DEFINITE INTEGRATION

Definition:

$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ is called the definite integral of f(x) between the limits a and b.

where $\frac{d}{dx}(F(x)) = f(x)$

Note : The word limit here is quite different as used in differential calculus.

Important Points:

(I) If $\int_a^b f(x)\,dx = 0$, then the equation $f(x) = 0$ has atleast one root in (a, b) provided f is continuous in (a, b).

Note that the converse is not true.

Illustration :

$\int_0^1 e^x(ax^2+bx+c)\,dx = 0 \Rightarrow e^x(ax^2+bx+c) = 0$ *has at least one root in (0, 1)*

$\Rightarrow ax^2+bx+c = 0$ *has at least one root in (0, 1) [e^x is always positive]*

(II) $\lim\limits_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right) = \int_a^b\left(\lim\limits_{n\to\infty} f_n(x)\right)dx$

Illustration :

If $\lim\limits_{n\to\infty}\int_{-\sqrt[3]{a}}^{\sqrt[3]{a}}\left(1-\frac{t^3}{n}\right)^n t^2\,dt = \frac{2\sqrt{2}}{3}$ *(n ∈ N), then the value of find 'a'.*

Sol. $L.H.S. = \int_{-a^{1/3}}^{a^{1/3}}\lim\limits_{n\to\infty}\left(1-\frac{t^3}{n}\right)^n t^2\,dt = \int_{-a^{1/3}}^{a^{1/3}} e^{-t^3}\,t^2\,dt$

$$= \left[-\frac{1}{3}e^{-t^3}\right]_{-a^{1/3}}^{a^{1/3}} = \frac{1}{3}\left[e^a - e^{-a}\right] \Rightarrow e^a - e^{-a} = 2\sqrt{2} \text{ or } a = \ln\left(\sqrt{2}+\sqrt{3}\right)$$

(III) $\int_a^b f(x)\cdot d(g(x)) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(x)\cdot g'(x)\,dx$.

(IV) If f(x) is continuous in (a, b), Then $\int_a^b \frac{d}{dx}(f(x)) = [f(x)]_a^b$ and if f(x) is discontinuous in (a, b) at $x = c \in (a, b)$, then $\int_a^b \frac{d}{dx}(f(x)) = [f(x)]_a^{c^-} + [f(x)]_{c^+}^b$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

$$\int_{-1}^{1}\left(\frac{d}{dx}\left(\cot^{-1}\frac{1}{x}\right)\right)=\cot^{-1}\frac{1}{x}\Big]_{-1}^{0^-}+\cot^{-1}\frac{1}{x}\Big]_{0^+}^{1}=\pi-\left(\frac{3\pi}{4}\right)+\frac{\pi}{4}=\frac{\pi}{2}$$

(V) If g (x) is the inverse of f (x) and f (x) has domain $x \in [a, b]$ where f (a) = c and f (b) = d then the value of $\int_a^b f(x)dx + \int_c^d g(y)dy = (bd - ac)$

Illustration :

Evaluate : $\int_0^1 e^{\sqrt{e^x}}\,dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)dx$

Sol. Consider $f : [0, 1] \to [e, e^{\sqrt{e}}]$, $f(x) = e^{\sqrt{e^x}}$ then $f^{-1}(x) = 2\ln(\ln x)$

$I = \int_0^1 e^{\sqrt{e^x}}\,dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)dx$ hence $I = 1\cdot e^{\sqrt{e}} - 0\cdot e = e^{\sqrt{e}}$

Evaluating definite integrals by finding antiderivatives :

Illustration :

Evaluate : $\int_3^8 \frac{\sin\sqrt{x+1}}{\sqrt{x+1}}\,dx$

Sol. $\int \frac{\sin\sqrt{x+1}}{\sqrt{x+1}}dx = -2\cos\sqrt{x+1}$

$\Rightarrow n\int_3^8 \frac{\sin\sqrt{x+1}}{\sqrt{x+1}}dx = \left[-2\cos\sqrt{x+1}\right]_3^8 = 2(\cos 2 - \cos 3)$

Illustration :

Evaluate : $\int_0^{\pi/4} \cos 2x\sqrt{4-\sin 2x}\,dx$

Sol. $\int \cos 2x\sqrt{4-\sin 2x}\,dx$, Put $4 - \sin 2x = t \Rightarrow = -2\cos 2x\,dx = dt$

Integral becomes $\int -\frac{1}{2}\sqrt{t}\,dt = -\frac{1}{3}(t)^{3/2} = -\frac{1}{3}(4-\sin 2x)^{3/2}$

$\Rightarrow \int_0^{\pi/4} \cos 2x\sqrt{4-\sin 2x}\,dx = \left[-\frac{1}{3}(4-\sin 2x)^{3/2}\right]_0^{\pi/4} = \frac{8-3\sqrt{3}}{3}$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Evaluate : $\int_0^1 x\ell n(1+2x)\ dx = \frac{3ln3}{8}$

Sol. $\int \underset{II}{x}\, ln(\underset{I}{1+2x})dx = ln\ (1+2x)\cdot\frac{x^2}{2} - \int\frac{2}{1+x}\cdot\frac{x^2}{2}dx = \frac{x^2}{2}\ ln\ (1+2x) - \int\left(\left(\frac{x}{2}-\frac{1}{4}\right)+\frac{1}{4}\left(\frac{1}{1+2x}\right)\right)dx$

$= \frac{x^2}{2}\ ln\ (1+2x) - \frac{x^2}{4}+\frac{1}{4}x - \frac{1}{8}\ ln\ (1+2x)\ \Rightarrow\ \int_0^1 x\, ln(1+2x)dx = \frac{3}{8}\ ln\ 3$

Illustration :

The value of the integral $\int_0^{2008}\left(3x^2-8028x+(2007)^2+\frac{1}{2008}\right)dx$ equals

(A) $(2008)^2$ (B) $(2009)^2$ (C) 2009 (D) 1

Sol. $\int\left(3x^2-8028x+(2007)^2+\frac{1}{2008}\right)dx = x^3 - 4014\ x^2 + (2007)^2\ x + \frac{x}{2008}$

$\Rightarrow$ value of integral $= \left[x^3-4014x^2+(2007)^2x+\frac{x}{2008}\right]_0^{2008} = 2009$

Illustration :

Evaluate : $\int_2^4 \frac{\sqrt{x^2-4}}{x^4}\ dx$

Sol. $\int\frac{\sqrt{x^2-4}}{x^2}dx = \int\frac{\sqrt{1-4/x^2}}{x^3}dx$ Put $1-\frac{4}{x^2}=t \Rightarrow 8x^{-3}\ dx = dt$ integral becomes

$\int\frac{1}{8}t^{1/2}\ dt = \frac{1}{12}t^{3/2} = \frac{1}{12}\left(1-\frac{4}{x^2}\right)^{3/2} \Rightarrow \int_2^4\frac{\sqrt{x^2-4}}{x^4}dx = \left[\frac{1}{12}\left(1-\frac{4}{x^2}\right)^{3/2}\right]_2^4 = \frac{\sqrt{3}}{32}$

Illustration :

Evaluate : $\int_0^{\pi/4} x\ sin^2 x\ dx$

Sol. Let $I = \int_0^{\pi/4} x\ sin^2 x\ dx = \int_0^{\pi/4}\frac{x}{2}(1-cos\ 2x)\ dx$

Integrating by parts, we have

$$I = \left[\frac{x}{2}\left(x-\frac{sin\ 2x}{2}\right)\right]_0^{\pi/4} - \int_0^{\pi/4}\frac{1}{2}\left(x-\frac{sin\ 2x}{2}\right)dx = \frac{\pi}{8}\left(\frac{\pi}{4}-\frac{1}{2}\right)-\left[\frac{x^2}{4}+\frac{cos\ 2x}{8}\right]_0^{\pi/4}$$

$$= \left(\frac{\pi^2}{32}-\frac{\pi}{16}\right)-\left(\frac{\pi^2}{64}-\frac{1}{8}\right) = \frac{\pi^2+8-4\pi}{64}.$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Find the value of $\int_0^{1/2} \frac{x\cos^{-1} x}{\sqrt{1-x^2}}dx$.

Sol. *Let* $I = \int_0^{1/2} \frac{x\cos^{-1} x}{\sqrt{1-x^2}}dx$

Let us put $x = \cos t$, $dx = -\sin t\, dt$. *Also, when* $x = 0$, *then* $t = \frac{\pi}{2}$ *and when* $x = \frac{1}{2}$, *then* $t = \frac{\pi}{3}$.

Thus, we have

$$I = \int_{\pi/2}^{\pi/3} \frac{t\cos t}{\sin t}(-\sin t\, dt) - \int_{\pi/2}^{\pi/3} t\cos t\, dt$$

$$= \left[-t\sin t\right]_{\pi/2}^{\pi/3} + \int_{\pi/2}^{\pi/3} \sin t\, dt = \frac{\pi}{2} - \frac{\pi}{3}\cdot\frac{\sqrt{3}}{2} - \left[\cos t\right]_{\pi/2}^{\pi/3} = \frac{\pi(\sqrt{3}-1)}{2\sqrt{3}} - \frac{1}{2}.$$

Practice Problem

Q.1 Let $\int_0^1 \frac{dx}{\sqrt{16+9x^2}} + \int_0^2 \frac{dx}{\sqrt{9+4x^2}} = \ln a$. Find a.

