INVERSE TRIGONOMETRIC FUNCTIONS

(A) GENERAL INTRODUCTION :

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. denote angles or real numbers whose sine is x, whose cosine is x and whose tangent is x, provided that the answers given are numerically smallest available. These are also written as arc sinx, arc cosx etc.

If there are two angles one positive & the other negative having same numerical value, then positive angle should be taken.

(B) PRINCIPAL VALUES AND DOMAINS OF INVERSE CIRCULAR FUNCTIONS :

S.No.	Function	Domain	Range
(i)	$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(ii)	$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
(iii)	$y = \tan^{-1}x$	$x \in R$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
(iv)	$y = \cot^{-1}x$	$x \in R$	$0 < y < \pi$
(v)	$y = \mathrm{cosec}^{-1}x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$
(vi)	$y = \sec^{-1}x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi$; $y \ne \frac{\pi}{2}$

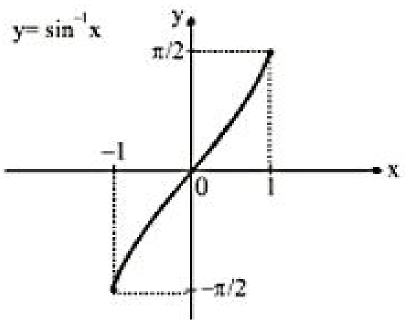
Note That : **(a)** 1st quadrant is common to all the inverse functions.

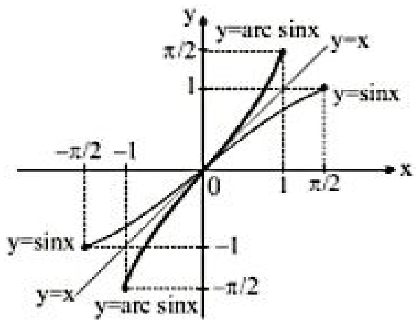
(b) 3rd quadrant is **not used** in inverse functions.

(c) 4th quadrant is used in the CLOCKWISE DIRECTION i.e. $-\frac{\pi}{2} \le y \le 0$.

Graphs of all 6 inverse circular functions :

(1) $y = \sin^{-1}x$, $|x| \le 1$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$





Note : Graph of $y = \sin^{-1}x$ and $y = \sin x$ are mirror image of each other about the line $y = x$.

Highlights : -

(i) $\sin^{-1}x$ is bounded in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

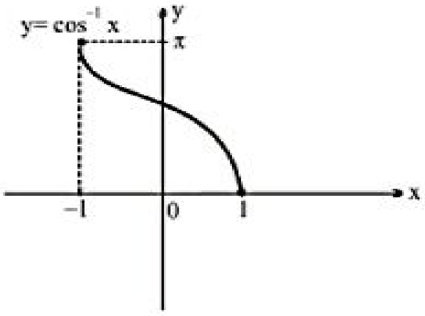
(ii) $\sin^{-1}x$ is an odd function. (symmetric about origin)

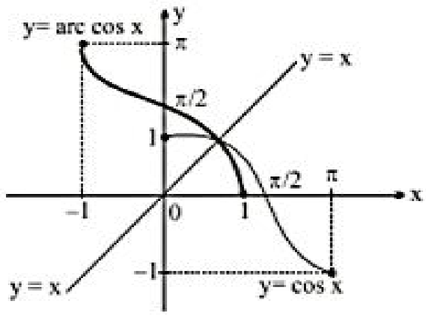
(iii) $\sin^{-1}x$ is an increasing function in its domain.

(iv) Maximum value of $\sin^{-1}x = \frac{\pi}{2}$, occurs at $x = 1$ and minimum value of $\sin^{-1}x = -\frac{\pi}{2}$, occurs at $x = -1$.

(v) $\sin^{-1}x$ is an aperiodic function.

(2) $y = \cos^{-1} x, |x| \le 1, \quad y \in [0, \pi]$





Note : Graph of $y = \cos^{-1}x$ and $y = \cos x$ are mirror image of each other about the line $y = x$.

Highlights : -

(i) $\cos^{-1}x$ is bounded in $[0, \pi]$.

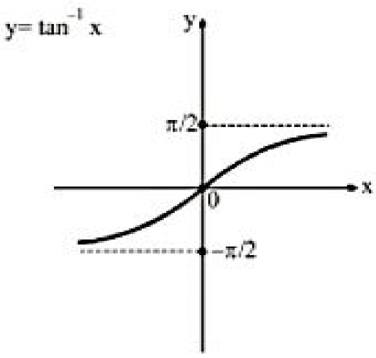
(ii) $\cos^{-1}x$ is a neither odd nor even function.

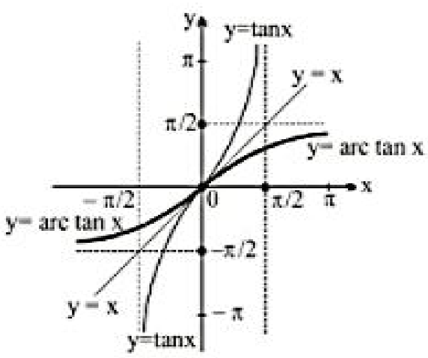
(iii) $\cos^{-1}x$ is a decreasing function in its domain.

(iv) Maximum value of $\cos^{-1}x = \pi$, occurs at $x = -1$ and minimum value of $\cos^{-1}x = 0$, occurs at $x = 1$.

(v) $\cos^{-1}x$ is an aperiodic function.

(3) $y = \tan^{-1} x, x \in R, \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$





Note : Graph of $y = \tan^{-1}x$ and $y = \tan x$ are mirror image of each other about the line $y = x$.

Highlights : -

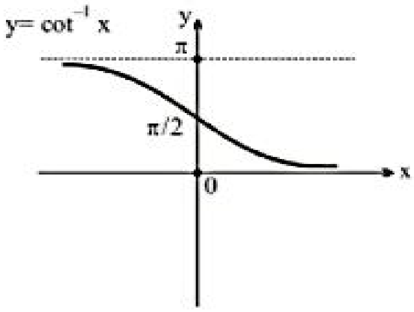
(i) $\tan^{-1}x$ is bounded in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

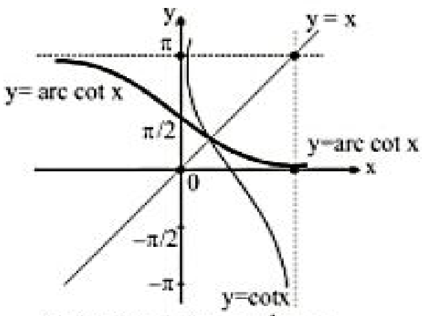
(ii) $\tan^{-1}x$ is an odd function. (symmetric about origin)

(iii) $\tan^{-1}x$ is an increasing function in its domain.

(iv) $\tan^{-1}x$ is an aperiodic function.

(4) $y = \cot^{-1} x$, $x \in R$, $y \in (0 , \pi)$





Note : Graph of $y = \cot^{-1}x$ and $y = \cot x$ are mirror image of each other about the line $y = x$.

Highlights : -

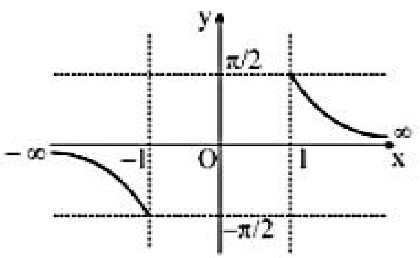
(i) $\cot^{-1}x$ is bounded in $(0, \pi)$.

(ii) $\cot^{-1}x$ is a neither odd nor even function.

(iii) $\cot^{-1}x$ is a decreasing function in its domain.

(iv) $\cot^{-1}x$ is an aperiodic function.

(5) $y = \text{cosec}^{-1}x$, $|x| \geq 1$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



Highlights : -

(i) $\text{cosec}^{-1}x$ is bounded in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

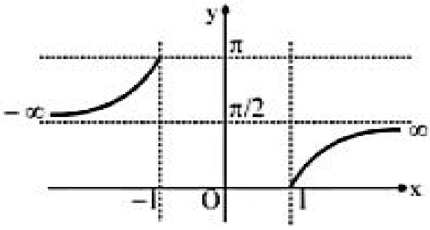
(ii) $\text{cosec}^{-1}x$ is an odd function. (symmetric about origin)

(iii) Maximum value of $\text{cosec}^{-1}x = \frac{\pi}{2}$, occurs at $x = 1$ and minimum value of $\text{cosec}^{-1}x = -\frac{\pi}{2}$, occurs at $x = -1$.

(iv) $\text{cosec}^{-1}x$ is a decreasing function.

(v) $\text{cosec}^{-1}x$ is an aperiodic function.

(6) $y = \sec^{-1} x, |x| \geq 1, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



Highlights : -

(i) $\sec^{-1}x$ is bounded in $[0, \pi]$.
(ii) $\sec^{-1}x$ is a neither odd nor even function.
(iii) Maximum value of $\sec^{-1}x = \pi$, occurs at $x = -1$ and minimum value of $\sec^{1} x = 0$, occurs at $x = 1$.
(iv) $\sec^{-1}x$ is an increasing function.
(v) $\sec^{-1}x$ is an aperiodic function.

Note :

(a) $\tan^{-1}(x)$ and $\cot^{-1}(x)$ are continuous and monotonic on R $\Rightarrow$ that their range is R
(b) If f (x) is continuous and has a range R $\not\Rightarrow$ it is monotonic. ***e.g.*** $y = x^3 - 3x$.

Illustration :

Find domain and range of the following

(a) $\sin^{-1}[x]$ *(b)* $\cos^{-1}\{x\}$
(c) $\sin^{-1}(e^x)$ *(d)* $f(x) = \tan^{-1}(\log_{4/5}(5x^2 - 8x + 4))$
(where [x] denotes the greatest integer function and {x} denotes the fractional part function.)

Sol.

(a) $\sin^{-1}[x]$ *defined when*
$-1 \leq [x] \leq 1 \Rightarrow -1 \leq x < 2$
domain : $x \in [-1, 2)$
In this domain [x] takes the values –1, 0, 1
$\Rightarrow$ *range of* $\sin^{-1}[x] = \{\sin^{-1} -1, \sin^{-1} 0, \sin^{-1} 1\}$

$$Range = \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$$

(b) $\cos^{-1}\{x\}$ *defined when*
$-1 \leq \{x\} \leq 1$
$\Rightarrow$ *domain :* $x \in R \quad (\because \{x\} \in [0, 1))$
$Range = \cos^{-1}[0, 1)$
$= (\cos^{-1} 1, \cos^{-1} 0]$

$$Range = \left(0, \frac{\pi}{2}\right]$$

(c) $\sin^{-1} e^x$ defined when

$-1 \le e^x \le 1 \Rightarrow e^x \ge -1$ holds always true

So $e^x \le 1 \Rightarrow x \le 0$

domain $x \in (-\infty, 0]$

In this domain $e^x \in (0, 1]$

$\Rightarrow$ Range of $\sin^{-1} e^x = \sin^{-1}(0, 1]$

$= \sin^{-1} (0, 1]$

$= (\sin^{-1} 0, \sin^{-1} 1]$

$$\text{Range } = \left(0, \frac{\pi}{2}\right]$$

(d) $f(x)$ is defined when $5x^2 - 8x + 4 > 0$

$\because a > 0, D < 0 \Rightarrow 5x^2 - 8x + 4 > 0$ is true for all $x \in R$

$\Rightarrow$ domain : $x \in R$

$$\text{Now, } \quad 5x^2 - 8x + 4 = 5\left[\left(x - \frac{4}{5}\right)^2 + \frac{4}{25}\right]$$

$$\Rightarrow 5x^2 - 8x + 4 \in \left[\frac{4}{5}, \infty\right) \text{ for } x \in R$$

$$\Rightarrow \text{Range of } f(x) = \tan^{-1}\left(\log_{4/5}\left[\frac{4}{5}, \infty\right)\right)$$

$$\text{Range } = \tan^{-1}(-\infty, 1] = \left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$$

Illustration :

Find the value of

(a) $\sin\left(2\sin^{-1}\dfrac{3}{5}\right)$ (b) $\cos(2\tan^{-1}2) + \sin(2\tan^{-1}3)$

(c) $\cos\left(\text{arc}\sin\dfrac{4}{5} - \text{arc}\cos\dfrac{4}{5}\right)$ (d) $\tan\left(2\cot^{-1}5 - \dfrac{\pi}{4}\right)$

Sol.

(a) $\sin^{-1}\dfrac{3}{5} = \theta \Rightarrow \sin\theta = \dfrac{3}{5}, \cos\theta = \dfrac{4}{5}$

$\sin(2\theta) = 2\sin\theta.\cos\theta$

$= 2\left(\dfrac{3}{5}\right)\left(\dfrac{4}{5}\right) = \dfrac{24}{25}$ $\left[\text{Ans. } \dfrac{24}{25}\right]$

(b) Let $\tan^{-1} 2 = \theta \Rightarrow \tan\theta = 2$

$\tan^{-1} 3 = \phi \Rightarrow \tan\phi = 3$

Now

$$\cos(2\theta) + \sin(2\phi) = \frac{1-\tan^2\theta}{1+\tan^2\theta} + \frac{2\tan\phi}{1+\tan^2\phi}$$

$$= \frac{1-(2)^2}{1+(2)^2} + \frac{2(3)}{1+(3)^2}$$

$$= \frac{-3}{5} + \frac{3}{5} = 0 \qquad [Ans.\ 0]$$

(c) Let $\sin^{-1}\frac{4}{5} = \theta \Rightarrow \sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}$

$\cos^{-1}\frac{4}{5} = \phi \Rightarrow \cos\phi = \frac{4}{5}, \sin\phi = \frac{3}{5}$

Now

$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$$= \frac{4}{5}\cdot\frac{3}{5} + \frac{4}{5}\cdot\frac{3}{5} = \frac{24}{25} \qquad \left[Ans.\ \frac{24}{25}\right]$$

(d) $let\ \cot^{-1}5 = \theta \Rightarrow \cot\theta = 5, \tan\theta = \frac{1}{5}$

Now

$$\tan\left(2\theta - \frac{\pi}{4}\right) = \frac{\tan 2\theta - 1}{1+\tan 2\theta}$$

$$= \frac{\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)-1}{1+\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)} = \frac{\left(\frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\right)-1}{1+\left(\frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\right)} = \frac{\frac{5}{12}-1}{1+\frac{5}{12}} = \frac{-7}{17} \qquad \left[Ans.\ \frac{-7}{17}\right]$$

Illustration :

Find the domain of definition of following functions.

(a) $f(x) = \arccos\frac{2x}{1+x}$

(b) $f(x) = \sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4-x)$

(c) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\frac{3-2x}{5}\right) + \log_6(2|x|-3) + \sin^{-1}(\log_2 x)$

Sol.

