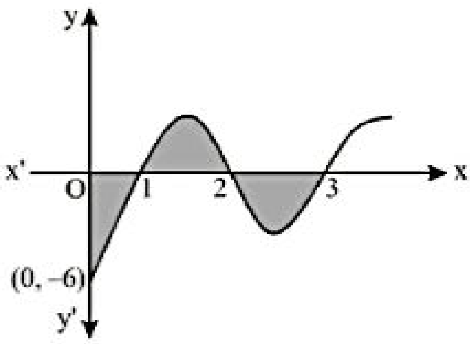
And the curve intersects the y-axis,

when $x = 0 \Rightarrow y = -6$

Thus, the graph of the given function for $0 \le x \le 3$ is as shown in figure.

Hence, the required area A = shaded area

$$= \left|\int_0^1 y\, dx\right| + \left|\int_1^2 y\, dx\right| + \left|\int_2^3 y\, dx\right|$$



$$\text{Since } \int y\, dx = \int (x-1)(x-2)(x-3)dx$$

$$= \int (x^3 - 6x^2 + 11x - 6)dx$$

$$= \frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x$$

$\therefore$ *from equation (1)*

$$A = \left|\left[\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right]_0^1\right| + \left|\left[\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right]_1^2\right| + \left|\left[\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right]_2^3\right|$$

$= | -9/4 | + (1/4) + | -1/4 | = 11/4$ *sq. units*

Illustration :

Consider the region formed by the linex $x = 0, y = 0, x = 2, y = 2$. Area enclosed by the curves $y = e^x$ and $y = \ln x$, within this region, is being removed. Then, find the area of the remaining region.

Sol. *Required area = shaded region*

$$= 2\int_0^{\ln 2} (2 - e^x)\,dx$$

$$= 2[2x - e^x]_0^{\ln 2}$$

$= 2(2 \ln 2 - 1)$ *sq. units*

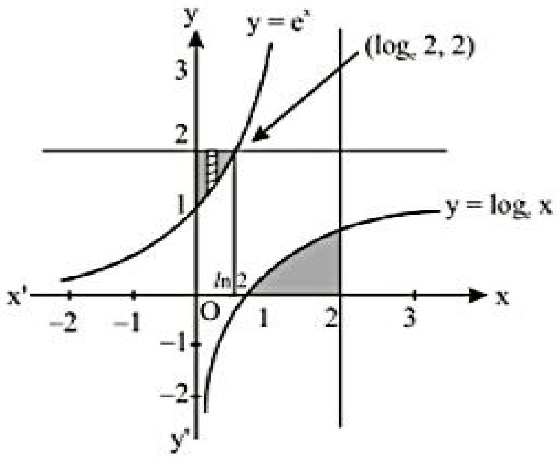


Illustration :

Find the area bounded by the curves $y = \sin x$ and $y = \cos x$ between two consecutive points of their intersection.

Sol. *Two consecutive points of intersection of $y = \sin x$ and $y = \cos x$ can be taken as $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$*

$$\therefore \text{ Required area} = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$$

$$= [-\cos x - \sin x]_{\pi/4}^{5\pi/4}$$

$$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$= 2\sqrt{2}$ *sq. units*

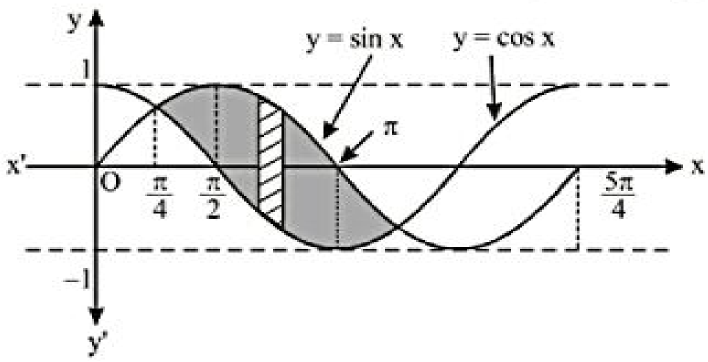


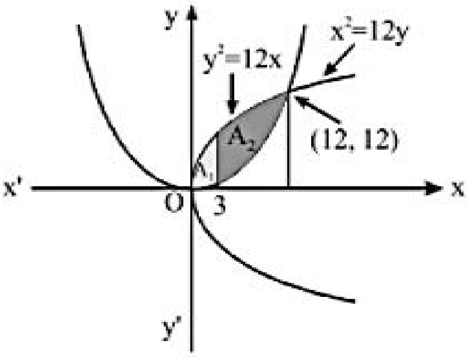
Illustration :

Find the ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$.

Sol. A_1 = area bounded by $y^2 = 12x$, $x^2 = 12y$ and line $x = 3$

$$= \int_0^3 \sqrt{12x}\,dx - \int_0^3 \frac{x^2}{12}\,dx$$

$$= \sqrt{12}\left|\frac{2x^{3/2}}{3}\right| - \left|\frac{x^3}{36}\right|_0^3 = \frac{45}{4} \text{ sq. units}$$



A_2 = area bounded by $y^2 = 12x$ and $x^2 = 12y$

$$= \frac{16(3)(3)}{3} = 48 \text{ sq. units}$$

$$\therefore \text{ required ratio} = \frac{\frac{45}{4}}{48 - \frac{45}{4}} = \frac{45}{147} = \frac{15}{49}$$

Illustration :

Find the area bounded by

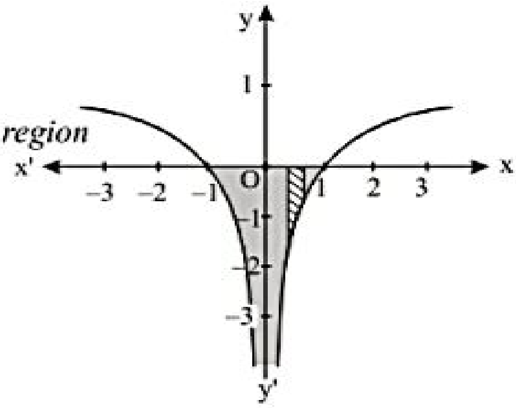
(i) $y = \log_e |x|$ and $y = 0$, *(ii) $y = |\log_e |x||$ and $y = 0$*

Sol.

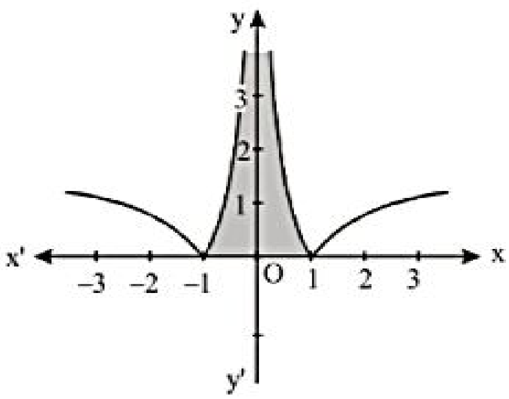
(i) $y = \log_e |x|$ and $y = 0$

From the figure, required area = area of the shaded region

$$= 2\left|\int_0^1 (\log_e x)dx\right| = 2\left|(x\log_e x - x)_0^1\right| = 2 \text{ sq. units}$$



(ii) $y = |\log_e |x||$ and $y = 0$



From the figure, required area = area of the shaded region = 1 + 1 = 2 sq. units.

Illustration :

Find the area included by the curve $y = \ln x$, x-axis and the two ordinate at $x = \frac{1}{e}$ and $x = e$.

Sol. $A = \left|\int_{1/e}^{1} \ln x\,dx\right| + \int_{1}^{e} \ln x\,dx = \left|[x(\ln x - 1)]_{1/e}^{1}\right| + [x(\ln x - 1)]_{1}^{e} = 2 - \frac{2}{e}$

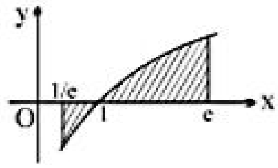
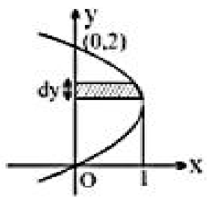


Illustration :

Find the area included by the curve $x = 2y - y^2$ and the y-axis

Sol. *Let $x = 2y - y^2$ and the y-axis*

$\frac{dx}{dy} = 2 - 2y = 0 \Rightarrow y = 1 \Rightarrow$ *curve bends at $y = 1$;*

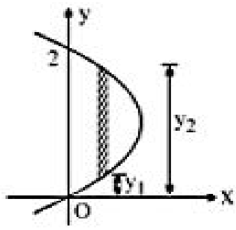


$$A = \int_0^2 x\,dy = \int_0^2 (2y - y^2)\,dy = y^2 - \frac{y^3}{3}\Bigg]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}\ \textbf{Ans.}$$

Alternative method :

This can also be done by taking vertical strip.

$y^2 - 2y + x = 0$



$y = \frac{2 \pm \sqrt{4 - 4x}}{2}$

$y = 1 + \sqrt{1-x}$ $\qquad (y_2)$

$y = 1 - \sqrt{1-x}$ (y_1)

$$A = \int_0^1 y\,dx = \int_0^1 2(\sqrt{1-x}\,)dx$$

Illustration :

For $b > a > 1$, the area enclosed by the curve $y = \ln x$, y axis and the straight lines $y = \ln a$ and $y = \ln b$ is

(A) $b - a$	*(B) $b(\ln b - 1) - a(\ln a - 1)$*
(C) $(\ln a)(b - a)$	*(D) $(\ln b)(\ln a)$*

Sol. *Required area* $= \int_{\ln a}^{\ln b} e^y dy = \left[e^y\right]_{\ln a}^{\ln b} = (b-a)$

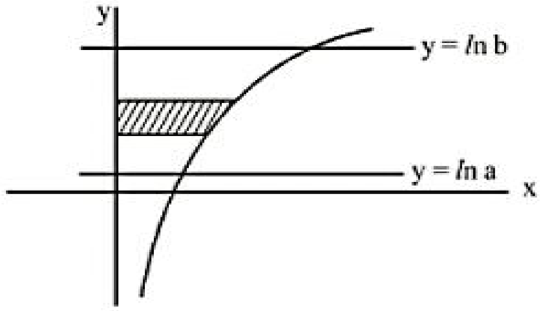


Illustration :

Find the area enclosed between y = sin x ; y = cos x and y-axis in the 1st quadrant

Sol. $A = \int_0^{\pi/4} (\cos x - \sin x)\, dx = \left[\sin x + \cos x\right]_0^{\pi/4} = \sqrt{2} - 1$

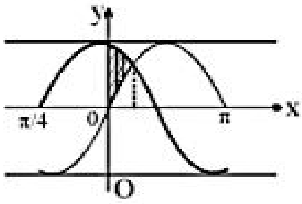
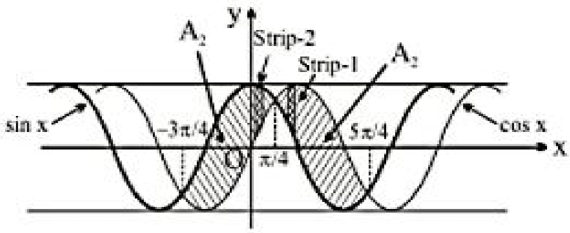


Illustration :

Curves y = sin x ; y = cos x intersect each other at infinite number of points enclosing regions of equal areas. Compute the area of one such equal region.