Q.2 Evaluate: $\int_0^{\ln 2} x e^{-x} dx$

Q.3 Evaluate: $\int_0^{\pi/4} \frac{\sin^2 x \cdot \cos^2 x}{\left(\sin^3 x + \cos^3 x\right)^2} dx$

Q.4 Evaluate: $\int_0^{\pi/2} \frac{dx}{1+\cos\theta\cdot\cos x}$ $\theta \in (0, \pi)$

Q.5 Suppose that the function f, g, f' and g' are continuous over [0, 1], $g(x) \neq 0$ for $x \in [0, 1]$, $f(0) = 0$, $g(0) = \pi$, $f(1) = \frac{2009}{2}$ and $g(1) = 1$. Find the value of the definite integral,

$$\int_0^1 \frac{f(x)\cdot g'(x)\{g^2(x)-1\} + f'(x)\cdot g(x)\{g^2(x)+1\}}{g^2(x)} dx.$$

Answer key

Q.1 $\left(2^{1/3}\cdot 3^{1/2}\right)$ Q.2 $1-\frac{2}{e}$ Q.3 $\frac{1}{6}$ Q.4 $\left(\frac{\theta}{\sin\theta}\right)$

Q.5 2009

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

PROPERTIES OF DEFINITE INTEGRAL :

(A) PROPERTIES :

P – 1 : $\int_a^b f(x)\,dx = \int_a^b f(t)dt$; **P – 2 :** $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$

P – 3 : $\int_a^b f(x) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided f has a piece wise continuity

or when f is not uniformly defined in (a, b)

Integral is broken at points of discontinuity or at the points where definition of 'f' changes.

Illustration :

Evaluate : $\int_0^{\pi}\sqrt{\frac{1+\cos 2x}{2}}\,dx$.

Sol. $\int_0^{\pi}|\cos x|dx = \int_0^{\pi/2}\cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx = [\sin x]_0^{\pi/2} + [-\sin x]_{\pi/2}^{\pi} = 2$

Illustration :

Evaluate : $\int_0^3 |5x-9|dx$.

Sol. $\int_0^3 |5x-9|dx = \int_0^{9/5}(9-5x)dx + \int_{9/5}^3 (5x-9)dx = \left[9x-\frac{5}{2}x^2\right]_0^{9/5} + \left[\frac{5x^2}{2}-9x\right]_{9/5}^3 = \frac{15}{2}$

Illustration :

Evaluate : $\int_{-1}^3 \left[x+\frac{1}{2}\right]dx$

Sol. $I = \int_{-1}^{-1/2}\left[x+\frac{1}{2}\right]dx + \int_{-1/2}^{1/2}\left[x+\frac{1}{2}\right]dx + \int_{1/2}^{3/2}\left[x+\frac{1}{2}\right]dx + \int_{3/2}^{5/2}\left[x+\frac{1}{2}\right]dx + \int_{5/2}^{3}\left[x+\frac{1}{2}\right]dx$

$= \int_{-1}^{-1/2} -1\,dx\;dx + \int_{-1/2}^{1/2} 0\,dx + \int_{1/2}^{3/2} 1\,dx + \int_{3/2}^{5/2} 2\,dx + \int_{5/2}^{3} 3\,dx = 4$

Illustration :

Evaluate : $\int_1^4 \ln[x]\,dx$, *[·] is the greatest integer function.*

Sol. *Let* $I = \int_1^4 \ln[x]dx = \int_1^2 \ln[x]dx + \int_2^3 \ln[x]dx + \int_3^4 \ln[x]dx$

$= 0 + \ln 2\int_2^3 dx + \ln 3\int_2^3 dx = \ln 2 + \ln 3 = \ln 6$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Find the value of following integrals

(a) $\int_{-3}^{1} |x+1|\,dx$ (b) $\int_{0}^{\pi} |\cos x - \sin x|\,dx$.

Sol.

(a) We have $|x+1| = -(x+1)$ for $x \le -1$; $= x+1$ for $x > -1$

Hence, we have

$$\int_{-3}^{1} |x+1|\,dx = \int_{-3}^{-1} -(x+1)\,dx + \int_{-1}^{1} (x+1)\,dx = \left[\frac{(x+1)^2}{2}\right]_{-3}^{-1} + \left[\frac{(x+1)^2}{2}\right]_{-1}^{1}$$

$$= -(0-2) + (2-0) = 4.$$

(b) We have

$$|\cos x - \sin x| = \sqrt{2}\left|\sin x\left(x - \frac{\pi}{4}\right)\right| = -\sqrt{2}\,\sin\left(x - \frac{\pi}{4}\right) \qquad 0 \le x \le \frac{\pi}{4}$$

$$= \sqrt{2}\,\sin\left(x - \frac{\pi}{4}\right) \qquad \frac{\pi}{4} < x \le \pi$$

Hence, we have

$$\int_{0}^{\pi} |\cos x - \sin x|\,dx = -\sqrt{2}\int_{0}^{\pi/4} \sin\left(x - \frac{\pi}{4}\right)dx + \sqrt{2}\int_{\pi/4}^{\pi} \sin\left(x - \frac{\pi}{4}\right)dx$$

$$= \sqrt{2}\left[\cos\left(x - \frac{\pi}{4}\right)\right]_{0}^{\pi/4} - \sqrt{2}\left[\cos\left(x - \frac{\pi}{4}\right)\right]_{\pi/4}^{\pi}$$

$$= \sqrt{2}\left[\cos 0 - \cos\frac{\pi}{4}\right] - \sqrt{2}\left[\cos\left(x - \frac{\pi}{4}\right) - \cos 0\right]$$

$$= \sqrt{2}\left(1 - \frac{1}{\sqrt{2}}\right) - \sqrt{2}\left(\frac{1}{\sqrt{2}} - 1\right) = \sqrt{2} - 1 + 1 + \sqrt{2} = 2\sqrt{2}.$$

P – 4 : $$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} (f(x) + f(-x))\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_{0}^{a} f(x)\,dx & \text{if } f(x) \text{ is even} \end{cases}$$

Proof: $I = \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$ Put $x = -t$ in first integral.

$$= \int_{a}^{0} f(-t)(-dt) + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(-t)\,dt + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} \{f(x) + f(-x)\}\,dx$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Show that $\int_{-1/2}^{1/2} \sec x \ln \frac{1-x}{1+x}\, dx = 0$.

Sol. Let $f(x) = \sec x \ln\left(\frac{1-x}{1+x}\right)$ then $f(-x) = \sec(-x) \ln\left(\frac{1+x}{1-x}\right) = -f(x)$

Illustration :

Show that $\int_{-1/2}^{1/2}\left([x] + \ln\left(\frac{1+x}{1-x}\right)\right)dx = -\frac{1}{2}$.

Sol. $I = \int_{-1/2}^{1/2}\left([x] + \ln\left(\frac{1+x}{1-x}\right)\right)dx = \int_{-1/2}^{1/2}[x]\,dx + \int_{-1/2}^{1/2}\ln\left(\frac{1+x}{1-x}\right)dx$

$= \int_{-1/2}^{1/2}[x]\,dx + 0 \qquad \left(\ln\left(\frac{1+x}{1-x}\right) \text{ is an odd function}\right)$

$= \int_{-1/2}^{0} -1\,dx + \int_{0}^{1/2} 0\,dx = -\frac{1}{2}$

Illustration :

The value of $\int_{-1}^{3}\left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$ is

(A) π (B) 2π (C) 3π (D) $5\pi/2$

Sol. $I = \int_{-1}^{3}\left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$

$= \int_{-1}^{1}\left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx + \int_{1}^{3}\left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$

$= 0 + \int_{1}^{3}\tan^{-1}\left(\frac{x}{x^2+1}\right) + \cot^{-1}\left(\frac{x}{x^2+1}\right)dx = 0 + \int_{1}^{3}\frac{\pi}{2}dx = 2 \times \frac{\pi}{2} = \pi$ **Ans.**

Illustration :

Evaluate : $\int_{-1}^{1} x^3 \tan(x^2)\, dx$

Sol. Since $x^3 \tan(x^2)$, $x \in [-1, 1]$ is an odd function, hence by property (4), we have

$$\int_{-1}^{1} x^3 \tan(x^2)\, dx = 0.$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Practice Problem

Evaluate the following definite integral

Q.1 $\int_{-\pi/2}^{0} |\sin x + \cos x| \, dx$

Q.2 $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2} \, dx$

Q.3 $\int_{-2}^{2} \frac{x^2-x}{\sqrt{x^2+4}} \, dx$

Q.4 $\int_{0}^{\sqrt{3}} \sin^{-1}\frac{2x}{1+x^2} \, dx$

Q.5 $\int_{-1}^{1} \frac{(2x^{332}+x^{998}+4x^{1668}\cdot\sin x^{691})}{1+x^{666}} \, dx$

Answer key

Q.1 $2(\sqrt{2}-1)$.

