BINOMIAL THEOREM

1.1 BINOMIAL EXPRESSION :

An algebric expression consisting of two different terms is called a binomial expression.

e.g. (1) $x + y$ (2) $x^3 + y^3$

But $(x + nx)$ is not a binomial, it is called a monomial.

1.2 BINOMIAL THEOREM :

In elementary algebra, the binomial theorem describes the algebric expansion of powers of a binomial expression.According to the theorem it is possible to expand the powers $(x + y)^n$ into a sum involving terms of the form ax^by^c, where exponents b and c are non-negative integers with $b + c = n$ and the coefficient 'a' of each term is a specific positive integer depending on n and b.

This theorem was given by Newton.

Binomial Theorem $\Rightarrow$ $(x + y)^n$

- If $n \in N$ (Form a finite series)
- Any index $n \notin N$

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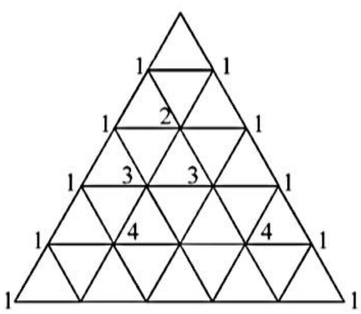
Binomial Theorem $\Rightarrow$ $(x + y)^n$

- If $n \in N$ (Form a finite series)
- Any index $n \notin N$
  (Form an infinite series)

1.3 HISTORICAL DEVELOPMENT :

Earlier people used to multiply the brackets to expand the given binomial of known index.

Then came the Pascal triangle



Note that

(a) The powers of x go down until it reaches reaches zero, starting value is n.

(b) The power of y goes up from zero untill it reaches n.

(c) The n^{th} row of the Pascals triangle will be the coefficients of the expanded binomial.

1.4 STATEMENT OF THE THEOREM :

$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots\ldots\ldots + {}^nC_rx^{n-r} \cdot y^r + \ldots\ldots\ldots + {}^nC_n\, x^{n-n}\, y^n.$

We observe $T_1 = {}^nC_0\, x^n$

$T_2 = {}^nC_1\, x^{n-1} \cdot y^1$

$\Rightarrow$ General term in the expansion of $(x + y)^n$ is

$T_{r+1} = {}^nC_r\, x^{n-r} \cdot y^r$

Where ${}^nC_r$ is called as combinatorial or binomial coefficient also denoted by $\binom{n}{r}$.

Also, $(x + y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} \cdot y^r$

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Proof-1:

Binomial theorem for a positive integral index is a special case of a very general result called symmetric product which states that

$(x + y_1)\, (x + y_2) \ldots\ldots\ldots (x + y_n) = x^n + S_1x^{n-1} + S_2\, x^{n-2} + S_3\, x^{n-3} + \ldots\ldots\ldots + S_n.$

where S_r $(r = 1, 2, 3, \ldots\ldots\ldots, n)$ denotes sum of product of quantities $y_1, y_2, \ldots\ldots\ldots\ldots, y_n$ taken r at a time.

Thus, $S_1 = y_1 + y_2 + \ldots\ldots\ldots\ldots y_r$ (${}^nC_1$ terms)

$S_2 = y_1y_2 + y_2y_3 + \ldots\ldots\ldots\ldots$ (${}^nC_2$ terms)

$S_3 = y_1\, y_2\, y_3 + y_2\, y_3\, y_4 \ldots\ldots\ldots\ldots$ (${}^nC_3$ terms)

and so on

Now put

$y_1 = y_2 = y_3 \ldots\ldots\ldots = y_n = y$

We get

$S_1 = y + y + y \ldots\ldots\ldots\ldots + y$ (${}^nC_1$ terms) $= {}^nC_1y$

$S_2 = y^2 + y^2 + \ldots\ldots\ldots\ldots + y^2$ (${}^nC_2$ terms) $= {}^nC_2\, y^2$

and so on

In this special case

$(x + y)^n = x^n + {}^nC_1\, x^{n-1} \cdot y + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots\ldots\ldots + {}^nC_n\, y^n.$

Proof-2: Combinatorial Proof

The coefficient of xy^2 in $(x + y)^3 = (x + y)(x + y)(x + y)$.

$= x\,x\,x + x\,x\,y + x\,y\,x + x\,y\,y + y\,x\,x + y\,x\,y + y\,y\,x + y\,y\,y$

$= x^3 + 3x^2y + 3xy^2 + y^3$

Equals ${}^3C_2 = 3$ because there are three x, y strings of length 3 with exactly two y's namely x y y, y x y, y y x.

Corresponding to the three 2 element subsets of $\{1, 2, 3\}$ namely $\{2, 3\}, \{1, 3\}, \{1, 2\}$

where each subset specifies the position of y in corresponding string.

Similarly in $(x + y)^n$

1.5 IMPORTANT POINTS OF EXPANSION :

(1) Number of terms in expansion of $(x + y)^n$ is $n + 1$ i.e., one more than index.

or

By begger's method n coins and 2 beggars

$\therefore \quad {}^{n+1}C_1 \Rightarrow (n + 1)$ times.

Illustration :

$\therefore \quad {}^{n+1}C_1 \Rightarrow (n + 1)$ times.

Illustration :

Find number of term is the expansion of $(x + y + z + w)^{10}$.

Sol. *Coin = 10, beggers = 4*

${}^{13}C_3$ terms. ***Ans.***

(2) Sum of indices of x and y in each term in the expansion of $(x + y)^n$ is n.

(3) New expansions by $(x + y)^n$

(a) We have $(x + y)^n = {}^nC_0x^n + {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + ... + {}^nC_n\, y^n$ ....(1)

i.e., $$(x + y)^n = \sum_{r=0}^{r=n} {}^nC_r\, x^{n-r} \cdot y^r.$$

(b) Replace y by $-y$

$(x - y)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + ... + {}^nC_r\, x^{n-r} \cdot y^r (-1)^r + ... + {}^nC_n\, (-1)^n \cdot y^n$

i.e., $$(x - y)^n = \sum_{r=0}^{r=n} {}^nC_r\, (-1)^r\, x^{n-r} \cdot y^r.$$

(c) Now replace x by 1 and y by x in Ist

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots\ldots + {}^nC_rx^r + \ldots\ldots + {}^nC_n\, y^n$.

Also, i.e., $(1 + x)^n = \sum_{r=0}^{r=n} {}^nC_r\, x^r$.

(d) $(1 - x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2x^2 + \ldots\ldots + {}^nC_r\,(-1)^r x^r + \ldots\ldots + {}^nC_n\,(-1)^n x^n$.

i.e., $(1 - x)^n = \sum_{r=0}^{r=n} (-1)^r\, {}^nC_r\, x^r$.

(e) Also remember

$(1 + x)^n + (1 - x)^n = 2\,[{}^nC_0 + {}^nC_2x^2 + \ldots\ldots\ldots]$

and $(1 + x)^n - (1 - x)^n = 2\,[{}^nC_1 + {}^nC_3\, x^3 + {}^nC_5\, x^5 + \ldots\ldots\ldots]$

$\therefore$ Coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$.

$T_{r+1} = {}^nC_r\, x^r$

$\therefore$ coefficient of $(r + 1)^{th}$ term = coefficient of $x^r = {}^nC_r$ in the expansion of $(1 + x)^n$.

For e.g., Find the coefficient of x^6 in $(1 + 3x + 3x^2 + x^3)^{15} = [(1 + x)^3]^{15} = (1 + x)^{45}$

$\therefore$ $T_{r+1} = {}^{45}C_rx^r$

$\therefore$ $r = 6$.

$\therefore$ coefficient is ${}^{45}C_6$.

$\therefore$ $T_{r+1} = {}^{45}C_rx^r$

$\therefore$ $r = 6$.

$\therefore$ coefficient is ${}^{45}C_6$.

Illustration :

Find the value of $\left(1+\sqrt{2}\right)^7 + \left(1-\sqrt{2}\right)^7 = ?$

Sol. $\left(1+\sqrt{2}\right)^7 + \left(1-\sqrt{2}\right)^7 = 2\left[{}^7C_0 1 + {}^7C_2\left(\sqrt{2}\right)^2 + \ldots\ldots + {}^7C_6\left(\sqrt{2}\right)^6\right] = 476.$

Illustration :

Find the value of $\left(\sqrt{3}+3\right)^5 - \left(\sqrt{3}-3\right)^5 = ?$

Sol. $\left(\sqrt{3}+3\right)^5 - \left(\sqrt{3}-3\right)^5 = \left(\sqrt{3}\right)^5\left[\left(1+\sqrt{3}\right)^5 - \left(1-\sqrt{3}\right)^5\right]$

$= 2\cdot 9\cdot\sqrt{3}\left[{}^5C_1\left(\sqrt{3}\right) + {}^5C_3\left(\sqrt{3}\right)^3 + {}^5C_5\left(\sqrt{3}\right)^5\right] = 2376.$

Illustration :

Find sum of series upto n terms $\displaystyle\sum_{r=0}^{n}(-1)^r\ {}^nC_r\left[\left(\frac{1}{2}\right)^r+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\text{........upto m terms}\right]$

Sol. $T_1 = \displaystyle\sum_{r=0}^{n}{}^nC_r(-1)^r\left(\frac{1}{2}\right)^r = \left(1-\frac{1}{2}\right)^n = \frac{1}{2^n}$

$$T_2 = \sum_{r=0}^{n}\left(\frac{3}{4}\right)^r(-1)^r\ {}^nC_r = \left(1-\frac{3}{4}\right)^n = \left(\frac{1}{4}\right)^n$$

$\vdots$

$T_1 + T_2 + T_3 +$ .... *upto m terms*

$\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^{2n}+\left(\frac{1}{2}\right)^{3n}$ .... *m terms*

$$= \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2^n}\right)^m}{1}\right) = \frac{2^{mn}-1}{(\quad)}.$$

$$= \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2^n}\right)^m}{1-\frac{1}{2^n}}\right) = \frac{2^{mn}-1}{\left(2^n-1\right)\ 2^{mn}}.$$

(4) ${}^nC_0, {}^nC_1, {}^nC_2,$............., ${}^nC_n$ are called binomial coefficient or combinatorial coefficients and may be simply written as $C_0, C_1, C_2,$ ...., C_n.

$$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1}\ y + {}^nC_2\, x^{n-2} y^2 + \text{........} + {}^nC_0\, x^{n-n} y^n.$$

Find the sum of all the combinatorial coefficient.

i.e., ${}^nC_0 + {}^nC_1 + {}^nC_2 +$ .... $+ {}^nC_n = 2^n$.

Put $x = 1$ and $y = 1$ to get sum of all the binomial coefficient.

$(x + 2y)^n$ find the sum of all the coefficients.

$(x + 2y)^2 = x^2 + 4xy + 4y^2$

$\therefore$ Sum of all coefficient $= 1 + 4 + 4 = 9$.

We can also get it by putting $x = y = 1$

$(1 + 2)^2 = 9$

$\therefore$ Sum of all binomial coefficients in

$(x + y)^n = 2^n \rightarrow$ In this case sum of coefficient = sum of binomial coefficient.

(5) Binomial coefficients of the term equidistant from beginning and end are equal.

$$(2x + 3y)^2 = {}^2C_0\,(2x)^2 + {}^2C_1\,(2x)^1\,(3y)^1 + {}^2C_2\,(2x)^0\,(3y)^2.$$

Now coefficient of 1st and last term is same ${}^2C_0 = {}^2C_2$.

(6) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(7) $\left(\frac{n+1}{r+1}\right){}^nC_r = {}^{n+1}C_{r+1}$

(8) Consecutive binomial coefficient.

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} \text{ i.e., } \frac{{}^5C_4}{{}^5C_3} = \frac{5-4+1}{4} = \frac{1}{2}.$$

Illustration :

Illustration :

If three consecutive coefficients in the expansion of $(1 + x)^n$ be 165,330 and 462. Find number of terms in the expansion of $(1 + x)^n$.

