QUADRATIC EQUATION

1. INTRODUCTION :

1.1 Polynomial :

An expression of the type $P_n(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n$ is called a polynomial of degree 'n', where all powers of x are non-negative integers and a_0 which is called **leading coefficient** of the polynomial should not be equal to zero.

⇒ If co-efficients $a_0, a_1, a_2 \ldots\ldots a_n$ are real then polynomial is called real polynomial and if co-efficients are in the form of $(a + ib)$ then it is called complex polynomial.

e.g., : $(2 + 3i)x^3 + 5x^2 + 6x + 3$ is called a complex polynomial.

If $n = 1$ then $P(x) = a_0x + a_1$ is called a linear polynomial.

If $n = 2$ then $P(x) = a_0x^2 + a_1x + a_2$ is called a quadratic polynomial.

If $n = 3$ then $P(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ is called a cubic polynomial.

If $n = 4$ then $P(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ is called a bi-quadratic polynomial.

⇒ $P_n(\alpha)$ means value of the polynomial $P_n(x)$ at $x = \alpha$.

If $P_n(\alpha) = 0$, then α is called as root or zero of the polynomial.

1.2 Remainder Theorem :

The remainder theorem states that if a polynomial $P(x)$ is divided by a linear function $x - k$, then the remainder is $P(k)$.

$$\frac{P(x)}{x-k} = Q(x) + \frac{R}{x-k} \text{ where } Q(x) \text{ is quotient and } R \text{ is remainder.}$$

⇒ $P(x) = Q(x)\,(x - k) + R$ at $x = k$, $P(k) = R$

1.3 Factor Theorem :

Let $P(x) = (x - k)\,Q(x) + R$

when $P(k) = 0$, $P(x) = (x - k)\,Q(x)$. Therefore, $P(x)$ is exactly divisible by $x - k$.

1.4 Quadratic Expression and Quadratic Equation :

A second degree expression in one variable contains the variable with an exponent of 2; but not higher power. Such expressions are called as quadratic expression.

⇒ e.g., : $y = ax^2 + bx + c$,

where a = leading coefficient & c = absolute term of quadratic polynomial.

⇒ If above is equated to zero called as quadratic equation.

e.g., : $ax^2 + bx + c = 0$; $a \neq 0$

⇒ If leading coefficient is 1 then polynomial is called **monic polynomial**.

Solving a quadratic equation means finding the values of x for which $ax^2 + bx + c$ vanishes and these values of x are also called the roots of quadratic equation.

1.5 Identity :

Let $ax^2 + bx + c = 0$ be a quadratic equation. Now, if this quadratic equation has more than two distinct roots then it becomes an identity and in this case $a = b = c = 0$.

Note: Identity is an equation which is true for all values of x.

Let us say α, β, γ are three distinct roots of the given quadratic equation. Then,

$ax^2 + bx + c = k(x - \alpha)(x - \beta)(x - \gamma)$, for some constant k.

⇒ $ax^2 + bx + c = k[x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma]$

Comparing the co-efficient of x^3 on both sides, we get $k = 0$

and $k = 0 \Rightarrow a = 0, b = 0$ and $c = 0$

⇒ If a quadratic equation is satisfied by more than two distinct values of x, then all the co-efficients must be zero. And when all the co-efficients are zero, quadratic equation is true for all $x \in R$ and hence, it becomes an identity.

Illustration :

For what values of p, the equation $(p + 2)(p - 1)x^2 + (p - 1)(2p + 1)x + p^2 - 1 = 0$ has more than

Illustration :

For what values of p, the equation $(p + 2)(p - 1)x^2 + (p - 1)(2p + 1)x + p^2 - 1 = 0$ has more than two roots.

Sol. *$(p + 2)(p - 1)x^2 + (p - 1)(2p + 1)x + p^2 - 1 = 0$ will have more than two roots if all the co-efficients are zero.*

$\Rightarrow (p + 2)(p - 1) = 0 \Rightarrow p = -2, 1$

and $(p - 1)(2p + 1) = 0 \Rightarrow p = 1, p = \frac{-1}{2}$

and $p^2 - 1 = 0 \Rightarrow p = 1, -1$

$\therefore$ *All the co-efficients are zero when p = 1.* ***Ans.***

2. SOLUTION OF QUADRATIC EQUATION :

2.1 Factorization Method :

Let $ax^2 + bx + c = a(x - \alpha)(x - \beta) = 0$

Then $x = \alpha$ and $x = \beta$ will satisfy the given equation

Hence factorize the equation and equating each to zero gives roots of equation.

e.g. $3x^2 - 2x - 1 = 0 \equiv (x - 1)(3x + 1) = 0$

$$x = 1, -\frac{1}{3}.$$

2.2 Hindu Method (Sri Dharacharya Method) :

$ax^2 + bx + c = 0$ means we have to find to those values of x for which $ax^2 + bx + c = 0$.
Finding roots of $ax^2 + bx + c = 0$; $a \neq 0$; $a, b, c \in R$.

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \qquad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence $\alpha = \dfrac{-b + \sqrt{D}}{2a}$ and $\beta = \dfrac{-b - \sqrt{D}}{2a}$ where $D = b^2 - 4ac$

3. RELATION BETWEEN ROOTS AND COEFFICIENT :

$ax^2 + bx + c = 0$; $\quad a \neq 0$; $a, b, c \in R$

If α, β are the roots then $\quad \alpha + \beta = -\frac{b}{a}$; $\alpha\beta = \frac{c}{a}$ and $\alpha - \beta = \pm\frac{\sqrt{D}}{a}$

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4. FORMATION OF A QUADRATIC EQUATION WHEN ROOTS ARE GIVEN :

Let α and β be the given roots of a quadratic equation, then

$$(x - \alpha)(x - \beta) = 0$$

$$x^2 - x(\alpha + \beta) + \alpha\beta = 0$$

$x^2 - x$ (sum of the roots) + Product of the roots = 0

Note : Some Transformation in terms of $\alpha + \beta$ and $\alpha\beta$:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$\alpha^4 + \beta^4 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2$$

Illustration :

If the product of the roots of the quadratic equation $mx^2 - 2x + (2m - 1) = 0$ *is 3, then the value of m*

(A) 1 (B) 2 (C) – 1 (D) 3

Sol. *Product of the roots* $\frac{c}{a} = 3 = \frac{2m-1}{m}$

$\therefore \quad 3m - 2m = -1 \qquad \Rightarrow m = -1$ ***Ans.***

Illustration :

If α, β *are roots of the* $ax^2 + bx + c = 0$*, then the value of* $\frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2}$ *is*

(A) $\frac{b^2-2ac}{ac}$ (B) $\frac{2ac-b^2}{ac}$ (C) $\frac{b^2-2ac}{a^2c^2}$ (D) $\frac{b^2}{a^2c}$

Sol. *Since* α, β *are the roots of the* $ax^2 + bx + c$

then $a\alpha^2 + b\alpha + c = 0$

$\alpha(a\alpha + b) + c = 0$

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then $a\alpha^2 + b\alpha + c = 0$

$\alpha(a\alpha + b) + c = 0$

$(a\alpha + b) = \frac{-c}{\alpha}$ ....(1)

Similarly

$(a\beta + b) = \frac{-c}{\beta}$ ....(2)

$\therefore \quad \frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2} = \frac{1}{(-c/\alpha)^2}+\frac{1}{(-c/\beta)^2}$

$\Rightarrow \quad \frac{\alpha^2}{c^2}+\frac{\beta^2}{c^2} = \frac{\alpha^2+\beta^2}{c^2} = \frac{(\alpha+\beta)^2-2\alpha\beta}{c^2} = \frac{\frac{b^2}{a^2}-\frac{2c}{a}}{c^2} = \frac{b^2-2ac}{a^2c^2}$ ***Ans.***

Illustration :

If the equation $(k - 2)x^2 - (k - 4)x - 2 = 0$ *has difference of roots as 3 then the value of k is*

(A) 1, 3 (B) 3, $\frac{3}{2}$ (C) 2, $\frac{3}{2}$ (D) $\frac{3}{2}$, 1

Sol. $(\alpha - \beta) = \sqrt{(\alpha+\beta)^2-4\alpha\beta}$

Now, $\alpha + \beta = \frac{(k-4)}{(k-2)}$, $\alpha\beta = \frac{-2}{k-2}$

$$\therefore \quad (\alpha - \beta) = \sqrt{\left(\frac{k-4}{k-2}\right)^2 + \frac{8}{(k-2)}} = \frac{\sqrt{k^2+16-8k+8(k-2)}}{(k-2)}$$

$$3 = \frac{\sqrt{k^2+16-8k+8(k-2)}}{(k-2)}$$

$$3k - 6 = \pm k$$

$$k = 3, \frac{3}{2}.$$ **Ans.**

Illustration :

If the roots of the quadratic equation $x^2 + mx + n = 0$ are $\tan 30^o$ and $\tan 15^o$, respectively, then find the value of $2 + n - m$.

Sol. *The equation $x^2 + mx + n = 0$ has roots $\tan 30^o$ and $\tan 15^o$.*
Therefore

$$\tan 30^o + \tan 15^o = -m \qquad ...(i)$$
$$\tan 30^o \tan 15^o = n \qquad ...(ii)$$

Now, $\tan 45^o = \tan(30^o + 15^o) \Rightarrow 1 = \frac{\tan 30^o + \tan 15^o}{1 - \tan 30^o \tan 15^o}$

$$\tan 30^o \tan 15^o = n \qquad ...(ii)$$

Now, $\tan 45^o = \tan(30^o + 15^o) \Rightarrow 1 = \frac{\tan 30^o + \tan 15^o}{1 - \tan 30^o \tan 15^o}$

$$\Rightarrow \quad 1 = \frac{-m}{1-n} \qquad \text{[Using (i) and (ii)]}$$
$$\Rightarrow \quad 1 - n = -m \quad \Rightarrow \quad n - m = 1$$
$$\Rightarrow \quad 2 + n - m = 3$$

Illustration :

If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then find the set of possible values of a.

Sol. *If α, β are roots of $x^2 + ax + 1 = 0$, then*

$$|\alpha - \beta| < \sqrt{5}$$
$$\Rightarrow \quad \left|\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\right| < \sqrt{5}$$
$$\Rightarrow \quad \left|\sqrt{a^2 - 4}\right| < \sqrt{5}$$
$$\Rightarrow \quad a^2 - 4 < 5$$
$$\Rightarrow \quad a^2 - 9 < 0$$
$$\therefore \quad a \in (-3, 3).$$

5. NATURE OF ROOTS :

Consider the quadratic equation

$$ax^2 + bx + c = 0$$

where a, b, c $\in$ R and a $\neq$ 0.

Roots of the equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now, we observe that the roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is generally denoted by D and is known as the **discriminant** of the quadratic equation which decides nature of the roots. We also observe the following results :

(i) If $D > 0 \Rightarrow$ roots are real and distinct.

(ii) If $D = 0 \Rightarrow$ roots are equal.

Note : From (i) and (ii) it is clear that for real roots of a quadratic equation D must be greater than or equal to zero. (i.e. $D \geq 0$)

(iii) If $D < 0 \Rightarrow$ roots are imaginary.

Important Note :

(iii) If $D < 0 \Rightarrow$ roots are imaginary.

Important Note :

(1) If co-efficients of the quadratic equation are rational then its irrational roots always occur in pair. If $p + \sqrt{q}$ is one of the roots then other root will be $p - \sqrt{q}$.

(2) If co-efficients of the quadratic equation are real then its imaginary roots always occur in complex conjugate pair. If $p + iq$ is one of the roots then other root will be $p - iq$.

6. ROOTS UNDER PARTICULAR CASES :

(i) Exactly one root is at infinity :

If exactly one root is ∞ and other root is finite, then co-efficient of x^2 must tend to zero and co-efficient of x must not be equal to zero.

Put $x = \frac{1}{y}$ in $ax^2 + bx + c = 0$, we get

$cy^2 + by + a = 0$ must have one root zero $\Rightarrow$ P = 0 i.e. $\frac{a}{c} = 0$

Hence, $a = 0$ and $-\frac{b}{c} \neq 0 \Rightarrow b \neq 0$.

original equation becomes $bx + c = 0$

(ii) Both the roots at infinity :

When both roots of the equation are infinity then, co-efficient of x^2 and co-efficient of x both must tend to zero and c must not be equal to zero. The equation $cy^2 + by + a = 0$ must have both roots zero.

i.e. $-\dfrac{b}{c} = 0$ and $\dfrac{a}{c} = 0 \Rightarrow b = 0$; $a = 0$ and $c \neq 0$.

In this case the equation becomes $y = c$.

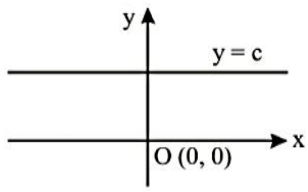

Illustration :

Find the value of P for which the equation $(P^3 - 3P^2 + 2P)x^2 + (P^3 - P)x + P^3 + 3P^2 + 2P = 0$

(i) *has exactly one root at infinity*

(ii) *has both the roots at infinity*

(iii) *becomes an identity*

Sol.

(i) ***Equation has exactly one root at infinity***

$a = 0,$ $\qquad b \neq 0$

$P = 0, 1, 2$ $\qquad P \neq \pm 1, 0$

hence $P = 2$

$a = 0,$ $\qquad b \neq 0$

$P = 0, 1, 2$ $\qquad P \neq \pm 1, 0$

hence $P = 2$

(ii) ***Equation has both the roots at infinity***

Sol. $a = 0,$ $\qquad b = 0$ $\qquad c \neq 0$

$P = 0, 1, 2$ $\qquad P = \pm 1, 0$ $\qquad P \neq -1, -2, 0$

hence $P = 1$

(iii) ***Equation becomes an identity***

Sol. $a = 0,$ $\qquad b = 0$ $\qquad c = 0$

$P = 0, 1, 2$ $\qquad P = \pm 1, 0$ $\qquad P = -1, -2, 0$

hence $P = 0$

Illustration :

If $a, b, c \in R$ *such that* $a + b + c = 0$ *and* $a \neq c$, *then prove that the roots of* $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ *are real and distinct.*

Sol. *at* $x = 1$, $\qquad f(x) = a + b + c = 0$

hence, $x = 1$ *is a root of the given equation.*

$$\text{Product of roots} = \frac{a+b-c}{b+c-a} = \frac{-2c}{-2a} = \frac{c}{a}$$

Since $c \neq a$ *hence other root is not unity.*

$\therefore$ *roots are real and distinct*

Illustration :

If $\cos\alpha$, $\sin\beta$, $\sin\alpha$ *are in G.P., then check the nature of roots of* $x^2 + 2\cot\beta \cdot x + 1 = 0$.

