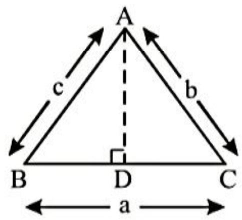
SOLUTIONS OF TRIANGLE

1. INTRODUCTION :

In any triangle, the three sides and the three angles are often called the elements of the triangle. When three elements of a triangle are given, the process of calculating its other three elements is called solution of the Triangle.

In any triangle ABC, the side BC, opposite to the angle A, is denoted by a; the sides CA and AB opposite to the angle B and C respectively are denoted by b & c.



In any triangle ABC

(i) $A + B + C = 180^\circ$, $\quad A, B, C > 0$

(ii) $a + b > c$, $\quad b + c > a$, $\quad c + a > b$

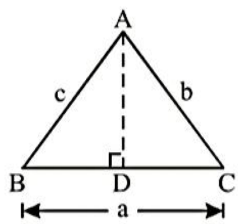
(iii) $|a - b| < c$, $\quad |b - c| < a$, $\quad |c - a| < b$.

(iv) $a, b, c > 0$

1.1 SINE RULE :

In any triangle ABC, $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$ i.e., the sines of the angle are proportional to the opposite sides.

Proof :-



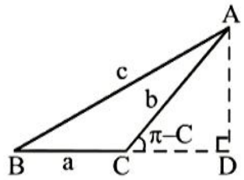
Consider the acute angle triangle ABC

Draw AD perpendicular to the opposite side BC

In the triangle ABD, we have $\dfrac{AD}{AB} = \sin B$, so that $AD = c \sin B$

In the triangle ACD, we have $\dfrac{AD}{AC}$

Consider the obtuse angle ΔABC



In a ΔABD, $\sin B = \frac{AD}{AB} = \frac{AD}{c}$

In a ΔACD, $\sin(\pi - C) = \frac{AD}{AC} = \frac{AD}{b}$

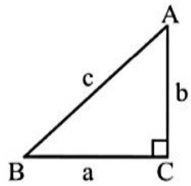
so, $c \sin B = b \sin(\pi - C) = b \sin C$

In a similar manner, by drawing a perpendicular from B upon CA, we have

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

Consider right angled triangle $C = 90^\circ$

If one of the angles C, be a right angle as in the figure, we have



$$\sin C = 1, \qquad \sin A = \frac{a}{c} \quad \text{and} \quad \sin B = \frac{b}{c}$$

So, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ [Because $\sin C = 1$]

Hence for any type of triangle $\frac{a}{\ } = \frac{b}{\ } = \frac{c}{\ }$

Hence for any type of triangle, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Illustration :

Prove that $a \cos \frac{B-C}{2} = (b + c) \sin\left(\frac{A}{2}\right)$.

Sol. $$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{2 \sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2 \sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \frac{\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}}$$

so $a \cos\left(\frac{B-C}{2}\right) = (b + c) \sin \frac{A}{2}$

Illustration :

Prove that $\frac{a-b}{a+b} = \tan\frac{A-B}{2}\cdot\cot\frac{A+B}{2}$.

Sol. $$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$

$$= \tan\left(\frac{A-B}{2}\right)\cot\left(\frac{A+B}{2}\right)$$

so $$\frac{a-b}{a+b} = \tan\left(\frac{A-B}{2}\right)\cot\left(\frac{A+B}{2}\right)$$

Illustration :

Prove that $\frac{a^2\sin(B-C)}{\sin B+\sin C} + \frac{b^2\sin(C-A)}{\sin C+\sin A} + \frac{c^2\sin(A-B)}{\sin A+\sin B} = 0$.

Sol. $$\frac{a^2\sin(B-C)}{\sin B+\sin C} = K^2\frac{\sin^2 A\sin(B-C)}{\sin B+\sin C} \qquad \because \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K \text{ (constant)}$$

$$= K^2\frac{\sin A\sin(B+C)\sin(B-C)}{\sin B+\sin C} = K^2\sin A\frac{(\sin^2 B-\sin^2 C)}{\sin B+\sin C}$$

$$= K^2\sin A\,(\sin B-\sin C)$$

Similarly $$\frac{b^2\sin(C-A)}{\sin C-\sin A} = K^2\sin B\,(\sin C-\sin A),\ \frac{c^2\sin(A-B)}{\sin A-\sin B} = K^2\sin C\,(\sin A-\sin B)$$

so sum of three terms = 0

Illustration :

Prove that $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\sin B} = \frac{a^2+b^2}{a^2+c^2}$

Sol. $$\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\sin B} = \frac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)}$$

$$= \frac{1-(\cos^2 A-\sin^2 B)}{1-(\cos^2 A-\sin^2 C)} = \frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C} = \frac{a^2+b^2}{a^2+c^2}$$

Illustration :

In any trianlge, if $\frac{a^2-b^2}{a^2+b^2}=\frac{sin(A-B)}{sin(A+B)}$. *Then prove that the triangle is either right angled or Isosceles.*

Sol. $\frac{a^2-b^2}{a^2+b^2}=\frac{sin^2 A-sin^2 B}{sin^2 A+sin^2 B}=\frac{sin(A+B)sin(A-B)}{sin^2 A+sin^2 B}$

So, $\frac{sin(A+B)sin(A-B)}{sin^2 A+sin^2 B}=\frac{sin(A-B)}{sin(A+B)}$

$\Rightarrow$ $sin\,(A-B)=0$ or $\frac{sin(A+B)}{sin^2 A+sin^2 B}=\frac{1}{sin(A+B)}$ or $\frac{sin\,C}{sin^2 A+sin^2 B}=\frac{1}{sin\,C}$

$\Rightarrow$ so $A = B$ or $sin^2 C = sin^2 A + sin^2 B$, $c^2 = a^2 + b^2$

so either isosceles or right angled triangle.

Illustration :

If $A = 75^o$, $B = 45^o$, *then prove that* $b + c\sqrt{2} = 2a$.

If $A = 75^o$, $B = 45^o$, *then prove that* $b + c\sqrt{2} = 2a$.

Sol. $A = 75^o$, $B = 45^o$ $\Rightarrow$ $C = 60^o$

$\frac{a}{sin\,75^o}=\frac{b}{sin\,45^o}=\frac{c}{sin\,60^o}$ so $b+\sqrt{2}\,c = \frac{sin\,45^o}{sin\,75^o}a+\sqrt{2}\frac{sin\,60^o}{sin\,75^o}a$

$$b + c\sqrt{2} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a+\sqrt{2}\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a=\frac{2}{\sqrt{3}+1}a+\frac{2\sqrt{3}a}{\sqrt{3}+1}=2a$$

Illustration :

If $\frac{cos\,A}{a}=\frac{cos\,B}{b}=\frac{cos\,C}{c}$ *and the side* $a = 2$, *then find the area of the triangle?*

Sol. $\frac{cos\,A}{a}=\frac{cos\,B}{b}=\frac{cos\,C}{c}$, $\frac{cos\,A}{sin\,A}=\frac{cos\,B}{sin\,B}=\frac{cos\,C}{sin\,C}$

so $cot\,A = cot\,B = cot\,C \Rightarrow$ equilateral triangle.

$Area = \frac{\sqrt{3}}{4}a^2=\frac{\sqrt{3}}{4}(2)^2=\sqrt{3}$

Practice Problem

Q.1 In a triangle ABC, if $\cos^2 A + \cos^2 B - \cos^2 C = 1$, then identify the type of the triangle?

Q.2 If angles A, B and C of a triangle ABC are in A.P. and if $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$, then find angle A.

Q.3 Prove that $b^2 \cos 2A - a^2 \cos 2B = b^2 - a^2$.

Q.4 Prove that $\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$

Q.5 If in any triangle angles are in the ratio 1 : 2 : 3, then prove that the corresponding sides are as $1 : \sqrt{3} : 2$.

Q.6 If in a triangle ABC, $a \sin A = b \sin B$, then prove that the triangle is an isosceles triangle.

Answer key

Q.1 Right angle triangle Q.2 75º

1.2 COSINE RULE :

In a ΔABC, we have $\boxed{\cos A = \frac{b^2+c^2-a^2}{2bc},\ \cos B = \frac{c^2+a^2-b^2}{2ca} \text{ and } \cos C = \frac{a^2+b^2-c^2}{2ab}}$

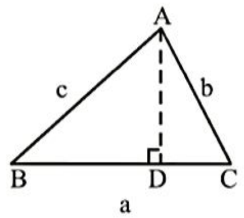
where a, b & c are sides and A, B & C are angle of the triangle.

Proof : Consider the acute angle ΔABC,

By geometry, we have

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD \qquad \ldots(i)$$

But $\frac{CD}{CA} = \cos C$, so that $CD = b \cos C$.

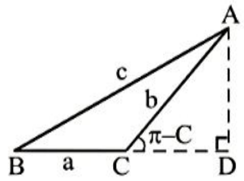


Hence (i) becomes $c^2 = a^2 + b^2 - 2ab \cos C$.

$$2ab \cos C = a^2 + b^2 - c^2 \text{, so } \cos C = \frac{a^2+b^2-c^2}{2ab}$$

Consider the obtuse angle ΔABC,

By geometry, we have $\quad AB^2 = BC^2 + CA^2 + 2BC \cdot CD \quad$ ...(ii)



But $\quad \dfrac{CD}{CA} = \cos(\angle ACD) = \cos(180^\circ - C) = -\cos C$

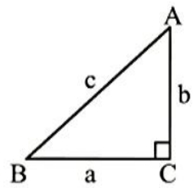
so $\quad CD = -b \cos C$

so equation (ii) becomes

$$c^2 = a^2 + b^2 + 2a(-b\cos C) = a^2 + b^2 - 2ab\cos C.$$

so once again, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Consider the right angle ΔABC,



If $\angle C = 90^\circ$, $\quad$ then $a^2 + b^2 = c^2$

so $\cos C = \dfrac{a^2+b^2+c^2}{2ab} = \dfrac{c^2-a^2}{2ab} = 0$, we know that $\cos 90^\circ = 0$

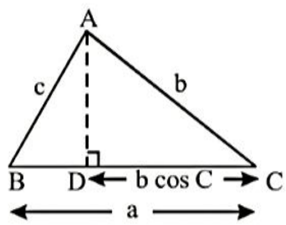
so here also our formula is valid. So we can say in any type of triangle ABC, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

similarly cos B, cos A can be proved.

Note :- There is another way to prove cosine law consider the triangle as shown in figure.

$$AD = AC\sin C, = b\sin C, \; CD = AC\cos C = b\cos C$$

so $\quad BD = BC - CD = a - b\cos C.$



ΔADB is a right angle triangle so

$$AB^2 = AD^2 + BD^2, \quad c^2 = (b\sin C)^2 + (a - b\cos C)^2$$

$$c^2 = b^2\sin^2 C + a^2 + b^2\cos^2 C - 2ab\cos C = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

Note :- (a) If the three sides of a triangle are known, we can find all the angles by using cosine rule.

(b) If in ΔABC, $a > b > c$, then $\angle A > \angle B > \angle C$ or vice-versa.

Illustration :

In any triangle ABC, prove that a (b cos C – c cos B) = $b^2 - c^2$.

Sol. $a(b\cos C - c\cos B) = ab\cos C - ac\cos B$

$$= ab\frac{(a^2+b^2-c^2)}{2ab} - ac\frac{(a^2+c^2-b^2)}{2ac}$$

$$= \frac{a^2+b^2-c^2-a^2-c^2+b^2}{2} = b^2-c^2$$

Illustration :

If $a = \sqrt{3}$, $b = \frac{1}{2}\left(\sqrt{6}+\sqrt{2}\right)$, $c = \sqrt{2}$, *then find* $\angle A$?

Sol. $$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{\frac{1}{4}\left(8+4\sqrt{3}\right)+2-3}{\sqrt{12}+\sqrt{4}} = \frac{\left(1+\sqrt{3}\right)}{2\left(1+\sqrt{3}\right)} = \frac{1}{2}$$

$$A = \frac{\pi}{3}$$

Illustration :

If the angle A, B, C of a triangle are in A.P. and sides a, b, c are in G.P., then prove that a^2, b^2, c^2 are in A.P.

Sol. *Given,* $2B = A + C = \pi - B \Rightarrow B = \frac{\pi}{3}$

Also, a, b, c are in G.P. $\Rightarrow b^2 = ac$

Now, $\cos B = \cos 60^o = \frac{1}{2} = \frac{c^2+a^2-b^2}{2ac} \Rightarrow ca = c^2 + a^2 - b^2$

$\Rightarrow b^2 = c^2 + a^2 - b^2 \Rightarrow 2b^2 = a^2 + c^2 \Rightarrow a^2, b^2, c^2$ *are in A.P.*

Illustration :

If in a triangle ABC, $\angle C = 60^o$, *then prove that* $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$.