Sol. $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}.$

Let three consecutive coefficients of terms are ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} = \frac{330}{165} = 2. \qquad ....(1)$$

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-(r+1)+1}{r+1} = \frac{n-r}{r+1}$$

$$\frac{n-r}{r+1} = \frac{462}{330} = \frac{n-r}{r+1} = \frac{7}{5}$$

$n = 11$ and $r = 4$.

Practice Problem

Q.1 If a, b, c, d are the coefficient of any four consecutive terms in the expansion of $(1+x)^n$, $n \in N$ prove that $\frac{a}{a+b}+\frac{c}{c+d}=\frac{2b}{b+c}$.

Q.2 $\left(x+\sqrt{2}\right)^4+\left(x-\sqrt{2}\right)^4$.

Q.3 If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_{2n}x^{2n}$.
Find (a) $a_0 + a_1 + a_2 + \ldots\ldots\ldots + a_{2n}$
(b) $a_0 - a_1 + a_2 - a_3 + \ldots\ldots\ldots + a_{2n}$.

Answer key

Q.2 $2x^4 + 24x^2 + 8$ Q.3 (a) 3^n (b) 1

2.1 IMPORTANT TERMS IN BINOMIAL :

(A) General term
(B) Term independent of x

2.1 IMPORTANT TERMS IN BINOMIAL :

(A) General term
(B) Term independent of x.
(C) Middle term

(A) GENERAL TERM :

$(T_{r+1})^{th}$ term is called as general term in $(x+y)^n$ and general term is given by

$$\boxed{T_{r+1} = {}^nC_r x^{n-r}.y^r}$$

Illustration :

Find fourth term in the expansion of $\left(2x-\frac{y}{2}\right)^7$.

Sol. $T_{r+1} = {}^7C_r\,(2x)^{7-r}\cdot\left(-\frac{y}{2}\right)^r$

Put $r = 3$ *to get* 4^{th} *term*

$T_4 = -\,{}^7C_3\,(2x)^4\cdot\left(\frac{y}{2}\right)^3 = -\,70x^4y^3$.

Illustration :

Find term involving x^3 & x^4 *in* $\left(2x^2 - \frac{1}{3x}\right)^6$

Sol. *Let* T_{r+1} *involving* x^3 *and* x^4 *then*

$$T_{r+1} = {}^6C_r\,(2x^2)^{6-r}\left(\frac{-1}{3x}\right)^r.$$

$$= {}^6C_r\,(2)^{6-r}\cdot\left(\frac{-1}{3}\right)^r x^{12-2r}\cdot\left(\frac{1}{x}\right)^r.$$

$$= {}^6C_r\,2^{6-r}\left(\frac{-1}{3}\right)^r x^{12-3r}$$

$\Rightarrow$ *Coeffiicent of* $x^3 \Rightarrow 12 - 3r = 3.$

$r = 3$ $\quad$ 4^{th} *term involve* x^3 *in expansion.*

$\Rightarrow$ *Coeff. of* $x^4 \Rightarrow 4 = 12 - 3r$

$r \neq \frac{8}{3}$. *So there is no term exist.*

Important : If r is non integral value then there is no term exist.

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Illustration :

Prove that coefficient of x^{50} *in* $(1 + x^2)^{25}(1 + x^{25})(1 + x^{40})(1 + x^{47})$ *is equal to* $1 + {}^{25}C_5$.

Sol. *As we are intersected in coefficient of* x^{50}, *we shall ignore all terms with exponent more than 50.*

So $(1 + x^2)^{25}(1 + x^{25})(1 + x^{40})(1 + x^{47})$

$= (1 + {}^{25}C_1x^2 + ... + {}^{25}C_{25}x^{50}) \times (1 + x^{25} + x^{40} + x^{45} + x^{47})$

$= {}^{25}C_{25} + {}^{25}C_5 = 1 + {}^{25}C_5.$

(B) TERM INDEPENDENT OF x :

It means term containing x^0.

Illustration :

Find term independent of x in $\left(x^2 + \frac{1}{x^2} - 2\right)^{10}$.

Sol. $\left(x^2 + \frac{1}{x^2} - 2\right)^{10} = \left(x - \frac{1}{x}\right)^{20}$

$\Rightarrow$ $T_{r+1} = {}^{20}C_r\,x^{20-r}(-1)^r\frac{1}{x^r} = {}^{20}C_r\,x^{20-2r}(-1)^r$

$\Rightarrow$ $20 - 2r = 0\,;\, r = 10$

$\Rightarrow$ 11^{th} *term is independent of x.*

(C) MIDDLE TERM :

Let T_m is middle term is expansion $(x+y)^n$ then

Case I : If n is odd, then number of terms will be even so there is two middle terms

$$\left(\frac{n+1}{2}\right)^{th} \textbf{ and } \left(\frac{n+3}{2}\right)^{th}.$$

Case II : If n is even, then number of terms will be odd so only one term is middle term $\left(\frac{n}{2}+1\right)^{th}$.

Note : Binomial coefficient of middle term is greatest.

Illustration :

Find coefficient of t^8 *in the expansion of* $(1 + 2t^2 - t^3)^9$.

Sol. $T_{r+1} = {}^9C_r\, t^{2r}\,(2-t)^r$

Now coefficient of t^8 *appears in* $r = 3$, $r = 4$.

$r = 3 \Rightarrow {}^9C_3\, t^6\,(2-t)^3 = {}^9C_3 \times t^6 \times {}^3C_p\, 2^{3-p}\,(-t)^p$

Put $p = 2 \Rightarrow {}^9C_3 \times {}^3C_2 \times 2^1 \times t^8 = 84 \times 6 \times t^8$

Similarly when $r = 4$

$r = 3 \Rightarrow {}^9C_3\, t^6\,(2-t)^3 = {}^9C_3 \times t^6 \times {}^3C_p\, 2^{3-p}\,(-t)^p$

Put $p = 2 \Rightarrow {}^9C_3 \times {}^3C_2 \times 2^1 \times t^8 = 84 \times 6 \times t^8$

Similarly when $r = 4$

${}^9C_4\, t^8\,(2-t)^4 = {}^9C_4 \times t^8 \times {}^4C_p \times {}^4C_p\, 2^{4-p}\,(-t)^p$

Put $p = 0 \Rightarrow {}^9C_4 \times {}^4C_0 \times 2^4 \times t^8 = 126 \times 16 \times t^8$

Hence coefficient of t^8 *in the expansion of* $(1 + 2t^2 - t^3)^9$ *is* $84 \times 6 + 126 \times 16$

$= 504 + 2016 = 2520$. ***Ans.***

Alternative:

We have

$$\left(\left(1+2t^2\right)-t^3\right)^9 = {}^9C_0\left(1+2t^2\right)^9 - {}^9C_1\left(1+2t^2\right)^8 \cdot t^3 + {}^9C_2\left(1+2t^2\right)^7 \cdot t^6 - {}^9C_3\left(1+2t^2\right)^6 \cdot t^9 + \ldots\ldots - {}^9C_9\left(t^3\right)^9$$

$\therefore$ *Coefficient of* t^8 *in the expansion of* $(1 + 2t^2 - t^3)^9$

$= {}^9C_0$ *(coefficient of* t^8 *in* $(1+2t^2)^9$*)* $- {}^9C_1$*(coefficient of* t^3 *in* $\left(1+2t^2\right)^7$*)*

$+ {}^9C_2$*(coefficient of* t^2 *in* $\left(1+2t^2\right)^7$*)*

$$= {}^9C_0 \cdot {}^9C_4\, 2^4 - 0 + {}^9C_2 \cdot {}^7C_1 \cdot 2 = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\cdot 16 + \frac{9\cdot 8}{2\cdot 1}\cdot 7\cdot 2$$

$= 9 \cdot 8 \cdot 7 \cdot 5 = 2520$. ***Ans.***

Illustration :

Prove that ${}^4C_4 \cdot {}^{100}C_6 + {}^4C_3 \cdot {}^{100}C_7 + {}^4C_2 \cdot {}^{100}C_8 + {}^4C_1 \cdot {}^{100}C_9 + {}^4C_0 \cdot {}^{100}C_{10} = {}^{104}C_{10}$.

Sol. $\underbrace{{}^{100}C_6 + {}^{100}C_7} + 3\left({}^{100}C_7 + {}^{100}C_8\right) + 3\left({}^{100}C_8 + {}^{100}C_9\right) + {}^{100}C_9 + {}^{100}C_{10}$

$= {}^{101}C_7 + 3\ {}^{101}C_8 + 3\ {}^{101}C_9 + {}^{101}C_{10}$

$= \underbrace{{}^{101}C_7 + {}^{101}C_8} + 2\left({}^{101}C_8 + {}^{101}C_9\right) + {}^{101}C_9 + {}^{100}C_{10}$

$= {}^{102}C_8 + 2 \cdot {}^{102}C_9 + {}^{102}C_{10} = {}^{102}C_8 + {}^{102}C_9 + {}^{102}C_9 + {}^{102}C_{10}$

$= {}^{103}C_9 + {}^{103}C_{10} = {}^{104}C_{10} \equiv {}^{104}C_{94}$.

Alternative : Proof by PNC

${}^4C_4 \cdot {}^{100}C_6 + {}^4C_3 \cdot {}^{100}C_7 + {}^4C_2 \cdot {}^{100}C_8 + {}^4C_1 \cdot {}^{100}C_9 + {}^4C_0 \cdot {}^{100}C_{10}$

Out of 104 students of which 100 are boys and 4 are girls, we have to select 10 students.

This can be done in ${}^{104}C_{10} = {}^{104}C_{94}$

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Hence, $x + y = 114$ *or* 198 ***Ans.***

Illustration :

Prove that middle term in the expansion of $(1 + x)^{2n}$ *is* $\dfrac{1 \cdot 3 \cdot 5 \,.......\, (2n-1) \cdot 2^n}{n\,!} \cdot x^n$.

Sol. $T_{\frac{2n}{2}+1} = T_{n+1} = {}^{2n}C_n x^n = \dfrac{(2n)!}{n!\,(n!)}\, x^n = \dfrac{2^n (n!)\,(1 \cdot 3 \,........\, \cdot 2n-1)}{n!\, n!}\, x^n$.

Illustration :

If sum of coefficient in the expansion of $(2 + 3cx + c^2x^2)^{12}$ *is zero. Find c.*

Sol. *Put* $x = 1$ *in the expansion, then*

$2 + 3c + c^2 = 0$

$c = -2,\ c = 1$. ***Ans.***

Illustration :

Find coeffiicnet of x^{15} *in* $(x - x^2)^{10}$.

Sol. $T_{r+1} = {}^{10}C_r\,(x)^{10-r}\,(-1)^r\,(x^2)^r = {}^{10}C_r\,(-1)^r\,(x)^{10+r}$

$10 + r = 15 : r = 5$ *i.e.,* 6^{th} *term*

$T_6 = {}^{10}C_5\,(-1)^5 \cdot x^{15} = -\,{}^{10}C_5\,x^{15}$. ***Ans.***

Illustration :

Find coefficient term independent of x in the expansion of $\left(x^2 + \frac{1}{x}\right)^{12}$.

Sol. $T_{r+1} = {}^{12}C_r\,(x^2)^{12-r}\left(\frac{1}{x}\right)^r = {}^{12}C_r\,(x)^{24-3r}$

$r = 5$ *and* 9^{th} *term* $\Rightarrow T_9$. ***Ans.***

Illustration :

Illustration :

Find last four terms in the expansion of $(x + 2x^2)^8$.

