VECTOR

VECTORS AND SCALARS :

The physical quantities (we deal with) are generally of two types:

Scalar Quantity:

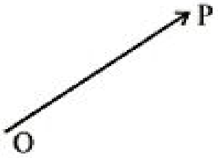
A quantity which has magnitude but no sense of direction is called scalar quantity (or scalar), e.g., mass, volume, density, speed etc.

Vector Quantity:

A quantity which has magnitude as well as a sense of direction in space and obey the laws of vector algebra is called a vector quantity, e.g., velocity, force, displacement etc.

Notation and Representation of Vectors :

Vectors are represented by $\vec{a}$, $\vec{b}$, $\vec{c}$ and their magnitude (modulus) are represented by a, b, c, or $|\vec{a}|$, $|\vec{b}|$, $|\vec{c}|$. The vectors are represented by directed line segments.



For example, line segment $\overrightarrow{OP}$ represents a vector with magnitude OP (length of line segment), arrow denotes its direction. O is initial point and P is terminal point, also called as head & tail of vector respectively.

KINDS OF VECTORS :

1. **Zero or null vector :** A vector whose magnitude is zero is called zero or null vector and it is denoted by 0 or $\vec{0}$. The initial and terminal points of the directed line segment representing zero vector are coincident and its direction is arbitrary.

2. **Unit vector :** A vector of unit magnitude is called a unit vector. A unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$. Thus

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\text{Vector a}}{\text{Magnitude of a}}$$

Note :

(i) $|\hat{a}| = 1$

(ii) Unit vectors parallel to x- axis, y- axis and z- axis are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively.

(iii) Two unit vectors may not be equal unless they have the same direction.

3. **Equal Vectors :** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal , if

(a) $|\vec{a}| = |\vec{b}|$

(b) they have the same sense of direction

4. **Co-initial vectors:** Vectors having same initial point.

5. **Free vectors:** All such vectors are those which when transformed into space from one point to another point without affecting their magnitude and direction, can be considered as equal. i.e. the physical effects produced by them remains unaltered. e.g. displacement, velocity

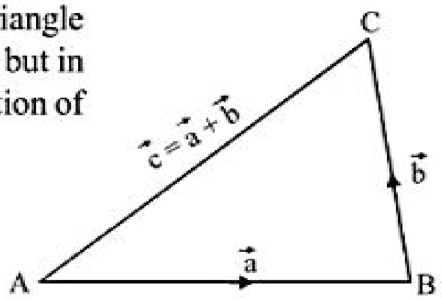
6. **Localised vectors:** e.g. force , different physical effect if line of application is changed.

Note : In mathematics we mainly deal with free vectors.

ADDITION OF VECTORS :

Triangle law of addition :

If two vectors are represented by two consecutive sides of a triangle then their sum is represented by the third side of the triangle but in opposite direction. This is known as the triangle law of addition of vectors.



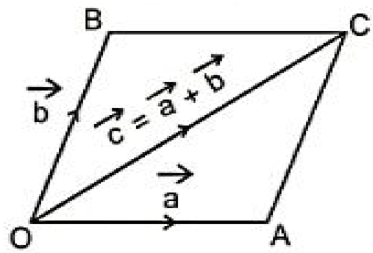
Thus, if $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, and $\overrightarrow{AC} = \vec{c}$

then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ i.e. $\vec{a} + \vec{b} = \vec{c}$

Converse of triangle law is also true.

Parallelogram Law of Addition :

If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram whose initial point is the same as the initial point of the given vectors. This is known as parallelogram law of addition of vectors.



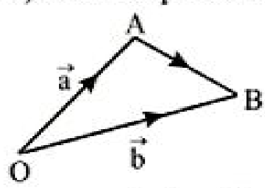
Thus if $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$

then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$

i.e. $\vec{a} + \vec{b} = \vec{c}$

POSITION VECTORS :

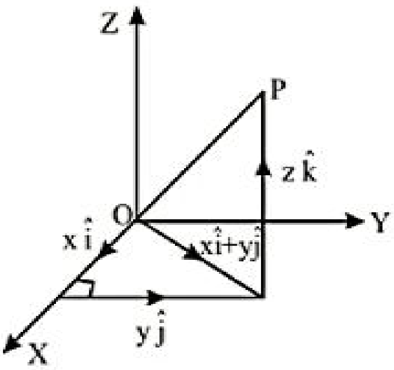
Let O be fixed point in space, then vector $\overrightarrow{OP}$ (P is any point in space) is called position vector of point P w.r.t. O. If A and B are any two point in space then



$\overrightarrow{AB}$ = p.v. of B – p.v. of A = $\overrightarrow{OB} - \overrightarrow{OA}$.

i.e. $\overrightarrow{AB} = \vec{b} - \vec{a}$

Note : Position vector of a point P (x, y, z) in terms of its cartesian coordinate is $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.



MULTIPLICATION OF A VECTOR BY SCALAR :

If $\vec{a}$ is a vector & m is a scalar, then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called SCALAR MULTIPLICATION. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

$m(\vec{a})=(\vec{a})m=m\vec{a}$ $\qquad$ $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$

$(m+n)\vec{a}=m\vec{a}+n\vec{a}$ $\qquad$ $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

DISTANCE BETWEEN TWO POINTS :

Let A and B be two given points whose coordinate are respectively (x_1, y_1, z_1) and (x_2, y_2, z_2)

If $\vec{a}$ and $\vec{b}$ are p.v. of A and B relative to point O, then

$$\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$$

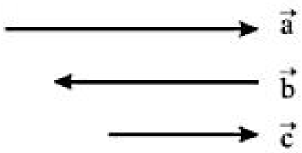
$$\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$$

Now $\overrightarrow{AB}=\overline{OB}-\overline{OA}=\vec{b}-\vec{a}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$

Distance between the points $\vec{A}$ and $\vec{B}$ = magnitude of $\overrightarrow{AB}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

COLLINEAR VECTORS OR PARALLEL VECTORS :

Vectors which are parallel to the same line are called collinear vectors or parallel vectors. Such vectors have either same direction or opposite direction. If they have the same direction they are said to be like vectors, and if they have opposite directions, they are called unlike vectors.



In the diagram $\vec{a}$ and $\vec{c}$ are like vectors whereas $\vec{a}$ and $\vec{b}$ are unlike vectors.

i.e. $\quad \vec{a}=k_1\vec{c}\ \ (k_1>0)$, $\ \vec{a}=k_2\vec{b}\ \ (k_2<0)$

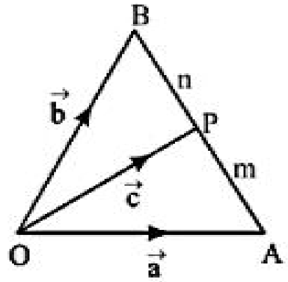
COPLANAR VECTORS :

If the directed line segments of some given vectors are parallel to the same plane then they are called coplanar vectors. It should be noted that two vectors are always coplanar but three or more vectors may or may not be coplanar.

SECTION FORMULAE :

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the position vector $\vec{c}$ of a point P dividing AB in the ratio $m:n$ is given by

$$\vec{c}=\frac{m\vec{b}+n\vec{a}}{m+n}$$

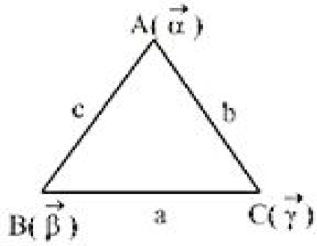


Particular Case :

1. Position vector of the mid point of AB is $\dfrac{\vec{a}+\vec{b}}{2}$

2. If the point P divides AB in the ratio m: n externally, then p.v. of P is given by $\vec{c} = \dfrac{m\vec{b}-n\vec{a}}{m-n}$

Using section Formulae we can prove that :



1. p.v. of the centroid of a triangle ABC $= \dfrac{\vec{\alpha}+\vec{\beta}+\vec{\gamma}}{3}$

(Concurrency of medians)

2. p.v. of incentre of the $\Delta = \dfrac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}$

(Concurrency of internal angle bisectors)

Excentres of the Δ are $\dfrac{-a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{-a+b+c}$; $\dfrac{a\vec{\alpha}-b\vec{\beta}+c\vec{\gamma}}{a-b+c}$ and $\dfrac{a\vec{\alpha}+b\vec{\beta}-c\vec{\gamma}}{a+b-c}$

3. p.v. of circumcentre of the $\Delta = \dfrac{\vec{\alpha}\sin 2A+\vec{\beta}\sin 2B+\vec{\gamma}\sin 2C}{\sum \sin 2A}$

(Concurrency of perpendicular bisectors of sides)

4. p.v. of orthocenter of the $\Delta = \dfrac{\vec{a}\tan A+\vec{b}\tan B+\vec{c}\tan C}{\sum \tan A}$

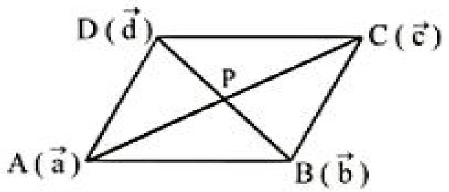
(Concurrency of altitudes)

Illustration :

ABCD is a parallelogram whose diagonals meet at P. If O is a fixed point, then $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}$ *equals-*

(A) $\overrightarrow{OP}$ (B) $2\overrightarrow{OP}$ (C) $3\overrightarrow{OP}$ (D) $4\overrightarrow{OP}$

Sol. *Since, P bisects both the diagonal AC and BD, so*



$\therefore\ \overrightarrow{OA}+\overrightarrow{OC}=2\overrightarrow{OP}$ and $\overrightarrow{OB}+\overrightarrow{OD}=2\overrightarrow{OP} \Rightarrow \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$ *Ans.[D]*

Illustration :

If $\vec{a}$, $\vec{b}$ are represented by the sides AB and BC of a regular hexagon ABCDEF, then vector represented by FA will be-

(A) $\vec{a}+\vec{b}$ (B) $\vec{b}-\vec{a}$ (C) $\vec{a}-\vec{b}$ (D) $2\vec{b}-\vec{a}$

Sol. $\because\ \overrightarrow{AC}=\vec{a}+\vec{b}$

$\overrightarrow{AD}=2\,\overrightarrow{BC}=2\vec{b}$ ($\because$ *By property of hexagon AD = 2BC*)

$\therefore\quad \overrightarrow{DC}=\overrightarrow{DA}+\overrightarrow{AC}$

$=-2\vec{b}+(\vec{a}+\vec{b})=\vec{a}-\vec{b}$

But $\overrightarrow{FA}=\overrightarrow{DC} \Rightarrow \overrightarrow{FA}=\vec{a}-\vec{b}$ *Ans.[C]*

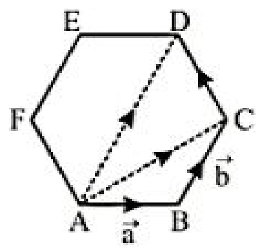


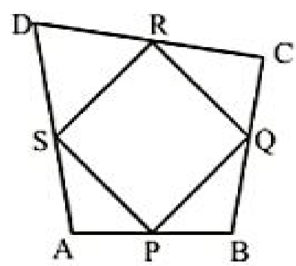
Illustration :

If the mid-points of the consecutive sides of a quadrilateral are joined, prove that the resulting quadrilateral is a parallelogram.

Sol. *Let ABCD be the given quadrilateral and P, Q, R, S be the mid-points of sides AB, BC, CD and AD respectively. Let O be origin of reference and let $\vec{a},\vec{b},\vec{c},\vec{d}$ be the position vectors of A, B, C and D respectively.*

Then, $\overrightarrow{OP}=\frac{1}{2}(\vec{a}+\vec{b})$, $\overrightarrow{OQ}=\frac{1}{2}(\vec{b}+\vec{c})$

$\overrightarrow{OR}=\frac{1}{2}(\vec{c}+\vec{d})$, $\overrightarrow{OS}=\frac{1}{2}(\vec{d}+\vec{a})$



Now, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\frac{1}{2}(\vec{b}+\vec{c}-\vec{a}-\vec{b})=\frac{1}{2}(\vec{c}-\vec{a})$

and $\overrightarrow{SR}=\overrightarrow{OR}-\overrightarrow{OS}=\frac{1}{2}(\vec{c}+\vec{d}-\vec{d}-\vec{a})=\frac{1}{2}(\vec{c}-\vec{a})$

$\therefore\quad \overrightarrow{PQ}=\overrightarrow{SR}$

Thus, PQ = SR and PQ || SR i.e., two opposite sides of PQRS are equal and parallel. Hence PQRS is a parallelogram.

Practice Problem

Q.1 If G is the centroid of ΔABC, prove that $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=0$. Further if G_1 be the centroid of another ΔPQR, show that $\overrightarrow{AP}+\overrightarrow{BQ}+\overrightarrow{CR}=3\overrightarrow{GG_1}$.

Q.2 Show that the points $A\left(2\hat{i}-\hat{j}+\hat{k}\right), B\left(\hat{i}-3\hat{j}-5\hat{k}\right)$ and $C\left(3\hat{i}-4\hat{j}-4\hat{k}\right)$ are the vertices of a right-angled triangle.

Q.3 If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and

$$\vec{A}=(x+4y)\vec{a}+(2x+y+1)\vec{b} \text{ and } \vec{B}=(y-2x+2)\vec{a}+(2x-3y-1)\vec{b}$$

find x and y such that $3\vec{A}=2\vec{B}$.

Answer key

Q.3 $x=2, y=-1$

RELATION BETWEEN TWO PARALLEL VECTORS :

1. If $\vec{a}$ and $\vec{b}$ be two parallel vectors, then there exists a non-zero scalar k such that $\vec{a}=k\vec{b}$ i.e. there exist two non- zero scalar quantities x and y so that $x\vec{a}+y\vec{b}=0$

2. If a and b be two non-zero non-parallel vectors then $x\vec{a}+y\vec{b}=0 \Rightarrow x=0$ and $y=0$

3. If $x\vec{a}+y\vec{b}=0 \Rightarrow \begin{cases} \vec{a}=0, \vec{b}=0 \\ \text{or} \\ x=0, y=0 \\ \text{or} \\ \vec{a} \parallel \vec{b} \end{cases}$

4. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then from the property of parallel vector, we have $\vec{a} \parallel \vec{b}$

$$\Rightarrow \quad \frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$$

Illustration :

The value of λ when $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{b}=8\hat{i}+\lambda\hat{j}+4\hat{k}$ are parallel is -

(A) 4 *(B) – 6* *(C) – 12* *(D) 1*

Sol. *Since* $\vec{a} \parallel \vec{b} \Rightarrow \frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$

$$\therefore \frac{2}{8}=-\frac{3}{\lambda}=\frac{1}{4} \Rightarrow \lambda=-12$$

Ans. [C]

VECTOR EQUATION OF A STRAIGHT LINE :

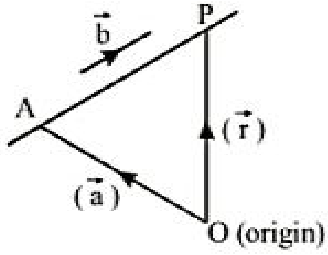
Vector equation of a straight line passing through a given point $A(\vec{a})$ and parallel to a given vector $\vec{b}$:

Let O be the origin. Let the line pass through a given point A whose position vector is $\vec{a}$, then $\overrightarrow{OA} = \vec{a}$

Let the given line be parallel to vector $\vec{b}$

Let $\vec{r}$ be the position vector any point P on the line, then

$$\overrightarrow{OP} = \vec{r}$$



Since $\overrightarrow{AP}$ is parallel to $\vec{b}$ $\quad\therefore\quad$ $\overrightarrow{AP} = t\vec{b}$, where t is a scalar.

Now $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ $\quad\therefore\quad$ $\boxed{\vec{r} = \vec{a} + t\vec{b}}$ ...(i)

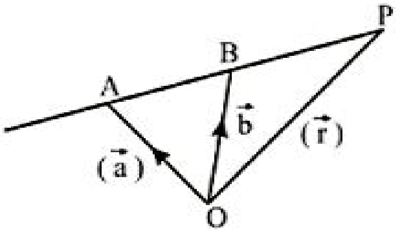
Since for different values of t, we get different positions of point P on the line, hence (i) is the vector equation of the required straight line.

Vector equation of straight line passing through two given point $A(\vec{a})$ and $B(\vec{b})$:

Let O be the origin. Let the line pass through two given point A and B whose position vectors referred to O be $\vec{a}$ and $\vec{b}$ respectively, then

$$\overrightarrow{OA} = \vec{a} \text{ and } \overrightarrow{OB} = \vec{b}$$

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - a$$



Clearly, the required line passes through $A(\vec{a})$

and is parallel to the vector $(\vec{b} - \vec{a})$.

Hence the vector equation of the required line is,

$$\vec{r} = \vec{a} + t(\vec{b} - \vec{a}) \quad \text{or} \quad \boxed{\vec{r} = (1-t)\vec{a} + t\vec{b}}$$

Important Note :

(i) Two lines in a plane are either intersecting or parallel conversely two intersecting or parallel lines must be in the same plane

(ii) However in space we can have two neither parallel nor intersecting lines. Such non coplanar lines are known as skew lines. If two lines are parallel and have a common point then they are coincident.

Vector equation of bisectors of angle between two straight lines :

Let OA and OB be the given straight line parallel to unit vectors $\hat{a}$ and $\hat{b}$ respectively. Take the point O as origin, and let Q be a point on the internal bisector of the angle AOB. From Q draw QR parallel to OA cutting OB at R.

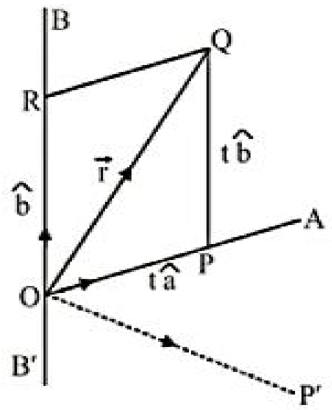
Now $\therefore$ $\angle AOQ = \angle BOQ$ (as OQ is the bisector)

and $\angle BOQ = \angle OQP$ (alternative angles)

$\therefore$ $\angle AOQ = \angle OQP$ $\therefore$ $OP = QP = t$ (say), where t is a scalar.

$\therefore$ $\overrightarrow{OP} = t\,\hat{a}$ and $\overrightarrow{PQ} = t\,\hat{b}$

$\overrightarrow{OQ} = \vec{r}$



Let $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

or $\vec{r} = t\hat{a} + t\vec{b}$

$\vec{r} = t\left(\hat{a} + \hat{b}\right)$

or, $\vec{r} = t\left(\dfrac{\vec{a}}{a} + \dfrac{\vec{b}}{a}\right)$

where $|\vec{a}| = a, |\vec{b}| = b$

This is the equation of internal bisector of $\angle AOB$.

Equation of external Bisector :

If OP' be the external bisector of $\angle OAB$, then OP' may be regarded as the internal bisector of the angle between the lines which are parallel to $\hat{a}$ and $-\hat{b}$. Hence its equaiton is

$$\vec{r} = t(\hat{a} - \hat{b}) \quad \text{or,} \quad \vec{r} = t\left(\frac{\hat{a}}{a} - \frac{\hat{b}}{a}\right)$$

Corollary : If the the lines intersect at E having position vector $\vec{\alpha}$, then the above equations becomes

$\vec{r} = \vec{\alpha} + t(\hat{a} + \hat{b})$ and $\vec{r} = \vec{\alpha} + t(\hat{a} - \hat{b})$ respectively.

Illustration :

Find the vector equation of the line through the point $2\vec{i} + \vec{j} - 3\vec{k}$ and parallel to the vector $\vec{i} + 2\vec{j} + \vec{k}$.

Sol. *Let the given point be $A(\vec{a})$ and given vector be $\vec{b}$ and O be the origin.*

Then, $\vec{a} = \overrightarrow{OA} = 2\hat{i} + \hat{j} - 3\hat{k}$ *and* $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$

Now vector equaiton of the line through A and parallel to $\vec{b}$ is

$\vec{r} = \vec{a} + t\vec{b}$, *where t is a scalar.*

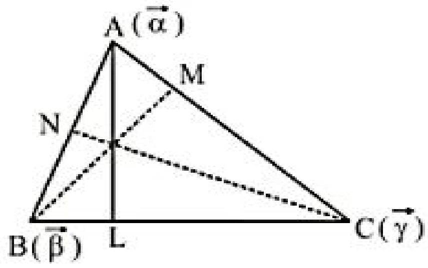
or, $\vec{r} = 2\hat{i} + \hat{j} - 3\hat{k} + t(\hat{i} + 2\hat{j} + \hat{k})$

Illustration :

Prove, by vector method that the internal bisectors of the angles of a triangle are concurrent.

Sol. *Let* $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ *be the position vectors of vertices A, B, C respectively of* ΔABC.

Let AL be the bisectors of $\angle ABC$



Then $\dfrac{BL}{LC} = \dfrac{BA}{AC} = \dfrac{c}{b}$

Thus L divides BC internally in the ratio $c : b$

$\therefore$ $\quad P.V.\text{ of } L = \dfrac{b\vec{\beta} + c\vec{\gamma}}{b+c}$

Now P.V. of the point which divide AL internally in the ratio $b + c : a$ *will be*

$$\frac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$$

[Here we have divided AL in the ratio $b + c : a$ *because* $b + c$ *occurs in the denominator of P.V. of L and there is* $b\vec{\beta}$ *and* $c\vec{\gamma}$ *therefore, there should also be* $a\vec{\alpha}$*.]*

Similarly, we can show that the position vectors of the points which divide bisectors BM of $\angle ABC$ *and CN of* $\angle ACB$ *in the ratio* $c + a : b$ *and* $a + b : c$ *respectively will be each* $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$.

Thus the point having position vector $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$ *lies on the three internal bisectors of the angles of the triangle ABC and hence internal bisectors of angles of a triangle are concurrent and the position vector of incentre of* ΔABC *will be* $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$.

