# TRIGONOMETRIC EQUATION

## 1. INTRODUCTION :

An equation involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation. e.g., $\cos x = \frac{1}{2}$ ; $\sin^2 x - 4\cos x = 1$.

The value of an unknown angle which satisfies the given trigonometric equation is called a solution or root of the equation. For example $2\sin\theta = \sqrt{3}$, Clearly $\theta = 60°$ and $120°$ are solutions of the equation between $0°$ and $360°$.

Now suppose $\tan\theta = 1$, then there will be many possible vlaues of $\theta$, so our main ojective is to write down all the solution in short form. Since all trigonometric functions are periodic and therefore solution of all trigonometrical equation can be generalized with the help of periodicity of trigonometrical function.

### 1.1. Principal solution of a Trigonometric Equation :

### 1.1. Principal solution of a Trigonometric Equation :

The solutions of a trigonometric equation $\sin\theta = \frac{1}{2}$ lying in the interval $[0, 2\pi]$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Thus principal solutions of $\sin\theta = \frac{1}{2}$ will be $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

### 1.2 General solution of a Trigonometric Equation :

(i) If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

(ii) If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$, $n \in I$.

(iii) If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

(iv) If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
(v) If $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
(vi) If $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

**Note :** $\alpha$ is called the principal angle.

**Proof :**

(i) $\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0 \Rightarrow 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \quad \cos\left(\frac{\theta+\alpha}{2}\right) = 0$ or $\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \quad \frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2}$ or $\frac{\theta-\alpha}{2} = m\pi$, where $m \in I$.

$\Rightarrow \quad \theta = (2m+1)\pi - \alpha$ or $\theta = 2m\pi + \alpha$, where $m \in I$.

$\Rightarrow \quad \theta = (2m+1)\pi + (-1)^{2m+1}\alpha$ or $\theta = 2m\pi + (-1)^{2m}\alpha$

$\Rightarrow \quad \theta = n\pi + (-1)^n\alpha$, $n \in I$.

(ii) $\cos\theta = \cos\alpha \Rightarrow \cos\alpha - \cos\theta = 0 \Rightarrow 2\sin\left(\frac{\alpha+\theta}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \quad \sin\frac{\theta+\alpha}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$, $\frac{\theta+\alpha}{2} = n\pi$ or $\frac{\theta-\alpha}{2} = n\pi$

$\Rightarrow \quad \theta = 2n\pi - \alpha$, or $\theta = 2n\pi + \alpha$ $n \in I$

$\Rightarrow \quad \theta = 2n\pi \pm \alpha$.

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$\Rightarrow \quad \theta = 2n\pi - \alpha$, or $\theta = 2n\pi + \alpha$ $n \in I$

$\Rightarrow \quad \theta = 2n\pi \pm \alpha$.

(iii) $\tan\theta = \tan\alpha \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \sin\theta\cos\theta - \cos\theta.\sin\alpha = 0$

$\Rightarrow \quad \sin(\theta - \alpha) = 0 \Rightarrow \theta - \alpha = n\pi$

$\Rightarrow \quad \theta = n\pi + \alpha$, where $n \in I$

(iv) $\sin^2\theta = \sin^2\alpha$

$\sin^2\theta - \sin^2\alpha = \sin(\theta+\alpha)\sin(\theta-\alpha) = 0$

$\sin(\theta+\alpha) = 0$ or $\sin(\theta-\alpha) = 0$

$\theta + \alpha = n\pi$ or $\theta - \alpha = n\pi$, $n \in I$

$\theta = n\pi \pm \alpha$, $n \in I$

(v) $\cos^2\theta = \cos^2\alpha \Rightarrow 1 - \sin^2\theta = 1 - \sin^2\alpha \Rightarrow \sin^2\theta - \sin^2\alpha$

$\theta = n\pi \pm \alpha$, $n \in I$

(vi) $\tan^2\theta = \tan^2\alpha \Rightarrow \tan\theta = \pm\tan\alpha = \tan(\pm\alpha)$

$\Rightarrow \quad \theta = n\pi \pm \alpha$, where $n \in I$

**Remember :**

| Trigonometric equation withtheir general solution | | |
|---|---|---|
| **Trigonometrical equation** | | **General Solution** |
| $\sin\theta = 0$ | $\Rightarrow$ | $\theta = n\pi$ |
| $\cos\theta = 0$ | $\Rightarrow$ | $\theta = (2n+1)\pi/2$ |
| $\tan\theta = 0$ | $\Rightarrow$ | $\theta = n\pi$ |
| $\sin\theta = 1$ | $\Rightarrow$ | $\theta = 2n\pi + \pi/2$ |
| $\sin\theta = \sin\alpha$ | $\Rightarrow$ | $\theta = n\pi + (-1)^n\alpha$ |
| $\cos\theta = \cos\alpha$ | $\Rightarrow$ | $\theta = 2n\pi \pm \alpha$ |
| $\tan\theta = \tan\alpha$ | $\Rightarrow$ | $\theta = n\pi + \alpha$ |
| $\sin^2\theta = \sin^2\alpha$ | $\Rightarrow$ | $\theta = n\pi \pm \alpha$ |
| $\tan^2\theta = \tan^2\alpha$ | $\Rightarrow$ | $\theta = n\pi \pm \alpha$ |
| $\cos^2\theta = \cos^2\alpha$ | $\Rightarrow$ | $\theta = n\pi \pm \alpha$ |
| $\sin\theta = \sin\alpha$<br>$\cos\theta = \cos\alpha$ | $\Rightarrow$ | $\theta = 2n\pi + \alpha$ |
| $\sin\theta = \sin\alpha$<br>$\tan\theta = \tan\alpha$ | $\Rightarrow$ | $\theta = 2n\pi + \alpha$ |
| $\tan\theta = \tan\alpha$<br>$\cos\theta = \cos\alpha$ | $\Rightarrow$ | $\theta = 2n\pi + \alpha$ |

$\cos\theta = \cos\alpha$

***Illustration :***

*Solve* $4\cos^2\theta - 4\sin\theta - 1 = 0,\ 0 \le \theta \le 2\pi.$

**Sol.** $4(1-\sin^2\theta) - 4\sin\theta - 1 = 0 \Rightarrow 4\sin^2\theta + 4\sin\theta - 3 = 0$

$$\Rightarrow \quad \sin\theta = \frac{1}{2}, -\frac{3}{2} \text{ (Not possible)}$$

$$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

*Thus we have two principal solutions.*

***Illustration :***

*Find the general value of $\theta$ satisfying both $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ simultaneously.*

**Sol.** $\sin\theta < 0, \tan\theta > 0$ *so $\theta$ is in third quadrant.*

*Now,* $\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}, \frac{7\pi}{6}$

*Common solution* $= \frac{7\pi}{6}$

*So general solution* $\theta = 2n\pi + \frac{7\pi}{6}$.

***Illustration :***

*Solve :* $\sqrt{3}\sec 2\theta = 2$.

**Sol.** $\cos 2\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{6}$

**Sol.** $\cos 2\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{6}$

$\theta = n\pi \pm \frac{\pi}{12}, n \in I$

***Illustration :***

*Solve :* $\tan(3\theta) = -1$.

**Sol.** $\tan 3\theta = \tan\left(-\frac{\pi}{4}\right)$

$\Rightarrow \quad 3\theta = n\pi + \left(-\frac{\pi}{4}\right) \Rightarrow \theta = \frac{n\pi}{3} - \frac{\pi}{12}, n \in I$

***Illustration :***

*Solve* $7\cos^2\theta + 3\sin^2\theta = 4$

**Sol.** $7(1 - \sin^2\theta) + 3\sin^2\theta = 4, \quad 4\sin^2\theta = 3$

$\sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2 = \sin^2\frac{\pi}{3}$

$\theta = n\pi \pm \frac{\pi}{3}, n \in I$

## 2. TYPES OF TRIGONOMETRIC EQUATION :

### 2.1 TYPE-I :

Solution of trigonometric equation by factorization or equation which are expressed in quadratic form or which can be expressed in quadratic form :

***Illustration :***

*Solve* $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$ *in* $[0, 2\pi]$.

