INDEFINITE INTEGRATION

DIFFERENTIALS:

Up to this point in our work, for $y = f(x)$ we have regarded dy/dx as a composite symbol for the derivative $f'(x)$, whose component parts, dy and dx, had no meaning by themselves. It is now convenient to modify this point of view and attach meaning to dy and dx, so that thereafter we can treat dy/dx as though it were a fraction in fact as well as in appearance. We shall not however enter into any discussions on it. We shall only state that,

for a function of a single variable $y = f(x)$, the differential of y denoted by dy is the product of the derivative of y (with respect to x) and the differential of x denoted by dx. Thus,

Diffrential of $y = f(x)$ is $dy = f'(x)dx$.

For $y = x^4$, $dy = 4x^3dx$, or simply $d(x^4) = 4x^3\,dx$. Thus
$d(\sin x) = \cos x\,dx$, $d(y^2) = 2y\,dy$, $d(\tan u) = \sec^2 u\,du$.

INTEGRATION AS ANTI-DERIVATIVE :

Simplest way to define integration is as an antiderivative the inverse of a derivative. Derivative of sin x is cos x then we may say that integral of cos x is sin x.

In general, if we consider

$$\frac{d}{dx} f(x) = \phi(x)$$

or, using differentials $d\,f(x) = \phi(x)\,dx$;
then an integral of $\phi(x)$ with respect to x or an integral of $\phi(x)\,dx$ is $f(x)$ and symbolically, we write,

$$\int \phi(x)\,dx = f(x)$$

where the symbol $\int$ which is an elongated S (the first letter of the word sum, or, of the Latin word Summa) is known as the sign of integration. Now we come to some formal definitions:

The actual process of finding the function, when its derivative or its differential is known, is called Integration as anti-derivative ; the function to which the integration is applied is called Integrand and the function obtained as a result of integration is said to be Integral. In the above case, $\phi(x)$ is the integrand and $f(x)$ is the integral.

The process of integrating many ordinary functions is simple, but in general, integration is more involved than differentiation, as will be evident from future discussions.

Summary:

If $\frac{d}{dx}[F(x)+C] = f(x)$ then $F(x) + C$ is called an antiderivative of $f(x)$ on [a, b] and is written as

$\int f(x)\,dx = F(x) + C.$

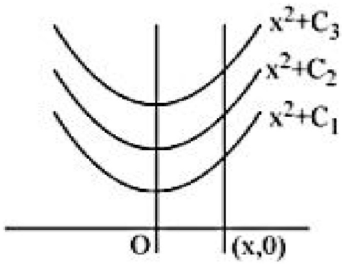
In this case we say that the function $f(x)$ is integrable on $[a, b]$. Note that every function is not integrable.

e.g. $f(x) = \begin{cases} 0 & \text{if } x \in Q \\ 1 & \text{if } x \notin Q \end{cases}$ is not integrable in $[0, 1]$. Every function which is continuous on a closed and bounded interval is integrable.

However for integrability function $f(x)$ may only be piece wise continuous in (a, b)

Notes on indefinite integration :

(1) **Geometrical interpretation :**



$$y = \int 2x\,dx = \frac{x^2}{2} + C$$

$$y = \int f(x)\,dx = F(x) + C$$

$\Rightarrow$ $F'(x) = f(x)$; $F'(x_1) = f(x_1)$

Hence $y = \int f(x)\,dx$ denotes a family of curves such that the slope of the tangent at $x = x_1$ on every member is same. i.e. $F'(x_1) = f(x)$ (when x_1 lies in the domain of $f(x)$) hence antiderivative of a function is not unique. If $g_1(x)$ and $g_2(x)$ are two antiderivatives of a function $f(x)$ on $[a, b]$ then they differ only by a constant i.e. $g_1(x) - g_2(x) = C$

(2) Antiderivative of a continuous function is differentiable

i.e. If $f(x)$ is continuous then $\int f(x)\,dx = F(x) + C \Rightarrow F'(x) = f(x) \Rightarrow F'(x)$ is always exists

$\Rightarrow$ $F(x)$ is differentiable

(3) If integrand is discontinuous at $x = x_1$ then its antiderivative at $x = x_1$ need not be discontinuous.

i.e. e.g. $\int x^{-1/3}dx$. Here $x^{-1/3}$ is discontinuous at $x = 0$.

but $\int x^{-1/3}dx = \frac{3}{2}x^{2/3} + C$ is continuous at $x = 0$

(4) If $\frac{d}{dx}(F(x) + C) = f(x) \Rightarrow \int f(x)\,dx = F(x) + C$ then only we say that $f(x)$ is integrable.

(5) Antiderivative of a periodic function need not be a periodic function

e.g. $f(x) = \cos x + 1$ is periodic but $\int (\cos x + 1)\,dx = \sin x + x + C$ is aperiodic.

Problems based on Indefinite integral as antiderivative :

Some times it is possible to convert given integral as a loving integral (Standrad integral) after simple manipulation

Evaluate the following integrals :

Illustration :

$$\int 2^x . e^x \, dx$$

Sol. $$\int 2^x e^x dx = \int (2e)^x dx = \frac{(2e)^x}{ln(2e)} + c$$

Illustration :

$$\int \frac{1-tan^2 x}{1+tan^2 x} \, dx$$

Sol. $$\int \frac{1-tan^2 x}{1+tan^2 x} \, dx = \int \frac{1-tan^2 x}{sec^2 x} dx = \int (cos^2 x - sin^2 x) dx = \int cos\, 2x\, dx = \frac{1}{2} sin\, 2x + c$$

Illustration :

$$\int \frac{\left(\sqrt{x}+1\right)\left(x^2-\sqrt{x}\right)}{x\sqrt{x}+x+\sqrt{x}} \, dx$$

Sol. $$\int \frac{\left(\sqrt{x}+1\right)\left(x^2-\sqrt{x}\right)}{x\sqrt{x}+x+\sqrt{x}} = \int \frac{\left(\sqrt{x}+1\right)\sqrt{x}\left(\left(\sqrt{x}\right)^3-1\right)}{\sqrt{x}\ (x+\sqrt{x}+1)} dx$$

$$= \int \frac{\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)\left(x+1+\sqrt{x}\right)}{(x+\sqrt{x}+1)} dx = \int (x-1) dx = \frac{x^2}{2} - x + c$$

Illustration :

$$\int \frac{\left(x^2+sin^2 x\right)sec^2 x}{1+x^2} \, dx$$

Sol. $$\int \frac{\left(x^2+sin^2 x\right)sec^2 x}{1+x^2} dx = \int \left(sec^2 x - \frac{1}{1+x^2} \right) dx = tan\ x - tan^{-1} x + c$$

Illustration :

$$\int \frac{sin\, 2x - sin\, 2k}{sin\, x - sin\, k + cos\, x - cos\, k} dx$$

Sol. $$\int \frac{sin\, 2x - sin\, 2k}{sin\, x - sin\, k + cos\, x - cos\, k} dx$$

$$= \int \frac{(sin\, x + cos\, x + sin\, k + cos\, k)(sin\, x + cos\, x - sin\, k - cos\, k)}{sin\, x - sin\, k + cos\, x - cos\, k} dx$$

$$= \int (sin\, x + cos\, x + sin\, k + cos\, k) dx \quad = (sin\ x - cos\ x) + (sin\ k + cos\ k)x + C$$

Practice Problem

Q.1 $\int \frac{1+\cos^2 x}{1+\cos 2x}\,dx$ Q.2 $\int \frac{1+\tan^2 x}{1+\cot^2 x}\,dx$ Q.3 $\int \frac{x^4+x^2+1}{2\left(1+x^2\right)}\,dx$

Q.4 $\int \frac{\sin 2x+\sin 5x-\sin 3x}{\cos x+1-2\sin^2 2x}\,dx$ Q.5 $\int \frac{2+3x^2}{x^2\left(1+x^2\right)}\,dx$

Answer key

Q.1 $\frac{1}{2}(\tan x+x)+C$ Q.2 $\tan x-x+C$ Q.3 $\frac{1}{2}\left[\frac{x^3}{3}+\tan^{-1}x\right]+C$

Q.4 $-2\cos x+C$ Q.5 $-\frac{2}{x}+\tan^{-1}x+C$

TECHNIQUES OF INTERGRATION :

Often it is not possible to convert an integral into loving integral just by simple manipulation. Then required some techniques to convert an integral into loving integral. This techniques are following.

- Substitution
- By part (product rule)
- Partial (fraction)
- Kuturputur & Misc.

