COMPLEX NUMBER

1. INTRODUCTION :

Indian mathematician Mahavira (850 A.D.) was first to mention in this work 'Ganitasara Sangraha' As in nature of things a negative (quantity) is not a square (quantity), it has, therefore, no square root'. Hence there is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$.

A symbol $\sqrt{-1}$, denoted by letter i was introduced by Swiss Mathematician, Leonhard Euler (1707-1783) in 1748 to provide solutions of equation $x^2 + 1 = 0$. i was regarded as a fictitious or imaginary number which could be manipulated algebrically like an ordinary real number, except that its square was -1. The letter i was used to denote $\sqrt{-1}$, possibly because i is the first letter of the Latin word 'imaginarius'.

To permit solutions of such polynomial equations, the set of complex numbers is introduced. We can consider a complex number as having the form $a + ib$ where a and b are real number.
It is denoted by z i.e. $z = a + ib$. 'a' is called as real part of z which is denoted by Re(z) and 'b' is called as imaginary part of z which is denoted by Im (z).

1.1 Classification of complex number :

In fact every complex can be classified as

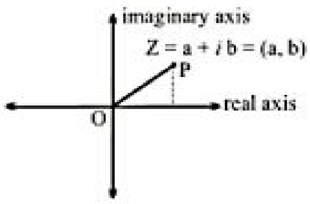
$z = a + ib$

purely real	purely imaginary	imaginary
if $b = 0$	if $a = 0$	if $b \neq 0$

Hence, $0 + 0i$ is both a purely real as well as a purely imaginary but not imaginary.

1.2 Geometrical representation of a complex number :

Master Argand had done a systematic studies on complex numbers and represented every complex number as a set of ordered pair (a, b) on a plane called complex plane / argand plane.



All complex numbers lying on the real axis were called as purely real and those lying on imaginary axis as purely imaginary.

1.3 Integral Powers of i :

We have $i = \sqrt{-1}$ so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$
or $i^{4n+1} = i$, $i^{4n+2} = -1$ for any $n \in I$,
$i^{4n+3} = -i$, $i^{4n} = 1$
Thus any integral power of i can be expressed as ± 1 or $\pm i$.

Illustration :

Find the value of $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}+2.$

Sol. $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}+2=\dfrac{i^{584}}{i^{574}}+2$

$= i^{10} + 2 = -1 + 2 = 1$

Note :

(a) The set R of real number is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) Zero is purely real as well as purely imaginary but not imaginary.

(c) $\sqrt{a}\sqrt{b}=\sqrt{a\,b}$ only if atleast one of a or b is non-negative.

(d) $z_1^2+z_2^2=0 \not\Rightarrow z_1=0=z_2$ i.e., $z_1 = 1+i$ and $z_2 = 1-i$

2. ALGEBRA OF COMPLEX NUMBER :

2.1 Equality of complex number :

Let there be two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

If $z_1 = z_2$ then $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

i.e., if $x_1 + iy_1 = x_2 + iy_2$

$\Rightarrow$ $x_1 = x_2$ and $y_1 = y_2$ simultaneously.

Illustration :

Find the real values of x and y for which the equation $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2yi)$ *is satisfied.*

Sol. *Given equation* $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2yi)$

$\Rightarrow$ $(x^4 - 3x^2) + i\,(2x - 3y) = 4 - 5i$

Equating real and imaginary parts, we get

$x^4 - 3x^2 = 4$ ... *(i)*

and $2x - 3y = -5$ ...*(ii)*

From (i) and (ii), we get $x = \pm 2$ *and* $y = 3, \dfrac{1}{3}$

Note : Inequality in complex numbers are never talked. If $x_1 + iy_1 > x_2 + iy_2$ has to be meaningful $\Rightarrow y_1 = y_2 = 0$. Equalities however in complex numbers are meaningful. Two complex numbers z_1 and z_2 are said to be equal if

Re z_1 = Re z_2 and Im (z_1) = Im (z_2) (i.e. they occupy the same position on complex plane)

2.2 Addition :

$z_1 + z_2 = (x_1 + i\,y_1) + (x_2 + i\,y_2) = (x_1 + x_2) + i\,(y_1 + y_2) \in C.$

It is easy to oberve that the sum of two complex numbers is a complex number whose real (imaginary) part is the sum of the real (imaginary) parts of the given numbers :

$$\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2);$$
$$\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$$

2.3 Subtraction :

$z_1 - z_2 = (x_1 + i\,y_1) - (x_2 + i\,y_2) = (x_1 - x_2) + i\,(y_1 - y_2) \in C.$

That is

$$\mathrm{Re}(z_1 - z_2) = \mathrm{Re}(z_1) - \mathrm{Re}(z_2);$$
$$\mathrm{Im}(z_1 - z_2) = \mathrm{Im}(z_1) - \mathrm{Im}(z_2).$$

2.4 Multiplication :

$z_1 \cdot z_2 = (x_1 + i\,y_1)\,(x_2 + i\,y_2) = (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1) \in C.$

In other words

$$\mathrm{Re}(z_1z_2) = \mathrm{Re}(z_1)\cdot\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\cdot\mathrm{Im}(z_2)$$

and $\quad \mathrm{Im}(z_1z_2) = \mathrm{Im}(z_1)\cdot\mathrm{Re}(z_2) + \mathrm{Im}(z_2)\cdot\mathrm{Re}(z_1)$

For a real number λ and a complex number $z = x + i\,y$.

$$\lambda \cdot z = \lambda\,(x + iy) = \lambda x + i\,\lambda y \in C$$

is the product of a real number with a complex number. The following properties are obvious :

(a) $\lambda\,(z_1 + z_2) = \lambda z_1 + \lambda z_2$

(b) $\lambda_1\,(\lambda_2 z) = (\lambda_1\lambda_2)z;$

(c) $(\lambda_1 + \lambda_2)z = \lambda_1 z + \lambda_2 z$ for all $z, z_1, z_2 \in C$ and $\lambda, \lambda_1, \lambda_2 \in R$.

Actually, relations (a) and (c) are special cases of the distributive law and relation (b) comes from the associative law of multiplication for complex numbers.

2.5 Division of Complex Number :

Let $\quad z_1 = x_1 + iy_1 \quad \& \quad z_2 = x_2 + iy_2$

Then $\quad \dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} \Rightarrow \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$

$$\Rightarrow \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{(x_2^2 + y_2^2)} \Rightarrow \left\{\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right\} + i\left\{\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right\}$$

$$\frac{z_1}{z_2} \Rightarrow \mathrm{Re}\left(\frac{z_1}{z_2}\right) + i\,\mathrm{Im}\left(\frac{z_1}{z_2}\right)$$

2.6 Square Root of Complex Number :

Let $z = x + iy$ be the given complex number and we have to obtain its square root.

Let $a + ib = (x + iy)^{1/2} \Rightarrow a^2 - b^2 + 2iab = x + iy$

$\Rightarrow x = a^2 - b^2$ and $y = 2ab$

$\Rightarrow x^2 = (a^2 + b^2)^2 - 4a^2b^2 \Rightarrow x^2 + y^2 = (a^2 + b^2)^2$

$\Rightarrow a^2 + b^2 = |z|$ ....(1) $\Rightarrow a^2 - b^2 = x$ ....(2)

$$\Rightarrow a^2 = \frac{|z|+x}{2} \Rightarrow a = \pm\sqrt{\frac{|z|+x}{2}} \qquad \Rightarrow b^2 = \frac{|z|-x}{2} \Rightarrow b = \pm\sqrt{\frac{|z|-x}{2}}$$

$$\therefore \sqrt{x+iy} = a + ib = \pm\left(\sqrt{\frac{|z|+\mathrm{Re}(z)}{2}} + i\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}}\right)$$

Replacing i by $-i$, we get

$$\sqrt{x-iy} = \pm\left(\sqrt{\frac{|z|+\mathrm{Re}(z)}{2}} - i\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}}\right)$$

Illustration :

Find the square root of $3 + 4i$

Sol. Let $\sqrt{3+4i} = a + ib \Rightarrow 3 + 4i = a^2 - b^2 + 2ab \Rightarrow a^2 - b^2 = 3,\ 2ab = 4$

$\therefore \quad a^2 + b^2 = \sqrt{(a^2-b^2)^2 + 4a^2b^2} = \sqrt{9+16} = 5$

$\therefore \quad a + i\,b = \pm (2 + i)$

Alternative method :

Using formula $\sqrt{3+4i} = \pm\left(\sqrt{\frac{5+3}{2}} + i\sqrt{\frac{5-3}{2}}\right) = \pm (2 + i)$

Illustration :

If $z = x + iy$, $z^{1/3} = a - ib$ and $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$, then find the value of k.

Sol. $(x + iy)^{1/3} = a - ib$

$\Rightarrow \quad x + iy = (a - ib)^3 = (a^3 - 3ab^2) + i\,(b^3 - 3a^2b)$

$\Rightarrow \quad x = a^3 - 3ab^2,\ y = b^3 - 3a^2b \Rightarrow \frac{x}{a} = a^2 - 3b^2$ and $\frac{y}{b} = b^2 - 3a^2$

$\Rightarrow \quad \frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - b^2 + 3a^2 = 4(a^2 - b^2)$

$\therefore \quad k = 4.$

Illustration :

Find the values of θ *if* $\frac{(3+2i\sin\theta)}{(1-2i\sin\theta)}$ *is purely real or purely imaginary.*

Sol. $z = \frac{3+2i\sin\theta}{1-2i\sin\theta}$

Multiplying numerator and denominator by conjugate,

$$z = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{1+4\sin^2\theta} = \frac{3-4\sin^2\theta+8i\sin\theta}{1+4\sin^2\theta}$$

Now z *is purely real if* $\sin\theta = 0$ *or* $\theta = n\pi,\ n \in I$. z *is purely imaginary if* $3 - 4\sin^2\theta = 0$

$$\Rightarrow \quad \sin\theta = \pm\frac{\sqrt{3}}{2} = \pm\sin\frac{\pi}{3} \quad \Rightarrow \quad \theta = n\pi \pm \frac{\pi}{3},\ n \in I.$$

Illustration :

Show that the polynomial $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$ *is divisible by* $x^3 + x^2 + x + 1$ *where* $p, q, r, s \in N$.

Sol. *Let* $f(x) = x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$

$x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1) = (x + i)(x - i)(x + 1)$

$f(i) = i^{4p} + i^{4q+1} + i^{4r+2} + i^{4s+3} = 1 + i^1 + i^2 + i^3 = 1 + i - 1 - i = 0$

$f(-i) = (-i)^{4p} + (-i)^{4q+1} + (-i)^{4r+2} + (-i)^{4s+3}$

$= 1 + (-i)^1 + (-i)^2 + (-i)^3 = 1 - i - 1 + i = 0$

$f(-1) = (-1)^{4p} + (-1)^{4q+1} + (-1)^{4r+2} + (-1)^{4s+3} = 0$

Thus by division theorem $f(x)$ *is divisible by* $x^3 + x^2 + x + 1$.

Illustration :

If the expression $(1 + ir)^3$ *is of the form of* $s(1 + i)$ *for some real 's' where 'r' is also real and* $i = \sqrt{-1}$, *then the value of 'r' can be*

(A) $\cot\frac{\pi}{8}$ (B) $\sec\pi$ (C) $\tan\frac{\pi}{12}$ (D) $\tan\frac{5\pi}{12}$

Sol. *We have* $(1 + ri)^3 = s(1 + i)$

$1 + 3ri + 3r^2i^2 + r^3i^3 = s(1 + i)$

$1 - 3r^2 + i(3r - r^3) = s + si \quad \Rightarrow \quad 1 - 3r^2 = s = 3r - r^3$

Hence $1 - 3r^2 = 3r - r^3$

$\Rightarrow \quad r^3 - 3r^2 - 3r + 1 = 0 \Rightarrow (r^3 + 1) - 3r(r + 1) = 0 \Rightarrow (r + 1)(r^2 + 1 - r - 3r) = 0$

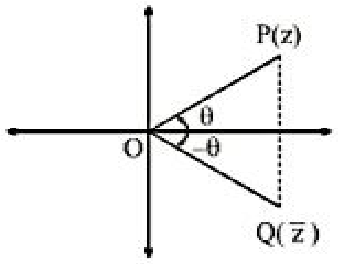
$\therefore \quad r = -1 \quad \text{or} \quad r^2 - 4r + 1 = 0$

$$\Rightarrow \quad r = \frac{4\pm\sqrt{16-4}}{2} = \frac{4\pm2\sqrt{3}}{2} \quad \Rightarrow \quad r = 2+\sqrt{3} \ \text{ or } \ 2-\sqrt{3} \quad \Rightarrow \textbf{B, C, D}$$

3. THREE IMPORTANT TERMS WITH RESPECT TO COMPLEX NUMBER :

3.1 Conjugate of Complex Number :

Conjugate of a complex number $z = a + ib$ is denoted and defined by $\bar{z} = a - ib$.
In a complex number if we replace i by $-i$, we get conjugate of the complex number. $\bar{z}$ is the mirror image of z about real axis on Argand's Plane.



Geometrical representation of conjugate of complex number

Note :

(a) $z + \bar{z} = 2\text{Re } z$

(b) $z - \bar{z} = 2i \text{ Im } z$

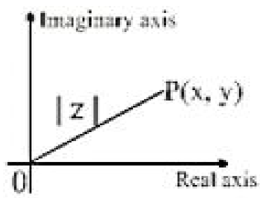
(c) $z\bar{z} = a^2 + b^2$, where $z = a + ib$

(d) If z lies in 1^{st} quadrant then $\bar{z}$ lies in 4^{th} quadrant and $-\bar{z}$ in the 2^{nd} Quad.

(e) If $x + iy = f(a + ib)$ then $x - iy = f(a - ib)$
Further , $g(x + iy) = f(a + ib) \Rightarrow g(x - iy) = f(a - ib)$
e.g. $\sin(\alpha + i\beta) = x + iy \Rightarrow \sin(\alpha - i\beta) = x - iy$

3.2 Modulus of Complex Number :

Modulus of complex number is a distance of the point on the argand plane representing the complex number z from the origin.



If P denotes a complex number $z = x + iy$

then $OP = |z| = \sqrt{x^2 + y^2}$

Note :

(i) $|z| > 0$.

(ii) All complex numbers having the same modulus lie on a circle with centre as origin and radius $r = |z|$.

Illustration :

Find the modulus of the following complex numbers

(i) $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ *(ii)* $\frac{\sqrt{3}+1}{2\sqrt{2}} - i\frac{\sqrt{3}-1}{2\sqrt{2}}$ *(iii)* $1 + \cos\alpha + i\sin\alpha,\ \alpha \in \left(\pi, \frac{3\pi}{2}\right)$

Sol. *(i)* $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$\therefore\ |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

(ii) $|z| \Rightarrow \sqrt{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2} \Rightarrow \sqrt{\frac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{8}} = \sqrt{1} = 1.$ ***Ans.***

(iii) $z = 1 + \cos\alpha + i\sin\alpha \qquad = 2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2} = 2\cos\frac{\alpha}{2}\left[\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right]$

$|z| = -2\cos\frac{\alpha}{2}$ ***Ans.*** $\left\{\frac{\alpha}{2} \in \left(\frac{\pi}{2},\frac{3\pi}{4}\right) \Rightarrow \cos\frac{\alpha}{2} < 0\right\}$

Illustration :

If the complex number z satisfying $z + |z| = 1 + 7i$ *then find the value of* $|z|^2$.

Sol. $z = x + iy$

$\therefore \quad x + iy + \sqrt{x^2+y^2} = 1 + 7i$

$x + \sqrt{x^2+y^2} = 1 \qquad \ldots (1)$

and $y = 7 \qquad \ldots (2)$

$\therefore \quad x + \sqrt{x^2+49} = 1$

$x^2 + 49 = 1 + x^2 - 2x$

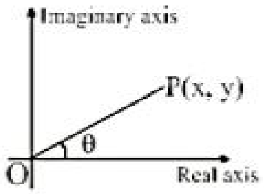
$2x = -48$

$x = -24$

$\therefore \quad |z|^2 = x^2 + y^2 = 625$ ***Ans.***

3.3 Argument of Complex Number :

Angle (θ) made by the line segment joining the point on the complex plane representing the complex number z to the origin from the positive real axis is called argument of complex number z which is denoted as arg(z) = θ.

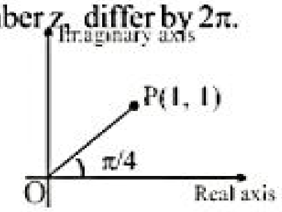


(i) General Argument :

If OP makes an angle θ with real axis then θ is called one of the argument of z.
General values of argument of z are given by
$2n\pi + \theta,\ n \in I$. Note that any two argument of the same complex number z, differ by 2π.

e.g. If $z = 1 + i$ then $\arg(z) = \frac{\pi}{4}$

$\therefore$ General value of argument of $z = 2n\pi + \frac{\pi}{4},\ n \in I$



Note that by specifying the modulus and argument, a complex number is completely defined. However for the complex number $0 + 0i$ the argument is not defined and this is the only complex number which is completely defined by talking in terms of its modulus. i.e., $|z| = 0$.

(ii) Amplitude (Principal value of argument) :

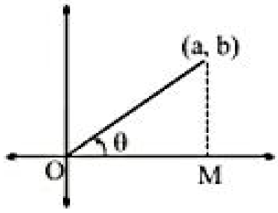
The unique value of θ such that $-\pi < \theta \leq \pi$ is called principal value of argument. Unless otherwise stated, amp z refers to the principal value of argument.

Working rule for finding principal argument of Complex number Z

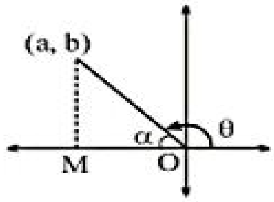
Let $z = a + ib$

First compute $\alpha = \tan^{-1}\left(\frac{|b|}{|a|}\right)$

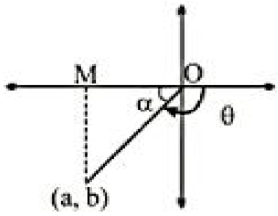
Case I: If z lies in I quadrant i.e. $a, b > 0$ then amp $(z) = \theta = \alpha$.



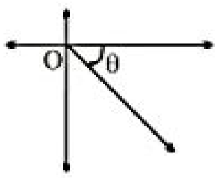
Case II : If z lies in II quadrant i.e. $a < 0, b > 0$ then $\text{amp}(z) = \theta = (\pi - \alpha)$



Case III: If z lies in III quadrant i.e. $a < 0, b < 0$ then $\text{amp}(z) = \theta = -(\pi - \alpha)$



Case IV: If z lies in IV quadrant i.e $a > 0, b < 0$ then $\text{amp}(z) = \theta = -\alpha$.



Note :

(i) If z is purely real positive complex number then $\text{amp}(z) = 0$.

(ii) If z is purely imaginary positive complex number then $\text{amp}(z) = \frac{\pi}{2}$.

(iii) If z is purely real negative complex number then $\text{amp}(z) = \pi$.

(iv) If z is purely imaginary neagative complex number then $\text{amp}(z) = \frac{-\pi}{2}$.

Illustration :

Find the amplitude of

(a) $-1-i\sqrt{3}$ (b) $\dfrac{1+\sqrt{3}i}{\sqrt{3}+i}$ (c) $\sin\alpha + i(1-\cos\alpha),\ 0<\alpha<\pi$

Sol.

(a) Let, $z=-1-i\sqrt{3}$. The $\alpha = \tan^{-1}\left|\dfrac{b}{a}\right| = \tan^{-1}\left|\dfrac{\sqrt{3}}{1}\right| = \dfrac{\pi}{3}$

Clearly, z is in third quadrant.

Therefore argument is $\theta = -(\pi-\alpha) = -\left(\pi-\dfrac{\pi}{3}\right) = -\dfrac{2\pi}{3}$.

(b) $$amp\left(\frac{1+\sqrt{3}i}{\sqrt{3}+i}\right) = amp\left(\frac{1+\sqrt{3}i}{\sqrt{3}+i}\times\frac{\sqrt{3}-i}{\sqrt{3}-i}\right) = amp\left(\frac{2\sqrt{3}+2i}{4}\right) = amp\left(\frac{\sqrt{3}+i}{2}\right)$$

Complex number $\dfrac{\sqrt{3}+i}{2}$ lies in 1^{st} quadrant

$$\therefore\ amp(z) = \theta = \alpha = \frac{\pi}{6}$$

(c) $z=\sin\alpha + i(1-\cos\alpha),\ 0<\alpha<\pi;$

$z = \sin\alpha + i(1-\cos\alpha)$

$$\Rightarrow\quad amp\,(z) = \tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\alpha}{2}\right) = \frac{\alpha}{2} \qquad \left\{0<\frac{\alpha}{2}<\frac{\pi}{2}\right\}$$

(iii) Least positive argument :

The value of θ such that $0<\theta\le 2\pi$ is called the least positive argument.

Illustration :

Find general argument, principal argument and least positive argument of the following complex numbers

(1) $z_1 = \sqrt{3} - 2i$ (2) $z_2 = -1 + i$ (3) $z_3 = -2 - 3i$

(4) $z_4 = \left(\sqrt{7} - 2\right)i$ (5) $z_5 = 2 - \sqrt{7}$ (6) $z_6 = \pi - e$

Sol.

