RELATION

This chapter deals with establishing binary relation between elements of one set and elements of another set according to some particular rule of relationship.

1. CARTESIAN PRODUCT:

The Cartesian product of two sets A, B is a non-void set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. This is denoted by $A \times B$

$\therefore \quad A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

e.g. $A = \{1, 2\}, \quad B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

Note : (i) $A \times B \neq B \times A$ (Non-commutative)

(ii) $n(A \times B) = n(A)\, n(B)$ and $n(P(A \times B)) = 2^{n(A)n(B)}$

(iii) $A = \phi$ and $B = \phi \Leftrightarrow A \times B = \phi$

(iv) If A and B are two non-empty sets having n elements in common then $(A \times B)$ and $(B \times A)$ have n^2 elements in common.

(v) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(vi) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(vii) $A \times (B - C) = (A \times B) - (A \times C)$

Illustration :

If $n(A) = 7$, $n(B) = 8$ and $n(A \cap B) = 4$, then match the following columns.

(i)	$n(A \cup B)$	*(a)*	56
(ii)	$n(A \times B)$	*(b)*	16
(iii)	$n((B \times A) \times A)$	*(c)*	392
(iv)	$n((A \times B) \cap (B \times A))$	*(d)*	96
(v)	$n((A \times B) \cup (B \times A))$	*(e)*	11

Sol. *(i)* $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 7 + 8 - 4 = 11$ *Ans. (e)*

(ii) $n(A \times B) = n(A)\, n(B) = 7 \times 8 = 56 = n(B \times A)$ *Ans. (a)*

(iii) $n((B \times A) \times A) = n(B \times A).\, n(A) = 56 \times 7 = 392$ *Ans. (c)*

(iv) $n((A \times B) \cap (B \times A)) = (n((A \cap B))^2 = 4^2 = 16$ *Ans. (b)*

(v) $n((A \times B) \cup (B \times A)) = n(A \times B) + n(B \times A) - n(A \times B) \cap (B \times A)$

$= 56 + 56 - 16 = 96$ *Ans. (d)*

Illustration :

If $A = \{2, 4\}$ and $B = \{3, 4, 5\}$, then $(A \cap B) \times (A \cup B)$ is

(1) $\{(2, 2), (3, 4), (4, 2), (5, 4)\}$

(2) $\{(2, 3), (4, 3), (4, 5)\}$

(3) $\{(2, 4), (3, 4), (4, 4), (4, 5)\}$

(4) $\{(4, 2), (4, 3), (4, 4), (4, 5)\}$

Sol. $A \cap B = \{4\}$ and $A \cup B = \{2, 3, 4, 5\}$

$\therefore\ (A \cap B) \times (A \cup B) = \{(4, 2), (4, 3), (4, 4), (4, 5)\}$ *Ans. (4)*

2. RELATION:

Every non-zero subset of $A \times B$ defined a relation from set A to set B.

If R is a relation from $A \to B$

$R : \{(a, b) \mid (a, b) \in A \times B \text{ and } a\,R\,b\}$

Highlights :

Let A and B be two non empty sets and $R : A \to B$ be a relation such that $R : \{(a, b) \mid (a, b) \in R,$ $a \in A \text{ and } b \in B\}$.

(i) 'b' is called image of 'a' under R.

(ii) 'a' is called pre-image of 'b' under R.

(iii) **Domain of R :** Collection of all elements of A which has a image in B or Set of all first entries in $A \times B$.

(iv) **Range of R :** Collection of all elements of B which has a pre-image in A or Set of all second entries in $A \times B$.

Note :

(1) It is not necessary that each and every element of set A has a image in Set B and each and every element of set B has a preimage in Set A

(2) Elements of set A having image in B is not necessary unique.

(3)Basically relation is the number of subsets of $A \times B$

<u>number of relations</u> = no. of ways of selecting a non zero subset of $A \times B$

$$= {}^{mn}C_0 + {}^{mn}C_1 + \dots\dots\dots + {}^{mn}C_{mn}$$
$$= 2^{mn}$$

Illustration :

Given $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 5\}$. A relation defined

$aRb \Rightarrow$ a and b are relatively prime or co-prime (i.e. HCF is 1),

find domain and range of R.

Sol. $R = \{(1, 2), (1, 4), (1, 5), (2, 5), (3, 2), (3, 4), (3, 5), (4, 5), (5, 2), (5, 4)\}$]

Domain of R $\{1,2,3,4,5\}$

Range of R $\{2,4,5\}$

Illustration :

A = {Jaipur, Patna, Kanpur, Lucknow} and B = {Rajasthan, Uttar Pradesh, Bihar}

$aRb \Rightarrow a$ is capital of b, $a \in A$ and $b \in B$

Sol. R = {(Jaipur, Rajasthan), (Patna, Bihar), (Lucknow, Uttar Pradesh) }

Illustration :

If $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8\}$

Relation is $aRb \Rightarrow a > b$, $a \in A$, $b \in B$

Sol. $R = \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)\}$

Domain $= \{3, 5, 7\}$

Range $= \{2, 4, 6\}$

Representation of a Relation :

1. **Roster form :** In this form we represent set of all ordered pairs (a,b) such that $(a,b) \in R$ where $a \in A$, $b \in B$
2. **Set builder notation :** Here we denote the relation by the rule which co relates the two set
3. **Arrow - diagram (Mapping):** This the pictorial notation of any relation .

Illustration :

Let $A = \{-2, -1, 4\}$ $B = \{1, 4, 9\}$

A relation from A to B i.e. a R b is defined as a is less than b.

This can be represented in the following ways.

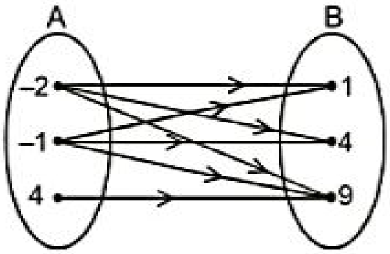
1. **Roster form :**

   $R = \{(-2, 1), (-2, 4), (-2, 9), (-1, 1), (-1, 4), (-1, 9), (4, 9)\}$

2. **Set builder notation :**

   $R = \{(a, b): a \in A \text{ and } b \in B, a \text{ is less than } b\}$

3. **Arrow - diagram :**



<u>Empty relation:</u> No elements of A is related to any elements of A.

<u>Universal relation:</u> Each elements of A is related to every element of A.

3. INVERSE RELATION:

If relation R is defined from A to B then the inverse relation would be defined from B to A, i.e

$R: A \to B \quad \Rightarrow \quad aRb$ where $a \in A, b \in B$

$R^{-1}: B \to A \quad \Rightarrow \quad bRa$ where $a \in A, b \in B$

Here Doman of R = Range of R^{-1}

and Range of R = Domain of R^{-1}

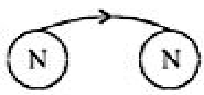
$\therefore \quad R^{-1} = \{(b, a) \mid (a, b) \in R\}$

A relation R is defined on the set of 1st ten natural numbers.

e.g. N is a set of first 10 natural nos. $\therefore N = \{1, 2, 3, ..., 10\}$ & $a, b \in N$

$aRb \quad \Rightarrow \quad a + 2b = 10$

$R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$



$R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

4. IDENTITY RELATION:

A relation defined on a set A is said to be an Identity relation if each & every element of A is related to itself & only to itself.

e.g. A relation defined on the set of natural nos. is

$aRb \quad \Rightarrow \quad a = b$ where a & $b \in N$

$R = \{(1, 1), (2, 2), (3, 3), ...\}$

R is an Identity relation

5. CLASSIFICATION OF RELATIONS:

Types of Relations

Reflexive | Symmetric | Transitive

(I) <u>Reflexive</u>: A relation defined on a set A is said to be an Identity relation if each & every element of A is related to itself.

i.e. if $(a, b) \in R$ then $(a, a) \in R$. However if there is a single ordered pair of $(a, b) \in R$ such $(a, a) \notin R$ then R is not reflexive.

e.g. A relation defined on (set of natural numbers)

aRb $\Rightarrow$ 'a' divides 'b' $a, b \in N$

R would always contain (a, a) because every natural number divides itself and hence it is a reflexive relation.