COLLINEARITY OF THREE POINTS :

1. If $\vec{a}, \vec{b}, \vec{c}$ be position vectors of three points A,B and C respectively and x, y, z be three scalars so that all are not zero, then the necessary and sufficient conditions for three points to be collinear is that

$$x\vec{a} + y\vec{b} + z\vec{c} = 0 \quad \text{where} \quad x + y + z = 0$$

2. Three points A, B and C are collinear, if $\overrightarrow{AB} = \lambda\,\overrightarrow{BC}$.

Illustration :

If $2\vec{a} - 3\vec{b}$, $\vec{b}$ *and* $\vec{a} - \vec{b}$ *are position vectors of three points A,B and C then they are -*

(A) Collinear *(B) Non- collinear* *(C) Can't say anything* *(D) None of these*

Sol. $\because 1\,(2\vec{a} - 3\vec{b}) + 1\,(\vec{b}) - 2\,(\vec{a} - \vec{b}) = 0$ *and* $1 + 1 - 2 = 0$

so the given vectors are collinear. *Ans.[A]*

Illustration :

If $A \equiv (2\hat{i} + 3\hat{j})$, $B \equiv (p\hat{i} + 9\hat{j})$ *and* $C \equiv (\hat{i} - \hat{j})$ *are collinear, then the value of p is-*

(A) 1/2 *(B) 3/2* *(C) 7/2* *(D) 5/2*

Sol. $\overrightarrow{AB} = (p-2)\,\hat{i} + 6\,\hat{j}$, $\overrightarrow{AC} = -\hat{i} - 4\hat{j}$

Now A,B,C are collinear $\Leftrightarrow \overrightarrow{AB} \,||\, \overrightarrow{AC} \Leftrightarrow \dfrac{p-2}{-1} = \dfrac{6}{-4} \Leftrightarrow p = 7/2$ *Ans. [C]*

PRODUCT OF VECTORS :

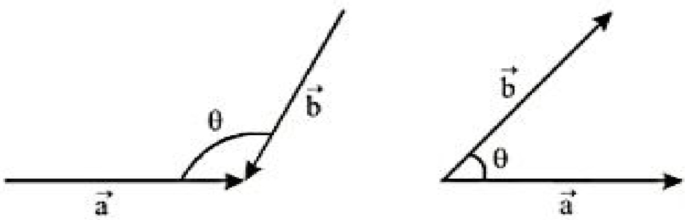
Product of two vectors is done by two methods when the product of two vectors results in a scalar quantity then it is called scalar product. It is also called as dot product because this product is represented by putting a dot.

When the product of two vectors results in a vector quantity then this product is called Vector Product. This product is represented by (x) sign so that it is also called as cross product.

Scalar or dot product of two vectors :

Definition : If $\vec{a}$ and $\vec{b}$ are two vectors and θ be the angle between their tails or heads, then their scalar product (or dot product) is defined as the number $|\vec{a}|\,|\vec{b}|\cos\theta$ where $|\vec{a}|$ and $|\vec{b}|$ are modulii of $\vec{a}$ and $\vec{b}$ respectively and $0 \le \theta \le \pi$. It is denoted by $\vec{a}.\vec{b}$. Thus

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$



Note :

(i) $\vec{a}.\vec{b} \in R$

(ii) $\vec{a}.\vec{b} \leq |\vec{a}||\vec{b}|$

(iii) $\vec{a}.\vec{b} > 0 \Rightarrow$ angle between a and $\vec{b}$ say $\theta \in \left[0, \frac{\pi}{2}\right)$.

$\vec{a}.\vec{b} < 0 \Rightarrow$ angle between $\vec{a}$ and $\vec{b}$ say $\theta \in \left(\frac{\pi}{2}, \pi\right]$.

$\vec{a}.\vec{b} = 0 \Rightarrow$ angle between $\vec{a}$ and $\vec{b}$ say $\theta = \frac{\pi}{2}$ or atleast one of $\vec{a}$ and $\vec{b}$ is zero vector.

(iv) The dot product of a zero and non- zero vector is a scalar zero i.e. $\vec{0}\cdot\vec{a} = 0$.

(v) If θ be angle between any two non zero vector $\vec{a}$ & $\vec{b}$ then $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$.

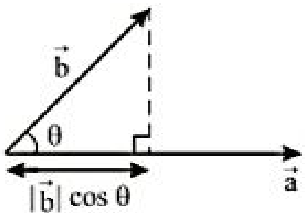
Geometrical Interpretation :

Geometrically, the scalar product of two vectors is equal to the product of the magnitude of one and the projection of second in the direction of first vector i.e. $\vec{a}.\vec{b} = |\vec{a}|\ \ (|\vec{b}|\cos\theta)$

$= |\vec{a}|$ (projection of $\vec{b}$ in the direction of $\vec{a}$)

Similarly $\vec{a}.\vec{b} = |\vec{b}|\ \ (|\vec{a}|\cos\theta)$

$= |b|$ (projection of $\vec{a}$ in the direction of $\vec{b}$)



Here projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$

Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

Illustration :

If angle between $\vec{a}$ and $\vec{b}$ is 120° and their magnitudes are respectively 2 and $\sqrt{3}$, then a.b equals-

(A) 3 *(B) $-\sqrt{3}$* *(C) $\sqrt{3}$* *(D) -3*

Sol. *We know that $\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\cos\theta$*

$= 2\sqrt{3}\ \cos 120°$

$= 2\sqrt{3}\ (-1/2) = -\sqrt{3}$ *Ans.[B]*

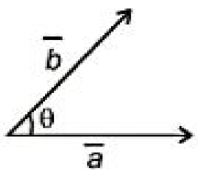
Illustration :

The projection of vector $\hat{i}+\hat{j}+\hat{k}$ *on the vector* $\hat{i}-\hat{j}+\hat{k}$ *is-*

(A) $\sqrt{3}$ *(B)* $1/\sqrt{3}$ *(C)* $2/\sqrt{3}$ *(D)* $2\sqrt{3}$

Sol. *Projection* $=\dfrac{(\hat{i}+\hat{j}+\hat{k}).(\hat{i}-\hat{j}+\hat{k})}{|\hat{i}-\hat{j}+\hat{k}|}=\dfrac{1-1+1}{\sqrt{1+1+1}}=\dfrac{1}{\sqrt{3}}$ *Ans.[B]*

Scalar product in particular cases :



1. $\vec{a}.\vec{b}=\vec{b}.\vec{a}$

2. $\vec{a}.(\vec{b}+\vec{c})=\vec{a}.\vec{b}+\vec{a}.\vec{c}$

3. $(m\vec{a}).\vec{b}=m(\vec{a}.\vec{b})=\vec{a}.(m\vec{b})$

4. If $\theta=0 \Rightarrow \vec{a}.\vec{b}=|\vec{a}||\vec{b}|$ (like vectors)

5. If $\theta=\pi \Rightarrow \vec{a}.\vec{b}=-|\vec{a}||\vec{b}|$ (unlike vectors)

6. If $\hat{a}$ and $\hat{b}$ are unit vectors then $\hat{a}.\hat{b}=\cos\theta$ (where θ is angle between them).

7. $\vec{a}.\vec{a}=|\vec{a}|^2 \Rightarrow |\vec{a}|=\sqrt{\vec{a}.\vec{a}}$

8. If $\vec{a}\perp\vec{b} \Rightarrow \vec{a}.\vec{b}=0$ but $\vec{a}.\vec{b}=0 \Rightarrow \vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ or $\vec{a}\perp\vec{b}$.

9. If $\hat{i},\hat{j}$ and $\hat{k}$ are unit vectors along the rectangular coordinate are OX, OY and OZ then

   $\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$, $\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$

10. $\vec{a}.\vec{b}=\vec{a}.\vec{c} \nRightarrow \vec{b}=\vec{c}$

    Infact $\vec{a}.\vec{b}=\vec{a}.\vec{c} \Rightarrow \vec{a}.(\vec{b}-\vec{c})=0 \Rightarrow \vec{a}=0$ or $\vec{b}=\vec{c}$ or $\vec{a}\perp(\vec{b}-\vec{c})$

11. $(\vec{a}.\vec{b}).\vec{c}$ is meaningless

Note : (a) $(\vec{a}.\vec{b}).\vec{b}$ is not defined

(b) $(\vec{a}+\vec{b})^2=|\vec{a}|^2+2\,\vec{a}.\vec{b}+|\vec{b}|^2$

(c) $(\vec{a}-\vec{b})^2=|\vec{a}|^2-2\vec{a}.\vec{b}+|\vec{b}|^2$

(d) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=|\vec{a}|^2-|\vec{b}|^2$

(e) $|\vec{a}+\vec{b}|=|\vec{a}|+|\vec{b}| \Rightarrow \vec{a}\parallel\vec{b}$

(f) $|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2 \Rightarrow \vec{a}\perp\vec{b}$

(g) $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}| \Rightarrow \vec{a}\perp\vec{b}$

Illustration :

Determine the values of c such that for all x (real) the vectors $cx\hat{i}-6\hat{j}+3\hat{k}$ *and* $x\hat{i}+2\hat{j}+2cx\hat{k}$ *make an obtuse angle with each other.*

Sol. *If* θ *is the angle between the given vectors then* $\cos\theta=\dfrac{cx^2-12+6cx}{\sqrt{c^2x^2+45}\sqrt{x^2+4c^2x^2+4}}$

If θ *is obtuse then* $\cos\theta<0 \Rightarrow \quad cx^2+6cx-12<0\ \forall\ x\in R$
Which is possible if $c<0$ *and* $36c^2+48c<0 \Rightarrow c<0$ *and* $12c(3c+4)<0$

$\Rightarrow c<0$ *and* $-\frac{4}{3}<c<0$ *(but for* $c=0,\ cx^2+6cx-12<0\ \ \forall\ x$*)*

Hence $-\frac{4}{3}<c\le 0$.

Illustration :

If $\vec{a},\vec{b},\vec{c}$ *are unit vectors such that* $\vec{a}$ *is perpendicular to the plane of* $\vec{b}$ *and* $\vec{c}$ *then find* $|\vec{a}+\vec{b}+\vec{c}|$ *when the angle between* $\vec{b}$ *and* $\vec{c}$ *is* $\pi/3$.

Sol. *We have* $|\vec{a}|=|\vec{b}|=|\vec{c}|=1, \vec{a}.\vec{b}=\vec{a}.\vec{c}=0$ *and* $\quad \vec{b}.\vec{c}=\cos\frac{\pi}{3}=\frac{1}{2}$

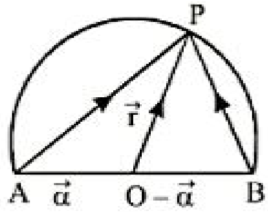
Now $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{b}.\vec{c}\ (\because \vec{a}.\vec{c}=\vec{a}.\vec{b}=0)$

$$=1+1+1+2.\frac{1}{2}=4 \quad\Rightarrow\quad |\vec{a}+\vec{b}+\vec{c}|=2$$

Illustration :

Prove that the angle in a semi circle is a right angle.

Sol. *Let O be the centre of the semi circle with AOB as its diameter. Let P be a point on the semi-circle, so that* $\angle APB$ *is an angle in the semi circle. Join OP. Let O be taken as origin. Let the position vectors of A, B and P be* $\vec{\alpha}, -\vec{\alpha}$ *and* $\vec{r}$ *respectively.*



Clearly, $\qquad OA=OB=OP$

Now $\quad \overrightarrow{AP}=\left(\vec{r}-\vec{\alpha}\right)$ *and* $\overrightarrow{BP}=\left(\vec{r}+\vec{\alpha}\right)$

$\therefore \quad \overrightarrow{AP}\cdot\overrightarrow{BP}=\left(\vec{r}-\vec{\alpha}\right)\cdot\left(\vec{r}+\vec{\alpha}\right)=r^2-\alpha^2=OP^2-OA^2=0$

[$\because$ *OP = OA]*

$\therefore \quad AP\perp BP$ *i.e.* $\angle APB=90^o$.

Scalar product in terms of components :

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{a} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then $\quad \vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

In particular

$$\vec{a}.\vec{a} = |\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2$$

For any vector $\vec{a}$,

$$\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$$

Illustration :

If $\vec{a} = 3\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 5\hat{k}$ then find $\vec{a}.\vec{b}$.

Sol. $\vec{a}.\vec{b} = (3)(1) + (2)(-2) + (1)(5) = 3 - 4 + 5 = 4.$

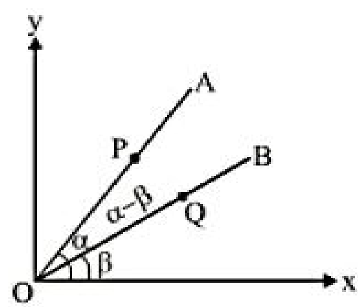
Illustration :

Prove by vector method the following formula of plane trigonometry

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Sol. *Let the unit vectors along OX and OY be $\hat{i}$ and $\hat{j}$ respectively. If OA and OB be any two lines in the same plane making angles α and β respectively, with OX then, $\angle PB = \alpha - \beta$*

Again, let $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent unit vectors along OA and OB respectively, so that their dot product is the consine of the angle between their directions.



Now, $\quad \overrightarrow{OP}\cdot\overrightarrow{OQ} = 1 \cdot 1 \cos(\alpha - \beta) = \cos(\alpha - \beta) \qquad ...(1)$

Since $\overrightarrow{OP}$ makes an angle α with x-axis.

$\therefore \quad \overrightarrow{OP} = \cos\alpha\hat{i} + \sin\alpha\hat{j}$

Similarly, $\overrightarrow{OQ} = \cos\beta\hat{i} + \sin\beta\hat{j}$

$\therefore \quad \overrightarrow{OP}\cdot\overrightarrow{OQ} = [\cos\alpha\hat{i} + \sin\alpha\hat{j}] \cdot [\cos\beta\hat{i} + \sin\beta\hat{j}]$

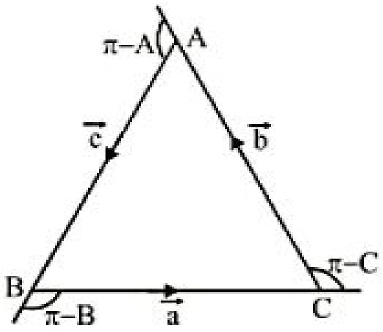
$\qquad = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta \qquad ...(2)$

From (1) and (2), we get $\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$

Illustration :

In any ΔABC, prove that $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$ *with the help of vectors.*

Sol. *In ΔABC,* $\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}=\vec{0}$



$\Rightarrow \quad \overrightarrow{AB} = -(\overrightarrow{BC}+\overrightarrow{CA})$

$\Rightarrow \quad \overrightarrow{AB}\cdot\overrightarrow{AB} = (\overrightarrow{BC}+\overrightarrow{CA})\cdot(\overrightarrow{BC}+\overrightarrow{CA})$

$\Rightarrow \quad \vec{c}\cdot\vec{c} = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$\Rightarrow \quad c^2 = a^2+b^2+2\,\vec{a}\cdot\vec{b}$

$\Rightarrow \quad c^2 = a^2+b^2+2ab\cos(\pi-C)$

$\Rightarrow \quad c^2 = a^2+b^2-2ab\cos C$

Illustration :

Prove by vector method that $(a_1b_1+a_2b_2+a_3b_3)^2 \le (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

Sol. *Let* $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ *and* $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

Then $\vec{a}.\vec{b} = a_1b_1+a_2b_2+a_3b_3$ ... (1)

Also $|\vec{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$ *and* $|\vec{b}| = \sqrt{b_1^2+b_2^2+b_3^2}$ ... (2)

If θ be the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \quad \because \cos^2\theta \le 1 \;\therefore \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)^2 \le 1$$

or $(\vec{a}\cdot\vec{b})^2 \le |\vec{a}|^2|\vec{b}|^2$

$\therefore \quad (a_1b_1+a_2b_2+a_3b_3)^2 \le (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

Angle between two vectors in terms of components :

If $\vec{a}$ and $\vec{b}$ be two vectors such that $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and θ be the angle between them, then

$$\cos\theta = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

Note : If $\vec{a}$ and $\vec{b}$ are perpendicular to each other then $a_1b_1+a_2b_2+a_3b_3=0$

Illustration :

Find the angle between the vectors $4\hat{i} + \hat{j} + 3\hat{k}$ *and* $2\hat{i} + 2\hat{j} - \hat{k}$.

Sol. *Let the required angle is* θ.

$$\therefore\ \theta = \cos^{-1}\left(\frac{4.2 + 1.2 + 3(-1)}{\sqrt{16+1+9}\sqrt{4+4+1}}\right) = \cos^{-1}\left(\frac{7}{3\sqrt{26}}\right)$$

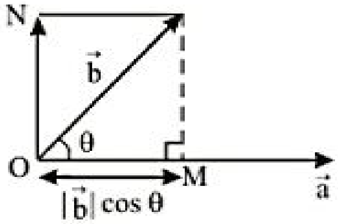
Components of $\vec{b}$ along & perpendicular to $\vec{a}$:

1. Component of $\vec{b}$ along $\vec{a}$ = $\overrightarrow{OM}$

$$= OM\ \hat{a} = (b\cos\theta)\hat{a}$$

$$= \frac{(ab\cos\theta)}{a}\hat{a} = \frac{(\vec{a}.\vec{b})}{a^2}\hat{a}$$

$$= \frac{(\vec{a}.\vec{b})}{|a|^2}.a$$



2. Component perpendicular to $\vec{a}$ = $\overrightarrow{ON}$

$$\because\ \vec{b} = \overrightarrow{ON} + \overrightarrow{OM}$$

$$\therefore\ \overrightarrow{ON} = \vec{b} - \overrightarrow{OM}$$

$$\overrightarrow{ON} = \vec{b} - \frac{(\vec{a}.\vec{b})}{a^2}\vec{a}.$$

Illustration :

Find the vector components of a vector $2\hat{i} + 3\hat{j} + 6\hat{k}$ *along and perpendicular to non-zero vector* $2\hat{i} + \hat{j} + 2\hat{k}$.

Sol. *Let* $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ *and* $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$

Now vector component of $\vec{a}$ *along* $\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\vec{b} = \frac{4+3+12}{9}(2\hat{i} + \hat{j} + 2\hat{k}) = \frac{19}{9}(2\hat{i} + \hat{j} + 2\hat{k})$$

and vector component of $\vec{a}$ perpendicular to $\vec{b}$

$$= \vec{a} - \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = (2\hat{i} + 3\hat{j} + 6\hat{k}) - \frac{19}{9}(2\hat{i} + \hat{j} + 2\hat{k}) = \frac{1}{9}(-20\hat{i} + 8\hat{j} + 16\hat{k})$$

Illustration :

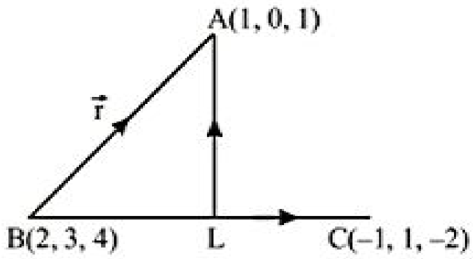
Find the perpendicular distance of the point A(1, 0, 1) to the line through the points B(2, 3, 4) and C(–1, 1, –2).

Sol. $\vec{r} = \overrightarrow{BA} = -\hat{i} - 3\hat{j} - 3\hat{k}$ and $\vec{a} = \overrightarrow{BC} = -3\hat{i} - 2\hat{j} - 6\hat{k}$

Now $\overline{BL}$ = *Projection vector of* $\vec{r}$ *on* $\vec{a}$

$$\frac{\vec{r}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} = \frac{3+6+18}{49}(-3\hat{i} - 2\hat{j} - 6\hat{k})$$

$$= \frac{-27}{49}(3\hat{i} + 2\hat{j} + 6\hat{k})$$



$$\overrightarrow{LA} = \overrightarrow{LB} + \overrightarrow{BA} = \overrightarrow{BA} - \overrightarrow{BL}$$

$$= \vec{r} - \frac{\vec{r}\cdot\vec{a}}{|\vec{a}|^2}\cdot\vec{a} = (-\hat{i} - 3\hat{j} - 3\hat{k}) + \frac{27}{49}(3\hat{i} + 2\hat{j} + 6\hat{k}) = \frac{1}{49}(32\hat{i} - 93\hat{j} + 15\hat{k})$$

$$\therefore \quad LA = \left|\overrightarrow{LA}\right| = \frac{\sqrt{9898}}{49}$$

Linear Combination of Vectors :

A vector $\vec{r}$ is said to be a linear combination of the vectors $\vec{a}, \vec{b}, \vec{c}$......
if $\exists$ scalars x,y,z, ... such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + ...$

Theorem in plane :

If $\vec{a}$ and $\vec{b}$ are two non zero non collinear vetors then any vector $\vec{r}$ coplanar with them can be expressed as a linear combination $\vec{r} = x\vec{a} + y\vec{b}$. (Explain using a sketch)

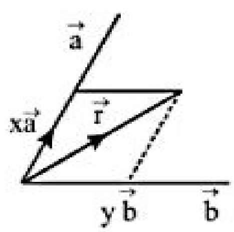
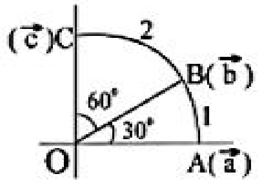


Illustration :

Arc AC of the quadrant of a circle with centre as origin and radius unity subtends a right angle at the origin. Point B divides the arc AC in the ratio 1 : 2. Express the vector $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$.



Sol. $\because\ \vec{c} = x\vec{a} + y\vec{b}$ ...(i)

Taking dot product with $\vec{a}$ in (i)

$$\vec{c}\cdot\vec{a} = \vec{a}\cdot\vec{a} + y\vec{a}\cdot\vec{b}$$

$$0 = x + y\frac{\sqrt{3}}{2} \qquad ...(ii)$$

Taking dot product with $\vec{c}$ in (i)

$$\vec{c}\cdot\vec{c} = x\vec{a}\cdot\vec{c} + y\vec{b}\cdot\vec{c}$$

$$1 = 0 + \frac{y}{2} \quad \Rightarrow \quad y = 2$$

from (ii) $x = -\sqrt{3}$ $\quad\therefore\quad$ $\vec{c} = -\sqrt{3}\,\vec{a} + 2\vec{b}$

Practice Problem

Q.1 Given that $\vec{a} = \hat{i} - \hat{j}$ and $\hat{b} = \hat{i} + 2\hat{j}$ are two vectors. Find a unit vector coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{a}$.