Sol. *By the cosine formula, we have*

$c^2 = a^2 + b^2 - 2ab\cos C, \Rightarrow c^2 = a^2 + b^2 - 2ab\cos 60^o = a^2 + b^2 - ab$

Now, $$\frac{1}{a+c}+\frac{1}{b+c}-\frac{3}{a+b+c}$$

$$= \left[\frac{(b+c)(a+b+c)+(a+c)(a+b+c)-3(a+c)(b+c)}{(a+b)(b+c)(a+b+c)}\right]$$

$$= \frac{(a^2+b^2-ab)-c^2}{(a+b)(b+c)(a+b+c)} = 0$$

so $$\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$$

Illustration :

If the sides of a triangle are $(x^2 + x + 1)$, $(x^2 - 1)$ & $(2x + 1)$, then find the largest angle?

Sol. Let $a = x^2 + x + 1,\qquad b = 2x + 1,\quad c = x^2 - 1$

$a > 0 \Rightarrow x \in R$

$b > 0 \Rightarrow x > -\frac{1}{2}$

$c > 0 \Rightarrow x < -1$ or $x > 1$

So, $x \in (1, \infty)$

$a - b = x^2 - x > 0 \qquad \Rightarrow \qquad a > b,$

$a - c = x + 2 > 0 \qquad \Rightarrow \qquad a > c$

So angle $\angle A$ is the largest angle

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(2x+1)^2 + (x^2-1)^2 - (x^2+x+1)^2}{2(2x+1)(x^2-1)}$$

$$= \frac{-\left(2x^3 + x^2 - 2x - 1\right)}{2(2x^3 + x^2 - 2x - 1)} = \frac{-1}{2}$$

$$\therefore \qquad \angle A = \frac{2\pi}{3}$$

Practice Problem

Q.1 In a triangle, the angles A, B, C are in A.P. show that $2\cos\frac{1}{2}(A - C) = \frac{a + c}{\sqrt{a^2 - ac + c^2}}$.

Q.2 If the sides of a triangle are a, b, $\sqrt{(a^2 + ab + b^2)}$, then find the greatest angle?

Q.3 If $a\cos A = b\cos B$, then prove that the triangle is isosceles or right angled.

Q.4 If in a triangle ABC, $\frac{\sin A}{4} = \frac{\sin B}{5} = \frac{\sin C}{6}$, then the value of $\cos A + \cos B + \cos C$?

Q.5 The sides of a triangle are 8 cm, 10 cm and 12 cm. Prove that the greatest angle is double the smallest angle.

Q.6 In a triangle ABC, if $\frac{b + c}{11} = \frac{c + a}{11} = \frac{a + b}{13}$. then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$

Answer key

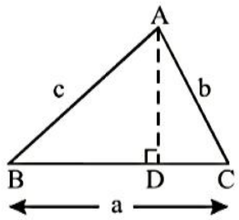
Q.2 $\dfrac{2\pi}{3}$ Q.4 $\dfrac{69}{48}$

1.3 PROJECTION FORMULA :

In any triangle with usual noations,

$$\boxed{\begin{aligned} a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos C \\ c &= a\cos B + b\cos A \end{aligned}}$$

Consider the acute angle ΔABC



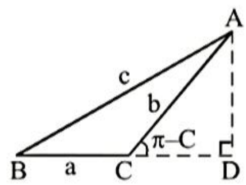
$\dfrac{BD}{BA} = \cos B$ so that $BD = c\cos B$

an $\dfrac{CD}{CA} = \cos C$ so that $BD = b\cos C$

Hence, $a = BC = BD + DC = c\cos B + b\cos C$

Hence, $a = BC = BD + DC = c\cos B + b\cos C$

Consider the obtuse angle ΔABC



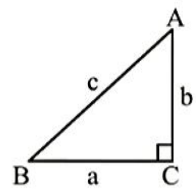
$\dfrac{BD}{BA} = \cos B,$ so that $BD = c\cos B$ and $\dfrac{CD}{CA} = \cos ACD$

$\dfrac{CD}{CA} = \cos(180^\circ - C) = -\cos C,$ $CD = -b\cos C$

Hence, in the case, $a = BC - BD = c\cos B - (-b\cos C)$

$a = c\cos B + b\cos C$

Consider the right angle ΔABC, $\angle C = 90^\circ$



then $a = c\cos B + 0 = c\cos B + b\cos 90^\circ$

so that in each case

$$a = b\cos C + c\cos B, \quad b = c\cos A + a\cos C, \quad c = a\cos B + b\cos A$$

Illustration :

In any triangle prove that $(b + c)\cos A + (c - a)\cos B + (a + b)\cos C = a + b + c$.

Sol. *L.H.S.* $= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (b\cos C + c\cos B)$

$= a + b + c.$

Illustration :

In any triangle $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$, *then find the relation between the sides of the triangle?*

Sol. $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2} \Rightarrow \quad a(1+\cos C) + c(1+\cos A) = 3b$

$a + c + a\cos C + c\cos A = 3b, \qquad a + c + b = 3b, \qquad a + c = 2b$

so, *a, b, c are in A.P.*

Practice Problem

Q.1 Prove that $a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B) = 3abc$.

Q.2 Prove that $a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)\cos C = 3abc$.

Q.3 In a ΔABC, prove that $c\cos(A-\alpha) + a\cos(C+\alpha) = b\cos\alpha$.

Q.4 Prove that $\frac{\cos C + \cos A}{c+a} + \frac{\cos B}{b} = \frac{1}{b}$.

1.4 TANGENT RULE (NAPIER ANALOGY) :

This rule is used when two sides and included angle are known.

$$\boxed{\begin{array}{l} \tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\left(\frac{A}{2}\right) \\ \tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\left(\frac{B}{2}\right) \\ \tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\left(\frac{C}{2}\right) \end{array}}$$

In any triangle, we have $\frac{b}{c} = \frac{\sin B}{\sin C}$

$$\therefore \qquad \frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}$$

$$= \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)} = \frac{\tan\left(\frac{B-C}{2}\right)}{\cot\left(\frac{A}{2}\right)}$$

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cdot\cot\frac{A}{2}$$

Illustration :

In any triangle ABC, if $b = \sqrt{3}$, $c = 1$ and $A = 30^o$. Find the value of a, c, A & C?

Sol. *We have* $$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$

$$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2} = \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)}\cdot\cot 15^o$$

$$= \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)}\cdot\frac{(\sqrt{3}+1)}{(\sqrt{3}-1)} \qquad \text{as } \tan 15^o = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\frac{B-C}{2} = 45^o, \quad B - C = 90^o,$$

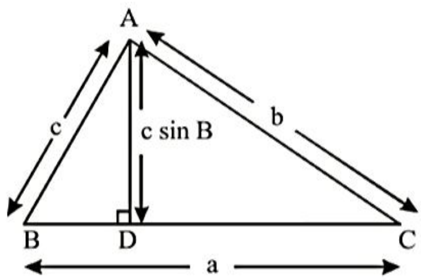
$A + B + C = 180^o$, so $B + C = 150^o$, $B = 120^o$, $C = 30^o$

Since $A = C$, *we have* $a = c = 1$.

1.5 AREA OF TRIANGLE :

1.5 AREA OF TRIANGLE :

$$\boxed{\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)}}$$



Different formulae for area of triangle are as follows :

$$AD = c\sin B$$

Area of triangle ABC is $\Delta = \frac{1}{2}(BC)(AD)$

$$\Delta = \frac{1}{2}(a)(c\sin B) \quad = \frac{1}{2}ac\sin B$$

$AD = b\sin C$, $\Delta = \frac{1}{2}(a)(b\sin C) = \frac{1}{2}ab\sin C.$

$$\Delta = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A$$

From our tenth class knowledge, we know area of triangle with sides a, b & c is denoted by Δ

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = \frac{a+b+c}{2}.$$

Illustration :

Prove that $\dfrac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2} = sin^2 A$

Sol. $\dfrac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$

$$= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$$

$$= \frac{4s(s-a)(s-b)(s-c)}{b^2c^2} = \frac{4\Delta^2}{b^2c^2} = \frac{4}{b^2c^2}\left(\frac{1}{2}bc\ \sin A\right)^2$$

$$= sin^2 A$$

Illustration :

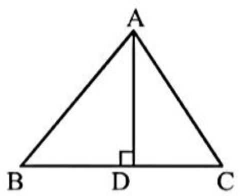
If the sides of a triangle are 17, 25, 28, then find the greatest length of the altitude.

If the sides of a triangle are 17, 25, 28, then find the greatest length of the altitude.

Sol. *Note that from geometry the greatest altitude is perpendicular to the shortest side.*

Let $a = 17, \quad b = 25, \quad c = 28$

Now $\Delta = \dfrac{1}{2} AD\ (BC) \quad \Rightarrow \quad AD = \dfrac{2\Delta}{17}$



where $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = 210$

$$AD = \frac{2 \times 210}{17} = \frac{420}{17}$$

Practice Problem

Q.1 If $c^2 = a^2 + b^2$, then prove that $4s\ (s-a)\ (s-b)\ (s-c) = a^2b^2$.

Q.2 In ΔABC, find the value of $\dfrac{b^2 \sin 2C + c^2 \sin 2B}{\Delta}$?

Answer key

Q.2 4

1.6 HALF ANGLE FORMULA :

1.6.1 Sine of half the angles in terms of the sides :-

$$\boxed{\begin{aligned}\sin\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}}\\ \sin\frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{ca}}\\ \sin\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}}\end{aligned}}$$

Proof:

In any triangle ABC, we know $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

But cos A can be wirtten as $\cos A = 1 - 2\sin^2\dfrac{A}{2}$

Hence, $2\sin^2\dfrac{A}{2} = 1-\cos A = 1-\dfrac{(b^2+c^2-a^2)}{2bc}$

$$= \frac{2bc-(b^2+c^2)+a^2}{2b^2} = \frac{a^2-(b^2+c^2-2bc)}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}$$

Let 2s stand for a + b + c

so $s = \dfrac{a+b+c}{2} = \text{semiperimeter.}$

$a+b-c = a+b+c-2c = 2s-2c = 2\,(s-c)$
$a-b+c = a+b+c-2b = 2s-2c = 2\,(s-b)$

so $2\sin^2\dfrac{A}{2} = \dfrac{2(s-c)2(s-b)}{2bc}$, $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

Similarly $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$ and $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

1.6.2 The cosines of half the angles in terms of the sides :

$$\boxed{\begin{aligned}\cos\frac{A}{2}&=\sqrt{\frac{s(s-a)}{bc}}\\ \cos\frac{B}{2}&=\sqrt{\frac{s(s-b)}{ac}}\\ \cos\frac{C}{2}&=\sqrt{\frac{s(s-c)}{ab}}\end{aligned}}$$

Proof:

We know $\cos A = 2\cos^2\frac{A}{2} - 1,\qquad 2\cos^2\frac{A}{2} = 1 + \cos A.$

$$2\cos^2\frac{A}{2} = 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{b^2+c^2+2bc-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc}$$

$$2\cos^2\frac{A}{2} = \frac{(b+c+a)(b+c-a)}{2bc}$$

$$a+b+c=2s \qquad a+b+c-2a = 2s-2a = 2(s-a)$$

$$2\cos^2\frac{A}{2} = \frac{2s(2)(s-a)}{2bc} = \frac{2s(s-a)}{bc}$$

$$2\cos^2\frac{A}{2} = \frac{2s(2)(s-a)}{2bc} = \frac{2s(s-a)}{bc}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$ Similarly

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

1.6.3 The tangent of half the angles in terms of the sides :

$$\boxed{\begin{aligned}\tan\frac{A}{2}&=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\Delta}{s(s-a)}\\ \tan\frac{B}{2}&=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}=\frac{\Delta}{s(s-b)}\\ \tan\frac{C}{2}&=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\frac{\Delta}{s(s-c)}\end{aligned}}$$

Proof :

$$\text{since,}\quad \tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$\tan\frac{A}{2}=\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\qquad \text{Similarly}$$

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}},\qquad \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Note :-

Since, in a triangle, A is always less than 180º, so $\frac{A}{2}$ is always less then 90º. Therefore, the sine, consine and tangent of $\frac{A}{2}$ (half angle) are therefore always positive.

Since, in a triangle, A is always less than 180º, so $\frac{A}{2}$ is always less then 90º. Therefore, the sine, consine and tangent of $\frac{A}{2}$ (half angle) are therefore always positive.