Sol. *(Strip-1)* $A_1 = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx = 2\sqrt{2}$



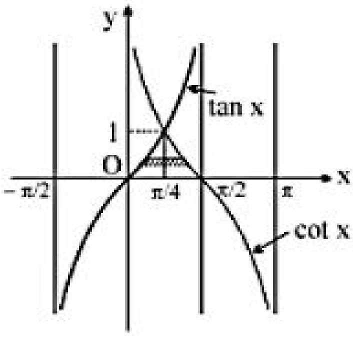
(Strip-2) $A_2 = \int_{-3\pi/4}^{\pi/4} (\cos x - \sin x)\, dx = 2\sqrt{2}$

So $A_1 = A_2 = 2\sqrt{2}$ *sq. unit*

Illustration :

Find the area enclosed by y = tan x ; y = cot x and x-axis in 1st quadrant.

Sol. $A = \int_0^{\pi/4} \tan x\, dx + \int_{\pi/4}^{\pi/2} \cot x\, dx$



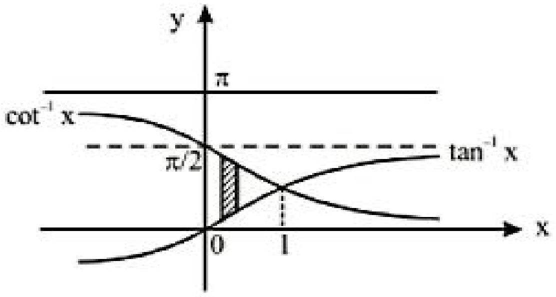
$A = 2\int_0^{\frac{\pi}{4}} \tan x\, dx = 2\left[\ln|\sec x|\right]_0^{\frac{\pi}{4}}$

$= 2\ln\sqrt{2} = \ln 2$

Illustration :

Compute the area enclosed between $y = \tan^{-1}x$; $y = \cot^{-1}x$ and y-axis.

Sol. $$A = \int_0^1 (\cot^{-1}x - \tan^{-1}x)dx$$



$$A = \int_0^{\frac{\pi}{4}} (\tan y)\,dy + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot y)\,dy = \ln 2$$

Illustration :

Area enclosed by $y = 9 - x^2$ and coordinates axes.

Sol. $$A = \int_0^3 (9 - x^2)dx = 18$$

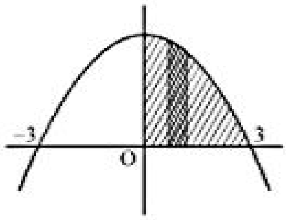


Illustration :

Compute the larger area bounded by $y = 4 + 3x - x^2$ and the coordinates axes.

Sol. $$A = \int_0^4 y\,dx = \int_0^4 (4 + 3x - x^2)dx$$

$$= \left[4x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{56}{3}$$

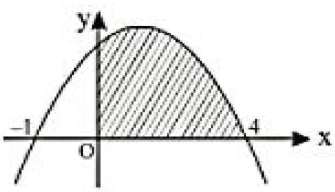


Illustration :

Find the area bounded by $y = \sin^{-1}x$, $y = \cos^{-1}x$ and x-axis.

Sol. *$y = \sin^{-1}x$, $y = \cos^{-1}x$ and the x-axis if vertical stripe is used, we get*

$$A = \int_0^{1/\sqrt{2}} \sin^{-1}x\,dx + \int_{1/\sqrt{2}}^{1} \cos^{-1}x\,dx$$

If horizontal strip is used, then

$$A = \int_0^{\pi/4} (\cos y - \sin y)dy$$

$$= [\sin y + \cos y]_0^{\pi/4}$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right] = \sqrt{2} - 1$$

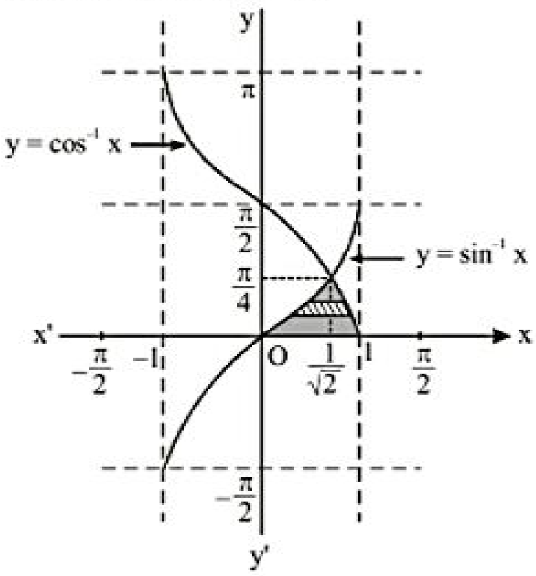


Illustration :

Find the area of the region in the 1^{st} quadrant bounded on the left by the y-axis, below by the line $y = \frac{x}{4}$, above left by the curve $y = 1 + \sqrt{x}$ and above right by the curve $y = \frac{2}{\sqrt{x}}$

Sol. *Required area* $= \int_0^1 \left(1 + \sqrt{x} - \frac{x}{4}\right)dx + \int_1^4 \left(\frac{2}{\sqrt{x}} - \frac{x}{4}\right)dx$

$$= \left(x + \frac{2}{3}x^{3/2} - \frac{x^2}{8}\right)_0^1 + \left(4\sqrt{x} - \frac{x^2}{8}\right)_1^4$$

$$= \left(1 + \frac{2}{3} - \frac{1}{8}\right) + \left(4 - \frac{15}{8}\right) = \frac{5}{3} + 2 = \frac{11}{3} \text{ sq. units.}$$

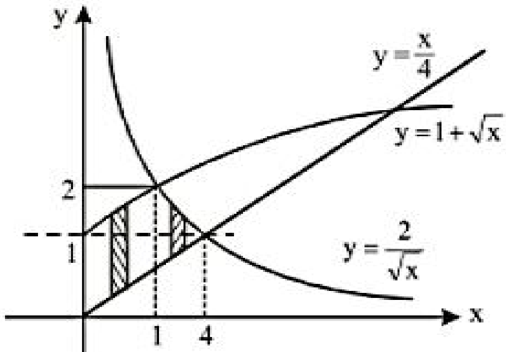


Illustration :

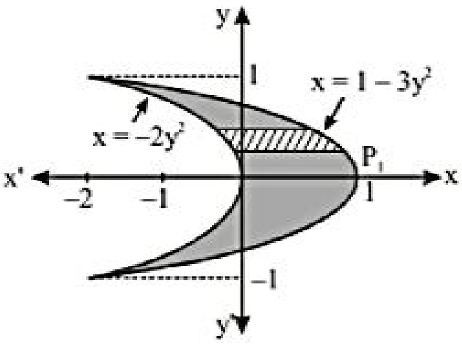
Find the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$.

Sol. *Solving the equation $x = -2y^2$, $x = 1 - 3y^2$, we find that ordinates of the point of intersection of the two curves as $y_1 = -1$, $y_2 = 1$. The points are (–2, –1) and (–2, 1).*
The required area (using horizontal strip)

$$A = 2\int_0^1 (x_1 - x_2)dy$$

$$= 2\int_0^1 \left[\left(1 - 3y^2\right) - \left(-2y^2\right)\right] dy$$

$$= 2\int_0^1 \left(1 - y^2\right) dy = 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3}$$



Practice Problem

Q.1 Find the area lying in the first quadrant and bounded by the curve $y = x^3$ and the line $y = 4x$.

Q.2 Find the area enclosed by the curves $x^2 = y$, $y = x + 2$ and x-axis.

Q.3 A curve is given by $y = \begin{cases} \sqrt{(4-x^2)}, & 0 \le x < 1 \\ \sqrt{(3x)}, & 1 \le x \le 3 \end{cases}$. Find the area lying between the curve and x-axis.

Q.4 Find the area of the region bounded by the limits $x = 0$, $x = \frac{\pi}{2}$ and $f(x) = \sin x$, $g(x) = \cos x$.

Q.5 Find the area bounded by the curve $y = \sin^{-1} x$ and the line $x = 0, |y| = \frac{\pi}{2}$.

Q.6 Find the area bounded by $y = \tan^{-1} x$, $y = \cot^{-1} x$ and y-axis is first quadrant.

Q.7 Find the area bounded by $y = \log_e x$, $y = -\log_e x$, $y = \log_e(-x)$ and $y = -\log_e(-x)$.

Q.8 Find the equation of the tangent to the parabola $x^2 = 4y$ with gradient unity. Also find the area enclosed by the curve, the tangent line and

(i) the y-axis (ii) the x-axis

Q.9 Pair of tangents are drawn from the point (3, 0) on the parabola $y = x^2$. Find the area enclosed by these tangents and the parabola.

Q.10 Compute the area included between the straight lines, $x - 3y + 5 = 0$; $x + 2y + 5 = 0$ and the circle $x^2 + y^2 = 25$.