Q.2 A line passes through a point with p.v. $\hat{i} - 2\hat{j} - \hat{k}$ and is parallel to the vector $\hat{i} - 2\hat{j} + 2\hat{k}$. Find the distance of a point P (5, 0, – 4) from the line.

Answer key

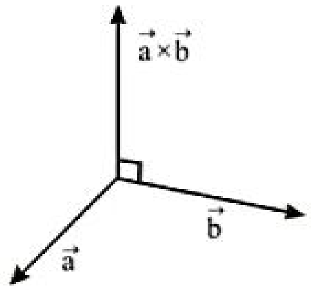
Q.1 $\pm\dfrac{\hat{i} + \hat{j}}{\sqrt{2}}$ Q.2 5

VECTOR OR CROSS PRODUCT OF TWO VECTORS :

Definition :

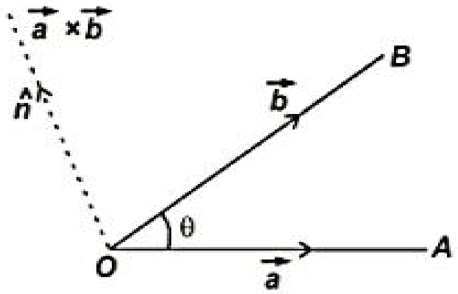
If $\vec{a}$ and $\vec{b}$ be two vectors and θ $(0 \le \theta \le \pi)$ be the angle between them, then their vector (or cross) product is defined to be a vector whose magnitude is $ab \sin\theta$ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}$ and $\vec{a} \times \vec{b}$ form a right handed system.

$\therefore \quad \vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$



Where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

Properties of Vector Product :



1. In general, $\vec{a} \times \vec{b} \ne \vec{b} \times \vec{a}$. In fact $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

2. For scalar m, $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$.

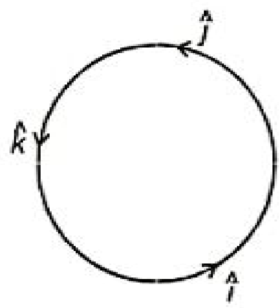
3. $\vec{a} \times (\vec{b} \pm \vec{c}) = \vec{a} \times \vec{b} \pm \vec{a} \times \vec{c}$

4. If $\vec{a} \,||\, \vec{b}$ then $\theta = 0$ or π $\Rightarrow$ $\vec{a} \times \vec{b} = \vec{0}$ (but $\vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \,||\, \vec{b}$). In particular $\vec{a} \times \vec{a} = \vec{0}$.

5. If $\vec{a} \perp \vec{b}$ then $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\,\hat{n}$ (or $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|$)

6. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ and $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$ and

   $\hat{k} \times \hat{i} = \hat{j}$ (use cyclic system)



7. Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is given by $\pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

8. If θ is angle between $\vec{a}$ and $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$

9. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

Illustration :

If $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ *and* $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$ *then* $\vec{a} \times \vec{b}$ *equals*

(A) $2\hat{i} - 2\hat{j} - \hat{k}$ (B) $\hat{i} - 10\hat{j} - 18\hat{k}$ (C) $\hat{i} + \hat{j} + \hat{k}$ (D) $6\hat{i} - 3\hat{j} + 2\hat{k}$

Sol. $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & -1 \\ 6 & -3 & 2 \end{vmatrix} = \hat{i}\ (4-3) - \hat{j}(4+6) + \hat{k}(-6-12) = \hat{i} - 10\hat{j} - 18\hat{k}$ *Ans.[B]*

Illustration :

If angle between $\hat{i} - 2\hat{j} + 3\hat{k}$ *and* $2\hat{i} + \hat{j} + \hat{k}$ *is* θ *then* $\sin\theta$ *equals-*

(A) $5/\sqrt{7}$ (B) $5/21$ (C) $5/2\sqrt{7}$ (D) $3/\sqrt{14}$

Sol. *We know that* $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$

Now $\vec{a} \times \vec{b} = -5i + 5\hat{j} + 5\hat{k}$

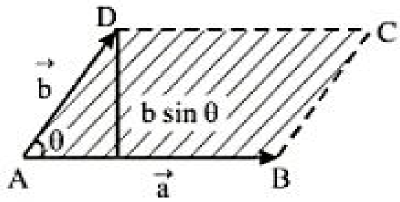
$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{(5)^2 + (5)^2 + (5)^2} = \sqrt{75} = 5\sqrt{3}$

$|\vec{a}| = \sqrt{1+4+9},\ |\vec{b}| = \sqrt{4+1+1}$

$\therefore \quad \sin\theta = \dfrac{5\sqrt{3}}{\sqrt{1+4+9}\sqrt{4+1+1}} = \dfrac{5\sqrt{3}}{\sqrt{14}\sqrt{6}} = \dfrac{5}{\sqrt{28}} = \dfrac{5}{2\sqrt{7}}$ *Ans.[C]*

Geometrical interpretation of vector product :

The vector product of the vectors $\vec{a}$ and $\vec{b}$ represents a vector whose modulus is equal to the area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



Area of parallelogram = base × height = ab sin θ = $|\vec{a} \times \vec{b}|$

Area of quadraliteral if its diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $= \dfrac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

Illustration :

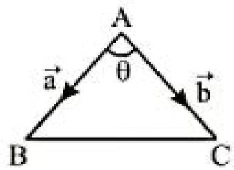
Find the area of a parallelogram whose two adjacent sides are represented by $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ *and* $b = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

Sol. *Area of parallelogram* $= |\vec{a} \times \vec{b}|$

$$\text{Now } \vec{a} \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = 8\hat{i} - 8\hat{j} - 8\hat{k}$$

$$\therefore \text{ Area} = |8\hat{i} - 8\hat{j} - 8\hat{k}| = 8\sqrt{3} \text{ units}$$

Area of a triangle :



1. Area of triangle ABC $= \frac{1}{2} ab \sin\theta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

2. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices of a ΔABC then its

Area $= \frac{1}{2}|(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a})|$ (think !)

Illustration :

Find the area of ΔABC if position vectors of its vertices A, B,C are $\hat{i} + \hat{j}$, $\hat{j} + \hat{k}$ *and* $\hat{k} + \hat{i}$ *respectively.*

Sol. $\overrightarrow{AB} = (\hat{j} + \hat{k}) - (\hat{i} + \hat{j}) = \hat{k} - \hat{i}$

$\overrightarrow{AC} = (\hat{k} + \hat{i}) - (\hat{i} + \hat{j}) = \hat{k} - \hat{j}$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$\therefore \text{ Area of } \Delta\ ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{1+1+1} = \sqrt{3}/2.$$

INTERPRETATION OF VECTOR PRODUCT AS VECTOR AREA :

1. Vector area of plane figures :

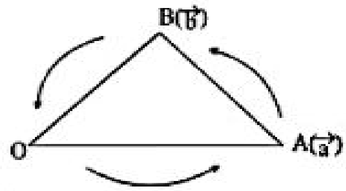
With every closed bound surface which has been described in a certain specific manner and whose boundaries do not cross, it is possible to associate a directed line segment $\vec{c}$ such that

(i) $|\vec{c}|$ = no. of units of area enclosed by the plane figure

(ii) The support of $\vec{c}$ is perpendicular to the area and

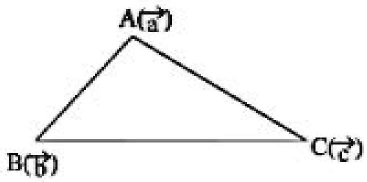
(iii) The sense of description of the boundaries and the direction of $\vec{c}$ is in accordance with the R.H. screw rule.

2. Vector area of a plane Δ (Triangle) :



Vector area of ΔOAB is $\vec{\Delta} = \frac{1}{2}(\vec{a} \times \vec{b})$

If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors then the vector area of ΔABC is



$$\vec{\Delta} = \frac{1}{2}\left[(\vec{c} - \vec{b}) \times (\vec{a} - \vec{b})\right]$$

$$\vec{\Delta} = \frac{1}{2}\left((\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})\right)$$

Note :

(i) If 3 points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear then $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$

(ii) Unit vector perpendicular to the plane of the ΔABC when $\vec{a}$, $\vec{b}$, $\vec{c}$ are the p.v. of its angular point is

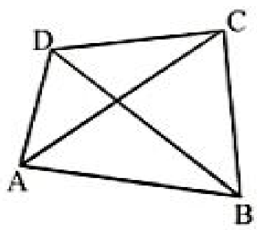
$\hat{n} = \pm \dfrac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{2\Delta}$, where $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the angular points of the triangle ABC.

(iii) Vector Area of a quadrilateral ABCD = Vector area of ΔABC + vector area of ΔACD

$$= \frac{1}{2}\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) + \frac{1}{2}\left(\overrightarrow{AC} \times \overrightarrow{AD}\right)$$

$$= \frac{1}{2}\left(\overrightarrow{AB} \times \overrightarrow{AC} - \overrightarrow{AD} \times \overrightarrow{AC}\right) = \frac{1}{2}\left(\overrightarrow{AB} - \overrightarrow{AD}\right) \times \overrightarrow{AC}$$

$$= \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{DA}\right) \times \overrightarrow{AC} \quad = \frac{1}{2}\overrightarrow{DB} \times \overrightarrow{AC}$$



$$\therefore \text{ Area of } \square ABCD = \frac{1}{2}\left|\overrightarrow{DB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$$

Illustration :

Using vector method, show that the points $A(2, -1, 3)$, $B(4, 3, 1)$ and $C(3, 1, 2)$ are collinear.

Sol. Let O be the origin

Given $A \equiv (2, -1, 3)$ $\therefore$ $\overrightarrow{OA} = 2\hat{i} - \hat{j} + 3\hat{k}$

$B \equiv (4, 3, 1)$ $\therefore$ $\overrightarrow{OB} = 4\hat{i} + 3\hat{j} + \hat{k}$

$C \equiv (3, 1, 2)$ $\therefore$ $\overrightarrow{OC} = 3\hat{i} + \hat{j} + 2\hat{k}$

Now, $\overrightarrow{AB} = (\text{P.V. of } B) - (\text{P.V. of } A) = \left(4\hat{i} + 3\hat{j} + \hat{k}\right) - \left(2\hat{i} - \hat{j} + 3\hat{k}\right) = \left(2\hat{i} + 4\hat{j} - 2\hat{k}\right)$

And $\overrightarrow{AC} = (\text{P.V. of } C) - (\text{P.V. of } A) = \left(3\hat{i} + \hat{j} + 2\hat{k}\right) - \left(2\hat{i} - \hat{j} + 3\hat{k}\right) = \left(\hat{i} + 2\hat{j} - \hat{k}\right)$

$$\therefore \quad \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -2 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{vmatrix} = \bar{0} \qquad [\because R_2 \text{ and } R_3 \text{ are identical}]$$

Thus, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors, having a common point A.
Hence, the points A, B, C are collinear.

Illustration :

AD, BE and CF are the medians of a triangle ABC intersecting in G. Show that

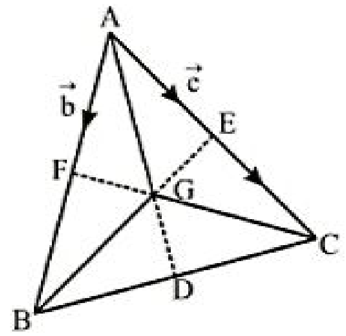
$$\Delta AGB = \Delta BGC = \Delta CGA = \frac{1}{3}\Delta ABC.$$

Sol. Let $\vec{b}, \vec{c}$ be the position vectors of B and C with respect to A as the origin of reference.

Therefore, the position vectors of D, E, F are $\frac{1}{2}(\vec{b} + \vec{c})$, $\frac{1}{2}\vec{c}$, $\frac{1}{2}\vec{b}$ respectively.

Also the position vector of the point G, the centroid, is

$$\frac{1}{3}(0 + \vec{b} + \vec{c}) = \frac{1}{3}(\vec{b} + \vec{c})$$



Therefore, area of $\Delta\, AGB = \frac{1}{2}\left(\overrightarrow{AB} \times \overrightarrow{AG}\right)$

$$= \frac{1}{2}\left|\vec{b} \times \frac{1}{3}\left(\vec{b} + \vec{c}\right)\right| = \frac{1}{6}\left|\vec{b} \times \vec{b} + \vec{b} \times \vec{c}\right|$$

$$= \frac{1}{6}\left|\vec{b} \times \vec{c}\right| = \frac{1}{3} \text{ area of } \Delta\, ABC$$

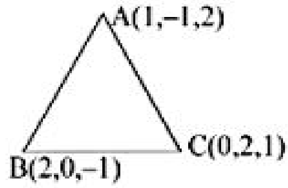
Similarly, we can show that area of $\Delta\, BGC = \frac{1}{3}$ area of $\Delta\, ABC$

and area of $\Delta\, CGA = \frac{1}{3}$ area of $\Delta\, ABC$

Practice Problem

Q.1 For a non zero vector $\vec{a}$, if $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$. Prove that $\vec{b}=\vec{c}$.

Q.2 Find



(i) A vector of magnitude $\sqrt{6}$ perpendicular to the plane ABC

(ii) Area of triangle ABC

(iii) Length of the altitude from A ($AB = AC = \sqrt{11}$)

Q.3 Let $\vec{OA}=\vec{a}$, $\vec{OB}=10\vec{a}+2\vec{b}$ and $\vec{OC}=\vec{b}$ where O, A & C are non–collinear points. Let 'p' denote the area of the quadrilateral OABC, and let 'q' denote the area of the parallelogram with OA and OC as adjacent sides. If p = kq. Find k.

Answer key

Q.2 (i) $\pm(2\hat{i}+\hat{j}+\hat{k})$; (ii) $2\sqrt{6}$; (iii) $2\sqrt{2}$ Q.3 6

Shortest distance between 2 skew lines :

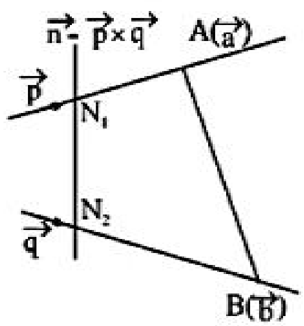
Note :

(i) 2 lines in a plane if not || must intersect and 2 lines in a plane if not intersecting must be parallel. Convertely 2 intersecting or parallel lines must be coplanar.

(ii) In space, however we come across situation when two lines neither intersect nor ||, Two such lines (like the flight paths of two planes) in space are known as skew lines or non coplanar lines.

(iii) S.D. between two such skew lines is the segment intercepted betweeen the two lines and perpendicular to both.

Method I: Two ways to determine the S.D.



$L_1 : \vec{r} = \vec{a} + \lambda\vec{p}$

$L_2 : \vec{r} = \vec{b} + \mu\vec{q}$

$\vec{n} = \vec{p}\times\vec{q}$

$\vec{AB} = (\vec{b}-\vec{a})$

$$\text{S.D.} = |\text{Projection of } \vec{AB} \text{ on } \vec{n}| = \left|\frac{\vec{AB}.\vec{n}}{|\vec{n}|}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

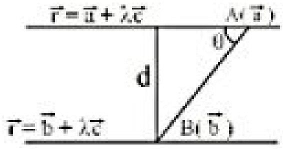
If S.D. = 0 $\Rightarrow$ lines are intersecting and hence coplanar.

Method II : p.v. of $N_1 = \vec{a} + \lambda\vec{p}$; p.v. of $N_2 = \vec{b} + \mu\vec{q}$

$$\overrightarrow{N_1N_2} = (\vec{b} - \vec{a}) + (\mu\vec{q} - \lambda\vec{p})$$

Now $\overrightarrow{N_1N_2} \cdot \vec{p} = 0$ and $\overrightarrow{N_1N_2} \cdot \vec{q} = 0$ (two linear equations to get the unique values of λ and μ.)
One p.v's of N_1 and N_2 are known we can also determine the equation to the line of shortest distance and the S.D.

Shortest Distance between two parallel lines :



$$d = |\vec{a} - \vec{b}| \sin\theta \Rightarrow \left| \frac{(\vec{a} - \vec{b}) \times \vec{c}}{|\vec{c}|} \right|$$

Illustration :

Find the shortest distance between the two lines whose vector equations are given by :

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \text{ and } \vec{r} = 2\hat{i} + 4\hat{j} + 5\hat{k} + \mu(3\hat{i} + 4\hat{j} + 5\hat{k})$$

Sol. *If the equations of the lines are $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, then shortest distance 'd' between them is given by*

$$d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right| \quad \text{...(i)}$$

Here $\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{a}_2 = 2\hat{i} + 4\hat{j} + 5\hat{k}$, $\vec{b}_2 = 3\hat{i} + 4\hat{j} + 5\hat{k}$

Now, $\vec{a}_2 - \vec{a}_1 = \left(2\hat{i} + 4\hat{j} + 5\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$

$= \hat{i} + 2\hat{j} + 2\hat{k}$ ...(ii)

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$= (15 + 16)\hat{i} - (10 - 12)\hat{j} + (8 - 9)\hat{k}$

$= \hat{i} + 2\hat{j} - \hat{k}$ ...(iii)

$\left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$ ...(iv)

and $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 \times \vec{b}_1) = \left(\hat{i} + 2\hat{j} + 2\hat{k}\right) \cdot \left(-\hat{i} + 2\hat{j} - \hat{k}\right)$

$= 1 \times (-1) + 2 \times 2 + 2 \times (-1) = 1$ ...(v)

Substituting the values from (iv) and (v) in (i), we get

$$d = \left| \frac{1}{\sqrt{6}} \right| = \frac{1}{\sqrt{6}}$$

Illustration :

Determine whether the following pair of lines intersect $\vec{r} = \hat{i} - \hat{j} + \lambda(2\hat{i} + \hat{k})$ *and* $\vec{r} = 2\hat{i} - \hat{j} + \mu(\hat{i} + \hat{j} - \hat{k})$.

Sol. *The shortest distance between the given pair of lines is given by*

$$d = \left| \frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

The two lines will intersect if and only if $d = 0$.

i.e. $(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$

Here, equation of first line is $\vec{r} = \hat{i} - \hat{j} + \lambda(2\hat{i} + \hat{k}) = \vec{a}_1 + \lambda\vec{b}_1$

where $\vec{a}_1 = \hat{i} - \hat{j}$ *and* $\vec{b}_1 = 2\hat{i} + \hat{k}$

Also equation of second line is $\vec{r} = 2\hat{i} - \hat{j} + \mu(\hat{i} + \hat{j} - \hat{k}) = \vec{a}_2 + \mu\vec{b}_2$

where $\vec{a}_2 = 2\hat{i} - \hat{j}$ *and* $\vec{b}_2 = \hat{i} + \hat{j} - \hat{k}$

Now, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -\hat{i} + 3\hat{j} + 2\hat{k}$ *and* $\vec{a}_2 - \vec{a}_1 = 2\hat{i} - \hat{j} - (\hat{i} - \hat{j}) = \hat{i}$

Since $(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = \hat{i}\cdot(-\hat{i} + 3\hat{j} + 2\hat{k}) = (-1)(1) + 3(0) + 2(0) = -1 \neq 0$

Hence the given lines do not intersect

PRODUCT OF THREE OR MORE VECTORS :

Scalar triple product :

Definition : If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then their scalar triple product is defined as the dot product of two vectors $\vec{a}$ and $\vec{b} \times \vec{c}$. It is generally denoted by $\vec{a}\cdot(\vec{b} \times \vec{c})$ or $[\vec{a}\ \vec{b}\ \vec{c}]$. It is read as box product of $\vec{a}, \vec{b}, \vec{c}$. Similarly other scalar triple products can be defined as $(\vec{b} \times \vec{c}).\vec{a}$, $(\vec{c} \times \vec{a}).\vec{b}$.

Note : Scalar triple product always results in a scalar quantity (number).

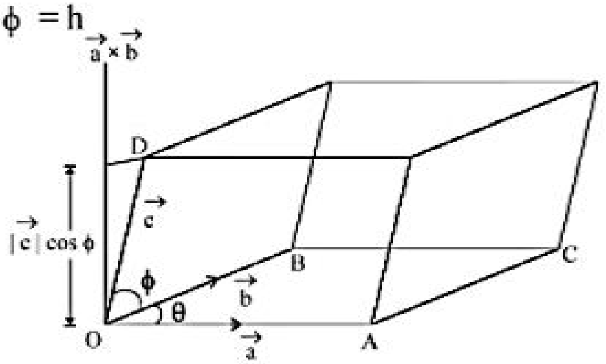
Geometrical Interpretation :

$(\vec{a} \times \vec{b}).\vec{c} = |\vec{a}||\vec{b}| \sin\theta\ \hat{n}.\vec{c} = |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\phi$ where $\theta = \vec{a} \wedge \vec{b}$; $\phi = \vec{n} \wedge \vec{c}$

but $|\vec{a}||\vec{b}| \sin\theta$ = area of $||^{gm}$ OACB and $|\vec{c}| \cos\phi = h$

There absolute value of scalar triple product of three vectors is equal to the volume of the parallelopiped whose three coterminous edges are represented by the given vectors.

Therefore $|(\vec{a} \times \vec{b})\cdot\vec{c}| = |[\vec{a}\ \vec{b}\ \vec{c}]|$ = Volume of the parallelopiped whose coterminous edges are $\vec{a}, \vec{b}$ and $\vec{c}$.



Formula for scalar Triple Product :

If $\vec{a} = a_1 i + a_2 j + a_3 k$, $\vec{b} = b_1 i + b_2 j + b_3 k$ and $\vec{c} = c_1 i + c_2 j + c_3 k$, then

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of Scalar Triple product :

1. The position of (.) and (×) can be interchanged i.e. $\vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$

2. $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$
Therefore if we don't change the cyclic order of a, b and c then the value of scalar triple product is not changed.