S.No	Complex No.	General Argument	Principal Argument	Least Positive Argument
1.	$\sqrt{3} - 2i$	$2n\pi - \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$	$-\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$	$2\pi - \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$
2.	$-1 + i$	$2n\pi + \frac{3\pi}{4}$	$\frac{3\pi}{4}$	$\frac{3\pi}{4}$
3.	$-2 - 3i$	$2n\pi - \pi + \tan^{-1}\left(\frac{3}{2}\right)$	$-\left(\pi - \tan^{-1}\left(\frac{3}{2}\right)\right)$	$\pi + \tan^{-1}\left(\frac{3}{2}\right)$
4.	$\left(\sqrt{7} - 2\right)i$	$2n\pi + \frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
5.	$\left(2 - \sqrt{7}\right)$	$2n\pi + \pi$	π	π
6.	$(\pi - e)$	$2n\pi$	0	2π

Practice Problem

Q.1 Find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$, $\forall$ n $\in$ I.

Q.2 If $\frac{(x-2)+(y-3)i}{1+i} = 1 - 3i$ find (x, y).

Q.3 If $f(x) = x^4 - 4x^3 + 4x^2 + 8x + 44$, find $f(3 + 2i)$.

Q.4 If $z = \frac{\sqrt{9+40i} + \sqrt{9-40i}}{\sqrt{9+40i} - \sqrt{9-40i}}$, find $|z|$ and z.

Q.5 A square $P_1P_2P_3P_4$ is drawn in the complex plane with P_1 at (1, 0) and P_3 at (3, 0). Let P_n denotes the point (x_n, y_n) n = 1, 2, 3, 4. Find the numerical value of the product of complex numbers $(x_1 + i\,y_1)(x_2 + i\,y_2)(x_3 + i\,y_3)(x_4 + i\,y_4)$.

Q.6 Solve the following equations over C and express the result in the form a + ib, a, b $\in$ R.
(a) $ix^2 - 3x - 2i = 0$ (b) $2(1+i)x^2 - 4(2-i)x - 5 - 3i = 0$

Q.7 Find the sum of the following series

$i + 2i^2 + 3i^3 + \ldots\ldots$ to 100 terms.

Q.8 If $z = (3 + 7i)(p + iq)$ where $p, q \in I - \{0\}$, is purely imaginary then minimum value of $|z|^2$ is

(A) 0 (B) 58 (C) $\frac{3364}{3}$ (D) 3364

Q.9 Match the equation in z, in **Column-I** with the corresponding values of arg(z) in **Column-II**.

	Column-I (equations in z)		**Column-II** (principal value of arg (z))
(A)	$z^2 - z + 1 = 0$	(P)	$-2\pi/3$
(B)	$z^2 + z + 1 = 0$	(Q)	$-\pi/3$
(C)	$2z^2 + 1 + i\sqrt{3} = 0$	(R)	$\pi/3$
(D)	$2z^2 + 1 - i\sqrt{3} = 0$	(S)	$2\pi/3$

Answer key

Q.1 0 Q.2 (6, 1) Q.3 5 Q.4 $\frac{\pi}{2}, \frac{4}{5}$ or $-\frac{\pi}{2}, \frac{5}{4}$

Q.5 15 Q.6 (a) $-i, -2i$; (b) $\frac{3-5i}{2}$ or $-\frac{1+i}{2}$ Q.7 $50(1-i)$

Q.8 D Q.9 (A) Q, R; (B) P, S; (C) Q, S; (D) P, R

4. REPRESENTATION OF A COMPLEX NUMBER IN DIFFERENT FORMS:

4.1 Cartesian Form / Algebraic Form :

Every complex number expressed in the form of $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$ is called cartesian form or algebraic form of complex number

for $z = x + iy$, $\text{Re}(z) = x$ and $\text{Im}(z) = y$

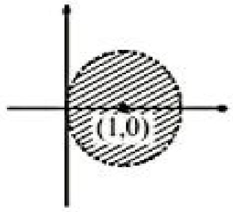
$$|z| = \sqrt{x^2 + y^2}, \quad \bar{z} = x - iy, \quad \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

Illustration :

If $Re\left(\frac{1}{z}\right) > \frac{1}{2}$ *then find the locus of z.*

Sol. Let $z = x + iy$

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}.$$



$\because \quad Re\left(\frac{1}{z}\right) > \frac{1}{2} \quad \Rightarrow \quad \frac{x}{x^2+y^2} > \frac{1}{2}$

$\Rightarrow \quad 2x > x^2 + y^2 \quad \Rightarrow \quad x^2 + y^2 - 2x < 0$

$\Rightarrow \quad (x-1)^2 + y^2 < 1.$

Hence, locus of z represents interior region of the circle whose centre is (1, 0) and radius is 1 unit.

Illustration :

Solve the equation $z^2 + |z| = 0$ *where* $z \in C$.

Sol. Let $z = x + iy$

$z^2 + |z| = 0$

$x^2 - y^2 + 2xy\,i + \sqrt{x^2+y^2} = 0$

Comparing real and imaginary parts

$\Rightarrow \quad x^2 - y^2 + \sqrt{x^2+y^2} = 0$ and $2xy = 0 \Rightarrow x = 0$ or $y = 0$

Case-I : When $x = 0$

$\Rightarrow \quad -y^2 + \sqrt{y^2} = 0 \quad \Rightarrow \quad y^2 = \sqrt{y^2} \quad \Rightarrow \quad y^2 = \pm y$

$\Rightarrow \quad y^2 \pm y = 0 \quad \Rightarrow \quad y(y \pm 1) = 0$

$y = 0,\ y = \pm 1$

$\therefore$ *Solutions are (0, 0), (0, 1) and (0, – 1)*

Case-II : When $y = 0$

$x^2 + \sqrt{x^2} = 0 \Rightarrow x^2 + |x| = 0 \Rightarrow x = 0$

$\therefore$ *Solution is (0, 0) already determined*

Hence, solutions of the equation are (0, 0), (0, 1), (0, – 1).

Illustration :

If z is a complex number satisfying the equation $|z - (1+i)|^2 = 2$ *and* $\omega = \frac{2}{z}$,

then the locus traced by 'ω' in the complex plane is

(A) $x - y - 1 = 0$ (B) $x + y - 1 = 0$ (C) $x - y + 1 = 0$ (D) $x + y + 1 = 0$

Sol. We have $|z - (1+i)|^2 = 2$

$\Rightarrow \quad (x-1)^2 + (y-1)^2 = 2 \qquad$ (Put $z = x + iy$)

$\Rightarrow \quad x^2 + y^2 = 2(x+y) \qquad$ ......(1)

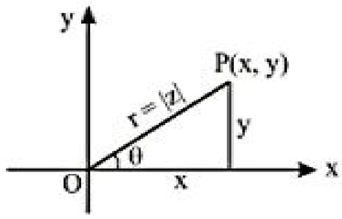
Let $\omega = h + ik = \frac{2}{z} = \frac{2}{x+iy} = \frac{2(x-iy)}{x^2+y^2}$, so

$$h = \frac{2x}{x^2+y^2}\ ,\ k = \frac{-2y}{x^2+y^2}$$

$\Rightarrow \quad h - k = \frac{2(x+y)}{x^2+y^2} = 1$ (from equation (1))

$\therefore$ Locus of the point $\omega(h, k)$ will be $x - y = 1$ Ans. (A)

4.2 Trigonometrical Form / Polar Form :



Let the given complex number be $z = x + iy$
r and θ be the modulus and amp (z) respectively.
From the figure $x = r\cos\theta, y = r\sin\theta$
$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$
Hence, $z = r(\cos\theta + i\sin\theta)$ is called polar / triangometrical form of the complex number.

Illustration :

Express the following complex number in polar form.

(i) $z_1 = -1 - i\sqrt{3}$ (ii) $z_2 = -2 + 3i$

Sol.

(i) $z_1 = -1 - i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$amp(z_1) = \theta = -\left(\pi - \frac{\pi}{3}\right) = \frac{-2\pi}{3}$$

$$r = \sqrt{1+3} = 2$$

$$\therefore \quad z_1 = -1 - i\sqrt{3} = 2\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right) = 2cis\left(\frac{-2\pi}{3}\right).$$

(ii) $z_2 = -2 + 3i$

$$amp(z_2) = \theta = \pi - \tan^{-1}\frac{3}{2}$$

$$r = \sqrt{4+9} = \sqrt{13}$$

$$\therefore \quad z_2 = -2 + 3i = \sqrt{13}\left(\cos\left(\pi - \tan^{-1}\frac{3}{2}\right) + i\sin\left(\pi - \tan^{-1}\frac{3}{2}\right)\right) = \sqrt{13}\ cis\left(\pi - \tan^{-1}\frac{3}{2}\right)$$

Illustration :

If $z = 1+\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}$ *then find modulus and amplitude of z.*

Sol. $z = 1+\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}$

$$2\cos^2\frac{3\pi}{5}+i\,2\sin\frac{3\pi}{5}\cdot\cos\frac{3\pi}{5} \quad\Rightarrow\quad 2\cos\frac{3\pi}{5}\left(\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right)$$

$$\Rightarrow\quad 2\cos\left(\pi-\frac{2\pi}{5}\right)\left(\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right) \quad\Rightarrow\quad -2\cos\frac{2\pi}{5}\left(\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right)$$

$$\Rightarrow\quad 2\cos\frac{2\pi}{5}\left(\cos\left(\pi-\frac{3\pi}{5}\right)-i\sin\left(\pi-\frac{3\pi}{5}\right)\right)$$

$$\Rightarrow\quad 2\cos\frac{2\pi}{5}\left(\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}\right) \quad\Rightarrow\quad 2\cos\frac{2\pi}{5}\left[\cos\left(\frac{-2\pi}{5}\right)+i\sin\left(\frac{-2\pi}{5}\right)\right]$$

$$\therefore\quad |z| = 2\cos\frac{2\pi}{5} \text{ and } amp(z) = \frac{-2\pi}{5}.$$

4.3 Exponential Form :

$z = x + iy = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$

$z = re^{i\theta}$ is called exponential form of the complex number.

where r is modulus of z and θ is amplitude of z.

Here, $\cos\theta + i\sin\theta = e^{i\theta}\ \forall\ \theta$ ....(1)

Replacing i by $-i$, we get

$\cos\theta - i\sin\theta = e^{-i\theta}\ \forall\ \theta$ ....(2)

Adding (1) and (2)

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \text{ which is purely real}$$

subtracting (2) from (1)

$$i\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} \text{ which is purely imaginary.}$$

Illustration :

If $z = -2e^{i\left(\frac{-\pi}{3}\right)}$ *then find modulus and amplitude of z.*

Sol. $z = -2\left(\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right)$

$$= -2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right) = -2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) = -1+i\sqrt{3} \text{ which lies in } 2^{nd} \text{ quadrant}$$

modulus $\quad r = \sqrt{1+3} = 2$

$$amp(z) = \theta = \pi - \tan^{-1}\left(\sqrt{3}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Illustration :

Find the real part of $z = e^{e^{i\theta}}$, $\theta \in R$

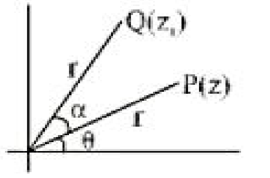
Sol. $z = e^{e^{i\theta}} = e^{(\cos\theta + i\sin\theta)} = e^{\cos\theta}\cdot e^{i\sin\theta} = e^{\cos\theta}\cdot(\cos(\sin\theta) + i\sin(\sin\theta))$

$\therefore \quad Re\,(z) = e^{\cos\theta}\cdot\cos(\sin\theta)$

4.4 Vectorial Representation of a Complex Number :

Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then,

$\overrightarrow{OP} = z \;\&\; |\overrightarrow{OP}| = |z|$.

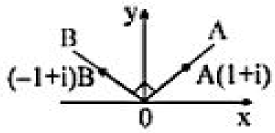


Geometrical meaning of $e^{i\alpha}$.

(i) If $\overrightarrow{OP} = z = r e^{i\theta}$ then $\overrightarrow{OQ} = z_1 = r e^{i(\theta+\alpha)} = z\,.\,e^{i\alpha}$.

If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}e^{i\alpha}$

(ii) If $z = \overline{OA} = 1 + i$ and $\alpha = \dfrac{\pi}{2}$ then $z_1 = \overline{OB} = i\,(1 + i) = -1 + i$



(iii) Using the vetorial concept and section formula complex numbers corresponding to centroid, incentre, orthocentre and circumcentre for a triangle whose vertices are z_1, z_2, z_3 can be deduced.

Practice Problem

Q.1 Find the locus of z if it satisfies the equation $(z-1)^2 + |z+1|^2 = 2$.

Q.2 Find all the possible solutions of the equation $z^2 = \bar{z}$ where $z \in C$.

Q.3 Find the set of all points on the complex plane for which $z^2 + z + 1$ is real and positive.

Q.4 If $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$ then find modulus and principal argument of $z = (1 + \cos 2\theta) + i\sin 2\theta$.

Answer key

Q.1 Point (0, 0)

Q.2 $(0, 0), (1, 0), \left(\dfrac{-1}{2}, \dfrac{\sqrt{3}}{2}\right), \left(\dfrac{-1}{2}, \dfrac{-\sqrt{3}}{2}\right)$

Q.3 $(\lambda, 0), \lambda \in R$ and $\left(\dfrac{-1}{2}, y\right)$ where $y \in \left(\dfrac{-\sqrt{3}}{2}, \dfrac{\sqrt{3}}{2}\right)$

Q.4 $-2\cos\theta, \theta - \pi$

5. PROPERTIES OF CONJUGATE, MODULUS AND ARGUMENT :

5.1 Properties of conjugate of complex Numbers :

(i) $(\bar{\bar{z}}) = z$ (ii) $|z| = |\bar{z}|$ (iii) $z + \bar{z} = 2\text{Re}(z)$

(iv) $z - \bar{z} = 2i\,\text{Im}(z)$ (v) If z is purely real $z = \bar{z}$ (vi) If z is purely imaginary $z = -\bar{z}$

(vii) $z\bar{z} = |z|^2 = |\bar{z}|^2$

(viii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ In general, $\overline{z_1 + z_2 + \ldots\ldots + z_n} = \bar{z}_1 + \bar{z}_2 + \ldots\ldots + \bar{z}_n$

(ix) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (x) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ (xi) $\overline{z^n} = (\bar{z})^n$ (xii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \left(\dfrac{\bar{z}_1}{\bar{z}_2}\right)$

(xiii) If $\alpha = f(z)$, then $\bar{\alpha} = \overline{f(z)} = f(\bar{z})$ where $\alpha = f(z)$ is a function in complex variable with real coefficients.

In other words if $f(x + iy) = a + ib$ then $f(x - iy) = a - ib$.

Explanation :

Let $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^2 + \ldots\ldots + a_n z$, where $a_0, a_1, a_2, \ldots\ldots, a_n$ are real numbers and z is a complex number. Then

$$f(\bar{z}) = a_0 + a_1\bar{z} + a_2(\bar{z})^2 + a_3(\bar{z})^3 + \ldots\ldots + a_n(\bar{z})^n = \overline{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots + a_n z^n}.$$

5.2 Properties of modulus of complex numbers :

(i) $|z| = 0 \Rightarrow z = 0 + i0$

(ii) $|z| = |-z| = |\bar{z}| = |iz|$

(iii) $-|z| \le \text{Re}(z) \le |z|$ and $-|z| \le \text{Im}(z) \le |z|$

(iv) $z\bar{z} = |z|^2$ If z is unimodular i.e. $|z| = 1$, then $\bar{z} = \dfrac{1}{z}$

(v) $|z_1 z_2| = |z_1||z_2|$ In general $|z_1 \cdot z_2 \cdot z_3 \ldots\ldots z_n| = |z_1||z_2||z_3| \ldots\ldots |z_n|$

(vi) $|z^n| = |z|^n$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $(z_2 \ne 0)$

(viii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

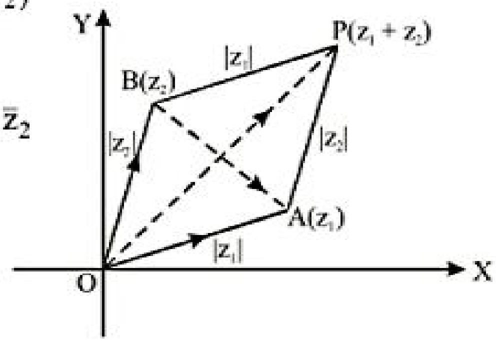
Proof :

$|z_1+z_2|^2+|z_1-z_2|^2 \Rightarrow (z_1+z_2)\left(\overline{z_1+z_2}\right)+(z_1-z_2)\left(\overline{z_1-z_2}\right)$

$\Rightarrow \quad (z_1+z_2)(\bar{z}_1+\bar{z}_2)+(z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$\Rightarrow \quad z_1\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_1-z_1\bar{z}_2-z_2\bar{z}_1+z_2\bar{z}_2$

$\Rightarrow \quad 2(|z_1|^2+|z_2|^2)$



OAPB is a parallelogram.

By the vector law of addition

$$\overrightarrow{OP}=z_1+z_2 \Rightarrow \left|\overrightarrow{OP}\right|=|z_1+z_2|$$

$$\overrightarrow{BA}=z_1-z_2 \Rightarrow \left|\overrightarrow{BA}\right|=|z_1-z_2|$$

$\therefore \quad OP^2+BA^2=OA^2+AP^2+PB^2+OB^2$

$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$

The above identity indicates the sum of squars of diagonals of a parallelogram is equal to sum of square of its all four sides.

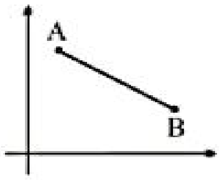
(ix) Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$

$|z_1-z_2|$ denotes the distance between two points on the complex plane representing z_1 and z_2.

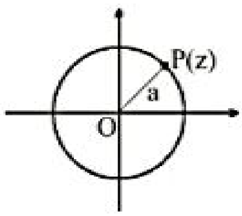
$AB=|z_1-z_2|$

$$=\left|x_1+iy_1-(x_2+iy_2)\right|$$

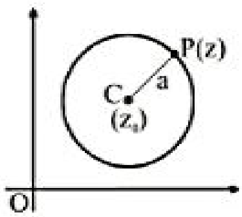
$$=\left|x_1-x_2+i(y_1-y_2)\right|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\ .$$



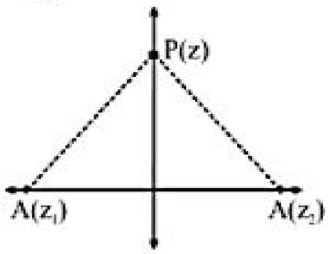
(x) $|z|=a,\ a\in R^+ \Rightarrow$ locus of z represents a circle whose centre is origin and radius is 'a'.



(xi) $|z-z_0|=a$ where z_0 is a fixed complex number and $a\in R^+ \Rightarrow$ locus of z represents a circle whose centre is z_0 and radius is a.



(xii) If $\left|\dfrac{z-z_1}{z-z_2}\right|=1$ where z_1 and z_2 are two fixed complex numbers then locus of z is the perpendicular bisector of joining the points representing z_1 and z_2.



(xiii) Triangle inequalities $\left| |z_1| - |z_2| \right| \le |z_1 \pm z_2| \le |z_1| + |z_2|$

Proof :

Method-I Algebraic Method :

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

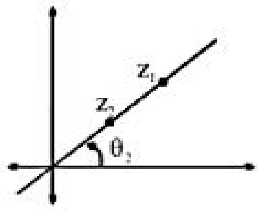
$z_1 + z_2 = r_1\cos\theta_1 + r_2\cos\theta_2 + i(r_1\sin\theta_1 + r_2\sin\theta_2)$

$$z_1 + z_2 = \sqrt{(r_1\cos\theta_1 + r_2\cos\theta)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2} = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

$|z_1 + z_2|$ will be maximum when $\cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \quad \theta_1 - \theta_2 = 2n\pi$

$\Rightarrow \quad \theta_1 = \theta_2 + 2n\pi,\ n \in I$

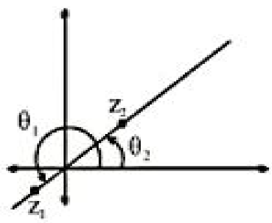


Hence, for the maximum value of $|z_1 + z_2|$ points representing complex numbers z_1, z_2 and the origin are collinear and z_1, z_2 must lie on the same side of the origin.

$|z_1 + z_2|$ will be minimum when $\cos(\theta_1 - \theta_2) = -1$

$\Rightarrow \quad \theta_1 - \theta_2 = 2n\pi + \pi,\ n \in I$

$\Rightarrow \quad \theta_1 = \theta_2 + 2n\pi + \pi,\ n \in I$



Hence, for the minimum value of $|z_1 + z_2|$ points representing the complex number z_1, z_2 and the origin are collinear and z_1, z_2 must lie on the opposite side of the origin.

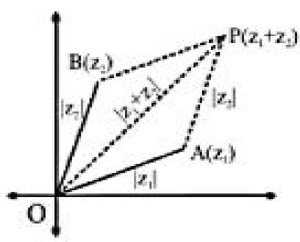
In the similar manner minimum and maximum values of $|z_1 - z_2|$ can also be determined.

Method-II Geometrical Method :

Let A and B represent complex numbers z_1 and z_2 respectively.

A parallelogram OAPB is completed $\overrightarrow{OP} = z_1 + z_2$

$\therefore \quad \left|\overrightarrow{OP}\right| = |z_1 + z_2|$



In the ΔOAP, from the fact

(i) Sum of two sides is always greater than third side.

