LOGARITHM

ASIC MATHEMATICS :

:mainder Theorem :

Let $p(x)$ be any polynomial of degree geater than or equal to one and 'a' be any real numl
If $p(x)$ is divided by $(x-a)$, then the remainder is equal to $p(a)$.

ictor Theorem :

Let $p(x)$ be a polynomial of degree greater than or equal to 1 and 'a' be a real number such t
$p(a)=0$, then $(x-a)$ is a factor of $p(x)$. Conversely, if $(x-a)$ is a factor of $p(x)$, then $p(a)=0$

te : Let $p(x)$ be any polynomial of degree greater than or equal to one. If leading coefficient of $p(x)$ i
then $p(x)$ is called monic. (Leading coefficient means coefficient of highest power.)

SOME IMPORTANT IDENTITIES :

(1) $(a+b)^2 = a^2 + 2ab + b^2 = (a-b)^2 + 4ab$

(2) $(a-b)^2 = a^2 - 2ab + b^2 = (a+b)^2 - 4ab$

(3) $a^2 - b^2 = (a+b)(a-b)$

(4) $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

(5) $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

(6) $a^3 + b^3 = (a+b)^3 - 3ab(a+b) = (a+b)(a^2 + b^2 - ab)$

(7) $a^3 - b^3 = (a-b)^3 + 3ab(a-b) = (a-b)(a^2 + b^2 + ab)$

(8) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$.

(9) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$

(10) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$$= \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$$

If $(a+b+c) = 0$, then $a^3 + b^3 + c^2 = 3abc$.

(11) $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a-b)(a+b)$

(12) If $a, b \geq 0$ then $(a-b) = \left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right)$

(13) $a^4 + a^2 + 1 = (a^4 + 2a^2 + 1) - a^2 = (a^2 + 1)^2 - a^2 = (a^2 + a + 1)(a^2 - a + 1)$

Definition of Indices :

The product of m factors each equal to a is represented by a^m. So, $a^m = a \cdot a \cdot a \ldots\ldots a$ (m times). Here a is called the base and m is the index (or power or exponent).

Law of Indices :

(1) $a^{m+n} = a^m \cdot a^n$, where m and n are rational numbers.

(2) $a^{-m} = \dfrac{1}{a^m}$, provided $a \neq 0$.

(3) $a^0 = 1$, provided $a \neq 0$.

(4) $a^{m-n} = \dfrac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$.

(5) $(a^m)^n = a^{mn}$.

(6) $a^{\frac{p}{q}} = \sqrt[q]{a^p}$

(7) $(ab)^n = a^n b^n$.

Intervals :

Intervals are basically subsets of R (the set of all real numbers) and are commonly used in solving inequaltities. If a , b $\in$ R such that $a < b$, then we can defined four types of intervals as follows :

Name	**Representation**	**Discription.**
Open interval	(a, b)	$\{x : a < x < b\}$ i.e., end points are not included.
Close interval	$[a, b]$	$\{x : a \leq x \leq b\}$ i.e., end points are also included. This is possible only when both a and b are finite.
Open-closed interval	$(a, b]$	$\{x : a < x \leq b\}$ i.e., a is excluded and b is included.
Closed-open interval	$[a, b)$	$\{x : a \leq x < b\}$ i.e., a is included and b is excluded.

Note :

(1) The infinite intervals are defined as follows :

(i) $(a, \infty) = \{x : x > a\}$

(ii) $[a, \infty) = \{x \mid x \geq a\}$

(iii) $(-\infty, b) = \{x : x < b\}$

(iv) $(-\infty, b] = \{x : x \leq b\}$

(v) $(-\infty, \infty) = \{x : x \in R\}$

(2) $x \in \{1, 2\}$ denotes some particular values of x, i.e., $x = 1, 2$.

(3) If their is no value of x, then we say $x \in \phi$ (i.e., null set or void set or empty set).

Proportion :

When two ratios are equal, then the four quantities compositing then are said to be proportional.

If $\frac{a}{b}=\frac{c}{d}$, then it is written as $a:b=c:d$ or $a:b::c:d$.

Note :

(1) a and d are known as extremes while b and c are known as means.

(2) Product of extremes = product of means.

(3) If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{b}{a}=\frac{d}{c}$ (Invertando)

(4) If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a}{c}=\frac{b}{d}$ (Alternando)

(5) If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a}{b}+1=\frac{c}{d}+1 \Rightarrow \frac{a+b}{b}=\frac{c+d}{d}$ (Componendo)

(6) If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a}{b}-1=\frac{c}{d}-1 \Rightarrow \frac{a-b}{b}=\frac{c-d}{d}$ (Dividendo).

(7) If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Componendo and dividendo)

(8) If $\frac{a}{b}=\frac{b}{c}$ then $b^2=ac$. Here b is called mean proportional of a and c.

Histrorical Development of Number System :

I. Natural Number's

Number's used for counting are called as Natural number's.

$\{1, 2, 3, 4, ...\}$

II. Whole number's

Including zero (0) | cypher | शून्य | duck |love| knot along with natural numbers called as whole numbers.

$w = \{0,1, 2, 3 ....\}$

i.e. $N \subset W$

0 is neither positive nor negative

III Integer's

Integer's given by

$$I = \{...........-2, -1, 0, 1, 2, 3.......\}$$

$$\text{i.e. } N \subset W \subset I$$

Type of Integer's

(a) None negative integers $\{0, 1, 2, 3, ...\}$

(b) Negative integers (I^-) $\{...........-3, -2, -1\}$

(c) Non positive integers $\{...........-3, -2 -1, 0\}$

(d) Positve integers (I^+) $\{1, 2, 3 ....\}$

IV. Rational Number's

Number's which are of the form p/q where p, q, $\in$ I & q $\neq$ 0 called as rational number's.

Rational numbers are also represented by recurring & terminating or repeating decimal's

e.g. $1.\overline{3} = 1.333$ ....

$$x = 1.3333 \ldots$$
$$10x = 13.33\ldots$$
$$9x = 12$$
$$x = \frac{4}{3}$$

Every rational is either a terminating or a recurring decimal

V Irrational number's

The number's which cannot be expressed in the form p/q (p,q $\in$ I) are called as irrational numbers. The decimal representation of these number is non-terminating and non repeating.

$$\sqrt{2} = 1.414 \ldots\ldots\ldots\ldots$$

π is an irrational number

VI Real Number's

Set of real number's is union of the set of rational number's and the set of irrational numbers.

$$\text{Real} \rightarrow \text{Rational} + \text{Irrational}$$
$$N \subset W \subset I \subset Q \subset R \subset Z$$

VII. Prime Number's

Number's which are devisible by 1 or itself

e.g. {2, 3, 5, 7, 11, 13 ....}

VIII Composite Number's

Number's which are multiples of prime are called composite number's

{4, 6, 8, 9 ....}

IX Coprime or relatively prime number's

The number's having heighest common factor 1 are called relatively prime.

e.g. (2, 9), (16, 25 ......)

X Twin primes :

The prime number's which having the diffrence of 2

e.g. (5, 3), (7, 5), (13,11) ....

1 is niether a prime nor a composite number.