Sol. *Above is equivalent to first four terms of* $(2x^2 + x)^8$

$T_9 = {}^8C_0\,(2x^2)^8 = 256\,x^{16}$

$T_8 = {}^8C_1\,(2x^2)^7\,x = 1024\,x^{15}$

$T_7 = {}^8C_2\,(2x^2)^6\,x^2 = 1792\,x^{14}$

$T_6 = {}^8C_3\,(2x^2)^5\,x^3 = 1792\,x^{13}$. ***Ans.***

Illustration :

In $\left(2x^2 - \frac{1}{3x}\right)^{11}$ *find term involving* x^6 *also find term independent of x.*

Sol. $T_{r+1} = {}^{11}C_r\,(2^{11-r})\,(x^2)^{11-r}\left(\frac{-1}{3}\right)^r \cdot \left(\frac{1}{x}\right)^r$

$= {}^{11}C_r \cdot (2)^{11-r}\,(-3)^{-r}\,(x)^{22-3r}$

$22 - 3r = 6 \Rightarrow t \neq \frac{16}{3}$ *and*

For x^6 $\qquad 22 - 3r = 6 \Rightarrow r = \dfrac{16}{3}$ $\qquad$ *(which is not possible)*

For independent of x $22 - 3r = 0 \Rightarrow r = \dfrac{22}{3}$ $\qquad$ *(which is not possible)*

Both do not exist. $\qquad$ ***Ans.***

Illustration :

In the expansion of $\left(4^{\frac{1}{3}} + \dfrac{1}{6^{\frac{1}{4}}}\right)^{20}$ *find number of rational terms*

Sol. $T_{r+1} = {}^{20}C_r \cdot (4)^{\frac{20-r}{3}} \cdot (6)^{\frac{-r}{4}} = {}^{20}C_r \cdot 2^{\frac{40-2r}{3}} \cdot \dfrac{1}{6^{(r/4)}} = {}^{20}C_r\,(2)^{\left\{\frac{160-11r}{12}\right\}}\,(3)^{\left\{-\frac{r}{4}\right\}}$ $\qquad$ $[0 \le r \le 20]$

$\therefore r = 8, 20.$

$\therefore$ *There are only two rational terms* (T_9 *and* T_{21}) $\qquad$ ***Ans.***

$\therefore r = 8, 20.$

$\therefore$ *There are only two rational terms* (T_9 *and* T_{21}) $\qquad$ ***Ans.***

Illustration :

Find greatest value of term independent of x in $\left(x\sin\alpha + \dfrac{\cos\alpha}{x}\right)^{10}$ $\alpha \in R.$

Sol. $T_{r+1} = {}^{10}C_r\,(x)^{10-r} \cdot (\sin\alpha)^{10-r}\,(\cos\alpha)^r \left(\dfrac{1}{x}\right)^r$

$\Rightarrow \qquad T_{r+1} = {}^{10}C_r\,(x)^{10-2r}\,(\sin\alpha)^{10-r}\,(\cos\alpha)^r$

$\Rightarrow \qquad 10 - 2r = 0\,;\ r = 5.$

$\Rightarrow \qquad {}^{10}C_5\,(\sin\alpha)^5 \cdot (\cos\alpha)^5$

$\Rightarrow \qquad T_{r+1} = {}^{10}C_5\,\dfrac{(\sin 2\alpha)^5}{32}$

$\therefore \qquad$ *Greatest value of* $T_{r+1} = \dfrac{{}^{10}C_5}{32}.$ $\qquad [\sin 2\alpha = 1]$ $\qquad$ ***Ans.***

Illustration :

The term independent of x in the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ *is*

(A) $T_5 = 210$ (B) $T_5 = -210$ (C) $T_4 = 180$ (D) $T_4 = -180$

Sol. *We have*

$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10} = \left(\frac{\left(x^{1/3}\right)^3+1^3}{x^{2/3}-x^{1/3}+1}-\frac{\left(x^{1/2}\right)^2-1^2}{x^{1/2}\left(x^{1/2}-1\right)}\right)^{10}$$

$$= \left(\left(x^{1/3}+1\right)-\frac{x^{1/2}-1}{x^{1/2}}\right)^{10} = \left(x^{1/3}+x^{-1/2}\right)^{10}$$

We have,

$$T_{r+1} = {}^{10}C_r\,(x^{1/3})^{10-r}\,(-x^{-1/2})^r \qquad ....(1)$$

$$= {}^{10}C_r\, x^{\frac{10-r}{3}-\frac{r}{2}}(-1)^r$$

It will be independent of x, if

$$= {}^{10}C_r\, x^{\frac{10-r}{3}-\frac{r}{2}}(-1)^r$$

It will be independent of x, if

$$\frac{10-r}{3}-\frac{r}{2} = 0.$$

$\Rightarrow \quad 20-2r-3r = 0 \quad \Rightarrow \quad r = 4$

Putting r = 4 in (1), we get

$$T_5 = {}^{10}C_4\,(-1)^4 \qquad = {}^{10}C_4 = 210.$$

Hence (a) is correct answer. ***Ans.***

Illustration :

The ninth term in the expansion of $\left\{3^{\log_3\sqrt{25^{x-1}+7}}+3^{-1/8\log_3\left(5^{x-1}+1\right)}\right\}^{10}$ *is equal to 180, then x is*

(A) 2 (B) 1 (C) 4 (D) None of these

Sol. *We have,*

$$\left\{3^{\log_3\sqrt{25^{x-1}+7}}+3^{-1/8\log_3\left(5^{x-1}+1\right)}\right\}^{10} = \left[\sqrt{25^{x-1}+7}+\left(5^{x-1}+1\right)^{-1/8}\right]^{10} \quad \left\{\because a^{\log_a N} = N\right\}$$

Here, $T_9 = 180 \Rightarrow {}^{10}C_8\left\{\sqrt{25^{x-1}+7}\right\}^{10-8}\left\{\left(5^{x-1}+1\right)^{-1/8}\right\} = 180$

$$\Rightarrow \quad {}^{10}C_8\,(25^{x-1}+7)\,(5^{x-1}+1)^{-1} = 180 \qquad \Rightarrow \quad 45\frac{\left(25^{x-1}+7\right)}{5^{x-1}+1} = 180$$

$$\Rightarrow \quad \frac{25^{x-1}+7}{5^{x-1}+1} = 4 \quad \Rightarrow \quad \frac{y^2+7}{y+1} = 4, \text{ where } y = 5^{x-1}$$

$$\Rightarrow \quad y^2 - 4y + 3 = 0 \quad \Rightarrow \quad y = 3, 1 \qquad \Rightarrow 5^{x-1} = 3 \text{ or } 5^{x-1} = 1 \quad \Rightarrow 5^x = 15 \text{ or } 5^x = 5.$$

Hence (b) is correct answer. **Ans.**

Illustration :

The coefficient of x^{50} *in th expansion*

$$(1+x)^{1000} + 2x\,(1+x)^{999} + 3x^2\,(1+x)^{998} + \ldots. + 1001x^{1000} \text{ is}$$

(A) ${}^{1002}C_{50}$ (B) ${}^{1002}C_{51}$ (C) ${}^{1005}C_{50}$ (D) ${}^{1005}C_{48}$

Sol. Let $S = (1+x)^{1000} + 2x\,(1+x)^{999} + 3x^2\,(1+x)^{998} + \ldots. + 1001x^{1000}$ ... (1)

Sol. Let $S = (1+x)^{1000} + 2x\,(1+x)^{999} + 3x^2\,(1+x)^{998} + \ldots. + 1001x^{1000}$ ... (1)

$$\Rightarrow \quad \frac{xS}{1+x} = x\,(1+x)^{999} + 2x^2\,(1+x)^{998} + \ldots\ldots. + 1000x^{1000} + 1001\,\frac{x^{1001}}{1+x}. \quad \ldots (2)$$

After subtraction we get

$$\Rightarrow \quad \frac{S}{1+x} = (1+x)^{100} + x\,(1+x)^{999} + x^2\,(1+x)^{998} + \ldots\ldots. + x^{1000} - 1001\,\frac{x^{1001}}{1+x}.$$

$$\Rightarrow \quad \frac{S}{1+x} = (1+x)^{1000}\left(\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{1-\frac{x}{1+x}}\right) - 1001\,\frac{x^{1001}}{1+x}$$

$$\Rightarrow \quad S = (1+x)^{1002} - x^{1001}\,(1+x) - 1001\,x^{1001}$$

$$S = (1+x)^{1002} - 1002x^{1001} - x^{1002}$$

Coefficient of x^{50} *in S = coefficient of* x^{50} *in* $(1+x)^{1002} = {}^{1002}C_{50}$. **Ans.**

Illustration :

Find coefficient of x^3 *in the expansion of* $(1 - x + x^2)^5$.

Sol. $(1 - x + x^2)^5 = \{1 + x(x - 1)\}^5 = {}^5C_0 + {}^5C_1 x\,(x - 1) + {}^5C_2 x^2\,(x - 1)^2 + {}^5C_3\, x^3\,(x - 1)^3 + \ldots\ldots\ldots\ldots\ldots$

Coefficient of $x^3 = -\,2\,{}^5C_2 - {}^5C_3 = -\,20.$ **Ans.**

Illustration :

Find coefficient of x^4 *in the expansion of* $(1 + x + x^2 + x^3)^{11}$.

Sol. $(1 + x + x^2 + x^3)^{11} = (1 + x)^{11}\,(1 + x^2)^{11}$

$= (1 + {}^{11}C_1 x + {}^{11}C_2\, x^2 + {}^{11}C_2 x^3 + {}^{11}C_3 x + {}^{11}C_4 x^4 + \ldots\ldots..) \times (1 + {}^{11}C_1 x^2 + {}^{11}C_2 x^4 + \ldots\ldots\ldots.)$

(The terms which gives x^4 *are)*

$= {}^{11}C_2 + ({}^{11}C_2 \times {}^{11}C_1) + {}^{11}C_4 = 55 + 605 + 330 = 990.$ **Ans.**

Illustration :

If $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots.. + a_{2n} x^{2n}$ $(n \in N)$, *then*

(a) $a_0 + a_1 + a_2 + a_3 + \ldots\ldots.. + a_{2n}$

(b) $a_0 - a_1 + a_2 - a_3 + \ldots\ldots.. + a_{2n}$

(c) $a_1 + a_3 + a_5 + \ldots\ldots.. + a_{2n-1}$

(d) $a_0 + a_2 + a_4 + \ldots\ldots.. + a_{2n}$

(e) $a_0^2 - a_1^2 + a_2^2 - a_3^2 \ldots\ldots.. + a_{2n}^2$

(f) $a_0 a_1 - a_1 a_2 + a_2 a_3 + \ldots\ldots..$

(g) $a_0 a_2 - a_1 a_3 + a_2 a_4 \ldots\ldots..$

Sol.

(a) $a_0 + a_1 + a_2 + a_3 + \ldots\ldots.. + a_{2n} = 3^n$ *put* $x = 1$

(b) $a_0 - a_1 + a_2 - a_3 + \ldots\ldots.. + a_{2n} = 1$ *put* $x = -\,1$

(c) *From (a) and (b)* $a_1 + a_3 + a_5 + \ldots\ldots.. + a_{2n-1} = \dfrac{3^n - 1}{2}$

(d) *From (a) and (b)* $a_0 + a_2 + a_4 + \ldots\ldots.. + a_{2n} = \dfrac{3^n + 1}{2}$

(e) $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots.. + a_{2n} x^{2n}$ ...(i)

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots\ldots + \frac{a_{2n}}{x^{2n}}$$

$$\Rightarrow \frac{1}{x^{2n}}\,(x^2 - x + 1)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots\ldots + \frac{a_{2n}}{x^{2n}} \qquad \ldots(ii)$$

From (i) & (ii)

$$(1 + x + x^2)^n \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = (a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n})\left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_{2n}}{x^{2n}}\right)$$

$$\Rightarrow \frac{1}{x^{2n}}(x^4 + x^2 + 1)^n = (a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n})\left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_{2n}}{x^{2n}}\right) \quad ...(iii)$$

Compairing constant terms on both sides.

$$a_0^2 - a_1^2 + a_2^2 - a_3^2 \; .... + a_{2n}^2 = a_n$$

(f) *Compairing coefficient of x in equation (iii) on both sides.*

$$a_0a_1 - a_1a_2 + a_2a_3 + ... = 0$$

(g) *Compairing coefficient of x^2 in equation (iii) on both sides.*

$$a_0a_2 - a_1a_3 + a_2a_4 \; .... = a_{n+1} \text{ or } a_{n-1}.$$

Practice Problem

Practice Problem

Q.1 Find coefficient of x^{20} in the expansion of $(1 + x^2)^{40}\left(x^2 + 2 + \frac{1}{x^2}\right)^{-5}$.