(a) $f(x) = \cos^{-1}\dfrac{2x}{1+x}$

$f(x)$ defined when $-1 \le \dfrac{2x}{1+x} \le 1$

Now $\dfrac{2x}{1+x} \ge -1 \Rightarrow \dfrac{2x}{1+x} + 1 \ge 0 \Rightarrow \dfrac{3x+1}{1+x} \ge 0$

$$\Rightarrow x \in (-\infty, -1) \cup \left[-\frac{1}{3}, \infty\right) \quad \text{... (1)}$$

and $\dfrac{2x}{1+x} \le 1 \Rightarrow \dfrac{2x}{1+x} - 1 \le 0 \Rightarrow \dfrac{x-1}{1+x} \le 0$

$\Rightarrow x \in (-1, 1]$ ... (ii)

from (i) and (ii)

$x \in \left[-\dfrac{1}{3}, 1\right]$ **Ans.** $x \in \left[-\dfrac{1}{3}, 1\right]$

(b) $f(x) = \sin^{-1}\left(\dfrac{x-3}{2}\right) - \log_{10}(4-x)$

$f(x)$ defined when $-1 \le \dfrac{x-3}{2} \le 1$ and $4 - x > 0$

Now $-1 \le \dfrac{x-3}{2} \le 1$

$\Rightarrow -2 \le x - 3 \le 2$

$\Rightarrow 1 \le x \le 5$

$\Rightarrow x \in [1, 5]$ ... (i)

and $4 - x > 0$

$\Rightarrow x < 4 \Rightarrow x \in (-\infty, 4)$ ... (ii)

$\Rightarrow$ from (i) and (ii)

$x \in [1, 4)$ **Ans.** $x \in [1, 4)$

(c) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\dfrac{3-2x}{5}\right) + \log_6(2|x|-3) + \sin^{-1}(\log_2 x)$

Now $3 - x \ge 0 \Rightarrow x \le 3 \Rightarrow x \in (-\infty, 3]$ ... (i)

$-1 \le \dfrac{3-2x}{5} \le 1 \Rightarrow -5 \le 3 - 2x \le 5$

$$\Rightarrow -2 \le 2x \le 8 \Rightarrow -1 \le x \le 4$$

$$\Rightarrow x \in [-1, 4] \quad \text{... (ii)}$$

$$2|x|-3>0 \Rightarrow |x|>\frac{3}{2} \Rightarrow x\in\left(-\infty,-\frac{3}{2}\right)\cup\left(\frac{3}{2},\infty\right) \quad \text{... (iii)}$$

$$-1\le \log_2 x \le 1 \Rightarrow \frac{1}{2}\le x\le 2 \Rightarrow x\in\left[\frac{1}{2},2\right] \quad \text{... (iv)}$$

from (i), (ii), (iii) and (iv)

$$x\in\left(\frac{3}{2},2\right] \qquad \textit{Ans. } \left(\frac{3}{2},2\right]$$

Illustration :

If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, *then the value of* $x^{2012}+y^{2012}+z^{2012}+\dfrac{6}{x^{2011}+y^{2011}+z^{2011}}$ *is equal to*

(A) 0 (B) 1 (C) –1 (D) 2

Sol. $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$ *is possible only when* $\cos^{-1}x=\cos^{-1}y=\cos^{-1}z=\pi$ *(all take their maximum value)*

$\Rightarrow \quad x=-1,\ y=-1,\ z=-1$

Now

$$x^{2012}+y^{2012}+z^{2012}+\frac{6}{x^{2011}+y^{2011}+z^{2011}}$$

$$=(-1)^{2012}+(-1)^{2012}+(-1)^{2012}+\frac{6}{(-1)^{2011}+(-1)^{2011}+(-1)^{2011}}$$

$$=3+\frac{6}{-3}=3-2=\mathbf{1} \qquad \textbf{Ans.}$$

Illustration :

Equation of the image of the line $x+y=\sin^{-1}(a^6+1)+\cos^{-1}(a^4+1)-\tan^{-1}(a^2+1)$, $a\in R$ *about x axis is given by*

(A) $x-y=0$ (B) $x-y=\dfrac{\pi}{2}$ (C) $x-y=\pi$ (D) $x-y=\dfrac{\pi}{4}$

Sol. $\because \quad \sin^{-1}$ *is defined for* $[-1, 1]$

$\therefore \quad a=0$

$\therefore \quad x+y=\sin^{-1}1+\cos^{-1}1-\tan^{-1}1=\dfrac{\pi}{4}$

Clearly image about x axis will be $x-y=\dfrac{\pi}{4}$ **Ans.**

Illustration :

If $\sin^{-1}\left(x^2-\frac{x^4}{3}+\frac{x^6}{9}\,..........\right)+\cos^{-1}\left(x^4-\frac{x^8}{3}+\frac{x^{12}}{9}\,..........\right)=\frac{\pi}{2}$, *where* $0 \le |x| < \sqrt{3}$, *then number of values of 'x' is equal to*

(A) 1 (B) 2 (C) 3 (D) 4

Sol. $\sin^{-1}\underbrace{\left(x^2-\frac{x^4}{3}+\frac{x^6}{9}......\right)}_{X}+\cos^{-1}\underbrace{\left(x^4-\frac{x^8}{3}+\frac{x^{12}}{9}......\right)}_{Y}=\frac{\pi}{2}$

Now $X = Y$

$$\frac{x^2}{1+\frac{x^2}{3}}=\frac{x^4}{1+\frac{x^4}{3}} \Rightarrow \frac{3}{3+x^2}=\frac{3x^2}{3+x^4} \Rightarrow 9+3x^4=9x^2+3x^4 \Rightarrow x^2=1$$

$\Rightarrow$ $x = 0, 1$ or -1

$\therefore$ *Number of values is equal to* **3. Ans.**

Illustration :

If $0 < \cos^{-1}x < 1$ *and* $1 + \cos^{-1}x + (\cos^{-1}x)^2 + \,.......\, \infty = 2$ *then* x *is equal to*

(A) $\frac{\pi}{4}$ (B) $\cos\frac{1}{2}$ (C) $\cos\frac{1}{\sqrt{2}}$ (D) $\frac{\pi}{6}$

Sol. *We have*

$1 + \cos^{-1}x + (\cos^{-1}x)^2 + \,.......\, \infty = 2$, *(which is an infinite geometric progression)*

$$\Rightarrow \frac{1}{1-\cos^{-1}x}=2 \Rightarrow \cos^{-1}x=\frac{1}{2} \Rightarrow x=\cos\frac{1}{2}\ \textbf{Ans.}$$

Illustration :

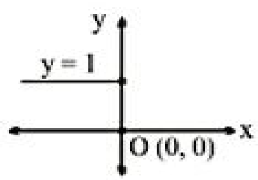
Let $f(x) = \frac{|\cos^{-1}(\text{sgn}\,x)|}{\cos^{-1}(\text{sgn}\,x)}$ *then which of the following is (are) not correct?*

*[**Note:** sgn x denotes signum function of x.]*

(A) Range of f(x) contains no integer.
(B) Graph of f(x) is symmetric about y-axis.
(C) The equation f(x) = 0 has two distinct real solutions.
(D) Inverse of f(x) is not defined.

Sol. *Clearly,* $D_f = (-\infty, 0]$

Now, $f(x) = 1\ \forall x \in (-\infty, 0]$ **Ans.**



Practice Problem

Q.1 Find domain and range of the following

(a) $\cos^{-1}[x]$ (b) $\sin^{-1}\{x\}$ (c) $\cot^{-1}(\operatorname{sgn} x)$ (d) $\cot^{-1}\log_{\frac{4}{5}}(5x^2-8x+4)$

(where [x] denotes the greatest integer function and {x} denotes the fractional part function.)

Q.2 Find the value of

(a) $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$ (b) $\sin(\tan^{-1}2)+\cos(\tan^{-1}2)$

Q.3 State which of the statements are True or False ?

(i) $y=\operatorname{sgn}(\cot^{-1}x)$ and $y=1$ identical. (ii) $e^{\ln(\tan^{-1}x)}$ and $\tan^{-1}x$ identical.

(iii) $e^{\ln(\cot^{-1}x)}$ and $\cot^{-1}x$ identical.

Q.4 Find domain of definition the functions $f(x)=\sqrt{\cos(\sin x)}+\sin^{-1}\frac{1+x^2}{2x}$

Q.5 The range of the function $y=\left(\frac{\cos^{-1}(3x-1)}{\pi}+1\right)^2$ is

(A) [1,4] (B) $[0,\pi]$ (C) $[1,\pi]$ (D) $[0,\pi^2]$

Q.6 If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then possible value(s) of a satisfying the equation

$x^{100}+y^{100}+z^{100}+\frac{a^2}{x^{50}+y^{50}+z^{50}}=\frac{10a}{3}$ are

(A) 1 (B) 4 (C) 9 (D) 16

Answer key

Q.1 (a) $D: x\in[-1,2)$ and $R\in\left\{0,\frac{\pi}{2},\pi\right\}$ (b) $D: x\in R$; $R:[0,\pi/2)$

(c) $D: x\in R$; $R:\left\{\frac{\pi}{2},\frac{\pi}{4},\frac{3\pi}{4}\right\}$ (d) $D: x\in R$; $R:\left[\frac{\pi}{4},\pi\right)$

Q.2 (a) $\frac{3-\sqrt{5}}{2}$, (b) $\frac{3}{\sqrt{5}}$ Q.3 (i) True ; (ii) False ; (iii) True

Q.4 $\{-1, 1\}$ Q.5 A Q.6 A,C

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTION :

Property-1 :

(i) $\sin(\sin^{-1} x) = x \ , \ -1 \le x \le 1$

(ii) $\cos(\cos^{-1} x) = x \ , \ -1 \le x \le 1$

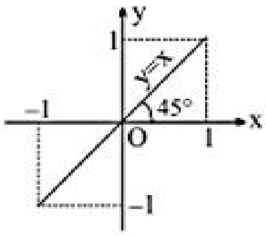
(iii) $\tan(\tan^{-1} x) = x \ , \ x \in R$

(iv) $\cot(\cot^{-1} x) = x \ , \ x \in R$

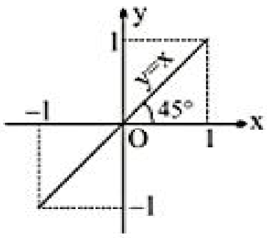
(v) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x \ , \ |x| \ge 1$

(vi) $\sec(\sec^{-1} x) = x \ , \ |x| \ge 1$

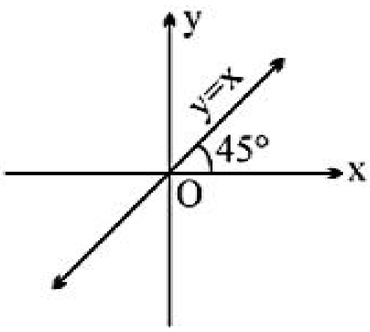
(1) $y=\sin(\sin^{-1}x)=x, x\in[-1,1], y\in[-1,1]$, y is aperiodic.



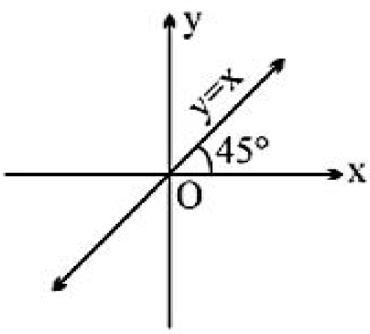
(2) $y=\cos(\cos^{-1}x)=x, x\in[-1,1], y\in[-1,1]$, y is aperiodic.



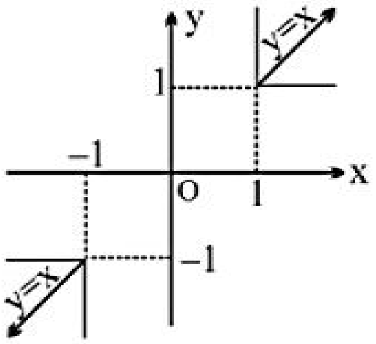
(3) $y=\tan(\tan^{-1}x)=x, x\in R, y\in R$, y is aperiodic.



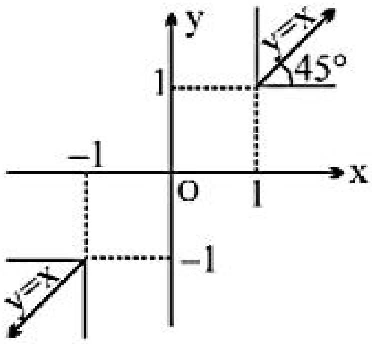
(4) $y=\cot(\cot^{-1}x)=x, x\in R, y\in R$, y is aperiodic.



(5) $y = \text{cosec}(\text{cosec}^{-1}x) = x, |x| \geq 1, |y| \geq 1,$ y is aperiodic.



(6) $y = \sec(\sec^{-1}x) = x, |x| \geq 1, |y| \geq 1,$ y is aperiodic.



Note that: (1, 2); (3, 4) and (5, 6) are identical function.

(vii) $\sin^{-1}(\sin x) = x\ ,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

(vii) $\cos^{-1}(\cos x) = x\ ;\ 0 \leq x \leq \pi$

(ix) $\tan^{-1}(\tan x) = x\ ;\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

(x) $\cot^{-1}(\cot x) = x\ ,\ 0 < x < \pi$

(xi) $\text{cosec}^{-1}(\text{cosec } x) = x\ ;\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0$

(xii) $\sec^{-1}(\sec x) = x\ ;\ 0 \leq x \leq \pi, x \neq \frac{\pi}{2}$

(7) $y = \sin^{-1}(\sin x), x \in R, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, Periodic with period 2π

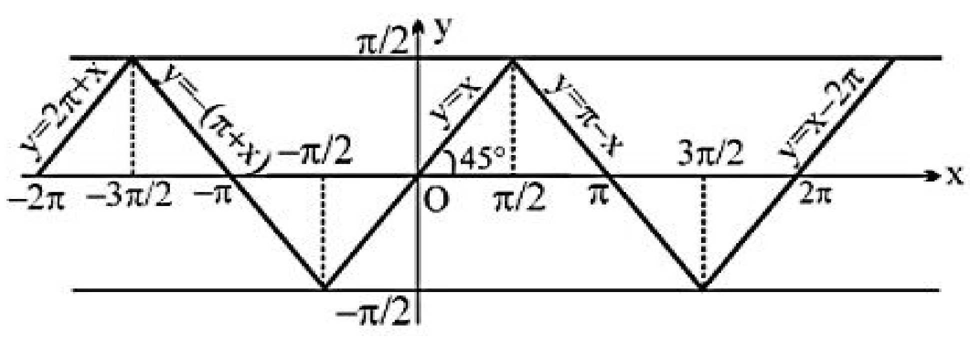


Illustration :

Find the value of following

(a) $\sin^{-1}(\sin 1)$ (b) $\sin^{-1}(\sin 2)$ (c) $\sin^{-1}(\sin 3)$

(d) $\sin^{-1}(\sin 4)$ (e) $\sin^{-1}(\sin 5)$ (f) $\sin^{-1}(\sin 10)$

Sol. (a) $\sin^{-1}(\sin 1) = 1 \quad (\because 1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(b) $\sin^{-1}(\sin 2) \neq 2 \quad (\because 2 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