SUBSTITUTION :

Theory : $I = \int f(x)\,dx$ and let $x=\phi(z)$

$$\frac{dI}{dx}=f(x) \;;\quad \frac{dx}{dz}=\phi'(z)$$

$$\Rightarrow \quad \frac{dI}{dz}=\frac{dI}{dx}\cdot\frac{dx}{dz}=f(x).\phi'(z) \text{ or } \frac{dI}{dz}=f(\phi(z))\;\phi'(z)$$

$$\boxed{\text{Hence } I=\int f(\phi(z))\phi'(z)\,dz} \quad ....(1)$$

Substitution is said to be appropriate if the integrand in (1) is a loving one. (standard integral)

If $\int [f(x)]^n f'(x)\, dx$ or $\int \frac{f'(x)}{[f(x)]^n} dx$

Start with $f(x)=t$

$$\boxed{\begin{array}{l}\int(\tan x)dx = \ln\sec x + C = -\ln(\cos x)+C; \\ \int(\cot x)dx = \ln(\sin x) \text{ (loving integrals)}\end{array}}$$

Proof : $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$

Put $\cos x = t$ to get $\int \frac{-dt}{t} = -\ln t + c = -\ln(\cos x) + c = \ln(\sec x) + c$

Illustration :

$$\int \frac{\cos(\ln x)}{x} dx$$

Sol. *Put* $\ln x = t \Rightarrow \frac{1}{x} dx = dt$

Integral becomes $\int \cos dt = \sin t + c = \sin(\ln x) + c$

Illustration :

$$\int \frac{x^3 dx}{1+x^8}$$

Sol. $x^4 = t \Rightarrow 4x^3\, dx = dt$

Integral becomes $\int \frac{\frac{1}{4}dt}{1+t^2} = \frac{1}{4}\tan^{-1} t + c = \frac{1}{4}\tan^{-1}(x^4) + c$

Illustration :

$$\int \frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} dx$$

Sol. *Put* $\ln\left(x+\sqrt{1+x^2}\right) = t \Rightarrow \frac{1+\frac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} dx = dt$ *or* $\frac{1}{\sqrt{1+x^2}} dx = dt$

Integral become $\int t\, dt = \frac{t^2}{2} + c = \frac{1}{2}\left\{\ln\left(x+\sqrt{1+x^2}\right)\right\}^2 + c$

Illustration :

$$\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$$

Sol. *Put* $\tan^{-1} x^3 = t \Rightarrow \frac{3x^2}{1+x^6} dx = dt$

Integral becomes $\int \frac{1}{3} t\, dt = \frac{1}{6} t^2 + c = \frac{1}{6} (\tan^{-1} x^3)^2 + c$

Illustration :

$$\int \frac{\tan\sqrt{x} \sec^2\sqrt{x}}{\sqrt{x}} dx$$

Sol. *Put* $\tan\sqrt{x} = t \Rightarrow \frac{\sec^2\sqrt{x}}{2\sqrt{x}} dx = dt$

Integral becomes $\int 2t\, dt = t^2 + c = \left(\tan\sqrt{x}\right)^2 + c$

$$\boxed{\left.\begin{array}{l}\int \sec x\, dx = \ln(\sec x + \tan x) + C \text{ or } \ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C; \\ \int \operatorname{cosec} x\, dx = \ln(\operatorname{cosec} x - \cot x) \text{ or } \ln\tan\frac{x}{2} + C\end{array}\right\} \text{(loving integrands)}}$$

Proof : $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

Put $\sec x + \tan x = t \Rightarrow (\sec x \tan x + \sec^2 x) dx = dt$

Integral becomes $\int \frac{dt}{t} = \ln t$

Illustration :

$$\int \frac{\operatorname{cosec}(\tan^{-1} x)}{1+x^2} dx$$

Sol. *Put* $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

integral becomes $\int \operatorname{cosec} t\, dt = \ln (\operatorname{cosec} t - \cot c) + c$

$= \ln [\operatorname{cosec} (\tan^{-1} x) - \cot (\tan^{-1} x)] + c = \ln \left[\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right] + c$

Illustration :

$$\int \frac{\cos 2x}{\sin x}\,dx$$

Sol. $\int \frac{1-2\sin^2 x}{\sin x}\,dx = \int (\text{cosec } x - 2\sin x)\,dx = \ln(\text{cosec } x - \cot x) + 2\cos x + c$

Illustration :

$$\int \frac{e^x(1+x)}{\cos(xe^x)}\,dx$$

Sol. *Put* $xe^x = t \Rightarrow e^x(1+x)dx = dt$

Integral becomes $\int \frac{dt}{\cos t} = \int \sec t\; dt = \ln(\sec t + \tan t) + c = \ln(\sec(xe^x) + \tan(xe^x)) + c$

General Substitution :

$\sqrt{a^2-x^2}$; $x = a\sin\theta$

$\sqrt{a^2+x^2}$; $x = a\tan\theta$

$\sqrt{x^2-a^2}$; $x = a\sec\theta$

$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$; $x^2 = a^2\cos 2\theta$

Examples :

$\sqrt{\frac{x}{a-x}}$; $\cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)$ [Ans. $\frac{x^2}{2}$] ; $\frac{1}{\sqrt{-2x^2+3x+5}}$;

$\frac{dx}{(x^2+4)\sqrt{4x^2+1}}$; $\int \frac{\sqrt{(9-x^2)^3}}{x^6}\,dx$; $\int \frac{\sqrt{x}\;dx}{\sqrt{a^3-x^3}}$

$$\int \frac{x^2}{\sqrt{a^6-x^6}}\,dx$$

note that $\int \sqrt{a^2+x^2}\,dx$ & $\int \sqrt{x^2-a^2}\,dx$

to be executed by parts.

$$\boxed{\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) \;\&\; \int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right) \text{ (loving integrals)}}$$

Loving Integrals:-

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$;

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$;

$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$;

$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a}\right) + c$;

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right) + c$

Illustration :

$$\int \frac{\sin 2x}{\sqrt{9-\sin^4 x}}\,dx$$

Sol. *Put* $\sin^2 x = t \Rightarrow \sin 2x\,dx = dt$

Integral becomes $\int \frac{dt}{\sqrt{9-t^2}} = \sin^{-1}\left(\frac{t}{3}\right)+c = \sin^{-1}\left(\frac{\sin^2 x}{3}\right)+c$

Illustration :

$$\int \frac{e^x dx}{\sqrt{e^{2x}-1}}$$

Sol. *Put* $e^x = t \Rightarrow e^x\,dx = dt$

Integral becomes $\int \frac{dt}{\sqrt{t^2-1}} = \ln\left(t+\sqrt{t^2-1}\right)+c = \ln\left(e^x+\sqrt{e^{2x}-1}\right)+c$

Illustration :

$$\int \frac{e^x}{4+e^{2x}}\,dx$$

Sol. *Put* $e^x = t$ *to get* $\int \frac{dt}{4+t^2} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+c \qquad = \frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right)+c$

NOTE :

For Integration of type $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ make ax^2+bx+c as perfect square.

for Integration of type $\int \frac{px+q}{ax^2+bx+c}dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ write $px+q = \lambda(2ax+b)+\mu$

Illustration :

$$\int \frac{e^x\,dx}{\sqrt{5-4e^x+e^{2x}}}$$

Sol. *Put* $e^x = t$ *to get* $\int \frac{dt}{\sqrt{5-4t+t^2}} = \int \frac{dt}{\sqrt{(t-2)^2+1}} = \ln\left(t-2+\sqrt{t^2-4+5}\right)+c$

Illustration :

$$\int \frac{4x+3}{3x^2+3x+1}dx$$

Sol. $4x + 3 = A\,(6x + 3) + B$ *by equating cofficients*

$$A = \frac{2}{3}\ \&\ B = 1 \qquad \Rightarrow \qquad \int \frac{4x+3}{3x^2+3x+1}dx = \frac{2}{3}\int \frac{6x+3}{3x^2+3x+1} + \int \frac{dx}{3x^2+3x+1}$$

$$= \frac{2}{3}\ln\,(3x^2 + 3x + 1) + \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{1}{12}} = \frac{2}{3}\ln\,(3x^2 + 3x + 1) + \frac{1}{3}\frac{1}{\sqrt{\frac{1}{12}}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{1}{\sqrt{12}}}\right)$$

Illustration :

$$\int \frac{x\ dx}{x^4+x^2+1}$$

Sol. Put $x^2 = t$ to get $\displaystyle \int \frac{\frac{1}{2}dt}{t^2+t+1} = \frac{1}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \frac{1}{2}\frac{1}{\sqrt{3/2}}\cdot\tan^{-1}\left(t+\frac{1}{2}\right) + c = \frac{1}{\sqrt{3}}\tan^{-1}\left(x^2+\frac{1}{2}\right) + c$$

Illustration :

$$\int \frac{5x+4}{\sqrt{x^2+2x+5}}\,dx.$$

Sol. Let $\displaystyle I = \int \frac{5x+4}{\sqrt{x^2+2x+5}}\,dx$

Let $5x + 4 = \lambda\,(2x + 2) + \mu$. *Comparing the coefficient's, we have*

$2\lambda = 5$ and $2\lambda + \mu = 4$ *gives* $\lambda = \frac{5}{2}$ and $\mu = -1$.

Hence, we have

$$I = \frac{5}{2}\int \frac{2x+2}{\sqrt{x^2+2x+5}}dx - \int \frac{dx}{\sqrt{x^2+2x+5}} = 5\sqrt{x^2+2x+5} - \int \frac{dx}{\sqrt{(x-1)^2+2^2}}$$

$$= 5\sqrt{x^2+2x+5} - \ln\left|\,x+1+\sqrt{x^2+2x+5}\,\right| + C.$$

Practice Problem

Q.1 $\int \frac{x^2}{\sqrt{1-4x^6}}dx = \frac{1}{6}\sin^{-1}\{f(x)\}+c$. Find f'(2).

Q.2 $\int \frac{dx}{x^2(1+x^5)^{4/5}} = -\left(1+\frac{1}{x^m}\right)^n + c$. Find mn?