Note: Every Identity relation is a reflexive relation but every reflexive relation need not be an Identity.

(II) **<u>Symmetric</u>:** A relation defined on a set is said to be symmetric if aRb $\Rightarrow$ bRa.

If $(a, b) \in R$ then (b, a) must be necessarily there in the same relation.

EXAMPLES:

A relation defined on the set of lines.

(1) aRb $\Rightarrow$ $a \parallel b$

It is a symmetric relation because if line is || to 'b' then the line 'b' is || to 'a'.

where $(a, b) \in L$ {L is a set of || lines}

(2) $L_1RL_2 \Rightarrow L_1 \perp L_2$ It is a symmetric relation

$L_1, L_2 \in L$ {L is a set of lines}

(3) aRb $\Rightarrow$ 'a' is borther of 'b' is not a symmetric relation as 'a' may be sister of 'b'.

(4) aRb $\Rightarrow$ 'a' is a cousin of 'b'. This is a symmetric relation.

If R is symmetric

(1) $R = R^{-1}$

(2) Rangle of R = Domain of R

(III) **<u>Transitive</u>:** A relation on set A is said to be transitive if aRb and bRc implies aRc

i.e. $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Here a, b, c need not be distinct.

EXAMPLES:

(1) A relation R defined on a set of natural numbers N with rule aRb $\Rightarrow$ $a < b$

R: {(1, 2), (1, 1)}.

In this relation a, b, c are not distinct but it is transitive. It is neither reflexive nor symmetric as (2, 1) is missing. Minium number of ordered pair that must be added to make it reflexive, symmetric and transitive is 2 i.e. (2, 1) and (2, 2).

(2) Only Transitive $R = \{(x, y) \mid x < y, \quad x \in N, y \in N\}$

Only Symmetric $R = \{(x, y) \mid x + y = 10, \quad x \in N, y \in N\}$

6. EQUIVALENCE RELATION:

If a relation is Reflexive, Symmetric and Transitive then it is said to be an equivalence relation.

EXAMPLES:

(1) A relation defined on N

$xRy \Rightarrow x = y$

R is an equivalence relation.

(2) Relation defined on the set of integer (I)

Prove that: $xRy \Rightarrow (x-y)$ is even is an equivalence relation.

Asking: $A = \{1, 2, 3, 4\}$; $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$ then

(A) $R \cap S \cap \overline{T}$ (B*) $R \cap T \cap \overline{S}$ (C) $S \cap T \cap \overline{R}$ (D) $R \cap S \cap T$

(3) $R = \{(1, 2), (2, 3)\}$ add minimum number of ordered pairs to make it an equivalence relation.

$\{(1, 1), (2, 2), (3, 3), (2, 1), (3, 2), (1, 3), (3, 1)\} = 7$

(4) $A = \{1, 2, 3, ..., 13, 14\}$

$R = \{(x, y) \mid 3x - y = 10\}$ $\overline{R} \cap \overline{S} \cap \overline{T}$

$R = \{(x, y) \mid x \text{ is factor of } y\}$ $R \cap \overline{S} \cap T$

$R = \{(x, y) \mid x \text{ is father of } y\}$ $\overline{R} \cap \overline{S} \cap \overline{T}$

MATHEMATICAL INDUCTION

INTRODUCTION :

In algebra, there are certain results that are formulated with n number of terms in them, where n is a natural number (i.e. a positive integer). Those results can be proved by a specific technique, known as the principle of mathematical induction. We use the symbol P(n) (read "P of n") to denote some proposition which depends on the positive integer n. For example, P(n) might denote the sum of the first n odd positive numbers, that is

$$1 + 3 + 5 + 7 + 9 + \ldots\ldots + (2n - 1) = n^2$$

where $n = 1, 2, 3, \ldots\ldots\ldots, n$.

• FIRST PRINCIPLE OF MATHEMATICAL INDUCTION :

The proof of the proposition P(n) by mathematical induction for all $n \in N$ consists of the following three steps :

Step-1. Verification step

Verify that the proposition P(n) is true for $n = 1$, i.e., the first natural number or the smallest positive integer. This is also called the basic step of the induction.

Step-2. Induction step

Assume that the proposition will also be true for some $n = k \geq 1$, i.e. we assume P(k) to be true. This is called the induction step.

Step-3. Generalization step

If P(k) is true, then prove that the proposition is also true for $n = (k + 1)$, which is the next positive integer (i.e. the next natural number), i.e. we have to prove that $P(k + 1)$ must also be true. In this step we prove that the implication $P(k) \Rightarrow P(k + 1)$ is true.

Next we generalize the result by saying that, since the proposition is proved to be true for $n = k + 1$, then it must also be true for $n = k$, and hence the proposition will be true for all n belonging to the set of natural numbers.

• SECOND PRINCIPLE OF MATHEMATICAL INDUCTION :

Step-1. Verification step

Verify that the proposition P(n) is true for $n = r$, where r is some fixed integer.

Step-2. Induction step

Assume that the proposition P(n) is true for $n = r, r + 1, r + 2, \ldots\ldots, m$.

Step-3. Generalization step

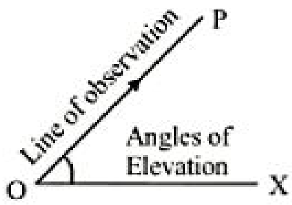
Prove that the proposition P(n) is true for $n = m + 1$. Thus, if true, we generalize the result by saying that since the proposition is true for $n = M + 1$, then it must also be true for $n = r, r + 1, r + 2, \ldots\ldots\ldots, m$ as assumed in Step 2. Thus, the proposition is true for all $n \geq r$ belonging to the set of natural numbers.

HEIGHT & DISTANCE

ANGLES OF ELEVATION AND DEPRESSION :

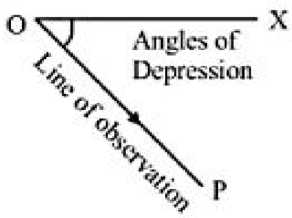
(i) Angle of elevation :

Let P be an object and OX be a horizontal line; an observer at O to perceive the object P has to elevate his eye from the direction OX to the direction OP. As such, we define $\angle XOP$ as the **angle of elevation** of P at O.



(ii) Angle of depression :

Let P be an object and OX be a horizontal line; an observer at O to perceive the object P has to depress his eye from the direction OX to the direction OP. As such, we define $\angle XOP$ as the **angle of depression** of P at O.



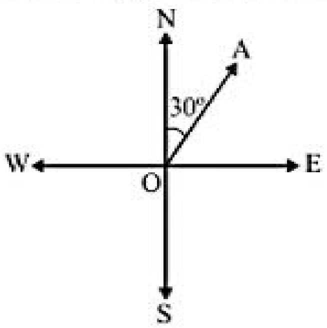
(iii) Angular elevation :

If PM is perpendicular to OX, then the angle XOP is also called the altitude or the **angular elevation** of MP at O.

(iv) Bearing of a point :

Let NS and EW stand for lines in the north-south and east-west directions respectively then the acute angle which the line OA makes with NS is called the **bearing** of the point A from O.