1.6.4 The sine of any angle of triangle in terms of the sides:

We know $\quad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

But by the previous discussion

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}\quad \text{and}\quad \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

So $\quad \sin A=\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$

$$=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2}{bc}\Delta$$

Illustration :

In any triangle prove that $(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}$

Sol. L.H.S. $= (2s)\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}+\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\right]$

$$= 2s\sqrt{\frac{(s-c)}{s}}\left[\sqrt{\frac{s-b}{s-a}}+\sqrt{\frac{s-a}{s-b}}\right]$$

$$= 2\sqrt{s(s-c)}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right] = \frac{2\sqrt{s(s-c)}\,c}{\sqrt{(s-a)(s-b)}}$$

$$= \frac{2c}{\tan\frac{c}{2}} = 2c\cot\frac{C}{2}$$

$$= \frac{2c}{\tan\frac{c}{2}} = 2c\cot\frac{C}{2}$$

Illustration :

If the sides of a triangle be in arithmetic progression, prove that the contangents of half the angles are also in arithmetic progression.

Sol. $a + c = 2b$ *(given)*

we have to prove that $\cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2}$

so $\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-c)}} = 2\sqrt{\frac{s(s-b)}{(s-c)(s-a)}}$

Multiplying both side by $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$

$$(s-a) + (s-c) = 2(s-b)$$

$a + c = 2b$ *(which is the given relation)*

Illustration :

If $\cos\frac{A}{2} = \sqrt{\frac{b+c}{2c}}$ *, then prove that* $a^2 + b^2 = c^2$.

Sol. $\cos\frac{A}{2} = \sqrt{\frac{b+c}{2a}}$

$\Rightarrow \quad \frac{s(s-a)}{bc} = \frac{b+c}{2c}$ *(squaring)*

$\Rightarrow \quad 2s\,(2s - 2a) = 2b\,(b + c)$

$\Rightarrow \quad (a + b + c)\,(b + c - a) = 2b^2 + 2bc$

$\Rightarrow \quad (b + c)^2 - a^2 = 2b^2 + 2bc$

$\Rightarrow \quad b^2 + c^2 + 2bc - a^2 = 2b^2 + 2bc$

$\Rightarrow \quad c^2 = a^2 + b^2$

Illustration :

Illustration :

If in a triangle ABC, if $\Delta = a^2 - (b - c)^2$*, then find the value of tan A?*

Sol. $\Delta = (a + b - c)\,(a - b + c) = (a + b + c - 2c)\,(a + b + c - 2b)$

$\Delta = (2s - 2c)\,(2s - 2b)$

$\Rightarrow \quad \Delta^2 = [2\,(s - b)\,2\,(s - c)]^2$

$\Rightarrow \quad s\,(s - a)\,(s - b)\,(s - c) = 16\,(s - b)^2\,(s - c)^2$

$\Rightarrow \quad \frac{(s-b)(s-c)}{s(s-a)} = \frac{1}{16}$

$\Rightarrow \quad \tan^2\frac{A}{2} = \frac{1}{16}$

$\Rightarrow \quad \tan\frac{A}{2} = \frac{1}{4}$

$\Rightarrow \quad \tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} = \frac{2\cdot\frac{1}{4}}{1-\frac{1}{16}} = \frac{8}{15}$

Practice Problem

Q.1 In any triangle ABC, prove that $(b + c - a)\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right) = 2a\cot\frac{A}{2}$

Q.2 In any triangle, if $\tan\frac{A}{2} = \frac{5}{6}$ and $\tan\frac{B}{2} = \frac{20}{37}$. Find $\tan\frac{C}{2}$ and prove that in this triangle $a + c = 2b$.

Q.3 If a, b and c be in A.P. Prove that $\cos A \cot\frac{A}{2}$, $\cos B \cot\frac{B}{2}$ and $\cos C \cot\frac{C}{2}$ are in A.P.

Q.4 If a, b and c are in H.P. Prove that $\sin^2\frac{A}{2}$, $\sin^2\frac{B}{2}$ and $\sin^2\frac{C}{2}$ are also in H.P.

Q.5 If a^2, b^2 and c^2 be in A.P. Prove that cot A, cot B and cot C are in A.P. also.

Answer key

Q.2 122

Q.2 $\frac{122}{205}$

1.7 m - n THEOREM :

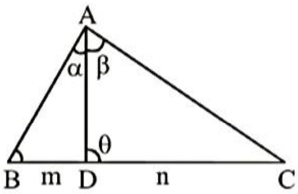
Let D be a point on the side BC of a ΔABC, such that BD : DC = m : n and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$. Prove that

(a) $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

(b) $(m + n)\cot\theta = n\cot B - m\cot C$.

Proof :

(a) Given $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$



$\angle ADB = 180^\circ - \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$.

So $\angle ABD = \theta - \alpha = B$, $\qquad C = 180^\circ - (\theta + \beta)$

From ΔABD, $\quad \frac{BD}{\sin\alpha} = \frac{AD}{\sin(\theta - \alpha)} \quad \ldots(i)$

From ΔADC, $\quad \frac{DC}{\sin\beta} = \frac{AD}{\sin(\theta + \beta)} \quad \ldots(ii)$

Dividing equation (i) by (ii)

$$\frac{BD}{DC}\cdot\frac{\sin\beta}{\sin\alpha}=\frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)} \quad \Rightarrow \quad \frac{m\sin\beta}{n\sin\alpha}=\frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)} \qquad \ldots(iii)$$

$$\frac{m\sin\beta}{n\sin\alpha}=\frac{\sin\theta\cos\beta+\cos\theta\sin\beta}{\sin\theta\cos\alpha-\cos\theta\sin\alpha}$$

$\Rightarrow$ $m\sin\beta\ (\sin\theta\cos\alpha-\cos\theta\sin\alpha)=n\sin\alpha\,(\sin\theta\cos\beta+\cos\theta+\sin\beta)$

Now dividing both sides by $\sin\alpha\sin\beta\sin\theta$

$\Rightarrow$ $m\cot\alpha-m\cot\theta=n\cot\beta+n\cot\theta$

$\Rightarrow$ $(m+n)\cot\theta=m\cot\alpha-n\cot\beta.$

(b) We have, $\angle CAD=180^o-(\theta+C)$

$\angle ABC=B,\quad \angle ACD=C,\quad \angle BAD=(\theta-B)$

Putting these values in equation (iii) we get

$m\sin(\theta+C)\sin B=n\sin C\sin(\theta-B)$

$m(\sin\theta\cos C+\cos\theta\sin C)\sin B=n\sin C(\sin\theta\cos B-\cos\theta\sin B)$

dividing both sides by $\sin\theta\sin B\sin C$

$\Rightarrow$ $m(\cot C+\cot\theta)=n(\cot B-\cot\theta)$

$\therefore$ $(m+n)\cot\theta=n\cot B-m\cot C$

Illustration :

In a triangle ABC, $\angle ABC=45^o$. Point 'D' is on BC so that 2BD = CD and $\angle DAB=15^o$. $\angle ACB$ in degree equals.

(A) 30^o *(B) 60^o* *(C) 75^o* *(D) 90^o*

Sol. *Applying m-n theorem, in ΔABC*

$(BD+DC)\cot 60^o=CD\cot 45^o-BD\cot C$

$\Rightarrow$ $3\cot 60^o=2\cot 45^o-\cot C$

$\Rightarrow$ $\cot C=2-\sqrt{3}$

$\Rightarrow$ $\boxed{C=75^o}$ *Ans.*

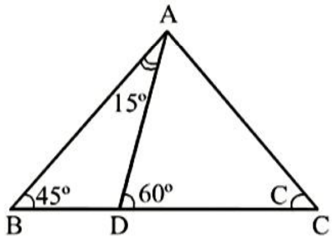


Illustration :

In a triangle ABC altitude AD, $\angle BAC = 45^o$, DB = 3 and CD = 2. The area of the ΔABC is?

(A) 6 (B) 15 (C) $\frac{15}{4}$ (D) 12

Sol. Let $\angle BAD = \alpha$

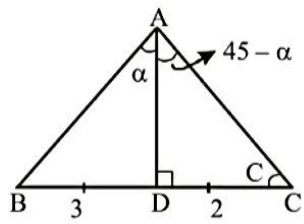
Applying (m – n) theorem, in ΔABC

$$(3 + 2) \cot 90^o = 3 \cot \alpha - 2 \cot (45^o - \alpha)$$

$$\Rightarrow \quad 0 = \frac{3}{\tan\alpha} - \frac{(1+\tan\alpha)}{(1-\tan\alpha)}$$

$$\Rightarrow \quad 3 - 3 \tan \alpha = 2 \tan \alpha + 2 \tan^2 \alpha$$

$$\Rightarrow \quad 2 \tan^2 \alpha + 5 \tan \alpha - 3 = 0$$



$$\Rightarrow \quad \tan \alpha = \frac{1}{2}, -3$$

$$\tan \alpha = \frac{1}{2} \qquad [\alpha \in (0, 45^o), \tan \alpha \in (0, 1)]$$

$$\tan \alpha = \frac{1}{2} \qquad [\alpha \in (0, 45^o), \tan \alpha \in (0, 1)]$$

$$\Delta ABD, \tan \alpha = \frac{3}{AD} = \frac{1}{2} \quad \Rightarrow \quad AD = 6$$

$$\text{Area of } \Delta ABC = \frac{1}{2} \times 5 \times 6 = 15 \text{ units}$$

Illustration :

If the median of a triangle ABC through A is perpendicular is AB then $\frac{\tan A}{\tan B}$ has the value equal to

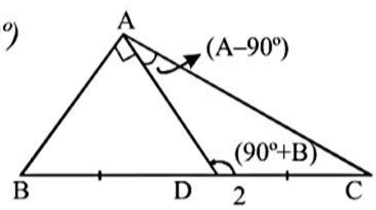
(A) $\frac{1}{2}$ (B) 2 (C) –2 (D) $-\frac{1}{2}$

Sol. Applying (m – n) theorem, ΔABC

$$(BD + CD) \cot (90^o + B) = BD \cot 90^o - CD \cot (A - 90^o)$$

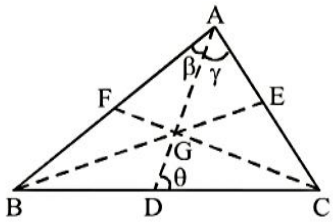
$$\Rightarrow \quad -2 \tan B = 0 + \tan A$$

$$\Rightarrow \quad \frac{\tan A}{\tan B} = -2 \qquad \textit{Ans.}$$



1.8 CENTROID AND MEDIANS OF ANY TRIANGLE :

If ABC be any triangle, and D, E and F respectively the middle points of BC, CA and AB the lines AD, BE and CF are called the medians of the triangle.



$$\boxed{\begin{aligned} AD &= \frac{1}{2}\sqrt{(2b^2+2c^2-a^2)} \\ BE &= \frac{1}{2}\sqrt{(2c^2+2a^2-b^2)} \\ CF &= \frac{1}{2}\sqrt{(2a^2+2b^2-c^2)} \end{aligned}}$$

From geometry, we know that the medians meet in a common point G, such that

$$AG = \frac{2}{3}AD, \quad BG = \frac{2}{3}BE \quad \text{and} \quad CG = \frac{2}{3}CF$$

The point 'G' is called the centroid of the triangle.

1.8.1 Length of the Medians :

$$AD^2 = AC^2 + CD^2 - 2ACCD \cos C = b^2 + \frac{a^2}{4} - 2b\frac{a}{2} \cos C$$

$$AD^2 = b^2 + \frac{a^2}{4} - ab \cos C.$$

and $\quad c^2 = b^2 + a^2 - 2ab \cos C$

so $\quad 2AD^2 - c^2 = b^2 - \frac{a^2}{2} \qquad$ so that $\qquad AD = \frac{1}{2}\sqrt{(2b^2+2c^2-a^2)}$

we can also write, $\quad AD = \frac{1}{2}\sqrt{(b^2+c^2+b^2+c^2-a^2)}$

$$AD = \frac{1}{2}\sqrt{(b^2+c^2+2bc\cos A)} \qquad \text{similarly}$$

$$BE = \frac{1}{2}\sqrt{(2c^2+2a^2-b^2)} \quad \text{and} \quad CF = \frac{1}{2}\sqrt{(2a^2+2b^2-c^2)}$$

1.8.2 Angles that the Median AD makes with the sides :

Let $\angle BAD = \beta$ and $\angle CAD = \gamma$, we have

$$\frac{\sin\gamma}{\sin C} = \frac{DC}{AD} = \frac{a}{2x}, \sin\gamma = \frac{a}{2x}\sin C, \text{ where } AD = x \text{ (say)}$$

$$x = AD = \frac{1}{2}\sqrt{(2b^2+2c^2-a^2)}, \ \sin\gamma = \frac{a\sin C}{\sqrt{(2b^2+2c^2-a^2)}}$$

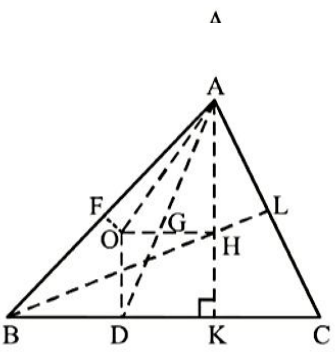
Similarly, $\sin\beta = \dfrac{a\sin B}{\sqrt{(2b^2+2c^2-a^2)}}$

Again, if the $\angle ADC = \theta$, we have

$$\frac{\sin\theta}{\sin C} = \frac{AC}{AD} = \frac{b}{x}, \sin\theta = \frac{b}{x}\sin C = \frac{2b\sin C}{\sqrt{(2b^2+2c^2-a^2)}}$$

Note : The centroid lies on the line segment joining the circumcentre to the orthocentre and divides the line segment in the ratio 1 : 2.

