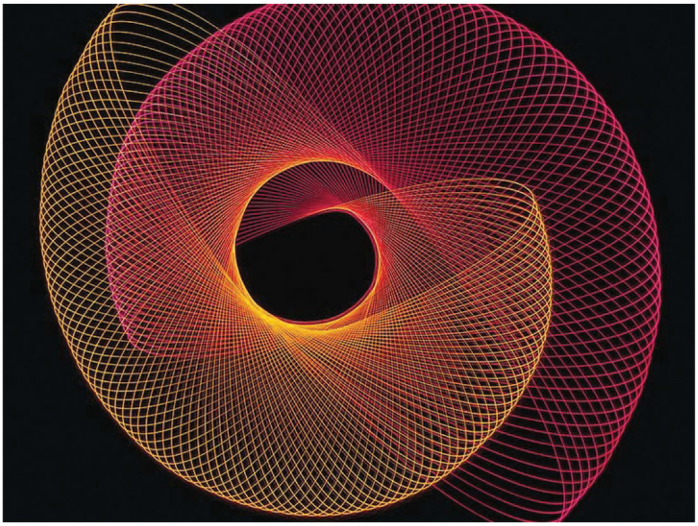
PRAMOTE DECHAUMPHAI

Calculus & Differential Equations with MATLAB



Alpha Science

Calculus and Differential Equations with MATLAB

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Pramote Dechaumphai

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There are so many kinds of books for me to read.
Some bring me joy;
Some give me knowledge and entertainment.
I therefore read every day.
Books present a diversity of subjects,
For readers to explore for their own pleasure.
Since there is so much that can be discovered in books,
I will continue reading for the rest of my life without getting bored.

—HRH Princess Maha Chakri Sirindhorn, composed when she was 12 years old

Translated by the Chalermprakiat Center of Translation and Interpretation, Faculty of Arts, Chulalongkorn University, as part of Chulalongkorn University Press's celebration of the 60th Birthday of HRH Princess Maha Chakri Sirindhorn

Preface

MATLAB is a software widely used to solve mathematical problems that arise in the fields of science and engineering. These problems require profound understanding in the fundamentals of calculus and differential equations. MATLAB can solve many types of calculus problems and differential equations symbolically for exact closed-form expressions. If their exact solutions are not available, approximate solutions are obtained by using numerical methods.

This book, *Calculus and Differential Equations with MATLAB*, explains how to use MATLAB to solve calculus and differential equation problems in a clear and easy-to-understand manner. Essential topics in the calculus and differential equation courses are selected and presented. These topics include: limit, differentiation, integration, series, special functions, Laplace and Fourier transforms, ordinary and partial differential equations. Numerous examples are used to present detailed derivation for their solutions. These solutions are carried out by hands as normally done in classes and verified by using the MATLAB commands. Students thus understand detailed mathematical process for obtaining solutions. They, at the same time, realize the software capability that can provide the same solutions effectively within a short time. The solutions are then plotted to provide clear understanding of their behaviors.

The author would like to thank Miss Kaiumporn Phong-khachorn for her fine typing and his graduate students for proof-reading the manuscript. He would like to thank Chulalongkorn University Press and Alpha Science International for printing and distributing this book. He appreciates his wife Mrs. Yupa Dechaumphai for her support while writing this book.

Pramote Dechaumphai

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Chapter 1

Symbolic Mathematics by MATLAB

1.1 Introduction

Calculus and differential equations are requirement courses for science and engineering students. However, some students may not realize the importance of these subjects and simply take them for fulfilling their degree requirement. Such subjects, in fact, are essential because they are basis toward studying higher level courses so that more realistic problems can be solved. Most of the commercial software for design and analyzing scientific and engineering problems today were developed based on the knowledge of mathematics and computational methods. Good background in calculus and differential equations is thus needed prior to using the high-level commercial software correctly.

As scientists or engineers, solutions obtained from solving mathematical problems must be further interpreted so that their physical meanings are understood. They prefer to obtain solutions without spending a lot of time deriving them. There are many software today that can provide solutions to a large class of mathematical problems. These software can be used for finding roots of algebraic equations, taking derivatives and integrating functions, including solving for solutions of many differential equations. As a simple example, the software can perform integration,

$$\int_0^3 x^2 \, dx$$

numerically and returns the result of 9 immediately. At the same time, if preferred, the software can provide symbolic answer, such as,

$$\int x^2 \, dx = \frac{x^3}{3}$$

The software capability thus helps learning calculus considerably.

The idea for developing symbolic mathematics software started in early 1970's when a group of researchers at Massachusetts Institute of Technology developed a software so called MACSYMA (Mac Symbolic Manipulation Program). It was quite astonishing at that time because the solutions can be shown in the form of symbolic expressions instead of numbers. Lately, many symbolic manipulation capabilities of the software have been improved and can be used to ease learning calculus and differential equations.

1.2 Symbolic Mathematics Software

In the past, students need to memorize formulas when learning calculus. Basic formulas are required for finding derivative or integral of a given function. Proper steps must be taken carefully so that the final solutions are derived correctly. Few examples for finding solutions after taking derivative and performing integration of some functions are highlighted below.

Example Find the second-order derivative of the function,

$$f = \frac{x}{3-4\cos x}$$

To determine the second-order derivative of the given function above, the standard first-derivative formula is applied twice. The final solution is relatively lengthy as,

$$\frac{d^2 f}{dx^2} = \frac{16x\cos^2 x + 4\cos x(3x-8\sin x) - 32x + 24\sin x}{(4\cos x-3)^3}$$

Deriving for the above solution by hands, mistake may occur. With the help of symbolic computer software, the solution can be obtained instantly without any error. In addition, if the tenth or other higher-order derivatives of the above function are needed, the software can provide correct solutions in a very short time as well.

Example Find the first- and second-order derivatives of a more complex function,

$$g = \frac{x\tan x - 3\tan x - 21x^3 + 7x^4}{x^3 - 3x^2 + 6x - 18}$$

Again, the symbolic computer software can be used to provide the first-order derivative as,

$$\frac{dg}{dx} = -\frac{2x\tan x + 504}{(x^2+6)^2} + \frac{\tan^2 x + 43}{x^2+6} + 7$$

and then the second-order derivative as,

$$\frac{d^2 g}{dx^2} = \frac{2016x - 48\tan x}{(x^2+6)^3} - \frac{4x\tan^2 x - 6\tan x + 88x}{(x^2+6)^2} + \frac{2\tan x\,(\tan^2 x + 1)}{x^2+6}$$

Derivation of these solutions by hands would take a long time and is likely to contain error.

<u>Example</u> Integration is another topic learned in calculus course that many students do not appreciate. This is because they have to memorize many formulas and do not know when it will be used for solving realistic problems. Examples of integration learned in calculus course include indefinite integral, such as,

$$\int \frac{dx}{x^2+9} \quad = \quad \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$$

Also the definite integral, such as,

$$\int_1^2 \frac{x\,dx}{(x+1)(2x-1)} \quad = \quad \frac{1}{2}\ln(3)-\frac{1}{3}\ln(2)$$

Students can obtain solutions above in a short time if they use symbolic computer software.

<u>Example</u> Symbolic computer software can help us to solve some other types of problems that require a long time to do by hands. For example, it can be used to factorize the function,

$$h \quad = \quad x^4+26x^3-212x^2-1578x+5859$$

to give, $$h \quad = \quad (x+7)(x-9)(x-3)(x+31)$$

within a second.

<u>Example</u> Solving differential equations is another topic that most students do not like. This is because they are many approaches to follow depending on the types of differential equations. As an example, a general solution of the first-order ordinary differential equation,

$$\frac{dy}{dx}+y \quad = \quad 5$$

is, $$y(x) \quad = \quad C\,e^{-x}+5$$

where C is a constant that can be determined from the given initial condition of the problem.

Some differential equations are more complicated, such as,

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y \quad = \quad 0$$

A general solution for the second-order ordinary differential equation above is,

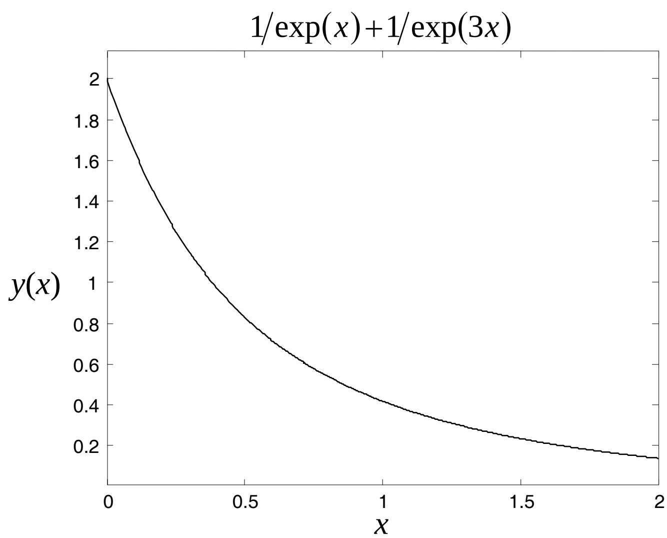
$$y(x) \quad = \quad C_1 e^{-x} + C_2 e^{-3x}$$

where C_1 and C_2 are constants that can be determined from the boundary conditions of the problem.

With the symbolic computer software, the above solutions can be obtained instantly. The software can also plot the solution behavior so that students understand its physical meaning clearly. For example, if the constants obtained after applying the boundary conditions are $C_1 = C_2 = 1$, the exact solution of the problem is,

$$y(x) \quad = \quad e^{-x} + e^{-3x}$$

The distribution of y that varies with x is plotted as shown in the figure.



1.3 History and Capability of MATLAB

MATLAB (MATrix LABoratory) was developed by Professor Cleve Moler, Head of Computer Science Department at the University of New Mexico in 1977. He wrote the LINPACK commands for solving algebraic equations and the EISPACK commands for analyzing eigenvalue problems, so that his students would not have to study FORTRAN language. Later, in 1983, Jack Little founded the Mathworks company to commercialize the software. The key capability of the software was to apply mathematical and computational methods through the use of matrices for solving academic problems. Soon, the software has received popularity mainly because of its ease of using.

In the past decade, MATLAB has included symbolic manipulation capability by linking its system operation with Maple and MuPad software. Such additional capability further increases the MATLAB popularity because a large class of mathematical problems can now be solved. The output solutions are in the forms of symbolic mathematical expressions instead of numbers. These solutions significantly help students in learning calculus and differential equation courses.

This book concentrates on how to use MATLAB to provide solutions in the forms of symbolic expressions similar to what we have learnt in classes. Selected topics which are important in calculus and differential equation courses are presented. Detailed derivations are illustrated prior to solving the same problems by using MATLAB commands. We will see that the same solutions are obtained instantly without any error from the software.

1.4 MATLAB Fundamentals

MATLAB is a huge computer software containing a large number of commands. Because this book concentrates on solving calculus problems and differential equations, only essential commands related to these topics are presented herein.

MATLAB assigns specific letters or names for some well-known quantities, such as,

- `i` and `j` denotes imaginary unit which is equal to $\sqrt{-1}$.
- `Inf` represents infinity which is $+\infty$, while $-\infty$ is denoted by `-Inf`.

- `NaN` refers to Not a Number, such as $0/0$ or `Inf/Inf`.
- `eps` is the acceptable tolerance which is 2.2204×10^{-16}.
- `pi` denotes π.

The value of π can be displayed up to n significant figures by using the command `vpa` (A, n) where A denotes the variable. As an example, the command for displaying the value of π with 200 significant figures is,

vpa

```
>> syms pi
>> vpa(pi,200)

ans =

3.141592653589793238462643383279502884197169399375
5105820974944592307816406286208998628034825342117
0679821480865132823066470938446095505822317253594
0812848111745028410270193852110555964462294895493
0382
```

Similarly, $\sqrt{2}$ can be displayed with 200 significant figures by using the command,

```
>> vpa('(2)^(1/2)', 200)

ans =

1.4142135623730950488016887242096980785696718753 7
6948073176679737990732478462107038850387534327641
5727350138462309122970249248360558507372126441214
9709993583141322266592750559275579995050115278206
05715
```

The command `syms` above is used to declare the specified variable as a symbol. For example, the variable x, y and t in the equation below,

$$u = 2x - 7y + t^2$$

can be declared as the three symbols by using the command,

syms

```
>> syms x y t
>> u = 2*x - 7*y + t^2
```

```
u =

t^2 + 2*x - 7*y
```

so that MATLAB won't expect the numerical values for them.

MATLAB contains several commands to manipulate and simplify algebraic expressions. These commands help reducing the complexity of the final symbolic expressions. Some useful commands are described herein.

The `collect` command expands the given expression and then collects similar terms together. For example,

$$f = (x+5)(x-3)$$

```
>> syms x
>> f = (x+5)*(x-3);
>> collect(f)

ans =

x^2 + 2*x - 15
```

collect

i.e., the final result is, $f = x^2 + 2x - 15$

The `expand` command expands and displays all the terms in the given function, e.g.,

$$g = \cos(x+y)$$

```
>> syms x y
>> g = cos(x+y);
>> expand(g)
ans =

cos(x)*cos(y) - sin(x)*sin(y)
```

expand

i.e., $g = \cos x \cos y - \sin x \sin y$

The `factor` command factorizes the given function to make it looks simpler. For example,

$$h = 6x^3 + 11x^2 - 16x - 21$$

```
>> h = 6*x^3 + 11*x^2 - 16*x - 21;
>> factor(h)                                    factor
ans =
(3*x + 7)*(2*x - 3)*(x + 1)
```

i.e., $$h = (3x+7)(2x-3)(x+1)$$

The `simplify` command simplifies the complex expression so that it is compact and easy to understand. As an example,

$$u = \frac{x+2y}{\dfrac{2}{x}+\dfrac{1}{y}}$$

```
>> u = (x + 2*y)/(2/x + 1/y);
>> simplify(u)                                  simplify
ans =
x*y
```

i.e., $$u = xy$$

The `simple` command is probably the most popular command because it combines the capability of the `collect`, `expand`, `factor` and `simplify` commands together. For example,

$$v = \frac{2x^5 - 5x^4 + 58x - 145}{2x - 5}$$

```
>> v = (2*x^5 - 5*x^4 + 58*x - 145)/(2*x - 5);
>> simple(v)
ans =                                           simple
x^4 + 29
```

i.e., the final solution is, $$v = x^4 + 29$$

It is noted that since the `simple` command contains several commands inside, detailed expressions during the simplification are appeared on the screen. These detailed expressions are omitted herein.

The `pretty` command is another useful command for transforming a symbolic expression into the rational form similar to those shown in textbooks. For example,

$$w = \frac{x \sin x}{x^2 + 12}$$

```
>> w = x*sin(x)/(x^2+12);
>> pretty(w)
```

pretty

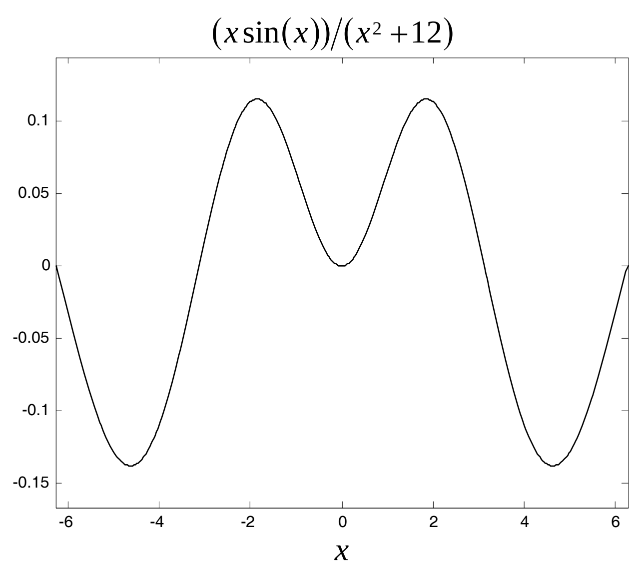
```
  x sin(x)
  --------
   2
  x  + 12
```

MATLAB contains the `ezplot` command that can be used to plot a given function effectively. As an example, if we would like to plot the *w* function above, we just simply enter the command,

```
>> ezplot(w)
```

ezplot

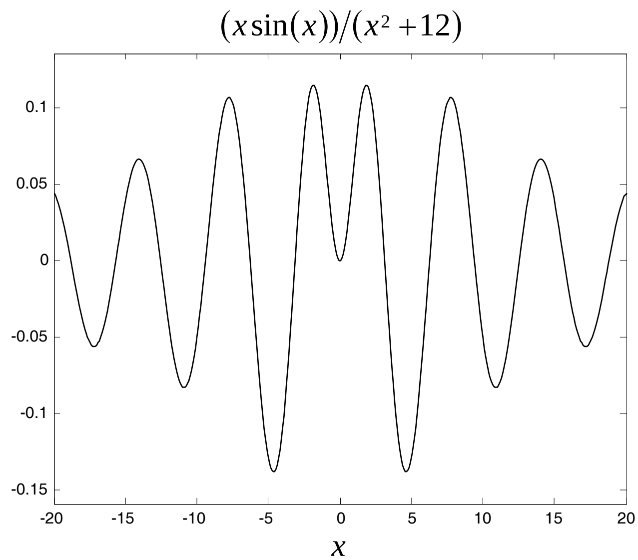
A plot of the *w* function will appear on the screen with the axis scaling adjusted automatically.



If we would like to plot the same *w* function within the interval of $-20 \le x \le 20$ along the *x*-axis, the above command is modified slightly to,

```
>> ezplot(w,[-20,20])
```

ezplot

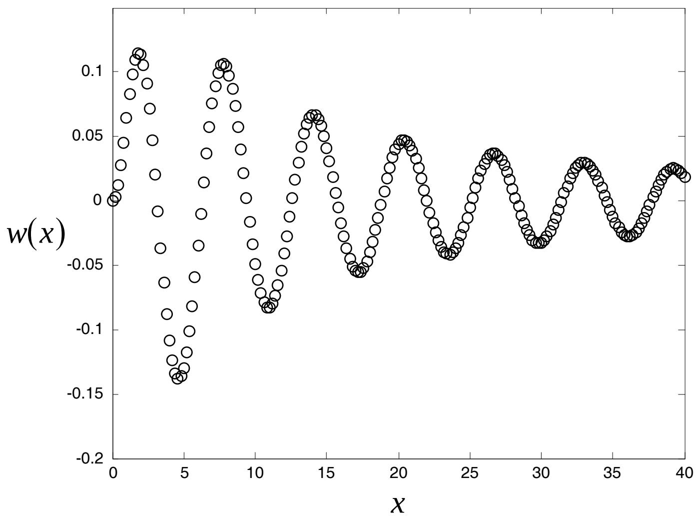


MATLAB contains the standard `plot` command that allows us to specify more details for plotting. For example, we want to compute the values of *w* function above at every *x*=0.2 for $0 \le x \le 40$. The values of *w* will be plotted as circle within the range of $-0.20 \le w \le 0.15$. The plot also includes labels on both horizontal and vertical axes as *x* and $w(x)$, respectively. In this case, the commands are as follows.

```
>> x = 0:.2:40;
>> w = x.*sin(x)./(x.^2+12);
>> plot(x,w,'ok')
>> axis([0,40,-.20,.15])
>> xlabel('x'), ylabel('w(x)')
```

plot

If preferred, we can include all the commands above in an m-file so that plotting details can be modified easily. In the argument of the `plot` command, ‘`o`’ denotes the circle while ‘`k`’ is for showing all circles in black. The plot generated from the commands is shown in the figure.



1.5 Concluding Remarks

In this chapter, the capability of symbolic manipulation software was introduced. The software helps finding solutions of basic problems learned in calculus and differential equation courses. These include: finding derivatives of functions, perform both definite and indefinite integrations, as well as solving for exact solutions of some differential equations. At present, there are many symbolic manipulation software suitable for learning and using in research work. Among them, MATLAB has received popularity due to its capability and ease of using.

Development history and essential features of MATLAB were briefly described. Few important commands were explained by using examples. These commands can help manipulating complex expressions and reduce them into simple forms. The solutions are plotted by using easy commands so that users can understand their physical meanings quickly. The chapter demonstrates advantages of using the symbolic manipulation software that can significantly reduce the effort for solving basic mathematical problems. We will appreciate these advantages in more details when we study essential calculus and differential equation topics in the following chapters.

Exercises

1. Study the symbolic manipulation capability in MATLAB by entering the command,

   ```
   >> doc symbolic
   ```

   Then, make conclusions on how to:

   (a) simplify expressions
   (b) plot symbolic functions

   by setting up examples.

2. Use the command `collect`, `expand`, `factor`, `simplify` or `simple` to reduce or determine the following quantities,

   (a) $(-3)^4$ (b) $16^{-3/4}$

   (c) $3(x+6)+4(2x-7)$ (d) $(x-3)(4x+1)$

   (e) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ (f) $3x^{3/2}-9x^{1/2}+6x^{-1/2}$

3. Use the command `simple` to simplify the following functions,

   (a) $\dfrac{x^2+3x+2}{x^2-x-2}$ (b) $\dfrac{2x^2-x-1}{x^2-9}\cdot\dfrac{x+3}{2x+1}$

   (c) $\dfrac{x^2}{x^2-4}-\dfrac{x+1}{x-2}$ (d) $\dfrac{\dfrac{y}{x}-\dfrac{x}{y}}{\dfrac{1}{y}-\dfrac{1}{x}}$

   (e) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$ (f) $\dfrac{(4x^3y^3)(3xy^2)^2}{\sqrt{xy}}$

4. Determine the product of the function f and g for each sub-problem. Then, employ the command `simple` to simplify their final expressions,

   (a) $f = x^2 - 2x \quad ; \quad g = (x+1)^2$

   (b) $f = \sqrt{x-1} \quad ; \quad g = \sqrt{x^2+2}$

   (c) $f = \sin(x - x^2) \quad ; \quad g = \cos(x^2 + x)$

   (d) $f = e^{x-2} \quad ; \quad g = \ln(x+3)$

   (e) $f = \sin(x^2 - x) \quad ; \quad g = \ln(x^2 + 2x)$

   (f) $f = e^x \quad ; \quad g = e^{-2x^2}$

5. Use the command `collect`, `expand`, `factor`, `simplify` or `simple` to yield simplest expressions of the following functions,

   (a) $f = 6x^4 + 28x^3 - 7x^2 + 14x - 5$

   (b) $g = (\cos x - \sin x)(\cos x + \sin x)(e^x + \sin x)(3x - 7)$

   (c) $h = \dfrac{x\tan x - 3\tan x - 21x^3 + 7x^4}{x^3 - 3x^2 + 6x - 18}$

   (d) $u = 3x + x^2 + 4\sqrt{x} + 2x\sqrt{x} + 2$

   (e) $v = \dfrac{\sin x\tan^2 x + 6\sin x\tan x + 9\sin x}{x\cos^2 x - x - 7\cos^2 x + 7}$

   (f) $w = \dfrac{15x + 3xy^2 + 5y^2 + y^4}{x^2(\cos x\sin y - \cos y\sin x) - \cos x\sin y - \cos y\sin x}$

6. Use the command `collect`, `expand`, `factor`, `simplify` or `simple` to prove that,

   (a) $x^5 - y^5 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$

   (b) $\sin(3x) = 3\sin x - 4\sin^3 x$

   (c) $\tan(4x) = \dfrac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}$

(d) $\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$

(e) $(e^x + e^{-x})\tanh x = e^x - e^{-x}$

7. Use the command `collect`, `expand`, `factor`, `simplify` or `simple` to show that,

   (a) $\tanh^2 x + \mathrm{sech}^2 x = 1$

   (b) $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

   (c) $\cosh \dfrac{x}{2} = \sqrt{\dfrac{\cosh x + 1}{2}}$

   (d) $\cosh(4x) = 8\cosh^4 x - 8\cosh^2 x + 1$

   (e) $\sinh^3 x = \dfrac{1}{4}\sinh(3x) - \dfrac{3}{4}\sinh x$

8. Use the `vpa` command to calculate and display the roots of the following prime numbers with 100 significant figures,

   (a) $\sqrt{7}$ (b) $(157)^{2/3}$

   (c) $(229)^{5/7}$ (d) $(443)^{8/9}$

   (e) $(587)^{1/13}$ (f) $(881)^{27/31}$

9. Use the `ezplot` command to plot the following functions,

   (a) $f = 3x^5 - 25x^3 + 60x$

   (b) $g = \dfrac{5x^2 + 8x - 3}{3x^2 + 2}$

   (c) $h = \dfrac{3}{4}(x^2 - 1)^{2/3}$

   (d) $u = \sin(\pi/x)$

   (e) $v = (x-2)^4 (x+1)^3 (x-1)$

10. Use the `plot` command to display the functions in Problem 9 again by showing essential details of their variations.

11. Study capabilities of the Mathematica and Maple software. Then, highlight their unique features and compare capabilities with MATLAB for manipulating symbolic expressions.

12. Use the `factor` command to simplify the function,

$$p = 3x^5 - 20x^3$$

Then, employ the `plot` command to display the variation of this function for (a) $-4 \le x \le 4$; $-500 \le p \le 500$ and (b) $0 \le x \le 4$; $-80 \le p \le 80$.

13. Use the `simplify` command to simplify the function,

$$q = x^4 - 2x^2 - 3$$

Then, use the `ezplot` command to display its variation for $-3 \le x \le 3$. Plot this function again but by using the `plot` command for $-3 \le x \le 3$ and $-6 \le q \le 6$.

14. Use the `ezplot` command to plot the following functions,

(a) $f = x^3 - x + 1$

(b) $g = x^4 - 3x^2 + x$

(c) $h = 3x^5 - 25x^3 + 60x$

Then, use the `plot` command with appropriate scaling in both the horizontal and vertical directions to clearly show their variations.

15. Use the `ezplot` command to display the following trigonometric functions,

(a) $\cos(2x)$ (b) $\sin\left(x - \dfrac{\pi}{2}\right)$

(c) $\cos(x + \pi)$ (d) $\tan(x + \pi)$

(e) $\cos\left(\dfrac{x}{6}\right)$ (f) $\cos\left(\pi + \dfrac{x}{2}\right)$

Then, use the `plot` command with appropriate scaling in both the horizontal and vertical directions to show their variations.

16. Given the function,

$$f = \sqrt[3]{1-x^3}$$

Find a proper command in MATLAB for determining the expression for f^{-1}. Then, plot to compare the variations between f and f^{-1}.

17. Given the function,

$$g = x\cos\left(\frac{10}{x}\right)$$

Use the `ezplot` command to plot its variation within the interval of $-2 \le x \le 2$. Then, use the `plot` command again to show the variation for $-0.5 \le x \le 0.5$. Suggest on how to plot the function when x approaches zero so that the variation is shown clearly.

Chapter 2

Calculus

2.1 Introduction

Calculus is an essential subject in mathematics required for science and engineering students. It contains two main topics which are the differentiation and integration of functions. The former one is based on understanding the determination of limits. Often, many students do not enjoy studying these topics because they have to memorize formulas for deriving solutions. Some solutions require a long time to derive by employing specific techniques. Furthermore, most students do not appreciate learning these topics because they don't know when the solutions will be used for realistic problems.

With the capability of the symbolic manipulation software today, solutions to calculus problems can be obtained rapidly.

Students can compare solutions obtained from the software with those derived by hands. So they will have more time to understand the meanings of the solutions. This chapter shows standard techniques to derive the solutions before using MATLAB commands to confirm the validity of them. Several examples will be presented with detailed derivation for the solutions. These solutions will also be plotted to increase understanding of their phenomena.

2.2 Limits

Limit of a function $f(x)$ when x approaches a, is defined by,

$$L = \lim_{x \to a} f(x)$$

Example Given a function,

$$f(x) = x^2$$

Then, $$L = \lim_{x \to a} f(x) = \lim_{x \to a} x^2 = a^2$$

We can use the `limit` command in MATLAB to obtain the solution by entering,

```
>> syms x a
>> f = x^2;
>> limit(f,x,a)

ans =

a^2
```

limit

Example Given a function,

$$g(x) = \frac{x^2 - a^2}{x - a}$$

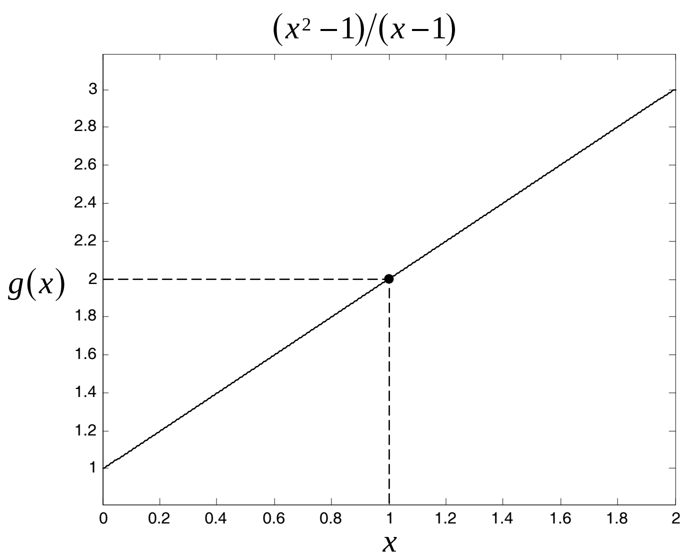
If we follow the simple procedure used above, we get,

$$L = \lim_{x \to a} g(x) = \lim_{x \to a} \frac{a^2 - a^2}{a - a} = \frac{0}{0}$$

The result cannot be determined and is not correct. To find the correct solution, we should observe the variation of this function $g(x)$ by plotting. If we assign the value of $a = 1$, then,

$$g(x) \quad = \quad \frac{x^2 - 1^2}{x - 1}$$

The plot of this function is shown in the figure. From the figure, the function $g(x)$ becomes 2 as x approaches 1.



The proper step to determine the limit of this problem is to first let $x = a + h$, where h is small. Then, substitute $x = a + h$ into the function $g(x)$ to get,

$$\begin{aligned} g(x) \quad &= \quad \frac{(a+h)^2 - a^2}{a + h - a} \quad = \quad \frac{2ah + h^2}{h} \\ &= \quad 2a + h \end{aligned}$$

Then, let h approaches zero, thus,

$$L \quad = \quad \lim_{x \to a} \left(\frac{x^2 - a^2}{x - a} \right) \quad = \quad 2a$$

The solution agrees with that shown in the graph at $a = 1$.

The same solution can be obtained instantly by using the `limit` command,

```
>> syms x a
>> g = (x^2-a^2)/(x-a);
>> limit(g,x,a)
```

limit

```
ans =

2*a
```

Finding limits of functions may require different methods depending on the function types. The examples below show standard techniques to determine limits for different types of functions.

Example Determine the limit,

$$L = \lim_{x\to 5} \frac{x^2-25}{x-5}$$

This example is similar to the preceding example. The factoring technique can be used to find the solution as follows,

$$\begin{aligned} L &= \lim_{x\to 5} \frac{(x-5)(x+5)}{x-5} &&= \lim_{x\to 5}(x+5) \\ &= 5+5 &&= 10 \end{aligned}$$

Again, the `limit` command is employed to obtain the same solution,

```
>> limit('(x^2-25)/(x-5)',x,5)
```

limit

```
ans =

10
```

Example Determine the limit,

$$L = \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$$

If we simply substitute $x=4$, we get $0/0$, which cannot be determined. A technique to find the limit of such function is to multiply its numerator and denominator by the conjugate value of the numerator, $\sqrt{x}+2$, before taking the limit. Detailed derivation is as follows,

$$
\begin{aligned}
L &= \lim_{x\to 4} \frac{\left(\sqrt{x}-2\right)}{(x-4)} \cdot \frac{\left(\sqrt{x}+2\right)}{\left(\sqrt{x}+2\right)} \\
&= \lim_{x\to 4} \frac{(x-4)}{(x-4)\left(\sqrt{x}+2\right)} \qquad = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} \\
&= \frac{1}{\sqrt{4}+2} \qquad\qquad\qquad\; = \frac{1}{4}
\end{aligned}
$$

Similarly, we can use the `limit` command to find such solution by entering,

```
>> limit('(sqrt(x)-2)/(x-4)',x,4)

ans =

1/4
```

<u>Example</u> Determine the limit,

$$
L = \lim_{x\to 0} \frac{\dfrac{1}{x+4}-\dfrac{1}{4}}{x}
$$

Again, if we substitute $x=0$ directly into the function, we get $0/0$. The technique of multiplying by the conjugate value as shown in the preceding example is not applicable. A different technique of multiplying the numerator and denominator by an appropriate function is needed. For this particular problem, the appropriate function is $4(x+4)$ and the detailed procedure is as follows,

$$
\begin{aligned}
L &= \lim_{x\to 0} \frac{\left(\dfrac{1}{x+4}-\dfrac{1}{4}\right)}{x} \cdot \frac{4(x+4)}{4(x+4)} \\
&= \lim_{x\to 0} \frac{4-(x+4)}{4x(x+4)} \qquad = \lim_{x\to 0} \frac{-x}{4x(x+4)} \\
&= \lim_{x\to 0} \frac{-1}{4(x+4)} \qquad\;\; = \frac{-1}{4(0+4)} \\
&= -\frac{1}{16}
\end{aligned}
$$

The same solution is obtained by using the `limit` command as,

```
>> limit('(1/(x+4)-1/4)/x',x,0)
```

limit

```
ans =

-1/16
```

If the given function is more complex, the same `limit` command still provides solution immediately as demonstrated by the following examples.

Example Determine the limit,

$$L = \lim_{x\to -4} \frac{4-x}{5-\sqrt{x^2-9}}$$

The solution is obtained by typing the `limit` command followed by the given function, variable and limit value as,

```
>> limit('(4-x)/(5-sqrt(x^2-9))',x,-4)

ans =

-8/(7^(1/2) - 5)
```

which leads to the solution,

$$\lim_{x\to -4} \frac{4-x}{5-\sqrt{x^2-9}} = \frac{8}{5-\sqrt{7}} = 3.3981$$

Example Determine the limit,

$$L = \lim_{x\to 0} \frac{e^{x^3}-1}{1-\cos\sqrt{x-\sin x}}$$

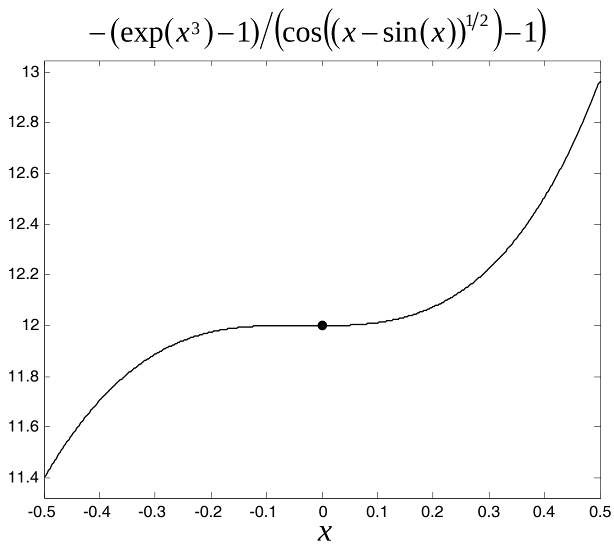
```
>> f = (exp(x^3)-1)/(1-cos(sqrt(x-sin(x))));
>> limit(f,x,0)

ans =

12
```

i.e., the solution is, $\lim_{x\to 0} \frac{e^{x^3}-1}{1-\cos\sqrt{x-\sin x}} = 12$

The solution can be verified by plotting such function through the use of the `ezplot` command as,

```
>> ezplot(f, [-.5,.5])
```

`ezplot`



<u>Example</u> Determine the limit of the function below as x approaches infinity,

$$L = \lim_{x\to\infty} \frac{9x+4}{\sqrt{3x^2-5}}$$

```
>> limit('(9*x +4)/(sqrt(3*x^2-5))',x,Inf)
ans =
3*3^(1/2)
>> double(ans)
ans =
    5.1962
```

`double`

i.e., $$\lim_{x\to\infty} \frac{9x+4}{\sqrt{3x^2-5}} = 3\sqrt{3} = 5.1962$$

Example The `limit` command can also be used to find the solution when the function contains two variables,

$$L = \lim_{\substack{x \to -1 \\ y \to 2}} \frac{x^2 y + xy^3}{(x+y)^3}$$

```
>> syms x y
>> f = (x^2*y+x*y^3)/(x+y)^3;
>> L = limit(limit(f,x,-1),y,2)
```

limit

```
L =

-6
```

i.e., $$\lim_{\substack{x \to -1 \\ y \to 2}} \frac{x^2 y + xy^3}{(x+y)^3} = -6$$

2.3 Differentiation

Differentiation is one of the most important topics in calculus. This is because the physics of most science and engineering problems are described by differential equations. Differential equations contain terms that are derivatives of the unknown variables. Finding for these unknown variables is the main objective for solving the differential equations. Thus, knowing how the derivatives of a function can be found is the first step toward learning the differential equations.

To find derivatives of a function as learned in classes, we need to apply basic formulas. Some formulas are easy to memorize while many others are difficult to recall. Furthermore, taking derivative of a complex function consumes a large amount of time and is likely to produce error.

Before using MATLAB command to find any derivative of a function, we start from understanding definition of the derivative. The derivative of a function $y(x)$ with respect to x is given by,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example Find the derivative of $y = f(x) = x^2$.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2x + \Delta x
\end{aligned}$$

As $\Delta x \to 0$, then,

$$\frac{dy}{dx} = 2x + 0$$

or,

$$\frac{dy}{dx} = 2x$$

MATLAB contains the `diff` command that can be used to find the derivative effectively. In this example, the entered commands and solution are,

```
>> syms x
>> y = x^2;
>> diff(y,x)

ans =

2*x
```

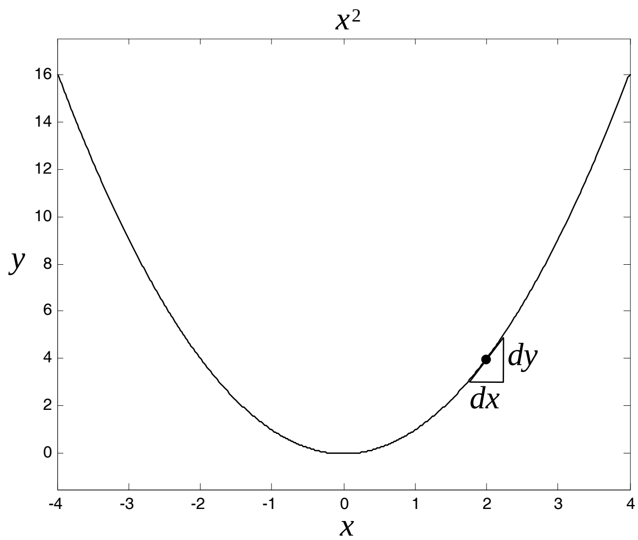
diff

Variation of the function $y = x^2$ can also be plotted easily as shown in the figure by using the command,

```
>> ezplot(y,[-4,4])
```

ezplot

The derivative dy/dx represents the slope at any x location. For example, at $x = 2$, the derivative $dy/dx = 2(2) = 4$.



Example Find the derivative of $y = f(x) = 2x - 3x^2$.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2(x+\Delta x) - 3(x+\Delta x)^2 - (2x - 3x^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\Delta x - 6x\Delta x - 3(\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2 - 6x - 3\Delta x \\
&= 2 - 6x - 3(0) \\
\frac{dy}{dx} &= 2 - 6x
\end{aligned}$$

Again, we can use the `diff` command to find the derivative of the function $y = f(x) = 2x - 3x^2$ as follows,

diff

```
>> syms x
>> y = 2*x-3*x^2;
>> diff(y,x)

ans =

2-6*x
```

<u>Example</u> Find the derivative of a constant function $y = f(x) = 5$.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{5-5}{\Delta x} \quad = \quad \lim_{\Delta x\to 0} 0\\ &= 0\end{aligned}$$

That is, the derivative of a constant is zero.

```
>> y = 5;
>> diff(y,x)

ans =

0
```

diff

<u>Example</u> Find the derivative of the function $y = f(x) = \sqrt{x}$ for $x > 0$.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\cdot\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}\\ &= \lim_{\Delta x\to 0}\frac{x+\Delta x-x}{\Delta x\left(\sqrt{x+\Delta x}+\sqrt{x}\right)}\\ &= \lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}\\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

We can use the `diff` command to obtain such solution directly,

```
>> syms x
>> y = sqrt(x);
>> diff(y,x)

ans =

1/(2*x^(1/2))
```

Example Find the derivative of the function $y = f(x) = \dfrac{x}{2x+3}$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{x+\Delta x}{2(x+\Delta x)+3} - \dfrac{x}{2x+3}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{x+\Delta x}{2(x+\Delta x)+3} \cdot \dfrac{2x+3}{2x+3} - \dfrac{x}{2x+3} \cdot \dfrac{2(x+\Delta x)+3}{2(x+\Delta x)+3}}{\Delta x}$$

After simplifying it, we obtain,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{3}{(2x+2\Delta x+3)(2x+3)}$$

$$= \frac{3}{(2x+0+3)(2x+3)}$$

or,

$$\frac{dy}{dx} = \frac{3}{(2x+3)^2}$$

If we use the `diff` command, we can get the same result instantly,

```
>> syms x
>> y = x/(2*x+3);
>> diff(y,x)

ans =

3/(2*x + 3)^2
```

diff

In general, the given function $y = f(x)$ is complicated. Finding its derivative by hands consumes a large amount of time and may produce error. The `diff` command can eliminate such difficulty as shown in the following examples.

<u>Example</u> Find the derivative of the function,

$$y = f(x) = x^4 - 8x^3 + 12x - 5$$

Then, evaluate the result numerically at $x = -2, 0$ and 2.

Again, we can employ the `diff` command to find the derivative as follows,

```
>> syms x
>> y = x^4 - 8*x^3 + 12*x - 5;
>> dydx = diff(y,x)

dydx =

4*x^3 - 24*x^2 + 12
```

i.e.,

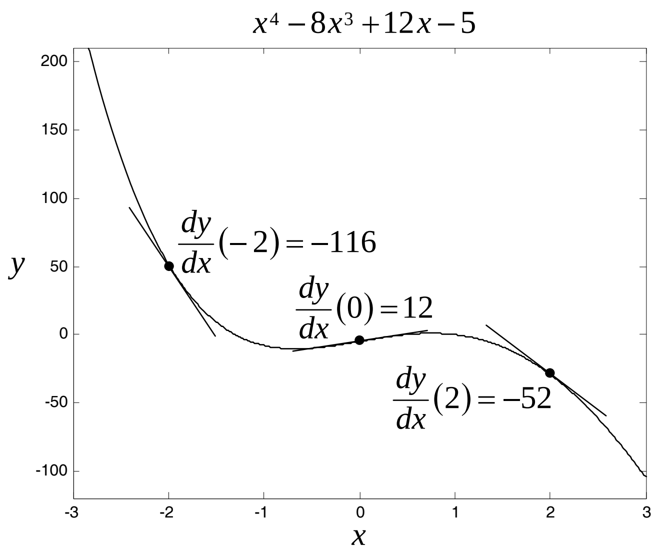
$$\frac{dy}{dx} = 4x^3 - 24x^2 + 12$$

The `subs` command can then be used to compute the numerical results at $x = -2, 0$ and 2 as follows,

```
>> subs(dydx,-2)

ans =

  -116

>> subs(dydx,0)

ans =

    12

>> subs(dydx,2)

ans =

   -52
```

subs

These computed derivatives represent the slopes at $x = -2, 0$ and 2 as shown in the figure.



Example Find the derivative of the function,

$$y = f(x) = \left[x + (x + \sin^2 x)^3\right]^4$$

If we use the `diff` command, the result is,

```
>> syms x
>> y = (x + (x + sin(x)^2)^3)^4;
>> dydx = diff(y,x)
dydx =
4*(x + (sin(x)^2 + x)^3)^3*(3*(sin(x)^2 + x)^2*
(2*cos(x)*sin(x) + 1) + 1)
```

The result above can be simplified by using the `simple` command to yield,

$$\frac{dy}{dx} = 4\left[x + (\sin^2 x + x)^3\right]^3 \left[3(\sin^2 x + x)^2(\sin 2x + 1) + 1\right]$$

It is noted that the `diff` command can also be used to find solutions of higher order derivatives. This is done by including the derivative order at the end of the command as shown in the following examples.

Example Find the second-order derivative of the function,

$$y = f(x) = 2x^3 + 7x^2 - 3x + 5$$

```
>> syms x
>> y = 2*x^3 + 7*x^2 - 3*x + 5;
>> diff(y,x,2)
```

diff

```
ans =

12*x + 14
```

i.e.,

$$\frac{d^2y}{dx^2} = 12x + 14$$

Example Find the second- and twentieth-order derivatives of the function,

$$y = f(x) = \frac{1 - \sqrt{\cos x}}{x\left(1 - \cos\sqrt{x}\right)}$$

MATLAB can determine derivatives of rather complex function above in a short time. We will use the `tic`, `toc` commands to measure the computational time.

For the second-order derivative, the result and computational time are,

```
>> syms x
>> y = (1-sqrt(cos(x)))/(x*(1-cos(sqrt(x))));
>> tic, diff(y,x,2), toc
```

tic, toc

```
ans =

(2*(cos(x)^(1/2)-1))/(x^3*(cos(x^(1/2))-1))-
cos(x)^(1/2)/ (2*x*(cos(x^(1/2))-
1))+(cos(x^(1/2))*(cos(x)^(1/2)-1))/
(4*x^2*(cos(x^(1/2))-1)^2)-
(5*sin(x^(1/2))*(cos(x)^(1/2) -
1))/(4*x^(5/2)*(cos(x^(1/2))-
1)^2)+sin(x)/(x^2*cos(x) ^(1/2)*(cos(x^(1/2))-
1))+(sin(x^(1/2))^2*(cos(x)^(1/2)-
```

```
1))/(2*x^2*(cos(x^(1/2))-1)^3)-
sin(x)^2/(4*x*cos(x)^(3/2) *(cos(x^(1/2))-1))-
(sin(x^(1/2))*sin(x))/(2*x^(3/2)*
cos(x)^(1/2)*(cos(x^(1/2))-1)^2)

Elapsed time is 0.055360 seconds.
```

That is, MATLAB uses only about 0.05 seconds to determine the second-derivative of the given function.

For the twentieth-order derivative, the computational time is,

```
>> tic, diff(y,x,20); toc
Elapsed time is 2.401116 seconds.
```

In this latter case, MATLAB requires about 2.4 seconds to determine the twentieth-order derivative of such complex function. All of the examples above clearly demonstrate the capability of MATLAB for finding derivatives of arbitrary functions in a short time.

The `diff` command can also be used to determine partial derivatives of functions that contain many variables. Examples for finding partial derivatives are shown below.

Example Given the function,

$$z = f(x,y) = x\,e^{-\left(x^2+y^2\right)}$$

Determine its partial derivatives with respect to x and y.

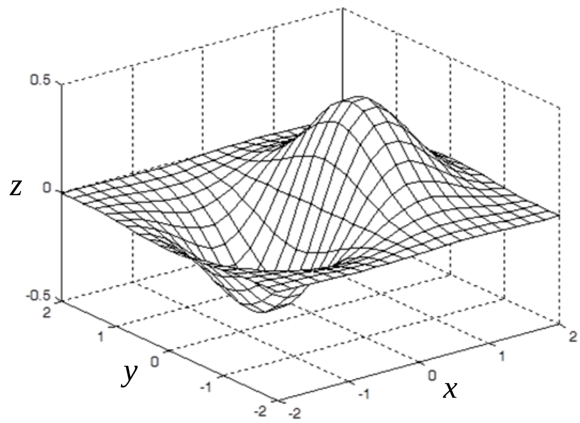
We can plot the variation of the function above by using the `meshgrid` and `mesh` commands.

```
xx = -2:.2:2; yy = -2:.2:2;
[x,y] = meshgrid(xx,yy);
z = x.*exp(-(x.^2+y.^2));
mesh(x,y,z,'EdgeColor','black')
```

meshgrid

mesh

The plot is shown in the figure.



The partial derivatives of the given z function with respect to x and y can be determined by using the `diff` command as follows.

```
>> syms x y
>> z = x*exp(-(x^2+y^2));
>> dzdx = diff(z,x)

dzdx =

-(2*x^2 - 1)/exp(x^2 + y^2)

>> dzdy = diff(z,y)

dzdy =

-(2*x*y)/exp(x^2 + y^2)
```

diff

The results are,

$$\frac{\partial z}{\partial x} = -(2x^2 - 1)e^{-(x^2+y^2)}$$

$$\frac{\partial z}{\partial y} = -(2xy)e^{-(x^2+y^2)}$$

The derivatives obtained above can be verified by the plot of the function $z = f(x, y)$. At $x = y = 0$, the values of $\partial z/\partial x = 1$ and $\partial z/\partial y = 0$ are obtained by using the `subs` command as follows,

```
>> subs(dzdx,{x,y},{0,0})
ans =
     1
>> subs(dzdy,{x,y},{0,0})
ans =
     0
```

subs

2.4 Integration

Integration is the inverse process of differentiation and is sometimes called anti-differentiation. It is rather a difficult topic to most students because they have to memorize many integration formulas. In addition, complicated functions require some special techniques and take a long time to integrate before reaching solutions. In the past, integrating handbooks can alleviate the difficulty in finding the integral results. At present, integration of a function can be obtained easily by using the symbolic computer software.

Integration of a function $f(x)$ is given by,

$$I = \int f(x)dx$$

As an example, the integration of the function $f(x) = x^2$ is,

$$I = \int x^2\,dx = \frac{x^3}{3} + C$$

where C is the integrating constant. We can verify the result obtained by taking its derivative as follows,

$$\frac{dI}{dx} = \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = \frac{3x^2}{3} + 0 = x^2$$

which gives back the original function $f(x)$.

It is noted that the integral sign $\int$ resembles the capital S denoting summation of the area under the function $f(x)$. The function $f(x)$ is called the integrand, while C is the integrating constant that can be determined from the specified condition of the problem.

Few basic integration formulas learned in calculus course are:

(a)	$\int x^n \, dx = \dfrac{x^{n+1}}{n+1}$	(b)	$\int \dfrac{1}{x} \, dx = \ln x$
(c)	$\int \sin x \, dx = -\cos x$	(d)	$\int \cos x \, dx = \sin x$
(e)	$\int \sec^2 x \, dx = \tan x$	(f)	$\int \operatorname{cosec}^2 x \, dx = -\cot x$
(g)	$\int \dfrac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x$	(h)	$\int \dfrac{1}{1+x^2} \, dx = \tan^{-1} x$
(i)	$\int a^x \, dx = \dfrac{a^x}{\ln a}$	(j)	$\int e^x \, dx = e^x$

where the integration constant C is omitted herein for simplicity. The above basic formulas were used to derive many other integrating formulas that are summarized in integration handbooks.

MATLAB has the `int` command that can be employed to perform integration of functions effectively. Examples on the use of such command are highlighted below.

<u>Example</u> $\int \sin x \, dx = -\cos x$

```
>> syms x
>> f = sin(x);
>> int(f)

ans =

-cos(x)
```

int

<u>Example</u> $\int \dfrac{1}{1+x^2} \, dx = \tan^{-1} x$

```
>> syms x
>> int(1/(1+x^2))

ans =

atan(x)
```

<u>Example</u> $$\int a^x \, dx = \frac{a^x}{\ln a}$$

```
>> syms x a
>> int('a^x')

ans =

a^x/log(a)
```

The integration formulas, such as those given in (a) - (j) above, can be applied to find the integrals of more complicated functions as follows.

<u>Example</u> Find the integral of,

$$I = \int \left(8x^3 - 5x^2 + 13x - 7\right) dx$$

```
>> syms x
>> f = 8*x^3 - 5*x^2 + 13*x - 7;
>> int(f)

ans =

2*x^4 - (5*x^3)/3 + (13*x^2)/2 - 7*x
```

i.e., the result is, $$I = 2x^4 - \frac{5}{3}x^3 + \frac{13}{2}x^2 - 7x$$

<u>Example</u> Find the integral of,

$$I = \int \left(3\sin 4x \cos 2x\right) dx$$

```
>> syms x
>> f = 3*sin(4*x)*cos(2*x);
>> int(f)
ans =

- (3*cos(2*x))/4 - cos(6*x)/4
```

i.e., the result is, $I = -\frac{1}{4}(3\cos(2x)+\cos(6x))$

Example Find the integral of,

$$I = \int \frac{e^{2x}+1}{e^x}\, dx$$

```
>> syms x
>> f = (exp(2*x)+1)/exp(x);
>> int(f)
```

int

```
ans =

2*sinh(x)
```

i.e., the result is, $I = 2\sinh x$

The basic integration formulas as shown in (a) - (j) can be further applied together with the use of some integration techniques. Some techniques are presented in details in the examples below.