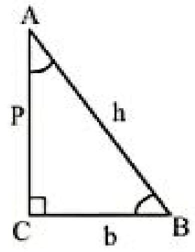
The bearing of A may be indicated precisely, by giving the size of the angle and specifying whether it is measured from ON (or OS) and whether to east (or west). For example, in figure, OA is in the direction 30° east of north. Thus, the bearing is written as N 30° E.



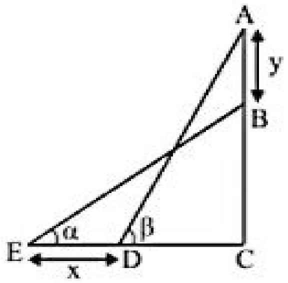
To express one side of a right angled triangle in terms of other sides :

Let AB = h (hypotenuse) and let $\angle ABC = \theta$. The side opposite to θ is a perpendicular p and the remaining side is base b.

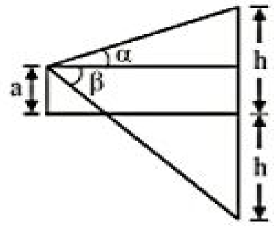
$b = p(\cot\theta)$
$p = b(\tan\theta)$
$p = h(\sin\theta)$
$b = h(\cos\theta)$



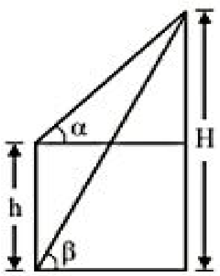
(a) If AD = BE, then $x = y\tan\left(\frac{\alpha+\beta}{2}\right)$



(b) $h = \frac{a\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$



(c) $h = \frac{H\sin(\beta-\alpha)}{\cos\alpha\cos\beta}$ and $H = \frac{h\cot\alpha}{\cos\alpha-\cos\beta}$



STATISTICS

1. DEFINITION :

STATISTICS : A set of concepts, rules and procedures that help us to :

→ **Organize** numerical information in the form of tables, graphs and charts;

→ **Understand** statistical techniques underlying decisions that affect our lives and well-being; and

→ **Make** informed decisions.

2. DATA :

Facts, observations and information that come from investigations.
Generally three types of data are used

(i) Ungrouped data, Raw data or individual series :

(ii) Discrete frequency or ungrouped data :
Definition :
Data consist of n distinct values $x_1, x_2 \ldots\ldots, x_n$ occuring with frequency $f_1, f_2, \ldots\ldots, f_n$ respectively. This data in tabular form is called discrete freqency distribution.

(iii) Continuous frequency or grouped data :
Definition :
A continuous frequency Distribution is a series in which the data are classified into different class intervals without gaps along with their respective frequencies.

3. MEASURES OF CENTRAL VALUE :

Measure of central value gives rough idea about where data points are centred. Mean, mode, median are three measure of central tendency.

(A) MEAN :

The mean is the most common measure of central tendency and the one that can be mathematically manipulated. It is defined as the **average** of a distribution is equal to the $\Sigma X / N$. Simply, the mean is computed by summing all the scores in the distribution (ΣX) and dividing that sum by the total number of scores (N).

(I) Arithmetic mean of individual series (Ungrouped data) :

If the series in this case be $x_1, x_2, x_3, \ldots\ldots, x_n$; then the arithmetic mean $\bar{x}$ is given by

i.e., $$\bar{x} = \frac{\text{Sum of the series}}{\text{Number of terms}} = \frac{x_1 + x_2 + x_3 + \ldots\ldots + x_n}{N} = \frac{1}{N}\sum_{i=1}^{n} x_i .$$

(II) Arithmetic mean for discrete frequency distribution :

If the terms of the given series be $x_1, x_2, \ldots\ldots\ldots, x_n$ and the corresponding frequencies be $f_1, f_2, \ldots\ldots, f_n$, then the arithmetic mean $\bar{x}$ is given by,

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots\ldots\ldots + f_nx_n}{N} = \frac{1}{N}\sum_{i=1}^{n} f_ix_i . \qquad \left(\sum_{i=1}^{n} f_i = N\right)$$

(III) Arithmetic mean for grouped or continuous frequency distribution :

$$\text{Arithmetic mean } (\bar{x}) = A + \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A),$$

where A = assumed mean, f = frequency and x − A = deviation of each item from the assumed mean.

(IV) Combined Arithmetic mean :

If $\bar{x}_i$ $(i = 1, 2, \ldots\ldots, k)$ are the means of k-component series of sizes n_i, $(i = 1, 2, \ldots, k)$ respectively, then the mean $\bar{x}$ of the composite series obtained on combining the component series is given by the formula

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots\ldots\ldots + n_k\bar{x}_k}{n_1 + n_2 + \ldots\ldots + n_k} = \frac{\sum_{i=1}^{n} n_i\bar{x}_i}{\sum_{i=1}^{n} n_i} .$$

(V) Weighted Arithmetic Mean :

Weighted arithmetic mean refers to the arithmetic mean calculated after assigning weights to different values of variable. It is suitable where the relative importance of difference items of variable is not same. Weighted Arithmetic Mean is give by

$$\bar{X}_W = \frac{\sum_{i=1}^{n} W_iX_i}{\sum_{i=1}^{n} W_i}$$

Properties of arithmetic mean :

If each of the values of a variable 'X' is increased of decreased by some constant k, then arithmetic mean also increased of decreased by k.

Similarly when the value of the variable 'X' are multiplied/divided by constant say k, arithmetic mean also multiplied /divided by the same quantity k.

Illustration :

The mean weight of 150 persons in a group is 60 kg. The mean weight of men in the group is 70 kg and that of the women is 55 kg. Find the number of men and women.

Sol. Number of person = 150; their mean weight = 60 kg;

mean weight of men $(\bar{x}_1) = 70$ kg and

mean weight of women $(\bar{x}_2) = 55$ kg

Let n_1 and n_2 be the number of men and number of women respectively.

We know that the total number of persons $(n_1 + n_2) = 150$ or $n_2 = 150 - n_1$.

We also know that the mean weight of all persons

$$(\bar{x}) = \frac{(n_1\bar{x}_1 + n_2\bar{x}_2)}{n_1 + n_2}$$

or $$60 = \frac{70n_1 + 55n_2}{150}$$

or $$3n_1 = (1800 - 1650) = 150$$

or $$n_1 = 50 \text{ and } n_2 = 100$$

Illustration :

Find the mean of the following data :

Marks obtained	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Number of students	2	3	8	14	8	3	2

Sol. <u>Method-1 :</u>

Marks obtained	Number of students	Mid - points	$f_i x_i$
10 – 20	2	15	30
20 – 30	3	25	75
30 – 40	8	35	280
40 – 50	14	45	630
50 – 60	8	55	440
60 – 70	3	65	195
70 – 80	2	75	150
	40		1800

$$N = \sum_{i=1}^{7} f_i = 40, \ \sum_{i=1}^{7} f_i x_i = 1800$$

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{7} f_i x_i = \frac{1800}{40} = 45$$

Method-2 :

Asumed mean $a = \frac{10+80}{2} = 45, \quad h = 10$

Marks obtained	Number of students	Mid - points	$d_i = \frac{x_i - 45}{10}$	$f_i d_i$
10 – 20	2	15	–3	–6
20 – 30	3	25	–2	–6
30 – 40	8	35	–1	–8
40 – 50	14	45	0	0
50 – 60	8	55	1	8
60 – 70	3	65	2	6
70 – 80	2	75	3	6
	40			0

$$\bar{x} = a + \frac{\sum_{i=1}^{7} f_i d_i}{N} = 45 + \frac{0}{40} \times 10 = 45$$

(B) MEDIAN :

(a) **Definition :** The median is the score that divides the distribution into halves; half of the scores are above the median and half are below it when the data are arranged in numerical order. The median is also referred to as the score at the 50^{th} percentile in the distribution.