3. If the cyclic order of vectors is changed, then sign of scalar triple product is changed i.e.

$$\vec{a}.[\vec{b} \times \vec{c}] = -\vec{a}.(\vec{c} \times \vec{b}) \text{ or } [\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$$

From (ii) and (iii) we have

$$[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}] = -[\vec{a}\,\vec{c}\,\vec{b}] = -[\vec{b}\,\vec{a}\,\vec{c}] = -[\vec{c}\,\vec{b}\,\vec{a}]$$

4. The scalar triple product of three vectors when two of them are equal or parallel, is zero i.e.

$$[\vec{a}\,\vec{b}\,\vec{b}] = [\vec{a}\,\vec{b}\,\vec{a}] = 0 \quad \text{(think !)}$$

5. The scalar triple product of three mutually perpendicular unit vectors is ± 1 Thus

$$[\hat{i}\;\hat{j}\;\hat{k}] = 1, [\hat{i}\;\hat{k}\;\hat{j}] = -1$$

6. If two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel then $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

7. $\vec{a}, \vec{b}, \vec{c}$ are three coplanar vectors if $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ i.e. the necessary and sufficient condition for three non-zero non-collinear vectors to be coplanar is

$$[\vec{a}\;\vec{b}\;\vec{c}] = 0$$

8. For any vectors $\vec{a}, \vec{b}, \vec{c}, d$

$$[\vec{a} + \vec{b}\;\vec{c}\;d] = [\vec{a}\;\;\vec{c}\;\;d] + [\vec{b}\;\;\vec{c}\;\;d]$$

9. For right handed system, $[\vec{a}\,\vec{b}\,\vec{c}] > 0$
and for left handed system, $[\vec{a}\,\vec{b}\,\vec{c}] < 0$; where $\vec{a}, \vec{b}, \vec{c}$ are non- coplanar

10. $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$

11. $[\vec{a}-\vec{b}\;\;\vec{b}-\vec{c}\;\;\vec{c}-\vec{a}]$ is always zero.

12. $[\vec{l}\;\vec{m}\;\vec{n}]\,[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$, where $\vec{l}, \vec{m}, \vec{n}$ & $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors.

13. $[\vec{a}\,\vec{b}\,\vec{c}]\ (\vec{p}\times\vec{q}) = \begin{vmatrix} \vec{p}.\vec{a} & \vec{q}.\vec{a} & \vec{a} \\ \vec{p}.\vec{b} & \vec{q}.\vec{b} & \vec{b} \\ \vec{p}.\vec{c} & \vec{q}.\vec{c} & \vec{c} \end{vmatrix}$

14. If $\vec{a} = a_1\ell + a_2 m + a_3 n$, $\vec{b} = b_1\ell + b_2 m + b_3 n$ and $\vec{c} = c_1\ell + c_2 m + c_3 n$, then

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\ell mn]$$

Illustration :

If $\vec{a} = 2\hat{i} - 3\hat{j}$, $\vec{b} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{k}$ represent three coterminous edges of a parallelopiped, then the volume of that parallelopiped is-

(A) 2 *(B) 4* *(C) 6* *(D) 10*

Sol. *Volume* $= \left|[\vec{a}\,\vec{b}\,\vec{c}]\right|$

$$= \left|\begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}\right| = |-2+9-3| = 4$$ *Ans.[B]*

Illustration :

For any three vectors $\vec{a}, \vec{b}, \vec{c}$ $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$ equals-

(A) $[\vec{a}\ \vec{b}\ \vec{c}]$ *(B) $2[\vec{a}\ \vec{b}\ \vec{c}]$* *(C) $[\vec{a}\,\vec{b}\,\vec{c}]^2$* *(D) 0*

Sol. $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}\ c+\vec{a}] = (\vec{a}+\vec{b}).\,[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$

$= (\vec{a}+\vec{b}).(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a})]$

$= (\vec{a}+\vec{b}).\,(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a})$ $[\because\ \vec{c}\times\vec{c} = 0]$

$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{a}\,\vec{b}\,\vec{a}] + [\vec{a}\,\vec{c}\,\vec{a}] + [\vec{b}\,\vec{b}\,\vec{c}] + [\vec{b}\,\vec{b}\,\vec{a}] + [\vec{b}\,\vec{c}\,\vec{a}]$

$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{b}\,\vec{c}\,\vec{a}] = 2\,[\vec{a}\,\vec{b}\,\vec{c}]$ *Ans.[B]*

Illustration :

If $\vec{a}, \vec{b}, \vec{c}$ be three non- zero vectors, then $|(\vec{a}\times\vec{b}).\,\vec{c}| = |\vec{a}|\,|\vec{b}|\,|\vec{c}|$, if-

(A) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = 0$ *(B) $\vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$*

(C) $\vec{c}.\vec{a} = \vec{a}.\vec{b} = 0$ *(D) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$*

Sol. $|(\vec{a}\times\vec{b}).\,c| = |\vec{a}|\,|\vec{b}|\,|\vec{c}| \Leftrightarrow |(a\times\vec{b})|\,|\vec{c}|\cos\theta = |\vec{a}|\,|\vec{b}|\,|\vec{c}|$

(where θ is the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$) $\Leftrightarrow |\vec{a}|\,|\vec{b}|\,|\vec{c}|\sin\phi\cos\theta = |\vec{a}|\,|\vec{b}|\,|\vec{c}|$

(where ϕ is the angle between $\vec{a}$ and $\vec{b}$) $\Leftrightarrow \sin\phi = 1, \cos\theta = 1 \Leftrightarrow \phi = 90^\circ, \theta = 0^\circ$

$\Leftrightarrow \vec{a}.\vec{b} = 0,\ \vec{a}.\vec{c} = 0,\ \vec{b}.\vec{c} = 0$ *Ans.[D]*

Illustration :

If vectors $a\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+b\hat{j}+\hat{k}$ *and* $\hat{i}+\hat{j}+c\hat{k}$ $(a \neq b \neq c \neq 1)$ *are coplanar, then*

$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$ *equals-*

(A) 1 *(B) 0* *(C) – 1* *(D) None of these*

Sol. *Since vectors are coplanar,*

$$\therefore \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 0 & 1-b & c-1 \end{vmatrix} = 0 \qquad [\text{Using } R_2 - R_1,\ R_3 - R_2]$$

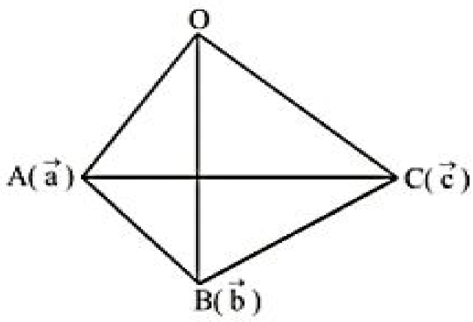
$$\Rightarrow a(b-1)(c-1) - (1-a)\{(c-1)-(1-b)\} = 0$$
$$\Rightarrow a(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$$
$$\Rightarrow (a-1+1)(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$$
$$\Rightarrow (1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = (1-a)(1-b)(1-c)$$
$$\Rightarrow \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} = 1$$

Ans.[A]

VOLUME OF TETRAHEDRON :

1. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices A, B and C with respect to O, then volume of tetrahedron OABC represented by V is given by

$$V = \frac{1}{3}\text{ Base area} \times \text{height}$$



$$\text{Base Area} = \frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|$$

Let $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a} = \vec{n}$

$$\therefore \quad \text{Base area} = \frac{1}{2}\left|\vec{n}\right|$$

Height = projection of $\vec{a}$ on $\vec{n}$

$$= \frac{\left|\vec{a}\cdot\vec{n}\right|}{\left|\vec{n}\right|} = \frac{\left|\vec{a}\cdot\left(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right)\right|}{\left|\vec{n}\right|} = \frac{\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|}{\left|\vec{n}\right|}$$

$$\therefore \quad V = \frac{1}{3}\cdot\frac{1}{2}\left|\vec{n}\right|\frac{\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|}{\left|\vec{n}\right|} = \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$$

2. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are position vectors of vertices A,B,C,D of a tetrahedron ABCD, then

$$\text{its volume} = \begin{cases} \frac{1}{6}\left|\left[\overrightarrow{AB} \quad \overrightarrow{AC} \quad \overrightarrow{AD}\right]\right| \\ \quad\quad \text{or} \\ \frac{1}{6}\left|[\vec{b}-\vec{a} \quad \vec{c}-\vec{a} \quad \vec{d}-\vec{a}]\right| \end{cases}$$

Illustration :

If the vertices of any tetrahedron be $\vec{a} = \hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} + \hat{k}$, $\vec{c} = 4\hat{i} + 3\hat{j} + 6\hat{k}$ and $\vec{d} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ then find its volume.

Sol. *Let the p.v. of the vertices A,B,C,D with respect to 0 are $\vec{a}, \vec{b}, \vec{c}$ and d respectively then*

$\overrightarrow{AB} = \vec{b} - \vec{a} = 3\hat{i} - \hat{j} - \hat{k}$,

$\overrightarrow{AC} = 4\hat{i} + 2\hat{j} + 4\hat{k}$ *and* $\overrightarrow{AD} = 2\hat{i} + 2\hat{j}$

Now volume of tetrahedron $= \frac{1}{6}\left[\overrightarrow{AB} \ \overrightarrow{AC} \ \overrightarrow{AD}\right] = \frac{1}{6}\begin{vmatrix} 3 & -1 & -1 \\ 4 & 2 & 4 \\ 2 & 2 & 0 \end{vmatrix} = -6$

$\therefore$ *Required volume = 6 units*

Practice Problem

Q.1 Find the value of p for which the vectors $(p+1)\hat{i} - 3\hat{j} + p\hat{k}$; $p\hat{i} + (p+1)\hat{j} - 3\hat{k}$ and $-3\hat{i} + p\hat{j} + (p+1)\hat{k}$ are linearly dependent/coplanar.

Q.2 Show that the lines $\vec{R} = \vec{R}_0 + t\vec{A}$ and $\vec{R} = \vec{R}_i + s\vec{B}$ intersect if $(\vec{R}_0 - \vec{R}_i)\cdot(\vec{A} \times \vec{B}) = 0$ i.e. $[\vec{R}_0 \, \vec{A} \, \vec{B}] = [\vec{R}_i \, \vec{A} \, \vec{B}]$.

Q.3 If $\vec{u} = 2\hat{i} - \hat{j} + \hat{k}$; $\vec{v} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{w}$ is a unit vector then find the maximum value of $[\vec{u} \ \vec{v} \ \vec{w}]$.

Answer key

Q.1 $p = 1$

Q.3 $[\vec{u} \ \vec{v} \ \vec{w}]_{max} = \sqrt{14}$]

VECTOR TRIPLE PRODUCT :

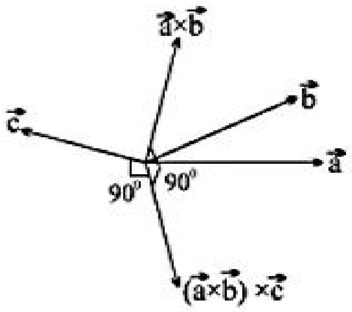
Definition : The vector triple product of three vectors $\vec{a}, \vec{b}, \vec{c}$ is defined as the vector product of two vectors $\vec{a}$ and $\vec{b}\times\vec{c}$. It is denoted by $\vec{a}\times(\vec{b}\times\vec{c})$.

$(\vec{a}\times\vec{b})\times\vec{c}$ is a vector which is coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$.

Hence $(\vec{a}\times\vec{b})\times\vec{c} = x\vec{a}+y\vec{b}$ ....(1) [linear combination of $\vec{a}$ and $\vec{b}$)

$\vec{c}\cdot(\vec{a}\times\vec{b})\times\vec{c} = x(\vec{a}.\vec{c})+y(\vec{b}.\vec{c})$

$0 = x(\vec{a}.\vec{c})+y(\vec{b}.\vec{c})$ ....(2)



$\therefore \dfrac{x}{\vec{b}.\vec{c}} = -\dfrac{y}{\vec{a}.\vec{c}} = \lambda$

$\therefore x = \lambda(\vec{b}.\vec{c})$ and $y = -\lambda(\vec{a}.\vec{c})$

Substituting the values of x and y in $(\vec{a}\times\vec{b})\times\vec{c} = \lambda(\vec{b}.\vec{c})\vec{a} - \lambda(\vec{a}.\vec{c})\vec{b}$

This is an identity and must be true for all values of $\vec{a}, \vec{b}, \vec{c}$

Put = $\vec{a} = \hat{i}$; $\vec{b} = \hat{j}$ and $\vec{c} = \hat{i}$

$(\hat{i}\times\hat{j})\times\hat{i} = \lambda(\hat{j}.\hat{i})\hat{i} - \lambda(\hat{i}.\hat{i})\hat{j}$

$\hat{j} = -\lambda\hat{j} \quad \Rightarrow \quad \lambda = -1$

hence $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

Properties :

1. Expansion formula for vector triple product is given by

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$(\vec{b}\times\vec{c})\times\vec{a} = (\vec{b}.\vec{a})\vec{c} - (\vec{c}.\vec{a})\vec{b}$.

2. $[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

Note that if $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors then $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ will also be non coplanar vectors.

3. Vector triple product is a vector quantity.

4. $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$

6 Unit vector coplanar with $\vec{a}$ & $\vec{b}$ and perpendicular to $\vec{c}$ is $\pm\dfrac{(\vec{a}\times\vec{b})\times\vec{c}}{|(\vec{a}\times\vec{b})\times\vec{c}|}$

Illustration :

$\hat{i} \times (\hat{j} \times \hat{k}) + \hat{j} \times (\hat{k} \times \hat{i}) + \hat{k} \times (\hat{i} \times \hat{j})$ *equals-*

(A) $\hat{i}$ (B) $\hat{j}$ (C) $\hat{k}$ (D) 0

Sol. $\hat{i} \times (\hat{j} \times \hat{k}) + \hat{j} \times (\hat{k} \times \hat{i}) + \hat{k} \times (\hat{i} \times \hat{j}) \Rightarrow \hat{i} \times \hat{i} + \hat{j} \times \hat{j} + \hat{k} \times \hat{k} = 0 + 0 + 0 = 0$ *Ans.[D]*

Illustration :

If $\vec{a}, \vec{b}, \vec{c}$ *are coplanar, then show that* $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}$ *and* $\vec{c} \times \vec{a}$ *are also coplanar.*

Sol. $[\vec{a}\,\vec{b}\,\vec{c}]$ *are coplanar* $\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0 \Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$

$\Rightarrow [\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}] = 0$

$\Rightarrow \vec{a} \times \vec{b}, \vec{b} \times \vec{c}$ *and* $\vec{c} \times \vec{a}$ *are coplanar*

Illustration :

Let $\vec{a}, \vec{b}, \vec{c}$ *such that* $|\vec{a}| = 1, |\vec{b}| = 1$ *and* $|\vec{c}| = 2$ *and if* $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = 0$ *then acute angle between* $\vec{a}$ *and* $\vec{c}$ *is -*

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) None of these

Sol. *If angle between* $\vec{a}$ *and* $\vec{c}$ *is* θ *then –*

$$\vec{a} \cdot \vec{c} = |\vec{a}|\,|\vec{c}| \cos\theta$$
$$= 1.2 \cos\theta = 2\cos\theta$$

but $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = 0$

$\Rightarrow (\vec{a} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{a})\vec{c} + \vec{b} = 0$

$\Rightarrow (2\cos\theta)\vec{a} - 1.\vec{c} = -\vec{b}$

$\Rightarrow [(2\cos\theta)\vec{a} - \vec{c}]^2 = [-\vec{b}]^2$

$\Rightarrow 4\cos^2\theta\, |\vec{a}|^2 - 2.(2\cos\theta)\, \vec{a}.\vec{c} + |\vec{c}|^2 = |\vec{b}|^2$

$\Rightarrow 4\cos^2\theta - 4\cos\theta\,(2\cos\theta) + 4 = 1 \quad \Rightarrow \quad 4(1 - \cos^2\theta) = 1 [\because |\vec{a}| = 1, |\vec{b}| = 1]$

$\Rightarrow \sin\theta = 1/2$

$\Rightarrow \theta = \frac{\pi}{6}$ *Ans.[C]*

Illustration :

Let $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k}$ *and* $\vec{b} = \hat{i}+\hat{j}$ *if vector* $\vec{c}$ *is such that* $\vec{a}\,.\,\vec{c} = |\vec{c}|$, $|\vec{c}-\vec{a}| = 2\sqrt{2}$ *and angle between (a × b) and c is the* 30^o *then* $|(\vec{a}\times\vec{b})\times\vec{c}|$ *is equal to -*

(A) $\frac{2}{3}$ *(B)* $\frac{3}{2}$ *(C) 2* *(D) 3*

Sol. $|\vec{c}-\vec{a}|^2 = (\vec{c}-\vec{a}).(\vec{c}-\vec{a}) = (2\sqrt{2})^2$

$\Rightarrow \quad \vec{c}^{\,2} + \vec{a}^2 - 2\vec{c}.\vec{a} = 8 \Rightarrow \vec{c}^{\,2} + (4+1+4) - 2\,\vec{c}.\vec{a} = 8$

$\Rightarrow \quad \vec{c}^{\,2} + 9 - 2|\vec{c}| = 8 \qquad [\because\ \vec{a}.\vec{c} = |\vec{c}|]$

$\Rightarrow \quad \vec{c}^{\,2} - 2|\vec{c}| + 1 = 0 \Rightarrow \vec{c}^{\,2} - 2\vec{c} + 1 = 0$

$\Rightarrow \quad (\vec{c}-1)^2 = 0 \Rightarrow \vec{c} = 1$

$|(\vec{a}\times\vec{b})\times\vec{c}| = |\vec{a}\times\vec{b}|\,|\vec{c}|\sin 30^o = 1\times\frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}|\vec{a}\times\vec{b}|$

But $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i}-2\hat{j}+\hat{k}$

$\therefore |\vec{a}\times\vec{b}| = \sqrt{4+4+1} = 3$

$\therefore |(\vec{a}\times\vec{b})\times\vec{c}| = \frac{3}{2}$ *Ans.[B]*

Scalar Product of four Vector :

(I) $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

Proof: $\underbrace{(\vec{a}\times\vec{b})}_{\vec{u}}\cdot(\vec{c}\times\vec{d}) = \vec{u}\cdot(\vec{c}\times\vec{d}) = (\vec{u}\times\vec{c})\cdot\vec{d}$ (Dot & cross are interchangeable in STP)

$$\left((\vec{a}\times\vec{b})\times\vec{c}\right)\cdot\vec{d} = \left((\vec{a}\times\vec{c})\vec{b} - (\vec{b}\times\vec{c})\vec{a}\right)\cdot\vec{d} = \left((\vec{a}\times\vec{c})(\vec{b}\times\vec{d}) - (\vec{b}\times\vec{c})(\vec{a}\times\vec{d})\right) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

$(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b}) = (\vec{a}\times\vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix} = (\vec{a})^2(\vec{b})^2 = (\vec{a}\cdot\vec{b})^2$ which is lagrange's identity.

Illustration :

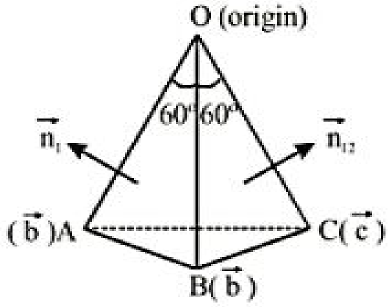
Prove that acute angle between the two plane faces of a regular tetrahedron is $\cos^{-1}\frac{1}{3}$.

Sol. *Let edge length of regular tetrahedron* $= 1$

$\vec{n}_1$ = *normal vector to plane OAB* = $\vec{a}\times\vec{b}$

$\vec{n}_2$ = *normal vector to plane OBC* = $\vec{b}\times\vec{c}$

$\therefore$ *acute angle between plane focus OAB & OBC is given as*



$$\cos\theta = \frac{|\vec{n}_1\,\vec{n}_2|}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{|(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})|}{|(\vec{a}\times\vec{b})|\,|(\vec{b}\times\vec{c})|} = \frac{\begin{vmatrix}\vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\end{vmatrix}}{\sin 60\cdot\sin 60^o}$$

$$= \frac{\begin{vmatrix}\cos 60^o & \cos 60^o\\ \cos 0^o & \cos 60^o\end{vmatrix}}{3/4} = \frac{\left|\frac{1}{4}-\frac{1}{2}\right|}{\frac{3}{4}} = \frac{1}{3} \Rightarrow \qquad \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Vector Product of Four Vector :

$\vec{V} = (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

$= \vec{u}\times(\vec{c}\times\vec{d}) \quad = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$ ....(1) (where $\vec{u} = \vec{a}\times\vec{b}$)

again $\vec{V} = (\vec{a}\times\vec{b})\times\underbrace{(\vec{c}\times\vec{d})}_{\vec{v}} = (\vec{a}.\vec{v})\vec{b} - (\vec{b}.\vec{v})\vec{a} = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$ ....(2)

from (1) and (2) $\quad [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$ ....(3)

Note that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = 0 \Rightarrow$ planes containing the vectors $\vec{a}$ & $\vec{b}$ and $\vec{c}$ & $\vec{d}$ are parallel.

$|||^{ly}$ $\quad (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 0 \Rightarrow$ the two planes are perpendicular.

(i) Equation (3) is suggestive that if $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors no 3 three of them are coplanar then each one of them can be expressed as a linear combination of other.