Let O and H be the circumcentre and orthocentre respectively.
Draw OD and HK perpendicular to BC.
Let AD and OH meet in G. By geometry ΔAGP and ΔOGD are similar



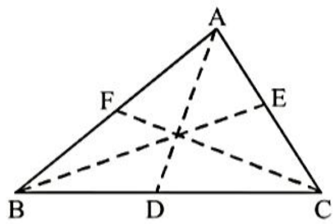
$$\frac{OG}{GP} = \frac{AG}{GD} = \frac{2}{1}$$

The centroid therefore lies on the line segment joining the circumcentre to the orthocentre and divides it in the ratio 1 : 2.

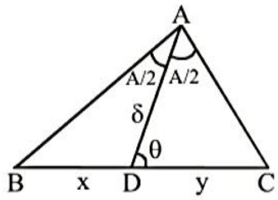
1.9 BISECTORS OF THE ANGLES :

If AD bisects the angle A and divide the base into portions x and y, we have by geometry. The length of bisectors will be as follows :

$$\boxed{\begin{aligned} AD &= \frac{2bc}{(b+c)}\cos\frac{A}{2} \\ BE &= \frac{2ca}{(c+a)}\cos\frac{B}{2} \\ CF &= \frac{2ab}{(a+b)}\cos\frac{C}{2} \end{aligned}}$$



$$\frac{x}{y} = \frac{AB}{AC} = \frac{c}{b}, \quad \frac{x}{c} = \frac{y}{b} \quad \Rightarrow \quad \frac{x+y}{b+c} = \frac{a}{b+c} \qquad \ldots(i)$$



giving x and y

Also, if δ be the length of AD and θ the angle it makes with BC, we have

$$\Delta ABD + \Delta ACD = \Delta ABC, \qquad \frac{1}{2} c\,\delta \sin\frac{A}{2} + \frac{1}{2} b\,\delta \sin\frac{A}{2} = \frac{1}{2} bc \sin A$$

$$\delta = \frac{bc}{b+c} \cdot \frac{\sin A}{\sin\frac{A}{2}} = \frac{2bc}{(b+c)} \cos\frac{A}{2}$$

$$\theta = \angle DAB + B = \frac{A}{2} + B.$$

Thus, we have the length of bisector and its inclination to BC.

1.10 CIRCUM CIRCLE :

To find the magnitude of R, the radius of the circum circle of any triangle ABC.

$$\boxed{2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}} \qquad \boxed{R = \frac{abc}{4\Delta}}$$

Proof : Consider any triangle ABC as shown in three figure

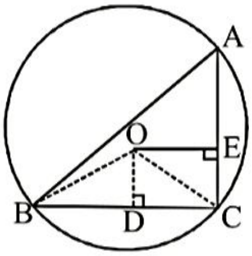


Figure (1)

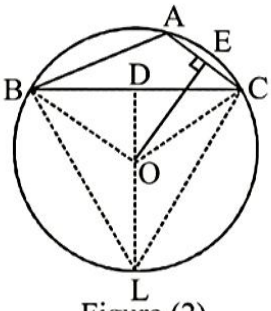


Figure (2)

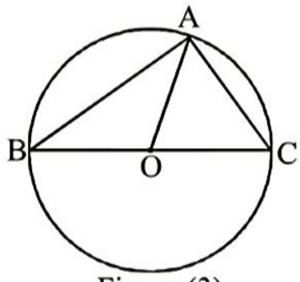


Figure (3)

Bisecting the two sides BC and CA in D and E respectivley and draw DO and EO perpendicular to BC and CA.

By geometry, O is the centre of the circumcircle. Join OB and OC.

The point O may either lie within the triangle as in figure (1) or without it as in figure (2) or upon one of the sides as in figure (3).

Taking the first figure, the two triangles BOD and COD are equal in all respects, so that

$$\angle BOD = \angle COD, \therefore \angle BOD = \frac{1}{2}(2\angle BAC) = \angle BAC = A$$

Also, $BD = BO \sin(BOD) = BO \sin A = R \sin A$ [as $R = BO$]

$$\frac{a}{2} = R \sin A$$

If A be obtuse, as in figure (2), we have

$$\angle BOD = \frac{1}{2}\angle BOC = \angle BLC = 180^\circ - A$$

$\sin(\angle BOD) = \sin(180^\circ - A) = \sin A$

and $\quad R = \dfrac{a}{2\sin A}$

If A be right angle as in figure (3) we have

$$R = OA = OC = \frac{a}{2} = \frac{a}{2\sin A} \text{ as } \sin A = \sin 90^\circ = 1$$

$$R = OA = OC = \frac{a}{2} = \frac{a}{2\sin A} \text{ as } \sin A = \sin 90^\circ = 1$$

so in all the three cases, we have

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

as we know $\quad \Delta = \dfrac{1}{2} bc \sin A$

As $\quad \sin A = \dfrac{a}{2R}$

$$\therefore \quad \Delta = \frac{1}{2} bc \frac{a}{2R} = \frac{abc}{4R}$$

$$\boxed{R = \frac{abc}{4\Delta}}$$

Note :

(a) In case of acute angle triangle, circumcentre lies within the triangle.

(b) In case of obtuse angle triangle, circumcentre lies outside the triangle.

(c) In case of right angle triangle, circumcentre lies on the mid point of hypotenuse.

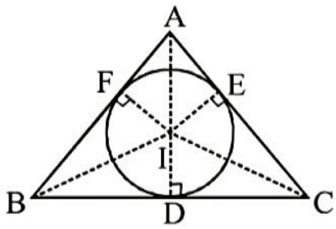
1.11 INCIRCLE :

Radius of incircle $\boxed{r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}}$

$$\boxed{r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

1.11.1 The value of r, the radius of the incircle of the triangle ABC :

Proof : Consider the triangle ABC as shown in figure. Bisect the two angles B and C by the two lines BI and CI meeting in I.
By geometry, I is the centre of the incircle, join IA, and draw ID, IE and IF perpendicular to the three sides.
The ID = IE = IF = r



We have Area of $\Delta IBC = \frac{1}{2}(ID)(BC) = \frac{1}{2}ra$

Area of $\Delta ICA = \frac{1}{2}(IE)(AC) = \frac{1}{2}r.b$, Area of $\Delta IAB = \frac{1}{2}(IF)(AB) = \frac{1}{2}r.c$

Hence by addition, we have

$$\frac{1}{2}r(a+b+c) = \text{Sum of Areas of } \Delta IBC, \Delta IAC, \Delta IBA$$

$$= \Delta ABC = \Delta$$

$$r = \frac{\Delta}{\frac{a+b+c}{2}} = \frac{\Delta}{s}$$

1.11.2 The same 'r' can be expressed in a different way also :

Consider the same figure ABC as shown above. The angles IBD and IDB are respectively equal to the angles IBF and IFB, so the two triangle IDB and IFB are equal in all respects.

Hence,	BD = BF,	so that	2BD = BD + BF
so also,	AE = AF,	so that	2AE = AE + AF
and	CE = CD,	so that	2CE = CE + CD

Hence, by addition, we have

$$2BD + 2AE + 2CE = (BD + CD) + (BF + AF) + (AE + CE)$$

$\therefore$ $\quad 2BD + 2b = a + b + c = 2s,$

Hence, $\quad BD = s - b = BF,\quad$ similarly $\quad CE = s - c,\quad AF = s - a$

$$\tan IBD = \frac{ID}{BD} = \frac{r}{(s-b)} = \tan\frac{B}{2}$$

$$r = (s-b)\tan\frac{B}{2} = (s-a)\tan\frac{A}{2} = (s-c)\tan\frac{C}{2}$$

1.11.3 A third value of r may be found as follows :

1.11.3 A third value of r may be found as follows :

$$a = BD + CD = ID\cot IBD + ID\cot ICD$$

$$= r\cot\frac{B}{2} + r\cot\frac{C}{2} = r\left[\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right]$$

$$a = r\frac{\sin\left(\frac{B+C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}} = r\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$r = a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} \qquad \left[\text{as} \quad \frac{a}{\sin A} = 2R\right]$$

As $\quad a = 2R\sin A = 2R\,2\sin\frac{A}{2}\cos\frac{A}{2}$

$\therefore$ $\quad \boxed{r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$

1.12 ESCRIBED CIRCLE :

1.12.1 To find the value of r_1, the radius of the escribed circle opposite the angle A of the triangle ABC :

$$\boxed{\begin{aligned} r_1 &= \frac{\Delta}{(s-a)} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \\ r_2 &= \frac{\Delta}{(s-b)} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} \\ r_3 &= \frac{\Delta}{(s-c)} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} \end{aligned}}$$

Proof : Produce AB and AC to L and M. Bisect the angles CBL and BCM by the lines BI_1 and CI_1 and let these lines meet in I_1.

Draw I_1D_1, I_1E_1 and I_1F_1 perpendicular to these lines respectively.

The two triangles I_1D_1B and I_1F_1B are equal in all respect, so that $I_1F_1 = I_1D_1$ similarly $I_1E_1 = I_1D_1$.

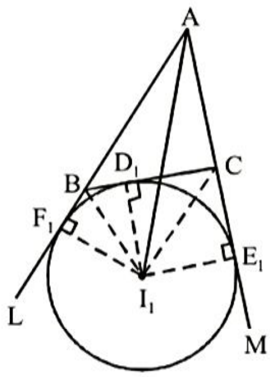
The three perpendicular I_1D_1, I_1E_1 and I_1F_1 being equal, the point I_1 is the centre of the required circle.

The three perpendicular I_1D_1, I_1E_1 and I_1F_1 being equal, the point I_1 is the centre of the required circle.

Now, the area ABI_1C is equal to the triangles ABC and I_1BC, it is also equal to the sum of the triangle I_1BA and I_1CA.

Hence, $\Delta ABC + \Delta I_1BC = \Delta I_1CA + \Delta I_1AB$

$$\therefore \quad \Delta + \frac{1}{2}(I_1D_1)(BC) = \frac{1}{2}(I_1E_1)(CA) + \frac{1}{2}(I_1F_1)(AB)$$



$$\therefore \quad \Delta + \frac{1}{2}r_1\,a = \frac{1}{2}\,r_1\cdot b + \frac{1}{2}r_1\,c$$

$$\Delta = \frac{r_1}{2}(b+c-a) = \frac{r_1}{2}(a+b+c-2a) = r_1\,(s-a)$$

$$r_1 = \frac{\Delta}{(s-a)} \qquad \text{similarly } r_2 = \frac{\Delta}{(s-b)},\ r_3 = \frac{\Delta}{(s-c)}$$

1.12.2 A Second Value of r_1 can be obtained :

Since AE_1 and AF_1 are tangents, $AE_1 = AF_1$

Similarly, $BF_1 = BD_1$ and $CE_1 = CD_1$

$$\therefore \quad 2AE_1 = AE_1 + AF_1 = AB + BF_1 + AC + CE_1$$
$$= AB + BD_1 + AC + CD_1$$
$$= AB + AC + BC = 2s$$

$$\therefore \quad AE_1 = s = AF_1$$

Also $BD_1 = BF_1 = AF_1 - AB = s - c$

similarly $CD_1 = CE_1 = AE_1 - AC = s - b$ $\quad \therefore \quad I_1E_1 = AE_1 \tan(I_1AE_1)$

so $r_1 = s \tan \dfrac{A}{2}$

1.12.3 A third value of r_1 may be obtained :

For, since I_1C bisects the angle BCE_1, we have

$$\angle I_1CD_1 = \frac{1}{2}(180^\circ - C) = 90^\circ - \frac{C}{2}$$

so, $\angle I_1BD_1 = 90^\circ - \dfrac{B}{2}$

$$\therefore \quad a = BC = BD_1 + D_1C = I_1D_1 \cot I_1BD_1 + I_1D_1 \cot I_1CD_1$$

$$= r_1 \left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = r_1 \left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = r_1 \frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$\Rightarrow \quad a \cos\frac{B}{2}\cos\frac{C}{2} = r_1 \cos\frac{A}{2}$$

$$r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} \text{ as } a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$$

$$\therefore \quad r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Similarly, $r_2 = 4R\sin\dfrac{B}{2}\cos\dfrac{C}{2}\cos\dfrac{A}{2}$

$$r_3 = 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$$

Important results regarding r_1, r_2 and r_3.