Sol. *We have* $\sin^2\beta = \cos\alpha\sin\alpha$

The discriminant of the given equation is

$$D = 4\cot^2\beta - 4$$

$$= 4\left[\frac{\cos^2\beta - \sin^2\beta}{\sin^2\beta}\right] = \frac{4(1-2\sin^2\beta)}{\sin^2\beta} = \frac{4(1-2\sin\alpha\cos\alpha)}{\sin^2\beta} = \left[\frac{2(\sin\alpha-\cos\alpha)}{\sin\beta}\right]^2 \geq 0$$

Illustration :

The roots of the quadratic equation $2x^2 - 7x + 4 = 0$ *are*

(A) Rational and different *(B) Rational and equation*

(C) Irrational and different *(D) Imaginary and different*

Sol. $b^2 - 4ac = 49 - 32 = 17 > 0$ *(not a perfect square)*

$\therefore$ *Its roots are irrational and different.*

Illustration :

The roots of the quadratic equation $x^2 - 2(a+b)x + 2(a^2+b^2) = 0$ *are*

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Sol. $A = 1,\ B = -2(a+b),\ C = 2(a^2+b^2)$

$$B^2 - 4AC = 1\,[2(a+b)]^2 - 4(1)\,(2a^2 + 2b^2)$$
$$= 4a^2 + 4b^2 + 8ab - 8a^2 - 8b^2$$
$$= -4a^2 - 4b^2 + 8\,ab$$
$$= -4\,(a-b)^2 < 0$$

So roots are imaginary and different.

Illustration :

The quadratic equation with rational coefficients whose one root is $2+\sqrt{3}$ *is*

(A) $x^2 - 4x + 1 = 0$ *(B)* $x^2 + 4x + 1 = 0$ *(C)* $x^2 + 4x - 1 = 0$ *(D)* $x^2 + 2x + 1 = 0$

Sol. *The required equation is*

$$x^2 - \left\{\left(2+\sqrt{3}\right)+\left(2-\sqrt{3}\right)\right\}x + \left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) = 0$$

or $x^2 - 4x + 1 = 0$ ***[Ans. 1]***

Practice Problem

Q.1 If α, β are roots of the equation $x^2 + px - q = 0$ and γ, δ are roots of $x^2 + px + r = 0$ then the value of $(\alpha - \gamma)(\alpha - \delta)$ is

(A) $p + r$ (B) $p - r$ (C) $q - r$ (D) $q + r$

Q.2 If α, β are roots of the equation $2x^2 - 35x + 2 = 0$, then the value of $(2\alpha - 35)^3 \cdot (2\beta - 35)^3$ is equal to

(A) 1 (B) 8 (C) 64 (D) none of these

Q.3 A certain polynomial $P(x)$, $x \in R$ when divided by $x - a$, $x - b$ and $x - c$ leaves remainders a, b and c, respectively. Then find the remainder when $P(x)$ is divided by $(x - a)(x - b)(x - c)$ where a, b, c are distinct.

Q.4 If $(a^2 - 1)x^2 + (a - 1)x + a^2 - 4a + 3 = 0$ be an identity in x, then find the value of a.

Q.5 For what value of m will the equation $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$ have equal roots.

Q.6 If p, q and r are odd integers, then prove that roots of $px^2 + qx + r = 0$ cannot be rational.

Answer key

Answer key

Q.1	D	Q.2	C	Q.3	x
Q.4	1	Q.5	$m = 2; \frac{-10}{9}$		

7. QUADRATIC EXPRESSION AND ITS GRAPH :

In $y = ax^2 + bx + c$, if $a, b, c \in R$ and $a \neq 0$. Graph of quadratic takes the shape of a parabola. The parabola opens upward or downward according as $a > 0$ or $a < 0$ respectively.

Figure - (i)

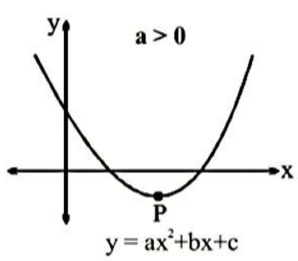
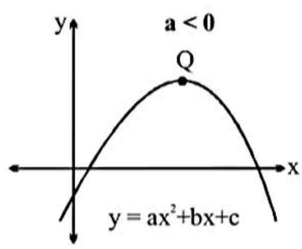


Figure - (ii)



The lowest point P in figure-(i) and highest point Q in figure-(ii) is called as vertex of parabola. Now for different values of a, b, c if graph $y = ax^2 + bx + c$ is plotted then following 6 different shapes are obtained.

Case-I : If $a > 0$ and $D > 0$

Then quatratic equation has two roots and graph cuts the x-axis at two distinct points.

(i) For $\alpha < x < \beta \Rightarrow$ y is negative.

(ii) For $x < \alpha$ or $x > \beta \Rightarrow$ y is positive.

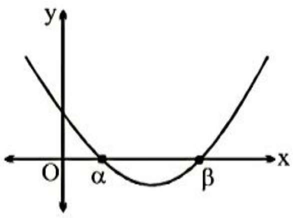
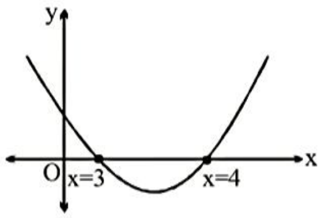


Illustration :

Draw graph of $y = x^2 - 7x + 12$ and find set of values of x where y is positive.



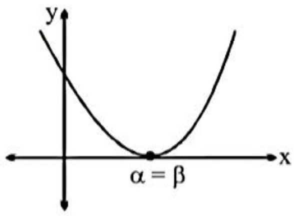
Sol. $y = x^2 - 7x + 12 = (x - 3)(x - 4)$.

Clearly, $y > 0$ if $x < 3$ or $x > 4$

i.e., $(-\infty, 3) \cup (4, \infty)$.

Case-II : If $a > 0$ and $D = 0$

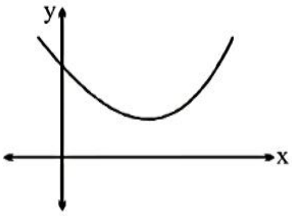
Then curve touches x-axis. Hence both zeroes of polynomial coincides.



In this type equation becomes $y = a(x - \alpha)^2$ and $y \geq 0$, for $x \in R$.

Case-III : If $a > 0$ and $D < 0$

Then curve completely lies above x-axis.



In this case imaginary roots appears and $y > 0$ for $x \in R$.

Illustration :

Find range of k for which graph of $y = x^2 - 3x + k$ lies completely above x-axis.

Sol. $D < 0$

$$9 - 4k < 0$$

$$k > \frac{9}{4}$$

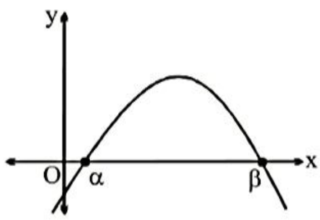
$$\left(\frac{9}{4}, \infty\right).$$ ***Ans.***

Case-IV : If $a < 0$ and $D > 0$

Then graph is downward and cuts the x-axis at two distinct points

Case-IV : If $a < 0$ and $D > 0$

Then graph is downward and cuts the x-axis at two distinct points.



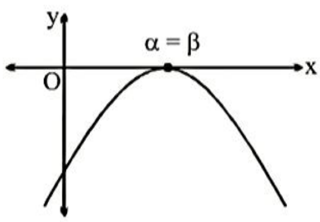
In this case

(a) $y > 0$, if $\alpha < x < \beta$

(b) $y < 0$, if $x < \alpha$ or $x > \beta$

Case-V : If $a < 0$ and $D = 0$

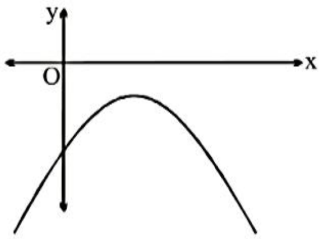
Then graph touches the x-axis from below.



In this case $x \in R$, $y \leq 0$ for $x \in R$.

Case-VI : If $a < 0$ and $D < 0$

Then graph lies completely below the x-axis and $y < 0$ for $x \in R$.



✍ **Important Note**

(1) The quadratic expression $ax^2 + bx + c$, where $a, b, c \in R$ is positive $\forall\ x \in R$, if $a > 0$ and $D < 0$ (Case-III).

(2) The quadratic expression $ax^2 + bx + c$, where $a, b, c \in R$ is negative $\forall\ x \in R$, if $a < 0$ and $D < 0$ (Case-VI).

Illustration :

A quadratic equation with rational coefficient has one of its roots as $2\sin^2\left(\frac{\pi}{5}\right)$ *if the sum of the*

$D < 0$ (Case-VI).

Illustration :

A quadratic equation with rational coefficient has one of its roots as $2\sin^2\left(\frac{\pi}{5}\right)$ *if the sum of the roots of quadratic equation is S and product of roots is P. Then P = KS implies that the value of K equals________.*

Sol. *One root* $= 2\sin^2\frac{\pi}{5} = 1 - \cos\frac{\pi}{5} = 1 - \left(\frac{\sqrt{5}-1}{4}\right) = \frac{5-\sqrt{5}}{4}$

$\therefore$ *Other root* $= \frac{5+\sqrt{5}}{4}$ *[As coefficients are rational, roots will be conjugate surds.]*

$\therefore$ *Sum of roots* $= S = \frac{5}{2}$

Product of roots $= P = \frac{20}{16} = \frac{5}{4}$

$$k = \frac{P}{S} = \frac{5/4}{5/2} = \frac{1}{2}.$$ ***Ans.***

Illustration :

If $x = 3+\sqrt{5}$ *find the value of* $x^4 - 12x^3 + 44x^2 - 48x + 17$.

Sol. $x = 3 + \sqrt{5} \quad \Rightarrow \quad x - 3 = \sqrt{5}$

$\Rightarrow \quad (x-3)^2 = 5 \quad \Rightarrow \quad x^2 - 6x + 4 = 0$

Now, $x^4 - 12x^3 + 44x^2 - 48x + 17 = (x^2 - 6x + 4)(x^2 - 6x + 4) + 1$

we know that, dividend = (divisior) (quotient) + R

But $x^2 - 6x + 4 = 0$

$\Rightarrow \quad x^4 - 12x^3 + 44x^2 - 48x + 17 = 1.$ ***Ans.***

Illustration :

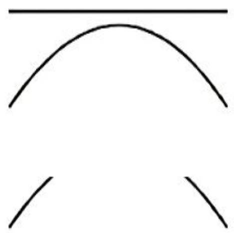
The quadratic equation $ax^2 + bx + c = 0$ *has no real root, then prove that*

$c\,(a + b + c) > 0.$

Sol. *As given equation has no real roots* $\Rightarrow D < 0$

$\therefore$ *Parabola either always lie above the x-axis or below the x-axis as shown*

$a < 0, D < 0$

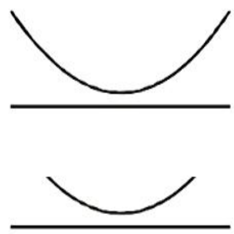


Here $f(x) = ax^2 + bx + c < 0 \ \forall x \in R$

$f(0) = c < 0$

$f(1) = a + b + c < 0$

$a > 0, D < 0$



Here $f(x) = ax^2 + bx + c > 0 \ \forall x \in R$

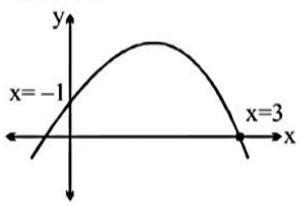
$f(0) = c > 0$

$f(1) = a + b + c > 0$

But in both the cases $c\,(a + b + c) > 0.$ ***Ans.***

Illustration :

Consider the graph of quadratic polynomial $y = ax^2 + bx + c$ *as shown below. Which of the following is/are correct.*



(A) $\dfrac{a-b+c}{abc} = 0$

(B) $abc\,(9a + 3b + c) < 0$

(C) $\dfrac{a+3b+9c}{abc} < 0$

(D) $ab\,(a - 3b + 9c) > 0$

Sol. *Clearly from given figure*

$a < 0, c > 0$

Also, $\dfrac{-b}{2a} > 0 \quad \Rightarrow \quad b > 0$

So, $abc < 0$

Also $f(-1) = a - b + c = 0, \qquad f(3) = 9a + 3b + c = 0$

Clearly $f\left(\dfrac{1}{3}\right) > 0 \quad \Rightarrow \dfrac{a}{9} + \dfrac{b}{3} + c \quad \Rightarrow \quad a + 3b + 9c > 0$

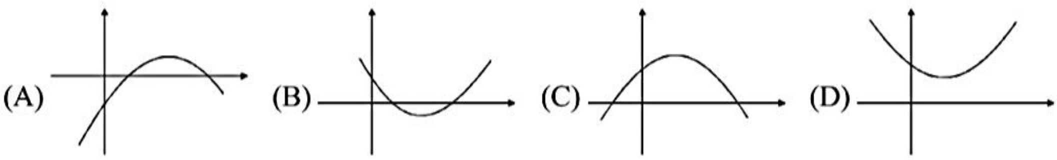
Also $f\left(\dfrac{-1}{3}\right) > 0 \quad \Rightarrow \dfrac{a}{9} - \dfrac{b}{3} + c > 0 \quad \Rightarrow \quad a - 3b + 9c > 0$

Now verify alternatives. ***Ans. A, C***

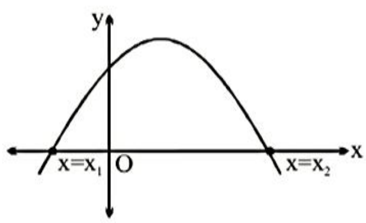
Practice Problem

Q.1 The best possible graph of the function $f(x) = Ax^2 + Bx + C$, where
$A = (\sec\theta - \cos\theta)(\csc\theta - \sin\theta)(\tan\theta + \cot\theta)$
$B = (\sin\theta + \csc\theta)^2 + (\cos\theta + \sec\theta)^2 - (\tan^2\theta + \cot^2\theta)$ and
$C = 12$, is represented by

$A = (\sec\theta - \cos\theta)(\csc\theta - \sin\theta)(\tan\theta + \cot\theta)$
$B = (\sin\theta + \csc\theta)^2 + (\cos\theta + \sec\theta)^2 - (\tan^2\theta + \cot^2\theta)$ and
$C = 12$, is represented by



Q.2 Consider the graph of quadratic trinomial $y = ax^2 + bx + c$ as shown below where x_1 and x_2 are roots of the equation $ax^2 + bx + c = 0$. Which of the following is/are correct?



(A) $a - b - c < 0$ (B) $bc < 0$
(C) $b > 0$ (D) b and c have the same sign different from a.

Q.3 The trinomial $ax^2 + bx + c$ has no real roots and $a + b + c < 0$. Comment on the sign of the number c.