$\Rightarrow \sin^{-1}(\sin 2) = \sin^{-1}(\sin(\pi - 2)) = \pi - 2 \quad (\because \pi - 2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(c) $\sin^{-1}(\sin 3) = \sin^{-1}(\sin(\pi - 3)) = \pi - 3 \quad (\because \pi - 3 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(d) $\sin^{-1}(\sin 4) = \sin^{-1}(\sin(\pi - 4)) = \pi - 4 \quad (\because \pi - 4 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(e) $\sin^{-1}(\sin 5) = \sin^{-1}(\sin(5 - 2\pi)) = 5 - 2\pi \quad (\because 5 - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(f) $\sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = 3\pi - 10 \quad (\because 3\pi - 10 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(8) $y = \cos^{-1}(\cos x) = x, x \in \mathbb{R}, y \in [0, \pi]$, periodic with period 2π

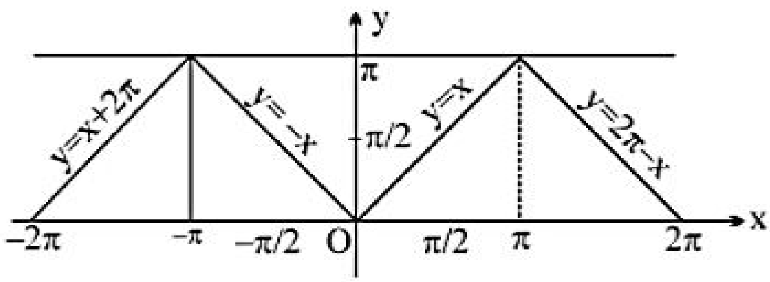


Illustration :

Find the value of following

(a) $\cos^{-1}(\cos 1)$ (b) $\cos^{-1}(\cos 2)$ (c) $\cos^{-1}(\cos 3)$

(d) $\cos^{-1}(\cos 4)$ (e) $\cos^{-1}(\cos 5)$ (f) $\cos^{-1}(\cos 10)$

Sol. (a) $\cos^{-1}(\cos 1) = 1$; $(\because 1 \in [0, \pi])$

(b) $\cos^{-1}(\cos 2) = 2$; $(\because 2 \in [0, \pi])$

(c) $\cos^{-1}(\cos 3) = 3$; $(\because 3 \in [0, \pi])$

(d) $\cos^{-1}(\cos 4) = \cos^{-1}(\cos(2\pi - 4)) = 2\pi - 4$; $(\because 2\pi - 4 \in [0, \pi])$

(e) $\cos^{-1}(\cos 5) = \cos^{-1}(\cos(2\pi - 5)) = 2\pi - 5$; $(\because 2\pi - 5 \in [0, \pi])$

(f) $\cos^{-1}(\cos 10) = \cos^{-1}(\cos(10 - 3\pi)) = 10 - 3\pi$. $(\because 10 - 3\pi \in [0, \pi])$

(9) $y = \tan^{-1}(\tan x) = x, x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}, \ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,
periodic with period π

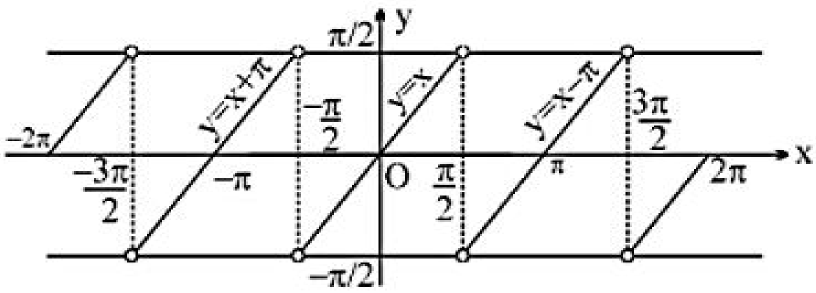


Illustration :

Find the value of following

(a) $\tan^{-1}(\tan 1)$ *(b) $\tan^{-1}(\tan 2)$* *(c) $\tan^{-1}(\tan 3)$*
(d) $\tan^{-1}(\tan 4)$ *(e) $\tan^{-1}(\tan 5)$* *(f) $\tan^{-1}(\tan 10)$*

Sol. *(a)* $\tan^{-1}(\tan 1) = 1 \quad \left(\because 1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(b) $\tan^{-1}(\tan 2) \neq 2 \quad \left(\because 2 \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$\Rightarrow \tan^{-1}(\tan 2) = \tan^{-1}(\tan(\pi - 2)) = \pi - 2 \qquad \left(\because \pi - 2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

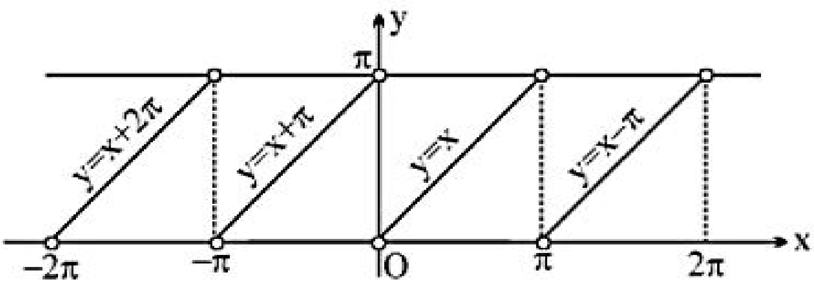
(c) $\tan^{-1}(\tan 3) = \tan^{-1}(\tan(\pi - 3)) = \pi - 3 \qquad \left(\because \pi - 3 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(d) $\tan^{-1}(\tan 4) = \tan^{-1}(\tan(\pi - 4)) = \pi - 4 \qquad \left(\because \pi - 4 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(e) $\tan^{-1}(\tan 5) = \tan^{-1}(\tan(5 - 2\pi)) = 5 - 2\pi \qquad \left(\because 5 - 2\pi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(f) $\tan^{-1}(\tan 10) = \tan^{-1}(\tan(3\pi - 10)) = 3\pi - 10 \qquad \left(\because 3\pi - 10 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

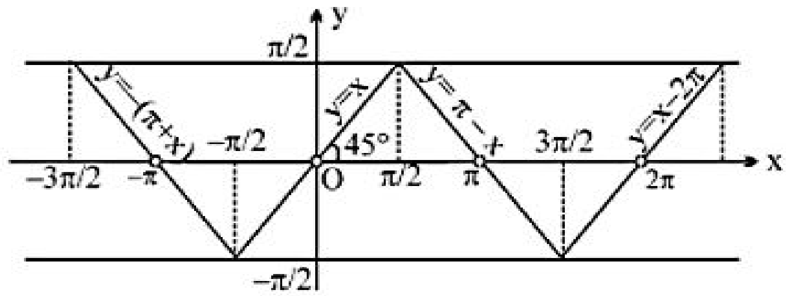
(10) $y = \cot^{-1}(\cot x) = x, x \in R - \{n\pi\}, \ y \in (0, \pi)$, periodic with π



(11) $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x), x \in R - \{n\pi, n \in I\}$,

$$y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$$

y is periodic with period 2π



(12) $y = \sec^{-1}(\sec x) = x$, y is periodic ;

$x \in R - \left\{(2n-1)\frac{\pi}{2}\ n \in I\right\}, y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

with period 2π

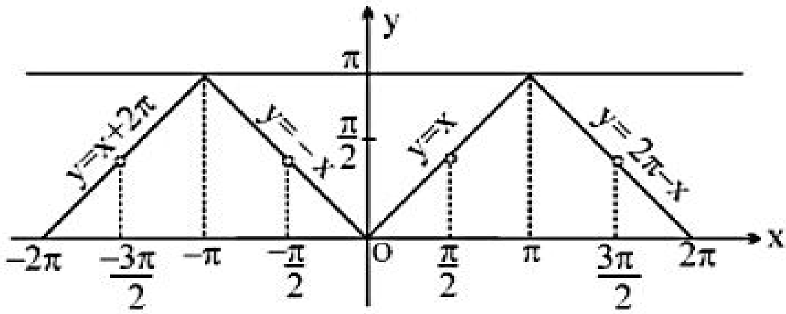


Illustration :

Find the integral solution of inequality $6x^2 - 5x < \cos^{-1}(\cos 5) - 2\sin^{-1}(\sin 3)$.

Sol. $6x^2 - 5x < \cos^{-1}(\cos(2\pi - 5)) - 2\sin^{-1}(\sin(\pi - 3))$

$6x^2 - 5x < 2\pi - 5 - 2\pi + 6$

$6x^2 - 5x < 1$

$6x^2 - 5x - 1 < 0$

$(6x + 1)(x - 1) < 0$

$\Rightarrow x \in \left(-\frac{1}{6}, 1\right)$

Integral solution is $x = 0$ *Ans.* *[0]*

Illustration :

If $\sin^{-1}(\sin 9) - \cos^{-1}(\cos 15)$ *can be written in the form* $a\pi - b$, *then find the value of* $a + b$. $(a, b \in N)$.

Sol. $\sin^{-1}(\sin 9) = \sin^{-1}\sin(3\pi - 9) = 3\pi - 9$ $\left(\because 3\pi - 9 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

$\cos^{-1}(\cos 15) = \cos^{-1}(\cos(15 - 4\pi)) = 15 - 4\pi$ $(\because 15 - 4\pi \in [0, \pi])$

$\Rightarrow \sin^{-1}(\sin 9) - \cos^{-1}(\cos 15) = (3\pi - 9) - (15 - 4\pi)$
$= 7\pi - 24 \Rightarrow a = 7, b = 24$
$a + b = 7 + 24 = 31.$ *Ans. 31*

Illustration :

Find the value of following

(a) $\sin^{-1}\sin\left(\frac{13\pi}{11}\right)$ (b) $\cos^{-1}\left(\sin\left(-\frac{\pi}{4}\right)\right)$ (c) $\sin^{-1}\left(\cos\frac{33\pi}{10}\right)$

Sol. (a) $\sin^{-1}\sin\left(\frac{13\pi}{11}\right) = \sin^{-1}\sin\left(\pi + \frac{2\pi}{11}\right) = \sin^{-1}\left(-\sin\left(\frac{2\pi}{11}\right)\right) = \sin^{-1}\sin\left(-\frac{2\pi}{11}\right) = -\frac{2\pi}{11}$ **Ans.**

(b) $\cos^{-1}\sin\left(-\frac{\pi}{4}\right) = \cos^{-1}\cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \cos^{-1}\cos\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4}$ **Ans.**

(c) $\sin^{-1}\cos\left(\frac{33\pi}{10}\right) = \sin^{-1}\cos\frac{13\pi}{10} = \sin^{-1}\left(-\cos\frac{3\pi}{10}\right) = \sin^{-1}\left(-\sin\left(\frac{5\pi}{10} - \frac{3\pi}{10}\right)\right)$

$$= \sin^{-1}\left(-\sin\frac{\pi}{5}\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{5}\right)\right) = -\frac{\pi}{5}$$ **Ans.**

Illustration :

If $\sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$

can be written in the from of $\frac{a\pi}{b}$ *(where a, b* $\in$ *N) then find the minimum value of a + b.*

Sol. $\sin^{-1}\left(\sin\left(\frac{33\pi}{7}\right)\right) = \sin^{-1}\left(\sin\left(5\pi - \frac{2\pi}{7}\right)\right) = \sin^{-1}\left(\sin\left(\frac{2\pi}{7}\right)\right) = \frac{2\pi}{7}$

$\cos^{-1}\left(\cos\left(\frac{46\pi}{7}\right)\right) = \cos^{-1}\left(\cos\left(6\pi + \frac{4\pi}{7}\right)\right) = \cos^{-1}\left(\cos\left(\frac{4\pi}{7}\right)\right) = \frac{4\pi}{7}$

$\tan^{-1}\left(-\tan\left(\frac{13\pi}{8}\right)\right) = \tan^{-1}\left(\tan\left(2\pi - \frac{13\pi}{8}\right)\right) = \tan^{-1}\left(\tan\left(\frac{3\pi}{8}\right)\right) = \frac{3\pi}{8}$

$\cot^{-1}\left(\cot\left(\frac{-19\pi}{8}\right)\right) = \cot^{-1}\left(\cot\left(3\pi - \frac{19\pi}{8}\right)\right) = \cot^{-1}\left(\cot\left(\frac{5\pi}{8}\right)\right) = \frac{5\pi}{8}$

$\Rightarrow \sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$

$= \frac{2\pi}{7} + \frac{4\pi}{7} + \frac{3\pi}{8} + \frac{5\pi}{8} = \frac{6\pi}{7} + \pi = \frac{13\pi}{7}$

$\Rightarrow a = 13, b = 7 \Rightarrow a + b = 13 + 7 = 20$ **Ans. 20**

Illustration :

The smallest positive integral value of n for which

$$(n-2)x^2 + 8x + n + 4 > \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)\ \forall x \in R, \text{ is}$$

(A) 4 (B) 5 (C) 6 (D) 7

Sol. We have

$\sin^{-1}(\sin 12) + \cos^{-1}(\cos 12) = -(4\pi - 12) + (4\pi - 12) = 0$

$\therefore \quad (n-2)x^2 + 8x + n + 4 > 0 \ \ \forall x \in R$

$\Rightarrow \quad (n-2) > 0 \Rightarrow n \geq 3$ and $(8)^2 - 4(n-2)(n+4) < 0$ or $n^2 + 2n - 24 > 0$

$\Rightarrow \quad n > 4 \Rightarrow n \geq 5$

So, $n_{smallest} = 5$. **Ans.**

Illustration :

The product of all real values of x satisfying the equation

$$\sin^{-1}\cos\left(\frac{2x^2+10|x|+4}{x^2+5|x|+3}\right) = \cot\left(\cot^{-1}\left(\frac{2-18|x|}{9|x|}\right)\right) + \frac{\pi}{2} \text{ is}$$

(A) 9 (B) – 9 (C) – 3 (D) – 1

Sol. $$\frac{\pi}{2} - \cos^{-1}\cos\left(\underbrace{\frac{2(x^2+5|x|+3)-2}{x^2+5|x|+3}}_{0<\downarrow<2}\right) = \cot\cot^{-1}\left(\frac{2}{9|x|} - 2\right) + \frac{\pi}{2}$$

$$\frac{\pi}{2} - 2 + \frac{2}{x^2+5|x|+3} = \frac{2}{9|x|} - 2 + \frac{\pi}{2}$$

$\Rightarrow \quad |x|^2 - 4|x| + 3 = 0$

$\quad |x| = 1, 3 \quad \Rightarrow \quad x = \pm 1, \pm 3$

$\Rightarrow \quad$ Product $= (1)(-1)(3)(-3) = 9$ **Ans.**

Illustration :

Which of the following is/are correct?