Given r_1, r_2, and r_3

(i) semiperimeter $= s = \sqrt{(r_1r_2 + r_2r_3 + r_3r_1)} = \sqrt{\Sigma r_1r_2}$ (ii) $\Delta = \dfrac{r_1r_2r_3}{\sqrt{\Sigma r_1r_2}}$

(iii) $r = \dfrac{r_1r_2r_3}{\Sigma r_1r_2}$ (iv) $R = \dfrac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{4\,\Sigma\, r_1r_2}$

(v) $a = \dfrac{r_1(r_2+r_3)}{\sqrt{\Sigma\, r_1r_2}}$, $b = \dfrac{r_2(r_3+r_1)}{\sqrt{\Sigma\, r_1r_2}}$, $c = \dfrac{r_3(r_1+r_2)}{\sqrt{\Sigma\, r_1r_2}}$ (vi) $\sin A = \dfrac{2r_1\sqrt{\Sigma r_2r_3}}{(r_1+r_2)(r_1+r_3)}$

Illustration :

If in a triangle, $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Sol. $r_1 = r_2 + r_3 + r \Rightarrow r_1 - r = r_2 + r_3$

$$\Rightarrow \quad \frac{\Delta}{(s-a)} - \frac{\Delta}{s} = \frac{\Delta}{(s-b)} + \frac{\Delta}{(s-c)}$$

$$\Rightarrow \quad \frac{\Delta a}{s(s-a)} = \frac{\Delta(2s-b-c)}{(s-b)(s-c)}$$

$\Rightarrow \quad s(s-a) = (s-b)(s-c)$

$\Rightarrow \quad s^2 - sa = s^2 - (b+c)s + bc$

$\Rightarrow \quad (b+c-a)s = bc,$ $\quad (b+c-a)\,2s = 2bc$

$(b+c-a)(b+c+a) = 2bc,$ $\quad (b+c)^2 - a^2 = 2bc,$ $\quad b^2 + c^2 - a^2 = 0$

$b^2 + c^2 = a^2$, *so the triangle is right angled.*

Illustration :

Prove that $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$.

Sol. $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

$$= 1 + \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R} = 1 + \frac{r}{R}$$

Illustration :

Prove that $\dfrac{a\cos A + b\cos B + c\cos C}{a+b+c} = \dfrac{r}{R}$

Sol. *We have*

$$a \cos A + b \cos B + c \cos C = 2R \sin A \cos A + 2R \sin B \cos B + 2R \sin C \cos C$$
$$= R (\sin 2A + \sin B + \sin 2C)$$
$$= 4R \sin A \sin B \sin C$$

$$a + b + c = 2R (\sin A + \sin B + \sin C) = 2R \cdot 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$$

$$\Rightarrow \quad \text{so L.H.S.} = \frac{4R\sin A\sin B\sin C}{8R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R} = \frac{r}{R}$$

Illustration :

Prove that $r_1 + r_2 + r_3 - r = 4R.$

Sol. $r_1 + r_2 + r_3 - r = \dfrac{\Delta}{s-a} + \dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-c} - \dfrac{\Delta}{s}$

$$= \Delta\left[\frac{(s-b)+(s-a)}{(s-a)(s-b)} + \frac{s-s+c}{3(s-c)}\right] = \Delta\left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right]$$

$$= \Delta c\ \frac{[s(s-c)+(s-a)(s-b)]}{s(s-a)(s-b)(s-c)} = \frac{\Delta c}{\Delta^2} = [2s^2 - s\,(a + b + c) + ab]$$

$$= \frac{c}{\Delta}[2s^2 - 2s^2 + ab] = \frac{abc}{\Delta} = 4R$$

Practice Problem

Prove that in a ΔABC,

Q.1 $\dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} = \dfrac{a^2+b^2+c^2}{\Delta^2}$

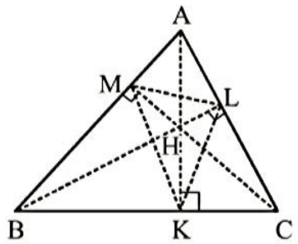
Q.2 $\dfrac{1}{bc} + \dfrac{1}{ca} + \dfrac{1}{ab} = \dfrac{1}{2Rr}$

Q.3 $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$

Q.4 $a(r r_1 + r_2 r_3) = b(r r_2 + r_3 r_1) = c(r r_3 + r_1 r_2)$

Q.5 $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$

1.13 ORTHOCENTRE AND PEDAL TRIANGLE OF ANY TRIANGLE :



Let ABC be any triangle and let AK, BL and CM to be perpendiculars from A, B and C upon the opposite sides of the triangle. It can be easily shown from geometry, that these three perpendiculars meet in a common point H. This point H is called the orthocentre of the triangle. The triangle KLM, which is formed by joining the feet of these perpendicular, is called the pedal triangle of ABC.

1.13.1 Distances of the orthocentre of the angular points of the triangle :

Consider an acute angle triangle ABC.

We have, $\quad HK = KB \tan (HBK) = KB \tan \ (90^\circ - C) = KB \cot C$

$$= AB \cos B \cot C = AB \cos B \frac{\cos C}{\sin C} = \frac{AB}{\sin C} \cos B \cos C$$

$$HK = \frac{c}{\sin C} \cos B \cos C = 2R \cos B \cos C.$$

sin C

Again, $\quad AH = \ AL \sec (KAC) = c \cos A \sec (90^\circ - C)$

$$= c \cos A \operatorname{cosec} C = \frac{c}{\sin C} \cos A = 2R \cos A$$

Similarly $\quad BH = 2R \cos B$ and $CH = 2R \cos C$.

The distances of the orthocentre from the angular point are therefore, $2R \cos A$, $2R \cos B$ and $2R \cos C$. Its distance from the sides a, b, c are $2R \cos B \cos C$, $2R \cos C \cos A$ and $2R \cos A \cos B$ respectively.

1.13.2 The sides and angles to the pedal triangle :

Consider an acute angle triangle ABC:

Since the angles HKC and HLC are right angles, the points H, L , C and K lie on a circle.

$\angle HKL = \angle HCL = 90^\circ - A$ Similarly.

H, K, B, M lie on a circle, and therefore,

$\angle HKM = \angle HBM = 90^\circ - A$

Hence $\quad \angle MKL = 180^\circ - 2A =$ the supplement of 2A.

$\angle KLM = 180^\circ - 2B, \quad \angle LMK = 180^\circ - 2C.$

Again, from the triangle ALM, we have

$$\frac{LM}{\sin A} = \frac{AL}{\sin(AML)} = \frac{AB\cos A}{\cos(HML)} = \frac{c\cos A}{\cos(HAL)} = \frac{c\cos A}{\sin C}$$

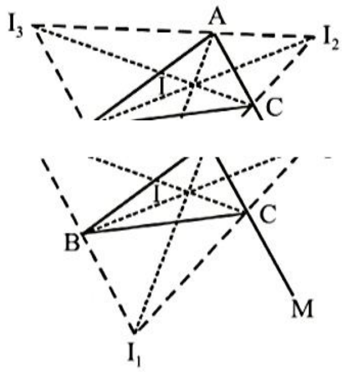
$$LM = \frac{c}{\sin C} \cos A \sin A = \frac{c}{\sin C} \sin A \cos A = a \cos A$$

so $\quad LM = a \cos A$, similarly $MK = b \cos B$, $KL = c \cos C$

The sides of the pedal triangle therefore a cos A, b cos B and c cos C; also its angles are the supplements of twice the angles of the triangle.

1.13.3 Excentric Triangle :

Let I be the centre of the incircle and I_1, I_2 and I_3 the centres of the escribed circles. Which are opposite to A, B and C respectively. IC bisects the angle ACB and I_1C bisects the angle



respectively. IC bisects the angle ACB and I_1C bisects the angle BCM.

$$\therefore \quad \angle ICI_1 = \angle ICB + \angle I_1CB = \frac{1}{2}\angle ACB + \frac{1}{2}\angle MCB$$

$$= \frac{1}{2}(\angle ACB + \angle MCB) = \frac{1}{2}(180^\circ) = 90^\circ$$

= A right angle.

Similarly, $\angle ICI_2$ is a right angle. Hence I_1CI_2 is a straight line to which IC is perpendicular. So, I_2AI_3 is a straight line to which IA is perpendiculars and I_3BI_1 is a straight line to which IB is perpendicular.

Also, since IA and I_1A both bisect the angle BAC, the three points A, I and I_1 are in a striaght line. Similarly, B I I_2 and C I I_3 are straight lines. Hence, I_1 I_2 I_3 is a triangle, which is such that A, B and C are the feet of the perpendiculars drawn from its vertices upon the opposite sides and such that I is the intersection of these perpendicular. i.e. ABC is its pedal triangle and I is its orthocentre. The triangle $I_1I_2I_3$ is often called the excentric triangle.

1.13.4 Prove that circumradii of ΔHBC, ΔHCA and ΔHAB and ΔABC are equal :

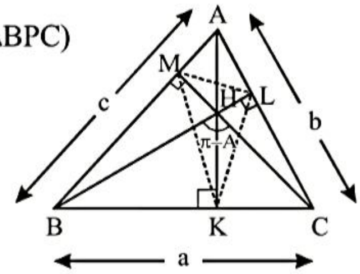
In ΔBHC, $\angle BHC = \pi - A$

In ΔBMC, $\angle B + \angle BCM + 90^\circ = 180^\circ$

$\angle BCM = 90^\circ - \angle B$, $\angle KHC = 90^\circ - (90^\circ - \angle B) = \angle B$

similarly $\angle KHB = \angle C$, so $\angle BHC = \angle B + \angle C = 180^\circ - \angle A$

In ΔBHC, $\dfrac{a}{\sin(\pi - A)} = 2R'$ (where R' is the circumcircle of ΔBPC)



$\dfrac{a}{\sin A} = 2R'$ But we know $\dfrac{a}{\sin A} = 2R$

so $2R' = 2R \Rightarrow R' = R$. So circum radii of ΔHBC, ΔHCA and ΔHAB and ΔABC are equal.

Note :- Radius of the circle circumscribing a pedal triangle in $\dfrac{R}{2}$.

Proof : $\dfrac{ML}{\sin(MKL)} = 2R' = \dfrac{a\cos A}{\sin(180^\circ - 2A)} = \dfrac{2R\sin A\cos A}{\sin 2A} = R$

Proof : $\dfrac{}{\sin(MKL)} = 2R' = \dfrac{}{\sin(180^\circ - 2A)} = \dfrac{}{\sin 2A} = R$

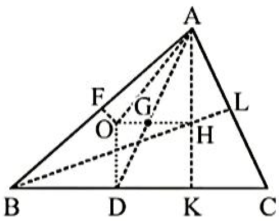
$$\boxed{R' = \frac{R}{2}}$$

1.13.5 Distance between the circumcentre and the orthocentre :

$$\boxed{OH = R\sqrt{(1 - 8\cos A\cos B\cos C)}}$$

If OF be perpendicular to AB, we have

$\angle OAF = 90^\circ - \angle AOF = 90^\circ - C$



Also, $\angle OAH = A - \angle OAF - \angle HAL$

$= A - 2(90^\circ - C) = A + 2C - 180^\circ$

$= A + 2C - (A + B + C) = C - B$

Also, $OA = R$, $HA = 2R\cos A$

$OH^2 = OA^2 + HA^2 - 2\,OA\,HA\cos OAP = R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$

$= R^2 + 4R^2\cos A[\cos A - \cos(C - B)]$

$= R^2 - 4R^2\cos A[\cos(B + C) + \cos(C - B)]$

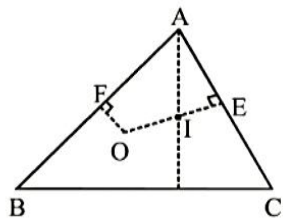
$OH^2 = R^2 - 8R^2\cos A\cos B\cos C$

$OH = R\sqrt{(1 - 8\cos A\cos B\cos C)}$

1.13.6 Distance between the cirumcentre and incentre :

$$\boxed{OI = \sqrt{R^2 - 2Rr}}$$

Let O be the circumcentre and OF be perpendicular to AB.
Let I be the incentre and IE be perpendicular to AC.
Then, as in the previous article,



$$\angle OAF = 90^\circ - C$$

$$\therefore \quad \angle OAI = \angle IAF - \angle OAF = \frac{A}{2} - (90^\circ - C)$$

$$= \frac{A}{2} + C - \frac{A+B+C}{2} = \frac{C-B}{2}$$

Also, $AI = \dfrac{IE}{\sin\frac{A}{2}} = \dfrac{r}{\sin\frac{A}{2}} = \dfrac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}$

so $\quad AI = 4R\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

$$\therefore \quad OI^2 = OA^2 + AI^2 - 2OA \cdot AI\cos(OAI)$$

$$OI^2 = R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\left(\frac{C-B}{2}\right)$$

$$OI^2 = R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\left(\frac{C-B}{2}\right)$$

$$\frac{OI^2}{R^2} = 1 + 16\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8\sin\frac{B}{2}\sin\frac{C}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\cos\left(\frac{B+C}{2}\right) = 1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$OI = R\sqrt{\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)}$$

we can write this in another form also.

$$OI^2 = R^2 - 8R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= R^2 - 2R\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= R^2 - 2Rr \qquad \text{as} \qquad r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

1.14 CYCLIC QUADRILATERAL AND REGULAR POLYGON :

1.14.1 Polygon :

(i) Sum of interior angles of a polygon = $(n-2) \times \pi$. where $n \geq 2$ and n denotes number of sides of a polygon.