(ii) If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are p.v.'s of four points then these four points are in the same plane if

$$[\vec{a}\,\vec{b}\,\vec{d}] - [\vec{a}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{c}\,\vec{d}] - [\vec{b}\,\vec{c}\,\vec{d}]$$

Illustration :

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}\times\vec{b} = \vec{c},\ \vec{b}\times\vec{c} = \vec{a}$, then

(A) $|\vec{b}| = 1, |\vec{c}| = |\vec{a}|$ *(B)* $|\vec{c}| = 1, |\vec{a}| = |\vec{b}|$

(C) $|\vec{b}| = 2, |\vec{c}| = 2|\vec{a}|$ *(D)* $|\vec{a}| = 1, |\vec{b}| = |\vec{c}|$

Sol. *Given* $\vec{a}\times\vec{b} = \vec{c}$ ...(i)

$\vec{b}\times\vec{c} = \vec{a}$ ...(ii)

From (i), $\vec{c} \perp \vec{a}$ and $\vec{c}\perp\vec{b}$ ...(iii)

From (ii), $\vec{a}\perp\vec{b}$ and $\vec{a}\perp\vec{c}$ ...(iv)

From (iii) and (iv), $\vec{a}, \vec{b}, \vec{c}$ are mutually perpedicular.
Taking cross product of (i) with (ii), we get,

$$(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c}) = \vec{c}\times\vec{a} \Rightarrow [\vec{a}\ \vec{b}\ \vec{c}]\vec{b} - [\vec{a}\,\vec{b}\,\vec{b}]\vec{c} = \vec{c}\times\vec{a}$$

$$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}]\vec{b} = \vec{c}\times\vec{a} \qquad [\because [\vec{a}\,\vec{b}\,\vec{b}] = 0]$$

$$\Rightarrow \left|[\vec{a}\vec{b}\vec{c}]\,\vec{b}\right| = \left|\vec{c}\times\vec{a}\right| \Rightarrow \left|[\vec{a}\vec{b}\vec{c}]\right|\left|\vec{b}\right| = \left|\vec{c}\times\vec{a}\right|$$

$$\Rightarrow |\vec{a}|\,|\vec{b}|\,|\vec{c}|\ |\vec{b}| = |\vec{c}|\ |\vec{a}| \Rightarrow |\vec{b}|^2 = 1 \Rightarrow |\vec{b}| = 1$$

From (i), $\vec{a}\times\vec{b} = \vec{c}$

$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{c}| \Rightarrow |\vec{a}|\ |\vec{b}| = |\vec{c}|$ *[$\because$ $\vec{a}$ and $\vec{b}$ are mutually perpendicular]*

$\Rightarrow |\vec{a}| = |\vec{c}|$ $[\because |\vec{b}| = 1]$

$\because$ $|\vec{b}| = 1$ and $|\vec{a}| = |\vec{c}|$ *Ans. [A]*

Practice Problem

Q.1 Prove : $\vec{d}.[\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}] = (\vec{b}.\vec{d})[\vec{a}\,\vec{c}\,\vec{d}]$

Q.2 Prove that : $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c}) = -2[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$

Q.3 If $\hat{a}\times(\hat{b}\times\hat{c}) = \frac{1}{2}\hat{b}$ where $\hat{b}$ and $\hat{c}$ are non collinear then find the angle between $\hat{a}$ and $\hat{b}$; between $\hat{a}$ and $\hat{c}$.

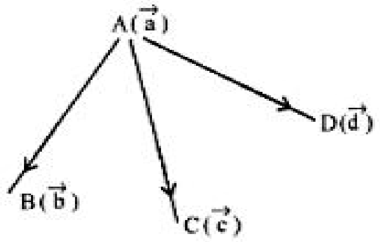
Q.4 Prove that $\hat{i}\times(\vec{a}\times\hat{i}) + \hat{j}\times(\vec{a}\times\hat{j}) + \hat{k}\times(\vec{a}\times\hat{k}) = 2\vec{a}$

Answer key

Q.3 $\pi/2$; $\pi/3$

Condition for coplanarity of four points :

4 points with pv's $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar iff $\exists$ scalars x, y, z and t not all simultaneously zero and satisfying $x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ where $x + y + z + t = 0$.



Case I : Let the four points A, B, C, D are in the same plane

$\Rightarrow$ the vectors $\vec{b} - \vec{a}, \vec{c} - \vec{a}$ and $\vec{d} - \vec{a}$ are in the same plane.

hence $\vec{d} - \vec{a} = l(\vec{b} - \vec{a}) + m(\vec{c} - \vec{a})$

or $\underbrace{(l + m - 1)}_{x}\vec{a} - \underbrace{l}_{y}\vec{b} - \underbrace{m}_{z}\vec{c} + \underbrace{1}_{t}\vec{d} = 0 \Rightarrow x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ where, $x + y + z + t = 0$ and x, y, z, t not all simultaneous zero.

Case II : Let $x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ where $x + y + z + t = 0$ and not all simultaneously zero

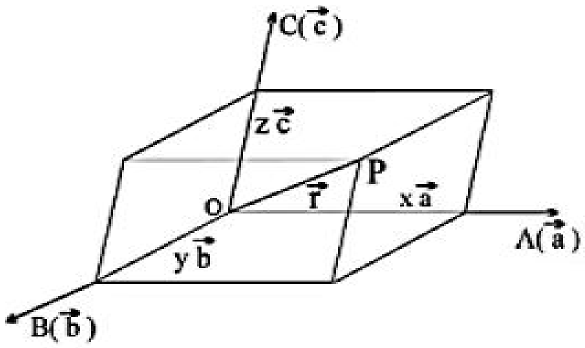
Let $t \neq 0$ $\quad (-y - z - t)\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0 \quad$ [putting $x = -y - z - t$]

$(\vec{d} - \vec{a})t + y(\vec{b} - \vec{a}) + z(\vec{c} - a) = 0$

$\Rightarrow \quad \vec{d} - \vec{a}, \vec{b} - \vec{a}$ and $\vec{c} - a$ are coplanar $\Rightarrow$ points A, B, C, D are coplanar

Theorem in space :

If $\vec{a}, \vec{b}, \vec{c}$ are 3 non zero non coplanar vectors then any vector $\vec{r}$ can be expressed as a linear combination : $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$ of $\vec{a}, \vec{b}, \vec{c}$



Examples :

Express the non coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ in terms of $\vec{b} \times \vec{c}, \vec{c} \times \vec{a}, \vec{a} \times \vec{b}$.

Since $[\vec{a}\,\vec{b}\,\vec{c}]^2 = [(\vec{a} \times \vec{b})\ (\vec{b} \times \vec{c})\ (\vec{c} \times \vec{a})]$

$\therefore$ If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar

$\Rightarrow$ $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}$ are also non coplanar.

$\vec{a} = x(\vec{a} \times \vec{b}) + y(\vec{b} \times \vec{c}) + z(\vec{c} \times \vec{a})$

Taking dot product with $\vec{a}$

$$\vec{a}^2 = y[\vec{a}\,\vec{b}\,\vec{c}] \Rightarrow y = \frac{(\vec{a})^2}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Taking dot product with $\vec{b}$

$$\vec{a}\cdot\vec{b} = z[\vec{b}\,\vec{c}\,\vec{a}] \Rightarrow z = \frac{\vec{a}\,\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Similarly taking dot product with $\vec{c}$

$$\vec{a}\cdot\vec{c} = x[\vec{a}\,\vec{b}\,\vec{c}] \Rightarrow x = \frac{\vec{a}\cdot\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$\therefore \quad \vec{a} = \frac{(\vec{a}\cdot\vec{c})(\vec{a}\times\vec{b}) + (\vec{a})^2(\vec{b}\times\vec{c}) + (\vec{a}\cdot\vec{b})(\vec{c}\times\vec{a})}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Practice Problem

Q.1 Express $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ in terms of 3 non coplanar vectors $\vec{a}, \vec{b}, \vec{c}$.

Q.2 Given the vector $\vec{a}$ and $\vec{b}$ orthogonal to each other find the vector $\vec{V}$ in terms of $\vec{a}$ and $\vec{b}$ satisfying $\vec{V}.\vec{a} = 0$; $\vec{V}.\vec{b} = 1$ and $[\vec{V}\ \vec{a}\ \vec{b}] = 1$

Answer key

Q.2 $\vec{V} = \dfrac{1}{\vec{b}^2}\vec{b} + \dfrac{1}{(\vec{a}\times\vec{b})^2}(\vec{a}\times\vec{b})$

Real definition of linearly independence :

If $\vec{V}_1, \vec{V}_2, \ldots\ldots\ldots \vec{V}_n$ are vectors and $\lambda_1, \lambda_2, \ldots\ldots\ldots \lambda_n$ are scalar and if the linear combination $\lambda_1\vec{V}_1 + \lambda_2\vec{V}_2 + \ldots\ldots\ldots + \lambda_n\vec{V}_n = 0$, necessarily implies $\lambda_1 = \lambda_2 = \ldots\ldots\lambda_n = 0$, we say that $\vec{V}_1, \vec{V}_2, \ldots\ldots\ldots \vec{V}_n$ are said to constitutes a linearly independent set of vectors.

Note:

(i) 2 non zero , non collinear vectors are linearly independent.

(ii) Three non zero , non coplanar vectors are linearly independent i.e. $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0 \Leftrightarrow \vec{a}, \vec{b}, \vec{c}$ are linearly independent.

(iii) Four or more vectors in 3D space are always linearly dependent.

Illustration :

Show that vectors $\vec{i}-3\vec{j}+2\vec{k}$, $2\vec{i}-4\vec{j}-\vec{k}$ *and* $3\vec{i}+2\vec{j}-\vec{k}$ *are linearly independent.*

Sol. *Let* $\vec{a}=\vec{i}-3\vec{j}+2\vec{k}$, $\vec{b}=2\vec{i}-4\vec{j}-\vec{k}$ *and* $\vec{c}=3\vec{i}+2\vec{j}-\vec{k}$

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\begin{vmatrix}1 & -3 & 2\\ 2 & 4 & -1\\ 3 & 2 & -1\end{vmatrix}\neq 0$$

Reciprocal system of vectors :

1. If $\vec{a},\vec{b},\vec{c}$ and $\vec{a}',\vec{b}',\vec{c}'$ are 2 sets of non coplanar vectors such that $\vec{a}.\vec{a}'=\vec{b}.\vec{b}'=\vec{c}.\vec{c}'=1$ and $\vec{a}.\vec{b}'=\vec{a}.\vec{c}'=\vec{b}.\vec{a}'=\vec{b}.\vec{c}'=\vec{c}.\vec{a}'=\vec{c}.\vec{b}'=0$, then $\vec{a},\vec{b},\vec{c}$ and $\vec{a}',\vec{b}',\vec{c}'$ are said to be constitute a reciprocal system of vectors.

2. Reciprocal system of vectors exists only in case of dot product.

3. It is possible to define $\vec{a}',\vec{b}',\vec{c}'$ in terms of $\vec{a},\vec{b},\vec{c}$ as.

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\ ;\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}\ ;\vec{c}'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}\ \left([\vec{a}\,\vec{b}\,\vec{c}]\neq 0\right)$$

Note: (i) $\vec{a}\times\vec{a}'+\vec{b}\times\vec{b}'+\vec{c}\times\vec{c}'=0$ i. e. $\dfrac{\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]}$

(ii) $(\vec{a}+\vec{b}+\vec{c}).(\vec{a}'+\vec{b}'+\vec{c}')=3$ (as $\vec{a}\cdot\vec{b}'=\vec{a}\cdot\vec{c}'=0$ etc)

(iii) If $[\vec{a}\,\vec{b}\,\vec{c}]=V$ then $[\vec{a}'\,\vec{b}'\,\vec{c}']=\dfrac{1}{V}$ $\Rightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{a}'\ \vec{b}'\ \vec{c}']=1$

(iv) $\vec{a}'\times\vec{b}'+\vec{b}'\times\vec{c}'+\vec{c}'\times\vec{a}'=\dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$

Isolating an known vectors

Satisfying a given relationship with some known vectors:

There is no general method for solving such equations, however dot or cross with known or unknown vectors or dot with $\vec{a}\times\vec{b}$, generally isolates the unknown vector. Use of linear combination also proves to be advantageous.

Illustration :

Find vector $\vec{r}$ *if* $\vec{r}\cdot\vec{a} = m$ *and* $\vec{r}\times\vec{b} = \vec{c}$, *where* $\vec{a}.\vec{b} \neq 0$.

Sol. $\vec{r}.\vec{a} = m \quad ...(i)$

and $\vec{r}\times\vec{b} = \vec{c} \quad ...(ii)$

From (ii), $\vec{a}\times(\vec{r}\times\vec{b}) = \vec{a}\times\vec{c}$

or $(\vec{a}.\vec{b})\vec{r} - (\vec{a}.\vec{r})\vec{b} = \vec{a}\times\vec{c}$

or $(\vec{a}.\vec{b})\vec{r} = \vec{a}\times\vec{c} + (\vec{a}\cdot\vec{r})\vec{b} = \vec{a}\times\vec{c} + m\vec{b}$

$\therefore \quad \vec{r} = \dfrac{1}{\vec{a}.\vec{b}}(\vec{a}\times\vec{c} + m\vec{b})$

Illustration :

Find $\vec{r}$ *such that* $t\vec{r} + \vec{r}\times\vec{a} = \vec{b}$, *where* $\vec{a}$ & $\vec{b}$ *ae non collinear vectors.*

Sol. *Given,* $t\vec{r} + \vec{r}\times\vec{a} = \vec{b} \quad ...(i)$

Since $\vec{a}, \vec{b}, \vec{a}\times\vec{b}$ *are non-coplanar vectors therefore* $\vec{r}$ *can be expressed as linear combination of* $\vec{a}, \vec{b}$ *and* $\vec{a}\times\vec{b}$

Let $\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})$

Putting the value of $\vec{r}$ *in (i), we get*

$$t[x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})] + x(\vec{a}\times\vec{a}) + y(\vec{b}\times\vec{a}) + z(\vec{a}\times\vec{b})\times\vec{a} = \vec{b}$$

$$t[x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})] + y(\vec{b}\times\vec{a}) + z[(\vec{a}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}] = \vec{b}$$

$$\Rightarrow \quad [tx - z(\vec{a}\cdot\vec{b})]\vec{a} + [ty + z(\vec{a}\cdot\vec{a}) - 1]\vec{b} + (tz - y)(\vec{a}\times\vec{b}) = 0$$

Equating the coefficients of $\vec{a}, \vec{b}$ *and* $\vec{a}\times\vec{b}$, *we get*

$tx - z(\vec{a}\cdot\vec{b}) = 0 \quad ...(iii)$

$ty + za^2 - 1 = 0 \quad ...(iv)$

$tz - y = 0 \quad ...(v)$

Solving (iii), (iv) and (v), we get

$$x = \frac{\vec{a}.\vec{b}}{t(t^2 + a^2)}, \quad y = \frac{t}{t^2 + a^2}, \quad z = \frac{1}{t^2 + a^2}$$

Putting the values of x, y, z in (ii), we have

$$\vec{r} = \frac{1}{t^2 + a^2}\left[\frac{(\vec{a}.\vec{b})\vec{a}}{t} + t\vec{b} + \vec{a}\times\vec{b}\right]$$

Second method :

Given $\quad t\vec{r}+\vec{r}\times\vec{a}=\vec{b} \quad$ ...(i)

$\therefore \quad \vec{r}\times\vec{a}=\vec{b}-t\vec{r} \quad$ ...(ii)

Taking corss product of both sides of (i) with $\vec{a}$, *we get*

$$t(\vec{r}\times\vec{a})+(\vec{r}\times\vec{a})\times\vec{a}=\vec{b}\times\vec{a} \quad \text{[Putting the value of } \vec{r}\times\vec{a}\text{]}$$

$$\Rightarrow \quad t(\vec{b}-t\vec{r})+(\vec{r}\times\vec{a})\vec{a}-a^2\vec{r}=\vec{b}\times\vec{a}$$

$$\Rightarrow \quad t\vec{b}-(t^2+a^2)\vec{r}+(\vec{r}\times\vec{a})\vec{a}=\vec{b}\times\vec{a} \quad \text{...(iii)}$$

In (i), taking dot product with $\vec{a}$, *we get*

$$r(\vec{r}\times a)=0=\vec{b}\times\vec{a} \qquad \therefore\ \vec{r}\times\vec{a}=\frac{\vec{a}\times\vec{b}}{t}$$

From (ii), $t\vec{b}-(t^2+a^2)\vec{r}+\dfrac{\vec{a}\times\vec{b}}{t}\vec{a}=\vec{b}\times\vec{a}$

$$\therefore \quad \vec{r}=\frac{1}{t^2+a^2}\left(\frac{\vec{a}\times\vec{b}}{t}\vec{a}+t\vec{b}+\vec{a}\times\vec{b}\right)$$

Illustration :

Solve the following simultaneous equations for $\vec{x}$ *and* $\vec{y}$ *:*

$$\vec{x}+\vec{y}=\vec{a}\,,\ \vec{x}\times\vec{y}=\vec{b} \text{ and } \vec{x}.\vec{a}=1$$

Sol. Given $\vec{x}+\vec{y}=\vec{a} \quad$ ...(i)

$\vec{x}\times\vec{y}=\vec{b} \quad$ ...(ii)

$\vec{x}.\vec{a}=1 \quad$ ...(iii)

Putting the value of $\vec{y}$ *from (i) in (ii), we get*

$$\vec{x}\times(\vec{a}-\vec{x})=\vec{b} \quad\Rightarrow\quad \vec{x}\times\vec{a}=\vec{b} \quad\Rightarrow\quad \vec{a}\times(\vec{x}\times\vec{a})=\vec{a}\times\vec{b}$$

$$\Rightarrow \quad a^2\vec{x}-(\vec{a}.\vec{x})\vec{a}=\vec{a}\times\vec{b} \quad\Rightarrow\quad a^2\vec{x}-\vec{a}=\vec{a}\times\vec{b}$$

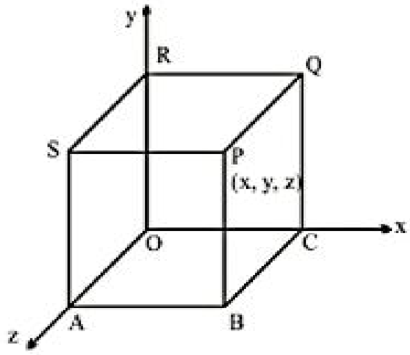
$$\therefore \quad \vec{x}=\frac{1}{\vec{a}}(\vec{a}+\vec{a}\times\vec{b}) \quad \text{and} \quad \vec{y}=\vec{a}-\vec{x}=\vec{a}-\frac{1}{a^2}(\vec{a}+\vec{a}\times\vec{b})$$

THREE DIMENSION

COORDINATES OF A POINT IN SPACE :

Consider a point P in space whose position is given by (x, y, z) where x, y, z are perpendicular distance from yz plane, zx plane and xy plane respectively.

If we assume $\hat{i}, \hat{j}, \hat{k}$ unit vectors along OX, OY, OZ respectively then the position vector of point P is $x\hat{i} + y\hat{j} + z\hat{k}$ or simply (x, y, z).



When a point lies on Co-ordinates

(i) x – axis $(\alpha, 0, 0)$

(ii) y – axis $(0, \beta, 0)$

(iii) z – axis $(0, 0, \gamma)$

(iv) XY – plane $(\alpha, \beta, 0)$

(v) XZ – plane $(\alpha, 0, \gamma)$

(vi) YZ – plane $(0, \beta, \gamma)$

Distance formulae :

Distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is equal to $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

Section formulae :

(1) Coordinates a point P which divides line joining A (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio m : n internally is given by $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$.

(2) Coordinates of a point P which divides line joining A (x_1, y_1, z_1) and B (x_2, y_2, z_2) in the ratio m : n externally is given by $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$.

(3) Coordinates of mid-point of line joining A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Direction cosines :

If α, β, γ are the angles which vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ makes with positive direction of the x, y, z axes respectively then α, β, γ are called direction angles and their cosines $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the vector and are generally denoted l, m, n respectively.
Thus $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

Note :

(i) If a line makes α, β, γ with positive direction of x, y, z axes respectively then direction cosines of line will be $\cos\alpha, \cos\beta, \cos\gamma$ or $-\cos\alpha, -\cos\beta, -\cos\gamma$.

(ii) A unit vector along the line whose direction cosines are $\cos\alpha, \cos\beta, \cos\gamma$ can be written as $(\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$.

(iii) If a vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ makes angles α, β, γ with positive direction of x, y, z axes respectively

then $\cos\alpha = \dfrac{\vec{a}\cdot\hat{i}}{|\vec{a}||\hat{i}|} = \dfrac{a_1}{|\vec{a}|}$, $\cos\beta = \dfrac{\vec{a}\cdot\hat{j}}{|\vec{a}||\hat{j}|} = \dfrac{a_2}{|\vec{a}|}$ and $\cos\gamma = \dfrac{\vec{a}\cdot\hat{k}}{|\vec{a}||\hat{j}|} = \dfrac{a_3}{|\vec{a}|}$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2 + a_2^2 + a_3^2}{|\vec{a}|^2} \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Also note that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

(iv) Direction cosines of x-axis are (1, 0, 0) or (–1, 0, 0).
Direction cosines of y-axis are (0, 1, 0) or (0, –1, 0).
Direction cosines of z-axis are (0, 0, 1) or (0, 0, –1).

Direction ratios :

If a, b, c are three numbers proportional to the direction cosines l, m, n of a straight line, then a, b, c are called its direction ratios. They are also called direction numbers or direction components.

Hence, we have $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c} = \lambda$ (say) $\Rightarrow l = a\lambda$, $m = b\lambda$, $n = c\lambda$

$$\because l^2 + m^2 + n^2 = 1 \Rightarrow (a^2 + b^2 + c^2)\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}} \text{ and } n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Note :

(i) Direction ratios of a line is not unique but infinite in number but direction cosines will be for a line will be only two. (l, m, n or $-l$, –m, –n)

(ii) A vector along the line with direction ratios a, b, c can be $a\hat{i} + b\hat{j} + c\hat{k}$.

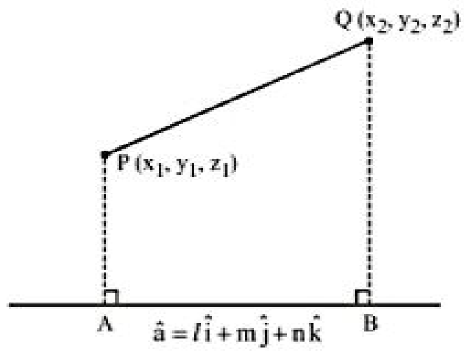
(iii) Direction ratios a line joining two points A and B are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

(iv) Projection of a Point on a Line :

Let P be a point and AB be a given line. Draw perpendicular PQ from P on AB which meets it at Q. This point Q is called projection of P on the line AB.