Answer key

Q.1 B Q.2 A, C, D Q.3 $c < 0$

8. SOLVING QUADRATIC AND RATIONAL INEQUALITIES (WAVY CURVE METHOD) :

While solving such inequations following steps to be taken.

(i) Factorise given-expression into linear factors

(ii) Make the coefficient of x positive in all factors

(iii) Plot the points where given expression vanishes or undefined (denominator becomes zero) on number line in increasing order

(iv) Start the number line from right to left taking positive or negative value.

While solving rational inequalities different situtations arive.

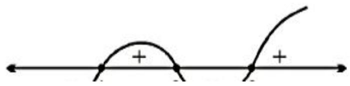
Type-1 : Inequalities involving non-repeated linear factors

Illustration :

$(x - 1)\ (x - 2)\ (x - 3) \geq 0$

Sol. *The set of values for which given expression is* ≥ 0

$[1, 2] \cup [3, \infty)$



$(x - 1)\ (x - 2)\ (x - 3) \geq 0$

Sol. *The set of values for which given expression is* ≥ 0

$[1, 2\,] \cup [3, \infty).$

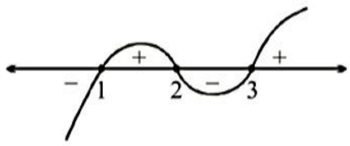
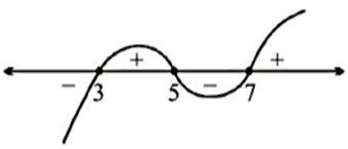


Illustration :

$(x - 3)\ (x - 5)\ (x - 7) < 0$

Sol. *In this problem given expression* < 0.



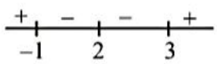
$\Rightarrow \quad (-\infty, 3) \cup (5, 7).$

Type-2 : Quadratic inequality involving Repeated linear factos.

Illustration :

$(x + 1)(x - 3)(x - 2)^2 \geq 0.$

Sol. $(x + 1)(x - 3)(x - 2)^2 \geq 0$



$x \in (-\infty, -1] \cup \{2\} \cup [3, \infty).$ ***Ans.***

Illustration :

$x(x + 6)(x + 2)^2(x - 3) > 0$

Sol. $x(x + 6)(x + 2)^2(x - 3) > 0$

$\Rightarrow x \in (-6, 0) \cup (3, \infty) - \{-2\}$ ***Ans.***

Illustration :

$(x - 1)^2(x + 1)^3(x - 4) < 0$

Sol. $(x - 1)^2(x + 1)^3(x - 4) < 0$

$x \in (-1, 1) \cup (1, 4).$ ***Ans.***

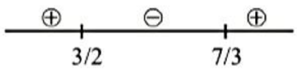
Type-3 : Quadratic / algebraic inequality of the type of $\frac{f(x)}{g(x)}$. (Rational inequality) involving modulus also.

Illustration :

$$\frac{2x-3}{3x-7} > 0$$

$$\frac{2x-3}{3x-7} > 0$$

Sol.



$$x \in \left(-\infty, \frac{3}{2}\right) \cup \left(\frac{7}{3}, \infty\right)$$ ***Ans.***

Illustration :

$$\frac{x^2 - 5x + 12}{x^2 - 4x + 5} > 3$$

Sol. *$x^2 - 4x + 5$ is always positive*

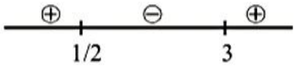
since $D = 16 - 20 = -4 < 0$

hence, we can cross multiply $x^2 - 4x + 5$ without changing the sign of inequality.

$x^2 - 5x + 12 > 3x^2 - 12x + 15$

$2x^2 - 7x + 3 < 0$

$(2x - 1)(x - 3) < 0$



$$x \in \left(\frac{1}{2}, 3\right)$$ ***Ans.***

Illustration :

$$\frac{x^2-5x+6}{x^2+x+1}<0$$

Sol. $$\frac{x^2-5x+6}{x^2+x+1}<0$$

As denominator is always positive

$\Rightarrow x^2 - 5x + 6 < 0 \Rightarrow (x-2)(x-3) < 0$

$\Rightarrow x \in (2, 3)$ **Ans.**

Illustration :

$$\frac{(x-1)^2(x+1)^3}{x^4(x-2)}<0$$

Sol. $$\frac{(x-1)^2(x+1)^3}{x^4(x-2)}<0$$

$x \in (-1, 0) \cup (0, 1) \cup (1, 2)$ **Ans.**

Illustration :

$x \in (-1, 0) \cup (0, 1) \cup (1, 2)$ **Ans.**

Illustration :

$$\frac{x+1}{x-1} \geq \frac{x+5}{x+1}$$

Sol. $$\frac{x+1}{x-1} \geq \frac{x+5}{x+1} \Rightarrow \frac{x+1}{x-1}-\frac{x+5}{x+1} \geq 0 \Rightarrow \frac{(x+1)^2-(x-1)(x+5)}{(x-1)(x+1)} \geq 0$$

$$\Rightarrow \frac{-2x+6}{(x-1)(x+1)} \geq 0 \Rightarrow \frac{x-3}{(x-1)(x+1)} \leq 0$$

$\Rightarrow x \in (-\infty, -1) \cup (1, 3]$. **Ans.**

Illustration :

$$\frac{2(x-4)}{(x-1)(x-7)} \geq \frac{1}{x-2}$$

Sol. $$\frac{2(x-4)}{(x-1)(x-7)} \geq \frac{1}{x-2}$$

$$\Rightarrow \quad \frac{2(x-4)}{(x-1)(x-7)}-\frac{1}{(x-2)} \geq 0$$

$$\Rightarrow \quad \frac{2(x-4)(x-2)-(x-1)(x-7)}{(x-1)(x-7)(x-2)} \geq 0$$

$$\Rightarrow \quad \frac{x^2-4x+9}{(x-1)(x-2)(x-7)} \geq 0$$

$\Rightarrow \quad x \in (1, 2) \cup (7, \infty)$ **Ans.**

Illustration :

$$\frac{x^2+6x-7}{|x+4|} < 0$$

Solve the inequality using method of interval.

Sol. $$\frac{x^2+6x-7}{|x+4|} < 0$$

$\Rightarrow \quad x^2 + 6x - 7 < 0 \Rightarrow (x + 7)(x - 1) < 0$

$\Rightarrow \quad x \in (-7, 1) - \{-4\}$ **Ans.**

Type-4 : Double inequality and biquadratic inequality.

$\Rightarrow \quad x^2 + 6x - 7 < 0 \Rightarrow (x + 7)(x - 1) < 0$

$\Rightarrow \quad x \in (-7, 1) - \{-4\}$ **Ans.**

Type-4 : Double inequality and biquadratic inequality.

Illustration :

$$1 < \frac{3x^2-7x+8}{x^2+1} \leq 2$$

Sol. $$1 < \frac{3x^2-7x+8}{x^2+1} \leq 2$$

Here, we need to make sure

$\Rightarrow \quad \dfrac{3x^2-7x+8}{x^2+1} > 1$ *and* $\dfrac{3x^2-7x+8}{x^2+1} \leq 2$

$\Rightarrow \quad 3x^2 - 7x + 7 > x^2 + 1$ *and* $3x^2 - 7x + 8 \leq 2x^2 + 2$

$\Rightarrow \quad 2x^2 - 7x + 7 > 0$ *and* $x^2 - 7x + 6 \leq 0$

Here, $a > 0$ *and* $D < 0$ *and* $(x - 1)(x - 6) \leq 0$

$\Rightarrow \quad x \in R$ *and* $x \in [1, 6]$

Taking intersection of both, we get $x \in [1, 6]$. **Ans.**

Practice Problem

Q.1 Solve the inequality -

(i) $x(x-2)(x+3) \geq 0$ (ii) $x(x-4)^2(x+6)^3(x-1) \leq 0$

Q.2 Solve the inequality $1 < \dfrac{x^2+3x+4}{x^2+4x+5} < 2$

Q.3 $\dfrac{(x+2)(x^2-2x+1)}{(4+3x-x^2)} \geq 0$

Q.4 $\dfrac{x^4-3x^3+2x^2}{x^2-x-30} > 0$

Q.5 Number of positive integral solution of $\dfrac{x^3(2x-3)^2(x-4)^6}{(x-3)^3(3x-8)^4} \leq 0$

(A) 1 (B) 2 (C) 3 (D) 4

Q.6 Find the set of values of a for which the quadratic polynomial

(i) $(a+4)x^2 - 2ax + 2a - 6 < 0 \quad \forall\ x \in R$

(ii) $(a-1)x^2 - (a+1)x + (a+1) > 0 \quad \forall\ x \in R$

Q.6 Find the set of values of a for which the quadratic polynomial

(i) $(a+4)x^2 - 2ax + 2a - 6 < 0 \quad \forall\ x \in R$

(ii) $(a-1)x^2 - (a+1)x + (a+1) > 0 \quad \forall\ x \in R$

Answer key

Q.1 (i) $[-3, 0] \cup [2, \infty)$; (ii) $[-6, 0] \cup [1, \infty)$ Q.2 $x \in (-\infty, -3)$

Q.3 $(-\infty, -2] \cup (-1, 4)$ Q.4 $(-\infty, -5] \cup (1, 2) \cup (6, \infty)$ Q.5 C

Q.6 (i) $(-\infty, -6)$; (ii) $(5/3, \infty)$

9. SYMMETRIC EXPRESSIONS :

The symmetric expressions of the roots α, β of an equation are those expressions in α and β, whice do not change by interchanging a and b. To find the value of such an expression, we generally express that in terms of $\alpha + \beta$ and $\alpha\beta$.

Some examples of symmetric expressions are

(i) $\alpha^2 + \beta^2$ (ii) $\alpha^2 + \alpha\beta + \beta^2$ (iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ (iv) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

(v) $\alpha^2\beta + \beta^2\alpha$ (vi) $\left(\dfrac{\alpha}{\beta}\right)^2 + \left(\dfrac{\beta}{\alpha}\right)^2$ (vii) $\alpha^3 + \beta^3$ (viii) $\alpha^4 + \beta^4$

10. FORMATION OF QUADRATIC EQUATION WHOSE ROOTS ARE SYMMETRIC EXPRESSION OF α AND β :

Let α and β be the roots of a quadratic equation $ax^2 + bx + c = 0$ then finding another quadratic equation whose roots are $2\alpha + 3, 2\beta + 3$.

Suppose $2\alpha + 3 = y \Rightarrow \alpha = \dfrac{y-3}{2}$

Put the value of α in the given equation ($\because$ α is its roots) and get a quadratic in y.

$$\frac{a\,(y-3)^2}{4} + \frac{b\,(y-3)}{2} + c = 0$$

$$a\,(y-3)^2 + 2b\,(y-3) + 4c = 0$$

$$ay^2 + 2y\,(b-3a) + 9a - 6b + 4c = 0$$

Replace y by x and get the desired equation.

$$ax^2 + 2x\,(b-3a) + 9a - 6b + 4c = 0.$$

Note: If α, β are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are

$$ay^2 + 2y\,(b-3a) + 9a - 6b + 4c = 0$$

Replace y by x and get the desired equation.

$$ax^2 + 2x\,(b-3a) + 9a - 6b + 4c = 0.$$

Note: If α, β are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are

(i) $-\alpha, -\beta \Rightarrow ax^2 - bx + c = 0$ (Replace x by $-x$)

(ii) $\dfrac{1}{\alpha}, \dfrac{1}{\beta} \Rightarrow cx^2 = bx + a = 0$ $\left(\text{Replace x by } \dfrac{1}{x}\right)$

(iii) $\alpha^n, \beta^n, n \in N \Rightarrow a\left(x^{\frac{1}{n}}\right)^2 + b\left(x^{\frac{1}{n}}\right) + c = 0$ $\left(\text{Replace x by } x^{\frac{1}{n}}\right)$

(iv) $k\alpha, k\beta \Rightarrow ax^2 + kbx + k^2c = 0.$ $\left(\text{Replace x by } \dfrac{x}{k}\right)$

(v) $k + a, k + b \Rightarrow a(x-k)^2 + b\,(x-k) + c = 0$ (Replace x by $(x-k)$)

(vi) $\dfrac{\alpha}{k}, \dfrac{\beta}{k} \Rightarrow k^2ax^2 + kbx + c = 0$ (Replace x by kx)

(vii) $\alpha^{\frac{1}{n}}, \beta^{\frac{1}{n}}$; $n \in N \Rightarrow a(x^n)^2 + b(x^n) + c = 0$ (Replace x by x^n)

Illustration :

If α, β *are the root of a quadratic equation* $x^2 - 3x + 5 = 0$ *then the equation whose roots are* $(\alpha^2 - 3\alpha + 7)$ *and* $(\beta^2 - 3\beta + 7)$ *is*

(A) $x^2 + 4x + 1 = 0$ (B) $x^2 - 4x + 4 = 0$ (C) $x^2 - 4x - 1 = 0$ (D) $x^2 + 2x + 3 = 0$

Sol. *Since* α, β *are the roots of equation* $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$

$\beta^2 - 3\beta + 5 = 0$

$\therefore$ $\alpha^2 - 3\alpha = -5$

$\beta^2 - 3\beta = -5$

putting in $(\alpha^2 - 3\alpha + 7)$ *and* $(\beta^2 - 3\beta + 7)$ ....(1)

$-5 + 7, -5 + 7$

$\therefore$ *2 and 2 are the roots*

$\therefore$ *the required equation is* $x^2 - 4x + 4 = 0.$ ***Ans.***

Illustration :

If α, β *are roots of the equation* $x^2 - 5x + 6 = 0$ *then the equation whose roots are* $\alpha + 3$ *and*

Illustration :

If α, β *are roots of the equation* $x^2 - 5x + 6 = 0$ *then the equation whose roots are* $\alpha + 3$ *and* $\beta + 3$ *is*

(A) $x^2 - 11x + 30 = 0$ (B) $(x - 3)^2 - 5(x - 3) + 6 = 0$

(C) Both (1) and (2) (D) None

Sol. *Let* $\alpha + 3 = x$

$\therefore$ $\alpha = x - 3$ *(Replace x by x – 3)*

So the required equation is

$(x - 3)^2 - 5(x - 3) + 6 = 0$ ......(1)

$x^2 - 6x + 9 - 5x + 6 = 0$

$x^2 - 11x + 30 = 0$ ......(2) ***[Ans. C]***

Illustration :

If α, β *are roots of the equation* $2x^2 + x - 1 = 0$ *then the eqution whose roots are* $\frac{1}{\alpha}, \frac{1}{\beta}$ *will be*

(A) $x^2 + x - 2 = 0$ (B) $x^2 + 2x - 8 = 0$ (C) $x^2 - x - 2 = 0$ (D) None of these

Sol. *From the given equation*

$$\alpha + \beta = \frac{-1}{2}, \ \alpha\beta = \frac{-1}{2}$$

The required equation is

$$x^2-\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)x+\frac{1}{\alpha\beta}=0 \quad \Rightarrow x^2-\left(\frac{\alpha+\beta}{\alpha\beta}\right)x+\frac{1}{\alpha\beta}=0 \Rightarrow \quad x^2-\left(\frac{\frac{-1}{2}}{\frac{-1}{2}}\right)x+\frac{1}{\frac{-1}{2}}=0$$

$\Rightarrow \quad x^2-x-2=0$ **[Ans. C]**

Short cut : Replace x by $\frac{1}{x}$ $\Rightarrow \quad 2\left(\frac{1}{x}\right)^2+\frac{1}{x} \quad 1=0 \Rightarrow x^2-x-2=0$

11. CONDITION OF COMMON ROOTS :

11.1 Condition for one common root :

Let $\quad a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ have a common root α.