**Sol.** $(1 + \cos x)(2\sin x - \cos x) - (1 - \cos^2 x) = 0$

$(1 + \cos x)[(2\sin x - \cos x) - (1 - \cos\ x)] = 0$, $(1 + \cos x)(2\sin x - 1) = 0$

$\sin x = \dfrac{1}{2}$, $\cos x = -1.$

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \pi.$

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \pi.$

***Illustration :***

*Find the general solutions of equation* $1 - \cos x \cot x = \cot x - \cos x.$

**Sol.** $(1 + \cos x)(1 - \cot x) = 0 \quad \Rightarrow \quad \cos x = -1$ *or* $\cot x = 1$

$\Rightarrow \quad (2n + 1)\pi, \quad n\pi + \dfrac{\pi}{4}, \ n \in I$

***Illustration :***

*Find the general solutions of equation* $3\cos^2 x - 10\cos x + 3 = 0.$

**Sol.** $(3\cos x - 1)(\cos x - 3) = 0 \ \Rightarrow \cos x = \dfrac{1}{3}, 3$ *but* $\cos x \neq 3$

$\therefore \quad \cos x = \dfrac{1}{3} \Rightarrow x = 2n\pi \pm \cos^{-1}\left(\dfrac{1}{3}\right)$

***Illustration :***

*Find the general solutions of equation* $2\sin^2 2x + 6\sin^2 x = 5$.

***Sol.*** $8\sin^2 x \,.\, \cos^2 x + 6\sin^2 x = 5$.

$\Rightarrow \quad 8\sin^4 x - 14\sin^2 x + 5 = 0 \quad \Rightarrow \quad (2\sin^2 x - 1)(4\sin^2 x - 5) = 0$

$\Rightarrow \quad \sin^2 x = \frac{1}{2}, \frac{5}{4} \Rightarrow \sin^2 x = \frac{1}{2} \Rightarrow n\pi \pm \frac{\pi}{4}, n \in I$

***Illustration :***

*Find the general solutions of equation* $(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$.

***Sol.*** $(1 - \tan\theta)\left(1 + \frac{2\tan\theta}{1+\tan^2\theta}\right) = (1 + \tan\theta) \quad \Rightarrow \quad \frac{(1-\tan\theta)(1+\tan\theta)^2}{1+\tan^2\theta} = (1 + \tan\theta)$

$\therefore \quad 1 + \tan\theta = 0 \;\; or \;\; \frac{(1-\tan^2\theta)}{1+\tan^2\theta} = 1$

$\tan\theta = -1 \;\; or \;\; \cos 2\theta = 1$

$\tan\theta = -1 \;\; or \;\; \cos 2\theta = 1$

$\theta = n\pi - \frac{\pi}{4} \quad or \quad 2\theta = 2n\pi$

$\therefore \quad \theta = n\pi - \frac{\pi}{4}, n\pi, \quad n \in I$

## ***Practice Problem***

Q.1 Find the most general values of θ satisfying

(i) $\tan^2\theta = \frac{1}{3}$ (ii) $\sec^2\theta = \frac{4}{3}$ (iii) $2\cot^2\theta = \operatorname{cosec}^2\theta$

Q.2 What is the most general value of θ satisfying both the equations $\cos\theta = -\frac{1}{\sqrt{2}}$ and $\tan\theta = 1$.

Q.3 Solve : $3\sin^2 x - \sin x \cos x - 4\cos^2 x = 0$

Q.4 Find the number of solution of the equation $16^{\sin^2 x} + 16^{\cos^2 x} = 10$, where $0 \le x \le 2\pi$.

Q.5 Find the general solution of $\cos 4x + 6 = 7\cos 2x$, and also find the sum of all the solution in $[0, 3\pi]$.

## Answer key

Q.1 (i) $n\pi \pm \frac{\pi}{6}$ (ii) $n\pi \pm \frac{\pi}{6}$ (iii) $n\pi \pm \frac{\pi}{4}$ Q.2 $2n\pi + \frac{5\pi}{4}$

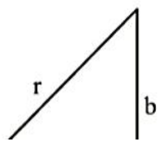
Q.3 $n\pi - \frac{\pi}{4}, n\pi + \alpha$, where $\tan\alpha = \frac{4}{3}$. Q.4 8 Q.5 $3\pi$

## 2.2 TYPE-II :

Solving trigonometric equations by introducing an Auxiliary argument. Equation of the form of $a\cos\theta + b\sin\theta = c$

To solve equation, we convert the equation to the form $\cos\theta = \cos\alpha$ or $\sin\theta = \sin\alpha$ etc.

For this let us suppose that $\left.\begin{array}{l} a = r\cos\phi \\ b = r\sin\phi \end{array}\right\} \Rightarrow \left\{\begin{array}{l} r = \sqrt{a^2+b^2} \\ \text{and } \tan\phi = \frac{b}{a}, \end{array}\right.$

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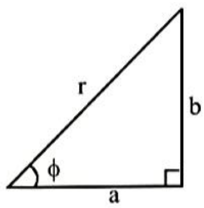



Substituting these values in the equation $a\cos\theta + b\sin\theta = c$, we have

$$r\cos\phi\cos\theta + r\sin\phi\sin\theta = c$$

$$\Rightarrow \quad r\cos(\theta - \phi) = c$$

$$\Rightarrow \quad \cos(\theta - \phi) = \frac{c}{r} = \frac{c}{\sqrt{a^2+b^2}} = \cos\beta \text{ (suppose)}$$

$$\Rightarrow \quad \theta - \phi = 2n\pi \pm \beta$$

$$\Rightarrow \quad \theta = 2n\pi + \phi \pm \beta, n \in Z$$

Here $\phi$ and $\beta$ are known as a, b and c are given.

Hence, we can solve the equation of this type by putting.

$a = r\cos\phi$ and $b = r\sin\phi$ provided $\left|\frac{c}{\sqrt{a^2+b^2}}\right| \le 1$ [$\because$ $\cos\beta$ lies between $-1$ and 1]

or $\frac{|c|}{\sqrt{a^2+b^2}} \le 1$ or $|c| \le \sqrt{a^2+b^2}$

***Illustration :***

*Find general value of* $\sin x + \cos x = \sqrt{2}$.

**Sol.** $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1, \qquad \sin x \sin\left(\frac{\pi}{4}\right) + \cos x \cos\left(\frac{\pi}{4}\right) = 1$

$\cos\left(x - \frac{\pi}{4}\right) = 1, \; x - \frac{\pi}{4} = 2n\pi, \qquad x = 2n\pi + \frac{\pi}{4}$

***Illustration :***

*Find general value of* $\sqrt{3}\ \cos x + \sin x = 2$.

**Sol.** $\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = 1, \qquad \cos x \cos\frac{\pi}{6} + \sin x \sin\left(\frac{\pi}{6}\right) = 1.$

$\cos\left(x - \frac{\pi}{6}\right) = 1 \qquad so \qquad x - \frac{\pi}{6} = 2n\pi, \; x = 2n\pi + \frac{\pi}{6}, \; n \in I.$

$\cos\left(x - \frac{\pi}{6}\right) = 1 \qquad so \qquad x - \frac{\pi}{6} = 2n\pi, \; x = 2n\pi + \frac{\pi}{6}, \; n \in I.$

***Illustration :***

*Find the general solutions of equation* $\sin x + \cos x = \frac{3}{2}$

**Sol.** $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{3}{2\sqrt{2}}$

$\sin\left(x + \frac{\pi}{4}\right) = \frac{3}{2\sqrt{2}} > 1$ *so no solution.*

***Illustration :***

*Find the general solutions of equation* $(\sec x - 1) = (\sqrt{2} - 1)\tan x$.

**Sol.** $\sec x + \tan x = \sqrt{2}\ \tan x + 1$

$\sec x - \tan x = \frac{1}{\sqrt{2}\tan x + 1}$

$$2\tan x = (\sqrt{2}\tan x + 1) - \frac{1}{(\sqrt{2}\tan x + 1)} = \frac{(\sqrt{2}\tan x + 1)^2 - 1}{(\sqrt{2}\tan x + 1)}$$

$$2\tan x = \frac{\sqrt{2}\tan x(\sqrt{2}\tan x + 2)}{(\sqrt{2}\tan x + 1)}$$

$\Rightarrow \quad \tan x = 0,\ x = m\pi,\ m \in I.$

or $\quad 2\sqrt{2}\tan x + 2 = 2\tan x + 2\sqrt{2}$

$\Rightarrow \quad \tan x\left(2\sqrt{2} - 2\right) = 2\sqrt{2} - 2,\ \ \tan x = 1$

$\Rightarrow \quad x = n\pi + \frac{\pi}{4},\ n \in I.$

## Practice Problem

Q.1 Find the general solution of the equation $4\cos x + 3\sin x = 5$.