Q.2 $\frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}$

Q.3 $4\sqrt{2} - 4\ln(\sqrt{2}+1)$

Q.4 $\frac{\pi\sqrt{3}}{3}$

Q.5 $\frac{\pi+4}{666}$

P – 5 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ or $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof : $I = \int_a^b (a+b-x)\,dx$ Put $a+b-x=t \Rightarrow -dx = dt$ & $I = \int_b^a f(t)(-dt)$

$= \int_a^b f(t)\,dt = \int_a^b f(x)\,dx$

P – 6 : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx \Rightarrow \left[\begin{array}{ll} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \end{array}\right.$

Proof : $I = \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$ Put $x = 2a - t$ in 2nd integral

$\Rightarrow I = \int_0^a f(x)dx + \int_a^0 f(2a-t)(-dt) = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$= \int_0^a \{f(x) + f(2a-x)\}\,dx$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Evaluate : $\int_{50}^{100} \frac{\ell nx}{\ell nx+\ell n(150-x)}dx$

Sol. $I = \int_{50}^{100} \frac{\ell nx}{\ell nx+\ell n(150-x)}dx$ *using P-5* $[x \to 100 + 50 - x]$

$$I = \int_{50}^{100} \frac{ln(150-x)}{ln(150-x)+ln\,x}dx$$

$$I + I = \int_{50}^{100} 1\,dx = 50 \Rightarrow I = 25$$

Illustration :

Evaluate : $\int_{0}^{\pi} \frac{dx}{1+2^{tanx}}$

Sol. $I = \int_{0}^{\pi} \frac{dx}{1+2^{tanx}}$ *using P-5* $[x \to \pi - x]$

$$I = \int_{0}^{\pi} \frac{dx}{1+2^{-tan\,x}} = \int_{0}^{\pi} \frac{2^{\tan x}}{1+2^{\tan x}}dx$$

$$\Rightarrow I + I = \int_{0}^{\pi} 1\,dx = \pi \text{ or } I = \frac{\pi}{2}$$

Illustration :

Evaluate : $\int_{-\pi/4}^{\pi/4} \frac{tan^2 x}{1+e^x}dx$

Sol. $I = \int_{-\pi/4}^{\pi/4} \frac{tan^2 x}{1+e^x}dx$ *using P-5* $[x \to 0 - x]$

$$\Rightarrow I = \int_{-\pi/4}^{\pi/4} \frac{tan^2 x}{1+e^{-x}}dx \text{ or } I + I = \int_{-\pi/4}^{\pi/4} tan^2 x\,dx = 2\int_{0}^{\pi/4}(sec^2 x - 1)dx$$

$$\Rightarrow I = [tan\,x - x]_0^{\pi/4} = 1 - \frac{\pi}{4}$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Find the value of following integral

(a) $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx$ (b) $\displaystyle\int_2^{3} \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$

Sol.

(a) $$\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

Also, we have by property P-5

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

Adding the above integrals, we have

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} 1\,dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

i.e. $I = \dfrac{\pi}{4}$.

(b) *Let* $I = \displaystyle\int_2^{3} \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$

Also, we have by property P-5

$$I = \int_2^{3} \frac{\sqrt{5-(5-x)}}{\sqrt{5-x}+\sqrt{5-(5-x)}}\,dx = \int_2^{3} \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$$

Adding the above integrals, we have

$$2I = \int_2^{3} \frac{\sqrt{5-x}\,\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx = \int_2^{3} 1\,dx = [x]_2^3 = 1$$

i.e. $I = \dfrac{1}{2}$.

Illustration :

Evaluate : $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x}$

Sol. $$\int_0^{\pi/2} \frac{dx}{1+\sin x} = \int_0^{\pi/2} \frac{dx}{1+\sin(\pi/2-x)} = \int_0^{\pi/2} \frac{dx}{1+\cos x}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sec^2\frac{x}{2}\,dx = \frac{1}{2}\left[\frac{\tan x/2}{1/2}\right]_0^{\pi/2} = 1.$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

$$I=\int_0^{2\pi} \sin^4 x\,dx = k\int_0^{\pi/2}\cos^4 x\,dx \quad \text{find value of } k?$$

Sol. Let $f(x) = \sin^4 x$ then $f(2\pi - x) = \sin^4(2\pi - x) = f(x)$

using P-6 $\quad I = 2\int_0^{\pi}\sin^4 x\,dx$, again using P-6 $\quad I = 4\int_0^{\pi/2}\sin^4 x\,dx$

using P-5 $\quad I = 4\int_0^{\pi/2}\cos^4 x\,dx$ **Ans. 4**

Illustration :

Evaluate : $\quad I = \int_0^{2\pi} x\cdot\cos^5 x\;dx$

Sol. Using P-5 $\quad I = \int_0^{2\pi}(2\pi - x)\cos^5 x\,dx$

$$\Rightarrow \quad I + I = \int_0^{2\pi}(x + 2\pi - x)\cos^5 x\,dx \quad \text{or } I = \int_0^{2\pi}\pi\cos^5 x\,dx = 2\pi\int_0^{\pi}\cos^5 x\,dx \quad \text{[using P-5]}$$

$= 2\pi \times 0 \quad$ [using P-6 as $\cos^5(\pi - x) = -\cos^5 x$]

Illustration :

Evaluate : $\displaystyle\int_0^{\pi}\frac{\pi - x}{a^2\cos^2 x + b^2\sin^2 x}\,dx$

Sol. Let $\displaystyle I = \int_0^{\pi}\frac{\pi - x}{a^2\cos^2 x + b^2\sin^2 x}\,dx$ using P-5

we have

$$I = \int_0^{\pi}\frac{\pi - (\pi - x)}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}\,dx = \int_0^{\pi}\frac{x}{a^2\cos^2 x + b^2\sin^2 x}\,dx$$

Adding the above integrals, we have

$$2I = \int_0^{\pi}\frac{\pi}{a^2\cos^2 x + b^2\sin^2 x}\,dx \qquad \text{i.e.} \qquad I = \frac{\pi}{2}\int_0^{\pi}\frac{\sec^2 x}{a^2 + b^2\tan^2 x}\,dx$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Hence, we have

$$I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx \qquad \text{[using P-6]}$$

Putting $\tan x = t$ *and* $\sec^2 x\, dx = dt$, *we have*

$$I = \pi \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} = \frac{\pi}{ab} \lim_{t \to \infty} \left[\tan^{-1}\left(\frac{bt}{a}\right) \right]_0^t = \frac{\pi}{ab} \cdot \frac{\pi}{2} = \frac{\pi^2}{ab}.$$

Practice Problem

Q.1 Prove that $\int_0^{\pi/4} \ln(1 + \tan x)\, dx = \frac{\pi}{8} \ln 2$

Q.2 $\int_{\pi/4}^{3\pi/4} \frac{x \sin x}{1 + \sin x} dx$

Q.3 $\int_0^{\pi} \frac{(ax+b)\sec x \tan x}{4 + \tan^2 x} dx$ (a,b>0)

Q.4 $\int_0^{\pi} \frac{(2x+3)\sin x}{(1+\cos^2 x)} dx$

Q.5 $\pi \int_0^{\pi} \frac{x^2 \sin 2x \cdot \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx$

Answer key

Q.2 $\pi[\frac{\pi}{4} - (\sqrt{2} - 1)]$ Q.3 $\frac{(a\pi + 2b)\pi}{3\sqrt{3}}$ Q.4 $\frac{\pi(\pi+3)}{2}$ Q.5 8

P–7 : $\int_0^{nT} f(x)\, dx = n \int_0^{T} f(x)\, dx$ where $f(T + x) = f(x)$ $n \in I$

Illustration :

Show that $\int_0^{1000} e^{x-[x]} dx = 1000(e-1)$

Sol. $I = \int_0^{1000} e^{x-[x]} dx = 1000 \int_0^{1} e^{x-[x]} dx$ $\left(e^{x-[x]} \text{ has period } 1\right)$

$$= 1000 \int_0^1 e^x dx = 1000 \left[e^x\right]_0^1 = 1000\,(e-1)$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Prove that : $\int_0^{n\pi+v} |\cos x|\,dx = (2n+2-\sin v)$; *where* $\frac{\pi}{2} < v < \pi$ & $n \in N$.