Example Find the integral of,

$$I = \int x^2\sqrt{x-1}\, dx$$

A technique is to let, $u^2 = x-1$

Then, $2\,u\,du = dx$ and $x = u^2+1$

Thus, the given integral above becomes,

$$\begin{aligned} I &= \int (u^2+1)^2 u\,(2u\,du) \\ &= \int (u^4+2u^2+1)(2u^2)\,du \\ &= \int (2u^6+4u^4+2u^2)\,du \\ &= \frac{2u^7}{7}+\frac{4u^5}{5}+\frac{2u^3}{3} \\ I &= \frac{2u^3}{105}(15u^4+42u^2+35) \end{aligned}$$

After substituting $u = \sqrt{x-1}$, the final solution is obtained,

$$\begin{aligned} I &= \frac{2(x-1)^{3/2}}{105}\left[15(x-1)^2 + 42(x-1) + 35\right] \\ &= \frac{2(x-1)^{3/2}}{105}\left(15x^2 + 12x + 8\right) \end{aligned}$$

The same solution can be obtained easily by using the `int` command,

```
>> syms x
>> f = x^2*sqrt(x-1);
>> int(f)
```

int

```
ans =

(2*(x - 1)^(3/2)*(15*x^2 + 12*x + 8))/105
```

Example Find the integral of,

$$I = \int \cos\left(x^4 + 2\right) x^3 \, dx$$

To integrate the above function, a technique different from the previous example is needed as follows,

$$\begin{aligned} I &= \int \cos\left(x^4 + 2\right) x^3 \, dx \\ &= \frac{1}{4}\int \cos\left(x^4 + 2\right)\left(4x^3\right) dx \\ &= \frac{1}{4}\int \cos\left(x^4 + 2\right) d\left(x^4 + 2\right) \\ &= \frac{1}{4}\sin\left(x^4 + 2\right) \end{aligned}$$

Again, the `int` command can provide the solution conveniently,

```
>> syms x
>> f = x^3*cos(x^4+2);
>> int(f)
```

int

```
ans =

sin(x^4 + 2)/4
```

If MATLAB cannot integrate the given function, the entering expression is returned with the message stating that the explicit integral could not be found. For example,

$$I \;=\; \int \frac{1}{\sqrt{1+x^2+x^4}}\,dx$$

```
>> syms x
>> int('1/sqrt(1+x^2+x^4)')
Warning: Explicit integral could not be found.

ans =

int(1/(x^4 + x^2 + 1)^(1/2), x)
```

Most of practical problems require solutions of the definite integrals for which the lower and upper limits of integration are specified. Techniques for finding definite integrals of some functions are shown in the examples below.

<u>Example</u> Find the definite integral of,

$$I \;=\; \int_0^4 \sqrt{2x+1}\,dx$$

A technique to find the integral above is to assign a new variable, $u = 2x+1$, so that $du = 2dx$. The lower and upper limits are changed into the form of new variable u, i.e., $u(0) = 2(0)+1 = 1$ and $u(4) = 2(4)+1 = 9$, respectively. Then,

$$\begin{aligned} I &= \int_1^9 \sqrt{u}\left(\frac{1}{2}du\right) &&= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^9 \\ &= \frac{1}{3}\left(9^{3/2}-1^{3/2}\right) &&= \frac{26}{3} \end{aligned}$$

Again, we can use the `int` command to find the same solution by simply entering,

```
>> syms x
>> int('sqrt(2*x+1)',0,4)

ans =

26/3
```

Example Find the definite integral of,

$$I = \int_0^\infty \frac{1}{1+x^2}\,dx$$

Using a formula in an integrating handbook, the above integral is,

$$\begin{aligned} I &= \lim_{a\to\infty}\int_0^a \frac{1}{1+x^2}\,dx = \lim_{a\to\infty}\left(\tan^{-1}x\right)\Big|_0^a \\ &= \lim_{a\to\infty}\left(\tan^{-1}a - \tan^{-1}0\right) = \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2} \end{aligned}$$

```
>> syms x
>> int('1/(1+x^2)',0,Inf)
```

int

```
ans =

pi/2
```

Multi-dimensional integration can also be performed by using the same `int` command as highlighted in the following examples.

Example Find the two-dimensional indefinite integral of,

$$\begin{aligned} I &= \iint f(x,y)\,dx\,dy \\ &= \iint x^2 y\,dx\,dy = \frac{x^3y^2}{6} \end{aligned}$$

```
>> syms x y
>> f = x^2*y;
>> I = int(int(f,x),y)
```

int

```
I =

(x^3*y^2)/6
```

<u>Example</u> Find the three-dimensional definite integral of,

$$I = \int_5^6 \int_3^4 \int_1^2 f(x,y,z)\, dx\, dy\, dz$$

$$= \int_5^6 \int_3^4 \int_1^2 x^3 y^2 z\, dx\, dy\, dz = \frac{2035}{8}$$

```
>> syms x y z
>> f = x^3*y^2*z;
>> I = int(int(int(f,x,1,2),y,3,4),z,5,6)

I =

2035/8
```

2.5 Taylor Series

Many solutions to differential equations are in form of infinite series. In calculus course, several types of infinite series are thus studied. Popular infinite series are in the power and polynomial form as shown below.

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$$

The function $f(x) = e^x$, $f(x) = \sin(x)$ and $f(x) = \cos(x)$ above can be derived in form of infinite series by first writing such $f(x)$ in the form,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$$

where $c_i,\ i = 1, 2, 3, \ldots, \infty$ are determined from the derivatives of the function as follows,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ... + nc_n(x-a)^{n-1} + ...$$
$$f''(x) = 1\cdot 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + ...$$
$$f'''(x) = 1\cdot 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + ...$$

At $x = a$, the above equations reduce to,

$$f'(a) = 1c_1$$
$$f''(a) = 1\cdot 2c_2$$
$$f'''(a) = 1\cdot 2\cdot 3c_3$$

Or, in general, $$f^{(n)}(a) = (n!)c_n$$

Thus, $$c_n = \frac{f^{(n)}(a)}{n!}$$

Then, the function $f(x)$ can be rewritten as,

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + ... + \frac{1}{n!}f^{(n)}(a)(x-a)^n + ...$$

The function $f(x)$ in this form is called the *Taylor series* at $x = a$.

It is noted that if $a = 0$, the Taylor series reduces to,

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + ... + \frac{1}{n!}f^{(n)}(0)x^n + ...$$

Or, $$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)x^n$$

which is known as the *Maclaurin series*.

MATLAB has the `taylor` command that can be used to display the function $f(x)$ in form of the Taylor and Maclaurin $(a = 0)$ series. The examples below demonstrate such capability.

Example Find the Maclaurin series for the function,

$$f(x) = e^x$$

Since the given function $f(x) = e^x$, then $f'(x) = e^x$, $f''(x) = e^x$,...,

$f^{(n)}(x)=e^x$. But $f^{(n)}(0)=e^0=1$ for any n, then the Maclaurin series is,

$$f(x) \quad = \quad e^x \quad = \quad \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad = \quad 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...$$

We can use the `taylor` command to generate such series. For example, the series with the first three terms can be displayed by entering,

```
>> syms x
>> f = exp(x);
>> taylor(f,x,3)

ans =

x^2/2 + x + 1
```

i.e., $$f(x) \quad = \quad e^x \quad \approx \quad 1+x+\frac{x^2}{2}$$

Similarly, the series with the first five terms can be obtained by typing,

```
>> taylor(f,x,5)

ans =

x^4/24 + x^3/6 + x^2/2 + x + 1
```

`taylor`

i.e., $$f(x) \quad = \quad e^x \quad \approx \quad 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$$

In the same manner, the Taylor series $x=a$ with the first three terms can be displayed by entering,

```
>> syms a
>> taylor(f,x,3,a)

ans =

exp(a) + (exp(a)*(a - x)^2)/2 - exp(a)*(a - x)
```

`taylor`

i.e., $$f(x) \quad = \quad e^x \quad \approx \quad e^a+e^a(x-a)+\frac{e^a}{2!}(x-a)^2$$

<u>Example</u> Find the Maclaurin series for the function,

$$f(x) = \cos x$$

Since,

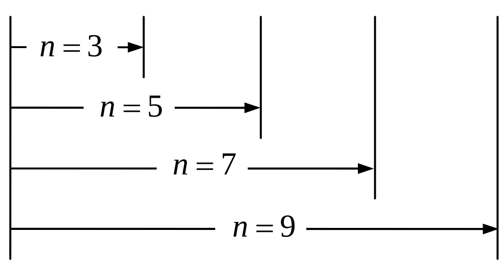
$$\begin{aligned} f(x) &= \cos x \quad ; \quad & f'(x) &= -\sin x \\ f''(x) &= -\cos x \quad ; \quad & f'''(x) &= \sin x \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x \quad ; \quad & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x \end{aligned}$$

At $x=0$, $\cos(0)=1$ and $\sin(0)=0$, then the derivatives of $f(x)$ are,

$$f^{(2n)}(0) = (-1)^n \qquad \text{and} \qquad f^{(2n+1)}(0) = 0$$

Thus, the Maclaurin series $(a=0)$ is,

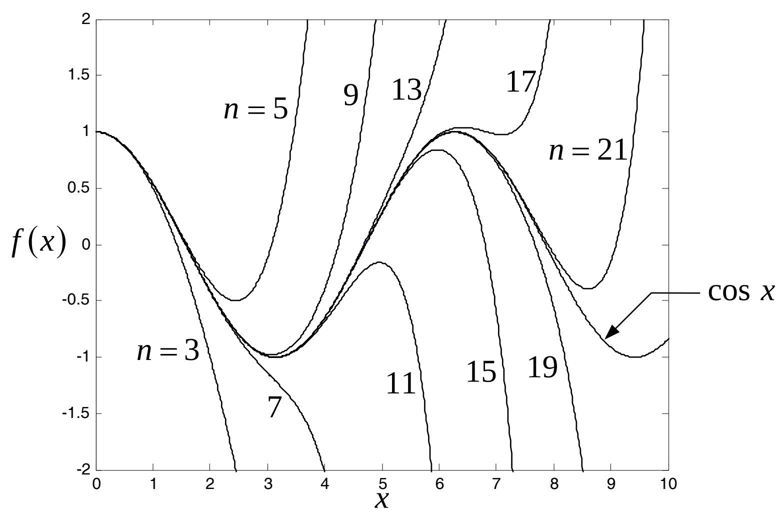
$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \dots + \frac{1}{n!} f^{(n)}(0)x^n + \dots \\ &= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + 0 + \frac{x^8}{8!} + \dots \end{aligned}$$



If we plot the function $f(x)$ by using the series above, we see that the series yield results that approach the solution of $f(x) = \cos x$ when more terms on right-hand-side of the series are included. This means the true variation of the cosine function can be obtained by computing the function $f(x) = \cos x$ and its derivatives at $x=0$ as $n \to \infty$. We can use the `plot` command to demonstrate convergence of the results as more terms are added into the series. Comparisons of these results with the true function $f(x) = \cos x$ are shown in the figure.

```
>> x0=0:.01:10; y0=cos(x0); syms x; y = cos(x);
   plot(x0,y0,'linewidth',2), axis([0,10,-2,2]);
   for n=[3:2:21]
     p=taylor(y,x,n), y1=subs(p,x,x0);
     line(x0,y1)
   end
```

plot



<u>Example</u> Use the `taylor` command to find the first three terms of the function,

$$f(x) = \frac{\cos x}{x^2 + x + 1}$$

at $x = 0$ and $x = 2$.

The first three terms of the Taylor series at $x = 0$ for the given function above can be found by entering,

```
>> syms x
>> f = cos(x)/(x^2+x+1);
>> taylor(f,x,3)

ans =

- x^2/2 - x + 1
```

taylor

i.e.,
$$f(x) = \frac{\cos x}{x^2+x+1} \approx 1 - x - \frac{x^2}{2}$$

These first three terms at $x = 2$ are,

```
>> taylor(f,x,3,2)
```

taylor

```
ans =

cos(2)/7 - (x - 2)*((5*cos(2))/49 + sin(2)/7) -
(x - 2)^2*((13*cos(2))/686 - (5*sin(2))/49)
```

i.e.,
$$\frac{\cos 2}{7} - (x-2)\left[\frac{5\cos 2}{49} + \frac{\sin 2}{7}\right] - (x-2)^2\left[\frac{13\cos 2}{686} - \frac{5\sin 2}{49}\right]$$

2.6 Other Series

MATLAB contains the `sum` and `symsum` commands that can be used to determine the series in the forms of numbers and symbols as demonstrated in the following examples.

Example Determine the result of the series,

$$S = \sum_{n=0}^{10} 2^n = 2^0 + 2^1 + 2^2 + ... + 2^{10}$$

We can employ the `sum` command by entering,

```
>> S = sum(2.^[0:10])
```

sum

```
S =

        2047
```

i.e.,
$$\sum_{n=0}^{10} 2^n = 2047$$

The `symsum` command can also be used to give the same result,

```
>> syms n
>> symsum(2^n,0,10)
```

symsum

```
ans =

2047
```

Example Determine the result of the series,,

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

We can employ the `symsum` command to find the result symbolically by entering,

```
>> syms n
>> symsum(1/n^2,n,1,Inf)

ans =

pi^2/6
```

i.e.,
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Result in the form of π symbol above appears in most of calculus textbooks. Such result can be determined numerically by typing,

```
>> format long; double(ans)

ans =

   1.644934066848226
```

double

It is noted that this series with the first hundred terms is,

```
>> format long; sum(1./([1:100].^2))

ans =

   1.634983900184892
```

which is different from the exact result starting from the second decimal place onward.

Example Prove the series,

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)} = \frac{1}{3}$$

Again, the `symsum` command can be used by entering,

```
>> syms n
>> symsum(1/((3*n-2)*(3*n+1)),n,1,Inf)
```

symsum

```
ans =

1/3
```

2.7 Concluding Remarks

In this chapter, we have studied and reviewed essential topics in calculus. These topics are the limitation, differentiation and integration of functions. As we learned in the calculus course, several formulas and techniques are needed in order to find solutions for these problems. Some simple formulas can be memorized while many others are collected in handbooks. Learning these topics, which represents the first step toward solving higher mathematical problems, is often difficult to most students.

MATLAB contains the `limit`, `diff` and `int` commands that can be used to find limitation, differentiation and integration of a given function, respectively. The effectiveness of these commands was demonstrated through examples by comparing the solutions with those carried out by hands. The commands can provide the solutions immediately so that students will have more time to understand physical behaviors of the problems. MATLAB also contains the `taylor` and `symsum` commands that can be used to generate the Taylor, Maclaurin and other series conveniently. These series can be expressed symbolically or computed numerically. Understanding these series is the basis for learning differential equations in the following chapters.

Few key commands presented in this chapter clearly demonstrate the capability and efficiency of MATLAB for solving calculus problems. These commands help students to verify their solutions derived in the traditional way as learned in classes. With the `plot` or `ezplot` command, these solutions can also be plotted easily to further increase understanding of the problems. The symbolic computer software today thus can alleviate difficulty in learning calculus and increase understanding of the subject considerably.

Exercises

1. Use the `limit` command to determine,

   (a) $\lim\limits_{x \to 1} \dfrac{x^6 - 1}{x^{10} - 1}$ (b) $\lim\limits_{x \to 0} \dfrac{9^x - 5^x}{x}$

   (c) $\lim\limits_{x \to 0} \dfrac{\sqrt{x+4} - 2}{x}$ (d) $\lim\limits_{x \to 0} \dfrac{\tan 3x}{\tan 5x}$

   (e) $\lim\limits_{x \to 0} \dfrac{\cos 2x - \cos x}{x^2}$ (f) $\lim\limits_{x \to 2} \dfrac{x^2 - 2x}{x^2 - 4x + 4}$

2. Use the `limit` command to determine,

   (a) $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{\sqrt{x} - 1}$ (b) $\lim\limits_{x \to -2} \dfrac{x^3 - 3x + 2}{x^2 + 2x}$

   (c) $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2 + 9} - 3}{x^2}$ (d) $\lim\limits_{x \to 0} x^2 \sin\left(\dfrac{1}{x}\right)$

   (e) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$ (f) $\lim\limits_{h \to 0} \dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$

   Then, verify the solutions by plotting with the use of `ezplot` command.

3. Use the `limit` command to determine,

   (a) $\lim\limits_{x \to \pi} \dfrac{\sin x}{2 + \cos x}$ (b) $\lim\limits_{x \to \infty} \left(2 - \dfrac{1}{x}\right)$

   (c) $\lim\limits_{x \to \infty} \dfrac{5x - 7}{4x - 3}$ (d) $\lim\limits_{x \to \infty} \dfrac{160x^{-0.4} + 90}{4x^{-0.4} + 15}$

   (e) $\lim\limits_{x \to \pi} \dfrac{\tan x - x}{\tan^2 x + 3}$ (f) $\lim\limits_{x \to -\infty} \sqrt{x^2 + 3x + 1} + x$

   Then, verify the solutions by plotting with the use of `ezplot` command.

4. Use the `limit` command to determine the limits of the functions containing two variable x and y,

   (a) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy}{\sqrt{xy+1}-1}$

   (b) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)e^{x^2+y^2}}$

   (c) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{1}{\sqrt{(1-x)^2+y^2}} + \dfrac{1}{\sqrt{(1+x)^2+y^2}}$

   (d) $\lim\limits_{\substack{x\to 1/\sqrt{y}\\ y\to \infty}} e^{-1/(x^2+y^2)} \dfrac{\sin^2 x}{x^2}\left(1+\dfrac{1}{y^2}\right)^{x+9y^2}$

5. Use the `diff` command to find the first-order derivatives of the following functions,

   (a) $2x^3 - x^4$ (b) $x(20-5x)$

   (c) $\dfrac{4x}{x^2+1}$ (d) $\dfrac{x^2-3x}{x-1}$

   (e) $x\sqrt{4-x^2}$ (f) $\sqrt{x^2+2x+5}$

   Then, verify the solutions with those derived by hands.

6. Use the `diff` command to find the first-order derivatives of the following functions,

   (a) $x-\sqrt{x}$ (b) $(x-1)^{2/3}-(x+1)^{2/3}$

   (c) $\dfrac{3x^2+2x-4}{2x^2-x+1}$ (d) $\dfrac{2x+3}{x^3-2x^2+4}$

   (e) $\sqrt{x}\ln x$ (f) $\sqrt{\sin x}$

   Then, use the `ezplot` command to verify the solutions by plotting at appropriate x locations.

7. Use the `diff` command to find the first-order derivatives of the following trigonometric functions,

(a) $\sin\left(x^3 - x^2\right)$ (b) $\left(\sin x + x\right)\left(x^3 - \ln x\right)$

(c) $\dfrac{e^x + x\sin x}{\tan x}$ (d) $\ln\left(\dfrac{x^2}{x-2}\right)$

(e) $x e^{x^2}$ (f) $\ln\left(e^{\cos x} + x\right)$

Then, use the `ezplot` command to plot comparing the variations of the functions and their derivatives.

8. Given,

$$u = \cos^{-1}\sqrt{\frac{x}{y}}$$

Use the `diff` command to show that,

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

9. Employ the `meshgrid` command to display the function $z = f(x, y)$,

(a) $$z = 100\,x(1-x)\,y(1-y)\tan^{-1}\left(10\left(\frac{x+y}{\sqrt{2}} - 0.8\right)\right)$$

for $0 \le x \le 1, 0 \le y \le 1$

(b) $$z = \left(x^2 - 2x\right)e^{-x^2 - y^2 - xy}$$

for $-3 \le x \le 3, -2 \le y \le 2$

Then, use the `diff` command to find the expressions of $\partial z/\partial x$ and $\partial z/\partial y$. Compute these expressions at $x = y = 0$ by using the `subs` command.

10. Use the `int` command to find solutions of the following integrals,

(a) $\int (1-x)^4\, dx$ (b) $\int (2x-3)^5\, dx$

(c) $\int \dfrac{1}{x+3}\, dx$ (d) $\int \dfrac{1}{(1-x)^2}\, dx$

(e) $\int \sin 3x\, dx$ (f) $\int \cos(\pi - x)\, dx$

Then, verify the solutions by finding their derivatives.

11. Use the `int` command to find solutions of the following integrals,

(a) $\int \frac{2x-5}{\sqrt{x^2-5x+3}}\,dx$ (b) $\int \frac{x}{1+x^2}\,dx$

(c) $\int \frac{\sin x}{3+5\cos x}\,dx$ (d) $\int \frac{x^2}{\sqrt{3x+4}}\,dx$

(e) $\int \frac{1+2x}{\sqrt{1-x^2}}\,dx$ (f) $\int \frac{\sec^2 x}{4+\tan^2 x}\,dx$

Then, verify the solutions by comparing with those derived by hands as learned in calculus course.

12. Use the `int` command to find the following integrals,

(a) $\int \frac{1}{\sqrt{1+x^4}}\,dx$ (b) $\int \frac{1}{x\sqrt{x^2-4}}\,dx$

(c) $\int \frac{\cos x}{(1-\sin x)^2}\,dx$ (d) $\int \tanh^2 x\,dx$

(e) $\int \frac{x^2-5}{2\sqrt{x}}\,dx$ (f) $\int \frac{x^3+8}{2x}\,dx$

13. Use the `int` command to determine the following definite integrals,

(a) $\int_0^1 \frac{2x+1}{\sqrt{5x-1}}\,dx$ (b) $\int_1^2 \frac{1}{x(x+1)^2}\,dx$

(c) $\int_0^3 \frac{x}{\sqrt{x^2+9}}\,dx$ (d) $\int_0^2 \frac{1}{x^2+4}\,dx$

(e) $\int_0^2 \sqrt{3x+5}\,dx$ (f) $\int_1^4 \sin 3x\,dx$

14. Use the `int` command to determine the following definite integrals,

(a) $\int_1^5 x^5\,dx$ (b) $\int_{-1}^4 \sqrt{x+4}\,dx$

(c) $\int_0^{\pi/2} \sin^2 x \, dx$ (d) $\int_0^{\pi/2} \sin 3x \cos x \, dx$

(e) $\int_0^2 \sqrt{4 - x^2} \, dx$ (f) $\int_1^2 x(x^2 - 1)^4 \, dx$

Then, verify the solutions with those derived by hands.

15. Use the `int` command to find the following multi-dimensional integrals,

    (a) $\iint (x^2 - 7xy + 5y^3) \, dx \, dy$

    (b) $\iint (\sin^2 x \cos x) \, dx \, dy$

    (c) $\iiint (x^2 y \sin(z) + 8\cos^2(y)) \, dx \, dy \, dz$

    (d) $\int_5^6 \int_3^4 \int_1^2 \left(\frac{x + y}{\sqrt{z}} \right) dx \, dy \, dz$

    Then, verify the solutions with those derived by hands.

16. Use the `taylor` command to express the first six terms of the Maclaurin series for the following functions,

    (a) e^{-5x} (b) $x e^x$

    (c) $x^2 \sin x$ (d) $x \cos \pi x$

    (e) $\cos \sqrt{x + 1}$ (f) $\sin^2 x$

17. Use the `taylor` command to express the first twenty terms of the Maclaurin series for the following functions,

    (a) $\sinh x$ (b) $\tan^{-1} x$

    (c) $\ln(1 + x)$ (d) $(e^x - 1 - x)/x^2$

    Then, plot to compare their variations with the true functions.

18. Use the `taylor` command to express the first five terms of the Taylor series at $x = a$ for the following functions,

    (a) $e^x \sin x$ (b) $\sqrt[3]{8 + x}$

    (c) $x \ln(1 + 2x)$ (d) $x^2 \cos^2 x$

    (e) $\frac{x}{\sqrt{x^2 + 4}}$ (f) $\frac{x - \sin x}{x^2}$

19. Use the `symsum` command to prove the following infinite series,

(a) $\dfrac{1}{1^4}+\dfrac{1}{3^4}+\dfrac{1}{5^4}+\dfrac{1}{7^4}+\ldots = \dfrac{\pi^4}{96}$

(b) $\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\ldots = \dfrac{\pi^2}{12}$

(c) $\dfrac{1}{1\cdot 3}+\dfrac{1}{2\cdot 4}+\dfrac{1}{3\cdot 5}+\dfrac{1}{4\cdot 6}+\ldots = \dfrac{3}{4}$

(d) $\dfrac{1}{1^2\cdot 3^2}+\dfrac{1}{3^2\cdot 5^2}+\dfrac{1}{5^2\cdot 7^2}+\dfrac{1}{7^2\cdot 9^2}+\ldots = \dfrac{\pi^2-8}{16}$

20. Use the `symsum` command to show that the following functions can be written in the form of infinite series,

(a) $\ln(1+x) = x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots \qquad -1<x<1$

(b) $e^x = 1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots \qquad -\infty<x<\infty$

(c) $\cosh x = 1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\dfrac{x^6}{6!}+\ldots \qquad -\infty<x<\infty$

(d) $\sin^{-1} x = x+\dfrac{1}{2}\dfrac{x^3}{3}+\dfrac{1\cdot 3}{2\cdot 4}\dfrac{x^5}{5}+\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\dfrac{x^7}{7}+\ldots \quad |x|<1$

21. Use the `symsum` command to determine the exact solutions of the following infinite series,

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{3^n}\right) = \left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\ldots$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(5n-4)(5n+1)} = \frac{1}{1\cdot 6}+\frac{1}{6\cdot 11}+\frac{1}{11\cdot 16}+\ldots$

Then, compute the series by using the `sum` command that contains only the first two hundred terms. Determine the percentage error by comparing the approximate solution with the exact solution for each case.

Chapter 3

Differential Equations

3.1 Introduction

Many phenomena surrounding us are explained by differential equations. Water flow in a river or air circulation in a room is described by the differential equations representing conservations of mass, momentums and energy. Solving these differential equations lead to solutions of flow velocity, pressure and temperature. Temperature distribution of a coffee cup is governed by a differential equation that describes the energy conservation at any position in the cup. Solving such differential equation leads to the solution of the temperature. Or, deformation of a beam under loading is governed by a differential equation representing the equilibrium condition at any location along the beam. Solving the differential equation gives the deformation shape as well as the stress. Solutions obtained from these differential equations thus help understanding the problem phenomena.

Most commercial scientific and engineering software for analysis and design are based on solving differential equations. Thus, understanding how the differential equations are solved is very important. The differential equation course is thus required for science and engineering students. Even though the course consists of topics just for solving simple forms of differential equations, it is still difficult to most students. This is because there are many specific techniques to memorize and follow in order to derive for the solutions.

With the symbolic computer software today, many differential equations learned in class can be solved easily. Students can compare solutions obtained from the software with those derived by hands, so that they will have more time to spend on understanding the solution behaviors. These will help them to appreciate and realize the importance in taking the differential equation course.

This chapter starts from explaining characteristics and types of differential equations typically learned in the course. Techniques for solving many types of differential equations are presented. The derived solutions are compared with those obtained from using MATLAB. The derived solutions may be in the forms of polynomials or some special functions. These solutions are plotted by using simple MATLAB commands to further increase understanding of their behaviors.

3.2 Characteristics of Differential Equations

In the differential equation course, we studied many types of differential equations. For an example,

$$3x\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + 4y \quad = \quad \cos x$$

In the differential equation above, y is the *dependent variable* that varies with the *independent variable x*. The dependent variable $y = y(x)$ is the solution to the differential equation. This equation is called the second-order differential equation according to the highest derivative order that appears in the equation.

A differential equation is *linear* if the coefficient of each term is a constant or function of x. The differential equation becomes *nonlinear* if the coefficient is function of y. Thus, the differential equation above is nonlinear since the coefficient of the first-order derivative term is $2y$. It is noted that exact solutions are not available or difficult to find for most nonlinear differential equations.

A differential equation is called *homogeneous* if the term on the right-hand-side of the equation is zero. The differential equation above is a *nonhomogeneous* equation because the right-hand-side term is $\cos x$.

In many science and engineering problems, the governing differential equations are often in some specific forms. For examples, the Airy equation,

$$\frac{d^2y}{dx^2} - x\,y = 0$$

the Bessel equation,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0$$

and the Legendre equation,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0$$

The solutions of these differential equations are normally in the forms of specific functions.

If a problem contains n dependent variables, then these variables must be solved from a system of n differential equations. For example, the two dependent variables, y_1 and y_2, are to be solved from the two differential equations,

$$7\frac{dy_1}{dx} + 3y_2 = 16$$

$$2\frac{dy_2}{dx} + 5y_1 = -9$$

The exact solutions of $y_1(x)$ and $y_2(x)$ are obtained by solving these two differential equations simultaneously.

All of the differential equations above are called *ordinary differential equations* because the dependent variable y is only function of the independent variable x. For most practical problems, the dependent variable y is function of many independent variables, e.g., x_1 and x_2,

$$\frac{\partial^2 y}{\partial x_1^2} + \frac{\partial^2 y}{\partial x_2^2} = 0$$

This latter differential equation is called *partial differential equation*.

3.3 Solutions of Differential Equations

One way to verify the solution is to substitute it back into the differential equation. The solution must satisfy the differential equation as shown in the examples below.

Example Show that,

$$y = y(x) = x^2 - \frac{1}{x}$$

is a solution of the second-order linear homogeneous differential equation,

$$\frac{d^2 y}{dx^2} - \frac{2}{x^2} y = 0$$

The first- and second-order derivatives of the solution are,

$$\frac{dy}{dx} = 2x + \frac{1}{x^2}$$

$$\frac{d^2 y}{dx^2} = 2 - \frac{2}{x^3}$$

By substituting these derivative terms into the left-hand-side of the differential equation,

$$\left(2 - \frac{2}{x^3}\right) - \frac{2}{x^2}\left(x^2 - \frac{1}{x}\right) = 2 - \frac{2}{x^3} - 2 + \frac{2}{x^3}$$

$$= 0$$

leading to the result of zero. This means the given solution is the solution of the differential equation.

We can use the `diff` command to verify such result as follows,

```
>> syms x
>> y = x^2 - 1/x;
>> diff(y,2) - 2*y/x^2
```

diff

```
ans =

0
```

<u>Example</u> Use the `diff` command to show that,

$$y = y(x) = C_1 e^{-x} + C_2 e^{2x}$$

where C_1 and C_2 are constants, is a solution of the second-order linear homogeneous differential equation,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

```
>> syms x C1 C2
>> y = C1*exp(-x) + C2*exp(2*x);
>> diff(y,2) - diff(y,1) - 2*y

ans =

0
```

It is noted that C_1 and C_2 could be any numerical value, the given solution is always the solution of the differential equation above.

Solutions of the differential equations may be in many forms from simple to complex functions. We will learn how to derive solutions in details in the following chapters. In this chapter, however, we will use the `dsolve` command to conveniently find the solutions. The objective herein is to show that the solutions could be in different forms, such as the trigonometric, polynomial and exponential functions.

<u>Example</u> Use the `dsolve` command to find solution of the first-order linear nonhomogeneous differential equation,

$$\frac{dy}{dx} = x^2$$

```
>> syms x
>> dsolve('Dy = x^2', 'x')
```

dsolve

```
ans =

x^3/3 + C3
```

i.e., $$y = \frac{x^3}{3} + C_3$$

where C_3 is a constant.

<u>Example</u> Use the `dsolve` command to find solution of the first-order linear nonhomogeneous differential equation,

$$\frac{dy}{dx} - y = x^2$$

```
>> syms x
>> dsolve('Dy - y = x^2', 'x')

ans =

C4*exp(x) - 2*x - x^2 - 2
```

i.e., $$y = C_4 e^x - 2x - x^2 - 2$$

where C_4 is a constant.

<u>Example</u> Use the `dsolve` command to find solution of the first-order nonlinear homogeneous differential equation,

$$\frac{dy}{dx} - y^2 = 0$$

```
>> syms x
>> dsolve('Dy - y^2 = 0', 'x')
```

dsolve

```
ans =

-1/(C5 + x)
```

i.e., $$y = -\frac{1}{C_5 + x}$$

where C_5 is a constant.

The differential equation in the preceding example is nonlinear,

$$\frac{dy}{dx} = y^2$$

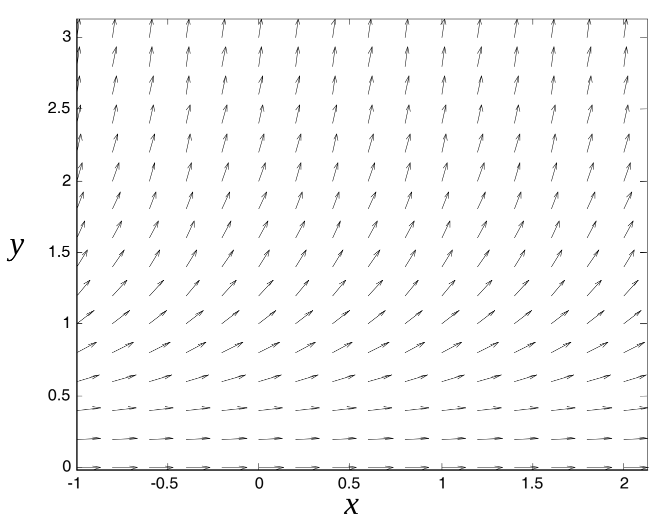
Solution of this differential equation can be plotted to show the direction field by using the `quiver` command. For example, the direction field for $-1 < x < 2$ and $0 < y < 3$ can be displayed by entering,

```
>> [x,y] = meshgrid(-1:0.2:2, 0:0.2:3);
>> dydx = y.^2;
>> vl = sqrt(1 + dydx.^2);
>> quiver(x, y, 1./vl, dydx./vl, 0.5), axis tight
```

meshgrid

quiver

where `vl` is the vector length. The plot shows the vectors representing the direction field of the solution $y(x)$ that varies with x.



Since the solution of the nonlinear differential equation above is,

$$y = -\frac{1}{C_5 + x}$$

where C_5 is a constant that depends on the initial condition. If the initial condition is given as $y(0)=1$, then the constant C_5 can be determined,

$$1 = -\frac{1}{C_5 + 0}$$

to give $C_5 = -1$. Thus, the exact solution corresponding to the given initial condition is,

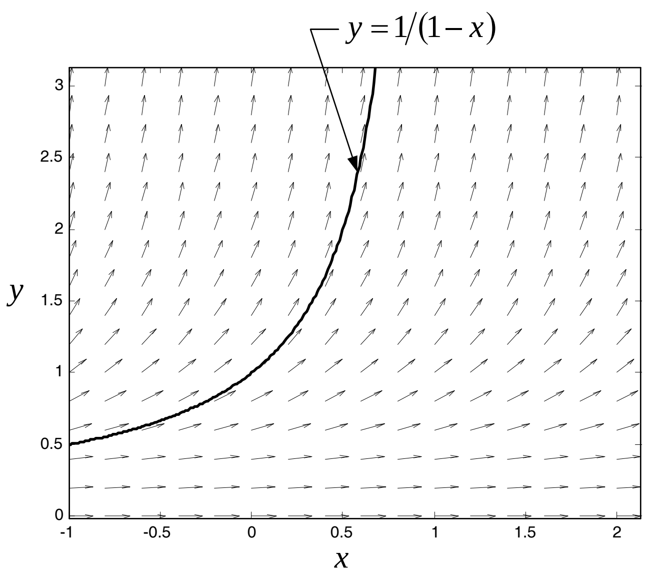
$$y = \frac{1}{1-x}$$

We can plot this exact solution by imposing it onto the direction field as,

```
>> hold on
>> xx=-1:.01:2;
>> yy = 1./(1.-xx);
>> plot(xx,yy,'linewidth',2)
```

plot

We can see that the exact solution (solid line) from the given initial condition is a solution of the direction field.



In the following examples, solutions to the differential equations are in the form of special functions, such as the error, Airy and Bessel functions. Details of these functions are given in Chapter 11.

Example Use the `dsolve` command to find solution of the first-order linear nonhomogeneous differential equation,

$$\frac{dy}{dx} = e^{-x^2}$$

```
>> syms x
>> dsolve('Dy = exp(-x^2)', 'x')
```

dsolve

```
ans =

C6 + (pi^(1/2)*erf(x))/2
```

i.e.,
$$y = C_6 + \frac{\sqrt{\pi}\, erf(x)}{2}$$

where C_6 is a constant.

Example Use the `dsolve` command to find solution of the second-order linear homogeneous differential equation,

$$\frac{d^2y}{dx^2} - xy = 0$$

```
>> syms x
>> dsolve('D2y - x*y', 'x')

ans =

C7*airyAi(x, 0) + C8*airyBi(x, 0)
```

i.e.,
$$y = C_7 Ai(x) + C_8 Bi(x)$$

where C_7 and C_8 are constants. The function $Ai(x)$ and $Bi(x)$ are the Airy and Bairy function, respectively. Values of these two functions can be determined at any x location as explained in Chapter 11.

Example Use the `dsolve` command to find solution of the second-order linear homogeneous differential equation,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - 4\right)y \quad = \quad 0$$

```
>> syms x
>> dsolve('x^2*D2y + x*Dy + (x^2-4)*y = 0', 'x')

ans =

C9*besselj(2, x) + C10*bessely(2, x)
```

i.e., $$y \quad = \quad C_9 J_2(x) + C_{10} Y_2(x)$$

where C_9 and C_{10} are constants. The function $J_2(x)$ and $Y_2(x)$ are the Bessel functions of the first and second kind, respectively. Again, values of these two functions at any x location can be determined as explained in Chapter 11.

Solutions of some differential equations may be in implicit form as shown in the following examples.

Example Use the `dsolve` command to find solution of the first-order nonlinear nonhomogeneous differential equation,

$$y\frac{dy}{dx} \quad = \quad x^2$$

```
>> syms x
>> dsolve('y*Dy = x^2', 'x')
```

`dsolve`

```
ans =

  2^(1/2)*(x^3/3 + C11)^(1/2)
 -2^(1/2)*(x^3/3 + C11)^(1/2)
```

The solution of y has to be determined from the equation,

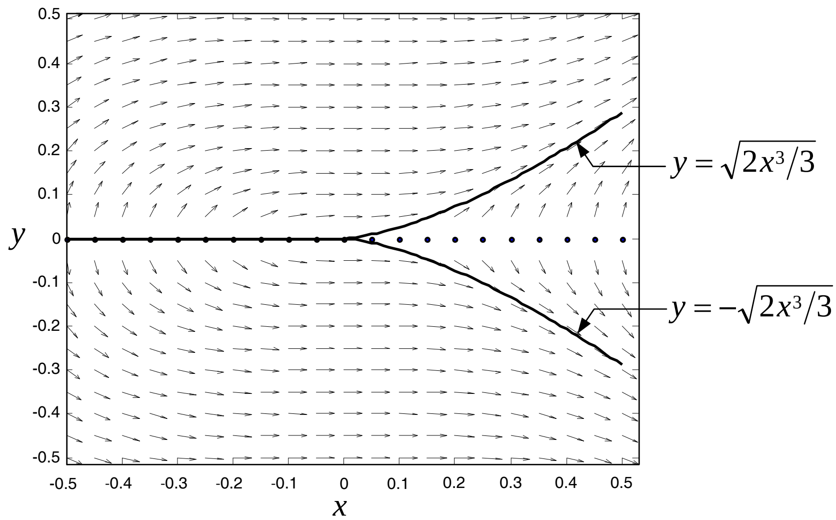
$$y^2 - \frac{2x^3}{3} - 2C_{11} \quad = \quad 0$$

which is in an implicit form.

If the initial condition is $y(0) = 0$, then $C_{11} = 0$ and the solution is,

$$y \quad = \quad \pm\sqrt{2x^3/3}$$

The solution of y can be plotted on the direction field of the general solution as shown in the figure.



<u>Example</u> Use the `dsolve` command to find solution of the first-order nonlinear homogeneous differential equation,

$$(y\cos y - \sin y + x)\frac{dy}{dx} \quad = \quad y$$

```
>> syms x
>> dsolve('(y*cos(y) - sin(y) + x)*Dy = y', 'x')

ans =

solve(- sin(y) – C12*y = -x, y)
```

The solution is in an implicit form of,

$$\sin y + C_{12} y \quad = \quad x$$

where C_{12} is a constant that can be determined from the given initial condition.

Example Use the `dsolve` command to find solution of the first-order nonlinear nonhomogeneous differential equation,

$$y^5 \frac{dy}{dx} - y^4 = 1$$

```
>> syms x
>> dsolve('y^5*Dy - y^4 = 1', 'x')
```

dsolve

```
ans =

solve(y^2 - atan(y^2) = 2*C13 + 2*x, y)
```

Again, the solution is in an implicit form of,

$$y^2 - \tan^{-1} y^2 = 2C_{13} + 2x$$

where C_{13} is a constant that can be determined from the given initial condition.

There are many differential equations that the explicit expressions of their solutions cannot be found as shown in the following examples.

Example Use the `dsolve` command to find solution of the first-order linear nonhomogeneous differential equation,

$$\frac{dy}{dx} = \frac{x^2}{\sqrt{1+x^4}}$$

```
>> syms x
>> dsolve('Dy = x^2/sqrt(1+x^4)', 'x')

ans =

C14 + int(x^2/(x^4 + 1)^(1/2), x)
```

In this case, MATLAB returns the solution in the integral form as,

$$y = \int \frac{x^2}{\sqrt{1+x^4}}\, dx + C_{14}$$

where C_{14} is an integrating constant.

<u>Example</u> Use the `dsolve` command to find solution of the first-order nonlinear differential equation,

$$\frac{dy}{dx} - y = e^{-y}$$

```
>> syms x
>> dsolve('Dy - y = exp(-y)', 'x')

ans =

solve(int(exp(y)/(y*exp(y) + 1), y) = C15 + x, y)
```

This means the solution must be determined from the implicit equation containing the integral term of y,

$$\int \frac{e^y}{y e^y + 1}\, dy = C_{15} + x$$

where C_{15} is an integrating constant.

<u>Example</u> Use the `dsolve` command to find solution of the second-order nonlinear homogeneous differential equation,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^2 = 0$$

dsolve

```
>> syms x
>> dsolve('D2y + Dy + y^2 = 0', 'x')
Warning: Explicit solution could not be found.
```

An explicit solution could not be found for the differential equation in this last example. It is noted that there is a large number of problems that their explicit solutions are not available. In this case, numerical methods must be applied to provide approximate solutions. Several numerical methods can provide very accurate solutions to different types of differential equations. We will see examples that demonstrate such capability in the latter chapters.

3.4 Concluding Remarks

Differential equations occur in many classes of science and engineering problems. They represent the nature of problems, such as the mass, momentums and energy must be conserved or a system must be in equilibrium. Solving the differential equations lead to solutions that help understanding their physical phenomena.

This chapter starts from describing the characteristics of the differential equations by using simple examples. These include linear, nonlinear, homogeneous, nonhomogeneous, first- and second-order differential equations. Finding solutions to the differential equations depends on their types. Exact solutions to some of these differential equations are easy to find while others are difficult or impossible to obtain.

This chapter presents how to use the `diff` command in MATLAB to find derivatives of the given functions. The `dsolve` command is then introduced to find solutions of the differential equations. Examples have shown that these commands help us to solve the differential equations conveniently. We will study different types of the differential equations in more details in the following chapters. Solutions of the differential equations will be derived by hands prior to using MATLAB to confirm them. We will thus understand how the solutions are derived and, at the same time, appreciate the capability of the symbolic computer software.

Exercises

1. Identify that each of the following equations is linear, nonlinear, homogeneous or nonhomogeneous differential equation,

(a) $$3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 1$$

(b) $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = \cos x$$

(c) $\dfrac{dy}{dx} + xy^2 = 0$

(d) $\dfrac{d^2y}{dx^2} + \cos(x+y) = \sin x$

(e) $(1+y^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 7y = e^x$

2. Identify that each of the following equations is linear, nonlinear, homogeneous or nonhomogeneous differential equation,

(a) $xy' + y = x$ (b) $x^2 y' + y = x^2$

(c) $y' = \dfrac{x^2 + y^2}{x^2 - y^2}$ (d) $y' = \dfrac{x + 2y}{2x - y}$

(e) $x^2 y'' + xy' + y = 0$ (f) $x^2 y' - 2y = 3x^2$

3. In each item below, use the `diff` command to verify that the left-hand-side function is a solution to the right-hand-side differential equation,

(a) $y = 3x + x^2$; $xy' - y = x^2$

(b) $y = e^{-3x}$; $y'' + 2y' - 3y = 0$

(c) $y = \cosh x$; $y'' - y = 0$

(d) $y = 1/x^2$; $x^2 y'' + 5xy' + 4y = 0$

(e) $y = x^2 - 1/x$; $x^2 y'' = 2y$

4. In each item below, use the `diff` command to verify that the left-hand-side function is a solution to the right-hand-side differential equation,

(a) $y = 3\sin 2x + 4\cos 2x$; $y'' + 4y = 0$

(b) $y^2 = e^{2x}$; $yy' = e^{2x}$

(c) $y = \sinh 2x + 2\cosh 2x$; $y'' - 4y = 0$

(d) $y^2 = x^2 - x$; $2xyy' = x^2 + y^2$

(e) $x + y = \tan^{-1} y$; $1 + y^2 + y^2 y' = 0$

5. In each item below, use the `diff` command to verify whether the left-hand-side function is a solution to the right-hand-side differential equation or not,

   (a) $y = 2\cos x - 3\sin x$; $y'' + y = 0$

   (b) $y = \sin x + x^2$; $y'' + y = x^2 + 2$

   (c) $y = \cos 2x$; $y' + xy = \sin 2x$

   (d) $y = e^{2x} - 3e^{-x}$; $y'' - y' - 2y = 0$

   (e) $y = 3\sin 2x + e^{-x}$; $y'' + 4y = 5e^{-x}$

6. Use the `diff` command to show that,

   (a) $y = x^2$ is an exact solution of the first-order linear homogeneous differential equation,

$$x\frac{dy}{dx} = 2y$$

   (b) $y = e^x - x$ is an exact solution of the first-order nonlinear nonhomogeneous differential equation,

$$\frac{dy}{dx} + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1$$

7. Use the `diff` command to show that,

$$y = \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2+1} + \ln\sqrt{x\sqrt{x^2+1}}$$

   is an exact solution of the first-order nonlinear homogeneous differential equation,

$$x\frac{dy}{dx} + \ln\left(\frac{dy}{dx}\right) = 2y$$

8. Use the `diff` command to show that each of the solutions below,

   (a) $y = e^x$

   (b) $y = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots + \dfrac{x^n}{n!}$

is an exact solution of the second-order linear homogeneous differential equation,

$$x\frac{d^2y}{dx^2} - (x+n)\frac{dy}{dx} + ny \quad = \quad 0$$

where n is any positive integer.

9. If A and B are constants, use the `diff` command to show that,

$$y_1(x) \quad = \quad (A+Bx)\,e^{3x}$$

and $$y_2(x) \quad = \quad (3A+B+3Bx)\,e^{3x}$$

are the solutions to the coupled differential equations,

$$\frac{dy_1}{dx} \quad = \quad y_2$$

and $$\frac{dy_2}{dx} \quad = \quad -9y_1 + 6y_2$$

10. Use the `dsolve` command to find solutions of the following differential equations,

(a) $(2x-y)\dfrac{dy}{dx} \quad = \quad 4y-3x$

(b) $x^2\dfrac{dy}{dx} \quad = \quad x^2 + xy + y^2$

(c) $2xy\dfrac{dy}{dx} \quad = \quad x^2 + 3y^2$

(d) $(2x+y)\dfrac{dy}{dx} \quad = \quad -4x-3y$

(e) $(x-y)\dfrac{dy}{dx} \quad = \quad x+3y$

11. Use the `dsolve` command to find solutions of the following differential equations,

(a) $\dfrac{d^2y}{dx^2} - y \quad = \quad 4-x$

(b) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y \quad = \quad 0$

(c) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y \;=\; 2e^x(1-x)$

(d) $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y \;=\; 0$

(e) $\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} - 7\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 6y \;=\; 0$

12. Use the `dsolve` command to find solutions of the following differential equations,

(a) $x^3y' + 4x^2y \;=\; e^{-x}$

(b) $y' + \dfrac{1}{x}y \;=\; 3\cos 2x$

(c) $y' \;=\; \dfrac{x}{3}y(4-y)$

(d) $x^2y' \;=\; x^2 + xy + y^2$

(e) $xy' + xy \;=\; 1-y$

13. Solution of the nonlinear differential equation,

$$\frac{dy}{dx} \;=\; y^2 + x$$

is in the form of Bessel function. Use the `dsolve` command to find such solution and plot it in the form of direction field.

14. Employ the `diff` command to show that,

(a) the equation $y^2 + x - 3 \;=\; 0$
is an implicit solution of the differential equation,

$$\frac{dy}{dx} \;=\; -\frac{1}{2y}$$

(b) the equation $x\,y^3 - x\,y^3\sin x \;=\; 1$

is an implicit solution of the differential equation,

$$\frac{dy}{dx} \;=\; \frac{(x\cos x + \sin x - 1)\,y}{3\,(x - x\sin x)}$$

15. Employ the `diff` command to find the first-order differential equations corresponding to the following implicit solutions,

(a) $xy - \ln y = 0$

(b) $y^3 - 3x + 3y = 5$

(c) $1 + x^2 y + 4y = 0$

(d) $y + \tan^{-1} y = x + \tan^{-1} x$

(e) $x^2 + y^2 - 6x + 10y + 34 = 0$

16. In each item below, use the `diff` command to verify whether the left implicit equation is the solution to the right differential equation,

(a) $x^2 + y^2 = 4 \quad ; \quad y' = \dfrac{x}{y}$

(b) $e^{xy} + y = x - 1 \quad ; \quad y' = \dfrac{e^{-xy} - y}{e^{-xy} + x}$

(c) $y - \ln y = x^2 + 1 \quad ; \quad y' = \dfrac{2xy}{y - 1}$

(d) $x + y + e^{xy} = 0 \quad ; \quad (1 + xe^{xy})y' + ye^{xy} = -1$

17. Employ the `diff` command (and other commands, if necessary) to show that the function $y(x)$ in the implicit equation form below,

(a) $3e^{-y^2} = 2e^{3x} + C$

is the solution of the nonlinear differential equation,

$$\frac{dy}{dx} + \frac{1}{y} e^{y^2 + 3x} = 0$$

(b) $\cos x \sin y = C$

is the solution of the nonlinear differential equation,

$$\frac{dy}{dx} = \frac{\tan y}{\cot x}$$

(c) $(x - 1)e^x = \dfrac{1}{y} + \dfrac{1}{2y^2} + C$

is the solution of the nonlinear differential equation,

$$\frac{dy}{dx} = -\frac{x\,y^3\,e^x}{y+1}$$

where C is a constant.

18. Employ the `quiver` command to plot the direction field of the differential equation,

$$\frac{dy}{dx} = x^2 - y$$

for the intervals of $-4 \le x \le 4$ and $-4 \le y \le 4$. Then use the `dsolve` command to solve the differential equation with the initial condition of $y(0) = 1$. Plot the solution by imposing it onto the direction field.

19. Employ the `quiver` command to display the direction field of the nonlinear differential equation,

$$\frac{dy}{dx} = 3y^{2/3}$$

for the intervals of $0 \le x \le 3$ and $-1 \le y \le 1$. Then use the `dsolve` command to solve the differential equation with the initial condition of $y(2) = 0$. Plot the solution by imposing it onto the direction field.

20. Use the `diff` command to show that the function of $y(x)$ in the implicit form,

$$\frac{\sqrt{2}}{8}\ln\left(\frac{y^2+\sqrt{2}y+1}{y^2-\sqrt{2}y+1}\right) + \frac{\sqrt{2}}{4}\left[\tan^{-1}\left(\sqrt{2}y+1\right) + \tan^{-1}\left(\sqrt{2}y-1\right)\right]$$
$$= x + C$$

where C is a constant, is a solution of the nonlinear nonhomogeneous differential equation,

$$\frac{dy}{dx} = y^4 + 1$$

Then, plot its direction field for the intervals of $-4 \le x \le 4$ and $-3 \le y \le 3$.

21. Use the `diff` command to show that the function of $y(x)$ in the implicit form,

$$\frac{x^3}{3} - \left(y + \frac{y^3}{3}\right) = C$$

where C is a constant, is a solution of the nonlinear nonhomogeneous differential equation,

$$\frac{dy}{dx} = \frac{x^2}{1 + y^2}$$

Plot the direction field for the intervals of $-4 \le x \le 4$ and $-4 \le y \le 4$. Then, redisplay the plot by imposing the solutions when $C = -7, -5, -3, 0, 3$ and 5 onto the direction field.

Chapter 4

First-Order Differential Equations

4.1 Introduction

The differential equation in the first-order form of,

$$\frac{dy}{dx} = f(x, y)$$

is probably the simplest form for all types of differential equations. The form is thus often used as the first step in learning how to solve differential equations. Exact solution of $y(x)$, which makes the differential equation satisfies, is not that difficult to find.

This chapter presents standard techniques for finding exact solutions of the first-order differential equations in the form above. Examples will be used to derive exact solutions by employing these techniques. The derived solutions will be verified by using MATLAB commands and plotted to show their variations. If the exact solutions are not available, numerical methods will be used to find the approximate solutions. This chapter will thus help readers to understand how to solve this type of differential equation symbolically and numerically. At the same time, readers will appreciate the current capability of the computer software that can reduce effort for finding solutions of the first-order differential equations.