When studying logarithms it is important to note that all the propertise of logarithms are consequences of the corresponding properties of power, which means that sudent should have a good working knowledge of powers are a foundation for tacking logarithms

RITHM :

Definition : Every positive real number N can be expressed in exponential form as

$N = a^x$ ....(1) ***e.g.*** $49 = 7^2$

where 'a' is also a positive real different than unity and is called the base and 'x' is called the expo
We can write the relation (1) in logarithmic form as

$\log_a N = x$ ....(2)

Hence the two relations

$$\left.\begin{array}{r} a^x = N \\ \text{and} \quad \log_a N = x \end{array}\right]$$

re identical where $N > 0, a > 0, a \neq 1$

Hence logarithm of a number to some base is the exponent by which the base must be raised in ord
get that number. Logarithm of zero does not exist and logarithm of (–) ve reals are not defined i
ystem of real numbers.

e a is raised what power to get N

Illustration :

Find value of

(i) $\log_{81} 27$ *(ii)* $\log_{10} 100$ *(iii)* $\log_{1/3} 9\sqrt{3}$

Sol.(i) *Let* $\log_{81} 27 = x$

$\Rightarrow 27 = 81^x$

$\Rightarrow 3^3 = 3^{4x}$ *gives* $x = 3/4$

(ii) *Let* $\log_{10} 100 = x$

$\Rightarrow 100 = 10^x$

$\Rightarrow 10^2 = 10^x$ *gives* $x = 2$

(iii) *Let* $\log_{1/3} 9\sqrt{3} = x$

$\Rightarrow 9\sqrt{3} = \left(\frac{1}{3}\right)^x$

$\Rightarrow 3^{5/2} = 3^{-x}$ *gives* $x = -5/2$

Note that :

(a) Unity has been excluded from the base of the logarithm as in this case

$\log_1 N$ will not be possible and if $N = 1$

then $\log_1 1$ will have infinitely many solutions and will not be unique which is necessary in the functional notation.

(b) $a^{\log_a N} = N$ is an identify for all $N > 0$ and $a > 0, a \neq 1$ e.g. $2^{\log_2 5} = 5$

(c) The number N in (2) is called the antilog of 'x' to the base 'a'. Hence If $\log_2 512$ is 9 then $\text{antilog}_2 9$ is equal to $2^9 = 512$

(d) Using the basic definition of log we have 3 important deductions :

(i) $\log_N N = 1$ i.e. logarithm of a number to the same base is 1.

(ii) $\log_{\frac{1}{N}} N = -1$ i.e. logarithm of a number to its reciprocal is – 1.

(iii) $\log_a 1 = 0$ i.e. logarithm of unity to any base is zero.

(basic constraints on number and base must be observed.)

(iv) $a^{\log_a n} = n$ is an identify for all N > 0 and a> 0 ; a ≠ 1 e.g. $2^{\log_2 5} = 5$

(e) Whenever the number and base are on the same side of unity then logarithm of that number to the base is (+ve), however if the number and base are located an diffrent side of unity then logarithm of that number to the base is (–ve)

e.g. (i) $\log_{10}100 = 2$

(ii) $\log_{1/10}100 = -2$

(f) For a non negative number 'a' & $n \geq 2, n \in N$ $\sqrt[n]{a} = a^{1/n}$

Illustration :

(i) $\log_{\sin 30^\circ}\cos 60^o = 1$ *(ii)* $\log_{3/4} 1.\overline{3} = -1$ *(iii)* $\log_{2-\sqrt{3}} 2+\sqrt{3} = -1$

(iv) $\log_5\sqrt{5\sqrt{5\sqrt{5.......}}}\infty = 1$

Sol. *Let* $\sqrt{5\sqrt{5\sqrt{5.....\infty}}} = x$

$\Rightarrow \sqrt{5x} = x \quad \Rightarrow x^2 = 5x \quad \Rightarrow x = 5 \quad \Rightarrow \log_5 5 = 1$

(v) *(log tan 1º) (log tan 2º) (log tan 3º) ....(log tan 89º) = 0*

Sol. *Since tan 45° = 1 thus log tan 45° = 0*

(vi) $7^{\log_7 x} + 2x + 9 = 0$

Sol. $3x + 9 = 0 \Rightarrow (x = -3)$ *as it makes initial problem undefined*

$x = \phi$

(vii) $2^{\log_2(x-3)} + 2(x-3) - 12 = 0$

Sol. $x - 3 + 2x - 6 - 12 = 0$

$3x = 21 \Rightarrow x = 7$

(viii) $\log_2(x-3) = 4$

Sol. $x - 3 = 2^4$

$x = 19$

Practice Problem

Q.1 Find the logarithms of the following numbers to the base 2:

(i) $\sqrt[3]{8}$ (ii) $2\sqrt{2}$ (iii) $\frac{1}{\sqrt[5]{2}}$ (iv) $\frac{1}{\sqrt[7]{8}}$

Q.2 Find the logarithms of the following numbers to the base $\frac{1}{3}$

(i) 81 (ii) $\sqrt[3]{3}$ (iii) $\frac{1}{\sqrt[7]{3}}$ (iv) $9\sqrt{3}$ (v) $\frac{1}{9\sqrt[4]{3}}$

Q.3 Find all number a for which each of the following equalities hold true?

(i) $\log_2 a = 2$ (ii) $\log_{10}(a(a+3)) = 1$

(iii) $\log_{1/3}(a^2 - 1) = -1$ (iv) $\log_2(a^2 - 5) = 2$

Q.4 Find all values of x for which the following equalities hold true?

(i) $\log_2 x^2 = 1$ (ii) $\log_3 x = \log_3(2-x)$ (iii) $\log_4 x^2 = \log_4 x$

(iv) $\log_{1/2}(2x+1) = \log_{1/2}(x+1)$ (v) $\log_{1/3}(x^2+8) = -2$

Q.5 If $2\left(\sqrt{3+\sqrt{5-\sqrt{13+\sqrt{48}}}}\right) = \sqrt{a} + \sqrt{b}$ where a and b are natural number find (a + b).

Answer key

Q.1 (i) 1, (ii) 3/2, (iii) – 1/5, (iv) – 3/7 Q.2 (i) – 4, (ii) – 1/3, (iii) 1/7, (iv) – 5/2, (v) 9/4

Q.3 (i) 4, (ii) –5, 2, (iii) –2, 2, (iv) –3, 3 Q.4 (i) $\sqrt{2}, -\sqrt{2}$, (ii) 1, (iii) 1, (iv) 0, (v) 1, – 1 Q.5 8

Answer key

Q.1 (i) 1, (ii) 3/2, (iii) – 1/5, (iv) – 3/7 Q.2 (i) – 4, (ii) – 1/3, (iii) 1/7, (iv) – 5/2, (v) 9/4

Q.3 (i) 4, (ii) –5, 2, (iii) –2, 2, (iv) –3, 3 Q.4 (i) $\sqrt{2}, -\sqrt{2}$, (ii) 1, (iii) 1, (iv) 0, (v) 1, – 1 Q.5 8

PRINCIPAL PROPERTISE OF LOGARITHM :

If m, n are arbitrary positive real numbers where

$a > 0$; $a \neq 1$

(1) $\log_a m + \log_a n = \log_a mn$ $(m > 0, n > 0)$

Proof: Let $x_1 = \log_a m$; $m = a^{x}$

$x_2 = \log_a n$; $n = a^{x_2}$

Now $mn = a^{x}$; a^{x_2}

$mn = a^{x_1+x_2}$

$x_1 + x_2 = \log_a mn$

$\log_a m + \log_a n = \log_a mn$

(2) $\log_a \frac{m}{n} = \log_a m - \log_a n$

$\frac{m}{n} = a^{x_1 - x_2}$

$x_1 - x_2 = \log_a \frac{m}{n}$

$\log_a m - \log_a n = \log_a \frac{m}{n}$

(3) $\log_a m^x = x\log_a m$

$\log_a m = p \quad ; \qquad m = a^p$

$m^x = a^{px}$

taking log both the side with base a

$$\log_a m^x = \log_a a^{px} = px = x\log_a m$$

(4) $\log_{a^x} m = \dfrac{1}{x}\log_a m$

Ex :(i) Find the solution of $\mathbf{\log_2 x^2 = 4}$ & $\mathbf{2\log_2 x = 4}$ and verify solutions.