Q.2 Find term independent of x in $\left[\sqrt{\frac{x}{3}} + \sqrt{\frac{3}{2x^2}}\right]^{10}$.

Q.3 Find the ratio of coefficients of x^{10} in $(1 - x^2)^{10}$ and term independent of x in $\left(x - \frac{2}{x}\right)^{10}$

Answer key

Q.1 ${}^{30}C_{25}$ Q.2 No term Q.3 $\frac{1}{32}$

3.1 NUMERICALLY GREATEST TERM IN $(x + y)^n$:

T_{r+1} term is said to be numerically greatest for a given value of x, y provided $T_{r+1} \geq T_r$ and $T_{r+1} \geq T_{r+2}$ $\Rightarrow$ $\dfrac{T_{r+1}}{T_r} \geq 1$ as well as $\dfrac{T_{r+1}}{T_{r+2}} \geq 1$

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r (x)^{n-r} y^r}{{}^nC_{r-1} (x)^{n-r+1} y^{r-1}} = \frac{(n-r+1)}{r} \underbrace{\left|\left(\frac{y}{x}\right)\right|}_{\text{Numerically greatest term is required}}$$

Illustration :

Find the greatest term in the expansion of $(7 - 5x)^{11}$ where $x = 2/3$.

Sol. *Since we have to find numerically greatest term,*

$\therefore$ *greatest term in $(7 - 5x)^{11}$ = greatest term in $(7 + 5x)^{11}$*

Let r th term be the greatest term in the expansion of $(7 + 5x)^{11}$

Now, $t_r = {}^{11}C_{r-1} (7)^{11-r+1} (5x)^{r-1} = {}^{11}C_{r-1}\, 7^{12-r} (5x)^{r-1}$ ...(i)

and $(r + 1)$*th term,* $t_{r+1} = {}^{11}C_r (7)^{11-r} (5x)^r$ ...(ii)

$$\therefore \quad \frac{t_r}{t_{r+1}} = \frac{{}^{11}C_{r-1}}{{}^{11}C_r} \cdot \frac{7^{12-r}(5x)^{r-1}}{7^{11-r}(5x)^r}$$

$$\text{or} \quad \frac{t_r}{t_{r+1}} = \frac{21r}{(12-r)10} \geq 1 \qquad \text{...(iii)}$$

$$\text{or} \quad 21r \geq 120 - 10r \quad \text{or} \quad r \geq 3\frac{27}{31}$$

Putting $(r - 1)$ in place of r in (iii), $\dfrac{t_{r-1}}{t_r} = \dfrac{21(r-1)}{\{12-(r-1)\}\cdot 10} = \dfrac{21r-21}{130-10r} \leq 1$

$$\text{or} \quad r \leq 4\frac{27}{31} \qquad \text{... (iv)}$$

From (iii) and (iv) $r = 4$

$$\therefore \quad \text{Greatest term} = t_4 = {}^{11}C_3 \cdot 7^8 \left(5 \cdot \frac{2}{3}\right)^3 = \frac{440}{9} \cdot 7^8 \cdot 5^3$$

Illustration :

Given T_4 *in the expansion of* $\left(2+\frac{3x}{8}\right)^{10}$ *has maximum numerical value find range of x.*

Sol. $T_4 > T_3$ *and* $T_4 > T_5$

$\Rightarrow \quad \frac{10-3+1}{3}\left|\frac{3x}{8\times 2}\right|$ *and* $\frac{10-4+1}{4}\left|\frac{3x}{16}\right| < 1$

$\Rightarrow \quad |x| > 2$ *and* $|x| < \frac{64}{21}.$

$\Rightarrow \quad x > 2 \; ; x < -2$ *and* $\frac{-64}{x} < x < \frac{64}{21}$

$$\underbrace{\qquad\qquad\qquad Inter\,sec\,tion \qquad\qquad\qquad}_{x\in\left(\frac{-64}{21},-2\right)\cup\left(2,\frac{64}{21}\right)}.$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{x\in\left(\frac{-64}{21},-2\right)\cup\left(2,\frac{64}{21}\right)}.$$

Illustration :

Find the index n of the binomial $\left(\frac{x}{5}+\frac{2}{5}\right)^n$ *if the* 9^{th} *term of the expansion has numerically the greatest coefficient* $(n \in N)$ *(at* $x = 1$*)*

Sol. *If* T_9 *is greatest term then*

$\frac{T_9}{T_8} > 1$ *and* $\frac{T_{10}}{T_9} < 1$

$\left(\frac{n-8+1}{8}\right)\frac{2}{5}\cdot 5 > 1$ *and* $\frac{2}{5}\cdot 5\left(\frac{n-9+1}{9}\right) < 1$

$2n - 14 > 8$ *and* $2n - 16 < 9$

$n > 11$ *and* $11 < n < 12.5$

$\therefore \quad n = 12.$ ***Ans.***

3.2 BINOMIAL COEFFICIENTS & THEIR PROPERTIES :

Propertices of ^nC_r

(1) $^nC_r = {}^nC_{n-r} \Rightarrow {}^nC_x = {}^nC_y$ has two solution $x = y$ or $x + y = n$.

(2) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(3) $^nC_r = \frac{n}{r}\ ({}^{n-1}C_{r-1})$

(4) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$.

Note :

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$$

$$\int_a^b x^n\, dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}$$

In the expansion of $(1+x)^n$; i.e.$(1+x)^n = {}^nC_0 + {}^nC_1x + ... + {}^nC_r\, x^r + ... + {}^nC_nx^n$

The coefficients $^nC_0$, $^nC_1$,^nC_n of various powers of x, are called binomial coefficients and they are written as

$$C_0, C_1, C_2, ..., C_n$$

Hence

$$(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_rx^r + ... + C_nx^n \qquad ...(1)$$

Where $C_0 = 1$, $C_1 = n$, $C_2 = \frac{n(n-1)}{2!}$

$$C_r = \frac{n(n-1)......(n-r+1)}{r!},\quad C_n = 1$$

Now, we shall obtain some important expressions involving binomial coefficients-

(a) Sum of Coefficient : putting x = 1 in (1), we get

$$C_0 + C_1 + C_2 + ... + C_n = 2^n \qquad ...(2)$$

(b) **Sum of coefficients with alternate signs :** putting $x = -1$ in(1)

We get

$$C_0 - C_1 + C_2 - C_3 + ... = 0 \qquad ...(3)$$

(c) **Sum of coefficients of even and odd terms:** from (3), we have

$$C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... \qquad ...(4)$$

i.e. sum of coefficients of even and odd terms are equal.

from (2) and (4)

$$\Rightarrow \quad C_0 + C_2 + ... = C_1 + C_3 + ... = 2^{n-1}$$

(d) **Sum of products of coefficients :** Replacing x by 1/x in (1)

We get

$$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ...+ \frac{C_n}{x^n} + ... \qquad ...(5)$$

Multiplying (1) by (5), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + ...)\ \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ...\right)$$

Now, comparing coefficients of x^r on both the sides, we get

$$C_0C_r + C_1C_{r+1} + ...+ C_{n-r}C_n = {}^{2n}C_{n-r}$$

$$= \frac{2n!}{(n+r)!(n-r)!} \qquad ...(6)$$

(e) **Sum of squares of coefficients :**

putting $r = 0$ in (6), we get

$$C_0^2 + C_1^2 + C_2^2 + ...+ C_n^2 = \frac{2n!}{n!n!}$$

(f) putting $r = 1$ in (6), we get

$$C_0C_1 + C_1C_2 + C_2C_3 + ...+ C_{n-1}C_n = {}^{2n}C_{n-1}$$

$$= \frac{2n!}{(n+1)!(n-1)!} \qquad ...(7)$$

(g) putting $r = 2$ in (6) , we get

$$C_0C_2 + C_1C_3 + C_2C_4 + ...+ C_{n-2}C_n = {}^{2n}C_{n-2}$$

$$= \frac{2n!}{(n+2)!(n-2)!} \qquad ... (8)$$

Illustration :

If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, *prove that* $C_0^2 - C_1^2 + C_2^2 \ldots\ldots + (-1)^n \cdot (-1)^{n/2} C_n^2 = 0$

or $(-1)^{n/2} \dfrac{n!}{(n/2)!\,(n/2)!}$ *according as n is odd or even.*

Sol. *Since*

$$(1-x)^n = C_0 - C_1x + C_2x^2 - \ldots\ldots + (-1)^n C_nx^n \qquad \ldots(i)$$

and $$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n \qquad \ldots(ii)$$

Multiplying (i) and (ii), we get

$$(1-x^2)^n = (C_0 - C_1x + C_2x^2 - \ldots\ldots + (-1)^n C_nx^n)\,(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n) \ldots(iii)$$

Now, coefficient of x^n *in R.H.S.*

$$= C_0^2 - C_1^2 + C_2^2 - \ldots\ldots + (-1)^n C_n^2$$

General term in L.H.S. $= T_{r+1} = {}^nC_r\,(-x^2)^r$

$$= {}^nC_r\,(-1)^r\,x^{2r}$$

Putting $2r = n$

$\therefore$ $r = n/2$

$\therefore$ $T_{(n/2)+1} = {}^nC_{n/2}\,(-1)^{n/2}\,x^n$

$\therefore$ *Coefficient of* x^n *in L.H.S.* $= {}^nC_{n/2}\,(-1)^{n/2}$

$$= (-1)^{n/2}\frac{n!}{(n/2)!(n/2)!} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\dfrac{n!}{(n/2)!(n/2)!} & \text{if } n \text{ is even} \end{cases}$$

But (iii) is an identity, therefore coefficient of x^n *in R.H.S. = coefficient of* x^n *in L.H.S.*

$$\Rightarrow \quad C_0^2 - C_1^2 + C_2^2 - \ldots\ldots + (-1)C_n^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\dfrac{n!}{(n/2)!(n/2)!} & \text{if } n \text{ is even} \end{cases}$$

Illustration :

Prove that ${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \ldots\ldots + {}^nC_r$ *if* $r < m$, $r < n$ *and m, n, r are positive integers.*

Sol. *Here sum of lower suffices of binomial coefficient in each term is r*

i.e. $r = r - 1 + 1 = r - 2 + 2 = \ldots\ldots = r = r$

since $(1+x)^m = {}^mC_0 + {}^mC_1x + \ldots\ldots + {}^mC_{r-2}x^{r-2} + {}^mC_{r-1}x^{r-1} + {}^mC_rx^r + \ldots\ldots + {}^mC_mx^m \quad \ldots(i)$

and $$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_rx^r + \ldots\ldots + {}^nC_nx^n \qquad \ldots(ii)$$

multiplying (i) and (ii), we get

$$(1 + x)^{m+n} = ({}^mC_0 + {}^mC_1x + \ldots\ldots + {}^mC_{r-2}x^{r-2} + {}^mC_{r-1}x^{r-1} + {}^mC_rx^r + \ldots\ldots$$

$$+ \ldots\ldots + {}^mC_mx^m) \times ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_rx^r + \ldots\ldots + {}^nC_nx^n) \qquad \ldots(iii)$$

Now coefficient of x^r in R.H.S.

$$= {}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots\ldots + {}^mC_0 \cdot {}^nC_r$$

$$= {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots\ldots + {}^nC_r$$

Coefficient of x^r in L.HS. $\qquad = {}^{m+n}C_r$

But (iii) is an identity, therefore coefficient of x^r in L.H.S. = coefficient of x^r R.H.S.

$${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots\ldots + {}^nC_r$$

(h) Use of Differentiation :