Hence $\quad a_1\alpha^2+b_1\alpha+c_1=0$

$\quad a_2\alpha^2+b_2\alpha+c_2=0$

by cross multiplication

$$\frac{\alpha^2}{b_1c_2-b_2c_1}=\frac{\alpha}{a_2c_1-a_1c_2}=\frac{1}{a_1b_2-a_2b_1}$$

by cross multiplication

$$\frac{\alpha^2}{b_1c_2-b_2c_1}=\frac{\alpha}{a_2c_1-a_1c_2}=\frac{1}{a_1b_2-a_2b_1}$$

$\therefore \quad \alpha=\frac{b_1c_2-b_2c_1}{a_2c_1-a_1c_2}=\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}$ Which is the required condition.

This is also the condition that the two quadratic functions $a_1x^2+b_1x\,y+c_1y^2$ and $a_2x^2+b_2x\,y+c_2y^2$ may have a common factor.

11.2 Condition for both the common roots :

If both roots of the given equations are common then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

Illustration :

Find the value of k for which the equations $3x^2+4kx+2=0$ and $2x^2+3x-2=0$ have a common root.

Sol. $2x^3+3x-2 \quad \Rightarrow \quad x=-2$ *or* $\frac{1}{2}$

when $x=-2$ *is common root.*

$\Rightarrow \quad 12-8k+2=0 \rightarrow 8k=14 \quad \Rightarrow \quad k=\frac{7}{4}$

when $x = \frac{1}{2}$ is a common root.

$$\Rightarrow \quad \frac{3}{4} + 2k + 2 = 0 \Rightarrow 2k = -\frac{11}{4} \quad \Rightarrow \quad k = \frac{11}{8}$$

Illustration :

If the quadratic equation $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ $(b \neq c)$ have a common root then prove that their uncommon roots are the roots of the equation $x^2 + x + bc = 0$

Sol. $\left.\begin{array}{l} x^2+bx+c=0 \\ x^2+cx+b=0 \end{array}\right|$ have common root

$\Rightarrow \quad (b-c)x = b-c \quad \Rightarrow \quad x = 1$ is the common root.

Let L be the root of $x^2 + bx + c = 0$

$\therefore$ Product of roots = $(1)(\alpha) = c \quad \Rightarrow \quad \alpha = C$

Similarly, Let β be the other root of $x^2 + cx + b = 0$

$\Rightarrow \quad \beta = b$

Sum of uncommon roots = $\alpha + \beta = b + c$

Product of uncommon roots = $\alpha\beta = bc$

Also, 1 is common root $\Rightarrow$ It must satisfy both the equations $\Rightarrow 1 + bt + c = 0 \Rightarrow b + c = -1$

$\therefore$ Required equation is $x^2 - (b + c)x + bc = 0$

Also, 1 is common root $\Rightarrow$ It must satisfy both the equations $\Rightarrow 1 + bt + c = 0 \Rightarrow b + c = -1$

$\therefore$ Required equation is $x^2 - (b + c)x + bc = 0$

$\Rightarrow \quad x^2 + x + bc = 0$

Illustration :

If $Q_1(x) = x^2 + (k - 29)x - k$ and $Q_2(x) = 2x^2 + (2k - 43)x + k$ both are factors of a cubic polynomial $P(x)$, then the largest value of k is

(A) 0 (B) 33 (C) 23 (D) 30

Sol. Two quadratic polynomials can be a factor of cubic polynomial only when they have atleast one root common

$\Rightarrow \quad x^2 + (b - 2a)x - k = 0 \quad \ldots (1)$

and $\quad 2x^2 + (2k - 43)x + k = 0 \quad \ldots (2)$ Must have a common roots

Multiple equation (1) by 2 and subtracting, we get

$$15x + 3k = 0 \Rightarrow x = \frac{-k}{5} \text{ is the common root}$$

and it must satisfy equation (1)

$$\Rightarrow \quad \frac{k^2}{25} + (k-29)\left(-\frac{k}{5}\right) - k = 0$$

$$\Rightarrow \quad \left(-\frac{k}{5}\right)\left[-\frac{k}{5} + k - 29 + 5\right] = 0$$

$\Rightarrow \quad k = 0$ or $k = 30$

Practice Problem

Q.1 If α and β are roots of $2x^2 - 7x + 6 = 0$, then the quadratic equation whose roots are $-\frac{2}{\alpha}, -\frac{2}{\beta}$ is

(A) $3x^2 + 7x + 4 = 0$
(B) $3x^2 - 7x + 4 = 0$
(C) $6x^2 + 7x + 2 = 0$
(D) $6x^2 - 7x + 2 = 0$

Q.2 If roots of quadratic equation $ax^2 + bx + c = 0$ are α and β then symmetric expression of its roots is

(A) $\frac{\alpha}{\beta} + \frac{\beta^2}{\alpha}$
(B) $\alpha^2\beta^{-2} + \alpha^{-2}\beta$
(C) $\alpha^2\beta + 2\alpha\beta^2$
(D) $\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$

Q.3 If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$, find the equation whose roots are α/β and β/α.

Q.4(a) Find the value of p and q if the equation $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots is common.

(b) If the equation $x^2 - 4x + 5 = 0$ and $x^2 + ax + b = 0$ have a common root find a and b, where $a, b \in R$.

Q.5 If the equations $4x^2\sin^2\theta - (4\sin\theta)x + 1 = 0$ and

common.

(b) If the equation $x^2 - 4x + 5 = 0$ and $x^2 + ax + b = 0$ have a common root find a and b, where $a, b \in R$.

Q.5 If the equations $4x^2\sin^2\theta - (4\sin\theta)x + 1 = 0$ and $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2(a^2 - b^2) = 0$ have a common root and the 2nd equation has equal root find the possible values of θ in $(0, \pi)$.

Q.6 If the quadratic equations $x^2 + ax + 12 = 0$ and $x^2 + bx + 15 = 0$ and $x^2 + (a+b)x + 36 = 0$ have a common positive root find a and b and the root of the equation.

Q.7 If $a, b, c \in R$ and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, then find a : b : c.

Answer key

Q.1 A

Q.2 B

Q.3 $3x^2 - 19x + 3$

Q.4 (a) $p = \frac{3}{2}$; $q = 4$; (b) $a = -4, b = 5$

Q.5 $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

Q.6 $a = -7$, $b = -8$; roots are (3, 4), (3, 5) and (3, 12).

Q.7 $a : b : c = 1 : 2 : 9$

12. MAXIMUM AND MINMUM VALUES OF QUADRATIC AND RATIONAL FUNCTIONS :

12.1 $y = ax^2 + bx + c$ attains its maximum value or minimum value at the point with abscissa $x = -\dfrac{b}{2a}$

according as $a < 0$ or $a > 0$.

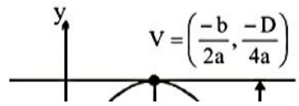
$$y = ax^2 + bx + c$$

$$y = a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$$

$$= a\left[x^2 + 2\cdot\frac{b}{2a}\cdot x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

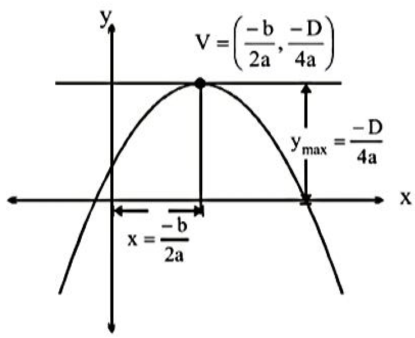
$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$



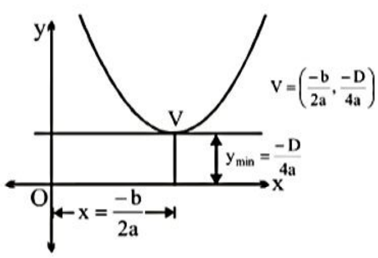
$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$



Now, If $a < 0$, then $y_{max} = \dfrac{-D}{4a}$ and it occurs at $x = \dfrac{-b}{2a}$.

If $a > 0$, then $y_{min} = \dfrac{-D}{4a}$ and it occurs at $x = \dfrac{-b}{2a}$.



where $D = b^2 - 4ac$

Note:

Maximum or minimum value can also be obtained by making a perfect square and then taking an interpretation.

Illustration :

Find the maximum value of $f(x) = -3x^2 + 6x + 5$.

Sol. *Since,* $a < 0$

$$\therefore f(x)_{max} = \frac{-D}{4a} = -\frac{36-4(-3)5}{4(-3)} = \frac{36+60}{12} = 8$$

Aliternative method :

$f(x) = -3(x^2 - 2x + 1) + 5 + 3 = -3(x-1)^2 + 8$

Clearly $f(x)_{max} = 8$ *at* $x = 1$.

Illustration :

Let $P(x) = ax^2 + bx + 8$ *is a quadratic polynomial. If the minimum value of* $P(x)$ *is 6 when* $x = 2$, *find the values of a and b.*

Sol. $-\frac{b}{2a} = 2$

$4a = -b \quad \ldots(i)$

$P(2) = 4a + 2b + 8 = 6$

2a

$4a = -b \quad \ldots(i)$

$P(2) = 4a + 2b + 8 = 6$

$4a + 2b = -2 \quad \ldots(ii)$

using (i) & (ii)

$b = -2, \quad a = \frac{1}{2}$

Illustration :

For $x \geq 0$, *what is the smallest possible value of the expression* $\log_{10}(x^3 - 4x^2 + x + 26) - \log_{10}(x + 2)$?

Sol. $y = \log_{10}(x^3 - 4x^2 + x + 26) - \log_{10}(x + 2)$

$x^3 - 4x^2 + x + 26 > 0, \qquad x + 2 > 0$

$(x + 2)(x^2 - 6x + 13) > 0$

$x + 2 > 0, \qquad x^2 - 6x + 13 > 0$

since $x^2 - 6x + 13$ *is always positive*

therefore $x > -2$

$$y = \log \frac{(x+2)(x^2-6x+13)}{(x+2)}$$

$$= \log_{10}(x^2 - 6x + 13)$$

$$y = \log_{10}[(x-3)^2 + 4]$$

$$y_{min} = \log_{10} 4$$

12.2 Range of functions expressed in the form of $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are either linear or quadratic polynomials.

TYPE-1 : $y = \frac{ax+b}{px+q}\ \frac{(\text{Linear})}{(\text{Linear})}$.

Illustration :

Find the range of the function. $\quad y = \frac{3x+2}{x-1}, \quad x \neq 1$

Sol. $y = \frac{3x-3+5}{x-1}$

$\Rightarrow \quad y = 3 + \frac{5}{x-1} \qquad \Rightarrow \quad y-3 = \frac{5}{x-1}$

$\Rightarrow \quad x-1 = \frac{5}{y-3} \qquad \Rightarrow \quad x = \frac{5}{y-3} + 1 = \frac{5+y-3}{y-3}$

$\Rightarrow \quad x-1 = \frac{5}{y-3} \qquad \Rightarrow \quad x = \frac{5}{y-3} + 1 = \frac{5+y-3}{y-3}$

$\Rightarrow \quad x = \frac{y+2}{y-3}, \quad y \neq 3$

for $\quad y = 3 \quad$ *x is not defined*

$\therefore \quad$ *range is* $R - \{3\}$

TYPE-2 $\quad y = \frac{ax+b}{px^2+qx+r}\ \frac{(\text{linear})}{(\text{quadratic})}$

Illustration :

If x is real then find the range of the function $y = \frac{x+2}{x^2+3x+6}$

Sol. $y = \frac{x+2}{x^2+3x+6}$

$\Rightarrow \quad x^2y + 3xy + 6y = x + 2$

$\Rightarrow \quad x^2y + x(3y-1) + 6y - 2 = 0$

$\because \quad$ *x is real* $\quad \therefore \quad D \geq 0$

$\Rightarrow \quad (3y-1)^2 - 4y\,(6y-2) \geq 0$

$\Rightarrow \quad 9y^2 - 6y + 1 - 24y^2 + 8y \geq 0$

$\Rightarrow \quad -15y^2 + 2y + 1 \geq 0$

$\Rightarrow \quad 15y^2 - 2y - 1 \leq 0$

$\Rightarrow \quad (5y + 1)(3y - 1) \leq 0$

$\Rightarrow \quad y \in \left[-\frac{1}{5}, \frac{1}{3}\right]$

TYPE-3 $\quad y = \dfrac{ax^2 + bx + c}{px^2 + qx + r}\left(\dfrac{\text{Quadratic}}{\text{Quadratic}}\right)$

Illustration :

If x is real then prove that $y = \dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$ *lies from* $\frac{1}{7}$ *to 7.*

Sol. $\dfrac{x^2 - 3x + 4}{x^2 + 3x + 4} = y$

$x^2(y - 1) + 3x(y + 1) + 4(y - 1) = 0 \qquad ...(i)$

For a quadratic equation, co-efficient of $x^2 \neq 0$

$\therefore \quad v \neq 1$

$x^2(y - 1) + 3x(y + 1) + 4(y - 1) = 0 \qquad ...(i)$

For a quadratic equation, co-efficient of $x^2 \neq 0$

$\therefore \quad y \neq 1$

$\because \quad x$ *is real* $\therefore \ D \geq 0$

$\Rightarrow \quad 9(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow \quad -7y^2 + 50y - 7 \geq 0 \quad \Rightarrow \quad 7y^2 - 50y + 7 \leq 0$

$\Rightarrow \quad (7y - 1)(y - 7) \leq 0$

$\Rightarrow \quad y \in \left[\frac{1}{7}, 7\right]$ *but* $y = 1$ *is not included.*

If $\quad y = 1 \Rightarrow \dfrac{x^2 - 3x + 4}{x^2 + 3x + 4} = 1$

$\Rightarrow \quad 6x = 0 \quad \Rightarrow \quad x = 0$

$\therefore \quad y = 1$ *is also one of the values in the range.*

Hence, $y \in \left[\frac{1}{7}, 7\right]$

TYPE-4 $\quad y = \dfrac{ax^2 + bx + c}{px^2 + qx + r}\left(\dfrac{\text{Quadratic}}{\text{Quadratic}}\right)$

when P(x) and Q(x) has exactly one linear factor is common.