Answer key

Q.1	4	Q.2	$\frac{5}{6}$	Q.3	$\frac{1}{6}\left(2\pi - \sqrt{3} + 36\right)$
Q.4	$2\left(\sqrt{2} - 1\right)$	Q.5	2	Q.6	$\log\sqrt{2}$
Q.7	4	Q.8	$x - y = 1$; P (2, 1) ; (i) 2/3; (ii) 1/6		
Q.9	18 sq. units	Q.10	$\frac{5}{4}$ $(5\pi + 14)$ sq. Units		

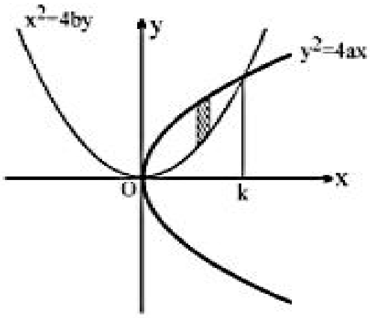
STANDARD AREAS TO BE REMEMBERED :

(1) Area bounded by the curve $y^2 = 4ax$; $x^2 = 4by$ is equal to $\frac{16\,ab}{3}$:

At point of intersection

$$\left(\frac{x^2}{4b}\right)^2 = 4ax \quad \Rightarrow \quad x^4 = 64\,ab^2\,x$$

$$\Rightarrow \quad x = 0,\ (64\,ab^2)^{1/3}$$



Let $\quad k = 4\,(ab^2)^{1/3}$

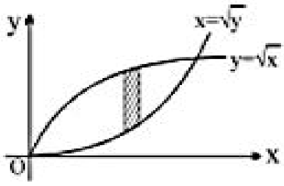
$$A = \int_0^k \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b}\right)dx$$

$$= \left[2\sqrt{a}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12b}\right]_0^k = \frac{4\sqrt{a}}{3}\,k^{\frac{3}{2}} - \frac{k^3}{12b} = \frac{4}{3}\sqrt{a}\,8\left(ab^2\right)^{\frac{1}{2}} - \frac{64(ab^2)}{12b}$$

$$= \frac{32}{3}ab - \frac{16}{3}ab = \frac{16ab}{3}$$

Illustration :

Find the area bounded by the curve $y = \sqrt{x}$; $x = \sqrt{y}$



Sol. $a = \frac{1}{4}$; $b = \frac{1}{4}$

Required area $= \frac{16ab}{3} = \frac{16\cdot\frac{1}{4}\cdot\frac{1}{4}}{3}$

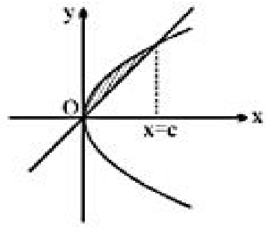
Area $= \frac{1}{3}$

(2) Area bounded by the parabola $y^2 = 4ax$ and $y = mx$ is equal to $\dfrac{8a^2}{3m^3}$:

$y^2 = 4ax$ and $y = mx$

At point of intersection

$$m^2x^2 = 4ax \quad \Rightarrow \quad x = 0,\ \frac{4a}{m^2}$$



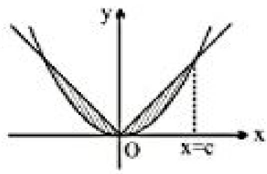
$$\text{Area} = \int_0^c \left(2\sqrt{a}\sqrt{x} - mx\right)dx \quad \text{where} \quad c = \frac{4a}{m^2}$$

$$= \left(2\sqrt{a}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{mx^2}{2}\right)_0^c = \frac{4\sqrt{a}}{3}c^{\frac{3}{2}} - \frac{mc^2}{2}$$

$$= \frac{4\sqrt{a}}{3}\cdot\frac{8a\sqrt{a}}{m^3} - \frac{m}{2}\cdot\frac{16a^2}{m^4} = \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{3m^3}$$

Illustration :

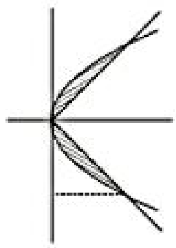
Find the area bounded by the curves $x^2 = y$; $y = |x|$.



Sol. $$Area = 2\left(\frac{8a^2}{3m^3}\right) = 16\,\frac{\left(\frac{1}{4}\right)^2}{3(1)^3} = \frac{1}{3}$$

Illustration :

Find the curve bounded $y^2 = x$; $x = |y|$.

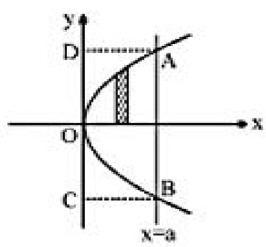


Sol. $$Area = 2\left(\frac{8a^2}{3m^3}\right) = \frac{16}{3}\cdot\frac{\left(\frac{1}{4}\right)^2}{(1)^3} = \frac{1}{3}$$

(3) Area enclosed by $y^2 = 4ax$ and its double ordinate at $x = a$:

(chord perpendicular to the axis of symmetry)

Required area = OABO



$$= 2\cdot\int_0^a \left(2\sqrt{ax}\right)dx = 4\sqrt{a}\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^a$$

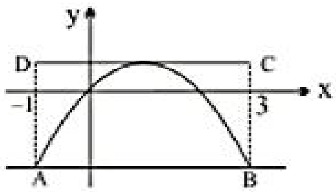
$$= \frac{8}{3}\sqrt{a}\cdot(a\sqrt{a}) = \frac{8a^2}{3}$$

Area of rectangle ABCD = $4a^2$

$\Rightarrow$ $\boxed{\text{Area of AOB} = \frac{2}{3}(\text{area} \square \text{ABCD})}$

Illustration :

Find the area bounded by the curve. $y = 2x - x^2$, $y + 3 = 0$

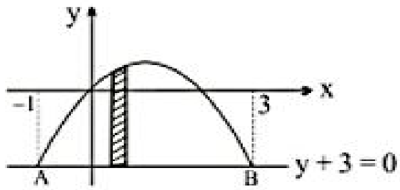


Sol. *For point of intersection of $y = 2x - x^2$ and $y + 3 = 0$*

Area (ABCD) = 4 × 4 = 16

Required area $= \frac{2}{3} \times 16 = \frac{32}{3}$

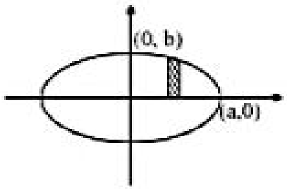
Alternative method :



By integration $A = \int_{-1}^{3} [(2x - x^2) - (-3)]\, dx = \frac{32}{3}$

(4) Whole area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to π ab :

$$A = 4\int_0^a \left(b\sqrt{1 - \frac{x^2}{a^2}} \right) dx$$



Put $x = a\sin\theta$

$$A = 4\int_0^{\pi/2} ab\cos^2\theta\, d\theta = 4ab\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta = 4ab\left(\frac{\pi}{4} \right) = \pi\, ab$$

Illustration :

Find the area of ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1.$

Sol. *Area of ellipse* $= \pi\, ab = \pi\,(4)\,(3) = 12\pi$

SHIFTING OF ORIGIN :

Since area remains invariant even if the coordinates axes are shifted, hence shifting of origin in many cases proves to be very convenient in computing the areas.

Illustration :

Area enclosed between the parabolas $y^2 - 2y + 4x + 5 = 0$ and $x^2 + 2x - y + 2 = 0$.

Sol. $y^2 - 2y + 1 \Rightarrow (y-1)^2 = -4(x+1)$ ... (1)

$x^2 + 2x + 1 = y - 1 \Rightarrow (x+1)^2 = (y-1)$ ... (2)

Let $y - 1 = Y$ and $x + 1 = X$

So equation $Y^2 = -4X$ and $X^2 = Y$

$$a = 1,\ b = \frac{1}{4}$$

$$\text{so required area} = \frac{16ab}{3} = \frac{16}{3} \cdot 1 \cdot \frac{1}{4} = \frac{4}{3}$$

Illustration :

Area enclosed between the ellipse $9x^2 + 4y^2 - 36x + 8y + 4 = 0$ and the line$3x + 2y - 10 = 0$ in the first quadrant.

Sol. $9x^2 + 4y^2 - 36x + 8y + 4 = 0$

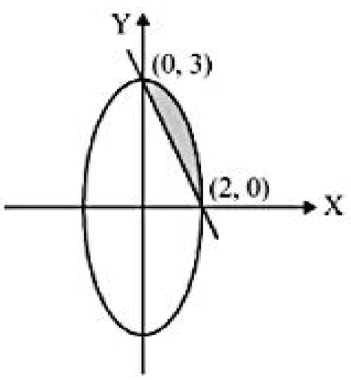
$\Rightarrow 9(x-2)^2 + 4(y+1)^2 = 36$

$$\Rightarrow \frac{(x-2)^2}{2^2} + \frac{(y+1)^2}{3^2} = 1 \qquad \text{... (1)}$$

Let $X = x - 2$ and $Y = y + 1$

So equation of ellipse will be

$$\frac{X^2}{2^2} + \frac{Y^2}{3^2} = 1$$



and equation of line $3x + 2y - 10 = 0$ ... (2)

$3(X+2) + 2(Y-1) - 10 = 0$

$3X + 2Y - 6 = 0$

So required area (shaded region)

$$= \frac{\pi ab}{4} - \frac{1}{2}(ab)$$

$$= \frac{\pi}{4}(2)(3) - \frac{1}{2}(2)(3) = \frac{3\pi}{2} - 3 = \frac{3(\pi - 2)}{2}$$

CURVE TRACING :

The approximate shape of a curve, the following procedure in order

(I) SYMMETRY :

(a) Symmetry about x-axis

If the equation of the curve remain unchanged by replacing y by $-y$ then the curve is symmetrical about the x-axis.

e.g., $y^2 = 4ax$.

(b) Symmetry about y-axis

If the equation of the curve remain unchanged by replacing x by $-x$ then the curve is symmetrical about the y-axis.

e.g., $x^2 = 4ay$

(c) Symmetry about both axes

If the equation of the curve remain unchanged by replacing x by $-x$ and y by $-y$ then the curve is symmetrical about the axis of 'x' as well as 'y'.

e.g., $x^2 + y^2 = a^2$

(d) Symmetry about the line y = x

If the equation of curve remains unchanged on interchaning 'x' and 'y', then the curve is symmetrical about the line $y = x$

e.g., $x^3 + y^3 = 3xy$.

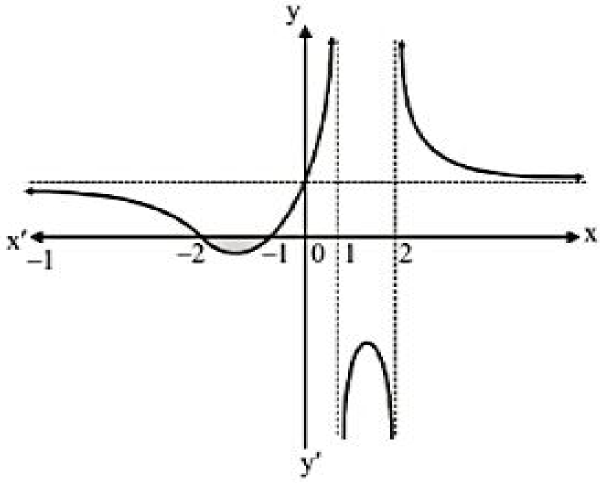
(II) Find the points where the curve crosses the x-axis and the y-axis.