Q.2 If $1 + \sin^3 x + \cos^3 x = \frac{3}{2}\sin 2x$, then find general solution of x.

Q.3 Solve : $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta\cos 3\theta$.

Q.4 Solve : $3\sqrt{3}\sin^3 x + \cos^3 x + 3\sqrt{3}\sin x + \cos x = 1$

Q.3 Solve : $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta\cos 3\theta$.

Q.4 Solve : $3\sqrt{3}\sin^3 x + \cos^3 x + 3\sqrt{3}\sin x + \cos x = 1$.

## Answer key

Q.1 $2n\pi + \alpha$, where $\tan\alpha = \frac{3}{4}$

Q.2 $(2n+1)\pi$, $(4n-1)\frac{\pi}{2}$

Q.3 $\frac{n\pi}{3} + \frac{\pi}{18}$, $\frac{-n\pi}{2} + \frac{\pi}{6}$

Q.4 $2n\pi$, $(3n-1)\frac{2\pi}{3}$

## 2.3 TYPE-III :

Solving equations by tranforming a sum of trigonometric functions into a product.

***Illustration :***

*General solution of $\sin x + \sin 5x = \sin 2x + \sin 4x$ is*

(A) $\frac{n\pi}{3}$ (B) $\frac{2n\pi}{3}$ (C) $2n\pi$ (D) $n\pi$

**Sol.** $2 \sin 3x \cos 2x = 2\sin 3x \cos x$

$\Rightarrow \quad \sin 3x = 0$ or $\cos 2x = \cos x$

$\Rightarrow \quad 3x = n\pi$ or $2x = 2n\pi \pm x$

$\therefore \quad \Rightarrow \quad x = \frac{n\pi}{3}, 2n\pi, \frac{2n\pi}{3}$

$\Rightarrow \quad x = \frac{n\pi}{3}$

## 2.4 TYPE-IV :

Solving equations by tranforming a product of trigonometric functions into a sum.

***Illustration :***

*Number of solutions of the trigonometric equation in $[0, \pi]$, $\sin 3\theta = 4\sin\theta \cdot \sin 2\theta \cdot \sin 4\theta$.*

(A) 4 (B) 6 (C) 8 (D) 10

**Sol.** $\sin 3\theta = 4 \sin \theta \sin (3\theta - \theta) \sin(3\theta + \theta) = 4 \sin\theta (\sin^2 3\theta - \sin^2\theta)$

$\Rightarrow \quad \sin 3\theta + 4 \sin^3\theta = 4 \sin \theta \sin^2 3\theta$

$\Rightarrow \quad 3 \sin \theta = 4 \sin \theta \sin^2 3\theta \quad \Rightarrow \quad \sin\theta = 0$ or $\sin^2 3\theta = \frac{3}{4}$

$\Rightarrow \quad \sin 3\theta + 4 \sin^3\theta = 4 \sin \theta \sin^2 3\theta$

$\Rightarrow \quad 3 \sin \theta = 4 \sin \theta \sin^2 3\theta \quad \Rightarrow \quad \sin\theta = 0$ or $\sin^2 3\theta = \frac{3}{4}$

$\Rightarrow \quad \theta = n\pi$ or $3\theta = n\pi \pm \frac{\pi}{3} \quad \Rightarrow \quad \theta = n\pi$ or $\theta = \frac{n\pi}{3} \pm \frac{\pi}{9}, n \in I$

$\Rightarrow \quad \theta = 0, \pi, \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$

***Illustration :***

*Find the number of solutions of the equations,*

$\sin x + 2 \sin 2x = 3 + \sin 3x$ in $[0, \pi]$

(A) No solution (B) Infinite solution (C) Exactly one solution (D) Two solutions

**Sol.** *We have,*

$\Rightarrow \quad \sin 3x - \sin x - 2 \sin 2x + 3 = 0$

$\Rightarrow \quad 2 \cos 2x \cdot \sin x - 4 \sin x \cdot \cos x + 3 = 0$

$\Rightarrow \quad \sin x (2 \cos 2x - 4 \cos x) + 3 = 0$

$\Rightarrow \quad \sin x (4 \cos^2 x - 4 \cos x - 2) + 3 = 0$

$\Rightarrow \quad \sin x (4 \cos^2 x - 4 \cos x + 1) + 3 - 3 \sin x = 0$

$\Rightarrow \quad \sin x (2 \cos x - 1)^2 + 3 (1 - \sin x) = 0 \quad \ldots(i)$

*since* $x \in [0, \pi]$, $\therefore$ $\sin x \geq 0$ *and* $1 - \sin x \geq 0$

$\therefore$ *each part in equation (i) must be zero.*

*i.e.* $\sin x\,(2\cos x - 1)^2 = 0$ *and* $3\,(1 - \sin x) = 0$

*from the second equation of system we have*

$\sin x = 1 \Rightarrow \cos x = 0$ *hence* $\sin x\,(2\cos^2 x - 1)^2 \neq 0$

$\therefore$ *not a single solution of the second equation is a solution of the first.*

*Hence the original equation has no real solution.*

***Illustration :***

*Find the number of solution of the equation in* $[0, 2\pi]$, $\tan(5\pi\cos\alpha) = \cot(5\pi\sin\alpha)$

**Sol.** $5\pi\cos\alpha = n\pi + \left(\frac{\pi}{2} - 5\pi\sin\alpha\right)$

$\Rightarrow$ $\sin\alpha + \cos\alpha = \frac{(2n+1)}{10}$ *as* $-\sqrt{2} \leq \sin\alpha + \cos\alpha \leq \sqrt{2}$

$n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, -7$

*For each value of* $n$, *we get two values of* $\alpha \in [0, 2\pi]$

$\therefore$ $2 \times 14 = 28$ *solutions.*

## *Practice Problem*

Q.1 Find the general values of x satisfying $\cos^2 x + \cos^2 2x + \cos^2 3x + \cos^2 4x = 2$.

Q.2 $\operatorname{cosec} x - \operatorname{cosec} 2x = \operatorname{cosec} 4x$. Find general vlaues of x ?

Q.3 Solve $\cos x \cos 2x \cos 3x = \frac{1}{4}$

Q.4 Solve $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$

Q.5 Solve $\sec 4\theta - \sec 2\theta = 2$.

## *Answer key*

Q.1 $(2m+1)\frac{\pi}{10}$, $(2n+1)\frac{\pi}{2}$, $(2k+1)\frac{\pi}{4}$ Q.2 $(2m-1)\frac{\pi}{7}$, $m \neq 7k - 3$, $k \in I$

Q.3 $m\pi \pm \frac{\pi}{3}$, $(2n+1)\frac{\pi}{8}$ Q.4 $(2n+1)\frac{\pi}{4}$ Q.5 $(2m+1)\frac{\pi}{2}$, $(2n+1)\frac{\pi}{10}$

## 2.5 TYPE-V :

**Solving equations by a change of variable or by substitution method :**

**(i)** Equations of the form $P(\sin x \pm \cos x, \sin x . \cos x) = 0$, where P(x, z) is a polynominal, can be solved by the change.$\cos x \pm \sin x = t$

$\Rightarrow$ $1 \pm 2\sin x.\cos x = t^2$. Consider the equation ; $\sin x + \cos x = 1 + \sin x . \cos x$.

**(ii)** Many equations can be solved by introducing a new variable e.g. consider the equation $\sin^4 2x + \cos^4 2x = \sin 2x. \cos 2x$.