Sol. $I = \int_0^{n\pi+v} |\cos x|\,dx = \int_0^{n\pi} |\cos x|\,dx + \int_{n\pi}^{n\pi+v} |\cos x|\,dx$ *(Put $x - n\pi = t$ in 2^{nd} integral)*

$$= n\int_0^{\pi} |\cos x|\,dx + \int_0^{v} |\cos(n\pi + t)|\,dt = 2n\int_0^{\pi/2} |\cos x|\,dx + \int_0^{v} |\cos t|\,dt$$

$$= 2n + \int_0^{\pi/2} \cos t\,dt + \int_{\pi/2}^{v} -\cos t\,dt = 2n + 1 + 1 - \sin v = 2n + 2 - \sin v$$

Illustration :

Evaluate : $\int_0^{10} \sqrt{1-\cos \pi x}\,dx$

Sol. *We have* $\sqrt{1-\cos \pi x} = \sqrt{2}\left|\sin \frac{\pi x}{2}\right|$

which is a periodic function, having period $T = \frac{1}{2} \times \frac{2\pi}{\pi/2} = 2.$

Hence, we have

$$\int_0^{10} \sqrt{1-\cos \pi x}\,dx = \int_0^{10} \sqrt{2}\left|\sin \frac{\pi x}{2}\right| dx = 5\sqrt{2}\int_0^{2} \left|\sin \frac{\pi x}{2}\right| dx$$

$$= 5\sqrt{2}\int_0^{2} \sin\left(\frac{\pi x}{2}\right) dx = 5\sqrt{2}\left[\frac{-\cos(\pi x/2)}{\pi/2}\right]_0^2 = \frac{20\sqrt{2}}{\pi}.$$

(B) DERIVATIVES OF ANTIDERIVATIVES (LEIBNITZ RULE) :

If f is continuous then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(g(x)).\,g'(x)$$ (integral of a continuous function is always differentiable)

Proof : Let $\int f(t)dt = F(t) + c$ then $\int_{g(x)}^{h(x)} f(t)dt = F(h(x)) - F(g(x))$

$$\Rightarrow \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = F'(h(x))\,h'(x) - F'(g(x))\,g'(x) = f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Let $G(x) = \int_{2}^{x^2} \frac{dt}{1+\sqrt{t}}$ $(x > 0)$. Find $G'(9)$.

Sol. $G'(x) = \frac{1}{1+\sqrt{x^2}} \cdot 2x - 0 = \frac{2x}{1+x} \quad \Rightarrow G'(9) = \frac{2\times 9}{1+9} = \frac{9}{5}$

Illustration :

If $f(x) = \int_{e^{2x}}^{e^{3x}} \frac{t}{\ell n t} dt$ $x > 0$. Find derivative of $f(x)$ w.r.t. $\ln x$ when $x = \ln 2$.

Sol. $f'(x) = \frac{e^{3x}}{\ln(e^{3x})} \cdot 3e^{3x} - \frac{e^{2x}}{\ln(e^{2x})} \cdot 2e^{2x} = \frac{e^{6x}}{x} - \frac{e^{4x}}{x}$

$f'(\ln 2) = \frac{e^{6\ln 2} - e^{4\ln 2}}{\ln 2} = \frac{2^6 - 2^4}{\ln 2} = \frac{48}{\ln 2}$

Illustration :

Evaluate : $\lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\,dx}{x\int_0^x \tan x\,dx}$.

Sol. $\lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\,dx}{x\int_0^x \tan x\,dx} = \lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\,dx}{x^3} \cdot \frac{x^2}{\int_0^x \tan x\,dx}$

Now, we have

$$\lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\,dx}{x^3} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{1-\cos 2x}{3x^2} = \frac{2}{3}$$

and $\lim_{x\to 0} \frac{x^2}{\int_0^x \tan x\,dx} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{2x}{\tan x} = 2.$

Hence, we have $L = \frac{4}{3}$.

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Practice Problem

Q.1 Evaluate $\lim\limits_{x \to 0} \dfrac{\int_0^{x^2} \cos t^2 \, dt}{x \sin x}$

Q.2 Evaluate $\int_0^{200\pi} \sqrt{1+\cos x}\, dx$

Q.3 Value of $\lim\limits_{x \to 0} \dfrac{\int_0^{x} x e^{t^2} dt}{1-e^{x^2}}$ is

(A) -1 (B) $-\dfrac{1}{2}$ (C) 0 (D) -2

Q.4 If $\int_0^{n\pi} \dfrac{x|\sin x|}{1+|\cos x|} dx$ $(n \in N)$ is equal to $100\pi \ln 2$, then find the value of n.

Q.5 If $y = x^{\int_1^x \ln t \, dt}$, find $\dfrac{dy}{dx}$ at $x = e$.

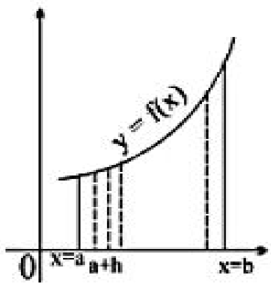
Q.6 Let $g(x) = x^c \cdot e^{2x}$ & let $f(x) = \int_0^x e^{2t} \cdot (3t^2+1)^{1/2} dt$. For a certain value of 'c', the limit of $\dfrac{f'(x)}{g'(x)}$ as $x \to \infty$ is finite and non zero. Determine the value of 'c' and the limit.

Answer key

Q.1	1	Q.2	$400\sqrt{2}$	Q.3	A	Q.4	10
Q.5	$1+e$	Q.6	$c = 1$ and $\lim\limits_{x\to\infty}$ will be $\dfrac{\sqrt{3}}{2}$				

(C) DEFINITE INTEGRAL AS A LIMIT OF SUM :

Fundamental theorem of integral calculus :



$$\int_a^b f(x)\,dx = \lim_{\substack{h \to 0 \\ n \to \infty}} h\left[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a + \overline{n-1}\,h)\right]$$

or $$\int_a^b f(x)\,dx = \lim_{\substack{h \to 0 \\ n \to \infty}} h \sum_{r=0}^{n-1} f(a+rh) \quad \text{where} \quad b - a = nh$$

Note: Evaluating a definite integral by evaluating the limit of a sum is called evaluating definite integral by first principle or by a b initio method.

Put $a = 0$ & $b = 1$ $\Rightarrow$ $nh = 1$, we have

$$\int_0^1 f(x)\,dx = \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right); \quad \text{replace } \frac{1}{n} \to dx;\ \Sigma \to \int\ ;\ \frac{r}{n} \to x$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Evaluate $\int_a^b \cos x\, dx$ *as the limit of a sum.*

Sol. *We have* $\int_a^b \cos x\, dx = \lim_{h\to 0}\sum_{r=1}^{n} h\, f(a+rh) = \lim_{h\to 0}\sum_{r=1}^{n} h\cos(a+rh)$

$$= \lim_{h\to 0} h\left[\cos(a+h)+\cos(a+2h)+....+\cos(a+nh)\right]$$

Now, let $S = \cos(a+h)+\cos(a+2h)+......+\cos(a+nh)$. *Multiplying both sides by* $2\sin\frac{h}{2}$, *we have*

$$\left(2\sin\frac{h}{2}\right)S = 2\sin\frac{h}{2}\cos(a+h)+2\sin\frac{h}{2}\cos(a+2h)+....+2\sin\frac{h}{2}\cos(a+nh)$$

$$= \sin\left(a+\frac{3}{2}h\right)-\sin\left(a+\frac{1}{2}h\right)+\sin\left(a+\frac{5}{2}h\right)-\sin\left(a+\frac{3}{2}h\right)$$

$$+......+\sin\left(a+\frac{2n+1}{2}h\right)-\sin\left(a+\frac{2n-1}{2}h\right)$$

$$= \sin\left(a+\frac{2n+1}{2}h\right)-\sin\left(a+\frac{1}{2}h\right)$$

$$= \sin\left(a+nh+\frac{h}{2}\right)-\sin\left(a+\frac{h}{2}\right) = \sin\left(b+\frac{h}{2}\right)-\sin\left(a+\frac{h}{2}\right)$$

Hence, we have

$$\int_a^b \cos x\, dx = \lim_{h\to 0}\frac{h\left[\sin\left(b+\frac{h}{2}\right)\right]-\sin\left(a+\frac{h}{2}\right)}{2\sin\left(\frac{h}{2}\right)} = \sin b-\sin a.$$

Illustration :

Find the value of $\lim_{n\to\infty}\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+......+\frac{1}{4n}$

Sol. $S = \lim_{n\to\infty}\sum_{r=0}^{3n}\frac{1}{n+r} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{3n}\frac{1}{1+\left(\frac{r}{n}\right)} = \int_0^3\frac{1}{1+x}d = \left[\ln(1+x)\right]_0^3 = \ln 4$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Evaluate $\lim\limits_{n\to\infty}\prod\limits_{r=1}^{n}\left(\frac{n+r}{n}\right)^{1/n}$.

Sol. *We have*

$$S=\lim_{n\to\infty}\prod_{r=1}^{n}\left(\frac{n+r}{n}\right)^{1/n}=Lim\left[\frac{n+1}{n}\cdot\frac{n+2}{n}\ldots\ldots\frac{n+n}{n}\right]^{1/n}$$

Taking in both sides, we have

$$\ln S=\lim_{n\to\infty}\frac{1}{n}\left[\ln\left(\frac{n+1}{n}\right)+\ln\left(\frac{n+2}{n}\right)+\ldots\ldots+\ln\left(\frac{n+n}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{n=1}^{n}\ln\left(1+\frac{r}{n}\right)=\int_0^1\ln(1+x)\,dx$$

$$=\ln 2-(1-\ln 2)=\ln 4-1=\ln\left(\frac{4}{e}\right)$$

$\therefore\qquad S=\frac{4}{e}.$

Illustration :

If $n\to\infty$ *, then find the limit of* $\frac{1}{n}\sum\limits_{r=1}^{n}\sin^{2k}\left(\frac{r\pi}{2n}\right)$.