4.2 Separable Equations

Separation of variables is a simple technique for solving the first-order differential equations. The dependent variable *y* is separated from the independent variable *x* so that they are on the opposite sides of the equation. By doing that, solutions of the differential equations can be found easily. For example,

$$\frac{dy}{dx} = (x+1)y$$

The variable *y* and *x* in the equation above can be separated so that they are on opposite sides as,

$$\frac{dy}{y} = (x+1)dx$$

Integration is then performed on both sides to obtain the solution of *y* as function of *x*. It is noted that the technique cannot apply if both variables are not separable, such as,

$$\frac{dy}{dx} = xy + 7$$

We will apply this technique to derive exact solutions of differential equations as shown in the examples below. The derived solutions will be verified by using the MATLAB commands.

Example Derive general solution of the first-order linear differential equation,

$$\frac{dy}{dx} = \frac{y-1}{x+3}$$

Then, find the exact solution for the case of $y(0) = 4$.

The given differential equation above is separable so that the variables *y* and *x* can be placed on the opposite sides as,

$$\frac{dy}{y-1} = \frac{dx}{x+3}$$

We can integrate both sides of the equation,

$$\int \frac{dy}{y-1} = \int \frac{dx}{x+3}$$

to get, $$\ln|y-1| = \ln|x+3| + C$$

where *C* is the integrating constant. If we apply the exponential function base *e* to both sides of the equation,

$$e^{\ln|y-1|} = e^{\ln|x+3|+C} = e^{C}e^{\ln|x+3|}$$

or, $$|y-1| = e^{C}|x+3| = C_1|x+3|$$

where $C_1 = e^C$. Then, for positive quantity,

$$y-1 = C_1(x+3)$$

i.e., the solution is, $$y = C_1(x+3)+1$$

Here, the constant C_1 is to be determined from the given initial condition.

We can employ the `dsolve` command in MATLAB to obtain the same solution as follows,

```
>> syms x y
>> dsolve('Dy = (y-1)/(x+3)', 'x')

ans =

C1*(x + 3) + 1
```

By applying the initial condition of $y(0)=4$, the constant C_1 can be determined as,

$$4 = C_1(0+3)+1$$

$$C_1 = 1$$

Thus, the exact solution of this problem is,

$$y = (1)(x+3)+1 = x+4$$

The same exact solution can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('Dy = (y-1)/(x+3)', 'y(0) = 4', 'x')
```

`dsolve`

```
ans =

x + 4
```

Example Derive general solution of the first-order nonlinear differential equation,

$$\frac{dy}{dx} = y^2 \sin x$$

Again, we separate the variables y and x so that they are on opposite sides of the equation as,

$$\frac{dy}{y^2} = \sin x\, dx$$

Then, perform integration on both sides to get,

$$\int \frac{dy}{y^2} = \int \sin x\, dx$$

$$-\frac{1}{y} = -\cos x + C$$

where C is the integrating constant. Thus, the general solution is,

$$y = \frac{1}{\cos x - C}$$

The same solution can be obtained by using the `dsolve` command as,

```
>> syms x y
>> dsolve('Dy = y^2*sin(x)', 'x')
```

`dsolve`

```
ans =

-1/(C2 - cos(x))
```

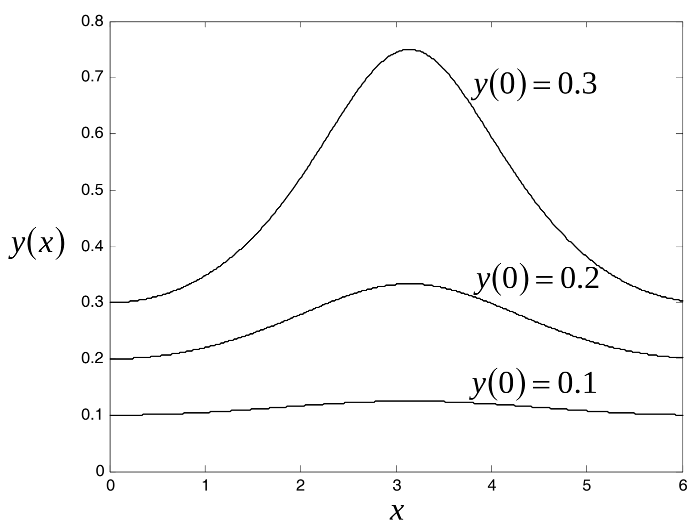
Depending on the initial conditions, the corresponding exact solutions can be determined. For examples,

$$y(0) = 0.1 \quad ; \quad y = \frac{1}{\cos x + 9}$$

$$y(0) = 0.2 \quad ; \quad y = \frac{1}{\cos x + 4}$$

$$y(0) = 0.3 \quad ; \quad y = \frac{1}{\cos x + \frac{7}{3}}$$

Variations of the exact solutions can be plotted by using the `ezplot` command as shown in the figure.



<u>Example</u> Solve the first-order nonlinear differential equation,

$$\frac{dy}{dx} = y^2 e^{-x}$$

We can separate the variables y and x so that they are on opposite sides of the equation as,

$$\frac{dy}{y^2} = e^{-x}\, dx$$

Then, perform integration on both sides to get,

$$\int \frac{dy}{y^2} = \int e^{-x}\,dx$$
$$-\frac{1}{y} = -e^{-x} + C$$

where C is the integrating constant. Thus, the general solution is,

$$y = \frac{1}{e^{-x} - C}$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('Dy = y^2*exp(-x)', 'x')
```

dsolve

```
ans =

-1/(C3 - 1/exp(x))
```

Example Derive exact solution of the first-order nonlinear differential equation,

$$y\frac{dy}{dx} = \cosh x + 3$$

with the initial condition of $y(0) = \pi$.

The variables y and x in the differential equation can be separated so that the equation becomes,

$$y\,dy = (\cosh x + 3)\,dx$$

Integration is performed on both sides of the equation to get,

$$\int y\,dy = \int (\cosh x + 3)\,dx$$
$$\frac{y^2}{2} = \sinh x + 3x + C$$

where C is the integrating constant. Thus, the general solution is,

$$y = \sqrt{2(\sinh x + 3x + C)}$$

The integrating constant is determined by applying the initial condition of $y(0) = \pi$,

$$\pi = \sqrt{2(0 + 0 + C)}$$

$$C = \frac{\pi^2}{2}$$

Hence, the exact solution is,

$$y = \sqrt{2\left(\sinh x + 3x + \frac{\pi^2}{2}\right)}$$

The same solution is obtained by using the `dsolve` command,

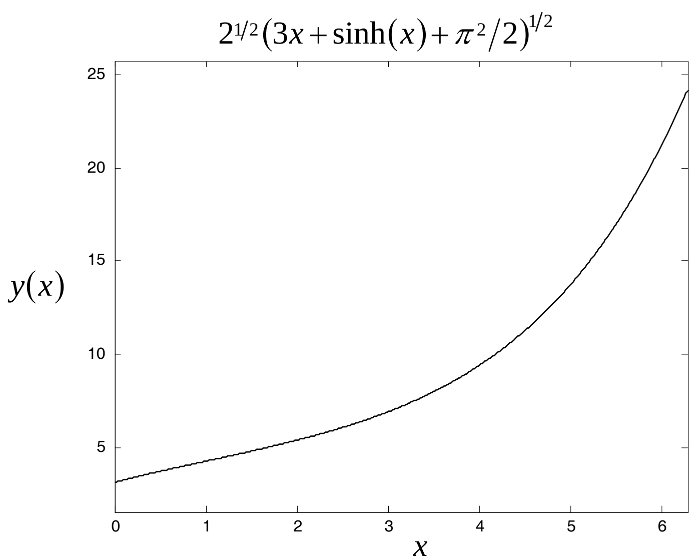
```
>> syms x y
>> dsolve('y*Dy = cosh(x) + 3', 'y(0) = pi', 'x')

ans =

2^(1/2)*(3*x + sinh(x) + pi^2/2)^(1/2)
```

Variation of the solution y can be plotted in the interval of $0 \le x \le 2\pi$ by using the `ezplot` command as,

```
>> ezplot(ans, [0 2*pi])
```

ezplot

The plot of the variation is shown in the figure.



Example Derive the exact solution of the first-order nonlinear differential equation,

$$2(y-1)\frac{dy}{dx} = 3x^2 + 4x + 2$$

with the initial condition of $y(0) = -1$.

The terms containing variables y and x are separable so that the differential equation can be written as,

$$2(y-1)dy = (3x^2 + 4x + 2)dx$$

By performing integration on both sides of the equation,

$$\int 2(y-1)dy = \int (3x^2 + 4x + 2)dx$$

we get,

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

where C is the integrating constant that can be found from the initial condition of $y(0) = -1$,

$$\begin{aligned} 1 + 2 &= 0 + 0 + 0 + C \\ C &= 3 \end{aligned}$$

Then, the exact solution is,

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

or,

$$y^2 - 2y - (x^3 + 2x^2 + 2x + 3) = 0$$

i.e.,

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$$

A proper solution is selected according to the initial condition,

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

which is the exact solution of this problem.

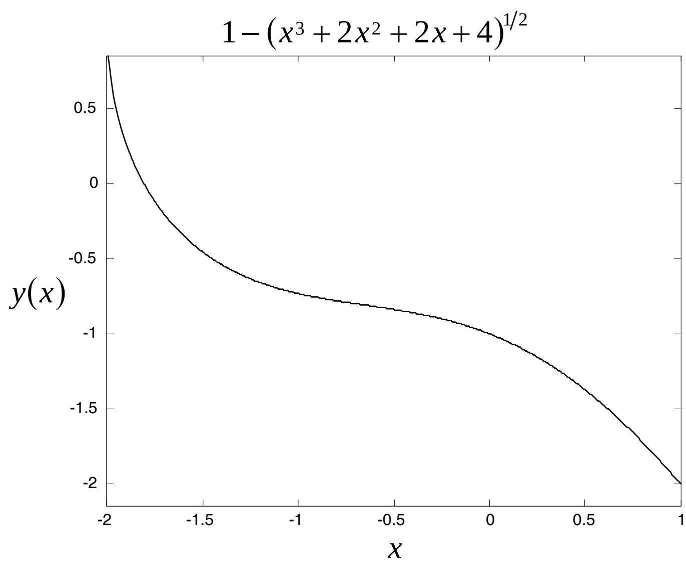
The above exact solution can also be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('2*(y-1)*Dy = 3*x^2+4*x+2',
          'y(0) = -1', 'x')

ans =

1 - (x^3 + 2*x^2 + 2*x + 4)^(1/2)
```

Note that the solution obtained from MATLAB may be lengthy. The `simple` command may help reducing the complexity of the solution. The `ezplot` command can then be used to display the variation of y with x as shown in the figure.

```
>> ezplot(ans, [-2 1])
```

ezplot



The exact solution above can also be verified by substituting it back into the governing differential equation,

```
>> y = 1 - (x^3 + 2*x^2 + 2*x + 4)^(1/2);
>> LHS = 2*(y-1)*diff(y,x)

LHS =

3*x^2 + 4*x + 2
```

diff

The differential equation must satisfy and the initial condition of $y(0) = -1$ must agree too,

```
>> subs(y,{x},{0})

ans =

-1
```

subs

4.3 Linear Equations

The first-order *linear* differential equation is probably the simplest equation among the others which is easy to solve. This section presents a popular technique of using the integration factor to solve for the solution. The general form of the first-order linear differential equation is,

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

where $a_1(x)$, $a_0(x)$ and $b(x)$ are constants or functions of x only. It is noted that, if the coefficient $a_0(x) = 0$, the solution can be obtained by integrating the differential equation directly as,

$$a_1(x)\frac{dy}{dx} = b(x)$$

i.e.,

$$y(x) = \int \frac{b(x)}{a_1(x)} dx + C$$

where C is the integrating constant.

However, when $a_0(x)$ is not zero, solution to the full form of the differential equation can still be derived conveniently. The full form of the first-order linear differential equation, after dividing through by $a_1(x)$, can be rewritten in the form,

$$\frac{dy}{dx} + P(x)y = Q(x)$$

If we multiply this differential equation by the integrating factor defined by,

$$\mu(x) = e^{\int P(x)\,dx}$$

then, the differential equation becomes,

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

or,

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

which can be integrated to give the solution directly. We will learn how to use the integrating factor to find solution of the first-order differential equation by using the following examples.

Example Use the method of integrating factor to solve the first-order linear homogeneous differential equation,

$$\frac{dy}{dx} - 3y = 0$$

Here, $P(x) = -3$, then the integrating factor is,

$$\mu(x) = e^{\int(-3)dx} = e^{-3x}$$

By multiplying the differential equation by the integrating factor above,

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0$$

or,

$$\frac{d}{dx}\left[e^{-3x}y\right] = 0$$

Then, perform integration to get,

$$e^{-3x}y = C$$

where C is the integrating constant. Thus, the general solution is,

$$y = Ce^{3x}$$

The same solution can be obtained by using the `dsolve` command as,

```
>> syms x y
>> dsolve('Dy - 3*y = 0', 'x')

ans =

C4*exp(3*x)
```

dsolve

Example Use the method of integrating factor to solve the first-order linear non-homogeneous differential equation,

$$\frac{dy}{dx} - 2y = 4 - x$$

Here, $P(x) = -2$, then the integrating factor is,

$$\mu(x) = e^{\int(-2)dx} = e^{-2x}$$

We first multiply the differential equation by the integrating factor,

$$e^{-2x}\frac{dy}{dx} - 2e^{-2x}y \;=\; 4e^{-2x} - xe^{-2x}$$

Or,
$$\frac{d}{dx}\left[e^{-2x}y\right] \;=\; 4e^{-2x} - xe^{-2x}$$

Then, perform integration on both sides of the equation to get,

$$e^{-2x}y \;=\; -2e^{-2x} + \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x} + C$$

where C is the integrating constant. Thus, the solution is,

$$y \;=\; -\frac{7}{4} + \frac{x}{2} + Ce^{2x}$$

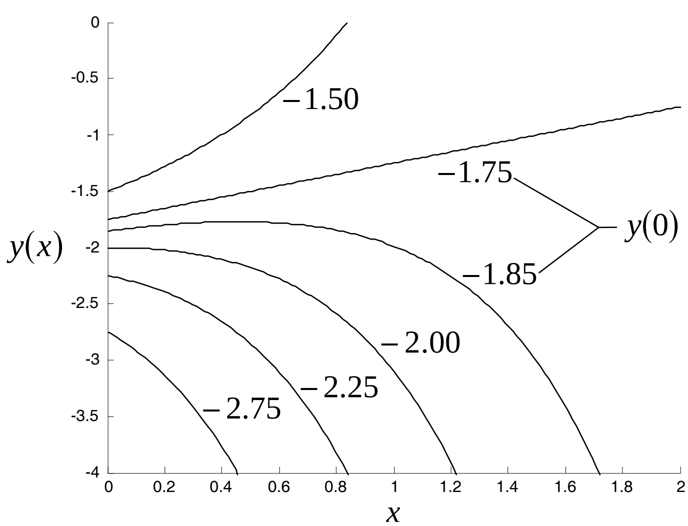
The same solution can be obtained by using the `dsolve` command as,

```
>> syms x y
>> dsolve('Dy - 2*y = 4 - x', 'x')
```

`dsolve`

```
ans =

x/2 + (C5*exp(2*x))/4 - 7/4
```

The integrating constant C (`C5/4` in MATLAB result above) is to be determined from the initial condition. Variations of $y(x)$ according to different initial conditions of $y(0) = -2.75$, -2.25, -2.00, -1.85, -1.75 and -1.50 are shown in the figure.



<u>Example</u> Use the method of integrating factor to find the exact solution of the first-order linear non-homogeneous differential equation,

$$\frac{dy}{dx} = 3x^2 - \frac{y}{x}$$

with the initial condition of $y(1) = 5$.

We start by writing the given differential equation in the form,

$$\frac{dy}{dx} + \frac{1}{x} y = 3x^2$$

Therefore, $P(x) = 1/x$, which leads to the integrating factor of,

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

We then multiply the differential equation by the integrating factor,

$$x\frac{dy}{dx} + y = 3x^3$$

or,

$$\frac{d}{dx}[x\,y] = 3x^3$$

and perform integration on both sides of the equation to get,

$$x\,y = \frac{3}{4}x^4 + C$$

where C is the integrating constant that can be determined from the initial condition of $y(1) = 5$ as follows,

$$(1)(5) = \frac{3}{4}(1)^4 + C$$

$$C = \frac{17}{4}$$

Thus, the exact solution of the problem is,

$$y(x) = \frac{3}{4}x^3 + \frac{17}{4x}$$

The same exact solution is obtained through the use of the `dsolve` command by entering,

```
>> syms x y
>> dsolve('Dy = 3*x^2 - y/x', 'y(1) = 5', 'x')
ans =
17/(4*x) + (3*x^3)/4
```

Example Use the method of integrating factor to derive the exact solution of the first-order linear non-homogeneous differential equation,

$$x\frac{dy}{dx} + 2y = 4x^2$$

with the initial condition of $y(1) = 2$.

Similar to the preceding example, we first write the differential equation in the standard form as,

$$\frac{dy}{dx} + \frac{2}{x}y = 4x$$

Therefore, $P(x) = 2/x$, which leads to the integrating factor of,

$$\mu(x) = e^{\int (2/x)\,dx} = e^{2\ln x} = x^2$$

Next, we multiply the differential equation by the integrating factor,

$$x^2\frac{dy}{dx} + 2xy = 4x^3$$

or,

$$\frac{dy}{dx}\left[x^2 y\right] = 4x^3$$

Then, perform integration on both sides of the equation to get,

$$x^2 y = x^4 + C$$

where C is the integrating constant that can be determined from the initial condition of $y(1) = 2$ as,

$$1^2(2) = (1)^4 + C$$

$$C = 1$$

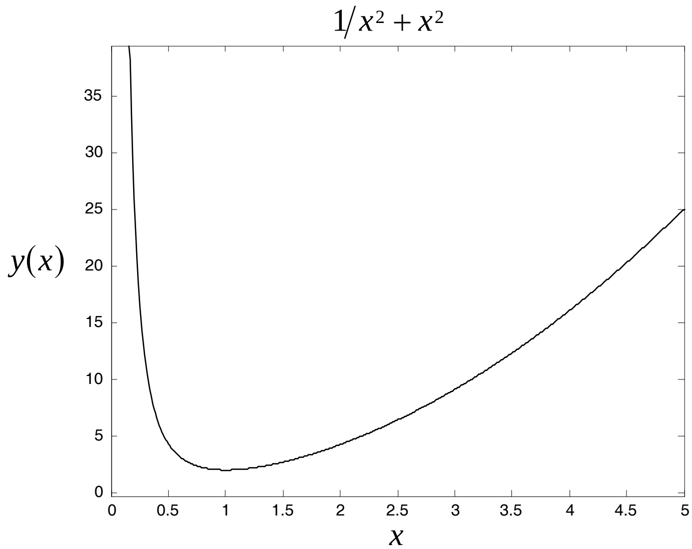
Thus, the exact solution of the problem is,

$$y(x) = x^2 + \frac{1}{x^2}$$

The same solution can be obtained by using the `dsolve` command as,

```
>> syms x y
>> dsolve('x*Dy + 2*y = 4*x^2', 'y(1) = 2', 'x')

ans =

1/x^2 + x^2
```

Variation of this exact solution $y(x)$ is plotted by the `ezplot` command as shown in the figure.



<u>Example</u> Use the method of integrating factor to solve the first-order linear non-homogeneous differential equation,

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x$$

The given differential equation is first written in the standard form for applying the integrating factor technique as,

$$\frac{dy}{dx} - \frac{2}{x}y = x^2\cos x$$

Therefore, $P(x) = -2/x$, so that the integrating factor is,

$$\mu(x) = e^{\int(-2/x)\,dx} = e^{\ln x^{-2}} = x^{-2}$$

By multiplying the integrating factor to the differential equation, we get,

$$x^{-2}\frac{dy}{dx} - 2x^{-3}y = \cos x$$

or,

$$\frac{d}{dx}\left(x^{-2}y\right) = \cos x$$

Then, performing integration on both sides to obtain,

$$x^{-2}y = \sin x + C$$

where C is the integrating constant. Thus, the general solution of the given differential equation is,

$$y(x) = x^2 \sin x + C\,x^2$$

Again, the same solution can be obtained by using the `dsolve` command as,

```
>> syms x y
>> dsolve('Dy/x - 2*y/x^2 = x*cos(x)', 'x')
ans =

x^2*sin(x) + C6*x^2
```

dsolve

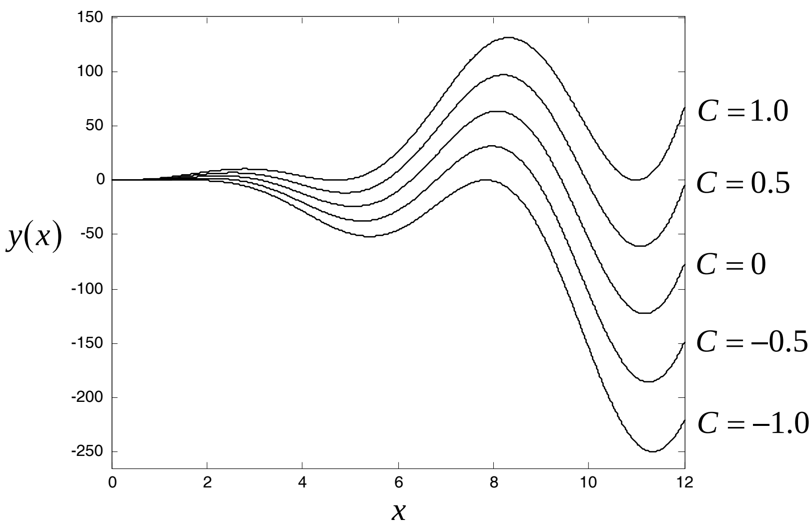
Variations of the solution, $y(x)$, depend on values of the integrating constant C (or `C6` from MATLAB above) as shown in the figure.

4.4 Exact Equations

If the differential equation is in the form of the so called *exact equation*, the idea explained below can be used to find its solution conveniently. We start from the differential equation in the form,

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

where $M(x,y)$ and $N(x,y)$ are functions of x and y. We can rewrite this differential equation as,



$$M(x,y)dx + N(x,y)dy = 0$$

But from definition of the total derivative of a function $\phi(x,y)$,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

If we could find the function $\phi(x,y)$ such that,

$$\frac{\partial \phi}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N(x,y)$$

Then, $$d\phi = 0$$

which means, $$\phi = C$$

where C is a constant.

This technique could be applied if the given differential equation contains the terms $M(x,y)$ and $N(x,y)$ that satisfy the conditions above. The differential equation containing the terms that meet such requirements is called the exact equation. We will demonstrate the approach for finding solutions to this differential equation form by using the examples below.

Example Find solution of the first-order linear homogeneous differential equation,

$$(1+x^2)\frac{dy}{dx}+2xy \quad = \quad 0$$

We start from writing the given differential equation in the form,

$$(2xy)dx+(1+x^2)dy \quad = \quad 0$$

If we compare it with the standard form of,

$$M(x,y)dx+\ N(x,y)dy \quad = \quad 0$$

we find that,

$$M(x,y) \quad = \quad 2xy \quad \text{and} \quad N(x,y) \quad = \quad 1+x^2$$

Then, if we choose a function,

$$\phi \quad = \quad y+x^2y$$

we notice that,

$$\frac{\partial\phi}{\partial x} = 2xy = M(x,y) \quad \text{and} \quad \frac{\partial\phi}{\partial y} = 1+x^2 = N(x,y)$$

Thus, $$d\phi \quad = \quad d(y+x^2y) \quad = \quad 0$$

or, $$y+x^2y \quad = \quad C$$

$$y(1+x^2) \quad = \quad C$$

So, we arrive at the solution of the given differential equation,

$$y \quad = \quad \frac{C}{1+x^2}$$

where C is a constant that can be determined from the initial condition.

It is noted that the approach explained above can be applied if,

$$\frac{\partial M(x,y)}{\partial y} \quad = \quad \frac{\partial N(x,y)}{\partial x}$$

Such as in the example above,

$$\frac{\partial M(x,y)}{\partial y} \quad = \quad \frac{\partial}{\partial y}(2xy) \quad = \quad 2x$$

and $$\frac{\partial N(x,y)}{\partial x} \quad = \quad \frac{\partial}{\partial x}(1+x^2) \quad = \quad 2x$$

We can verify that the derived solution is correct by using the `diff` command. If we let the integrating constant $C = 1$ and substitute the solution into the left-hand-side of the differential equation,

```
>> syms x y
>> y = 1/(1+x^2);
>> LHS = (1+x^2)*diff(y,x) + 2*x*y
```

diff

```
LHS =

0
```

we obtain the result of zero which is equal to the value on the right-hand-side of the equation.

We can also use the `dsolve` command to solve the given differential equation by entering,

```
>> syms x y
>> dsolve('(1+x^2)*Dy + 2*x*y = 0', 'x')

ans =

C7/(x^2 + 1)
```

If the initial condition is $y(0) = 1$, then the complete problem statement of this example is,

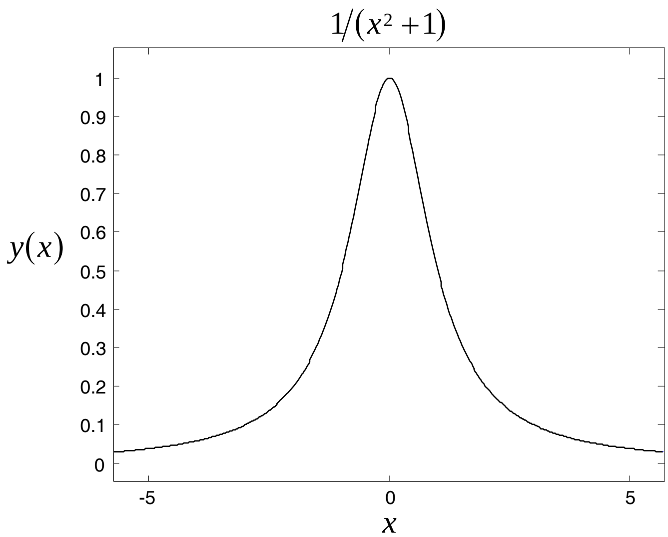
$$(1+x^2)\frac{dy}{dx} + 2xy = 0, \qquad y(0) = 1$$

which has the exact solution of,

$$y(x) = \frac{1}{1+x^2}$$

```
>> syms x y
>> dsolve('(1+x^2)*Dy + 2*x*y = 0', 'y(0) = 1',
'x')

ans =

1/(x^2 + 1)
```

The `ezplot` command can be used to plot the solution of y that varies with x as shown in the figure.



Example Find solution of the first-order differential equation which is in the exact equation form,

$$\frac{dy}{dx} = -\frac{2xy^2+1}{2x^2y}$$

We start by writing the given differential equation in the form,

$$M(x,y)dx + N(x,y)dy = 0$$

i.e.,
$$(2xy^2+1)dx + 2x^2y\,dy = 0$$

It is in the form of the exact equation because,

$$\frac{\partial M(x,y)}{\partial y} = 4xy = \frac{\partial N(x,y)}{\partial x}$$

If we choose the function,

$$\phi = x^2y^2 + x$$

we find that,

$$\frac{\partial \phi}{\partial x} = 2xy^2+1 = M(x,y)$$

and
$$\frac{\partial \phi}{\partial y} = 2x^2y = N(x,y)$$

This means, from the definition of the total derivative,

$$d\phi = d\left(x^2y^2+x\right) = 0$$

or,
$$x^2y^2+x = C$$

where C is a constant. Thus, the general solution of the given differential equation is,

$$y = \pm\frac{\sqrt{C-x}}{x}$$

The same solution can be obtained by using the `dsolve` command as,

```
>> syms x y
>> dsolve('Dy = -(2*x*y^2+1)/(2*x^2*y)', 'x')

ans =

  (C8 - x)^(1/2)/x
 -(C8 - x)^(1/2)/x
```

dsolve

In the two examples above, we chose functions $\phi(x,y)$ so that $\partial\phi/\partial x = M(x,y)$ and $\partial\phi/\partial y = N(x,y)$. In practice, the proper functions $\phi(x,y)$ can be derived by using the following procedure.

Since $\partial\phi/\partial x = M(x,y)$, then,

$$\phi(x,y) = \int M(x,y)dx + g(y)$$

If we take derivative of ϕ with respect to y,

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + \frac{\partial g}{\partial y} = N(x,y)$$

i.e.,
$$\frac{\partial g}{\partial y} = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx$$

which can be integrated to find $g(y)$ so that the function ϕ is obtained. We will use this technique to find solutions of the differential equations in the following examples.

Example Find solution of the first-order differential equation,

$$\left(2xy - \sec^2 x\right)dx + \left(x^2 + 2y\right)dy = 0$$

The given differential equation is the form of the exact equation,

$$M(x,y)dx + N(x,y)dy = 0$$

because
$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

The function $\phi(x,y)$ can be found by using the technique explained above as follows,

$$\phi(x,y) = \int (2xy - \sec^2 x)dx + g(y)$$
$$= x^2 y - \tan x + g(y)$$

Then,
$$\frac{\partial \phi}{\partial y} = x^2 + \frac{\partial g}{\partial y} = N(x,y) = x^2 + 2y$$

or,
$$\frac{\partial g}{\partial y} = 2y$$

We integrate to get, $g = y^2$

So that, $\phi(x,y) = x^2 y - \tan x + y^2$

From the definition of the total derivative and the property of the exact equation,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = M(x,y)dx + N(x,y)dy = 0$$

Thus,

$$d(x^2 y - \tan x + y^2) = (2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0$$

Or, $\phi = x^2 y - \tan x + y^2 = C$

The solution of $y(x)$ in the equation above is in an implicit form.

We can also use the `dsolve` command to solve for the solution of $y(x)$. The solution obtained from MATLAB is lengthy but can be reduced by using the `simple` command.

```
>> syms x y
>> dsolve('Dy = -(2*x*y-sec(x)^2)/(x^2+2*y)',
          'x');
>> simple(ans)
ans =

   ((4*sin(x))/cos(x) - 4*C9 + x^4 + 4*i)^(1/2)/2
     - x^2/2 - ((4*sin(x))/cos(x) - 4*C9 + x^4
     + 4*i)^(1/2)/2 - x^2/2
```

The solution obtained above contains the constant `C9`. Such solution can be used to determine the function ϕ for which it must be a constant. If we let `C9` = 1, then ϕ becomes,

```
>> y = ((4*sin(x))/cos(x)-4*1+x^4+4*i)^(1/2)/2 –
         x^2/2;
>> phi = x^2*y - tan(x) + y^2;
>> simple(phi)
ans =

i - 1
```

simple

which is a constant as expected. We can check whether the solution $y(x)$ is correct by substituting it back into the differential equation,

$$\frac{(x^2+2y)}{(2xy-\sec^2 x)}\frac{dy}{dx} = -1$$

```
>> syms x y
>> y = ((4*sin(x))/cos(x)-4*1+x^4+4*i)^(1/2)/2 –
         x^2/2;
>> LHS = ((x^2+2*y)/(2*x*y-sec(x)^2))*diff(y,x);
>> simple(LHS)
ans =

-1
```

In the first step of finding the function ϕ as explained above, $M(x,y)$ must be integrated. The derivation process would be lengthy if $M(x,y)$ is complicated. We can start the process by integrating $N(x,y)$ if it is simpler,

$$\phi(x,y) = \int N(x,y)dy + h(x)$$

and then find $h(x)$. This latter process is shown in the following examples.

<u>Example</u> Solve the first-order differential equation,

$$(1+e^x y + xe^x y)dx + (xe^x + 2)dy = 0$$

The given differential equation is in the form of the exact equation,

$$M(x,y)dx + N(x,y)dy = 0$$

because
$$\frac{\partial M}{\partial y} = e^x + xe^x = \frac{\partial N}{\partial x}$$

To find the function $\phi(x,y)$, we integrate $N(x,y)$ with respect to y,

$$\begin{aligned}\phi(x,y) &= \int (xe^x + 2)dy + h(x)\\ &= xe^x y + 2y + h(x)\end{aligned}$$

Then,
$$\frac{\partial \phi}{\partial x} = e^x y + xe^x y + \frac{\partial h}{\partial x}$$

Also,
$$\frac{\partial \phi}{\partial x} = M(x,y) = 1 + e^x y + xe^x y$$

By comparing these two equations,

$$\frac{\partial h}{\partial x} = 1$$

so,
$$h = x$$

Then,
$$\phi(x,y) = xe^x y + 2y + x$$

Since,
$$d\phi = 0$$

Thus,
$$xe^x y + 2y + x = C$$

i.e., the solution is,

$$y(x) = \frac{C-x}{(xe^x+2)}$$

where C is a constant.

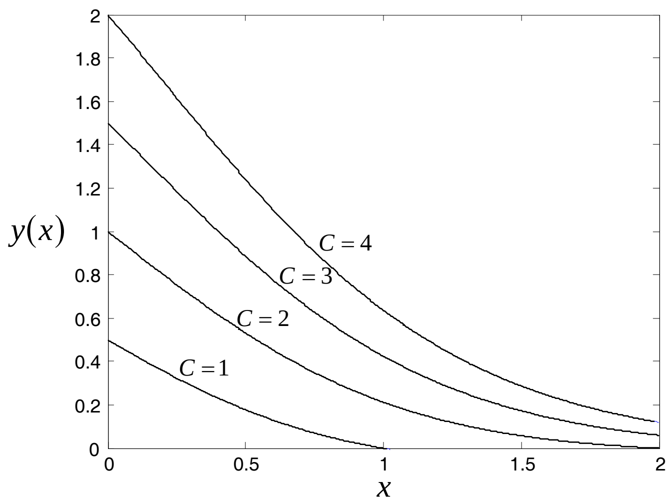
We can use the `dsolve` command to solve the given differential equation which is in the form,

$$\frac{dy}{dx} = -\frac{1+e^x y + xe^x y}{xe^x + 2}$$

by entering,

```
>> syms x y
>> dsolve('Dy = -(1+exp(x)*y+x*exp(x)*y)/
          (x*exp(x)+2)', 'x')

ans =

C10/(x*exp(x) + 2) - x/(x*exp(x) + 2)
```

MATLAB gives the same solution as we derived earlier. The solution depends on values of the constant C as shown in the figure.



If the given differential equation is not in the exact equation form at the beginning, we may multiply it by an integrating factor so that it becomes an exact equation. Then, we can use the same procedure to find the solutions as demonstrated in the following examples.

Example Use the exact equation technique to solve the first-order differential equation,

$$(2x^2 + y)dx + (x^2 y - x)dy = 0$$

The given differential equation is not in an exact equation form of,

$$M(x,y)dx + N(x,y)dy = 0$$

because $\dfrac{\partial M}{\partial y} = 1$ which is not equal to $\dfrac{\partial N}{\partial x} = (2xy - 1)$.

If we multiply both sides by the integrating factor (only function of x in this case),

$$\mu(x) = \frac{1}{x^2}$$

we get, $$\left(2 + \frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0$$

The differential equation becomes an exact equation because,

$$\frac{\partial M}{\partial y} = \frac{1}{x^2} = \frac{\partial N}{\partial x}$$

Then, we can use the exact equation technique to find the solution similar to the preceding two examples. The solution of $y(x)$ is obtained in an implicit form,

$$2x - \frac{y}{x} + \frac{y^2}{2} = C$$

We can use the `dsolve` command to provide the same solution of $y(x)$ but in the explicit form as,

```
>> syms x y
>> dsolve('Dy = -(2*x^2+y)/(x^2*y-x)', 'x')
ans =

  ((- 4*x^3 - 2*C11*x^2 + 1)^(1/2) + 1)/x
 -((- 4*x^3 - 2*C11*x^2 + 1)^(1/2) - 1)/x
```

dsolve

In general, the integrating factor μ may be function of x and y. However, if the integrating factor μ is only function of x or y, we can find it by determining,

$$P \quad = \quad \frac{\partial M/\partial y - \partial N/\partial x}{N}$$

If the result of P is only function of x, then the integrating factor μ is,

$$\mu(x) \quad = \quad e^{\int P(x)dx}$$

But if this is not the case, we determine,

$$Q \quad = \quad \frac{\partial N/\partial x - \partial M/\partial y}{M}$$

If the result of Q is only function of y, then the integrating factor μ is,

$$\mu(y) \quad = \quad e^{\int Q(y)dy}$$

We will demonstrate this technique in the example below.

<u>Example</u> Determine the integrating factor for solving the first-order differential equation,

$$(3xy + y^2)dx + (x^2 + xy)dy \quad = \quad 0$$

Note that the given differential equation is not in the form of the exact equation,

$$M(x,y)dx + N(x,y)dy \quad = \quad 0$$

because $\dfrac{\partial M}{\partial y} = 3x + 2y$ which is not equal to $\dfrac{\partial N}{\partial x} = 2x + y$.

We first find the integrating factor by determining,

$$\begin{aligned} P \quad &= \quad \frac{\partial M/\partial y - \partial N/\partial x}{N} \quad = \quad \frac{3x + 2y - 2x - y}{x^2 + xy} \\ &= \quad \frac{x + y}{x(x+y)} \quad = \quad \frac{1}{x} \end{aligned}$$

Then, the integrating factor is,

$$\mu(y) \quad = \quad e^{\int 1/x\,dx} \quad = \quad x$$

After multiplying this integrating factor into the differential equation, we get,

$$(3x^2 y + xy^2)dx + (x^3 + x^2 y)dy = 0$$

The differential equation is now in the exact equation form because,

$$\frac{\partial M}{\partial y} = 3x^2 + 2xy = \frac{\partial N}{\partial x}$$

We start deriving the solution to the differential equation by finding the function $\phi(x, y)$ from,

$$\begin{aligned} \phi(x,y) &= \int M(x,y)dx + g(y) \\ &= \int (3x^2 y + xy^2)dx + g(y) \\ \phi(x,y) &= x^3 y + \frac{x^2 y^2}{2} + g(y) \end{aligned}$$

Then, $$\frac{\partial \phi}{\partial y} = x^3 + x^2 y + \frac{\partial g}{\partial y} = N(x,y) = x^3 + x^2 y$$

or, $$\frac{\partial g}{\partial y} = 0$$

i.e., $$g = C_1$$

where C_1 is a constant.

Hence, $$\phi(x,y) = x^3 y + \frac{x^2 y^2}{2} + C_1$$

But from the definition of the total derivative and the exact equation,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = M(x,y)dx + N(x,y)dy = 0$$

Then, $$d\left(x^3 y + \frac{x^2 y^2}{2} + C_1\right) = 0$$

or, $$x^3 y + \frac{x^2 y^2}{2} + C_1 = C_2$$

i.e., $$x^3 y + \frac{x^2 y^2}{2} = C$$

where C_2 and C are constants. The solution is in an implicit form that can be further determined for $y(x)$.

The `dsolve` command can again be used to find the solution of the differential equation. The obtained solution is in an explicit form as follows,

```
>> syms x y
>> dsolve('Dy = -(3*x*y+y^2)/(x^2+x*y)', 'x')

ans =

  x*((exp(C12 - 4*log(x)) + 1)^(1/2) - 1)
 -x*((exp(C12 - 4*log(x)) + 1)^(1/2) + 1)
```

It is noted that the explicit solution $y(x)$ obtained from MATLAB is in fact identical to the implicit solution derived earlier. This can be verified by substituting the explicit solution $y(x)$ into the left-hand-side of the implicit expression as follows,

syms

```
>> syms C12
>> y = x*((exp(C12 - 4*log(x)) + 1)^(1/2) - 1);
>> LHS = x^3*y + x^2*y^2/2

LHS =

exp(C12)/2
```

The result is a constant which is equal to the right-hand-side of the implicit expression.

4.5 Special Equations

Solutions of the first-order differential equations can be found easily if the equations are in some special forms. For example, the differential equation in the form,

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

where only the dy/dx term appears on the left-hand-side while every term on the right-hand-side is in the form of y/x. The example below shows the procedure to find the solution for this type of differential equation.

Example Solve the first-order nonlinear differential equation,

$$x\frac{dy}{dx} = \frac{y^2}{x} + y$$

We start from dividing the differential equation by x,

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

If we let $y = ux$, where $u = u(x)$, then the differential equation becomes,

$$x\frac{du}{dx} + u = u^2 + u$$

or,

$$\frac{du}{dx} = \frac{u^2}{x}$$

We can separate the variables u and x, so that they are on opposite sides of the equation, and then perform integration,

$$\int \frac{du}{u^2} = \int \frac{dx}{x}$$

to get,

$$-\frac{1}{u} = \ln|x| + C$$

where C is the integrating constant. Thus,

$$u = -\frac{1}{\ln|x| + C}$$

By substituting $u = y/x$ back, we obtain the solution $y(x)$ as,

$$y(x) = -\frac{x}{\ln|x| + C}$$

The same solution is also obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('x*Dy = y^2/x + y', 'x')

ans =

x/(C13 - log(x))
```

dsolve

The technique above can be applied to other differential equations with the term y/x on the right-hand-side of the equation. For examples,

$$\frac{dy}{dx} = \frac{x^3}{y^3}$$

and,

$$\frac{dy}{dx} = \sin\left(\frac{y}{x}\right) - \frac{x}{y}$$

etc.

Bernoulli equation is the first-order differential equation that can be solved conveniently by changing the variable. The Bernoulli equation is in the form,

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where $P(x)$ and $Q(x)$ are continuous functions of x, while n is an integer. If $n = 0$ or 1, the Bernoulli equation reduces from the nonlinear to linear equation.

The idea for solving the nonlinear Bernoulli equation is to transform it into a linear one. The techniques we learned earlier can then be applied to solve for solutions. Such idea is summarized and demonstrated by the examples below.

If we divide the Bernoulli equation by y^n,

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

and introduce a new variable v in form of y as,

$$v = y^{1-n}$$

with,

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Then, the Bernoulli equation, after dividing by y^n, above becomes a linear differential equation in the form,

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

Example Solve the first-order nonlinear differential equation which is in form of the Bernoulli equation,

$$\frac{dy}{dx} - 5y \quad = \quad -\frac{5}{2}\,xy^3$$

Here, $P(x) = -5$, $Q(x) = -5x/2$ and $n = 3$. We first divide the given differential equation by y^3 to give,

$$y^{-3}\frac{dy}{dx} - 5y^{-2} \quad = \quad -\frac{5}{2}\,x$$

Then, we assign the new variable $v = y^{-2}$, so that $dv/dx = -2y^{-3}dy/dx$. The nonlinear differential equation becomes linear as,

$$-\frac{1}{2}\frac{dv}{dx} - 5v \quad = \quad -\frac{5}{2}\,x$$

or,
$$\frac{dv}{dx} + 10v \quad = \quad 5\,x$$

To solve such linear differential equation, we find the integrating factor, which is,

$$e^{\int 10\,dx} \quad = \quad e^{10x}$$

Then, multiply it into the linear differential equation to get,

$$e^{10x}\frac{dv}{dx} + 10e^{10x}v \quad = \quad 5\,xe^{10x}$$

or,
$$\frac{d}{dx}\left(e^{10x}v\right) \quad = \quad 5\,xe^{10x}$$

After performing integration on both sides of the equation, we get,

$$e^{10x}v \quad = \quad \frac{10x-1}{20}e^{10x} + C$$

or,
$$v \quad = \quad \frac{x}{2} - \frac{1}{20} + Ce^{-10x}$$

where C is the integrating constant. The final solution is obtained after we substitute the variable $v = y^{-2}$ back into the above solution,

$$\frac{1}{y^2} = \frac{x}{2} - \frac{1}{20} + Ce^{-10x}$$

i.e.,

$$y = \frac{\pm 1}{\sqrt{\frac{x}{2} - \frac{1}{20} + Ce^{-10x}}}$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('Dy - 5*y = -5*x*y^3/2', 'x')
```

dsolve

```
ans =

  1/(x/2 + C14/exp(10*x) - 1/20)^(1/2)
 -1/(x/2 + C14/exp(10*x) - 1/20)^(1/2)
```

Example Solve the first-order nonlinear differential equation which is in the form of the Bernoulli equation,

$$\frac{dy}{dx} + \frac{1}{x} y = 3x^2 y^3$$

Here, $P(x) = 1/x$, $Q(x) = 3x^2$ and $n = 3$. We start from dividing the given differential equation by y^3 to give,

$$y^{-3}\frac{dy}{dx} + \frac{1}{x} y^{-2} = 3x^2$$

Then, we assign a new variable of $v = y^{-2}$ so that $dv/dx = -2y^{-3}dy/dx$ and the differential equation above becomes,

$$-\frac{1}{2}\frac{dv}{dx} + \frac{1}{x} v = 3x^2$$

or,

$$\frac{dv}{dx} - \frac{2}{x} v = -6x^2$$

The integrating factor is determined from,

$$e^{\int -(2/x)dx} = e^{\ln\left(x^{-2}\right)} = x^{-2}$$

After multiplying the differential equation by the integrating factor, we get,

$$x^{-2}\frac{dv}{dx} - 2x^{-3}v = -6$$

or,
$$\frac{d}{dx}\left(x^{-2}v\right) = -6$$

Then, perform integration on both sides to give,

$$x^{-2}v = -6x + C$$

where C is the integrating constant. Thus,

$$v = -6x^3 + Cx^2$$

The final solution is obtained after substituting $v = y^{-2}$ into the above equation,

$$y^{-2} = -6x^{-3} + Cx^2$$

or,
$$y = \frac{\pm 1}{\sqrt{-6x^3 + Cx^2}}$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('Dy + y/x = 3*x^2*y^3', 'x')
```

dsolve

```
ans =

  1/(C15*x^2 - 6*x^3)^(1/2)
 -1/(C15*x^2 - 6*x^3)^(1/2)
```

The *Riccati equation* is another nonlinear differential equation that can be transformed into a linear equation. The general form of the Riccati equation is,

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$$

Transformation from the nonlinear to linear equation is by changing the variable,

$$y = S(x) + \frac{1}{z}$$

where $z = z(x)$. Solving the differential equations in this form is demonstrated by the following example.

Example Solve the first-order nonlinear differential equation which is in the form of the Riccati equation,

$$\frac{dy}{dx} = \frac{1}{x}y^2 + \frac{1}{x}y - \frac{2}{x}$$

If we change the variable $y(x)$ into the new variable $z(x)$ by using the relation,

$$y = 1 + \frac{1}{z}$$

then,

$$\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

Thus, the original Riccati equation becomes,

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{1}{x}\left(1+\frac{1}{z}\right)^2 + \frac{1}{x}\left(1+\frac{1}{z}\right) - \frac{2}{x}$$

or,

$$\frac{dz}{dx} + \frac{3}{x}z = -\frac{1}{x}$$

which is in the form of a linear differential equation. Next, we multiply the equation by the integrating factor of x^3 to give,

$$x^3\frac{dz}{dx} + 3x^2 z = -x^2$$

or,

$$\frac{d}{dx}(x^3 z) = -x^2$$

Then, perform integration on both sides of the equation of get,

$$x^3 z = -\frac{x^3}{3} + C$$

or,

$$z = -\frac{1}{3} + \frac{C}{x^3}$$

where C is the integrating constant. Thus, the final solution to the given differential Riccati equation is,

$$y = 1 + \frac{1}{z} = 1 + \frac{1}{-\frac{1}{3} + \frac{C}{x^3}}$$

or,

$$y(x) = \frac{3C + 2x^3}{3C - x^3}$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('Dy = y^2/x + y/x - 2/x', 'x')

ans =

- 3/(x^3*exp(3*C16) - 1) - 2
```

where the constant `exp(3*C16)` is equivalent to $1/3C$ in the derived solution.

4.6 Numerical Methods

There are many first-order differential equations that cannot be derived for exact solutions in the form of explicit expressions. For these differential equations, numerical methods are used to find their approximate solutions. MATLAB contains many commands, such as `ode23`, `ode45` and `ode23s` that can provide approximate solutions with high accuracy. Accuracy of the solutions strongly depends on the time steps which are adjusted automatically. We will employ the examples below to demonstrate the use of these commands. We will also compare the numerical solutions with the exact solutions for the cases when the exact solutions are available in order to demonstrate the numerical solution accuracy obtained from using these commands.

<u>Example</u> Employ the `ode23` command in MATLAB to solve the first-order linear nonhomogeneous differential equation,

$$\frac{dy}{dx} = e^{-2x} \qquad\qquad 0 \le x \le 2$$

with the initial condition of $y(0) = 0$.

This initial value problem has exact solution in the form of explicit expression that can be found by using the `dsolve` command,

```
>> syms x y
>> dsolve('Dy = exp(-2*x)', 'y(0)=0', 'x')

ans =

1/2 - 1/(2*exp(2*x))
```

dsolve

i.e., the exact solution is,

$$y_{exact} = \frac{1}{2}\left(1 - e^{-2x}\right)$$

The `ode23` command in MATLAB uses the combined second- and third-order Runge-Kutta method to numerically solve the first-order differential equation in the general form of,

$$\frac{dy}{dx} = f(x, y)$$

To use the `ode23` command, we have to supply the function $f(x, y)$ by using the `inline` command as,

```
>> f = inline('exp(-2*x)', 'x', 'y')

f =

     Inline function:
     f(x,y) = exp(-2*x)
```

inline

The `inline` command consists of the function $f(x, y)$, independent variable *x* and dependent variable *y*, respectively. To find numerical solution, the `ode23` command is then used by entering,

```
>> [x,y] = ode23(f,[0:.1:2], 0)
```

ode23

The numbers in the square bracket denote the starting *x*, interval for printing output and ending *x*, respectively. The number 0 at the end of the command denotes the initial condition of $y(0) = 0$.

We can create a set of commands so that the computed numerical solutions can be compared with the exact solution by a plot as follows,

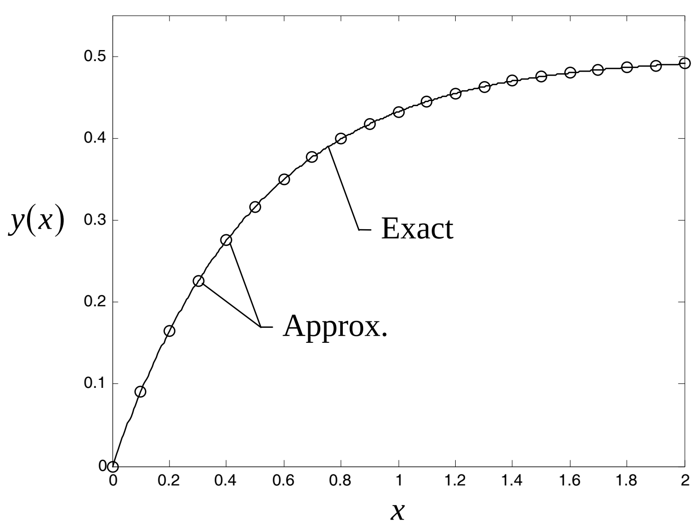
```
>> f = inline('exp(-2*x)','x','y')
>> [x,y] = ode23(f,[0:.1:2], 0)
>> plot(x,y,'ko')
>> axis([0 2 0 0.55])
>> xlabel('x'), ylabel('y(x)')
>> hold on
>> x = 0:.005:2
>> ye = (1 - exp(-2*x))/2
>> plot(x,ye,'k')
```

axis

hold on

plot

The generated plot as shown in the figure indicates that the numerical solution is very accurate as compared to the exact solution. This is because the `ode23` command adjusts the time step automatically so that the relative tolerance is always less than 1×10^{-3} during the computation.



<u>Example</u> Employ the `ode45` command in MATLAB to solve the first-order linear nonhomogeneous differential equation,

$$\frac{dy}{dx} = 2\cos(2x) - \sin x \qquad 0 \le x \le 10$$

with the initial condition of $y(0) = 0$.

Matlab also contains the `ode45` command that employs the combined fourth- and fifth-order Runge-Kutta method to solve for numerical solution to the first order differential equation. The `ode45` command can provide higher solution accuracy than the `ode23` command. We will use this `ode45` command to solve for numerical solution and compare it with the exact solution. The exact solution can be determined by employing the `dsolve` command as,

```
>> syms x y
>> dsolve('Dy = 2*cos(2*x) - sin(x)',
          'y(0)=0', 'x')

ans =

sin(2*x) + cos(x) - 1
```

i.e., the exact solution is,

$$y_{exact} \quad = \quad \sin(2x) + \cos x - 1$$

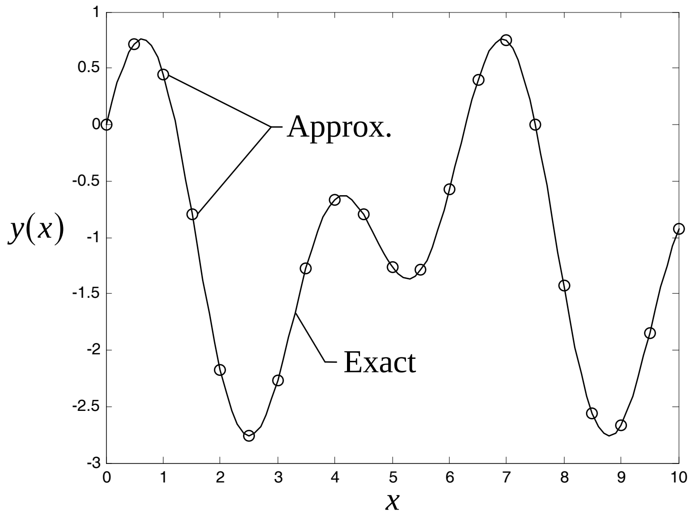
We can prepare a set of commands to determine the numerical solution and compare it with the exact solution by using a plot as follows,

```
>> f = inline('2*cos(2*x) - sin(x)','x','y')
>> [x,y] = ode45(f,[0:.5:10], 0)
>> plot(x,y,'ko')
>> axis([0 10 -3 1])
>> xlabel('x'), ylabel('y(x)')
>> hold on
>> x = 0:.1:10
>> ye = sin(2*x) + cos(x) - 1
>> plot(x,ye,'k')
```

`ode45`

`xlabel`

The plot as shown in the figure demonstrates that the numerical solution obtained from using the `ode45` command compares very well with the exact solution.



Example Employ the `ode45` command in MATLAB to solve the first-order linear nonhomogeneous differential equation,

$$\frac{dy}{dx} = \sin y - \cos x + e^{-3y} \qquad 0 \le x \le 8$$

with the initial condition of $y(0) = 0.2$.

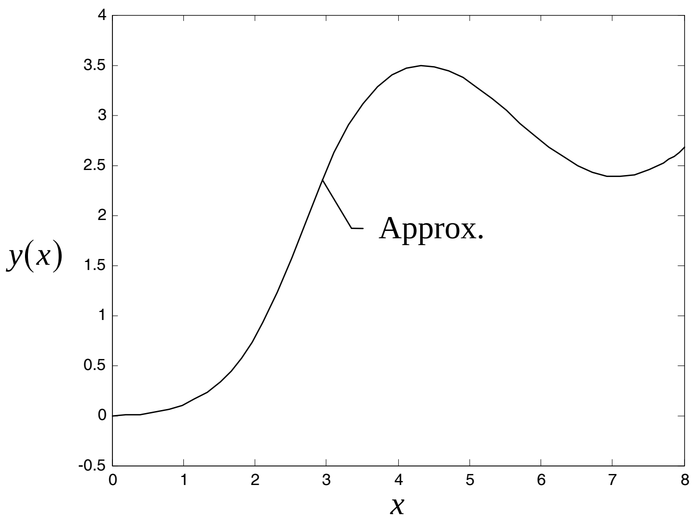
For this problem, if we try to find the exact solution by using the `dsolve` command as we did in the preceding examples,

```
>> syms x y
>> dsolve('Dy = sin(y) – cos(x) + exp(-3*y)',
          'y(0) = 0.2', 'x')
Warning: Explicit solution could not be found.
```

We found that MATLAB cannot provide an exact solution in explicit form. We need to employ the numerical method to find the approximate solution by using a set of commands below. The plot of the computed solution is shown in the figure. It is noted that exact solutions to most of the differential equations are not available. The numerical method is thus an important tool to provide us the approximate solutions of the differential equations.