$(1+x)^n = C_0+C_1x + C_2x^2+.....+ C_rx^r + ...+ C_nx^n$ $\qquad$ ...(1)

Differentiating both sides of (1) w.r.t. x, we get

$$n(1+ x)^{n-1} = C_1 + 2C_2x + 3C_3\, x^2 +......+ nC_n\, x^{n-1}$$

Now putting x = 1 and x = – 1 respectively

$$C_1+ 2C_2 +3C_3+ .....+ nC_n = n.2^{n-1} \qquad .....(9)$$

Now putting x = 1 and x = – 1 respectively

$$C_1+ 2C_2 +3C_3+ .....+ nC_n = n.2^{n-1} \qquad .....(9)$$

and $\qquad C_1 – 2C_2+ 3C_3$ - ....= 0 $\qquad$ ...(10)

(i) Adding (2) and (9)

$$C_0+ 2C_1 + 3C_2 + ...+ (n+1)\, C_n = 2^{\,n-1}\, (n+2) \qquad ...(11)$$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, then prove that

$$C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n = n\cdot 2^{n-1}.$$

Sol. *Given series is*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$$

then differentiating both sides w.r. to x, we get

$$\Rightarrow \qquad n\,(1 + x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1}$$

Putting x = 1, we get

$$n\,.\,2^{n-1} = C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n$$

or $\qquad C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n = n.2^{n-1}$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, *then prove that*

$$C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n + 1)C_n = (n + 2)2^{n-1}.$$

Sol. *Given series is*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$$

Now replacing by x^1 *and multiplying both sides by* x^1, *then*

$$x\,(1 + x)^n = C_0x + C_1x^2 + C_2x^3 + \ldots\ldots + C_nx^{n+1}$$

Now differentiating both sides w.r.t. x, we get

$$x \,.\, n\,(1 + x)^{n-1} + (1 + x)^n \,.\, 1 = C_0 + 2C_1x + 3C_2x^2 + \ldots\ldots + (n + 1)C_nx^n$$

Putting $x = 1$, *we get*

$$n\,(2)^{n-1} + 2^n = C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n + 1)C_n$$

or $\quad C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n + 1)C_n = (n + 2)2^{n-1}$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ *then prove that*

$$C_0 + 3C_1 + 5C_2 + \ldots\ldots + (2n + 1)\,C_n = (n + 1)2^n$$

Sol. *The given series is*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$$

Now replacing x by x^2

then $\quad (1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + \ldots\ldots + C_nx^{2n}$

multiplying both sides by x^1 , *we get*

$$x\,(1 + x^2)^n = C_0x + C_1x^3 + C_2x^5 + \ldots\ldots\ C_nx^{2n+1}$$

then differentiating both sides w.r. to x, we get

$$x \,.\, n\,(1 + x^2)^{n-1} \,.\, 2x + (1 + x^2)^n \,.1 = C_0 + 3C_1x^2 + 5C_2x^4 + \ldots\ldots + (2n + 1)C_nx^{2n}$$

Putting $x = 1$, *then we get*

$$n.1^{n-1} \,.\, 2 + 2^n = C_0 + 3C_1 + 5C_2 + \ldots\ldots\ (2n + 1)C_n$$

or $\quad C_0 + 3C_1 + 5C_2 + \ldots\ldots\ (2n + 1)C_n = (n + 1)\,2^n$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ *then prove that*

$$(1.2)C_2 + (2.3)C_3 + \ldots\ldots\ ((n - 1)\,.\,n)\,C_n = n\,(n - 1)\,2^{n-2}$$

Sol. *The given series is*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$$

Differentiating both sides w.r. to x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1}$$

again differentiating both sides w.r. to x, we get

$$n(n-1)(1+x)^{n-2} = 0 + 0 + (1.2)C_2 + (2.3)C_3x + \ldots\ldots + ((n-1).n)C_nx^{n-2}$$

Putting x = 1, then

$$n(n-1)(1+1)^{n-2} = (1.2)C_2 + (2.3)C_3 + \ldots\ldots + (n-1)n\, C_n$$

or $$(1.2)C_2 + (2.3)C_3 + \ldots\ldots + (n-1)n \,.\, C_n = n(n-1)\,2^{n-2}\,.$$

Illustration :

If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$ *then prove that*

$$C_0 + 2C_1 + 3C_2 - 4C_3 + \ldots\ldots + (-1)^n(n+1)\,C_n = 0$$

Sol. *The given series is*

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$$

Multiplying both sides by x, then

$$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + C_3x^4 + \ldots\ldots + C_nx^{n+1}$$

$$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + C_3x^4 + \ldots\ldots + C_nx^{n+1}$$

Differentiating both sides w.r. to x, then we get

$$x.n(1+x)^{n-1} + (1+x)^n \,.\, 1 = C_0 + 2C_1x + 3C_2x^2 + 4C_3x^3 + \ldots\ldots + (n-1)C_nx^n$$

Putting x = –1, then we get

$$0 = C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots\ldots + (-1)^n(n+1)\,C_n$$

or $$C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots\ldots\ldots\ldots + (-1)^n(n+1)C_n = 0$$

Illustration :

If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$ *then prove that*

$$C_1 - 2C_2 + 3C_3 - \ldots\ldots + (-1)^{n-1}\,n\,C_n = 0$$

Sol. $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$

Differentiating both sides w.r. to x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1}$$

Putting x = –1, then we get

$$0 = C_1 - 2C_2 + 3C_3 - \ldots\ldots + (-1)^{n-1}\,nC_n$$

or $$C_1 - 2C_2 + 3C_3 - \ldots\ldots + (-1)^{n-1}\,n\,C_n = 0$$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$, *then prove that*

$$C_0 - 3C_1 + 5C_2 - \ldots\ldots + (-1)^n (2n + 1) C_n = 0$$

Sol. *The given series is*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n \text{ replacing } x \text{ by } x^2 \text{ then}$$

$$(1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + \ldots\ldots + C_nx^{2n}$$

Multiplying both sides by x, then

$$x\,(1 + x^2)n^2 = C_0x + C_1x^3 + C_2x^5 + \ldots\ldots + C_nx^{2n+1}$$

Differentiating both sides w.r. to x, we get

$$x.n\,(1 + x^2)^{n-1}\,2n + (1 + x^2)^n .1 = C_0 + 3C_1x^2 + 5C_2x^4 + \ldots\ldots + (2n + 1)C_nx^{2n}$$

Putting x = i in both sides, we get

$$0 + 0 = C_0 - 3C_1 + 5C_2 - \ldots\ldots + (2n + 1)\,(-1)^n\,C_n$$

or $$C_0 - 3C_1 + 5C_2 - \ldots\ldots + (-1)^n (2n + 1) C_n = 0$$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$ *then prove that*

$$C_1^2 + 2C_2^2 + 3C_3^2 + \ldots\ldots nC_n^2 = \frac{(2n-1)!}{((n-1)!)^2}$$

$$C_1^2 + 2C_2^2 + 3C_3^2 + \ldots\ldots nC_n^2 = \frac{[illegible]}{((n-1)!)^2}$$

Sol. *Given* $$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_2x^3 + \ldots\ldots + C_nx^n$$

Differentiating both sides w.r.t. to x, we get

$$n\,(1 + x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1}$$

$$\Rightarrow \quad n\,(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1} \quad \ldots(i)$$

and $$(x + 1)^n = C_0x^n + C_1\,x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \ldots\ldots + C_n \quad \ldots(ii)$$

Multiplying (i) and (ii), we get

$$n\,(1 + x)^{2n-1} = (C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1})$$

$$\times\,(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \ldots\ldots + C_n) \quad \ldots(iii)$$

Now, coefficient of x^{n-1} *on R.H.S.*

$$C_1^2 + 2C_2^2 + 3C_3^2 + \ldots\ldots + nC_n^2$$

and coefficient of x^{n-1} *on L.H.S.* $= n.\ {}^{2n-1}C_{n-1}$

$$= n \cdot \frac{(2n-1)!}{(n-1)!\,n!} = \frac{(2n-1)!}{(n-1)!\,(n-1)!} = \frac{(2n-1)!}{\{(n-1)!\}^2}$$

but (iii) is an identity, therefore the coefficient of x^{n-1} *in R.H.S. = coefficient of* x^{n-1} *in L.H.S.*

$$\Rightarrow \quad C_1^2 + 2C_2^2 + 3C_3^2 + \ldots\ldots + nC_n^2 = \frac{(2n-1)!}{\{(n-1)!\}}$$

(j) Use of Integration :

$(1+x)^n = C_0+C_1x + C_2x^2+.....+ C_rx^r + ...+ C_nx^n$ ...(1)

Integrating (1) w.r.t. x between the limits 0 to 1, we get,

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]$$

$$\Rightarrow \quad C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1} \qquad \ldots(12)$$

Integrating (1) w.r.t. x between the limits –1 to 0, we get

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^0 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_{-1}^0$$

$$\Rightarrow \quad C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \ldots + \frac{(-1)^n.C_n}{n+1} = \frac{1}{(n+1)} \qquad \ldots(13)$$

Illustration :

If $C_r = {}^nC_r$, *then prove that* $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$.

Sol. *Consider the expansion*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n \qquad \ldots(i)$$

Integrating both sides of (i) within limits 0 to 1, we get

$$\int_0^1 (1 + x)^n\, dx = \int_0^1 (C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n)dx$$

$$\Rightarrow \quad \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow \quad \frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1}$$

Hence $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Illustration :

Prove that $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots\ldots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

Sol. *Consider the expansion* $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$
Integrating both sides of (i) within limits –1to 0, we get

$$\int_{-1}^{0} (1 + x)^n \, dx = \int_{-1}^{0} (C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n) \, dx$$

$$\Rightarrow \quad \left[\frac{(1+x)^n}{n+1}\right]_{-1}^{0} = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\Rightarrow \quad \frac{1-0}{n+1} = 0 - \left(-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \ldots\ldots(-1)^{n+1}\frac{C_n}{n+1}\right)$$

$$\Rightarrow \quad \frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots\ldots(-1)^n\frac{C_n}{n+1}$$

Hence $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots\ldots + (-1)^n\frac{C_n}{n+1} = \frac{1}{n+1}$

Illustration :

Prove that $\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \ldots\ldots = \frac{2^n}{n+1}$

Sol. *Consider the expansion*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots\ldots + C_nx^n \ \ldots(i)$$

Integrating both sides of (i) within limits –1 to 1, we get

$$\int_{-1}^{1} (1 + x)^n \, dx = \int_{-1}^{1} (C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots\ldots + C_nx^n) \, dx$$

$$= \int_{-1}^{1} (C_0 + C_2x^2 + C_4x^4 + \ldots\ldots) \, dx + \int_{-1}^{1} (C_1x + C_3x^3 + \ldots\ldots) \, dx$$

$$= 2\int_{-1}^{1} (C_0 + C_2x^2 + C_4x^4 + \ldots\ldots)dx + 0 \qquad \textit{(By prop. of difinite integral)}$$

(since second integral contains odd function)

$$\Rightarrow \quad \left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{1} = 2\left(C_0x + \frac{C_2x^3}{3} + \frac{C_4x^5}{5} + \ldots\ldots\right)\Big]_{0}^{1}$$

$$\Rightarrow \quad \frac{2^{n+1}}{n+1} = 2\left(C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots\ldots\right)$$

Hence $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots\ldots = \frac{2^n}{n+1}$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, *prove that*

$$3C_0 + 3^2\ \frac{C_1}{2} + \frac{3^3 C_2}{3} + \frac{3^4 C_3}{4} + \ldots\ldots + \frac{3^{n+1} C_n}{n+1} = \frac{4^{n+1} - 1}{n+1}$$

Sol. *Consider the expansion*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$$