Calculation of median :

(i) **Individual series :** If the data is raw, arrange in ascending or descending order. Let n be the number of observations.

If n is odd, Median = value of $\left(\frac{n+1}{2}\right)^{th}$ item.

If n is even, Median $= \frac{1}{2}\left[\text{value of } \left(\frac{n}{2}\right)^{th} \text{ item + value of } \left(\frac{n}{2}+1\right)^{th} \text{ item}\right]$.

(ii) **Discrete series :** In this case, we first find the cumulative frequencies of the variable arranged in ascending or descending order and the median is given by Median $= \left(\frac{n}{2}+1\right)^{th}$ observation, where n is the cumulative frequency.

(iii) **For grouped or continuous distributions :** In this case, following formula can be used.

$$\textbf{Median} = l + \frac{\left(\frac{N}{2} - C\right)}{f} \times i$$

where l = Lower limit of the median class

f = Frequency of the median class

N = The sum of all frequencies

i = The width of the median class

C = The cumulative frequency of the class preceding to median class.

(b) **Quartile :** As median, divides a distribution into two equal parts, similarly the quartiles, quantiles, deciles and percentiles divide the distribution respectively into 4, 5, 10 and 100 equal parts. The j^{th} quartile is given by $Q_j = l + \left(\frac{j\frac{N}{10} - C}{f}\right) i$.

Illustration :

The marks obtained by 10 students in an examination are 22, 26, 14, 30, 18, 11, 35, 41, 12, 32. What is the median mark?

Sol. Number of students (n) = 10 and marks obtained by them = 22, 26, 14, 30, 18, 11, 35, 41, 12, 32

Arranging the given marks in the ascending order, we get 11, 12, 14, 18, 22, 26, 30, 32, 35, 41.

Since the number of students is even, therefore median of their marks

$$= \text{Arithmetic mean of } \left(\frac{10}{2}\right) \text{ and } \left(\frac{10+2}{2}\right) \text{ marks}$$

$$= \text{Arithmetic mean of } 5^{th} \text{ and } 6^{th} \text{ marks}$$

$$= \frac{22+26}{2} = 24 \qquad \textbf{Ans.}$$

Illustration :

Calculate the median of the following data:

Wages per week (in Rs)	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Number of worker	4	6	10	20	10	6	4

Sol. Calculation of Mean Deviation from Median

Wages per week (in Rs.)	Mid-Values (x_i)	Frequency (f_i)	Cumulative Frequency
10-20	15	4	4
20-30	25	6	10
30-40	35	10	20
40-50	45	20	40
50-60	55	10	50
60-70	65	6	56
70-80	75	4	60
		$N = \Sigma f_i = 60$	

Here, $N = 60$. So $\frac{N}{2} = 30$.

The cumulative frequency just greater than $\frac{N}{2} = 30$ is 40 and the corresponding class is 40-50.

So, 40-50 is the median class.

$\therefore \quad l = 40, f = 20, h = 10, F = 20$

Now, $\text{Median} = l + \frac{\frac{N}{2} - F}{f} \times h = 40 + \frac{30 - 20}{20} \times 10 = 55$ **Ans.**

(C) MODE :

Mode is the most frequent score in the distribution. A distribution where a single score is most frequent has one mode and is called unimodal. When there are ties for the most frequent score, the distribution is bimodal if two scores tie or multimodal if more than two scores tie.

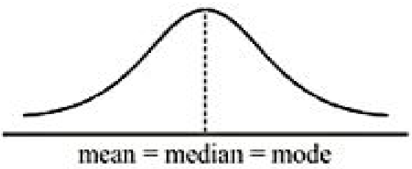
Mode for continuous series

$$\text{Mode} = l_1 + \left[\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right] \times i$$

Where, l_1 = The lower limit of the model class
f_1 = The frequency of the model class
f_0 = The frequency of the class preceding the model class
f_2 = The frequency of the class succeeding the model class
i = The size of the model class.

Symmetric distribution :

A distribution is a symmetric distribution if the values of mean, mode and median coincide. In a symmetric distribution frequencies are symmetrically distributed on both sides of the centre point of the frequency curve.

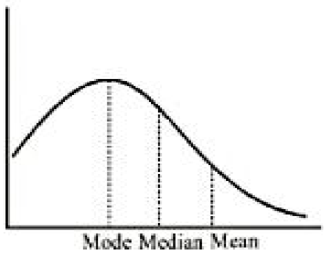


A distribution which is not symmetric is called a skewed distribution. In a moderately asymmetric distribution, the interval between the mean and the median is approximately one-third of the interval between the mean and the mode i.e., when have the following empirical relation between them,
Mean – Mode = 3 (Mean – Median) ⇒ Mode = 3 Median – 2 Mean. it is known as Empirical relation.

Positively skewed :

A distribution is positively skewed when is has a tail extending out to the right (larger numbers) When a distribution is positively skewed, the mean is greater than the median reflecting the fact that the mean is sensitive to each score in the distribution and is subject to large shifts when the sample is small and contains extreme scores.

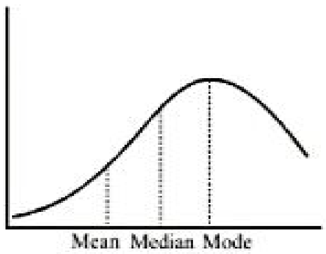
Mean > Median > Mode



Negatively skewed :

A negatively skewed distribution has an extended tail pointing to the left (smaller numbers) and reflects bunching of numbers in the upper part of the distribution with fewer scores at the lower end of the measurement scale.

Mean < Median < Mode.



In a moderately asymmetric distribution, the interval between the mean and the median is approximately one-third of the interval between the mean and the mode i.e., when have the following empirical relation between them,

Empirical formula : mode = 3 median – 2 mean

$$\text{Coefficient of skewness} = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

Limitations of central values :

An average, such as the mean or the median only locates the centre of the data and does not tell us anything about the spread of the data.

4. MEASURES OF SPREAD OR DISPERSION :

Measures of variability provide information about the degree to which individual scores are clustered about or deviate from the average value in a distribution i.e.,

The degree to which numerical data tend to spread about an average value is called the dispersion of the data. The four measure of dispersion are

(i)	Range	(ii)	Mean deviation
(iii)	Variance	(iv)	Standard deviation

Important Note :

(a) A small value for a measure of dispersion indicate that the data are clustered closely (the mean is therefore representative of the data).

(b) A large value of dispersion indicates that the mean is not reliable (it is not representative of the data).

(i) Range :

The simplest measure of variability to compute and understand is the range. The range is the difference between the highest and lowest score in a distribution. Because it is based solely on the most extreme scores in the distribution and does not fully reflect the pattern of variation within a distribution, the range is a very limited measure of variability.

$$\text{Coefficient of range} : \frac{L - S}{L + S}$$

L = Largest value

S = Smallest value

(ii) Mean deviation :

The arithmetic average of the deviations (all taking positive) from the mean, median or mode is known as mean deviation.

(a) Mean deviation from ungrouped data (or individual series)

Mean deviation = $\frac{1}{N}\sum_{i=1}^{n}|x_i - M|$.

Where $\sum_{i=1}^{n}|x_i - M|$ is the sum of modulus of the deviation of the variate from the mean (mean, median or mode) and N is the number of terms.

(b) Mean deviation from continuous series :

Here first of all we find the mean from which deviation is to be taken. Then we find the deviation $|x_i - M|$ of each variate from the mean M and multiply these deviations by the corresponding frequency

So, Mean deviation = $\frac{1}{N}\sum_{i=1}^{n}f_i|x_i - M|$, where $N = \sum_{i=1}^{n}f_i$.

Illustration :

The scores of a batsman in ten innings are : 38, 70, 48, 34, 42, 55, 63, 46, 54, 44. Find the mean deviation about the median.