```
>> f = inline('sin(y)-cos(x)+exp(-3*y)',
              'x', 'y')
>> [x,y] = ode45(f,[0 8], 0)
>> plot(x,y,'k')
>> axis([0 8 -0.5 4])
>> xlabel('x'), ylabel('y(x)')
```

ode45



There are some differential equations that their solutions change abruptly with x. These differential equations are classified as the *stiff equations*. Approximate solutions obtained from standard numerical methods may not be accurate because of their sudden changes. MATLAB contains commands such as `ode15s`, `ode23s` and `ode23t` to accurately capture the solutions. Nature of the solution to the stiff differential equation and the use of these latter commands are demonstrated in the following example.

<u>Example</u> Employ the `ode23s` command to solve the stiff differential equation which is in the form,

$$\frac{dy}{dx} = 20e^{-100(x-2)^2} - 0.6y \qquad 0 \le x \le 4$$

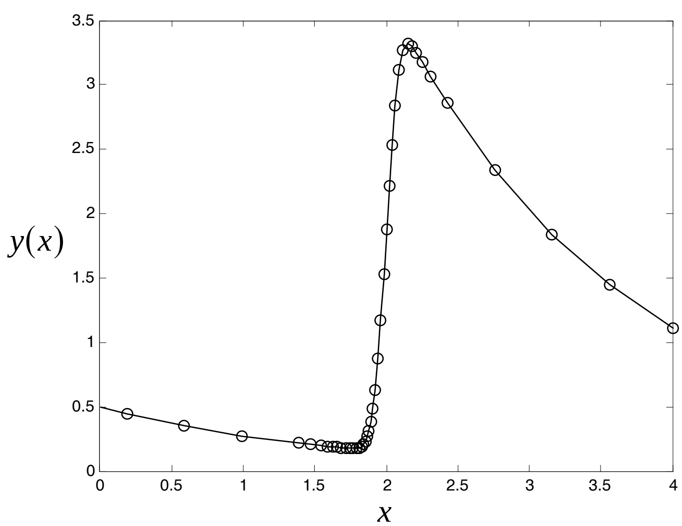
with the initial condition of $y(0) = 0.5$.

A set of commands that consists of `inline`, `ode23s` and `plot` for solving this problem are shown below. The plot indicates that there is a sudden change of the solution $y(x)$ at $x = 2$. In this region, MATLAB uses small time steps to accurately capture the sudden change of the solution. Larger time steps are used in other regions where the solution gradients are small to reduce the computational time.

```
>> f = inline('20*exp(-100*(x-2)^2) - 0.6*y',
              'x','y')
>> [x,y] = ode23s(f,[0 4], 0.5)
>> plot(x,y,'k')
>> axis([0 4 0 3.5])
>> xlabel('x'), ylabel('y(x)')
>> hold on
>> plot(x,y,'ko')
```

ode23s

hold on



4.7 Concluding Remarks

In this chapter, we learned several techniques for finding exact solutions of the first-order differential equations. Depending on the forms of differential equations, proper techniques should be selected and applied to solve for solutions. These include the technique of separating variables, the technique of using integrating factors for linear and exact equations. Some specific techniques were also introduced to solve differential equations that are in special forms, such as the Bernoulli and Riccati equations. The same solutions were obtained by using the `dsolve` command in MATLAB. These solutions were also plotted by the `plot` command to increase understanding of their behaviors.

Numerical methods for solving the differential equations were introduced. MATLAB contains several commands such as `ode23`, `ode45` and `ode23s` that can be used to provide accurate solutions. Approximate solutions obtained from the numerical methods were compared with the exact solutions to highlight their efficiency. Since there are many first-order differential equations that the exact solutions cannot be found, these commands thus provide an alternative way to obtain the approximate solutions with high accuracy.

Exercises

1. Use the `dsolve` command to find solutions of the first-order differential equations,

   (a) $\dfrac{dy}{dx} = \dfrac{x^2}{y}$ (b) $\dfrac{dy}{dx} = \dfrac{x^2}{y(1+x^3)}$

   (c) $\dfrac{dy}{dx} = \dfrac{(3x^2-1)}{(3+2y)}$ (d) $x\dfrac{dy}{dx} = \sqrt{1-y^2}$

   (e) $\dfrac{dy}{dx} = y^2e^{-x}$

   Then, use the separable equation technique to verify the solutions.

2. Use the `dsolve` command to find solutions of the first-order differential equations,

(a) $\dfrac{dy}{dx} - 3x^2 y = 0$ (b) $x\dfrac{dy}{dx} - \dfrac{1}{y^3} = 0$

(c) $\dfrac{dy}{dx} + y^2 \sin x = 0$ (d) $\dfrac{1}{x}\dfrac{dy}{dx} - 2\cos^2 y = 0$

(e) $e^x \dfrac{dy}{dx} - (x+1)y^2 = 0$

Then, use the separable equation technique to verify the solutions.

3. Use the `dsolve` command to solve the initial value problems,

(a) $\dfrac{dy}{dx} = x^3(1-y), \quad y(0) = 3$

(b) $\dfrac{dy}{dx} = 6x^3 e^{-2y}, \quad y(1) = 0$

(c) $\dfrac{dy}{dx} = (1+y^2)\tan x, \quad y(0) = \sqrt{3}$

(d) $\dfrac{1}{2}\dfrac{dy}{dx} = \sqrt{y+1}\cos x, \quad y(\pi) = 0$

(e) $\sqrt{1-x^2}\, y^2 \dfrac{dy}{dx} = \sin^{-1} x, \quad y(0) = 1$

Derive the solutions by using the technique of separable equation. Then, use the `ezplot` command to plot the solution of $y(x)$ that varies with x.

4. Solve the initial value problems below by using the `dsolve` command,

(a) $\dfrac{dy}{dx} = \dfrac{2x}{y + x^2 y}, \quad y(0) = -2$

(b) $\dfrac{dy}{dx} = \dfrac{3x^2 - e^x}{2y-5}, \quad y(0) = 1$

(c) $\dfrac{dy}{dx} = \dfrac{e^{-x} - e^x}{3+4y}, \quad y(0) = 1$

(d) $\dfrac{dy}{dx} = \dfrac{xy^3}{\sqrt{1+x^2}}, \quad y(0) = 1$

(e) $\dfrac{dy}{dx} = \sqrt{1+\sin x}\,(1+y^2), \quad y(0) = 1$

Then, for each problem, derive the exact solution by using the technique of separable equation. Verify each solution by showing that both the differential equation and initial condition are satisfied.

5. Use the integrating factor technique to find solutions of the first-order differential equations below. Verify the solutions by comparing with those obtained by employing the `dsolve` command.

(a) $\dfrac{dy}{dx} - \dfrac{y}{x} = 2x + 1$ (b) $\dfrac{dy}{dx} - y = e^{3x}$

(c) $\dfrac{dy}{dx} + 4y = x^2 e^{-4x}$ (d) $x\dfrac{dy}{dx} + 2y = \dfrac{1}{x^3}$

(e) $x\dfrac{dy}{dx} + 3(y + x^2) = \dfrac{\sin x}{x}$

6. Employ the `dsolve` command to solve the first-order linear differential equations,

(a) $\dfrac{dy}{dx} + 3y = x + e^{-2x}$ (b) $\dfrac{dy}{dx} + y = xe^{-x} + 1$

(c) $2\dfrac{dy}{dx} + y = 3x^2$ (d) $x\dfrac{dy}{dx} + 2y = \sin x$

(e) $x\dfrac{dy}{dx} - y = x^2 e^{-x}$

Then, derive their solutions by using the technique of integrating factor.

7. Use the technique of integrating factor to solve the initial value problems that are governed by the first-order differential equations and their initial conditions,

(a) $\dfrac{dy}{dx} - y = 2x\, e^{2x}, \qquad y(0) = 1$

(b) $\dfrac{dy}{dx} + 3y = xe^{-3x}, \qquad y(1) = 0$

(c) $\dfrac{dy}{dx} + \dfrac{2}{x}y = \dfrac{\cos x}{x^2}, \qquad y(\pi) = 0$

(d) $x\dfrac{dy}{dx} + 2y = \sin x, \qquad y(\pi/2) = 1$

(e) $x^3\dfrac{dy}{dx} + 4x^2 y = e^{-x}, \qquad y(-1) = 0$

Then, employ the `dsolve` command to solve these problems again. Plot the solution of $y(x)$ that varies with x by using the `ezplot` command.

8. Solve the following initial value problems by using the `dsolve` command,

(a) $\dfrac{dy}{dx} + 4y = e^{-x}, \qquad y(0) = 2$

(b) $\dfrac{dy}{dx} + \dfrac{3y}{x} = 3x - 2, \qquad y(1) = 1$

(c) $\dfrac{dy}{dx} + \dfrac{y}{x-2} = 3x, \qquad y(3) = 4$

(d) $\dfrac{dy}{dx} + \dfrac{5y}{9x} = 3x^3 + x, \qquad y(-1) = 0$

(e) $\sin x\dfrac{dy}{dx} + y\cos x = x\sin x, \qquad y(\pi/2) = 2$

In each problem, verify the solution by comparing with that obtained from the technique of exact equation with integrating factor. Use the `ezplot` command to plot the solutions.

9. Check whether the differential equations below are in the exact equation form by finding their derivatives with the `diff` command,

(a) $(2xy+3)dx + (x^2-1)dy = 0$

(b) $(1+\ln y)dx + \dfrac{x}{y}dy = 0$

(c) $(\cos x\cos y + 2x)dx + (\sin x\sin y + 2y)dy = 0$

(d) $e^x(y-x)dx + (1+e^x)dy = 0$

(e) $\left(\dfrac{1}{x} + 2xy^2\right)dx - (2x^2y - \cos y)dy = 0$

10. Solve the differential equations in Problem 9 that are in the exact equation form. Show the derivation of the exact solutions in details. Then, repeat the problems by using the `dsolve` command to verify the derived solutions.

11. Show that the following differential equations are the exact equations before deriving for their solutions. Then, use the `dsolve` command to find solutions and compare with the solutions derived. Hint: Property of the exact equation, $\partial M/\partial y = \partial N/\partial x$, can be verified conveniently by using the `diff` command.

    (a) $(2x+4)dx + (3y-8)dy = 0$

    (c) $(4xy^2-5)dx + (4x^2y+2)dy = 0$

    (d) $\left(\dfrac{y}{x}+5x\right)dx + (\ln x - 1)\,dy = 0$

    (e) $(3x^2y+e^y)dx + (x^3+xe^y-2y)dy = 0$

    (f) $(\tan x - \sin x \sin y)dx + (\cos x \cos y)dy = 0$

12. Derive the exact solutions of the following initial value problems by using the exact equation technique. Verify the derived solutions with those obtained from using the `dsolve` command.

    (a) $(3x^2y)dx + (y+x^3)dy = 0,\quad y(0) = 0$

    (b) $(x+y)^2\,dx + (2xy+x^2-1)dy = 0,\quad y(1) = 1$

    (c) $(e^xy+1)dx + (e^x-1)dy = 0,\quad y(1) = 1$

    (d) $(2x\sin y)dx + (x^2\cos y)dy = 0,\quad y(1) = 1$

    (e) $(\sin x\cos x - xy^2)dx + y(1-x^2)dy = 0,\quad y(0) = 2$

13. Solve the following differential equations by using the exact equation technique with the integrating factors. Compare the derived solutions with those obtained from using the `dsolve` command. Hint: The integrating factors for (d) and (e) are y^2 and xe^x, respectively.

(a) $(3x^2 + y)dx + (x^2y - x)dy = 0$
(b) $(x^4 - x + y)dx - x\,dy = 0$
(c) $(2y^2 + 2y + 4x^2)dx + (2xy + x)dy = 0$
(d) $(2xy^4 + x)dx + 4x^2y^3\,dy = 0$
(e) $(x+2)\sin y\,dx + x\cos y\,dy = 0$

14. Solve the following differential equations by using the exact equation technique with the integrating factors. Compare the derived solutions with those obtained from using the `dsolve` command.
    (a) $y\,dx + (1-x)dy = 0$
    (b) $(xy + y^2 + y)dx + (x + 2y)dy = 0$
    (c) $(2y^2 + 3x)dx + (2xy)dy = 0$
    (d) $(10 - 6y + e^{-3x})dx - 2\,dy = 0$
    (e) $(x+2)\sin y\,dx + x\cos y\,dy = 0$

15. Solve the following initial value problems that are governed by the differential equations and initial conditions. Derive their exact solutions by using the exact equation technique with the integrating factors. Compare the derived solutions with those obtained by using the `dsolve` command.
    (a) $x\,dx + (x^2y + 4y)dy = 0, \quad y(1) = 0$
    (b) $(x^2 + y^2 - 5)dx - (y + xy)dy = 0, \quad y(0) = 1$
    (c) $xy\,dx + (2x^2 + 3y^2 - 20)dy = 0, \quad y(0) = 1$

16. Derive the solutions of the differential equations that contain terms in the form of y/x as follows,
    (a) $\dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{y}{x}$ (b) $\dfrac{dy}{dx} + xy = xy^2$
    (c) $\dfrac{dy}{dx} = \dfrac{y^2}{x^2} - \dfrac{y}{x} + 1$ (d) $x^3\dfrac{dy}{dx} = x^2y - y^3$
    (e) $x\dfrac{dy}{dx} = x\cos\left(\dfrac{y}{x}\right) + y$

Then, use the `dsolve` command to solve for solutions of these differential equations again and compare them with the derived solutions.

17. Derive solutions of the following Bernoulli differential equations. Compare the derived solutions with those obtained by using the `dsolve` command.

(a) $\dfrac{dy}{dx} = -\dfrac{1}{x}y^2 + \dfrac{2}{x}y$ (b) $x\dfrac{dy}{dx} + y = xy^2$

(c) $\dfrac{dy}{dx} + xy = xy^4$ (d) $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3y^3$

(e) $xy^2\dfrac{dy}{dx} + y^3 = x\cos x$

18. Solve the Riccati differential equation,

$$\frac{dy}{dx} + 2x^2y - 2xy^2 = 1$$

by changing the variable,

$$y = x + \frac{1}{z}$$

Show the derivation of the exact solution in detail. Then, verify the solution by comparing with that obtained from using the `dsolve` command.

19. Solve the Riccati differential equation,

$$\frac{dy}{dx} = x^3(y-x)^2 + \frac{y}{x}$$

by changing the variable,

$$y = x + \frac{1}{z}$$

Show the derivation of the exact solution in detail. Then, verify the solution by comparing with that obtained from using the `dsolve` command.

20. Employ the `dsolve` and `ode23` commands to find the exact and approximate solutions of the initial value problem governed by the first-order linear homogeneous differential equation,

$$\frac{dy}{dx} + 2xy = 0 \qquad 0 \le x \le 1$$

with the initial condition of $y(0) = 1$. Plot to compare the two solutions of $y(x)$ that vary with x in the interval $0 \le x \le 1$.

21. Employ the `dsolve` and `ode45` commands to find the exact and approximate solutions of the initial value problem governed by the first-order nonlinear nonhomogeneous differential equation,

$$2xy\frac{dy}{dx} - y^2 = x^2 \qquad 1 \le x \le 2$$

with the initial condition of $y(1) = 2$. Plot to compare the two solutions of $y(x)$ that vary with x in the interval $1 \le x \le 2$.

22. Employ the `dsolve` and `ode45` commands to find the exact and approximate solutions of the initial value problem governed by the first-order nonlinear nonhomogeneous differential equation,

$$y\frac{dy}{dx} + xy^2 = 5x \qquad 0 \le x \le 2$$

with the initial condition of $y(0) = 2$. Plot to compare the two solutions of $y(x)$ that vary with x in the interval $0 \le x \le 2$.

23. Use the `dsolve` command to find exact solution of the initial value problem governed by the stiff differential equation,

$$\frac{dy}{dx} = 20e^{-100(x-2)^2} - 0.6y \qquad 0 \le x \le 4$$

with the initial condition of $y(0) = 0.5$. Then, employ the `ode23` and `ode23s` commands to find the approximate solutions. Plot to compare these solutions with the exact solution and provide comments on the numerical solution accuracy.

24. Use the `dsolve` command to find exact solution of the initial value problem governed by the stiff differential equation,

$$\frac{dy}{dx} = y^2 - y^3 \qquad 0 \le x \le 2000$$

with the initial condition of $y(0) = 0.001$. Then, employ the `ode23`, `ode45` and `ode23s` commands to find the approximate solutions. Comment on the numerical solution accuracy of these solutions that do not have the exact solution to compare.

Chapter 5

Second-Order Linear Differential Equations

5.1 Introduction

We learned several techniques for solving many types of the first-order differential equations in Chapter 4. We found that proper techniques should be applied according to the types of differential equations in order to reach for the solutions. We also found that exact solutions for many differential equations cannot be derived in closed-form expressions. Numerical methods are needed to find the approximate solutions.

In this chapter, we will learn how to solve the second-order linear differential equations. This type of differential equation arises in several scientific and engineering problems, such as heat transfer, fluid flow, wave propagation, electro-magnetic field, etc. Solving these second-order differential equations is simpler than the first-order differential equations because there are only few standard techniques that are easy to follow and understand. We will start from solving the homogeneous second-order differential equations before extending to the nonhomogeneous equations. Several examples are used and, at the same time, the derived solutions are verified by employing MATLAB commands. At the end of the chapter, we will study how to apply the numerical methods to solve these second-order differential equations. The numerical methods provide approximate solutions when the exact solutions are complicated or not available.

5.2 Homogeneous Equations with Constant Coefficients

The second-order linear homogeneous differential equation with constant coefficients can be conveniently solved for solution. Such differential equation is in the form,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

where a and b are constants, while the right-hand-side of the equation is zero. A solution to this differential equation is $e^{\lambda x}$ for which λ is a number.

To find the values of λ, we substitute this solution $e^{\lambda x}$ into the differential equation,

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

then, divide by $e^{\lambda x}$ to get,

$$\lambda^2 + a\lambda + b = 0$$

The result is in form of quadratic equation that can be used to find values of λ. We call this quadratic equation as the *characteristic* or *auxiliary equation*. The roots of this equation are,

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

Depending on the values of the coefficients a and b, there are three possible cases for the values of the roots. We will consider these cases in details in the next three sections.

It is noted that the function in the form of $e^{\lambda x}$ is a solution of the second-order differential equation. For example,

$$y(x) \quad = \quad e^{-2x} + e^{-3x}$$

where $\lambda_1 = -2$ and $\lambda_2 = -3$, is a solution of the second-order homogeneous differential equation,

$$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y \quad = \quad 0$$

We can verify this by determining the derivatives of the solution and substitute them into the equation, as follows.

Since, $$\frac{dy}{dx} \quad = \quad -2e^{-2x} - 3e^{-3x}$$

and, $$\frac{d^2 y}{dx^2} \quad = \quad +4e^{-2x} + 9e^{-3x}$$

then, $$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y$$

$$= \quad \left(4e^{-2x} + 9e^{-3x}\right) + 5\left(-2e^{-2x} - 3e^{-3x}\right) + 6\left(e^{-2x} + e^{-3x}\right) \quad = \quad 0$$

which is equal to the value on the right-hand-side of the equation.

The `diff` command can help us to verify such equation conveniently,

```
>> syms x y
>> y = exp(-2*x) + exp(-3*x);
>> LHS = diff(y,x,2) + 5*diff(y,x) + 6*y

LHS =

0
```

diff

If we consider these two functions e^{-2x} and e^{-3x}, we see that their variations are different and do not depend on each other. Or, in the other word, they are *linearly independent*.

We can also ensure that the two functions are linearly independent by using the Wronski's test. The two functions y_1 and y_2 are linearly independent if the *determinant* defined by,

$$W \quad = \quad \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad = \quad y_1 y_2' - y_2 y_1'$$

is not zero. As in this example, $y_1 = e^{-2x}$ and $y_2 = e^{-3x}$, then,

$$W = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -e^{-5x}$$

which is not zero. The determinant, so called the *Wronskian*, can be easily obtained by using the `det` command,

```
>> y1 = exp(-2*x);
>> y2 = exp(-3*x);
>> W = [y1 y2; diff(y1,x) diff(y2,x)];
>> det(W)
```

det

```
ans =

-1/exp(5*x)
```

Here, the solution,

$$y(x) = e^{-2x} + e^{-3x}$$

of the differential equation,

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

can be also obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y + 5*Dy + 6*y = 0', 'x')
```

dsolve

```
ans =

C1/exp(2*x) + C2/exp(3*x)
```

i.e.,
$$y(x) = C_1 e^{-2x} + C_2 e^{-3x}$$

where C_1 and C_2 are constants. If these two constants are determined from the two initial conditions, such as $y(0) = 0$ and $y'(0) = 1$, we call the problem as the *initial value problem*. But if they are determined from the two boundary conditions, such as $y(0) = 0$ and $y(5) = 1$, we call it as the *boundary value problem*. We will solve the initial value problem in this chapter, while solving the boundary value problem will be shown in Chapter 9.

5.3 Solutions from Distinct Real Roots

As explained in the preceding section, the second-order linear homogeneous differential equation with constant coefficients,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by \quad = \quad 0$$

has a solution in the form of $e^{\lambda x}$. By substituting this solution into the differential equation above, we obtain the characteristic equation,

$$\lambda^2 + a\lambda + b \quad = \quad 0$$

which leads to the two roots of,

$$\lambda_1, \lambda_2 \quad = \quad \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

If $\sqrt{a^2-4b} > 0$, the two roots are distinct real numbers as,

$$\lambda_1 \quad = \quad \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) \quad \text{and} \quad \lambda_2 \quad = \quad \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

which lead to the solution of,

$$y(x) \quad = \quad C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

where C_1 and C_2 are constants.

We will learn how to solve the differential equation when its solution consists of the distinct real roots by using the following examples.

<u>Example</u> Derive the general solution of the second-order differential equation,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y \quad = \quad 0$$

After assuming the solution in the form of $e^{\lambda x}$ and substitute it into the differential equation, we get,

$$\lambda^2 e^{\lambda x} - 5\lambda e^{\lambda x} + 4e^{\lambda x} \quad = \quad 0$$

Then, we divide it by $e^{\lambda x}$ to obtain the characteristic equation,

$$\lambda^2 - 5\lambda + 4 \quad = \quad 0$$

Or, $$(\lambda - 1)(\lambda - 4) = 0$$

i.e., $$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 4$$

Thus, the general solution is,

$$y = C_3 e^{x} + C_4 e^{4x}$$

where C_3 and C_4 are constants.

We can employ the `dsolve` command to obtain the same solution,

```
>> syms x y
>> dsolve('D2y - 5*Dy + 4*y = 0', 'x')
```

dsolve

```
ans =

C3*exp(x) + C4*exp(4*x)
```

<u>Example</u> Derive the general solution of the second-order differential equation,

$$6\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

Similar to the preceding example, by assuming the solution in the form of $e^{\lambda x}$, the characteristic equation is,

$$6\lambda^2 + \lambda - 2 = 0$$

Or, $$(2\lambda - 1)(3\lambda + 2) = 0$$

i.e., $$\lambda_1 = \frac{1}{2} \quad \text{and} \quad \lambda_2 = -\frac{2}{3}$$

Thus, the general solution is,

$$y = C_5 e^{x/2} + C_6 e^{-2x/3}$$

Again, the `dsolve` command can be used to find the solution,

```
>> syms x y
>> dsolve('6*D2y + Dy - 2*y = 0', 'x')
```

dsolve

```
ans =
```

```
C5*exp(x/2) + C6/exp((2*x)/3)
```

Example Derive the general solution of the second-order differential equation,

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} = 0$$

If we follow the same procedure as in the preceding two examples, we obtain the characteristic equation as,

$$\lambda^2 - 7\lambda = 0$$

Or,
$$\lambda(\lambda - 7) = 0$$

i.e.,
$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 7$$

Thus, the general solution is,

$$y = C_7 e^{0x} + C_8 e^{7x} = C_7 + C_8 e^{7x}$$

The `dsolve` command can be employed to give the same solution,

```
>> syms x y
>> dsolve('D2y - 7*Dy = 0', 'x')

ans =

C7 + C8*exp(7*x)
```

dsolve

Example Derive the exact solution of the initial value problem governed by the second-order differential equation,

$$4\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 3y = 0$$

with the initial conditions of $y(0) = 1$ and $y'(0) = 1/2$.

From the given differential equation, the characteristic equation is,

$$4\lambda^2 - 8\lambda + 3 = 0$$

Or,
$$(2\lambda - 1)(2\lambda - 3) = 0$$

i.e.,
$$\lambda_1 = \frac{1}{2} \quad \text{and} \quad \lambda_2 = \frac{3}{2}$$

Then, the general solution is,

$$y = C_9 e^{x/2} + C_{10} e^{3x/2}$$

Also,
$$y' = \frac{1}{2} C_9 e^{x/2} + \frac{3}{2} C_{10} e^{3x/2}$$

where C_9 and C_{10} are constants that can be determined from the given initial conditions as follows,

$$y(0) = 1 = C_9 + C_{10}$$

$$y'(0) = \frac{1}{2} = \frac{1}{2} C_9 + \frac{3}{2} C_{10}$$

to give, $C_9 = 1$ and $C_{10} = 0$

Hence, the exact solution for this initial value problem is,

$$y = e^{x/2}$$

We can use the `dsolve` command to find the same exact solution,

```
>> syms x y
>> dsolve('4*D2y - 8*Dy + 3*y = 0', 'y(0) = 1',
          'Dy(0) = 1/2', 'x')
ans =

exp(x/2)
```

dsolve

This exact solution can be plotted by using the `ezplot` command,

```
ezplot(ans, [0 5])
```

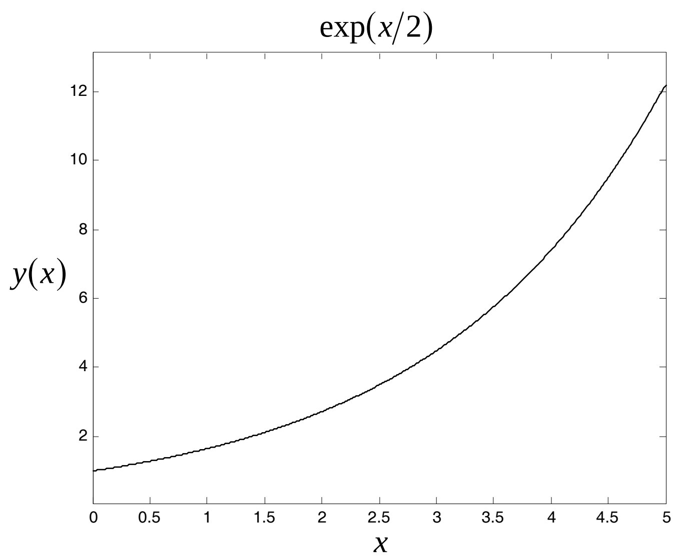
Variation of the solution *y* that varies with *x* is shown in the figure.

5.4 Solutions from Repeated Real Roots

From the second-order homogeneous differential equation with constant coefficients,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

We assume the solution in the form of $e^{\lambda x}$ and substitute it into the differential equation, we obtain the characteristic equation,



$$\lambda^2 + a\lambda + b = 0$$

which leads to the two roots of,

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

If the coefficients a and b in the differential equation are given such that,

$$a^2 - 4b = 0$$

then, the two roots are the same real number,

$$\lambda_1 = \lambda_2 = -\frac{a}{2}$$

So that, the general solution is,

$$y = e^{-ax/2}$$

But the general solution must consist of two functions because the differential equation is second order. Thus, we need to find the second function. If we assume the second function in the form,

$$y = u(x)e^{-ax/2}$$

Then, $$y' = u' e^{-ax/2} - \frac{a}{2} u\, e^{-ax/2}$$

and, $$y'' = u'' e^{-ax/2} - a u' e^{-ax/2} + \frac{a^2}{4} u\, e^{-ax/2}$$

By substituting these y, y' and y'' into the differential equation,

$$\left(u'' e^{-ax/2} - a u' e^{-ax/2} + \frac{a^2}{4} u e^{-ax/2}\right) + a\left(u' e^{-ax/2} - \frac{a}{2} u e^{-ax/2}\right)$$
$$+ b\left(u e^{-ax/2}\right) = e^{-ax/2}\left[u'' + u\left(b - \frac{a^2}{4}\right)\right] = 0$$

Because $(b - a^2/4) = 0$ in this case and $e^{-ax/2}$ could not be zero, this means,

$$u'' = 0$$

Or, $$u(x) = cx + d$$

where c and d are constant.

Hence, $$y = (cx + d)e^{-ax/2}$$

is another function that could be the solution of the differential equation. For simplicity, we may select $c = 1$ and $d = 0$, so that the general solution for the case of repeated roots is,

$$y = e^{-ax/2} + xe^{-ax/2}$$

We will learn how to find general solutions of the differential equations when the roots of their characteristic equations are repeated by using the following examples.

Example Derive the general solution of the second-order differential equation,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

We first assume the solution in the form of $e^{\lambda x}$ and substitute it into the differential equation to get,

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + e^{\lambda x} = 0$$

Then, we divide it by $e^{\lambda x}$ to obtain the characteristic equation,

$$\lambda^2 + 2\lambda + 1 = 0$$

Or, $$(\lambda + 1)(\lambda + 1) = 0$$

which leads to the two roots of $\lambda_1 = -1$ and $\lambda_2 = -1$. Since the roots are repeated, thus the general solution is,

$$y = C_{11} e^{-x} + C_{12} x e^{-x}$$

where C_{11} and C_{12} are constants.

The same solution is obtained by using the `dsolve` command as,

```
>> syms x y
>> dsolve('D2y + 2*Dy + y = 0', 'x')
```

dsolve

```
ans =

C11/exp(x) + (C12*x)/exp(x)
```

<u>Example</u> Derive the general solution of the second-order differential equation,

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

The characteristic equation corresponding to the given differential equation is,

$$\lambda^2 - 6\lambda + 9 = 0$$

Or, $$(\lambda - 3)(\lambda - 3) = 0$$

which leads to the repeated real roots of,

$$\lambda_1 = \lambda_2 = 3$$

Thus, the general solution is,

$$y = C_{13} e^{3x} + C_{14} x e^{3x}$$

where C_{13} and C_{14} are constants.

Again, the `dsolve` command can provide the same solution,

```
>> syms x y
>> dsolve('D2y - 6*Dy + 9*y = 0', 'x')
```

dsolve

```
ans =

C13*exp(3*x) + C14*x*exp(3*x)
```

<u>Example</u> Solve the initial value problem governed by the second-order differential equation,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

with the initial conditions of $y(0) = 1$ and $y'(0) = 3$.

The corresponding characteristic equation is,

$$\lambda^2 + 4\lambda + 4 = 0$$

Or,

$$(\lambda + 2)(\lambda + 2) = 0$$

which leads to the two repeated roots of $\lambda_1 = -2$ and $\lambda_2 = -2$.

Thus, the general solution is,

$$y = C_{15}\, e^{-2x} + C_{16}\, xe^{-2x}$$

so that,

$$y' = -2C_{15}\, e^{-2x} - 2C_{16}\, xe^{-2x} + C_{16}\, e^{-2x}$$

where C_{15} and C_{16} are constants that can be determined from the two given initial conditions,

$$y(0) = 1 = C_{15} + 0$$

$$y'(0) = 3 = -2C_{15} - 0 + C_{16}$$

to give,

$$C_{15} = 1 \quad \text{and} \quad C_{16} = 5$$

Hence, the exact solution to this initial value problem is,

$$y = e^{-2x} + 5xe^{-2x}$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y + 4*Dy + 4*y = 0', 'y(0) = 1',
          'Dy(0) = 3', 'x')

ans =

1/exp(2*x) + (5*x)/exp(2*x)
```

The above exact solution can also be verified by substituting it back into the differential equation and using the `diff` command,

```
>> y = 1/exp(2*x) + (5*x)/exp(2*x);
>> LHS = diff(y,x,2) + 4*diff(y,x) + 4*y
```

diff

```
LHS =

0
```

The exact solution must also satisfy the two initial conditions. This can be checked by using the `subs` command,

```
>> y = 1/exp(2*x) + (5*x)/exp(2*x);
>> subs(y,{x},{0})
```

subs

```
ans =

     1

>> dydx = diff(y,x);
>> subs(dydx,{x},{0})
```

subs

```
ans =

     3
```

<u>Example</u> Solve the initial value problem governed by the second-order differential equation,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{y}{4} \quad = \quad 0$$

with the initial conditions of $y(0)=1$ and $y'(0)=1/3$.

From the given differential equation, the corresponding characteristic equation is,

$$\lambda^2 - \lambda + \frac{1}{4} \quad = \quad 0$$

Or, $$\left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{1}{2}\right) \quad = \quad 0$$

i.e., $$\lambda_1 \; = \; \frac{1}{2} \qquad \text{and} \qquad \lambda_2 \; = \; \frac{1}{2}$$

Since the roots are repeated, then the general solution is,

$$y = C_{17}\, e^{x/2} + C_{18}\, xe^{x/2}$$

So that,
$$y' = \frac{1}{2}C_{17}\, e^{x/2} + \frac{1}{2}C_{18}\, xe^{x/2} + C_{18}\, e^{x/2}$$

where C_{17} and C_{18} are constants which can be determined from the two initial conditions,

$$y(0) = 1 = C_{17} + 0$$

$$y'(0) = \frac{1}{3} = \frac{1}{2}C_{17} + 0 + C_{18}$$

to give,
$$C_{17} = 1 \quad \text{and} \quad C_{18} = -\frac{1}{6}$$

Thus, the exact solution to this initial value problem is,

$$y = e^{x/2} - \frac{x}{6}e^{x/2}$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y - Dy + y/4 = 0', 'y(0) = 1',
          'Dy(0) = 1/3', 'x')

ans =

exp(x/2) - (x*exp(x/2))/6
```

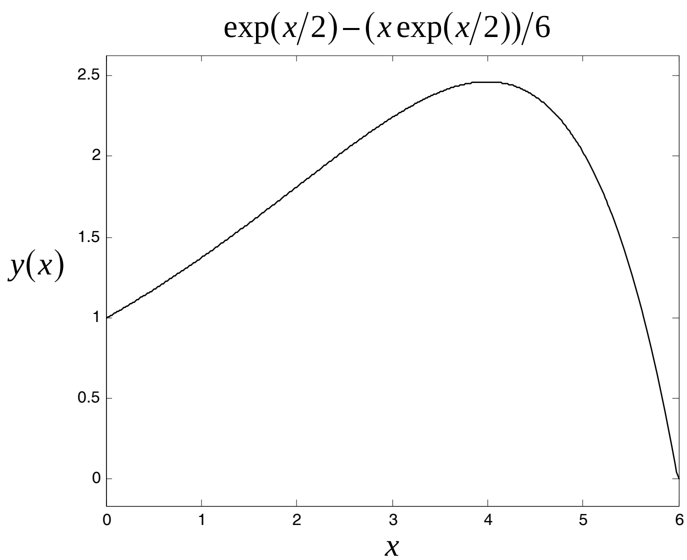
The `ezplot` command can then be used to plot the variation of y that varies with x as shown in the figure.

5.5 Solutions from Complex Roots

From the second-order homogeneous differential equation with constant coefficients in the form,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

By assuming the solution in the form of $e^{\lambda x}$ and substitute it into the differential equation, we obtain the characteristic equation,



$$\lambda^2 + a\lambda + b \quad = \quad 0$$

which leads to two roots of,

$$\lambda_1, \lambda_2 \quad = \quad \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

If $a^2 - 4b < 0$, then the two roots are conjugate complex numbers $\alpha + i\beta$ and $\alpha - i\beta$ where α might be zero but β is not. In this case, the general solution of the differential equation is,

$$y \quad = \quad A e^{(\alpha + i\beta)x} + B e^{(\alpha - i\beta)x}$$

where A and B are constants.

By using the Euler's formula,

$$e^{i\beta x} = \cos(\beta x) + i \sin(\beta x)$$

and
$$e^{-i\beta x} = \cos(\beta x) - i \sin(\beta x)$$

then, the general solution above becomes,

$$\begin{aligned} y(x) &= e^{\alpha x}\left(A e^{i\beta x} + B e^{-i\beta x}\right) \\ &= A e^{\alpha x}(\cos(\beta x) + i \sin(\beta x)) + B e^{\alpha x}(\cos(\beta x) - i \sin(\beta x)) \\ &= (A + B) e^{\alpha x} \cos(\beta x) + i(A - B) e^{\alpha x} \sin(\beta x) \end{aligned}$$

This solution can be written in a simpler form if we select $A = B = 1/2$, then $y(x) = e^{\alpha x}\cos(\beta x)$. At the same time, if we select $A = 1/2i = -B$, we get $y(x) = e^{\alpha x}\sin(\beta x)$. Since the two solutions must be linearly independent, thus we can write the general solution in the form of,

$$y(x) = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x)$$

where C_1 and C_2 are constants.

In conclusion, for the case when the roots of the characteristic equation are complex conjugate numbers,

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right) = \alpha \pm i\beta$$

then, the general solution is in the form,

$$y(x) = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x)$$

We will learn how to derive solutions of the differential equations when the two roots of their characteristic equations are complex conjugate numbers by using following examples.

<u>Example</u> Derive the general solution of the second-order differential equation,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$$

By assuming the solution in the form of $e^{\lambda x}$ and substitute it into the differential equation, we get,

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + 5e^{\lambda x} = 0$$

After dividing it through by $e^{\lambda x}$, we obtain the characteristic equation,

$$\lambda^2 - 2\lambda + 5 = 0$$

The two roots of this equation are complex conjugate numbers,

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(2 \pm \sqrt{4 - 20}\right) = 1 \pm 2i$$

By comparing the coefficients with those in the equation derived earlier, we find that,

$$\alpha \quad = \quad 1 \qquad \text{and} \qquad \beta \quad = \quad 2$$

Thus, the general solution of the differential equation is,

$$y \quad = \quad C_{19}\, e^{x} \cos(2\,x) + C_{20}\, e^{x} \sin(2\,x)$$

where C_{19} and C_{20} are constants.

The same solution can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y - 2*Dy + 5*y = 0', 'x')
```

dsolve

```
ans =

C19*cos(2*x)*exp(x) + C20*sin(2*x)*exp(x)
```

Example Derive the general solution of the second-order differential equation,

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 2y \quad = \quad 0$$

The characteristic equation corresponding to the above differential equation is,

$$\lambda^2 + 2\lambda + 2 \quad = \quad 0$$

which leads to the two roots of complex conjugates,

$$\lambda_1, \lambda_2 \quad = \quad \frac{1}{2}\left(-2 \pm \sqrt{4-8}\right) \quad = \quad -1 \pm i$$

Here, $\qquad \alpha \quad = \quad -1 \qquad \text{and} \qquad \beta \quad = \quad 1$

Thus, the general solution is,

$$y \quad = \quad C_{21}\, e^{-x} \cos x + C_{22}\, e^{-x} \sin x$$

where C_{21} and C_{22} are constants.

Again, the same general solution can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y + 2*Dy + 2*y =0', 'x')
```

dsolve

```
ans =

(C21*cos(x))/exp(x) + (C22*sin(x))/exp(x)
```

The two constants of C_{21} and C_{22} are determined from the initial conditions of the problem. For example, the given initial conditions are,

$$y(0) \quad = \quad 1 \qquad \text{and} \qquad \frac{dy}{dx}(0) \quad = \quad 2$$

Since, $\quad y \quad = \quad C_{21}\, e^{-x} \cos x + C_{22}\, e^{-x} \sin x$

then,

$$\frac{dy}{dx} = C_{21}\left(-e^{-x} \cos x - e^{-x} \sin x\right) + C_{22}\left(-e^{-x} \sin x + e^{-x} \cos x\right)$$

From the given initial conditions, the two equations become,

$$1 \quad = \quad C_{21}(1) + C_{22}(0)$$

and

$$2 \quad = \quad C_{21}(-1 - 0) + C_{22}(0 + 1)$$

By solving these two equations, the two constants can be determined,

$$C_{21} \quad = \quad 1 \qquad \text{and} \qquad C_{22} \quad = \quad 3$$

Hence, the exact solution of the differential equation with the initial conditions is,

$$y \quad = \quad e^{-x} \cos x + 3e^{-x} \sin x$$

The `dsolve` command can also be used to find the exact solution of this initial value problem by entering,

```
>> syms x y
>> dsolve('D2y + 2*Dy + 2*y =0', 'y(0) = 1',
          'Dy(0) = 2', 'x')
```

dsolve

```
ans =

cos(x)/exp(x) + (3*sin(x))/exp(x)
```

Example Solve the initial value problem governed by the second-order differential equation,

$$16\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 145y \quad = \quad 0$$

with the initial conditions of $y(0) = -2$ and $y'(0) = 1$.

The characteristic equation that corresponds to the differential equation is,

$$16\lambda^2 - 8\lambda + 145 \quad = \quad 0$$

The two roots are complex conjugate numbers,

$$\lambda_1, \lambda_2 \quad = \quad \frac{8 \pm \sqrt{64 - 9280}}{32} \quad = \quad \frac{1}{4} \pm 3i$$

By comparing the coefficients of the roots with the equation derived earlier, we find that,

$$\alpha \quad = \quad \frac{1}{4} \qquad \text{and} \qquad \beta \quad = \quad 3$$

Thus, the general solution to the differential equation is,

$$y \quad = \quad C_{23}\, e^{x/4} \cos(3x) + C_{24}\, e^{x/4} \sin(3x)$$

where C_{23} and C_{24} are constants that can be determined from the given initial conditions of the problem. For example, if the initial conditions are given as $y(0) = -2$ and $y'(0) = 1$, the exact solution can be determined as follows.

First, we need to find the derivative of the solution y,

$$\begin{aligned} y' \quad &= \quad C_{23}\left(\frac{1}{4}e^{x/4} \cos(3x) - 3e^{x/4} \sin(3x)\right) \\ &\qquad + C_{24}\left(\frac{1}{4}e^{x/4} \sin(3x) + 3e^{x/4} \cos(3x)\right) \end{aligned}$$

We apply the two initial conditions to equations of y and y', respectively,

$$-2 \quad = \quad C_{23}(1) + C_{24}(0)$$

and
$$1 = C_{23}\left(\frac{1}{4} - 0\right) + C_{24}\,(0+3)$$

Then, we solve for the two constants of C_{23} and C_{24} to get,

$$C_{23} = -2 \qquad \text{and} \qquad C_{24} = \frac{1}{2}$$

Thus, the exact solution to the differential equation with the given initial conditions is,

$$y = -2e^{x/4}\cos(3x) + \frac{1}{2}e^{x/4}\sin(3x)$$

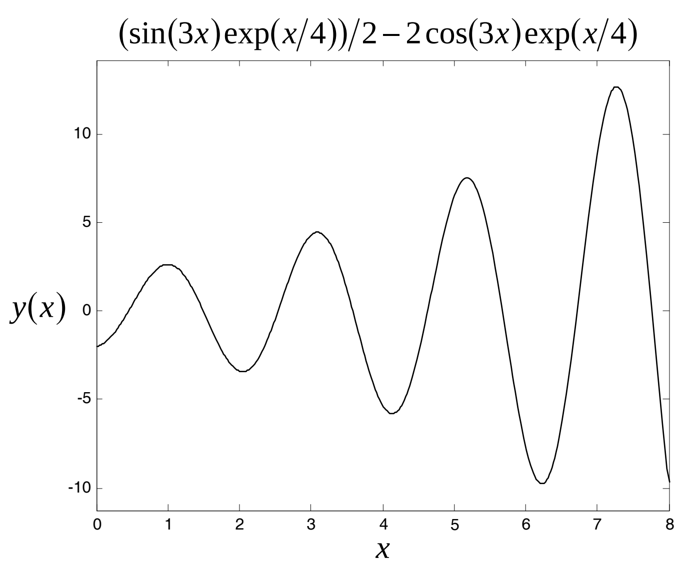
The same exact solution can be obtained by using the `dsolve` command by entering,

```
>> syms x y
>> dsolve('16*D2y-8*Dy+145*y = 0', 'y(0) = -2',
          'Dy(0) = 1', 'x')
```

dsolve

```
ans =

(sin(3*x)*exp(x/4))/2 - 2*cos(3*x)*exp(x/4)
```

The `ezplot` command can then be used to plot the variation of y that varies with x as shown in the figure.



Example Solve the initial value problem governed by the second-order differential equation,

$$64\frac{d^2y}{dx^2}+16\frac{dy}{dx}+1025y = 0$$

with the initial conditions of $y(0)=-1$ and $y'(0)=3$.

From the given differential equation, the corresponding characteristic equation is,

$$64\lambda^2+16\lambda+1025 = 0$$

The two roots are in the form of complex conjugates as,

$$\lambda_1,\lambda_2 = \frac{-16\pm\sqrt{256-262{,}400}}{128} = -\frac{1}{8}\pm 4i$$

By comparing the coefficients with those in the equation derived earlier, we find that,

$$\alpha = -\frac{1}{8} \quad \text{and} \quad \beta = 4$$

Then, the general solution to the differential equation is,

$$y = C_{25}\,e^{-x/8}\cos(4x)+C_{26}\,e^{-x/8}\sin(4x)$$

where C_{25} and C_{26} are the constants that can be determined from the initial conditions. The process starts from finding the derivative of the solution y,

$$\begin{aligned} y' = \; & C_{25}\left(-\frac{1}{8}e^{-x/8}\cos(4x)-4e^{-x/8}\sin(4x)\right) \\ & +C_{26}\left(-\frac{1}{8}e^{-x/8}\sin(4x)+4e^{-x/8}\cos(4x)\right) \end{aligned}$$

and by applying the initial conditions of $y(0)=-1$ and $y'(0)=3$ into these two equations to get,

$$\begin{aligned} -1 &= C_{25}(1)+C_{26}(0) \\ 3 &= C_{25}\left(-\frac{1}{8}-0\right)+C_{26}(0+4) \end{aligned}$$

Then, we solve these two equations to obtain,

$$C_{25} = -1 \qquad \text{and} \qquad C_{26} = \frac{23}{32}$$

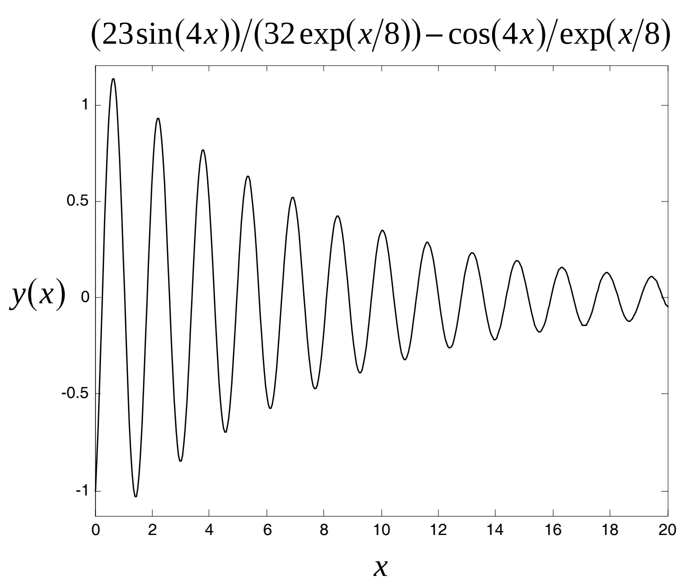
Thus, the exact solution of this initial value problem is,

$$y = -e^{-x/8}\cos(4x) + \frac{23}{32}e^{-x/8}\sin(4x)$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('64*D2y + 16*Dy + 1025*y = 0',
          'y(0) = -1', 'Dy(0) = 3', 'x')

ans =

(23*sin(4*x))/(32*exp(x/8)) - cos(4*x)/exp(x/8)
```

The `ezplot` command can then be used to plot the variation of y that varies with x as shown in the figure.



5.6 Nonhomogeneous Equations

In this section and the next two sections, we will concentrate on finding the solutions of the second-order linear nonhomogeneous differential equations in the form,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

where the term on the right-hand-side of the equation is only function of x. The coefficients a and b for the terms on the left-hand-side of the equation are constants. Since the differential equation is linear, then the general solution consists of two parts,

$$y = y_h + y_p$$

where y_h is the solution of the homogeneous differential equation we learned earlier. The function y_p is called the *particular solution* that normally depends on the form of the function $f(x)$ on the right-hand-side of the differential equation.

As an example, a general solution of the nonhomogeneous differential equation,

$$\frac{d^2y}{dx^2} + y = 3\sin(2x)$$

is

$$\begin{aligned} y &= y_h + y_p \\ &= A\sin x + B\cos x - 2\sin x(1+\cos x) \end{aligned}$$

The first part of the solution,

$$y_h = A\sin x + B\cos x$$

where A and B are constants, is the solution of the homogeneous differential equation,

$$\frac{d^2y}{dx^2} + y = 0$$

The second part of the solution,

$$y_p = -2\sin x(1+\cos x)$$

is the particular solution that depends on the form of the given function $f(x) = 3\sin(2x)$ on the right-hand-side of the differential equation.

We can verify that the solution above is, in fact, the general solution of the nonhomogeneous differential equation. This can be done by using the `diff` command to evaluate the terms on the left-hand-side of the differential equation as follows,

```
>> syms x A B
>> y = A*sin(x) + B*cos(x) - 2*sin(x)*(1+cos(x));
>> LHS = diff(y,x,2) + y

LHS =

3*sin(2*x)
```

The result is equal to the right-hand-side function of the differential equation.

The same idea is applied when the given nonhomogeneous is more complex. For example,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 6e^x$$

Again, the general solution consists of two parts,

$$y = y_h + y_p = Ae^{-x} + Be^{-2x} + e^x$$

The first part is the homogeneous solution,

$$y_h = Ae^{-x} + Be^{-2x}$$

of the homogeneous differential equation,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

where A and B are constants. While the second part,

$$y_p = e^x$$

is the particular solution. Again, we can use the `diff` command to verify the general solution above by evaluating the terms on the left-hand-side of the differential equation as follows,

```
>> syms x A B
>> y = A*exp(-x) + B*exp(-2*x) + exp(x);
>> LHS = diff(y,x,2) + 3*diff(y,x) + 2*y

LHS =

6*exp(x)
```

The result is identical to the function $f(x)$ on the right-hand-side of the differential equation.

There are several methods for finding the particular solution of y_p. The two popular methods, which are the method of undetermined coefficients and the variation of parameters, are presented in the next two sections.

5.7 Method of Undetermined Coefficients

The method of undetermined coefficients is a simple method for finding the particular solution of the nonhomogeneous differential equation. The idea is to seek for the particular solution y_p that contains terms which are in similar forms with the function $f(x)$ on the right-hand-side of the equation. The method works well when the function $f(x)$ is in the forms of the exponential, polynomials, sine and cosine functions. The method of undetermined coefficients is demonstrated by using the following examples.

Example Derive the general solution of the second-order nonhomogeneous differential equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 3e^{2x}$$

The homogeneous solution is derived from the given differential equation in the homogeneous form of,

$$\frac{d^2y_h}{dx^2} - 3\frac{dy_h}{dx} - 4y_h = 0$$

to give,

$$y_h = C_{27}e^{4x} + C_{28}e^{-x}$$

where C_{27} and C_{28} are constants. Or, it can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y - 3*Dy - 4*y = 0', 'x')
```

dsolve

```
ans =

C27*exp(4*x) + C28/exp(x)
```

The particular solution y_p is to be determined such that the differential equation in the form,

$$\frac{d^2y_p}{dx^2} - 3\frac{dy_p}{dx} - 4y_p = 3e^{2x}$$

is satisfied. In this case, let us assume,

$$y_p = Ae^{2x}$$

where A is an constant. We assume the particular solution in the exponential form because their derivatives are also in the exponential form as,

$$\frac{dy_p}{dx} = 2Ae^{2x} \quad \text{and} \quad \frac{d^2y_p}{dx^2} = 4Ae^{2x}$$

By substituting these derivatives into the differential equation above, we get,

$$\begin{aligned} (4A - 6A - 4A)e^{2x} &= 3e^{2x} \\ -6Ae^{2x} &= 3e^{2x} \\ A &= -\frac{1}{2} \end{aligned}$$

Then, the particular solution is,

$$y_p = -\frac{1}{2}e^{2x}$$

Thus, the general solution of the nonhomogeneous differential equation is,

$$y = C_{27}e^{4x} + C_{28}e^{-x} - \frac{1}{2}e^{2x}$$

where C_{27} and C_{28} are constants.