Integrating both sides of (i) within limits 0 to 3, we get :

$$\int_0^3 (1 + x)^n\, dx = \int_0^3 (C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n)\, dx$$

$$\Rightarrow \quad \left.\frac{(1+x)^{n+1}}{n+1}\right]_0^3 = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \frac{C_3x^4}{4} + \ldots\ldots + \left.\frac{C_nx^{n+1}}{n+1}\right]_0^3$$

$$\Rightarrow \quad \frac{4^{n+1} - 1}{n+1} = 3C_0 + \frac{3^2 C_1}{2} + \frac{3^3 C_2}{3} + \frac{3^4 C_3}{4} + \ldots\ldots \frac{3^{n+1} C_n}{n+1}$$

Hence $3C_0 + \dfrac{3^2 C_1}{2} + \dfrac{3^3 C_2}{3} + \dfrac{3^4 C_3}{4} + \ldots\ldots + \dfrac{3^{n+1} C_n}{n+1} = \dfrac{4^{n+1} - 1}{n+1}$

Illustration :

Prove that ${}^{15}C_4 + 2\,{}^{15}C_5 + {}^{15}C_6 = {}^{17}C_6$

Sol. ${}^{15}C_4 + 2\,{}^{15}C_5 + {}^{15}C_6 = ({}^{15}C_4 + {}^{15}C_5) + ({}^{15}C_5 + {}^{15}C_6) = {}^{16}C_5 + {}^{16}C_6 = {}^{17}C_6$ **Ans.**

Illustration :

$$\left(\frac{{}^nC_0 + {}^nC_1}{{}^nC_0}\right)\left(\frac{{}^nC_1 + {}^nC_2}{{}^nC_1}\right) \ldots\ldots\ldots\ldots \left(\frac{{}^nC_{n-1} + {}^nC_n}{{}^nC_{n-1}}\right) = \frac{(15)^{14}}{14\,!}$$ *find value of n .*

Sol. *Given product series may be written as* $\displaystyle\prod_{r=1}^{n}\left(\frac{{}^nC_{r-1} + {}^nC_r}{{}^nC_{r-1}}\right)$

$$= \prod_{r=1}^{n} \frac{{}^{n+1}C_r}{{}^nC_{r-1}} = \prod_{r=1}^{n} \frac{n+1}{r} = \frac{(n+1)}{1} \times \frac{(n+1)}{2} \times \frac{(n+1)}{3} \ldots\ldots\ldots \frac{(n+1)}{n} = \frac{(n+1)^n}{n\,!} \Rightarrow n = 14. \ \textbf{Ans.}$$

Illustration :

Find the sum of $S = {}^nC_0 + 2\,({}^nC_1) + 3\,({}^nC_2) + \ldots\ldots\ldots\ldots + (n+1)\,({}^nC_n)$.

Sol. *Given sum may be written as*

$$S = \sum_{r=1}^{n}(r+1)\left({}^nC_r\right) = \sum_{r=0}^{n} r\cdot\left({}^nC_r\right) + \sum_{r=0}^{n}\left({}^nC_r\right) = \sum_{r=0}^{n} n\cdot\left({}^{n-1}C_{r-1}\right) + 2^n$$

$$= n\sum_{r=0}^{n}{}^{n-1}C_{r-1} + 2^n = n\cdot 2^{n-1} + 2^n. \ \textbf{Ans.}$$

Alternate Method :

Consider $\quad x(1+x)^n = {}^nC_0\,x + {}^nC_1\,x^2 \ldots\ldots + {}^nC_n\,x^{n+1}$

Differtentate w.r.t. x $\quad (1+x)^n + nx(1+x)^{n-1} = {}^nC_0 + 2\,{}^nC_1\,x + \ldots\ldots + (n+1)\,{}^nC_n\,x^n$

Put x = 1 to get $\quad {}^nC_0 + 2\cdot({}^nC_1) + 3\,({}^nC_2) + \ldots\ldots\ldots\ldots + (n+1)\,({}^nC_n) = n\cdot 2^{n-1} + 2^n$

Ans.

Illustration :

Find value of sum $\frac{{}^nC_0}{2} + \frac{{}^nC_1}{3} + \frac{{}^nC_2}{4}$ *up to n term*

Find value of sum $\frac{{}^nC_0}{2} + \frac{{}^nC_1}{3} + \frac{{}^nC_2}{4} \ldots\ldots$ *up to n term*

Sol. $S = \frac{{}^nC_0}{2} + \frac{{}^nC_1}{3} + \frac{{}^nC_2}{4} \ldots\ldots$

$$= \sum_{r=0}^{n}\frac{{}^nC_r}{r+2}$$

Consider $\quad x(1+x)^n = {}^nC_0\,x + {}^nC_1\,x^2 + {}^nC_2\,x^3 + \ldots\ldots + {}^nC_n\,x^{n+1}$

Intergrate both side w.r.t x

$$\int_0^1 x(1+x)^n\,dx = \int_0^1\left({}^nC_0x + {}^nC_1x^2 + \ldots\ldots + {}^nC_nx^{n+1}\right)dx$$

or $$\left[\frac{x\cdot(1+x)^{n+1}}{n+1}\right]_0^1 - \int_0^1\frac{(1+x)^{n+1}}{n+1}\,dx = \left[{}^nC_0\frac{x^2}{2} + {}^nC_1\frac{x^3}{3}\ldots\ldots\right]_0^1$$

or $$\frac{2^{n+1}}{n+1} - \left(\frac{2^{n+2}}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)}\right) = \frac{{}^nC_0}{2} + \frac{{}^nC_1}{3} + \ldots\ldots + \frac{{}^nC_n}{n+2}$$

$$\Rightarrow \quad S = \frac{2^{n+1}}{n+1} - \frac{2^{n+2}}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}$$

Illustration :

Find sum of series $S = {}^nC_1 + 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 + \ldots\ldots + n\,{}^nC_n$

Sol. $S = \sum_{r=1}^{n} r \cdot {}^nC_r = \sum_{r=1}^{n} r \cdot \frac{n}{r}\,{}^{n-1}C_{r-1} = n\sum_{r=1}^{n} {}^{n-1}C_{r-1}$

$$= n \cdot 2^{n-1}$$

Alternate Method :

$S = 0 \cdot {}^nC_0 + 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + \ldots\ldots\ n \cdot {}^nC_n$

$S = n \cdot {}^nC_n + (n-1)^n\, C_{n-1} + \ldots\ldots + 0 \cdot {}^nC_0$

$\Rightarrow \quad 2S = n \cdot {}^nC_0 + n \cdot {}^nC_1 + \ldots\ldots + n \cdot {}^nC_n$

or $\quad S = \frac{n}{2}\,({}^nC_0 + {}^nC_1 + \ldots\ldots + {}^nC_n)$

$\Rightarrow \quad S = \frac{n}{2}\,2^n$

An Important Result :

An Important Result :

For sums involving product of two binomial coefficients use

$${}^nC_0\,{}^mC_k + {}^nC_1\,{}^mC_{k-1} + {}^nC_2\,{}^mC_{k-2} \ldots\ldots\ldots\ldots + {}^nC_k\,{}^mC_0 = {}^{m+n}C_k$$

Proof: Consider two series

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2\,x^2 + \ldots\ldots. + {}^nC_k\,x^k + \ldots\ldots. + {}^nC_n\,x^n \qquad \ldots\ldots.(1)$

$(1 + x)^m = {}^mC_mx^m + \ldots\ldots\ldots + {}^mC_k\,x^k + {}^mC_{k-1}\,x^{k-1} + \ldots\ldots\ldots + {}^mC_0 \qquad \ldots\ldots.(2)$

Here second series is written in reverse order.

Multiplying (1) and (2) and equate coefficients of x^k on both sides to get

$${}^{m+n}C_k = {}^nC_0\,{}^mC_k + {}^nC_1 \cdot {}^mC_{k-1} + \ldots\ldots\ldots + {}^nC_k\,{}^mC_0.$$

Illustration :

Find the sum $S = {}^nC_0{}^nC_1 + {}^nC_1{}^nC_2 + \ldots\ldots\ldots + {}^nC_{n-1}{}^nC_n$.

Sol. $S = {}^nC_0\,{}^nC_{n-1} + {}^nC_1\,{}^nC_{n-2} + \ldots\ldots\ldots\ldots + {}^nC_{n-1}{}^nC_0 \qquad [{}^nC_r = {}^nC_{n-r}]$

If sum of lower suffices is constant then

$\Rightarrow \quad S = {}^{n+n}C_{n-1} = {}^{2n}C_{n-1} = {}^{2n}C_{n+1}$. ***Ans.***

Illustration :

Find the sum $S = 1 \cdot ({}^nC_0)^2 + 2 \cdot ({}^nC_1)^2 + 3 \cdot ({}^nC_2)^2 + \ldots\ldots\ldots + (n+1)\,({}^nC_n)^2$

Sol. *Given sum may be written as*

$$S = \sum_{r=1}^{n} r \cdot \left({}^nC_r\right)^2 \quad or \quad S = \sum_{r=1}^{n} \left(r \cdot {}^nC_r\right) {}^nC_r \quad or \quad S = \sum_{r=1}^{n} n \cdot {}^{n-1}C_{r-1} \cdot {}^nC_r$$

$$or \qquad S = n \sum_{r=1}^{n} {}^{n-1}C_{r-1} \cdot {}^nC_r = n \sum_{r=1}^{n} {}^{n-1}C_{r-1}\ {}^nC_{n-r} \;\Rightarrow\; S = n \cdot {}^{2n-1}C_{n-1}\,.\ \textbf{Ans.}$$

Illustration :

If $(1+x)^n = C_0 + C_1x + \ldots\ldots + C_nx^n$, *then the value of* $\sum_{r=0}^{n}\sum_{s=0}^{n}(C_r + C_s)$ *is equal to*

(A) $(n+1)\,2^{n+1}$ (B) $(n-1)2^{n+1}$ (C) $(n+1)\,2^n$ (D) none of these

Sol. *We have,* $\sum_{r=0}^{n}\sum_{s=0}^{n}(C_r + C_s)$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n} C_r + \sum_{r=0}^{n}\sum_{s=0}^{n} C_s = \sum_{s=0}^{n}\left(\sum_{r=0}^{n} C_r\right) + \sum_{r=0}^{n}\left(\sum_{s=0}^{n} C_s\right) = \sum_{s=0}^{n} 2^n + \sum_{r=0}^{n} 2^n$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n} C_r + \sum_{r=0}^{n}\sum_{s=0}^{n} C_s = \sum_{s=0}^{n}\left(\sum_{r=0}^{n} C_r\right) + \sum_{r=0}^{n}\left(\sum_{s=0}^{n} C_s\right) = \sum_{s=0}^{n} 2^n + \sum_{r=0}^{n} 2^n$$

$$= (n+1)\,2^n + (n+1)\,2^n = (n+1)\,2^{n+1}$$

Hence (A) is correct answer.

Illustration :

If $(1+x)^n = C_0 + C_1x + \ldots\ldots + C_nx^n$, *then the value of* $\sum\sum_{0\le r<s\le n} C_rC_s$ *is equal to*

(A) $\frac{1}{2}[2^{2n} - {}^{2n}C_n]$ (B) $\frac{1}{4}[2^{2n} - {}^{2n}C_n]$ (C) $\frac{1}{2}[2^{2n} + {}^{2n}C_n]$ (D) $\frac{1}{2}[2^n - {}^{2n}C_n]$

Sol. *We have*

$$\sum_{r=0}^{n}\sum_{s=0}^{n} C_rC_s = \left(\sum_{r=0}^{n} C_r^2\right) + 2\sum\sum_{0\le r<s\le n} C_rC_s$$

$$\Rightarrow \qquad 2^{2n} = {}^{2n}C_n + 2\sum\sum_{0\le r<s\le n} C_rC_s$$

$$\Rightarrow \qquad \sum\sum_{0\le r<s\le n} C_rC_s = \frac{1}{2}[\,2^{2n} - {}^{2n}C_n\,]$$

Hence (A) is correct answer.