(v) Projection of a Line Segment Joining Two Points on a Line :

Projection of the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m, n is $AB = |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$.



Proof :

Vector $\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

A unit vector along another line $\hat{a} = l\hat{i} + m\hat{j} + n\hat{k}$.

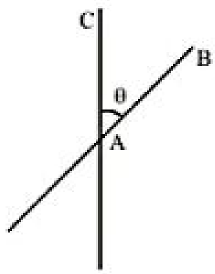
$\therefore$ Projection AB = Projection of $\overrightarrow{PQ}$ on $\hat{a}$ $= \dfrac{|\overrightarrow{PQ} \cdot \hat{a}|}{|\hat{a}|}$

$= |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$

Angle between two lines :

If direction ratios of two lines are a_1, b_1, c_1 and a_2, b_2, c_2 then acute angle between two lines is given by

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$



Proof : Vector along lines can be taken as $\hat{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\hat{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.

Acute angle between lines = acute angle between vectors $\vec{a}$ and $\vec{b}$.

$$\therefore \cos\theta = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

If direction cosines of lines are l_1, m_1, n_1 and l_2, m_2, n_2 then acute angle between them is given by $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$.

Note :

(i) If lines are perpendiculars (i.e. vectors along them are also perpendicular) then $a_1a_2 + b_1b_2 + c_1c_2 = 0$ or $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

(ii) If lines are parallel (i.e. vectors along them area also parallel) then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ or $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$.

Illustration :

Find the coordinates of the point which divides the line joining points (2, 3, 4) and (3, – 4, 7) in ratio 5 : 3 internally.

Sol. *Let the coordinates of the required point be (x, y, z) then* $x = \dfrac{2(3)+3(5)}{3+5} = \dfrac{21}{8}$;

$$y = \frac{3(3)-4(5)}{3+5} = \frac{-11}{8}\ ;\ z = \frac{4(3)+7(5)}{3+5} = \frac{47}{8}$$

$\therefore$ *The required point is* $\left(\dfrac{21}{8}, \dfrac{-11}{8}, \dfrac{47}{8}\right)$.

Illustration :

Find unit vector(s) with $\cos\alpha = \dfrac{1}{2}$ *and* $\cos\beta = \dfrac{1}{2}$, *where* α, β *are angles made by unit vector with positive direction x, y axes respectively.*

Sol. *Let unit vectors makes angle* γ *with positive z-axis.*

$\therefore$ *Unit vector will be* $(\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$

$$\because\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Rightarrow \frac{1}{4} + \frac{1}{4} + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2\gamma = \frac{1}{2} \Rightarrow \cos\gamma = \pm\frac{1}{\sqrt{2}}$$

$\therefore$ *Unit vector will be :* $\dfrac{\hat{i}}{2} + \dfrac{\hat{j}}{2} \pm \dfrac{\hat{k}}{\sqrt{2}}$.

Illustration :

Find the direction cosines of two lines which are connected by the relations $l - 5m + 3n = 0$ *and* $7l^2 + 5m^2 - 3n^2 = 0$.

Sol. *The given relations are* $l - 5m + 3n = 0 \Rightarrow l = 5m - 3n$ ....(1)

and $7l^2 + 5m^2 - 3n^2 = 0$ ....(2)

Putting the value of l from (1) in (2), we get

$7(5m - 3n)^2 + 5m^2 - 3n^2 = 0$

or $(2m - n)(3m - 2n) = 0 \Rightarrow \dfrac{m}{n} = \dfrac{1}{2}$ *or* $\dfrac{2}{3}$

When $\frac{m}{n} = \frac{1}{2}$ *i.e.* $n = 2m \Rightarrow l = 5m - 3n = -m$ *or* $\frac{l}{m} = -1$

thus $\frac{m}{n} = \frac{1}{2}$ *and* $\frac{l}{m} = -1$ *giving* $\frac{l}{-1} = \frac{m}{1} = \frac{n}{2}$

$$or,\ \frac{l}{-1} = \frac{m}{1} = \frac{n}{2} = \frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(-1)^2+1^2+2^2\}}} = \frac{1}{\sqrt{6}}$$

So, direction cosines of one line are $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$.

Again when $\frac{m}{n} = \frac{2}{3}$

$$\Rightarrow \frac{l}{m} = \frac{1}{2} \text{ giving } \frac{l}{1} = \frac{m}{2} = \frac{n}{3} = \frac{1}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}}$$

$\therefore$ *The direction cosines of the other line are* $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

Illustration :

Find the direction ratios and direction cosines of the line joining the points A (6, – 7, – 1) and B (2, – 3, 1).

Sol. *Direction ratios of AB are* $(4, -4, -2) = (2, -2, -1)$

$$a^2 + b^2 + c^2 = 9$$

Direction cosines are $\left(\pm\frac{2}{3}, \mp\frac{2}{3}, \mp\frac{1}{3}\right)$.

Illustration :

Find direction cosines of a line perpendicular to two lines whose drs are 1, 2, 3 and – 2, 1, 4.

Sol. *Vector along lines can be taken as* $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + \hat{j} + 4\hat{k}$

$$\therefore \quad \vec{a} \times \vec{b} = 5(\hat{i} - 2\hat{j} + \hat{k})$$

$$\therefore \quad dcs\ of\ line = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \text{ or } \left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right).$$

Illustration :

Find the projection of the line segment joining the points (–1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3.

Sol. *The direction cosines ℓ, m, n of the line are given by*

$$\frac{\ell}{6} = \frac{m}{2} = \frac{n}{3} = \frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{6^2+2^2+3^2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$$

$$\therefore \quad \ell = \frac{6}{7},\ m = \frac{2}{7},\ n = \frac{3}{7}$$

The required projection is given by

$$= |\ \ell(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)\ |$$

$$= \left|\frac{6}{7}[2-(-1)+\frac{2}{7}(5-0)+\frac{3}{7}(1-3)\right|$$

$$= \left|\frac{6}{7}\times 3+\frac{2}{7}\times 5+\frac{3}{7}\times(-2)\right|$$

$$= \left|\frac{18}{7}+\frac{10}{7}-\frac{6}{7}\right| = \left|\frac{18+10-6}{7}\right| = \frac{22}{7}.$$ ***Ans.***

Practice Problem

Q.1 If points P, Q are (2, 3, 4), (1, –2, 1), then prove that OP is perpendicular to OQ where O is (0, 0, 0).

Q.2 A line OP makes with the x-axis an angle of measure 120° and with y-axis an angle of measure 60°. Find the angle made by the line with the z-axis.

Q.3 What are the d.c's of the lines equally inclined to the axes ?

Q.4 A line makes angle $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is equal to

(A) 1 (B) $\frac{4}{3}$ (C) $\frac{3}{4}$ (D) $\frac{4}{5}$

Q.5 If a variable line in two adjacent positions has direction cosines l, m, n and $l + \delta l$, $m + \delta m$, $n + \delta n$, show that the small angle $\delta\theta$ between the two positions is given $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

Answer key

Q.2 $\gamma = 45°$ or $135°$ Q.3 $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ Q.4 B

PLANES :

Definition :
A plane is a surface such that a line joining any two points on the surface lies completely on it.

General equation of plane :

A linear equation in three variables of the type **$ax + by + cz + d = 0$** represents the general equation of a plane.
where a, b, c are not simultaneously zero.

Dividing by d we get $\left(\frac{a}{d}\right)x + \left(\frac{b}{d}\right)y + \left(\frac{c}{d}\right)z + 1 = 0.$

Thus equation of plane involves only three arbitrary constants. Hence in order to determine a unique plane 3 independent conditions are needed.

Note :
(i) Equation of xy plane is $z = 0$.
(ii) Equation of yz plane is $x = 0$.
(iii) Equation of zx plane is $y = 0$.

Division by Coordinate Planes :

The ratios in which the line segment PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is divided by coordinate planes are as follows.

(i) by yz - plane : $-\frac{x_1}{x_2}$ ratio

(ii) by zx - plane : $-\frac{y_1}{y_2}$ ratio

(iii) by xy - plane : $-\frac{z_1}{z_2}$ ratio

Illustration :

Find the ratio in which the line joining the points (3, 5, –7) and (–2,1,8) is divided by yz-plane.

Sol. *Let the line joining the points (3, 5, –7) and (–2, 1, 8) divides yz-plane in the ratio $\lambda : 1$, then coordinates of the dividing point will be*

$$\left(\frac{-2\lambda+3}{\lambda+1}, \frac{\lambda+5}{\lambda+1}, \frac{8\lambda-7}{\lambda+1}\right)$$

Now above points lies on the yz-plane, so its x-coordinate should be zero i.e.

$$\frac{-2\lambda+3}{\lambda+1} = 0 \Rightarrow \lambda = \frac{3}{2}$$

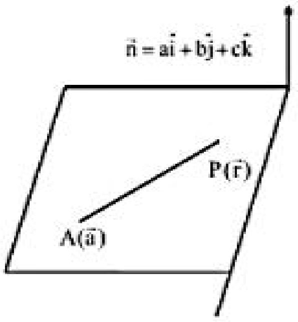
Hence yz-plane divides line joining the given points in the ratio $\frac{3}{2} : 1$ or 3 : 2. ***Ans.***

DIFFERENT FORMS OF THE EQUATIONS OF PLANES :

1. A point in the plane and a vector normal to it is given :

Let a point $A(\vec{a})$ lies in the plane and a vector normal to it is $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$.

$P(\vec{r})$ is a moving point whose locus is plane then for every position of vector $\overrightarrow{AP}$, vector $\vec{n}$ will be perpendicular to it.



$\therefore\ \overrightarrow{AP} \cdot \vec{n} = 0$

$\Rightarrow\ (\vec{r} - \vec{a}) \cdot \vec{n} = 0 \ \Rightarrow\ \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\Rightarrow\ \vec{r} \cdot \vec{n} = d$ is general equation of plane in vector form.

It is also known as equation of plane in dot (or scalar) product form.

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ the equation of plane will be **$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.**

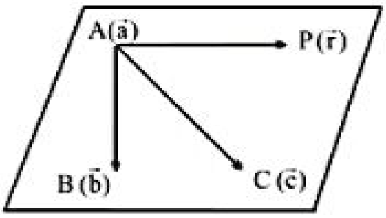
This is equation of plane containing point (x_0, y_0, z_0) and perpendicular to vector $\vec{a}\hat{i} + b\hat{j} + c\hat{k}$.

Note :

If equation of a plane is $ax + by + cz + d = 0$ then a, b, c are direction ratio of normal to the plane.

2. Plane passing through three given points :

Let three points $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ lies in the plane and point $P(\vec{r})$ is moving point whose locus is plane.



$\because\ \overrightarrow{AP}, \overrightarrow{AB}$ and $\overrightarrow{AC}$ are coplanar.

$\therefore\ \boxed{\left[\vec{r} - \vec{a} \quad \vec{b} - \vec{a} \quad \vec{c} - \vec{a}\right] = 0}$

In represents equation of plane passing through three points.

If $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, $C = (x_3, y_3, z_3)$ and $P(x, y, z)$ then equation of plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

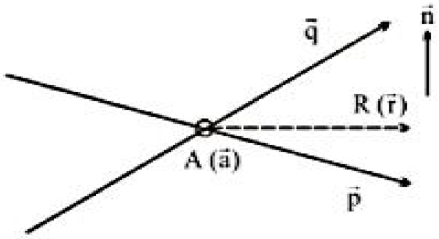
3. Plane containing two intersecting lines :

Let the equations of two lines are $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{b}$.

Now, $n = \vec{p} \times \vec{q}$ is a vector perpendicular to the plane.

Hence equation of plane is $(\vec{r} - \vec{a}) \cdot (\vec{p} \times \vec{q}) = 0$

$\Rightarrow [\vec{r} - \vec{a} \quad \vec{p} \quad \vec{q}] = 0 \Rightarrow [\vec{r} \quad \vec{p} \quad \vec{q}] \Rightarrow [\vec{a} \quad \vec{p} \quad \vec{q}]$



Since vectors $\vec{r} - \vec{a}, \vec{p}, \vec{q}$ are coplanar.

Therefore $\vec{r} - \vec{a} = \lambda\vec{p} + \mu\vec{q} \Rightarrow \vec{r} = \vec{a} + \lambda\vec{p} + \mu\vec{q}$

It represents equation of a plane containing point $\vec{a}$ and parallel to two non-collinear vectors $\vec{p}$ and $\vec{q}$.

This is also known as parametric equation.

Illustration :

Express the equation of a plane $\vec{r} = \hat{i} - 2\hat{j} + \lambda\left(2\hat{i} - \hat{j} + 3\hat{k}\right) + \mu\left(3\hat{i} + 4\hat{j} - \hat{k}\right)$ *in*

(a) *cartesian form.*

(b) *Scalar product form.*

Sol.

(a) *Clearly plane is passing through the point* $\hat{i} - 2\hat{j}$ *and parallel to vectors* $2\hat{i} - \hat{j} + 3\hat{k}$ *and* $3\hat{i} + 4\hat{j} - \hat{k}$.

$\therefore$ *Equation of plane is*

$$\begin{vmatrix} x-1 & y+2 & z-0 \\ 2 & -1 & 3 \\ 3 & 4 & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad (x - 1)(-11) - (y + 2)(-11) + z(11) = 0$

$\Rightarrow \quad x - 1 - y - 2 - z = 0 \Rightarrow x - y - z = 3 \quad \Rightarrow \; x(1) + y(-1) + z(-1) = 3$

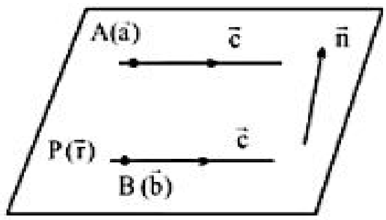
(b) *Therefore equation of plane is scalar product form is* $\vec{r} \cdot \left(\hat{i} - \hat{j} - \hat{k}\right) = 3$. ***Ans.***

4. Equation of plane containing two parallel lines :

Let lines be $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{b}$

vector normal to plane is

$$\vec{n} = (\vec{a} - \vec{c}) \times \vec{b}$$



$\therefore$ equation of plane is

$$(\vec{r} - \vec{a}) \cdot (\vec{a} - \vec{c}) \times \vec{b} = 0$$

Alternatively : Vectors $\vec{r} - \vec{a}$, $\vec{c} - \vec{a}$ and $\vec{b}$ are coplanar

$\therefore$ equation of plane is

$$\left[\vec{r} - \vec{a} \quad \vec{c} - \vec{a} \quad \vec{b}\right] = 0.$$

Illustration :

Find the equation of the plane passing through the point (2, – 1, 3) which is the foot of the perpendicular drawn from the origin to the plane.

Sol. *The direction ratios of the normal to the plane are 2, – 1, 3.*
The equation of required plane is $2(x-2) - 1(y+1) + 3(z-3) = 0$
$\Rightarrow 2x - y + 3z - 14 = 0.$

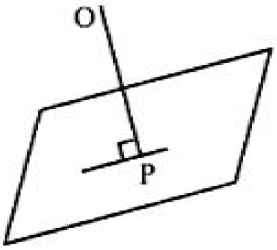


Illustration :

Find the equation of the plane through (2, 3, – 4), (1, – 1, 3) and parallel to x-axis.

Sol. *The equation of the plane passing through (2, 3, – 4) is* $a(x-2) + b(y-3) + c(z+4) = 0$ ....(1)
Since (1, – 1, 3) lies on it, we have $a + 4b - 7c = 0$ ....(2)
Since required plane is parallel to x-axis i.e. perpendicular to YZ plane i.e.

$$1 \cdot a + 0 \cdot b + 0 \cdot c = 0 \Rightarrow a = 0 \Rightarrow 4b - 7c = 0 \Rightarrow \frac{b}{7} = \frac{c}{4}$$

$\therefore$ *Equation of required plane is* $7y + 4z = 5$.

Illustration :

Two planes are given by equations $x + 2y - 3z = 0$ *and* $2x + y + z + 3 = 0$. *Find*
(i) *DC's of their normals and the acute angle between them.*
(ii) *DC's of their line of intersection.*
(iii) *Equation of the plane perpendicular to both of them through the point (2, 2, 1)*

Sol. $\vec{n}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$, $\quad \vec{n}_2 = 2\hat{i} + \hat{j} + \hat{k}$

(i) $\cos\theta = \dfrac{\vec{n}_1 - \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \dfrac{1}{2\sqrt{21}}$

(ii) $\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 5\hat{i} - 7\hat{j} - 3\hat{k}$

DC's of line of intersection of the plane $\left(\pm\frac{5}{\sqrt{83}}, \mp\frac{7}{\sqrt{83}}, \mp\frac{3}{\sqrt{83}}\right)$

(iii) $\left(\vec{r} - (2\hat{i} + 2\hat{j} + \hat{k})\right) \cdot (\vec{n}_1 \times \vec{n}_2) = 0$
$\Rightarrow \quad 5(x-2) - 7(y-2) - 3(z-1) = 0$
$\Rightarrow \quad 5x - 7y - 3z + 7 = 0$

5. Normal form of the plane :

A unit vector $\hat{n}$ normal to the plane from origin is known and perpendicular distance of the plane from the origin is d.

Projection of $\vec{r}$ on $\hat{n} = d$

$\Rightarrow \quad \vec{r} \cdot \hat{n} = d \quad$ ....(1)

Note : $d > 0$, as d is distance of the plane from origin.

Cartesian form of the plane is

$$lx + my + nz = d$$

where l, m, n are dcs of normal to plane.

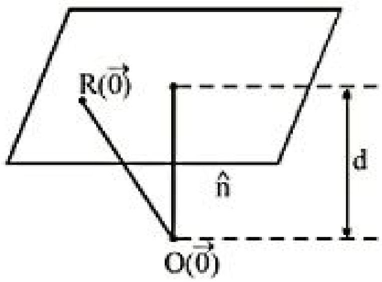


Illustration :

Let equation of plane be $\vec{r} \cdot (6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$ then find perpendicular distance of plane from origin and also find direction cosines of this perpendicular.

Sol. *Plane is* $\vec{r} \cdot (6\hat{i} - 3\hat{j} - 2\hat{k}) = -1 \Rightarrow \vec{r} \cdot \left(\frac{-6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right) = \frac{1}{7}$

$\therefore$ *perpendicular distance from origin* $= \frac{1}{7}$ *and dcs of perpendicular* $= \left(\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}\right)$.

Illustration :

Find the vector equation of plane which is at a distance of 8 units from the origin and which is normal to the vector $2\hat{i} + \hat{j} + 2\hat{k}$

Sol. *Here, d = 8 and* $\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$

$$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

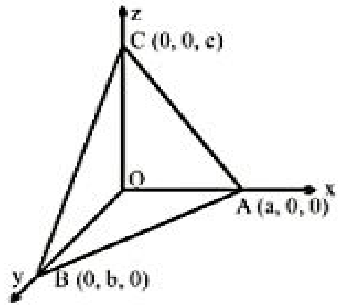
Hence, the required equation of plane is,

$$\vec{r} \cdot \hat{n} = d$$

$$\Rightarrow \quad \vec{r}\left(\frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}\right) = 8$$

$$or \quad \vec{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 24$$

6. Intercept form the plane :



Equation of plane in the intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

where $a = $ x-intercept,
$b = $ y-intercept,
$c = $ z-intercept

Proof :
Equation of plane passing through three points A (a, 0, 0), B (0, b, 0) and C (0, 0, c) will be

$$\begin{vmatrix} x-a & y-0 & z-0 \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0.$$

$\Rightarrow (x-a)\,bc - y\,(-ac-0) + z\,(0+ab) = 0$
$\Rightarrow xbc + yac + zab = abc$

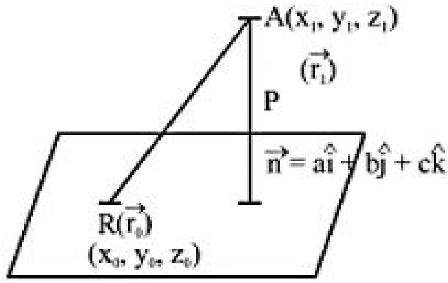
$$\Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Note : Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{BC}\right| = \frac{1}{2}\left|(b\hat{j}-a\hat{i})\times(c\hat{k}-b\hat{j})\right| = \frac{1}{2}\left|bc\hat{i}+ac\hat{j}+ab\hat{k}\right|$

$$= \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2} = \sqrt{\left(\frac{ab}{2}\right)^2+\left(\frac{bc}{2}\right)^2+\left(\frac{ca}{2}\right)^2}$$

$\therefore$ Area of $\Delta ABC = \sqrt{(\text{area of }\Delta OAB)^2 + (\text{area of }\Delta OBC)^2 + (\text{area of }\Delta OCA)^2}$

PERPENDICULAR DISTANCE OF A POINT FROM A PLANE :



Let equation of plane is $\vec{r}\cdot\vec{n} = d$ then perpendicular distance from the point A ($\vec{r}_1$) on the plane = projection of $\overrightarrow{RA}$ on $\vec{n}$.

$$\Rightarrow p = \frac{|(\vec{r}_1-\vec{r}_0)\cdot\vec{n}|}{|\vec{n}|} = \frac{|\vec{r}_1\cdot\vec{n}-\vec{r}_0\cdot\vec{n}|}{|\vec{n}|}$$

$$\Rightarrow p = \frac{|\vec{r}_1\cdot\vec{n}-d|}{|\vec{n}|}$$

If equation of plane is $ax+by+cz+d=0$ then perpendicular distance from point (x_1, y_1, z_1) is given by $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

Note :

(i) Planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

(a) parallel but not identical if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \neq \frac{d_1}{d_2}$

(b) perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(c) identical if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$

(ii) The equation of a plane parallel to the plane $ax + by + cz + d$ is $ax + by + cz + k = 0$, where k is an arbitrary constant and is determined by the given condition.