The same solution can be found by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y - 3*Dy - 4*y = 3*exp(2*x)', 'x')
```

dsolve

```
ans =

C27*exp(4*x) - exp(2*x)/2 + C28/exp(x)
```

<u>Example</u> Derive the general solution of the second-order non-homogeneous differential equation,

$$\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} - 4y \quad = \quad 4x^2$$

Since the terms on the left-hand-side of the differential equation are the same as those in the preceding example, thus, the homogeneous solution is,

$$y_h \quad = \quad C_{27}e^{4x} + C_{28}e^{-x}$$

The particular solution y_p is to be determined such that the differential equation,

$$\frac{d^2 y_p}{dx^2} - 3\frac{dy_p}{dx} - 4y_p \quad = \quad 4x^2$$

is satisfied. We may assume the particular solution in the form of polynomials,

$$y_p \quad = \quad Ax^2 + Bx + C$$

where A, B and C are constants. By substituting this particular solution y_p into the differential equation, we get,

$$(2A) - 3(2Ax + B) - 4(Ax^2 + Bx + C) \quad = \quad 4x^2$$

Or, $$(-4A)x^2 + (-6A - 4B)x + (2A - 3B - 4C) \quad = \quad 4x^2$$

Comparing the coefficients leads to the three equations,

$$\begin{aligned} -4A &= 4 \\ -6A - 4B &= 0 \\ 2A - 3B - 4C &= 0 \end{aligned}$$

Solving these three equations gives, $A = -1$, $B = 3/2$ and $C = -13/8$. Then, the particular solution is,

$$y_p \quad = \quad -x^2 + \frac{3}{2}x - \frac{13}{8}$$

Hence, the general solution of the given nonhomogeneous differential equation is,

$$y \quad = \quad C_{27}e^{4x} + C_{28}e^{-x} - x^2 + \frac{3}{2}x - \frac{13}{8}$$

The same solution can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y - 3*Dy - 4*y = 4*x^2', 'x')
ans =

(3*x)/2 - x^2 + C27*exp(4*x) + C28/exp(x) - 13/8
```

dsolve

Example Derive the general solution of the second-order non-homogeneous differential equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y \quad = \quad 2\sin x$$

Since the terms on the left-hand-side of the given differential equation are the same as those in the preceding example, then the homogeneous solution is,

$$y_h \quad = \quad C_{27}e^{4x} + C_{28}e^{-x}$$

The particular y_p must satisfy the differential equation,

$$\frac{d^2y_p}{dx^2} - 3\frac{dy_p}{dx} - 4y_p \quad = \quad 2\sin x$$

Because the derivatives of the sine function are in the form of sine and cosine functions, thus we need to assume the particular solution in the form,

$$y_p \quad = \quad A\sin x + B\cos x$$

where A and B are constants. If we substitute this particular solution y_p into the differential equation, we get,

$$(-A\sin x - B\cos x) - 3(A\cos x - B\sin x) - 4(A\sin x + B\cos x) = 2\sin x$$

Or, $$(-A + 3B - 4A)\sin x + (-B - 3A - 4B)\cos x = 2\sin x$$

Then, comparing the coefficients leads to two equations,

$$-A + 3B - 4A = 2$$

and $$-B - 3A - 4B = 0$$

which can be solved to give $A = -5/17$ and $B = 3/17$. Thus, the particular solution is,

$$y_p = -\frac{5}{17}\sin x + \frac{3}{17}\cos x$$

Hence, the general solution of the nonhomogeneous differential equation is,

$$y = C_{27}e^{4x} + C_{28}e^{-x} - \frac{5}{17}\sin x + \frac{3}{17}\cos x$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y - 3*Dy - 4*y = 2*sin(x)', 'x')

ans =

(3*cos(x))/17 - (5*sin(x))/17 + C27*exp(4*x) +
C28/exp(x)
```

dsolve

<u>Example</u> Solve the initial value problem governed by the second-order nonhomogeneous differential equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 3e^{2x} + 4x^2 + 2\sin x$$

with the initial conditions of $y(0) = 0$ and $y'(0) = 1$.

The right-hand-side of the differential equation is the combination of the functions in the preceding three examples. Thus general solution is,

$$y = C_{27}e^{4x} + C_{28}e^{-x} - \frac{1}{2}e^{2x} - x^2 + \frac{3}{2}x - \frac{13}{8} - \frac{5}{17}\sin x + \frac{3}{17}\cos x$$

We can use the `dsolve` command to verify that the above solution is a general solution to the given differential equation,

```
>> syms x y
>> dsolve('D2y - 3*Dy - 4*y = 3*exp(2*x) + 4*x^2
           + 2*sin(x)', 'x')

ans =

(3*x)/2 - exp(2*x)/2 + (3*cos(x))/17 -
(5*sin(x))/17 - x^2 + C27*exp(4*x) +
C28/exp(x) - 13/8
```

Since the derivative of the general solution is,

$$y' = 4C_{27}e^{4x} - C_{28}e^{-x} - e^{2x} - 2x + \frac{3}{2} - \frac{5}{17}\cos x - \frac{3}{17}\sin x$$

Then, by applying the initial conditions of $y(0) = 0$ and $y'(0) = 1$, we obtain,

$$0 = C_{27} + C_{28} - \frac{1}{2} - 0 + 0 - \frac{13}{8} + \frac{3}{17}$$

and

$$1 = 4C_{27} - C_{28} - 1 - 0 + \frac{3}{2} - \frac{5}{17} - 0$$

These two equations give $C_{27} = 373/680$ and $C_{28} = 7/5$

Hence, the exact solution of the initial value problem is,

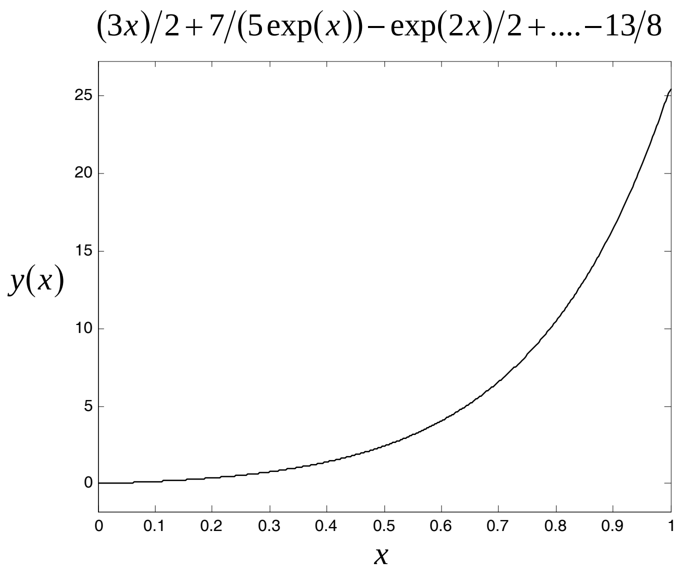
$$y = \frac{373}{680}e^{4x} + \frac{7}{5}e^{-x} - \frac{1}{2}e^{2x} - x^2 + \frac{3}{2}x - \frac{13}{8} - \frac{5}{17}\sin x + \frac{3}{17}\cos x$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y - 3*Dy - 4*y = 3*exp(2*x) + 4*x^2
   + 2*sin(x)', 'y(0) = 0', 'Dy(0) = 1', 'x')

ans =

(3*x)/2 + 7/(5*exp(x)) - exp(2*x)/2 +
(373*exp(4*x))/680 + (3*cos(x))/17 -
(5*sin(x))/17 - x^2 - 13/8
```

dsolve

The `ezplot` command can then be used to plot the variation of y that varies with x as shown in the figure.



5.8 Variation of Parameters

The method of undermined coefficients for solving the nonhomogeneous differential equation described in the preceding section works well when the right-hand-side functions $f(x)$ are in the form of exponential, polynomials, sine and cosine functions. When the functions $f(x)$ are in the other forms, it may be difficult to guess for the solutions. In this section, we will learn another method so called the *variation of parameters* to find the particular solutions. The advantage of this latter method is that we do not have to guess the solutions in some specific forms as we did in the method of undermined coefficients. However, since the method involves integration, it may be difficult if the functions $f(x)$ are complicated.

We will learn this method by considering the same form of the second-order nonhomogeneous differential equation used earlier,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

Its general solution consists of two parts,

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the solution of the homogeneous differential equation,

$$\frac{d^2y_h}{dx^2} + a\frac{dy_h}{dx} + by_h = 0$$

which can be written in the form,

$$y_h(x) = Ay_1(x) + By_2(x)$$

where A and B are constants.

The idea of the method of variation of parameters is to assume the particular solution $y_p(x)$ in the same form as the homogeneous solution $y_h(x)$, i.e.,

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

where $v_1(x)$ and $v_2(x)$ are the functions to be determined so that $y_p(x)$ satisfies the above nonhomogeneous differential equation.

By taking the first-order derivative of the particular solution y_p above, the result contains four terms,

$$y_p' = (v_1'y_1 + v_2'y_2) + (v_1y_1' + v_2y_2')$$

This means if we determine its second-order derivative, the result will contain many more terms. Furthermore, the second-order derivative also includes the terms such as v_1'' and v_2''. To avoid this, let us assume the value in the first bracket of y_p' to be zero,

$$v_1'y_1 + v_2'y_2 = 0$$

So that the expression of y_p' reduces to,

$$y_p' = v_1y_1' + v_2y_2'$$

Then, if we take derivative of this reduced form of y_p', we get,

$$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''$$

By substituting the expressions of y_p, y_p' and y_p'' into the differential equation, we obtain,

$$\begin{aligned} f(x) &= y_p'' + ay_p' + by_p \\ &= (v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y'') + a(v_1y_1' + v_2y_2') + b(v_1y_1 + v_2y_2) \\ &= (v_1'y_1' + v_2'y_2') + v_1(y_1'' + ay_1' + by_1) + v_2(y_2'' + ay_2' + by_2) \\ &= v_1'y_1' + v_2'y_2' + 0 + 0 \end{aligned}$$

i.e.,
$$v_1'y_1' + v_2'y_2' = f$$

In conclusion, the process above leads to two equations of,

$$y_1v_1' + y_2v_2' = 0$$

and
$$y_1'v_1' + y_2'v_2' = f$$

with the two unknown functions of v_1' and v_2'. These two functions can be determined by using the Cramer's rule to give,

$$v_1' = \frac{-fy_2}{y_1y_2' - y_1'y_2}$$

and
$$v_2' = \frac{fy_1}{y_1y_2' - y_1'y_2}$$

Then, the two functions of v_1 and v_2 can be obtained by performing integration,

$$v_1 = \int \frac{-fy_2}{y_1y_2' - y_1'y_2}\,dx$$

and
$$v_2 = \int \frac{fy_1}{y_1y_2' - y_1'y_2}\,dx$$

From the process explained above, we can see that the method of variation of parameters can provide the particular solution directly. Since the process involves integration to find the functions v_1 and v_2, difficulty may arise if the function $f(x)$ on the right-hand-side of the differential equation is complicated. We will learn how to use the method of variation of parameters to solve the nonhomogeneous differential equations through the following examples.

<u>Example</u> Use the method of variation of parameters to solve the second-order nonhomogeneous differential equation,

$$\frac{d^2y}{dx^2} + y = \tan x$$

We can employ the technique presented in the preceding sections to derive the homogeneous solution of the homogeneous differential equation,

$$\frac{d^2y_h}{dx^2} + y_h = 0$$

to get, $$y_h = C_{29}\cos x + C_{30}\sin x$$

where C_{29} and C_{30} are constants. This homogeneous solution can also be obtained conveniently by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y + y = 0', 'x')

ans =

C29*cos(x) + C30*sin(x)
```

dsolve

It is noted that the homogeneous solution obtained above is in the form,

$$y_h = C_{29}y_1 + C_{30}y_2$$

i.e., $$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x$$

Since the particular solution for the method of variation of parameters is assumed in the form,

$$y_p = v_1y_1 + v_2y_2$$

Thus, $$y_p = v_1\cos x + v_2\sin x$$

where the functions v_1 and v_2 are determined from,

$$v_1 = \int \frac{-f\,y_2}{y_1y_2' - y_1'y_2}\,dx \quad \text{and} \quad v_2 = \int \frac{f\,y_1}{y_1y_2' - y_1'y_2}\,dx$$

Therefore, by substituting the function $f(x)$ and performing integration, we obtain the two functions v_1 and v_2 as follows,

$$
\begin{aligned}
v_1 &= \int \frac{-(\tan x)(\sin x)}{(\cos x)(\cos x)-(-\sin x)(\sin x)}\,dx \\
&= \int \frac{-(\tan x)(\sin x)}{\sin^2 x + \cos^2 x}\,dx \\
&= -\int (\tan x)(\sin x)\,dx \\
v_1 &= \ln\left(\tan\left(\frac{x}{2}\right)-1\right) - \ln\left(\tan\left(\frac{x}{2}\right)+1\right) + \sin x
\end{aligned}
$$

and

$$
\begin{aligned}
v_2 &= \int \frac{(\tan x)(\cos x)}{(\cos x)(\cos x)-(-\sin x)(\sin x)}\,dx \\
&= \int \sin x\,dx \\
v_2 &= -\cos x
\end{aligned}
$$

Thus, the particular solution is,

$$
\begin{aligned}
y_p &= \left[\ln\left(\tan\left(\frac{x}{2}\right)-1\right) - \ln\left(\tan\left(\frac{x}{2}\right)+1\right) + \sin x\right]\cos x - \sin x \cos x \\
&= \left[\ln\left(\tan\left(\frac{x}{2}\right)-1\right) - \ln\left(\tan\left(\frac{x}{2}\right)+1\right)\right]\cos x
\end{aligned}
$$

Hence, the general solution of the given nonhomogeneous differential equation is,

$$y = y_h + y_p$$

$$y = C_{29}\cos x + C_{30}\sin x + \left[\ln\left(\tan\left(\frac{x}{2}\right)-1\right) - \ln\left(\tan\left(\frac{x}{2}\right)+1\right)\right]\cos x$$

The same solution can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y + y = tan(x)', 'x')

ans =

log(tan(x/2)-1)*cos(x) - log(tan(x/2)+1)*cos(x) +
C29*cos(x) + C30*sin(x)
```

Example Use the method of variation of parameters to solve the second-order nonhomogeneous differential equation,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4x^2$$

The homogeneous solution can be derived from the homogeneous differential equation,

$$\frac{d^2y_h}{dx^2} - \frac{dy_h}{dx} - 2y_h = 0$$

to get,

$$y_h = C_{31}e^{2x} + C_{32}e^{-x}$$

where C_{31} and C_{32} are constants. Such homogeneous solution can also be found by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y - Dy - 2*y = 0', 'x')

ans =

C31*exp(2*x) + C32/exp(x)
```

dsolve

This homogeneous solution is in the form of,

$$y_h = C_{31}y_1 + C_{32}y_2$$

i.e., $y_1 = e^{2x}$ and $y_2 = e^{-x}$

In the method of variation of parameters, the particular solution is assumed in the same form as the homogeneous solution,

$$y_p = v_1y_1 + v_2y_2$$

Thus,

$$y_p = v_1e^{2x} + v_2e^{-x}$$

Then, the functions v_1 and v_2 can be determined as follows,

$$\begin{aligned} v_1 &= \int \frac{-f\,y_2}{y_1y_2' - y_1'y_2}\,dx \\ &= \int \frac{-(4x^2)(e^{-x})}{(e^{2x})(-e^{-x}) - (2e^{2x})(e^{-x})}\,dx \\ &= \int \frac{4}{3}x^2e^{-2x}\,dx \end{aligned}$$

$$v_1 = -\frac{e^{-2x}}{3}\left(2x^2 + 2x + 1\right)$$

and

$$\begin{aligned} v_2 &= \int \frac{f\, y_1}{y_1 y_2' - y_1' y_2}\, dx \\ &= \int \frac{(4x^2)(e^{2x})}{(e^{2x})(-e^{-x}) - (2e^{2x})(e^{-x})}\, dx \\ &= -\int \frac{4}{3} x^2 e^x\, dx \\ v_2 &= -\frac{4e^x}{3}\left(x^2 - 2x + 2\right) \end{aligned}$$

Hence, the general solution of the given nonhomogeneous differential equation is,

$$\begin{aligned} y &= y_h + y_p \\ &= C_{31}e^{2x} + C_{32}e^{-x} - \frac{e^{-2x}}{3}\left(2x^2 + 2x + 1\right)e^{2x} - \frac{4e^x}{3}\left(x^2 - 2x + 2\right)e^{-x} \\ y &= C_{31}e^{2x} + C_{32}e^{-x} - 2x^2 + 2x - 3 \end{aligned}$$

The same solution can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y - Dy - 2*y = 4*x^2', 'x')    dsolve

ans =

2*x - 2*x^2 + C31*exp(2*x) + C32/exp(x) - 3
```

<u>Example</u> Use the method of variation of parameters to solve the initial value problem governed by the second-order nonhomogeneous differential equation,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (x+1)e^{2x}$$

with the initial conditions of $y(0) = 0$ and $y'(0) = 0$. Plot the solution of *y* that varies with *x* in the interval of $0 \le x \le 1$.

The homogeneous solution of the homogeneous differential equation,

$$\frac{d^2y_h}{dx^2} - 4\frac{dy_h}{dx} + 4y_h = 0$$

is,

$$y_h = C_{33}e^{2x} + C_{34}xe^{2x}$$

where C_{33} and C_{34} are constants. The homogeneous solution above is in the form of,

$$y_h = C_{33}y_1 + C_{34}y_2$$

Then,

$$y_1 = e^{2x} \quad \text{and} \quad y_2 = xe^{2x}$$

In the method of variation of parameters, the particular solution is assumed in the same form as the homogeneous solution,

$$y_p = v_1y_1 + v_2y_2$$

Thus,

$$y_p = v_1e^{2x} + v_2xe^{2x}$$

where the functions v_1 and v_2 are determined by integrating the functions that are in the form of $f(x)$. Here,

$$\begin{aligned} v_1 &= \int \frac{-f\,y_2}{y_1y_2' - y_1'y_2}\,dx \\ &= \int \frac{-(x+1)e^{2x}(xe^{2x})}{(e^{2x})(e^{2x}+2xe^{2x}) - (2e^{2x})(xe^{2x})}\,dx \\ v_1 &= -\frac{x^3}{3} - \frac{x^2}{2} \end{aligned}$$

and

$$\begin{aligned} v_2 &= \int \frac{f\,y_1}{y_1y_2' - y_1'y_2}\,dx \\ &= \int \frac{(x+1)e^{2x}(e^{2x})}{(e^{2x})(e^{2x}+2xe^{2x}) - (2e^{2x})(xe^{2x})}\,dx \\ v_2 &= \frac{x^2}{2} + x \end{aligned}$$

Hence, the general solution of the given nonhomogeneous differential equation is,

$$y = y_h + y_p$$

$$y = C_{33}e^{2x} + C_{34}xe^{2x} - \left(\frac{x^3}{3}+\frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2}+x\right)xe^{2x}$$

Or, $$y = e^{2x}\left(C_{33} + xC_{34} + \frac{x^3}{6}+\frac{x^2}{2}\right)$$

and then, $$y' = e^{2x}\left(2C_{33} + C_{34} + 2xC_{34} + \frac{x^3}{3}+\frac{3x^2}{2}+x\right)$$

where C_{33} and C_{34} are constants that can be determined from the initial conditions as follows,

$$y(0) = 0 = (1)(C_{33}+0+0+0)$$

$$y'(0) = 0 = (1)(2C_{33}+C_{34}+0+0+0+0)$$

which give $C_{33}=0$ and $C_{34}=0$. Hence, the solution of this initial value problem is,

$$y = e^{2x}\left(\frac{x^3}{6}+\frac{x^2}{2}\right) = \frac{x^2e^{2x}}{6}(x+3)$$

The same solution is obtained by using the `dsolve` command by entering,

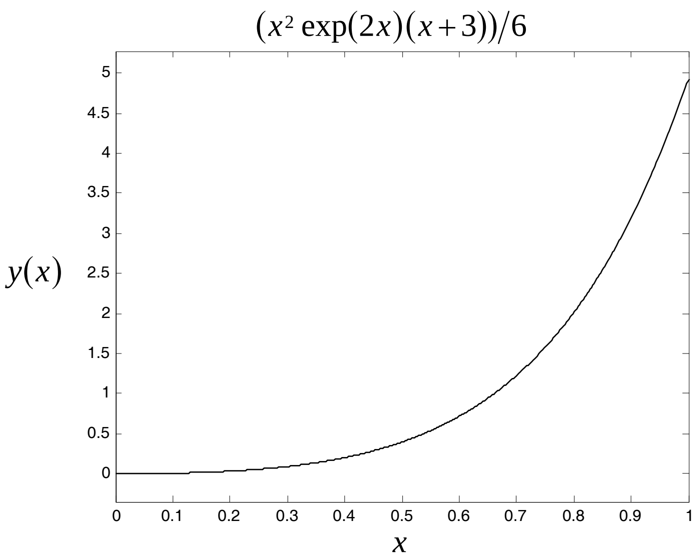
```
>> syms x y
>> dsolve('D2y - 4*Dy + 4*y = (x+1)*exp(2*x)',
          'y(0)=0', 'Dy(0)=0', 'x')
ans =

x^2*exp(2*x)*(x + 3))/6
```

dsolve

The `ezplot` command is then used to plot the variation of y that varies with x within the interval of $0 \le x \le 1$ as shown in the figure.

5.9 Numerical Methods

For most of the initial value problems, their exact solutions in closed-form expressions cannot be derived easily. This is because the function $f(x)$ on the right-hand-side of the nonhomogeneous differential equation is often complicated. Furthermore, if the coefficients of the derivative terms on the left-



hand side of the equation are function of y, the differential equation becomes nonlinear. Exact closed-form solutions for most nonlinear differential equations are not available. Thus, the numerical methods are needed to provide approximate solutions.

In this section, we will use the `ode23` and `ode45` commands to find approximate solutions of the second-order differential equations. The commands employ the Runge-Kutta method with variable time-stepping to provide accurate solutions. The following examples demonstrate how to use these commands to solve general initial value problems.

Example Employ the `ode23` command to solve the initial value problem that is governed by the second-order nonhomogeneous differential equation,

$$36\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 37y = 0 \qquad 0 \le x \le 10$$

with the initial conditions of $y(0) = 0$ and $y'(0) = 1$.

It is noted that this initial value problem has the exact solution of,

$$y = e^{x/6}\sin x$$

which can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('36*D2y - 12*Dy + 37*y = 0',
          'y(0) = 0', 'Dy(0) = 1', 'x')
```

dsolve

```
ans =

exp(x/6)*sin(x)
```

If the exact solution is not available, we can use the numerical methods to solve for the approximate solution. In the preceding chapter, we learned how to use the numerical methods to solve the first-order differential equation. We can follow the same procedure but we have to firstly separate the second-order differential equation into two first-order differential equations. We start from writing the given second-order differential equation in the form,

$$36\frac{d^2y_1}{dx^2} - 12\frac{dy_1}{dx} + 37y_1 \quad = \quad 0 \qquad\qquad 0 \le x \le 10$$

with the initial conditions of,

$$y_1(0) \quad = \quad 0 \qquad \text{and} \qquad \frac{dy_1}{dx}(0) \quad = \quad y_2(0) \quad = \quad 1$$

i.e., we assign the new variables $y_1 = y$ and $y_2 = \dfrac{dy}{dx}$.

Then, the second-order differential equation above becomes the two first-order differential equations as follows,

$$\frac{dy_1}{dx} \quad = \quad y_2$$

and

$$\frac{dy_2}{dx} \quad = \quad (12y_2 - 37y_1)/36$$

To conveniently solve these two first-order differential equations simultaneously, a MATLAB m-file should be created. As an example, an m-file with the name of `example1.m` consists of the statements,

function

```
function yex1 = example1(x,y)
yex1 = [y(2); (12*y(2)-37*y(1))/36];
```

The first statement defines the function name while the second statement contains descriptions of the two first-order differential equations. We then use the `ode23` command to solve this problem by typing on the Command Window as follows,

ode23

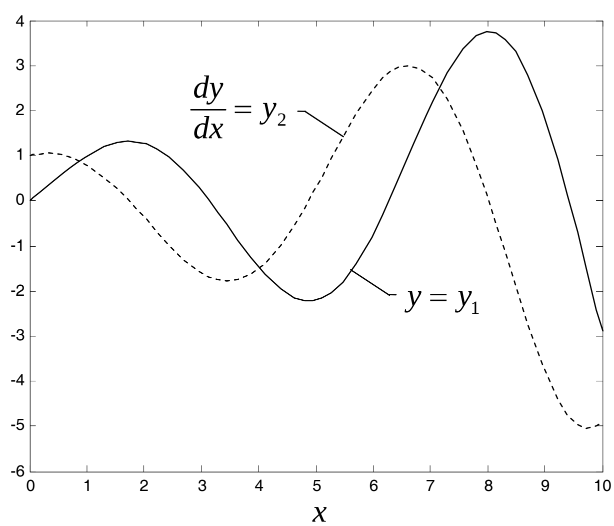
```
>> [x,y] = ode23('example1', [0 10], [0 1]);
```

MATLAB will determine the values of y_1 and y_2 at different x locations, and print the values of x, y_1 and y_2 on the monitor screen. In the `ode23` command above, the numbers in the first square bracket denote the interval of $0 \le x \le 10$, while the numbers in the second square bracket represent the initial conditions of $y_1(0) = 0$ and $y_2(0) = 1$, respectively.

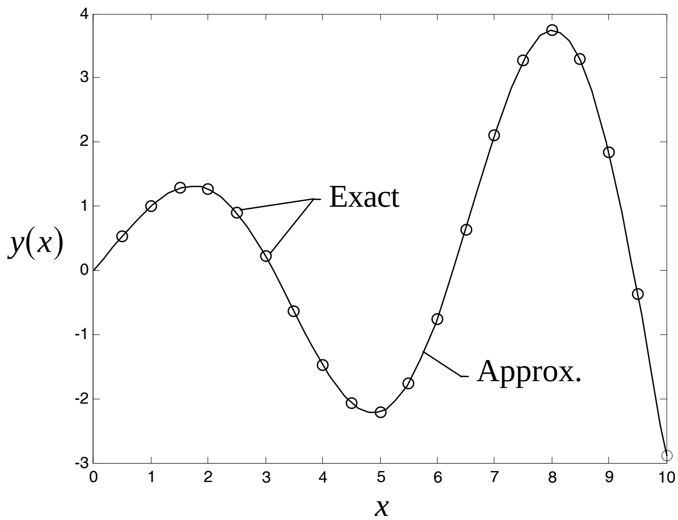
The computed solutions of y_1 and y_2 that vary with x can be plotted by using the `plot` command,

```
>> plot(x,y(:,1),'-k',x,y(:,2),'--k')
```

plot



The computed solution y_1 from the `ode23` command is compared with the exact solution as shown in the figure. The figure (scale is enlarged) indicates that the numerical method using the `ode23` command can provide very accurate solution.



Example Employ the `ode45` command to solve the initial value problem governed by the second-order nonhomogeneous differential equation,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 20\cos x \qquad 0 \le x \le 12$$

with the initial conditions of $y(0) = 0$ and $y'(0) = 0$.

The `ode45` command uses the combined fourth- and fifth-order Runge-Kutta method to solve for approximate solution of the first-order differential equation. To apply the command, we first write the given second-order differential equation in the form of the unknown variable y_1 as,

$$\frac{d^2y_1}{dx^2} + 4\frac{dy_1}{dx} + 3y_1 = 20\cos x \qquad 0 \le x \le 12$$

This second-order differential equation can be separated into two first-order differential equations as follows,

$$\frac{dy_1}{dx} = y_2$$

and

$$\frac{dy_2}{dx} = 20\cos x - 4y_2 - 3y_1$$

together with the initial conditions of,

$$y_1(0) = 0 \quad \text{and} \quad \frac{dy_1}{dx}(0) = y_2(0) = 0$$

Similar to the preceding example, we create an m-file and name it as `example2.m`,

function

```
function yex2 = example2(x,y)
yex2 = [y(2); 20*cos(x)-4*y(2)-3*y(1)];
```

We then employ the `ode45` command to solve the two first-order differential equations simultaneously by typing on the Command Window as,

ode45

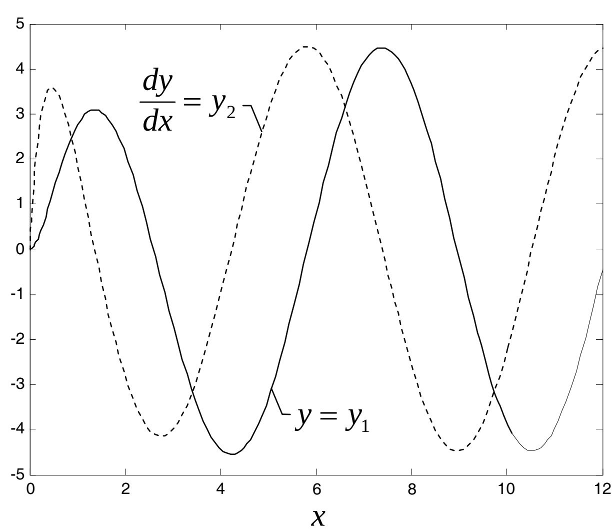
```
>> [x,y] = ode45('example2', [0 12], [0 0]);
```

In the `ode45` command above, the numbers in the first square bracket denote the interval of $0 \le x \le 12$, while the numbers in the second square bracket represent the two initial conditions of $y_1(0) = 0$ and $y_2(0) = 0$, respectively.

The computed solutions of y_1 and y_2 that vary with *x* can be plotted by using the `plot` command,

plot

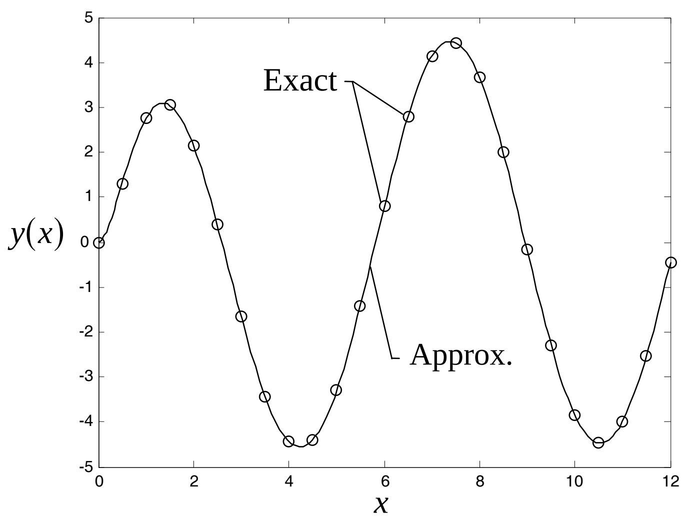
```
plot(x,y(:,1),'-k',x,y(:,2),'--k')
```



It is noted that this initial value problem has exact solution which can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y+4*Dy+3*y = 20*cos(x)', 'y(0) = 0',
          'Dy(0) = 0', 'x')

ans =

3/exp(3*x) - 5/exp(x) + 2*cos(x) + 4*sin(x)
```

The approximate solution from the numerical method using the `ode45` command is compared with the exact solution as shown in the figure. The figure indicates that the numerical method can provide high solution accuracy as compared to the exact solution. The method will be useful when the exact solution to the differential equation is not available as demonstrated in the following example.



Example Employ the `ode45` command to find the approximate solution of the initial value problem governed by the second-order nonlinear differential equation,

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0 \qquad\qquad 0 \le x \le 20$$

with the initial conditions of $y(0) = 1$ and $y'(0) = 0$.

For this problem, the exact solution could not be found by using the `dsolve` command,

dsolve

```
>> syms x y
>> dsolve('D2y + y*Dy + y = 0', 'y(0) = 1',
          'Dy(0) = 0', 'x')
Warning: Explicit solution could not be found.
> In dsolve at 101

ans =

[ empty sym ]
```

We will use the `ode45` command to determine the approximate solution. Similar to the preceding examples, we start from writing the differential equation in the form of the unknown variable y_1 as,

$$\frac{d^2 y_1}{dx^2} + y_1 \frac{dy_1}{dx} + y_1 = 0 \qquad 0 \le x \le 20$$

Then, we separate this second-order differential equation into two first-order differential equations as,

$$\frac{dy_1}{dx} = y_2$$

and

$$\frac{dy_2}{dx} = -y_1 y_2 - y_1$$

together with the initial conditions of,

$$y_1(0) = 1 \quad \text{and} \quad y_2(0) = 0$$

We then create an m-file, `example3.m`, which defines the two first-order differential equations,

function

```
function yex3 = example3(x,y)
yex3 = [y(2); -y(1)*y(2) - y(1)];
```

To solve the problem, we use the `ode45` command by typing on the Command Window as follows,

ode45

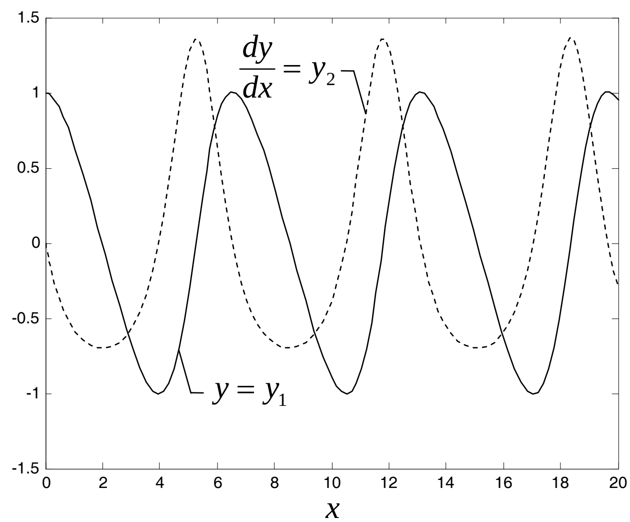
```
>> [x,y] = ode45('example3', [0 20], [1 0]);
```

In the `ode45` command above, the numbers in the first square bracket denote the interval of $0 \le x \le 20$, while the numbers in the second square bracket represent the two initial conditions of $y_1(0) = 1$ and $y_2(0) = 0$, respectively.

The computed solutions of y_1 and y_2 that vary with x can be plotted by using the `plot` command as shown in the figure.

```
>> plot(x,y(:,1),'-k',x,y(:,2),'--k')
```

plot



5.10 Concluding Remarks

In this chapter, several methods for solving the second-order linear differential equations with constant coefficients were presented. For homogeneous differential equations, the exact solutions depend on the characteristic equations. The roots of the characteristic equations could be distinct real numbers, repeated real numbers or conjugate complex numbers. For nonhomogeneous differential equations, two methods were explained. These are the methods of undetermined coefficients and variation of parameters. The method of undetermined coefficients assumes the particular solution in the same form as the nonhomogeneous function. The method works well only when the nonhomogeneous functions are in the forms of exponential, polynomials, sine and cosine functions. Thus, for more general nonhomogeneous functions, the method of variation of parameters should be used. This later method provides the particular solutions directly. However, since the method involves integration of the nonhomogeneous function, difficulty may arise if such function is complicated.

Several examples have been employed to show detailed derivation of the exact solutions by using the above methods. In all examples, the derived exact solutions to the differential equations were verified by using the MATLAB commands. Variations of the solutions were plotted to help understanding of their physical meanings. Numerical methods for solving these second-order linear differential equations were also introduced. Accurate approximate solutions were obtained by using MATLAB commands. The main advantage of these commands is that they can provide approximate solutions when the exact closed-form solutions are not available. Thus, for a given differential equation, finding exact closed-form solution should be tried first. If the exact solution is not available, the numerical method is an effective tool to provide approximate solution. Examples have shown that the MATLAB commands can provide approximate solutions with high accuracy and efficiency.

Exercises

1. In each problem below, show that,

   (a) $y = e^{-2x} + 3e^{-4x}$ is the solution of
   $y'' + 6y' + 8y = 0$

   (b) $y = 2e^{5x} + 4e^{3x}$ is the solution of
   $y'' - 8y' + 15y = 0$

(c) $y \;=\; e^{x/6} + 7e^{-x}$ is the solution of
$6y'' + 5y' - y \;=\; 0$

(d) $y \;=\; 2e^{x/3} + 5e^{-4x}$ is the solution of
$3y'' + 11y' - 4y \;=\; 0$

(e) $y \;=\; e^{9x/8} + e^{-7x/3}$ is the solution of
$24y'' + 29y' - 63y \;=\; 0$

2. Find the determinant (Wronskian) of each solution in Problem 1. Then, verify these determinants by using the `det` command.

3. Employ the `diff` command to show that,

$$y \;=\; 3e^{-2x} - 7xe^{-2x}$$

is the exact solution of the second-order homogeneous differential equation,

$$y'' + 4y' + 4y \;=\; 0$$

Then, determine its determinant by using the `det` command.

4. Employ the `diff` command to show that,

$$y \;=\; e^{x}\cos 2x + e^{x}\sin 2x$$

is the exact solution of the second-order homogeneous differential equation,

$$y'' - 2y' + 5y \;=\; 0$$

Then, determine its determinant by using the `det` command.

5. Find the second-order homogeneous differential equations that correspond to the following solutions,

(a) $e^{x} + 2e^{2x}$ (b) $3e^{-x} + e^{3x}$

(c) $3 + 5e^{4x}$ (d) $7e^{-x} + 8e^{-4x}$

(e) $e^{3x} + 2e^{5x}$ (f) $2e^{2x} + 3e^{-3x}$

6. Solve the following second-order homogeneous differential equations when their roots are distinct real numbers,

(a) $y'' + y' - 12y = 0$

(b) $y'' + 2y' - 35y = 0$

(c) $6y'' - 5y' - 4y = 0$

(d) $15y'' - 23y' - 22y = 0$

(e) $20y'' + 73y' + 63y = 0$

Show the derivation of their solutions in detail. Then, use the `dsolve` command to verify the derived solutions.

7. Solve the following initial value problems that are governed by the second-order homogeneous differential equations and initial conditions. The characteristic equations of these differential equations contain the roots that are distinct real numbers.

   (a) $y'' - y' - 6y = 0$
   $y(0) = 0\,, \quad y'(0) = 1$

   (b) $2y'' + 3y' - 5y = 0$
   $y(0) = 1\,, \quad y'(0) = 0$

   (c) $9y'' + 3y' - 20y = 0$
   $y(0) = 0\,, \quad y'(0) = 2$

   (d) $7y'' - 24y' - 31y = 0$
   $y(0) = \pi\,, \quad y'(0) = 2\pi$

   Show the derivation of their solutions in detail. Then, use the `dsolve` command to verify the derived solutions and the `ezplot` command to plot *y* that varies with *x*.

8. Solve the following second-order homogeneous differential equations when their roots are repeated real numbers,

   (a) $y'' - 10y' + 25y = 0$

   (b) $4y'' + 4y' + y = 0$

   (c) $9y'' - 12y' + 4y = 0$

   (d) $4y'' + 28y' + 49y = 0$

   (e) $25y'' - 30y' + 9y = 0$

Show the derivation of their solutions in detail. Then, use the `dsolve` command to verify the derived solutions.

9. Solve the following initial value problems that are governed by the second-order homogeneous differential equations and initial conditions. The characteristic equations of these differential equations contain the roots that are repeated real numbers.

   (a) $y'' + 6y' + 9y = 0$
   $y(0) = 0\,,\quad y'(0) = 2$

   (b) $4y'' - 28y' + 49y = 0$
   $y(0) = 1\,,\quad y'(0) = 0$

   (c) $16y'' - 8y' + y = 0$
   $y(0) = 0\,,\quad y'(0) = 1$

   (d) $49y'' + 28y' + 4y = 0$
   $y(0) = 1\,,\quad y'(0) = 0$

   Show the derivation of their solutions in detail. Then, use the `dsolve` command to verify the derived solutions.

10. Solve the following second-order homogeneous differential equations when their roots are conjugate complex numbers,

    (a) $y'' - 2y' + 6y = 0$

    (b) $y'' + 6y' + 13y = 0$

    (c) $y'' + 4y' + 5y = 0$

    (d) $y'' + 3y' + 18y = 0$

    (e) $y'' - 6y' + 10y = 0$

    Show the derivation of their solutions in detail. Then, use the `dsolve` command to verify the derived solutions.

11. Solve the following initial value problems that are governed by the second-order homogeneous differential equations and initial conditions. The characteristic equations of these differential equations contain the roots that are conjugate complex numbers.

(a) $y'' + 4y' + 5y = 0$
$y(0) = 1 , \quad y'(0) = 0$

(b) $y'' - 6y' + 45y = 0$
$y(0) = 0 , \quad y'(0) = 2$

(c) $y'' - 6y' + 13y = 0$
$y(\pi/2) = 0 , \quad y'(\pi/2) = 2$

(d) $y'' - y' + y = 0$
$y(0) = 1 , \quad y'(0) = 4$

Show the derivation of their solutions in detail. Then, use the `dsolve` command to verify the derived solutions and the `ezplot` command to plot *y* that varies with *x*.

12. Employ the `dsolve` command to solve the second-order homogeneous differential equations with constant coefficients as follows,

(a) $y'' - 4y' + 4y = 0$

(b) $y'' - 9y' + 20y = 0$

(c) $4y'' - 12y' + 9y = 0$

(d) $y'' + y' = 0$

(e) $y'' - 6y' + 25y = 0$

In each problem, verify the obtained solution with that derived by hands. Or substitute it into the differential equation to check whether the differential equation is satisfied.

13. Employ the `dsolve` command to solve the initial value problems governed by the second-order differential equations and initial conditions below,

(a) $y'' - 6y' + 9y = 0$
$y(0) = 0 , \quad y'(0) = 3$

(b) $y'' - 5y' + 6y = 0$
$y(1) = 1, \quad y'(1) = 2$

(c) $y'' + 4y' + 5y = 0$
$y(0) = 1 , \quad y'(0) = 0$

(d) $y'' + 8y' - 9y = 0$
$y(1) = 2, \quad y'(1) = 0$

In each case, verify that the solution satisfies the differential equation and initial conditions.

14. Use the method of undetermined coefficients to solve the second-order nonhomogeneous differential equations,

(a) $y'' - y' - 2y = 2x^2$

(b) $y'' - 5y' + 6y = e^{-x} + \cos x$

(c) $y'' - 6y' + 9y = \sin(3x)$

(d) $y'' + 4y' + 4y = 3e^{x}$

(e) $4y'' - 8y' + 3y = x^2 + \sin x$

Show the derivation in detail. Compare the derived solutions with those obtained by using the `dsolve` command.

15. Use the method of undetermined coefficients to solve the initial value problems that are governed by the second-order nonhomogeneous differential equations and initial conditions,

(a) $9y'' + 3y' - 20y = \sin x + \cos x$
$y(0) = 1, \quad y'(0) = 0$

(b) $16y'' - 8y' + y = x^2 + 2x$
$y(0) = 0, \quad y'(0) = 1$

(c) $y'' + 4y = \cos(2x)$
$y(0) = 0, \quad y'(0) = 0$

(d) $y'' - 6y' + 13y = e^{x}$
$y(0) = 1, \quad y'(0) = 0$

Show detailed derivation of the exact solutions. Then, verify these solutions with those obtained from using the `dsolve` command. In each case, employ the `ezplot` command to plot the variation of y that varies with x.

16. Solve the following second-order nonhomogeneous differential equations by the method of variation of parameters. Show the derivation of their solutions in detail. Then, verify the solutions with those obtained by using the `dsolve` command.

    (a) $y'' + 4y = \cot x$

    (b) $y'' - y = \cosh x$

    (c) $y'' + 2y' + y = e^{-x} \ln x$

    (d) $y'' + 3y' + 2y = \sin e^{x}$

    (e) $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$

17. Solve the following initial value problems by using the method of variation of parameters,

    (a) $y'' - y = xe^{x/2}$
    $y(0) = 1, \quad y'(0) = 0$

    (b) $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$
    $y(0) = 1, \quad y'(0) = 0$

    (c) $y'' - 4y' + 4y = (6x^2 - 3x)e^{2x}$
    $y(0) = 1, \quad y'(0) = 0$

    (d) $y'' - 4y' + 5y = \sin x + 3x$
    $y(0) = 1, \quad y'(0) = 0$

    Show detailed derivation of the exact solutions. Compare these exact solutions with those obtained by using the `dsolve` command. Then, employ the `ezplot` command to plot their variations for $0 \le x \le 1$.

18. Use the `ode23` command to find the approximate solution of the initial value problem governed by the second-order homogeneous differential equation,

    $$y'' + 5y' + 6y = 0 \qquad\qquad 0 \le x \le 2$$

    with the initial conditions of $y(0) = 2$ and $y'(0) = 5$. Repeat solving the problem but by using the `dsolve` command to find the exact solution. Plot to compare the two solutions for $0 \le x \le 2$.

19. Use the `ode23` command to find the approximate solution of the initial value problem governed by the second-order nonhomogeneous differential equation,

$$y'' - 3y' + 2y \quad = \quad e^{3x} \qquad\qquad 0 \le x \le 2$$

with the initial conditions of $y(0) = 1$ and $y'(0) = -1$. Then, use the `dsolve` command to find the exact solution. Plot to compare the two solutions for $0 \le x \le 2$.

20. Use the `ode45` command to find the approximate solution of the initial value problem governed by the second-order nonhomogeneous differential equation,

$$y'' + 2y' + 5y \quad = \quad 4e^{-x}\cos(2x) \qquad 0 \le x \le 5$$

with the initial conditions of $y(0) = 1$ and $y'(0) = 0$. Repeat solving the problem but by using the `dsolve` command to find the exact solution. Plot to compare the two solutions for $0 \le x \le 5$.

21. Use the `ode45` command to find the approximate solution of the initial value problem governed by the second-order nonhomogeneous differential equation,

$$4y'' - y \quad = \quad x - 2 - 5\cos x - e^{-x/2} \qquad 0 \le x \le 5$$

with the initial conditions of $y(0) = 2$ and $y'(0) = 1$. Repeat solving the problem but by using the `dsolve` command to find the exact solution. Plot to compare the two solutions for $0 \le x \le 5$.

22. Use the `ode45` command to solve for the approximate solution of the Bessel equation,

$$x^2y'' + xy' + x^2y \quad = \quad 0 \qquad\qquad 1 \le x \le 6$$

with the initial conditions of $y(1)=0$ and $y'(1)=1$. Plot this approximate solution for the interval of $1 \le x \le 6$. Then, check whether the `dsolve` command can provide exact solution. If it can, plot to compare the two solutions.

23. Employ the `ode23` command to find the approximate solution of the second-order nonlinear equation,

$$y'' - y^4 = 0 \qquad 0 \le x \le 2$$

with the initial conditions of $y(0)=1$ and $y'(0)=-3$. Plot this approximate solution for the interval of $0 \le x \le 2$. Then, check whether the `dsolve` command can provide exact solution. If the exact solution can be obtained, plot to compare the two solutions.

24. Employ the `ode45` command to solve the initial value problem that is governed by the second-order nonlinear differential equation,

$$y'' - (1-y^2)y' + y = 0 \qquad 0 \le x \le 20$$

with the initial conditions of $y(0)=1$ and $y'(0)=1$. Plot the computed variations of $y(x)$ and $y'(x)$ in the interval of $0 \le x \le 20$. Then, use the `ode23s` command to solve this problem again when the differential equation changes slightly to,

$$y'' - 1000(1-y^2)y' + y = 0 \qquad 0 \le x \le 6000$$

Plot the solution of $y(x)$ in the interval of $0 \le x \le 6000$. Provide comments on the accuracy of the approximate solution $y(x)$ from its abrupt change within a small interval of *x*.

Chapter 6

Higher-Order Linear Differential Equations

6.1 Introduction

Once we understand how to solve the second-order differential equations explained in the preceding chapter, we can follow the same procedure to solve the higher-order differential equations. The higher-order linear differential equations arise in many scientific and engineering problems, such as the deflection of a beam under loading and the laminar flow behavior over a flat plate. Exact solutions obtained from solving these differential equations help us to understand more about the problem behaviors.

We will start by solving the higher-order homogeneous differential equations with constant coefficients. The roots of their characteristic equations could be distinct real, repeated real, complex conjugate, or mixed numbers. Exact solutions obtained from different types of these roots are in different forms. After that, we will extend the idea to solve the nonhomogeneous differential equations. In all cases, we will verify the derived solutions by using the MATLAB commands.

For more complicated differential equations, their exact solutions may not be derived in closed-form expressions. In these cases, we will employ MATLAB commands that use numerical methods to solve for approximate solutions. Plotting commands will also be used to display variations of the computed solutions. The techniques presented in this chapter will help us to understand clearly on how to solve the higher-order differential equations.

6.2 Homogeneous Equations with Constant Coefficients

General form of the n^{th}-order homogeneous linear differential equation with constant coefficients is,

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots.. + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y \quad = \quad 0$$

where a_n, a_{n-1}, ..., a_2, a_1 and a_0 are the constant coefficients while the right-hand-side of the equation is zero. A solution of the above differential equation is in the form of $e^{\lambda x}$ where λ is a number. We can prove this by substituting it into the differential equation to get,

$$\left(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots\ldots.. + a_2 \lambda^2 + a_1 \lambda + a_0\right) e^{\lambda x} \quad = \quad 0$$

After dividing through by $e^{\lambda x}$, the characteristic equation is obtained,

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots\ldots.. + a_2 \lambda^2 + a_1 \lambda + a_0 \quad = \quad 0$$

which can be solved for n values of the root λ.

For example, the third-order homogeneous differential equation,

$$\frac{d^3 y}{dx^3} - 6 \frac{d^2 y}{dx^2} + 11 \frac{dy}{dx} - 6y \quad = \quad 0$$

has the characteristic equation in the form,

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

or,
$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

which leads to the three values of root λ as,

$$\lambda_1 = 1, \qquad \lambda_2 = 2 \qquad \text{and} \qquad \lambda_3 = 3$$

It is noted that MATLAB contains the `roots` command that can be used to factorize the algebraic equation above conveniently,

roots

```
>> roots([1 -6 11 -6])

ans =

    3.0000
    2.0000
    1.0000
```

Thus, the exact solution of the third-order differential equation is,

$$y(x) = e^x + e^{2x} + e^{3x}$$

We can verify this exact solution by taking their derivatives and substituting them into the left-hand-side of the differential equation,

$$\frac{dy}{dx} = e^x + 2e^{2x} + 3e^{3x}$$
$$\frac{d^2y}{dx^2} = e^x + 4e^{2x} + 9e^{3x}$$
$$\frac{d^3y}{dx^3} = e^x + 8e^{2x} + 27e^{3x}$$

to get,

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = \left(e^x + 8e^{2x} + 27e^{3x}\right)$$
$$-6\left(e^x + 4e^{2x} + 9e^{3x}\right) + 11\left(e^x + 2e^{2x} + 3e^{3x}\right) - 6\left(e^x + e^{2x} + e^{3x}\right) = 0$$

The result is zero which is equal to the right-hand-side of the equation. It is also noted that the `diff` command can be used to find the derivatives of the solution easily as follows,

syms

```
>> syms x y
>> y = exp(x) + exp(2*x) + exp(3*x);
>> LHS = diff(y,x,3) - 6*diff(y,x,2) +
         11*diff(y,x) - 6*y

LHS =

0
```

The general solution $y(x)$ contains the three functions of $y_1 = e^x$, $y_2 = e^{2x}$ and $y_3 = e^{3x}$ which are linearly independent. This can be verified by finding the determinant (Wronskian) that must not be zero as follows,

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x}$$

The determinant above can be determined conveniently by using the `det` command,

det

```
>> W=[y1 y2 y3; diff(y1,x) diff(y2,x) diff(y3,x);
   diff(y1,x,2) diff(y2,x,2) diff(y3,x,2)];
>> det(W)

ans =

2*exp(6*x)
```

The roots λ from the characteristic equation may be distinct real, repeated real or conjugate complex roots which lead to different forms of the solutions. We will learn how to derive the solutions according to different types of the roots λ in the following sections.