Q.3 $\int \frac{x+3}{\sqrt{4x^2+4x-3}}dx = \frac{1}{4}\sqrt{f(x)}+\frac{5}{2}\int\frac{dx}{\sqrt{4x^2+4x-3}}$ then find minimum value of f(x)?

Evaluate following integrals :

Q.4 $\int \frac{1}{x(\ln x)^3}dx$ Q.5 $\int \frac{x+2}{\sqrt{x-3}}dx$ Q.6 $\int \frac{\sqrt{x}}{a\sqrt{x}+b}dx$

Answer key

Q.1 24 Q.2 1 Q.3 –4 Q.4 $\frac{-1}{2\ln^2 x}+C$

Q.5 $\frac{2}{3}(x-3)^{3/2}+10\sqrt{x-3}+C$ Q.6 $\frac{1}{a^3}\left(a\sqrt{x}+b\right)^2-\frac{4b}{a^3}\left(a\sqrt{x}+b\right)+\frac{2b^2}{a^3}\ln\left|a\sqrt{x}+b\right|+C$

INTEGRATION BY PARTS :

Theory: If f (x) and g (x) are derivable functions then

$$\frac{d}{dx}[f(x).g(x)] = f(x).g'(x)+g(x).f'(x)$$

$$\therefore \quad \int \underset{\text{I}}{f(x)}.\underset{\text{II}}{g'(x)}\,dx = f(x).g(x) - \int g(x).f'(x)\,dx$$

$$I = \int \underbrace{f(x)}_{\text{I}}\cdot\underbrace{g(x)}_{\text{II}}dx$$

$$= 1^{st}\text{ function} \times \text{integral of } 2^{nd} - \int (\text{diff. co-eff. of } 1^{st}) \times (\text{integral of } 2^{nd})\,dx$$

Remember **ILATE** for deciding the choice of the first and second function which is not arbitrary.

Here I for inverse trigonometric function
L for Logarithmic function
A for Algebraic function
T for Trigonometric function
E for Exponential Function

Illustration :

$$\int x \cos x \, dx$$

Sol. *Take x as first and cos x as 2^{nd} function.*

$$\int x \cos x \, dx = x \sin x - \int 1 + \sin x \, dx = x \sin x + \cos x + c.$$

Illustration :

$$\int x \tan^{-1} x \, dx$$

Sol. $$\int \underbrace{x}_{II} \underbrace{\tan^{-1} x}_{I} \, dx = (\tan^{-1} x) \frac{x^2}{2} - \int \frac{x^2}{2(1+x^2)} dx$$

$$= \frac{1}{2} x^2 (\tan^{-1} x) - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} dx = \frac{1}{2} x^2 (\tan^{-1} x) - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$$

Illustration :

$$\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$$

Sol. *Put* $\sin^{-1} x = \theta$ *or* $x = \sin \theta \Rightarrow dx = \cos \theta \, d\theta$

Integral becomes $\int \frac{\theta . \cos\theta \, d\theta}{\cos^3 \theta} = \int \underbrace{\theta}_{I} . \underbrace{\sec^2 \theta}_{II} d\theta$

$$= \theta . \tan\theta - \int 1 . \tan\theta \, d\theta = \theta \tan\theta - \ln(\sec\theta) + c. = (\sin^{-1} x) . \frac{x}{\sqrt{1-x^2}} + \ln(\sqrt{1-x^2}) + c$$

Illustration :

$$\int \sin(\ln x) dx$$

Sol. *Put* $\ln x = t$ *to get* $I = \int \underset{I}{e^t} \underset{II}{\sin t} \, dt = e^t(-\cos t) - \int e^t(-\cos t) dt$

$$= -\cos t \cdot e^t + \int e^t \cos t \, dt = -\cos t \cdot e^t + \left[e^t \sin t - \int e^t \sin t \, dt \right]$$

$$\Rightarrow \quad I = -\cos t \cdot e^t + e^t \sin t - I \quad or \quad I = \frac{1}{2} e^t (\sin t - \cos t) + c$$

$$= \frac{1}{2} x (\sin (\ln x) - \cos (\ln x)) + c$$

Illustration :

$$\int x^2 e^{3x} dx$$

Sol. $I = \int \underset{I}{x^2} \underset{II}{e^{3x}} dx = \frac{x^2 e^{3x}}{3} - \int 2x \cdot \frac{e^{3x}}{3} dx$

$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int \underset{I}{x} \underset{II}{e^{3x}} dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{3}\left[x \cdot \frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} dx\right]$$

$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + c$$

Example : $I = \underset{I}{e^{ax}} \underset{II}{\cos(bx+c)} = e^{ax} \frac{\sin(bx+c)}{b} - \int a e^{ax} \frac{\sin(bx+c)}{b} dx$

$$= e^{ax} \frac{\sin(bx+c)}{b} - \frac{a}{b} \int \underset{I}{e^{ax}} \underset{II}{\sin(bx+c)} dx$$

$$\Rightarrow I = \frac{1}{b} e^{ax} \sin(b+c) + \frac{a}{b^2} e^{ax} \cos(bx+c) - \frac{a^2}{b^2} \int e^{ax} \cos(bx+c) dx$$

$$\text{or } I = \frac{1}{b} e^{ax} \sin(bx+c) + \frac{a}{b^2} e^{ax} \cos(bx+c) - \frac{a^2}{b^2} I$$

$$I = \frac{1}{a^2+b^2} e^{ax} [a \cos(bx+c) + b \sin(bx+c)]$$

Two Classic Integrands :

(a) $\int e^x (f(x) + f'(x)) dx = e^x f(x) + C$ **&** **(b)** $\int (f(x) + x f'(x)) dx = x\ f(x) + C$

Proof :

(a) $\int e^x (f(x) + f'(x)) dx = \int e^x f(x) dx + \int \underset{I}{e^x} \underset{II}{f'(x)} dx$

$$= \int e^x f(x) dx + e^x f(x) - \int e^x f(x) dx + c = e^x f(x) + c$$

(b) $\int f(x) dx + x f'(x) dx = \int f(x) dx + \int \underset{I}{x} \underset{II}{f'(x)} dx$

$$= \int f(x) dx + x f(x) - \int f(x) dx = x f(x) + c$$

Illustration :

$$\int \frac{x e^x}{(1+x)^2} dx$$

Sol. $\int \frac{x e^x}{(1+x)^2} dx = \int e^x \frac{1+x-1}{(1+x)^2} dx = \int e^x . \left\{\frac{1}{1+x} + \frac{-1}{(1+x)^2}\right\} = e^x \frac{1}{1+x} + c$

Illustration :

$$\int [\sin(\ln x) + \cos(\ln x)dx$$

Sol. *Put* $\ln x = t$ *to get* $\int e^t(\sin t + \cos t)dt = e^t .\sin t + c = x \sin(\ln x) + c$

Illustration :

$$\int \frac{e^x}{x}(1 + x \cdot \ln x)dx$$

Sol. $\int \frac{e^x}{x}(1 + x \cdot \ln x)dx = \int e^x\left(\frac{1}{x} + \ell n\, x\right)dx = e^x \ln x + c.$

Illustration :

$$\int \frac{x^2 e^x}{(x+2)^2}dx$$

Sol. $\int e^x \frac{x^2}{(x+2)^2}dx = \int e^x \frac{x^2 - 4 + 4}{(x+2)^2}dx = \int e^x\left\{\frac{x^2}{(x+2)^2} + \frac{4}{(x+2)^2}\right\}dx$

$$= \int e^x\left\{\frac{x-2}{x+2} + \frac{4}{(x+2)^2}\right\}dx = \int e^x\left(\frac{x-2}{x+2}\right) + c$$

Illustration :

$$\int (\sin x + x \cos x)dx$$

Sol. $\int (\sin x + x \cos x)dx = x \sin x + c$

Illustration :

$$\int (2\ln x + (\ln x)^2)dx$$

Sol. $\int (2\ln x + (\ln x)^2)dx = \int\left(x \cdot \frac{2\ln x}{x} + (\ln x)^2\right)dx = x \cdot (\ln n)^2 + c$

Illustration :

$$\int\left(\ell n(\ell nx) + \frac{1}{\ell n^2 x}\right)dx$$

Sol. *Put* $\ln x = t$ *to get* $\int e^t\left(\ln t + \frac{1}{t^2}\right)dt = \int e^t\left(\ln t - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right)$

$$= e^t\left(\ln t - \frac{1}{t}\right) + c = e^{\ln x}\left(\ln(\ln x) - \frac{1}{\ln x}\right) + c = x\left[\ln(\ln x) - \frac{1}{\ln x}\right] + c$$

Practice Problem

Q.1 $\int \tan^{-1} x\, dx = x\, g(x) - \frac{1}{2} ln(1+x^2) + c$. Find number of point of discontinuties of $g(x)$?

Q.2 $\int (ln\, x)^2 dx = Ax\,(ln\, x)^2 - Bx\, ln\, x + cx + D$. Find value of $A + B + C$?