(iii) Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is equal to

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

(iv) 3 planes $a_rx + b_ry + c_rz = d_r \quad r = 1, 2, 3$

(a) Can intersect at a point ≡ system of equations in 3 variables having unique solution.

(b) Can intersect coaxially ≡ system of equations in 3 variables having infinite solutions.

(c) May not have a common point ≡ system of equations in 3 variables having no solution.

Illustration :

A variable plane passes through a fixed point (α, β, γ) and meets the axes in A, B, C. Show that the locus of the point of intersection of the planes through A, B and C parallel to the co-ordinate planes is $\alpha x^{-1} + \beta y^{-1} + \gamma z^{-1} = 1$.

Sol. *Let the equation of the variable plane be* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ....(1)

where a, b, c are parameters

The plane (1) passes through the point (α, β, γ).

$$\therefore \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1 \qquad ....(2)$$

The plane (1) meets the co-ordinate axes in the points A, B and C whose co-ordinates are respectively given by (a, 0, 0), (0, b, 0) and (0, 0, c). The equations of the planes through A, B and C are $x = a$, $y = b$, $z = c$ respectively ....(3)

The locus of the point of intersection of these planes is obtained by eliminating the parameters a, b, c between the equation (2), (3). Putting the values of a, b, c from (3) in (2), the required locus is given by $\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 1$ *or* $\alpha x^{-1} + \beta y^{-1} + \gamma z^{-1} = 1$.

EQUATION OF PLANES BISECTING THE ANGLES BETWEEN TWO GIVEN PLANES :

Cartesian Form :

The equation of the planes bisecting the angles between the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are

$$\frac{(a_1x + b_1y + c_1z + d_1)}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Note :

If angle between bisector plane and one of the plane is less than 45° then it is acute angle bisector otherwise it is obtuse angle bisector.

Vector Form :

The equation of the planes bisecting the angles between the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ are

$$\frac{|\vec{r} \cdot \vec{n}_1 - d_1|}{|\vec{n}_1|} = \frac{|\vec{r} \cdot \vec{n}_2 - d_2|}{|\vec{n}_2|}$$

or $$\frac{|\vec{r} \cdot \vec{n}_1 - d_1|}{|\vec{n}_1|} = \pm \frac{|\vec{r} \cdot \vec{n}_2 - d_2|}{|\vec{n}_2|}$$

or $$\vec{r} \cdot (\hat{n}_1 \pm \hat{n}_2) = \frac{d_1}{|\vec{n}_1|} \pm \frac{d_2}{|\vec{n}_2|}$$

FAMILY OF PLANES :

The equation of a plane passing through the lines of intersection of $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $(a_1x + b_1y + c_1z + d_1) + \lambda (a_2x + b_2y + c_2z + d_2) = 0$, where λ is a constant.

Vectorially :

Equation of a plane passing through the line of intersection of planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by

$$(\vec{r}.\vec{n}_1 - d_1) + \lambda(\vec{r}.\vec{n}_2 - d_2) = 0$$

Illustration :

Find the equation of plane containing the line of intersection of the plane $x + y + z - 6 = 0$ and $2x + 3y + 4z + 5 = 0$ and passing through (1,1,1)

Sol. *The equation of the plane through the line of intersection of the given planes is ,*

$$(x + y + z - 6) + \lambda\,(2x + 3y + 4z + 5) = 0 \qquad \text{...(i)}$$

If it is passes through (1, 1, 1)

$$\Rightarrow \quad (1 + 1 + 1 - 6) + \lambda\,(2 + 3 + 4 + 5) = 0 \;\Rightarrow\; \lambda = \frac{3}{14}$$

Putting $\lambda = 3/14$ in (i) we get

$$(x + y + z - 6) + \frac{3}{14}\,(2x + 3y + 4z + 5) = 0$$

$$\Rightarrow \quad 20x + 23y + 26z - 69 = 0$$

ANGLE BETWEEN TWO PLANES :

1. Vector form :

The angle between the two planes is defined as the angle between their normals.
Let θ be the angle between planes;

$\vec{r}\,.\,\vec{n}_1 = d_1$ and $\vec{r}\,.\vec{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

2. Cartesian form :

The angle θ between the planes $a_1\,x + b_1\,y + c_1\,z + d_1 = 0$ and $a_2\,x + b_2\,y + c_2\,z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

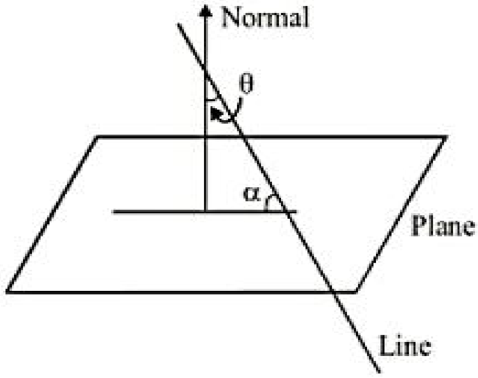
3. Two planes are perpendicular iff

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \;\&\; \text{Parallel if } \vec{n}_1 \times \vec{n}_2 = 0$$

Angle between a line and a plane :

The angle between a line and a plane is the complement of the angle between the line and the normal to the plane

If α, β, γ be the direction ratios of the line and $ax + by + cz + d = 0$ be the equation of plane and θ be the angle between the line and the plane.



$$\Rightarrow \quad \cos(90° - \theta) = \frac{a\alpha + b\beta + c\gamma}{\sqrt{a^2+b^2+c^2}\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

$$\text{or} \quad \sin\theta = \frac{a\alpha + b\beta + c\gamma}{\sqrt{a^2+b^2+c^2}\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

Vector form :

If θ is the angle between the line; $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r}.\vec{n} = d$

$$\Rightarrow \quad \sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}$$

Illustration :

Find the angle between the planes $2x - y + z = 11$ and $x + y + 2z = 3$.

Sol. $$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)\left(a_2^2+b_2^2+c_2^2\right)}}$$

$$\Rightarrow \quad \cos\theta = \frac{2\cdot 1 + (-1)\cdot 1 + 1\cdot 2}{\sqrt{2^2+(-1)^2+1^2}\sqrt{1^2+1^2+2^2}} = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

Illustration :

Find the equation of plane passing through the intersection of planes $2x - 4y + 3z + 5 = 0$, $x + y + z = 6$ and parallel to straight line having direction cosines $(1, -1, -1)$.

Sol. *Equation of required plane be $(2x - 4y + 3z + 5) + \lambda(x + y - z - 6) = 0$*

i.e. $(2 + \lambda)x + (-4 + \lambda)y + z(3 - \lambda) + (5 - 6\lambda) = 0$

This plane is parallel to a straight line. So, $al + bm + cn = 0$

$1(2 + \lambda) + (-1)(-4 + \lambda) + (-1)(3 - \lambda) = 0$ *i.e.* $\lambda = -3$

$\therefore$ *Equation of required plane is* $-x - 7y + 6z + 23 = 0$.

i.e. $x + 7y - 6z - 23 = 0$.

Illustration :

Find the angle between the line $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z-2}{4}$ *and the plane* $2x + y - 3z + 4 = 0$

Sol. *The given line is parallel to the vector* $\vec{b} = 3\hat{i} + 2\hat{j} + 4\hat{k}$ *and the given plane is normal to the vector*

$$\vec{n} = 2\hat{i} + \hat{j} - 3\hat{k}$$

$$\sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|} = \frac{(3\hat{i}+2\hat{j}+4\hat{k}).(2\hat{i}+\hat{j}-3\hat{k})}{\sqrt{3^2+2^2+4^2}\sqrt{2^2+1^2+3}}$$

$$= \frac{6+2-12}{\sqrt{29}\sqrt{14}} = \frac{-4}{\sqrt{406}} \quad \therefore \quad \theta = \sin^{-1}\left(\frac{-4}{\sqrt{406}}\right)$$

Practice Problem

Q.1 The ratio in which yz-plane divides the line joining (2, 4, 5) and (3, 5, 7)
(A) – 2 : 3 (B) 2 : 3 (C) 3 : 2 (D) – 3 : 2

Q.2 The points (0, – 1, – 1), (– 4, 4, 4), (4, 5, 1) and (3, 9, 4) are
(A) collinear (B) coplanar (C) forming a square (D) none of these

Q.3 The distance of centroid from x-axis of the triangle formed by the points (2, –4, 3), (3, –1, –2) and (–2, 5, 8) is-
(A) 1 (B) 0 (C) 3 (D) $\sqrt{10}$

Q.4 The locus of a point, which moves in such a way that its distance from the origin is thrice the distance from xy-plane is -
(A) $x^2 - 8y^2 - 8z^2 = 0$ (B) $x^2 - 8y^2 + z^2 = 0$
(C) $-8x^2 + y^2 + z^2 = 0$ (D) $x^2 + y^2 - 8z^2 = 0$

Q.5 A non-zero vector $\bar{a}$ is parallel to the line of intetsection of the plane determined by the vectors $\hat{i}, \hat{i}+\hat{j}$ and the plane determined by the vectors $\hat{i}-\hat{j}, \hat{i}+\hat{k}$. The angle between $\bar{a}$ and $\hat{i}-2\hat{j}+2\hat{k}$ is.
(A) π/3 (B) π/4 (C) π/6 (D) None of these

Answer key

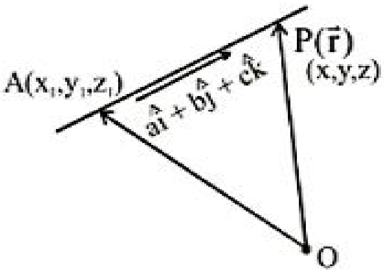
Q.1	A	Q.2	B	Q.3	C	Q.4	D	Q.5	B

STRAIGHT LINES :

Symmetric Form :

1. Equation of a straight line passing through (x_1, y_1, z_1) and having drs as a, b, c is

$$\boxed{\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda}$$



Proof :

A vector parallel to line will be $a\hat{i}+b\hat{j}+c\hat{k}$.

A vector along the line can be written as

$$\overrightarrow{AP}=(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}$$

$\because$ vector $\overrightarrow{AP}$ is parallel to $a\hat{i}+b\hat{j}+c\hat{k}$

$$\therefore \quad \boxed{\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda}$$

- Any point on this line can be taken as $(x_1+\lambda a, y_1+\lambda b, z_1+\lambda c)$.
- If dcs of line be l, m, n then its equation will $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=\lambda$ and any point on this line can be taken as $(x_1+\lambda a, y_1+\lambda b, z_1+\lambda c)$.

2. Equation of straight line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) will be

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Note :

(i) (a) Equation of x-axis is $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$ (or) $y=z=0$

(b) Equation of y-axis is $\frac{x}{0}=\frac{y}{1}=\frac{z}{0}$ (or) $x=z=0$

(c) Equation of z-axis $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$ (or) $x=y=0$

Here zero in denominator represents that line is perpendicular to that axis.

(ii) Line $\frac{x-2}{3}=\frac{y+1}{-2}$ and $z=2$ is written as $\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-2}{0}$

This line is perpendicular to z-axis or parallel to xy plane at a distance of 2 units.

Unsymmetrical form of straight line:

The equations $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ together represents a line in unsymmetrical form. This represent equation of line of intersection of planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$.

Procedure to convert Unsymmetrical Form of straight line to Symmetrical Form :

Let the direction ratios of the line of intersection (AB) of two planes

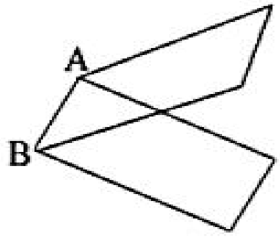
$a_1x + b_1y + c_1z + d_1 = 0$ .....(1) and $a_2x + b_2y + c_2z + d_2 = 0$ .....(2) are a, b, c

Direction ratios of normal to plane (1) are a_1, b_1, c_1 and

Direction ratios of normal to plane (2) are a_2, b_2, c_2

Line AB lies in both the planes (1) and (2)

hence normals to (1) and (2) are perpendicular to AB.



Hence $aa_1 + bb_1 + cc_1 = 0$ and $aa_2 + bb_2 + cc_2 = 0$

these two will give the proportional values of a, b, c.

Let the line AB cuts the xy plane at $(x_1, y_1, 0)$

Hence $a_1x_1 + b_1y_1 = -d_1$ and $a_2x_1 + b_2y_1 = -d_2$ This will give a point on the line AB

$\therefore$ equation of AB is $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-0}{c}$.

Illustration :

Find the angle between the line $x - 2y + z = 0 = x + 2y - 2z$ and $x + 2y + z = 0 = 3x + 9y + 5z$.

Sol. *Let a_1, b_1, c_1 be the direction ratios of the line $x - 2y + z = 0$ and $x + 2y - 2z = 0$. Since it lies in both the planes, therefore, it is $\perp$ to the normals to the two planes.*

$$\therefore \quad a_1 - 2b_1 + c_1 = 0$$
$$a_1 + 2b_1 - 2c_1 = 0$$

Solving these two equations by cross-multiplication, we have

$$\frac{a_1}{4-2} = \frac{b_1}{1+2} = \frac{c_1}{2+2} \quad or \quad \frac{a_1}{2} = \frac{b_1}{3} = \frac{c_1}{4}$$

Let a_2, b_2, c_2 be the direction ratios of the line $x + 2y + z = 0 = 3x + 9y + 5z$. Then the discussed above

$$a_2 + 2b_2 + c_2 = 0 \qquad 3a_2 + 9b_2 + 5c_2 = 0$$

$$\Rightarrow \quad \frac{a_2}{10-9} = \frac{b_2}{3-5} = \frac{c_2}{9-6} \quad or \quad \frac{a_2}{1} = \frac{b_2}{-2} = \frac{c_2}{3}$$

Let θ be the angle between the given lines. Then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$= \frac{(1)(2)+(-2)(3)+(3)(4)}{\sqrt{2^2+3^2+4^2}\sqrt{1^2+(-2)^2+(3)^2}}$$

$$= \frac{2-6+12}{\sqrt{29}\sqrt{14}} = \frac{8}{\sqrt{406}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{8}{\sqrt{406}}\right)$$

Illustration :

Find the coordinates of the point where the line joining the points (2, – 3, 1) and (3, – 4, – 5) cuts the plane 2x + y + z = 7.

Sol. The direction ratios of the line are $3 - 2, -4 - (-3), -5 - 1$ i.e. $1, -1, -6$

Hence equation of the line joining the given points is $\frac{x-2}{1} = \frac{y+3}{-1} = \frac{z-1}{-6} = r$ (say)

Coordinates of any point on this line are $(r + 2, -r - 3, -6r + 1)$

If this point lies on the given plane $2x + y + z = 7$, then

$$2(r + 2) + (-r - 3) + (-6r + 1) = 7 \Rightarrow r = -1$$

Coordinates of the point are $(-1 + 2, -(-1) - 3, -6(-1) + 1)$ i.e. $(1, -2, 7)$.

Illustration :

Find in symmetrical form the equations of the line 3x + 2y – z – 4 = 0 = 4x + y – 2z + 3.

Sol. The equation of the line in unsymmetrical form are $3x + 2y - z - 4 = 0$, $4x + y - 2z + 3 = 0$ .....(1)

Let l, m, n be the direction cosines of the line. Since the line is common to both the planes, it is perpendicular to the normals to both the planes. Hence, $3l + 2m - n = 0$, $4l + m - 2n = 0$

Solving these we get, $\frac{l}{-4+1} = \frac{m}{-4+6} = \frac{n}{3-8}$

i.e. $\frac{l}{-3} = \frac{m}{2} = \frac{n}{-5} = \frac{1}{\sqrt{(-3)^2 + 2^2 + (-5)^2}} = \frac{1}{\sqrt{38}}$

So, direction cosines of the lines are $\frac{-3}{\sqrt{38}}, \frac{2}{\sqrt{38}}, \frac{-5}{\sqrt{38}}$

Now to find the coordinates of a point on a line. Let us find out the point where it meets the plane $z = 0$. Putting $z = 0$ in the equation given by (1), we have $3x + 2y - 4 = 0$, $4x + y + 3 = 0$.

Solving these, we get $x = -2$, $y = 5$

So, one point of the line is $(-2, 5, 0)$

$\therefore$ Equation of the line in symmetrical form is $\frac{x+2}{\frac{-3}{\sqrt{38}}} = \frac{y-5}{\frac{2}{\sqrt{38}}} = \frac{z-0}{\frac{-5}{\sqrt{38}}}$ i.e. $\frac{x+2}{-3} = \frac{y-5}{2} = \frac{z}{-5}$.

Illustration :

Find the equation of the plane which contains the two parallel lines $\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1}$ *and* $\frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}$.

Sol. The equations of the two parallel lines are

$$\frac{x+1}{3} = \frac{y-2}{2} = \frac{z-0}{1} \qquad ....(1)$$

and

$$\frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1} \qquad ....(2)$$

the equation of any plane through the line (1) is

$a(x + 1) + b(y - 2) + cz = 0$ ....(3)

where $3a + 2b + c = 0$ ....(4)

the line (2) will also lie on the plane (3) if the point $(3, -4, 1)$ lies on the plane (3), and for this we have $a(3 + 1) + b(-4 - 2) + c = 0$ or $4a - 6b + c = 0$ ....(5)

Solving (4) and (5), we get $\frac{a}{8} = \frac{b}{1} = \frac{c}{-26}$

Putting the values of a, b, c in (3), the required equation of the plane is $8x + y - 26z + 6 = 0$.

Illustration :

Find the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$, $\frac{x+3}{3} = \frac{y+7}{2} = \frac{z-6}{4}$.

Also find the equation of line of shortest distance.

Sol. Given lines are $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = r_1$ (say) ....(1)

$$\frac{x+3}{3} = \frac{y+7}{2} = \frac{z-6}{4} = r_2 \quad (say) \qquad ....(2)$$

any point on line (1) is $P(3r_1 + 3, 8 - r_1, r_1 + 3)$ and on line (2) is

$Q(-3 - 3r_2, 2r_2 - 7, 4r_2 + 6)$.

If PQ is line of shortest distance, then direction ratios of

$PQ = (3r_1 + 3) - (-3 - 3r_2), (8 - r_1) - (2r_2 - 7), (r_1 + 3) - (4r_2 + 6)$

i.e. $3r_1 + 3r_2 + 6, -r_1 - 3r_2 + 15, r_1 - 4r_2 - 3$

As PQ is perpendicular to lines (1) and (2)

$\therefore$ $3(3r_1 + 3r_2 + 6) - 1(-r_1 - 2r_2 + 15) + 1(r_1 - 4r_2 + 3) = 0$

$\Rightarrow$ $11r_1 + 7r_2 = 0$ ....(3)

and $-3(3r_1 + 3r_2 + 6) + 2(-r_1 - 2r_2 + 15) + 4(r_1 - 4r_2 + 3) = 0$

i.e. $7r_1 + 11r_2 = 0$ ....(4)

On solving equations (3) and (4), we get $r_1 = r_2 = 0$.

So, point $P(3, 8, 3)$ and $Q(-3, -7, 6)$

$\therefore$ Length of shortest distance $PQ = \sqrt{\{(-3-3)^2 + (-7-8)^2 + (6-3)^2\}} = 3\sqrt{30}$.

Direction ratios of shortest distance line is $2, 5, -1$.

$\therefore$ Equation of shortest distance line $\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$.

Illustration :

Find the distance of the point (1, – 2, 3) from the plane x – y + z = 5 measured parallel to the line

$$\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}.$$

Sol. *Here we are not to find perpendicular distance of the point from the plane but distance measured along with the given line. The method is as follow :*

The equation of the line through the point (1, – 2, 3) and parallel to given line is

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}=r\ (say)$$

The coordinate of any point on it is (2r + 1, 3r – 2, –6r + 3).

If this point lies in the given plane then

$$2r+1-(3r-2)+(-6r+3)=5 \Rightarrow -7r=-1 \text{ or } r=\frac{1}{7}$$

$$\therefore \text{ Point of intersection is } \left(\frac{9}{7},\frac{-11}{7},\frac{15}{7}\right).$$

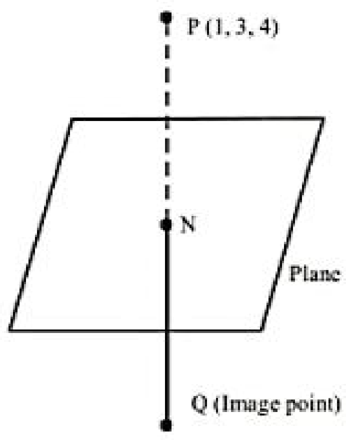
$$\therefore \text{ The required distance = the distance between the points } (1,-2,3) \text{ and } \left(\frac{9}{7},\frac{-11}{7},\frac{15}{7}\right)$$

$$=\sqrt{\left(1-\frac{9}{7}\right)^2+\left(-2+\frac{11}{7}\right)^2+\left(3-\frac{15}{7}\right)^2}=\frac{\sqrt{49}}{7}=1 \text{ Unit.}$$

Illustration :

Find the image of the point (1, 3, 4) in the plane 2x – y + z + 3 = 0.

Sol. *As it is clear from the figure that PQ will be perpendicular to the plane and foot of this perpendicular is mid point of PQ i.e. N.*



So, direction ratios of line PQ is 2, – 1, 1

$$\Rightarrow \text{Equation of line } PQ=\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=r\ (say)$$

Any point on line PQ is (2r + 1, – r + 3, r + 4)

If this point lies on the plane, then

$$2(2r+1)-(-r+3)+(r+4)+3=0$$

$$\Rightarrow \quad r=-1$$

∴ Coordinate of foot of perpendicular N = (–1, 4, 3).

As N is middle point of PQ.

$$\therefore\ -1=\frac{1-x_1}{2},\ 4=\frac{3+y_1}{2},\ 3=\frac{4+z_1}{2} \Rightarrow x_1=-3,\ y_1=5,\ z_1=2$$

∴ Image of point P (1, 3, 4) is the point Q (– 3, 5, 2).