(ii) Absolute value of the difference of two sides is always less than third side.

$$\therefore \quad \left| |z_1| - |z_2| \right| \le |z_1 + z_2| \le |z_1| + |z_2|$$

Note : Sign of equality holds when z_1, z_2 and the origin are collinear.

5.3 Properties of Argument of complex number :

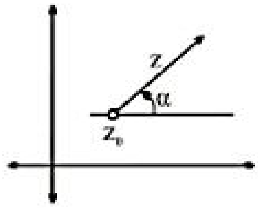
(i) amp $(z_1 z_2)$ = amp (z_1) + amp (z_2) + $2k\pi$, $k \in I$
In general amp $(z_1 \cdot z_2 \ldots\ldots z_n)$ = amp (z_1) + amp (z_2) + ... amp (z_n) + $2k\pi$, $k \in I$

(ii) amp $\left(\frac{z_1}{z_2}\right)$ = amp (z_1) – amp (z_2) + $2k\pi$, $k \in I$

(iii) amp (z^n) = n amp (z) + $2k\pi$, $k \in I$

Note :
In the above properties $2k\pi$, $k \in I$ is added in RHS and k is choosen in such a way so that value of the expresion in RHS belongs to $(-\pi, \pi]$

(iv) If amp $(z - z_0) = \alpha$ where z_0 is a fixed complex number then locus of z denotes a ray emanating from z_0 (z_0 is not included) and making an angle α from positive real axis.



Note : For any complex number z
amp (z) + amp $(-\bar{z}) = \pi$ or amp(z) + amp(–1) + amp $(\bar{z}) = \pi$

Illustration :

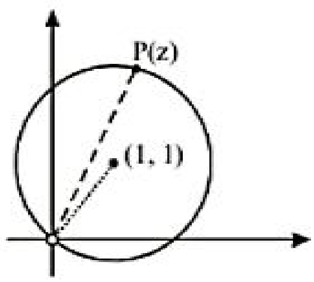
Represent the locus of z satisfying given equation or inequation on the argand's plane.

(i) $|z - 1 - i| = \sqrt{2}$ *and* $amp(z) = \frac{\pi}{3}$ *(ii)* $|amp\,(z - 2 - i)| < \frac{\pi}{3}$

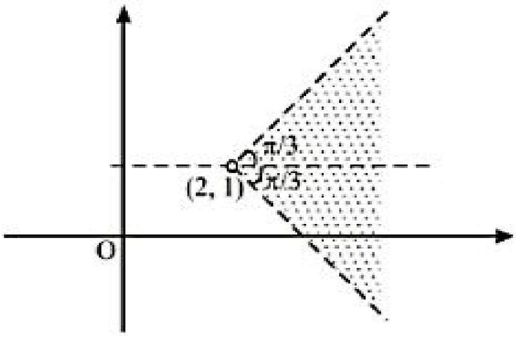
(iii) $3 \le |z - 4| \le 4$ *and* $|amp\,(z - 4)| \le \frac{\pi}{4}$

Sol.

(i) The complex number z represting the point P is only the solution satisfying both the equations simultaneously



(ii) *Complex numbers corresponding to the points lie in the shaded region satisfy the given inequality.*



(iii) *All the complex number* z *satisfying both the inequalities simultaneously lies in the shaded reagion*

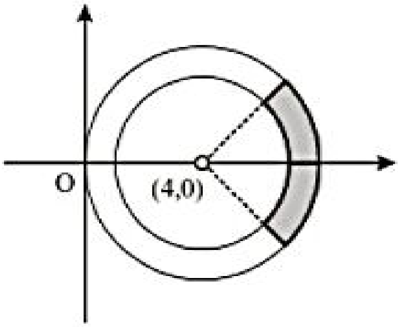


Illustration :

If $|z_1 - 1| \le 1$, $|z_2 - 2| \le 2$, $|z_3 - 3| \le 3$, *then find the greatest value of* $|z_1 + z_2 + z_3|$.

Sol. $|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6|$

$$\le |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6 \le 1 + 2 + 3 + 6 = 12.$$

Illustration :

If $|z_1| = |z_2| = |z_3| = 1$, *prove that* $|z_1 + z_2 + z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right|$.

Sol. *We know that* $|z| = |\bar{z}|$

$\Rightarrow \quad |z_1 + z_2 + z_3| = |\bar{z}_1 + \bar{z}_2 + \bar{z}_3|$

$$= \left|\frac{\bar{z}_1 . z_1}{z_1} + \frac{\bar{z}_2 . z_2}{z_2} + \frac{\bar{z}_3 . z_3}{z_3}\right| = \left|\frac{|z_1|^2}{z_1} + \frac{|z_2|^2}{z_2} + \frac{|z_3|^2}{z_3}\right|$$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| \qquad \left(\because |z_1|^2 = |z_2|^2 = |z_3|^2 = 1\right).$$

Illustration :

Consider two pairs of non-zero conjugate complex numbers (z_1, z_2) and (z_3, z_4).

Find the value of $\arg\left(\frac{z_1}{z_3}\right) + \arg\left(\frac{z_2}{z_4}\right)$.

Sol. $\arg\left(\frac{z_1}{z_3}\right) + \arg\left(\frac{z_2}{z_4}\right) = \arg\left(\frac{z_1 z_2}{z_3 z_4}\right)$

$$= \arg\left(\frac{|z_1|^2}{|z_3|^2}\right) \qquad (\text{as } z_2 = \bar{z}_1 \text{ and } z_4 = \bar{z}_3)$$

$= 0$ *(as argument of a positive real number is zero).*

Illustration :

If z and w are complex numbers satisfying $\bar{z} + i\bar{w} = 0$ and $Amp(zw) = \pi$, then $Amp(z)$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{-\pi}{2}$ (C) $\frac{\pi}{2}$ (D) $\frac{3\pi}{4}$

Sol. Given $\bar{z} + i\bar{w} = 0 \Rightarrow \bar{z} = -i\bar{w}$ or $z = i\,w$

$amp(z) - amp(w) = amp\ i = \frac{\pi}{2}$ ....(1)

also $amp(zw) = \pi$

$amp(z) + amp(w) = \pi$ ....(2)

(1) + (2), gives $2\,amp(z) = \frac{3\pi}{2} \Rightarrow amp(z) = \frac{3\pi}{4}$; Also $amp(w) = \frac{\pi}{4}$

Illustration :

If $Arg\,(z + a) = \frac{\pi}{6}$ *and* $Arg\,(z - a) = \frac{2\pi}{3}$; $a \in R^+$, *then*

(A) *z is independent of a* (B) $|\,a\,| = |\,z + a\,|$

(C) $z = a\ Cis\ \frac{\pi}{6}$ (D) $z = a\ Cis\ \frac{\pi}{3}$

Sol. *Refer the figure z lies on the point of intersection of the rays from A and B. ΔACB is a right angle and OBC is an equilateral triangle*

$\Rightarrow OC = a \Rightarrow z = a\ Cis\ \frac{\pi}{3} \Rightarrow (D)$

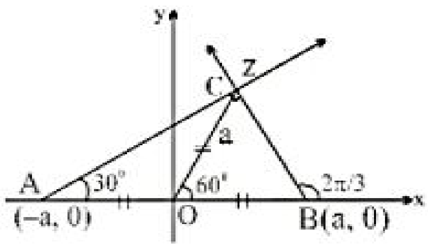


Illustration :

If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ *and* $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, *then find the value of* $|z_1 + z_2 + z_3|$.

Sol. $|z_1| = 1 \Rightarrow z_1\bar{z}_1, |z_2| = 2 \Rightarrow z_2\bar{z}_2 = 4, |z_3| = 3 \Rightarrow z_3\bar{z}_3 = 9.$

Also, $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

$\Rightarrow \quad |z_1z_2z_3\bar{z}_3 + z_1z_2z_3\bar{z}_2 + \bar{z}_1z_2z_3| = 12 \quad \Rightarrow \quad |z_1z_2z_3||\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |z_1||z_2||z_3||\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12 \quad \Rightarrow \quad 6|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 2 \quad \Rightarrow \quad |z_1 + z_2 + z_3| = 2.$

Illustration :

Find modulus and principal argument of the complex number

$$z = (1 + i)\left(1 - i\sqrt{3}\right)(-2 - 2i)\,(i)\,(3).$$

Sol. $amp \text{ of } (1 + i) = \tan^{-1} 1 = \dfrac{\pi}{4}$

$amp \text{ of } \left(1 - i\sqrt{3}\right) = -\tan^{-1}\left|\sqrt{3}\right| = -\dfrac{\pi}{3}$

$amp \text{ of } (-2 - 2i) = -(\pi - \tan^{-1} 1) = -\dfrac{3\pi}{4}$

$amp \text{ of } i = \dfrac{\pi}{2}$

$amp \text{ of } 3 = 0$

$amp\,(z) = \dfrac{\pi}{4} - \dfrac{\pi}{3} + \left(\dfrac{-3\pi}{4}\right) + \dfrac{\pi}{2} + 0 \Rightarrow \quad \dfrac{-\pi}{3}$

$|z| = |1 + i|\left|1 - i\sqrt{3}\right||-2 - 2i||i||3|$

$\quad = \sqrt{2}\cdot 2\cdot\sqrt{2}\cdot 1\cdot 3 = 24.$ ***Ans.***

Illustration :

If $|z - (5 + 7i)| = 9$ *then find the greatest and least value of* $|z - 2 - 3i|$.

Sol. $|z - 2 - 3i| = |z - (5 + 7i) + (3 + 4i)|$

$||z - (5 + 7i)| - |3 + 4i|| \le |z - (5 + 7i) + 3 + 4i| \le |z - (5 + 7i)| + |3 + 4i|$

$|9 - 5| \le |z - 2 - 3i| \le 9 + 5$

$4 \le |z - 2 - 3i| \le 14.$ ***Ans.***

Alternative method :

Z lies on the circle with centre C(5, 7) and radius equal to 9.
Now, we have to find maximum and minimum distance of the point A (2, 3) from the circle.
Maximum distance, $AP = AC + CP = 5 + 9 = 14$
Minimum distance, $AQ = CQ - AQ = 9 - 5 = 4$

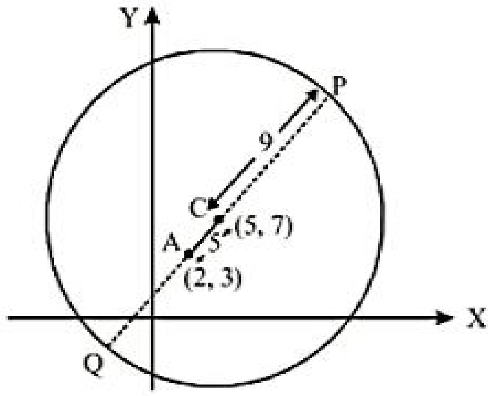


Illustration :

Find the greatest and least value of $|z|$ *is* z *satisfies* $\left|z - \frac{4}{z}\right| = 2$.

Sol. $\left||z| - \frac{4}{|z|}\right| \le \left|z - \frac{4}{z}\right| \le |z| + \frac{4}{|z|}$

$$\left||z| - \frac{4}{|z|}\right| \le 2 \le |z| + \frac{4}{|z|}$$

Put $|z| = r,\ r > 0$

$$\left|r - \frac{4}{r}\right| \le 2 \le r + \frac{4}{r}$$

$$2 \le r + \frac{4}{r} \quad \Rightarrow \quad r^2 - 2r + 4 \ge 0 \text{ always true} \quad \Rightarrow \quad r > 0$$

$$\left|r - \frac{4}{r}\right| \le 2$$

$$-2 \le r - \frac{4}{r} \le 2$$

$$r^2 + 2r - 4 \ge 0 \quad \Rightarrow \quad (r+1)^2 - 5 \ge 0 \quad \Rightarrow \quad \left(r + 1 - \sqrt{5}\right)\underbrace{\left(r + 1 + \sqrt{5}\right)}_{Positive} \ge 0$$

$$r \ge \sqrt{5} - 1$$

$$r^2 - 2r - 4 \le 0 \quad \Rightarrow \quad (r-1)^2 - 5 \le 0 \quad \Rightarrow \quad \left(r - 1 - \sqrt{5}\right)\left(r - 1 + \sqrt{5}\right) \le 0$$

$$\Rightarrow \quad r \in \left[1 - \sqrt{5}, \sqrt{5} + 1\right]$$

$$\therefore \quad r \in \left[\sqrt{5} - 1, \sqrt{5} + 1\right] \quad \textbf{Ans.}$$

Illustration :

Let z_1 & z_2 *be complex numbers such that* $z_1 \neq z_2$ *and* $|z_1| = |z_2|$, *then show that* $\frac{z_1+z_2}{z_1-z_2}$ *is purely imaginary.*

Sol. $|z_1| = |z_2|$, $Re\left(\frac{z_1+z_2}{z_1-z_2}\right) = \frac{1}{2}\left(\frac{z_1+z_2}{z_1-z_2} + \frac{\bar{z}_1+\bar{z}_2}{\bar{z}_1-\bar{z}_2}\right) = \frac{1}{2}\left(\frac{2|z_1|^2-2|z_2|^2}{|z_1-z_2|^2}\right) = 0 =$ *purely imaginary.*

Illustration :

It $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ *is unimodulus and* z_2 *is not unimodulus, then find* $|z_1|$.

Sol. $\left|\frac{z_1-2z_2}{2-z_1z_2}\right| = 1 \quad \Rightarrow \quad |z_1-2z_2|^2 = |2-z\bar{z}_2|^2$

$\Rightarrow \quad (z_1-2z_2)(\bar{z}_1-2\bar{z}_2) = (2-\bar{z}_1z_2)(2-z_1\bar{z}_2)$

$\Rightarrow \quad z_1\bar{z}_1 - 2\bar{z}z_2 - 2z_1\bar{z}_2 + 4z_2\bar{z}_2 = 4 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1\, z_2\bar{z}_2$

$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2\,|z_2|^2 \qquad \Rightarrow \qquad (|z_2|^2-1)(|z_1|^2-4) = 0$

$\Rightarrow \quad |z_1| = 2$ *or* $|z_2| \neq 1$. ***Ans.***

Illustration :

It is given that complex numbers z_1 *and* z_2 *satisfy* $|z_1| = 2$ *and* $|z_2| = 3$. *If the included angle of their corresponding vectors is 60° then* $\left|\frac{z_1+z_2}{z_1-z_2}\right|$ *can be expressed as* $\frac{\sqrt{N}}{7}$ *where N is natural number then N equals*

(A) 126 *(B) 119* *(C) 133* *(D) 19*

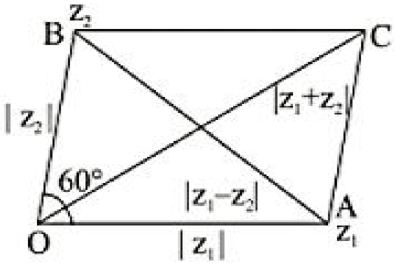
Sol. *Using cosine rule*

$$|z_1+z_2| = \sqrt{|z_1|^2+|z_2|^2-2|z_1||z_2|\cos 120^\circ}$$

$$= \sqrt{4+9+2\cdot 3} = \sqrt{19}$$

and $$|z_1-z_2| = \sqrt{|z_1|^2+|z_2|^2-2|z_1||z_1|\cos 60^\circ}$$

$$= \sqrt{4+9-6} = \sqrt{7}$$



$$\therefore \quad \left|\frac{z_1+z_2}{z_1-z_2}\right| = \sqrt{\frac{19}{7}} = \frac{\sqrt{133}}{7} \quad \Rightarrow N = 133 \;\; \textit{Ans.}$$

Illustration :

Let $z_1, z_2 \in C$ be complex numbers such that $|z_1 + z_2| = \sqrt{3}$ and $|z_1| = |z_2| = 1$. Compute $|z_1 - z_2|$.

Sol. $|z_1 + z_2| = \sqrt{3}$ and $|z_1| = |z_2| = 1$

By squaring

$(|z_1 + z_2|)^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = 3$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2) = 3$

or $\quad 2Re\,(z_1\bar{z}_2) = 1$

$\therefore \quad |z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$= |z_1|^2 + |z_2|^2 - 2Re(z_1\bar{z}_2) = 1 + 1 - 1$

$|z_1 - z_2|^2 = 1$

$\therefore |z_1 - z_2| = 1$

Illustration :

If $z = \dfrac{1 + i\sqrt{3}}{2i\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)}$ *then find* $|z|$ *& amp (z)*

Sol. $z = \dfrac{1 + i\sqrt{3}}{(-\sqrt{3} + i)} \quad \Rightarrow \quad |z| = \dfrac{|1 + i\sqrt{3}|}{|-\sqrt{3} + i|} = 1$

$amp\ z = amp\left(1 + i\sqrt{3}\right) - amp\left(-\sqrt{3} + i\right) \quad \Rightarrow \quad \dfrac{\pi}{3} - \dfrac{5\pi}{6} \Rightarrow \dfrac{-\pi}{2}$. ***Ans.***

Practice Problem

Q.1 If $\dfrac{z-1}{z+1}$ is purely imaginary, then prove that $|z| = 1$.

Q.2 Prove that $z = x + iy$ which satisfy the equation $\left|\dfrac{z-5i}{z+5i}\right| = 1$ lie on the axis of x.

Q.3 If $\arg z = \dfrac{\pi}{4}$ and $|z + 3 - i| = 4$ then find z.

Q.4 If z is any complex number such that $|z + 4| \le 3$, then the least value and greatest value of $|z + 1|$ are.

Q.5 If $z\left(2 - 2\sqrt{3}i\right)^2 = i\left(\sqrt{3} + i\right)^4$, then find the principal argument of z.

Q.6 If $\arg(z) = 50°$ then find principal argument of (z^{100}).

Q.7 Show that all the roots of the equation $z^n \cos\theta_0 + z^{n-1}\cos\theta_1 + \ldots\ldots + \cos\theta_n = 2$ where $\theta_0, \theta_1, \theta_2 \ldots\ldots\theta_n \in R$ lies out side the circle $|z| = \frac{1}{2}$

Q.8 Find z if it satisfies $\text{amp}(z+2) = \frac{\pi}{4}$ and $\text{amp}(z-3+2i) = \frac{3\pi}{4}$.

Q.9 If $\text{amp}\left(\frac{(8+i)(7+i)}{5-i}\right)^6 = 6\tan^{-1}\left(\frac{1}{\lambda}\right)$ then find λ.

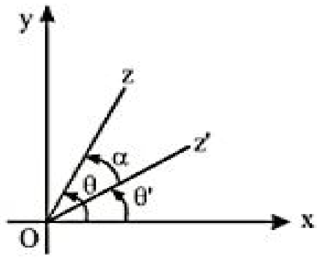
Q.10 If $|z_1| = |z_2| = 1$ and $z_1 z_2 \neq -1$ then prove that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

Answer key

Q.3	$z = 1 + i$	Q.4	0, 5	Q.5	$-\frac{\pi}{6}$	Q.6	$-40°$
Q.8	$-\frac{1}{2} + i\frac{3}{2}$	Q.9	2				

6. CONCEPT OF ROTATION :

If z and z' are two complex numbers then argument of $\frac{z}{z'}$ is the angle through which Oz' must be turned in order that it may lie along Oz.

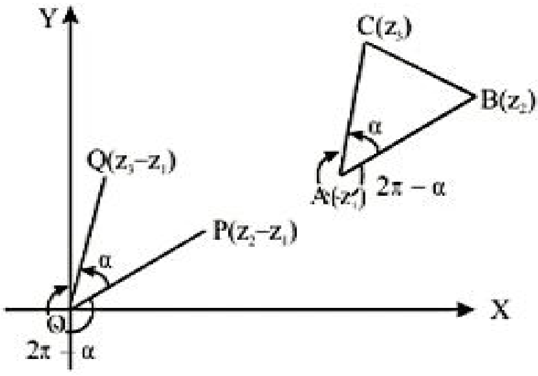


$$\frac{z}{z'} = \frac{|z|e^{i\theta}}{|z'|e^{i\theta'}} = \frac{|z|}{|z'|}e^{i(\theta-\theta')} = \frac{|z|}{|z'|}e^{i\alpha}$$

In general, let z_1, z_2, z_3 be the three vertices of a triangle ABC described in the counter-clock wise sense. Draw OP and OQ parallel and equal to AB and AC respectively. Then the point P is $z_2 - z_1$ and Q is $z_3 - z_1$ and

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{OQ}{OP}(\cos\alpha + i\sin\alpha)$$

$$= \frac{CA}{BA}\cdot e^{i\alpha} = \frac{|z_3 - z_1|}{|z_2 - z_1|}\cdot e^{i\alpha}$$



Note that arg. $(z_3 - z_1) - \arg(z_2 - z_1) = \alpha$ is the angle through which OP must be rotated in the anti-clockwise direction so that it concides with OQ.

Here we can write $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} \cdot e^{-i(2\pi - \alpha)}$ also. In case we are rotating OP in clockwise direction by an angle $(2\pi - \alpha)$. Since the rotation is in clockwise direction, we are taking negative sign with angle $(2\pi - \alpha)$.