***Illustration :***

*Find general value of x satisfying the equaiton $\sin^4 2x + \cos^4 2x = \sin 2x \cos 2x$.*

***Sol.*** *Let* $\sin 2x . \cos 2x = y$.

$\Rightarrow$ $\sin^4 2x + \cos^4 2x = \sin 2x. \cos 2x$

$\Rightarrow$ $(\sin^2 2x + \cos^2 2x) - 2 \sin^2 2x \cos^2 2x = \sin 2x \cos 2x$

$\Rightarrow$ $1 - 2 y^2 = y \Rightarrow 2y^2 + y - 1 = 0$

$\Rightarrow$ $y = -1, \frac{1}{2}$

$\Rightarrow$ $\sin 2x . \cos 2x = y = -1, \frac{1}{2}$

$\Rightarrow$ $\sin 2x . \cos 2x = y = -1, \frac{1}{2}$

$\Rightarrow$ $\sin 4x = -2$ *(rejected)*, $1$

$\Rightarrow$ $\sin 4x = 1$ $\qquad \Rightarrow$ $4x = 2n\pi + \frac{\pi}{2}, n \in I$

$\Rightarrow$ $x = (4n + 1)\frac{\pi}{8}$

***Illustration :***

*Solve the equation :* $\sin x + \cos x - 2\sqrt{2} \sin x \cos x = 0$.

***Sol.*** *Let* $(\sin x + \cos x) = t$ *and using the equation*

$\sin x \cdot \cos x = \frac{t^2 - 1}{2}$, *we get*

$$t - 2\sqrt{2}\left(\frac{t^2 - 1}{2}\right) = 0$$

$\Rightarrow$ $$\sqrt{2}\, t^2 - t - \sqrt{2} = 0$$

*The numbers* $t_1 = \sqrt{2}$, $t_2 = -\frac{1}{\sqrt{2}}$ *are roots of this quadritic equation.*

*Thus the solution of the given equation reduces to the solution of two trigonometric equations :*

$$\sin x + \cos x = \sqrt{2}$$

or $$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1$$

or $$\sin x \cdot \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = 1$$

$$\Rightarrow \quad \sin\left(x+\frac{\pi}{4}\right) = 1$$

$$\Rightarrow \quad x + \frac{\pi}{4} = (4n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad x = 2n\pi + \frac{\pi}{4}$$

$$\sin x + \cos x = -\frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{2}$$

$$\sin x \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = -\frac{1}{2}$$

$$\sin\left(x+\frac{\pi}{4}\right) = -\frac{1}{2}$$

$$x + \frac{\pi}{4} = n\pi + (-1)^n \cdot \left(-\frac{\pi}{6}\right)$$

$$x = n\pi + (-1)^{n+1}\frac{\pi}{6} - \left(\frac{\pi}{4}\right)$$

***Illustration :***

*Solve the equation :* $\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x.$

**Sol.** *Using half-angle formulae we can represent the given equation in the form;*

$16$

**Sol.** *Using half-angle formulae we can represent the given equation in the form;*

$$\left(\frac{1-\cos 2x}{2}\right)^5 + \left(\frac{1+\cos 2x}{2}\right)^5 = \frac{29}{16}\cos^4 2x$$

*Put* $\cos 2x = t$,

$$\left(\frac{1-t}{2}\right)^5 + \left(\frac{1+t}{2}\right)^5 = \frac{29}{16}t^4$$

$$\Rightarrow \quad 24t^4 - 10t^2 - 1 = 0$$

$$\Rightarrow \quad (12t^2+1)(2t^2-1) = 0$$

*whose only real root is,* $t^2 = \frac{1}{2}$.

$$\therefore \quad \cos^2 2x = \frac{1}{2}$$

$$\Rightarrow \quad 1 + \cos 4x = 1$$

$$\Rightarrow \quad \cos 4x = 0$$

$$\Rightarrow \quad 4x = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad x = \frac{n\pi}{4} + \frac{\pi}{8};\ n \in \text{Integer}$$

***Illustration :***

*Solve :* $\tan\theta + \tan 2\theta + \tan 3\theta = 0$

**Sol.** $(\tan\theta + \tan 2\theta) + \left(\dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta}\right) = 0$

$\Rightarrow \quad (\tan\theta + \tan 2\theta)\,(2 - \tan\theta\ \tan 2\theta) = 0$

$\Rightarrow \quad \tan 2\theta + \tan\theta = 0 \quad or \quad \tan\theta\tan 2\theta = 2$

*when* $\tan 2\theta + \tan\theta = 0$

$\tan 2\theta = -\tan\theta = \tan(-\theta)$

$2\theta = n\pi - \theta \quad \Rightarrow \quad \theta = \dfrac{n\pi}{3}$

*when* $\tan\theta\tan 2\theta = 2$

$\therefore \quad \tan\theta\ \dfrac{2\tan\theta}{1 - \tan^2\theta} = 2$

$\Rightarrow \quad \tan^2\theta = 1 - \tan^2\theta$

$\Rightarrow \quad \tan^2\theta = \dfrac{1}{\ }$

$\Rightarrow \quad \tan^2\theta = \dfrac{1}{2}$

$\Rightarrow \quad \theta = n\pi \pm \tan^{-1}\left(\dfrac{1}{2}\right)$

***Illustration :***

*Solve for x :* $\sin 3\alpha = 4\sin\alpha\sin(x + \alpha)\sin(x - \alpha)$

**Sol.** $\sin 3\alpha = 4\sin\alpha\,(\sin^2 x - \sin^2\alpha)$

$\Rightarrow \quad \sin 3\alpha + 4\sin^3\alpha = 4\sin\alpha - \sin^2 x$

$\Rightarrow \quad 3\sin\alpha = 4\sin\alpha\sin^2 x$

$\Rightarrow \quad \sin\alpha = 0 \quad or \quad \sin^2 x = \dfrac{3}{4}$

$\Rightarrow \quad \alpha = n\pi \text{ or } x = n\pi \pm \dfrac{\pi}{3}$

***Illustration :***

*Solve for x :* $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$.

**Sol.** $\tan\theta + \tan 2\theta = \sqrt{3}\ (1 - \tan\theta\tan 2\theta)$

$\Rightarrow \quad \tan 3\theta = \sqrt{3}$

$\therefore \quad 3\theta = n\pi + \dfrac{\pi}{3}$

$\theta = \dfrac{n\pi}{3} + \dfrac{\pi}{9}$

***Illustration :***

*Solve the equation :* $4\cot 2\theta = \cot^2\theta - \tan^2\theta$

**Sol.** $\dfrac{4(1-\tan^2\theta)}{2\tan\theta} = \dfrac{(1-\tan^2\theta)(1+\tan^2\theta)}{\tan^2\theta}$

$\Rightarrow \quad 1 - \tan^2\theta = 0 \quad or \quad 2\tan\theta = 1 + \tan^2\theta$

$\Rightarrow \quad \tan^2\theta = 1 \quad or \quad (\tan\theta - 1)^2 = 0$

**Sol.** $\dfrac{}{2\tan\theta} = \dfrac{}{\tan^2\theta}$

$\Rightarrow \quad 1 - \tan^2\theta = 0 \quad or \quad 2\tan\theta = 1 + \tan^2\theta$

$\Rightarrow \quad \tan^2\theta = 1 \quad or \quad (\tan\theta - 1)^2 = 0$

$\Rightarrow \quad \tan^2\theta = 1 \quad \Rightarrow \quad \theta = n\pi \pm \dfrac{\pi}{4}$

***Illustration :***

*Find the general solution of equation* $\cos 2\theta = (\sqrt{2}+1)\left(\cos\theta - \dfrac{1}{\sqrt{2}}\right)$

**Sol.** $\Rightarrow \quad (2\cos^2\theta - 1) = (\sqrt{2} + 1)\ \dfrac{(\sqrt{2}\cos\theta - 1)}{\sqrt{2}}$

$\Rightarrow \quad \sqrt{2}\cos\theta - 1 = 0 \quad or \quad \sqrt{2}\cos\theta + 1 = \dfrac{\sqrt{2}+1}{\sqrt{2}}$

$\Rightarrow \quad \cos\theta = \dfrac{1}{\sqrt{2}} \quad or \quad \cos\theta = \dfrac{1}{2}$

$\Rightarrow \quad \theta = 2n\pi \pm \dfrac{\pi}{4} \quad or \quad \theta = 2n\pi \pm \dfrac{\pi}{3}$

## *Practice Problem*

Q.1 Prove that the equation $x^3 - 2x + 1 = 0$ is satisfied by putting for x, either of the values.

$\sqrt{2}\sin 45^\circ$, $2\sin 18^\circ$ and $2\sin 234^\circ$.

Q.2 If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, prove that $\cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

Q.3 If $\sin(\pi\cot\theta) = \cos(\pi\tan\theta)$, prove that either cosec $2\theta$ or $\cot 2\theta$ is equal to $n + \frac{1}{4}$, where n is a positive or negative integer.