Illustration :

If x is real then find the range of the function $y = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$.

Sol. $y = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$

$\Rightarrow \quad \dfrac{(x-1)(x-2)}{(x+3)(x-2)}, \qquad x \neq 2$

$\Rightarrow \quad y = \dfrac{x-1}{x+3} \qquad \ldots(i)$

$\Rightarrow \quad xy + 3y = x - 1$

$\Rightarrow \quad x(1-y) = 3y + 1 \quad \Rightarrow \quad x = \dfrac{3y+1}{1-y}, \quad y \neq 1$

y is not defined at x = 2

$\therefore$ *on putting x = 2 in (i)*

$y = \dfrac{2-1}{2+3} = \dfrac{1}{5}$

Hence, range of the function is $R - \left\{\dfrac{1}{5}, 1\right\}$

$2+3 \quad 5$

Hence, range of the function is $R - \left\{\dfrac{1}{5}, 1\right\}$

TYPE-5 $\quad y = \dfrac{ax^2 + px + c}{px^2 + qx + r}$ when y takes all real values.

Illustration :

Prove that $y = \dfrac{(x+1)(x-2)}{x(x+3)}$ *can have any value in* $(-\infty, \infty)$ *for* $x \in R$

Sol. $y = \dfrac{(x+1)(x+2)}{x(x+3)}$

$\Rightarrow \quad x^2y + 3xy = x^2 - x - 2$

$\Rightarrow \quad x^2(y-1) + x(3y+1) + 2 = 0$

$\because \quad$ *x is real* $\qquad \therefore \quad D \geq 0$

$\Rightarrow \quad (3y+1)^2 - 4.2(y-1) \geq 0$

$\Rightarrow \quad 9y^2 + 6y + 1 - 8y + 8 \geq 0$

$\Rightarrow \quad 9y^2 - 2y + 9 \geq 0$

$\Rightarrow \quad D < 0$

$\therefore \quad$ *It is true for all* $y \in R$

$\therefore \quad$ *Range of the given expression is R.*

Illustration :

Find all possible values of 'a' for which the expression $\dfrac{ax^2-7x+5}{5x^2-7x+a}$ *may be capable of all values, x being any real quantity.*

Sol. *Let* $y = \dfrac{ax^2-7x+5}{5x^2-7x+a}$

$\Rightarrow \quad 5x^2y - 7xy + ay = ax^2 - 7x + 5$

$\Rightarrow \quad x^2(5y-a) - 7x(y-1) + ay - 5 = 0$

$\because \quad x$ is real $\quad \therefore \quad D \geq 0$

$\Rightarrow \quad 49(y-1)^2 - 4(5y-a)(ay-5) \geq 0$

$\Rightarrow \quad 49(y^2-2y+1) - 4(5ay^2 - 25y - a^2y + 5a) \geq 0$

$\Rightarrow \quad y^2(49-20a) + 2y(1+2a^2) + 49 - 20a \geq 0$

Which is true for all $y \in R$

$\therefore \quad D \leq 0$ & leady coefficient $\quad 49 - 2a > 0$

$4(1+2a^2)^2 - 4(49-20a)^2 \leq 0$

$(1+2a^2+41-20a)(1+2a^2-49+20a) \leq 0$

$(a^2-10a+25)(a^2+10a-24) \leq 0$

$\therefore \quad D \leq 0$ & leady coefficient $\quad 49 - 2a > 0$

$4(1+2a^2)^2 - 4(49-20a)^2 \leq 0$

$(1+2a^2+41-20a)(1+2a^2-49+20a) \leq 0$

$(a^2-10a+25)(a^2+10a-24) \leq 0$

$(a-5)^2(a+12)(a-2) \leq 0$

+ – + +

–12 2 5

$a \in [-12, 2] \cup \{5\}$

but $\quad a < \dfrac{49}{2}$

$\therefore \quad a \in [-12, 2]$

Now when $a = -12$

$$y = \frac{-12x^2-7x+5}{5x^2-7x-12} = \frac{-(12x^2+7x-5)}{5x^2-7x-12} = -\frac{(12x-5)(x+1)}{(5x-12)(x+1)}$$

Here $(x+1)$ *is a common factor in numerator and denominator*

$\therefore$ *y does not take all real numbers.*

Similarly for $a = 2$ *numerator and denominator contains a common linear factor and again y does not take all real numbers.*

Hence, $a \in (-12, 2)$

13. RESOLVING A GENERAL QUADRATIC EXPRESSION IN x AND y INTO TWO LINEAR FACTORS :

$$f(x, y) = ax^2 + 2bxy + by^2 + 2gx + 2fy + C$$

Writing the above equation as a quadrating equation is x,

$$ax^2 + 2x\,(hy + g) + by^2 + 2fy + C = 0$$

Solving for x, we get

$$x = \frac{-(hy+g) \pm \sqrt{(hy+g)^2 - a(by^2 + 2fy + c)}}{a}$$

$\Rightarrow$ $\quad ax + hy + g = \pm\sqrt{y^2(h^2 - ab) + 2y(hg - af) + (g^2 - ac)}$

Now f(x, y) can be writing as product of two linear factors only when quantity under radical sign is a perfect square.

As quantity under radical sign is a quadratic equation in y. Therefore, it will be perfect square only when $D = 0$

$\Rightarrow$ $\quad (hg - af)^2 - (h^2 - ab)\,(g^2 - ac) = 0$

$\Rightarrow$ $\quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

which is the require condition

$\Rightarrow$ $\quad (hg - af)^2 - (h^2 - ab)\,(g^2 - ac) = 0$

$\Rightarrow$ $\quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

which is the require condition

Illustration :

Prove that the expression $2x^2 + 3xy + y^2 + 2y + 3x + 1$ can be factorised into two linear factors. Find them.

Sol. $2x^2 + 3xy + y^2 + 2y + 3x + 1$

Comparing given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, we get

$$a = 2,\ h = \frac{3}{2},\ b = 1 g = \frac{3}{2},\ f = 1,\ c = 1$$

Clearly, $\quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow$ $\quad$ *Given expression can be factorized into two linear factor*

To find the factors, form the quadratic in x

$$2x^2 + 3x\,(y + 1) + (y + 1)^2$$

$\Rightarrow$ $\quad x = \dfrac{-3(y+1) \pm \sqrt{(y+1)^2}}{y}$ $\quad\Rightarrow\quad 4x = -3(y + 1) \pm (y + 1)$

$\Rightarrow$ $\quad 4x = -2(y + 1)$ $\quad$ *or* $\quad 4x = -4(y + 1)$

$\Rightarrow$ $\quad 2x + y + 1 = 0$ $\quad$ *or* $\quad x + y + 1 = 0$

Illustration :

If the equation $x^2 + 16y^2 - 3x + 2 = 0$ is satisfied by real values of x and y then prove that $1 \le x \le 2$ and $-1/8 \le y \le 1/8$.

Sol. $x^2 - 3x + 16y^2 + 2 = 0$ As $x \in R \Rightarrow D \ge 0$

$\Rightarrow 9 - 64y^2 - 8 \ge 0 \qquad \Rightarrow 64y^2 - 1 \le 0$

$\Rightarrow (8y - 1)(8y + 1) \le 0 \qquad \Rightarrow y \in \left[-\frac{1}{8}, \frac{1}{8}\right]$

Again, $-(x^2 - 3x + 2) = 16y^2$

As R.H.S. $\ge 0 \qquad \Rightarrow -(x^2 - 3x + 2) \ge 0$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow x \in [1, 2]$

Practice Problem

Q.1 If x is real, prove that the expression $y = \dfrac{x^2 + 2x - 11}{2(x-3)}$ can have all numerical values except which lie between 2 and 6.

Q.2 If x is real then find the range of $y = \dfrac{x^2 + 3x - 4}{x^2 + 7x + 12}$.

Q.3 Show that in the equation, $x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0$, for every real value of x there is a real

$x^2 + 7x + 12$

Q.3 Show that in the equation, $x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0$, for every real value of x there is a real
value of y, and for every value of y there is a real value of x.

Q.4 Find the range of the function $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$.

Answer key

Q.2 $R - \{1, 5\}$ **Q.4** $[4, \infty)$

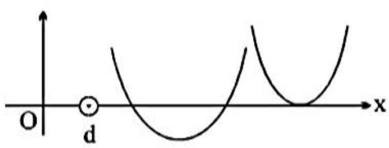
14. LOCATION OF ROOTS :

This article deals with an elegant approach of solving problems on quadratic equations when the roots are located / specified on the number line with variety of constraints :

Consider $f(x) = ax^2 + bx + c$ with $a > 0$.

TYPE–1 : Both roots of the quadratic equation are greater than a specified number say (d). The necessary and sufficient condition for this are :

(i) $a > 0$; (ii) $D \ge 0$; (iii) $f(d) > 0$; (iv) $-\dfrac{b}{2a} > d$



Note : If $a < 0$ then intercept accordingly.

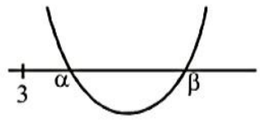
Illustration :

Find all the values of the parameter d for which both roots of the equation

$x^2 - 6dx + (2 - 2d + 9d^2) = 0$ *exceed the number 3.*

Sol. $x^2 - 6dx + (2 - 2d + 9d^2) = 0$

if both roots exceed 3



Conditions

(i) $D > 0$

$(6d)^2 - 4(2 - 2d + 9d^2) > 0$

$36d^2 - 8 + 8d - 36d^2 > 0$

$d > 1$

(ii) $f(3) > 0$

$9 - 18d + (2 - 2d + 9d^2) > 0$

$(9d - 11)(d - 1) > 0$

$d \in (-\infty, 1) \cup \left(\frac{11}{9}, \infty\right)$

$-b$

$9 - 18d + (2 - 2d + 9d^2) > 0$

$(9d - 11)(d - 1) > 0$

$d \in (-\infty, 1) \cup \left(\frac{11}{9}, \infty\right)$

(iii) $\frac{-b}{2a} > 3$

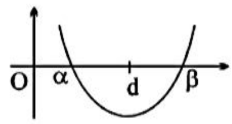
$-\left(\frac{-6d}{2}\right) > 3$

$d > 1$

Taking infersection of all the above three conditions we get

$d \in \left(\frac{11}{9}, \infty\right)$

TYPE–2: Both roots lie on either side of a fixed number say (d). Alternatively one root is greater than d and other less than d or d lies between the roots of the given equation.



Conditions for this

$$\left.\begin{array}{ll} & \text{(i) } a > 0 \\ \text{and} & \text{(ii) } f(d) < 0 \end{array}\right\} \text{ or } \quad \begin{array}{ll} \text{(i)} & a < 0 \\ \text{(ii)} & f(d) > 0 \end{array}$$

Note that no consideration for discriminant will be useful here.

Illustration :

Find the value of k for which one root of the equation of $x^2 - (k + 1)x + k^2 + k - 8 = 0$ exceed 2 and other is smaller than 2.

Sol. $x^2 - (k + 1)x + k^2 + k - 8 = 0$

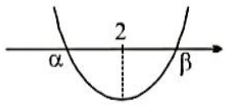
$a > 0$, hence $f(2) < 0$

$4 - (K + 1)\,2 + k^2 + k - 8 < 0$

$k^2 - k - 6 < 0$

$(k - 3)(k + 2) < 0$

$k \in (-2, 3)$



TYPE–3 : Exactly one root lies in the interval (d, e) when d < e.

Conditions for this are :

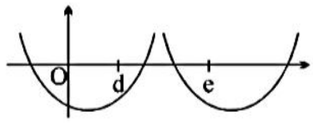
(i) $a \neq 0$;

(ii) $f(d) \cdot f(e) < 0$

(iii) An another case arises

when $f(d) \cdot f(e) = 0$

then we have to check end points

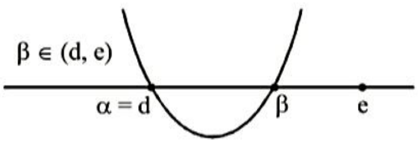


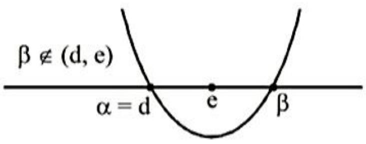
(iii) An another case arises

when $f(d) \cdot f(e) = 0$

then we have to check end points

For $f(d) = 0$, i.e., one root is "d"





Check if other root lies between "d" and "e" or not. If yes then we will include that point otherwise we will exclude that point

similarly for $f(e) = 0$

we will check for the other root and find out if it lies between "d" and "e" or not.

Note : If $f(d)\,f(e) < 0$ then exactly one root lies in the interval (d, e) but not the converse.

Illustration :

Find the set of values of m for which exactly one root of the equation

$x^2 + mx + (m^2 + 6m) = 0$ *lie in* $(-2, 0)$ *[Ans.* $(-6, -2) \cup (-2, 0)$*]*

Sol. $x^2 + mx + (m^2 + 6m) = 0$

If exactly one root lies in $(-2, 0)$ *then* $f(-2)\,f(0) < 0$

$(m^2 + 4m + 4)(m^2 + 6m) < 0$

$m \in (-6, -2) \cup (-2, 0)$

We have to find out the conditions when one of the root is –2, or 0.

Case I : *if one root is –2*

then $f(-2) = 0$

$m = -2$

$x^2 - 2x - 8 = 0$

$x = 4, -2$, *no root lie in* $(-2, 0)$ *for* $m = -2$.

Case II : *If one root is zero.*

then $m = 0$, *or* -6

If $m = 0,\quad x^2 = 0$ *both the roots are zero and no root lies in* $(-2, 0)$

If $m = -6,\quad x = 0, 6$ *no root lies in* $(2, 0)$

Hence $m \in (-6, -2) \cup (-2, 0)$

TYPE–4 :

When both roots are confined between the number d and e (d < e). Conditions for this are

(i) $a > 0$; (ii) $D \geq 0$; (iii) $f(d) > 0$; (iv) $f(e) > 0$

$$d < -\frac{b}{2a} < e$$

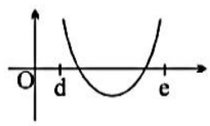


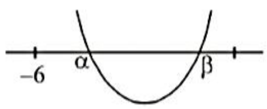
Illustration :

If α, β *are the roots of the quadratic equation*
$x^2 + 2(k-3)x + 9 = 0$ $(\alpha \neq \beta)$. *If* $\alpha, \beta \in (-6, 1)$ *then find the values of k.*

Sol. $x^2 + 2(k-3)x + 0 = 0$

$\alpha, \beta \in (-6, 1)$

Since heading coefficient is 1, hence



(i) $D \geq 0$

$4(k-3)^2 - 4 \times 9 \geq 0$

$(k-6)k \geq 0$

$k \in (-\infty, 0] \cup [6, \infty)$

(ii) $f(-6) > 0$

$36 - 12(k-3) + 9 > 0$

$36 + 36 - 12k + 9 > 0$

$12k < 81$

$k < \dfrac{27}{4}$

(iii) $f(1) > 0$

$1 + 2k - 3 + 9 > 0$

$2k > 6$

$k > 3$

(iv) $-6 < \frac{-b}{2a} < 1$

$-6 < 3 - k < 1$

$2 < k < 9$

Taking intersection of above four condition we get $k \in \left[6, \frac{27}{4}\right)$

TYPE–5 :

One root is greater than e and the other root is less than d.