(A) $\cos(\cos(\cos^{-1} 1)) < \sin(\sin^{-1}(\sin(\pi - 1)) < \sin(\cos^{-1}(\cos(2\pi - 2))$

(B) $\cos(\cos(\cos^{-1} 1)) < \sin(\cos^{-1}(\cos(2\pi - 2))) < \sin(\sin^{-1}(\sin(\pi - 1))) < \tan(\cot^{-1}(\cot 1))$

(C) $\sum_{t=1}^{5000}\cos^{-1}(\cos(2t\pi - 1)) = \sum_{t=1}^{2500}\cot^{-1}(\cot(t\pi + 2))$ where $t \in I$

(D) $\cot^{-1}\cot\,\mathrm{cosec}^{-1}\mathrm{cosec}\,\sec^{-1}\sec\tan\tan^{-1}\cos\cos^{-1}\sin^{-1}\sin 4 = 4 - \pi$

Sol. ***for (A) and (B)***

$\cos(\cos^{-1} 1) = 1 \quad \Rightarrow \quad \cos(\cos(\cos^{-1} 1)) = \cos 1$

$\sin^{-1}(\sin(\pi - 1)) = \pi - (\pi - 1) = 1 \quad \Rightarrow \quad \sin(\sin^{-1}(\sin(\pi - 1))) = \sin 1$

$\cos^{-1}(\cos(2\pi - 2)) = \cos^{-1}(\cos 2) = 2 \quad \Rightarrow \quad \sin(\cos^{-1}(\cos(2\pi - 2))) = \sin 2$

$\tan(\cot^{-1}(\cot 1)) = \tan 1$

It is easy to compare $\cos 1,\ \sin 1,\ \sin 2,\ \tan 1$

$\cos 1 < \sin 1 < \sin 2 < \tan 1 \Rightarrow$ ***(A)** is correct*

for (C)

$\cos^{-1}\cos x$ is periodic and even

$\cos^{-1}\cos(2t\pi - 1) = \cos^{-1}(\cos 1) = 1\ (t \in I)$

$$\sum_{t=1}^{5000} \cos^{-1}\cos(2t\pi - 1) = 5000$$

now $\cot^{-1}\cot(t\pi + 2) = 2$ *[$\cot^{-1}\cot x$ is periodic with period π]*

$$\therefore \quad \sum_{t=1}^{2500} \cot^{-1}\cot(t\pi + 2) = 5000 \Rightarrow$$ ***(C)** is correct*

(D) $\sin^{-1}\sin 4 = \pi - 4$

$\cos\cos^{-1}(\pi - 4) = \pi - 4$

$\tan\tan^{-1}(4 - \pi) = \pi - 4$

$\sec^{-1}\sec(\pi - 4) = 4 - \pi$

$\operatorname{cosec}^{-1}\operatorname{cosec}(4 - \pi) = 4 - \pi$

$\cot^{-1}\cot(4 - \pi) = 4 - \pi \quad \Rightarrow$ ***(D)** is correct*

Practice Problem

Q.1 Find the value of following

(i) $\sin^{-1}[\cos 2\cot^{-1}(\sqrt{2} - 1)]$ (ii) $\sin^{-1}(\sin 7) + \cos^{-1}\cos(13)$

(iii) $\sin^{-1}\left(\sin\dfrac{10\pi}{7}\right)$ (iv) $\cos^{-1}\left(\sin\left(-\dfrac{\pi}{9}\right)\right)$

Q.2 If $3 \le a < 4$ then the value of $\sin^{-1}(\sin[a]) + \tan^{-1}(\tan[a]) + \sec^{-1}(\sec[a])$, where [x] denotes greatest integer function less than or equal to x, is equal to

(A) 3 (B) $2\pi - 9$ (C) $2\pi - 3$ (D) $9 - 2\pi$

Q.3 The value of $\sin^{-1}(\cos 2) - \cos^{-1}(\sin 2) + \tan^{-1}(\cot 4) - \cot^{-1}(\tan 4) + \sec^{-1}(\operatorname{cosec} 6) - \operatorname{cosec}^{-1}(\sec 6)$ is

(A) 0 (B) 3π (C) $8 - 3\pi$ (D) $5\pi - 16$

Paragraph for question nos. 4 to 6

For $x \in \left(0, \dfrac{\pi}{4}\right)$,

Let $S_n = \sum_{r=1}^{2n} \sin\left(\sin^{-1} x^{3r-2}\right)$, $C_n = \sum_{r=1}^{2n} \cos\left(\cos^{-1} x^{3r-1}\right)$ and $T_n = \sum_{r=1}^{2n} \tan\left(\tan^{-1} x^{3r}\right)$

where $n \in N$ and $n \ge 3$.

Q.4 The correct order of S_n, C_n and T_n is given by

(A) $S_n > T_n > C_n$ (B) $S_n < C_n < T_n$ (C) $S_n < T_n < C_n$ (D) $S_n > C_n > T_n$

Q.5 The value of $\lim_{n\to\infty}(S_n + C_n + T_n)$ is equal to

(A) $\frac{1}{1-x}$ (B) $\frac{x}{1-x}$ (C) $\frac{1}{1-x}$ (D) $\frac{x}{1+x}$

Q.6 The value of 'x' for which $S_n = C_n + T_n$, is

(A) $\sin\frac{\pi}{5}$ (B) $2\sin\frac{\pi}{5}$ (C) $2\sin\frac{\pi}{10}$ (D) $\sin\frac{\pi}{10}$

Answer key

Q.1 (i) $-\frac{\pi}{4}$; (ii) $20-6\pi$; (iii) $-\frac{3\pi}{7}$; (iv) $\frac{11\pi}{18}$

Q.2 A Q.3 D Q.4 D Q.5 B Q.6 C

Property-2 :

(1) $\operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$; $|x| \geq 1$

(2) $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$, $|x| \leq 1$, $x \neq 0$

(3) $\sec^{-1}x = \cos^{-1}\frac{1}{x}$; $|x| \geq 1$

(4) $\cos^{-1}x = \sec^{-1}\frac{1}{x}$, $|x| \leq 1$, $x \neq 0$

(5) $\cot^{-1}x = \tan^{-1}\frac{1}{x}$; $x > 0$

$= \pi + \tan^{-1}\frac{1}{x}$; $x < 0$

Note : (i) $\operatorname{cosec}^{-1}x$ and $\sin^{-1}\frac{1}{x}$ are identical function.

(ii) $\sin^{-1}x$ and $\operatorname{cosec}^{-1}\frac{1}{x}$ are not identical because domain of $\sin^{-1}x$ and $\operatorname{cosec}^{-1}\frac{1}{x}$ is not equal.

(iii) $\sec^{-1}x$ and $\cos^{-1}\frac{1}{x}$ are identical function.

(iv) $\cos^{-1}x$ and $\sec^{-1}\frac{1}{x}$ are not identical because domain of $\cos^{-1}x$ and $\sec^{-1}\frac{1}{x}$ is not equal.

Illustration :

Are $tan(cot^{-1}x)$ and $cot(tan^{-1}x)$ are identical ?

Sol. *[True], as both functions have same graph.*

Illustration :

Find the value(s) of x satisfying the equation

$$cot^{-1}\frac{x^2-1}{2x}+tan^{-1}\frac{2x}{x^2-1}=\frac{2x}{3}$$

Sol. Case-(i) $\frac{x^2-1}{2x}>0$

$$cot^{-1}\frac{x^2-1}{2x}+tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3} \quad (\because cot^{-1}x=tan^{-1}\frac{1}{x};\ x>0)$$

$$\Rightarrow tan^{-1}\frac{2x}{x^2-1}+tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3} \quad \Rightarrow \quad tan^{-1}\frac{2x}{x^2-1}=\frac{\pi}{3}$$

$$\Rightarrow \frac{2x}{x^2-1}=\sqrt{3} \quad \Rightarrow \sqrt{3}\,x^2-2x-\sqrt{3}=0 \Rightarrow x=\frac{2\pm 4}{2\sqrt{3}} \quad \Rightarrow \quad x=\frac{-1}{\sqrt{3}},\ \sqrt{3}$$

Case-(ii)

$$\frac{x^2-1}{2x}<0$$

$$cot^{-1}\frac{x^2-1}{2x}+tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi+tan^{-1}\frac{2x}{x^2-1}+tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3} \quad (\because cot^{-1}x=\pi+tan^{-1}\frac{1}{x};\ x<0)$$

$$\Rightarrow \quad tan^{-1}\frac{2x}{x^2-1}=\frac{-\pi}{6} \quad \Rightarrow \quad \frac{2x}{x^2-1}=\frac{-1}{\sqrt{3}} \quad \Rightarrow \quad x^2+2\sqrt{3}\,x-1=0$$

$$\Rightarrow \quad x=-\sqrt{3}\pm 2 \quad \Rightarrow \quad x=-(2+\sqrt{3}),\ 2-\sqrt{3}$$

From case (i) and (ii)

$$\Rightarrow \quad x=\sqrt{3}\ ,-\frac{1}{\sqrt{3}}\ ,-(2+\sqrt{3}),\ (2-\sqrt{3}) \quad \textit{Ans.}$$

Property-3 :

(i) $\sin^{-1}(-x) = -\sin^{-1}x \quad , \quad -1 \le x \le 1$
(ii) $\tan^{-1}(-x) = -\tan^{-1}x \quad , \quad x \in R$
(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x \quad , \quad -1 \le x \le 1$
(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x \quad , \quad x \in R$
(v) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x \quad , \quad |x| \ge 1$
(vi) $\sec^{-1}(-x) = \pi - \sec^{-1}x \quad , \quad |x| \ge 1$

Property-4 :

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \quad -1 \le x \le 1$ **(ii)** $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad x \in R$

(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, \quad |x| \ge 1$

Illustration :

Find the value of x if

(a) $4\sin^{-1}x + \cos^{-1}x = \frac{3\pi}{4}$; (b) $5\tan^{-1}x + 3\cot^{-1}x = \frac{7\pi}{4}$

Sol.

(a) $4\sin^{-1}x + \frac{\pi}{2} - \sin^{-1}x = \frac{3\pi}{4}$

$\Rightarrow 3\sin^{-1}x = \frac{\pi}{4} \quad \Rightarrow \quad \sin^{-1}x = \frac{\pi}{12} \quad \Rightarrow \quad x = \sin\frac{\pi}{12} \quad \Rightarrow \quad x = \frac{\sqrt{3}-1}{2\sqrt{2}}$ ***Ans.***

(b) $5\tan^{-1}x + 3\left(\frac{\pi}{2} - \tan^{-1}x\right) = \frac{7\pi}{4}$

$2\tan^{-1}x = \frac{7\pi}{4} - \frac{3\pi}{2} \quad \Rightarrow \quad 2\tan^{-1}x = \frac{\pi}{4} \quad \Rightarrow \quad \tan^{-1}x = \frac{\pi}{8}$

$\Rightarrow x = \tan\frac{\pi}{8} \quad \Rightarrow \quad x = \sqrt{2}-1$ ***Ans.***

Illustration :

Find the maximum and minimum values of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$

Sol. $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = (\sin^{-1}x + \cos^{-1}x)((\sin^{-1}x)^2 + (\cos^{-1}x)^2 - \sin^{-1}x \cdot \cos^{-1}x)$

$= \frac{\pi}{2}((\sin^{-1}x) + (\cos^{-1}x))^2 - 3\sin^{-1}x \cdot \cos^{-1}x)$

$= \frac{\pi}{2}\left[\left(\frac{\pi}{2}\right)^2 - 3\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)\right] = \frac{\pi}{2}\left[\frac{\pi^2}{4} - \frac{3\pi}{2}\sin^{-1}x + 3(\sin^{-1}x)^2\right]$

$$= \frac{3\pi}{2}\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x + \frac{\pi^2}{12}\right] \quad = \frac{3\pi}{2}\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{48}\right]$$

Maximum value occurs when $\sin^{-1}x = -\frac{\pi}{2}$

$$\Rightarrow \textit{Maximum value} = \frac{3\pi}{2}\left[\left(\frac{-\pi}{2} - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{48}\right] = \frac{3\pi}{2}\cdot\frac{7\pi^2}{12} = \frac{7\pi^3}{8}$$

Minimum value occurs when $\sin^{-1}x = \frac{\pi}{4}$

$$\Rightarrow \textit{Minimum value} = \frac{3\pi}{2}\cdot\left[\frac{\pi^2}{48}\right] = \frac{\pi^3}{32}$$

Illustration :

Find the range of $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$.

Sol. $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$

domain : $x \in [-1, 1]$

$$\Rightarrow f(x) = \frac{\pi}{2} + [\tan^{-1}-1, \tan^{-1}1] = \frac{\pi}{2} + \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

Property-5 :

(1) $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

Proof:

Let $\tan^{-1}x = A$ and $\tan^{-1}y = B$, where $A, B \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{x+y}{1-xy}$

$$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}\tan(A+B)$$

$= \tan^{-1}\tan\alpha$, where $\alpha \in (-\pi, \pi)$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}(\tan\alpha)$$

$$=\begin{cases}\alpha+\pi, & -\pi<\alpha<-\frac{\pi}{2}\\ \alpha, & -\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}\\ \alpha-\pi, & \frac{\pi}{2}<\alpha<\pi\end{cases}=\begin{cases}\tan^{-1}x+\tan^{-1}y+\pi, & -\pi<\tan^{-1}x+\tan^{-1}y<-\frac{\pi}{2}\\ \tan^{-1}x+\tan^{-1}y, & -\frac{\pi}{2}\le\tan^{-1}x+\tan^{-1}y\le\frac{\pi}{2}\\ \tan^{-1}x+\tan^{-1}y-\pi, & \frac{\pi}{2}<\tan^{-1}x+\tan^{-1}y<\pi\end{cases}$$

Case-I :

$$-\pi<\tan^{-1}x+\tan^{-1}y<-\frac{\pi}{2} \quad\Rightarrow\quad x<0, y<0$$

Also, $\tan^{-1}x<-\frac{\pi}{2}-\tan^{-1}y$

$$\Rightarrow\quad \tan^{-1}x<-\left(\frac{\pi}{2}-\tan^{-1}(-y)\right)\Rightarrow\ x<-\left(-\frac{1}{y}\right)\Rightarrow\ x<\frac{1}{y}\Rightarrow\ xy>1$$

Case-II :

$$\frac{\pi}{2}<\tan^{-1}x+\tan^{-1}y<\pi\ \Rightarrow\ x, y>0$$

Also, $\tan^{-1}x>\frac{\pi}{2}-\tan^{-1}y \quad\Rightarrow\ \tan^{-1}x>\tan^{-1}\frac{1}{y}\Rightarrow x>\frac{1}{y}\ \Rightarrow\ xy>1$

Case-III :

$$-\frac{\pi}{2}\le\tan^{-1}x+\tan^{-1}y\le\pi/2\ \Rightarrow\ xy<1$$

(2) $x>0$ and $y>0$, $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}$ (with no other restriction)

(Remember)

(i) $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$

(ii) $\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

(iii) $\dfrac{\tan^{-1}1+\tan^{-1}2+\tan^{-1}3}{\cot^{-1}1+\cot^{-1}2+\cot^{-1}3}=2$

Sol.(i) $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\tan^{-1}1+\left(\pi+\tan^{-1}\frac{2+3}{1-2.3}\right)$

$$=\tan^{-1}1+(\pi+\tan^{-1}(-1))$$

$$=\frac{\pi}{4}+\pi-\frac{\pi}{4}=\pi$$

(ii) $$\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3} = \tan^{-1}1+\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$= \tan^{-1}1+\tan^{-1}\left(\frac{5}{5}\right) = \tan^{-1}1+\tan^{-1}1 = \frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$$

(iii) $$\frac{\tan^{-1}1+\tan^{-1}2+\tan^{-1}3}{\cot^{-1}1+\cot^{-1}2+\cot^{-1}3}=\frac{\tan^{-1}1+\tan^{-1}2+\tan^{-1}3}{\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}} = \frac{\pi}{\left(\frac{\pi}{2}\right)} = 2$$

Illustration :

If $\tan^{-1}2 + \tan^{-1}4 = \cot^{-1}(\lambda)$ *then find* λ.