Sol. $\log_2 x^2 = 4 \qquad\qquad 2\log_2 x = 4$

$\Rightarrow \quad x^2 = 16 \qquad\qquad \Rightarrow \quad \log_2 x = 2$

$\Rightarrow \quad x = \pm 4$ (two solution) $\qquad \Rightarrow \quad x = 4$ only possible soln.

Ex: (ii) $\log_2 x^2 + 2\log_2 x = 8$

Sol. $4\log_2 x = 8 \qquad (x > 0)$

$\log_2 x = 2$

$x = 4$

BASE CHANGING THEOREM :

Can be stated as "quotient of the logarithm of two numbers is independent of their common base."

Symbolically, $\dfrac{\log_c a}{\log_c b} = \log_b a$

proof Let $\log_c a = x \; ; \quad \log_c b = y \quad \& \quad \log_b a = z$

$a = c^x \; ; \qquad b = c^y , \qquad a = b^z$

$c^x = b^z$

$a = c^x \; ; \qquad b = c^y , \qquad a = b^z$

$c^x = b^z$

$c^x = c^{yz} \quad \Rightarrow \quad x = yz$

i.e $\quad z = \dfrac{x}{y}$

$$\boxed{\log_b a = \frac{\log_c a}{\log_c b}}$$

e.g. Find value of $\log_{64} 16$

$$\log_{64} 16 = \frac{\log_4 16}{\log_4 64} = \frac{2}{3}$$

Case-I : $\log_b a = \dfrac{1}{\log_a b}$

We have proved that $\dfrac{\log_c a}{\log_c b} = \log_b a$

put $\quad c = a$

Similarly $\quad \dfrac{\log_a a}{\log_a b} = \log_b a$

or $\quad \boxed{\dfrac{1}{\log_a b} = \log_b a}$

e.g. prove $\quad \dfrac{1}{\log_a abc} + \dfrac{1}{\log_b abc} + \dfrac{1}{\log_c abc} = 1$

Sol. $\quad \log_{abc} a + \log_{abc} b + \log_{abc} C = \log_{abc} abc = 1$

Case-II : $(\log_b a).(\log_c b).(\log_d c) = \log_d a$

Proof $\dfrac{\log a}{\log b} \times \dfrac{\log b}{\log c} \cdot \dfrac{\log c}{\log d} = \dfrac{\log a}{\log b} = \log_d a$

e.g. $(\log_3 5).(\log_{25} 27) = \dfrac{3}{2}$

Case-III : Very imp form

$a^{\log_b c} = c^{\log_b a}$

Proof $a^{\log_b c} = a^{(\log_b c)\ (\log_a c)\ (\log_c a)} = a^{\log_a c\ (\log_b c.\ \log_c a)} = c^{\log_b a}$

$\Rightarrow \boxed{a^{\log_b c} = c^{\log_b a}}$

Illustration :

$$2^{-\log_{\frac{1}{2}} 7} = 7$$

Sol. $2^{\log_2 7} = 7$

Illustration :

$$8^{\frac{-1}{\log_3 2}} = \frac{1}{27} = 8^{\frac{-1}{\log_3 2}}$$

Sol. $8^{-\log_2 3} = 8^{\log_2 \frac{1}{3}} = \dfrac{1}{27}$

Illustration :

$(\log_2 3).(\log_3 4).(\log_4 5)....\log_n (n+1) = 10$. *Find* $n = ?$

Sol. $\log_2 (n+1) = 10$

$n + 1 = 1024 \ ; n = 1023$

Illustration :

$$\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80}.$$

Sol. $\log 2 + 16 \log 16 - 16 \log 15 + 12 \log 25 - 12 \log 24 + 7 \log 81 - 7 \log 80$

$= \log 2 + 64 \log 2 - 16 \log 5 - 16 \log 3 + 24 \log 5 - 12 \times 3 \log 2 - 12 \log 3$

$+ 28 \log 3 - 7 \log 5 - 28 \log 2$

$= \log 2 + \log 5 = \log 10 = 1$

Illustration :

$$\frac{1}{\log_3 2} + \frac{2}{\log_9 4} + \frac{3}{\log_{27} 8} = 0$$

Sol. $\log_2 3 + 2 \log_4 9 - 3 \log_8 27$

$= 3 \log_2 3 - 3 \log_2 3 = 0$

Illustration :

Let $a > 1$ be a real number then solve

$$a^{2\log_2 x} = 5 + 4x^{\log_2 a}$$

Sol. $(x^{\log_2 a})^2 - 4(x^{\log_2 a}) - 5 = 0$; $t^2 - 4t - 5 = 0$

$t = 5$ $x^{\log_a 2} = 5$ $(\log_2 a).\log_5 x = 1$

Take log w.r.t. base 5.

$\log_5 x = \log_a 2$

$x = (5)^{\log_a 2}$

Illustration :

Prove that $2^{\sqrt{\log_2 3}} = 3^{\sqrt{\log_3 2}}$

Sol. $2^{\sqrt{\log_2 3}} = 2^{(\log_2 3).\frac{1}{\sqrt{\log_2 3}}} = \left(2^{\log_2 3}\right)^{\frac{1}{\sqrt{\log_2 3}}} = 3^{\frac{1}{\sqrt{\log_2 3}}} = 3^{\sqrt{\log_3 2}}$

Illustration :

If $a > 0$; $c > 0$; $b = \sqrt{ac}$; $ac \neq 1$

Prove that $\dfrac{\log_a N}{\log_c N} = \dfrac{\log_a N - \log_b N}{\log_b N - \log_c N}$

Sol. L.H.S. $= \dfrac{\log_a N}{\log_c N} = \dfrac{\log_N c}{\log_N a} = \log_a c$

R.H.S. $= \dfrac{\log_a N - \log_b N}{\log_b N - \log_c N}$

$= \dfrac{(\log_N b - \log_N a)}{\log_N c - \log_N b} \times \dfrac{\log_N c}{\log_N b} \times \dfrac{\log_N b}{\log_N a}$

$= \dfrac{\log_N b/a}{\log_N c/b} \times \log_a c$

$= \log_{c/b} b/a \times \log_a c$

$= \left(\log_{c/a}^{c/a}\right) \log_a c$

$= \log_a c$ Ans.

$$\begin{pmatrix} b = \sqrt{ac} \\ b^2 = ac \\ \dfrac{b}{a} = \dfrac{c}{b} \end{pmatrix}$$