6.3 Solutions from Distinct Real Roots

We will use the following examples to show the derivation of exact solutions for the higher-order differential equations when roots of their characteristic equations are distinct real numbers.

Example Find solution of the third-order homogeneous differential equation,

$$12\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} - 6\frac{dy}{dx} - y = 0$$

By assuming the solution in the form of $e^{\lambda x}$ and substituting it into the differential equation, we get,

$$12\lambda^3 e^{\lambda x} - 5\lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} - e^{\lambda x} = 0$$

After dividing through by $e^{\lambda x}$, the characteristic equation in the form of an algebraic equation is obtained,

$$12\lambda^3 - 5\lambda^2 - 6\lambda - 1 = 0$$

Or,

$$(\lambda - 1)(3\lambda + 1)(4\lambda + 1) = 0$$

i.e., $\lambda_1 = 1$, $\lambda_2 = -1/3$ and $\lambda_3 = -1/4$.

The `factor` command can be used to factorize the above algebraic equation,

factor

```
>> syms lambda
>> factor(12*lambda^3- 5*lambda^2 - 6*lambda - 1)

ans =

(lambda - 1)*(3*lambda + 1)*(4*lambda + 1)
```

Thus, the general solution of the given third-order differential equation is,

$$y = C_1 e^{x} + C_2 e^{-x/3} + C_3 e^{-x/4}$$

where C_1, C_2 and C_3 are constants. The same solution is obtained by using the `dsolve` command,

```
>> dsolve('12*D3y - 5*D2y - 6*Dy - y = 0', 'x')

ans =

C1*exp(x) + C2/exp(x/3) + C3/exp(x/4)
```

Example Find solution of the fourth-order homogeneous differential equation,

$$\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = 0$$

Similar to the preceding example, we assume the solution in the form of $e^{\lambda x}$ and substitute it into the differential equation to obtain the characteristic equation,

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

Or,
$$(\lambda - 1)(\lambda + 1)(\lambda - 2)(\lambda + 2) = 0$$

i.e., $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$ and $\lambda_4 = -2$. Thus, the general solution is,

$$y = C_4 e^{x} + C_5 e^{-x} + C_6 e^{2x} + C_7 e^{-2x}$$

where C_4, C_5, C_6 and C_7 are constants. The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D4y - 5*D2y + 4*y = 0', 'x')
```

dsolve

```
ans =

C4*exp(x) + C5/exp(x) + C6*exp(2*x) + C7/exp(2*x)
```

Example Solve the initial value problem governed by the fourth-order differential equation,

$$\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$$

with the initial conditions of $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$ and $y'''(0) = 0$.

We first solve for a general solution of the given differential equation. Assuming the solution in the form of $e^{\lambda x}$ leads to the characteristic equation,

$$\lambda^4 + \lambda^3 - 7\lambda^2 - \lambda + 6 \quad = \quad 0$$

The `factor` command can be used to factorize the algebraic equation above,

```
>> syms lambda
>> factor(lambda^4+lambda^3-7*lambda^2-lambda+6)

ans =

(lambda-1)*(lambda-2)*(lambda+1)*(lambda+3)
```

i.e.,
$$(\lambda - 1)(\lambda - 2)(\lambda + 1)(\lambda + 3) \quad = \quad 0$$

So that $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -1$ and $\lambda_4 = -3$. Thus, the general solution is,

$$y \quad = \quad C_8 e^{x} + C_9 e^{2x} + C_{10} e^{-x} + C_{11} e^{-3x}$$

where C_8, C_9, C_{10} and C_{11} are constants that can be determined from the initial conditions as follows,

$$\begin{aligned}
y(0) &= 1; & C_8 + C_9 + C_{10} + C_{11} &= 1 \\
y'(0) &= 0; & C_8 + 2C_9 - C_{10} - 3C_{11} &= 0 \\
y''(0) &= 0; & C_8 + 4C_9 + C_{10} + 9C_{11} &= 0 \\
y'''(0) &= 0; & C_8 + 8C_9 - C_{10} - 27C_{11} &= 0
\end{aligned}$$

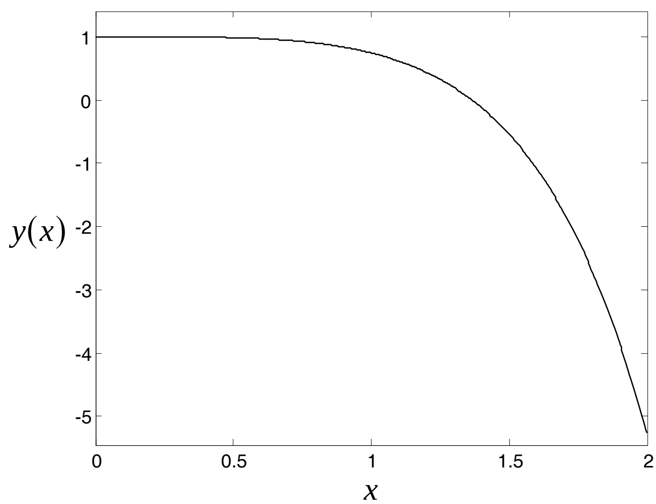
The results are $C_8 = 3/4$, $C_9 = -1/5$, $C_{10} = 1/2$ and $C_{11} = -1/20$. Hence, the exact solution is,

$$y \quad = \quad \frac{3}{4}e^{x} - \frac{1}{5}e^{2x} + \frac{1}{2}e^{-x} - \frac{1}{20}e^{-3x}$$

The same exact solution can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D4y+D3y-7*D2y-Dy+6*y=0','y(0)=1',
          'Dy(0)=0','D2y(0)=0','D3y(0)=0','x')

ans =

1/(2*exp(x)) - exp(2*x)/5 - 1/(20*exp(3*x)) +
(3*exp(x))/4
```

The solution of y that varies with x can be plotted by using the `ezplot` command as shown in the figure.



It is noted that MATLAB contains commands that can convert the solutions into other computer language directly. This is convenient when the solution is to be used further in other programs. For example, the solution above is converted to Fortran language by using the `fortran` command,

fortran

```
>> fortran(ans)

ans =

  t0 = exp(-x)*(1.0D0/2.0D0)-exp(x*2.0D0)*(1.0D0/5.0D0)-exp(x*-3.0D0
 +)*(1.0D0/2.0D1)+exp(x)*(3.0D0/4.0D0)
```

In the Fortran language above, the executing statements must be in between the 7^{th} and 72^{nd} column. The continuation line (last line above) that contains more executing statements is indicated by the plus sign in the 6^{th} column.

6.4 Solutions from Repeated Real Roots

In the preceding chapter when the roots of the second-order differential equations are repeated, we have learnt how to derive for their solutions. For example, if the roots are $\lambda_1 = 3$ and $\lambda_2 = 3$, then the general solution is in the form of,

$$y = e^{3x} + xe^{3x}$$

We will follow the same procedure to derive solutions of the higher-order differential equations when their roots are repeated. We will show detailed derivation by using the following examples.

<u>Example</u> Find solution of the third-order homogeneous differential equation,

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$$

We assume the solution in the form of $e^{\lambda x}$, substitute it into the differential equation and divide through by $e^{\lambda x}$ to get the characteristic equation as,

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

Or $$(\lambda - 1)(\lambda - 1)(\lambda - 1) = 0$$

i.e., $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Thus, the general solution is,

$$y = C_{12}e^x + C_{13}xe^x + C_{14}x^2e^x$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D3y - 3*D2y + 3*Dy - y = 0', 'x')

ans =

C12*exp(x) + C13*x*exp(x) + C14*x^2*exp(x)
```

dsolve

<u>Example</u> Find solution of the fourth-order homogeneous differential equation,

$$\frac{d^4y}{dx^4} + 6\frac{d^3y}{dx^3} + 13\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 4y = 0$$

Similar to the preceding example, by assuming the solution in the form of $e^{\lambda x}$, this leads to the characteristic equation,

$$\lambda^4 + 6\lambda^3 + 13\lambda^2 + 12\lambda + 4 \quad = \quad 0$$

The `factor` command is used to factorize the algebraic equation above,

```
>> syms lambda
>> factor(lambda^4+6*lambda^3+13*lambda^2+12*lambda+4)

ans =

(lambda + 2)^2*(lambda + 1)^2
```

which gives, $$(\lambda+1)^2(\lambda+2)^2 \quad = \quad 0$$

then, $$\lambda_1 = \lambda_2 = -1 \quad \text{and} \quad \lambda_3 = \lambda_4 = -2$$

Thus, the general solution of the fourth-order differential equation is,

$$y \quad = \quad C_{15}e^{-x} + C_{16}xe^{-x} + C_{17}e^{-2x} + C_{18}xe^{-2x}$$

where C_{15}, C_{16}, C_{17} and C_{18} are constants. The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D4y + 6*D3y + 13*D2y + 12*Dy +
           4*y = 0', 'x')

ans =

C15/exp(x) + (C16*x)/exp(x) + C17/exp(2*x) +
(C18*x)/exp(2*x)
```

<u>Example</u> Solve the initial value problem governed by the fourth-order differential equation,

$$\frac{d^4y}{dx^4} - 8\frac{d^2y}{dx^2} + 16y \quad = \quad 0$$

with the initial conditions of $y(0)=0$, $y'(0)=1$, $y''(0)=0$ and $y'''(0)=0$.

Similar to the preceding example, we start by assuming the solution in the form of $e^{\lambda x}$ and substituting into the differential equation. This leads to the characteristic equation in the form of algebraic equation as,

$$\lambda^4 - 8\lambda^2 + 16 = 0$$

Or,
$$(\lambda - 2)^2 (\lambda + 2)^2 = 0$$

Then, the roots are,

$$\lambda_1 = \lambda_2 = 2 \quad \text{and} \quad \lambda_3 = \lambda_4 = -2$$

Thus, the general solution is,

$$y = C_{19}e^{2x} + C_{20}xe^{2x} + C_{21}e^{-2x} + C_{22}xe^{-2x}$$

where C_{19}, C_{20}, C_{21} and C_{22} are constants that can be determined from the given initial conditions as follows,

$$\begin{array}{rclrrrrrrrrr}
y(0) & = & 0; & C_{19} & + & 0 & + & C_{21} & + & 0 & = & 0 \\
y'(0) & = & 1; & 2C_{19} & + & C_{20} & - & 2C_{21} & + & C_{22} & = & 1 \\
y''(0) & = & 0; & C_{19} & + & C_{20} & + & C_{21} & - & C_{22} & = & 0 \\
y'''(0) & = & 0; & 2C_{19} & + & 3C_{20} & - & 2C_{21} & + & 3C_{22} & = & 0
\end{array}$$

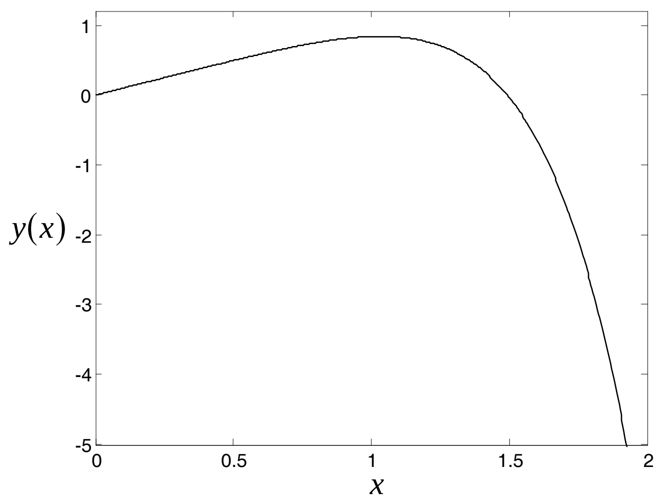
The four equations above give the values of the four constants as $C_{19} = 3/8$, $C_{20} = -1/4$, $C_{21} = -3/8$ and $C_{22} = -1/4$. Hence, the exact solution of this initial value problem is,

$$y = \frac{3}{8}e^{2x} - \frac{1}{4}xe^{2x} - \frac{3}{8}e^{-2x} - \frac{1}{4}xe^{-2x}$$

The same exact solution is obtained by using the `dsolve`,

```
>> syms x y
>> dsolve('D4y - 8*D2y + 16*y = 0', 'y(0)=0',
          'Dy(0)=1', 'D2y(0)=0', 'D3y(0)=0', 'x')

ans =

(3*exp(2*x))/8 - 3/(8*exp(2*x)) - x/(4*exp(2*x))
- (x*exp(2*x))/4
```

The solution of y that varies with x is plotted as shown in the figure.



6.5 Solutions from Complex Roots

Roots of the characteristic equations from the higher-order differential equations may be in the form of the conjugate complex numbers. In this case, derivation of the general solutions is more complicated as shown by the following examples.

Example Find solution of the fourth-order homogeneous differential equation,

$$\frac{d^4y}{dx^4} + 5\frac{d^2y}{dx^2} + 4y = 0$$

By assuming the solution in the form of $e^{\lambda x}$, substituting it into the differential equation and dividing through by $e^{\lambda x}$, we obtain the characteristic equation as,

$$\lambda^4 + 5\lambda^2 + 4 = 0$$

which is,

$$(\lambda^2 + 1)(\lambda^2 + 4) = 0$$

Then, $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = 2i$ and $\lambda_4 = -2i$, where $i = \sqrt{-1}$. Thus, the general solution of the fourth-order differential equation is,

$$y \quad = \quad Ae^{ix} + Be^{-ix} + Ce^{2ix} + De^{-2ix}$$

where A, B, C and D are constants. From the Euler's formula,

$$e^{i\beta x} \quad = \quad \cos(\beta x) + i\sin(\beta x)$$

and

$$e^{-i\beta x} \quad = \quad \cos(\beta x) - i\sin(\beta x)$$

the general solution can be written in the form of sine and cosine functions as,

$$y \quad = \quad C_{23}\cos(x) + C_{24}\sin(x) + C_{25}\cos(2x) + C_{26}\sin(2x)$$

where C_{23}, C_{24}, C_{25}, C_{26} are functions of the constants A, B, C, D above.

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D4y + 5*D2y + 4*y = 0', 'x')
```

dsolve

```
ans =

C23*cos(x) + C24*sin(x) + C25*cos(2*x) +
C26*sin(2*x)
```

<u>Example</u> Solve the initial value problem governed by the fourth-order differential equation,

$$\frac{d^4y}{dx^4} + 34\frac{d^2y}{dx^2} + 225y \quad = \quad 0$$

with the initial conditions of $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$ and $y'''(0) = 0$.

Similar to the preceding example, we assume the solution in the form $e^{\lambda x}$ and substitute it into the differential equation. This leads to the characteristic equation as,

$$\lambda^4 + 34\lambda^2 + 225 \quad = \quad 0$$

The `factor` command can be used to factorize the algebraic equation above,

```
>> syms lambda
>> factor(lambda^4 + 34*lambda^2 + 225)

ans =

(lambda^2 + 9)*(lambda^2 + 25)
```

factor

to get, $$(\lambda^2+9)(\lambda^2+25) = 0$$

Then, $\lambda_1 = 3i$, $\lambda_2 = -3i$, $\lambda_3 = 5i$ and $\lambda_4 = -5i$ where $i = \sqrt{-1}$. Thus, the general solution of the fourth-order differential equation is,

$$y = Ae^{3ix} + Be^{-3ix} + Ce^{5ix} + De^{-5ix}$$

where A, B, C and D are constants. By applying the Euler's formula, the general solution above can be written in the form of sine and cosine functions as,

$$y = C_{27}\cos(3x) + C_{28}\sin(3x) + C_{29}\cos(5x) + C_{30}\sin(5x)$$

where C_{27}, C_{28}, C_{29} and C_{30} are to be determined from the four initial conditions as follows,

$$\begin{array}{rclrcrcrcrcl}
y(0) & = & 1; & C_{27} & + & 0 & + & C_{29} & + & 0 & = & 1 \\
y'(0) & = & 0; & 0 & + & 3C_{28} & + & 0 & + & 5C_{30} & = & 0 \\
y''(0) & = & 0; & -9C_{27} & + & 0 & - & 25C_{29} & + & 0 & = & 0 \\
y'''(0) & = & 0; & 0 & - & 27C_{28} & + & 0 & - & 125C_{30} & = & 0
\end{array}$$

These four equations give the values of the four constants as $C_{27} = 25/16$, $C_{28} = 0$, $C_{29} = -9/16$ and $C_{30} = 0$. Hence, the exact solution of this initial value problem is,

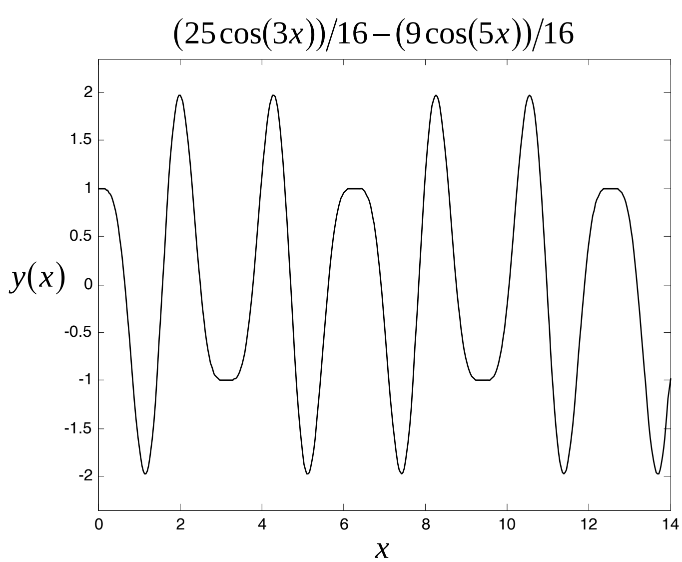
$$y = \frac{25}{16}\cos(3x) - \frac{9}{16}\cos(5x)$$

The same exact solution is obtained by using the `dsolve` command,

dsolve

```
>> syms x y
>> dsolve('D4y + 34*D2y + 225*y = 0', 'y(0)=1',
          'Dy(0)=0', 'D2y(0)=0', 'D3y(0)=0', 'x')

ans =

(25*cos(3*x))/16 - (9*cos(5*x))/16
```

Such exact solution can be plotted easily by using the `ezplot` command. The solution of y that varies with x is shown in the figure.



6.6 Solutions from Mixed Roots

There are cases when the roots of the characteristic equation from the given differential equations are mixed between the real and complex numbers. The same procedure explained earlier can be applied to derive for solutions as shown in the following examples.

Example Find solution of the fourth-order homogeneous differential equation,

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

The characteristic equation corresponds to the differential equation is,

$$\lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1 = 0$$

or,

$$(\lambda - 1)^2(\lambda^2 + 1) = 0$$

i.e., $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = i$ and $\lambda_4 = -i$ where $i = \sqrt{-1}$. Since the roots consist of repeated real and conjugate complex numbers, then the general solution is in the form,

$$y = Ae^x + Bxe^x + Ce^{ix} + De^{-ix}$$

where A, B, C and D are constants. After applying the Euler's formula, the solution can be written in the form of sine and cosine functions as,

$$y = C_{31}e^x + C_{32}xe^x + C_{33}\cos(x) + C_{34}\sin(x)$$

where C_{31}, C_{32}, C_{33} and C_{34} are constants.

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D4y-2*D3y+2*D2y-2*Dy+y = 0',  'x')

ans =

C31*exp(x)+C32*x*exp(x)+C33*cos(x)+C34*sin(x)
```

Example Solve the initial value problem governed by the fourth-order differential equation,

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

with the initial conditions of $y(0) = 0$, $y'(0) = 2$, $y''(0) = 0$ and $y'''(0) = 0$.

Since the given differential equation is the same as that in the preceding example, then the general solution is,

$$y = C_{31}e^{x} + C_{32}xe^{x} + C_{33}\cos(x) + C_{34}\sin(x)$$

where C_{31}, C_{32}, C_{33} and C_{34} are to be determined from the given initial conditions of,

$$\begin{array}{rclcccccccccc} y(0) & = & 0; & C_{31} & + & 0 & + & C_{33} & + & 0 & = & 0 \\ y'(0) & = & 2; & C_{31} & + & C_{32} & + & 0 & + & C_{34} & = & 2 \\ y''(0) & = & 0; & C_{31} & + & 2C_{32} & - & C_{33} & + & 0 & = & 0 \\ y'''(0) & = & 0; & C_{31} & + & 3C_{32} & + & 0 & - & C_{34} & = & 0 \end{array}$$

By solving the four equations above, the four constants are $C_{31} = -1$, $C_{32} = 1$, $C_{33} = 1$ and $C_{34} = 2$. Hence, the exact solution of this initial value problem is,

$$y = -e^{x} + xe^{x} + \cos(x) + 2\sin(x)$$

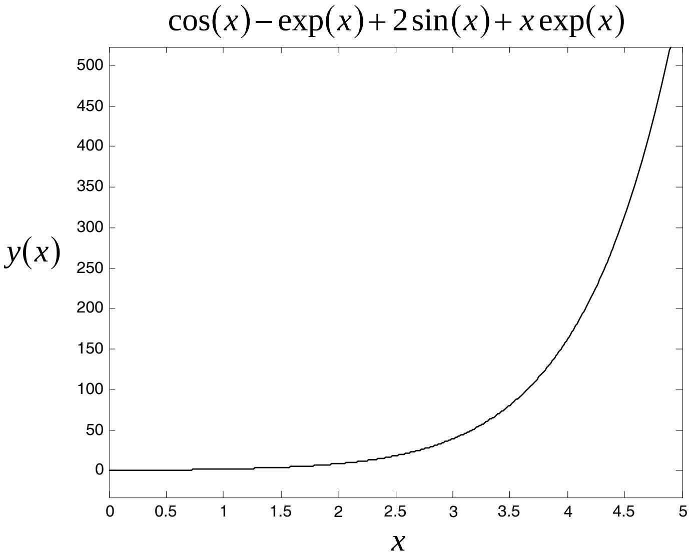
The same exact solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D4y - 2*D3y + 2*D2y - 2*Dy + y = 0',
'y(0)=0', 'Dy(0)=2', 'D2y(0)=0', 'D3y(0)=0', 'x')

ans =

cos(x) - exp(x) + 2*sin(x) + x*exp(x)
```

The solution y that varies with x is plotted by using the `ezplot` command as shown in the figure.

6.7 Nonhomogeneous Equations

The higher-order nonhomogeneous differential equations that we will learn how to solve for their solutions are in the form,



$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + ... + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y \quad = \quad f(x)$$

where the coefficients a_n, a_{n-1}, ..., a_2, a_1 and a_0 are constants. The function $f(x)$ on the right-hand-side of the equation may be in form of the polynomial, exponential, sine and cosine functions. The general solution of such differential equation consists of two parts,

$$y \quad = \quad y_h + y_p$$

where y_h is the homogeneous solution of the homogeneous differential equation and y_p is the particular solution. We will employ the method of undetermined coefficients learned in the preceding chapter to find the particular solution. The entire process for deriving general solutions of the higher-order nonhomogeneous differential equations will be demonstrated by using the following examples.

Example Find solution of the third-order nonhomogeneous differential equation,

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 3x+1$$

The corresponding characteristic equation of the homogeneous differential equation above is,

$$\lambda(\lambda-3)(\lambda+2) = 0$$

which leads to the three roots of, $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = -2$.

Then, the homogeneous solution is,

$$y_h = C_{35} + C_{36}e^{3x} + C_{37}e^{-2x}$$

where C_{35}, C_{36} and C_{37} are constants. The same homogeneous solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D3y - D2y - 6*Dy = 0', 'x')
```

dsolve

```
ans =

C35 + C36*exp(3*x) + C37/exp(2*x)
```

In the process of finding the particular solution y_p using the method of undetermined coefficients, since the function $f(x)$ on the right-hand-side of the differential equation is $3x+1$, we should assume the solution in form of the polynomials. The assumed polynomials should be second order so that after taking derivatives according to the terms on the left-hand-side of the equation will yield the first-order polynomials (term $3x$) on the right-hand-side of the equation. Thus, we assume the particular solution in the form,

$$y_p = Ax^2 + Bx + C$$

where A, B and C are constants.

We substitute the assumed y_p into the differential equation to get,

$$(0)-(2A)-6(2Ax+B) \quad = \quad 3x+1$$
$$(-12A)x+(-2A-6B) \quad = \quad 3x+1$$

By comparing coefficients, the two algebraic equations are obtained,

$$-12A \quad = \quad 3$$

and
$$-2A-6B \quad = \quad 1$$

These two equations are solved to give $A=-1/4$ and $B=-1/12$. Then, the particular solution is,

$$y_p \quad = \quad -\frac{1}{4}x^2-\frac{1}{12}x$$

Hence, the exact solution of the third-order nonhomogeneous differential equation is,

$$y \quad = \quad C_{35}+C_{36}e^{3x}+C_{37}e^{-2x}-\frac{x^2}{4}-\frac{x}{12}$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D3y - D2y - 6*Dy = 3*x + 1', 'x')
ans =
C35 + C36*exp(3*x) + C37/exp(2*x) - x^2/4 - x/12
```

dsolve

where C_{35}, C_{36} and C_{37} are constants.

Example Find solution of the fourth-order nonhomogeneous differential equation,

$$\frac{d^4y}{dx^4}-5\frac{d^2y}{dx^2}+4y \quad = \quad 10\sin x$$

The corresponding characteristic equation of the homogeneous differential equation above is,

$$\lambda^4-5\lambda^2+4 \quad = \quad 0$$

or,
$$(\lambda-1)(\lambda+1)(\lambda-2)(\lambda+2) \quad = \quad 0$$

The four roots are $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$ and $\lambda_4 = -2$ which lead to the homogeneous solution,

$$y_h \quad = \quad C_{38}e^{x} + C_{39}e^{-x} + C_{40}e^{2x} + C_{41}e^{-2x}$$

Since the function $f(x)$ on the right-hand-side of the differential equation is in the form of sine function, thus we should assume the particular solution in the form of both sine and cosine functions as,

$$y_p \quad = \quad A\cos x + B\sin x$$

where A and B are constants. By substituting the assumed particular solution y_p into the differential equation,

$$(A\cos x + B\sin x) - 5(-A\cos x - B\sin x) + 4(A\cos x + B\sin x)$$

$$= \quad 10\sin x$$

or, $\quad (A + 5A + 4A)\cos x + (B + 5B + 4B)\sin x \quad = \quad 10\sin x$

i.e., $\quad (10A)\cos x + (10B)\sin x \quad = \quad 10\sin x$

and comparing the coefficients, we obtain $A = 0$ and $B = 1$. Then, the particular solution is,

$$y_p \quad = \quad \sin x$$

Thus, the general solution of the fourth-order nonhomogeneous differential equation is,

$$y \quad = \quad C_{38}e^{x} + C_{39}e^{-x} + C_{40}e^{2x} + C_{41}e^{-2x} + \sin x$$

The same solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D4y - 5*D2y + 4*y = 10*sin(x)', 'x')

ans =

C38*exp(x) + C39/exp(x) + C40*exp(2*x) +
C41/exp(2*x) + sin(x)
```

<u>Example</u> Find solution of the third-order nonhomogeneous differential equation,

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y \quad = \quad 6e^x$$

Similar to the preceding example, we first assume the homogeneous solution in the form of $e^{\lambda x}$, substitute into the differential equation and divide through by $e^{\lambda x}$ to get the characteristic equation,

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 \quad = \quad 0$$

or,

$$(\lambda - 1)(\lambda - 1)(\lambda - 1) \quad = \quad 0$$

which leads to the three roots of $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = 1$. Then, the homogeneous solution is,

$$y_h \quad = \quad C_{42}e^x + C_{43}xe^x + C_{44}x^2e^x$$

In the process of finding the particular solution, since e^x, xe^x and x^2e^x are solutions of the third-order differential equation, thus we should assume the particular solution in the form,

$$y_p \quad = \quad Ax^3e^x$$

After substituting the assumed solution y_p into the differential equation, we obtain,

$$6Ae^x \quad = \quad 6e^x$$

i.e.,

$$A \quad = \quad 1$$

so that,

$$y_p \quad = \quad x^3e^x$$

It is noted that the `diff` command can alleviate the task of substituting y_p into the left-hand-side of the differential equation,

```
>> syms A
>> yp = A*x^3*exp(x);
>> LHS = diff(yp,x,3)-3*diff(yp,x,2)+
         3*diff(yp,x)-yp

LHS =

6*A*exp(x)
```

diff

Hence, the general solution of the third-order nonhomogeneous differential equation is,

$$y = C_{42}e^{x} + C_{43}xe^{x} + C_{44}x^{2}e^{x} + x^{3}e^{x}$$

The same solution is obtained by using the `dsolve` command,

```
>> dsolve('D3y-3*D2y+3*Dy-y = 6*exp(x)', 'x')

ans =

C42*exp(x) + C43*x*exp(x) + C44*x^2*exp(x) +
x^3*exp(x)
```

Example Solve the initial value problem governed by the third-order differential equation,

$$\frac{d^{3}y}{dx^{3}} + \frac{d^{2}y}{dx^{2}} + \frac{13}{2}\frac{dy}{dx} = e^{-x}$$

with the initial conditions of $y(0) = 1$, $y'(0) = 2$ and $y''(0) = -1$.

We start from finding the homogeneous solution from the homogeneous differential equation. By assuming the solution in the form of $e^{\lambda x}$, the corresponding characteristic equation is,

$$\lambda^{3} + \lambda^{2} + \frac{13}{2}\lambda = 0$$

or,

$$\lambda\left(\lambda^{2} + \lambda + \frac{13}{2}\right) = 0$$

So that the three roots are,

$$\lambda_{1} = 0, \qquad \lambda_{2},\, \lambda_{3} = -\frac{1}{2} \pm \frac{5}{2}i$$

Then, the homogeneous solution is,

$$y_{h} = C_{45} + C_{46}e^{-x/2}\cos(5x/2) + C_{47}e^{-x/2}\sin(5x/2)$$

where C_{45}, C_{46} and C_{47} are constants to be determined from the initial conditions.

The particular solution is assumed in the form,

$$y_{p} = Ae^{-x}$$

By substituting the assumed particular solution into the differential equation, we obtain,

$$-\frac{13}{2}Ae^{-x} \quad = \quad e^{-x}$$

i.e.,
$$A \quad = \quad -\frac{2}{13}$$

It is noted that the `diff` command can help finding the result of the three derivative terms on the left-hand-side of the differential equation,

```
>> syms A
>> yp = A*exp(-x);
>> LHS = diff(yp,x,3) + diff(yp,x,2) +
         (13/2)*diff(yp,x)

LHS =

-(13*A)/(2*exp(x))
```

diff

Thus, the general solution of the third-order nonhomogeneous differential equation is,

$$y \quad = \quad C_{45} + C_{46}e^{-x/2}\cos(5x/2) + C_{47}e^{-x/2}\sin(5x/2) - \frac{2}{13}e^{-x}$$

where C_{45}, C_{46} and C_{47} are constants that can be determined from the initial conditions as follows,

$$\begin{array}{rclcccccccccc}
y(0) & = & 1; & C_{45} & + & C_{46} & & & - & \frac{2}{13} & = & 1 \\
y'(0) & = & 2; & 0 & - & \frac{1}{2}C_{46} & + & \frac{5}{2}C_{47} & + & \frac{2}{13} & = & 2 \\
y''(0) & = & -1; & 0 & - & 6C_{46} & - & \frac{5}{2}C_{47} & - & \frac{2}{13} & = & -1
\end{array}$$

These three equations are solved to give $C_{45} = 17/13$, $C_{46} = -2/13$ and $C_{47} = 46/65$. Hence, the solution of the initial value problem is,

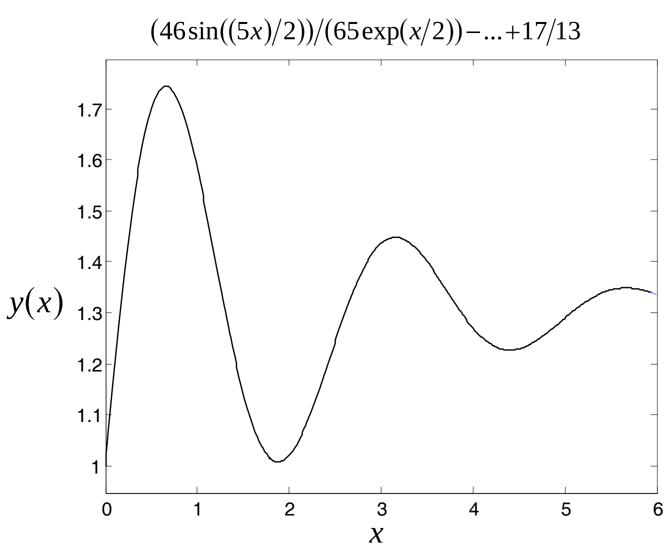
$$y = \frac{17}{13} - \frac{2}{13}e^{-x/2}\cos(5x/2) + \frac{46}{65}e^{-x/2}\sin(5x/2) - \frac{2}{13}e^{-x}$$

The same solution is obtained by using the `dsolve` command,

dsolve

```
syms x y
>> dsolve('D3y + D2y + (13/2)*Dy = exp(-x)',
          'y(0)=1','Dy(0)=2','D2y(0)=-1','x')

ans =

(46*sin((5*x)/2))/(65*exp(x/2)) -
(2*cos((5*x)/2))/(13*exp(x/2)) - 2/(13*exp(x)) +
17/13
```

The solution of y that varies with x can be plotted easily by using the `ezplot` command as shown in the figure.



6.8 Numerical Methods

For all examples presented earlier in this chapter, the higher-order differential equations are linear. The coefficients of the derivative terms are constants so that their exact solutions are not difficult to find. The differential equations become nonlinear if these coefficients are function of y. Exact solutions are not available for most nonlinear differential equations.

In this section, we will employ numerical methods to find approximate solutions for both linear and nonlinear differential equations. MATLAB contains many commands that can provide approximate solutions with high accuracy. Several examples are presented to demonstrate the capability for finding solutions, especially for nonlinear differential equations where their exact closed-form solutions are not available.

<u>Example</u> Use the `ode23` command to solve the initial value problem governed by the third-order homogeneous differential equation,

$$10\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 10\frac{dy}{dx} - y \quad = \quad 0 \qquad\qquad 0 \le x \le 10$$

with the initial conditions of $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 1$. Then, plot to compare the approximate solution with the exact solution.

The `ode23` command uses the combined second- and third-order Runge-Kutta method to solve the first-order differential equation. Since the governed differential equation is third order, thus we need to separate it into three first-order differential equations. This can be done by first writing the given differential equation in the form,

$$10\frac{d^3y_1}{dx^3} - \frac{d^2y_1}{dx^2} + 10\frac{dy_1}{dx} - y_1 \quad = \quad 0$$

If we assign,

$$\frac{dy_1}{dx} \quad = \quad y_2 \qquad \text{and} \qquad \frac{dy_2}{dx} \quad = \quad y_3$$

then, the given differential equation becomes,

$$\frac{dy_3}{dx} \quad = \quad (y_1 - 10y_2 + y_3)/10$$

To solve these three first-order differential simultaneously, it is more convenient to create an m-file. The m-file, `example1.m`, consists of the three first-order differential equations as follows,

function

```
function yex1=example1(x,y)
yex1 = [y(2); y(3); (y(1)-10*y(2)+y(3))/10];
```

We then employ the `ode23` command by typing on the Command Window as,

ode23

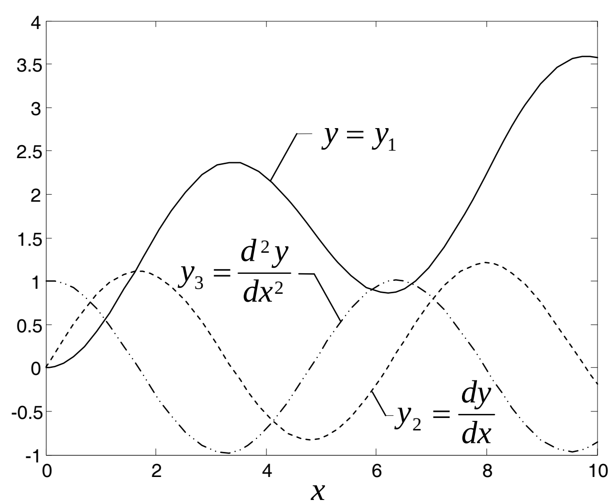
```
>> [x,y] = ode23('example1', [0 10], [0 0 1])
```

MATLAB will compute the values of y_1, y_2 and y_3 at different x locations, and at the same time, print the values of x, y_1, y_2 and y_3 on the screen monitor. In the `ode23` command, the values in the first square bracket denote the interval of $0 \le x \le 10$, while the values in the second square bracket are the three initial conditions of $y_1(0) = 0$, $y_2(0) = 0$ and $y_3(0) = 1$, respectively.

The computed solutions of y_1, y_2 and y_3 that vary with x can be plotted by using the `plot` command as shown in the figure.

```
>> plot(x,y(:,1),'-k',x,y(:,2),'--k',x,y(:,3),
        '-.k')
```

plot



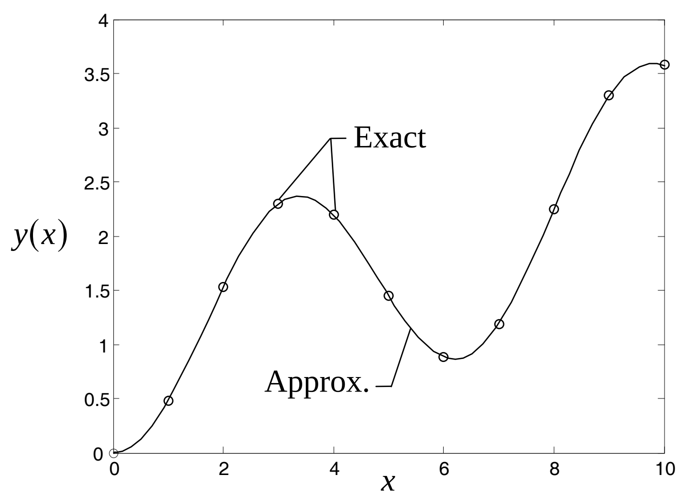
The exact solution of this initial value problem can also be found by using the `dsolve` command,

```
>> dsolve('10*D3y - D2y + 10*Dy – y = 0',
   'y(0)=0','Dy(0)=0','D2y(0)=1','x')
```

dsolve

```
ans =

(100*exp(x/10))/101 - (100*cos(x))/101 -
(10*sin(x))/101
```

The approximate solution using the `ode23` command is compared with the exact solution as shown in the figure. The figure shows that both solutions agree very well.



Example Use the `ode45` command to solve the initial value problem governed by the fourth-order nonhomogeneous differential equation,

$$49\frac{d^4y}{dx^4}+442\frac{d^2y}{dx^2}+9y \quad = \quad \sin x-\frac{x}{2} \qquad\qquad 0\le x\le 200$$

with the initial conditions of $y(0)=10$, $y'(0)=0$, $y''(0)=0$ and $y'''(0)=0$. Then, plot to compare the approximate solution with the exact solution.

The `ode45` command employs the fourth- and fifth-order Runge-Kutta method to solve the first-order differential equation. Thus, before applying the method, the given fourth-order differential equation is separated into four first-order differential equations. This can be done by first writing the given differential in the form of the unknown y_1 as,

$$49\frac{d^4y_1}{dx^4} + 442\frac{d^2y_1}{dx^2} + 9y_1 \quad = \quad \sin x - \frac{x}{2}$$

Then, by assigning,

$$\frac{dy_1}{dx} \quad = \quad y_2, \qquad \frac{dy_2}{dx} \quad = \quad y_3 \qquad \text{and} \qquad \frac{dy_3}{dx} \quad = \quad y_4$$

the fourth-order differential equation becomes,

$$\frac{dy_4}{dx} \quad = \quad \left(\sin x - \frac{x}{2} - 9y_1 - 442y_3\right) \Big/ 49$$

To solve the four first-order differential equations simultaneously, we create an m-file, `example2.m`, that contains the descriptions as follows,

function

```
function yex2=example2(x,y)
yex2 = [y(2); y(3); y(4);
        (sin(x) - x/2 - 9*y(1) - 442*y(3))/49];
```

Then, we can use the `ode45` command by typing on the Window Command as,

```
>> [x,y] = ode45('example2',[0 200], [10 0 0 0]);
```

The values of y_1, y_2, y_3 and y_4 at different x locations will be determined. In the `ode45` command, the values in the first square bracket denote the interval of $0 \le x \le 200$, while the values in the second square bracket represent the four initial conditions of $y_1(0) = 10$, $y_2(0) = 0$, $y_3(0) = 0$ and $y_4(0) = 0$, respectively.

It is noted that the `dsolve` command can be used to find the exact solution for this initial value problem,

```
>> dsolve('49*D4y+442*D2y+9*y = sin(x)-x/2',
   'y(0)=10','Dy(0)=0','D2y(0)=0','D3y(0)=0','x')

ans =

(441*cos(x/7))/44 - cos(3*x)/44 - x/18 +
sin(3*x)/19008 + 1715*sin(x/7)/4224 - sin(x)/384
```

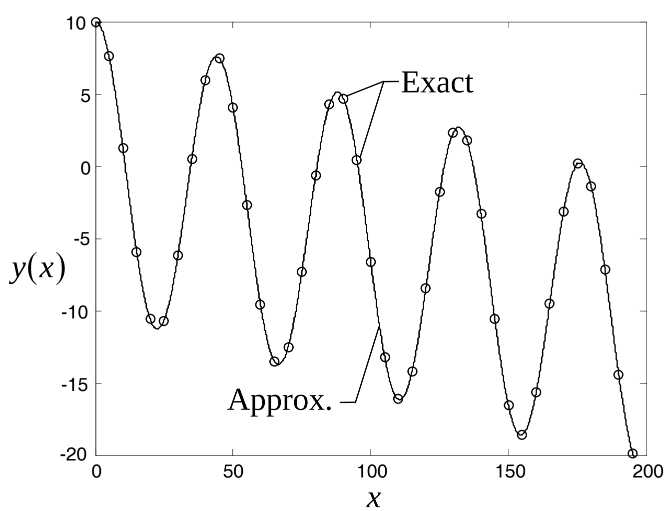
The exact and approximate solutions are plotted together by using the following commands,

```
>> xe=0:5:200;
>> ye=(441*cos(xe./7))/44-cos(3.*xe)/44- xe./18 +
      sin(3.*xe)/19008 + (1715*sin(xe./7))/4224 –
      sin(xe)./384;
>> plot(xe,ye,'ok')
>> axis([0 200 -20 10])
>> hold on
>> plot(x,y(:,1),'-k')
```

axis

plot

The plot indicates that the approximate solution obtained from the `ode45` command is very accurate as shown in the figure.



Example For many higher-order linear differential equations, their exact solutions are not available. As an example, MATLAB cannot provide exact solution if the term $9y$ on the left-hand-side of the differential equation in the preceding example is changed to $18y$, i.e.,

$$49\frac{d^4y}{dx^4}+442\frac{d^2y}{dx^2}+18y \quad = \quad \sin x-\frac{x}{2} \qquad\qquad 0\le x\le 200$$

with the same initial conditions of $y(0)=10,\ y'(0)=0,\ y''(0)=0$ and $y'''(0)=0$,

```
>> dsolve('49*D4y+442*D2y+18*y = sin(x)-x/2',
   'y(0)=10','Dy(0)=0','D2y(0)=0','D3y(0)=0','x')

Warning: Explicit solution could not be found.
> In dsolve at 101

ans =

[ empty sym ]
```

In this case, the `ode45` command can still be used to find the approximate solution. We can follow the same procedure by creating an m-file, `example3.m`, as follows,

```
function yex3=example3(x,y)
yex3 = [y(2); y(3); y(4);
        (sin(x) - x/2 - 18*y(1) - 442*y(3))/49];
```

function

Then, type the `ode45` command on the Command Window as,

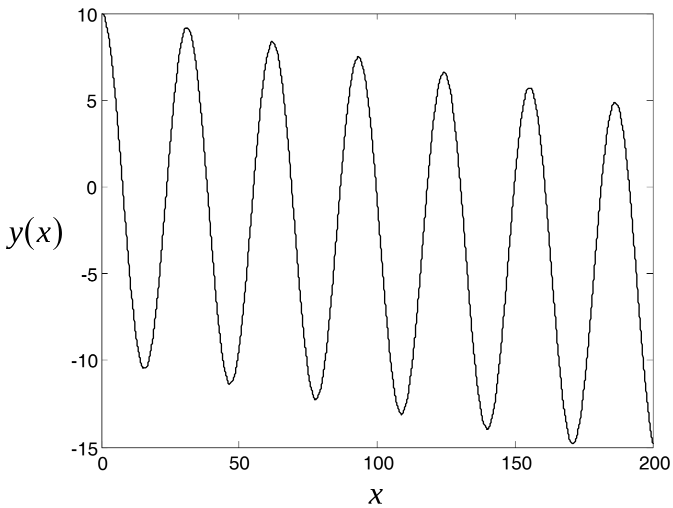
```
>> [x,y] = ode45('example3',[0 200], [10 0 0 0]);
```

ode45

The approximate solution is obtained as shown in the figure by using the `plot` command as,

```
>> plot(x,y(:,1),'-k')
```

plot



6.9 Concluding Remarks

In this chapter, we started from the general procedure for solving the higher-order homogeneous linear differential equations. The coefficients of the derivative terms are constants so that the exact closed-form solutions can be derived. The forms of the solutions depend on the characteristic equations arisen from the differential equations. The characteristic equations produce roots that could be distinct real, repeated real and conjugate complex roots including combination of them. We have learnt how to derive the solutions and, at the same time, verify them by using MATLAB commands.

For the nonhomogeneous differential equations, we used the method of undetermined coefficients to find the particular solutions. Several examples have been employed to show detailed derivation of the solutions. The same approach was used to solve the initial value problems when their initial conditions are given in addition to the differential equations.

The numerical methods were introduced in the last section to solve the initial value problems. Examples have shown that the numerical methods in MATLAB can provide approximate solutions with very high accuracy. The main advantage of using the numerical methods is that approximate solutions can be obtained when the exact solutions are not available. This is true for most realistic problems when their differential equations are complicated and usually in nonlinear form.

Exercises

1. In each sub-problem below, show that the given solution is the general solution of the corresponding differential equation,

   (a) $$y = 4e^{x} + 7e^{-2x} - 2e^{3x}$$ $$y''' - 2y'' - 5y' + 6y = 0$$

   (b) $$y = e^{2x} - 3xe^{2x} + 5x^2e^{2x}$$ $$y''' - 6y'' + 12y' - 8y = 0$$

   (c) $$y = 2\cos x - \sin x + 5\cos(2x) - 3\sin(2x)$$ $$y^{IV} + 5y'' + 4y = 0$$

   (d) $$y = 2e^{2x} - 3xe^{2x} + 4\cos(x) - 5\sin(x)$$ $$y^{IV} - 4y''' + 5y'' - 4y' + 4y = 0$$

   (e) $$y = 5e^{3x} + 7xe^{3x} - 3e^{-5x} + 2xe^{-5x}$$ $$y^{IV} + 4y''' - 26y'' - 60y' + 225y = 0$$

2. Employ the `det` command to determine the determinant (Wronskian) of the solution for each sub-problem in Problem 1. Note that the `diff` command can help finding derivatives of the solution.

3. Use the `diff` command to show that,

   $$y = 4e^{x/2} + 3e^{-x/3} + 5e^{-3x/2} + 7e^{2x/3}$$

   is the exact solution of the fourth-order homogeneous differential equation,

$$36y^{IV} + 24y''' - 47y'' + y' + 6y \quad = \quad 0$$

Then, employ the `det` command to show that the Wronskian is nonzero.

4. Use the `diff` command to show that,

$$y \quad = \quad e^{3x/2} + 5xe^{3x/2} + 3e^{-x/5} + 2xe^{-x/5}$$

is the exact solution of the fourth-order homogeneous differential equation,

$$100y^{IV} - 260y''' + 109y'' + 78y' + 9y \quad = \quad 0$$

Then, employ the `det` command to show that the Wronskian is nonzero.

5. Use the `diff` command to show that,

$$y \quad = \quad 2\cos(x/2) + 3\sin(x/2) + 4\cos(7x/3) + 5\sin(7x/3)$$

is the exact solution of the fourth-order homogeneous differential equation,

$$36y^{IV} + 205y'' + 49y \quad = \quad 0$$

Then, employ the `det` command to show that the Wronskian is nonzero.

6. Use the `diff` command to show that,

$$y \quad = \quad 23 + 7e^{x} - 3xe^{x} - 5\cos(x) + 4\sin(x)$$

is the exact solution of the fifth-order homogeneous differential equation,

$$y^{V} - 2y^{IV} + 2y''' - 2y'' + y' \quad = \quad 0$$

7. Find the third-order differential equations corresponding to the following solutions,

(a) $y \;=\; e^{x} + 3e^{-x} + 7e^{2x}$

(b) $y \;=\; 2e^{3x} + 4xe^{3x} - 5x^{2}e^{3x}$

(c) $y \;=\; 6\sin(2x) + 5\cos(2x) - 4\sin(x/2) + \cos(x/2)$

(d) $y = 3e^{-2x} + 4xe^{-2x} + \sin x + 2\cos x$

(e) $y = e^{x/2} + xe^{x/2} + e^{7x/2} + xe^{7x/2}$

Then, verify the differential equations by substituting these solutions into them.

8. Find the fourth-order differential equations corresponding to the following solutions,

(a) $y = e^{x/3} + e^{x} - e^{-2x} + e^{3x}$

(b) $y = 2e^{2x} + xe^{2x} + 3e^{-x} + 4xe^{-x}$

(c) $y = 5\sin x + 6\cos x - 7\sin(2x) - 9\cos(2x)$

(d) $y = 3e^{x} + 4xe^{x} + 2\sin x + 3\cos x$

(e) $y = e^{-7x/3} + 5xe^{-7x/3} + 3\sin(3x) + 4\cos(3x)$

Then, verify them by comparing with the solutions obtained from the `dsolve` command.

9. Solve the higher-order homogeneous differential equations when roots of the characteristic equations are distinct real numbers,

(a) $y''' + 6y'' - 9y' - 14y = 0$

(b) $y^{IV} + y''' - 7y'' - y' + 6y = 0$

(c) $y^{IV} - 16y''' + 86y'' - 176y' + 105y = 0$

(d) $36y^{IV} + 24y''' - 47y'' + y' + 6y = 0$

(e) $360y^{V} - 786y^{IV} - 23y''' + 714y'' - 367y' + 42y = 0$

Show derivation of the solutions in detail and verify them by comparing with the solutions obtained from the `dsolve` command.

10. Solve the initial value problems governed by the higher-order homogeneous differential equations when roots of the characteristic equations are distinct real numbers,

(a) $y''' - 6y'' + 11y' - 6y = 0$

$y(0) = 1, \; y'(0) = 0, \; y''(0) = 0$

(b) $30y''' - 79y'' + 59y' - 12y = 0$
$y(0) = 0\,,\ \ y'(0) = 1\,,\ \ y''(0) = 0$

(c) $y^{IV} + 14y''' + 71y'' + 154y' + 120y = 0$
$y(0) = 1\,,\ \ y'(0) = 0\,,\ \ y''(0) = 0\,,\ \ y'''(0) = 0$

(d) $120y^{IV} - 2y''' - 263y'' - 20y' + 21y = 0$
$y(0) = 1\,,\ \ y'(0) = 0\,,\ \ y''(0) = 0\,,\ \ y'''(0) = 0$

Show derivation of the solutions in detail and verify them by using the `dsolve` command. Then, use the `ezplot` command to plot the variation of *y* in the interval of $0 \le x \le 2$.