(III) Find $\frac{dy}{dx}$ and examine, if possible, the intervals where $f(x)$ is increasing or decreasing and also its stationary points.

(IV) Examine y when $x \to \infty$ or $x \to -\infty$.

Illustration :

Draw a rough sketch of the curve, $y = \dfrac{x^2+3x+2}{x^2-3x+2}$ *and find the area of the bounded region between the curve and x-axis.*



Sol. $f(x) = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$

Graph will cut x-axis $x = -1$ *and* $x = -2$.
It is discontinuous at $x = 1$ *and* $x = 2$.

$\lim\limits_{x\to\pm\infty} f(x) \to 1,\ \lim\limits_{x\to 1^-} f(x) \to +\infty.$

$\lim\limits_{x\to 1^+} f(x) \to -\infty.$

$\lim\limits_{x\to 2^-} f(x) \to -\infty.$

$\lim\limits_{x\to 2^+} f(x) \to +\infty,\ f(0) = 1.$

Now we have to find area of the shaded region. The required area

$$= \left|\int_{-2}^{-1} f(x)\,dx\right| = \left|\int_{-2}^{-1}\left(\frac{x^2+3x+2}{x^2-3x+2}\right)dx\right| = \left|\int_{-2}^{-1}\left(1+\frac{6x}{(x-1)(x-2)}\right)dx\right|$$

$$= \left|\left[x\Big|_{-2}^{-1} + 6\int_{-2}^{-1}\left(\frac{2}{x-2}-\frac{1}{x-1}\right)dx\right]\right|$$

$$= |\,1 + 6\,[2\ln|x-2| - \ln|x-1|\,]_{-2}^{-1}|$$

$$= |\,1 + 6\,[2(\ln 3 - \ln 4) - (\ln 2 - \ln 3)]| = |\,1 + 6\,[3\ln 3 - 5\ln 2]|$$

$$= 6\ln\left(\frac{32}{27}\right) - 1 \text{ sq. units}$$

Illustration :

Find the area bounded by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ *and the straight line* $y = 3x - 15$.

Sol. *The given curves are*

$y = -x^2 + 6x - 5$ *or* $(x-3)^2 = -(y-4)$ ...(i)

which is a parabola with vertes at A_1 *(3, 4) and axis parallel to negative y-axis. It intersects the x-axis at the point P(1, 0) and Q(5, 0)*

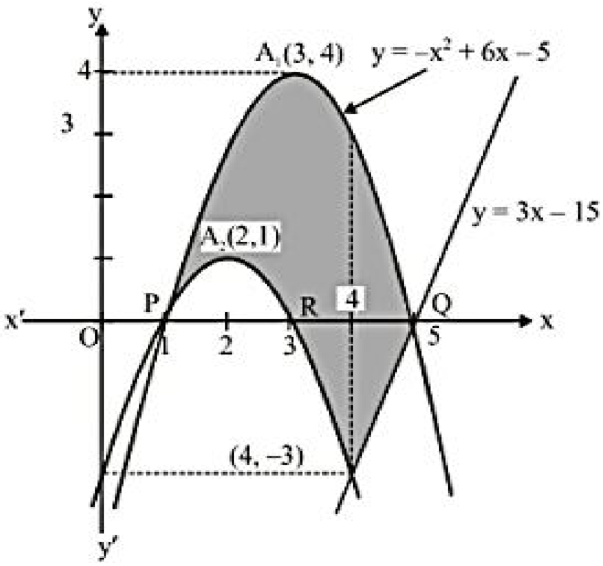
$y = -x^2 + 4x - 3$ or $(x-2)^2 = -(y-1)$ ...(ii)

which is parabola with vertex at $A_2(2, 1)$ and axis parallel to negative y-axis. It intersects the x-axis at the points P(1, 0) and R(3, 0).

And $y = 3x - 15$. ...(iii)

Solving, the points of intersection of (i), (ii) is (1, 0); (i), (iii) are (–2, –21) and (5, 0) and (ii), (iii) are (–3, –24) and (4, –3).

Thus, the required area is the shaded area in the diagram.



$$\text{Required area} = \left|\int_1^4 (y_1 - y_2)dx\right| + \left|\int_4^5 (y_1 - y_3)dx\right|$$

$$= \left|\int_1^4 [(-x^2+6x-5)-(-x^2+4x-3)]dx\right| + \left|\int_4^5 [(-x^2+6x-5)-(3x-15)]dx\right|$$

$$= \left|\int_1^4 (2x-2)dx\right| + \left|\int_4^5 (-x^2+3x+10)dx\right|$$

= 9 + 19/6 = 73/6 sq. units.

Illustration :

The area of the region enclosed by the curves $y = x \log x$ and $y = 2x - 2x^2$ is

(A) $\frac{7}{12}$ sq. units (B) $\frac{1}{2}$ sq. units (C) $\frac{5}{12}$ sq. units (D) None of these

Sol. *Curve tracing : $y = x \log_e x$*

Clearly, $x > 0$,

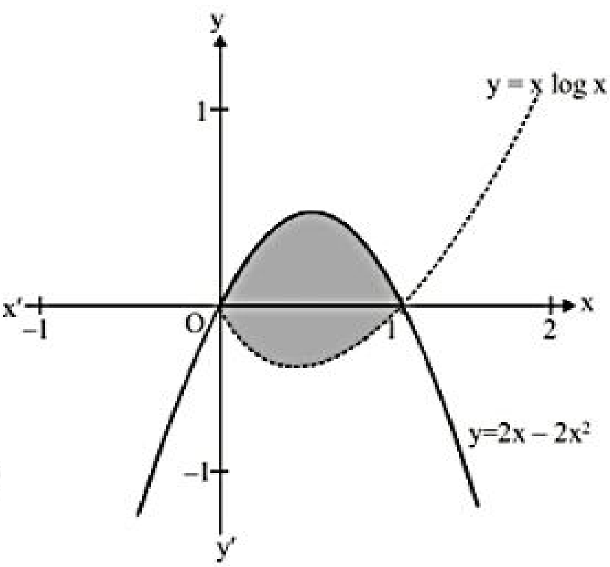
For $0 < x < 1$, $x \log_e x < 0$,

and for $x > 1$, $x \log_e x > 0$

Also $x \log_e x = 0 \Rightarrow x = 1$

Further, $\frac{dy}{dx} = 0 \Rightarrow 1 + \log_e x = 0$

$\Rightarrow x = 1/e$, which is a point of minima.



$$\text{Required area} = \int_0^1 (2x - 2x^2)dx - \int_0^1 x \log x \, dx$$

$$= \left[x^2 - \frac{2x^3}{3}\right]_0^1 - \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1$$

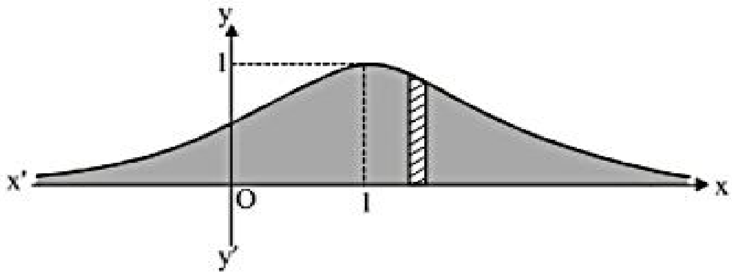
$$= \left(1 - \frac{2}{3}\right) - \left[0 - \frac{1}{4} - \frac{1}{2}\lim_{x \to 0} x^2 \log x\right] = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Illustration :

Area bounded by $y = \dfrac{1}{x^2 - 2x + 2}$ *and x-axis is*

(A) 2π sq. units (B) $\dfrac{\pi}{2}$ sq. units (C) 2 sq. units (D) π sq. units

Sol. $y = \dfrac{1}{(x-1)^2 + 1}$



$$\text{Area} = 2\int_1^{\infty} \frac{1}{(x-1)^2 + 1} dx$$

$$= 2\left[\tan^{-1}(x-1)\right]_1^{\infty} = \pi \text{ sq. units}$$

Illustration :

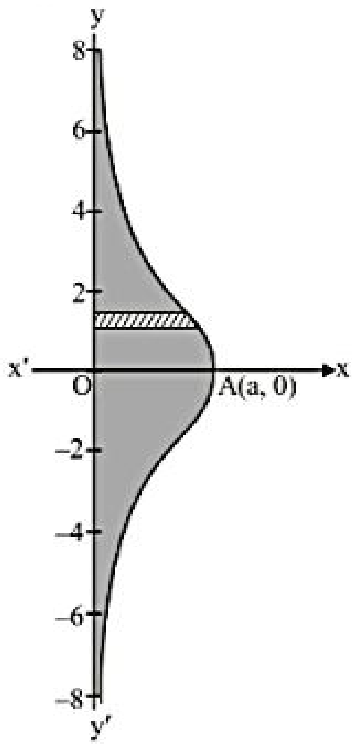
Area bounded by the curve $xy^2 = a^2(a - x)$ *and y-axis is*

(A) $\dfrac{\pi r^2}{2}$ sq. units (B) πa^2 sq. units (C) $3\pi a^2$ sq. units (D) None of these

Sol. $xy^2 = a^2(a - x)$

$$\Rightarrow x = \frac{a^3}{y^2 + a^2}$$

The given curve is symmetrical about x-axis, and meets it at (a, 0).
The line x = 0, i.e., y-axis is an asymptote.



$$\text{Area} = 2\int_0^{\infty} x\, dy = 2\int_0^{\infty} \frac{a^3}{y^2 + a^2} dx$$

$$= 2a^3 \frac{1}{a}\left[\tan^{-1}\frac{y}{a}\right]_0^{\infty}$$

$$= 2a^2 \frac{\pi}{2} = \pi a^2 \text{ sq. units.}$$

Illustration :

The area between the curve $y = 2x^4 - x^2$, *the x-axis and the ordinates of the two minima of the curve is*

(A) $\frac{11}{60}$ *sq. units* (B) $\frac{7}{120}$ *sq. units* (C) $\frac{1}{30}$ *sq. units* (D) $\frac{7}{90}$ *sq. units*

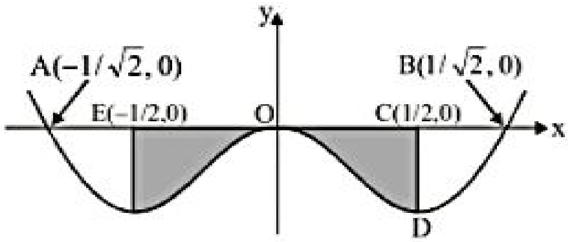
Sol. *The curve is* $y = 2x^4 - x^2 = x^2(2x^2 - 1)$

The curve is symmetrical about of axis of y.