Note :
If a complex number (z) is mutliplied by i, it means z has been rotated through an angle $\frac{\pi}{2}$ in anticlockwise sense.

e.g $\quad z = 1 + i$

$\quad z' = (1 + i)\, e^{i\pi/2} = (1 + i)\, i = -1 + i.$

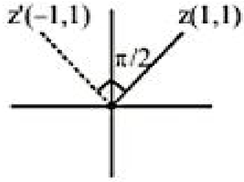
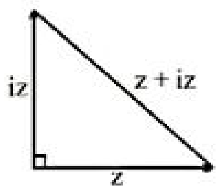
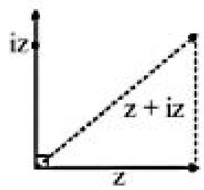


Illustration :

If area of triangle whose sides are reprsented by z, iz and z + iz is 200 square units then | z | is





Sol. $z = x + iy\ (x, y)$

$iz = ix + i^2y\ (-y, x)$

$\frac{1}{2}|z||iz| = 200$

$|z|^2 = 400$

$|z| = 20.$ ***Ans.***

Illustration :

Consider a square ABCD such that z_1, z_2, z_3 and z_4 represent its vertices A, B, C and D respectively. Express 'z_3' and 'z_4' in terms of z_1 and z_2.

Sol. *Consider the rotation of AB about A through an angle $\frac{\pi}{4}$. We get*

$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i\pi/4}$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \Rightarrow z_3 = z_1 + (z_2 - z_1)(1 + i)$$

Similarly, $\quad \frac{z_4 - z_1}{z_2 - z_1} = \left|\frac{z_4 - z_1}{z_2 - z_1}\right|\left|\frac{z_4 - z_1}{z_2 - z_1}\right| e^{i\pi/2} = i$

$\Rightarrow \quad z_4 = z_1 + i\,(z_2 - z_1).$

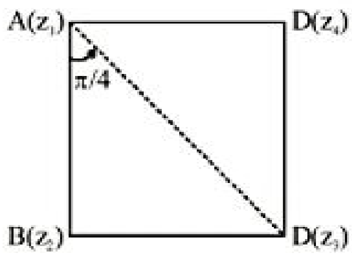


Illustration :

If z_1, z_2, z_3 are the vertices of an equilateral triangle then prove that

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

and if z_0 is the circumcentre of the triangle then also prove that $3z_0^2 = z_1^2 + z_2^2 + z_3^2$.

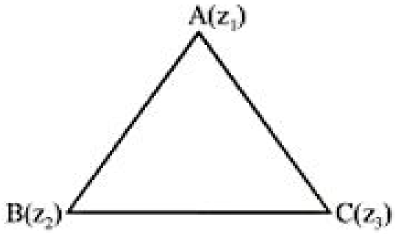
Sol. *Since ΔABC is equilateral*

$\therefore AB = BC = CA$

$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$ ... (1)

also $\angle CBA = \angle ACB = \alpha$

Rotate BC about B (z_2) in anticlockwise sense then



$$\Rightarrow \quad \frac{z_1 - z_2}{z_3 - z_2} = \frac{|z_1 - z_2|}{|z_3 - z_2|} \cdot e^{i\alpha} \qquad \text{... (2)}$$

again rotate side AB about A(z_1) in anticlockwise sense then

$$\Rightarrow \quad \frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_2 - z_1}{|z_2 - z_1|} \cdot e^{i\alpha} \qquad \text{... (3)}$$

By equation (2) and (3)

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_3 - z_1}{z_2 - z_1}$$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1 \qquad \text{...(4)}$$

and if z_0 is the circumcentre for equilateral triangle

$$z_0 = \frac{z_1 + z_2 + z_3}{3} \qquad \text{... (5)}$$

or by equation (5)

$$9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1)$$

or by equaiton (4)

$$z_1^2 + z_2^2 + z_3^2 = 3z_0^2$$

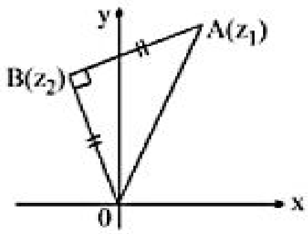
Illustration :

If z_1 & z_2 be non zero complex numbers satisfying the equation, $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$ then find the geometrical nature of the triangle whose vertices are the origin and the points representing z_1 & z_2

(A) an isosceles right angled triangle *(B) a right angled triangle which is not isosceles*

(C) an equilateral triangle *(D) an isosceles triangle which is not right angled.*

Sol. $\frac{z_1}{z_2} = z \Rightarrow z^2 - 2z + 2 = 0 \Rightarrow z = 1 \pm i$



$\Rightarrow \quad \frac{z_1}{z_2} = 1 \pm i \Rightarrow z_1 = z_2 \pm z_2 i \Rightarrow z_1 - z_2 = \pm z_2 i$

$\Rightarrow \quad z_1 - z_2$ *is perpendicular to* z_2 *and* $|z_1 - z_2| = |z_2|$

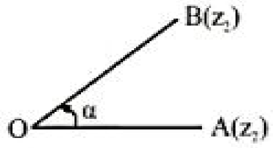
Illustration :

Let z_1 and z_2 be the roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex number. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2\left(\frac{\alpha}{2}\right)$.

Sol. $z_1 + z_2 = -p$ ....(1)

and $z_1 z_2 = q$ ....(2)

Also $z_2 = z_1 e^{i\alpha}$ ....(3)



Now $p^2 = (z_1 + z_2)^2 = (z_1 + z_1 e^{i\alpha})^2$

$= z_1^2 (1 + 2e^{i\alpha} + e^{i2\alpha})$

$= qe^{-i\alpha}(1 + 2e^{i\alpha} + e^{i2\alpha}) = q(2 + e^{i\alpha} + e^{i\alpha})$

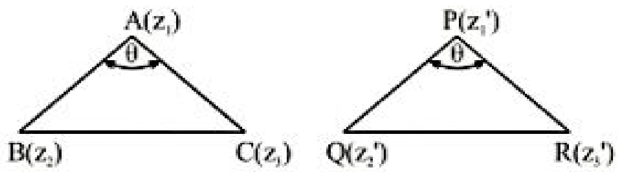
$= q(2 + 2\cos\alpha) = 4q\cos^2\frac{\alpha}{2}$.

Illustration :

If z_1, z_2, z_3 and z_1', z_2', z_3' represent the vertices of two similar triangles ABC and PQR, respectively then prove that

$$\left|\frac{\bar{z}_1'}{\bar{z}_2 - \bar{z}_1}\right|\left|\frac{z_2 - z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2 - z_1}\right|\left|\frac{\bar{z}_3 - \bar{z}_1}{\bar{z}_3'}\right| \geq 1$$

Sol.



Since ΔABC and ΔPQR are similar, $\frac{AC}{AB} = \frac{PR}{PQ}$ and $\angle BAC = \angle QPR = \theta$.

In ΔABC, $\frac{z_3 - z_1}{z_2 - z_1} = \frac{AC}{AB}e^{i\theta}$ ......(1)

In ΔPQR, $\frac{z_3' - z_1'}{z_2' - z_1'} = \frac{PR}{PQ}e^{i\theta}$ .....(2)

From (1) and (2), $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{z_3' - z_1'}{z_2' - z_1'}$

$\Rightarrow \quad z_1'(z_2 - z_3) + z_2'(z_3 - z_1) = z_3'(z_2 - z_1)$

$\Rightarrow \quad |z_1'(z_2 - z_3) + z_2'(z_3 - z_1)| = |z_3'(z_2 - z_1)|$

$\Rightarrow \quad |z_1'(z_2 - z_3)| + |z_2'(z_3 - z_1)| \geq |z_3'(z_2 - z_1)|$

$$\Rightarrow \quad \left|\frac{z_1'}{z_2 - z_1} \cdot \frac{z_2 - z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2 - z_1}\right| \left|\frac{z_3 - z_1}{z_3'}\right| \geq 1$$

$$\Rightarrow \quad \left|\frac{z_1'}{z_2 - z_1}\right| \left|\frac{z_2 - z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2 - z_1}\right| \left|\frac{z_3 - z_1}{z_3'}\right| \geq 1$$

$$\Rightarrow \quad \left|\frac{\bar{z}_1'}{\bar{z}_2 - \bar{z}_1}\right| \left|\frac{z_2 - z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2 - z_1}\right| \left|\frac{\bar{z}_3 - \bar{z}_1}{\bar{z}_3'}\right| \geq 1.$$

Practice Problem

Q.1 Complex number z_1, z_2, z_3 are the vertices A_1, B_1, C respectively of an isosceles right angle triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

Q.2 Find the centre of the arc represented by $\arg\left[\dfrac{(z-3i)}{(z-2i+4)}\right] = \dfrac{\pi}{4}$.

Q.3 If z_r (r = 1, 2, ……, 6) are vertices of a regular hexagon and $\sum_{r=1}^{6} z_r^2 = kz_0^2$ then find the value of k.

Q.4 If z_1, z_2, z_3 are collinear then prove that $z_1|z_2 - z_3| - z_2|z_3 - z_1| + z_3|z_1 - z_2| = 0$.

Q.5 P is a point on the arranged diagram. On the circle with OP an diameter, two points Q & R are taken such that $\angle OOQ = \angle QOR = \theta$, if O is the origin and P, Q and R are respresented by the complex number z_1, z_2 and z_3 respectively show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$.

Answer key

Q.2 $\left(\dfrac{9i-5}{2}\right)$ Q.3 k = 6

7. DEMOIVRE'S THEOREM (DMT) :

Case-I :

Statement :
If n is any integer then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)......(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 + ...\theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + ...\theta_n)$

Case-II :

Statement : If $p, q \in Z$ and $q \neq 0$ then

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where $k = 0, 1, 2, 3, ..., q-1$

Note : When index 'n' is integer then $(\cos\theta + i\sin\theta)^n$ has exactly one value which is $\cos n\theta + i\sin n\theta$ but when n is rational number (say p/q, $q \neq 0$) other than integer then $(\cos\theta + i\sin\theta)^{p/q}$ has exactly q different values.

Illustration :

Prove that $\left(\dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \cos n\left(\dfrac{\pi}{2} - \theta\right) + i\sin n\left(\dfrac{\pi}{2} - \theta\right)$.

Sol. $\dfrac{(1+\sin\theta) + i\cos\theta}{(1+\sin\theta) - i\cos\theta}$

$$\Rightarrow \left[\frac{1+\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right) - i\sin\left(\frac{\pi}{2}-\theta\right)}\right]^n \quad \Rightarrow \quad \left(\frac{1+\cos\alpha + i\sin\alpha}{1+\cos\alpha - i\sin\alpha}\right)^n \text{ where } \alpha = \frac{\pi}{2} - \theta$$

$$\Rightarrow \left(\frac{2\cos^2\frac{\alpha}{2} + i2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2} - i\,2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)^n = \left(\frac{\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}}\right)^n \Rightarrow \left(\frac{e^{i\alpha/2}}{e^{-i\alpha/2}}\right)^n \Rightarrow (e^{i\alpha})^n = e^{in\alpha}$$

$$\Rightarrow \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right).$$

Aliter : Let $\sin\theta + i\cos\theta = z$

$$\left(\frac{1+z}{1+\bar{z}}\right)^n \Rightarrow \left(\frac{1+z}{1+\frac{1}{z}}\right)^n, \ \bar{z} = \frac{1}{z} \Rightarrow \quad z^n \Rightarrow (\sin\theta + i\cos\theta)^n$$

$$\Rightarrow \quad \left(\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right)^n \Rightarrow \quad \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right).$$

Illustration :

If $z = \left(1+i\sqrt{3}\right)^6 + \left(1-i\sqrt{3}\right)^6$ *the find* $|z|$.

Sol. $1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$1 - i\sqrt{3} = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$\therefore \ z = \left(1+i\sqrt{3}\right)^6 + \left(1-i\sqrt{3}\right)^6 = 2^6\left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^6 + \left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^6\right]$$

$$= 2^6\,[\cos 2\pi + i\sin 2\pi + \cos 2\pi - i\sin 2\pi] = 2^7$$

$$\therefore \ |z| = 2^7$$

Application of Demoivre's Theorem :

To find roots of a complex quantity is the main application of DMT.
Working rule for finding roots of a complex quantity.

Illustration :

Find all the roots of the equation $z^4 - (1 + i) = 0$.

Sol. $z^4 = 1 + i \quad \Rightarrow \quad z = (1 + i)^{1/4}$

Step-1 : *Express* $1 + i$ *in the polar form*

$$z = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^{1/4} = 2^{1/8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/4}$$

Step-2 : *Add* $2m\pi$ *in the principal argument.*

$$z = 2^{1/8}\left(\cos\left(2m\pi + \frac{\pi}{4}\right) + i\sin\left(2m\pi + \frac{\pi}{4}\right)\right)^{1/4}$$

Step-3 : *Apply DMT*

$$z = z^{1/8}\left(\cos\left(2m\pi + \frac{\pi}{4}\right)\cdot\frac{1}{4} + i\sin\left(2m\pi + \frac{\pi}{4}\right)\cdot\frac{1}{4}\right)$$

Step-4 : *Put m = 0, 1, 2, 3 to get all four roots of the equation*

$$m = 0,\ z_1 = 2^{1/8}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right), \qquad m = 1,\ z_2 = 2^{1/8}\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right)$$

$$m = 2,\ z_3 = 2^{1/8}\left(\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right), \qquad m = 3,\ z_4 = 2^{1/8}\left(\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right).$$

7.1 Cube Roots of unity :

$$z^3 - 1 = 0 \Rightarrow z = (1)^{1/3} = (\cos 0 + i\sin 0)^{1/3} = (\cos 2m\pi + i\sin 2m\pi)^{1/3}$$

$$= \cos\frac{2m\pi}{3} + i\sin\frac{2m\pi}{3},\ m = 0, 1, 2$$

$$m = 0,\ z_1 = \cos 0 + i\sin 0 = 1$$

$$m = 1,\ z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{-1}{2} + i\frac{\sqrt{3}}{2} = \omega$$

$$m = 2,\ z_3 = \cos\frac{4\pi}{3}, + i\sin\frac{4\pi}{3} = \frac{-1}{2} - i\frac{\sqrt{3}}{2} = \omega^2$$

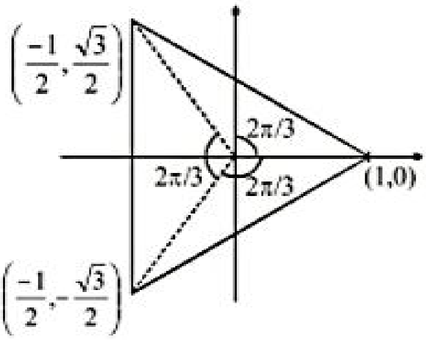
Note :

(i) $\omega^3 = 1$

(ii) $\omega^{3n} = 1, n \in I$

(iii) $1 + \omega + \omega^2 = 0$

(iv) $1 + \omega^r + \omega^{2r} = \begin{cases} 0 & \text{if r is not a multiple of 3.} \\ 3 & \text{if r is a multiple of 3.} \end{cases}$



(v) Representation of cube roots of unity on argand plane. Cube roots of unity form an equilateral Δ whose side is $\sqrt{3}$ units.

(vi) Some important facts

- $a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a + b)(a^2 + (\omega + \omega^2)ab) + b^2)$
  $= (a + b)[a^2 + ab\omega + ab\omega^2 + b^2\omega^3] = (a + b)[a(a + b\omega) + bw^2(a + b\omega)]$
  $\Rightarrow (a + b)(a + b\omega^2)(a + b\omega)$
- $a^3 - b^3 = (a - b)(a - b\omega^2)(a - b\omega)$
- $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

Illustration :

Find the sum of series $1\cdot(2-\omega)(2-\omega^2)+2\cdot(3-\omega)(3-\omega^2)+\ldots\ldots+(n-1)(n-\omega)(n-\omega^2)$ *where* ω *is one of the imaginary cube root of unity.*

Sol.
$$\begin{aligned} T_n &= (n-1)(n-\omega)(n-\omega^2)\\ &= (n-1)(n^2-n(\omega+\omega^2)+\omega^3)\\ &= (n-1)(n^2+n+1)\\ T_n &= n^3-1 \end{aligned}$$

$$\therefore\ S_n = \sum_1^n T_n = \sum n^3 - \sum 1 = \left(\frac{n(n+1)}{2}\right)^2 - n \quad \textit{Ans.}$$

Illustration :

If $x = 1 - i\sqrt{3}$, $y = 1 + i\sqrt{3}$ *and* $z = 2$ *then prove that* $x^p + y^p = z^p$ *where P is a prime number* > 3.

Sol. $x = \left(1-i\sqrt{3}\right) = -2\left(\dfrac{-1+i\sqrt{3}}{2}\right) = -2\omega$

$\therefore\ x^p = (-2\omega)^p = -(2)^p\,\omega^p$

$y = 1 + i\sqrt{3} = -2\left(\dfrac{-1-i\sqrt{3}}{2}\right) = -2\,\omega^2$

$\therefore\ y^p = (-2\omega^2)^p = -(2)^p\,\omega^{2p}$

$x^p + y^p = -2^p(\omega^p + \omega^{2p}) = -(2)^p(-1) = 2^p = z^p$

Illustration :

If $(a+\omega)^{-1}+(b+\omega)^{-1}+(c+\omega)^{-1}=2\omega^{-1}$ *and* $(a+\omega^2)^{-1}+(b+\omega^2)^{-1}+(c+\omega^2)^{-1}=2\omega^{-2}$ *where* ω *is the complex cube root of unity then show that* $(a+1)^{-1}+(b+1)^{-1}+(c+1)^{-1}=2$. $a, b, c \in R$.

Sol. *Let* $\dfrac{1}{a+x}+\dfrac{1}{b+x}+\dfrac{1}{c+x}=\dfrac{2}{x}$

then $\dfrac{1}{a+x}+\dfrac{1}{b+x}=\dfrac{2}{x}-\dfrac{1}{c+x}$

$\Rightarrow\ x^3 - x(ac+bc+ab) - 2abc = 0$ *(Let the roots of the euqation are* ω_1, ω_2 *and* α*)*

If equation has three roots then

sum of roots $= 0$

$\therefore\ \omega + \omega^2 + \alpha = 0$ *or* $\alpha = 1$

$$\therefore\ \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=\frac{2}{1}$$

Illustration :

Let z_1 and z_2 be the roots of the equation $x^2 - x + 1 = 0$. Compute
(i) $x_1^{2000} + x_2^{2000}$ (ii) $x_1^{1999} + x_2^{1999}$ (iii) $x_1^n + x_2^n$ for all $n \in N$.

Sol. *For equation $x^2 - x + 1 = 0$*
solutions are $-\omega$ and $-\omega^2$
$x_1 = -\omega$ and $x_2 = -\omega^2$
(i) $x_1^{2000} + x_2^{2000} = (-\omega)^{2000} + (-\omega^2)^{2000}$
$\Rightarrow \omega^2 + \omega = -1$
(ii) $x_1^{1999} + x_2^{1999} = (-\omega)^{1999} + (-\omega^2)^{1999}$
$\Rightarrow -(\omega + \omega^2) = +1$
(iii) $x_1^n + x_2^n = (-\omega)^n + (-\omega^2)^n,\ n \in N$
$n = 3\lambda \Rightarrow -1 - 1 = -2$
$n = 3\lambda + 1 \Rightarrow \omega + \omega^2 = -1$
$n = 3\lambda + 2 \Rightarrow \omega^2 + \omega = -1$ or 1

7.2 n, nth root of unity :

$$z^n - 1 = 0 \quad \Rightarrow \quad z = (1)^{1/n} = (\cos 0 + i \sin 0)^{1/n} = (\cos 2m\pi + i \sin 2m\pi)^{1/n}$$

$$\Rightarrow \quad \cos\left(\frac{2m\pi}{n}\right) + i\sin\left(\frac{2m\pi}{n}\right), \quad m = 0, 1, 2, 3, ..., (n-1)$$

$m = 0, z_1 = 1$

$$m = 1, z_2 = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) = e^{i\frac{2\pi}{n}} = \alpha \quad \text{(Let)}$$

$$m = 2, z_3 = \cos\left(\frac{4\pi}{n}\right) + i\sin\left(\frac{4\pi}{n}\right) = e^{i\frac{4\pi}{n}} = \alpha^2$$

$\vdots$
$\vdots$

$$m = n-1, z_n = \cos\ \cos\left(\frac{2(n-1)\pi}{n}\right) + i\sin\left(\frac{2(n-1)\pi}{n}\right) = e^{\frac{i2(n-1)\pi}{n}} = \alpha^{n-1}.$$

(1) $1, \alpha, \alpha^2, \alpha^3 + ..., \alpha^{n-1}$ are n, nth roots of unity which are in G.P. with common ratio α where $\alpha = e^{\frac{i2\pi}{n}}$.