Evaluate the following integrals :

Q.3 $\int \frac{ln\,(x^2+a^2)}{x^2} dx$ Q.4 $\int \frac{x}{1+\sin x} dx$ Q.5 $\int \frac{e^{\tan^{-1}x}(1+x+x^2)}{1+x^2} dx$

Q.6 $\int \frac{e^x(1+x+x^3)}{(1+x^2)^{3/2}} dx$

Answer key

Q.1 0 Q.2 5 Q.3 $\frac{-ln\,(x^2+a^2)}{x} + \frac{2}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Q.4 $-x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right) + 2\,ln\;\; ln\left|\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C$ Q.5 $x\,e^{\tan^{-1}x}$ Q.6 $\frac{e^x x}{\sqrt{x^2+1}}$

PARTIAL FRACTION :

This technique is used if a rational function is being integrated whose denomenator can be is factorised. If degree of numerator is greater then degree of denomenator then first devide numerator by denomenator

$$\boxed{\text{Loving Integrands } \int \frac{dx}{a^2-x^2} = \frac{1}{2a} ln\left(\frac{a+x}{a-x}\right) \;\&\; \int \frac{dx}{x^2-a^2} = \frac{1}{2a} ln\left(\frac{x-a}{x+a}\right)}$$

Illustration :

$$\int \frac{x^2+2}{(x+1)(x^2-1)} dx$$

Sol. *We have* $\frac{x^2+2}{(x+1)(x^2-1)} = \frac{x^2+2}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$

i.e. $x^2 + 2 = A(x^2-1) + B(x-1) + C(x+1)^2$

Comparing the coeff. s, we have

$$A + C = 1,\; B + 2C = 0 \;\text{ and }\; -A - B + C = 2 \quad \text{gives} \quad A = \frac{1}{4}, B = \frac{-3}{2} \text{ and } C = \frac{3}{4}$$

Hence, we have

$$I = \int \frac{x^2+2}{(x+1)(x^2-1)} dx = A\int \frac{dx}{x+1} + B\int \frac{dx}{(x+1)^2} + C\int \frac{dx}{x-1}$$

$$= \frac{1}{4} ln\,|x+1| + \frac{3}{2(x+1)} + \frac{3}{4} ln\,|x-1| + C$$

Illustration :

$$\int \frac{(1+x)^3}{(1-x)^3}dx$$

Sol. *We have* $\frac{(1+x)^3}{(1-x)^3} = \frac{x^3+3x^2+3x+1}{-x^3+3x^2-3x+1}+1-1 = \frac{6x^2+2}{-x^3+3x^2-3x+1}-1 = -1-\frac{6x^2+2}{(x-1)^3}$

Note : Before decomposing into partial fractions, we must ensure that the degree of the numerator is less than the degree of the denominator. Take special note of the method of performing division. Adding 1 to the given fraction cancels out the x^3 them in the numerator, thereby reducing the degree of the numerator.

Now we have $\frac{6x^2+2}{(x-1)^3} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}$

i.e. $6x^2+2 = A(x-1)^2+B(x-1)+C$

Comparing the coeff.s, we have

$A = 6, -2A+B = 0$ and $A-B+C = 2$ gives $A = 6, B = 12$ and $C = 8$

Hence, we have

$$I = \int \frac{(1+x)^3}{(1-x)^3}dx = -1\int dx - A\int \frac{dx}{x-1} - B\int \frac{dx}{(x-1)^2} - C\int \frac{dx}{(x-1)^3}$$

$$= -x - 6\ ln\,|x-1| + \frac{12}{x-1} + \frac{4}{(x-1)^2} + C.$$

Illustration :

$$\int \frac{1}{(x+1)(x^2+1)^2}dx$$

Sol. *We have* $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

i.e. $1 = A(x^2+1)^2 + (Bx+C)(x^3+x^2+x+1) + (Dx+E)(x+1)$

Comparing the coeff.s, we have

$A+B = 0,\ B+C = 0,\ 2A+B+C+D = 0$

$B+C+D+E = 0$ *and* $A+C+E = 1$

gives $A = \frac{1}{4}\ B = \frac{-1}{4}, C = \frac{1}{4}, D = \frac{-1}{2}$ *and* $E = \frac{1}{2}$

Hence, we have

$$I = \int \frac{1}{(x+1)(x^2+1)^2}dx = \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{x-1}{x^2+1}dx - \frac{1}{2}\int \frac{x-1}{(x^2+1)^2}dx$$

$$= \frac{1}{4}\ ln\,|x+1| - \frac{1}{8}\int \frac{2x}{x^2+1}dx + \frac{1}{4}\int \frac{dx}{x^2+1} - \frac{1}{4}\int \frac{2x}{(x^2+1)^2}dx + \frac{1}{2}\int \frac{dx}{(x^2+1)^2}$$

$$= \frac{1}{4}\ ln\,|x+1| - \frac{1}{8}\ ln\,(x^2+1) + \frac{1}{4}\tan^{-1}x + \frac{1}{4(x^2+1)} + \frac{1}{2}I_1$$

To evaluate I_1, put $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$. Thus, we have

$$I_1 = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] = \frac{1}{2}\left[\tan^{-1}x + \frac{x}{x^2+1}\right]$$

Hence, we have

$$I = \frac{1}{4}\ln|x+1| - \frac{1}{8}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{x+1}{4(x^2+1)} + C$$

Decomposition of Fractions Involving Even Powers of x only :

Illustration :

$$\int \frac{x^2+1}{(x^2+2)(x^2+3)}\,dx$$

Sol. *Before decomposing such fractions into partial fractions, it is more convenient to write them as*

$$\frac{y+1}{(y+2)(y+3)} = \frac{A}{y+2} + \frac{B}{y+3} \qquad \text{[writing } x^2 = y]$$

i.e. $y + 1 = A(y+3) + B(y+2)$

Comparing the coeff.s, we have

$A + B = 1$ and $3A + 2B = 1$ gives $A = -1$ and $B = 2$

Thus, we have

$$f(x) = \frac{x^2+1}{(x^2+2)(x^2+3)} \quad \Rightarrow \quad \int \frac{x^2+1}{(x^2+2)(x^2+3)}\,dx$$

$$= \int \frac{-1}{x^2+2}\,dx + \int \frac{2}{x^2+3}\,dx = \frac{-1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

Substitution after manipulation (Kuturputur) :

Illustration :

$$\int \frac{dx}{x(x^n+1)}$$

Sol. $$\int \frac{dx}{x(x^n+1)} = \int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)}$$

Put $1 + \frac{1}{x^n} = t$ *to get* $\int \frac{-\frac{1}{n}dt}{t} = -\frac{1}{n}\ln t + c = -\frac{1}{n}\ln(1+x^{-n}) + c$

Illustration :

$$\int \frac{x^7}{(1-x^2)^5}dx$$

Sol. $\displaystyle\int \frac{x^7 dx}{x^{10}\left(\frac{1}{x^2}-1\right)^5} = \int \frac{dx}{x^3\left(\frac{1}{x^2}-1\right)^5}$

Put $\displaystyle\frac{1}{x^2}-1 = t$ *to get* $\displaystyle\int \frac{-\frac{1}{2}dt}{t^5}dt = \frac{-1}{2}\cdot\frac{-1}{4}t^4 + c = \frac{1}{8}\frac{1}{(x^{-2}-1)^4}+c$

Illustration :

$$\int \frac{xdx}{(1-x^4)^{3/2}}$$

Sol. $\displaystyle I = \int \frac{x\,dx}{x^6\left(\frac{1}{x^4}-1\right)^{3/2}} = \int \frac{dx}{x^5\left(\frac{1}{x^4}-1\right)^{3/2}}$ *Put* $\displaystyle\frac{1}{x^4}-1 = t \Rightarrow -4\frac{1}{x^5}dx = dt$

$$\Rightarrow I = \int \frac{-\frac{1}{4}dt}{t^{3/2}} = \frac{1}{2}\frac{1}{\sqrt{t}}+c = \frac{1}{2}\frac{1}{\sqrt{\frac{1}{x^4}-1}}$$

Practice Problem

Q.1 $\displaystyle\int \frac{x+1}{x(1+xe^x)}dx = ln|f(x)|+c.$ Find value of $f(ln\,2)$?

Q.2 $\displaystyle\int \frac{x}{(x-1)(x^2+4)}dx = \frac{1}{5}\log(x-1)+\frac{1}{5}\int\frac{\lambda x+\mu}{x^2+4}dx$ Find value of $\lambda+\mu$.