The curve passes through the origin and the tangent at the origin is $y = 0$, *i.e., x-axis.*

The turning points of the curve are given by

$$\frac{dy}{dx} = 8x^3 - 2x = 0 \Rightarrow 2x(4x^2 - 1) = 0$$



$$\Rightarrow \quad x = 0, \pm 1/2$$

Now, $\frac{d^2y}{dx^2} = 24x^2 - 2$

Obviously, $\frac{d^2y}{dx^2}$ *is +ve when* $x = \pm\frac{1}{2}$ *and –ve when* $x = 0$

$x = -1/2$ *and* $x = 1/2$

At $x = -1/2$, *min* $y = -1/8$.

The curve intersects the axes at $O(0, 0)$, $A(-1/\sqrt{2}, 0)$ *and* $B(1/\sqrt{2}, 0)$.

Thus, the graph of the curve is known in the figure

Here, $y \le 0$, *as x varies from* $x = -1/2$ *to* $x = 1/2$

$\therefore$ *The required area = 2 Area OCDO*

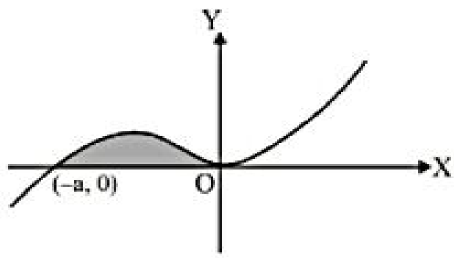
$$= 2\left|\int_0^{1/2} y\,dx\right| = 2\left|\int_0^{1/2}(2x^4 - x^2)\,dx\right| = \frac{7}{120} \text{ sq. units.}$$

Illustration :

The area bounded by the curve $a^2y = x^2(x + a)$ *and x-axis is*

(A) $\frac{a^2}{3}$ *sq. units* (B) $\frac{a^2}{4}$ *sq. units* (C) $\frac{3a^2}{4}$ *sq. units* (D) $\frac{a^2}{12}$ *sq. units*

Sol. *The curve is* $y = \frac{x^2(x+1)}{a^2}$, *which is a cubic polynomial.*



Since $\frac{x^2(x+a)}{a^2} = 0$ *has repeated root* $x = 0$, *it touches x-axis at (0, 0) and intersects at (–a, 0).*

$$\text{Required area} = \int_{-a}^{0} y\,dx = \int_{-a}^{0}\left[\frac{x^2(x+a)}{a^2}\right]dx = \frac{a^2}{12} \text{ sq. units.}$$

Illustration :

The area of the loop of the curve, $ay^2 = x^2(a-x)$ is

(A) $4a^2$ sq. units (B) $\frac{8a^2}{15}$ sq. units (C) $\frac{16a^2}{9}$ sq. units (D) None of these

Sol. $ay^2 = x^2(a-x) \Rightarrow y = \pm x\sqrt{\frac{a-x}{a}}$

Curve tracing : $y = x\sqrt{\frac{a-x}{a}}$

We must have $x \le a$

For $0 < x \le a$, $y > 0$ and for $x < 0$, $y < 0$

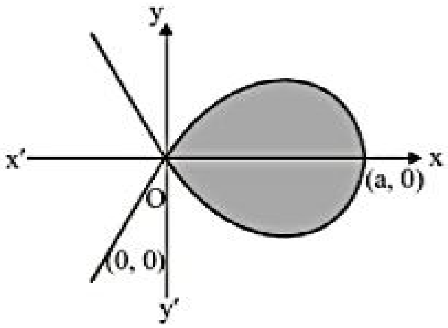
Also $y = 0 \Rightarrow x = 0, a$

Curve is symmetrical about x-axis.

When $x \to -\infty$, $y \to -\infty$

Also, it can be verified that y has only one point of maxima for $0 < x < a$.

$$\text{Area} = 2\int_0^a x\sqrt{\frac{a-x}{a}}\,dx$$



$$\sqrt{\frac{a-x}{a}} = t \Rightarrow 1 - \frac{x}{a} = t^2 \Rightarrow x = a(1-t^2)$$

$$\Rightarrow A = 2\int_1^0 a(1-t^2)t(-2at)\,dt$$

$$= 4a^2\int_0^1 (t^2 - t^4)\,dt = 4a^2\left[\frac{t^3}{3} - \frac{t^5}{5}\right]_0^1 = 4a^2\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{8a^2}{15} \text{ sq. units.}$$

Illustration :

The area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1-y^2}$ is

(A) 1 sq. units (B) $\frac{4}{3}$ sq. units (C) $\frac{2}{3}$ sq. units (D) 2 sq. units

Sol. $A = 2\int_0^1 \left[y\sqrt{1-y^2} - (y^2-1) \right] dy$

$= 2$ sq. units

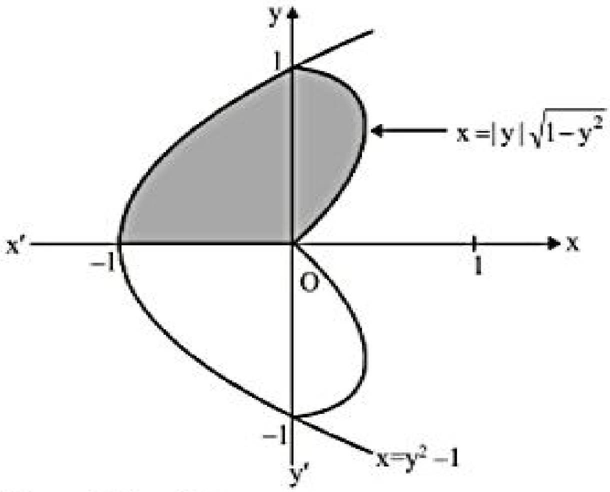


Illustration :

The area bounded by the loop of the curve $4y^2 = x^2(4-x^2)$ is

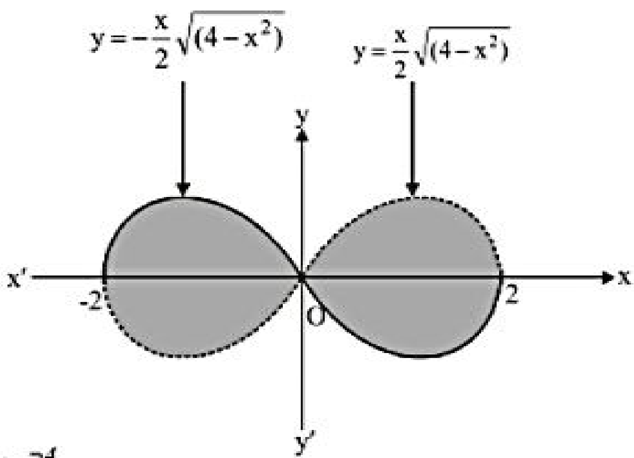
(A) $\frac{7}{3}$ sq. units (B) $\frac{8}{3}$ sq. units (C) $\frac{11}{3}$ sq. units (D) $\frac{16}{3}$ sq. units

Sol. $4y^2 = x^2(4-x^2)$

$\Rightarrow y = \pm\frac{1}{2}\sqrt{x^2(4-x^2)}$

$\Rightarrow y = \pm\frac{x}{2}\sqrt{(4-x^2)}$

$\therefore$ Area $(A) = 4\int_0^2 \frac{x}{2}\sqrt{(4-x^2)}dx$



Let $4 - x^2 = t \quad \Rightarrow \quad -2x\,dx = dt$

$\Rightarrow \quad A = \frac{-4}{4}\int_4^0 \sqrt{t}\,dt = \int_0^4 \sqrt{t}\,dt = \left[\frac{t^{3/2}}{3/2}\right]_0^4 = \frac{2}{3} \times [\sqrt{64} - 0]$

$\Rightarrow \quad A = \frac{16}{3}$ sq. units.

Illustration :

The area enclosed by the curves, $xy^2 = a^2(a-x)$ and $(a-x)y^2 = a^2x$ is

(A) $(\pi - 2)a^2$ sq. units (B) $(4-\pi)a^2$ sq. units

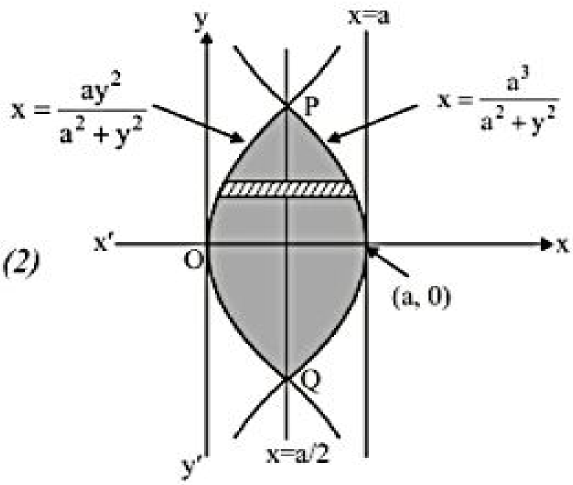
(C) $\pi\frac{a^2}{3}$ sq. units (D) None of these

Sol. The two curves are

$xy^2 = a^2(a - x) \Rightarrow x = \dfrac{a^3}{a^2 + y^2}$ ...(1)

and $(a - x)y^2 = a^2x$

$\Rightarrow x = \dfrac{ay^2}{a^2 + y^2} = \dfrac{ay^2 + a^3 - a^3}{a^2 + y^2} = a - \dfrac{a^3}{a^2 + y^2}$ ...(2)



Curve (1) is symmetrical about x-axis, and have y-axis as the asymptote.

Curve (2) is symmetrical about x-axis, tangent at origin as y-axis and the asymptote $x = a$.