Sol. *Let* $P=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\sin^{2k}\left(\frac{r\pi}{2n}\right)=\int_0^1\sin^{2k}\left(\frac{\pi}{2}x\right)dx$

Put $\frac{\pi}{2}x=t\qquad\therefore\qquad dx=\frac{2}{\pi}dt$

$$=\frac{2}{\pi}\int_0^{\pi/2}\sin^{2k}t\,dt\quad=\frac{2}{\pi}\frac{(2k-1)(2k-3)(2k-5)\ldots\ldots3.1}{2k(2k-2)(2k-4)\ldots\ldots4.2}\cdot\frac{\pi}{2}$$

$$=\frac{2k(2k-1)(2k-2)(2k-3)\ldots\ldots3.2.1}{[2k(2k-2)(2k-4)\ldots\ldots4.2]^2}$$

Hence $P=\frac{2k!}{(2^k k!)^2}=\frac{2k!}{2^{2k}(k!)^2}$

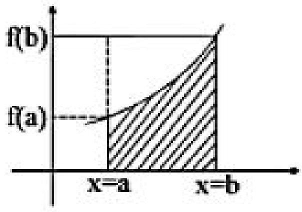
Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

(D) ESTIMATION OF DEFINITE INTEGRAL AND GENERAL INEQUALITIES IN INTEGRATION:

Not all integrals can be evaluated using the technique discussed so far. In this situation we try to obtain the interval in which value of integral may lie by using following method.

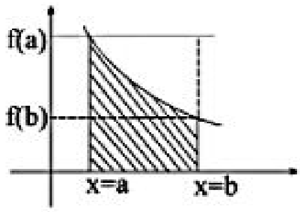
(a) For a monotonic increasing function in (a, b)

$$(b-a)\,f(a) < \int_a^b f(x)\,dx < (b-a)\,f(b)$$



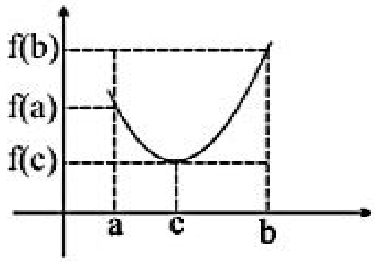
(b) For a monotonic decreasing function in (a, b)

$$f(b).\,(b-a) < \int_a^b f(x)\,dx < (b-a)\,f(a)$$



(c) For a non monotonic function in (a, b)

$$f(c)\,(b-a) < \int_a^b f(x)\,dx < (b-a)\,f(b)$$



(d) In addition to this note that

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$ inequality holds when f(x) lies completely above the x-axis

(e) If $h(x) \le f(x) \le g(x)\ \forall\ x \in [a, b]$ then $\int h(x)dx < \int f(x)dx < \int g(x)dx$

Illustration :

Show that $1 < \displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\,dx < \frac{\pi}{2}$.

Sol. $f(x) = \dfrac{\sin x}{x}$ or $f'(x) = \dfrac{x\cos x - \sin x}{x^2} = \dfrac{\cos x}{x^2}[x - \tan x]$

$\Rightarrow f'(x) < 0$ hence $f(x)_{min} = \dfrac{2}{\pi}$ $f(x)_{max} = 1$.

$\Rightarrow \dfrac{2}{\pi}\left(\dfrac{\pi}{2} - 0\right) < I < 1\left(\dfrac{\pi}{2} - 0\right)$ or $1 < I < \dfrac{\pi}{2}$.

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Show that $\frac{1}{4} \le \int_0^1 \frac{dx}{1+x^2+2x^5} \le 1$.

Sol. *Consider the following function* $f(x) = \frac{1}{1+x^2+2x^5}$, $x \in [0, 1]$

In the interval [0, 1], f(x) is strictly decreasing, therefore we have

$$f(1) \le f(x) \le f(0) \qquad \text{i.e.} \qquad \frac{1}{4} \le f(x) \le 1$$

Hence, we have

$$(1-0)\,\frac{1}{4} \le \int_0^1 f(x)\,dx \le (1-0)\,1 \qquad \text{[by property (7)]}$$

i.e. $\frac{1}{4} \le \int_0^1 f(x)\,dx \le 1$ *which is the desired result.*

Illustration :

Prove that $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi\sqrt{2}}{8}$

Sol. $4-2x^2 \le 4-x^2-x^3 \le 4-x^2$

$$\Rightarrow \frac{1}{\sqrt{4-2x^2}} \ge \frac{1}{\sqrt{4-x^2-x^3}} \ge \frac{1}{\sqrt{4-x^2}} \quad \Rightarrow \int_0^1 \frac{dx}{\sqrt{4-x^2}} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \int_0^1 \frac{dx}{\sqrt{4-2x^2}}$$

$$\Rightarrow \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 < I < \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1 \quad or \quad \frac{\pi}{6} < I < \frac{\sqrt{2}}{8}\pi$$

Practice Problem

Q.1 $\lim_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\dots\dots\dots+2\sqrt{n}}{n\sqrt{n}}$.

Q.2 $\lim_{n\to\infty} \frac{[(n+1)(n+2)\dots\dots\dots(n+n)]^{1/n}}{n}$

Q.3 Prove that $\frac{e-1}{3} < \int_1^e \frac{dx}{2+\ln x} < \frac{e-1}{2}$

Q.4 Prove the inequalities: $2\,e^{-1/4} < \int_0^2 e^{x^2-x}\,dx < 2e^2$.

Answer key

Q.1 $\frac{16}{3}$

Q.2 $\frac{4}{e}$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

(E) WALLI'S THEORM :

$$\int_0^{\pi/2} \sin^n x \cos^m x \, dx = \frac{[(n-1)(n-3)....1 \text{ or } 2]\,[(m-1)(m-3)...1 \text{ or } 2]}{(m+n)(m+n-2)...1 \text{ or } 2} K$$

(m, n are non-negative integer)

$$\text{where } K = \begin{cases} \dfrac{\pi}{2} & \text{if } m, n \text{ both are even} \\ 1 & \text{otherwise} \end{cases}$$

Illustration :

Evaluate : $\int_0^{2\pi} x \sin^6 x \cos^4 x dx$

Sol. $I = \int_0^{2\pi} x \sin^6 x \cos^4 x dx$ *using P5*

$$I = \int_0^{2\pi} (2\pi - x) \sin^6 x \cos^4 x \, dx \Rightarrow I + I = \int_0^{2\pi} 2\pi \sin^6 x \cos^4 x \, dx$$

$$\text{or } I = \pi \int_0^{2\pi} \sin^6 x \cos^4 x \, dx \quad \text{Using P6 twice}$$

$$I = 4\pi \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = 4\pi \frac{(5 \cdot 3 \cdot 1)(3 \cdot 1)}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} = \frac{3\pi^2}{128} \quad \textit{Ans.}$$

Illustration :

Evaluate : $\int_0^{3\pi/2} \cos^4 3x \cdot \sin^2 6x dx$

Sol. $I = \int_0^{3\pi/2} \cos^4 3x \cdot \sin^2 6x dx = \int_0^{3\pi/2} 4 \sin^2 (3x) \cos^6 (3x) \, dx$

Put $3x = t$ *to get*

$$I = \int_0^{9\pi/2} 4 \cdot \sin^2 t \cdot \cos^6 t \frac{1}{3} dt = \frac{4}{3}\left[\int_0^{4\pi} \sin^2 t \cos^6 t \, dt + \int_{4\pi}^{9\pi/2} \sin^2 t \cos^6 t \, dt\right]$$

$$= \frac{4}{3}\left[4\int_0^{\pi} \sin^2 t \cos^6 t \, dt + \int_{4\pi}^{9\pi/2} \sin^2 t \cos^6 t \, dt\right] \quad (\sin^2 t \cos^6 t \text{ has period } \pi)$$

$$= \frac{4}{3}\left[8\int_0^{\pi/2} \sin^2 t \cos^6 t \, dt + \int_0^{\pi/2} \sin^2 t \cos^6 t \, dt\right] \quad (\text{using P-6 in } 1^{st} \text{ integral and } t - 4\pi = z \text{ in } 2^{nd})$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Illustration :

Evaluate : $\int_0^{\pi/2} \cos^7 x \, dx$

Sol. $I = \int_0^{\pi/2} \cos^7 x \, dx = \dfrac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3 \cdot 1}$.

REDUCTION METHOD :

For integration of type $\int_a^b (f(x))^n dx$

where 'n' is big natural number it is possible to reduce 'n' by some methods specially by parts.