LOGARITHMIC EQUATIONS :

Illustration :

Prove that $x^2 + 7^{\log_7 x} - 2 = 0$

Sol. $\Rightarrow x^2 + x - 2 = 0$ $\left[a^{\log_a x} = x(>0)\right]$

$= (x + 2)(x - 1) = 0$

$= x = -2$ or 1

$=$ By definition, $x > 0$

$\therefore$ $x = 1$

Illustration :

Find the value of x : $(x+1)^{\log_{10}(x+1)} = 100(x+1)$

Sol. *By definition* $(x+1) > 0$

$\therefore$ *Taking log with base 10 both side*

$= \log (x+1)^{\log_{10}(x+1)} = \log 100(x+1)$

$= \log_{10}(x+1).\log_{10}(x+1) = \log_{10} 100 + \log_{10}(x+1)$

$= (\log_{10}(x+1))^2 - \log(x+1) = 2$

$=$ *let* $\log_{10}(x+1) = y$

$= y^2 - y - 2 = 0$

$= (y-2)(y+1) = 0$

$= y = 2 \text{ or } -1$

$= \log_{10}(x+1) = 2 \quad \text{or} \quad \log_{10}(x+1) = -1$

$\therefore\ x + 1 = 100 \quad \text{or} \quad x + 1 = 10^{-1} = 0.1$

$x = 99 \quad \text{or} \quad x = -0.9$

Illustration :

Find the value of x : $3^{\log_3^2 x} + x^{\log_3 x} = 162$

Sol. $\left(3^{\log_3 x}\right)^{\log_3 x} + x^{\log_3 x} = 162$

$\Rightarrow x^{\log_3 x} + x^{\log_3 x} = 162 \qquad \left[a^{\log_a x} = x\right]$

$\Rightarrow x^{\log_3 x} = 81$

Taking log both side with base 3

$\Rightarrow x \quad = 81$

Taking log both side with base 3

$\Rightarrow \quad (\log_3 x)^2 = \log_3 81 = 4 \quad \Rightarrow \quad \log_3 x = \pm 2$

$\Rightarrow \quad x = 9 \text{ or } \dfrac{1}{9}$

Illustration :

Find the value of x : $\log_5(5^{1/x} + 125) = \log_5(6) + 1 + \dfrac{1}{2x}$.

Sol. $= \log_5(5^{1/x} + 125) = \log_5 6 + \left(1 + \dfrac{1}{2x}\right)$

$= \log_5\left[\dfrac{5^{1/x} + 125}{6}\right] = \left(1 + \dfrac{1}{2x}\right)$

$= 5^{1/x} + 125 = (5^1 \cdot 5^{1/2x}) \cdot 6$

$= 5^{1/x} + 125 = 5 \cdot 6 \cdot 5^{1/2x}$

$=$ *let* $5^{\frac{1}{2x}} = y$

$= y^2 - 30y + 125 = 0$

$= (y-25)(y-5) = 0$

$= y = 5 \quad \text{or} \quad 25$

$\therefore\ 5^{\frac{1}{2x}} = 5^1 \quad \text{or} \quad 5^2$

$\therefore\ x = \dfrac{1}{2} \quad \text{or} \quad \dfrac{1}{4}$

Note : [If given problem is $log_5\left(\sqrt[x]{5}+125\right) = \log_5(6)+1+\frac{1}{2x}$ then the equation will have no solution since for $\sqrt[x]{5}$, $x \in N$ and $N \geq 2$]

Illustration :

Prove that : $log_2 7$ is irrational.

Sol. *Let $log_2 7$ is Rational.*

So $log_2 7 = a/b$ where (a & b are integers & $b \neq 0$)

$$= \frac{log_{10} 7}{log_{10} 2} = \frac{a}{b}$$

$$= b\ log_{10} 7 = a\ log_{10} 2$$

$$= 7^b = 2^a \qquad \text{(7 \& 2 are co-prime)}$$

So there is no such type of integers for a & b so there is a contradiction.

Common and natural logarithm :

$\log_{10}N$ is referred as a common logarithm and $\log_e N$ is called as natural logarithm or logarithm of N to the base Napierian and is popularly written as $ln\,N$. Note that e is an irrational quantity lying between 2.7 to 2.8 which you will study later. **Note that** $e^{ln\,x} = x$

Characteristic and Mantissa :

We observe that $\log_{10}10 = 1$ and $\log_{10}100 = 2$.
Hence logarithm of a number lying between 10 to 100 = 1 + a positive quantity

$\log_{10}(0.1) = -1$ and $\log_{10}(0.01) = -2$

hence log (a number between 0.01 to 0.1) = – 2 + a positive quantity

Hence the common logarithm of a number consists of two parts, integral and fractional, of which the integral part may be zero or an integer (+ve or –ve) and the fractional part, a decimal, less than one and always positive.

The integral part is called the <u>characteristic</u> and the decimal part is called the <u>mantissa</u>.

e.g. $\log_{10}33.8 = 1.5289 \Rightarrow 33.8 = 10^{1.5289} = 10.10^{0.5289}$

$\log_{10}0.338 = -1 + 0.5289 = \bar{1}.5289$

It should be noted that, if the characteristic of the logarithm of N is

$$\left.\begin{array}{lll} 1 & \Rightarrow & \text{that N has two significant digits before decimal.} \\ 2 & \Rightarrow & \text{that N has three significant digits before decimal.} \end{array}\right\} \textit{very Important}$$

(Hence number of significant digit in N = p + 1 if p is the non negative characteristic of log N.)

if characteristic

– 1 ⇒ N has no zeros after decimal before a significant digit starts

– 2 ⇒ N has 1 zero after decimal before a significant digit starts and so on.

Using log 2 = 0.3010 and log 3 = 0.4771, and log 7 = 0.8451

Illustration :

Find the number of digits $(2.5)^{200}$

Sol. *Let* $N = (2.5)^{200}$
Taking log both side with base 10

$$\log_{10} N = 200 \log_{10} (2.5) = 200 \log_{10}\left(\frac{5}{2}\right) = 200 [\log_{10} 5 - \log_{10} 2]$$

$$= 200 [1 - 2 \log_{10} 2] = 200 [1 - 2 \times 0.3010] = 200 [0.3990] = 79.80$$

Characteristic $= 79$
Number of digits $= 79 + 1 = 80$

Illustration :

Find the number of digits 6^{50}

Sol. *Let* $N = 6^{50}$
Taking log both side with base 10

$$\log N = 50 [\log_{10} 6]$$

$$= 50 [\log_{10} 2 + \log_{10} 3] = 50 [0.3010 + 0.4771] = 50 [0.7781] = 38.9050$$

Characteristic $= 38$
Number of digits $= 38 + 1 = 39$

Number of digits $= 38 + 1 = 39$

Illustration :

Find the number of digits 5^{25}

Sol. *Let* $N = 5^{25}$
Taking log both side with base 10

$$\log N = 25 \log_{10} 5 = 25 [1 - \log_{10} 2] = 25 [1 - 0.3010] = 17.4750$$

Characteristic $= 17$
Number of digits $= 17 + 1 = 18$

Illustration :

Find the number of zeros after decimal before a significant figure start in $\left(\frac{9}{8}\right)^{-100}$