(ii) Sum of exterior angles of a polygon is 2π.

(iii) **Convex polygon :** If the highest interior angle is less than 180° then it is called convex polygon.

(iv) **Concave polygon :** Highest interior angle is more than 180° then it is concave polygon.

1.14.2 Cyclic Quadrilateral :

A cyclic quadrilateral is a quadrilateral which can be inscribed by a circle.

Note : The sum of the opposite angles of a cyclic quadrilateral is 180°.

In a cyclic quadrilateral sum of the products of the opposite sides is equal to the product of the diagonals.

Regular Polygon

A regular polygon is a polygon which has all its sides as well as its angles equal. If the polygon has n-

$(n-2)\pi$

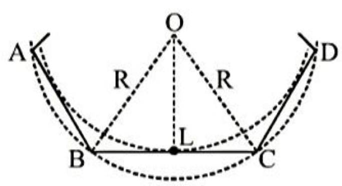
A regular polygon is a polygon which has all its sides as well as its angles equal. If the polygon has n-

sides, sum of its internal angles is $(n-2)\pi$ and each angle is $\frac{(n-2)\pi}{n}$.

Note : In the regular polygon, the centroid, the circumcentre and the in-centre are the same.

To find the perimeter (P) and Area(A) of a regular polygon inscribed in a circle of radius 'R'.

Let AB, BC and CD be three successive sides of the polygon and O be the centre of both the incircle and the circumcircle of the polygon.



$$\angle BOC = \frac{2\pi}{n}, \text{ so } \angle BOL = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}$$

If 'a' be the side of the polygon, we have

$$a = BC = 2BL = 2R\sin(\angle BOL) = 2R\sin\left(\frac{\pi}{n}\right)$$

So, $$R = \frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{n}\right)$$

Again $$a = 2BL = 2OL\tan(\angle BOL),\ OL = \frac{a}{2\tan\left(\frac{\pi}{n}\right)} = \frac{a}{2}\cot\left(\frac{\pi}{n}\right) \Rightarrow r = \frac{a}{2}\cot\left(\frac{\pi}{n}\right)$$

where R : Radius of circle circumscribing the polygon = OB = OC
r : Radius of circle inscribed in the polygon = OL

Perimeter $P = nBC = n(2BL) = 2n\,R\sin(\angle BOL) = 2n\,R\sin\left(\frac{\pi}{n}\right) = 2nR\sin\left(\frac{\pi}{n}\right)$

Area $A = n$ Area of $\Delta BOC = n\,\frac{1}{2}\,R\cdot R\cdot\sin(\angle BOC) = \frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$

Illustration :

Prove that the area of a regular polygon of 2n sides inscribed in a circle is the geometric mean of the areas of the inscribed and circumscribed polygons of n-sides.

Sol. *Let 'a' be the radius of the circle.*

Then S_1 = *Area of regular polygon of n-sides inscribed in the circle* $= \frac{1}{2}na^2\sin\left(\frac{2\pi}{n}\right)$

S_2 = *Area of regular polygon of n-sides circumscribing the circle* $= na^2\tan\left(\frac{\pi}{n}\right)$

S_3 = *Area of regular polygon of 2n-sides inscribed in the circle* $= na^2\sin\left(\frac{\pi}{n}\right)$

S_3 = *Area of regular polygon of 2n-sides inscribed in the circle* $= na^2\sin\left(\frac{\pi}{n}\right)$

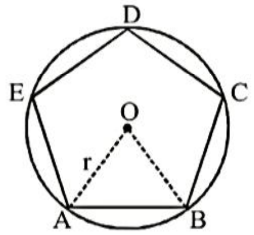
$\therefore$ *Geometric mean of* S_1 *and* $S_2 = \sqrt{\frac{1}{2}na^2\;2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)na^2\tan\left(\frac{\pi}{n}\right)} = na^2\sin\left(\frac{\pi}{n}\right) = S_3$

Illustration :

If the area of the circle is A_1 *and the area of the regular pentagon inscribed in the circle is* A_2. *then find the ratio* $\frac{A_1}{A_2}$ *?*

Sol. *In* ΔOAB,

$OA = OB = r$ *and* $\angle AOB = \frac{360°}{5} = 72°$



$\therefore$ *Area of* $\Delta AOB = \frac{1}{2}(r)(r)\sin 72° = \frac{r^2}{2}\cos 18°$

A_2 = *Area of pentagon* $= \frac{5r^2}{2}\cos 18°$

A_1 = *Area of circle* $= \pi r^2$ *so* $\frac{A_1}{A_2} = \frac{2\pi}{5}\sec\left(\frac{\pi}{10}\right)$. ***Ans.***

Illustration :

The length of each side of a regular dodecagen is 20 cm. Find (1) The radius of its inscribed circle (2) The radius of its circumscribing circle (3) its area ?

Sol. *The angle subtended by a side at the centre of the polygon* $= \dfrac{360°}{12} = 30°$.

Hence, $\dfrac{20}{2} = r \tan 15° = R \sin 15°$

So, $r = 10 \cot 15° = 10\left(2+\sqrt{3}\right)$ cm, $R = \dfrac{10}{\sin 15°} = 10\left(\sqrt{6}+\sqrt{2}\right)$ cm

Area $= \dfrac{1}{2} \times 20 \times r \times 12 = 10 \times r \times 12 = 1200\left(2+\sqrt{3}\right)$ cm. **Ans.**

Practice Problem

Q.1 Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each is 24 cm.

Q.1 Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each is 24 cm.

Q.2 If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as 2 : 3.

Q.3 Prove that the sum of the radii of the circles, which are respectively inscribed in and circumscribed about a regular polygon of n-sides is $\dfrac{a}{2} \cot \dfrac{\pi}{2n}$, where a is side of the polygon.

Q.4 Of two regular polygon of n-sides, one circumscribes and other is inscribed in a given circle. Prove that the perimeters of the circumscribing polygon, the circle, and the inscribed polygon are in the ratio $\sec\dfrac{\pi}{n} : \dfrac{\pi}{n}\operatorname{cosec}\dfrac{\pi}{n} : 1$ and that the areas of the polygons are in the ratio $\cos^2\dfrac{\pi}{n} : 1$.

Q.5 Given that the area of a polygon of n-sides circumscribed about a circle is to the area of the circumscribed polygon of 2n sides 3 : 2, find n ?

Answer key

Q.1 1.8866 cm Q.5 5

1.15 SOLUTION OF TRIANGLES (Ambiguous cases) :

When three elements of a triangle are known, the other three elements can be evaluated. This process is called solution of triangles. Note following points

(i) If the three sides a, b, c are given, angle A is a obtained from

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ or } \cos A = \frac{b^2+c^2-a^2}{2bc}$$

B and C can be obtained similarly.

(ii) If two sides b and c and the included angle A are given, then

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2} \quad \text{gives} \quad \frac{B-C}{2} \quad \text{also} \quad \frac{B+C}{2} = 90° - \frac{A}{2},$$

so that B and C can be evaluated. The third side is given by

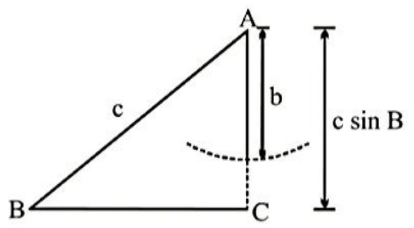
$$a = \frac{b\sin A}{\sin B} \text{ or } a^2 = b^2 + c^2 - 2bc\cos A.$$

(iii) If two sides b and c and the angle B (opposite to side b) are given, then $\sin C = \frac{c}{b}\sin B$,

(iii) If two sides b and c and the angle B (opposite to side b) are given, then $\sin C = \frac{}{b}\sin B$,

$A = 180° - (B+C)$ and $a = \frac{b\sin A}{\sin B}$ give the remaining elements.

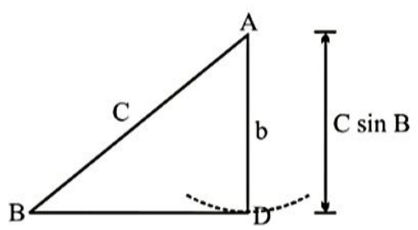
Here in this segment there are many cases of possibility of triangle. We will study them one by one.



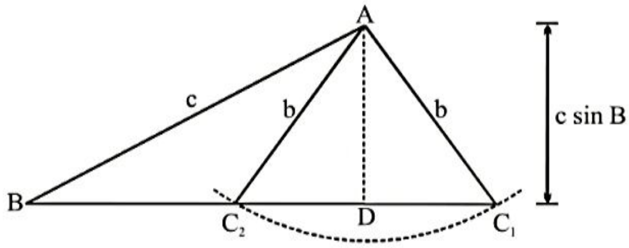
Case-I : $b < c\sin B$,

We draw the side c and angle B. Such kind of triangle is not possible.

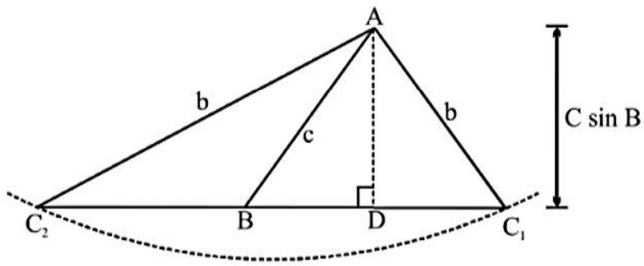
Case-II : $b = c\sin B$ and B is an acute angle, then there is only one triangle possible.



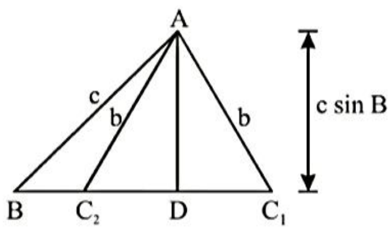
Case-III : If $b > c \sin B$, $b < c$ and B is an acute angle, then there are two values of angle C.



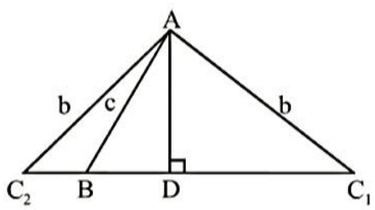
Case-IV : $b > c \sin B$, $c < b$ and B is an acute angle, then there is only one triangle possible.



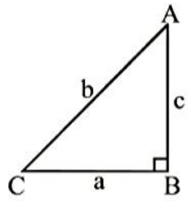
Case-V : $b > c \sin B$, $c > b$ and B is an obtuse angle. For any choice of point C, b will be greater than c which is a contradiction as $c > b$ (given). So there is no triangle possible. Because B is obtuse .



Case-VI: $b > c \sin B$, $c < b$ and B is an obtuse angle. We can see that the circle with A as centre and b as radius will cut the line only in one point. So, one triangle is possible.



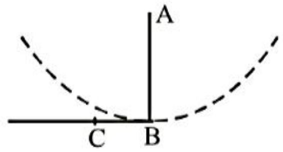
Case-VII: b > c and B = 90°



Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.

Case-VIII: b ≤ c and B = 90°

The circle with A as centre and b as radius will not cut the line in any point. So, no triangle is possible. Point 'C' will coincide with point 'B'



Alternative method : By applying the cosine rule, we have

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}.$$

⇒ $a^2 - (2\,c\cos B)\,a + c^2 - b^2 = 0$

⇒ $a = c\cos B \pm \sqrt{\left[(c\cos B)^2 - (c^2 - b^2)\right]}$

⇒ $a = c\cos B \pm \sqrt{\left[b^2 - (c\sin B)^2\right]}$

This equation leads to the following cases :

Case-I : If $b < c\sin B$, no such triangle is possible.