11. Solve the higher-order homogeneous differential equations when roots of the characteristic equations are repeated real numbers,

(a) $y''' - 6y'' + 12y' - 8y = 0$

(b) $y^{IV} - 2y''' - 3y'' + 4y' + 4y = 0$

(c) $y^{IV} + 12y''' + 54y'' + 108y' + 81y = 0$

(d) $y^{IV} + 2y''' - 39y'' - 40y' + 400y = 0$

(e) $y^{V} + 4y^{IV} + y''' - 10y'' - 4y' + 8y = 0$

Show derivation of the solutions in detail and verify them by comparing with the solutions obtained from the `dsolve` command.

12. Solve the initial value problems governed by the higher-order homogeneous differential equations when roots of the characteristic equations are repeated real numbers,

(a) $y''' + 12y'' + 48y' + 64y = 0$
$y(0) = 2\,,\ \ y'(0) = 0\,,\ \ y''(0) = 0$

(b) $y^{IV} - 2y''' - 11y'' + 12y' + 36y = 0$
$y(0) = 1\,,\ \ y'(0) = 0\,,\ \ y''(0) = 0\,,\ \ y'''(0) = 0$

(c) $y^{IV} - 8y''' + 24y'' - 32y' + 16y = 0$
$y(0) = 0\,,\;\; y'(0) = 1\,,\;\; y''(0) = 0\,,\;\; y'''(0) = 0$

(d) $36y^{IV} + 12y''' - 23y'' - 4y' + 4y = 0$
$y(0) = 3\,,\;\; y'(0) = 0\,,\;\; y''(0) = 0\,,\;\; y'''(0) = 0$

Show derivation of the solutions in detail and verify them by using the `dsolve` command. Then, use the `ezplot` command to plot the variation of y in the interval of $0 \le x \le 2$.

13. Solve the higher-order homogeneous differential equations when roots of the characteristic equations are conjugate complex numbers,

(a) $y^{IV} + 5y'' + 4y = 0$

(b) $y^{IV} + 34y'' + 225y = 0$

(c) $36y^{IV} + 277y'' + 441y = 0$

(d) $144y^{IV} + 481y'' + 225y = 0$

Show derivation of the solutions in detail and verify them by comparing with the solutions obtained from the `dsolve` command.

14. Solve the initial value problems governed by the higher-order homogeneous differential equations when roots of the characteristic equations are conjugate complex numbers,

(a) $y^{IV} + 13y'' + 36y = 0$
$y(0) = 1\,,\;\; y'(0) = 0\,,\;\; y''(0) = 0\,,\;\; y'''(0) = 0$

(b) $y^{IV} + 25y'' + 144y = 0$
$y(0) = 0\,,\;\; y'(0) = 1\,,\;\; y''(0) = 0\,,\;\; y'''(0) = 0$

(c) $36y^{IV} + 241y'' + 100y = 0$
$y(0) = 0\,,\;\; y'(0) = 1\,,\;\; y''(0) = 2\,,\;\; y'''(0) = 0$

(d) $144y^{IV} + 1465y'' + 3136y = 0$
$y(0) = 0\,,\;\; y'(0) = 1\,,\;\; y''(0) = 2\,,\;\; y'''(0) = 3$

Show derivation of the solutions in detail and verify them by using the `dsolve` command. Then, use the `ezplot` command to plot the variation of y in the interval of $0 \le x \le 10$.

15. Solve the higher-order homogeneous differential equations when roots of the characteristic equations are distinct real, repeated real or conjugate complex numbers,

(a) $y''' + 9y'' - 48y' - 448y = 0$

(b) $y^{IV} - 4y''' + 13y'' - 36y' + 36y = 0$

(c) $y^{IV} + 16y''' + 94y'' + 240y' + 225y = 0$

(d) $36y^{IV} + 36y''' + 25y'' + 16y' + 4y = 0$

(e) $400y^{IV} - 600y''' + 289y'' - 96y' + 36y = 0$

Show derivation of the solutions in detail and verify them by comparing with the solutions obtained from the `dsolve` command.

16. Solve the initial value problems governed by the higher-order homogeneous differential equations when roots of the characteristic equations are distinct real, repeated real or conjugate complex numbers,

(a) $12y''' + 64y'' + 7y' - 245y = 0$
$y(0) = 1, \; y'(0) = 0, \; y''(0) = 0$

(b) $64y^{IV} - 192y''' + 148y'' - 12y' + 9y = 0$
$y(0) = 0, \; y'(0) = 1, \; y''(0) = 0, \; y'''(0) = 1$

(c) $100y^{IV} - 120y''' + 61y'' - 30y' + 9y = 0$
$y(0) = 2, \; y'(0) = 0, \; y''(0) = 0, \; y'''(0) = 0$

(d) $441y^{IV} + 126y''' + 205y'' + 56y' + 4y = 0$
$y(0) = 0, \; y'(0) = 1, \; y''(0) = 0, \; y'''(0) = 2$

Show derivation of the solutions in detail and verify them by using the `dsolve` command. Then use the `ezplot` command to plot the variation of y in the interval of $0 \le x \le 2$.

17. Employ the `dsolve` command to find general solutions of the following higher-order differential equations,

    (a) $18y^{IV} - 117y''' - 101y'' + 1117y' - 357y \quad = \quad 0$

    (b) $24y^{V} - 214y^{IV} - 103y''' + 784y'' - 161y' - 90y \quad = \quad 0$

    (c) $100y^{VI} - 100y^{V} - 271y^{IV} + 146y''' + 214y'' + 6y' + 9y \quad = \quad 0$

    (d) $24y^{VII} + 140y^{VI} + 126y^{V} - 311y^{IV} - 503y''' - 246y'' - 40y' \quad = \quad 0$

    (e) $120y^{VIII} + 634y^{VII} - 239y^{VI} - 3296y^{V} + 2221y^{IV} + 4876y''' - 7196y'' + 3456y' - 576y \quad = \quad 0$

    Then, verify them by substituting into the differential equations.

18. Derive the general solutions of the higher-order nonhomogeneous differential equations,

    (a) $y''' - 3y'' - 10y' + 24y \quad = \quad e^{-7x}$

    (b) $y''' - 10y'' + 25y' \quad = \quad x^4 - x^2 + 9$

    (c) $4y^{IV} + 13y'' + 9y \quad = \quad x^2 \sin x$

    (d) $9y^{IV} - 18y''' - 110y'' - 50y' - 375y \quad = \quad xe^{x} + 1$

    (e) $81y^{IV} + 216y''' + 288y'' + 384y' + 256y \quad = \quad e^{-x}\cos x + x^2$

    Show derivation of the solutions in detail and verify them by comparing with the solutions obtained from the `dsolve` command.

19. Solve the initial value problems governed by the higher-order nonhomogeneous differential equations and the initial conditions,

    (a) $6y''' + 49y'' + 44y' - 35y \quad = \quad \cos(2x) + 1$

    $y(0) = 0\,, \quad y'(0) = 0\,, \quad y''(0) = 0$

(b) $100y^{IV} + 229y'' + 9y = \sin x$

$y(0) = 0\,,\;\; y'(0) = 0\,,\;\; y''(0) = 0\,,\;\; y'''(0) = 0$

(c) $64y^{IV} + 96y''' + 52y'' + 24y' + 9y = x^2 - e^x$

$y(0) = 0\,,\;\; y'(0) = 5\,,\;\; y''(0) = 3\,,\;\; y'''(0) = 1$

(d) $324y^{IV} + 261y'' + 25y = x^2 - e^x$

$y(0) = 0\,,\;\; y'(0) = 1\,,\;\; y''(0) = 2\,,\;\; y'''(0) = 3$

Show derivation of the solutions in detail and verify them by comparing with the solutions obtained from the `dsolve` command. Then use the `ezplot` command to plot the variation of *y* in the interval of $0 \le x \le 10$.

20. Employ the `ode23` command to determine approximate solution of the initial value problem governed by the third-order homogeneous differential equation,

$$2y''' - 5y'' - 31y' + 84y = 0 \qquad\qquad 0 \le x \le 1$$

with the initial conditions of $y(0) = 0$, $y'(0) = 1$ and $y''(0) = 2$. Then, use the `dsolve` command to find the exact solution. Plot to compare the two solutions in the interval of $0 \le x \le 1$.

21. Employ the `ode23` command to determine approximate solution of the initial value problem governed by the fourth-order homogeneous differential equation,

$$24y^{IV} + 46y''' - 63y'' - 60y' + 25y = 0 \qquad 0 \le x \le 2$$

with the initial conditions of $y(0) = 0$, $y'(0) = -1$, $y''(0) = -2$ and $y'''(0) = -3$. Then, use the `dsolve` command to find the exact solution. Plot to compare the two solutions in the interval of $0 \le x \le 2$.

22. Employ the `ode45` command to determine approximate solution of the initial value problem governed by the fourth-order nonhomogeneous differential equation,

$$42y^{IV} - 29y''' - 516y'' - 157y' - 12y = e^x + \cos x$$
$$0 \le x \le 2$$

with the initial conditions of $y(0) = y'(0) = y''(0) = y'''(0) = 0$. Then, use the `dsolve` command to find the exact solution. Plot to compare the two solutions in the interval of $0 \le x \le 2$.

23. Employ the `ode45` command to determine approximate solution of the initial value problem governed by the fourth-order nonhomogeneous differential equation,

$$4y^{IV} + 61y'' + 225y = 5x^2 - 24x \qquad 0 \le x \le 10$$

with the initial conditions of $y(0) = y'(0) = y''(0) = y'''(0) = 0$. Then, use the `dsolve` command to find the exact solution. Plot to compare the two solutions in the interval of $0 \le x \le 10$.

24. Check whether the `dsolve` command can find the exact solution of the initial value problem governed by the fourth-order nonhomogeneous nonlinear differential equation,

$$y\, y^{IV} + 61y'' + 225y = 5x^2 - 24x \qquad 0 \le x \le 2$$

with the initial conditions of $y(0) = y'(0) = y''(0) = y'''(0) = 0$. If the exact solution could not be found, use the `ode45` command to solve for the approximate solution. Plot to show the solution of *y* that varies with *x* in the interval of $0 \le x \le 2$.

Chapter 7

Laplace Transforms

7.1 Introduction

Several practical problems are governed by the higher-order nonhomogeneous differential equations. Functions on the right-hand-side of the differential equations do have physical meanings. These functions may represent an external force of a mechanical system or an impressed voltage in an electrical system. Magnitudes of these functions may change abruptly in the form of a unit impulse or square wave. The method for solving the nonhomogeneous differential equations presented in the preceding chapters is not suitable for solving these types of problems. The Laplace transform that we will learn in this chapter can solve these problems effectively.

Laplace transform is a topic that creates difficulty in solving differential equations to most students. This is mainly because there are several transform formulas that need to memorize. The Laplace transformation process includes both forward and inverse transformations. The forward transformation can be carried out without much effort, while the inverse transformation is rather difficult and limited to few functions.

With the help of MATLAB commands, the task of transformations can be carried out easily. We will start the chapter by learning the definition of the Laplace transform. We will use examples to understand the transformation process. Results of the transformations will be verified by using MATLAB commands. At the end of the chapter, we will learn how to apply the method of Laplace transform to solve some realistic problems governed by the nonhomogeneous differential equations. These include the mass-spring-damper system subjected to different types of loadings. Solutions will be plotted to increase understanding of the system behaviors.

7.2 Definitions

The Laplace transform is defined by the integration from $t = 0$ to ∞ of the product between the functions e^{-st} and $f(t)$ as,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

where $f(t)$ is the function of *t* for $t \geq 0$. The result is denoted by,

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

In the opposite way, the function $f(t)$ is called the inverse Laplace transform of $F(s)$,

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

The function $f(t)$ may be in different forms. For examples, if $f(t) = 1$, the Laplace transform is,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

If $f(t) = e^{at}$ where a is a constant, then the Laplace transform is,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\,e^{at}\,dt$$

$$= \frac{1}{a-s}e^{-(s-a)t}\Big|_0^\infty = \frac{1}{s-a}$$

for $s - a > 0$.

If $f(t) = \sin at$, then the Laplace transform can be obtained as follows,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt$$

$$= \frac{e^{-st}}{-s}\sin at\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos at\,dt$$

$$= \frac{a}{s^2+a^2}$$

Note that the result above is from integrating by parts, i.e., if we let,

$$L_c = \mathcal{L}\{\cos at\} \qquad \text{and} \qquad L_s = \mathcal{L}\{\sin at\}$$

then,

$$L_c = \int_0^\infty e^{-st}\cos at\,dt = \frac{e^{-st}}{-s}\cos at\Big|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin at\,dt$$

$$= \frac{1}{s} - \frac{a}{s}L_s$$

$$L_s = \int_0^\infty e^{-st}\sin at\,dt = \frac{e^{-st}}{-s}\sin at\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos at\,dt = \frac{a}{s}L_c$$

By substituting L_c from the upper into lower equation, we get,

$$L_s = \frac{a}{s}\left(\frac{1}{s} - \frac{a}{s}L_s\right) \qquad \text{or} \qquad L_s\left(1 + \frac{a^2}{s^2}\right) = \frac{a}{s^2}$$

Thus, $$L_s = \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

Similarly, if we substitute L_s from the lower equation into L_c in the upper equation, we get,

$$L_c = \frac{1}{s} - \frac{a}{s}\left(\frac{a}{s}L_c\right) \qquad \text{or} \qquad L_c\left(1 + \frac{a^2}{s^2}\right) = \frac{1}{s}$$

i.e.,
$$L_c = \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

This means the Laplace transform has the property of linearity as,

$$\mathcal{L}\{a\, f(t) + b\, g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

where a and b are constants. As an example, the hyperbolic sine function is,

$$\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$$

then,

$$\begin{aligned}
\mathcal{L}\{\sinh at\} &= \frac{1}{2}\left(\mathcal{L}\{e^{at}\} - \mathcal{L}\{e^{-at}\}\right) \\
&= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) \\
&= \frac{a}{s^2 - a^2}
\end{aligned}$$

Similarly, the hyperbolic cosine function is,

$$\cosh at = \frac{1}{2}\left(e^{at} + e^{-at}\right)$$

then,

$$\begin{aligned}
\mathcal{L}\{\cosh at\} &= \frac{1}{2}\left(\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{-at}\}\right) \\
&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) \\
&= \frac{s}{s^2 - a^2}
\end{aligned}$$

Laplace transform of a given function $f(t)$ can be obtained without much difficulty. Many mathematical textbooks have shown tables of the Laplace transform for different functions. Same results are obtained by using MATLAB commands as will be shown in the following sections.

7.3 Laplace Transform

MATLAB contains the `laplace` command that can be used to obtain result of the Laplace transform of a given function $f(t)$. For examples, if we want the Laplace transform of the function,

$$f(t) = e^{at}$$

we just enter the following,

syms

laplace

```
>> syms t s a
>> laplace(exp(a*t))

ans =

-1/(a - s)
```

i.e., the result is,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\} = -\frac{1}{a-s}$$

Or, if we want the Laplace transform of the function,

$$f(t) = \sin at$$

we enter the command,

```
>> laplace(sin(a*t))

ans =

a/(a^2 + s^2)
```

i.e., the result is,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\} = \frac{a}{a^2+s^2}$$

Similarly, if we want the Laplace transform of the function,

$$f(t) = \cos at$$

we enter the command,

laplace

```
>> laplace(cos(a*t))
ans =
s/(a^2 + s^2)
```

i.e., the result is,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos at\} = \frac{s}{a^2+s^2}$$

The Laplace transforms of the hyperbolic sine and cosine functions,

$$\mathcal{L}\{\sinh at\} \qquad \text{and} \qquad \mathcal{L}\{\cosh at\}$$

can be obtained conveniently by entering the commands,

```
>> laplace(sinh(a*t))
ans =
-a/(a^2 - s^2)
>> laplace(cosh(a*t))
ans =
-s/(a^2 - s^2)
```

i.e., the results are,

$$-\frac{a}{a^2-s^2} \qquad \text{and} \qquad -\frac{s}{a^2-s^2}$$

These results agree with those derived earlier.

The `laplace` command can be used to provide the Laplace transforms of other functions, such as,

$$\mathcal{L}\{t^2\} \qquad \text{and} \qquad \mathcal{L}\{t^7\}$$

```
>> laplace(t^2)
ans =
2/s^3
>> laplace(t^7)
ans =
5040/s^8
```

laplace

i.e., the results are,

$$\frac{2}{s^3} \quad \text{and} \quad \frac{5040}{s^8}$$

which agree with the formula,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

The command can be also used to obtain the Laplace transforms of more complex functions. For example,

$$f(t) = \sin(at)\cosh(at) - \cos(at)\sinh(at)$$

```
>> laplace(sin(a*t)*cosh(a*t)-cos(a*t)*sinh(a*t))
ans =
(4*a^3)/(4*a^4 + s^4)
```

The `simple` command may be used to reduce the complexity of the result above to yield,

$$\mathcal{L}\{\sin(at)\cosh(at)-\cos(at)\sinh(at)\} = \frac{4a^3}{4a^4+s^4}$$

It is noted that the `laplace` command employs the property of linearity as mentioned earlier. As an example,

$$f(t) = 1+5t$$

```
>> laplace(1 + 5*t)
ans =
1/s + 5/s^2
```

i.e., the result is,

$$\mathcal{L}\{1+5t\} = \frac{1}{s}+\frac{5}{s^2}$$

Or, as another example,

$$f(t) = 4e^{-3t} - 10\sin 2t$$

```
>> laplace(4*exp(-3*t) - 10*sin(2*t))

ans =

4/(s + 3) - 20/(s^2 + 4)
```

i.e., the result is,

$$\mathcal{L}\{4e^{-3t} - 10\sin 2t\} = \frac{4}{s+3} - \frac{20}{s^2+4}$$

7.4 Inverse Laplace Transform

MATLAB contains the `ilaplace` command to provide an inverse Laplace transform of a given function $F(s)$. For example, if we want to find the inverse Laplace transform of the function,

$$F(s) = \frac{1}{s}$$

we enter the commands,

```
>> syms t s a b
>> ilaplace(1/s)

ans =

1
```

ilaplace

i.e., we obtain the result of,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

Or, as another example, to find the inverse Laplace transform of,

$$F(s) = \frac{1}{s-a}$$

```
>> ilaplace(1/(s-a))
ans =
exp(a*t)
```

i.e., the result is,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

And another example,

$$F(s) = \frac{a}{a^2+s^2}$$

```
>> ilaplace(a/(a^2 + s^2))
ans =
sin(a*t)
```

ilaplace

i.e.,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{a}{a^2+s^2}\right\} = \sin at$$

Also, the inverse Laplace transform of the function,

$$F(s) = \frac{s}{s^2-a^2}$$

```
>> ilaplace(s/(s^2 - a^2))
ans =
cosh(a*t)
```

i.e.,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$$

Note that the results obtained may be in form of long expressions. The `simple` command can be used to simplify their complexity.

Many differential equation textbooks provide tables of inverse Laplace transforms for some popular forms of $F(s)$.

The `ilaplace` command can reduce effort to find inverse Laplace transforms of complicated functions $F(s)$. For example,

$$F(s) = \frac{3s+7}{s^2-2s-3}$$

The process to perform the inverse transformation of this function is as follows,

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{3s+7}{(s-1)^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-4}\right\} \\
&= 3\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2-4}\right\} + 5\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2-4}\right\} \\
&= 3e^t \cosh 2t + 5e^t \sinh 2t \\
&= e^t\left(3\cosh 2t + 5\sinh 2t\right) \\
&= 4e^{3t} - e^{-t}
\end{aligned}$$

In the process shown above, terms must be arranged properly so that results could be found from the inverse Laplace transform table. The same result is obtained easily by using the `ilaplace` command,

```
>> ilaplace((3*s + 7)/(s^2 - 2*s - 3))
ans =
4*exp(3*t) - 1/exp(t)
```

As another example when $F(s)$ is complicated,

$$F(s) = \frac{s}{(s^2+1)^2}$$

The inverse Laplace transform can be obtained easily by using the `ilaplace` command,

```
>> ilaplace(s/((s^2+1)^2))
```

ilaplace

```
ans =
(t*sin(t))/2
```

i.e., the result of $f(t)$ is,

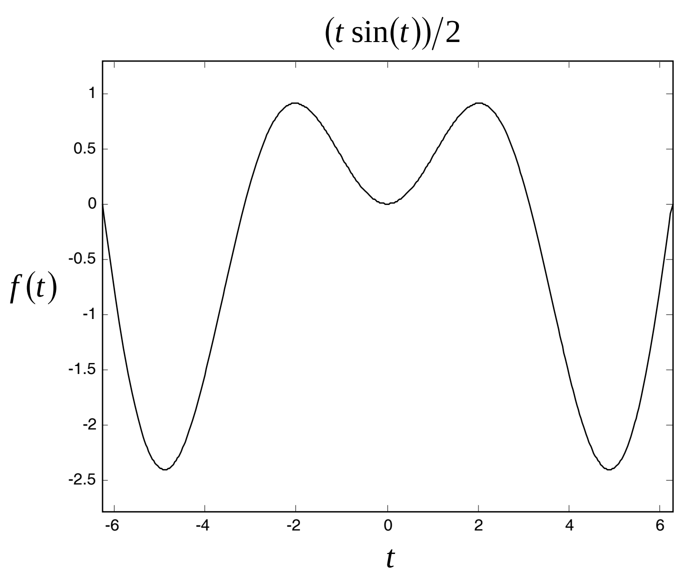
$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$= \frac{1}{2}t\sin t$$

Variation of the $f(t)$ function is plotted using the `ezplot` command as shown in the figure below.

```
>> ezplot(ans)
```

ezplot



The `ilaplace` command can provide the inverse Laplace transform even though the given function $F(s)$ is quite complicated,

$$F(s) = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$$

```
>> ilaplace((s^2+6*s+9)/((s-1)*(s-2)*(s+4)))
ans =
(25*exp(2*t))/6 + 1/(30*exp(4*t)) - (16*exp(t))/5
```

i.e., the result of $f(t)$ is,

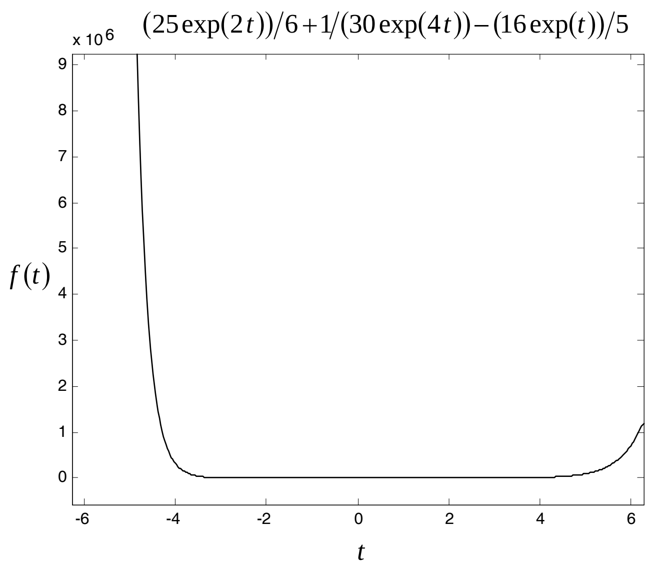
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

$$= \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t} - \frac{16}{5}e^{t}$$

which can be plotted by using the `ezplot` command as shown in the figure.

```
>> ezplot(ans)
```

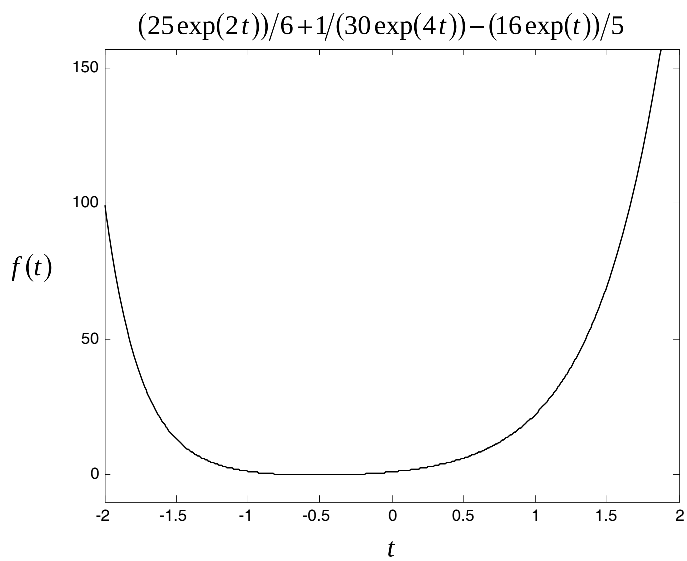
ezplot



The figure indicates a sudden change of $f(t)$ in the interval of $-5 \le t \le -4$. If we want to amplify the change that occurs in the interval of $-2 \le t \le 2$, we can modify the `ezplot` command slightly by including the desired interval for plotting as follow,

```
>> ezplot(ans, [-2, 2])
```

The variation of $f(t)$ in the interval of $-2 \le t \le 2$ is now shown in the figure.



7.5 Solving Differential Equations

In this section, we will learn how to apply the method of the Laplace transform to solve the initial value problems. We will see that the transformation changes the differential equation to an algebraic equation which is easier to solve. It should be noted that the Laplace transform for the first-order derivative of function $f(t)$ is,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Similarly, the transform for the second-order derivative of function $f(t)$ is,

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

The transformation for the higher-order derivatives of the function $f(t)$ can be determined in the same way. Application of the Laplace transform method for solving the initial value problems is demonstrated by using the following examples.

Example Solve the initial value problem governed by the second-order nonhomogeneous differential equation,

$$y'' + y = t$$

with the initial conditions of $y(0)=1$ and $y'(0)=-2$. We start from finding the Laplace transform of the differential equation,

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{t\}$$

which gives,

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + \mathcal{L}\{y\} = \mathcal{L}\{t\}$$

Since $\mathcal{L}\{t\} = 1/s^2$ can be found by using the `laplace` command,

```
>> syms t s y
>> laplace(t)

ans =

1/s^2
```

syms

laplace

With the given initial conditions of $y(0)=1$ and $y'(0)=-2$, then the equation above becomes,

$$s^2 Y - s + 2 + Y = \frac{1}{s^2}$$

or,

$$Y = \left(\frac{1}{s^2} + s - 2\right)\Big/\left(s^2+1\right)$$

The inverse Laplace transform of the Y function above is,

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^2} + s - 2\right)\Big/\left(s^2+1\right)\right\}$$

which can be obtained by using the `ilaplace` command,

```
>> ilaplace((1/s^2+s-2)/(s^2+1))
ans =
t + cos(t) - 3*sin(t)
```

ilaplace

Thus, the solution of the initial value problem is,

$$y = t + \cos t - 3\sin t$$

It is noted that if we solve this problem by hands, we need to arrange the terms of s on the right-hand-side of the Y equation properly so that we can find their inverse Laplace transforms from the transform table as follows,

$$\begin{aligned} Y &= \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1} \\ &= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1} \\ &= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1} \end{aligned}$$

Then,

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right\}$$

Thus, the solution is,

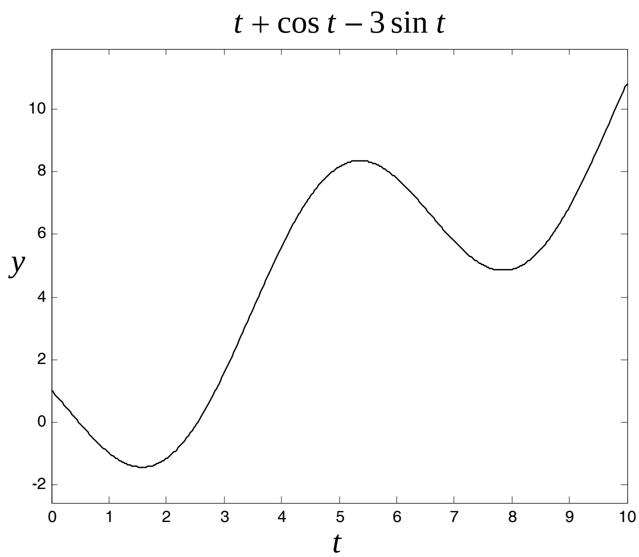
$$y = t + \cos t - 3\sin t$$

Such solution can be plotted by using the `ezplot` command as shown in the figure.

ezplot

```
>> ezplot(ans, [0, 10])
```

The plot shows that both the initial conditions of $y(0) = 1$ and $y'(0) = -2$ are satisfied. This example demonstrates the advantage of the Laplace transform method for solving the initial value problem conveniently. Results of the Laplace and inverse Laplace transforms are obtained by using the `laplace` and `ilaplace` commands, respectively.



Example Solve the initial value problem governed by the second-order nonhomogeneous differential equation,

$$y'' - 3y' + 2y = 4e^{2t}$$

with the initial conditions of $y(0) = -3$ and $y'(0) = 5$.

We start from finding the Laplace transform of the differential equation,

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 4\mathcal{L}\{e^{2t}\}$$

Here $\mathcal{L}\{e^{2t}\} = 1/(s-2)$ which is obtained from using the `laplace` command,

laplace

```
>> laplace(exp(2*t))

ans =

1/(s - 2)
```

so that the transformed equation is,

$$\left(s^2Y - sy(0) - y'(0)\right) - 3\left(sY - y(0)\right) + 2Y = \frac{4}{s-2}$$

After applying the initial conditions, the transformed equation becomes,

$$\left(s^2Y + 3s - 5\right) - 3\left(sY + 3\right) + 2Y \; = \; \frac{4}{s-2}$$

which gives,

$$Y \; = \; \left(\frac{4}{s-2} - 3s + 14\right) \Big/ \left(s^2 - 3s + 2\right)$$

The solution is then obtained by inverse transformation,

$$y \; = \; \mathcal{L}^{-1}\{Y\} \; = \; \mathcal{L}^{-1}\left\{\left(\frac{4}{s-2} - 3s + 14\right) \Big/ \left(s^2 - 3s + 2\right)\right\}$$

which can be done by using the `ilaplace` command,

```
>> ilaplace(((4/(s-2))-3*s+14)/(s^2-3*s+2))
ans =
4*exp(2*t) - 7*exp(t) + 4*t*exp(2*t)
```

`ilaplace`

Hence, the solution of the initial value problem is,

$$y \; = \; 4e^{2t} - 7e^{t} + 4t\,e^{2t}$$

The solution of y that varies with t is plotted by using the `ezplot` command in the interval of $0 \le t \le 1$ as shown in the figure.

```
>> ezplot(ans, [0, 1])
```

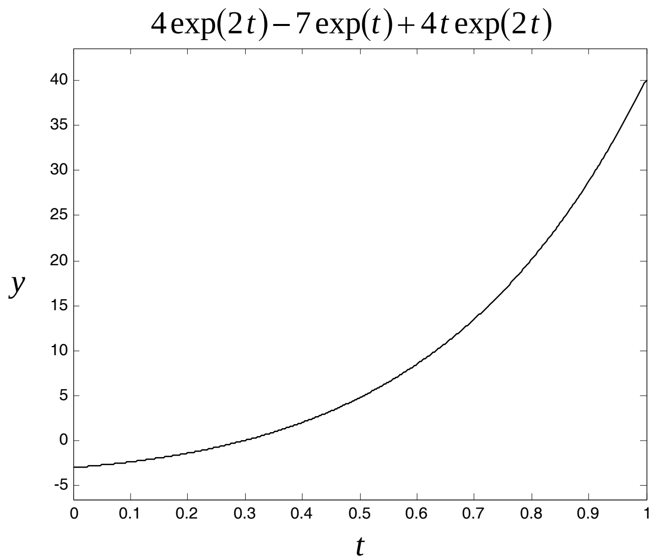
<u>Example</u> Solve the initial value problem governed by the fourth-order homogeneous differential equation,

$$y^{IV} - y \; = \; 0$$

with the initial conditions of $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$ and $y'''(0) = 1$.

Again, we start by performing the Laplace transform of the given differential equation,

$$\mathcal{L}\{y^{IV}\} - \mathcal{L}\{y\} \; = \; 0$$



to give,

$$s^4 Y - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0) - Y \;=\; 0$$

Then, we apply the initial conditions to obtain,

$$s^4 Y - 0 - 0 - 0 - 1 - Y \;=\; 0$$

Thus,

$$Y \;=\; \frac{1}{s^4 - 1}$$

After that, we can find the inverse Laplace transform by using the `ilaplace` command,

```
>> ilaplace(1/(s^4-1))

ans =

sinh(t)/2 - sin(t)/2
```

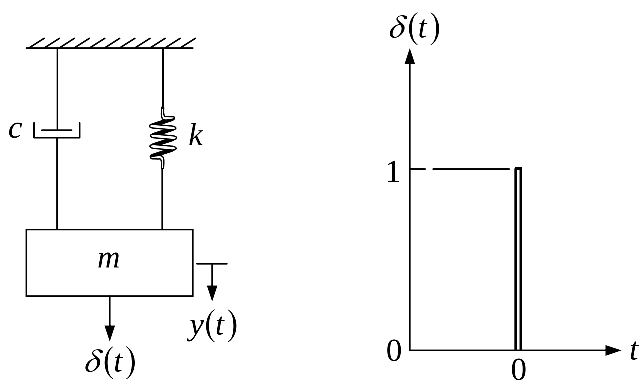
Hence, the solution of this initial value problem is,

$$y \;=\; \frac{1}{2}(\sinh t - \sin t)$$

<u>Example</u> Motion of the mass in the mass-spring-damper system as shown in the figure is governed by the second-order nonhomogeneous differential equation,

$$2y'' + y' + 2y \;=\; \delta(t)$$

with the initial conditions of $y(0)=0$ and $y'(0)=0$. The notation $\delta(t)$ on the right-hand-side of the differential equation is the Dirac delta function representing a unit impulse applied on the mass. The coefficients in the differential equation are equivalent to the mass of $m=2$, the damping coefficient of $c=1$ and the spring stiffness of $k=2$.



We can use the method of Laplace transform to solve for the mass motion which is the displacement in the vertical direction $y(t)$ that varies with time t. Similar to the preceding examples, we start from transforming the differential equation,

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} \;=\; \mathcal{L}\{\delta(t)\}$$

which leads to,

$$2(s^2Y - s\,y(0) - y'(0)) + (sY - y(0)) + 2Y \;=\; 1$$

It is noted that transformation of the Dirac delta function on the right-hand-side of the differential equation can be obtained conveniently by using the `laplace` command,

`laplace`

```
>> laplace(dirac(t))
ans =
1
```

We apply the given initial conditions of zero displacement and velocity at time $t = 0$ to give,

$$2\left(s^2 Y - 0 - 0\right) + \left(sY - 0\right) + 2Y \;=\; 1$$

Thus,

$$Y \;=\; \frac{1}{2s^2 + s + 2}$$

Then, we perform inverse transformation to find the solution of the displacement $y(t)$ that varies with time t,

$$y(t) \;=\; \mathscr{L}^{-1}\{Y\} \;=\; \mathscr{L}^{-1}\left\{\frac{1}{2s^2 + s + 2}\right\}$$

This is done by using the `ilaplace` command,

```
>> ilaplace(1/(2*s^2+s+2))
```

`ilaplace`

```
ans =

(2*15^(1/2)*sin((15^(1/2)*t)/4))/(15*exp(t/4))
```

Hence, the solution for the motion of the mass is,

$$y(t) \;=\; \frac{2}{\sqrt{15}} e^{-t/4} \sin\left(\frac{\sqrt{15}}{4} t\right)$$

The motion is plotted for the interval of $0 \le t \le 15$ by using the command >> `ezplot(ans, [0, 15])` as shown in the figure. At the early time, the displacement is large from the unit impulse that applies on the mass. The displacement decreases as the time increases due to the damping effect.

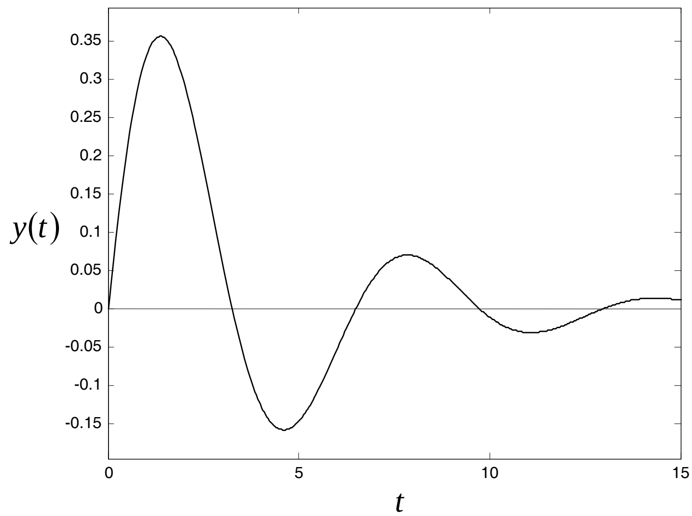
If the system does not include the damper $(c = 0)$, the governing differential equation becomes,

$$2y'' + 2y \;=\; \delta(t)$$

After performing transformation, we get,

$$Y \;=\; \frac{1}{2s^2 + 2}$$

Then, we find the solution by performing the inverse transformation,



$$y(t) = \mathscr{L}^{-1}\{Y\} = \mathscr{L}^{-1}\left\{\frac{1}{2s^2+2}\right\}$$

which can be done by using the `ilaplace` command,

```
>> ilaplace(1/(2*s^2+2))
ans =
sin(t)/2
```

ilaplace

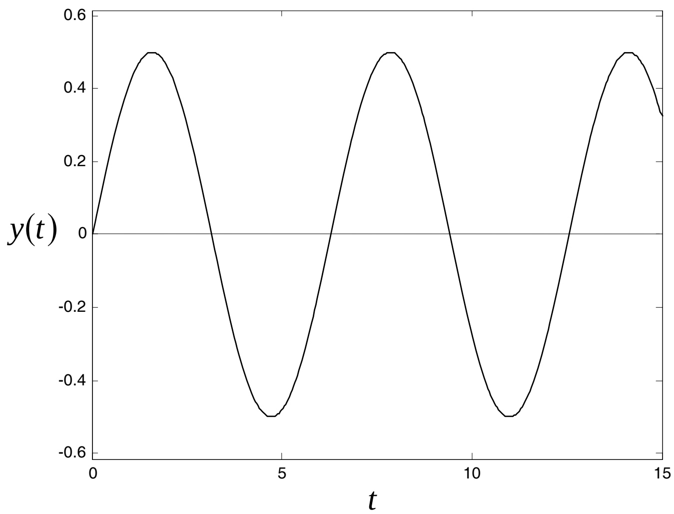
This leads to the solution of the motion when there is no damper,

$$y(t) = \frac{\sin t}{2}$$

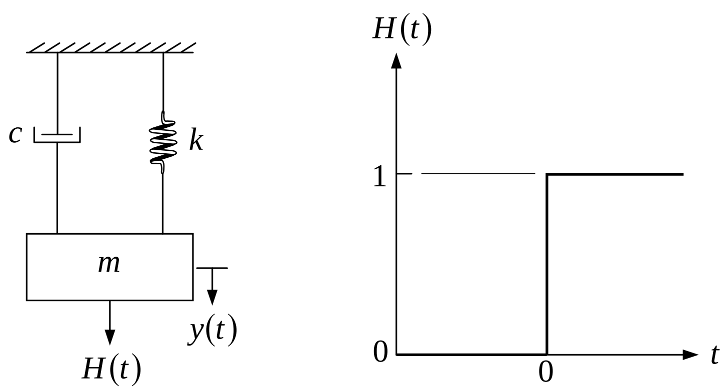
The motion can be plotted by using the `ezplot` command,

```
>> ezplot(ans, [0, 15])
```

The figure shows that the mass moves up and down as the sine function about $y = 0$.



Example Determine the displacement $y(t)$ of the mass if the mass-spring-damper system in the preceding example is subjected to a unit step (Heaviside) function as shown in the figure.



For the mass-spring-damper system with $m = 2$, $c = 1$ and $k = 2$ as explained in the preceding example, the corresponding differential equation is,

$$2y'' + y' + 2y \ = \ H(t)$$

The forcing function on the right-hand-side of the equation is,

$$H(t) \ = \ \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

This differential equation is to be solved with the initial conditions of zero displacement and velocity at $t = 0$, i.e., $y(0) = 0$ and $y'(0) = 0$.

By considering the differential equation, we can see that as $t \to \infty$, $y'' \to 0$ and $y' \to 0$, then the displacement at the equilibrium position is $y = 1/2 = 0.5$.

To determine the dynamic response of the mass, the Laplace transformation is applied to the differential equation,

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} \;=\; \mathcal{L}\{H(t)\}$$

which gives,

$$2\left(s^2 Y - s\,y(0) - y'(0)\right) + (sY - y(0)) + 2Y \;=\; \frac{1}{s}$$

The transformation of the unit step function on the right-hand-side of the equation is obtained easily by using the `laplace` command,

laplace

```
>> laplace(heaviside(t))

ans =

1/s
```

Then, the initial conditions for the displacement and velocity of $y(0) = 0$ and $y'(0) = 0$ are imposed to give,

$$2\left(s^2 Y - 0 - 0\right) + (sY - 0) + 2Y \;=\; \frac{1}{s}$$

i.e.,

$$Y \;=\; \frac{1}{s(2s^2 + s + 2)}$$

The inverse transformation is performed to yield the solution of the displacement *y* that varies with time *t*,

$$y(t) \;=\; \mathcal{L}^{-1}\{Y\} \;=\; \mathcal{L}^{-1}\left\{\frac{1}{s(2s^2 + s + 2)}\right\}$$

This can be done by using the `ilaplace` command,

```
>> ilaplace(1/(s*(2*s^2+s+2)))

ans =

1/2 - (cos((15^(1/2)*t)/4) +
(15^(1/2)*sin((15^(1/2)*t)/4))/15)/(2*exp(t/4))
```

which is,

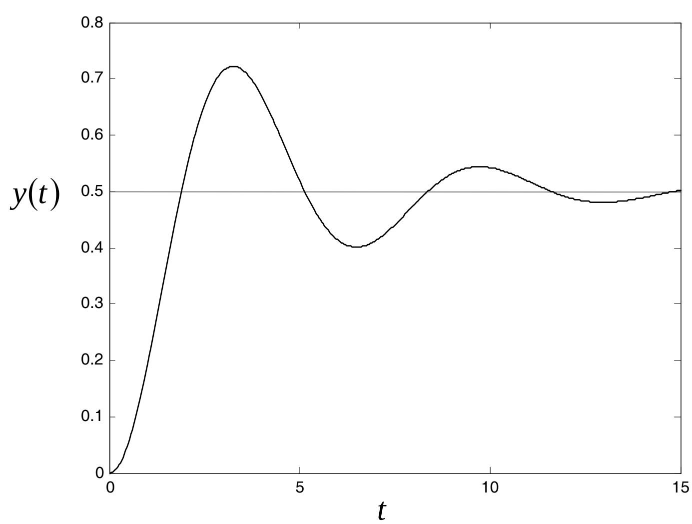
$$y(t) \;=\; \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{2\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

The vertical movement y that varies with time t is plotted by using the `ezplot` command in the interval of $0 \le t \le 15$ as shown in the figure,

ezplot

```
>> ezplot(sol, [0, 15, 0, .8])
```

where `sol` is the same as `ans` under the quote sign (' '). It can be seen from the figure that the system approaches the equilibrium configuration at the displacement of $y = 0.5$ as expected.



If we increase the damping coefficient from $c = 1$ to $c = 2$, then the differential equation becomes,

$$2y'' + 2y' + 2y \;=\; H(t)$$

This leads to the value of Y after performing transformation,

$$Y = \frac{1}{2s(s^2+s+1)}$$

Then, the displacement solution is,

$$y(t) = \mathscr{L}^{-1}\{Y\} = \mathscr{L}^{-1}\left\{\frac{1}{2s(s^2+s+1)}\right\}$$

The `ilaplace` command is used to find the inverse Laplace transform,

```
>> ilaplace(1/(2*s*(s^2+s+1)))
```

`ilaplace`

```
ans =

1/2 - (cos((3^(1/2)*t)/2) +
(3^(1/2)*sin((3^(1/2)*t)/2))/3)/(2*exp(t/2))
```

i.e.,

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{2\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

The solution above is plotted as shown in the figure. The figure shows that, when the damping coefficient is larger, the mass-spring-damper system reaches the equilibrium configuration sooner with smaller magnitude of oscillation.

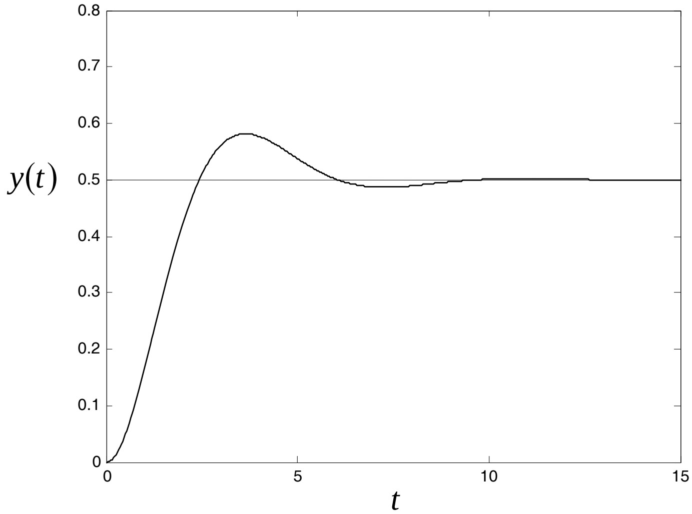
The technique of Laplace transform can also be applied to solve a set of differential equations. This is explained by using the examples as follows.

<u>Example</u> Solve a set of two first-order differential equations,

$$\begin{aligned} x' - 2x + 3y &= 0 \\ y' - y + 2x &= 0 \end{aligned}$$

for the solution of $x(t)$ and $y(t)$ with the initial conditions of $x(0) = 8$ and $y(0) = 3$.

We start by finding the Laplace transform of the first differential equation,



$$\begin{aligned}\mathscr{L}\{x'\} - 2\mathscr{L}\{x\} + 3\mathscr{L}\{y\} &= 0\\ (sX - x(0)) - 2X + 3Y &= 0\end{aligned}$$

and apply the initial condition to obtain,

$$sX - 8 - 2X + 3Y = 0$$

or, $$(s-2)X + 3Y = 8$$

Similarly, we find the Laplace transform of the second differential equation,

$$\begin{aligned}\mathscr{L}\{y'\} - \mathscr{L}\{y\} + 2\mathscr{L}\{x\} &= 0\\ (sY - y(0)) - Y + 2X &= 0\end{aligned}$$

and apply the initial condition to obtain,

$$sY - 3 - Y + 2X = 0$$

or, $$(s-1)Y + 2X = 3$$

Thus, the Laplace transformation leads to the two algebraic equations,

$$\begin{aligned}(s-2)X + 3Y &= 8\\ 2X + (s-1)Y &= 3\end{aligned}$$

which can be solved for function X and Y. We can use the `solve` command to solve them as follows,

```
>> syms x y s
>> eq1 = '(s-2)*x + 3*y = 8';
>> eq2 = '2*x + (s-1)*y = 3';
>> [xx yy] = solve(eq1,eq2)
```

solve

```
xx =

-(8*s - 17)/(- s^2 + 3*s + 4)

yy =

-(3*s - 22)/(- s^2 + 3*s + 4)
```

to obtain,
$$X = -\frac{8s-17}{-s^2+3s+4}$$
and
$$Y = -\frac{3s-22}{-s^2+3s+4}$$

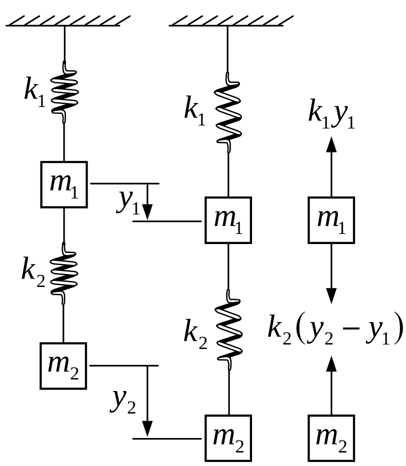
Then, we perform the inverse transformation,
$$x(t) = \mathcal{L}^{-1}\{X\} = \mathcal{L}^{-1}\left\{-\frac{8s-17}{-s^2+3s+4}\right\}$$
and
$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{-\frac{3s-22}{-s^2+3s+4}\right\}$$
by using the `ilaplace` command,

```
>> ilaplace(-(8*s - 17)/(- s^2 + 3*s + 4))
ans =
5/exp(t) + 3*exp(4*t)
>> ilaplace(-(3*s - 22)/(- s^2 + 3*s + 4))
ans =

5/exp(t) - 2*exp(4*t)
```

ilaplace

Hence, the solutions of the coupled first-order differential equations are,
$$x(t) = 5e^{-t} + 3e^{4t}$$
and
$$y(t) = 5e^{-t} - 2e^{4t}$$

<u>Example</u> A mass-spring system consisting of two masses and two springs is shown in the figure. The two masses are m_1 and m_2, while the two springs have the spring stiffness of k_1 and k_2, respectively. The motion of the two masses (y_1 and y_2) are governed by the two differential equations from the Newton's second law.



<u>Mass m_1</u>

$$m_1 y_1'' \;=\; -k_1 y_1 + k_2(y_2 - y_1)$$

<u>Mass m_2</u>

$$m_2 y_2'' \;=\; -k_2(y_2 - y_1)$$

If $m_1 = 1$, $m_2 = 1$, $k_1 = 6$ and $k_2 = 4$, then the differential equations that describe the motions of $y_1(t)$ and $y_2(t)$ at any time t are,

$$y_1'' + 10y_1 - 4y_2 \;=\; 0$$

and

$$y_2'' - 4y_1 + 4y_2 \;=\; 0$$

The coupled differential equations above will be solved with the initial conditions of $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 0$ and $y_2'(0) = -1$.

We start by transforming the first differential equation,

$$\mathcal{L}\{y_1''\} + 10\mathcal{L}\{y_1\} - 4\mathcal{L}\{y_2\} \;=\; 0$$

$$\left(s^2 Y_1 - s\, y_1(0) - y_1'(0)\right) + 10Y_1 - 4Y_2 \;=\; 0$$

After applying the given initial conditions, we obtain,

$$\left(s^2 Y_1 - 0 - 1\right) + 10Y_1 - 4Y_2 \;=\; 0$$

or,

$$\left(s^2 + 10\right)Y_1 - 4Y_2 \;=\; 1$$

Similarly, we transform the second differential equation,

$$\mathcal{L}\{y_2''\} - 4\mathcal{L}\{y_1\} + 4\mathcal{L}\{y_2\} = 0$$
$$(s^2 Y_2 - s\,y_2(0) - y_2'(0)) - 4Y_1 + 4Y_2 = 0$$

After applying the given initial conditions, we obtain,

$$\left(s^2 Y_2 - 0 + 1\right) - 4Y_1 + 4Y_2 = 0$$

or,

$$4Y_1 - (s^2 + 4)Y_2 = 1$$

Thus, the two algebraic equations after applying the transformation and initial conditions are,

$$(s^2 + 10)Y_1 - 4Y_2 = 1$$
$$4Y_1 - (s^2 + 4)Y_2 = 1$$

We can use the `solve` command to solve for Y_1 and Y_2 as follows,

```
>> syms y1 y2 s
>> eq1 = '(s^2+10)*y1 - 4*y2 = 1';
>> eq2 = ' 4*y1 - (s^2+4)*y2 = 1';
>> [yy1 yy2] = solve(eq1,eq2)

yy1 =

s^2/(s^4 + 14*s^2 + 24)

yy2 =

-(s^2 + 6)/(s^4 + 14*s^2 + 24)
```

solve

i.e.,

$$Y_1 = \frac{s^2}{s^4 + 14s^2 + 24}$$
$$Y_2 = -\frac{s^2 + 6}{s^4 + 14s^2 + 24}$$

Then, we find the inverse Laplace transforms,

$$y_1(t) = \mathcal{L}^{-1}\{Y_1\} = \mathcal{L}^{-1}\left\{\frac{s^2}{s^4 + 14s^2 + 24}\right\}$$

$$y_2(t) = \mathcal{L}^{-1}\{Y_2\} = \mathcal{L}^{-1}\left\{-\frac{s^2 + 6}{s^4 + 14s^2 + 24}\right\}$$

by using the `ilaplace` command,

```
>> ilaplace(s^2/(s^4 + 14*s^2 + 24))
```

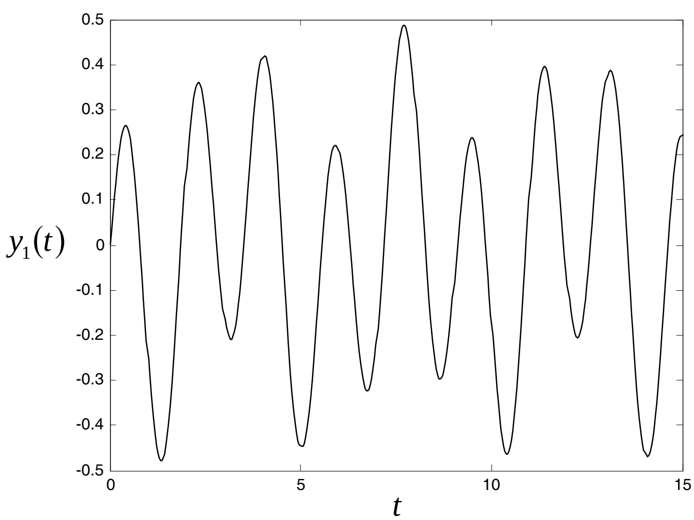
ilaplace

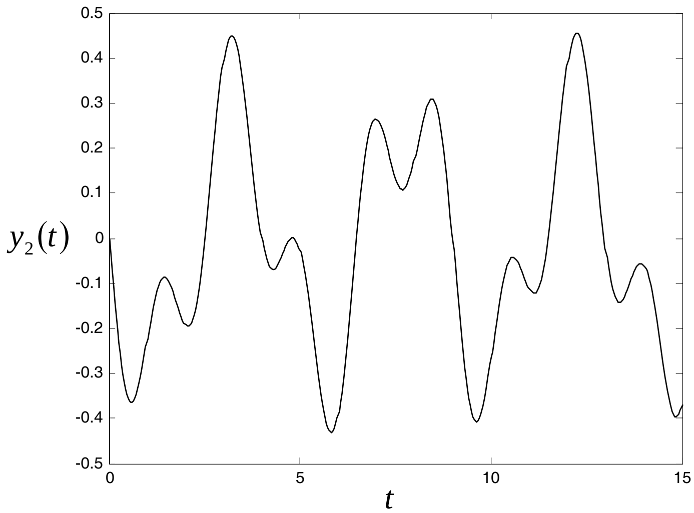
```
ans =

(3^(1/2)*sin(2*3^(1/2)*t))/5 -
(2^(1/2)*sin(2^(1/2)*t))/10

>> ilaplace(-(s^2 + 6)/(s^4 + 14*s^2 + 24))

ans =

- (2^(1/2)*sin(2^(1/2)*t))/5 -
(3^(1/2)*sin(2*3^(1/2)*t))/10
```

The solutions of $y_1(t)$ and $y_2(t)$ representing the displacements of mass m_1 and m_2 that vary with time t are,

$$\begin{aligned} y_1(t) &= \frac{\sqrt{3}}{5}\sin\left(2\sqrt{3}\,t\right) - \frac{\sqrt{2}}{10}\sin\left(\sqrt{2}\,t\right) \\ y_2(t) &= -\frac{\sqrt{2}}{5}\sin\left(\sqrt{2}\,t\right) - \frac{\sqrt{3}}{10}\sin\left(2\sqrt{3}\,t\right) \end{aligned}$$

These displacements $y_1(t)$ and $y_2(t)$ of the mass m_1 and m_2 are plotted as shown in the figures.