Illustration :

Let $I_n = \int_0^1 (1-x^a)^n dx$. *Find the ratio* I_n/I_{n+1}.

Sol. *We have* $I_{n+1} = \int_0^1 (1-x^a)^n dx$

$= \left[x(1-x^a)^{n+1}\right]_0^1 + (n+1)a \int_0^1 x^a (1-x^a)^n dx$

[taking 1 as one function and integrating by parts]

$= (n+1)a \int_0^1 (x^a - 1 + 1)(1-x^a)^n dx = (n+1)a \int_0^1 (1-x^a)^n dx - (n+1)a \int_0^1 (1-x^a)^{n+1} dx$

$= (n+1)a \, I_n - (n+1) \, a \, I_{n+1}$

Simplifying, we have $\dfrac{I_n}{I_{n+1}} = 1 + \dfrac{1}{(n+1)a}$.

Illustration :

Given $I_n = \int_0^{\pi/4} (\tan x)^n dx \quad (n \in N)$

Prove that $I_n + I_{n-1} = \dfrac{1}{n-1}$ $(n \geq 3)$. *Hence find value of* I_6.

Sol. $I_n = \int_0^{\pi/4} (\tan x)^n dx$, $\quad I_{n-2} = \int_0^{\pi/4} (\tan x)^{x-2} dx$

$I_n + I_{n-1} = \int_0^{\pi/4} (\tan x)^n + (\tan x)^{n-2} dx$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Put $\tan x = t$ *to get* $I_n + I_{n-2} = \int_0^1 t^{n-2}\,dt = \left.\dfrac{t^{n-1}}{n-1}\right|_0^1 = \dfrac{1}{n-1}$

Also $I_2 = \int_0^{\pi/4} \tan^2 x\,dx = \int_0^{\pi/4} (\sec^2 x - 1)dx = [\tan x - x]_0^{\pi/4} = 1 - \dfrac{\pi}{4}$

Using $I_n + I_{n-2} = \dfrac{1}{n-1}$, $I_4 + I_2 = \dfrac{1}{3}$ *and* $I_6 + I_4 = \dfrac{1}{5}$

$\Rightarrow$ $I_6 = I_2 - \dfrac{2}{\sqrt{5}}$ *or* $I_6 = 1 - \dfrac{\pi}{4} - \dfrac{2}{15} = \dfrac{13}{15} - \dfrac{\pi}{4}$

(F) SOME INTEGRALS WHICH CANNOT BE FOUND IN TERMS OF KNOWN ELEMENTRY FUNCTIONS :

(1) $\int \dfrac{\sin x}{x}\,dx$ (2) $\int \dfrac{\cos x}{x}\,dx$ (3) $\int \sqrt{\sin x}\,dx$ (4) $\int \sin x^2\,dx$

(5) $\int \cos x^2\,dx$

(6) $\int x \tan x\,dx$ (7) $\int e^{-x^2}\,dx$ (8) $\int e^{x^2}\,dx$ (9) $\int \dfrac{x^3}{1+x^5}\,dx$

(10) $\int (1+x^2)^{1/3}\,dx$ (11) $\int \dfrac{dx}{\ln x}$ (12) $\int \sqrt{1+k^2\sin^2 x}\,dx \quad k \in R$

Practice Problem

Q1 Let $U_{10} = \int_0^{\frac{\pi}{2}} x \sin^{10} x\,dx$, then find the value of $\left(\dfrac{100U_{10} - 1}{U_8}\right)$.

Q.2 If $U_n = \int_0^{\pi/2} \dfrac{\sin^2 nx}{\sin^2 x}\,dx$, then show that $U_1, U_2, U_3, \ldots, U_n$ constitute an AP. Hence or otherwise find the value of U_n.

Q.3 Evaluate : $\int_0^{4\pi} \sin^4 x \cos^2 x\,dx$

Answer key

Q1 90 Q.2 $U_n = \dfrac{n\pi}{2}$ Q.3 $\dfrac{\pi}{4}$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Solved Examples

Q.1 Prove that $\int_0^\infty \frac{dx}{\left[x+\sqrt{(1+x^2)}\right]^2} = \frac{n}{n^2-1} (n>1)$

Sol. L.H.S. $= \int_0^\infty \frac{dx}{\left(x+\sqrt{1+x^2}\right)^n}$ Put $x+\sqrt{(1+x^2)} = t$ ......(i)

$\therefore \quad \left(1+\frac{x}{\sqrt{1+x^2}}\right)dx = dt$ when $x \to 0$ then $t \to 1$ $\quad x \to \infty$ then $t \to \infty$

$\Rightarrow \quad \frac{t\,dx}{\sqrt{1+x^2}} = dt \quad \Rightarrow \quad dx = \frac{\sqrt{(1+x^2)}}{t}dt \quad \Rightarrow \quad \sqrt{1+x^2} - x = \frac{1}{t}$ ......(ii)

Adding (i) & (ii) we get $2\sqrt{1+x^2} = t+\frac{1}{t}$

$\Rightarrow \quad \sqrt{(1+x^2)} = \frac{t^2+1}{2t} \quad \therefore \quad dx = \frac{(t^2+1)}{2t^2}dt$

Hence L.H.S. $= \int_1^\infty \frac{(t^2+1)dt}{2t^2t^n} = \frac{1}{2}\int_1^\infty \left(\frac{1}{t^n}+\frac{1}{t^{n+2}}\right)dt = \frac{1}{2}\left[-\frac{1}{(n-1)t^{n-1}} - \frac{1}{(n+1)t^{n+1}}\right]_1^\infty$

$= \frac{1}{2}\left[(-0-0)-\left(-\frac{1}{n-1}-\frac{1}{n+1}\right)\right] = \frac{1}{2}\left[\frac{1}{n-1}+\frac{1}{n+1}\right] \quad (\because\ n>1)$

$= \frac{n}{n^2-1} =$ R.H.S.

Q.2 If $f(x) = \int_0^x \frac{e^t}{t}dt, x>0$. Prove that $\int_1^x \frac{e^t dt}{(t+a)} = e^{-a}[f(x+a)-f(1+a)]$.

Sol. R.H.S. $= e^{-a}[f(x+a)-f(1+a)] = e^{-a}\left[\int_0^{x+a}\frac{e^t}{t}dt - \int_0^{1+a}\frac{e^t}{t}dt\right] = e^{-a}\left[\int_0^{x+a}\frac{e^t}{t}dt + \int_{1+a}^0 \frac{e^t}{t}dt\right]$

$= e^{-a}\int_{1+a}^{a+x}\frac{e^t}{t}dt$ Put $t = a+y \quad \therefore \quad dt = dy$

$= e^{-a}\int_1^x \frac{e^{a+y}}{(a+y)}dy = e^{-a}\cdot e^a \int_1^x \frac{e^y}{(a+y)}dy = \int_1^x \frac{e^y}{a+y}dy = \int_1^x \frac{e^y}{(a+t)}dt$ (by Prop.)

= L.H.S.

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Q.3 Evaluate $\int_{-\pi}^{\pi}\left|x\sin[x^2-\pi]\right|dx$, $[\cdot]$ is the greatest integer function.

Sol. Let $I=\int_{-\pi}^{\pi}\left|x\sin[x^2-\pi]\right|dx \;=2\int_{0}^{\pi}\left|x\sin[x^2-\pi]\right|dx$ [it is even function]

$$I=2\left[\int_0^{\sqrt{\pi-3}}\left|x\sin[x^2-\pi]\right|dx+\int_{\sqrt{\pi-3}}^{\sqrt{\pi-2}}\left|x\sin[x^2-\pi]\right|dx+\int_{\sqrt{\pi-2}}^{\sqrt{\pi-1}}\left|x\sin[x^2-\pi]\right|\right.$$

$$+\int_{\sqrt{\pi-1}}^{\sqrt{\pi}}\left|x\sin[x^2-\pi]\right|dx+\ldots\ldots+\int_{\sqrt{\pi+6}}^{\pi}\left|x\sin[x^2-\pi]\right|dx$$

$$=2\left[\int_0^{\sqrt{\pi-3}}x\sin 4\,dx+\int_{\sqrt{\pi-3}}^{\sqrt{\pi-2}}x\sin 3\,dx+\int_{\sqrt{\pi-2}}^{\sqrt{\pi-1}}x\sin 2\,dx+\int_{\sqrt{\pi-1}}^{\sqrt{\pi}}x\sin 1\,dx+0\right.$$

$$+\int_{\sqrt{\pi-1}}^{\sqrt{\pi+2}}x\sin 1\,dx+0+\int_{\sqrt{\pi-1}}^{\sqrt{\pi+2}}x\sin 1\,dx+\ldots\ldots+\int_{\sqrt{\pi-6}}^{\pi}x\sin 6\,dx$$

$$=\{\sin 4\,(\pi-3)+\sin 3(1)+\sin 2(1)+\sin 1(1)+\ldots\ldots+\sin 1\,(1)+\ldots\ldots$$
$$+\sin 1(1)+\ldots\ldots+\sin 6(\pi^2-\pi-6)\}$$
$$=2\sin 1+2\sin 2+3\sin 3+(\pi-2)\sin 4+\sin 5+(\pi^2-\pi-6)\sin 6$$

Q.4 Show that $\frac{\pi}{3\sqrt{3}}\le\int_0^1\frac{dx}{1+x^2+2x^5}\le\frac{\pi}{4}$.