## 2.6 TYPE-VI :

Solving equations with the use of boundedness of the function.

Solving equations with the use of boundedness of the function.

**Remember :-**

$-1 \le \sin x \le 1,\ -1 \le \cos x \le 1,\ \tan x \in R, \cot x \in R.$

$|\operatorname{cosec} x| \ge 1,\ |\sec x| \ge 1.$

***Illustration :***

*Solve for x :* $\cos x + \cos 2x + \cos 3x = 3.$

**Sol.** $\cos x = 1$ *and* $\cos 2x = 1$ *and* $\cos 3x = 1$

*when* $\cos x = 1 \Rightarrow x = 2n\pi,\ n \in I$

*when* $\cos 2x = 1 \Rightarrow x = \frac{2n\pi}{2} = n\pi,\ n \in I$

*when* $\cos 3x = 1 \Rightarrow 3x = 2n\pi$

$\Rightarrow x = \frac{2n\pi}{3},\ n \in I$

$2n\pi,\ n \in I$ *Ans.*

***Illustration :***

*Solve for x :* $\sin^3 x - \cos^3 x = 1 + \sin x \cos x.$

**Sol.** $(\sin x - \cos x)(\sin^2 x + \cos^2 x + \sin x \cos x) = 1 + \sin x \cos x$

$\Rightarrow \quad (\sin x - \cos x)(1 + \sin x \cos x) = 1 + \sin x \cos x$

$\Rightarrow \quad \sin x - \cos x = 1 \qquad or \qquad 1 + \sin x \cos x = 0$

$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$ | $2 \sin x \cos x = -2$

$\Rightarrow \quad x + \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4}$ | $\sin 2x = -2$

$\Rightarrow \quad x = 2n\pi \pm \frac{3\pi}{4} - \frac{\pi}{4}$ | $\Rightarrow$ No solution.

$\Rightarrow \quad x = (2n - 1)\pi,\ 2n\pi + \frac{\pi}{2}$

***Illustration :***

***Illustration :***

*Solve for x :* $\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right)\cos x = 0$

**Sol.** $\sin\left(\frac{5x}{4}\right) + \cos x = 2$

$\Rightarrow \quad \sin\left(\frac{5x}{4}\right) = 1 \quad and \quad \cos x = 1$

$\Rightarrow \quad \frac{5x}{4} = (4n + 1)\frac{\pi}{2}$ | $x = 2m\pi$

$\Rightarrow \quad x = (4n + 1)\frac{2\pi}{5}$ | $x = 0, \pm 2\pi, \pm 4\pi$

*Period of the given equation is* $8\pi$.

$\therefore$ *consider* $x \in [0, 8\pi)$

$\Rightarrow \quad x = \frac{2\pi}{5}, 2\pi, \frac{18\pi}{5}, \frac{26\pi}{5}, \frac{34\pi}{5}$

*common solution* $= 2\pi$

$\therefore \quad 8n\pi + 2\pi = 2\pi(4n + 1) \qquad$ *Ans.*

***Illustration :***

*If $x, y \in [0, 2\pi]$ then find the total number of order pairs $(x, y)$ satisfying the equation $\sin x \cos y = 1$.*

**Sol.** *We have $\sin x \cos y = 1$*

$\Rightarrow \quad \sin x = 1, \cos y = 1 \quad$ *or* $\quad \sin x = -1, \cos y = -1$

*If* $\quad \sin x = 1, \cos y = 1 \quad \Rightarrow \quad x = \frac{\pi}{2}, y = 0, 2\pi$

*If* $\quad \sin x = -1, \cos y = -1 \quad \Rightarrow \quad x = \frac{3\pi}{2}, y = \pi$

*Then the possible ordered pairs are* $\left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 2\pi\right), \left(\frac{3\pi}{2}, \pi\right)$.

## *Practice Problem*

## *Practice Problem*

Q.1 Solve : $\cos^{50} x - \sin^{50} x = 1$.

Q.2 Solve : $\sin^2 x + \cos^2 y = 2\sec^2 z$ for x, y, z.

Q.3 Solve : $1 + \sin x \sin^2 \frac{x}{2} = 0$

Q.4 Solve : $12 \sin x + 5 \cos x = 2y^2 - 8y + 21$, to get the values of x and y.

Q.5 Solve for x and y : $1 - 2x - x^2 = \tan^2 (x + y) + \cot^2 (x + y)$

## *Answer key*

Q.1 $n\pi$    Q.2 $x = n\pi + \frac{\pi}{2}, y = m\pi, z = k\pi$.    Q.3 $x \in \phi$

Q.4 $2n\pi + \alpha$, where $\tan \alpha = \frac{12}{5}$    Q.5 $x = -1, y = n\pi \pm \frac{\pi}{4} + 1$

## 2.7 TYPE-VII :

Solution of trigonometric equation of the form $f(x) = \sqrt{\phi(x)}$.

(i) $f(x) \geq 0, \phi(x) \geq 0$ (ii) $f^2(x) = \phi(x)$

***Illustration :***

*Solve for x,* $\sqrt{(1-\cos x)} = \sin x$.

**Sol.** $\sin x \geq 0$ *and* $1 - \cos x \geq 0$ ...(i)

$\therefore \quad 1 - \cos x = \sin^2 x = 1 - \cos^2 x$

$\Rightarrow \quad (1 - \cos x)[1 - (1 + \cos x)] = 0 \quad \Rightarrow \quad \cos x = 0, 1$

*when* $\cos x = 0, \quad x = 2n\pi \pm \dfrac{\pi}{2}$ *but* $\sin x \geq 0 \Rightarrow x = 2n\pi + \dfrac{\pi}{2}$

*when* $\cos x = 1, \quad x = 2n\pi$

*when* $\cos x = 0, \quad x = 2n\pi \pm \dfrac{\pi}{2}$ *but* $\sin x \geq 0 \Rightarrow x = 2n\pi + \dfrac{\pi}{2}$

*when* $\cos x = 1, \quad x = 2n\pi$

*Both are satisfying in equality (i)*

# 3 TRIGONOMETRIC INEQUALITIES AND SYSTEM OF INEQUALITY:

To solve the trigonometric inequalities of the type $f(x) \leq a$ or $f(x) \geq a$, where f(x) is some trigonometric ratio we take following steps

1. Draw the graph of f(x) in an interval length equal to the fundamental period of f(x).

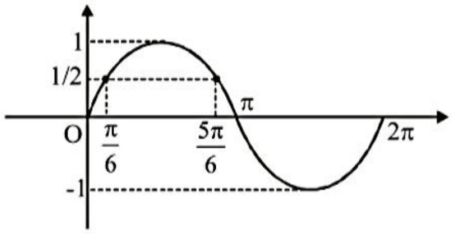
2. Draw the line $y = a$.

3. Take the portion of the graph for which the inequality is satisfied.

4. To generalise, add $p \cdot n, n \in I$ and in the final solution where p is the fundamental period of f(x).

***Illustration :***

*Solve :* $\sin x > +\frac{1}{2}$.




**Sol.** $\sin x = \frac{1}{2}, x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin x > \frac{1}{2}$ *for* $\frac{\pi}{6} < x < \frac{5\pi}{6}$ *fundamental period of* $\sin x$ *is* $2\pi$. *So adding* $2n\pi$ *on both sides,*

$2n\pi + \frac{\pi}{6} < x < 2n\pi + \frac{5\pi}{6}$.

***Illustration :***

*Solve :* $2\sin^2\theta - \sin\theta \geq 0$, *where* $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$.