The two curves intersect at the point $P(a/2, a)$ and $Q(a/2, -a)$.

$$\text{Required area} = 2\int_0^a \left(\frac{a^3}{a^2+y^2} - \frac{ay^2}{a^2+y^2}\right)dy = 2a\int_0^a \frac{a^2-y^2}{a^2+y^2}dy = 2a\left[2\int_0^a \frac{a^2}{a^2+y^2}dy - \int_0^a dy\right]$$

$$= 2a\left[2a\tan^{-1}\left(\frac{y}{a}\right)_0^a - a\right] = 2a\left[2a\frac{\pi}{4} - a\right] = a^2(\pi - 2)$$

Illustration :

The area bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$ is

(A) $\dfrac{2}{e}$ sq. units (B) $1 - \dfrac{2}{e}$ sq. units (C) $\dfrac{1}{e}$ sq. units (D) $1 - \dfrac{1}{e}$ sq. units

Sol. Curve tracing : $y = x\,e^x$

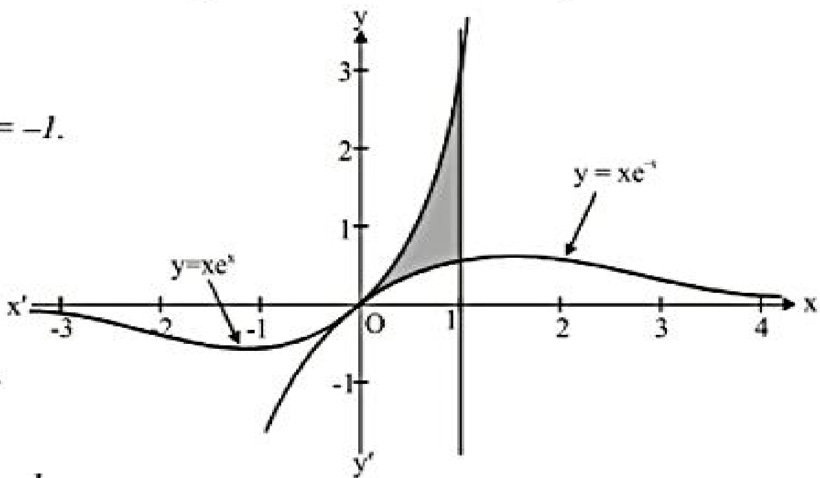
Let $\dfrac{dy}{dx} = 0 \Rightarrow e^x + xe^x = 0 \Rightarrow x = -1$.

Also, at $x = -1$,

$\dfrac{dy}{dx}$ changes sign from –ve to +ve

hence, $x = -1$ is a point of minima.

When $x \to \infty$, $y \to \infty$



Also $\lim\limits_{x\to-\infty} xe^x = \lim\limits_{x\to-\infty} \dfrac{x}{e^{-x}} = \lim\limits_{x\to-\infty} \dfrac{1}{e^{-x}} = 0$

With similary types of arguments, we can draw the graph of $y = x\,e^{-x}$.

$$Required\ area = \int_0^1 xe^x\,dx - \int_0^1 xe^{-x}\,dx \ = \left[xe^x\right]_0^1 - \int_0^1 e^x\,dx - \left(\left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx\right)$$

$$= e - (e-1) - \left(-e^{-1} - \left(e^{-1} - 1\right)\right) = \frac{2}{e}\ sq.\ units$$

Illustration :

The area bounded by the two branches of curve $(y-x)^2 = x^3$ and the straight line $x = 1$ is

(A) $\frac{1}{5}$ *sq. units* (B) $\frac{3}{5}$ *sq. units* (C) $\frac{4}{5}$ *sq. units* (D) $\frac{8}{4}$ *sq. units*

Sol. $(y-x)^2 = x^3$, *where* $x \ge 0$

$\Rightarrow y - x = \pm x^{3/2}$

$\Rightarrow y = x + x^{3/2}$ ...(1)

$y = x - x^{3/2}$ ...(2)

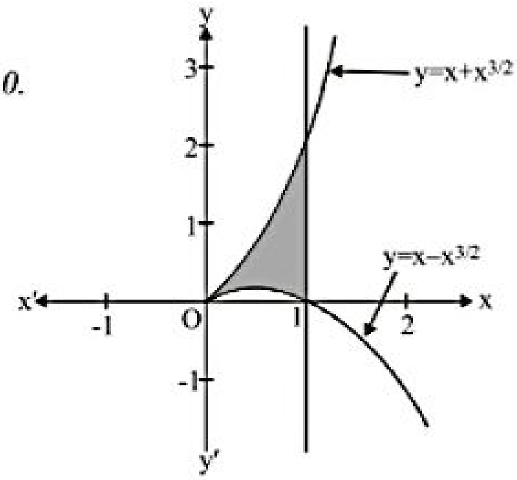
Function (1) is an increasing function.

Function (2) meets x-axis, where $x - x^{3/2} = 0$ or $x = 0, 1$.

Also, for $0 < x < 1$, $x - x^{3/2} > 0$ and for $x > 1$, $x - x^{3/2} < 0$.

When $x \to \infty$, $y \to -\infty$,

From these information, we can plot the graph as below:



$$Required\ area = \int_0^1 \left[\left(x + x^{3/2}\right) - \left(x - x^{3/2}\right)\right] dx$$

$$= 2\int_0^1 x^{3/2}\,dx = 2\left[\frac{x^{5/2}}{5/2}\right]_0^1 = \frac{4}{5}\ sq.\ units$$

Illustration :

(a) *Sketch and find the area bounded by the curve* $\sqrt{|x|}+\sqrt{|y|} = \sqrt{a}$ *and* $x^2 + y^2 = a^2$ *(where* $a > 0$*).*

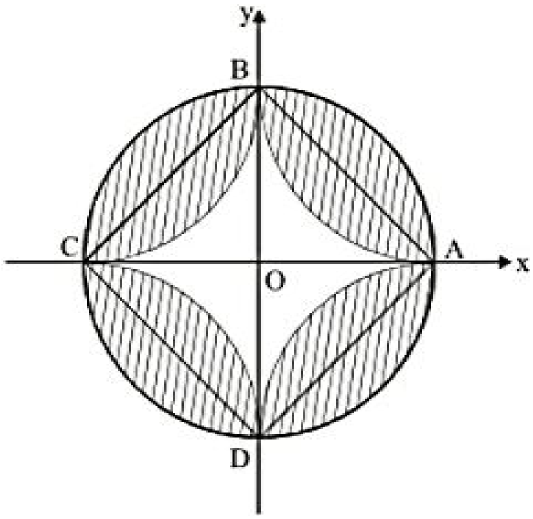
(b) *If curve* $|x| + |y| = a$ *divides the area in two parts, then find their ratio in first quadrant only.*

Sol. $\sqrt{|x|}+\sqrt{|y|}=\sqrt{a}$

$x = 0 \Rightarrow \ y = \pm a$

$y = 0 \Rightarrow \ x = \pm a$

(a) Required area



$$4\int_0^a \sqrt{a^2-x^2}\,dx - 4\int_0^a (\sqrt{a}-\sqrt{x})^2 dx$$

$$= \pi a^2 - 4\int_0^a (\sqrt{a}-\sqrt{x})^2 dx$$

$$= \pi a^2 - 4\int_0^a [a+x-2\sqrt{a}\sqrt{x}]dx \ = \pi a^2 - 4\left[a^2+\frac{a^2}{2}-2\sqrt{a}\frac{2}{3}a^{3/2}\right]$$

$$= \pi a^2 - \left[\frac{3a^2}{2}-\frac{4}{3}a^2\right] = \pi a^2 - 4\frac{a^2}{6} \ = \left(\pi-\frac{2}{3}\right)a^2 \textit{ sq. units.}$$

(b) *Area included between curves and circle in Ist quadrant* $= \frac{1}{4}\pi a^2 - \frac{1}{2}a^2 = \frac{(\pi-2)a^2}{4}$

Area included between $|x| + |y| = a$ *and curve* $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$ *in Ist quadrant*

$$\frac{1}{4}\left(\pi-\frac{2}{3}\right)a^2 - \left(\frac{\pi}{4}-\frac{1}{2}\right)a^2 = \frac{a^2}{3}$$

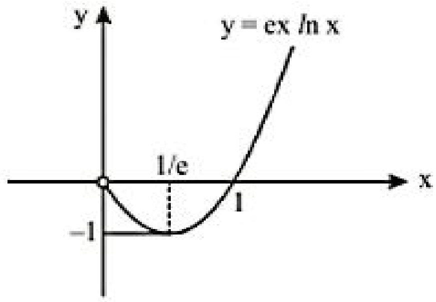
Area ratio $= \dfrac{4}{3(\pi-2)}$

Illustration :

Area enclosed between the curves $y = ex \cdot \ln x$ *and* $y = \dfrac{\ln x}{ex}$

Sol. $y = ex \ln x$

$$\frac{dy}{dx} = e(1 + \ln x) \qquad \therefore \qquad \frac{dy}{dx} = 0 \qquad \Rightarrow \qquad x = e^{-1}$$

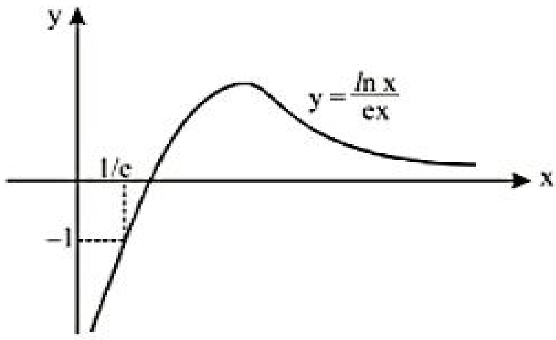


$$\frac{d^2y}{dx^2} = \frac{e}{x} \qquad \therefore \qquad \left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{e}} > 0$$

$$\Rightarrow \qquad \text{minimum at } x = \frac{1}{e}$$

$$y = \frac{\ln x}{ex}$$

$$\frac{dy}{dx} = \frac{1}{e}\left(\frac{1 - \ln x}{x^2}\right) \qquad \therefore \qquad \frac{dy}{dx} = 0 \quad \text{when} \quad x = e$$



$$\left.\frac{dy}{dx}\right|_{x=e^+} < 0, \ \left.\frac{dy}{dx}\right|_{x=e^-} > 0$$

$$\Rightarrow \qquad \text{at } x = e, \qquad y = \frac{\ln x}{ex} \text{ has local maxima}$$

$$\text{Required area} = \int_{1/e}^{1} \left(\frac{\ln x}{ex} - ex\ln x\right) dx$$

$$= \left[\frac{(\ln x)^2}{2e} - e\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\right]_{\frac{1}{e}}^{1} = \frac{e}{4} - \left[\frac{1}{2e - e} - e\left(\frac{-1}{2e^2} - \frac{1}{4e^2}\right)\right]$$

$$= \frac{e}{4} - \left[\frac{1}{2e} + \frac{3}{4e}\right] = \frac{e}{4} - \frac{5}{4e} = \left(\frac{e^2 - 20}{4e}\right)$$

Illustration :

Area enclosed by the curve $(y - \sin^{-1}x)^2 = x - x^2$.