Sol. We have $1+x^2+2x^5\ge 1+x^2$

and $1+x^2+2x^5\le 1+x^2+2x^2=1+3x^2$ $[x^5<x^2 \text{ on } [0,1]]$

Hence, we have $\frac{1}{1+3x^2}\le\frac{1}{1+x^2+2x^5}\le\frac{1}{1+x^2}$

i.e. $\int_0^1\frac{dx}{1+3x^2}\le\int_0^1\frac{dx}{1+x^2+2x^5}\le\int_0^1\frac{dx}{1+x^2}$ [by property (8)]

i.e. $\left[\frac{\tan^{-1}\sqrt{3}\,x}{\sqrt{3}}\right]_0^1\le\int_0^1\frac{dx}{1+x^2+2x^5}\le\left[\tan^{-1}x\right]_0^1$ i.e. $\frac{\pi}{3\sqrt{3}}\le\int_0^1\frac{dx}{1+x^2+2x^5}\le\frac{\pi}{4}$

which is the desired result.

Q.5 Evaluate $\int_{-1}^{1}\frac{|\sin x|}{\sin x}dx$

Sol. We have $\frac{|\sin x|}{\sin x}=-1,\ -1\le x\le 0\ =1,\ 0\le x\le 1\Rightarrow\frac{|\sin x|}{\sin x}$, $x\in[-1,1]$ is an odd function

Hence, by property (4), we have $\int_{-1}^{1}\frac{|\sin x|}{\sin x}=0$.

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Q.6 Evaluate : $\int_0^2 [x^2-1]dx$ where [x] represents integral part of x.

Sol. We have $[x^2-1] = -1,\ 0 \le x < 1 \quad = 0,\ 1 \le x < \sqrt{2} \quad = 1,\ \sqrt{2} \le x < \sqrt{3} \quad = 2,\ \sqrt{3} \le x < 2.$

Hence, we have $\int_0^2 [x^2-1]dx = \int_0^1 -1\,dx + \int_0^{\sqrt{2}} 0\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 1\,dx + \int_{\sqrt{3}}^{2} 1\,dx$

$= [-x]_0^1 + 0 + [x]_{\sqrt{2}}^{\sqrt{3}} + [x]_{\sqrt{3}}^{2} = -1 + \sqrt{3} - \sqrt{2} + 2 - \sqrt{3} = 1 - \sqrt{2}.$

Q.7 Evaluate : $\int_1^3 \frac{dx}{\sqrt{x+1}-\sqrt{x-1}}$

Sol. We have $\int_1^3 \frac{dx}{\sqrt{x+1}-\sqrt{x-1}} = \int_1^3 \frac{\sqrt{x+1}-\sqrt{x-1}}{2}dx = \frac{1}{2}\left[\frac{(x+1)^{3/2}}{3/2} + \frac{(x-1)^{3/2}}{3/2}\right]_1^3$

$= \frac{1}{3}\left[(x+1)^{3/2} + (x-1)^{3/2}\right]_1^3 = \frac{1}{3}\left[4^{3/2} + 2^{3/2} - 2^{3/2}\right] = \frac{8}{3}.$

Q.8 Evaluate the definite integral $\int_0^{\pi/2} \ln(\tan x + \cot x)dx$

Sol. Let $\quad I = \int_0^{\pi/2} \ln(\tan x + \cot x)dx = \int_0^{\pi/2} \ln\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)dx = \int_0^{\pi/2} \ln\left(\frac{1}{\sin x \cos x}\right)dx$

$= \int_0^{\pi/2} \ln\left(\frac{2}{\sin 2x}\right)dx = \int_0^{\pi/2} \ln 2\,dx - \int_0^{\pi/2} \ln(\sin 2x)dx = \frac{\pi}{2}\ln 2 - \int_0^{\pi/2} \ln(\sin 2x)dx.$

Let us put $2x = y$ and $2\,dx = dy$ in the second integral on the RHS. Also, when $x = 0$, then $y = 0$ and when $x = \frac{\pi}{2}$, then $y = \pi$. Hence, we have $\int_0^{\pi/2} \ln(\sin 2x)dx = \frac{1}{2}\int_0^{\pi/2} \ln(\sin y)dy$

$= \frac{1}{2}\cdot 2\int_0^{\pi/2} \ln(\sin y)dy \qquad [\ln(\sin y) = \ln \sin \pi - y]$

$= -\frac{\pi}{2}\ln 2$

Hence, we have $I = \frac{\pi}{2}\ln 2 + \frac{\pi}{2}\ln 2 = \pi \ln 2.$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Q.9 Find f(x) if it satisfies the relation $f(x) = e^x + \int_0^1 \left(x + ye^x\right) f(y)\, dy$.

Sol. We have $f(x) = e^x + x\int_0^1 f(y)\,dy + e^x \int_0^1 y\, f(y)\, dy$

$$= e^x\left(1 + \int_0^1 y\, f(y)\, dy\right) + x\int_0^1 f(y)\, dy = ae^x + bx \text{ (say)}$$

where a, b are constants, given by $a = 1 + \int_0^1 y\, f(y)\, dy = 1 + \int_0^1 y\left(ae^y + by\right)dy$

$$= 1 + \left[(y-1)\, e^y\right]_0^1 + \left[\frac{by^3}{3}\right]_0^1 = 1 + a + \frac{b}{3}$$

and $b = \int_0^1 f(y)\, dy = \int_0^1 \left(ae^y + by\right)dy = \left[ae^y + \frac{by^2}{3}\right]_0^1 = a(e-1) + \frac{b}{2}$

Solving, we have $b = -2$ and $a = \dfrac{-3}{2(e-1)}$

Hence, we have $f(x) = \dfrac{-3e^x}{2(e-1)} - 3x$.

Q.10 If $b = \int_0^1 \frac{e^t}{t+1}\, dt$, then show that $\int_{a-1}^{a} \frac{e^{-t}}{t-a-1}\, dt = be^{-a}$.

Sol. We have $I = \int_{a-1}^{a} \frac{e^{-t}}{t-a-1}\, dt = \int_{a-1}^{-a} \frac{e^{y}}{-y-a-1}(-dy)$ [putting $t = -y$]

$$= e^{-a}\int_{a-1}^{-a} \frac{e^{y+a}}{y+a+1}\, dy = e^{-a}\int_{a-1}^{-a} \frac{e^{u}}{u+1}\, du \quad [\text{putting } y + a = u]$$

$$= -e^{-a}\int_0^1 \frac{e^u}{u+1}\, dx = -be^{-a} \quad \left[\int_0^1 \frac{e^t}{t+1}\, dt = b \text{ given}\right]$$

Q.11 Evaluate the definite integral $\int_0^1 \frac{1-x^2}{1+x^2} \cdot \frac{dx}{\sqrt{1+x^4}}$

Sol. We have $I = \int_0^1 \frac{1-x^2}{1+x^2} \cdot \frac{dx}{\sqrt{1+x^4}} = \int_0^1 \frac{\frac{1}{x^2} - 1}{x + \frac{1}{x}} \cdot \frac{dx}{\sqrt{x^2 + \frac{1}{x^2}}} = \int_0^1 \frac{-d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)\sqrt{\left(x + \frac{1}{x}\right)^2 - 2}}$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

$$= \int_{\infty}^{2} \frac{-dt}{t\sqrt{t^2-2}} \quad [\text{Puting } x + \frac{1}{x} = t] \qquad = \int_{2}^{\infty} \frac{t\,dt}{t^2\sqrt{t^2-2}} \quad [\text{Putting } t^2 - 2 = u^2]$$

$$= \int_{\sqrt{2}}^{\infty} \frac{u\,du}{u(u^2+2)} = \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{u}{\sqrt{2}}\right]_{\sqrt{2}}^{\infty} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$$

Q.12 If $I_1 = \int_0^{n\pi} f\left(\sin^4 x\right)dx$ and $I_2 = \int_0^{\pi} f\left(\sin^4 x\right)dx$. Find the value of $\frac{I_1}{I_2}$.

Sol. We have

$$I = \frac{\int_0^{n\pi} f(\sin^4 x)\,dx}{\int_0^{\pi} f(\sin^4 x)\,dx} = \frac{\int_0^{2n\left(\frac{\pi}{2}\right)} f(\sin^4 x)\,dx}{\int_0^{2\left(\frac{\pi}{2}\right)} f(\sin^4 x)\,dx} = \frac{2n\int_0^{\pi/2} f(\sin^4 x)\,dx}{2\int_0^{\pi/2} f(\sin^4 x)\,dx} \quad [\text{period of } \sin^4 x \text{ is } \frac{\pi}{2}] \ = n.$$

Q.13 Prove that $\int_{1/2}^{2} (\ln x)^2\,dx < \int_{1/2}^{2} |\ln x|\,dx$

Sol. In the interval $\left[\frac{1}{4},\frac{1}{2}\right]$, $|\ln x|$ is a fraction. hence, we have $(\ln x)^2 < |\ln x|$

i.e. $\int_{1/2}^{2} (\ln x)^2\,dx < \int_{1/2}^{2} |\ln x|\,dx$.