Sol. *Let* $N = \left(\frac{9}{8}\right)^{-100}$

Taking log both side with base 10

$$\log_{10}\left(\frac{9}{8}\right)^{-100} = -100 [\log_{10} 9/8] = -100 [\log_{10} 9 - \log_{10} 8] = -100 [2 \log_{10} 3 - 3 \log_{10} 2]$$

$$= -100 [2 \times 0.4771 - 3 \times 0.3010] = -100 [0.0512] = -5.12$$

Characteristic $= -6$
Number of zero after decimal before a significant figure start $= 5$

ustration :

Find the number of zeros after decimal before a significant figure start in 3^{-50}.

l. *Let $N = 3^{-50}$*

Taking log both side with base 10

$$\log_{10}3^{-50} = -50\log_{10}3 = -50(0.4771) = -23.8550$$

Characteristic = – 24

Number of zeros after decimal befores a significant figure start = 23

ustration :

Find the number of zeros after decimal before a significant figure start in $(0.35)^{12}$

l. *Let $N = (0.35)^{12}$*

Taking log both side with base 10

$$\log_{10}(0.35)^{12} = 12\log_{10}(0.35) = 12\log_{10}\frac{7}{20} = 12[\log_{10}7 - \log_{10}20]$$

$$= 12[\log_{10}7 - \log_{10}2 - 1] = 12[0.8451 - 0.3010 - 1] = -5.4708$$

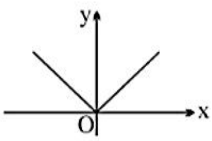
Characteristic = –6

Number of zeros after decimal befores a significant figure start = 5]

bsolute value function :

Absolute value function :

(a) $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



(b) $\sqrt{x^2} = |x|$

(c) $\log x^{2n} = 2n\log|x|$, where $n \in I$

General Note : Equations of the form

$[a(x)]^{b(x)} = [a(x)]^{c(x)}$ (Variable exponent on a variable base)

with the set of permissible values defined by the condition $a(x) > 0$, can be reduced to the equivalent equation

$$b(x)\log_d[a(x)] = c(x)\log_d[a(x)]$$

by taking logarithms of its both sides. The last equation is equivalent to two equations.

$\log_d[a(x)] = 0,\ b(x) = c(x)$.

e.g. $|x-2|^{10x^2-1} = |x-2|^{3x}$

Sol. Taking log both side w.r.t. base 10

$$\log_{10}|x-2|^{10x^2-1} = \log|x-2|^{3x}$$

$\Rightarrow (10x^2 - 1)\log|x-2| = 3x\log|x-2|$

$\Rightarrow \log|x-2|(10x^2 - 3x - 1) = 0$

$\Rightarrow \log|x-2| = 0$ or $10x^2 - 3x - 1 = 0$

$\Rightarrow |x-2| = 1$ or $(2x-1)(5x+1) = 0$

$\Rightarrow x - 2 = \pm 1$

$x = 3 ; 1$ $\quad x = 1/2, -1/5$

Illustration :

Solve for x : $|3x-2| + x = 11$

Sol. $|3x-2| = 11 - x$ $[11 - x \geq 0]$

Case 1 : $3x - 2 \geq 0$

$\Rightarrow$ $3x - 2 = 11 - x$

$\Rightarrow$ $4x = 13$

$\Rightarrow$ $x = 13/4$

Case 2 : $3x - 2 < 0$

$\Rightarrow$ $-(3x-2) = 11 - x$

$\Rightarrow$ $-3x + 2 = 11 - x$

$\Rightarrow$ $2x = -9$

$\Rightarrow$ $x = -9/2$

Illustration :

Solve for x : $|x| - |x-2| = 2$

Sol. Case 1: $x \geq 2$ $\Rightarrow$ $(x) - (x-2) = 2$ $\Rightarrow$ $2 = 2$

for all $x \geq 2$

Case 2 : $0 \leq x < 2$

$\Rightarrow$ $x + x - 2 = 2$ $\Rightarrow$ $2x = 4$ $\Rightarrow$ $x = 2$

No solution $(0 \leq x < 2)$

Case 3 : $x < 0$

$\Rightarrow -x + x - 2 = 2$ $\Rightarrow$ $-2 = 2$

Not possible

$\therefore$ $x \in [2, \infty]$

$\Rightarrow -x + x - 2 = 2$ $\Rightarrow$ $-2 = 2$

Not possible

$\therefore$ $x \in [2, \infty]$

Illustration :

Solve for x : $|x-3|^{3x^2-10x+3} = 1$

Sol. *Taking log both side with 10*

$\Rightarrow$ $\log |x-3|^{3x^2-10x+3} = \log 1$

$\Rightarrow$ $(3x^2 - 10x + 3) \log_{10} |x-3| = 0$ $[x \neq 3]$

$\Rightarrow$ $(3x-1)(x-3) \log_{10} |x-3| = 0$

$\Rightarrow$ $x = \frac{1}{3}, \; |x-3| = 1$

again when $x > 3$

$\Rightarrow$ $x - 3 = 1$

$\Rightarrow$ $x = 4$

when $x < 3$

$-(x-3) = 1$

$\Rightarrow$ $x = 2$

$\therefore$ $x = \frac{1}{3}, 2, 4$

Illustration :

Solve for x : $\log_4(x^2 - 1) - \log_4(x-1)^2 = \log_4\sqrt{(4-x)^2}$

Sol. $\log_4(x^2 - 1) - \log_4(x-1)^2 = \log_4\sqrt{(4-x)^2}$ $\quad [x \ne [-1, 1] \cup \{4\}]$

$\log_4(x-1)(x+1) - \log_4(x+1)^2 = \log_4|4-x|$

$$\log_4\left[\frac{(x-1)(x+1)}{(x+1)^2}\right] = \log_4|4-x|$$

$\Rightarrow \quad \dfrac{x+1}{x-1} = |4-x|$

$\Rightarrow \quad (x+1) = (x-1)|x-4| \quad (\because |4-x| = |x-4|)$

Case (i) : $\quad x \ge 4$

$\Rightarrow \quad (x+1) = (x-1)(x-4)$

$\Rightarrow \quad x + 1 = x^2 - 5x + 4$

$\Rightarrow \quad x^2 - 6x + 3 = 0$

$\Rightarrow \quad x = \dfrac{6 \pm \sqrt{24}}{2}$

$x = 3 \pm \sqrt{6}$

$x \ne 3 - \sqrt{6}$ (not in domain)

$\Rightarrow \quad x = 3 + \sqrt{6}$

Case (ii) : $\quad x < 4$

$\Rightarrow (x+1) = (x-1)(4-x)$

$\Rightarrow x + 1 = 4x - x^2 - 4 + x$

$\Rightarrow x^2 - 4x + 5 = 0$

Discriminant < 0

$\because x^2 - 4x + 5 \ne 0$

= no solution

from case (i) and case (ii)

$= \quad x = 3 + \sqrt{6}$

$x = 3 \pm \sqrt{6}$ $\quad \because x^2 - 4x + 5 \ne 0$

$x \ne 3 - \sqrt{6}$ (not in domain) $\quad$ = no solution

$\Rightarrow \quad x = 3 + \sqrt{6}$

from case (i) and case (ii)