Conditions are :

(i) $f(d) < 0$ and $f(e) < 0$ if $(a > 0)$

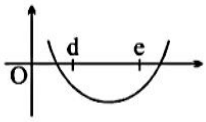


Illustration :

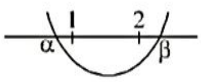
Illustration :

Find all the values of k for which one root of the quadratic equation $(k - 5)x^2 - 2kx + k - 4 = 0$ *is smaller than 1 and the other root exceed 2.*

Sol. $(k - 5)x^2 - 2kx + k - 4 = 0$

Case I *(i)* $k - 5 > 0$

$k > 5$



(ii) $f(1) < 0$

$(k - 5) - 2k + k - 4 < 0$

$-9 < 0$

(iii) $f(2) < 0$

$4(k - 5) - 4k + k - 4 < 0$

$k - 24 < 24$

$k < 24$

$k \in (5, 24)$

Case II : *(i)* $k - 5 < 0$

$k < 5$

(ii) $f(1) > 0$

$-9 > 0$ *that is not possible.*

Hence solution is $k \in (5, 24)$.

Practice Problem

Q.1 Let α be a real root of the quadratic equation $ax^2 + bx + c$ and β be a real root of the equation $-x^2 + bx + c = 0$. Show that there exists a root γ of the equation $\frac{a}{2}x^2 + bx + c = 0$ that lie between α and β. $(\alpha, \beta \neq 0)$.

Q.2 Find all the values of a for which both roots of the equation $x^2 + x + a = 0$ exceed the quantity a.

Q.3 Find the set of values of a for which zeroes of the quadratic polynomial
$(a^2 + a + 1)x^2 + (a - 1)x + a^2$ are located on either side of 3.

Q.4 Find all possible values of a for which exactly one root of the quadratic equation $x^2 - (a + 1)x + 2a = 0$ lie in the interval (0, 3).

Q.5 If $x^2 + 2ax + a < 0 \ \forall \ x \in [1, 2]$, then find the values of a.

Answer key

Answer key

Q.2 $(-\infty, -2)$ Q.3 ϕ Q.4 $(-\infty, 0] \cup (6, \infty)]$ Q.5 $a \in \left(-\infty, -\frac{4}{5}\right)$

15. THEORY OF EQUATIONS :

Relation between roots and coefficients of polynomial equation :

15.1 For Quadratic Equation :

If α and β are roots of a quadratic equation $ax^2 + bx + c = 0$ then

$ax^2 + bx + c = a(x - \alpha)(x - \beta)$

$\Rightarrow ax^2 + bx + c = a[x^2 - x(\alpha + \beta) + \alpha\beta]$

Comparing co-efficients on both sides, we get

$$-a(\alpha + \beta) = b \Rightarrow \alpha + \beta = \frac{-b}{a}$$

$$\alpha + \beta = c \Rightarrow \alpha\beta = \frac{c}{a}$$

15.2 For Cubic Equation :

If α, β and γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ then

$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)$$

$$ax^3 + bx^2 + cx + d = a[x^3 - (\Sigma \alpha)x^2 + (\Sigma \alpha\beta)x^2 - \alpha\beta\gamma]$$

Comparing co-efficients on both sides, we get

$$\therefore \quad \alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

and $\quad \alpha\beta\gamma = -\dfrac{d}{a}$

15.3 For Bi-quadratic Equation :

If α, β, γ and δ are roots of a bi-quadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ then

$$ax^4 + bx^3 + cx^2 + dx + e = a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$$

$$ax^4 + bx^3 + cx^2 + dx + e = a[x^4 - (\Sigma \alpha)x^3 + (\Sigma \alpha\beta)x^2 - (\Sigma \alpha\beta\gamma)x + \alpha\beta\gamma\delta]$$

Comparing co-efficients on both sides, we get

$$\therefore \quad \alpha + \beta + \gamma + \delta = \frac{-b}{a}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

a

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = \frac{-d}{a}$$

and $\quad \alpha\beta\gamma\delta = \dfrac{e}{a}$

NOTE:

A polynomial equations of degree odd with real coefficient must have at least one real root as imaginary roots always occur in pair of conjugates.

Illustration :

Find the

(i) sum of the squares and

(ii) sum of the cubes of the roots of the cubic equation, $x^3 - px^2 + qx - r = 0$ α β γ

Sol. *Given* $x^3 - px^2 + qx - r = 0$

Let the root be α, β, γ

$\alpha + \beta + \gamma = p, \qquad \Sigma\alpha\beta = q$

$\alpha\beta\gamma = r$

(i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\Sigma\alpha\beta$

$= p^2 - 2q$

(ii) $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma + (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$

$= 3\alpha\beta\gamma + (\alpha + \beta + \gamma)[(\alpha + \beta + \gamma)^2 - 3\Sigma\alpha\beta]$

$= 3r + p[p^2 - 3q]$

Illustration :

Solve the cubic $4x^3 + 16x^2 - 9x - 36 = 0$, the sum of its two roots being equal to zero.

Sol. *Given* $4x^3 + 16x^2 - 9x - 36 = 0$

Let the roots of equation be $\alpha_1 - \alpha_1\beta$

$$\alpha - \alpha + \beta = \frac{-16}{4}$$

$$\beta = -4$$

Product of roots $\alpha(-\alpha)(\beta) = \dfrac{-(-36)}{4}$

$$\alpha^2 = \frac{9}{4} \quad \Rightarrow \quad \alpha = \pm\frac{3}{2}$$

Product of roots $\alpha(-\alpha)(\beta) = \dfrac{-(-36)}{4}$

$$\alpha^2 = \frac{9}{4} \quad \Rightarrow \quad \alpha = \pm\frac{3}{2}$$

roots are $-\dfrac{3}{2}, \dfrac{3}{2}, -4.$

Illustration :

If a, b, c are the roots of cubic $x^3 - x^2 + 1 = 0$ then find the value of $a^{-2} + b^{-2} + c^{-2}$.

Sol. $x^2 - x^2 + 1 = 0$ *if a, b, c are roots*

then $a + b + c = 1, \quad ab + bc + ca = 0, \quad abc = 1$

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 - 2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)$$

$$= \left(\frac{ab + bc + ca}{abc}\right)^2 - \left(\frac{a + b + c}{abc}\right)$$

$$= 0 - 2\left(\frac{1}{-1}\right) = 2$$

Illustration :

If a polynomial is defined as $P(x) = 2x^5 + ax^4 + bx^3 + cx^2 + dx + e$ *such that* $P(0) = 4$, $P(1) = 5$, $P(2) = 8$, $P(3) = 13$ *and* $P(4) = 20$. *Find the value of* $P(5)$

Sol. *Consider the polynomial*

$$P(x) = Q(x) + x^2 + 4$$

$$P(0) = Q(0) + 4 = 4$$

$$Q(0) = 0$$

$$P(1) = Q(1) + 5 \quad \Rightarrow \quad Q(1) = 0$$

Similarly $Q(2) = 0$, $Q(3) = 0$ *and* $Q(4) = 0$

hence, $Q(x) = 2x\,(x-1)\,(x-2)\,(x-3)\,(x-4)$

$$\therefore \quad P(x) = 2x\,(x-1)\,(x-2)\,(x-3)\,(x-4) + x^2 + 4$$

$$P(5) = 2\cdot5\cdot4\cdot3\cdot2\cdot1 + 25 + 4 = 269$$

Practice Problem

Q.1 $\alpha, \beta, \gamma, \delta$ are the roots of the equation $\tan\left(\frac{\pi}{4} + x\right) = 3\tan 3x$ no two of which have equal tangents, find

Practice Problem

Q.1 $\alpha, \beta, \gamma, \delta$ are the roots of the equation $\tan\left(\frac{\pi}{4} + x\right) = 3\tan 3x$ no two of which have equal tangents, find the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$.

Q.2 Find the cubic each of whose roots is greater by unity than a root of the equation

$$x^3 - 5x^2 + 6x - 3 = 0.$$

Q.3 Form a cubic whose roots are the cubes of the roots of $x^3 + 3x^2 + 2 = 0$.

Q.4 The length of the sides of a triangle are the 3 distinct roots of the equation $4x^3 - 24x^2 + 47x - 30 = 0$, If the area of triangle is Δ, find the value of 100Δ.

Answer key

Q.1 Zero Q.2 $y^3 - 8y^2 + 19y - 1 = 0$ Q.3 $y^3 + 33y^2 + 12y + 8 = 0$

Q.4 150 where $\Delta = 3/2$

16. LOG INEQUALITIES :

(1) For $a > 1$, If $\log_a x > \log_a y$, then $\Rightarrow$ $x > y$ that is if base is greater than unity the inequality remains unchanged when log is removed.

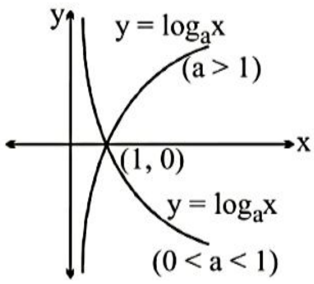
(2) For $0 < a < 1$. If $\log_a x > \log_a y$, then $\Rightarrow$ $x < y$ that is if base is less than unity then inequality sign is reversed when log is removed.

(3) If $a > 1$. $\log_a x < P \Rightarrow x < a^P$.

(4) If $a > 1$. $\log_a x > P \Rightarrow x > a^P$.

(5) If $a < 1$. $\log_a x < P \Rightarrow x > a^P$.

(6) If $a < 1$. $\log_a x > P \Rightarrow x < a^P$.



$\Rightarrow$ If base is less than unity then value of log x decreases as x-increases.

$\Rightarrow$ If base is greater than unity then value of log x increases as x increases.

To solve log inequality when base is constant :

(1) First define log i.e. find the condition that base is positive, expression inside log is positive and base is not

To solve log inequality when base is constant :

(1) First define log i.e. find the condition that base is positive, expression inside log is positive and base is not equal to unity. Term it as initial condition

(2) Check weather base is greater than unity or less than unity.

(3) If base is greater than unity then remove the log without changing the inequality and if base is less than unity then reverse the inequality when log is removed.

(4) Solve the inequality and term it as final condition.

(5) Take the intersection of initial and final conditions.

Illustration :

$$\log_{\frac{1}{3}}(5x-1) > 0$$

Sol. *First define logarithm*

$$5x - 1 > 0$$

$$5x > 1$$

$$x > \frac{1}{5} \rightarrow \quad \textit{(initial condition)}$$

Now as base is less than unity then on removing log, inequality will be reversed $\log_{\frac{1}{3}}(5x-1) > 0$

$$5x - 1 < \left(\frac{1}{3}\right)^0$$

$$5x - 1 < 1$$

$$x < \frac{2}{5} \rightarrow \quad \textit{(final condition)}$$

Taking intersection of initial and final condition, we get $x \in \left(\frac{1}{5}, \frac{2}{5}\right)$. ***Ans.***

Illustration :

$\log_{0.5}(x^2 - 5x + 6) > -1$

Sol. *Expression inside log should be positive.*

$$x^2 - 5x + 6 > 0$$

$$(x - 2)(x - 3) > 0$$

$$x < 2 \text{ or } x > 3$$

$$x \in (-\infty, 2) \cup (3, \infty) \rightarrow \quad \textit{(initial condition)}$$

Now, $x^2 - 5x + 6 < (0.5)^{-1}$ *(inequality is reversed since base is less than one.)*

$$x^2 - 5x + 6 < 2$$

$$x \in (-\infty, 2) \cup (3, \infty) \rightarrow \quad \textit{(initial condition)}$$

Now, $x^2 - 5x + 6 < (0.5)^{-1}$ *(inequality is reversed since base is less than one.)*

$$x^2 - 5x + 6 < 2$$

$$x^2 - 5x + 4 < 0$$

$$(x - 1)(x - 4) < 0$$

$$1 < x < 4 \quad \rightarrow \quad \textit{(final condition)}$$

Taking intersection of initial condition and final condition we get $x \in (1, 2) \cup (3, 4)$. ***Ans.***

Illustration :

$\log^2_{0.5} x + \log_{0.5} x - 2 \le 0$

Sol. *Put* $\log_{0.5} x = t$

$$t^2 + t - 2 \le 0$$

$$(t + 2)(t - 1) \le 0$$

$$-2 \le t \le 1$$

$$-2 \le \log_{0.5} x \le 1$$

$\Rightarrow \quad x > 0 \rightarrow \quad$ *(initial condition)*

Now, since base is less than unity hence inequality will be reversed when log is removed.

$$(0.5)^{-2} \ge x \ge (0.5)^1$$

$$4 \ge x \ge 0.5$$

$$x \in [0.5, 4] \rightarrow \quad \textit{(final condition)}$$

Taking intersection of initial and final condition, we get $x \in [0.5, 4]$. ***Ans.***

To solve log inequality when base is also variable :

(1) Define log i.e. find the condition that expression inside log is positive, base is positive and base is not equal to unity. Term it as initial condition.

(2) Take the case I, when base is greater than unity call it condition "I".

(3) Solve the inequality as per case I that is remove the log without changing the inequality and term it as condition I(a).

(4) Take the intersection of condition I, condition I(a) and initial condition and term it as condition A.

(5) Take the case II, when base is less than unity, term it as condition II.

(6) Solve the inequality as per case II i.e. reverse the inequality on removing the log and term it as condition II(a).

(7) Take the intersection of condition II, condition II(a) and initial condition. Call it condition B.

(8) Take the union of condition formed in step 4 and condition formed in step 7 that is find the union of "A" and "B".