Q.14 Show that $\int_0^{k\pi} \sin\left[\frac{2x}{\pi}\right]dx = \frac{\pi}{2}\cdot\frac{\sin k \sin(k+1/2)}{\sin(1/2)}$.

Sol. We have $I = \int_0^{k\pi} \sin\left[\frac{2x}{\pi}\right]dx = \int_0^{\pi/2} \sin 0\,dx + \int_{\pi/2}^{2\pi/2} \sin 1\,dx + \int_{2\pi/2}^{3\pi/2} \sin 2\,dx + \ldots\ldots + \int_{2\pi/2}^{2k\pi/2} \sin(2k-1)\,dx$

$$= \frac{\pi}{2}[\sin 1 + \sin 2 + \sin 3 + \ldots\ldots + \sin(2k-1)]$$

$$= \frac{\pi}{2}\,\frac{\left[\sin\frac{1}{2}\sin 1 + \sin\frac{1}{2}\sin 2 + \sin\frac{1}{2}\sin 3 + \ldots + \sin\frac{1}{2}\sin(2k-1)\right]}{\sin\frac{1}{2}}$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

$$= \frac{\left[\cos\frac{1}{2} - \cos\frac{3}{2} + \cos\frac{3}{2} - \cos\frac{5}{2} + ... + \cos\left(2k - \frac{3}{2}\right) - \cos\left(2k + \frac{1}{2}\right)\right]}{2\sin\frac{1}{2}}$$

$$= \frac{\pi}{2} \cdot \frac{\cos\frac{1}{2} - \cos\left(2k + \frac{1}{2}\right)}{2\sin\frac{1}{2}} = \frac{\pi}{2} \cdot \frac{\sin k \sin(k + 1/2)}{\sin(1/2)}$$

Q.15 Let $I_n = \int_0^1 x^n \tan^{-1} x\, dx$. Show that $(n+1)I_{n-1} + (n-1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n}$.

Sol. We have $I_n = \int_0^1 x^n \tan^{-1} x\, dx = \int_0^{\pi/4} \theta (\tan\theta)^n \sec^2\theta\, d\theta$ [Putting $x = \tan\theta$]

$$= \left[\frac{\theta(\tan\theta)^{n+1}}{n+1}\right]_0^{\pi/4} - \int_0^{\pi/4} \frac{(\tan\theta)^{n+1}}{n+1} d\theta = \frac{\pi/4}{n+1} - \int_0^{\pi/4} \frac{(\tan\theta)^{n-1}(\sec^2\theta - 1)}{n+1} d\theta$$

$$= \frac{\pi/4}{n+1} - \frac{1}{n+1} \int_0^{\pi/4} (\tan\theta)^{n-1} \sec^2\theta\, d\theta + \frac{1}{n+1} \int_0^{\pi/4} (\tan\theta)^{n-1}\, d\theta$$

$$= \frac{\pi/4}{n+1} - \frac{1}{n+1}\left[\frac{(\tan\theta)^n}{n}\right]_0^{\pi/4} + \left(\frac{1}{n+1}\right) I$$

$$= \frac{\pi/4}{n+1} - \frac{1}{n(n+1)} + \left(\frac{1}{n+1}\right) I \qquad \text{and} \qquad I_{n-2} = \int_0^{\pi/4} \theta (\tan\theta)^{n-2} \sec^2\theta\, d\theta$$

$$= \left[\frac{\theta(\tan\theta)^{n-1}}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \frac{(\tan\theta)^{n-1}}{n-1} d\theta = \frac{\pi/4}{n-1} \left(\frac{1}{n-1}\right) I$$

Eliminating I from equations (1) and (2), we have

$$(n+1)I_{n-1} + (n-1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n} \qquad \text{which is the desired result.}$$

Q.16 Evaluate : $\lim_{n\to\infty} \prod_{r=1}^{n} \frac{(n^2 + r^2)^{1/n}}{n^2}$

Sol. We have $S = \lim_{n\to\infty} \prod_{r=1}^{n} \frac{(n^2 + r^2)^{1/n}}{n^2}$

$$= \lim_{n\to\infty} \left[\left(\frac{n^2 + 1^2}{n^2}\right)\left(\frac{n^2 + 2^2}{n^2}\right) ...... \left(\frac{n^2 + n^2}{n^2}\right)\right]^{1/n}$$ Taking in both sides, we have

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

$$\ln S = \lim_{n\to\infty} \frac{1}{n}\left[\ln\left(1+\frac{1^2}{n^2}\right)+\ln\left(1+\frac{2^2}{n^2}\right)+......+\ln\left(1+\frac{n^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\ln\left(1+\frac{r^2}{n^2}\right) = \int_0^1 \ln\left(1+x^2\right)dx = \left[x\ln\left(1+x^2\right)\right]_0^1 - \int_0^1 x\left(\frac{2x}{1+x^2}\right)dx$$

$$= \ln 2 - -\int_0^1 2\left(1-\frac{1}{1+x^2}\right)dx = \ln 2 - 2\left[x - \tan^1 x\right]_0^1$$

$$= \ln 2 - 2\left(1-\frac{\pi}{4}\right) = \ln 2 + \frac{\pi-4}{2} \qquad \text{gives} \quad S = 2e^{\left(\frac{\pi-4}{2}\right)}.$$

Q.17 Evaluate the definite integrals $\int_0^{\pi/2} \sin x \ln(\cos x)\,dx$

Sol. We have

$$I = \int_0^{\pi/2} \sin x \ln(\cos x)\,dx = \left[-\cos x \ln(\cos x)\right]_0^{\pi/2} + \int_0^{\pi/2} \cos x.\frac{-\sin x}{\cos x}dx$$

$$= \lim_{x\to\pi/2}\frac{-\ln(\cos x)}{\sec x} + \left[\cos x\right]_0^{\pi/2} = \lim_{x\to\pi/2}\frac{\tan x}{\sec x\tan x} - 1$$

$$= \lim_{x\to\pi/2} \cos x - 1 = -1.$$

Q.18 Evaluate the following limits, using definite integral :

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+.....+\frac{1}{\sqrt{n^2(n-1)^2}}\right]$$

Sol. We have

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+.....+\frac{1}{\sqrt{n^2(n-1)^2}}\right]$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{\sqrt{n^2-r^2}} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{1-(r/n)^2}} \qquad \text{[Omitting one term will not affect the limit]}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_0^1 = \frac{\pi}{2}.$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)

Q.19 Prove the following results, using definite integral :

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}\,(a\sqrt{n}-b\sqrt{r})^2}=\frac{2}{a(a-b)}$$

Sol. We have

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}\,(a\sqrt{n}-b\sqrt{r})^2}=\frac{2}{a(a-b)}$$

$$=\int_0^1\frac{dx}{\sqrt{x}\,(a-b\sqrt{x})^2}=\frac{-2}{b}\int_a^{a-b}\frac{dt}{t^2}\qquad [\text{Putting } a-b\sqrt{x}=t]$$

$$=\frac{2}{b}\left[\frac{1}{t}\right]_a^{a-b}=\frac{2}{b}\left(\frac{1}{a-b}-\frac{1}{a}\right)=\frac{2}{a(a-b)}$$

Q.20 Evaluate : $\lim_{n\to\infty}\left[\frac{\sqrt{n}}{\sqrt{n^3}}+\frac{\sqrt{n}}{\sqrt{(n+2)^3}}+\frac{\sqrt{n}}{\sqrt{(n+4)^3}}+\frac{\sqrt{n}}{\sqrt{(n+8)^3}}+.....\right]$

Sol. We have

$$\lim_{n\to\infty}\left[\frac{\sqrt{n}}{\sqrt{n^3}}+\frac{\sqrt{n}}{\sqrt{(n+2)^3}}+\frac{\sqrt{n}}{\sqrt{(n+4)^3}}+\frac{\sqrt{n}}{\sqrt{(n+8)^3}}+.....\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{\sqrt{n}}{\sqrt{(n+2r)^3}}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{(n+2r)^3}}\qquad [\text{omitting one term will not affect the limit}]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{n^{3/2}}{(n+2r)^{3/2}}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\left\{1+2\left(\frac{r}{n}\right)\right\}^{3/2}}$$

$$=\int_0^1\frac{1}{(1+2x)^{3/2}}dx=\left[\frac{(1+2x)^{-1/2}}{-1/2}\cdot\frac{1}{2}\right]_0^1=\left[\frac{1}{\sqrt{1+2x}}\right]_1^0=1-\frac{1}{\sqrt{3}}.$$

Print to PDF without this message by purchasing novaPDF (http://www.novapdf.com/)