3.3 AN IMPORTANT CONCEPT :

Finding nature of integral part of expression.

$$N = \left(a + \sqrt{b}\right)^n \qquad (n \in N)$$

Step-1: Consider $N' = \left(a - \sqrt{b}\right)^n$ or $\left(\sqrt{b} - a\right)^n$ according as $a > \sqrt{b}$ or $\sqrt{b} > a$.

Step-2: Use $N + N'$ or $N - N'$ such that result is integer.

Step-3: Use fact $N = I + f$

'I' stands for $[N]$ and 'f' for $\{N\}$.

Illustration :

For $n \in N$ prove that integral part of $N = \left(3+\sqrt{7}\right)^n$ is an odd integer.

Sol. *Consider* $N' = \left(3-\sqrt{7}\right)^n$

$$N = {}^nC_0\, 3^n + {}^nC_1\, 3^{n-1}\sqrt{7} + {}^nC_2\, 3^{n-2}\left(\sqrt{7}\right)^2 + \text{.........} - {}^nC_n\left(\sqrt{7}\right)^n.$$

$$N' = {}^nC_0\, 3^n - {}^nC_1\, 3^{n-1}\sqrt{7} + {}^nC_2\, 3^{n-2}\left(\sqrt{7}\right)^2 + \text{............} - {}^nC_n\left(\sqrt{7}\right)^n.$$

Using $N + N' = 2\,[{}^nC_0\, 3^n + {}^nC_2\, 3^{n-2}\left(\sqrt{7}\right)^2 + \text{.........}]$

$\Rightarrow \quad N + N' = 2k \quad (k \in I) \quad \Rightarrow \quad I + f + N' = 2k \qquad \{N = I + f\}$

$0 < N' < 1 \Rightarrow 0 < f + N' < 2$

But $f + N'$ *it self an integer* $f + N' = 1 \Rightarrow I + 1 = 2k$ *or* $I = 2k - 1$.

Hence proved.

Illustration :

Show that integral part of $P = \left(3\sqrt{3}+5\right)^{2n+1}$ *$(n \in N)$ is an even number.*

Sol. *Consider* $P' = \left(3\sqrt{3}-5\right)^{2n+1}$ *here* $0 < P' < 1$

Using $P - P' = 2\left[{}^{2n+1}C_1\left(3\sqrt{3}\right)^{2n}5' + {}^{2n+1}C_3\left(3\sqrt{3}\right)^{2n-2}(5)^3 + \ldots\ldots\right]$

$\Rightarrow \quad I + f - P' = 2k \; (k \in N) \qquad\qquad \{P = I + f\}$

$-1 < f - P' < 1$ *but* $f - P'$ *is an integer* $\Rightarrow f - P' = 0 \Rightarrow I = 2k.$

Illustration :

Let $N = \left(7+4\sqrt{3}\right)^n = I + f \quad (n \in N)$*, then find the value of* $(1-f)\,N$.

Sol. *Consider* $N' = \left(7-4\sqrt{3}\right)^n$

Using $N + N' = 2\,[{}^nC_0\,7^n + {}^nC_2\,7^{n-2}\left(4\sqrt{3}\right)^2 + \ldots\ldots]$

or $\quad I + f + N' = 2k \qquad (k \in I)$

Illustration :

Let $N = \left(7+4\sqrt{3}\right)^n = I + f \quad (n \in N)$*, then find the value of* $(1-f)\,N$.

Sol. *Consider* $N' = \left(7-4\sqrt{3}\right)^n$

Using $N + N' = 2\,[{}^nC_0\,7^n + {}^nC_2\,7^{n-2}\left(4\sqrt{3}\right)^2 + \ldots\ldots]$

or $\quad I + f + N' = 2k \qquad (k \in I)$

or $\quad f + N' = 2k - I$

but $\quad 0 < f + N' < 2 \Rightarrow f + N' = 1$ *or* $f = 1 - N'$.

$$(1-f)N = N'N = \left(7-4\sqrt{3}\right)^n\left(7+4\sqrt{3}\right)^n = (49-48)^n = 1.$$

Solved Examples

Single correct question

Q.1 Value of middle term of expansion $\left(\frac{2}{3}x^2 - \frac{3}{2x}\right)^{20}$ for x = p is $32 \cdot {}^{20}C_{10}$ value of 'p' is

(A) 2 (B) $\frac{1}{2}$ (C) $\sqrt{2}$ (D) $\frac{1}{4}$

Sol. Here n = 20 so no of terms will be 21 then middle term $t_m = t_{10+1} = {}^{20}C_{10}x^{10}$

Q.2 If $b_1, b_2, \ldots\ldots b_n$ are in G.P. with common ratio '2' then $b_1\,{}^nC_1 + b_2\,{}^nC_2 + \ldots\ldots + b_n \cdot {}^nC_n =$

(A) $\frac{b_1}{2}(3^n)$ (B) $b_1(3^n)$ (C) $b_1(3^n - 1)$ (D) $\frac{b_1}{2}(3^n - 1)$

Sol. $b_1\,{}^nC_1 + b_2\,{}^nC_2 + \ldots\ldots + b_n\,{}^nC_n = b_1({}^nC_1 + 2({}^nC_2) + 2^2({}^nC_3) + \ldots\ldots + 2^{n-1}\,{}^nC_n)$

$= \frac{b_1}{2}[(1+2)^n - 1] = \frac{b_1}{2}(3^n - 1)$

Q.3 If $(1 + ax)^n = 1 + 8x + 24x^2 \ldots\ldots$ then $\frac{a-n}{a+n}$ is equal to

(A) 3 (B) –3 (C) $-\frac{1}{3}$ (D) $\frac{1}{3}$

Sol. $1 + ax \cdot {}^nC_1 + {}^nC_2(ax)^2 + \ldots\ldots = 1 + 8x + 24x^2 + \ldots\ldots\ldots$

By comparison

$\Rightarrow$ ${}^nC_1\, a = 8$ ....(1) and ${}^nC_2\, a^2 = 24$ ....(2)

By equation (1) and (2), we get

n = 4 and a = 2.

Q.4 In the expansion $\left(x^2 + \frac{2}{x}\right)^n$ $(n \in N)$ has 13^{th} term independent of x, then sum of even divisors of n is equal to –

(A) 39 (B) 63 (C) 13 (D) 26

Sol. $t_{13} = {}^{n}C_{12} \cdot 2^{12}\, x^{2n-36}$ $\Rightarrow$ $2n - 36 = 0$ or $n = 18$

$n = 18 = 2^1 \times 3^2$

sum of divisors (even) $= (2^1)(3^0 + 3^1 + 3^2)$

$= 2 \times 13 = 26$

Q.5 Cofficient of x^7 in the expansion of $(1 - x^4)(1 + x)^9$ is –

(A) 27 (B) –24 (C) 48 (D) –48

Sol. $(1 - x^4)(1 + x)^9 = (1 + x)^9 - x^4(1 + x)^9$

cofficient of $x^7 = {}^{9}C_7 - {}^{9}C_3 = -48$

Q.6 Cofficient of x^{100} in $1 + (1 + x) + (1 + x)^2 + (1 + x)^3 \ldots\ldots (1 + x)^n$ $(n > 100)$ is ${}^{201}C_{101}$, then n =

(A) 100 (B) 200 (C) 101 (D) None of these

Sol. Coefficient of $x^{100} = {}^{100}C_{100} + {}^{101}C_{100} + \ldots\ldots + {}^{n}C_{100}$

$= {}^{n+1}C_{101} = {}^{201}C_{101}$ $\Rightarrow$ $n = 200$

Multiple correct type question

Q.7 Cofficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 42, value of'n'

Multiple correct type question

Q.7 Cofficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 42, value of'n' is always less than or equal to –

(A) 55 (B) 54 (C) 56 (D) 51

Sol. ${}^{n}C_{r-1} : {}^{n}C_r : {}^{n}C_{r+1} = 1 : 7 : 42$ $\Rightarrow$ $n = 55$ and $r = 7$

Q.8 $\alpha = \left\{\dfrac{3^{200}}{8}\right\}$ where {x} factional of x then possible of middle term in $(2 + 5x)^{72\alpha}$ is –

(A) 5^{th} (B) 7^{th} (C) 8^{th} (D) 6^{th}

Sol. $3^{200} = 9^{100} = (1 + 8)^{100} = 1 + [{}^{100}C_1\, 8 + {}^{100}C_2\, 8^2 \ldots\ldots]$

$\Rightarrow \left\{\dfrac{3^{200}}{8}\right\} = \dfrac{1}{8}$

$(2 + 5x)^{72\alpha} = (2 + 5x)^9$

Q.9 Third term in the expansion of $\left(x + x^{\log_{10} x}\right)^5$ is 10^6 then possible value of x are –

(A) 1 (B) 10 (C) $10^{-5/2}$ (D) 10^6

Sol. $T_3 = {}^5C_2\, x^3 \left(x^{\log_{10} x}\right)^2 = 10 \cdot x^{3+2\log_{10} x} = 10^6$

$\Rightarrow \quad x^{3+2\log_{10} x} = 10^5$ taking long on both sides given

$$(3 + 2\log_{10} x)\log_{10} x = 5 \quad \Rightarrow \quad \log_{10} x = 1 \quad \text{or} \quad -\frac{5}{2}$$

Q.10 If $(1 + x + x^2 + x^3) = C_0 + C_1 x + C_2 x^2 + \ldots\ldots + C_{3n} x^{3n}$ then which of following are correct –

(A) $C_0 + C_1 + C_2 + \ldots\ldots + C_{3n} = 2^{2n}$ (B) $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_2 + C_5$

(C) $C_0 = C_{3n}, C_1 = C_{3n-1}, C_2 = C_{3n-2}$ (D) None of these

Sol. Put x = 1 to get $4^n = C_0 + C_1 + C_2 + \ldots\ldots + C_{3n} \ldots$

Put x = –1 to get $\Rightarrow C_0 + C_2 + C_4 \ldots\ldots = C_1 + C_3 + C_5 \ldots\ldots$

Paragraph type

Paragraph type

Paragraph for question nos. 11 to 13

Let P be a product given by $P = (x + \alpha_1)(x + \alpha_2) \ldots\ldots (x + \alpha_n)$ and $S_1 = \alpha_1 + \alpha_2 + \ldots\ldots + \alpha_n = \sum_{i=1}^{n} \alpha_i$

$S_2 = \sum\sum_{i<j} \alpha_i \alpha_j$ $\quad S_3 = \sum\sum\sum_{i<j<k} \alpha_i \alpha_j \alpha_k$ and so on then $P = x^n + S_1 x^{n-1} + S_2 x^{n-2} \ldots\ldots + S_n$

Q.11 Cofficient of x^8 in $(x + 2)^2 (x + 3)^3 (x + 4)^4$ is –

(A) 26 (B) 27 (C) 28 (D) 29

Q.12 Cofficient of x^{203} in $(x - 1)(x^2 - 2)(x^3 - 3) \ldots\ldots (x^{20} - 20)$ is –

(A) 11 (B) 12 (C) 13 (D) 15

Q.13 Cofficient of x^{98} in $(x - 1)(x - 2) \ldots\ldots (x - 100)$ is

(A) $1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2$

(B) $(1 + 2 + 3 + \ldots\ldots + 100)^2 - (1^2 + 2^2 + \ldots\ldots + 100^2)$

(C) $\frac{1}{2}[(1 + 2 + \ldots\ldots + 100)^2 - (1^2 + 2^2 + \ldots\ldots + 100^2)]$

(D) None of these

Sol. (i) $(x^4 + 4 + 4x)\,(x^3 + 3c_1\,x^2 + \ldots\ldots)\,(x^4 + {}^4C_1\,x^3 + \ldots\ldots)$ expand suitably

(ii) $(x - 1)\,(x^2 - 2)\,(x^3 - 3)\ \ldots\ldots\ (x^{20} - 20)$

$$= x^{210}\left[\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\ldots\ldots\left(1-\frac{20}{x^{20}}\right)\right]$$