Illustration :

Illustration :

$log_x(x^3 - x^2 - 2x) < 3$

Sol. *Defining log*

$x^3 - x^2 - 2x > 0$

$x (x^2 - x - 2) > 0$

$x (x - 2) (x + 1) > 0$

$x \in (-1, 0) \cup (2, \infty)$

Base > 0

$x > 0$

$x \in (2, \infty) \quad \rightarrow \quad$ *(initial condition)*

Case I : *when base is greater than 1.*

$x > 1$ *condition I*

$x^3 - x^2 - 2x < x^3$

$x^2 + 2x > 0$

$x (x + 2) > 0$

$x \in (-\infty, -2) \cup (0, \infty) \quad \rightarrow \quad$ *condition I(a)*

Taking intersection of initial condition, condition I and condition I(a)
we get $x \in (2, \infty) \rightarrow$ *(condition "A").*

Case II : *when base is less than 1*

$0 < x < 1$ condition II

$\log_x(x^3 - x^2 - 2x) < 3$

$x^3 - x^2 - 2x > x^3$

$x^2 + 2x < 0$

$x(x + 2) < 0$

$-2 < x < 0$ condition II (a)

Taking intersection of initial condition, condition II and condition II(a)
we get $x \in \phi \rightarrow$ *(condition "B").*
Taking union of condition "A" and condition "B" we get $x \in (2, \infty)$*]*

Illustration :

$\log_{2x}(x^2 - 5x + 6) < 1$

Sol. $x^2 - 5x + 6 > 0$

$\log_{2x}(x^2 - 5x + 6) < 1$

Sol. $x^2 - 5x + 6 > 0$

$(x - 2)(x - 3) > 0$

$x \in (-\infty, 2) \cup (3, \infty)$

$2x > 0$

$x > 0$

$x \in (0, 2) \cup (3, \infty) \rightarrow$ (initial condition)

Case I $1 < 2x$

$\frac{1}{2} < x \rightarrow$ (condition I)

Since base is greater than 1.
inequality remains unchanged

$x^2 - 5x + 6 < (2x)^1$

$x^2 - 7x + 6 < 0$

$(x - 6)(x - 1) < 0$

$1 < x < 6 \rightarrow$ condition I(a)

taking intersection of initial condition, condition I and condition I(a),
we get $x \in (1, 2) \cup (3, 6) \rightarrow$ *condition (A)*

Case II $\quad 0 < 2x < 1$

$$0 < x < \frac{1}{2} \rightarrow \text{(condition II)}$$

$$x^2 - 5x + 6 > 2x$$
$$x^2 - 7x + 6 > 0$$
$$(x - 6)(x - 1) > 0$$
$$x \in (-\infty, 1) \cup (6, \infty) \rightarrow \quad \text{(condition II(a)}$$

Taking intersection of initial condition, condition II and condition II(a),

we get $x \in \left(0, \frac{1}{2}\right) \rightarrow$ *condition B*

Taking union of condition "A" and condition "B", we get $x \in \left(0, \frac{1}{2}\right) \cup (1, 2) \cup (3, 6)$. ***Ans.***

Practice Problem

Q.1 $\quad \log_{0.5}\left(\log_6 \frac{x^2 + x}{x + 4}\right) < 0.$ $\qquad$ Q.2 $\quad \log_{x^2} \frac{4x - 5}{|x - 2|} \geq \frac{1}{2}.$

Practice Problem

Q.1 $\quad \log_{0.5}\left(\log_6 \frac{x^2 + x}{x + 4}\right) < 0.$ $\qquad$ Q.2 $\quad \log_{x^2} \frac{4x - 5}{|x - 2|} \geq \frac{1}{2}.$

Q.3 $\quad 2\log_5 x - \log_x 125 < 1.$ $\qquad$ Q.4 $\quad \log_{x^2}(2 + x) < 1.$

Q.5 $\quad \left(\frac{1}{2}\right)^{\log_2(x^2 - 1)} > 1.$

Answer key

Q.1 $\quad x \in (-4, -3) \cup (8, \infty)$ $\qquad$ Q.2 $\quad x \in [\sqrt{6} - 1, 2) \cup (2, 5].$

Q.3 $\quad x \in \left(0, \frac{1}{5}\right) \cup \left(1, \sqrt{125}\right)$ $\qquad$ Q.4 $\quad x \in (-2, -1) \cup (2, \infty)$

Q.5 $\quad x \in (-\sqrt{2}, -1) \cup (1, \sqrt{2})$

Solved Examples

Q.1 Find the values of 'x' for which the inequality $-1 \le \dfrac{3x^2-7x+8}{x^2+1} \le 1$, is satisfied.

Sol. $-1 \le \dfrac{3x^2-7x+8}{x^2+1} \le 1$

$\Rightarrow \quad \dfrac{3x^2-7x+8}{x^2+1} \ge -1 \quad$ and $\quad \dfrac{3x^2-7x+8}{x^2+1} \le 1$

$\Rightarrow \quad 4x^2 - 7x + 9 \ge 0 \quad$ and $\quad 2x^2 - 7x + 7 \le 0$

$\Rightarrow \quad x \in R \quad$ and $\quad x \in \phi$

Taking $x \in \phi$.

Q.2 $(x^2 + 3x + 1)(x^2 + 3x - 3) \ge 5$

Sol. $(x^2 + 3x + 1)(x^2 + 3x - 3) \ge 5$

Let $x^2 + 3x = t$

$\Rightarrow \quad (t + 1)(t - 3) \ge 5 \Rightarrow t^2 - 2t - 8 \ge 0$

$\Rightarrow \quad (t - 4)(t + 2) \ge 0$

$\Rightarrow \quad t \in (-\infty, -2] \cup [4, \infty)$

$\Rightarrow \quad x^2 + 3x \le -2 \quad$ or $\quad x^2 + 3x \ge 4$

$\Rightarrow \quad x^2 + 3x + 2 \le 0 \quad$ or $\quad x^2 + 3x \ge 4$

$\Rightarrow \quad (x + 1)(x + 2) \le 0 \quad$ or $\quad (x + 4)(x - 1) \ge 0$

$\Rightarrow \quad x \in [-2, -1] \quad$ or $\quad x \in (-\infty, -4] \cup [1, \infty)$

$\Rightarrow \quad x^2 + 3x + 2 \le 0 \quad$ or $\quad x^2 + 3x \ge 4$

$\Rightarrow \quad (x + 1)(x + 2) \le 0 \quad$ or $\quad (x + 4)(x - 1) \ge 0$

$\Rightarrow \quad x \in [-2, -1] \quad$ or $\quad x \in (-\infty, -4] \cup [1, \infty)$

Taking union

$x \in (-\infty, -4] \cup [-2, -1] \cup [1, \infty)$. Ans.

Q.3 If both the roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ exceed 3, then

(A) $a < 1$ (B) $a > \dfrac{11}{9}$ (C) $a > \dfrac{3}{2}$ (D) $a < \dfrac{5}{2}$

Sol. The quadratic equation $f(x) = x^2 - 6ax + 2 - 2a + 9a^2 = 0$ ......(1)

will have real roots if $D = 36a^2 - 4(2 - 2a + 9a^2) \ge 0$

$\Rightarrow \quad -8(1 - a) \ge 0 \quad$ or $\quad a \ge 1$ ......(2)

The roots of (1) will exceed 3 if

$$\frac{-b}{2a} = -\left(\frac{-6a}{2}\right) = 3a > 3 \text{ or } a > 1 \qquad ......(3)$$

and $\quad f(3) = 9 - 18a + 2 - 2a + 9a^2 > 0$

$\Rightarrow \quad 9a^2 - 20a + 11 > 0 \quad \Rightarrow (9a - 11)(a - 1) > 0 \quad \Rightarrow \left(a - \dfrac{11}{9}\right)(a - 1) > 0$

$\Rightarrow \quad a < 1 \;$ or $\; a > \dfrac{11}{9}$ ......(4)

Thus (2), (3) and (4) will hold simultaneously if $a > \dfrac{11}{9}$. Ans.

Q.4 If one root of the equation $x^2 + ax + b = 0$ is also a root of $x^2 + mx + n = 0$, show that its other root is a root of $x^2 + (2a - m)x + a^2 - am + n = 0$.

Sol. Let α be a root of the equation $x^2 + ax + b = 0$ which is also a root of $x^2 + mx + n = 0$.

Let β be the other root of $x^2 + ax + b = 0$, then $\alpha + \beta = -a$.

We have $\alpha = -a - \beta$

Since a is a root of $x^2 + mx + n = 0$, we get

$$(-a - \beta)^2 + m(-a - \beta) + n = 0$$

or $\quad \beta^2 + 2a\beta + a^2 - ma - m\beta + n = 0$

or $\quad \beta^2 + (2a - m)\beta + a^2 - ma + n = 0$

Thus, β is a root of $x^2 + (2a - m)x + a^2 - ma + n = 0$. Ans.

Q.5 Let a, b, c be positive integers and consider all the quadratic equations of the form $ax^2 - bx + c = 0$ which have two distinct real roots in the open interval $0 < x < 1$. Find the least positive integer a for which such a quadratic equation exists.

Sol. Let α, β be two distinct real roots of $f(x) = ax^2 - bx + c = 0$ lying in $(0, 1)$.

Then $f(x) = a(x - \alpha)(x - \beta)$

Now, $\quad f(0)\,f(1) = a^2\,\alpha(1 - \alpha)\,\beta(1 - \beta)$

Now, $\quad f(0)\,f(1) = a^2\,\alpha(1 - \alpha)\,\beta(1 - \beta)$

As $\quad 0 < \alpha < 1,\ 0 < \alpha(1 - \alpha) \le \frac{1}{4}$, with equality holding for $\alpha = \frac{1}{2}$.

Since $\quad 0 < \alpha, \beta < 1$ and $\alpha \ne \beta,\ 0 < \alpha(1 - \alpha)\,\beta(1 - \beta) < \frac{1}{16}$

$$\Rightarrow \quad 0 < f(0)\,f(1) < \frac{a^2}{16} \qquad ....(1)$$

As a, b, c are positive integers.

$$f(0)\,f(1) = c(a - b + c) \ge 1 \qquad [\because\ f(0)\,f(1) > 0] \qquad ....(2)$$

(1) and (2) imply $\frac{a^2}{16} > 1$, that is $a \ge 5$. Since the roots of $f(x) = 0$ are real and distinct, its discriminant $= b^2 - 4ac > 0$.

$$\Rightarrow \quad b^2 > 4ac \ge 20 \qquad [\because\ c \ge 1]$$

Hence, the minimum possible value of b is 5. Let us try the least values of a, b and c, that is $a = 5$, $b = 5$ and $c = 1$. It is easy to check that $5x^2 - 5x + 1 = 0$ has two distinct real roots lying between 0 and 1. Thus, the least positive integral value of a is 5. Ans.

Q.6 For what values of the parameter m is the inequality $\left|\frac{x^2+mx+1}{x^2+x+1}\right| < 3$ satisfied for all real values of x?

Sol. The inequality is equivalent to $-3 < \frac{x^2+mx+1}{x^2+x+1} < 3$

Since, $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} > 0$, we have

$$-3\,(x^2+x+1) < x^2+mx+1 < 3\,(x^2+x+1)$$

Thus, for each $x \in R$

$\therefore \quad 4x^2 + (m+3)\,x + 4 > 0$ ....(1)

and $\quad 2x^2 - (m-3)\,x + 2 > 0$ ....(2)

Since, the co-efficient of x^2 in L.H.S. of (1) = 4 > 0, the inequality (1) will be valid if

$(m+3)^2 - 64 < 0$, i.e. if $(m+11)\,(m-5) < 0$

or $\quad -11 < m < 5$ ....(3)

Since, the co-efficient of x^2 in L.H.S. of (2) = 2 > 0, the inequality (2) will be valid if

$(m-3)^2 - 16 < 0$ i.e. if $(m+1)\,(m-7) < 0$

or $\quad -1 < m < 7$ ....(4)

The conditions (3) and (4) will hold simultaneously if $-1 < m < 5$. Ans.

or $\quad -1 < m < 7$ ....(4)

The conditions (3) and (4) will hold simultaneously if $-1 < m < 5$. Ans.

Q.7 For what real values of a is one of the equation $(2a+1)x^2 - ax + a - 2 = 0$ greater and the other is smaller than unity?

Sol. The quadratic equation $(2a+1)x^2 - ax + (a-2) = 0$ ....(1)

will have real roots if $D = a^2 - 4\,(2a+1)\,(a-2) \geq 0$

$\Rightarrow \quad a^2 - 4\,(2a^2 - 3a - 2) \geq 0$

$\Rightarrow \quad a^2 - 4\,(2a^2 - 3a - 2) \geq 0 \quad \Rightarrow \quad 7a^2 - 12a - 8 \leq 0$

$\Rightarrow \quad a^2 - \frac{12a}{7} - \frac{8}{7} \leq 0 \quad \Rightarrow \quad \left(a - \frac{6}{7}\right)^2 \leq \frac{8}{7} + \frac{36}{49} = \frac{92}{49}$

$\Rightarrow \quad \frac{6}{7} - \frac{2\sqrt{23}}{7} \leq a \leq \frac{6}{7} + \frac{2\sqrt{23}}{7}$

Next, if α_1, α_2 are the roots of the given equation, then the desired condition is satisfied if and only if $(1-\alpha_1)\,(1-\alpha_2) < 0$

$\Rightarrow \quad 1 - (\alpha_1 + \alpha_2) + \alpha_1\,\alpha_2 < 0 \quad \Rightarrow \quad 1 - \frac{a}{2a+1} + \frac{a-2}{2a+1} < 0$

$\Rightarrow \quad \frac{2a+1-a+a-2}{2a+1} < 0 \Rightarrow \frac{2a-1}{2a+1} < 0 \Rightarrow \frac{(2a-1)\,(2a+1)}{(2a+1)^2} < 0$

$\Rightarrow \quad \left(a+\frac{1}{2}\right)\left(a-\frac{1}{2}\right) < 0 \qquad \Rightarrow \quad \frac{-1}{2} < a < \frac{1}{2}$

But $\frac{6}{7}-\frac{2\sqrt{23}}{7} < \frac{-1}{2}$ and $\frac{1}{2} < \frac{6}{7}+\frac{2\sqrt{23}}{7}$

Therefore, the required values of a lie in the range $\frac{-1}{2} < a < \frac{1}{2}$. Ans.

Q.8 Solve th equation $2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$.

Sol.