Evaluate the following indefinite integrals :

Q.3 $\displaystyle\int \frac{dx}{x^4(x^3+1)^2}$

Q.4 $\displaystyle\int \frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}dx$

Answer key

Q.1 $\log_{(4e)}4$

Q.2 3

Q.3 $\displaystyle-\frac{1}{3}\left(t-\frac{1}{t}-2\log t\right)+c$ where $\displaystyle t=\frac{1}{x^3}+1$

Q.4 $\displaystyle\frac{3}{2}ln\left(x^2-2x+5\right)+\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)-ln|x-1|+C$

INTEGRALS OF TRIGONOMETRIC FUNCTIONS :

Type – 1 : $\int \frac{dx}{a+b\sin^2 x} \Big/ \int \frac{dx}{a+b\cos^2 x} \Big/ \int \frac{dx}{a\sin^2 x+b\cos^2 x+c\sin x\cos x} \Big/ \frac{dx}{(a\cos x+b\sin x)^2}$

Multiply N^r and D^r by $\sec^2 x$ or $\operatorname{cosec}^2 x$ and proceed

Type – 2 : $\int \frac{dx}{a+b\sin x} \Big/ \int \frac{dx}{a+b\cos x} \Big/ \int \frac{dx}{a+b\sin x+c\cos x}$

Convert sin x and cos x into their corresponding tangent to half the angles and

put $\tan\frac{x}{2}=t$

Type–3 : $\int \frac{a\sin x+b\cos x+c}{\ell\sin x+m\cos x+n}dx$; $\quad N^r = A(D^r)+B\left(\frac{d}{dx}D^r\right)+C$

Type–4 : $\int \frac{x^2+1}{x^4+kx^2+1}dx$ **or** $\int \frac{x^2-1}{x^4+kx^2+1}dx$

Divide N^r and D^r by x^2 and take suitable substituion

Illustration :

$$\int \frac{dx}{4-5\sin^2 x}$$

***Sol.** $\int \frac{dx}{4-5\sin^2 x} = \int \frac{\operatorname{cosec}^2 x}{4\operatorname{cosec}^2 x-5}dx = \int \frac{\operatorname{cosec}^2 x}{4\cot^2 x-1}dx$

Put $\cot x = t$ *to get* $\int \frac{-dt}{4t^2-1} = \frac{-1}{4}\int \frac{dt}{t^2-\left(\frac{1}{2}\right)^2} = \frac{-1}{4}\ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)+c = -\frac{1}{4}\ln\left(\frac{2\cot x-1}{2\cot x+1}\right)+c$

Illustration :

$$\int \frac{dx}{5+4\cos x}$$

Sol. $\int \frac{dx}{5+4\cos x} = \int \frac{dx}{5+4\frac{\left(1-\tan^2\frac{x}{2}\right)}{\left(1+\tan^2\frac{x}{2}\right)}} = \int \frac{\sec^2\frac{x}{2}dx}{9+\tan^2\frac{x}{2}}$

Put $\tan\frac{x}{2}=t$ *to get* $\int \frac{2dt}{9+t^2} = \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+c$

Illustration :

$$\int \frac{1}{\sin x - 3\cos x - 1} dx$$

Sol. Let $I = \int \frac{1}{\sin x - 3\cos x - 1} dx$

Putting $\tan\frac{x}{2} = t$, we have $I = \int \frac{1}{\frac{2t}{1+t^2} - \frac{3(1-t^2)}{1+t^2}} \cdot \frac{2dt}{1+t^2} = \int \frac{2dt}{2t - 3(1-t^2) - (1+t^2)}$

$$= \int \frac{2dt}{2t^2 + 2t - 4} = \int \frac{dt}{t^2 + t - 2} = \int \frac{dt}{(t-1)(t+2)} = \frac{1}{3}\int \left[\frac{1}{t-1} - \frac{1}{t+2}\right] dt = \frac{1}{3} \ln \left|\frac{t-1}{t+2}\right| + C$$

$$= \frac{1}{3} \ln \left|\frac{\tan\frac{x}{2} - 1}{\tan\frac{x}{2} + 2}\right| + C$$

Illustration :

Evaluate the indefinite integral $\int \frac{\sin x + 3\cos x + 1}{\sin x - 3\cos x - 1} dx$.

Sol. Let $\sin x + 3\cos x + 1 = \lambda(\sin x - 3\cos x - 1) + \mu(\cos x + 3\sin x) + \nu$

Comparing the coefficient of $\sin x$, $\cos x$ and constant term we have

$$\lambda + 3\mu = 1, \; -3\lambda + \mu = 3 \text{ and } -\lambda + \nu = 1 \quad \text{kgives } \lambda = -\frac{4}{5}, \; \mu = \frac{3}{5} \text{ and } \nu = \frac{1}{5}$$

Thus, we have

$$I = \frac{-4}{5}\int 1\,dx + \frac{3}{5}\int \frac{\cos x + 3\sin x}{\sin x - 3\cos x - 1} dx + \frac{1}{5}\int \frac{dx}{\sin x - 3\cos x - 1}$$

$$= \frac{-4}{5}x + \frac{3}{5} \ln|\sin x - 3\cos x - 1| + \frac{1}{5} I_1$$

Now, we have

$$I_1 = \int \frac{1}{\frac{2t}{1+t^2} - \frac{3(1-t^2)}{1+t^2} - 1} \cdot \frac{2dt}{1+t^2} \qquad \left[\text{Putting } \tan\frac{x}{2} = t\right]$$

$$= \int \frac{2dt}{2t - 3(1-t^2) - (1+t^2)} = \int \frac{2dt}{2t^2 + 2t - 4} = \int \frac{dt}{(t-1)(t+2)} = \frac{1}{3}\int \left(\frac{1}{t-1} - \frac{1}{t+2}\right) dt$$

$$= \frac{1}{3} \ln \left|\frac{t-1}{t+2}\right| = \frac{1}{3} \ln \left|\frac{\tan(x/2) - 1}{\tan(x/2) + 2}\right|$$

Hence, we have

$$I = \frac{-4}{5}x + \frac{3}{5} \ln|\sin x - 3\cos x - 1| + \frac{1}{15} \ln \left|\frac{\tan(x/2) - 1}{\tan(x/2) + 2}\right| + C$$

Illustration :

$$\int \frac{dx}{(3\sin x - 4\cos x)^2}$$

Sol. $I = \int \frac{dx}{(3\sin x - 4\cos x)^2} = \int \frac{\sec^2 x\, dx}{(3\tan x - 4)^2}$ Put $3\tan x - 4 = t \Rightarrow 3\sec^2 x\, dx = dt$

$$\Rightarrow I = \int \frac{\frac{1}{3}dt}{t^2} = -\frac{1}{3}\frac{1}{(3\tan x - 4)} + c$$

Illustration :

$$\int \frac{x^2+1}{x^4+7x^2+1}dx$$

Sol. $\int \frac{x^2+1}{x^4+7x^2+1}dx = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+7+\frac{1}{x^2}}$ Put $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$ & $x^2 + \frac{1}{x^2} = t^2 + 2$

To get $\int \frac{dt}{t^2+9} = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + c = \frac{1}{3}\tan^{-1}\left(\frac{x^2-1}{3x}\right) + c$

Illustration :

$$\int \frac{x^2+1}{x^4+x^2+1}dx$$

Sol. $I = \int \frac{x^2+1}{x^4+x^2+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3}dx$

$= \int \frac{dt}{t^2+3}$ $\left[\text{Putting } x - \frac{1}{x} = t \text{ and } \left(1+\frac{1}{x^2}\right)dx = dt\right]$

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + C$

Illustration :

$$\int \frac{x^2+2}{x^4-5x^2+4}dx$$

Sol. $I = \int \frac{x^2+2}{x^4-5x^2+4}dx = \int \frac{1+\frac{2}{x^2}}{x^2+\frac{4}{x^2}-5}dx = \int \frac{1+\frac{2}{x^2}}{\left(x-\frac{2}{x}\right)-1}dx$

$= \int \frac{dt}{t^2-1}$ $\left[\text{Putting } x - \frac{2}{x} = t \text{ and } \left(1+\frac{2}{x^2}\right)dx = dt\right]$

$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\ln\left|\frac{x^2-x-2}{x^2+x-2}\right| + C$

INTEGRATION OF IRRATIONAL ALGEBRAIC FUNCTION :

Type–1 : $\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(\beta-x)}}$ $(\beta > \alpha)$ (Start: $x = \alpha\cos^2\theta + \beta\sin^2\theta$)

Type – 2 : $\int \frac{dx}{(ax+b)\sqrt{px+q}}$; e.g. $\int \frac{dx}{(2x+1)\cdot\sqrt{4x+3}}$

Put $px + q = t^2$

Type – 3 : $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$; e.g. $\int \frac{dx}{(x+1)\cdot\sqrt{1+x-x^2}}$

Put $ax + b = \frac{1}{t}$

Type – 4 : $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; Put $px + q = t^2$

e.g. $\int \frac{dx}{\left(x^2+5x+2\right)\sqrt{x-2}}$ this reduces to $2\int \frac{dt}{t^4+9t^2+16}$

Type – 5 : $\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$

Case-I: When $(ax^2 + bx + c)$ breaks up into two linear factors, e.g.

$$I = \int \frac{dx}{(x^2-x-2)\sqrt{x^2+x+1}} \text{ then}$$

$$= \int\left(\frac{A}{x-2}+\frac{B}{x+1}\right)\frac{1}{\sqrt{x^2+x+1}}\,dx = A\int \underbrace{\frac{dx}{(x-2)\sqrt{x^2+x+1}}}_{\text{put } x-2=1/t} + B\int \underbrace{\frac{dx}{(x+1)\sqrt{x^2+x+1}}}_{\text{put } x+1=1/t}$$

Case-II: If $ax^2 + bx + c$ is a perfect square say $(lx + m)^2$ then put $lx + m = 1/t$

Case-III: If $b = 0$; $q = 0$ e.g. $\int \frac{dx}{(ax^2+b)\sqrt{px^2+r}}$ then put $x = \frac{1}{t}$ or the trigonometric substitution are also helpful.

e.g. $\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$.