Illustration :

Determine whether each statement is true or false

(a) Two lines parallel to a third line are parallel.
(b) Two lines perpendicular to a third line are parallel.
(c) Two planes parallel to a third plane are parallel.
(d) Two planes perpendicular to a third plane are parallel.
(e) Two lines parallel to a plane are parallel.
(f) Two lines perpendicular to a plane are parallel.
(g) Two planes parallel to a line are parallel.
(h) Two planes perpendicular to a line are parallel.
(i) Two planes either intersect or are parallel.
(j) Two lines either intersect or are parallel.
(k) A plane and a line either intersect or are parallel.

Ans. (a) True, (b) False, (c) True, (d) False, (e) False, (f) True,
(g) False, (h) True, (i) True, (j) False, (k) True

Illustration :

Find the equation of a plane passing through the point A(3, –2, 1) and perpendicular to the vector $4\vec{i}+7\vec{j}-4\vec{k}$. *If PM be perpendicular from the point P(1, 2, –1) to this plane, find its length.*

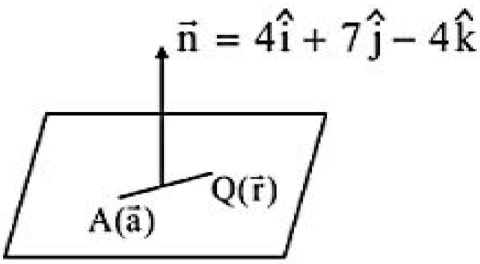
Sol. Let O be the origin, then $\vec{a}=\overrightarrow{OA}=3\vec{i}-2\vec{j}+\vec{k}$

Let $\vec{n}=4\vec{i}+7\vec{j}-4\vec{k}$

Let Q(x, y, z) be any point on the plane

then, $\vec{r}=\overrightarrow{OQ}=x\vec{i}+y\vec{j}+z\vec{k}$

Now $\vec{r}-\vec{a}=(x-3)\vec{i}+(y+2)\vec{j}+(z-1)\vec{k}$



Now, equation of the required plane passing point $A(\vec{a})$

and perpendicular to $\vec{n}$ is

$$(\vec{r}-\vec{a})\cdot\vec{n}=0$$

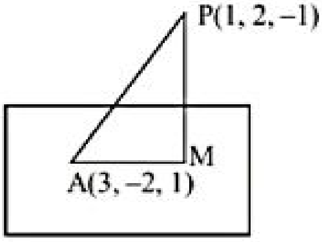
$\therefore$ $4(x-3)+7(y+2)-4(z-1)=0$

or, $4x-3y-4z+6=0$. This is the required equation of the plane.

Now $PM \perp AM$

$\therefore$ Unit vector parallel to $\overrightarrow{PM}$

$$\vec{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{4\vec{i} + 7\vec{j} - 4\vec{k}}{9}$$



Now PM = *length of projection of* $\overline{PA}$ *on* $\overline{PM}$

$$= \left|\overline{PA}\cdot\hat{n}\right| = \left|\left(2\vec{i} - 4\vec{j} + 2\vec{k}\right)\cdot\left(\frac{4\vec{i} + 7\vec{j} - 4\vec{k}}{9}\right)\right|$$

Let $\quad \overline{\alpha} = \overline{OP} = \overline{i} + 2\overline{j} - \overline{k}$

Second method :

Alternatively, $PM = \left|\dfrac{(\vec{\alpha} - \vec{a})\cdot\vec{n}}{n}\right|$

Here, $\vec{\alpha} - \vec{a} = \overrightarrow{AP} = -2\vec{i} + 4\vec{j} - 2\vec{k}$ *and* $\vec{n} - 4\vec{i} + 7\vec{j} - 4\vec{k}$

$$PM = \left|\frac{-8 + 28 + 8}{9}\right| = \frac{28}{9}$$

Illustration :

Find the shortest distance and the vector equation of the line of shortest distance between the lines given by $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$ *and* $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(3\hat{i} + 2\hat{j} + 4\hat{k})$

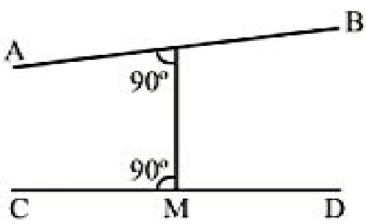
Sol. *Given lines are* $\quad \vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k}) \quad$ ...(i)

and $\quad \vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(3\hat{i} + 2\hat{j} + 4\hat{k}) \quad$ ...(ii)

Equations of lines (i) and (ii) in cartesian from are

$$AB : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda \qquad \text{...(iii)}$$

and $$CD : \frac{x-3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu \qquad \text{...(iv)}$$



Let $\quad L \equiv (3\lambda + 3, -\lambda + 8, \lambda + 3)$

and $\quad M \equiv (-3\mu - 3, 2\mu - 7, 4\mu + 6)$

Direction ratios of LM are

$$3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3$$

Since $LM \perp AB$

$\therefore \quad 3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$

or $\quad 11\lambda + 7\mu = 0 \qquad \ldots(v)$

Again $LM \perp CD$

$\therefore \quad -3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$

or $\quad -7\lambda - 29\mu = 0 \qquad \ldots(vi)$

Solving (v) and (vi), we get $\lambda = 0, \mu = 0$

$\therefore \quad L \equiv (3, 8, 3), \; M \equiv (-3, -7, 6)$

Hence shortest distance $LM = \sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = \sqrt{270} = 3\sqrt{30}$ *units*

Vector equation of LM is

$$\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + t(6\hat{i} + 15\hat{j} - 3\hat{k})$$

Note : *Cartesian equation of LM is* $\frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}$

Practice Problem

Q.1 The shortest distance between the two straight lines $\frac{x-4/3}{2} = \frac{y+6/5}{3} = \frac{z-3/2}{4}$ and $\frac{5y+6}{8} = \frac{2z-3}{9} = \frac{3x-4}{5}$ is

(A) $\sqrt{29}$ (B) 3 (C) 0 (D) $6\sqrt{10}$

Q.2 A straight line passes through the point $(2, -1, -1)$. It is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z-5}{1}$. The equations of the straight line are

(A) $\frac{x-2}{4} = \frac{y+1}{1} = \frac{z+1}{1}$ (B) $\frac{x+2}{4} = \frac{y-1}{1} = \frac{z-1}{1}$

(C) $\frac{x-2}{-1} = \frac{y+1}{1} = \frac{z+1}{3}$ (D) $\frac{x+2}{-1} = \frac{y-1}{1} = \frac{z-1}{3}$

Q.3 The cosine of angle between any two diagonal of a cube is -

(A) 1/3 (B) 1/2 (C) 2/3 (D) $1/\sqrt{3}$

Q.4 The equation of the plane containing the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and the point (0, 7, – 7) is-

(A) $x+y+z=2$ (B) $x+y+z=3$ (C) $x+y+z=0$ (D) None of these

Q.5 The distance of the point (1, – 2, 3) from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z-1}{-6}$ is-

(A) 1 (B) 2 (C) 4 (D) None of these

Q.6 The equation of the plane through the point (2, –1, –3) and parallel to the lines $\frac{x-1}{3}=\frac{y+2}{2}=\frac{z}{-4}$ and $\frac{x}{2}=\frac{y-1}{-3}=\frac{z-2}{2}$ is

Answer key

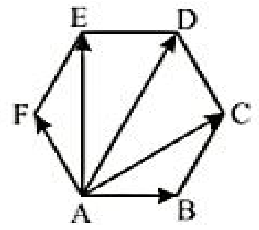
Q.1	C	Q.2	C	Q.3	A	Q.4	C	Q.5	A
Q.6	0								

Solved Examples

Q.1 If ABCDEF is a regular hexagon and $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = k\,\overrightarrow{AD}$, then k equals-

(A) 2 (B) 3 (C) 6 (D) 5

Sol. $\because \overrightarrow{AB} = \overrightarrow{ED}$ and $\overrightarrow{AF} = \overrightarrow{CD}$,



so $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$

$= \overrightarrow{ED} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{CD}$

$= (\overrightarrow{AC} + \overrightarrow{CD}) + (\overrightarrow{AE} + \overrightarrow{ED}) + \overrightarrow{AD}$

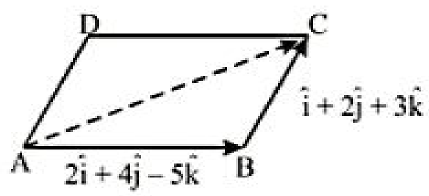
$= \overrightarrow{AD} + \overrightarrow{AD} + \overrightarrow{AD} = 3\,\overrightarrow{AD}$

$\therefore$ k = 3 Ans.[B]

Q.2 The length of diagonal AC of a parallellogram ABCD whose two adjacent sides AB and AD are represented respectively by $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$ is-

(A) 3 (B) 4 (C) 5 (D) 7

Sol. $\because \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$



$= 3\hat{i} + 6\hat{j} - 2\hat{k}$

$\therefore$ Length of the diagonal $\overrightarrow{AC} = |\overrightarrow{AC}|$

$= \sqrt{3^2 + 6^2 + (-2)^2} = 7$ Ans.[D]

Q.3 If the middle points of sides BC, CA & AB of triangle ABC are respectively D,E,F then position vector of centre of triangle DEF, when position vector of A,B, C are respectively $\hat{i} + \hat{j}$, $\hat{j} + \hat{k}$, $\hat{k} + \hat{i}$ is-

(A) $\frac{1}{3}(\hat{i} + \hat{j} + \hat{k})$ (B) $(\hat{i} + \hat{j} + \hat{k})$ (C) $2(\hat{i} + \hat{j} + \hat{k})$ (D) $\frac{2}{3}(\hat{i} + \hat{j} + \hat{k})$

Sol. The position vector of points D,E,F are respectively

$\frac{\hat{i} + \hat{j}}{2} + \hat{k}$, $\hat{i} + \frac{\hat{k} + \hat{j}}{2}$ and $\frac{\hat{i} + \hat{k}}{2} + \hat{j}$

So, position vector of centre of ΔDEF

$= \frac{1}{3}\left[\frac{\hat{i} + \hat{j}}{2} + \hat{k} + \hat{i} + \frac{\hat{k} + \hat{j}}{2} + \frac{\hat{i} + \hat{k}}{2} + \hat{j}\right] = \frac{2}{3}\,[\hat{i} + \hat{j} + \hat{k}]$ Ans.[D]

Q.4 Let position vectors of points A,B,C and D are respectively $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$. If the points are coplanar, then the value of λ is-

(A) $-\frac{146}{17}$ (B) $\frac{146}{17}$ (C) 0 (D) None of these

Sol. $\overrightarrow{AB} = -\hat{i}+5\hat{j}-3\hat{k}$

$\overrightarrow{AC} = -4\hat{i}+3\hat{j}+3\hat{k}$ and $\overrightarrow{AD} = \hat{i}+7\hat{j}+(\lambda+1)\hat{k}$

If A,B,C,D are coplanar, then vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, then

$$\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=0 \quad \text{or} \quad \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix}=0$$

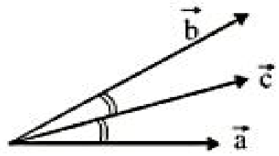
$\Rightarrow \lambda = \frac{-146}{17}$ Ans.[A]

Q.6 The vector $\vec{c}$, directed along the internal bisector of the angle between the vectors $7\hat{i}-4\hat{j}-4\hat{k}$ and $-2\hat{i}-\hat{j}+2\hat{k}$ with $|\vec{c}|=5\sqrt{6}$ is-

(A) $\frac{5}{3}\hat{i}-7\hat{j}+2\hat{k}$ (B) $\frac{5}{3}(5\hat{i}+5\hat{j}+2\hat{k})$ (C) $\frac{5}{3}(\hat{i}+7\hat{j}+2\hat{k})$ (D) None of these

Sol. Let $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$ and $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$

$$\vec{c}=\lambda(\hat{a}+\hat{b})=\lambda\left(\frac{7\hat{i}-4\hat{j}-4\hat{k}}{9}+\frac{-2\hat{i}-\hat{j}+2\hat{k}}{3}\right)=\lambda\left(\frac{\hat{i}-7\hat{j}+2\hat{k}}{9}\right)$$



$|\vec{c}| = 5\sqrt{6} \Rightarrow \lambda = \pm 15 \Rightarrow c = \pm\frac{5}{3}\left(\hat{i}-7\hat{j}+2\hat{k}\right)$ Ans.[A]

Q.7 If moduli of vectors $\vec{a},\vec{b},\vec{c}$ are 3,4 and 5 respectively and $\vec{a}$ and $\vec{b}+\vec{c}$, $\vec{b}$ and $\vec{c}+\vec{a}$, $\vec{c}$ and $\vec{a}+\vec{b}$ are perpendicular to each other, then modulus of $\vec{a}+\vec{b}+\vec{c}$ is -

(A) $5\sqrt{2}$ (B) $2\sqrt{5}$ (C) 50 (D) 20

Sol. $\because \vec{a} \perp (\vec{b}+\vec{c}) \Rightarrow \vec{a}.\vec{b}+\vec{a}.\vec{c}=0$

Similarly $\vec{b}\perp(\vec{c}+\vec{a}) \Rightarrow \vec{b}.\vec{c}+\vec{b}.\vec{a}=0$ and $\vec{c}\perp(\vec{a}+\vec{b}) \Rightarrow \vec{c}.\vec{a}+\vec{c}.\vec{b}=0$

$\therefore \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=0$

Now $|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 9 + 16 + 25 = 50$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$ Ans.[A]

Q.8 If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ then angle between a and b is -

(A) 60° (B) 30° (C) 90° (D) 180°

Sol. $|a+b| = |\vec{a}-\vec{b}|$

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$

$\Rightarrow \quad 4\,\vec{a}.\vec{b} = 0$

$\Rightarrow \quad \vec{a}.\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$ Ans.[C]

Q.9 If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c} = 0$, then-

(A) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c}$ (B) $\vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

(C) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ (D) None of these

Sol. $\because \vec{a}+\vec{b}+\vec{c} = 0 \Rightarrow \vec{c} = -(\vec{a}+\vec{b})$

$\therefore \vec{b}\times\vec{c} = -\vec{b}\times(\vec{a}+\vec{b}) = -\vec{b}\times\vec{a} - \vec{b}\times\vec{b} = \vec{a}\times\vec{b}$

Similarly $\vec{c}\times\vec{a} = \vec{a}\times\vec{b}$

$\therefore \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ Ans.[C]

Q.10 If $\ell\hat{i} + m\hat{j} + n\hat{k}$ is a unit vector which is perpendicular to vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$ then $|\ell|$ is equal to-

(A) $-\dfrac{3}{\sqrt{155}}$ (B) $\sqrt{\dfrac{3}{155}}$ (C) $\dfrac{3}{\sqrt{155}}$ (D) None of these

Sol. Vector $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$

$$= \frac{(2\hat{i}-\hat{j}+\hat{k})\times(3\hat{i}+4\hat{j}-\hat{k})}{|(2\hat{i}-\hat{j}+\hat{k})\times(3\hat{i}+4\hat{j}-\hat{k})|} = \frac{\hat{i}(1-4)-\hat{j}(-2-3)+\hat{k}(8+3)}{\sqrt{9+25+121}} = \frac{-3\hat{i}+5\hat{j}+11\hat{k}}{\sqrt{155}}$$

$\therefore |\ell| = \left|\dfrac{-3}{\sqrt{155}}\right| = \dfrac{3}{\sqrt{155}}$ Ans.[C]

Q.11 The unit vector perpendicular to the plane passing through points $P(\hat{i}-\hat{j}+2\hat{k})$, $Q(2\hat{i}-\hat{k})$ and $R(2\hat{j}+\hat{k})$ is-

(A) $2\hat{i}+\hat{j}+\hat{k}$ (B) $\sqrt{6}(2\hat{i}+\hat{j}+\hat{k})$ (C) $\frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k})$ (D) $\frac{1}{6}(2\hat{i}+\hat{j}+\hat{k})$

Sol. $\overrightarrow{PQ}=(2\hat{i}-\hat{k})-(\hat{i}-\hat{j}+2\hat{k}) = \hat{i}+\hat{j}-3\hat{k}$

$\overrightarrow{PR}=(2\hat{i}+\hat{k})-(\hat{i}-\hat{j}+2\hat{k}) = -\hat{i}+3\hat{j}-\hat{k}$

Now $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix} = 8\hat{i}+4\hat{j}+4\hat{k} \Rightarrow \left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \sqrt{64+16+16} = 4\sqrt{6}$

$\therefore$ reqd. unit vector $=\dfrac{4(2\hat{i}+\hat{j}+\hat{k})}{4\sqrt{6}} = \dfrac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k})$ Ans.[C]

Q.12 The area of parallelogram whose diagonals are respectively $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$ is-

(A) $5\sqrt{2}$ (B) $5\sqrt{3}$ (C) $2\sqrt{5}$ (D) $3\sqrt{5}$

Sol. Area of parallelogram $=\frac{1}{2}|\vec{a}\times\vec{b}|$

where $\vec{a} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = \hat{i}-3\hat{j}+4\hat{k}$

now $\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -2\\ 1 & -3 & 4\end{vmatrix} = -2\hat{i}-14\hat{j}-10\hat{k}$

$\therefore$ Area of parallelogram

$= \frac{1}{2}|-2\hat{i}-14\hat{j}-10\hat{k}| = \sqrt{1+49+25} = 5\sqrt{3}$ Ans.[B]

Q.13 If $\hat{i}-\hat{j}+2\hat{k}$, $2\hat{i}+\hat{j}-\hat{k}$ and $3\hat{i}-\hat{j}+2\hat{k}$ are position vectors of vertices of a triangle, then its area is-

(A) 26 (B) 13 (C) $2\sqrt{13}$ (D) $\sqrt{13}$

Sol. If A,B,C are given vertices, then

$\overrightarrow{AB} = \hat{i}+2\hat{j}-3\hat{k}$, $\overrightarrow{AC} = 2\hat{i}$

$\therefore\ \overrightarrow{AB}\times\overrightarrow{AC} = (\hat{i}+2\hat{j}-3\hat{k})\times 2\hat{i} = -4\hat{k}-6\hat{j} \Rightarrow \left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{16+36} = 2\sqrt{13}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{13}$ Ans.[D]

Q.14 If A,B,C,D are any four points, then $|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$ equals-

(A) Area of ΔABC (B) 2(Area of ΔABC)

(C) 3(Area of ΔABC) (D) 4 (Area of ΔABC)

Sol. Let a, b, c and d be position vectors of points A,B,C and D respectively, then

$\overrightarrow{AB}\times\overrightarrow{CD} = (\vec{b}-\vec{a})\times(\vec{d}-\vec{c}) = \vec{b}\times\vec{d}-\vec{b}\times\vec{c}-\vec{a}\times\vec{d}+\vec{a}\times\vec{c}$

Similarly

$\overrightarrow{BC}\times\overrightarrow{AD} = \vec{c}\times\vec{d}-\vec{c}\times\vec{a}-\vec{b}\times\vec{d}+\vec{b}\times\vec{a}$

$\overrightarrow{CA}\times\overrightarrow{BD} = \vec{a}\times\vec{d}-\vec{a}\times\vec{b}-\vec{c}\times\vec{d}+\vec{c}\times\vec{b}$

Therefore given expression

$= |2(\vec{b}\times\vec{a}-\vec{b}\times\vec{c}+\vec{a}\times\vec{c})| = 2\,|(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})|$

$= 4$ (Area of ΔABC) Ans.[D]

Q. 15 a,b,c,d are the position vectors of four coplanar points A, B, C and D respectively. If $(\vec{a}-\vec{d}).(\vec{b}-\vec{c}) = 0 = (\vec{b}-\vec{d}).(\vec{c}-\vec{a})$, then for the ΔABC,D is-

(A) incentre (B) orthocentre (C) circumcentre (D) centroid

Sol. $(\vec{b}-\vec{d}).(\vec{c}-\vec{a}) = 0 \Rightarrow (\vec{a}-\vec{d}) \perp (\vec{b}-\vec{c}) \Rightarrow \overrightarrow{AD} \perp \overrightarrow{BC}$

Similarly $(\vec{b}-\vec{d}).(\vec{c}-\vec{a}) = 0 \Rightarrow \overrightarrow{BD} \perp \overrightarrow{AC}$

$\therefore$ D is the orthocentre of ΔABC. Ans.[B]

Q.16 For any vector $\vec{a}$, $\vec{u} = \hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$ equals-

(A) $2\vec{a}$ (B) $-2\vec{a}$ (C) $\vec{a}$ (D) $-\vec{a}$

Sol. $\vec{u} = (\hat{i}.\hat{i})\,\vec{a}-(\hat{i}.\vec{a})\,\hat{i}+(\hat{j}.\hat{j})\,\vec{a}-(\hat{j}.\vec{a})\,\hat{j}+(\hat{k}.\hat{k})\,\vec{a}-(\hat{k}.\vec{a})\,\hat{k}$

$= \vec{a}-a_1\hat{i}+\vec{a}-a_2\hat{j}+\vec{a}-a_3\hat{k}$ $[\vec{a} = a_1 i + a_2\hat{j} + a_3\hat{k}\ (\text{say})]$

$\therefore\ u = 3\vec{a}-\vec{a} = 2\vec{a}$ Ans.[A]

Q.17 Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} = 2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on a is $\sqrt{2/3}$ will be-

(A) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (B) $2\hat{i} + 3\hat{j} + 3\hat{k}$ (C) $-2\hat{i} - \hat{j} + 5\hat{k}$ (D) $2\hat{i} + \hat{j} + 5\hat{k}$

Sol. Let the required vector $\vec{r} = \vec{b} + t\vec{c}$

$\Rightarrow r = (1 + t)\,\hat{i} + (2 + t)\,\hat{j} - (1 + 2t)\,\hat{k}$

Also projection of $\vec{r}$ on $\vec{a} = \sqrt{2/3}$

$$\Rightarrow \frac{\vec{r}.\vec{a}}{|\vec{a}|} = \sqrt{2/3} \Rightarrow \frac{2(1+t)-(2+t)-(1+2t)}{\sqrt{6}} = \sqrt{\frac{2}{3}} \Rightarrow -t - 1 = 2 \Rightarrow t = -3$$

$\therefore \vec{r} = -2\hat{i} - \hat{j} + 5\hat{k}$ Ans.[C]