**Sol.** $\sin\theta\,(2\sin\theta - 1) \geq 0$ *which is possible. only where*

**Sol.** $\sin\theta\,(2\sin\theta - 1) \geq 0$ *which is possible. only where*

$\sin\theta \geq \frac{1}{2}$ *or* $\sin\theta \leq 0$

$\sin\theta \geq \frac{1}{2} \quad \Rightarrow \quad \frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$

$\sin\theta \leq 0 \quad \Rightarrow \quad \pi \leq \theta \leq \frac{3\pi}{2}$

$\theta \in \left[\frac{\pi}{2}, \frac{5\pi}{6}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$

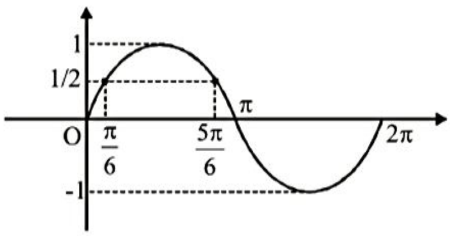



***Illustration :***

*Solve :* $\sin\theta + \sqrt{3}\cos\theta \geq 1, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

**Sol.** $\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta \geq \frac{1}{2}$

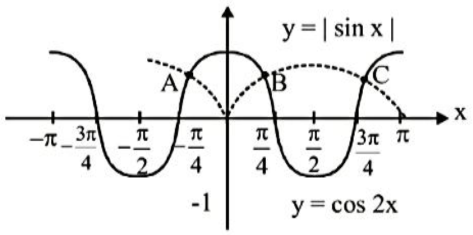
$\sin\left(\theta + \frac{\pi}{3}\right) \geq \frac{1}{2}$

$\frac{\pi}{6} \leq \theta + \frac{\pi}{3} \leq \frac{5\pi}{6} \quad \Rightarrow \quad -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$

***Illustration :***

*Solve :* $\cos 2x > |\sin x|,\ x \in \left(-\frac{\pi}{2}, \pi\right)$

**Sol.**




*For points B and C*

$\cos 2x = \sin x,\ 2\sin^2 x + \sin x - 1 = 0,\ \sin x = -1,\ \frac{1}{2}$

$\sin x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6}, \ \frac{5\pi}{6}.$

*Since* $\cos 2x$ *and* $|\sin x|$ *are even fucntion so x-coordinate of point A* $= -\frac{\pi}{6}$

*From graph* $\Rightarrow x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$

*From graph* $\Rightarrow x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$

## ***Practice Problem***

Q.1 Find the general solution of the following inequations

(i) $\sin x > 0$ (ii) $\sin x > \frac{1}{2}$ (iii) $\sin x \le \frac{1}{2}$

(iv) $\log_2\left(\sin\frac{x}{2}\right) < -1$ (v) $\cos x < -\frac{1}{2}$ (vi) $\tan x > 0$

Q.2 $\tan^2 x - \left(\sqrt{3}+1\right)\tan x + \sqrt{3} < 0.$

Q.3 Solve the inequality, $\sin x \ge \cos 2x.$

Q.4 $\sqrt{5 - 2\sin x} \ge 6\sin x - 1.$

## *Answer key*

Q.1 (i) $\bigcup\limits_{n\in I}(2n\pi, 2n\pi+\pi)$

(ii) $\bigcup\limits_{n\in I}\left(2n\pi+\frac{\pi}{6}, 2n\pi+\frac{5\pi}{6}\right)$

(iii) $\bigcup\limits_{n\in I}\left[2n\pi, 2n\pi+\frac{\pi}{6}\right]\bigcup\limits_{n\in I}\left[2n\pi+\frac{5\pi}{6}, 2n\pi+2\pi\right]$

(iv) $\bigcup\limits_{n\in I}\left(4n\pi, 4n\pi+\frac{\pi}{3}\right)\bigcup\limits_{n\in I}\left(4n\pi+\frac{5\pi}{3}, 4n\pi+2\pi\right)$

(v) $\bigcup\limits_{n\in I}\left(2n\pi+\frac{2\pi}{3}, 2n\pi+\frac{4\pi}{3}\right)$

(vi) $\bigcup\limits_{n\in I}\left(n\pi, n\pi+\frac{\pi}{2}\right)$

(v) $\bigcup\limits_{n\in I}\left(2n\pi+\frac{2\pi}{3}, 2n\pi+\frac{4\pi}{3}\right)$

(vi) $\bigcup\limits_{n\in I}\left(n\pi, n\pi+\frac{\pi}{2}\right)$

Q.2 $\bigcup\limits_{n\in I}\left(n\pi+\frac{\pi}{4}, n\pi+\frac{\pi}{3}\right)$

Q.3 $\bigcup\limits_{n\in I}\left[2n\pi+\frac{\pi}{6}, 2n\pi+\frac{5\pi}{6}\right], 2n\pi-\frac{\pi}{2}.$

Q.4 $\bigcup\limits_{n\in I}\left[2n\pi-\frac{7\pi}{6}, 2n\pi+\frac{\pi}{6}\right]$

# Solved Examples

Q.1 If $\cos p\,\theta + \cos q\,\theta = 0$, then the different vlaues of $\theta$ are in A.P., where the common difference is

(A) $\frac{\pi}{p+q}$ (B) $\frac{\pi}{p-q}$ (C) $\frac{2\pi}{p\pm q}$ (D) $\frac{3\pi}{p\pm q}$

Sol. $\cos p\theta = -\cos q\,\theta = \cos(\pi - q\,\theta)$

$p\theta = 2n\pi \pm (\pi - q\theta)$, $(p \pm q)\,\theta = (2n \pm 1)\pi$

$$\theta = \frac{(2n\pm1)\pi}{(p\pm q)} = \frac{r\pi}{(p\pm q)}, \text{ where } r = -3, -1, 1, 3, \ldots$$

$$\Rightarrow \quad \theta = \ldots, \frac{-3\pi}{(p\pm q)}, \frac{-\pi}{(p\pm q)}, \frac{\pi}{(p\pm q)}, \frac{3\pi}{(p\pm q)}, \ldots$$

So common difference $= \frac{2\pi}{(p\pm q)}$

Q.2 : If $3\tan^2\theta - 2\sin\theta = 0$, then $\theta$ is equal to

$n\pi$ $\pi$ $\pi$ $\pi$

Q.2 : If $3\tan^2\theta - 2\sin\theta = 0$, then $\theta$ is equal to

(A) $\frac{n\pi}{4}$ (B) $n\pi + (-1)^n \frac{\pi}{6}$ (C) $n\pi + (-1)^n \frac{\pi}{3}$ (D) $n\pi + \frac{\pi}{3}$

Sol. $\frac{3\sin^2\theta}{\cos^2\theta} - 2\sin\theta = 0$, $\cos\theta \neq 0$

$\Rightarrow \quad 3\sin^2\theta - 2\sin\theta\,(\cos^2\theta) = 0$, $3\sin^2\theta - 2\sin\theta\,(1 - \sin^2\theta) = 0$

$\Rightarrow \quad \sin\theta\,(2\sin^2\theta + 3\sin\theta - 2) = 0$

$\Rightarrow \quad \sin\theta\,(2\sin\theta - 1)(\sin\theta + 2) = 0 \Rightarrow \sin\theta = 0, \frac{1}{2}, -2$ (rejected)

$\Rightarrow \quad \theta = n\pi,\ n\pi + (-1)^n \frac{\pi}{6}$

Q.3 If $0 \le x \le 2\pi$, then the number of solutions of equation $3(\sin x + \cos x) - 2(\sin^3 x + \cos^3 x) = 8$ is

(A) 0 (B) 1 (C) 2 (D) 4

Sol. $3(\sin x + \cos x) - 2(\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x \cos x) = 8$

$\Rightarrow \quad (\sin x + \cos x)[3 - 2 + 2\sin x\cos x] = 8$

$\Rightarrow \quad (\sin x + \cos x)[1 + 2\sin x\cos x] = 8$

$\Rightarrow \quad (\sin x + \cos x)^3 = 8, \quad \sin x + \cos x = 2$

No solution

Q.4 If $\frac{1}{6}\sin\theta, \cos\theta, \tan\theta$ are in G.P. then $\theta$ is equal to $(n \in I)$.