Sol. $$\tan^{-1}2 + \tan^{-1}4 = \pi + \tan^{-1}\left(\frac{2+4}{1-2.4}\right) = \pi + \tan^{-1}\left(\frac{6}{-7}\right)$$

$$= \pi - \tan^{-1}\frac{6}{7} = \pi - \cot^{-1}\frac{7}{6} = \cot^{-1}\left(-\frac{7}{6}\right) \Rightarrow \lambda = -\frac{7}{6} \text{ Ans.}$$

Illustration :

If $\alpha = \tan^{-1}5 - \tan^{-1}3 + \tan^{-1}\frac{7}{9}$ *and* $\beta = \tan^{-1}\frac{2}{11} + \cot^{-1}\frac{24}{7} + \tan^{-1}\frac{1}{3}$, *then*

(A) $\alpha = \beta$ *(B)* $\alpha > \beta$ *(C)* $\alpha < \beta$ *(D)* $\alpha + \beta = \pi/2$

Sol. $$\alpha = \tan^{-1}5 - \tan^{-1}3 + \tan^{-1}\frac{7}{9}$$

$$= \tan^{-1}\left(\frac{5-3}{1+5.3}\right) + \tan^{-1}\frac{7}{9} = \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{7}{9}$$

$$= \tan^{-1}\left(\frac{\frac{1}{8}+\frac{7}{9}}{1-\frac{1}{8}\cdot\frac{7}{9}}\right) = \tan^{-1}\left(\frac{65}{65}\right) = \tan^{-1}1 = \frac{\pi}{4}$$

$$\beta = \tan^{-1}\frac{2}{11} + \cot^{-1}\frac{24}{7} + \tan^{-1}\frac{1}{3}$$

$$= \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} + \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{2}{11} + \tan^{-1}\left(\frac{\frac{7}{24}+\frac{1}{3}}{1-\frac{7}{24}\cdot\frac{1}{3}}\right)$$

$$= tan^{-1}\frac{2}{11} + tan^{-1}\left(\frac{45}{65}\right) \quad = tan^{-1}\frac{2}{11} + tan^{-1}\frac{9}{13}$$

$$= tan^{-1}\left(\frac{\frac{2}{11}+\frac{9}{13}}{1-\frac{2}{11}\cdot\frac{9}{13}}\right) = tan^{-1}\left(\frac{125}{125}\right) = tan^{-1}\frac{\pi}{4} \Rightarrow \quad \alpha = \beta$$

Illustration :

Find the value of $cos^{-1}\sqrt{\frac{2}{3}} - cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}$.

Sol. $cos^{-1}\sqrt{\frac{2}{3}} - cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}} = tan^{-1}\frac{1}{\sqrt{2}} - tan^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{6}}\right) = tan^{-1}\frac{1}{\sqrt{2}} - tan^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{3}\cdot\sqrt{2}}\right)$

$$= tan^{-1}\frac{1}{\sqrt{2}} - (tan^{-1}\sqrt{3} - tan^{-1}\sqrt{2}) = cot^{-1}\sqrt{2} - tan^{-1}\sqrt{3} + tan^{-1}\sqrt{2}$$

$$= \frac{\pi}{2} - tan^{-1}\sqrt{3} = cot^{-1}\sqrt{3} = \frac{\pi}{6}\ \textbf{Ans.}$$

Illustration :

Find the value of $tan^{-1}\left(\frac{1}{2}tan\,2A\right) + tan^{-1}(cot\,A) + tan^{-1}(cot^3A)$, *for* $0 < A < \frac{\pi}{4}$.

Sol. *For* $0 < A < \frac{\pi}{4}$, $cot\,A > 1 \Rightarrow (cot\,A)(cot^3A) > 1$

Then $tan^{-1}\left(\frac{1}{2}tan\,2A\right) + tan^{-1}(cotA) + tan^{-1}(cot^3A)$

$$= tan^{-1}\left(\frac{tan\,A}{1-tan^2A}\right) + \pi + tan^{-1}\left(\frac{cot\,A+cot^3A}{1-cot^4A}\right)$$

$$= tan^{-1}\left(\frac{tan\,A}{1-tan^2A}\right) + \pi + tan^{-1}\left(\frac{cot\,A}{1-cot^2A}\right)$$

$$= tan^{-1}\left(\frac{tan\,A}{1-tan^2A}\right) + \pi + tan^{-1}\left(\frac{tan\,A}{tan^2A-1}\right) = \pi\ \textbf{Ans.}$$

Property-6 :

(I) $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \text{if } x \geq 0; y \geq 0 \text{ and } x^2 + y^2 \leq 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \text{if } x \geq 0; y \geq 0 \text{ and } x^2 + y^2 > 1 \end{cases}$

note that $x^2 + y^2 \leq 1 \Rightarrow 0 \leq \sin^{-1}x + \sin^{-1}y \leq \frac{\pi}{2}$

Let $\sin^{-1}x = \alpha$ and $\sin^{-1}y = \beta$; $\alpha, \beta \in \left[0, \frac{\pi}{2}\right]$

now $x^2 + y^2 \leq 1$

$\sin^2\alpha + \sin^2\beta \leq 1 \Rightarrow \sin^2\alpha \leq \cos^2\beta$

$\sin^2\alpha \leq \sin^2\left(\frac{\pi}{2} - \beta\right) \Rightarrow \alpha \leq \frac{\pi}{2} - \beta \Rightarrow \alpha + \beta \leq \frac{\pi}{2}$

$0 \leq \sin^{-1}x + \sin^{-1}y \leq \frac{\pi}{2}$]

and $x^2 + y^2 > 1 \Rightarrow \frac{\pi}{2} < \sin^{-1}x + \sin^{-1}y < \pi$

This formula should normally be used in establishing the identities.

e.g. find whether $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{12}{13}$ is acute / obtuse will be unduly difficult using the above.

However if we convert it into $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{12}{5}$ it becomes simple.

(II) ||| ly we have $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right), x > 0; y > 0$

and $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right), x > 0, y > 0, x < y$

Illustration :

Solve the equation $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$.

Sol. $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$.

$\sin^{-1}2x = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}x = \sin^{-1}\left[\frac{\sqrt{3}}{2}\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right]$

$$\Rightarrow 2x = \frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{x}{2}$$

$$\Rightarrow \left(\frac{5x}{2}\right)^2 = \frac{3}{4}(1-x^2) \Rightarrow 28x^2 = 3$$

$$\Rightarrow x = \sqrt{\frac{3}{28}} = \frac{1}{2}\sqrt{\frac{3}{7}} \text{ Ans.} \qquad \left(\because x = -\frac{1}{2}\sqrt{\frac{3}{7}} \text{ makes L.H.S. of (1) negative}\right)$$

Illustration :

If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ *then value of* $x^2 + y^2 + z^2 + 2xyz$ *is equal to*

(A) 1 *(B) –1* *(C) 0* *(D) 3*

Sol. $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi \Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$$\Rightarrow \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos^{-1}(-z)$$

$$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$$

$$\Rightarrow xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$$

squaring both sides

$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = \mathbf{1.}$

Property-7 :

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-(xy+yz+zx)}\right]$$

where $x > 0, y > 0, z > 0$ and $xy + yz + zx < 1$ and $xy < 1, yz < 1, zx < 1$

Solution

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y}{1-xy}\right) + \tan^{-1}z$$

$$= \tan^{-1}\left(\frac{\frac{x+y}{1-xy}+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right) = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-(x+y)z}\right) = \tan^{-1}\left(\frac{x+y+z-xyz}{1-(xy+yz+zx)}\right)$$

Practice Problem

Q.1 Find the minimum value of $(\sec^{-1}x)^2 + (\operatorname{cosec}^{-1}x)^2$.

Q.2 If two angles of a triangle are $\tan^{-1}(2)$ and $\tan^{-1}(3)$, then find the third angle.

Q.3 Find x satisfying $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cos^{-1}(x) = \frac{\pi}{4}$.

Q.4 Value of $\sin^{-1}\left(\frac{3}{\sqrt{73}}\right) + \cos^{-1}\left(\frac{11}{\sqrt{146}}\right) + \cot^{-1}\left(\sqrt{3}\right)$ is equal to

(A) π (B) $\pi/2$ (C) $5\pi/12$ (D) $\pi/3$

Q.5 $\cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right)$ is equal to

(A) $\frac{\pi}{3}$ for $x \in \left[\frac{1}{2}, 1\right]$ (B) $\frac{\pi}{3}$ for $x \in \left[0, \frac{1}{2}\right]$

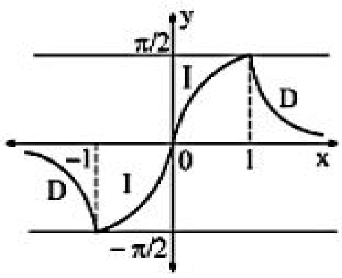
(C) $2\cos^{-1}x - \cos^{-1}\frac{1}{2}$ for $x \in \left[\frac{1}{2}, 1\right]$ (D) $2\cos^{-1}x - \cos^{-1}\frac{1}{2}$ for $x \in \left[0, \frac{1}{2}\right]$

Answer key

Q.1 $\frac{\pi^2}{8}$ Q.2 $\frac{\pi}{4}$ Q.3 $x = \frac{3}{\sqrt{10}}$ Q.4 C Q.5 A, D

SIMPLIFICATION & TRANSFORMATION OF INVERSE FUNCTIONS BY ELEMENTRY SUBSTITUTION AND THEIR GRAPHS :

(1) $$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & -1 \le x \le 1 \\ \pi - 2\tan^{-1}x & \text{if} \quad x \ge 1 \\ -\pi - 2\tan^{-1}x & x \le -1 \end{cases}$$



Proof :

Let $x = \tan\theta,\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \theta = \tan^{-1}x$

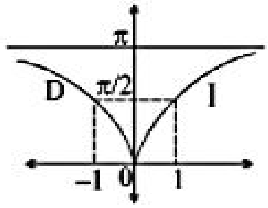
Now, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = \sin^{-1}(\sin\alpha)$, where $\alpha \in (-\pi, \pi)$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}(\sin\alpha)$$

$$= \begin{cases} -\alpha-\pi, & -\pi < \alpha < -\frac{\pi}{2} \\ \alpha, & -\frac{\pi}{2} \le \alpha \le -\frac{\pi}{2} \\ -\alpha+\pi, & \frac{\pi}{2} < \alpha < \pi \end{cases} = \begin{cases} -2\tan^{-1}x-\pi, & -\pi < 2\tan^{-1}x < -\frac{\pi}{2} \\ 2\tan^{-1}x, & -\frac{\pi}{2} \le 2\tan^{-1}x \le \frac{\pi}{2} \\ -2\tan^{-1}x+\pi, & \frac{\pi}{2} < 2\tan^{-1}x < \pi \end{cases}$$

$$= \begin{cases} -2\tan^{-1}x-\pi, & -\frac{\pi}{2} < \tan^{-1}x < -\frac{\pi}{4} \\ 2\tan^{-1}x, & -\frac{\pi}{4} \le \tan^{-1}x \le \frac{\pi}{4} \\ -2\tan^{-1}x+\pi, & \frac{\pi}{4} < \tan^{-1}x < \frac{\pi}{2} \end{cases} = \begin{cases} -2\tan^{-1}x-\pi, & x < -1 \\ 2\tan^{-1}x, & -1 \le x \le 1 \\ -2\tan^{-1}x+\pi, & x > 1 \end{cases}$$

(2) $\cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & x \ge 0 \\ -2\tan^{-1}x & x < 0 \end{cases}$



Proof :

Let $x = \tan\theta,\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \theta = \tan^{-1}x$

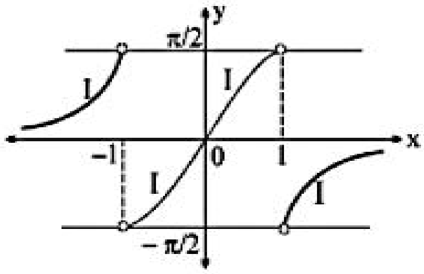
Now, $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = \cos^{-1}(\cos\alpha)$, where $\alpha \in (-\pi, \pi)$

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}(\cos\alpha)$$

$$= \begin{cases} -\alpha, & -\pi < \alpha < 0 \\ \alpha, & 0 \le \alpha < \pi \end{cases} = \begin{cases} -2\tan^{-1}x, & -\pi < 2\tan^{-1}x < 0 \\ 2\tan^{-1}x, & 0 \le 2\tan^{-1}x < \pi \end{cases}$$

$$= \begin{cases} -2\tan^{-1}x, & -\frac{\pi}{2} < \tan^{-1}x < 0 \\ 2\tan^{-1}x, & 0 \le \tan^{-1}x < \frac{\pi}{2} \end{cases} = \begin{cases} -2\tan^{-1}x, & x < 0 \\ 2\tan^{-1}x, & x \ge 0 \end{cases}$$

(3) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} \pi + 2\tan^{-1}x & x < -1 \\ 2\tan^{-1}x & -1 < x < 1 \\ 2\tan^{-1}x - \pi & x > 1 \end{cases}$



Proof:

Let $x = \tan\theta,\ \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \Rightarrow \theta = \tan^{-1}x$

Now, $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \tan^{-1}\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = \tan^{-1}(\tan\alpha)$, where $\alpha \in (-\pi, \pi)$

$\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \tan^{-1}(\tan\alpha)$

$$= \begin{cases} \alpha + \pi, & -\pi < \alpha < -\dfrac{\pi}{2} \\ \alpha, & -\dfrac{\pi}{2} \le \alpha \le \dfrac{\pi}{2} \\ \alpha - \pi, & \dfrac{\pi}{2} < \alpha < \pi \end{cases} \qquad = \begin{cases} 2\tan^{-1}x + \pi, & -\pi < 2\tan^{-1}x < -\dfrac{\pi}{2} \\ 2\tan^{-1}x, & -\dfrac{\pi}{2} \le 2\tan^{-1}x \le \dfrac{\pi}{2} \\ 2\tan^{-1}x - \pi, & \dfrac{\pi}{2} < 2\tan^{-1}x < \pi \end{cases}$$

$$= \begin{cases} 2\tan^{-1}x + \pi, & -\dfrac{\pi}{2} < \tan^{-1}x < -\dfrac{\pi}{4} \\ 2\tan^{-1}x, & -\dfrac{\pi}{4} \le \tan^{-1}x \le \dfrac{\pi}{4} \\ 2\tan^{-1}x - \pi, & \dfrac{\pi}{4} < \tan^{-1}x < \dfrac{\pi}{2} \end{cases} \qquad = \begin{cases} \pi + 2\tan^{-1}x & x < -1 \\ 2\tan^{-1}x & -1 \le x \le 1 \\ 2\tan^{-1}x - \pi & x > 1 \end{cases}$$

Highlights :-

(a) $f(x) = \sin^{-1}\dfrac{2x}{1+x^2} + 2\tan^{-1}x = \pi$ if $x \geq 1$

(b) $f(x) = \sin^{-1}\dfrac{2x}{1+x^2} + 2\tan^{-1}x = -\pi$ if $x \leq -1$

Illustration :