Sol. Arranging the data in ascending order, we have

34, 38, 42, 44, 46, 48, 54, 55, 63, 70

Here n = 10. So, median is the A.M. of 5th and 6th observations.

$\therefore$ Median, $M = \left(\frac{46+48}{2}\right) = 47$

Calculation of Mean Deviation

x_i	$\lvert d_i\rvert = \lvert x_i - 47\rvert$
38	9
70	23
48	1
34	13
42	5
55	8
63	16
46	1
54	7
44	3
Total	$\Sigma \lvert d_i\rvert = 86$

$\therefore$ M.D. = $\frac{1}{n}\Sigma|d_i| = \frac{86}{10} = 8.6$ **Ans.**

Illustration :

Calculate the mean deviation from the median of the following data:

Age	16 – 20	21 – 25	26 – 30	31 – 35	36 – 40	41 – 45	46 – 50	51 – 55
Number	5	6	12	14	26	12	16	9

Since given data is not continuous frequency distribution but we can make it continuous frequency distribution by subtracting lower limit by 0.5 and adding 0.5 to upper limit of every group.

Sol. Calculation of Mean Deviation from Median

Age	Mid-Values (x_i)	Frequency (f_i)	Cumulative Frequency	$\lvert d_i \rvert = \lvert x_i - 38 \rvert$	$f_i \lvert d_i \rvert$
15.5-20.5	18	5	5	20	100
20.5-25.5	23	6	11	15	90
25.5-30.5	28	12	23	10	120
30.5-35.5	33	14	37	5	70
35.5-40.5	38	26	63	0	0
40.5-45.5	43	12	75	5	60
45.5-50.5	48	16	91	10	160
50.5-55.5	53	9	100	15	135
		$N = \Sigma f_i = 100$			$\Sigma f_i \lvert d_i \rvert = 735$

Here, $N = 100$. So $\frac{N}{2} = 50$.

The cumulative frequency just greater than $\frac{N}{2} = 50$ is 63 and the corresponding class is 35.5-40.5.

So, 35.5-40.5 is the median class.

$\therefore \quad l = 35.5,\ f = 26,\ h = 5,\ C = 37$

Now, $$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times h = 35.5 + \left(\frac{50-37}{26}\right) \times 5 = 38$$ **Ans.**

$$\text{Mean Deviation from median} = \frac{\Sigma f_i \lvert d_i \rvert}{N} = \frac{735}{100} = 7.35$$ **Ans.**

(iii) Variance or Var(X) or σ^2 :

The variance is a measure based on the deviations of individual scores from the mean. As noted in the definition of the mean, however, simply summing the deviations will result in a value of 0. The get around this problem the variance is based on squared deviations of scores about the mean. When the deviations are squared, the rank order and relative distance of scores in the distribution is preserved while negative values are eliminated. Then to control for the number of subjects in the distribution, the sum of the squared deviations, $\sum(X-\overline{X})^2$, is divided by N(population). *The average of the sum of the squared deviations is called the variance.*

(a) Variance of individual observations :

If $x_1, x_2, \ldots\ldots, x_n$ are n values of a variable X, then

$$\mathrm{Var}(X) = \frac{1}{n}\left|\sum_{i=1}^{n}(x_i-\overline{X})^2\right| = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2$$

= Mean of squares – Squares of Mean

(b) Variance of discrete frequency distribution :

If $x_1, x_2, \ldots\ldots, x_n$ are n values of a variable X and corresponding frequencies of them are $f_1, f_2, \ldots\ldots f_n$

$$\mathrm{Var}(X) = \frac{1}{N}\left|\sum_{i=1}^{n}f_i(x_i-\overline{X})^2\right| = \frac{1}{N}\sum_{i=1}^{n}f_ix_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n}f_ix_i\right)^2 \qquad \left(\sum_{i=1}^{n}f_i = N\right)$$

(c) Variance of a grouped or continuous frequency distribution :

$$\mathrm{Var}(X) = h^2\left[\frac{1}{N}\sum f_iu_i^2 - \left(\frac{1}{N}\sum f_iu_i\right)^2\right] \qquad u_i = \frac{x_i-\overline{X}}{h}$$

where h = Class width

Properties :

(1) If $x_1, x_2, x_3, \ldots, x_n$ be n values of a variable X. If these values are changed to $x_1+a, x_2+a, \ldots x_n+a$, where $a \in R$, then the variance remains unchanged.

(2) If $x_1, x_2, \ldots\ldots, x_n$ values of a variable X and let 'a' be a non-zero real number. Then, the variance of the observation $ax_1, ax_2, \ldots\ldots, ax_n$ is $\mathbf{a^2\,Var(X)}$.

(iv) Standard Deviation :

The standard deviation (s or σ) is defined as the positive square root of the variance. The variance is a measure in squared units and has little meaning with respect to the data. Thus, the standard deviation is a measure of variability expressed in the same units as the data. The standard deviation is very much like a mean or an "average" of these deviations.

Combined Standard Deviation :

If there are two sets of observations containing n_1 & n_2 items with respective mean $\bar{x}_1$ & $\bar{x}_2$ and standards deviations σ_1 & σ_2, then the mean $\bar{x}$ and the standard deviations of $n_1 + n_2$ observations, taken together, are

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$\sigma^2 = \frac{1}{n_1 + n_2}\left[n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)\right]$$

where $d_1 = \bar{x} - \bar{x}_1$, $d_2 = \bar{x} - \bar{x}_2$

Illustration :

Calculate the mean and standard deviation of first n natural numbers.

Sol. Here $x_i = i = i = 1, 2, \ldots\ldots, n$. Let $\overline{X}$ be the mean and σ be the S.D. Then,

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}(1 + 2 + 3 + \ldots\ldots + n)$$

$$\Rightarrow \quad \overline{X} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

and $$\sigma^2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^{\,2}\right) - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \quad \Rightarrow \quad \sigma^2 = \frac{1}{n}(1^2 + 2^2 + \ldots\ldots + n^2) - \left(\frac{n+1}{2}\right)^2$$

$$\Rightarrow \quad \sigma^2 = \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2 \quad \Rightarrow \quad \sigma^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12} \ \textbf{Ans.}$$

Illustration :

The mean and variance of 7 observations are 8 and 16 respectively. If 5 of the observations are 2, 4, 10, 12, 14, find the remaining two observations.

Sol. Let x and y be the remaining two observation. Then,

Mean = 8

$$\Rightarrow \quad \frac{2+4+10+12+14+x+y}{7} = 8 \quad \Rightarrow \quad 42 + x + y = 56$$

$$\Rightarrow \quad x + y = 14 \qquad \text{.....(i)}$$

Variance = 16

$$\Rightarrow \quad \frac{1}{7}(2^2 + 4^2 + 10^2 + 12^2 + 14^2 + x^2 + y^2) - (\text{Mean})^2 = 16$$

$$\Rightarrow \quad \frac{1}{7}(4 + 16 + 100 + 144 + 196 + x^2 + y^2) - 64 = 16 \Rightarrow 460 + x^2 + y^2 = 7 \times 80$$

$$\Rightarrow \quad x^2 + y^2 = 100 \qquad \text{.....(ii)}$$

Now, $(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$

$\Rightarrow \quad 196 + (x - y)^2 = 2 \times 100 \Rightarrow (x - y)^2 = 4 \Rightarrow x - y = \pm 2$

If $x - y = 2$, then $x + y = 14$ and $x - y = 2 \Rightarrow x = 8, y = 6$

If $x - y = -2$, then $x + y = 14$ and $x - y = -2 \Rightarrow x = 6, y = 8$

Hence, the remaining two observations are 6 and 8.