Case-II: Let $b = c\sin B$. There are further following two cases :

(a) B is an obtuse angle ⇒ cos B is negative. There exists no solution triangle.

(b) B is an acute angle ⇒ cos B is positive. There exists only one such triangle.

Case-III: Let b > c sin B. There are further following two cases :

(a) B is an acute angle $\Rightarrow$ cos B is positive. In this cases two values of a will exists if and only if

$c \cos B > \sqrt{\left(b^2 - (c \sin B)^2\right)}$ or $c > b$ $\Rightarrow$ two such triangles are possible.

If $c < b$, only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow$ cos B is negative. In this case, triangle will exists if and only if

$\sqrt{\left(b^2 - (c \sin B)^2\right)} > |\cos B|$ $\Rightarrow$ $b > c$. So in this case only one triangle is possible. If $b < c$ there exists no such triangle.

Illustration :

In a triangle ABC, the sides b, c and angle B are given such that a has two values a_1 and a_2. Then prove that $|a_1 - a_2| = \sqrt{(b^2 - c^2 \sin^2 B)}$.

Sol. $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} \Rightarrow a^2 - 2ac \cos B + c^2 - b^2 = 0$

$\Rightarrow \quad a_1 + a_2 = 2c \cos B, \quad a_1 a_2 = c^2 - b^2$

$\Rightarrow \quad a_1 + a_2 = 2c \cos B, \quad a_1 a_2 = c^2 - b^2$

$\Rightarrow \quad (a_1 - a_2)^2 = (a_1 + a_2)^2 - 4a_1 a_2 = 4c^2 \cos^2 B - 4(c^2 - b^2)$

$= 4b^2 - 4c^2 \sin^2 B = 4(b^2 - c^2 \sin^2 B)$

$\Rightarrow \quad |a_1 - a_2| = 2\sqrt{(b^2 - c^2 \sin^2 B)}$.

Illustration :

In a ΔABC, a, c and A are given and b_1, b_2 are two values of the third side b such that $b_2 = 2b_1$. Then prove that $\sin A = \sqrt{\left(\dfrac{9a^2 - c^2}{8c^2}\right)}$.

Sol. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} \Rightarrow b^2 - 2bc \cos A + (c^2 - a^2) = 0$

$b_1 + b_2 = 2c \cos A, \quad b_1 b_2 = c^2 - a^2$

$\Rightarrow \quad 3b_1 = 2 \cos A, \qquad 2b_1^2 = c^2 - a^2$

$$2\left(\frac{2c \cos A}{3}\right)^2 = c^2 - a^2 \Rightarrow 8c^2 (1 - \sin^2 A) = 9c^2 - 9a^2$$

$$\sin A = \sqrt{\left(\frac{(9a^2 - c^2)}{8c^2}\right)}.$$

Practice Problem

Q.1 If in a triangle ABC, $a = \left(1+\sqrt{3}\right)$ cm, $b = 2$ cm, and $\angle C = 60°$, then find the other two angles and the third side ?

Q.2 If $A = 30°$, $a = 7$, $b = 8$ in ΔABC, then find the number of triangles that can be constructed.

Q.3 If $b = 3$, $c = 4$ and $B = \frac{\pi}{3}$ in ΔABC, then find the number of triangles that can be constructed.

Answer key

Q.1 $A = 75°$, $B = 45°$, $c = \sqrt{6}$ cm Q.2 2 Q.3 0

Solved Examples

Q.1 In ΔABC, if $\dfrac{\sin A}{c\sin B}+\dfrac{\sin B}{c}+\dfrac{\sin C}{b}=\dfrac{c}{ab}+\dfrac{b}{ac}+\dfrac{a}{bc}$, then the value of angle A, is

(All symbols used have their usual meaning in a triangle.)

(A) 120° (B) 90° (C) 60° (D) 30°

Sol. We have

$$\frac{\sin A}{c\sin B}+\frac{\sin B}{c}+\frac{\sin C}{b}=\frac{c}{ab}+\frac{b}{ac}+\frac{a}{bc}$$

$$\Rightarrow \quad \frac{a}{cb}+\frac{b\sin B+c\sin C}{bc}=\frac{c^2+b^2}{abc}+\frac{a}{bc}$$

$$\Rightarrow \quad a\,(b\sin B+c\sin C)=b^2+c^2$$

$$\Rightarrow \quad a\left(b\cdot\frac{b}{2R}+c\frac{c}{2R}\right)=b^2+c^2$$

$$\Rightarrow \quad \frac{a(b^2+c^2)}{2R}=b^2+c^2$$

$$\Rightarrow \quad \frac{a(b^2+c^2)}{2R}=b^2+c^2$$

$$\Rightarrow \quad a=2R$$

$\Rightarrow$ ΔABC is a right angle triangle, $\angle A=90^\circ$.

Q.2 In a triangle ABC, $3\sin A+4\cos B=6$ and $3\cos A+4\sin B=1$, then $\angle C$ can be

(A) 30° (B) 60° (C) 90° (D) 150°

Sol. Given

$$3\sin A+4\cos B=6 \quad \ldots(i)$$

$$3\cos A+4\sin B=1 \quad \ldots(ii)$$

Squaring and adding equation (i) & (ii)

$$(3\sin A+4\cos B)^2+(3\cos A+4\sin B)^2=36+1$$

$$\Rightarrow \quad 9+16+24(\sin A\cos B+\cos A\sin B)=37$$

$$\Rightarrow \quad \sin(A+B)=\frac{1}{2}$$

$$\Rightarrow \quad A+B=30^\circ \quad \text{or} \quad 150^\circ$$

when $A+B=30^\circ$ then $(3\sin A+4\cos B)<3\sin 30^\circ+4\cos 30^\circ<6$

so $A+B=150^\circ$

$\therefore$ $\angle C=30^\circ$

Q.3 In a triangle ABC, angle A is greater than B, if the measures of angles A and B satisfy the equation $3 \sin x - 4 \sin^3 x = k$, $0 < k < 1$, then measure of angle C is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

Sol. We have $3 \sin x - 4 \sin^3 x = k$

$\Rightarrow \quad \sin 3x = k$

$\Rightarrow \quad \sin 3A = k, \sin 3B = k$

$\Rightarrow \quad \sin 3A - \sin 3B = 0$

$$\Rightarrow \quad 2\sin\left(\frac{3A-3B}{2}\right)\cos\left(\frac{3A+3B}{2}\right) = 0$$

$$\Rightarrow \quad \cos\frac{3}{2}(A+B) = 0 \qquad \because A > B$$

$$\Rightarrow \quad \frac{3}{2}(A+B) = \frac{\pi}{2}$$

$$\Rightarrow \quad A + B = \frac{\pi}{3}$$

$$\Rightarrow \quad C = \frac{2\pi}{3} \qquad \text{Ans.}$$

Q.4 If in a triangle ABC, $\sin A = \sin^2 B$ and $2 \cos^2 A = 3 \cos^2 B$, then the ΔABC is

(A) right angled (B) obtuse angled (D) isosceles (D) equilateral

Sol. $\sin A = \sin^2 B$ ...(i)

$2 \cos^2 A = 3 \cos^2 B$ ...(ii)

$\Rightarrow \quad 2(1 - \sin^2 A) = 3(1 - \sin A)$

$$\Rightarrow \quad \sin A = 1, \frac{1}{2}$$

But $\sin A \neq 1$ from equation (i)

$$\therefore \quad \sin A = \frac{1}{2} \quad \Rightarrow \quad A = 30^\circ \text{ or } 150^\circ$$

$$\sin^2 A = \frac{1}{2} \quad \Rightarrow \quad B = 45^\circ \text{ or } 135^\circ$$

in each can triangle is obtuse angled.

Q.5 In a triangle with sides a, b and c, a semicircle touching the sides AC and CB is inscribed whose diameter lies on AB. Then, the radius of the semicircle is

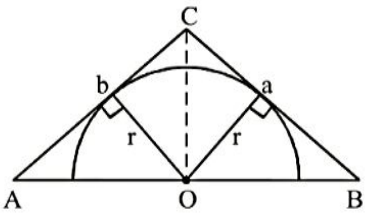
(A) $a / 2$ (B) Δ / s

(C) $\dfrac{2\Delta}{a+b}$ (D) $\dfrac{2abc}{(s)(a+b)} \cos\dfrac{A}{2} \cos\dfrac{B}{2} \cos\dfrac{C}{2}$

Sol. Let radius of semicircle is r.

Area of ΔACB = Area of ΔAOC + Area of ΔBOC

$\Rightarrow \qquad \Delta = \dfrac{1}{2}ar + \dfrac{1}{2}br$



$\Rightarrow \qquad r = \left(\dfrac{2\Delta}{a+b}\right)$

$$\frac{2abc}{s(a+b)}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{2abc}{s(a+b)}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}}$$

$$- \quad 2abc \quad \sqrt{s(b-a)(s-b)(s-c)} \quad - \quad 2\Delta \quad -$$

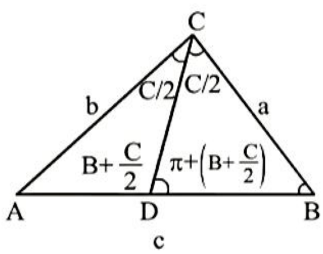
$$= \frac{2abc}{(a+b)}\frac{\sqrt{s(b-a)(s-b)(s-c)}}{abc} = \frac{2\Delta}{a+b} = r$$

Q.6 If in a triangle ABC, CD is the angular bisector of the angle ACB then CD is equal to

(A) $\dfrac{a+b}{2ab}\cos\dfrac{C}{2}$ (B) $\dfrac{a+b}{ab}\cos\dfrac{C}{2}$ (C) $\dfrac{2ab}{a+b}\cos\dfrac{C}{2}$ (D) $\dfrac{b\sin A}{\sin\left(B+\dfrac{C}{2}\right)}$

Sol. We know that $CD = \dfrac{2ab}{a+b}\cos\dfrac{C}{2}$

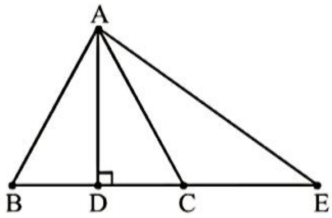
$\therefore \qquad \angle ADC = B + \dfrac{C}{2}$



Applying SINE Rule in ΔACD

$$\frac{CD}{\sin A} = \frac{b}{\sin\left(B+\dfrac{C}{2}\right)} \quad \Rightarrow \quad CD = \frac{b\sin A}{\sin\left(B+\dfrac{C}{2}\right)}$$

Q.7 In triangle ABC, |AB| = |AC|. Points D and E lie on ray BC such that |BD| = |DC| and |BE| > |CE|. Suppose that tan ∠EAC. tan ∠EAD and tan ∠EAB form geometric progression, and that cot ∠DAE, cot ∠CAE and cot ∠DAB from an arithmetic progression. If |AE| = 10, then which of the following is/are true.



(A) $\angle DEA = \frac{\pi}{4}$ (B) $\cot \angle DAC = 3$

(C) $\cot \angle CAE = 2$ (D) Area of the triangle ABC $= \frac{50}{3}$

Sol. Let $\angle EAD = \alpha$

and $\angle BAD = \angle DAC = \beta$

$\therefore$ $\angle EAC = \alpha - \beta$

tan ∠EAC, tan ∠EAD and tan ∠EAB form a geometric progression.

$\Rightarrow$ $\tan^2\alpha = \tan(\alpha - \beta) \cdot \tan(\alpha + \beta)$

$$\Rightarrow \quad \tan^2\alpha = \left(\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\right)\left(\frac{\tan\alpha + \tan\beta^3}{1 - \tan\alpha\tan\beta}\right) \Rightarrow \tan^2\alpha = \frac{\tan^2\alpha - \tan^2\beta}{1 - \tan^2\alpha\ \tan^2\beta}$$

$\Rightarrow$ $\tan^2\alpha - \tan^4\alpha\tan^2\beta = \tan^2\alpha - \tan^2\beta$

$\Rightarrow$ $\tan\alpha = 1$ $\Rightarrow$ $\alpha = 45^\circ$

$\Rightarrow$ ΔADE is an isosceles triangle

$\Rightarrow$ $\tan^2\alpha - \tan^4\alpha\tan^2\beta = \tan^2\alpha - \tan^2\beta$

$\Rightarrow$ $\tan\alpha = 1$ $\Rightarrow$ $\alpha = 45^\circ$

$\Rightarrow$ ΔADE is an isosceles triangle.