(A) $2n\pi \pm \frac{\pi}{3}$ (B) $2n\pi \pm \frac{\pi}{6}$ (C) $n\pi + (-1)^n \frac{\pi}{3}$ (D) $n\pi + \frac{\pi}{3}$

Sol. $\cos^2\theta = \frac{1}{6}\sin\theta\tan\theta \quad \Rightarrow \quad 6\cos^3\theta = 1 - \cos^2\theta$

$\Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0 \quad \Rightarrow \quad (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}, \ n \in I$

Q.5 If $\sin x + \cos x = \sqrt{\left(y + \frac{1}{y}\right)}$, $x \in [0, \pi]$ then

(A) $x = \frac{\pi}{4}, y = 1$ (B) $y = 0$ (C) $y = 2$ (D) $x = \frac{3\pi}{4}$

Sol. $\sqrt{y + \frac{1}{y}} \geq \sqrt{2}$ assuming $y > 0$

Sol. $\sqrt{y + \frac{1}{y}} \geq \sqrt{2}$ assuming $y > 0$

But $|\sin x + \cos x| \leq \sqrt{2}$ so $y = 1$ & $x = \frac{\pi}{4}$

Q.6 Let $\theta \in [0, 4\pi]$ satisfying the equation $(\sin\theta + 2)(\sin\theta + 3)(\sin\theta + 4) = 6$. If the sum of all the values of $\theta$ is of the term $k\pi$ then value of k is

(A) 6 (B) 5 (C) 4 (D) 2

Sol. since, L.H.S. $\geq 6$ and R.H.S. $= 6$, so equality holds

only if $\sin\theta = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{2}, \frac{7\pi}{2}$

$\therefore \quad \text{sum} = 5\pi \quad \Rightarrow \quad k = 5$

Q.7 For $n \in I$, the general solution of the equation $\left(\sqrt{3} - 1\right)\sin\theta + \left(\sqrt{3} + 1\right)\cos\theta = 2$ is

(A) $\theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ (B) $\theta = n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{12}$

(C) $\theta = 2n\pi \pm \frac{\pi}{4}$ (D) $\theta = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{12}$

Sol. $$\frac{(\sqrt{3}-1)}{2\sqrt{2}}\sin\theta+\frac{(\sqrt{3}+1)}{2\sqrt{2}}\cos\theta=\frac{1}{\sqrt{2}}$$

$\Rightarrow$ $$\sin\frac{\pi}{12}\sin\theta+\cos\frac{\pi}{12}\cos\theta=\cos\frac{\pi}{4}$$

$$\cos\left(\theta-\frac{\pi}{12}\right)=\cos\frac{\pi}{4}, \qquad \theta-\frac{\pi}{12}=2n\pi\pm\frac{\pi}{4}$$

$$\theta=2n\pi\pm\frac{\pi}{4}+\frac{\pi}{12}$$

Q.8 If $\tan\left(\frac{p\pi}{4}\right)=\cot\left(\frac{q\pi}{4}\right)$ and $n \in I$, then

(A) $p+q=0$ (B) $p+q=2n+1$ (C) $p+q=2n$ (D) $p+q=2(2n+1)$

Sol. $$\tan\left(\frac{p\pi}{4}\right)=\tan\left(\frac{\pi}{2}-\frac{q\pi}{4}\right)$$

$\Rightarrow$ $$\frac{p\pi}{4}=n\pi+\frac{\pi}{2}-\frac{q\pi}{4} \quad \Rightarrow \quad \frac{(p+q)}{4}=n+\frac{1}{2}$$

$\Rightarrow$ $$\frac{p\pi}{4}=n\pi+\frac{\pi}{2}-\frac{q\pi}{4} \quad \Rightarrow \quad \frac{(p+q)}{4}=n+\frac{1}{2}$$

$\Rightarrow$ $(p+q)=2(2n+1)$

Q.9 The number of solution of the equation $\sin x+\sin 2x+\sin 3x=\cos x+\cos 2x+\cos 3x$, $0\le x\le 2\pi$ is

(A) 7 (B) 5 (C) 4 (D) 6

Sol. $(\sin x+\sin 3x)+\sin 2x=(\cos x+\cos 3x)+\cos 2x$

$\Rightarrow$ $2\sin 2x\cos x+\sin 2x=2\cos 2x\cos x+\cos 2x$

$\Rightarrow$ $\sin 2x(2\cos x+1)=\cos 2x(2\cos x+1)$

$\Rightarrow$ $$\cos x=-\frac{1}{2}, \quad \tan 2x=1$$

$$x=2n\pi\pm\frac{2\pi}{3}, 2x=n\pi+\frac{\pi}{4}$$

$\Rightarrow$ $$x=2n\pi\pm\frac{2\pi}{3}, \frac{n\pi}{2}+\frac{\pi}{8}$$

$0\le x\le 2\pi$

$$x=\frac{\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{9\pi}{8}, \frac{4\pi}{3}, \frac{13\pi}{8}$$

Q.10 The general solution of the equation $\sin 3\alpha = 4 \sin \alpha \sin (x + \alpha) \sin (x - \alpha)$ is $(\alpha \neq n\pi)$.

(A) $n\pi \pm \frac{\pi}{4} \ \forall n \in I$ (B) $n\pi \pm \frac{\pi}{3} \ \forall n \in I$

(C) $n\pi \pm \frac{\pi}{9} \ \forall n \in I$ (D) $n\pi \pm \frac{\pi}{12} \ \forall n \in I$

Sol. $\sin 3\alpha = 4 \sin \alpha (\sin^2 x - \sin^2 \alpha)$

$\Rightarrow$ $3 \sin \alpha - 4 \sin^3 \alpha = 4 \sin \alpha \sin^2 x - 4 \sin^3 \alpha$

$\Rightarrow$ $3 \sin \alpha = 4 \sin \alpha \sin^2 x$

If $\sin \alpha \neq 0$ $\quad \sin^2 x = \frac{3}{4}, \quad \sin x = \pm \frac{\sqrt{3}}{2}$

$x = n\pi \pm \frac{\pi}{3} \ \forall n \in I$

If $\sin \alpha = 0$; i.e. $\alpha = n\pi$, then equation becomes an identiy.

Q.11 Find the general solution of $\sin^2 x + \frac{1}{4} \sin^2 3x = \sin x \sin^2 3x$

Sol. $\sin^2 x - \sin x \sin^2 3x + \frac{1}{4} \sin^2 3x = 0$

Sol. $\sin^2 x - \sin x \sin^2 3x + \frac{1}{4} \sin^2 3x = 0$

$\Rightarrow$ $\left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \frac{1}{4} \sin^2 3x (1 - \sin^2 3x) = 0$

$\Rightarrow$ $\left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \frac{1}{4} \sin^2 3x \cos^2 3x = 0$

$\Rightarrow$ $\left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \frac{1}{16} \sin^2 6x = 0$

$\Rightarrow$ $\sin x - \frac{1}{2} \sin^2 3x = 0$ $\quad$ and $\quad \sin 6x = 0$

$\Rightarrow$ $2 \sin x = \sin^2 3x$ $\quad$ and $\quad \sin 6x = 0 \quad \Rightarrow \quad x = \frac{k\pi}{6} = k \in I.$

$$\sin^2\left(3\left(\frac{k\pi}{6}\right)\right) = \sin^2\left(\frac{k\pi}{2}\right) = \begin{cases} 1, & \text{if k is odd} \\ 0, & \text{if k is even} \end{cases}$$

$\Rightarrow$ $\sin x = 0$ $\quad$ or $\quad \frac{1}{2}$

$\Rightarrow$ $x = n\pi$ $\quad$ or $\quad x = n\pi + (-1)^n \frac{\pi}{6}, \quad n \in I$

Q.12 Find the general solution of $\tan\left(\frac{\pi}{2}\cos\theta\right) = \cot\left(\frac{\pi}{2}\sin\theta\right)$

Sol. $\tan\left(\frac{\pi}{2}\cos\theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{2}\sin\theta\right)$

$\Rightarrow \quad \frac{\pi}{2}\cos\theta = n\pi + \frac{\pi}{2} - \frac{\pi}{2}\sin\theta \quad \Rightarrow \quad \sin\theta + \cos\theta = (2n+1)$

$\Rightarrow \quad \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) = (2n+1) \quad \Rightarrow \quad n = 0, -1$ are the only possibility

$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}$, when $\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$

$\Rightarrow \quad \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}, \quad \theta = 2n\pi \quad$ or $\quad 2n\pi + \frac{\pi}{2}$

when $\cos\left(\theta - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} = \cos\left(\frac{3\pi}{4}\right)$, $\theta - \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4}$

$\theta = (2n+1)\pi \quad$ or $\quad 2n\pi - \frac{\pi}{2}$

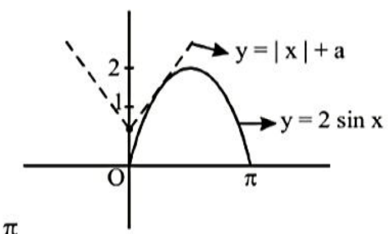
$\theta = (2n+1)\pi \quad$ or $\quad 2n\pi - \frac{\pi}{2}$

So $\quad \theta = 2n\pi, (2n+1)\pi, \left(2n \pm \frac{1}{2}\right)\pi$

$\theta = m\pi, \left(2n \pm \frac{1}{2}\right)\pi$ Ans.