$= \quad x = 3 + \sqrt{6}$

Illustration :

Solve for x : $\quad 2\log_3(x-2) + \log_3(x-4)^2 = 0$

Sol. $2\log_3(x-2) + \log_3(x-4)^2 = 0$ $\quad [x - 2 > 0, x \ne 4]$

$\Rightarrow \quad 2\log_3(x-2) + 2\log_3|x-4| = 0$

$\Rightarrow \quad 2(\log_3(x-2)|x-4|) = 0$

$\Rightarrow \quad (x-2)|x-4| = 1$

Case (i) : $\quad x \ge 4$

$\Rightarrow \quad (x-2)(x-4) = 1 \quad \Rightarrow \quad x^2 - 6x + 7 = 0$

$\Rightarrow \quad x = \dfrac{6 \pm \sqrt{6}}{2} \quad \Rightarrow \quad x = 3 \pm \sqrt{2}$

$\Rightarrow \quad x = 3 - \sqrt{2}$ (Not in domain)

$\Rightarrow \quad x = 3 + \sqrt{2}$

Case (ii): $\quad x < 4$

$\Rightarrow \quad (x-2)(4-x) = 1 \quad \Rightarrow \quad 4x - x^2 - 8 + 2x = 1$

$\Rightarrow \quad x^2 - 6x + 9 = 0 \quad \Rightarrow \quad (x-3)^2 = 0$

$x = 3$

from case (i) and case (ii)

$x\,\{3, 3 + \sqrt{2}\}$

Solved Examples

Q.1 Find the value of x satisfying $\log_{10}(2^x + x - 41) = x(1 - \log_{10}5)$.

Sol. We have,

$$\log_{10}(2^x + x - 41) = x(1 - \log_{10}5)$$

$$\Rightarrow \quad \log_{10}(2^x + x - 41) = x\log_{10}2 = \log_{10}(2^x)$$

$$\Rightarrow \quad 2^x + x - 41 = 2^x \Rightarrow x = 41. \text{ Ans.}$$

Q.2 If the product of the roots of the equation, $x^{\left(\frac{3}{4}\right)(\log_2 x)^2 + \log_2 x - \left(\frac{5}{4}\right)} = \sqrt{2}$ is $\frac{1}{\sqrt[b]{a}}$ (where a, b $\in$ N) then the value of $(a + b)$.

Sol. Take log on both the sides with base 2

$$\left(\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right)\log_2 x = \frac{1}{2}$$

$\log_2 x = y$

$$3y^3 + 4y^2 - 5y - 2 = 0 \Rightarrow \quad 3y^2(y-1) + 7y(y-1) + 2(y-1) = 0$$

$$\Rightarrow \quad (y-1)(3y^2 + 7y + 2) = 0 \quad \Rightarrow \quad (y-1)(3y+1)(y+2) = 0$$

$$\Rightarrow \quad y = 1 \text{ or } y = -2 \text{ or } y = \frac{-1}{3}$$

$$\therefore \quad x = 2;\ \frac{1}{4};\ \frac{1}{2^{1/3}} \Rightarrow x_1x_2x_3 = \frac{1}{\sqrt[3]{16}} \Rightarrow a + b = 19$$

Q.3 For $0 < a \neq 1$, find the number of ordered pair (x, y) satisfying the equation $\log_{a^2}|x + y| = \frac{1}{2}$ and

$$\therefore \quad x = 2;\ \frac{1}{4};\ \frac{1}{2^{1/3}} \Rightarrow x_1x_2x_3 = \frac{1}{\sqrt[3]{16}} \Rightarrow a + b = 19$$

Q.3 For $0 < a \neq 1$, find the number of ordered pair (x, y) satisfying the equation $\log_{a^2}|x + y| = \frac{1}{2}$ and

$\log_a y - \log_a|x| = \log_{a^2}4$.

Sol. We have $\log_{a^2}|x + y| = \frac{1}{2} \quad \Rightarrow \quad |x + y| = a \quad \Rightarrow \quad x + y = \pm a$ .....(1)

Also, $\log_a\left(\frac{y}{|x|}\right) = \log_{a^2}4 \quad \Rightarrow \quad y = 2|x|$ .....(2)

If $x > 0$, then $x = \frac{a}{3},\ y = \frac{2a}{3}$

If $x < 0$, then $y = 2a,\ x = -a$

$\therefore$ possible ordered pairs $= \left(\frac{a}{3}, \frac{2a}{3}\right)$ and $(-a, 2a)$

Q.4 The system of equations

$$\log_{10}(2000xy) - \log_{10}x \cdot \log_{10}y = 4$$

$$\log_{10}(2yz) - \log_{10}y \cdot \log_{10}z = 1$$

and $\log_{10}(zx) - \log_{10}z \cdot \log_{10}x = 0$

has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $(y_1 + y_2)$.

Sol. From (1),

$$3 + \log_{10}(2xy) - \log_{10}x \cdot \log_{10}y = 4 \quad \text{....(i)}$$

or $\log_{10}(xy) - \log_{10}x \cdot \log_{10}y = 1 - \log_{10}(2)$

From (2)

$$\log_{10}(yz) - \log_{10}y \cdot \log_{10}z = 1 - \log_{10}(2) \qquad ....(ii)$$

From (i) and (ii), we get

$$\log x + \log y - \log x \cdot \log y = \log y + \log z - \log y \cdot \log z$$

$\Rightarrow \quad \log x\,(1 - \log y) = \log z\,(1 - \log y) \;\Rightarrow\; (\log x - \log z)(1 - \log y) = 0$

$\therefore \quad$ Either, $\log x = \log z$

or $\quad \log_{10}y = 1 \;\Rightarrow\; y = 10$

but $y = 10$ does not satisfy equation (1), hence rejected.

$\therefore \quad \log x = \log z$

From (3), we get

$$(\log_{10}x)^2 = 2(\log_{10}x) \;\Rightarrow\; \log_{10}x\,[\log_{10}x - 2] = 0$$

$\therefore \quad x = 1$ or $x = 100$

if $\;x = z = 1$ then $y = 5 \;\Rightarrow\; (x_1, y_1, z_1) \equiv (1, 5, 1)$

$\quad x = z = 100$ then $y = 20 \;\Rightarrow\; (x_2, y_2, z_2) \equiv (100, 20, 100)$

Hence $(y_1 + y_2) = 25$ Ans.