If $f(x) = \sin^{-1}\dfrac{2x}{1+x^2} + 2\tan^{-1}x$ *then find*

(a) $f(100)$ *(b)* $\cos(f(-10))$

Sol. *We know that*

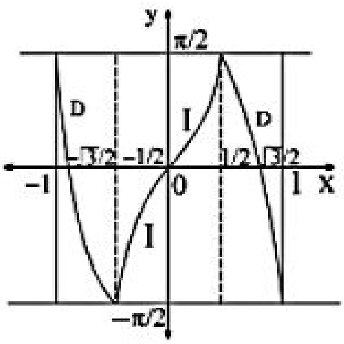
$$f(x) = \sin^{-1}\frac{2x}{1+x^2} + 2\tan^{-1}x = \pi \text{ if } x \geq 1$$

$$f(x) = \sin^{-1}\frac{2x}{1+x^2} + 2\tan^{-1}x = -\pi \text{ if } x \leq -1$$

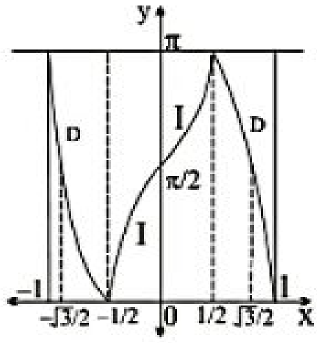
$\Rightarrow$ *(a)* $f(100) = \pi$

(b) $\cos(f(-10)) = \cos(-\pi) = \mathbf{-1}$

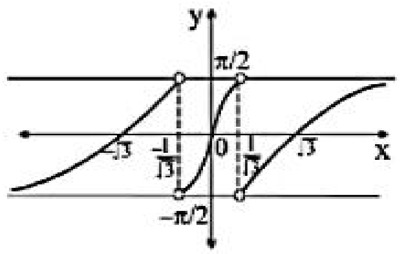
(4) $\sin^{-1}(3x-4x^3) = \begin{cases} -(\pi+3\sin^{-1}x) & \text{if } -1\le x\le -1/2 \\ 3\sin^{-1}x & \text{if } -1/2\le x\le 1/2 \\ \pi-3\sin^{-1}x & \text{if } 1/2\le x\le 1 \end{cases}$;



(5) $\cos^{-1}(4x^3-3x) = \begin{cases} 3\cos^{-1}x-2\pi & \text{if } -1\le x\le -1/2 \\ 2\pi-3\cos^{-1}x & \text{if } -1/2\le x\le 1/2 \\ 3\cos^{-1}x & \text{if } 1/2\le x\le 1 \end{cases}$;



(6) $\tan^{-1}\dfrac{3x-x^3}{1-3x^2} = \begin{cases} 3\tan^{-1}x & \text{if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ -\pi+3\tan^{-1}x & \text{if } x > \dfrac{1}{\sqrt{3}} \\ \pi+3\tan^{-1}x & \text{if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$



* **(4, 5, 6 to be proved similarly as 1, 2, 3)**

(C) IDENTITIES INVOLVING INVRSE TRIGONOMETRIC FUNCTIONS:

(I) $2\tan^{-1}\left(\tan\left(\dfrac{\pi}{4}-\alpha\right)\tan\dfrac{\beta}{2}\right) = \cos^{-1}\left(\dfrac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}\right)$

Proof: Let $x = \tan\left(\dfrac{\pi}{4}-\alpha\right)\tan\dfrac{\beta}{2}$

$$x = \frac{1-\tan\alpha}{1+\tan\alpha}\tan\frac{\beta}{2} \Rightarrow x = \left(\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\right)\frac{\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}}$$

$$x^2 = \left(\frac{1-\sin 2\alpha}{1+\sin 2\alpha}\right)\frac{\sin^2\frac{\beta}{2}}{\cos^2\frac{\beta}{2}}$$

$$x^2 = \frac{(1-\sin 2\alpha)(1-\cos\beta)}{(1+\sin 2\alpha)(1+\cos\beta)} = \frac{1-\sin 2\alpha-\cos\beta+\sin 2\alpha\cdot\cos\beta}{1+\sin 2\alpha+\cos\beta+\sin 2\alpha\cdot\cos\beta}$$

$$\frac{x^2-1}{x^2+1}=\frac{-(\sin 2\alpha+\cos\beta)}{(1+\sin 2\alpha.\cos\beta)}$$

(By applying componendo and dividendo)

$$\frac{1-x^2}{1+x^2}=\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha.\cos\beta}$$

We know that

$$2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\alpha\right)\tan\frac{\beta}{2}\right)=\cos^{-1}\left(\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}\right)$$

(II) $\tan^{-1}x=2\tan^{-1}[\text{cosec}(\tan^{-1}x)-\tan(\cot^{-1}x)]$ $\qquad (x\neq 0)$

Sol. R.H.S. $2\tan^{-1}[\text{cosec}(\tan^{-1}x)-\tan(\cot^{-1}x)]$

$$=2\tan^{-1}\left[\text{cosec}(\tan^{-1}x)-\tan\left(\frac{\pi}{2}-\tan^{-1}x\right)\right]$$

$=2\tan^{-1}[\text{cosec}(\tan^{-1}x)-\cot\tan^{-1}x]$

let $\tan^{-1}x=\theta$

$\Rightarrow 2\tan^{-1}[\text{cosec}\,\theta-\cot\theta]$

$$=2\tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]=2\tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}\right]=2\tan^{-1}\tan\frac{\theta}{2}=2\left(\frac{\theta}{2}\right)$$

$=\theta=\tan^{-1}x$ $\qquad$ L.H.S.

(D) EQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS :

Illustration :

Solve the equation $\tan^{-1}\frac{x-1}{x+2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$.

Sol. $\tan^{-1}\frac{x-1}{x+2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x+2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x+2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4} \quad \Rightarrow \left[\frac{2x(x+2)}{x^2+4+4x-x^2+1}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x(x+2)}{4x+5}=1 \Rightarrow 2x^2+4x=4x+5 \quad \Rightarrow x=\pm\sqrt{\frac{5}{2}}$$

But for $x=-\sqrt{\frac{5}{2}}$, *L.H.S. is negative. Hence* $x=\sqrt{\frac{5}{2}}$.

Illustration :

Find the x *satisfying the equation* $2\cot^{-1}2 - \cos^{-1}\frac{4}{5} = \text{cosec}^{-1}x$.

Sol. $2\cot^{-1}2 - \cos^{-1}\frac{4}{5} = 2\tan^{-1}\frac{1}{2} - \cos^{-1}\frac{4}{5}$

$$= \tan^{-1}\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} - \tan^{-1}\frac{3}{4} \quad = \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}-\frac{3}{4}}{1+\frac{4}{3}\cdot\frac{3}{4}}\right) = \tan^{-1}\left(\frac{7}{24}\right) = \text{cosec}^{-1}\left(\frac{25}{7}\right)$$

$$= \text{cosec}^{-1}x = \text{cosec}^{-1}\left(\frac{25}{7}\right)$$

$x = \frac{25}{7}$ *Ans.*

Illustration :

Find the x *satisfying the equation* $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$

Sol. $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$

$\Rightarrow 2\cos^{-1}\{\cot(2\tan^{-1}x)\} = n\pi,\ n \in I$

$\Rightarrow \cos^{-1}\{\cot(2\tan^{-1}x)\} = \frac{n\pi}{2} \quad \Rightarrow \cot(2\tan^{-1}x) = \cos\frac{n\pi}{2}$

$\cos\frac{n\pi}{2}$ *can take the values* ± 1, 0 *for* $n \in I$.

Case-I : *When* $\cos\frac{n\pi}{2} = \pm 1$

$\Rightarrow \cot(2\tan^{-1}x) = \pm 1 \quad \Rightarrow 2\tan^{-1}x = n\pi \pm \frac{\pi}{4}$

$\Rightarrow \tan^{-1}x = \frac{n\pi}{2} \pm \frac{\pi}{8} \quad \Rightarrow x = \tan\left(\frac{n\pi}{2} \pm \frac{\pi}{8}\right) \quad \Rightarrow x = \pm\tan\frac{\pi}{8},\ \pm\cot\frac{\pi}{8}$

$x = \pm(\sqrt{2}-1), \pm(\sqrt{2}+1)$

$\Rightarrow x = \pm(\sqrt{2} \pm 1)$

Case-II : *When* $\cos\frac{n\pi}{2} = 0$

$$\Rightarrow \cot(2\tan^{-1}x) = 0 \quad \Rightarrow \quad 2\tan^{-1}x = n\pi + \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1} x = \frac{n\pi}{2} + \frac{\pi}{4} \quad \Rightarrow x = \tan\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) \quad \Rightarrow x = \tan\frac{\pi}{4}, -\cot\frac{\pi}{4}$$

$x = \pm 1$

Case-I and Case-II

$x = \pm 1, \pm(\sqrt{2} \pm 1)$ *Ans.*

(E) SIMULTANEOUS EQUATIONS AND INEQUATIONS INVOLVING I.T.F. :

Illustration :

Find the x *satisfying the inequality* $\cos^{-1}x > \cos^{-1}x^2$.

Sol. $\cos^{-1}x > \cos^{-1}x^2$

$\Rightarrow x^2 - x > 0 \Rightarrow x(x-1) > 0 \qquad \Rightarrow x \in (-\infty, 0) \cup (1, \infty)$

$\because \cos^{-1}x$ *defined for* $x \in [-1, 1]$

$\Rightarrow x \in [-1, 0)$ *Ans.*

Illustration :

Solve the inequality satisfying

$$\text{arc}\tan^2 x - 3\,\text{arc}\tan x + 2 > 0$$

where [ ] denotes the greatest integer function.

Sol. $(\tan^{-1}x)^2 - 3\tan^{-1}x + 2 > 0$

$\Rightarrow (\tan^{-1}x - 1)(\tan^{-1}x - 2) > 0$

$\because \tan^{-1}x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow (\tan^{-1}x - 2)$ *is always negative*

$\Rightarrow (\tan^{-1}x - 1)(\tan^{-1}x - 2) > 0$ *holds true only when* $\tan^{-1}x - 1 < 0$

$\Rightarrow \tan^{-1}x < 1 \quad \Rightarrow x < \tan 1$

$\Rightarrow x \in (-\infty, \tan 1)$ ***Ans.***

(F) SUMMATION OF SERIES :

Illustration :

Prove that

$$\tan^{-1}\frac{2}{2+1^2+1^4}+\tan^{-1}\frac{4}{2+2^2+2^4}+\tan^{-1}\frac{6}{2+3^2+3^4}+\ldots\text{upto } n \text{ terms} = \tan^{-1}(n(n+1)+1)-\frac{\pi}{4}$$

Sol. $$T_r = \tan^{-1}\left(\frac{2r}{2+r^2+r^4}\right) = \tan^{-1}\left(\frac{2r}{1+(r^2+1)^2-r^2}\right)$$

$$= \tan^{-1}\left[\frac{(r^2+r+1)-(r^2-r+1)}{1+(r^2+r+1)(r^2-r+1)}\right]$$

$$= \tan^{-1}(r^2+r+1)-\tan^{-1}(r^2-r+1)$$

$$S_n = \sum_{r=1}^{n} T_r = (\tan^{-1}3-\tan^{-1}1)+(\tan^{-1}7-\tan^{-1}3)+(\tan^{-1}13-\tan^{-1}7)$$

$$+\ldots\ldots+(\tan^{-1}(n^2+n+1)-\tan^{-1}(n^2-n+1))$$

$$S_n = \tan^{-1}(n^2+n+1)-\tan^{-1}1 = \tan^{-1}(n(n+1)+1)-\frac{\pi}{4}$$

Illustration :

Prove that

$$\tan^{-1}\frac{x}{1+(1\times 2)x^2}+\tan^{-1}\frac{x}{1+(2\times 3)x^2}+\ \ldots\ldots\ldots+\tan^{-1}\frac{x}{1+n(n+1)x^2} = \tan^{-1}(n+1)x-\tan^{-1}x.$$

Sol. $$T_r = \tan^{-1}\frac{x}{1+r(r+1)x^2} = \tan^{-1}\left(\frac{(r+1)x-rx}{1+r(r+1)x^2}\right) = \tan^{-1}(r+1)x-\tan^{-1}rx$$

$$S_n = \sum_{r=1}^{n} T_r = (\tan^{-1}2x-\tan^{-1}x)+(\tan^{-1}3x-\tan^{-1}2x)+\ldots\ldots+(\tan^{-1}(n+1)x-\tan^{-1}nx)$$

$$S_n = \tan^{-1}(n+1)x-\tan^{-1}x$$

Illustration :

The value of $cosec^{-1}\sqrt{5} + cosec^{-1}\sqrt{65} + cosec^{-1}\sqrt{325} + ...\infty$ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

Sol. $cosec^{-1}\sqrt{5} + cosec^{-1}\sqrt{65} + cosec^{-1}\sqrt{325} + ...\infty.$

$$= tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{8} + tan^{-1}\frac{1}{18} + ...\ \infty$$

$$T_r = tan^{-1}\frac{1}{2r^2} = tan^{-1}\frac{2}{4r^2} = tan^{-1}\left(\frac{(2r+1)-(2r-1)}{1+(2r-1)(2r+1)}\right)$$

$T_r = tan^{-1}(2r + 1) - tan^{-1}(2r - 1)$

$$S_n = \sum_{r=1}^{n} T_r = (tan^{-1} 3 - tan^{-1} 1) + (tan^{-1} 5 - tan^{-1} 3) + ... + (tan^{-1} (2n + 1)x - tan^{-1} (2n - 1))$$

$= tan^{-1} (2n + 1)x - tan^{-1} 1$

when $n \to \infty$

$$S_\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\ \textbf{Ans.}$$

Illustration :

The value of $S = \sum_{n=1}^{\infty} tan^{-1}\left(\frac{4n}{n^4 - 2n^2 + 2}\right)$ is equal to

(A) $3tan^{-1} 1$ (B) $cot^{-1} sgn(x^2 + 1)$ (C) $\frac{\pi}{4}$ (D) $tan^{-1}2 + tan^{-1} 3$

Sol. $S = \sum_{n=1}^{\infty} tan^{-1}\left(\frac{4n}{n^4 - 2n^2 + 2}\right) = \sum_{n=1}^{\infty} tan^{-1}\left(\frac{4n}{1+(n^2-1)^2}\right)$

$$= \sum_{n=1}^{\infty} tan^{-1}\left(\frac{(n+1)^2-(n-1)^2}{1+(n+1)^2(n-1)^2}\right) = \sum_{n=1}^{\infty}\left[tan^{-1}(n+1)^2 - tan(n-1)^2\right]$$

$= (tan^{-1} 2^2 - tan^{-1} 0) + (tan^{-1} 3^2 - tan^{-1} 1^2) + ... + (tan^{-1} n^2 - tan^{-1} (n-2)^2) + (tan^{-1} (n+1)^2 - tan^{-1} (n-1)^2)$

$\Rightarrow S = tan^{-1} (n+1)^2 + tan^{-1} n^2 - (tan^{-1} 1^2 + tan^{-1} 0)$

$$= \left(\frac{\pi}{2} + \frac{\pi}{2}\right) - \left(\frac{\pi}{4} + 0\right)\ (\because n \to \infty) = \frac{3\pi}{4}\ \textbf{Ans.}$$

Practice Problem

Q.1 If $(x-1)(x^2+1) > 0$, then find the value of $\sin\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} - \tan^{-1}x\right)$

Q.2 If $\cos^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1}3x$, then find the value of x.