$AD = DE = \frac{AE}{\sqrt{2}} = 5\sqrt{2}$

ΔACD, $DC = AD\tan\beta$

Areaof $\Delta ABC = AD \cdot CD = AD^2\tan\beta$

cot ∠DAE, cot ∠CAE, cot ∠DAB are in AP

$\therefore$ $2\cot(45^\circ - \beta) = \cot 45^\circ + \cot\beta$

$$\frac{2(\cot 45^\circ \cot\beta + 1)}{\cot 45^\circ - \cot\beta} = 1 + \cot\beta$$

By solving $\cot\beta = 3$ $\therefore$ Area of $\Delta ABC = \left(\frac{50}{3}\right)$ unit^2

Q.8 If the angles A, B, C of a triangle are in A.P. and sides a, b, c are in G.P. then a^2, b^2, c^2 are in

(A) A.P. (B) H.P. (C) G.P. (D) None of these

Sol. $2B = A + C$, $B = 60^\circ$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{1}{2}, \qquad a^2 + c^2 - b^2 = ac$$

as $ac = b^2$ so $a^2 + c^2 - b^2 = b^2$, $a^2 + c^2 = 2b^2$

Q.9 If $\frac{r}{r_1} = \frac{r_2}{r_3}$ then

(A) A = 90° (B) B = 90° (C) C = 90° (D) None of these

Sol. $\frac{r}{r_1} = \frac{r_2}{r_3} \Rightarrow r\,r_3 = r_1\,r_2 \Rightarrow \frac{\Delta}{s}\frac{\Delta}{(s-c)} = \frac{\Delta}{(s-a)}\frac{\Delta}{(s-b)}$

$\Rightarrow \quad (s-a)(s-b) = s(s-c)$

$\frac{(s-a)(s-b)}{s(s-c)} = 1, \qquad \tan^2\frac{c}{2} = 1, \qquad \tan\frac{c}{2} = 1$

$c = 90°$

Q.10 The ratio of the area of a regular polygon of n-sides inscribed in a circle to that of the the polygon of same number of sides circumscribing the same circle is 3 : 4. Then the value of n is?

(A) 6 (B) 4 (C) 8 (D) 12

Sol. Let 'a' be the radius of the circle, then the ratio of the area of the regular polygen of n sides inscribed and circumscribing the same circle is

$$\frac{s_1}{s_2} = \frac{\frac{1}{2}na^2\sin\left(\frac{2\pi}{n}\right)}{na^2\tan\left(\frac{\pi}{n}\right)} = \frac{3}{4} \qquad \Rightarrow \qquad \cos^2\left(\frac{\pi}{n}\right) = \frac{3}{4}$$

$\cos\frac{\pi}{n} = \frac{\sqrt{3}}{2}$ so $\frac{\pi}{n} = \frac{\pi}{6}$ Ans. n = 6

Q.11 If in a ΔABC, $\sin^3 A + \sin^3 B + \sin^3 C = 3\sin A \sin B \sin C$. Find the value of determinant.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Sol. $\sin^3 A + \sin^3 B + \sin^3 C = 3\sin A \sin B \sin C$

$\Rightarrow \quad \sin A = \sin B = \sin C$

$\Rightarrow \quad a = b = c$

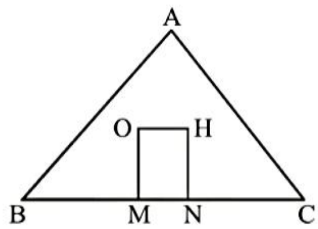
$$\therefore \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Q.12 In a scalene acute ΔABC, it is known that line joining circumcentre and orthocentre is parallel to BC.

Prove that the angle $A \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Sol. Distance of circumcentre (O)

from BC = Distance of orthocentre (H) from BC



$OM = HN$

$R \cos A = 2R \cos B \cos C$

$\Rightarrow \quad \cos A = 2 \cos B \cos C = \cos (B + C) + \cos (B - C)$

$\Rightarrow \quad \cos A = -\cos A + \cos (B - C)$

$\Rightarrow \quad \cos (B - C) = 2 \cos A$

$\therefore \quad 0 < \cos (B - C) < 1 \quad \Rightarrow \quad 0 < 2 \cos A < 1$

$\Rightarrow \quad \cos A \in \left(0, \frac{1}{2}\right)$

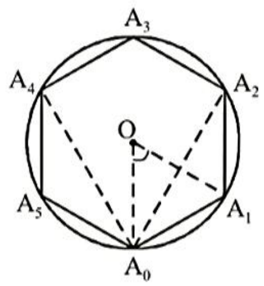
$\therefore \quad A \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

$\therefore \quad A \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

Q.13 If A_0, A_1, A_2, A_3, A_4 and A_5 be the consecutive vertices of a regular hexagon inscribed in a unit circle. Then find the product of length of (A_0A_1), (A_0A_2) and (A_0A_4).

Sol. ΔOA_0A_1

$\angle A_0OA_1 = \frac{2\pi}{6} = \frac{\pi}{3}$



$\cos\frac{\pi}{3} = \frac{1^2 + 1^2 - (A_0A_1)^2}{2\cdot 1\cdot 1} = \frac{1}{2}$

$A_0A_1 = 1$

$\Delta A_0OA_2 \qquad \cos\frac{2\pi}{3} = \frac{1^2 + 1^2 - (A_0A_2)^2}{2\cdot 1\cdot 1} = \frac{-1}{2}, A_0A_1 = \sqrt{3}$

Similarly $\qquad A_0A_4 = \sqrt{3}$

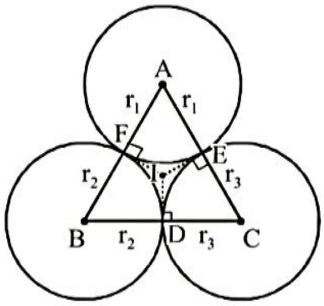
$\therefore \quad (A_0A_1)(A_0A_2)(A_0A_4) = 3$

Q.14 Three circles with radius r_1, r_2, r_3 touch one another externally. The tangents at their points of contact meet at a point whose distance from a point of contact is 2. The the value of $\left(\dfrac{r_1\, r_2\, r_3}{r_1 + r_2 + r_3}\right)$ is equal to

Sol. $a = r_2 + r_3$, $\quad b = r_3 + r_1$, $\qquad c = r_1 + r_2$

We have given $ID = IE = IF = 2$

$$2 = \frac{\text{Area of } \Delta ABC}{\text{semi perimeter of } \Delta ABC}$$



$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(r_1 + r_2 + r_3)r_1 r_2 r_3}$$

$$2 = \frac{\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}}{(r_1 + r_2 + r_3)} = \sqrt{\frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}}$$

$$\Rightarrow \qquad \frac{r_1 r_2 r_3}{r_1 + r_2 + r_3} = 4 \qquad \text{Ans.}$$

Q.15 If in the triangle ABC, O is the circumcentre and R is the circumradius and R_1, R_2, R_3 are the circumradii

Q.15 If in the triangle ABC, O is the circumcentre and R is the circumradius and R_1, R_2, R_3 are the circumradii of the triangles OBC, OCA and OAB respectively, then prove that $\dfrac{a}{R_1} + \dfrac{b}{R_2} + \dfrac{c}{R_3} = \dfrac{abc}{R^3}$.

Sol. ΔBOC, using SINE rule

$$2R_1 = \frac{a}{\sin 2A}$$

$$\therefore \qquad \frac{a}{R_1} = 2 \sin 2A$$

Similarly $\qquad \dfrac{b}{R_2} = 2\sin 2B, \qquad \dfrac{c}{R_3} = 2\sin^2 C$

$$\therefore \qquad \frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = 2(\sin 2A + \sin 2B + \sin 2C)$$

$$= 2(4 \sin A \sin B \sin C)$$

$$= 8 \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{R^3}$$

Q.16 If x, y and z are respectively the distances of the vertices of the ΔABC from its orthocentre then prove that

(i) $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=\dfrac{abc}{xyz}$ (ii) $x+y+z=2(R+r)$

Sol. $x = 2R\cos A$
$y = 2R\cos B$
$z = 2R\cos C$

(i) $\dfrac{a}{x}=\dfrac{2R\sin A}{2R\cos A}=\tan A$

similarly $\dfrac{b}{y}=\tan B,\quad \dfrac{c}{z}=\tan C$

$\therefore\quad \dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=\tan A+\tan B+\tan C=\tan A\tan B\tan C$

$$=\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$$

(ii) $x+y+z=2R(\cos A+\cos B+\cos C)=2R\left(1+4\sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right)\right)$

$$=2\left(R+4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)$$

$$=2(R+r)$$

Q.17 If I be the in centre of ΔABC, then prove that $IA\cdot IB\cdot IC=abc\ \tan\left(\dfrac{A}{2}\right)\tan\left(\dfrac{B}{2}\right)\tan\left(\dfrac{C}{2}\right)$.

Sol. $IA = r\operatorname{cosec}\dfrac{A}{2},\qquad IB = r\operatorname{cosec}\dfrac{B}{2},\qquad IC = r\operatorname{cosec}\dfrac{C}{2}$

$$IA\cdot IB\cdot IC=\frac{r^3}{\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}=\frac{\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$=64R^3\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}$$

$$=64R^3\left(\frac{abc}{4\Delta}\right)^3\cdot\frac{(s-b)(s-c)}{bc}\cdot\frac{(s-c)(s-a)}{ca}\cdot\frac{(s-a)(s-b)}{ab}$$

$$=\frac{abc}{\Delta^3}\cdot\left(\frac{\Delta^2}{r}\right)^2=\frac{abc\Delta}{s^2}$$

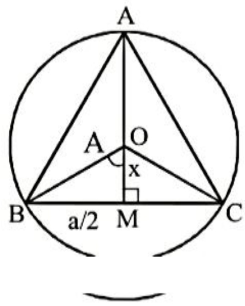
$$abc\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) = abc\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\sqrt{\frac{(s-a)(s-c)}{s(s-c)}}$$

$$= \frac{abc}{s^2}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \left(\frac{abc\Delta}{s^2}\right)$$

Q.18 If x, y, z are respectively be the perpendicular from the circumcentre to the sides of ΔABC then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$.

Sol. ΔBOM



$$\tan A = \frac{a/2}{x} = \frac{a}{2x}$$

similarly $\tan(B) = \frac{a}{2y}$

$\tan(C) = \frac{a}{2z}$

in a triangle $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\Rightarrow \quad \frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z} = \frac{a}{2x}\frac{b}{2y}\cdot\frac{c}{2z}$$

$$\Rightarrow \quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z} = \frac{abc}{4xyz}$$

Q.19 If two times the square of the diameter of the circumcircle of a triangle is equal to the sum of the squares of its sides then prove that the triangle is right angled.

Sol. In ΔABC

We have to prove that $8R^2 = a^2 + b^2 + c^2$

using SINE Rule

$$8R^2 = (2R\sin A)^2 + (2R\sin B)^2 + (2R\sin C)^2$$

$$\Rightarrow \quad 2 = \sin^2 A + \sin^2 B + \sin^2 C$$

$$\Rightarrow \quad 2 = \frac{1-\cos 2A}{2} + \frac{1-\cot 2B}{2} + \frac{1-\cos 2C}{2}$$

$$\Rightarrow \quad 4 = 3 - [-1 - 4\cos A\cos B\cos C]$$

$\Rightarrow \quad \cos A \cos B \cos C = 0$

$\Rightarrow \quad$ one of A, B or C must be 90º

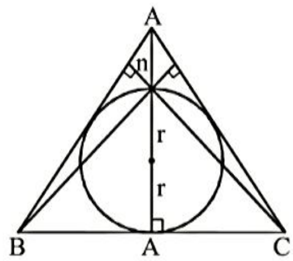
$\Rightarrow \quad$ right angle triangle.

Q.20 In an isosceles ΔABC, if the altitudes intersect on the inscribed circle then find the secant of the vertical angle 'A'.

Sol. $2r$ = distance of orthocentre from BC

$$2r = 2R \cos B \cos C$$

$$2\left(4R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = 2R \cos B \cos C$$



$$\Rightarrow \quad (\cos A + \cos B + \cos C - 1) = \cos B \cos C$$

$\because \quad B = C \qquad \therefore \quad B = \left(\dfrac{\pi - A}{2}\right)$

$$\cos A + 2 \cos B - 1 = \cos^2 B$$

$$\cos A + 2 \cos B - 1 = \cos^2 B$$

$$\Rightarrow \quad \cos A + 2 \cos\left(\frac{\pi - A}{2}\right) - 1 = \cos^2\left(\frac{\pi - A}{2}\right)$$

$$\Rightarrow \quad 1 - 2\sin^2\frac{A}{2} + 2\sin\frac{A}{2} - 1 = \sin^2\frac{A}{2}$$

$$\Rightarrow \quad \sin\frac{A}{2} = \frac{2}{3}$$

$$\Rightarrow \quad \cos A = 1 - 2\sin^2\frac{A}{2} = 1 - 2\left(\frac{2}{3}\right)^2 = \frac{1}{9} \qquad \therefore \quad \sec A = 9 \qquad \text{Ans.}$$