Illustration :

$$\int \frac{dx}{(x+2)\sqrt{x+1}}$$

Sol. $I = \int \frac{dx}{(x+2)\sqrt{x+1}}$ Put $x + 1 = t^2 \Rightarrow I = \int \frac{2t\,dt}{(t^2+1)t} = 2\tan^{-1} + c = 2\tan^{-1}\left(\sqrt{x+1}\right) + c$

Illustration :

$$\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$$

Sol. $I = \int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$ Put $x = \frac{1}{t}$ to get $I = \int \frac{-\frac{1}{t^2}dt}{\left(\frac{1}{t^2}+4\right)\sqrt{\frac{4}{t^2}+1}} = \int \frac{-t\,dt}{(1+4t^2)\sqrt{4+t^2}}$

Again put $4 + t^2 = z^2 \Rightarrow tdt = zdz \Rightarrow I = \int \frac{-zdz}{\{1+4(z^2-4)\}z}$

$$= \int \frac{-dz}{4z^2-15} = \frac{1}{4}\int \frac{dz}{z^2-15/4} = -\frac{1}{4}\cdot\frac{1}{\sqrt{15}}\ln\left|\frac{2z-\sqrt{15}}{2z+\sqrt{15}}\right| + c$$

Illustration :

$$\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$$

Sol. $I = \int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$ Put $x + 1 = \frac{1}{t} \Rightarrow I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t}-\left(\frac{1}{t}-1\right)^2}} = \int \frac{-dt}{\sqrt{3t-1-t^2}}$

$$= \int \frac{-dt}{\sqrt{\frac{5}{4}-\left(t-\frac{3}{2}\right)^2}} = -\sin^{-1}\left(\frac{2t-3}{\sqrt{5}}\right) + c = -\sin^{-1}\left(\frac{\frac{2}{x+1}-3}{\sqrt{5}}\right) + c$$

$$= \sin^{-1}\left(\frac{3x+1}{\sqrt{5}(x+1)}\right) + c$$

Practice Problem

Q.1 $\int \frac{dx}{5\sin^2 x+4} = \frac{1}{6}\tan^{-1} f(x)+c$. Find value of $f'(x)$ at $x = \frac{\pi}{4}$.

Q.2 $\int \frac{x^2-3}{x^4+2x^2+9}\,dx = \frac{1}{4}\ln|f(x)|+c$. Find $f(0)$.

Evaluate the following indefinite integrals :

Q.3 $\int \frac{1}{(\cos x+2\sin x)^2}\,dx$

Q.4 $\int \frac{x^x(x^{2x}+1)(\ln x+1)}{x^{4x}+1}\,dx$

Q.5 $\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$

Q.6 $\int \sqrt{\frac{5-x}{x-2}}\,dx$

Q.7 $\int \sqrt{2x^2-x+1}\,dx$

Answer key

Q.1 3

Q.2 1

Q.3 $\frac{-1}{2(1+2\tan x)}+C$

Q.4 $\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{x^x-\frac{1}{x^x}}{\sqrt{2}}\right)\right]$

Q.5 $c-\tan^{-1}(\cot^2 x)$

Q.6 $\sqrt{(x-2)(5-x)}+3\sin^{-1}\sqrt{\frac{x-2}{3}}$

Q.7 $\frac{1}{2}\left(x-\frac{1}{4}\right)\sqrt{2x^2-x+1}+\frac{7}{16\sqrt{2}}\ln\left|x-\frac{1}{4}+\frac{\sqrt{2x^2-x+1}}{\sqrt{2}}\right|+C$

STANDARD RESULTS (Must be memorised):

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}+c \quad n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b)+c$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}+c$

(iv) $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ell n a}\ (a>0)+c$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$

(vi) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\ln\sec(ax+b)+c$

(viii) $\int \cot(ax+b)\,dx = \frac{1}{a}\ln\sin(ax+b)+c$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+c$

(x) $\int \mathrm{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)+c$

(xi) $\int \sec(ax+b) \cdot \tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$

(xii) $\int \operatorname{cosec}(ax+b) \cdot \cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$

(xiii) $\int \sec x\,dx = \ln(\sec x + \tan x) + c$ **OR** $\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(xiv) $\int \operatorname{cosec} x\,dx = \ln(\operatorname{cosec} x - \cot x) + c$ **OR** $\ln\tan\frac{x}{2} + c$ **OR** $-\ln(\operatorname{cosec} x + \cot x)$

(xv) $\int \sinh x\,dx = \cosh x + c$

(xvi) $\int \cosh x\,dx = \sinh x + c$

(xvii) $\int \operatorname{sech}^2 x\,dx = \tanh x + c$

(xviii) $\int \operatorname{cosech}^2 x\,dx = -\coth x + c$

(xix) $\int \operatorname{sech} x \,.\, \tanh x\,dx = -\operatorname{sech} x + c$

(xx) $\int \operatorname{cosech} x \,.\, \coth x\,dx = -\operatorname{cosech} x + c$

(xxi) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

(xxii) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xxiii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xxiv) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left[x + \sqrt{x^2+a^2}\right]$ **OR** $\sinh^{-1}\frac{x}{a} + c$

(xxv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left[x + \sqrt{x^2-a^2}\right]$ **OR** $\cosh^{-1}\frac{x}{a} + c$

(xxvi) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x} + c$

(xxvii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} + c$

(xxviii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxix) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$

(xxx) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c$

(xxxi) $\int e^{ax} \,.\, \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxxii) $\int e^{ax} \,.\, \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

Solved Examples

Q.1 Evaluate the following indefinte integrals :

(i) $\int\left(\sqrt{x}+\frac{2}{x}-\sin x\right)dx$ (ii) $\int\left(\frac{1}{2x+3}+\sin(2x+3)\right)dx$

Sol.(i) $I=\int\left(\sqrt{x}+\frac{2}{x}-\sin x\right)dx=\int\sqrt{x}\,dx+2\int\frac{1}{x}dx-\int\sin x\,dx$

$=\frac{x^{1/2+1}}{1/2+1}+2\,ln\,|x|-(-\cos x)+C=\frac{2}{3}x^{3/2}+2\,ln\,|x|+\cos x+C.$

(ii) $I=\int\left(\frac{1}{2x+3}+\sin(2x+3)\right)dx=\int\frac{dx}{2x+3}+\int\sin(2x+3)dx$

$=\frac{ln\,|2x+3|}{2}-\frac{\cos(2x+3)}{2}+C$

Q.2 Evaluate the following indefinite integrals :

(i) $\int\frac{\sqrt{x}}{x+1}dx$ (ii) $\int\frac{dx}{e^x+e^{-x}}$

Sol.(i) $\int\frac{\sqrt{x}}{x+1}dx=\int\frac{t}{t^2+1}\cdot 2t\,dt$ $\left[\text{Putting } \sqrt{x}=t \text{ and } dx=2t\,dt\right]$

$=\int\frac{2t^2}{t^2+1}dt=2\int\frac{t^2+1-1}{t^2+1}dt=2\int dt-2\int\frac{dt}{t^2+1}=2t-2\tan^{-1}t+C=2\sqrt{x}-2\tan^{-1}\sqrt{x}+C.$

(ii) $\int\frac{dx}{e^x+e^{-x}}=\int\frac{1}{t+1/t}\cdot\frac{dt}{t}$ $\left[\text{Putting } e^x=t \text{ and } dx=\frac{dt}{t}\right]$

$=\int\frac{dt}{t^2+1}=\tan^{-1}t+C=\tan^{-1}(e^x)+C.$

Q.3 Evaluate the following integrals :

(i) $\int x^2\sin x\,dx$ (ii) $\int(x^2+5x)e^{2x}dx$

Sol.(i) We have

$I=\int x^2\sin x\,dx=x^2(-\cos x)-\int(-\cos x)2x\,dx$ [integrating by parts]

$=-x^2\cos x+2\int x\cos x\,dx$

Using parts again, we have $\int x\cos x\,dx=x\sin x-\int\sin x\,dx=x\sin x+\cos x$

Hence, we have $I=-x^2\cos x+2x\sin x+2\cos x+C.$

(ii) $$I=\int (x^2+5x)e^{2x}\,dx=(x^2+5x)\frac{e^{2x}}{2}-\int\frac{e^{2x}}{2}(2x+5)\,dx$$

$$=\frac{(x^2+5x)e^{2x}}{2}-(2x+5)\frac{e^{2x}}{4}+\int\frac{e^{2x}}{4}2\,dx$$

$$=\frac{(x^2+5x)e^{2x}}{2}-\frac{(2x+5)e^{2x}}{4}+\frac{e^{2x}}{4}+C$$

Q.4 Evaluate the following indefinte integrals :

(i) $\int\frac{(x-1)^3}{\sqrt{x}}dx$ (ii) $\int\frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}dx$

Sol.(i) $$\int\frac{(x-1)^3}{\sqrt{x}}dx=\int\frac{x^3-3x^2+3x-1}{\sqrt{x}}dx=\int (x^{5/2}-3x^{3/2}+3x^{1/2}-x^{-1/2})dx$$

$$=\frac{x^{7/2}}{7/2}-\frac{3x^{5/2}}{5/2}+\frac{3x^{3/2}}{3/2}-\frac{x^{1/2}}{1/2}+C=\frac{2}{7}x^{7/2}-\frac{6}{5}x^{5/2}+2x^{3/2}-2x^{1/2}+c$$

(ii) $$\int\frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}dx=\int \tan x\sec x\,dx+\int\cot x\csc x\,dx=\sec x-\csc x+C.$$

Q.5 Evaluate the following indefinite integrals :