Q.5 A circle has a radius of $\log_{10}(a^2)$ and a circumference of $\log_{10}(b^4)$. The value of $\log_a b$ is equal to

(A) $\frac{1}{4\pi}$ (B) $\frac{1}{\pi}$ (C) π (D) 2π

Sol. $C = 4\log_{10}b = 2\pi r$

$\therefore \quad 4\log_{10}b = 2\pi \cdot 2\log_{10}a \qquad (\text{as } r = 2\log_{10}a)$

$$\frac{\log_{10} b}{\log_{10} a} = \pi$$

$\therefore \quad 4\log_{10}b = 2\pi \cdot 2\log_{10}a \qquad (\text{as } r = 2\log_{10}a)$

$$\frac{\log_{10} b}{\log_{10} a} = \pi$$

$\therefore \quad \log_a b = \pi \qquad$ Ans. (C)

Q.6 If $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{(\log_{10} n) - 1}{2}$ then the value of 'n' is

(A) 24 (B) 36 (C) 20 (D) 12

Sol. Given $\log_{10}\left(\frac{\sin 2x}{2}\right) = -1$

$$\Rightarrow \quad \frac{\sin 2x}{2} = \frac{1}{10} \;\Rightarrow\; \sin 2x = \frac{1}{5} \qquad ....(1)$$

Also $\quad \log_{10}(\sin x + \cos x) = \dfrac{\log_{10}\left(\frac{n}{10}\right)}{2}$

$$\Rightarrow \quad \log_{10}(\sin x + \cos x)^2 = \log_{10}\left(\frac{n}{10}\right)$$

$$\Rightarrow\; 1 + \sin 2x = \frac{n}{10} \;\Rightarrow\; 1 + \frac{1}{5} = \frac{n}{10}$$

$$\Rightarrow\; \frac{6}{5} = \frac{n}{10} \Rightarrow n = 12 \qquad \text{Ans. (D)}$$

Q.7 The ratio $\dfrac{2^{\log_{2^{1/4}} a}-3^{\log_{27}\left(a^2+1\right)^3}-2a}{7^{4\log_{49} a}-a-1}$ simplifies to

(A) a^2-a-1 (B) a^2+a-1 (C) a^2-a+1 (D) a^2+a+1

Hint: $\dfrac{2^{\log_{2^{1/4}} a}-3^{\log_{27}\left(a^2+1\right)^3}-2a}{7^{4\log_{49} a}-a-1}=\dfrac{2^{4\log_2 a}-3^{3\log_{3^3}(a^2+1)}}{7^{4\log_{7^2} a}-a-1}$

$=\dfrac{a^4-a^2-2a-1}{a^2-a-1}$ $[a^4-(a-1)^2$ Ans. (D)

Q.8 The number of values of x satisfying the equation $2^{\log_5 16\cdot\log_4 x+\log_{x_{\sqrt{2}}} 5}+5^x+x^{\log_3 4+5}+x^5=0$ is :

(A) 0 (B) 1 (C) 2 (D) 3

Sol. $2^{\log_5 16\cdot\log_4 x+\log_{x_{\sqrt{2}}} 5}+5^x+x^{\log_3 4+5}+x^5=0$

$2^{2\log_5 4\cdot\log_4 x+x\log_2 5}+5^x+x^{\log_5 4}\cdot x^5+x^5=0$

$2^{2\log_5 x}\,2^{x\log_2 5}+5^x+x^{2\log_5 2}\cdot x^5+x^5=0$

$\left(2^{\log_5 x}\right)^2\cdot 5^x+5^x+\left(2^{\log_5 x}\right)^2\cdot x^5+x^5=0$

$5^x\left[\left(2^{\log_5 x}\right)^2+1\right]+x^5\left[\left(2^{\log_5 x}\right)^2+1\right]=0$

$\left(5^x+x^5\right)\left[\left(2^{\log_5 x}\right)^2+1\right]=0$

$5^x\left[\left(2^{\log_5 x}\right)^2+1\right]+x^5\left[\left(2^{\log_5 x}\right)^2+1\right]=0$

$\left(5^x+x^5\right)\left[\left(2^{\log_5 x}\right)^2+1\right]=0$

$5^x+x^5=0$ $\left(2^{\log_5 x}\right)^2+1=0$

This possible only when x will be –ve No solution

while according to question $x \geq 2$

$\therefore$ number of values of x = zero Ans. (A)

Q.9 The number $N=6\log_{10}2+\log_{10}31$, lies between two successive integers whose sum is equal to

(A) 5 (B) 7 (C) 9 (D) 10

Hint: $N=\log_{10}64+\log_{10}31=\log_{10}1984$

$\therefore$ $3<N<4$ $\Rightarrow$ 7 Ans. Ans. (B)

Q.10 If $2^a=7^b$ then number of ordered pairs (a, b) of real numbers is

(A) zero (B) one (C) two (D) more than 2

Sol. $2^a=7^b \Rightarrow a=b=0$ if a and b are integers

In case a and b are not integers then

$2^{\log_2 7}=7^b \Rightarrow a=\log_2 7$ and $b=1$

or $2^{\log_2 49}=7^b \Rightarrow a=\log_2 49$ and $b=2$

or $2^a=7^{\log_7 2}$

$\Rightarrow a=1$ and $b=\log_7 2$

$\therefore$ Infinite solutions. Ans.(D)

Q.11 The number $2^{2\log_2\left(3^{3\log_3 4}\right)}$ simplifies as :

(A) 12 (B) 16 (C) 24 (D) 72

Sol. $2^{2\log_2\left(3^{3\log_3 4}\right)} = 2^{2\log_2\left(3^{\log_3 4^3}\right)}$

$= 2^{2\log_2\left(4^3\right)} = 2^{\log_2\left(4^3\right)^2} = 2^{\log_2\left((2)^2\right)^6} = 2^{12}$. Ans. (A)

Q.12 If $\log_2\log_3\log_4\log_5 A = x$, then the value of A is

(A) 120^x (B) 2^{60x} (C) $2^{3^{4^{5^x}}}$ (D) $5^{4^{3^{2^x}}}$

Sol. $\log_3\log_4\log_5 A = 2^x$

$\log_4\log_5 A = 3^{2^x}$

$\log_5 A = 4^{3^{2^x}}$

$A = 5^{4^{3^{2^x}}}$.

Q.13 Suppose $\log_a 2 = m$, $\log_a 3 = r$, $\log_a 5 = s$ and $\log_a 11 = t$. The value of $\log_a 990$, is

(A) 2mrst (B) m + 2r + s + t (C) m + r + s + t (D) m + 2r +5 + t

Sol. $a^m = 2$; $a^r = 3$; $a^s = 5$ and $a^t = 11$

now, $\log_a 990 = \log_a 11 + 2\log_a 3 + \log_a 2 + \log_a 5 = t + 2r + m + s$. Ans. (D)

Q.14 Suppose $\log_{32} p = q$ and $\log_2 r = q$ for positive numbers p, q and r. Which of the following must be true?

I. $5\log_2 r = \log_2 p$ II. $p = \frac{q}{5}$ III. $5\log_{32} r = \log_2 p$

(A) Only (I) must be true (B) Only (II) must be true

(C) Only (III) must be true (D) none of the statements must be true

Q.15 Solve following log equation $\log_4(2\log_3(1 + \log_2(1 + 3\log_3 x))) = \frac{1}{2}$

Sol. $\Rightarrow \frac{1}{2}\log_2(2\log_3(1 + \log_2(1 + 3\log_3 x))) = \frac{1}{2}$

$\Rightarrow 2\log_3(1+\log_2(1+3\log_3 x))) = 2^1 = 2$ $\Rightarrow (1 + \log_2(1 + 3\log_3 4) = 3^1$

$\Rightarrow \log_2(1 + 3\log_3 x) = 3 - 1 = 2$ $\Rightarrow 1 + 3\log_3 x = 2^2$

$\Rightarrow 3\log_3 x = 3$ $\Rightarrow \log_3 x = 1$

$\Rightarrow x = 3^1 = 3$ $\therefore$ Final solution $x \in \{3\}$

Q.16 Solve following log equation $3^{\log_3 \log\sqrt{x}} - \log x + \log^2 x - 3 = 0$