<u>Case I:</u> Let $x + 2 \geq 0$. In this case $|x + 2| = x + 2$ and the equation becomes

$2^{x+2} - 2^{x+1} - 1 - |2^{x+1} - 1| = 0$

$\Rightarrow \quad 2^{x+1}(2 - 1) - 1 - |2^{x+1} - 1| = 0$

$\Rightarrow \quad 2^{x+1} - 1 = |2^{x+1} - 1|$

$\Rightarrow \quad 2^{x+1} - 1 \geq 0$ or $x + 1 \geq 0$ or $x \geq -1$

<u>Case II:</u> Let $x + 2 < 0$. In this case $|x + 2| = -x - 2$ and the equation becomes

$2^{-(x+2)} - |2^{x+1} - 1| = 2^{x+1} + 1$ ......(1)

Put $2^{x+1} = y$. Then (1) becomes

$\frac{1}{2y} - |y - 1| = y + 1$

$\frac{1}{2y} - |y - 1| = y + 1$

or $2y^2 + 2y + |y - 1|2y = 1$ ......(2)

If $y \geq 1$, then L.H.S. of (2) > 1. If $y < 1$, then (2) becomes $2y^2 + 2y - 2y(y - 1) - 1 = 0$

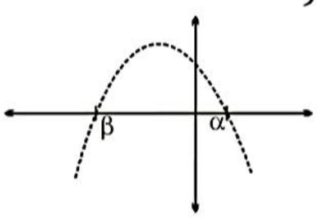
$\Rightarrow \quad 4y = 1$ or $y = \frac{1}{4}$

Also $2^{x+1} = 2^{-2}$ or $x = -3$. Ans.

Q.9 If in the quadratic equation $ax^2 + bx + c = 0$, $a < 0$, $b < 0$, $c > 0$ also α and β are its's roots where $\alpha > \beta$

(A) $|\alpha| > |\beta|$ (B) $|a| = |\beta|$ (C) $|\beta| > |\alpha|$ (D) $|\beta| > |\alpha| \geq \alpha > \beta$

Sol. $\left.\begin{array}{l} \alpha + \beta = -\frac{b}{a} = \text{negative} \\ \alpha\beta = \frac{c}{a} = \text{negative} \end{array}\right\}$ both roots are of opposite sign and root in greater magnitude is negative.



Q.10 If $a \in R_+$ and the roots of the equation $ax^2 - 3x + c = 0$, are two consecutive odd positive inters, then

(A) $a \in (1, \infty)$ (B) $a \in (1, 4)$ (C) $a \in \left(0, \frac{3}{4}\right]$ (D) $a \in (0, \infty)$

Sol. Let α and $\alpha + 2$ be two consecutive odd positive integers

$\therefore$ $a\alpha^2 - 3\alpha + c = 0,\ a(\alpha + 2)^2 - 3(\alpha + 2) + c = 0$

$\Rightarrow$ $(a\alpha^2 - 3\alpha + c) + 4a\alpha + 4a - 6 = 0$

$\Rightarrow$ $4a\alpha + 4a - 6 = 0$, since $a\alpha^2 - 3\alpha + c = 0$

$\Rightarrow$ $3 = 2a(1 + \alpha)$ where $\alpha \ge 1$

$\Rightarrow$ $\frac{3}{2a} - 1 = \alpha \Rightarrow \frac{3}{2a} - 1 \ge 1 \Rightarrow \frac{3}{2a} \ge 2$

$\Rightarrow$ $a \le \frac{3}{4}$. Ans.

Q.11 For what real values of a does the range of the function $y = \frac{x-1}{a - x^2 + 1}$ not contain any values belonging to the interval $[-1, -1/3]$?

$a - x^2 + 1$

to the interval $[-1, -1/3]$?

Sol. $y = \frac{x-1}{a - x^2 + 1}$

$ay - x^2y + y = x - 1$

$x^2y + x - (1 + ay + y) = 0$

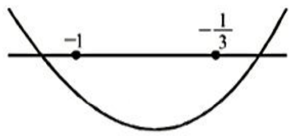
$\because$ x is real $\therefore$ $D \ge 0$

$1 + 4y\,(1 + ay + y) \ge 0$

$4y^2\,(a + 1) + 4y + 1 \ge 0$

Case-I $a + 1 > 0$

$f(-1) < 0$ and $f\left(\frac{-1}{3}\right) < 0$

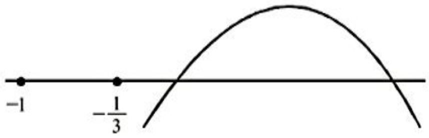


$a < \frac{-1}{4}$ and $a < \frac{-1}{4}$

but $a > -1$ $\therefore$ $a \in \left(-1, \frac{-1}{4}\right)$

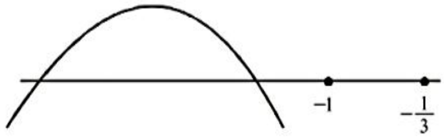
Case-II $\quad a + 1 = 0 \quad \Rightarrow \quad a = -1$

$$4y + 1 \geq 0 \quad \Rightarrow \quad y \geq \frac{-1}{4}$$



Case-III $\quad a + 1 < 0 \quad \Rightarrow \quad a < -1$

$$f(-1) < 0 \quad \text{and} \quad f\left(-\frac{1}{3}\right) < 0$$



$$\Rightarrow \quad a < -\frac{1}{4} \qquad a \in (-\infty, -1)$$

Finally, from Case-I, II and II

$$a \in \left(-\infty, -\frac{1}{4}\right) \qquad \text{Ans.}$$

Q.12 For what real values of a do the roots of $x^2 - 2x - a^2 + 1 = 0$ lie between the roots $x^2 - 2(a + 1)x + a(a - 1) = 0$.

Sol. As $y = x^2 - 2x - a^2 + 1$ and $y = x^2 - 2(a + 1)x + a(a - 1)$ are upward opening parabolas, the roots α, β of $x^2 - 2x - a^2 + 1 = 0$ will lie between the roots of $f(x) = x^2 - 2(a + 1)x + a(a - 1) = 0$ if $f(\alpha) < 0$ and $f(\beta) < 0$.

α, β of $x^2 - 2x - a^2 + 1 = 0$ will lie between the roots of $f(x) = x^2 - 2(a + 1)x + a(a - 1) = 0$ if $f(\alpha) < 0$ and $f(\beta) < 0$.

The roots of $x^2 - 2x - a^2 + 1 = 0$ ....(1)

are $\quad x = 1 \pm a$

and the root x of $x^2 - 2(a + 1)x + a(a - 1) = 0$

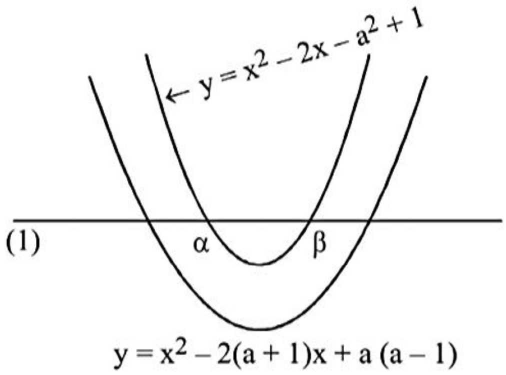
will be real and distinct if $4(a + 1)^2 - 4a(a - 1) > 0$

i.e. if $\quad 3a + 1 > 0$ or $a > \frac{-1}{3}$

We have $f(1 + a)$ and $f(1 - a) < 0$

$\Rightarrow \quad (1 + a)^2 - 2(a + 1)(1 + a) + a(a - 1) < 0$

and $\quad (1 - a)^2 - 2(a + 1)(1 - a) + a(a - 1) < 0$



$\Rightarrow \quad a > \frac{-1}{3}$ and $4a^2 - 3a - 1 < 0$

$\Rightarrow \quad a > \frac{-1}{3}$ and $(4a + 1)(a - 1) < 0$

$\Rightarrow \quad a > \frac{-1}{3}$ and $\frac{-1}{4} < a < 1$

Thus, $\frac{-1}{4} < a < 1$. $\qquad$ Ans.

Q.13 For what real values of the parameter a does the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$ have at least two distinct negative roots?

Sol. The equation (1) does not have 0 as a root.

Dividing (1) by x^2, we can rewrite it as

$$x^2 + \frac{1}{x^2} + 2a\,c + 1 = 0 \quad \Rightarrow \quad \left(x + \frac{1}{x}\right)^2 + 2a\left(x + \frac{1}{x}\right) - 1 = 0 \qquad ......(2)$$

Put $x + \frac{1}{x} = y$. Then we can write (2) as

$$y^2 + 2ay - 1 = 0 \qquad ......(3)$$

Since, the discriminant of (3) is $4a^2 + 4$, which is positive, (3) has two distinct real roots.

These roots are given by

$$y = \frac{-2a \pm \sqrt{4a^2 + 4}}{2} = -a \pm \sqrt{a^2 + 1}$$

Since, $-a + \sqrt{a^2 + 1} > 0$, the equation

2

Since, $-a + \sqrt{a^2 + 1} > 0$, the equation

$$x + \frac{1}{x} = -a + \sqrt{a^2 + 1}$$

has either positive roots or non-real complex roots. Since $-a - \sqrt{a^2 + 1} < 0$, both the roots of the equation

$$x + \frac{1}{x} = -a - \sqrt{a^2 + 1} \qquad ....(4)$$

We either negative or non-real complex. In case (4) has negative roots, we can rewrite (4) as

$$\left(\sqrt{-x} - \frac{1}{\sqrt{-x}}\right)^2 = a - 2 + \sqrt{a^2 + 1}$$

or $\left(\sqrt{-x} - \frac{1}{\sqrt{-x}}\right)^2 > 0$, we must have, $a - 2 + \sqrt{a^2 + 1} > 0$ i.e. $\sqrt{a^2 + 1} > |2 - a|$.

Since, $a^2 + 1 = (a + 2)^2 + 4a - 3$, $\sqrt{a^2 + 1} > |2 - a|$ if and only if $(a - 3) > 0$ that is, if and only if $a > \frac{3}{4}$.

Q.14 Find all real roots of the equation $\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$ where p is a real parameter.

Sol. As $\sqrt{x^2 - p} \geq 0,\ 2\sqrt{x^2 - 1} \geq 0$, we get from (1) that

$$x = \sqrt{x^2 - p} + 2\sqrt{x^2 - 1} \geq 0$$

Therefore, all the roots of (1) are non-negative. If $p < 0$, then $\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} > |x| \geq x$

$$\Rightarrow \quad \sqrt{x^2 - p} + 2\sqrt{x^2 - 1} > |x| \geq x$$

Thus, the equation (1) has no solution for $p < 0$. In other for (1) to have a solution we must $p \geq 0$.
We rewriter (1) as

$$2\sqrt{x^2 - 1} = x - \sqrt{x^2 - p}$$

and square it to obtain

$$4(x^2 - 1) = x^2 + x^2 - p - 2x\sqrt{x^2 - p}$$

$$\Rightarrow \quad 2x^2 + p - 4 = -2x\sqrt{x^2 - p}$$

Squaring again, we get

$$4x^4 + 4x^2(p - 4) + (p - 4)^2 = 4x^2(x^2 - p)$$

Squaring again, we get

$$4x^4 + 4x^2(p - 4) + (p - 4)^2 = 4x^2(x^2 - p)$$

$$\Rightarrow \quad x^2(8p - 16) + (p - 4)^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{(p-4)^2}{8(2-p)}$$

As $x^2 > 0$, we get $2 - p > 0 \Rightarrow 0 \leq p < 2$
For $0 \leq p < 2$, we get

$$x = \frac{|p-4|}{2\sqrt{2}\sqrt{2=p}}$$

Putting this value of x in (1), we get

$$\sqrt{\frac{(p-4)^2}{8(2-p)} - p} + 2\sqrt{\frac{(p-4)^2}{8(2-p)} - 1} = \frac{p-4}{2\sqrt{2}\sqrt{2-p}}$$

$$\Rightarrow \sqrt{p^2 - 8p + 16 + 8p^2} + 2\sqrt{p^2 - 8p + 16 - 16 + 8p} = 4 - p$$

$$\Rightarrow \sqrt{(3p-4)^2} + 2\sqrt{p^2} = 4 - p$$

$$\Rightarrow |3p - 4| = -(3p - 4) \Rightarrow 3p - 4 \leq 0$$

Thus, $p \leq \frac{4}{3}$. hence, $0 \leq p \leq \frac{4}{3}$, and for this value of p, $x = \frac{p-4}{2\sqrt{2}\sqrt{2-p}}$. Ans.

Q.15 Find all the values of the parameter a for which exactly one root of the equation $e^a x^2 - e^{2a} x + e^a - 1 = 0$ lies in the interval (1, 2).

Sol. Let $f(x) = e^a x^2 - e^{2a} x + e^a - 1$.

Note that $y = f(x)$ is an upward opening parabola. Exactly one root of (1) will lie in the interval (1, 2) if $f(1)\, f(2) < 0$.

We have $\quad f(1) = e^a - e^{2a} + e^a - 1 = -(e^a - 1)^2$

Note that $\quad f(1) < 0$ if $a \neq 0$

Thus, we must have $f(2) > 0$.

$\Rightarrow \quad f(2) = e^a (2^2) - e^{2a} (2) + e^a - 1 > 0$

$\Rightarrow \quad e^a$ lies between the roots of $2y^2 - 5y + 1 = 0$

i.e. between $\dfrac{5-\sqrt{17}}{4}$ and c

$$\text{Now,}\quad \frac{5-\sqrt{17}}{4} < e^a < \frac{5-\sqrt{17}}{4}$$

$$\Rightarrow \quad \log_e\left(\frac{5-\sqrt{17}}{4}\right) < a < \log_e\left(\frac{5+\sqrt{17}}{4}\right)$$

As $a \neq 0$, we get

As $a \neq 0$, we get

$$a \in \left(\log_e\left(\frac{5-\sqrt{17}}{4}\right), 0\right) \cup \left(0, \log_e\left(\frac{5+\sqrt{17}}{4}\right)\right). \qquad \text{Ans.}$$

Q.16 Let x, y, z be real variables satisfying the equations $x + y + z = 6$ and $xy + yz + zx = 7$. Find the range in which variables can lie.

Sol. Eliminating z from the two given equations, we get $xy + (y + x)\{6 - (x + y)\} = 7$

$\Rightarrow \quad -(x + y)^2 + xy + 6(x + y) - 7 = 0$

$\Rightarrow \quad y^2 + y(x - 6) + x^2 - 6x + 7 = 0$

As y is real, we get $(x - 6)^2 - 4(x^2 - 6x + 7) \geq 0$

$\Rightarrow \quad x^2 - 12x + 36 - 4x^2 + 24x - 28 \geq 0$

$\Rightarrow \quad 3x^2 - 12x - 8 \leq 0$

Thus, x must lie between the roots of $3x^2 - 12x - 8 = 0$, that is, between

$$\frac{12-\sqrt{144+96}}{6} \leq x \leq \frac{12+\sqrt{144+96}}{6}$$

$$\Rightarrow 2 - \frac{2\sqrt{15}}{3} \leq x \leq 2 + \frac{2\sqrt{15}}{3}.$$

$$\text{Hence, x, y, z}\left[2-\frac{2}{3}\sqrt{15}, 2+\frac{2}{3}\sqrt{15}\right]. \qquad \text{Ans.}$$