(i) $\int \sin x\sin 2x\sin 3x\,dx$ (ii) $\int \sin^3 x\cos 3x\,dx$

Sol.(i) $$I=\int \sin x\sin 2x\sin 3x\,dx=\int\left(\frac{\cos x-\cos 3x}{2}\right)\sin 3x\,dx$$

$$=\frac{1}{2}\int(\cos x\sin 3x-\sin 3x\cos 3x)\,dx=\frac{1}{4}\int(\sin 4x+\sin 2x-\sin 6x)\,dx$$

$$=\frac{1}{4}\left(\frac{-\cos 4x}{4}-\frac{\cos 2x}{2}+\frac{\cos 6x}{6}\right)+C$$

(ii) $$I=\int \sin^3 x\cos 3x\,dx=\int\left(\frac{3\sin x-\sin 3x}{4}\right)\cos 3x\,dx$$ $[\sin 3x=3\sin x-4\sin^3 x]$

$$=\frac{3}{4}\int\sin x\cos 3x\,dx-\frac{1}{4}\int\sin 3x\cos 3x\,dx$$

$$=\frac{3}{8}\int(\sin 4x-\sin 2x)\,dx-\frac{1}{8}\int\sin 6x\,dx$$

$$=\frac{3}{8}\left(\frac{-\cos 4x}{4}\right)+\frac{3}{8}\left(\frac{\cos 2x}{2}\right)+\frac{1}{8}\left(\frac{\cos 6x}{6}\right)+C$$

Q.6 Evaluate the following indefinite integral $\int \frac{\sin 2x}{a^2+b^2\sin^2 x}dx$.

Sol. $I=\int \frac{\sin 2x}{a^2+b^2\sin^2 x}dx$ Put $\sin^2 x = t \Rightarrow \sin 2x\, dx = dt$

$$\Rightarrow I=\int \frac{dt}{a^2+b^2t^2}=\frac{1}{b^2}\int \frac{dt}{t^2+(a/b)^2}=\frac{1}{b^2}\left(\frac{1}{a/b}\right)\tan^{-1}\left(\frac{t}{a/b}\right)+c$$

$$=\frac{1}{ab}\tan^{-1}\left(\frac{b\sin^2 x}{a}\right)+c$$

Q.7 Evaluate the indefinite integral $\int x^2 \cos x\, dx$.

Sol. $I=\int x^2\cos x\,dx = x^2\sin x - \int \sin x\cdot 2x\,dx = x^2\sin x - 2I_1$

where $I_1=\int x\sin dx = x(-\cos x)-\int(-\cos x)\cdot 1\,dx = -x\cos x+\sin x$

Hence, we have $I = x^2\sin x + 2x\cos x - 2\sin x + C$

Q.8 Evaluate the following indefinite integrals :

(i) $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ (ii) $\int x(1+x^2)e^{x^2}dx$

Sol.(i) $I=\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx=\int e^x\frac{1+x^2-2x}{(1+x^2)}dx$

$$=\int e^x\left[\frac{1}{1+x^2}+\frac{-2x}{(1+x^2)^2}\right]dx=\frac{e^x}{1+x^2}+C$$

(ii) $I=\int x(1+x^2)e^{x^2}dx=\frac{1}{2}\int(t+1)e^t\,dt$ [Putting $x^2=t$ and $2x\,dx=dt$]

$$=\frac{1}{2}te^t+C=\frac{1}{2}x^2e^{x^2}+C$$

Q.9 Evaluate the indefinite integral $\int \frac{1}{2+\sin 2x+\cos 2x}$.

Sol. Let $I=\int \frac{1}{2+\sin 2x+\cos 2x}dx$

Putting $\tan x = t$, we have

$$I=\int \frac{1}{2+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{dt}{1+t^2}=\int\frac{1}{t^2+2t+3}dt$$

$$=\int\frac{1}{(t+1)^2+2}dt=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right)+C=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1+\tan x}{\sqrt{2}}\right)+C$$

Q.10 Evaluate the indefinite integral $\int \frac{dx}{(x+1)\sqrt{2x-3}}$

Sol. Let $I = \frac{dx}{(x+1)\sqrt{2x-3}}$

Putting $2x-3=t^2$, i.e. $x = \frac{t^2+3}{2}$ and $dx = t\,dt$, we have

$$I = \int \frac{tdt}{\left(\frac{t^2+3}{2}+1\right)t} = \int \frac{2dt}{t^2+5} = \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + C = \frac{2}{\sqrt{5}}\tan^{-1}\sqrt{\frac{2x-3}{5}} + C$$

Q.11 Evaluate the indefinite integral $\int \frac{\cos^4 x\,dx}{\sin^3 x\left(\sin^5 x+\cos^5 x\right)^{3/5}}$

Sol. Let $I = \int \frac{\cos^4 x\,dx}{\sin^3 x\left(\sin^5 x+\cos^5 x\right)^{3/5}} = \int \frac{\cot^4 x\,\text{cosec}^2 x\,dx}{\left(1+\cot^5 x\right)^{3/5}}$

Putting $1+\cot^5 x = t$ and $-5\cot^4 x\,\text{cosec}^2 x\,dx = dt$, we have

$$I = \int \frac{-dt}{5t^{3/5}} = \frac{-1}{2}t^{2/5} + C = \frac{-1}{2}\left(1+\cot^5 x\right)^{2/5} + C.$$

Q.12 Evaluate the indefinite integral $\int \cos 2x \, ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)dx$.

Sol. Let $I = \int \cos 2x \, ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)dx$

Integrating by parts, we have

$$I = \frac{\sin 2x}{2}ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right) - \int \frac{\sin 2x}{2}\cdot\frac{\cos x-\sin x}{\cos x+\sin x}\cdot\frac{(\cos x-\sin x)^2+(\cos x+\sin x)^2}{(\cos x-\sin x)^2}dx$$

$$= \frac{\sin 2x}{2}ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right) - \int \frac{\sin 2x}{\cos 2x}dx = \frac{\sin 2x}{2}ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right) + \frac{1}{2}ln\left|\cos 2x\right| + C.$$

Q.13 Evaluate the indefinite integral $\int \frac{\sec x\,dx}{\sqrt{\sin(2x+a)+\sin a}}$.

Sol. Let $I = \int \frac{\sec x\,dx}{\sqrt{\sin(2x+a)+\sin a}} = \int \frac{\sec x\,dx}{\sqrt{\sin 2x\cos a+(1+\cos 2x)\sin a}}$

$$= \int \frac{\sec x\,dx}{\sqrt{2\cos x(\sin x\cos a+\cos x\sin a)}} = \int \frac{\sec x\,dx}{\sqrt{2\cos^2 x\cos a\,(\tan x+\tan a)}}$$

$$= \frac{1}{\sqrt{2\cos a}}\int \frac{\sec^2 x\,dx}{\sqrt{\tan x+\tan a}}$$

Putting $\tan x + \tan a = t$ and $\sec^2 x\,dx = dt$, we have

$$I = \frac{1}{\sqrt{2\cos a}}\int\frac{dt}{\sqrt{t}} = \frac{1}{\sqrt{2\cos a}}\cdot 2\sqrt{t} + C = \sqrt{\frac{2}{\cos a}}\,\sqrt{\tan x + \tan a} + C.$$

Q.14 Evaluate the integral $\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Sol. Let $I = \int\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2}\frac{dx}{x} = \int\left(\frac{1-\cos 2\theta}{1+\cos 2\theta}\right)^{1/2}\frac{d\left(\cos^2 2\theta\right)}{\cos^2 2\theta}$

Putting $x = \cos^2 2\theta$ and $dx = 2\cos 2\theta\,(-2\sin 2\theta)\,d\theta$, we have

$$I = \int\frac{\sin\theta}{\cos\theta}\cdot\frac{-8\sin\theta\cos\theta\,d\theta}{\cos 2\theta} = -8\int\frac{\sin^2\theta}{\cos 2\theta}\,d\theta$$

$$= 4\int\left(\frac{\cos 2\theta - 1}{\cos 2\theta}\right)d\theta = 4\int(1-\sec 2\theta)\,d\theta = 4\theta - 2\ln\left|\sec 2\theta + \tan 2\theta\right| + C$$

$$= 2\cos^{-1}\sqrt{x} - 2\,ln\left|\frac{1}{\sqrt{x}} + \sqrt{\frac{1}{x} - 1}\right| + C = 2\cos^{-1}\sqrt{x} - 2\,ln\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right| + C.$$

Q.15 Evaluate the integral $\int\frac{dx}{x^{11}\sqrt{1+x^4}}$.

Sol. Let $I = \int\frac{dx}{x^{11}\sqrt{1+x^4}} = \int\frac{x\,dx}{x^{12}\sqrt{1+x^4}}$

Putting $x^2 = \tan\theta$ and $2x\,dx = \sec^2\theta\,d\theta$, we have

$$I = \frac{1}{2}\int\frac{\sec^2\theta\,d\theta}{\tan^6\theta\sqrt{1+\tan^2\theta}} = \frac{1}{2}\int\frac{\sec^2\theta\,d\theta}{\tan^6\theta\sec\theta} = \frac{1}{2}\int\frac{\cos^5\theta}{\sin^6\theta}d\theta = \frac{1}{2}\int\frac{\cos^4\theta}{\sin^6\theta}.\cos\theta\,d\theta$$

Put $\sin\theta = t$

$$= \frac{1}{2}\int\frac{(1-t^2)^2}{t^6}dt = \frac{1}{2}\int\frac{t^2 - 2t^2 + 1}{t^6}dt = \frac{1}{2}\int(t^{-2} - 2t^{-4} + t^{-6})dt$$

$$= \frac{-1}{2t} + \frac{1}{3t^3} - \frac{1}{10t^5} + C = \frac{-1}{2\sin\theta} + \frac{1}{3\sin^3\theta} - \frac{1}{10\sin^5\theta} + C$$

Here $\sin\theta = \frac{x^2}{\sqrt{1+x^4}}$.