Illustration :

Find the variance and standard deviation for the following distribution:

Classes	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100
Frequency	3	7	12	15	8	3	2

Sol. Calculation of Variance and Standard Deviation

Class	Frequency (f_i)	Mid-point (x_i)	$y_i = \frac{x_i - 65}{10}$	y_i^2	$f_i y_i$	$f_i y_i^2$
30-40	3	35	–3	9	–9	27
40-50	7	45	–2	4	–14	28
50-60	12	55	–1	1	–12	12
60-70	15	65	0	0	0	0
70-80	8	75	1	1	8	8
80-90	3	85	2	4	6	12
90-100	2	95	3	9	6	18
	N = 50				–15	105

Therefore $\bar{x} = A + \frac{\sum f_i y_i}{50} \times h = 65 - \frac{15}{50} \times 10 = 62$

variance $\sigma^2 = \frac{h^2}{N^2}\left[N\sum f_i y_i^2 - \left(\sum f_i y_i\right)^2\right] = \frac{(10)^2}{(50)^2}\left[50 \times 105 - (-15)^2\right] = \frac{1}{25}[5250 - 225] = 201$

and standard deviation $(\sigma) = \sqrt{201} = 14.18$ **Ans.**

Illustration :

The mean and standard deviation of 20 observations are found to be 10 and 2 respectively. On rechecking, it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases:

(i) If the wrong item is omitted.

Sol. We have , $n = 20$, $\overline{X} = 10$ and $\sigma = 2$

$$\therefore \quad \overline{X} = \frac{1}{n}\Sigma x_i \quad \Rightarrow \quad \Sigma x_i = n\overline{X} = 20 \times 10 = 200 \quad \Rightarrow \quad \text{Incorrect } \Sigma x_i = 200$$

$$\text{and,} \quad \sigma = 2 \quad \Rightarrow \quad \sigma^2 = 4 \quad \Rightarrow \quad \frac{1}{n}\Sigma x_i^2 - (\text{Mean})^2 = 4$$

$$\Rightarrow \quad \frac{1}{20}\Sigma x_i^2 - 100 = 4 \quad \Rightarrow \quad \Sigma x_i^2 = 104 \times 20 \quad \Rightarrow \quad \text{Incorrect } \Sigma x_i^2 = 2080$$

(i) When 8 is omitted from the data.

If 8 is omitted from the data, then 19 observations are left.

Now Incorrect $\Sigma x_i = 200 \Rightarrow$ Correct $\Sigma x_i + 8 = 200 \Rightarrow$ Correct $\Sigma x_i = 192$

and Incorrect $\Sigma x_i^2 = 2080 \Rightarrow$ Correct $\Sigma x_i^2 + 8^2 = 2080 \Rightarrow$ Correct $\Sigma x_i^2 = 2016$

$$\therefore \quad \text{Correct mean} = \frac{192}{19} = 10.10$$

$$\Rightarrow \quad \text{Correct variance} = \frac{1}{19}(\text{Correct } \Sigma x_i^2) - (\text{Correct mean})^2$$

$$\Rightarrow \quad \text{Correct variance} = \frac{2016}{19} - \left(\frac{192}{19}\right)^2$$

$$\text{Correct variance} = \frac{38304 - 36864}{361} = \frac{1440}{361}$$

$$\therefore \quad \text{Correct standard deviation} = \sqrt{\frac{1440}{361}} = \frac{12\sqrt{10}}{19} = 1.997$$

Analysis of Frequency Distributions :

Measures of dispersion are unable to compare two or more series which are measured in different units even if they have the same mean. Thus, we require those measures which are independent of the units. The measure of variability which is independent of units is called coefficient of variation (C.V.). The coefficient of variation is defined as

$$\text{C.V.} = \frac{\sigma}{\overline{X}} \times 100$$

where σ and $\overline{X}$ are the standard deviation and mean of the data.

For comparing the variability of two series, we calculate the coefficient of variation for each series. The series having greater C.V. is said to be more variable or conversely less consistent, less uniform less stable or less homogeneous than the other and the series having lesser C.V. is said to be more consistent (or homogeneous) than the other.

Illustration :

The following values are calculated in respect of heights and weights of the students of a section of Class XI :

	Height	Weight
Mean	162.6 cm	52.36
Variance	127.69 cm^2	23.1361 kg^2

Can we say that the weights show greater variation than the heights ?

Sol. To compare the variability, we have to calculate their coefficients of variation

Given Variance of height = 127.69 cm^2

Therefore Standard deviation of height $\sqrt{127.69}$ cm = 11.3 cm

Also Variance of weight = 23.1361 kg^2

Therefore Standard deviation of weight = $\sqrt{23.1361}$ kg = 4.81 kg

Now, the coefficient of variations (C.V.) are given by

$$\text{(C.V.) in heights} = \frac{\text{Standard Deviation}}{\text{Mean}} \times 100$$

$$= \frac{11.3}{162.6} \times 100 = 6.95$$

and $$\text{(C.V.) in weight} = \frac{4.81}{52.36} \times 1000 = 9.18$$

Clearly C.V. in weights is greater than the C.V. in heights

Therefore, we can say that weights show more variability than heights.

IMPORTANT DEFINITIONS :

1. Raw Data :

Data collected in original form.

2. Frequency :

The number of times a certain value or class of values occurs.

3. Frequency Distribution :

The organization of raw data in table form with classes and frequencies.

4. Categorical Frequency Distribution :

A frequency distribution in which the data is only nominal or ordinal.

5. Ungrouped Frequency Distribution :

A frequency distribution of numerical data. The raw data is not grouped.

6. Grouped Frequency Distribution :

A frequency distribution where several numbers are grouped into one class.

7. Class Limits :

Separate one class in a grouped frequency distribution from another. The limits could actually appear in the data and have gaps between the upper limit of one class and the lower limit of the next.

8. Class Boundaries :

Separate one class in a grouped frequency distribution from another. The boundaries have one more decimal place than the raw data and therefore do not appear in the data. There is no gap between the upper boundary of one class and the lower boundary of the next class. The lower class boundary is found by subtracting 0.5 units from the lower class limit and the upper class boundary is found by adding 0.5 units to the upper class limit.

9. Class Width :

The difference between the upper and lower boundaries of any class. The class width is also the difference between the lower limits of two consecutive classes or the upper limits of two consecutive classes. It is not the difference between the upper and lower limits of the same class.

10. Class Mark (Midpoint) :

The number in the middle of the class. It is found by adding the upper and lower limits and dividing by two. It can also be found by adding the upper and lower boundaries and dividing by two.

11. Cumulative Frequency :

The number of values less than the upper class boundary for the current class. This is a running total of the frequencies.

12. Relative Frequency :

The frequency divided by the total frequency. This gives the percent of values falling in that class.

13. Cumulative Relative Frequency (Relative Cumulative Frequency) :

The running total of the relative frequencies or the cumulative frequency divided by the total frequency. Gives the percent of the values which are less than the upper class boundary.

14. Histogram :

A graph which displays the data by using vertical bars of various heights to represent frequencies. The horizontal axis can be either the class boundaries, the class marks, or the class limits.

15. Frequency Polygon :

A line graph. The frequency is placed along the vertical axis and the class midpoints are placed along the horizontal axis. These points are connected with lines.

16. Ogive :

A frequency polygon of the cumulative frequency or the relative cumulative frequency. The vertical axis the cumulative frequency or relative cumulative frequency. The horizontal axis is the class boundaries. The graph always starts at zero at the lowest class boundary and will end up at the total frequency (for a cumulative frequency) or 1.00 (for a relative cumulative frequency).

MATHEMATICAL REASONING

STATEMENT:

A sentence is called a mathematically acceptable statement if it is either true or false but not both. A statement is neither imperative, nor interrogative nor exclamatory. A sentence which is a request, or a command is not a statement .