Sol. $(y - \sin^{-1} x)^2 = x - x^2$

$y = \sin^{-1} x \pm \sqrt{x - x^2} \Rightarrow$ domain $x \in [0, 1]$

Area enclosed by the curve

$$= \int_0^1 \left(\sin^{-1} + \sqrt{x - x^2}\right) - \left(\sin^{-1} x - \sqrt{x - x^2}\right) dx = 2\int_0^1 \sqrt{x - x^2}\, dx$$

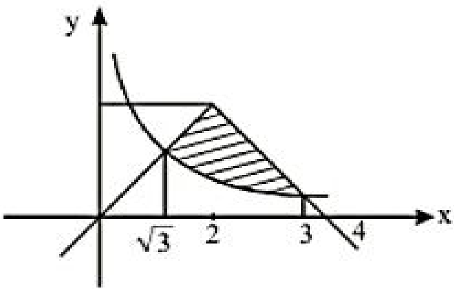
$$= 2\int_0^1 \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2} \quad = 2\left[\frac{1}{2}\left(x - \frac{1}{2}\right)\sqrt{x - x^2} + \frac{1}{2}\left(\frac{1}{4}\right)\sin^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$= 2\left[\left(0 + \frac{1}{8}\frac{\pi}{2}\right) - \left(0 + \frac{1}{8}\left(-\frac{\pi}{2}\right)\right)\right] = 2\left(\frac{\pi}{16} + \frac{\pi}{16}\right) = \frac{\pi}{4}$$

Illustration :

Area of the closed figure bounded by the curves $y = 2 - |2 - x|$ and $y = \dfrac{3}{|x|}$

Sol. $y_1 = 2 - |2 - x| = \begin{cases} x; & x \le 2 \\ 4 - x; & x > 2 \end{cases}$



and $\quad y_2 = \begin{cases} \dfrac{3}{x}; & x > 0 \\ -\dfrac{3}{x}; & x < 0 \end{cases}$

so $\quad \dfrac{3}{x} = x \quad \Rightarrow \quad x = \sqrt{3}$

and $\quad \dfrac{3}{x} = 4 - x \quad \Rightarrow \quad 3 = 4x - x^2 \quad \Rightarrow \quad x^2 - 4x + 4 = 1$

$\Rightarrow \quad x - 2 = \pm 1 \quad \Rightarrow \quad x = 3$

so required area $= \displaystyle\int_{\sqrt{3}}^{2} \left(x - \frac{3}{x}\right) dx + \int_2^3 \left(4 - x - \frac{3}{x}\right) dx$

$$= \left[\frac{x^2}{2} - 3\ln x\right]_{\sqrt{3}}^2 + \left[4x - \frac{x^2}{2} - 3\ln x\right]_2^3$$

$$= 2 - 3\ln 2 - \frac{3}{2} + 3\ln\left(\sqrt{3}\right) + \left(12 - \frac{9}{2} - 3\ln 3\right) - \left(8 - \frac{4}{2} - 3\ln 2\right)$$

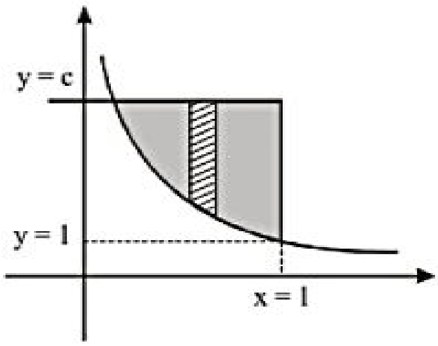
$$= 2 - 3\ln 2 - \frac{3}{2} + \frac{3}{2}\ln 3 + 4 - \frac{5}{2} - 3\ln 3 + 3\ln 2$$

$$= \frac{1}{2} + \frac{3}{2} - \frac{3}{2}\ln 3 = \frac{4 - 3\ln 3}{2}$$

DETERMINATION OF PARAMETERS :

Illustration :

Find the value of c for which the area of the figure bounded by the curves $y = \frac{4}{x^2}$; $x = 1$ *and* $y = c$ *is equal to* $\frac{9}{4}$.



Sol. *So required area* $= \int_{2/\sqrt{c}}^{1}\left(c - \frac{4}{x^2}\right)dx = \left[cx + \frac{4}{x}\right]_{2/\sqrt{c}}^{1}$

$$Area = c\left(1 - \frac{2}{\sqrt{c}}\right) + 4 - 2\sqrt{c} = c - 4\sqrt{c} + 4 = \frac{9}{4}$$

$$\Rightarrow \quad (\sqrt{c} - 2)^2 = \frac{9}{4} \quad \Rightarrow \quad \sqrt{c} = 2 \pm \frac{3}{2}$$

$$\sqrt{c} = \frac{1}{2}, \frac{7}{2}$$

$$c = \frac{1}{4}, \frac{49}{4}$$

Illustration :

If the area bounded by $y = x^2 + 2x - 3$ *and the line* $y = kx + 1$ *is the least, find k and also the least area.*

Sol. x_1 *and* x_2 *are the roots of the equation*

$x^2 + 2x - 3 = kx + 1$, *or*

$x^2 + (2 - k)x - 4 = 0$

$$\Rightarrow \quad \left.\begin{aligned} x_1 + x_2 &= k - 2 \\ x_1 x_2 &= -4 \end{aligned}\right\}$$

$$A = \int_{x_1}^{x_2}\left[(kx + 1) - (x^2 + 2x - 3)\right]dx$$

$$= \left[(k-2)\frac{x^2}{2} - \frac{x^3}{2} + 4x\right]_{x_1}^{x_2} = \left[(k-2)\frac{x_2^2 - x_1^2}{2} - \frac{1}{3}\left(x_2^3 - x_1^3\right) + 4(x_2 - x_1)\right]$$

$$= (x_2 - x_1)\left[\frac{(k-2)^2}{2} - \frac{1}{3}\left((x_2+x_1)^2 - x_1x_2\right) + 4\right]$$

$$= \sqrt{(x_2+x_1)^2 - 4x_1x_2}\left[\frac{(k-2)^2}{2} - \frac{1}{3}\left((k-2)^2+4\right) + 4\right]$$

$$= \frac{\sqrt{(k-2)^2+16}}{6}\left[\frac{1}{6}(k-2)^2 + \frac{8}{3}\right]$$

$$= \frac{\left[(k-2)^2+16\right]^{3/2}}{6}$$

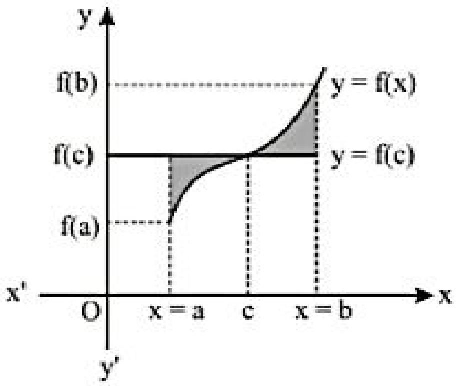
which is least when $k = 2$ and $A_{least} = 32/3$ sq. units.

VARIABLE AREA GREATEST AND LEAST VALUE :

An important concept :

If $y = f(x)$ is a monotonic function in (a, b) then the area bounded by the ordinates at $x = a$, $x = b$, $y = f(x)$ and $y = f(c)$, [where $c \in (a, b)$] is minimum when $c = \dfrac{a+b}{2}$.

Proof: $A = \int_a^c (f(c) - f(x))\,dx + \int_c^b (f(x) - f(c))\,dx$



$$= f(c)(c-a) - \int_a^c (f(x))\,dx + \int_c^b (f(x))\,dx - f(c)(b-c)$$

$$A = [2c - (a+b)]\, f(c) + \int_c^b (f(x))\,dx - \int_a^c (f(x))\,dx$$

Differetiating w.r.t. c,

$$\frac{dA}{dc} = [2c - (a+b)]\, f'(c) + 2\,f(c) + 0 - f(c) - (f(c))$$

for maxima and minima $\dfrac{dA}{dc} = 0$

$\Rightarrow \quad f'(x)[2c - (a+b)] = 0$ (as $f'(c) \neq 0$)

hence $c = \frac{a+b}{2}$

Also $c < \frac{a+b}{2}, \frac{dA}{dc} < 0$ and $c > \frac{a+b}{2}, \frac{dA}{dc} > 0$.

Hence A is minimum when $c = \frac{a+b}{2}$.

Note : Let f(x) be the bijective functon and g(x) be the inverse of it then area bounded by y = g(x), and the ordinate at x = a and x = b is same as area bounded by y = f(x) and the abscissa at y = a and y = b as f(x) and g(x) are mirror image with respect to line y = x.

Illustration :

If the area bounded by $f(x) = \frac{x^3}{3} - x^2 + a$ *and the straight lines* $x = 0$; $x = 2$ *and the x-axis is minimum then find the value of 'a'.*

Sol. $f(x) = \frac{x^3}{3} - x^2 + a$

$f'(x) = x^2 - 2x = x(x-2) < 0$ *(note that f(x) is monotonic in (0, 2))*

Hence for the minimum and f(x) must cross the x-axis at $\frac{0+2}{2} = 1$

Hence $f(1) = \frac{1}{3} - 1 + a = 0$

$\Rightarrow a = \frac{2}{3}$

Illustration :

The value of the parameter a for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$ *and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, is the least, is*

(A) 2 *(B) 0* *(C) – 1* *(D) 1*

Sol. $f(x) = x^3 + 3x^2 + x + a$

$f'(x) = 3x^2 + 6x + 1 = 0$

$\Rightarrow \quad x = -1 \pm \frac{\sqrt{6}}{3}$

Hence, f(x) cuts the x-axis at $\frac{1}{2}\left[\left(-1+\frac{\sqrt{6}}{3}\right)+\left(-1-\frac{\sqrt{6}}{3}\right)\right] = -1.$

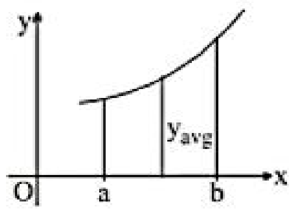
$f(-1) = -1 + 3 - 1 + a = 0$

$a = -1.$

AVERAGE VALUE OF A FUNCTION :

Average value of the function in $y = f(x)$ w.r.t. x over an interval $a \le x \le b$ is defined as

$$y_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



Note :

(i) Average value can be + ve, – ve or zero.