Q.16 Evaluate the integral $\int \cos^{-1}\left(x+\sqrt{x^2+2}\right)dx$.

Sol. Let $I = \int \cos^{-1}\left(x+\sqrt{x^2+2}\right)dx$

Put $\quad x+\sqrt{x^2+2} = \cos t \quad$ and $\quad \left(1+\frac{x}{\sqrt{x^2+2}}\right)dx = \sin t\,dt$

i.e. $\quad dx = -\frac{\sqrt{x^2+2}}{x+\sqrt{x^2+2}}.\sin t\,dt = \frac{\sqrt{x^2+2}}{\cos t}\cdot \sin t\,dt$

Now, rationalising LHS of equation (1), we have $\quad \sqrt{x^2+2} - = \frac{2}{\cos t}$

Adding equations (1) and (2), we have $\quad \sqrt{x^2+2} = \frac{1}{2}\left(\cos t + \frac{2}{\cos t}\right)$

Thus, we have $\quad dx = \frac{-1}{2}\left(\cos t + \frac{2}{\cos t}\right)\frac{\sin t}{\cos t}dt$

Hence, we have

$$I = \frac{-1}{2}\int t\left(\cos t + \frac{2}{\cos t}\right)\frac{\sin t}{\cos t}dt = \frac{-1}{2}\int t\sin t\,dt + \int t\left(\frac{-\sin t}{\cos^2 t}\right)dt$$

$$= \frac{-1}{2}\left[t(-\cos t) + \int \cos t\,dt\right] + \left[t\left(\frac{-1}{\cos t}\right) - \int \frac{-1}{\cos t}dt\right]$$

$$= \frac{1}{2}t\cos t - \frac{1}{2}\sin t - \frac{t}{\cos t} + \ln|\sec t + \tan t| + C \quad \text{where} \quad \cos t = x+\sqrt{x^2+2}.$$

Q.17 Evaluate the integral $\int\left[\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}} + \frac{\ln\left(1+\sqrt[6]{x}\right)}{\sqrt[3]{x}+\sqrt{x}}\right]dx$

Sol. $I = \int\left[\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}} + \frac{\ln\left(1+\sqrt[6]{x}\right)}{\sqrt[3]{x}+\sqrt{x}}\right]dx$

The common denominator of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{6}$ is 12.

Putting $x = t^{12}$ and $dx = 12t^{11}\,dt$, we have

$$I = \int\left[\frac{1}{t^4+t^3} + \frac{\ln(1+t^2)}{t^4+t^6}\right]12t^{11}dt = 12\left[\int\frac{t^8}{t+1}dt + \int\frac{t^7\ln(1+t^2)}{1+t^2}\right] = 12\,(I_1+I_2)$$

Now, we have

$$I_1 = \int\frac{t^8-1+1}{t+1}dt = \int(t-1)\left(t^2+1\right)\left(t^4+1\right)dt + \int\frac{dt}{t+1}$$

$$= \int(t-1)\left(t^6+t^4+t^2+1\right)dt + \ln|t+1| = \int\left(t^7-t^6+t^5-t^4+t^3+t-1\right)dt + \ln|t+1|$$

$$= \frac{t^8}{8} - \frac{t^7}{7} + \frac{t^6}{6} + \frac{t^5}{5} + \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - t + \ln|t+1| \quad \text{and} \quad I_2 = \int(t^2)^3\ln\left(1+t^2\right)\frac{t\,dt}{1+t^2}$$

Putting $\ln(1+t^2) = u$ and $\frac{2t\,dt}{1+t^2} = du$, we have

$$I_2 = \frac{1}{2}\int \left(e^u - 1\right)^3 u\,du = \frac{1}{2}\int \left(e^{3u} - 3e^{2u} + 3e^u - 1\right) u\,du$$

$$= \frac{1}{2}\left(\frac{e^{3u}}{3} - \frac{3e^{2u}}{2} + 3e^u - u\right)u - \frac{1}{2}\int\left(\frac{e^{3u}}{3} - \frac{3e^{2u}}{2} + 3e^u - u\right)du \qquad \text{[integrating by parts]}$$

$$= \frac{u}{2}\left(\frac{e^{3u}}{3} - \frac{3e^{2u}}{2} + 3e^u - u\right) - \frac{1}{2}\left(\frac{e^{3u}}{9} - \frac{3e^{2u}}{4} + 3e^u - \frac{u^2}{2}\right)$$

$$= \frac{u}{12}\left(2e^{3u} - 9e^{2u} + 18e^u + 18e^u - 6u\right) - \frac{1}{36}\left(4e^{3u} - 27e^{2u} + 108e^u - 18u^2\right)$$

where $e^u = t^2 - 1 = x^{1/6} - 1$.

Q.18 Evaluate the following indefinite integrals $\int\sqrt{\frac{\sin(x-a)}{\sin(x+a)}}\,dx$

Sol. Let $I = \int\sqrt{\frac{\sin(x-a)}{\sin(x+a)}}\,dx = \int\frac{\sin(x-a)}{\sqrt{\sin(x+a)\sin(x-a)}}\,dx$

$$= \int\frac{\sin x\cos a - \cos x\sin a}{\sqrt{\sin^2 x - \sin^2 a}}\,dx = I_1 - I_2$$

Now, we have

$$I_1 = \int\frac{\cos a\sin x\,dx}{\sqrt{\sin^2 x - \sin^2 a}} = \cos a\int\frac{\sin x\,dx}{\sqrt{\cos^2 a - \cos^2 x}}$$

$$= \cos a\int\frac{-dt}{\sqrt{\cos^2 a - t^2}} \qquad \text{[putting } \cos x = t\text{]}$$

$$= \cos a\cos^{-1}\left(\frac{t}{\cos a}\right) = \cos a\cos^{-1}\left(\frac{\cos x}{\cos a}\right)$$

and $$I_2 = \int\frac{\sin a\cos x\,dx}{\sqrt{\sin^2 x - \sin^2 a}} = \sin a\int\frac{dt}{\sqrt{t^2 - \sin^2 a}} \qquad \text{[putting } \sin x = t\text{]}$$

$$= \sin a\ln\left|t + \sqrt{t^2 - \sin^2 a}\right| = \sin a\ln\left|\sin x + \sqrt{\sin^2 x - \sin^2 a}\right|$$

Hence, we have

$$I = \cos a\cos^{-1}\left(\frac{\cos x}{\cos a}\right) + \sin a\,ln\left|\sin x + \sqrt{\sin^2 x - \sin^2 a}\right| + C.$$

Q.19 Evaluate the following indefinite integral $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

Sol. $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = \int e^x\left[\frac{1}{1+\cos x}+\frac{\sin x}{1+\cos x}\right]dx$

$$= \int e^x\left[\frac{1}{2}\sec^2\left(\frac{x}{2}\right)+\tan\left(\frac{x}{2}\right)\right]dx = e^x\tan\left(\frac{x}{2}\right)+C$$

Q.20 Evaluate the following indefinite integral $\int \frac{(x^2-1)\,dx}{x\sqrt{x^4+3x^2+1}}$

Sol. Let $I = \int \frac{(x^2-1)dx}{x\sqrt{x^4+3x^2+1}} = \int \frac{x\,dx}{\sqrt{x^4+3x^2+1}} - \int \frac{dx}{x\sqrt{x^4+3x^2+1}} = I_1 - I_2$

Putting $x^2 = t$ and $2x\,dx = dt$, we have

$$I_1 = \frac{1}{2}\int \frac{dt}{\sqrt{t^2+3t+1}} = \frac{1}{2}\int \frac{dt}{\sqrt{\left(t+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}}$$

$$= \frac{1}{2}\cosh^{-1}\left(\frac{2t+3}{\sqrt{5}}\right) = \frac{1}{2}\cosh^{-1}\left(\frac{2x^2+3}{\sqrt{5}}\right) \quad \text{and} \quad I_2 = \int \frac{dx}{x^3\sqrt{1+\frac{3}{x^2}+\frac{1}{x^4}}}$$

$$= \frac{-1}{2}\int \frac{dt}{\sqrt{t^2+3t+1}} \qquad \left[\text{putting } \frac{1}{x^2} = t \text{ and } \frac{-2dx}{x^3} = dt\right]$$

$$= \frac{-1}{2}\cosh^{-1}\left(\frac{2t+3}{\sqrt{5}}\right) = \frac{-1}{2}\cosh^{-1}\left(\frac{3x^2+2}{\sqrt{5}x^2}\right)$$

Hence, we have

$$I = \frac{-1}{2}\cosh^{-1}\left(\frac{x^3+3}{\sqrt{5}}\right)+\frac{1}{2}\cosh^{-1}\left(\frac{3x^2+2}{\sqrt{5}x^2}\right)+C.$$