Q.13 Prove that the equation $2\sin x = |x| + a$ has no solution for $a \in \left(\frac{3\sqrt{3} - \pi}{3}, \infty\right)$

Sol. The equation $2\sin x = |x| + a$ will have a solution so long as the line $y = |x| + a$ intersects or at least touches the curve, $y = 2\sin x$. So $\frac{dy}{dx} = 2\cos x = 1$.




$\Rightarrow \quad \cos x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{3}$

Hence, solution will not be possible if $\quad x + a > 2\sin x$ at $x = \frac{\pi}{3}$

$\frac{\pi}{3} + a > 2\cdot\frac{\sqrt{3}}{2} \Rightarrow a > \frac{3\sqrt{3} - \pi}{3}$

Q.14 Find the smallest positive root of the equation $\sqrt{\sin(1-x)} = \sqrt{\cos x}$ ?

Sol. $\sin(1-x) \ge 0$ and $\cos x \ge 0$

$\sin(1-x) = \cos x, \quad \cos\left(\frac{\pi}{2} - (1-x)\right) = \cos x$

$\Rightarrow \quad \frac{\pi}{2} - 1 + x = 2n\pi \pm x \quad \Rightarrow \quad x = \dfrac{2n\pi - \frac{\pi}{2} + 1}{2}$

For $n = 2$, $\quad x = \frac{7\pi}{4} + \frac{1}{2}$ which is the smallest positive root of the given equation.

Q.15 Solve the equation $\tan^4 x + \tan^4 y + 2\cot^2 x \cot^2 y = 3 + \sin^2(x+y)$ for the values of x and y.

Sol. $\tan^4 x + \tan^4 y + 2\cot^2 x \cot^2 y - 2 = 1 + \sin^2(x+y)$

$(\tan^2 x - \tan^2 y)^2 + 2(\tan x \tan y - \cot x \cot y)^2 = -1 + \sin^2(x+y)$

L.H.S. $\ge 0$ and R.H.S. $\le 0 \quad \Rightarrow \quad$ L.H.S. = R.H.S. = 0

Sol. $\tan^4 x + \tan^4 y + 2\cot^2 x \cot^2 y - 2 = 1 + \sin^2(x+y)$

$(\tan^2 x - \tan^2 y)^2 + 2(\tan x \tan y - \cot x \cot y)^2 = -1 + \sin^2(x+y)$

L.H.S. $\ge 0$ and R.H.S. $\le 0 \quad \Rightarrow \quad$ L.H.S. = R.H.S. = 0

$\Rightarrow \quad \tan^2 x = \tan^2 y \quad$ and $\quad \tan x \tan y = \cot x \cot y$

$\Rightarrow \quad \tan^2 x \tan^2 y = 1 \quad$ and $\quad \sin^2(x+y) = 1$

$\Rightarrow \quad \tan^2 x = \tan^2 y = 1$

$x = n\pi \pm \frac{\pi}{4},\ y = n\pi \pm \frac{\pi}{4},\ x + y = 2n\pi \pm \frac{\pi}{2}$

So, $\quad x = y = n\pi \pm \frac{\pi}{4}$. $\quad$ Ans.

Q.16 Solve : $\sin x + \sin y = \sin(x+y)$ and $|x| + |y| = 1$

Sol. $2\sin\left(\frac{x+y}{2}\right)\left[\cos\left(\frac{x-y}{2}\right) - \cos\left(\frac{x+y}{2}\right)\right] = 0$

$\Rightarrow \quad 4\sin\left(\frac{x+y}{2}\right)\sin\frac{x}{2}\cdot\sin\frac{y}{2} = 0$

a. $\sin\left(\frac{x+y}{2}\right)=0 \quad \Rightarrow x+y=2n\pi \quad \Rightarrow \quad x+y=0, n\in I$

b. $\sin\frac{x}{2}=0 \quad \Rightarrow x=2m\pi, m\in I \quad \Rightarrow \quad x=0$

c. $\sin\frac{y}{2}=0 \quad \Rightarrow y=2p\pi, p\in I \quad \Rightarrow \quad y=0$

Now, In $|x|+|y|=1$, if $x=0$, then $|y|=1 \quad \Rightarrow \quad y=\pm 1$

If $y=0$, then $|x|=1 \quad \Rightarrow \quad x=\pm 1$

If $y=-x$, then $|x|+|-x|=1 \Rightarrow x=\pm\frac{1}{2}$ and $y=\mp\frac{1}{2}$

Solutions are $(0, 1), (0, -1), (1, 0), (-1, 0), \left(\frac{1}{2}, \frac{-1}{2}\right)$ and $\left(\frac{-1}{2}, \frac{1}{2}\right)$.

Q.17 Solve the inequality $\sin^4\left(\frac{x}{3}\right)+\cos^4\left(\frac{x}{3}\right) > \frac{1}{2}$

Q.17 Solve the inequality $\sin^4\left(\frac{x}{3}\right)+\cos^4\left(\frac{x}{3}\right) > \frac{1}{2}$

Sol. $1-2\sin^2\left(\frac{x}{3}\right)\cos^2\left(\frac{x}{3}\right) > \frac{1}{2}$

$\Rightarrow 1-\frac{1}{2}\sin^2\left(\frac{2x}{3}\right) > \frac{1}{2}$

$\Rightarrow \quad \sin^2\left(\frac{2x}{3}\right) < 1$

which is always true except when $\sin^2\left(\frac{2x}{3}\right)=1$

$\frac{2x}{3}=n\pi\pm\frac{\pi}{2} \quad$ or $\quad x=\frac{3n\pi}{2}\pm\frac{3\pi}{4}$

So, solution of x is $R\sim\{x : x=\frac{3n\pi}{2}\pm\frac{3\pi}{4}, n\in I\}$.

Q.18 Solve $3\tan 2x - 4\tan 3x = \tan^2 3x \tan 2x$.

Sol. $3(\tan 2x - \tan 3x) = \tan 3x\,(1 + \tan 3x \tan 2x)$

$$\Rightarrow 3\left(\frac{\tan 2x - \tan 3x}{1 + \tan 3x \tan 2x}\right) = \tan 3x \Rightarrow -3\tan(3x - 2x) = \tan 3x$$

$$\Rightarrow -3\tan x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \quad \Rightarrow -3 = \frac{3 - \tan^2 x}{1 - 3\tan^2 x}$$

or $\tan x = 0$ so $x = n\pi$, $\tan^2 x = \dfrac{3}{5}$

$x = n\pi \pm \tan^{-1}\sqrt{\dfrac{3}{5}}$, $n \in I$

$x = n\pi$, $n \in I$. Ans.

Q.19 Find the general solution of the equation $\sin^{100}x - \cos^{100}x = 1$.

Sol. $\sin^{100}x = 1 + \cos^{100}x$ L.H.S. $\le 1$, R.H.S. $\ge 1$

So, L.H.S. = R.H.S. = 1

$x = n\pi$, $n \in I$. Ans.

Q.19 Find the general solution of the equation $\sin^{100}x - \cos^{100}x = 1$.

Sol. $\sin^{100}x = 1 + \cos^{100}x$ L.H.S. $\le 1$, R.H.S. $\ge 1$

So, L.H.S. = R.H.S. = 1

$\cos^{100}x = 0$, $\sin^{100}x = 1$ ; $x = n\pi \pm \dfrac{\pi}{2}$. Ans.]

Q.20 Find the set of all x in $(-\pi, \pi)$ satisfying $|4\sin x - 1| < \sqrt{5}$.

Sol. $-\sqrt{5} < 4\sin x - 1 < \sqrt{5}$, $\dfrac{-(\sqrt{5}-1)}{4} < \sin x < \dfrac{(\sqrt{5}-1)}{4}$

$$\Rightarrow \quad -\sin\left(\frac{\pi}{10}\right) < \sin x < \cos\left(\frac{2\pi}{10}\right) \Rightarrow \sin\left(\frac{-\pi}{10}\right) < \sin x < \sin\left(\frac{3\pi}{10}\right)$$

$$x \in \left(\frac{-\pi}{10}, \frac{3\pi}{10}\right)$$