(iii) Cofficient x^{98} in S_2 in prodct $(x - 1)\,(x - 2)\ \ldots\ldots\ (x - 100)$

$$=\frac{1}{2}\,[(1 + 2 + \ldots\ldots + 100)^2 - (1^2 + 2^2 + \ldots\ldots + 100^2)]$$

Match the Column

Q.14

	Column-I		Column-II
(A)	Cofficient of x^7 and x^8 are equal in the expansion $\left(3+\frac{x}{2}\right)^n$ value of n is	(P)	9
(B)	${}^nC_2 = 990$ then n is divisible by	(Q)	45
(B)	${}^nC_2 = 990$ then n is divisible by	(Q)	45
(C)	Cofficient of x^2 and x^3 are equal in expansion of $\left(3+\frac{9x}{7}\right)^m$ then m is grater than or equal to	(R)	7
		(S)	55
(D)	${}^pC_6 = {}^{p-1}C_5 + 1$ then p is less than	(T)	15

Sol. (A) ${}^nC_7\cdot\frac{3^{n-7}}{2^7} = {}^nC_8\cdot\frac{3^{n-8}}{2^8} \Rightarrow \quad n = 55$

(B) ${}^nC_2 = 990 \quad \Rightarrow \quad n = 45$

(C) ${}^mC_2\,\frac{3^{m+2}}{7^2} = {}^mC_3\,\frac{3^{m+3}}{7^3} \quad \Rightarrow \quad m = 9$

(D) ${}^pC_6 = {}^{p-1}C_5 + 1 \quad$ or $\quad {}^{p-1}C_6 + {}^{p-1}C_5 = {}^{p-1}C_5 + 1$

$\Rightarrow\ {}^{p-1}C_6 = 1 \quad$ or $\quad p = 7$

Reasoning type question

Q.15 **Statement-1:** ${}^{20}C_0\,{}^{15}C_r + {}^{20}C_1\,{}^{15}C_{r-1} + \ldots\ldots + {}^{20}C_r\,{}^{15}C_0$ has maximum value ${}^{35}C_{18}$.

Statement-2: ${}^{n}C_0\,{}^{m}C_s + {}^{n}C_1\,{}^{m}C_{s-1} + {}^{n}C_2\,{}^{m}C_{s-2}\ \ldots\ldots + {}^{n}C_s\,C_0 = {}^{m+n}C_s$

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

Sol. S_2 is true as $\sum\sum {}^{n}C_{r_1}\,{}^{m}C_{r_2}$ where $r_1 + r_2 = k$ (constant) is equal to ${}^{m+n}C_k$. Sum given in S_1 is equal to ${}^{35}C_r$.

Q.16 **Statement-1:** If ${}^{100}C_r$, ${}^{100}C_{r+1}$, ${}^{100}C_{r+2}$ and ${}^{100}C_{r+3}$ are in AP then r = 49.

Statement-2: Four consecutive binomial cofficient can never be in AP.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

Sol. $2 \cdot {}^{100}C_{r+1} = {}^{100}C_r + {}^{100}C_{r+2}$

$$\Rightarrow \qquad 2 = \frac{r+1}{100-r} + \frac{99-r}{r+2}$$

which has no solution

Appendix - I

THE PRINCIPLE OF MATHEMATICAL INDUCTION :

Mathematical induction is a technique for proving any statement any theorem, or a formula that is asserted about every natureal number,.

By "every", or "all" natural numbers, we mean any one that we might possibly name.

For example,

$$1 + 2 + 3 + \ldots\ldots + n = \frac{1}{2n}(n+1)$$

This asserts that the sum of consecutive numbers from 1 to n is given by the formula on the right. We want to prove that this will be true for $n = 1$, $n = 2$, $n = 3$, and so on. Now we can test the formula for any given number, say $n = 3$.

$$1 + 2 + 3 = \frac{1}{2} \cdot 3 \cdot 4 = 6$$

Which is true. It is also true for $n = 4$.

Which is true. It is also true for $n = 4$.

$$1 + 2 + 3 + 4 = \frac{1}{2} \cdot 4 \cdot 5 = 10.$$

But how are we to prove this rule for every value of n ?

The method of proof is the following. It is called the principle of mathematical induction.

AXIOM OF INDUCTION :

(1) when a statement is true for a natural number $n = k$,
then it will also be true for its successor, $n = k + 1$,
and

(2) the statement is true for $n = 1$.
then the statement will be true for every natural number n.

To prove a statement by induction, we must prove parts 1 and 2 above. For, when the statement is true for n= 1, then according to 1, it will also be true for 2. But that implies it will be true for 3 which implies it will be true for 4. And so on. It will be true for any natural number that we migth name.

Appendix - II

BINOMIAL THEOREM FOR ANY INDEX :

When n is a negative integer or a fraction then the expansion of a binomial is possible only when

(i) Its first term is 1, and

(ii) Its second term is numerically less than 1.

Thus when $n \notin N$ and $|x| < 1$, then it states

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \text{........} + \frac{n(n-1)(n-r+1)}{r!}x^r + \text{.....}\infty\infty$$

1.1 General Term :

$$T_{r+1} = \frac{n(n-1)(n-2)\text{.....}(n-r+1)}{r!}\,.\,x^r$$

Note :

(i) In this expansion the coefficient of different terms can not be expressed as ${}^nC_0, {}^nC_1, {}^nC_2$... because **n** is not a positive integer.

(ii) In this case there are infinite terms in the expansion.

(ii) In this case there are infinite terms in the expansion.

1.2 Some Important Expansions :

If $|x| < 1$ and $n \in Q$ but $n \notin N$, then

(a) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \text{....} + \frac{n(n-1)\text{...}(n-r+1)}{r!}x^r + \text{......}$

(b) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \text{.....} + \frac{n(n-1)\text{...}(n-r+1)}{r!}(-x)^r + \text{....}$

(c) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \text{.......} + \frac{n(n+1)\text{...}(n+r-1)}{r!}x^r + \text{..}$

(d) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \text{.....} + \frac{n(n+1)\text{...}(n+r-1)}{r!}(-x)^r + \text{..}$

By putting n = 1, 2, 3 in the above results (c) and (d), we get the following results-

(e) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \text{.....} + x^r + \text{.....}$

General term $T_{r+1} = x^r$

(f) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \text{....}(-x)^r + \text{.....}$

General term $T_{r+1} = (-x)^r$

(g) $(1-x)^{-2} = 1+2x+3x^2+4x^3+.....+(r+1)x^r+.....$

General term $T_{r+1} = (r+1)\,x^r$

(h) $(1+x)^{-2} = 1-2x+3x^2-4x^3+.....+(r+1)\,(-x)^r+.....$

General term $T_{r+1} = (r+1)\,(-x)^r$.

(i) $(1-x)^{-3} = 1+3x+6x^2+10\,x^3+......+\dfrac{(r+1)(r+2)}{2!}x^r+.....$

General term $= \dfrac{(r+1)(r+2)}{2!}x^r$.

(j) $(1+x)^{-3} = 1-3x+6x^2-10x^3+.....+\dfrac{(r+1)(r+2)}{2!}(-x)^r+.....$

General term $= \dfrac{(r+1)(r+2)}{2!}\,(-x)^r$.

Illustration :

If $|x| < 2/3$ then the fourth term in the expansion of $\left(1+\dfrac{3}{2}x\right)^{1/2}$ is–

(A) $\dfrac{27}{128}x^3$ (B) $-\dfrac{27}{128}x^3$ (C) $\dfrac{81}{256}x^3$ (D) $-\dfrac{81}{256}x^3$

Sol. $T_4 = \dfrac{1/2(1/2-1)(1/2-2)}{3!}.\left(\dfrac{3x}{2}\right)^3 = \dfrac{27}{128}x^3$ **Ans.[A]**

(A) $\dfrac{27}{128}x^3$ (B) $-\dfrac{27}{128}x^3$ (C) $\dfrac{81}{256}x^3$ (D) $-\dfrac{81}{256}x^3$

Sol. $T_4 = \dfrac{1/2(1/2-1)(1/2-2)}{3!}.\left(\dfrac{3x}{2}\right)^3 = \dfrac{27}{128}x^3$ **Ans.[A]**

Illustration :

The term independent of x in the expansion of $\left(\dfrac{1-x}{1+x}\right)^2$ is–

(A) 4 (B) 3 (C) 2 (D) 1

Sol. $(1-x)^2\,(1+x)^{-2}$

$\Rightarrow$ $(1-2x+x^2)\,(1-2x+x^2+....)$

$\Rightarrow$ so term independent of $x = 1$. **Ans. [D]**

Illustration :

The coefficient of x^5 in the expansion of $(1-x)^{-6}$ is –

(A) 1260 (B) – 1260 (C) – 252 (D) 252

Sol. x^5 occurs in T_6 of the expansion, so

$$T_6 = T_{5+1} = \frac{6.7.8.9.10}{5!}\,x^5 = 252\,x^5$$

$\therefore$ Coefficient of $x^5 = 252$ **Ans.[D]**

1.3 Applications of binomial theorem :

(I) **With the help of binomial theorem, we can find out the value of sq. root, cube root and 4th root etc. of the given number upto any decimal places.**

Illustration :

The value of cube root of 1001 upto five decimal places is –

(A) 10.03333 (B) 10.00333 (C) 10.00033 (D) None of these

Sol. $(1001)^{1/3} = (1000+1)^{1/3} = 10\left(1+\frac{1}{1000}\right)^{1/3} = 10 \left\{1+\frac{1}{3}.\frac{1}{1000}+\frac{1/3(1/3-1)}{2!}\frac{1}{1000^2}+.....\right\}$

$= 10\ \{1 + 0.0003333 - 0.00000011+....\}$

$= 10.00333$ **Ans.[B]**

(II) **To find the sum of Infinite series :**

We can compare the given infinite series with the expansion of $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}$

We can compare the given infinite series with the expansion of $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}$ $x^2 + ...$ and by finding the value of x and n and putting in $(1+x)^n$ the sum of series is determined.

Illustration :

The sum of $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+.....\infty$ is–

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $2^{3/2}$

Sol. Comparing with $1+nx+\frac{n(n-1)}{2!}x^2+......$

$n\,x = 1/4$ ...(1)

and $\dfrac{n(n-1)x^2}{2!} = 1.3/4.8$

or $\dfrac{nx(nx-x)}{2!} = \dfrac{3}{32} \Rightarrow \dfrac{1}{4}\left(\dfrac{1}{4}-x\right) = \dfrac{3}{16}$ (by (1))

$\Rightarrow \left(\dfrac{1}{4}-x\right) = \dfrac{3}{4} \Rightarrow x = \dfrac{1}{4} - \dfrac{3}{4} = -\dfrac{1}{2}$ ... (2)

putting the value of x in (1)

$n(-1/2) = 1/4 \quad \Rightarrow n = -1/2$

$\therefore$ sum of series $= (1+x)^n$

$= (1-1/2)^{-1/2} = (1/2)^{-1/2} = \sqrt{2}$ **Ans.[A]**

(III) Approximation:

Illustration :

If x is so small so that its square and higher power can be neglected.

If x is so small so that its square and higher power can be neglected.

Find the value of $\dfrac{\left(1+\dfrac{2x}{3}\right)^{-5} + (4+2x)^{1/2}}{(4+x)^{3/2}}$.

Sol. $\dfrac{\left(1+\dfrac{2x}{3}\right)^{-5} + (4+2x)^{1/2}}{(4+x)^{3/2}} = \dfrac{\left(1-\dfrac{10x}{3}\right) + 2\left(1+\dfrac{x}{4}\right)}{8\left(1+\dfrac{3x}{8}\right)} = \dfrac{1}{8}\left(3-\dfrac{10x}{3}+\dfrac{x}{2}\right)\left(1+\dfrac{3x}{8}\right)^{-1}$

$= \dfrac{3}{8}\left(1-\dfrac{17x}{18}\right)\left(1-\dfrac{3x}{8}\right) = \dfrac{3}{8}\left(1-\dfrac{17x}{18}-\dfrac{3x}{8}\right) = \dfrac{72-95x}{24\times 8}$