Q.3 Solve the equation $[\sin^{-1}x] > [\cos^{-1}x]$
where [ ] denotes the greatest integer function.

Q.4 Value of x satisfying the equation $2\cot^{-1}2 + \cos^{-1}(3/5) = \operatorname{cosec}^{-1}x$ is
(A) ϕ (B) 1 (C) 25/7 (D) 25/24

Q.5 Sum of the series $\cot^{-1}(2a^{-1}+a) + \cot^{-1}(2a^{-1}+3a) + \cot^{-1}(2a^{-1}+6a) + \cot^{-1}(2a^{-1}+10a) + \ldots\ldots$ to ∞ is equal to $(a > 0)$
(A) $\tan^{-1}\left(\frac{a}{2}\right)$ (B) $\cot^{-1}\left(\frac{a}{2}\right)$ (C) $a + \frac{1}{a}$ (D) 2a

Q.6 If $\cos^{-1}x - \sin^{-1}x = \cos^{-1}x\sqrt{3}$ then value(s) of x satisfying
(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $-\frac{1}{2}$

Answer key

Q.1 -1 Q.2 $x \in \left(\frac{1}{3}, \infty\right)$ Q.3 $x \in (\sin 1, 1)$ Q.4 A

Q.5 B Q.6 A,B,D

Solved Examples

Q.1 Domain of definition of the function $f(x) = \sqrt{3\cos^{-1}(4x) - \pi}$ is equal to

(A) $\left[\frac{-1}{4}, \frac{1}{8}\right]$ (B) $\left[\frac{1}{8}, 1\right]$ (C) $\left[\frac{1}{8}, \frac{1}{4}\right]$ (D) $\left[-1, \frac{1}{8}\right]$

Sol. For domain of $f(x) = \sqrt{3\cos^{-1}(4x) - \pi}$, we must have

$$\cos^{-1} 4x \geq \frac{\pi}{3} \Rightarrow 4x \leq \frac{1}{2} \quad \Rightarrow x \leq \frac{1}{8} \quad ....(1)$$

$$\text{Also } -1 \leq 4x \leq 1 \Rightarrow \frac{-1}{4} \leq x \leq \frac{1}{4} \quad .... (2)$$

$\therefore$ From (1) and (2), we get $x \in \left[\frac{-1}{4}, \frac{1}{8}\right]$

Q.2 If $a\sin^{-1} x - b\cos^{-1} x = c$, then the value of $a\sin^{-1} x + b\cos^{-1} x$ (whenever exists) is equal to

(A) 0 (B) $\frac{\pi ab + c(b-a)}{a+b}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi ab + c(a-b)}{a+b}$

Sol. We have $b\sin^{-1} x + b\cos^{-1} x = \frac{b\pi}{2}$ ...... (1)

and $a\sin^{-1} x - b\cos^{-1} x = c$ ...... (2) (given)

$\therefore$ On adding (1) and (2), we get $(a+b)\sin^{-1} x = \frac{b\pi}{2} + c$

$$\Rightarrow \quad \sin^{-1} x = \frac{\frac{b\pi}{2} + c}{a+b}. \quad \text{Similarly } \cos^{-1} x = \frac{\frac{a\pi}{2} - c}{a+b}$$

Hence $(a\sin^{-1} x + b\cos^{-1} x) = \frac{\pi ab + c(a-b)}{a+b}$

Q.3 If $0 < \cos^{-1} x < 1$ and $1 + \sin(\cos^{-1} x) + \sin^2(\cos^{-1} x) + \sin^3(\cos^{-1} x) + ... \infty = 2$, then x equals

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{2\sqrt{3}}$

Sol. We have $1 + \sin(\cos^{-1} x) + \sin^2(\cos^{-1} x) + ... \infty = 2$

$$\Rightarrow \frac{1}{1 - \sin(\cos^{-1} x)} = 2 \Rightarrow \frac{1}{2} = 1 - \sin(\cos^{-1} x) \Rightarrow \sin(\cos^{-1} x) = \frac{1}{2}$$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{6} \quad \Rightarrow \quad x = \frac{\sqrt{3}}{2}$$ **Ans.**

Q.4 If $\tan^{-1}\left(x+\frac{3}{x}\right)-\tan^{-1}\left(x-\frac{3}{x}\right)=\tan^{-1}\frac{6}{x}$, then the value of $5x^8-4x^4+7$ equals

(A) 397 (B) 393 (C) 376 (D) 379

Sol. We have $\tan^{-1}\left(x+\frac{3}{x}\right)-\tan^{-1}\left(x-\frac{3}{x}\right)=\tan^{-1}\frac{6}{x}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\left(x+\frac{3}{x}\right)-\left(x-\frac{3}{x}\right)}{1+\left(x+\frac{3}{x}\right)\left(x-\frac{3}{x}\right)}\right)=\tan^{-1}\frac{6}{x} \quad \Rightarrow \quad x^2-\frac{9}{x^2}=0 \quad \Rightarrow \quad x^4=9$$

Hence $(5x^8-4x^4+7)=5(81)-4(9)+7=405-36+7=412-36=376.$

Q.5 The value of $\tan^{-1}\frac{4}{7}+\tan^{-1}\frac{4}{19}+\tan^{-1}\frac{4}{39}+\tan^{-1}\frac{4}{67}+........\infty$ equals

(A) $\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$ (B) $\tan^{-1}1+\cot^{-1}3$

(C) $\cot^{-1}1+\cot^{-1}\frac{1}{2}+\cot^{-1}\frac{1}{3}$ (D) $\cot^{-1}1+\tan^{-1}3$

Sol. Let

$$S=7+19+39+67+......+T_n$$
$$S=0+\ \ 7+19+39+..........+T_{n-1}+T_n$$

(Subtracting) – – – – – – –

$$T_n=7+12+20+28+......+(T_n-T_{n-1})$$

$$=7+\frac{(n-1)}{2}\left[24+8(n-2)\right]=4n^2+3$$

$$\therefore\ T_n'=\tan^{-1}\frac{4}{4n^2+3}=\tan^{-1}\frac{1}{n^2+\frac{3}{4}}=\tan^{-1}\frac{1}{1+\left(n^2-\frac{1}{4}\right)}$$

$$=\tan^{-1}\left[\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right]=\tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\left(n-\frac{1}{2}\right)$$

Hence $S_\infty=\sum_{n=1}^{\infty}T_n'=\frac{\pi}{2}-\tan^{-1}\frac{1}{2}=\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{2}=\tan^{-1}1+\cot^{-1}3$

Q.6 Number of values of x satisfying the equation

$$\tan^{-1}\left(x-\frac{x^3}{4}+\frac{x^5}{16}-......\right)+\cot^{-1}\left(x+\frac{x^2}{2}+\frac{x^3}{4}+......\right)=\frac{\pi}{2} \quad \text{for } 0<|x|<2, \text{ is}$$

(A) 0 (B) 1 (C) 2 (D) 3

Sol. We must have $x-\frac{x^3}{4}+\frac{x^5}{16}-........=x+\frac{x^2}{2}+\frac{x^3}{4}+......$

$$\Rightarrow \frac{x}{1+\frac{x^2}{4}}=\frac{x}{1-\frac{x}{2}} \Rightarrow \frac{4x}{4+x^2}=\frac{2x}{2-x} \Rightarrow 2x^2(x+2)=0$$

$\therefore$ $x = 0, -2$ (As $0<|x|<2$)

Clearly, no value of x satisfies given equation.

Q.7 Number of integral ordered pairs (x, y) satisfying the equation $\text{arc tan}\frac{1}{x}+\text{arc tan}\frac{1}{y}=\text{arc tan}\frac{1}{10}$, is

(A) 1 (B) 2 (C) 3 (D) 4

Sol. Since $\tan(\text{arc tan } a) = a \ \forall\ a \in R = a\ \forall\ a \in R$,

Take tan both side

$$\tan\left(\text{arc tan}\frac{1}{x}+\text{arc tan}\frac{1}{y}\right)=\tan\left(\text{arc tan}\frac{1}{10}\right)$$

$$\Rightarrow \frac{\frac{1}{x}+\frac{1}{y}}{1-\frac{1}{xy}}=\frac{1}{10}=(x-10)(y-10)=101$$

The following four ordered pair of integer numbers are solutions of this equation :

(11, 111) ; (111, 11), (9, –91) $\Rightarrow$ ordered pairs **Ans.**

Q.8 For $n \in N$, if $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{n}=\frac{\pi}{4}$ then n is equal to

(A) 43 (B) 47 (C) 49 (D) 51

Sol. We have, $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{12}}\right)=\tan^{-1}\left(\frac{7}{11}\right)$

Again, $\tan^{-1}\frac{7}{11}+\tan^{-1}\frac{1}{5}=\tan^{-1}\left(\frac{\frac{7}{11}+\frac{1}{5}}{1-\frac{7}{55}}\right)=\tan^{-1}\left(\frac{46}{48}\right)=\tan^{-1}\frac{23}{24}$

$\therefore$ $\tan^{-1}\frac{1}{n}=\tan^{-1}1-\tan^{-1}\frac{23}{24}=\tan^{-1}\left(\frac{1-\frac{23}{24}}{1+\frac{23}{24}}\right)=\tan^{-1}\left(\frac{1}{47}\right) \Rightarrow n = 47$ **Ans.**

Q.9 If $0 < \cos^{-1}x < 1$ and $1 + \cos^{-1}x + (\cos^{-1}x)^2 + ... \infty = 2$ then x is equal to

(A) $\frac{\pi}{4}$ (B) $\cos\frac{1}{2}$ (C) $\cos\frac{1}{\sqrt{2}}$ (D) $\frac{\pi}{6}$

Sol. We have

$1 + \cos^{-1}x + (\cos^{-1}x)^2 + ... \infty = 2$, (which is an infinite geometric progression)

$\Rightarrow \frac{1}{1-\cos^{-1}x} = 2 \Rightarrow \cos^{-1}x = \frac{1}{2} \Rightarrow x = \cos\frac{1}{2}$ **Ans.**

Q.10 The domain of definition of function $f(x) = \sqrt{\cos^{-1}x - 2\sin^{-1}x}$ is equal to

(A) $\left[\frac{-1}{2},\frac{1}{2}\right]$ (B) $\left[\frac{1}{2},1\right]$ (C) $\left[-1,\frac{1}{2}\right]$ (D) $\left[-1,\frac{\sqrt{3}}{2}\right]$

Sol. We have $f(x) = \sqrt{\cos^{-1}x - 2\sin^{-1}x}$

Clearly, for domain of f(x), $\cos^{-1}x - 2\sin^{-1}x \geq 0$

$\Rightarrow \frac{\pi}{2} \geq 3\sin^{-1}x \Rightarrow \sin^{-1}x \leq \frac{\pi}{6} \Rightarrow x \leq \frac{1}{2}$

So, $D_f = \left[-1,\frac{1}{2}\right]$ **Ans.**

Paragraph for question nos. 11 to 13

In ΔABC, if $\angle B = \sec^{-1}\left(\frac{5}{4}\right) + \text{cosec}^{-1}\sqrt{5}$, $\angle C = \text{cosec}^{-1}\left(\frac{25}{7}\right) + \cot^{-1}\left(\frac{9}{13}\right)$ and $c = 3$.

(All symbols used have their usual meaning in a triangle.)

Q.11 tan A, tan B, tan C are in

(A) A.P. (B) G.P. (C) H.P. (D) neither A.P, G.P. nor H.P.

Q.12 The distance between orthocentre and centroid of triangle with sides a^2, $b^{\frac{4}{3}}$ and c is equal to

(A) $\frac{5}{2}$ (B) $\frac{5}{3}$ (C) $\frac{10}{3}$ (D) $\frac{7}{2}$

Q.13 Which of the following is rational with respect to ΔABC?

(A) r_1 (B) r_2 (C) r_3 (D) Δ

Sol. $\angle B = \sec^{-1}\left(\frac{5}{4}\right) + \text{cosec}^{-1}\sqrt{5} = \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\frac{\frac{3}{4}+\frac{1}{2}}{1-\frac{3}{4}\cdot\frac{1}{2}} = \tan^{-1}2$

$\angle C = \text{cosec}^{-1}\left(\frac{25}{7}\right) + \cot^{-1}\left(\frac{9}{13}\right) = \tan^{-1}\left(\frac{7}{24}\right) + \tan^{-1}\left(\frac{13}{9}\right) = \tan^{-1}\left(\frac{\frac{7}{24}+\frac{13}{9}}{1-\frac{7}{24}\cdot\frac{13}{9}}\right) = \tan^{-1}3$

$\therefore \angle A = \pi - \angle B - \angle C = \pi - \tan^{-1}2 - \tan^{-1}3 = \tan^{-1}1$

$\therefore \sin A = \frac{1}{\sqrt{2}}, \sin B = \frac{2}{\sqrt{5}}$ and $\sin C = \frac{3}{\sqrt{10}}$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \sqrt{2}a = \frac{\sqrt{5}b}{2} = \frac{c\sqrt{10}}{3}$$

Hence $a = \sqrt{5}$ and $b = 2\sqrt{2}$, $c = 3$

(i) $\tan A = 1$, $\tan B = 2$, $\tan C = 3$ are in A.P.

(ii) The triangle with sides a^2, $b^{\frac{4}{3}}$ and c will have side-length 5, 4 and 3 respectively

$\therefore$ Distance between orthocentre and centroid $= \frac{2}{3}$ (circumradius) $= \frac{\text{hypotenuse}}{3} = \frac{5}{3}$ Ans.

(iii) Area of ΔABC, $\Delta = \frac{1}{2} ab \sin C = \frac{1}{2}(\sqrt{5})(2\sqrt{2})\left(\frac{3}{\sqrt{10}}\right) = 3$

Also $s = \frac{1}{2}(a+b+c) = \frac{1}{2}\left(\sqrt{5}+2\sqrt{2}+3\right)$

$\therefore s-a = \frac{1}{2}\left(-\sqrt{5}+2\sqrt{2}+3\right)$, $s-b = \frac{1}{2}\left(\sqrt{5}-2\sqrt{2}+3\right)$ and $s-c = \frac{1}{2}\left(\sqrt{5}+2\sqrt{2}-3\right)$

$\because$ Δ is rational

$\therefore$ Each of values $\frac{\Delta}{s-a}$, $\frac{\Delta}{s-b}$ and $\frac{\Delta}{s-c}$ i.e. r_1, r_2 and r_3 (respectively) will be irrational.