(2) Sum of n, nth roots of unity is awlays zero
$$1 + \alpha + \alpha^2 + \alpha^3 + ...\ \alpha^{n-1} = 0$$

Proof: $S = \dfrac{1-\alpha^n}{1-\alpha}$

$$\alpha = e^{\frac{i2\pi}{n}} \Rightarrow \alpha^n = \left(e^{\frac{i2\pi}{n}}\right)^n = e^{i\,2\pi} = 1$$

$$\therefore S = \frac{1-1}{1-\alpha} = 0$$

(3) Sum of p^{th} powers of n, n^{th} roots of unity $= \begin{cases} n, \text{ if } p \text{ is an integral multiple of } n \\ 0, \text{ if } p \text{ is not an integral multiple of } n \end{cases}$

Proof: $1^p + (\alpha)^p + (\alpha^2)^p + (\alpha^3)^p + ... + (\alpha^{n-1})^p$

Case-I: When 'p' is an integral multiple of n, $p = n\lambda, \lambda \in I$

$$\alpha^p = \left(e^{\frac{i2\pi}{n}}\right)^p = e^{\frac{i2\pi}{n}p} = e^{i2\lambda\pi} = 1$$

$\alpha^{2p} = 1, \alpha^{3p} = 1, ..., \alpha^{(n-1)p} = 1$

$\therefore \quad 1 + \alpha^p + \alpha^{2p} + \alpha^{3p} + ... + \alpha^{(n-1)p}$

$\Rightarrow \quad 1 + 1 + 1 + 1 + ... + 1 = n$

Case-2 : When p is not an integral multiple of n :

$\Rightarrow \quad 1 + \alpha^p + \alpha^{2p} + \alpha^{3p} + ... + \alpha^{(n-1)p}$

$$= \frac{1-(\alpha^p)^n}{1-\alpha^p} \Rightarrow \frac{1-1}{1-\alpha^p} = 0 \qquad \{\because \alpha^{np} = 1\}$$

Illustration :

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *and* α_5 *are 5, 5^{th} roots of unity then find the value of*

(i) $\alpha_1^4 + \alpha_2^4 + \alpha_3^4 + \alpha_4^4 + \alpha_5^4 = 0$

(ii) $\alpha_1^{20} + \alpha_2^{20} + \alpha_3^{20} + \alpha_4^{20} + \alpha_5^{20} = 5$

Sol.

(i) $\because$ *4 is not a multiple of 5.*

$$\therefore \quad \sum_{i=1}^{5} \alpha_i^4 = 0$$

(ii) *20 is a multiple of 5.*

$$\therefore \quad \sum_{i=1}^{5} \alpha_i^{20} = 5$$

Illustration :

Evaluate : $\sum_{\lambda=1}^{12}\left(\sin\frac{2\pi\lambda}{13}-i\cos\frac{2\pi\lambda}{13}\right)$

Sol. $\sum_{\lambda=1}^{12}\left(\sin\frac{2\pi\lambda}{13}-i\cos\frac{2\pi\lambda}{13}\right) \Rightarrow (-i)\sum_{\lambda=1}^{12}\left(\cos\frac{2\pi\lambda}{13}+i\sin\frac{2\pi\lambda}{13}\right)$

$\Rightarrow \quad (-i)\sum_{\lambda=1}^{12} e^{\frac{i2\pi\lambda}{13}} \Rightarrow (-i)\left[1+e^{\frac{i2\pi}{3}}+e^{\frac{i4\pi}{13}}+e^{\frac{i6\pi}{13}}+.....+e^{\frac{i24\pi}{13}}-1\right]$

$\Rightarrow \quad (-i)(-1)=i.$ ***Ans.***

Illustration :

If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n-1}$ *are non- real* n^{th} *roots of unity then prove that*

(i) $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\,.........(1-\alpha_{n-1})=n$

(ii) $(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)+......+(1+\alpha_{n-1})=\begin{cases}0, \text{ if } n \text{ is even}\\ 1, \text{ if } n \text{ is odd}\end{cases}$

Sol.

(i) $z^{n-1}=(z-1)(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)\,......\,(z-\alpha_{n-1})$

$\frac{z^n-1}{z-1}=(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)\,........(z-\alpha_{n-1})$

$1+z+z^2+........\,z^{n-1}=(z-\alpha_1)(z-\alpha_2)\,.......(z-\alpha_{n-1})$

Put $z=1$

$\Rightarrow \quad \underbrace{1+1+1+...........1}_{n\ times}=(1-\alpha_1)(1-\alpha_2)\,.........(1-\alpha_{n-1})$

Aliternative method :

$\Rightarrow \quad (1-\alpha_1)(1-\alpha_2)\,.........\,(1-\alpha_{n-1})=n$

$\lim_{z\to 1}\frac{z^n-1}{z-1}=(1-\alpha_1)(1-\alpha_2)\,.......\,(1-\alpha_{n-1})$

$\Rightarrow \quad \lim_{z\to 1}\frac{nz^n-1}{1}=(1-\alpha_1)(1-\alpha_2)\,........\,(1-\alpha_{n-1})$

$\Rightarrow \quad (1-\alpha_1)(1-\alpha_2)\,.........\,(1-\alpha_{n-1})=n$

(ii) $\dfrac{z^n - 1}{z - 1} = (z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \,\ldots\ldots\, (z - \alpha_{n-1})$

Put $z = -1$

$(-1 - \alpha_1)(-1 - \alpha_2)(-1 - \alpha_3) \,\ldots\ldots\ldots (-1 - \alpha_{n-1}) = \dfrac{(-1)^n - 1}{-1-1}$

$\Rightarrow \quad (-1)^{n-1}[(1 + \alpha_n)(1 + \alpha_2) + \ldots\ldots\ldots (1 + \alpha_{n-1})] = \dfrac{(-1)^n - 1}{-2}$

$\Rightarrow \quad (1 + \alpha_1)(1 + \alpha_2) \ldots\ldots (1 + \alpha_{n-1}) = \dfrac{(-1)^n - 1}{-2} \; \dfrac{1}{(-1)^{n-1}} \Rightarrow \begin{bmatrix} 0, \text{even} \\ 1, \text{odd} \end{bmatrix}$

Illustration :

If z_1, z_2, z_3, z_4 & z_5 are roots of the equation $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ then find them also evaluate

(i) $\sum_{i=1}^{5} z_i$ (ii) $\sum_{i=1}^{5} z_i^4$ (iii) $\sum_{i=1}^{5} z_i^{12}$ (iv) $\prod_{i=1}^{5}(2 - z_i)$

Sol. $(z^5 + z^4 + z^3 + z^3 + z^2 + z + 1)(z - 1) = 0$

$\Rightarrow z^6 - 1 = 0 \Rightarrow z = (1)^{1/6} \Rightarrow \quad z = e^{\frac{i\,2m\pi}{6}}, \; m = 0, 1, 2, \ldots\ldots, 5$

Let roots of the equation are $1, z_1, z_2, z_3 \ldots\ldots z_5$.

$z_1 = e^{\frac{i2\pi}{6}} = e^{\frac{i\pi}{3}}, \; z_2 = e^{\frac{i4\pi}{6}} = e^{\frac{i2\pi}{3}}, \; z_3 = e^{\frac{i6\pi}{6}} = e^{i\pi}, \; z_4 = e^{\frac{i8\pi}{6}} = e^{\frac{i4\pi}{3}}, \; z_5 = e^{\frac{i\,10\pi}{6}} = e^{\frac{i\,5\pi}{3}}$

(i) $1 + z_1 + z_2 + z_3 + z_4 + z_5 = 0 \Rightarrow \sum_{i=1}^{5} z_i = -1$

(ii) $1^4 + z_1^4 + \ldots\ldots\ldots + z_5^4 = 0 \Rightarrow \sum_{i=1}^{5} z_i^4 = -1$ $\{\because$ 4 is not a mutliple of 6$\}$

(iii) $\sum_{i=1}^{5} z_i^{12}$

$\Rightarrow 1 + z_1^{12} + z_2^{12} + \ldots\ldots + z_5^{12} = 6$ $\{\because$ 12 is multiple of 6$\}$

$\Rightarrow z_1^{12} + z_2^{12} + \ldots\ldots + z_5^{12} = 5$

(iv) $\prod_{i=1}^{5}(2-z_i)$

$z^6 - 1 = (z-1)(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)$

Put $z = 2$

$$\frac{2^6-1}{2-1} = (2-z_1)(2-z_2) \,\ldots\ldots\, (2-z_5) \Rightarrow \prod_{i=1}^{5}(2-z_i) = 63 \quad \textit{Ans.}$$

Representation of all the roots of the equation on argand plane :

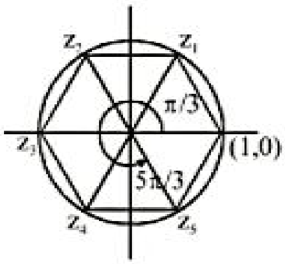


Illustration :

Factorize $z^7 + 1$ *into linear and quadratic factors with real coefficients.*

Sol. $z^7 + 1 = 0$

$$\Rightarrow z = (-1)^{1/7} = (\cos \pi + i \sin \pi)^{1/7} = (\cos(2m\pi + \pi) + i\sin(2m\pi + \pi))^{1/7}$$

$$= \cos\left(\frac{2m\pi+\pi}{7}\right) + i\sin\left(\frac{2m\pi+\pi}{7}\right) = e^{i\left(\frac{2m\pi+\pi}{7}\right)}, \; m = 0, 1, 2, \ldots\ldots, 6$$

$z_1 = e^{i\pi/7}, z_2 = e^{i3\pi/7}, z_3 = e^{i5\pi/7}, z_4 = e^{i7\pi/7}, z_5 = e^{i9\pi/7}, z_6 = e^{i11\pi/7}, z_7 = e^{i13\pi/7}$

$z_7 = e^{i13\pi/7}, = e^{-i\pi/7} = \bar{z}_1$

$z_6 = e^{i11\pi/7}, = e^{-3i\pi/7} = \bar{z}_2$

$z_5 = e^{i9\pi/7}, = e^{-i5\pi/7} = \bar{z}_3$

Now $z^7 + 1 = (z-z_1)(z-z_2) \ldots\ldots (z-z_7)$

$= (z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-\bar{z}_3)(z-\bar{z}_2)(z-\bar{z}_1)$

$= \left(z^2 - z(z_1+\bar{z}_1) + |z_1|^2\right)\left(z^2 - z(z_2+\bar{z}_2) + |z_2|^2\right)\left(z^2 - z(z_3+\bar{z}_3) + |z_3|^2\right)(z-z_4)$

$$= \left(z^2 - z \cdot 2\cos\frac{\pi}{7} + 1\right)\left(z^2 - z\, 2\cos\frac{3\pi}{7} + 1\right)\left(z^2 - z\cdot 2\cos\frac{5\pi}{7} + 1\right)(z+1)$$

$$= (z+1)\left(z^2 - 2\,z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2\,z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2\,z\cos\frac{5\pi}{7} + 1\right)$$

Practice Problem

Q.1 Find the value of the following expression $\left[\dfrac{1-\cos\dfrac{\pi}{10}+i\sin\dfrac{\pi}{10}}{1-\cos\dfrac{\pi}{10}-i\sin\dfrac{\pi}{10}}\right]^5$.

Q.2 If $x^2 - x + 1 = 0$ then the value of $\sum_{n=1}^{5}\left(x^n + \dfrac{1}{x^n}\right)^2$.

Q.3 If α is a non real fifth root of unity, then find the value of $3^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is.

Q.4 $z_1, z_2, z_3, \ldots\ldots, z_{n-1}$ are the non real n^{th} roots of unity, then the value of

$$\frac{1}{(3-z_1)}+\frac{1}{(3-z_2)}+\ldots\ldots+\frac{1}{(3-z_{n-1})}$$

Q.5 Find the number of roots of the equation $z^{10} - z^5 - 992 = 0$ whose real part is negative.

Q.6 Least positive argument of the 4^{th} root of the complex number $2 - i\sqrt{12}$ is

(A) $\pi/6$ (B) $5\pi/12$ (C) $7\pi/12$ (D) $11\pi/12$

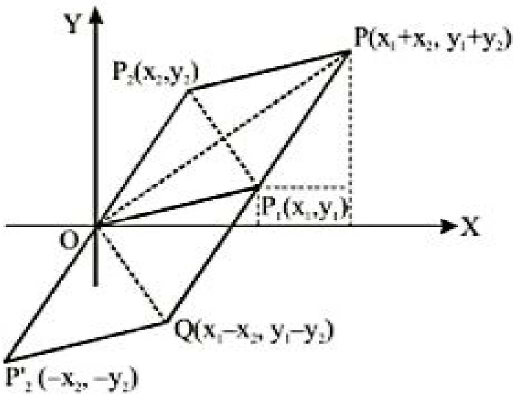
Answer key

Q.1	-1	Q.2	8	Q.3	9	Q.4	$\dfrac{n\cdot 3^{n-1}}{3^n-1}-\dfrac{1}{2}$
Q.5	5	Q.6	B				

8. GEOMETRY OF COMPLEX NUMBER :

8.1 Geometrical meaning of addition and subtraction :

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers represented by the point $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively. By definition $z_1 + z_2$ should be represented by the point $(x_1 + x_2, y_1 + y_2)$. This point is the vertex which completes the parallelogram with the line segments joining the origin with OP_1 and OP_2 as the adjacent sides.



$\Rightarrow \quad |z_1 + z_2| = OP$

Also by definition $z_1 - z_2$ should be represented by the point $(x_1 - x_2, y_1 - y_2)$. This point is the vertex which completes the parallelogram with the line segments joining the origin with OP_1 and OP'_2 (where the point P'_2 represents $-z_2$; the point $-z_2$ can be obtained by producing the directed line P_2O by length $(|z_2|)$ asthe adjacent sides.

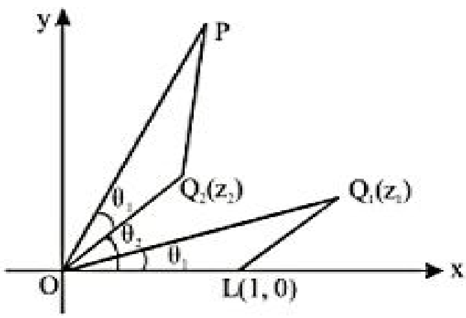
$\Rightarrow \quad |z_1 - z_2| = OQ = P_2P_1$

8.2 Geometrical Meaning of product and Division :

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex numbers represented by Q_1 and Q_2.

(i) Construction for the point representing the product z_1z_2 :

Let L be the point on OX which represents unity, so that OL = 1. Draw the triangle OQ_2P directly similar to the triangle OLQ_1. Then point P represents the product z_1z_2.



Explanation :

Due to similar triangles

$$\frac{OP}{OQ_1} = \frac{OQ_2}{OL}, \text{ that is } \frac{OP}{r_1} = \frac{r_2}{1} \Rightarrow OP = r_1r_2$$

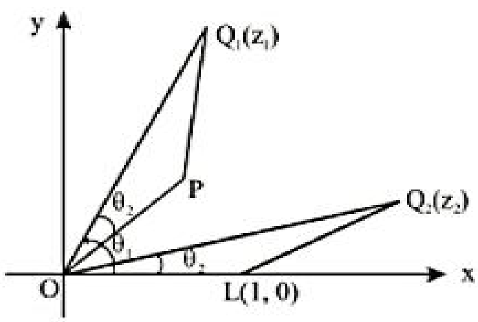
Also $\angle Q_2OP = \angle LOQ_1 = \theta_1 \Rightarrow \angle LOP = \theta_1 + \theta_2$

Since $z_1z_2 = r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$, P represents z_1z_2.

(ii) Construction for the point representing the quotient z_1/z_2 :

Draw the triangle OQ_1P directly similar to the triangle OQ_2L

Then P represents the quotient $\frac{z_1}{z_2}$



Explanation :

From the last construction,

$$\frac{OQ_1}{OQ_2} = \frac{OP}{OL} \Rightarrow \frac{r_1}{r_2} = \frac{OP}{1}$$

number represented by $P \cdot z_2 = z_1$ $\Rightarrow$ number represented by $P = \frac{z_1}{z_2}$

Remark :

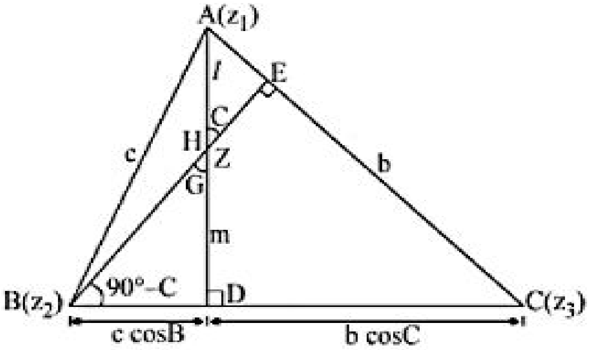
(i) If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then $z_1z_2 = r_1r_2 = e^{i(\theta_1+\theta_2)}$ and $\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{i(\theta_1-\theta_2)}$

(ii) $\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) + \arg(z_2)$

8.3 Centroid, Incentre, Orthocentre & Circumcentre of a triangle on a complex plane :



(i) Centroid 'G' $= \dfrac{z_1 + z_2 + z_3}{3}$

(ii) Incentre 'I' $= \dfrac{a z_1 + b z_2 + c z_3}{a + b + c}$

(iii) Orthocentre :

$$Z_D = \frac{b \cos C \ z_2 + c \cos B \ z_3}{a}$$

Now $AE = c \cos A$;

$$l = AE \ \text{cosec}\, C = c \cos A \ \text{cosec}\, C$$

$$l = 2R \cos A \qquad \left(\frac{c}{\sin C} = 2R\right)$$

and $\quad m = c \cos B \cot C \quad$ or $\quad m = 2R \cos B \cos C$

Hence $Z_H = \dfrac{m z_1 + \ell Z_D}{\ell + m}$

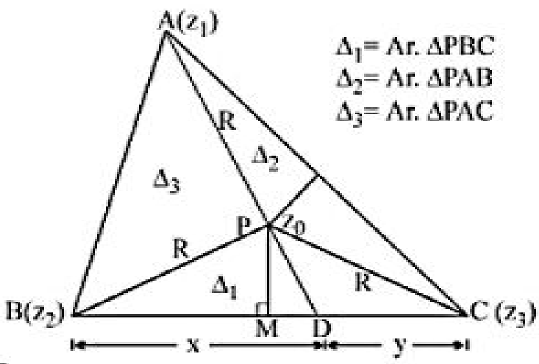
$$= \frac{2R\cos B\cos C z_1 + 2R\cos A\left(\dfrac{b\cos C z_2 + c\cos B z_1}{a}\right)}{2R(\cos A + \cos B\cos C)}$$

$$= \frac{a \cos B \cos C z_1 + b \cos A \cos C z_2 + c \cos A \cos B z_3}{a\left(-\cos(B+C) + \cos B \cos C\right)}$$

$$= \frac{z_1(\sin A \cos B \cos C) + (\sin B \cos C \cos A) z_2 + (\sin C \cos A \cos B) z_3}{\sin A (\sin B \sin C)}$$

$$Z_H = \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\Sigma \tan A} \quad \text{or} \quad \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\Pi \tan A}$$

(iv). Circumcentre :



We have z_0 being equidistant from the vertices gives,

$$|z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0|$$

Consider, $|z_1 - z_0|^2 = |z_2 - z_0|^2$

$$(z_1 - z_0)(\bar{z}_1 - \bar{z}_0) = (z_2 - z_0)(\bar{z}_2 - \bar{z}_0)$$

$$\bar{z}_1 (z_1 - z_0) - \bar{z}_2 (z_2 - z_0) = \bar{z}_0 \left[(z_1 - z_0) - (z_2 - z_0)\right]$$

$$\bar{z}_1 (z_1 - z_0) - \bar{z}_2 (z_2 - z_0) = \bar{z}_0 (z_1 - z_2) \quad ....(1)$$

Similarly 1st & 3rd gives

$$\bar{z}_1 (z_1 - z_0) - \bar{z}_3 (z_3 - z_0) = \bar{z}_0 (z_1 - z_3) \quad ....(2)$$

dividing (1) by (2) eliminate $\bar{z}_0$ and get z_0.

8.4 Different forms of equation of a straight line :

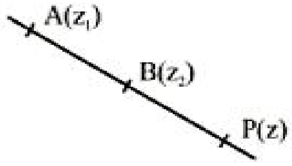
(i) Equation of straight line with line the help of coordinate geometry :

Writing $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ etc. in $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$ and re-arranging terms, we find that the equation of the line through z_1 and z_2 is given by.

$$\frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1} \quad \text{or} \quad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

(ii) Equation of a straight line with the help of rotation formula :

Let $A(z_1)$ and $B(z_2)$ be any two points lying on any line and we have to obtain the equation of this line. For this purpose let us take any point



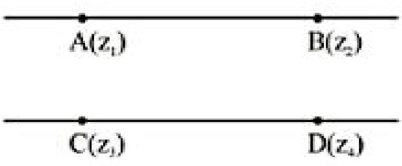
P(z) lying on this line. Since $\arg\left(\frac{z-z_1}{z_2-z_1}\right) = 0$ or π,

$\frac{z-z_1}{z_2-z_1}$ is purely real.

$$\Rightarrow \quad \frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$$

Conditon for which two lines which are parallel or perpendicular :

(a) For parallel :

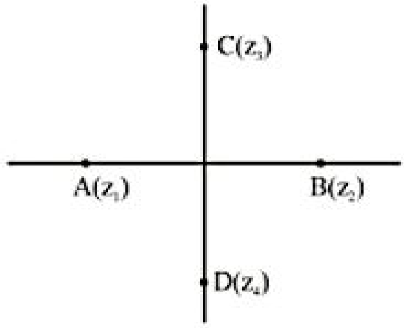


Since AB and CD lines are parallel

$$\therefore \quad \arg\left(\frac{z_1-z_2}{z_3-z_4}\right) = 0 \text{ or } \pi \Rightarrow \frac{z_1-z_2}{z_3-z_4} \text{ is purely real} \Rightarrow \boxed{\frac{z_1-z_2}{z_3-z_4} = \frac{\bar{z}_1-\bar{z}_2}{\bar{z}_3-\bar{z}_4}}$$

(b) For perpendicular :

Since lines AB and CD are perpendicular



$$\arg\left(\frac{z_1-z_2}{z_3-z_4}\right) = \pm\frac{\pi}{2}$$

$$\Rightarrow \frac{z_1-z_2}{z_3-z_4} \text{ is purely imaginary}$$

$$\Rightarrow \boxed{\frac{z_1-z_2}{z_3-z_4} = -\left(\frac{\bar{z}_1-\bar{z}_2}{\bar{z}_3-\bar{z}_4}\right)}$$

(c) General equation of the line :

From equation (i) we get, $z(\bar{z}_2 - \bar{z}_1) - z_1\bar{z}_2 + z_1\bar{z}_1 = \bar{z}(z_2 - z_1) - \bar{z}_1 z_2 + z_1\bar{z}_1$

$\Rightarrow \quad z(\bar{z}_2 - \bar{z}_1) + \bar{z}(\bar{z}_1 - \bar{z}_2) + \bar{z}_1 z_2 - z_1\bar{z}_2 = 0$

Here $\bar{z}_1 z_2 - z_1\bar{z}_2$ is a purely imaginary number as $\bar{z}_1 z_2 - z_1\bar{z}_2 = 2i\,\text{Im}(\bar{z}_1 z_2)$.