Ex : The following are the statements

(a) 6 is less than 8

(b) 2 is an odd number

(c) Every square is a rectangle

(d) New Delhi is in India

Note : A sentence which is both true and false simultaneously is not a statement. Such a sentence is called a **paradox**.

OPEN STATEMENT :

A declarative sentence containing variable (s) is an open statement if it becomes a statement when the variable(s) is (are) replaced by some definite value (s).

e.g. s = x is an integer

s is true if x is integer and false if x is not integer.

COMPOUND STATEMENTS :

A compound statement is a statement which is made up of two or more statements. In this case, each statement is called a component statement.

e.g. All rational numbers are real and all real numbers are complex.

The component statement are

p : all rational numbers are real

q : all real numbers are complex numbers

CONJUNCTION :

Compound statement are combined by the word "and "($\wedge$) the resulting statement is called a conjunction denoted as $p \wedge q$.

e.g. A point occupies a position and its location can be determined.

The component statement are

p : A point occupies a position

q : Its location can be determined

Both statements are true.

Imp. : Do not think that a statement with "And" is always a compound statement.

e.g. A mixture of alcohol and water can be separated by chemical methods.

(Here "And" refers to two things).

Note :

(i) The compound statement with 'And' is true if all its component statements are true.

(ii) The compound statement with 'And' is false if any of its component statements is false (this includes the case that some of its component statements are false or all of its component statements are false).

The following truth table shows the truth values of $p \wedge q$ (p and q) and $q \wedge p$ (q and p).

Truth Table ($p \wedge q$, $q \wedge p$)			
p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F
Rule : $p \wedge q$ is true only when p and q are true.			

Remark : The above truth table shows that $p \wedge q = q \wedge p$.

DISJUNCTION OR ALTERNATION :

Compound statements p and q are combined by the connective 'OR' ($\vee$), then the compound statement denoted as $p \vee q$ (p or q) so formed is called a disjunction.
e.g. Two lines in a plane either intersect at one point or they are parallel.

Sometimes we use the connective 'either ... or ... ' to obtain $p \vee q$ and read $p \vee q$ as 'either p or q'.

Note :

(i) A compound statement with an 'Or' is true when one component statement is true or both the component statement are true.

(ii) A compound statement with an 'Or' is false when both the component statements are false.

Truth Table ($p \vee q$, $q \vee p$)			
p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F
Rule : $p \vee q$ is false only when both p and q are false.			

Imp. : e.g. A student who has taken biology or chemistry can apply for M.Sc. microbiology program. This means that student who have taken both biology and chemistry or only biology or only chemistry can apply for the microbiology program. This is example of **inclusive "Or"**. In this case truth table is same as $p \vee q$.

Imp. : e.g. Student can take French or Sanskrit as their third language.
This means that student have to choose only one subject from French and Sanskrit. It exclude the case when one student can choose both subject. This is case of **exclusive "Or"**. This is represented as $p \veebar q$ or $p \otimes q$. Truth table for exclusive or is as follows.

Truth Table ($p \veebar q$)		
p	q	$p \veebar q$ or $p \otimes q$
T	T	F
T	F	T
F	T	T
F	F	F
Rule : $p \veebar q$ is true only when one of p and q is true and the other is false.		

NEGATION (OR DENIAL) :

The denial of a statement is called the negation of the statement denoted as ~ .
e.g. p : Everyone in Germany speaks German.
~ p : it is false that everyone in Germany speaks German.

While forming the negation of a statement, phrases like, "It is not the case" or "It is false that" are also used.
e.g. p : All integers are rational numbers".
~ p : Atleast one integer is not a rational number.

If p is true then ~ p must be false and if p is false then ~ p must be true

Truth Table (~ p)		
p	~ p	~ (~ p)
T	F	T
F	T	F
Rule : ~ is true only when p is false.		

It may be noticed that ~ (~p) = p. Also p and ~ p are contrary.
e.g. the statements 'x is an even number' and ' x is an odd number' are contrary if x is a whole number because both the statements cannot have the same truth value.

Imp. : If may be observed that negation is not a binary operation, it is a unary operation i.e. a modifier.
1. ~ p is true iff p is false.
2. ~ p is false iff p is true.

Quantifiers :

Quantifiers are phrases like "There exists" and "for all".

Negation of Quantifiers

(i) P = There exist a number which is equal to its square .

~ P = There does not exist a number which is not equal to its square.

(ii) P = For every real number x, x is less than x + 1.

~ P = There exist a number for which x is not less than x + 1.

IMPLICATION :

There are three types of implications :

(i) "If ...... then "

(ii) "Only if"

(iii) "If and only if"

(1) "If ...... then " type of compound statement is called **conditional statement**.

The statement 'if p then q' is denoted by $p \to q$ (to be read as 'p implies q') or by $p \Rightarrow q$. Note that $p \to q$ also means

(i) p is sufficient for q (ii) q is necessary for p (iii) p only if q

(iv) p lead to q (v) q if p (vi) q when p

(vii) if p, then q

e.g. p : a number is a multiple of 9

q : a number is a multiple of 3.

Then $p \to q$ or $p \Rightarrow q$

$p \to q$ is false only when p is true and q is false. Truth table for $p \to q$ is as follows.

Truth Table ($p \to q$, $q \to p$)			
p	**q**	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T
Rule : $p \to q$ is false only when p is true and q is false.			

(2) "If and only if " type of compound statement is called **Biconditional or equivalence** or **'double implication'**. Symbolically 'p iff q' is represented by $p \leftrightarrow q$ or by $p \Leftrightarrow q$.

(i) p is a necessary and sufficient condition for q.

(ii) q is necessary and sufficient condition for p.

(iii) If p then q and if q then p

(iv) q if and only if p.

e.g. p : If the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q : If a number is divisible by 3, then the sum of its digits is divisible by 3.

A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

The following are other illustrations which actually do not appear to be so but they infact are biconditional.

(i) If you work hard only then you can succeed.

(ii) You can go on leave only if your boss permits. The truth table for biconditional is as follows:

Truth Table ($p \leftrightarrow q$, $q \leftrightarrow p$)			
p	**q**	$p \leftrightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T
Rule : $p \leftrightarrow q$ is true only when both p and q have the same truth value.			

CONTRAPOSITIVE AND CONVERSE :

Contrapositive and converse are certain other statements which can be formed from a given statement with "if..........then".

Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

e.g. If a number is multiple of 6 then it is multiple of 2.

Contrapositive If a number is not multiple of 2 then it is not multiple of 6.

Converse of $p \rightarrow q$ is $q \rightarrow p$

e.g. If the angles of a triangle are equal then it is equilateral triangle.

Converse is if triangle is equilateral then angles of triangle are equal.

Truth Table ($p \to q$)				
p	**q**	**$p \to q$**	**Contrapositive ($\sim q \to \sim p$)**	**Converse ($q \leftrightarrow p$)**
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Note : Truth table for $p \to q$ is same as its contrapositive.

NEGATION OF IMPLICATION :

If p and q are two statements, then

$\sim(p \Rightarrow q) = p \wedge \sim q$ $\quad [\because \; p \Rightarrow q \equiv \sim p \wedge q]$

Proof :

p	q	$p \Rightarrow q$	$\sim(p \Rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Negation of Biconditional statement or equivalence theorem :

If p and q are two statement, then

$$\sim(p \Leftrightarrow q) = (p \wedge \sim q) \vee (q \wedge \sim p)$$

Proof :

$$p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$$

$$\therefore \quad \sim(p \Leftrightarrow q) = \{(p \Rightarrow q) \wedge (q \Rightarrow p)\}$$

$$= [\sim(p \Rightarrow q)] \vee [\sim(q \Rightarrow p)]$$

$$= (p \wedge \sim q) \vee (q \wedge \sim p)$$

TAUTOLOGIES AND FALLACIES :

The compound statements (or propositions) which are true for any truth value of their components are called **'Tautologies'**.

e.g. '$p \vee \sim p$' is a tautology, p being logical statement. This is illustrated by the truth table given below which shows only Ts in the last column.