Q.14 If $\left(\sin^{-1}x\right)^2 + \left(\sin^{-1}y\right)^2 + \left(\sin^{-1}z\right)^2 = \frac{3\pi^2}{4}$, then the value of $(x-y+z)$ can be

(A) 1 (B) – 1 (C) 3 (D) – 3

Sol. As $\frac{-\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}\ \forall -1 \le x \le 1$

$\therefore \left(\sin^{-1}x\right)^2 + \left(\sin^{-1}y\right)^2 + \left(\sin^{-1}z\right)^2 = \frac{\pi^2}{4} + \frac{\pi^2}{4} + \frac{\pi^2}{4}$ is possible if $x, y, z \in \{-1, 1\}$

$\therefore$ Possible values of $x-y+z$ from the ordered triplet (x, y, z) are as follows :

(x, y, z)	**(x – y + z)**
(– 1, –1, –1)	– 1
(– 1, 1, 1)	– 1
(1, – 1, 1)	3
(1, 1, – 1)	– 1
(1, 1, 1)	1
(1, – 1, – 1)	– 1
(– 1, 1, – 1)	–3
(– 1, – 1, 1)	1

Hence set of values of $(x-y+z)$ is $\{\pm 1, \pm 3\}$

Q.15 Let $f(x) = \sin^{-1}|\sin x| + \cos^{-1}(\cos x)$. Which of the following statement(s) is/are **TRUE** ?

(A) $f(f(3)) = \pi$

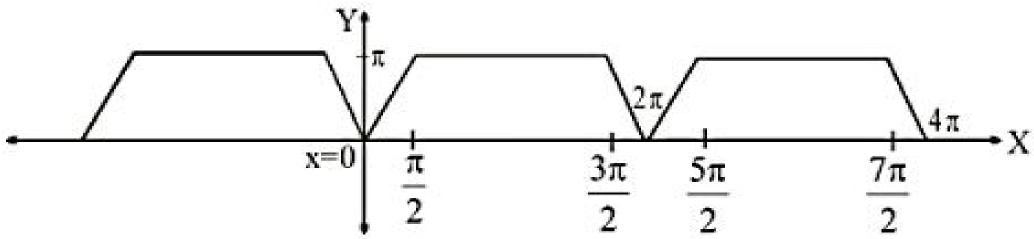
(B) $f(x)$ is periodic with fundamental period 2π.

(C) $f(x)$ is neither even nor odd.

(D) Range of $f(x)$ is $[0, 2\pi]$

Sol. $$f(x) = \begin{cases} 2x & ; 0 \le x \le \frac{\pi}{2} \\ \pi & ; \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ 4\pi - 2x & ; \frac{3\pi}{2} < x \le 2\pi \end{cases}$$

Clearly f(x) is periodic function with period 2π. The graph of $f(x)$ is shown below.



Q.16 If $f(x) = \sin^{-1}x \cdot \cos^{-1}x \cdot \tan^{-1}x \cdot \cot^{-1}x \cdot \sec^{-1}x \cdot \operatorname{cosec}^{-1}x$, then which of the following statement(s) **hold(s) good** ?

(A) The graph of $y = f(x)$ does not lie above x axis.

(B) The non-negative difference between maximum and minimum value of the function $y = f(x)$ is $\frac{3\pi^6}{64}$.

(C) The function $y = f(x)$ is not injective.

(D) Number of non-negative integers in the domain of f(x) is two.

Sol. Domain of $\sin^{-1}x$ and $\cos^{-1}x$, each is $[-1, 1]$ and that of $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$, each is $(-\infty, -1] \cup [1, \infty)$

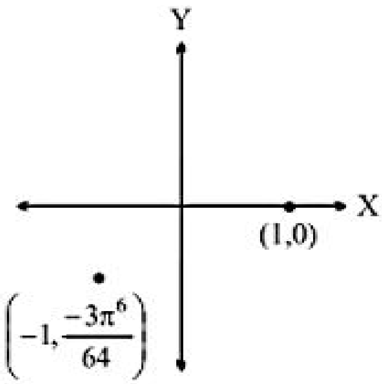
$\therefore$ Domain of $f(x)$ must be $\{-1, 1\}$ $\quad\therefore$ Range of f(x) will be $\{f(-1), f(1)\}$

where $f(-1) = \sin^{-1}(-1) \cdot \cos^{-1}(-1) \cdot \tan^{-1}(-1) \cdot \cot^{-1}(-1) \cdot \sec^{-1}(-1) \cdot \operatorname{cosec}^{-1}(-1)$

$$= \left(\frac{-\pi}{2}\right) \cdot (\pi) \cdot \left(\frac{-\pi}{4}\right) \cdot \left(\frac{3\pi}{4}\right) \cdot (\pi) \cdot \left(\frac{-\pi}{2}\right) = \frac{-3\pi^6}{64} \text{ and } f(1) = 0 \ \{\text{as } \cos^{-1} 1 = 0\}$$

(i) Thus, the graph of f(x) is a two point graph which doesn't lie above x - axis.

(ii) $f(x)_{max} = 0$ and $f(x)_{min} = \frac{-3\pi^6}{64}$



Hence $\left|f(x)_{max} - f(x)_{min}\right| = \frac{3\pi^6}{64}$

(iii) f(x) is one-one hence injective.

(iv) Domain is $\{-1, 1\}$

$\therefore$ Number of non-negative integers in the domain of f(x) is one.

Q.17 Consider $f(x) = \tan^{-1}\left(\frac{(\sqrt{12}-2)x^2}{x^4+2x^2+3}\right)$ and m and M are respectively minimum and maximum values of f(x) and x = a (a > 0) is the point in the domain of f(x) where f(x) attains its maximum value.

	Column I		Column II
(A)	If $\sin^{-1}2\sqrt{x} = 3\tan^{-1}(\tan(m+M))$ then 8x equals	(P)	0
(B)	If $\cos^{-1}x+\cos^{-1}y = 3\left\{\tan^{-1}\left(\tan\frac{7M}{2}\right)+\tan^{-1}\left(m+\tan\frac{3\pi}{8}\right)\right\}$ then $(x+y)$ equals	(Q)	2
(C)	The value of $\tan\left(\sec^{-1}\left(\frac{2}{a^2}\right)+M\right)$ equals	(R)	– 2
(D)	If α and β are roots of the equation $x^2-(\tan(3\sin^{-1}(\sin M)))x+a^4=0$, then $\alpha\beta-(\alpha+\beta)$ equals	(S)	1
		(T)	–1

Sol. We have $f(x) = \tan^{-1}\left(\frac{2(\sqrt{3}-1)}{x^2+\frac{3}{x^2}+2}\right)$

As $x^2+\frac{3}{x^2} \geq 2\sqrt{3}$ (Using A.M. – G.M. inequality)

$\Rightarrow x^2+\frac{3}{x^2}+2 \geq 2+2\sqrt{3}$

$\therefore$ $f(x)|_{max} = \tan^{-1}\left(\frac{2(\sqrt{3}-1)}{2(\sqrt{3}+1)}\right) = \frac{\pi}{12} = M$, which occurs at $x^2 = \frac{3}{x^2} \Rightarrow x = 3^{\frac{1}{4}} = a$

$f(x)|_{min} = 0 = m$, which occurs at $x = 0$

(A) $\sin^{-1}(2\sqrt{x}) = 3\tan^{-1}\left(\tan\frac{\pi}{12}\right) = \frac{\pi}{4}$

$2\sqrt{x} = \frac{1}{\sqrt{2}} \Rightarrow 8x = 1$

(B) $\cos^{-1}x+\cos^{-1}y = 3\left[\tan^{-1}\left(\tan\frac{7\pi}{24}\right)+\tan^{-1}\left(0+\tan\frac{3\pi}{8}\right)\right] = 3\left[\frac{7\pi}{24}+\frac{3\pi}{8}\right] = 3\left(\frac{16\pi}{24}\right) = 2\pi$

$\therefore$ $x = y = -1 \Rightarrow x+y = -2$

(C) $\tan\left(\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)+\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{6}+\frac{\pi}{12}\right) = 1$

(D) $x^2 - \tan\left(3\sin^{-1}\left(\sin\frac{\pi}{12}\right)\right)x + 3 = 0$

$x^2 - x + 3 = 0 \begin{cases} \alpha \\ \beta \end{cases}$

$\therefore\quad \alpha + \beta = 1$

$\therefore\quad \alpha\beta = 3$

Hence $\alpha\beta - (\alpha + \beta) = 2$

Q.18 Let $\alpha = 3\cos^{-1}\left(\frac{5}{\sqrt{28}}\right) + 3\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\beta = 4\sin^{-1}\left(\frac{7\sqrt{2}}{10}\right) - 4\tan^{-1}\left(\frac{3}{4}\right)$

then which of the following does not hold(s) good?

(A) $\alpha < \pi$ but $\beta > \pi$.

(B) $\alpha > \pi$ but $\beta < \pi$.

(C) Both α and β are equal.

(D) $\cos(\alpha + \beta) = 0$.

Sol. $\alpha = 3\tan^{-1}\left(\frac{\sqrt{3}}{5}\right) + 3\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = 3\left[\tan^{-1}\frac{\frac{\sqrt{3}}{5} + \frac{\sqrt{3}}{2}}{1 - \frac{3}{10}}\right] = 3\tan^{-1}\left(\frac{7\sqrt{3}}{7}\right) = \pi.$

$$\beta = 4\left[\tan^{-1}7 - \tan^{-1}\frac{3}{4}\right] = 4\left[\tan^{-1}\frac{7 - \frac{3}{4}}{1 + \frac{21}{4}}\right] = 4\tan^{-1}\left(\frac{25}{25}\right) = \pi.$$

Q.19 If range of the function $f(x) = \sin^{-1}x + 2\tan^{-1}x + x^2 + 4x + 1$ is $[p, q]$ then find the value of $(p + q)$.

Sol. We have $f(x) = \sin^{-1}x + 2\tan^{-1}x + x^2 + 4x + 1$

Clearly domain of $f(x)$ is $[-1, 1]$.

Also $f(x)$ is increasing function in its domain.

$$\therefore\quad p = f_{min.}(x) = f(-1) = -\frac{\pi}{2} + 2\left(\frac{-\pi}{4}\right) + 1 - 4 + 1 = -\pi - 2.$$

$$q = f_{max.}(x) = f(1) = \frac{\pi}{2} + 2\left(\frac{\pi}{4}\right) + 1 + 4 + 1 = \pi + 6.$$

$\therefore\quad$ Range of $f(x)$ is $[-\pi - 2, \pi + 6]$

Hence $(p + q) = 4$

Note : Vertex of $y = x^2 + 4x + 1$ is at $x = -2$ and hence in the domain $(x^2 + 4x + 1)$ is increasing.

Q.20 Let α, β, γ and δ be the roots of equation $x^4 - 3x^3 + 5x^2 - 7x + 9 = 0$. If the value of $|\tan(\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma + \tan^{-1}\delta)| = \frac{a}{b}$ where a and b are coprime to each other, then find the value of $(a^b + b^a + a^a + b^b + ab)$.

Sol. From given equation, we have

$S_1 = \Sigma\alpha = 3,\quad S_2 = \Sigma\alpha\beta = 5$

$S_3 = \Sigma\alpha\beta\gamma = 7$ and $S_4 = \alpha\beta\gamma\delta = 9$

Let $\tan^{-1}\alpha = A$, $\tan^{-1}\beta = B$, $\tan^{-1}\gamma = C$ & $\tan^{-1}\delta = D$

Now $|\tan(\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma + \tan^{-1}\delta)|$

$= |\tan(A+B+C+D)| = \left|\dfrac{S_1 - S_3}{1 - S_2 + S_4}\right| = \left|\dfrac{3-7}{1-5+9}\right| = \dfrac{4}{5} = \dfrac{a}{b}$

Hence a = 4 and b = 5

So $(a^b + b^a + a^a + b^b + ab) = 4^5 + 5^4 + 4^4 + 5^5 + 4.5 = 1024 + 625 + 256 + 3125 + 20 = 5050$ **Ans.**

Q.21 How many terms of the sequence $\cot^{-1} 3, \cot^{-1} 7, \cot^{-1} 13, \cot^{-1} 21, \ldots\ldots$ must be taken to have their sum equal to $\dfrac{1}{2}\cos^{-1}\left(\dfrac{24}{145}\right)$.

Sol. $T_1 = \tan^{-1}\dfrac{1}{3} = \tan^{-1}2 - \tan^{-1}1$; $T_2 = \tan^{-1}\dfrac{1}{7} = \tan^{-1}3 - \tan^{-1}2$; $T_3 = \tan^{-1}\dfrac{1}{13} = \tan^{-1}4 - \tan^{-1}3$

Clearly $T_n = \tan^{-1}(n+1) - \tan^{-1}(n)$

Hence $S_n = \tan^{-1}(n+1) - \tan^{-1}1 = \tan^{-1}\left(\dfrac{n+1-1}{1+(n+1)\cdot 1}\right) = \left(\tan^{-1}\dfrac{n}{n+2}\right) = \dfrac{1}{2}\cos^{-1}\left(\dfrac{24}{145}\right)$

$\Rightarrow 2\left(\tan^{-1}\dfrac{n}{n+2}\right) = \cos^{-1}\left(\dfrac{24}{145}\right)$ $\quad \left(\text{Using } 2\tan^{-1}x = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) \forall x \ge 0\right)$

$\Rightarrow \cos^{-1}\left(\dfrac{2(n+1)}{n^2+2n+2}\right) = \cos^{-1}\left(\dfrac{24}{145}\right) \Rightarrow \left(\dfrac{2(n+1)}{n^2+2n+2}\right) = \left(\dfrac{24}{145}\right)$

$\Rightarrow 12(n+1)^2 - 144(n+1) - (n+1) + 12 = 0 = ((n+1)-12)(12(n+1)-1) = 0$

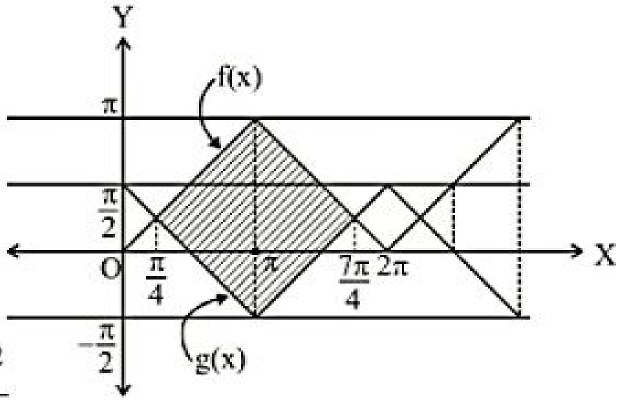
$\therefore n+1 = 12, \dfrac{1}{12} \qquad \therefore n = 11, \dfrac{-11}{12} \qquad \because n \in N \quad \therefore n \ne \dfrac{-11}{12}$

Hence, n = 11 **Ans.**

Q.22 If the area enclosed by the curves $f(x) = \cos^{-1}(\cos x)$ and $g(x) = \sin^{-1}(\cos x)$ in $x \in \left[\dfrac{9\pi}{4}, \dfrac{15\pi}{4}\right]$ is $\dfrac{a\pi^2}{b}$ (where a and b are coprime), then find (a + b).

Sol. We have $g(x) = \sin^{-1}(\cos x) = \dfrac{\pi}{2} - \cos^{-1}(\cos x)$

Both the curves bound the regions of same area in $\left[\dfrac{\pi}{4}, \dfrac{7\pi}{4}\right], \left[\dfrac{9\pi}{4}, \dfrac{15\pi}{4}\right]$ and so on



$\therefore$ Required area = area of shaded square $= \dfrac{9\pi^2}{8} = \dfrac{a\pi^2}{b}$

$\therefore$ a = 9 and b = 8

Hence a + b = 17 **Ans.**