Let $\quad \bar{z}_1 z_2 - z_1\bar{z}_2 = ib, \quad b \in R$

$\Rightarrow \quad z(\bar{z}_2 - \bar{z}_1) + \bar{z}(\bar{z}_1 - \bar{z}_2) + ib = 0 \quad \Rightarrow \quad z\,i(\bar{z}_1 - \bar{z}_2) + \bar{z}\,i(z_2 - z_1) + b = 0$

Let $\quad a = i(z_2 - z_1) \quad \Rightarrow \quad \bar{a} = i(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow \quad \bar{a}z + a\bar{z} + b = 0$

This is the general equation of a line in the complex plane.

8.5 Complex slope of a line :

(a) Complex slope of a line $\bar{a}z + a\bar{z} + b = 0$ is defined as

$$\omega = -\frac{a}{\bar{a}}$$

(b) Complex slope of a line passing through $A(z_1)$ & $B(z_2)$ is given by

$$\omega = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$$

(c) Complex slope of a line making an angle θ from positive real axis is given by

$$\omega = e^{i2\theta}$$

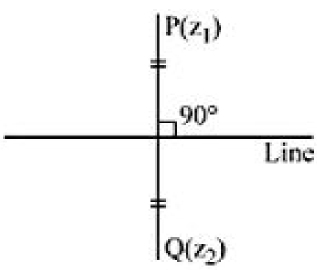
Note : Let ω_1 and ω_2 be the complex slopes of two lines

(i) If $\omega_1 = \omega_2$ then lines are parallel

(ii) If $\omega_1 + \omega_2 = 0$ then lines are perpendicular.

8.6 Reflection Points For A Line (Image of a point in a line) :

Two given points P & Q denoted by complex numbers z_1 and z_2 respectively are the reflection points in a given straight $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if the given line is the right bisector of the line segment PQ. Two points $P(z_1)$ & $Q(z_2)$ will be the reflection points of each other in the given straight line, iff $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$ where 'r' is real and α is non zero complex constant.

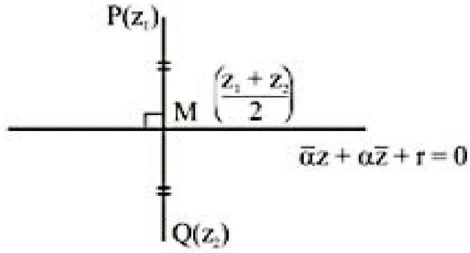


Proof : M is the mid point of PQ which lies on the line mirror

$$\therefore \quad \bar{\alpha}\left(\frac{z_1+z_2}{2}\right)+\alpha\left(\frac{\bar{z}_1+\bar{z}_2}{2}\right)+r=0$$

$$\bar{\alpha}\, z_1+\bar{\alpha}\, z_2+\alpha\, \bar{z}_1+\alpha\, \bar{z}_2+2r=0 \quad \ldots(i)$$

Line PQ and the given line both are perpendicular



$$\therefore \quad \frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}+\left(-\frac{\alpha}{\bar{\alpha}}\right)=0$$

$$z_1\bar{\alpha}-z_2\bar{\alpha}=\alpha\bar{z}_1-\alpha\bar{z}_2$$

$$\alpha\bar{z}_1+\bar{\alpha}z_2=\alpha\bar{z}_2+\bar{\alpha}z_1 \quad \ldots(ii)$$

From (i) & (ii)

$$2(\bar{\alpha}z_2+\alpha\bar{z}_1)+2r=0$$

$$\Rightarrow \quad \alpha\bar{z}_1+\bar{\alpha}z_2+r=0$$

Illustration :

Find the image of the point P(1 – i) in the line mirror $(1+i)z-(1-i)\bar{z}+2i=0$

Sol. *The given line mirror is*

$$(1+i)z-(1-i)\bar{z}+2i=0$$

$$(-1+i)z-(1+i)\bar{z}-2=0$$

$$(1-i)z+(1+i)\bar{z}+2=0$$

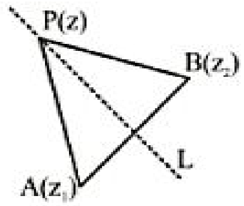
Let z_2 *be the image of the given point* $P(1-i)$

$$\Rightarrow \quad (1-i)z_2+(1+i)(1+i)+2=0 \qquad \Rightarrow \quad (1-i)z_2+1-1+2i+2=0$$

$$\Rightarrow \quad z_2=\frac{-2(1+i)}{1-i}=\frac{-2(1-1+2i)}{2}=-2i$$

8.7 Equation of perpendicular bisector :

Consider a line segment joining A(z_1) and B(z_2). Let the line 'L' be it's perpendicular bisector. If P(z) be any point on the 'L'. we have



$$PA=PB \Rightarrow |z-z_1|=|z-z_2|$$

$$\Rightarrow \quad |z-z_1|^2=|z-z_2|^2$$

$$\Rightarrow \quad (z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2)$$

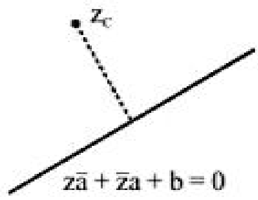
$$\Rightarrow \quad z\bar{z}-z\bar{z}_1-\bar{z}_1\bar{z}+z\bar{z}-z\bar{z}_2-z_2\bar{z}+z_2\bar{z}_2$$

$$\Rightarrow \quad z(\bar{z}_2-\bar{z}_1)+\bar{z}(\bar{z}_2-\bar{z}_1)+z_1\bar{z}_1-z_2\bar{z}_2=0$$

8.8 Distance of a given point from a given line :

Let the given line be $z\bar{a} + \bar{z}a + b = 0$, and the given point be z_c

Say $\quad z_c = x_c + iy_c$.



Replacing z by $x + iy$, in the given equation,

we get, $x(a + \bar{a}) + iy(\bar{a} - a) + b = 0$

Distance of (x_c, y_c) from this line is,

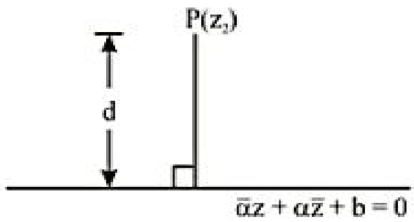
$$\frac{|x_c(a+\bar{a}) + iy_c(\bar{a}-a) + b|}{\sqrt{(a+\bar{a})^2 - (a-\bar{a})^2}} = \frac{|z_c\bar{a} + \bar{z}_c a + b|}{\sqrt{4(\text{Re}(a))^2 + 4(\text{im}(a))^2}} = \frac{|z_c\bar{a} + \bar{z}_c a + b|}{2|a|}$$

Illustration :

Consider a line $(1 + i)z - (1 - i)\bar{z} + 2i = 0$. Find the distance of a point $2i$ from the above line.

Sol. *The given line can be written as*

$$(1 - i)z + (1 + i)\bar{z} + 2 = 0$$



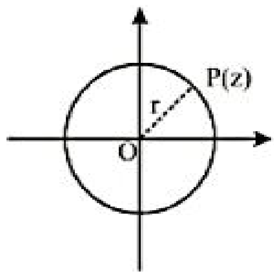
Applying distance formula

$$d = \frac{|z_c\bar{\alpha} + \bar{z}_c\alpha + b|}{2|\alpha|}$$

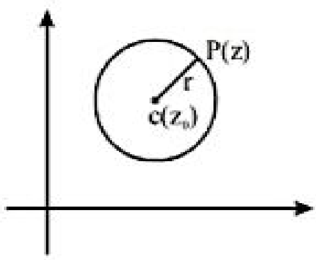
$$= \left|\frac{(1-i)2i + (1+i)(-2i) + 2}{2\sqrt{2}}\right| = \left|\frac{2i + 2 - 2i + 2 + 2}{2\sqrt{2}}\right| = \frac{3}{\sqrt{2}} \quad \textit{Ans.}$$

8.9 Different forms of equation of a circle :

(i) $|z| = r$, $\; r \in R^+$ then locus of z represent a circle whose centre is the origin and radius is equal to r.



(ii) $|z - z_0| = r$, $\; r \in R^+$ then locus of z represents a circle whose centre is z_0 and radius is equal to r.



(iii) Equation $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represnt a circle whose centre is $-a$ and radius is equal $\sqrt{|a|^2 - b}$.

Equation of the circle with centre z_0 and radius 'r'.

$|z - z_0| = r \Rightarrow |z - z_0|^2 = r^2 \quad \Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2 \Rightarrow \quad z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + |z_0|^2 - r^2 = 0$

Putting $-z_0 = a$ and $|z_0|^2 - r^2 = b$ equation becomes

$$z\bar{z} + \bar{a}z + a\bar{z} + b = 0$$

$\therefore$ Centre $z_0 = -a$ and radius $r = \sqrt{|z_0|^2 - b} = \sqrt{|a|^2 - b}$

Illustration

Find the centre and radius of the circle

$$z\bar{z} - (3 - 4i)z - (3 + 4i)\bar{z} + 9 = 0$$

$$z\bar{z} + (-3 + 4i)z + (-3 - 4i)\bar{z} + 9 = 0$$

Sol. *Here,* $a = -3 - 4i,\ b = 9$

$\therefore$ *centre,* $-a = 3 + 4i$ *or* $(3, 4)$

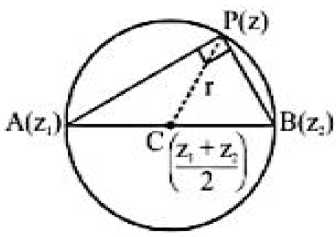
radius $r = \sqrt{|a|^2 - b} = \sqrt{|3 + 4i|^2 - 9} = \sqrt{25 - 9} = 4$

(iv) Diametric form of equation of circle :

Let $A(z_1)$ and $B(z_2)$ are the extremities of diameter of a circle and $P(z)$ be a variable point then

(a) $CP = r$ (r = radius of the circle)

$$\left|z - \frac{z_1 + z_2}{2}\right| = \left|\frac{z_1 - z_2}{2}\right|$$



(b) AP & BP are perpendicular

$\therefore \omega_1 + \omega_2 = 0 \quad \Rightarrow \quad \frac{z - z_1}{\bar{z} - \bar{z}_1} + \frac{z - z_2}{\bar{z} - \bar{z}_2} = 0$

(c) $AP^2 + BP^2 = AB^2$

$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

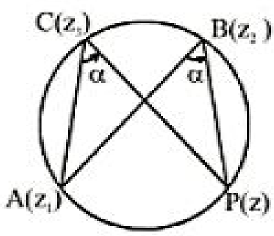
(v) Equation of circle passing through three non-collinear points :

Let $A(z_1)$, $B(z_2)$ and $C(z_3)$ be three given non-collinear points and $P(z)$ be a variable point.

Angle substended by the arc AP at B and C are equal (say 'α')

Using rotation theorem :

$$\frac{z - z_2}{z_1 - z_2} = \lambda_1 e^{i\alpha} \qquad \text{...(i)}$$



$$\frac{z - z_3}{z_1 - z_3} = \lambda_2 e^{i\alpha} \qquad \text{...(ii)}$$

(i) ÷ (ii)

$$\frac{z-z_2}{z_1-z_2}\cdot\frac{z_1-z_3}{z-z_3}=\frac{\lambda_1}{\lambda_2}=\text{Purely real}$$

$$\therefore\quad \left(\frac{z-z_2}{z-z_3}\right)\left(\frac{z_1-z_3}{z_1-z_2}\right)=\left(\frac{\bar{z}-\bar{z}_2}{\bar{z}-\bar{z}_3}\right)\left(\frac{\bar{z}_1-\bar{z}_3}{\bar{z}_1-\bar{z}_2}\right)$$

The above equaiton represents a circle passing through $A(z_1)$, $B(z_2)$ and $C(z_3)$.

Imp. Note :

Let z_1 and z_2 be two given complex numbers and z be any complex number such that, $\arg\left(\frac{z-z_1}{z-z_2}\right)=\alpha$, where $\alpha\in(0,\pi)$

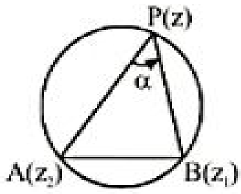


Figure-I

Then 'z' would be lie on an arc of segment of a circle on z_1z_2, containing angle α. Clearly if $\alpha\in\left(0,\frac{\pi}{2}\right)$, z would lie on the major arc (excluding the points z_1 and z_2) and if $\alpha\in\left(\frac{\pi}{2},\pi\right)$. 'z' would lie on the minor arc (excluding the points z_1 and z_2).

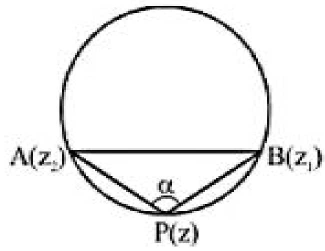


Illustration :

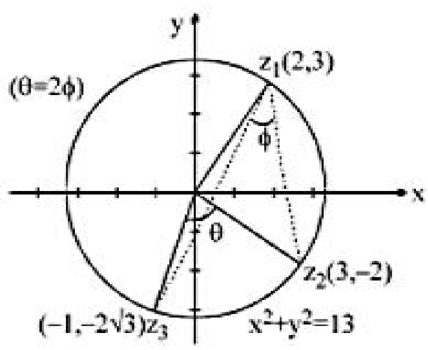
If $z_1 = 2 + 3i$, $z_2 = 3 - 2i$ and $z_3 = -1 - 2\sqrt{3}i$ then which of the following is true?

(A) $\arg\left(\frac{z_3}{z_2}\right)=\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$ (B) $\arg\left(\frac{z_3}{z_2}\right)=\arg\left(\frac{z_2}{z_1}\right)$

(C) $\arg\left(\frac{z_3}{z_2}\right)=2\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$ (D) $\arg\left(\frac{z_3}{z_2}\right)=\frac{1}{2}\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$

Sol. *Note that $|z_1| = |z_2| = |z_3| = \sqrt{13}$*

Hence z_1, z_2, z_3 lies on a circle with centre (0, 0) and $r = \sqrt{13}$ as shown



now $\quad Arg\frac{z_2}{z_3}=2Arg\frac{z_2-z_1}{z_3-z_1}$

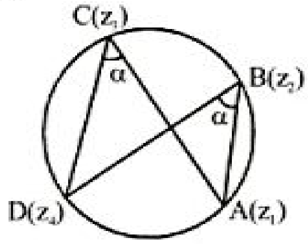
$\therefore\quad Arg\frac{z_3}{z_2}=2Arg\frac{z_3-z_1}{z_2-z_1}\quad\Rightarrow\quad$ *(C)*

8.10 Condition of concyclic of four points :

Four points $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ taken in order are cycyclic then

$$\frac{z_1 - z_2}{z_4 - z_2} = \lambda_1 e^{i\alpha} \qquad \ldots(i)$$

$$\frac{z_1 - z_3}{z_4 - z_3} = \lambda_2 e^{i\alpha} \qquad \ldots(ii)$$



From (i) & (ii)

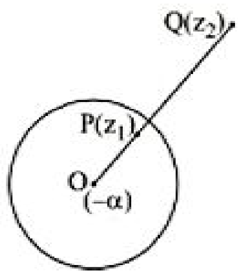
$$\frac{z_1 - z_2}{z_4 - z_2} \cdot \frac{z_4 - z_3}{z_1 - z_3} = \text{Purely real (Which is the required conditon)}$$

8.11 Inverse Point w.r.t. a circle :

Two points $P(z_1)$ & $Q(z_2)$ are said to be the inverse point of each other w.r.t. the circle if

(i) O, P, Q are collinear and lie on the same side of the centre 'O' and

(ii) $(OP) \cdot (OQ) = \rho^2$ when ρ is the radius of the circle.

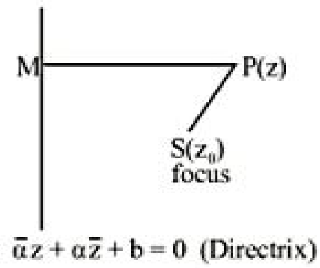


The points z_1 & z_2 will be the inverse point of each other w.r.t. the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ if

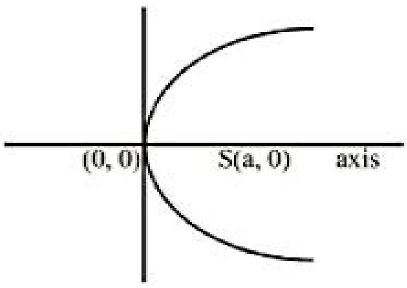
$$z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$$

8.12 General locii on complex plane :

(a) (i) If $|z - z_0| = \left|\dfrac{\bar{\alpha}z + \alpha\bar{z} + b}{2|\alpha|}\right|$ then locus of z is a parabola whose focus is z_0 and directrix is the line $\bar{\alpha}z + \alpha\bar{z} + b = 0$ provided $\bar{\alpha}z_0 + \alpha\bar{z}_0 + b \neq 0$



(ii) If $(z - \bar{z})^2 + 8a(z + \bar{z}) = 0$ then locus of z represents a parabola whose focus is (a, 0), vertex is (0, 0) and real axis is the axis of parabola.



Put $\quad z = x + iy$

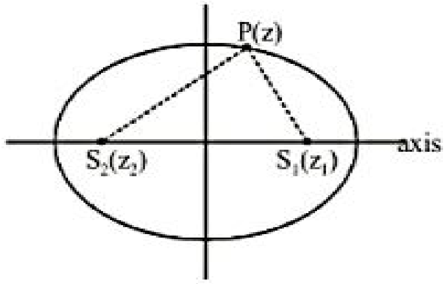
$\Rightarrow \quad (i\,2y)^2 + 8a.2x = 0$

$\Rightarrow \quad -4y^2 + 16ax = 0$

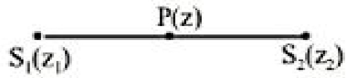
$\Rightarrow \quad y^2 = 4ax$

(b) $|z - z_1| + |z - z_2| = k$

(i) $k > |z_1 - z_2|$ then locus of z represents an ellipse whose foci are z_1 and z_2.



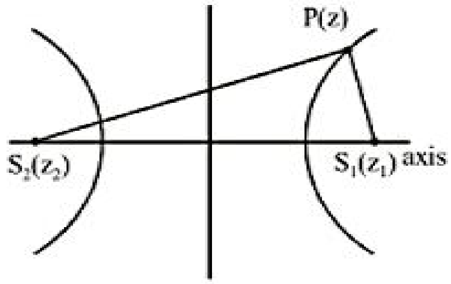
(ii) $k = |z_1 - z_2|$ then locus of z is line segment joining z_1 and z_2.



$PS_1 + PS_2 = S_1S_2 = |z_1 - z_2| \Rightarrow$ P lies on the line segement joining $S_1(z_1)$ and $S_2(z_2)$.

(c) $||z - z_1| - |z - z_2|| = k$

(i) $k < |z_1 - z_2|$ then locus of z represents a hyperbola whose foci are z_1 and z_2.



$|PS_1 - PS_2| < S_1S_2$

$||z - z_1| - |z - z_2| < |z_1 - z_2|$

(ii) $k = |z_1 - z_2|$ then locus of z is union of two rays emanting from z_1 and z_2.

$|PS_1 - PS_2| = S_1S_2$

$||z - z_1| - |z - z_2| = |z_1 - z_2|$

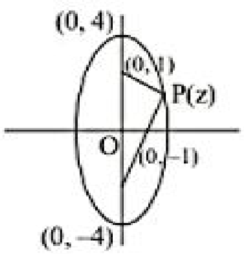
$\therefore$ P(z) lies on the rays emanting either z_1 or z_2.

Illustration :

If z is a complex number satisfying the equation $|z + i| + |z - i| = 8$, on the complex plane then maximum value of $|z|$ is

(A) 2 *(B) 4* *(C) 6* *(D) 8*

Sol. *If $|z + i| + |z - i| = 8$,*



$PF_1 + PF_2 = 8$

$\therefore \quad |z|_{max} = 4 \quad \Rightarrow \quad$ ***(B)***

Illustration :

Number of complex numbers satisfying the relation $|z+\bar{z}|+|z-\bar{z}|=2$ *and* $|z+i|+|z-i|=2$, *is*

(A) 1 *(B) 2* *(C) 3* *(D) 4*

Sol. *We have* $\left|\frac{z+\bar{z}}{2}\right|+\left|\frac{z-\bar{z}}{2}\right|=1 \Rightarrow |x|+|y|=1$

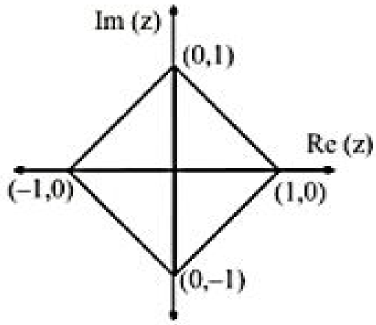
Also, $|z-i|+|z+i| = 2$

$\Rightarrow$ *A line segment between*

(0, 1) and (0, –1).