(ii) If the function is defined in $(0, \infty)$ then

$$y_{avg} = \lim_{b\to\infty} \frac{1}{b}\int_0^b f(x)\,dx \qquad \text{provided the limit exists.}$$

Root mean square value **(RMS)** is defined as

$$\rho = \left[\frac{1}{b-a}\int_a^b f^2(x)\,dx\right]^{\frac{1}{2}}$$

Illustration :

Compute the average value of $f(x) = \frac{\cos^2 x}{\sin^2 x + 4\cos^2 x}$ *in* $\left[0, \frac{\pi}{2}\right]$

Sol. *Average value of f(x)* $= \frac{\cos^2 x}{\sin^2 x + 4\cos^2 x}$

$$y_{average} = \frac{1}{\left(\frac{\pi}{2}-0\right)}\int_0^{\pi/2}\frac{\cos^2 x}{(\sin^2 x + 4\cos^2 x)}dx = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{(\tan^2 x + 4)}dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x(\tan^2 x+4)}dx = \frac{2}{\pi}\int_0^{\pi/2} \frac{\sec^2 x\, dx}{(1+\tan^2 x)(4+\tan^2 x)}$$

Put $t = \tan x$ so $dt = \sec^2 x\, dx$

$$y_{average} = \frac{2}{\pi}\int_0^{\infty} \frac{dt}{(t^2+1)(4+t^2)} = \frac{2}{3\pi}\int_0^{\infty}\left[\frac{1}{t^2+1} - \frac{1}{t^2+4}\right]dt$$

$$= \frac{2}{3\pi}\left[\tan^{-1} t - \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{\infty} = \frac{2}{3\pi}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{2}{3\pi}\,\frac{\pi}{4} = \frac{1}{6} \qquad \textit{Ans.}$$

DETERMINATION OF FUNCTION :

The area function A(x) satisfies the differential equation $\frac{dA(x)}{dx} = f(x)$ with initial condition $A(a) = 0$ i.e. derivative of the area function is the function itself.

Note :

If $F(x)$ is any integral of $f(x)$ then,

$A(x) = \int f(x)\,dx = [\,F(x) + c\,]$ $\qquad A(a) = 0 = F(a) + c \;\Rightarrow\; c = -F(a)$

hence $A(x) = F(x) - F(a)$. Finally by taking $x = b$ we get, $A(b) = F(b) - F(a)$.

Illustration :

The area from 0 to x under a certain graph is given to be $A = \sqrt{1+3x} - 1,\ x \ge 0$;

(a) *Find the average rate of change of A w.r.t. x as x increases from 1 to 8.*

(b) *Find the instantaneous rate of change of A w.r.t. x at x = 5.*

(c) *Find the ordinate (height) y of the graph as a function of x.*

(d) *Find the average value of the ordinate (height) y, w.r.t. x as x increases from 1 to 8*

Sol. $A = \sqrt{1+3x} - 1 = \int_0^x f(x)\,dx$

(a) $$\left.\frac{dA}{dx}\right|_{avg} = \frac{1}{(8-1)}\int_1^8 \left(\frac{dA}{dx}\right)dx$$

$$= \frac{1}{7}\left(\sqrt{1+3x} - 1\right)_1^8 = \frac{1}{7}(4-1) = \frac{3}{7}$$

(b) $\left.\frac{dA}{dx}\right|_{x=5} = \frac{3}{2\sqrt{1+3x}} = \frac{3}{2\sqrt{1+3(5)}} = \frac{3}{8}$

(c) $A(x) = \int_0^x f(x)\,dx = \sqrt{1+3x} - 1$

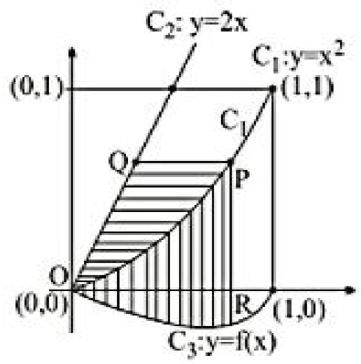
Differentiating w.r.t. x

$f(x) = \frac{3}{2\sqrt{1+3x}}$

(d) $y_{avg} = \frac{1}{(8-1)} \int_1^8 f(x)\,dx = \frac{1}{7}\left(\sqrt{1+3x}-1\right)_1^8 = \frac{3}{7}$

Illustration :

Let C_1 & C_2 be the graphs of the functions $y = x^2$ & $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 & C_3 at Q & R respectively (see figure). If for every position of P (on C_1), the areas of the shaded regions OPQ & ORP are equal, determine the function f(x). [JEE '98, 8]

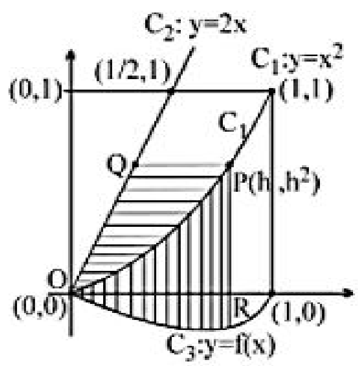


Sol. *Let $P(h, h^2)$ be a point on the curve C_1.*

$\Rightarrow \quad R(h,\ f(h))$

Area OPQO = Area OPRO

$$\int_0^{h^2}\left(\sqrt{y}-\frac{y}{2}\right)dy = \int_0^h\left(x^2 - f(x)\right)dx$$



Differenting w.r.t. h

$$\left(\sqrt{h^2}-\frac{h^2}{2}\right).2h = h^2 - f(h)$$

$\Rightarrow \quad 2h^2 - h^3 = h^2 - f(h)$

$\Rightarrow \quad f(h) = h^3 - h^2$

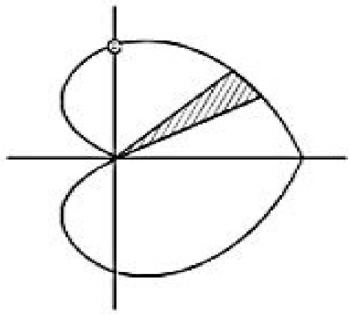
$\Rightarrow \quad f(x) = x^3 - x^2$

AREA ENCLOSED IN CASE ONE CURVE ARE EXPRESSED IN POLAR FORM :

Area of any curve $= \frac{1}{2}\int r^2 d\theta$

Illustration :

Find the area of the cardioid $r = a(1 + \cos\theta)$



Sol. $A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{a^2}{2}\int_0^{2\pi} 4\cos^4\frac{\theta}{2}\, d\theta$ put $\frac{\theta}{2} = t$

$A = a^2\int_0^{\pi} 4\cos^4 t\, dt = 8 \times \frac{3\pi a^2}{16} = \left(\frac{3\pi a^2}{2}\right)$

AREA IN RESPECT OF CURVE REPRESENTED PARAMETRICALLY :

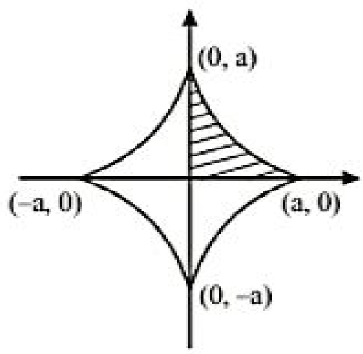
Illustration :

Find the area enclosed by the curves $x = a\sin^3 t$ and $y = a\cos^3 t$

Sol. $x^{2/3} + y^{2/3} = a^{2/3}$

Required area $= 4\int_0^{a}\left(a^{2/3} - x^{2/3}\right)^{3/2} dx$

Put $x = a\sin^3 t$; $dx = 3a\sin^2 t\cos t\, dt$



Area $= 4\int_0^{\pi/2}\left(a^{2/3} - a^{2/3}\sin^2 t\right)^{3/2} 3a\sin^2 t\cos t\, dt$

$A = 12a^2\int_0^{\pi/2}\sin^2 t\cos^4 t\, dt \quad ...(1)$

$A = 12a^2\int_0^{\pi/2}\sin^4 t\cos^2 t\, dt \quad ...(2)$

Adding (i) and (ii)

$$A = \frac{12a^2}{2}\int_0^{\pi/2} \cos^2 t \sin^2 t\ (\cos^2 t + \sin^2 t)\ dt$$

$$= 6a^2 \int_0^{\pi/2} \sin^2 t \cos^2 t\ dt \ = \ 6a^2 \int_0^{\pi/2} \frac{\sin^2 2t}{4} \ = \ \frac{3a^2}{2}\int_0^{\pi/2}\left(\frac{1-\cos 4t}{2}\right)dt$$

$$= \frac{3a^2}{4}\left(t - \frac{\sin 4t}{4}\right)_0^{\pi/2} \ = \ \frac{3a^2}{4}\left(\frac{\pi}{2}\right) = \frac{3\pi a^2}{8}$$

Practice Problem

Q.1 For what value of k is the area of the figure bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is the least. Determine the least area.

Q.2 Find the area enclosed by the parabola $(y-2)^2 = x - 1$ and the tangent to it at (2, 3) & x-axis.

Q.3 Area enclosed between the smaller arc of the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ and the parabola $y = -x^2 + 2x + 1 - 2\sqrt{3}$.

Q.4 Find the area of the figure bounded by the parabola $y = ax^2 + 12x - 14$ and the straight line $y = 9x - 32$ if the tangent drawn to the parabola at the point $x = 3$ is known to make an angle $\pi - \tan^{-1}6$ with the x-axis.

Q.5 Find the area bounded by the curve g (x), the x-axis and the ordinate at $x = -1$ and $x = 4$ where g (x) is the inverse of the function $f(x) = \frac{x^3}{24} + \frac{x^2}{8} + \frac{13x}{12} + 1$

Q.6 $f(x) = x^3 + 3x + 2$ and $g(x)$ is the inverse of it. Then compute the area bounded by g(x), x-axis and the ordinate at $x = -2$ and $x = 6$.

Q.7 Find the value of the parameter 'a' for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$, and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, is the least.

Q.8 Find the average value of y^2 w.r.t. x for the curve $ay = b\sqrt{a^2 - x^2}$ between $x = 0$ & $x = a$. Also find the average value of y w.r.t. x^2 for $0 \le x \le a$.

Answer key

Q.1	$k = 0, A = \frac{20\sqrt{5}}{3}$	Q.2	0009	Q.3	$4\left(\frac{8 - 3\sqrt{3} + 2\pi}{3}\right)$
Q.4	$\frac{125}{2}$	Q.5	$\frac{16}{3}$	Q.6	$\frac{9}{2}$
Q.7	$a = -1$	Q.8	(i) $a = \frac{2b^2}{3}$, (ii) $b = \frac{2b}{3}$		