Sol. $\log\sqrt{x} - \log x + \log^2 x - 3 = 0$

$$\Rightarrow \quad \frac{1}{2}\log_{10} x - \log_{10} x + (\log_{10} x)^2 - 3 = 0$$

$$\Rightarrow \quad \log_{10}x - 2\log_{10} x + 2(\log_{10}x)^2 - 6 = 0 \Rightarrow \quad 2(\log_{10}x)^2 - \log_{10}x - 6 = 0$$

$$\Rightarrow \quad 2(\log_{10}x)^2 - 4\log_{10}x + 3\log_{10}x - 6 = 0 \Rightarrow \quad (2\log_{10}x + 3)(\log_{10}x - 2) = 0$$

$$\Rightarrow \quad \log_{10}x = 2 \quad \text{or} \quad \log_{10}x = -\frac{3}{2}$$

$$\Rightarrow \quad x = 10^2 = 100 \quad \text{or} \quad x = 10^{-3/2}$$

Q.17 Solve following log equation $(x-2)^{\log^2(x-2)+\log(x-2)^5-12} = 10^{2\log(x-2)}$

Sol. $\log\ (x-2)^{\log^2(x-2)+\log(x-2)(5-12)} = \log 10^{2\log(x-2)}$ (Let $\log(x-2) = t$)

$$\Rightarrow \quad \log(x-2)(\log^2(x-2) + \log(x-2)^5 - 12)) = \log_{10}10^{2\log(x-2)}$$

$$\Rightarrow \quad t[t^2 + 5t - 12)] = 2t \quad \Rightarrow \quad t^2 + 5t^2 - 14t = 0 \quad \Rightarrow \quad t(t^2 + 5t - 14) = 0$$

$$\Rightarrow \quad t(t+7)(t-2) = 0 \quad \Rightarrow \quad t = 0, -7, 2 \quad \Rightarrow \quad \log(x-2) = 0, -7, 2$$

$$\Rightarrow \quad x = 3, 2 + 10^{-7}, 102$$

Q.18 Solve following log equation $x^{\frac{\log x+5}{3}} = 10^{5+\log x}$

Q.18 Solve following log equation $x^{\frac{\log x+5}{3}} = 10^{5+\log x}$

Sol. $\log x \,.\, \dfrac{\log x + 5}{3} = \log_{10} 10^{5+\log x}$ $\quad \Rightarrow \log x \,.\, \dfrac{\log x + 5}{3} = 5 + \log x$

$$\Rightarrow \log^2 x + 5\log_{10}x = 15 + 3\log_{10} x \quad \Rightarrow \log^2 x + 2\log_{10}x - 15 = 0$$

$$\Rightarrow \log_{10}x(\log_{10}x + 5) - 3(\log_{10}x + 5) = 0 \quad \Rightarrow (\log_{10}x - 3)(\log_{10}x + 5) = 0$$

$$\Rightarrow \log_{10}x = 3 \qquad \log_{10}x = -5$$

$$x = 10^3 \qquad x = 10^{-5}$$

Q.19 Solve following log equation $x^{\frac{\log x+7}{4}} = 10^{(\log x)+1}$

Sol. taking log both the side with base 10

$$\Rightarrow \frac{\log x + 7}{4}.\log x = \log x + 1 \quad \Rightarrow \log^2_{10} x + 3\log_{10}x - 4 = 0$$

$$\Rightarrow \log^2_{10} x + 4\log_{10}x - \log_{10}x - 4 = 0 \quad \Rightarrow (\log_{10}x + 4)(\log_{10}x - 1) = 0$$

$$\Rightarrow x = 10^{-4} \text{ and } x = 10^1$$

Q.20 Solve following log equation $\log^2 x - 3\log x = \log(x^2) - 4$

Sol. $\log^2 x - 3\log x - 2\log x + 4 = 0 \quad \Rightarrow \log_{10}{}^2 x - 5\log_{10}x + 4 = 0$

$$\Rightarrow \log_{10}{}^2 x - \log_{10}x - 4\log_{10}x + 4 = 0 \quad \Rightarrow (\log_{10}x - 4)(\log_{10}x - 1) = 0$$

$$\Rightarrow \log_{10}x = 4 \qquad \log_{10}x = 1$$

$$\Rightarrow x = 10^4 \qquad x = 10$$

$$\Rightarrow x = 10 \text{ or } 10^4$$

Q.21 Solve following log equation $\log_{1/3}x - 3\sqrt{\log_{1/3} x} + 2 = 0$

Sol. $-\log_3 x - 3\sqrt{-\log_3 x} + 2 = 0$

$\Rightarrow (2 - \log_2 x)^2 = \left(3\sqrt{-\log_3 x}\right)^2$

$\Rightarrow 4 + \log_2{}^2 x - 4\log_3 x = -9\log_3 x$

$\Rightarrow (\log_3 x + 4)(\log_3 x + 1) = 0$

$\Rightarrow \log_3 x = -4 \qquad \log_3 x = -1$

$\Rightarrow x = 3^{-4} \qquad x = 3^{-1}$

$\Rightarrow x = \frac{1}{81}, \frac{1}{3}$

Q.22 Solve the value of x : $2(\log_x \sqrt{5})^2 - 3\log_x \sqrt{5} + 1 = 0$

Sol. $\Rightarrow \frac{2}{(\log_{\sqrt{5}} x)^2} - \frac{3}{\log_{\sqrt{5}} x} + 1 = 0$

$\Rightarrow \log^2_{\sqrt{5}} x - 2\log_{\sqrt{5}} x - \log_{\sqrt{5}} x + 2 = 0$

$\Rightarrow \log_{\sqrt{5}} x(\log_{\sqrt{5}} x - 2) - (\log_{\sqrt{5}} x - 2) = 0$

$\Rightarrow (\log_{\sqrt{5}} x - 1)(\log_{\sqrt{5}} x - 2) = 0$

$\Rightarrow \log_{\sqrt{5}} x = 1$ or $\log_{\sqrt{5}} x = 2$

$x = \sqrt{5}$ or $x = \left(\sqrt{5}\right)^2 = 5$

$\Rightarrow \log_{\sqrt{5}} x = 1$ or $\log_{\sqrt{5}} x = 2$

$x = \sqrt{5}$ or $x = \left(\sqrt{5}\right)^2 = 5$

23 Solve the value of x : $(\log 100\,x)^2 + (\log 10\,x)^2 = 14 + \log\frac{1}{x}$

1. $= (\log 100 + \log x)^2 + (\log 10 + \log x)^2 = 14 - \log x$

$= (2 + \log x)^2 + (1 + \log x)^2 = 14 - \log x$

$=$ put $\log x = t$

$= (2 + t)^2 + (1 + t)^2 = 14 - t$

$= 2t^2 + 7t - 9 = 0$

$= 2t^2 + 9t - 2t - 9 = 0$

$= t(2t + 9) - 1(2t + 9) = 0$

$\Rightarrow t = 1 \qquad$ or $\qquad t = -\frac{9}{2}$

$= \log_{10} x = 1 \qquad$ or $\qquad \log_{10} x = -\frac{9}{2}$

$x = 10 \qquad$ or $\qquad x = +10^{-\frac{9}{2}} = \sqrt{10^{-9}}$