So, number of solution is 2

i.e., $z=i$ *and* $-i$



9. LOGARITHM OF COMPLEX QUANTITY :

Let u and z be two complex number such that

$u = e^z$ where $u = \alpha + i\beta$ and $z = x + iy$ or $\alpha + i\beta = e^{x+iy}$

then z is called the logarithm of u to the base e

$z = \log_e u$

or $x + iy = \log_e(\alpha + i\beta)$

Now $\alpha + i\beta = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y)$

Hence corresponding to some vlaues of x and y there is one and only one value of $\alpha + i\beta$.

Now we will prove that for a given value of u, there can be infinite values of z.

Here $z = \log u$

or $x + iy = \log(\alpha + i\beta)$

$= \log r(\cos\theta + i\sin\theta)$

[where $\alpha = r\cos\theta$, $\beta = r\sin\theta$ and $r = \sqrt{(\alpha^2+\beta^2)}$, $\theta = \tan^{-1}\beta/\alpha$]

$= \log r\{\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)\}$

$= \log re^{i(\theta+2n\pi)}$, where $n \in I$

$= \log r + i(\theta + 2n\pi)$

$= \log\sqrt{(\alpha^2+\beta^2)} + i(2n\pi + \tan^{-1}\beta/\alpha) \ldots (1)$

From (1) it is clear that for different values of n, we have different values of x + iy.

Hence logrithim of a complex quantity is a multi-valued function and it is expressed as Log u i.e.,

$$\boxed{\text{Log } u = \text{Log}(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\beta/\alpha + 2n\pi i}$$

If we put n = 0, we obtain the principal value of log u.

Hence the principal value of $\boxed{\log u = \frac{1}{2}\log(\alpha^2+\beta^2) + i\theta}$ and $\theta\,(= \tan^{-1}\beta/\alpha)$ should be such that $-\pi < \theta \le \pi$.

Illustration :

Separate into real and imaginary parts the following :

(i) $\log(1+i)$ (ii) $\log(-5)$

Sol.

(i) $\log(1+i) = \log(1+i) + 2n\pi i.$

Let $1 = r\cos\theta$ and $1 = r\sin\theta$

then $r = \sqrt{1^2+1^2} = \sqrt{2}$

and $\tan\theta = 1$ or $\theta = \pi/4$

$\therefore$
$$\begin{aligned}\log(1+i) &= \log(r\cos\theta + ir\sin\theta)\\ &= \log r(\cos\theta + i\sin\theta)\\ &= \log\sqrt{2}(\cos\pi/4 + ir\sin\pi/4)\\ &= \log(\sqrt{2}e^{i\pi/4})\\ &= \log\sqrt{2} + \log e^{i\pi/4}\\ &= \frac{1}{2}\log 2 + \frac{1}{4}i\pi\end{aligned}$$

Here
$$\begin{aligned}\text{Log}(1+i) &= \frac{1}{2}\log 2 + \frac{1}{4}i\pi + 2n\pi i\\ &= \frac{1}{2}\log 2 + i(2n\pi + \pi/4)\end{aligned}$$

(ii) $\text{Log}(-5) = \log(-5) + 2n\pi i$

Now
$$\begin{aligned}\log(-5) &= \log(-5 + i.0)\\ &= \log(r\cos\theta + ir\sin\theta)\end{aligned}$$

where $r\cos\theta = -5,\ r\sin\theta = 0,$

$\therefore$ $r = \sqrt{[(-5)^2 + 0^2]} = 5,\ r\sin\theta = 0,$

$\therefore$
$$\begin{aligned}\log(-5) &= \log 5(\cos\pi + i\sin\pi)\\ &= \log(5e^{i\pi})\\ &= \log 5 + i\pi\end{aligned}$$

Hence
$$\begin{aligned}\text{Log}(-5) &= \log 5 + i\pi + 2n\pi i\\ &= \log 5 + i(2n+1)\pi\end{aligned}$$

Illustration :

If $\tan\log(x+iy) = a + ib$ *and* $a^2 + b^2 \neq 1$, *then prove that*

$$\tan\log(x^2+y^2) = \frac{2a}{1-a^2-b^2}.$$

Sol. Here $\tan\log(x+iy) = a + ib$

$\therefore$ $\log(x+iy) = \tan^{-1}(a+ib)$ ... (1)

and $\log(x-iy) = \tan^{-1}(a-ib)$ ... (2)

$\therefore$ $\log(x+iy) + \log(x+iy) = \tan^{-1}(a+ib) + \tan^{-1}(a-ib)$

or $\quad \log\left\{ (x+iy)(x-iy) = \tan^{-1}\left\{\dfrac{(a+ib)+(a-ib)}{1-(a+ib)(a-ib)}\right\}\right\}$

or $\quad \log(x^2+y^2) = \tan^{-1}\left\{\dfrac{2a}{1-a^2-b^2}\right\}$

or $\quad \tan\log(x^2+y^2) = \dfrac{2a}{1-a^2-b^2}$

Illustration :

Prove that : $i^i = \exp\{-(4n+1)\pi/2\}$

Sol. $i^i = \exp(i \log i)$

$= \exp\{i(\log i + 2n\pi i)\}$

$= \exp[i\{2n\pi i + \log(0+1.i)\}]$

$= \exp[i\{2n\pi i + \log(\cos\pi/2 + i\sin\pi/2)\}]$

$= \exp[i(2n\pi i + \log e^{(i\pi)/2}]$

$= \exp\{i(2n\pi i + i\pi/2)\} = \exp(-2n\pi - \pi/2)$

$= \exp\{-(4n+1)\pi/2\}$

where $n = 0, 1, 2, 3$ ....

Illustration :

If $i^{i^{i\ldots\infty}} = A + iB$ *(principal values only being considered then prove that)*

$$\tan\left(\frac{\pi A}{2}\right) = \frac{B}{A} \text{ and } A^2 + B^2 = e^{-B\pi}$$

Sol. *Here* $i^{i^{i\ldots\infty}} = A + iB$

$\therefore \quad i^{A+iB} = A + iB$

$\Rightarrow \quad (A+iB)\log\{\cos(\pi/2) + i\sin(\pi/2)\} = \log(A+iB)$

$\Rightarrow \quad (A+iB)\log e^{(i\pi/2)} = \dfrac{1}{2}\log(A^2+B^2) + i\tan^{-1}(B/A)$

$\Rightarrow \quad (A+iB)(i\pi/2) = \dfrac{1}{2}\log(A^2+B^2) + i\tan^{-1}(B/A)$

Equating real and imaginary parts of both sides

$$-\left(\frac{B\pi}{2}\right) = \frac{1}{2}\log(A^2+B^2)$$

$\Rightarrow \quad A^2 + B^2 = e^{-B\pi}$

and $\quad A\pi/2 = \tan^{-1}(B/A) \Rightarrow B/A = \tan(A\pi/2)$

$\Rightarrow \quad \tan\dfrac{A\pi}{2} = \dfrac{B}{A}$. *Hence proved.*

Practice Problem

Q.1 Find the equation of the line passing through $1+i$ and $2+i$ on argand plane and also find the complex numbers corresponding to the points on the line which are at a distance of $\sqrt{3}$ units from $1+i\cdot 0$.

Q.2 Find the equation of a line passing through $A(z_1)$ and perpendicular to OA where O is the origin.

Q.3 Locate the complex number $z=x+iy$ which satisfies the inequality $\log_{\sqrt{3}}\dfrac{|z^2|-|z|+1}{2+|z|}<2$.

Q.4 If A & B represent the complex numbers z_1 and z_2 such that $|z_1+z_2|=|z_1-z_2|$ then find the circumcentre of ΔOAB, where O is the origin.

Q.5 Locate the points representing complex number z for which

(i) $\dfrac{\pi}{3}<\arg z\le\dfrac{3\pi}{2}$ (ii) $\arg\left(\dfrac{z-1-i}{z-2}\right)=\dfrac{\pi}{3}$

Q.6 Find the equation of a circle which touches the line $iz+\bar{z}+1+i=0$ and the lines $(z-i)z=(2+i)\bar{z}$ and $(2+i)z+(i-2)\bar{z}-4i=0$ are the normals to the circle.

Q.7 If $|z+2|+|z|\le 8$, then the range of values of $|z-4|$.

Q.8 Prove the following

(a) $\text{Log}(-i)=2n\pi i-\dfrac{1}{2}i\pi$ (b) $\log(-i)=-\dfrac{1}{2}\pi i$ (c) $\log_i i=\dfrac{4m+1}{4n+1}$, m, n ∈ i

Q.9 $i\log\dfrac{x-i}{x+i}=\pi-2\tan^{-1}x$

Answer key

Q.1 $1+\sqrt{2}+i,\ 1-\sqrt{2}+i$ Q.2 $z\bar{z}_1+z_1\bar{z}-2|z_1|^2=0$

Q.3 z lies in the interior region of the circle whose centre is the origin and radius is equal to 5.

Q.4 $\dfrac{z_1+z_2}{2}$

Q.6 $|z-(1-2i)|=2\sqrt{2}$ Q.7 $[1,9]$

SOLVED EXAMPLES

Q.1 The value of $169\, e^{i\left(\pi + \sin^{-1}\frac{12}{13} + \cos^{-1}\frac{5}{13}\right)}$ is

(A) $119 - 120\, i$ (B) $120 + 119\, i$ (C) $119 + 120\, i$ (D) None of these

Sol. $169\, e^{i\left(\pi+\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{5}{13}\right)}$

$$= -169\left[\cos\left(\cos^{-1}\frac{5}{13}\right) + i\sin\left(\cos^{-1}\frac{5}{13}\right)\right]\left[\cos\left(\sin^{-1}\frac{12}{13}\right) + i\sin\left(\sin^{-1}\frac{12}{13}\right)\right]$$

$$= -169\left[\frac{5}{13} + i\frac{12}{13}\right]\left[\frac{5}{13} + i\frac{12}{13}\right]$$

$= [119 - 120\, i] = -i\,[120 + 119\, i]$

Hence (A) is the correct answer.

Q.2 If $A(z_1), B(z_2)$ and $C(z_3)$ be the vertices of a triangle ABC in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$, then the value of z_2 is equal are

(A) $z_3 + i(z_1 + z_3)$ (B) $z_3 - i(z_1 - z_3)$ (C) $z_3 + i(z_1 - z_3)$ (D) None of these

Sol. $\frac{AB}{BC} = \sqrt{2}$

Considering the rotation about B, we get

$$\frac{z_1 - z_2}{z_3 - z_2} = \left|\frac{z_1 - z_2}{z_3 - z_2}\right| e^{i\pi/4} = \frac{AB}{BC} e^{i\pi/4}$$

$\Rightarrow \quad \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 1 + i \quad \Rightarrow \quad z_1 - z_2 = \{1 + i\}\,(z_3 - z_2)$

$\Rightarrow \quad z_1 - (1 + i)\, z_3 = z_2(1 - 1 - i) = -iz_3$

$\Rightarrow \quad z_2 = -z_3\,(z_3 + z_1)$

Q.3 If $f(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, then

(A) g(x) is divisible by $(x - 1)$ but not by h(x)
(B) h(x) is divisible by $(x - 1)$ but not by g(x)
(C) both g(x) and h(x) are divisible by $(x - 1)$
(D) None of these

Sol. $f(x) = g(x^3) + xh\,(x^3)$

Let $f_1(x) = 1 + x + x^2$

Clearly, the roots of $f_1(x) = 0$ and ω and ω^2 (where is a non-real cube root of unity). As $f_1(x)$ divides f(x).

$\Rightarrow \quad f(\omega) = 0,\ \ f(\omega^2) = 0 \quad \Rightarrow \quad g(\omega^3) + \omega h(\omega^3) = 0 \quad$ and $\quad g(\omega^6) + \omega^2 n(\omega^6) = 0$

$\Rightarrow \quad g(1) + \omega h(1) = 0,\quad g(1) + \omega^2 h(1) = 0 \quad \Rightarrow \quad 2g(1) + h(1)\ (\omega + \omega^2) = 0$

$\Rightarrow \quad 2g(1) - h(1) = 0 \quad \Rightarrow \quad h(1) = 2g(1)$

$\Rightarrow \quad g(1) + \omega.\ 2g(1) = 0 \quad \Rightarrow \quad g(1)\ (1 + 2\omega) = 0 \ \Rightarrow \ g(1) = 0 \ \Rightarrow \ g(1) = 0$

$\Rightarrow$ $x = 1$ is the root of $g(x) = 0$ and $h(x) = 0$. Thus $g(x)$ and $h(x)$ both are divisible by $x - 1$.

Hence (C) is the correct answer.

Q.4 Find z such that $|z - 2 + 2i| \le 1$ and z has

(i) least absolute value (ii) numerically least amplitude.

Sol.

(i) Given equation represents a circle with centre $(2, -2)$ and radius 1.

Distance of $|z - 2 + 2i| = 1$ will be minimum from origin along the line which is normal to the circle passing through the origin, so it will passing through the centre of the circle also. $OP = \left(2\sqrt{2} - 1\right)$

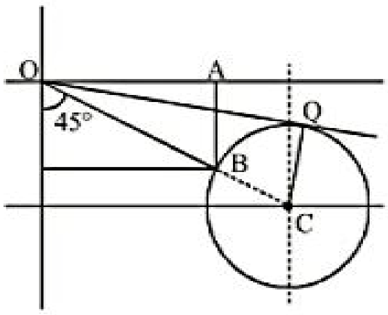
$OA = OP \cos 45°$ and $OB = OP \sin 45°$ so that

$$z = \frac{\left(2\sqrt{2}-1\right)}{\sqrt{2}} - i\left(\frac{2\sqrt{2}-1}{\sqrt{2}}\right)$$

(ii) We have to find the complex number z represent by Q where OQ is a tangent to the circle

Now $OQ = \sqrt{OC^2 - CQ^2} = |2 - 2i|^2 = 2\sqrt{2^2 + (-2)^2 - 1} = \sqrt{7}$

$\therefore \quad |z| = \sqrt{7}$



$\angle QOX = |\angle COX - \angle COQ|$

$$= -\left|\frac{\pi}{4} - \sin^{-1}\frac{1}{|2-2i|}\right| = -\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right)$$

$$\Rightarrow \quad \text{amp } z = -\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right)$$

amp $z = |z|\ \{\cos(\text{amp } z) + i \sin(\text{amp } z)\}$

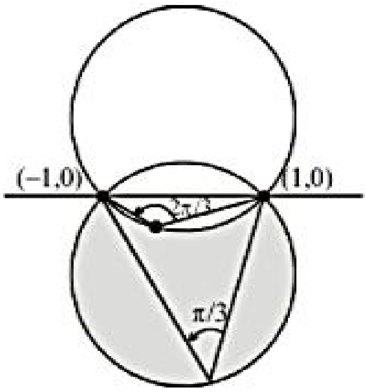
$$= \text{amp } z = \left|\cos\left\{-\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right)\right\} + i\sin\left\{-\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right)\right\}\right|$$

$$= \text{amp } z = \left|\cos\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right) - i\sin-\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right)\right|.$$

Q.5 Plot the region represented by $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ in the Argand plane.

Sol. Let us take $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$,

Clearly z lies on the minor arc of the circle passing through (1, 0) and (– 1, 0).



Similarly, $\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{3}$ means that 'z' is lying on the major arc of the circle passing through (1, 0) and (–1, 0). Now if we take any point in the region included between the two arcs, say $P_1(z_1)$,

we get $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$.

Thus $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ represents the shaded region (excluding the points (1, 0) and (– 1, 0).

Q.6 Consider the complex numbers $z_1 = 10 + 6i$ and $z_2 = 4 + 2i$. If $\arg\left(\frac{z-z_1}{z-z_2}\right) = \frac{\pi}{4}$, find the centre and radius of the circle traced by the complex number z.

Sol. Let $O(z_0)$ be the centre of the circle.

We hve $\angle AOB = \frac{\pi}{2}$. $AB = |z_1 - z_2| = |6 + 4i| = \sqrt{52}$.

Let $OA = OB = r \Rightarrow AB = r\sqrt{2} \Rightarrow r = \sqrt{26}$

Also $\frac{z_2 - z_0}{z_1 - z_0} = e^{-i\pi/2} = -i \Rightarrow z_2 - z_0 = -i(z_1 - z_0)$

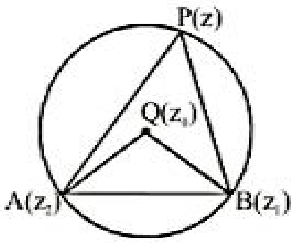
$\Rightarrow \quad z_0 = \frac{1}{2}(z_2 - iz_2 + iz_1 + z_1) = 5 + 7i.$

Alternative:

We have $\frac{\pi}{4} = \arg\frac{z-z_1}{z-z_2} = \arg(z - z_1) - \arg(z - z_2)$

$\Rightarrow \quad \tan^{-1} 1 = \tan^{-1}\frac{y-6}{x-10} - \tan^{-1}\frac{y-2}{x-4}$

$$\Rightarrow \quad \tan^{-1} 1 = \tan^{-1}\left[\frac{\frac{y-6}{x-10} - \frac{y-2}{x-4}}{1+\frac{(y-6)(y-2)}{(x-10)(x-4)}}\right]$$



$$\Rightarrow \quad 1 = \frac{6y-4x+4}{x^2+y^2-14x-8y+52}$$

$$\Rightarrow \quad x^2 + y^2 - 10x - 14y + 48 = 0$$

or $\quad (x-5)^2 + (y-7)^2 = 26 = \left(\sqrt{26}\right)^2$

or $\quad |x-5+i(y-7)| = \sqrt{26} \Rightarrow |z-5-7i| = \sqrt{26}$.

Q.7 Find all complex numbers z for which $\arg\left(\frac{3z-6-3i}{2z-8-6i}\right) = \frac{\pi}{4}$ and $|z-3+i| = 3$.

Sol. Now $\frac{3z-6-3i}{2z-8-6i} = \frac{3(x+iy)-6-3i}{2(x+iy)-8-6i}$

$$= \frac{3x-6+i(3y-3)}{(2x-8)+i(2y-6)} \frac{[(2x-8)-i(2y-6)]}{[(2x-8)-i(2y-6)]}$$

$$= \frac{6x^2+6y^2-36x-24y+66}{(2x-8)^2+(2y-6)^2} + \frac{(12x-12y-12)}{(2x-8)^2+(2y-6)^2} = a + ib \quad \text{(say)}$$

Since $\arg(a+ib) = \frac{\pi}{4} \therefore \tan\frac{\pi}{4} = \frac{b}{a} \Rightarrow a = b$

$\Rightarrow \quad 6x^2 + 6y^2 - 36x - 24y + 66 = 12x - 12y - 12$

$\Rightarrow \quad x^2 + y^2 - 8x - 2y + 13 = 0 \qquad$ ...(i)

Again $|z-3+i| = 3 \Rightarrow |x+iy-3+i| = 3$

$\Rightarrow \quad (x-3)^2 + (y+1)^2 = 9 \Rightarrow x^2 + y^2 - 6x + 2y + 1 = 0 \qquad$ ...(ii)

(i) – (ii), $\Rightarrow -2x - 4y + 12 = 0$

$\Rightarrow x = -2y + 6 \qquad$ ...(iii)

Putting the value of x in (ii), we get

$(-2y+6)^2 + y^2 - 6(-2y+6) + 2y + 1 = 0$

$\Rightarrow \quad 5y^2 - 10y + 1 = 0$

$\therefore \quad y = -\frac{10 \pm 4\sqrt{5}}{10} = 1 \pm \frac{2}{\sqrt{5}} \qquad \therefore \quad x = -2y + 6 = 4 \mp \frac{-4}{\sqrt{5}}$

$\therefore \quad z = x + iy = 4 \mp \frac{4}{\sqrt{5}} + i\left(1 \pm \frac{2}{\sqrt{5}}\right)$

Q.8 If $|a_n| < 2$ for $n = 1, 2, 3, \ldots\ldots$ and $1 + a_1 z + a_2 z^2 + \ldots\ldots + a_N z^N = 0$, show that z does not lie in the interior of the circle $|z| = \frac{1}{3}$

Sol. Given $|a_n| < 2$ for $n = 1, 2, 3, \ldots\ldots$

Again given $1 + a_1 z + a_2 z^2 + \ldots\ldots + a_N z^N = 0$

$\Rightarrow \quad -1 = a_1 z + a_2 z^2 + \ldots\ldots + a_N z^N$

$\Rightarrow \quad |-1| = |a_1 z + a_2 z^2 + \ldots\ldots + a_N z^N| = |a_1|\ |z| + |a_2 z^2| + \ldots\ldots |a_N|\ |z^N|$

$= |a_1|\ |z| + |a_2|\ |z|^2 + \ldots\ldots |a_N|\ |z|^N < 2\,|z| + 2|z|^2 + \ldots\ldots 2|z|^N$ [from (i)]

$< 2\,|z| + 2|z|^2 + \ldots\ldots + \text{to } \infty$ [From (iii), $|z| \neq 0$]

Case-I : When $|z| < 1$.