Truth Table ($p \vee \sim p$)		
p	$\sim p$	$p \wedge \sim p$
T	F	T
F	T	T

The negation of tautology is called a **fallacy** or a **contradiction** i.e. a proposition which is false for any truth value of their components is called a fallacy. For example '$p \wedge \sim p$' is a fallacy, p being any logical statement. This is illustrated by the truth table given below which shows only Fs in the last column.

Truth Table ($p \wedge \sim p$)		
p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Note :

(i) $p \vee q$ is true iff at least one of p and q is true.

(ii) $p \underline{\vee} q$ is true iff exactly one of p and q is true and the other is false.

(iii) $p \wedge q$ is true iff both p and q are true.

(iv) A tautology is always true.

(v) A fallacy is always false.

ALGEBRA OF STATEMENTS :

Statements satisfy many laws some of which are given below -

(1) **Idempotent Laws :** If p is any statement then

(i) $p \vee p = p$ (ii) $p \wedge p = p$

(2) **Associative Laws :** If p, q, r are any three statements, then

(i) $p \vee (q \vee r) = (p \vee q) \vee r$ (ii) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

(3) **Commutative Laws :** If p, q are any two statements, then

(i) $p \vee q = q \vee p$ (ii) $p \wedge q = q \wedge p$

(4) **Distributive Laws :** If p, q, r are any three statements, then

(i) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ (ii) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(5) **Identity Laws :** If p is any statement, t is tautology and c is a contradiction, then

(i) $p \vee t = t$ (ii) $p \wedge t = p$ (iii) $p \vee c = p$ (iv) $p \wedge c = c$

(6) **Complement Laws :** If t is a tautology, c is a contradiction and p is any statement, then

(i) $p \vee (\sim p) = t$ (ii) $p \wedge (\sim p) = c$ (iii) $\sim t = c$ (iv) $\sim c = t$

(7) **Involution Law :** If p is any statement, then $\sim(\sim p) = p$.

(8) **De-morgan's Law :** If p and q are two statements, then

(i) $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$ (ii) $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$

(i) **Proof :**

$\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

(ii) **Proof :**

$\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

SOLVED EXAMPLES

Q.1 By mathematical induction, $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)}$ is equal to

(A) $\frac{n(n+1)}{4(n+2)(n+3)}$ (B) $\frac{n(n+3)}{4(n+1)(n+2)}$ (C) $\frac{n(n+2)}{4(n+1)(n+3)}$ (D) None of these

Sol. Let $P(n): \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

(i) For n = 1

$$\text{L.H.S.} = \frac{1}{1\cdot 2\cdot 3} = \frac{1}{6} \text{ and R.H.S.} = \frac{1(1+3)}{4(1+1)(1+2)} = \frac{1}{6}$$

$\therefore$ P(1) is true.

(ii) Let P(k) be true, then

$$P(k): \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)} \qquad \text{.........(1)}$$

(iii) For n = k + 1,

$$P(k+1): \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$\text{L.H.S.} = \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \quad \text{[From Eq. (1)]}$$

$$= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)} = \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)} = \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)} = \text{R.H.S.}$$

Hence, P(k + 1) is true.

Hence, by the principle of mathematical induction for all $n \in N$, P(n) is true. **Ans. (B)**

Q.2 For all $n \in N$, $(3)(5^{2n+1}) + 2^{3n+1}$ is divisible by

(A) 17 (B) 19 (C) 21 (D) 23

Sol. $P(n) : 3(5^{2n+1}) + 2^{3n+1}$

$P(1) : 3(5^3) + 2^4 = 3(125) + 16$

$= 375 + 16 = 391 = 17(23)$

So, P(1) is divisible by 17.

Let $P(k) : 3(5^{2k+1}) + 2^{3k+1} = 17m$

$\therefore \quad P(k+1) : 3(5^{2k+3}) + 2^{3k+4} = 17\lambda$

L.H.S. of $P(k+1) = 3(5^{2k+3}) + 2^{3k+4}$

$= 3(5^{2k+1})(5^2) + (2^{3k+1})(2^3)$

$= (17m - 2^{3k+1})(25) + 8(2^{3k+1})$ [Assuming P(k) to be true]

$= 17(25m) - 25(2^{3k+1}) + 8(2^{3k+1})$

$= 17(25m) - 17(2^{3k+1})$

$= 17(25m - 2^{3k+1}) = 17\lambda$

Thus, P(k + 1) is divisible by 17 whenever P(k) is divisible by 17. Hence P(n) is divisible by 17 for all $n \in N$. **Ans. (A)**

Q.3 If $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$, then n is

(A) a positive integer (B) an even positive integer

(C) an odd positive integer (D) none of these

Sol. $P(n) : x^{2n-1} + y^{2n-1}$

$P(1) : x^1 + y^1 = x + y$, which is divisible by $x + y$

Let $P(k) : x^{2k-1} + y^{2k-1} = (x + y)\, m$

$\therefore \quad P(k+1) : x^{2k+1} + y^{2k+1} = (x + y)\, \lambda$

L.H.S. of $P(k+1) = x^{2k+1} + y^{2k+1}$

$= x^2(x^{2k-1}) + y^2(y^{2k-1})$

$= x^2[m(x + y) - y^{2k-1}] + y^2(y^{2k-1})$ [Assuming P(k) to be true]

$= (x + y)(mx^2) - y^{2k-1}(x^2 - y^2)$

$= (x + y)[mx^2 - y^{2k-1}(x - y)] = (x + y)\lambda$

Thus, P(k + 1) is divisible by (x + y) whenever P(k) is divisible by (x + y). Hence P(n) is divisible by x + y for all $n \in N$, i.e. for all positive integers. **Ans. (A)**

Q.4 When $P(n) = 9^n - 8^n$ is divided by 8, then the remainder is

(A) 2 (B) 3 (C) 1 (D) 7

Sol. $P(n) = 9^n - 8^n$

$\therefore$ $P(1) = 9 - 8 = 1$

$\therefore$ $P(1) - 1 = 0$ which is divisible by 8

$\therefore$ 1 is the remainder when P(n) is divided by 8

Now, $P(2) = 9^2 - 8^2 = 17 = 16 + 1$.

Remainder is 1, when divided by 8. **Ans.(C)**

Q.5 Let $P(n) = 2^{3n} - 7n - 1$, then P(n) is divisible by

(A) 63 (B) 36 (C) 49 (D) 25

Sol. $P(n) : 2^{3n} - 7n - 1$

$P(1) : 2^3 - 7 - 1 = 8 - 7 - 1 = 0$

$P(2) : 2^6 - 7(2) - 1 = 64 - 14 - 1 = 49$.

which is divisible by 49

Let $P(k) : 2^{3k} - 7k - 1 = 49m$

$\therefore$ $P(k+1) = 2^{3k+3} - 7(k+1) - 1 = 49\lambda$

L.H.S. of $P(k+1) = 2^{3k+3} - 7(k+1) - 1$

$= (2^{3k})(2^3) - 7k - 7 - 1$

$= (49m + 7k + 1)(8) - 7k - 8$ [Assuming P(k) to be true]

$= 49(8m) + 56k + 8 - 7k - 8$

$= 49(8m + k) = 49\lambda$.

Thus, $P(k+1)$ is divisible by 49 whenever P(k) is divisible by 49. Hence P(n) is divisible by 49 for all $n \in N$. **Ans.(C)**