From (iii), $\frac{2|z|}{1-|z|} \Rightarrow 1 - |z| < 2|z| \Rightarrow 3|z| > 1 \Rightarrow |z| > \frac{1}{3}$

Case-II : When $|z| \geq 1$. I this case $|z| > \frac{1}{3}$ [$\because |z| \geq 1$]

Hence z does not lie in the interior of the circle $|z| = \frac{1}{3}$.

Note : Above question can also be asked as : if $|a_n| < 2$ for $1 \leq n \leq N$, then prove that there exists no z in the interior of the circle $|z| = \frac{1}{3}$ such that

$$1 + a_1 z + a_2 z^2 + \ldots\ldots + a_N z^N = 0.$$

Q.9 Consider a triangle formed by the points $A\left(\frac{2}{\sqrt{3}} e^{i\frac{\pi}{2}}\right)$, $B\left(\frac{2}{\sqrt{3}} e^{-i\frac{\pi}{6}}\right)$, $C\left(\frac{2}{\sqrt{3}} e^{-i\frac{5\pi}{6}}\right)$. Let P (z) be any point on it's in-circle. Prove that $AP^2 + BP^2 + CP^2 = 5$.

Sol. Let $\quad z_1 = \frac{2}{\sqrt{3}} e^{i\frac{\pi}{2}}, z_2 = \frac{2}{\sqrt{3}} e^{-i\frac{\pi}{6}}, z_3 = \frac{2}{\sqrt{3}} e^{-i\frac{5\pi}{6}}$.

Clearly the points lie on the circle $|z| = \frac{2}{\sqrt{3}}$

If l the length of side of the ΔABC, $AD = l \sin 60° = l\frac{\sqrt{3}}{2} \Rightarrow OD = \frac{1}{3}\ AD = \frac{l\sqrt{3}}{3}$

Now, $\quad OA = \frac{2}{3}\ AD = \frac{2}{3} \cdot \frac{l\sqrt{3}}{2} = \frac{2}{\sqrt{3}}$

$\Rightarrow \quad l = 2$ and $OD = \frac{2\sqrt{3}}{6} = \frac{1}{\sqrt{3}}$

Let P(z) be any point on the in circle, is $|z| = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad z = \frac{1}{\sqrt{3}} e^{i\theta}$

Now, $AP^2 = |z - z_1|^2 = |z|^2 + |z_1|^2 - (z\bar{z}_1 + \bar{z}z_2)$

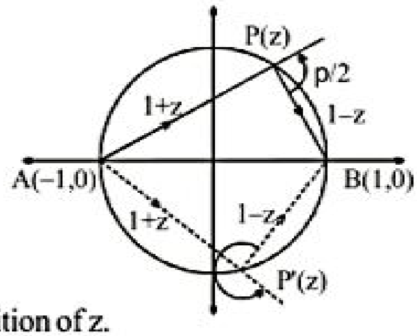
Similarly, $BP^2 = |z|^2 + |z_2|^2 - (z\bar{z}_2 + \bar{z}z_2)$ and $CP^2 = |z|^2 + |z_3|^2 - (z\bar{z}_3 + \bar{z}z_3)$

$AP^2 + BP^2 + CP^2 = 3|z|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 - z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) - \bar{z}(z_1 + z_2 + z_3)$

$= 1 + 3 \cdot \frac{4}{3} - z(0) - \bar{z}(0) = 5.$

Q.10 If $|z| = 1$, then prove that the points represented by $\sqrt{\frac{1+z}{1-z}}$ lie on one or other of two fixed perpendicular straight lines.

Sol. Since $|z| = 1$. z lies on a unit circle having centre at the origin



$$\arg\left(\frac{1+z}{1-z}\right) = +\frac{\pi}{2} \text{ or } +\frac{3\pi}{2}$$

$$\Rightarrow \quad \frac{1+z}{1-z} = ke^{i\pi/2} \text{ or } ke^{i3\pi/2}$$

where k is a real parameter and its value depends upon the position of z.

Let $\alpha = \sqrt{\frac{1+z}{1-z}} \quad \Rightarrow \quad \alpha = \sqrt{k}\, e^{i\pi/4}$ or $\sqrt{k}\, e^{i3\pi/4}$.

$\Rightarrow$ α lies on one or other of the two perpendicular lines.

Q.11 Dividing f(z) by $z - i$, we obtain the remainder i and dividing it by $z + i$, we get remainder $1 + i$. Find the remainder upon the division of f(z) by $z^2 + 1$.

Sol. $z - i = 0 \quad \Rightarrow \quad z = i$

Remainder when f(z) is divided by $(z - i) = f(i)$

Similarly remainder when f(z) is divided by $(z + i) = f(-i)$ ...(i)

According to question $f(i) = i$

and $f(-i) = 1 + i$ ...(ii)

Since $z^2 + 1$ is a quadratic expression, therefore remainder when f(z) is divided by $z^2 + 1$ will be in general a linear expression.

Let g(z) be the quotient and $az + b$ the reaminder when g(z) is divided by $z^2 + 1$.

then $f(z) = g(z)(z^2 + 1) + az + b$ ...(iii)

$\therefore$ $f(i) = g(i)(i^2 + 1) + ai + b = ai + b$ ...(iv)

and $f(-i) = g(-i)(i^2 + 1) - ai + b = -ai + b$ ...(v)

From (i) and (iv), we have $b + ai = i$ ...(vi)

From (ii) and (v), we have $b - ai = 1 + i$ ...(vii)

Solving (vi) and (vii) we get , $b = \frac{1}{2} + i$ and $a = \frac{i}{2}$

Hence required remainder $= az + b = \frac{1}{2} iz + \frac{1}{2} + i$

Paragraph for question nos. 12 to 14

Let A, B, C be three sets of complex numbers as defined below.

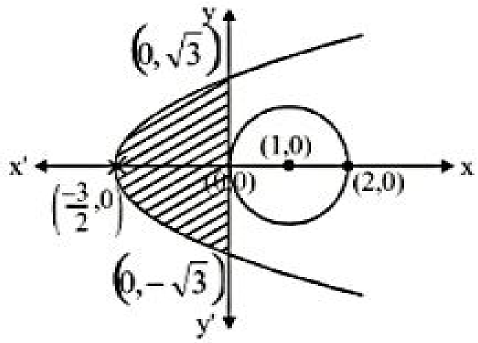
$$A = \{z : |z+1| \le 2 + \text{Re}(z)\},\quad B = \{z : |z-1| \ge 1\} \text{ and } C = \left\{z : \left|\frac{z-1}{z+1}\right| \ge 1\right\}$$

Q.12 The number of point(s) having integral coordinates in the region $A \cap B \cap C$ is
(A) 4 (B) 5 (C) 6 (D) 10

Q.13 The area of region bounded by $A \cap B \cap C$ is
(A) $2\sqrt{3}$ (B) $\sqrt{3}$ (C) $4\sqrt{3}$ (D) 2

Q.14 The real part of the complex number in the region $A \cap B \cap C$ and having maximum amplitude is
(A) -1 (B) $\frac{-3}{2}$ (C) $\frac{1}{2}$ (D) -2

Sol. For A, $|z+1| \le 2 + \text{Re}(z)$
$\Rightarrow \quad (x+1)^2 + y^2 \le 4 + 4x + x^2$
$\Rightarrow \quad y^2 \le 3 + 2x$
$\Rightarrow \quad y^2 \le 2\left(x + \frac{3}{2}\right)$ .....(1)



For B, $|z-1| \ge 1$
$\Rightarrow \quad (x-1)^2 + y^2 \ge 1$ ......(2)
For C, $|z-1|^2 \ge |z+1|^2$
$\Rightarrow \quad (z-1)(\bar{z}-1) \ge (z+1)(\bar{z}+1)$
$\Rightarrow \quad (z\bar{z} - \bar{z} - z + 1) \ge (z\bar{z} + \bar{z} + z + 1)$
$\Rightarrow \quad z + \bar{z} \le 0$
i.e. $\quad x \le 0$ .....(3)

(i) $(-1,0), (-1,1), (-1,-1), (0,0), (0,1), (0,-1)$ but $z = -1$ is not in the domain in set C
$\therefore$ Total number of point(s) having integral coordinates in the region $A \cap B \cap C$ is 5.

(ii) Required area $= 2\int_{-\frac{3}{2}}^{0}\sqrt{2\left(x+\frac{3}{2}\right)}\,dx = 2\sqrt{3}$ (square units)

(iii) Clearly $z = \frac{-3}{2} + i0$ is the complex number in the region $A \cap B \cap C$ and having maximum amplitude.

$\therefore \quad \text{Re}(z) = \frac{-3}{2}$

Paragraph for question nos. 15 to 17

Let $A(z_1)$ be the point of intersection of curves $\arg(z - 2 + i) = \frac{3\pi}{4}$ and $\arg\left(z + i\sqrt{3}\right) = \frac{\pi}{3}$.

$B(z_2)$ be the point on the curve $\arg\left(z + i\sqrt{3}\right) = \frac{\pi}{3}$ such that $|z_2 - 5|$ is minimum and $C(z_3)$ be the centre of circle $|z - 5| = 3$.

[Note : $i^2 = -1$**]**

Q.15 The area of triangle ABC is equal to

(A) $4\sqrt{3}$ (B) $\frac{3\sqrt{3}}{2}$ (C) $2\sqrt{3}$ (D) 4

Q.16 The equation of straight line passing through origin and perpendicular to line joining $A(z_1)$ and $B(z_2)$ on the complex plane is equal to

(A) $z = \lambda\left(2 + i\sqrt{3}\right)$ (B) $z = \lambda\left(-\sqrt{3} + i\right)$ (C) $z = \lambda\left(1 + i\sqrt{3}\right)$ (D) $z = \lambda\left(\sqrt{3} + i\right)$

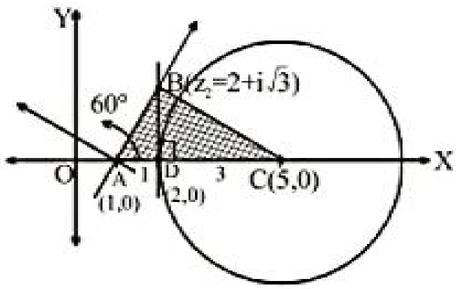
(where λ is real parameter.)

Q.17 If $|z - z_1| = 1$ and $\omega = \text{Re}(z + 2)$ then ω lie on

(A) real axis. (B) line not passing through the origin.
(C) line segment joining (2, 0) and (4, 0). (D) circle centred at (1, 0) and radius 2.

Sol.(i) Clearly, $A(z_1)$ is the point of intersection of $\arg(z - 2 + i) = \frac{3\pi}{4}$ and $\arg\left(z + i\sqrt{3}\right) = \frac{\pi}{3}$

$\Rightarrow \quad z_1 = 1$



Also, $B(z_2)$ is the point on $\arg\left(z + i\sqrt{3}\right) = \frac{\pi}{3}$

such that $|z_2 - 5| = \text{minimum}$, so $z_2 = 2 + i\sqrt{3}$.

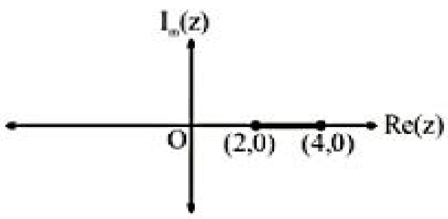
Ans $C(z_3)$ be the centre of the circle $|z - 5| = 3 \Rightarrow z_3 = 5$.

Hence, area $(\Delta ABC) = \frac{1}{2}(AC)(AD) = \frac{1}{2}(4)\left(\sqrt{3}\right) = 2\left(\sqrt{3}\right)$ (square units.)

(ii) Clearly, required equation of straight line is

$z = \lambda\, i\,(z_2 - z_1) = \lambda\, i\left(2 + i\sqrt{3} - 1\right) = \lambda\, i\left(1 + i\sqrt{3}\right) = \lambda\left(-\sqrt{3} + i\right)$.

(iii) We have



$|z - 1| = 1 \Rightarrow z = 1 + e^{i\theta}, \theta \in [0, 2\pi)$

$\Rightarrow \quad z = 1 + e^{i\theta} = 1 + \cos\theta + i\sin\theta$

$\therefore \quad \omega = \text{Re}(z + 2) = 3 + \cos\theta = \text{purely real}$

Clearly, locus of ω is the line segment joining (2, 0) and (4, 0). **Ans.**

Q.18 z_1 & z_2 are two distinct points in an argand plane. If $a|z_1| = b|z_2|$, (where $a, b \in R$) then prove that the point $\frac{a z_1}{b z_2} + \frac{b z_2}{a z_1}$ lies on the line segment [–2, 2] of the real axis.

Sol. Assuming $\arg z_1 = \theta$ and $\arg z_2 = \theta + \alpha$.

$$\frac{a z_1}{b z_2} + \frac{b z_2}{a z_1} = \frac{a|z_1|e^{i\theta}}{b|z_2|e^{i(\theta+\alpha)}} + \frac{b|z_2|e^{i(\theta+\alpha)}}{a|z_1|e^{i\theta}} = e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha$$

Alternatively: Let $\alpha = \frac{az_1}{bz_2}$; $\quad \frac{1}{\alpha} = \frac{bz_2}{az_1}$; Also $|\alpha| = \frac{|az_1|}{|bz_2|} = \frac{a|z_1|}{b|z_2|} = 1 \Rightarrow \alpha = \frac{1}{\overline{\alpha}}$

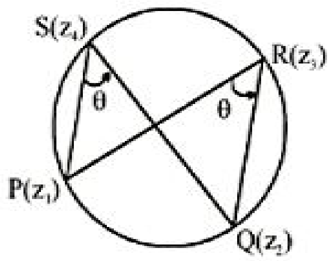
$$\Rightarrow \alpha + \frac{1}{\alpha} = \alpha + \overline{\alpha} = 2\,\text{Re.}(\alpha) = 2\cos\alpha$$

Q.19 If z_1, z_2, z_3 are complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.

Sol. Since $P(z_1)$, $Q(z_2)$, $P(z_3)$ and $S(z_4)$ are concyclic points,

$$\angle PSQ = \angle PRQ \qquad \Rightarrow \qquad \arg\frac{z_2 - z_4}{z_1 - z_4} = \arg\frac{z_2 - z_3}{z_1 - z_3}$$

$$\Rightarrow \qquad \arg\left[\left(\frac{z_2 - z_4}{z_1 - z_4}\right)\left(\frac{z_1 - z_3}{z_2 - z_3}\right)\right] = 0$$



$$\Rightarrow \qquad \frac{(z_2 - z_4)}{(z_1 - z_4)} \cdot \frac{(z_1 - z_3)}{(z_2 - z_3)} = \text{real}$$

If $\quad z_4 = 0 + i0$, then $\frac{z_2}{z_1} \cdot \frac{z_1 - z_3}{z_2 - z_3} = \text{real}$ ....(1)

We have $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$ from which $z_3 = \frac{z_1 z_2}{2z_2 - z_1}$ .....(2)

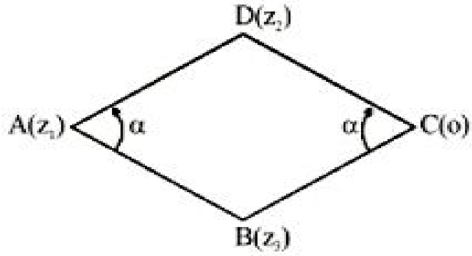
From (1) and (2), $\frac{z_2}{z_1} \times \dfrac{z_1 - \frac{z_1 z_2}{2z_2 - z_1}}{z_2 - \frac{z_1 z_2}{2z_2 - z_1}} = \text{real}$

$$\Rightarrow \qquad \frac{z_2 - z_1}{2(z_2 - z_1)} = \text{real} \Rightarrow \frac{1}{2} = \text{real, which is true.}$$

Therefore, z_1, z_2, z_3 and the origin are concylic.

Alternative

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} \quad \Rightarrow \quad \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1}$$



$$\Rightarrow \quad \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_1 z_3} \quad \Rightarrow \quad \frac{z_2 - z_1}{z_3 - z_1} = -\frac{z_2}{z_3}$$

$$\Rightarrow \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(-\frac{z_2}{z_3}\right) = \pi + \arg\left(\frac{z_2}{z_3}\right)$$

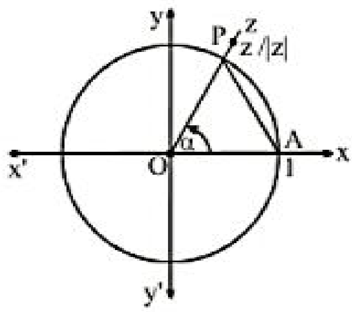
$\Rightarrow \quad \alpha = \pi - \beta \Rightarrow \alpha + \beta = \pi$

$\Rightarrow$ points A, B, C, D are concyclic.

Q.20 Prove the following inequalities geometrically and analytically :

(a) $\left|\frac{z}{|z|} - 1\right| \le |\arg z|$ (b) $|z - 1| \le \left||z| - 1\right| + |z||\arg z|$.

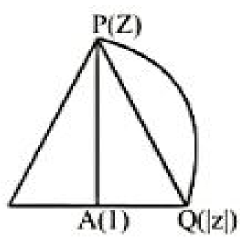
Sol.(a) The quantity $\frac{z}{|z|}$ lies on a unit circle centered at the origin.



It is clear that $\left|\frac{z}{|z|} - 1\right| = AP \le \text{arc (AP)} = \alpha = \arg(z)$

If α is negative, this gives $\left|\frac{z}{|z|} - 1\right| \le |\arg z|$.

To show this analytically, put $z = r(\cos\alpha + i\sin\alpha)$,



where $r = |z|$. Then $\left|\frac{z}{|z|} - 1\right| = |(\cos\alpha - 1) + i\sin\alpha|$

$$= \left[(\cos\alpha - 1)^2 + \sin^2\alpha\right]^{1/2} = [2(1 - \cos\alpha)]^{1/2}$$

$$= \left(4\sin^2\frac{\alpha}{2}\right)^{1/2} = 2\left|\sin\frac{\alpha}{2}\right| \le 2\left|\frac{\alpha}{2}\right| = |\alpha| = |\arg z| \cdot [\sin\theta \le \theta \text{ if } \theta \ge 0]$$

(b) Referring to the show figure and usign the result of part (a),

we have $|z-1| = AP \le AQ + QP = ||z|-1| + |z-|z||$

$$\Rightarrow \quad |z-1| \le ||z|-1| + |z|\left|\frac{z}{|z|}-1\right| \le ||z|-1| + |z||\arg z|.$$

Analytically, we can get this result as follows :

$$|z-1| = |(z-|z|)| + (|z|-1) \le |z-|z|| + ||z|-1|$$

$$\Rightarrow \quad |z-1| \le ||z|-1| + |z|\left|\frac{z}{|z|}-1\right| \le ||z|-1| + |z||\arg z|.$$

Q.21 If $z = 2+t+i\sqrt{3-t^2}$, where t is real and $t^2 < 3$, show that the modulus of $\frac{(z+1)}{(z-1)}$ is independent of t. Also show that the locus of the points z for different values of t is a circle and fnd its centre and radius.

Sol. We have $\frac{z+1}{z-1} = \frac{3+i+i\sqrt{3-t^2}}{1+t+i\sqrt{3-t^2}}$

$$\therefore \quad \left|\frac{z+1}{z-1}\right|^2 = \frac{(3+t)^2+3-t^2}{(1+t^2)^2+3-t^2} = \frac{6(t+2)}{2(t+2)} = 3 \qquad \text{(independent of r)}$$

Let $z = x+iy$, then $x+iy = 2+t+i\sqrt{3-t^2}$

Equating real and imaginary parts, we get

$x = 2+t$ ...(i) and $y = \sqrt{3-t^2}$ ...(ii)

[In order to find the locus of z we will have a eliminate t]

Putting the value of t from (i) and (ii), we get

$y^2 = 3-(x-2)^2$ or $(x-2)^2 + y^2 = 3$

Thus locus of z is a circle whose centre is (2, 0) and radius is $\sqrt{3}$.

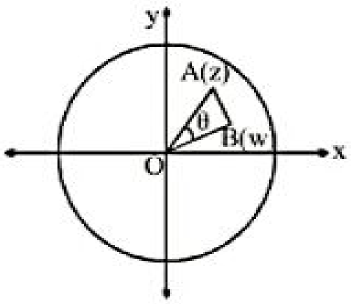
Q.22 $|z| \le 1$, $|w| \le 1$, show that $|z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$.

Sol. Let us consider a unit circle will its centre as the origin.

Let $\angle AOX = \theta_1$ and $\angle BOX = \theta_2$

$\therefore \arg(z) = \theta_1$ and $\arg(\omega) = \theta_2$.

$z = \overline{OA}$, $w = \overline{OB}$



Now in ΔOAB,

$$\cos\theta = \frac{|\overline{OA}|^2 + |\overline{OB}|^2 - |\overline{BA}|^2}{2|\overline{OA}||\overline{OB}|^2}$$

$$\Rightarrow |\overline{BA}|^2 = |\overline{OA}|^2 + |\overline{OB}|^2 - 2|\overline{OA}||\overline{OB}|\cos\theta$$

$$\Rightarrow \quad |z - w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos\theta = (|z| - |w|)^2 - 4|z||w|\sin^2\left(\frac{\theta}{2}\right)$$

We know $\sin\frac{\theta}{2} \le \frac{\theta}{2}$ $\quad \left[0 \le \frac{\theta}{2} < \frac{\pi}{2}\right]$

Hence $|z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$