7.6 Concluding Remarks

In this chapter, we have studied the method of Laplace transform for solving differential equations. There are many practical problems that are governed by the nonhomogeneous differential equations for which the forcing functions on the right-hand-side of the equations are in form of the impulse or step functions. The standard methods learned in the preceding chapters are not suitable for solving this type of problems while the method of Laplace transformation can handle them very well.

We started from understanding the definitions of the Laplace transform and inverse Laplace transform. Several examples were presented to show the transformations of many functions. At the same time, the `laplace` and `ilaplace` commands are used to confirm the derived results. The method of Laplace transform was then applied to solve the nonhomogeneous differential equations for the problems mentioned above. The problems may be governed by a single differential equation or coupled equations. The method of Laplace transform changes the differential equations into algebraic equations so that they can be solved easier. Several examples have demonstrated the advantage of the method to provide solutions to this type of problems effectively.

Exercises

1. Use the `laplace` command to find $F(s)$ which are the Laplace transforms of the following functions,

   (a) $f(t) = t^{-1/2}$ (b) $f(t) = t^{1/2}$

   (c) $f(t) = a^2 t^{-1/2}$ (d) $f(t) = (a^2 + b^2)t^{1/2}$

   (e) $f(t) = e^{at} \cos bt$ (f) $f(t) = t \cos bt$

   where a and b are constants.

2. Use the `laplace` command to find $F(s)$ which are the Laplace transforms of the following functions,

   (a) $f(t) = t^2 - 2t$ (b) $f(t) = (t^2 - 3)^2$

   (c) $f(t) = \sin^2 4t$ (d) $f(t) = \sin\left(3t - \frac{1}{2}\right)$

   (e) $f(t) = e^{2t} \cosh t$ (f) $f(t) = t^2 e^{3t}$

3. Use the `laplace` command to find $F(s)$ which are the Laplace transforms of the following functions,

   (a) $f(t) = e^{2t} \cos 3t$ (b) $f(t) = t^3 + 4t^2 + 3$

   (c) $f(t) = t^3 \sin at$ (d) $f(t) = t^4 e^{2t}$

   (e) $f(t) = t e^{at} \sin bt$ (f) $f(t) = t e^{at} \cos bt$

   where a and b are constants.

4. Use the `laplace` command to find $F(s)$ which are the Laplace transforms of the following functions,

   (a) $f(t) = 6 \sin 2t - 5 \cos 2t$

   (b) $f(t) = 3 \cosh 5t - 4 \sinh 5t$

   (c) $f(t) = (5e^{2t} - 3)^2$

   (d) $f(t) = (1 + t e^{-t})^3$

   (e) $f(t) = (e^{-at} - e^{-bt})/t$

   (f) $f(t) = (\sin at - \cos bt)t$

   where a and b are constants.

5. Use the `ilaplace` command to find $f(t)$ which are the inverse Laplace transforms of the following functions,

(a) $F(s) = \dfrac{s}{(s-a)^2}$ (b) $F(s) = \dfrac{1}{(s-a)(s-b)}$

(c) $F(s) = \dfrac{s}{(s-a)(s-b)}$ (d) $F(s) = \dfrac{s}{(s-a)(s^2+b^2)}$

(e) $F(s) = \dfrac{s}{(s^2+a^2)^2}$ (f) $F(s) = \dfrac{s}{s^4-b^4}$

where *a* and *b* are constants.

6. Use the `ilaplace` command to find $f(t)$ which are the inverse Laplace transforms of the following functions,

(a) $F(s) = 3.8te^{2.4t}$ (b) $F(s) = -3t^4e^{-0.5t}$

(c) $F(s) = \dfrac{7}{(s-1)^3}$ (d) $F(s) = \dfrac{\pi}{(s+\pi)^2}$

(e) $F(s) = \dfrac{15}{s^2+4s+29}$ (f) $F(s) = \dfrac{2s-56}{s^2-4s-12}$

7. Use the `ilaplace` command to find $f(t)$ which are the inverse Laplace transforms of the following functions,

(a) $F(s) = \dfrac{6}{(s+2)^4}$ (b) $F(s) = \dfrac{4}{s^2+16}$

(c) $F(s) = \dfrac{s+1}{s^2+2s+5}$ (d) $F(s) = \dfrac{3s+1}{s^2-4s+20}$

(e) $F(s) = \dfrac{1}{s^4-1}$ (f) $F(s) = \dfrac{1}{\sqrt{2s+3}}$

8. Use the `ilaplace` command to find $f(t)$ which are the inverse Laplace transforms of the following functions,

(a) $F(s) = \dfrac{s^2}{(s+1)(s^2+2s+5)}$

(b) $F(s) = \dfrac{2}{(s+2)^4} + \dfrac{3}{s^2+16} + \dfrac{5(s+1)}{s^2+2s+5}$

(c) $F(s) = \dfrac{2s-5}{(3s-4)(2s+1)^3}$

(d) $F(s) = \dfrac{3s^2-4s+2}{(s^2+2s+4)^2(s-5)}$

(e) $F(s) = \dfrac{s^2+20s+31}{(s+2)^2(s-3)}$

(f) $F(s) = \dfrac{5s+4}{s^3} - \dfrac{2s-18}{s^2+9} + \dfrac{24-30\sqrt{s}}{s^4}$

9. Use the method of Laplace transform to solve the second-order nonhomogeneous differential equation,

$$y'' + 4y = 9t$$

with the initial conditions of $y(0) = 0$ and $y'(0) = 7$.

10. Use the method of Laplace transform to solve the second-order nonhomogeneous differential equation,

$$y'' - 3y' + 2y = t$$

with the initial conditions of $y(0) = 1$ and $y'(0) = 0$.

11. Use the method of Laplace transform to solve the second-order nonhomogeneous differential equation,

$$y'' + 2y' + 5y = e^{-t}\sin 2t$$

with the initial conditions of $y(0) = 2$ and $y'(0) = -1$.

12. Use the method of Laplace transform to solve the third-order nonhomogeneous differential equation,

$$y''' - y = e^{t}$$

with the initial conditions of $y(0) = y'(0) = y''(0) = 0$.

13. Use the method of Laplace transform to solve the fourth-order nonhomogeneous differential equation,

$$y^{IV} + 2y'' + y = \sin t$$

with the initial conditions of $y(0) = y'(0) = y''(0) = y'''(0) = 0$.

14. Use the method of Laplace transform to solve the fourth-order homogeneous differential equation,

$$y^{IV} - 4y = 0$$

with the initial conditions of $y(0) = 2$, $y'(0) = 0$, $y''(0) = -4$, $y'''(0) = 0$. Then, verify the solution by substituting it into the differential equation and initial conditions to check whether they are satisfied.

15. Use the method of Laplace transform to solve the fourth-order homogeneous differential equation,

$$y^{IV} - 4y''' + 6y'' - 4y' + y = 0$$

with the initial conditions of $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 1$.

16. Use the method of Laplace transform to solve the second-order nonhomogeneous differential equation,

$$y'' + 4y = \delta(t)$$

where $\delta(t)$ is the Dirac delta function at time $t = 0$ with the initial conditions of $y(0) = 0$ and $y'(0) = 0$. Plot the solution of $y(t)$ in the interval of $0 \le t \le 10$.

17. Use the method of Laplace transform to solve the second-order nonhomogeneous differential equation,

$$2y'' + y' + 4y = \delta(t)$$

where $\delta(t)$ is the Dirac delta function at time $t = 0$ with the initial conditions of $y(0) = 0$ and $y'(0) = 0$. Plot the solution of $y(t)$ and check whether it satisfies the initial conditions.

18. Use the method of Laplace transform to solve the second-order nonhomogeneous differential equation,

$$y'' + 4y = H(t)$$

where $H(t)$ is the Heaviside function at time $t = 0$ with the initial conditions of $y(0) = 0$ and $y'(0) = 0$. Plot the solution of $y(t)$ in the interval of $0 \le t \le 10$.

19. Use the method of Laplace transform to solve the set of two first-order homogeneous differential equations,

$$\begin{aligned} x' &= -x + y \\ y' &= 2x \end{aligned}$$

with the initial conditions of $x(0) = 0$ and $y(0) = 1$.

20. Use the method of Laplace transform to solve the set of two first-order nonhomogeneous differential equations,

$$\begin{aligned} 2x' + y' - 2x &= 1 \\ x' + y' - 3x - 3y &= 2 \end{aligned}$$

with the initial conditions of $x(0) = 0$ and $y(0) = 0$. Then, verify the solutions by substituting them into the differential equations and initial conditions to check whether they are satisfied.

21. Use the method of Laplace transform to solve the set of two second-order homogeneous differential equations,

$$\begin{aligned} x'' + x - y &= 0 \\ y'' + y - x &= 0 \end{aligned}$$

with the initial conditions of $x(0) = 0$, $x'(0) = -2$, $y(0) = 0$ and $y'(0) = 1$.

22. Use the method of Laplace transform to solve the set of two second-order nonhomogeneous differential equations,

$$\begin{aligned} x'' + y'' &= t^2 \\ x'' - y'' &= 4t \end{aligned}$$

with the initial conditions of $x(0) = 8$, $x'(0) = 0$, $y(0) = 0$ and $y'(0) = 0$.

Chapter 8

Fourier Transforms

8.1 Introduction

Fourier transform is based on the knowledge of the Fourier series used for analyzing some practical problems. The Fourier series consists of the sine and cosine functions suitable for representing different types of periodic motions, such as the swinging pendulums, telecommunication signals, electric motor vibration, etc. Analysis solutions can provide comprehensive understanding of their behaviors.

Similar to the Laplace transform method learned in the preceding chapter, the Fourier transform can be used to solve some specific types of differential equations. This includes the non-homogeneous differential equations when the functions on the right-hand-side of the equations change abruptly with time.

Before applying the method of Fourier transform to solve such differential equations, the definitions of both the Fourier transform and inverse Fourier transform will be introduced. Examples will be used to show how to perform transformation for different types of functions. At the same time, MATLAB commands will also be employed to confirm the results. The discrete and fast Fourier transforms which have been used in many current applications will be explained by examples.

8.2 Definitions

Fourier transform is a valuable tool in transforming data and functions from the time domain into the frequency domain. It has been used widely in the field of telecommunication that involves waves and signals. The idea came from the Fourier series which contain the continuous periodic functions, such as sine and cosine functions. The sine function is generally written in the form,

$$A \sin(2\pi\, \omega t + \phi)$$

where A is the amplitude, ω is the frequency measured by the cycle or period per second, t is time, and ϕ is called the phase which may not be zero at time $t = 0$. The cosine function is in the same form except the phase is shifted by $\pi / 2$. Therefore, the periodic function $f(t)$ as mentioned can be written in a general form as,

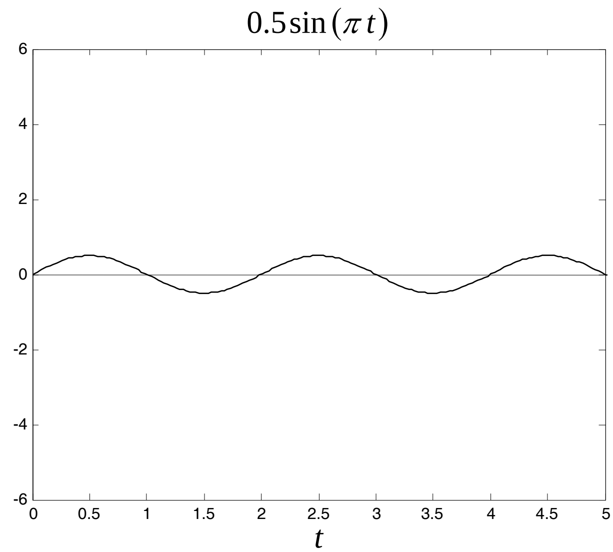
$$f(t) \;=\; \sum_{k=1}^{n} \left(A_k \cos(2\pi\, \omega_k\, t) + B_k \sin(2\pi\, \omega_k\, t)\right)$$

To understand the concept more clearly, let's consider the periodic function given by,

$$f(t) \;=\; 0.5\sin(\pi t) + 2\sin(4\pi t) + 4\cos(2\pi t)$$

If we plot the first term on the right-hand-side of the periodic function above in the interval of $0 \le t \le 5$,

```
>> ezplot('0.5*sin(pi*t)', [0 5 -6 6])
```

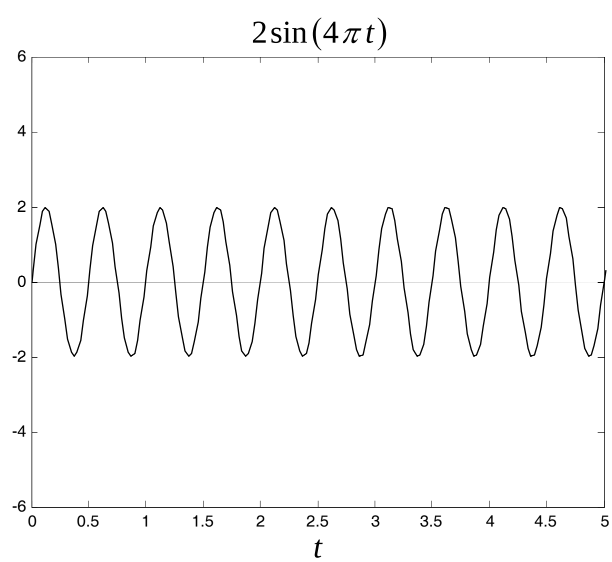


We observe that the magnitude of oscillation is 0.5 and the frequency is half cycle per second.

Similarly, if we plot the second term in the same interval of $0 \le t \le 5$,

```
>> ezplot('2*sin(4*pi*t)', [0 5 -6 6])
```

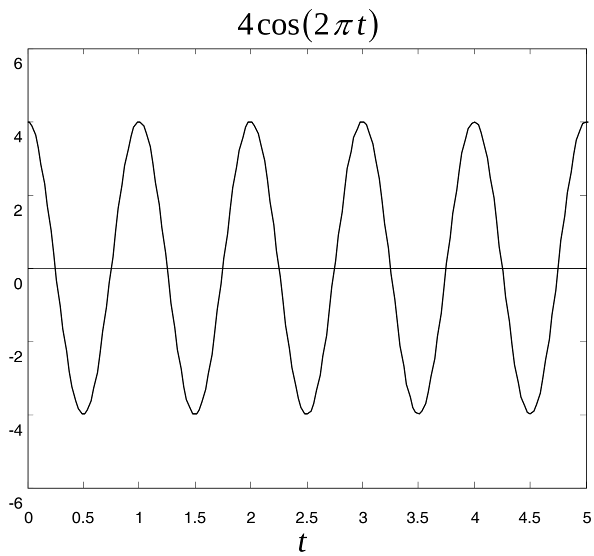
ezplot



we see that the magnitude is now 2 and the frequency is two cycles per second.

Also, if we plot the last term of the equation in the same interval of $0 \le t \le 5$,

```
>> ezplot('4*cos(2*pi*t)', [0 5 -6 6])
```

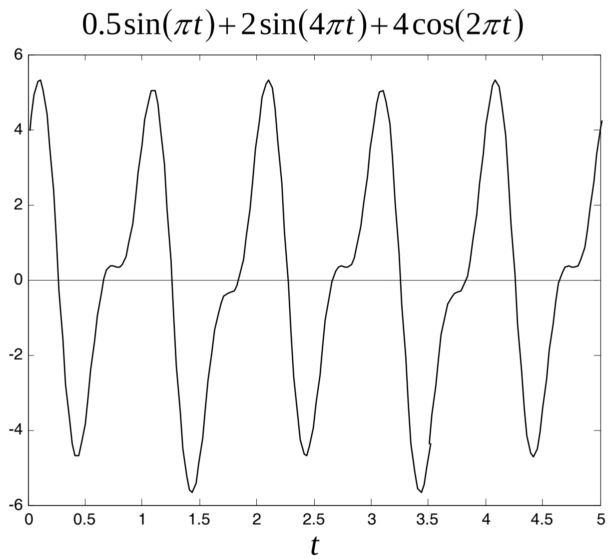


we see that the magnitude is 4 and the frequency is one cycle per second.

By combining the three terms together, we can plot the oscillating behavior of the function $f(t)$ by using the `ezplot` command as shown in the figure.

```
>> ezplot('0.5*sin(pi*t) + 2*sin(4*pi*t) +
           4*cos(2*pi*t)', [0 5 -6 6])
```

In general, the Fourier series could be in any form that may include a large number of terms. The Fourier transform method will help us to identify magnitudes and frequencies of the given functions. It is noted that the analysis of Fourier transform method is usually carried out by using the complex numbers. This is because the involved equations will be in simpler forms. Both sine and cosine functions can be written in the form of complex numbers by using the Euler's formula,



$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{and} \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

where $i = \sqrt{-1}$ is the imaginary unit number. The two equations above lead to,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

After substituting these $\cos\theta$ and $\sin\theta$ functions in the form of complex numbers, the expression of $f(t)$ reduces to a more compact form as,

$$f(t) = \sum_{k=-n}^{n} C_k \, e^{2\pi i \omega_k t}$$

where C_k is the magnitude and ω_k is the frequency.

8.3 Fourier Transform

In MATLAB, the Fourier transform of a function $f(t)$ is defined by,

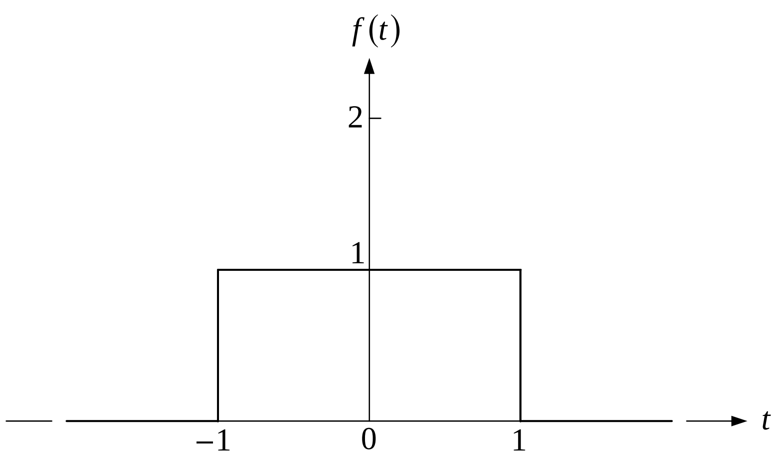
$$\mathscr{F}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} \, dt$$

The definition differs from those defined in most textbooks which have the factor of $\sqrt{2\pi}$ as denominator.

Fourier transform requires performing integration from $-\infty$ to ∞ of the product between $f(t)$ and $e^{-i\omega t}$. For example, the given function $f(t)$ is,

$$f(t) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

which has its variation as shown in the figure.



Then, the Fourier transform is,

$$\begin{aligned} \mathscr{F}(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \\ &= \int_{-1}^{1} (1) e^{-i\omega t}\, dt \\ &= \left. \frac{e^{-i\omega t}}{-i\omega} \right|_{-1}^{1} = \frac{e^{-i\omega} - e^{i\omega}}{-i\omega} \\ &= \frac{-2i \sin\omega}{-i\omega} = \frac{2 \sin\omega}{\omega} \end{aligned}$$

We can use the `int` command to perform integration symbolically,

```
>> syms w t
>> f = 1;
>> int(f*exp(-i*w*t),t, -1, 1)
```

int

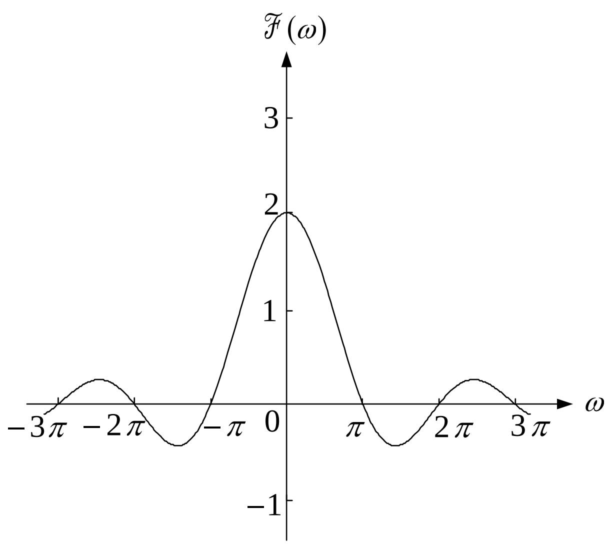
```
ans =

2*sin(w)/w
```

The result is plotted by using the command,

```
>> ezplot('2*sin(w)/w', [-10 10 -1 3])
```

ezplot

The figure shows the result of Fourier transform along ω-axis in the π scale.



MATLAB contains the `fourier` command that can be used to transform $f(t)$ from the time domain to $\mathcal{F}(\omega)$ in the frequency domain. The definition of Fourier transform used in MATLAB is,

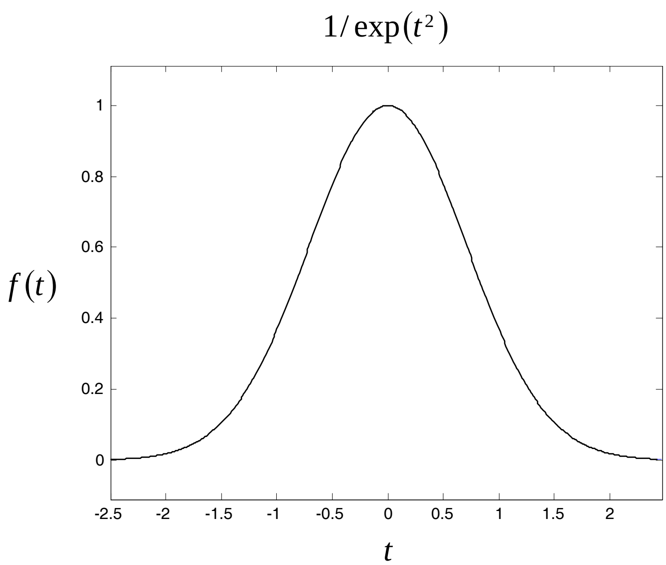
$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

For example,

$$f(t) = e^{-t^2}$$

```
>> syms t w
>> f = exp(-t^2);
>> ezplot(f)
```

Variation of the function with time *t* is shown in the figure.



We can use the `fourier` command to find Fourier transform,

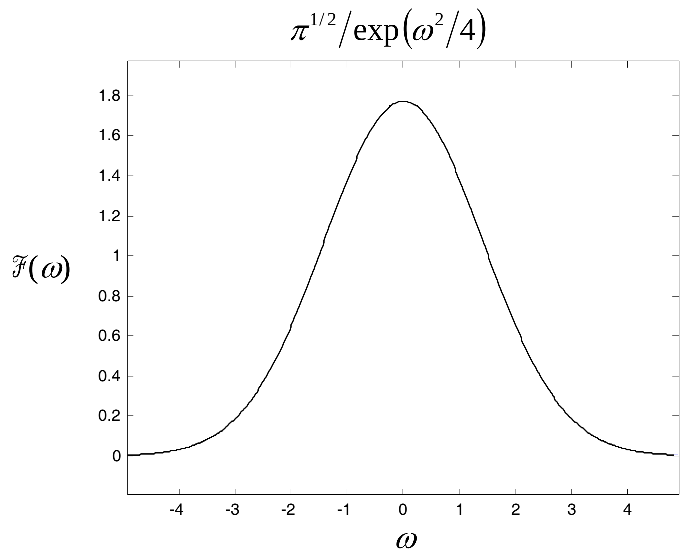
fourier

```
>> fourier(f)
ans =
pi^(1/2)/exp(w^2/4)
```

i.e., $$\mathcal{F}(\omega) = \sqrt{\pi}\, e^{-\omega^2/4}$$

The result varies with the frequency ω that can be plotted using the `ezplot` command as shown in the figure.

```
>> ezplot(ans)
```



If the function $f(t)$ is given by,

$$f(t) = e^{-|t|}$$

```
>> syms t w
>> f = exp(-abs(t));
>> ezplot(f)
```

which varies with time t as shown in the figure.

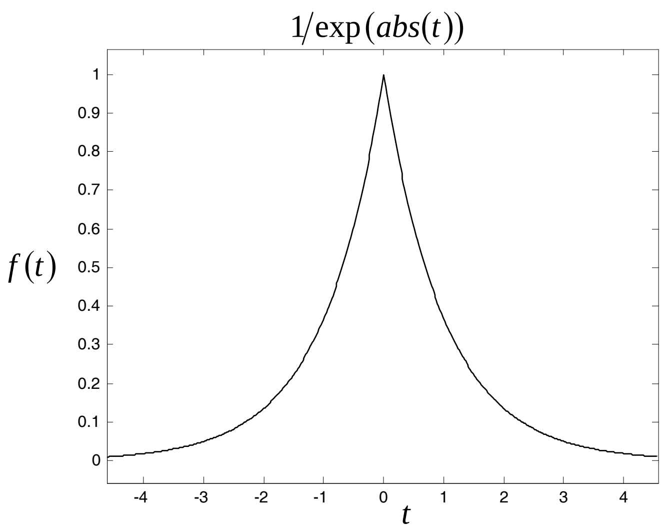
The Fourier transform is obtained by using the `fourier` command with its variation as shown in the figure.

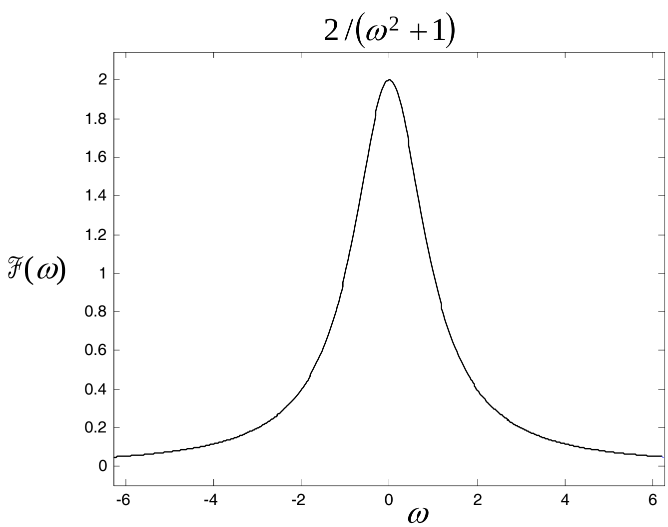
fourier

```
>> fourier(f)

ans =

2/(w^2 + 1)

>> ezplot(ans)
```

As another example, if the function $f(t)$ is given by,

$$f(t) = t e^{-|t|}$$





The Fourier transform is obtained by using the `fourier` command,

```
>> syms t w
>> f = t*exp(-abs(t));
>> fourier(f)
```

`fourier`

```
ans =

-(4*w*i)/(w^2 + 1)^2
```

i.e.,
$$\mathscr{F}(\omega) \;=\; -\frac{4\omega i}{\left(\omega^2+1\right)^2}$$

8.4 Inverse Fourier Transform

MATLAB also contains the `ifourier` command to find the inverse Fourier transform of the function $\mathscr{F}(\omega)$ from the frequency domain to the function $f(t)$ in time domain. The inverse Fourier transform in MATLAB is defined by,

$$f(t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{F}(\omega)\, e^{i\omega t}\, d\omega$$

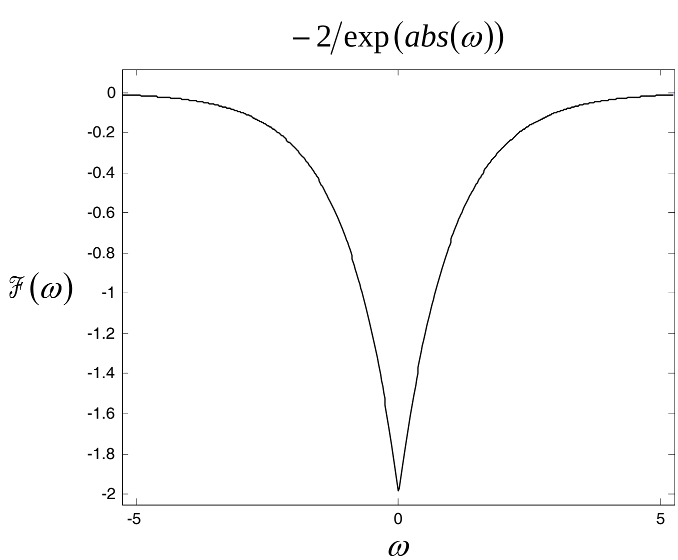
For example, if the function in the frequency domain is,

$$\mathscr{F}(\omega) \;=\; -2e^{-|\omega|}$$

```
>> syms w t
>> F = -2*exp(-abs(w));
>> ezplot(F)
```

ezplot

which varies with the frequency as shown in the figure.



We can use the `ifourier` command to find the inverse Fourier transform and plot its variation as follows,

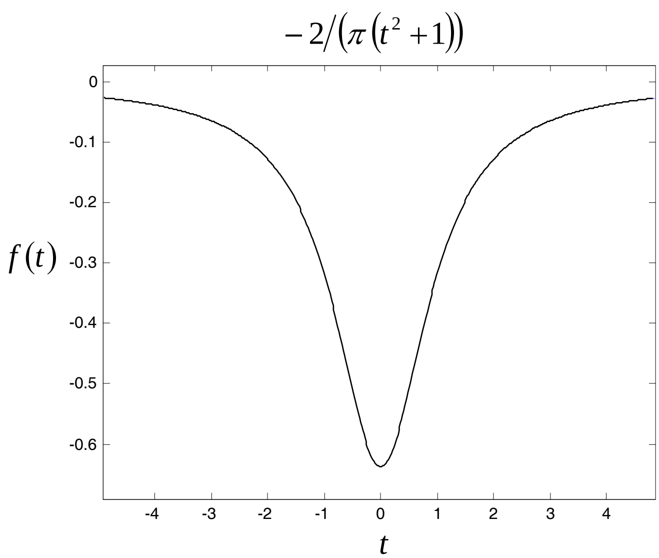
```
>> ifourier(F,t)
ans =
-2/(pi*(t^2 + 1))
>> ezplot(ans)
```

ifourier

i.e., the result of inverse Fourier transform is,

$$f(t) = -\frac{2}{\pi(t^2+1)}$$

with the variation as shown in the figure.



It is noted that the symbol *t* at the end of the `ifourier` command is for assigning it as the independent variable. Otherwise, MATLAB will use the default symbol of *x* as the independent variable.

If the function in the frequency domain is given by,

$$\mathscr{F}(\omega) = 6\sqrt{\pi}\, e^{-\omega^2/4}$$

The inverse Fourier transform can be found by entering,

```
>> F = (6*pi^(1/2))/exp(w^2/4)
>> f = ifourier(F,t)
```

ifourier

```
f =

6/exp(t^2)
```

i.e., the result in the time domain is,

$$f(t) = 6e^{-t^2}$$

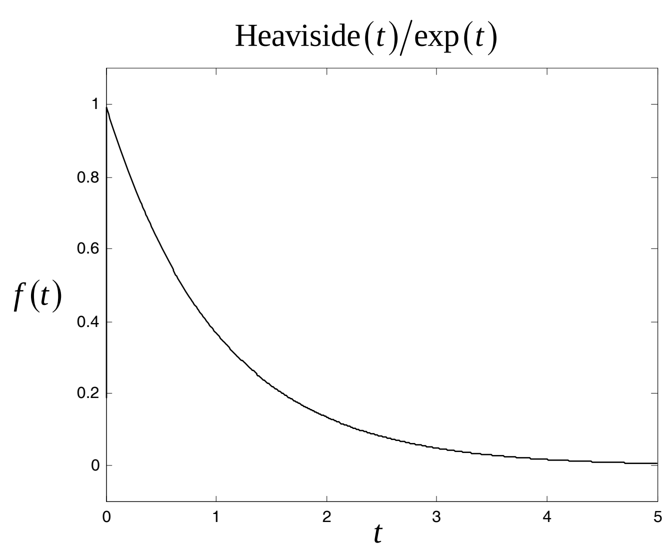
As another example, if the function in the frequency domain is given by,

$$\mathscr{F}(\omega) = \frac{1}{1+i\omega}$$

The inverse Fourier transform can be found by entering,

```
>> F = 1/(1 + i*w);
>> f = ifourier(F,t)

f =

heaviside(t)/exp(t)
```

The result of $f(t)$ in the time domain is shown in the figure.



If the given function contains a quantity that must always be positive value, we have to inform that to MATLAB. As an example, the value of a in the function below must be a positive quantity, we need to declare it in the `syms` command as,

```
>> syms a positive
```

positive

For example, the function in the frequency domain is,

$$\mathscr{F}(\omega) = \frac{e^{-\omega^2/4a}}{\sqrt{2a}} \qquad a > 0$$

where a must be greater than zero. If we perform the inverse transformation without declaring that $a > 0$ as follows,

```
>> syms w t a
>> F = exp(-w^2/(4*a))/sqrt(2*a);
>> f = ifourier(F)

f =

(2^(1/2)*transform::fourier(1/exp(w^2/(4*a)), w,
x))/ (4*pi*a^(1/2))
```

MATLAB could not find the inverse Fourier transform, i.e., there is no result for arbitrary a value.

But if we start by declaring that a is only positive value as,

```
>> syms w t; syms a positive
>> F = exp(-w^2/(4*a))/sqrt(2*a);
>> f = ifourier(F)

f =

2^(1/2)/(2*pi^(1/2)*exp(a*t^2))
```

ifourier

we obtain the correct result of the function in the time domain, i.e.,

$$f(t) = \frac{e^{-at^2}}{\sqrt{2\pi}}$$

8.5 Fast Fourier Transform

All of the functions $f(t)$ we learned in the preceding sections are continuous and easy to handle. For practical problems, a large number of data points may be generated from experiments. These data points are discrete and not continuous. The technique of the Fourier transform can still be applied to analyze such data. We call the analysis of this latter case as the *discrete Fourier analysis*. The transformation process is known as the *Discrete Fourier Transform* or DFT.

To understand the process, we start from a periodic function $f(t)$ with the period of 2π. If we have *N* data points with equal intervals, the distance between each pair of data points is,

$$t_k = \frac{2\pi k}{N} \qquad k = 0, 1, 2, ..., N-1$$

The idea is to create a function $f(t)$ to represent the variation of these data points in the form of,

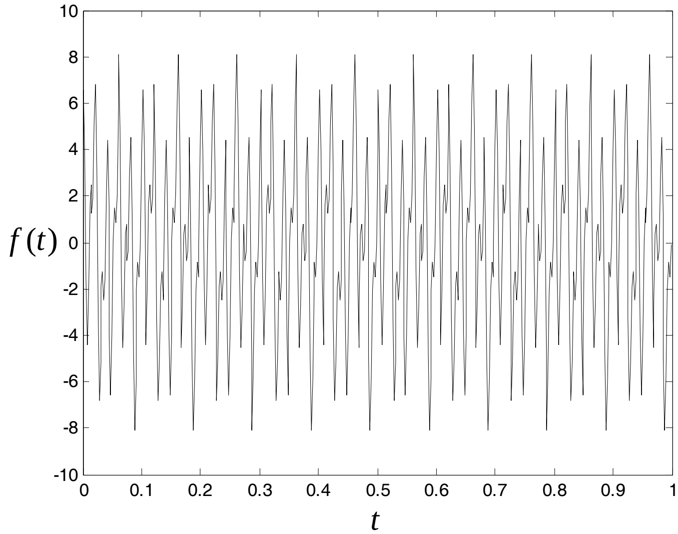
$$f(t_k) = \sum_{n=0}^{N-1} C_n e^{i n t_k}$$

where the coefficients C_n, $n = 0, 1, 2, ..., N-1$, are to be determined. It was found that the time used for determining these coefficients varies with N^2. Later, an improved method so called the *Fast Fourier Transform* or FFT was developed. The method significantly reduces the computational time to $N \log N$. This latter method was implemented on MATLAB which can be used by calling the `fft` command. We will learn how to use this `fft` command to study the signal frequency through a simple example as follow.

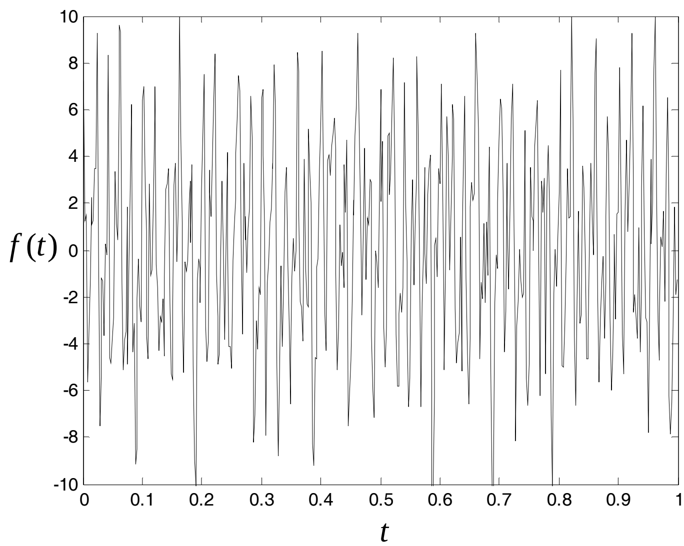
Suppose we have a signal composing of the three sine and cosine functions,

$$f(t) = 2\sin\left(2\pi(20)t\right) + 4\cos\left(2\pi(50)t\right) + 3\sin\left(2\pi(100)t\right)$$

It is noted that the frequencies of these three sine and cosine functions are 20, 50 100 Hz, respectively. The variation of the continuous $f(t)$ function is plotted from 0 to 1 as shown in the figure.



To include the noise similar to that occurs in the data from experiment, we first represent the continuous function $f(t)$ by 500 discrete data points and then add arbitrary noise values randomly into it by using the built-in `randn` command. The variation of the function $f(t)$ after adding noise is shown in the figure.

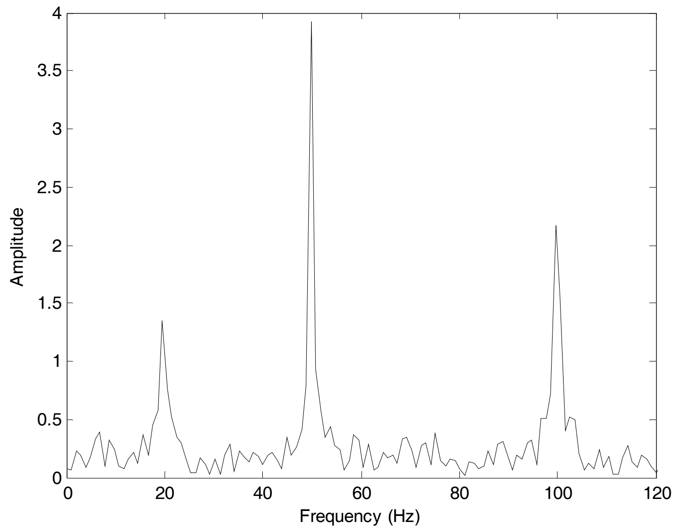


The variations of the original continuous function $f(t)$ and after it was represented by 500 data points with noise as seen from the figures are slightly different. It is difficult to identify the three frequencies of 20, 50 100 Hz from these two figures. We will use the fast Fourier transform through the `fft` command to find these three frequencies. The MATLAB file below shows the commands to plot the original continuous $f(t)$ function and to convert it into 500 data points. Noise is added randomly into these data points before plotting their variation again. The data points are transformed using the `fft` command to determine the amplification. The amplification is then plotted versus the frequency as shown in the figure. The figure shows that the three frequencies of 20, 50 100 Hz are clearly identified after using the `fft` command. This example thus demonstrates the advantage of the fast Fourier transform that can find frequencies of the data with noise normally obtained from experiment.

```
% Number of data points, data point interval,
% and discrete times
n = 500; T = 1/n; t = (0:n-1)*T;
% Discrete data points composing of 3 waves
ft = 2.*sin(2*pi*20*t) + 4.*cos(2*pi*50*t) + ...
     3.*sin(2*pi*100*t);
plot(t(1:n),ft(1:n),'k'); axis([0 1 -10 10])
% Add noise randomly into these discrete data
% points
fn = ft + 2.*randn(size(t));
plot(t(1:n),fn(1:n),'k'); axis([0 1 -10 10])
% Perform fast Fourier transform
nfft = 2^nextpow2(n);
amp = fft(fn,nfft)/n;
fre = n/2*linspace(0,1,nfft/2+1);
plot(fre,2*abs(amp(1:nfft/2+1)),'k')
xlabel('Frequency (Hz)'); ylabel('Amplitude');
     axis([0 120 0 4])
```

randn

fft



8.6 Solving Differential Equations

The method of Fourier transform can be used to solve the differential equations that are in some specific forms. The Fourier transform for the first-order derivative of function $f(t)$ is,

$$\mathcal{F}\{f'(t)\} = (\omega i)\,F(\omega)$$

Similarly, the Fourier transform for the second-order derivative of function $f(t)$ is,

$$\mathcal{F}\{f''(t)\} = (\omega i)^2\,F(\omega)$$

The Fourier transform for the higher-order derivatives of function $f(t)$ can be obtained in the same fashion, i.e.,

$$\mathcal{F}\{f^{n}(t)\} = (\omega i)^n\,F(\omega)$$

We will use examples to demonstrate the method of Fourier transform to solve for solutions of some specific types of differential equations as follows.

Example Solve the first-order nonhomogeneous differential equation,

$$y' - 4y = e^{-4t} H(t)$$

where $H(t)$ is the Heaviside function which is equal to one when $t \geq 0$ and is equal to zero when $t < 0$. The differential equation can be rewritten as,

$$y' - 4y = \begin{cases} e^{-4t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

We start by performing the Fourier transform of the differential equation,

$$\mathcal{F}\{y'\} - 4\mathcal{F}\{y\} = \mathcal{F}\{e^{-4t} H(t)\}$$

We can use the `fourier` command to find the Fourier transform for the term on the right-hand-side of the equation as,

```
>> syms t w
>> RHS = exp(-4*t)*heaviside(t);
>> fourier(RHS)

ans =

1/(4 + w*i)
```

fourier

Thus, the Fourier transform of the differential equation is,

$$(\omega i)F - 4F = \frac{1}{(\omega i + 4)}$$

or,
$$F = \frac{1}{(\omega i + 4)(\omega i - 4)}$$

Then, we perform inverse transformation by using the `ifourier` command,

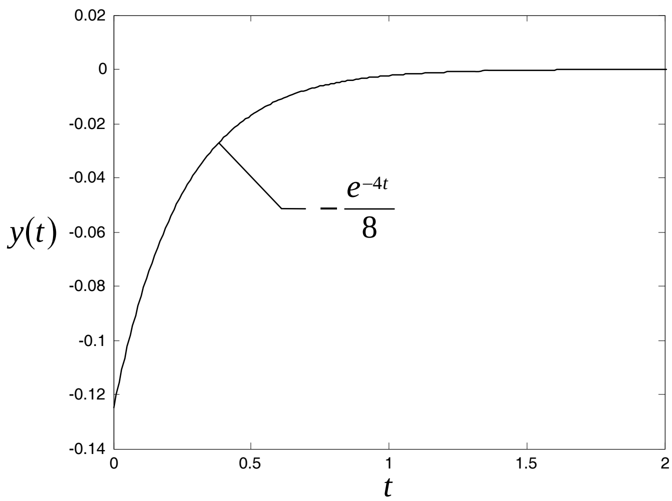
```
>> F = 1/((w*i+4)*(w*i-4));
>> ifourier(F,t)

ans =

-heaviside(t)/(8*exp(4*t))
```

ifourier

The result above was simplified by the `simple` command so that it becomes more compact,

$$y(t) = -\frac{e^{-4t}}{8}$$

The solution of y that varies with t when $t \geq 0$ is shown in the figure.



It is noted that the solution can be verified by substituting it into the left-hand-side of the differential equation,

```
>> syms t y
>> y = -exp(-4*t)/8;
>> dy = diff(y,t);
>> LHS = dy - 4*y

LHS =

1/exp(4*t)
```

diff

The result is equal to the right-hand-side of the differential equation when $t \geq 0$.

<u>Example</u> Solve the second-order nonhomogeneous differential equation,

$$y'' + 3y' + 2y = \delta(t)$$

where $\delta(t)$ is the Dirac delta function which is equal to one at $t = 0$ and equal to zero at any other *t*.

We start by performing Fourier transform of the differential equation,

$$\mathscr{F}\{y''\} + 3\mathscr{F}\{y'\} + 2\mathscr{F}\{y\} \;=\; \mathscr{F}\{\delta(t)\}$$

The `fourier` command can be used to find the Fourier transform of the Dirac delta function,

```
>> syms t w
>> fourier(dirac(t))
```

fourier

```
ans =

1
```

Thus, the differential equation after transformation becomes,

$$(\omega i)^2 F + 3(\omega i)F + 2F \;=\; 1$$

i.e.,

$$F \;=\; \frac{1}{-\omega^2 + 3\omega i + 2}$$

Then, we perform inverse transformation by using the `ifourier` command to obtain the solution,

```
>> F = 1/(-w^2 + 3*w*i + 2);
>> ifourier(F,t)
```

ifourier

```
ans =

heaviside(t)*(exp(t) - 1))/exp(2*t)
```

i.e., the solution of this differential equation is,

$$y(t) \;=\; e^{-2t}\,(e^t - 1)\,H(t)$$

Again, we can verify the solution by substituting it into the left-hand-side of the differential equation,

```
>> y = (heaviside(t)*(exp(t) - 1))/exp(2*t);
>> dy = diff(y,t);
>> d2y = diff(dy,t);
>> LHS = d2y + 3*dy + 2*y

LHS =

dirac(t)
```

The result is the Dirac delta function which is equal to the right-hand-side of the differential equation.

8.7 Concluding Remarks

In this chapter, we have learnt the method of Fourier transform to solve for solutions of some differential equations. Such differential equations arise in many applications for which the loads are applied to the problems instantly. Definitions of the Fourier transform and its inverse transformation were first explained. The Fourier transform changes the given function from the time domain into another function in the frequency domain. In the opposite way, the inverse Fourier transform changes the function in the frequency domain back to the time domain. The `fourier` and `ifourier` commands in MATLAB were used to obtain the transformation results as demonstrated by examples.

For discrete data collected from experiments, the discrete Fourier analysis was applied. The discrete Fourier transform was explained by employing the `fft` command of the fast Fourier transform. The command helps us to identify the frequency contents from the signal with noise. In the last section, the method of Fourier transform was applied to solve differential equations. Examples have shown that the method can provide solutions effectively when the nonhomogeneous functions on the right-hand-side of the differential equations are in form of the Dirac delta and Heaviside functions.

Exercises

1. Use the `fourier` command to find the Fourier transform $\mathcal{F}(\omega)$ of the following functions,

   (a) $f(t) = H(t)$ (b) $f(t) = e^{-t}H(t)$

   (c) $f(t) = e^{-t^2/2}$ (d) $f(t) = \dfrac{1}{1+t^2}$

   (e) $f(t) = 5e^{-3(t-5)^2}$ (f) $f(t) = e^{-5|t|}$

   where $H(t)$ is the Heaviside function.

2. Use the `fourier` command to find the Fourier transform $\mathcal{F}(\omega)$ of the following functions,

   (a) $f(t) = t^2 e^{-|t|}$ (b) $f(t) = 3e^{-4|t+2|}$

   (c) $f(t) = \dfrac{t}{9+t^2}$ (d) $f(t) = \dfrac{5e^{3it}}{t^2-4t+13}$

   (e) $f(t) = 3te^{-9t^2}$ (f) $f(t) = 25te^{-2t}H(t)$

3. Use the `fourier` command to find the Fourier transform $\mathcal{F}(\omega)$ of the following functions,

   (a) $f(t) = 1-|t|$ (b) $f(t) = e^{-7|t|}$

   (c) $f(t) = e^{-16t^2}$ (d) $f(t) = te^{-t^2/2}$

   (e) $f(t) = \sin(3t^2)$ (f) $f(t) = \cos(\pi+5t^2)$

4. Use the `ifourier` command to find the inverse Fourier transform $f(t)$ of the following functions,

   (a) $\mathcal{F}(\omega) = \dfrac{1}{1+\omega i}$ (b) $\mathcal{F}(\omega) = \dfrac{2}{\omega i}$

   (c) $\mathcal{F}(\omega) = \pi e^{|-\omega i|}$ (d) $\mathcal{F}(\omega) = \dfrac{1}{\omega^2+4}$

   (e) $\mathcal{F}(\omega) = \dfrac{6}{\omega^2+9}$ (f) $\mathcal{F}(\omega) = \dfrac{3}{(2-\omega i)(1+\omega i)}$

5. Use the `ifourier` command to find the inverse Fourier transform $f(t)$ of the following functions,

(a) $\mathcal{F}(\omega) = \dfrac{1}{(1+\omega i)^2}$

(b) $\mathcal{F}(\omega) = \dfrac{1}{(1+\omega i)(2+\omega i)}$

(c) $\mathcal{F}(\omega) = \dfrac{1}{(1+\omega i)(6+5\omega i-\omega^2)}$

(d) $\mathcal{F}(\omega) = \dfrac{5(4+\omega i)}{(9+8\omega i-\omega^2)}$

(e) $\mathcal{F}(\omega) = 7e^{-(\omega+4)^2/32}$

(d) $\mathcal{F}(\omega) = \dfrac{e^{(2\omega-6)i}}{4-(3-\omega)i}$

6. Use the `ifourier` command to find the inverse Fourier transform $f(t)$ of the following functions,

(a) $\mathcal{F}(\omega) = 9e^{-(\omega+4)^2/32}$

(b) $\mathcal{F}(\omega) = \pi e^{-3|\omega|}$

(c) $\mathcal{F}(\omega) = \dfrac{e^{2\omega i}}{5+\omega i}$

(d) $\mathcal{F}(\omega) = \dfrac{e^{-5|\omega|}}{\omega i}$

(e) $\mathcal{F}(\omega) = \dfrac{e^{(20-4\omega)i}}{3-(5-\omega)i}$

(f) $\mathcal{F}(\omega) = \dfrac{e^{(2\omega-6)i}}{5-(3-\omega)i}$

7. A signal is given in the form of two sine functions,

$$f(t) = \sin\left(2\pi(15)t\right) + 2\sin\left(2\pi(60)t\right)$$

Use the `randn` command to add noise into the signal similar to that explained in section 8.5. Then, apply the fast Fourier transform to find the two frequencies from the amplitude versus frequency plot.

8. A signal is given in the form of a sine and two cosine functions,

$$f(t) = 2\cos\left(2\pi(30)t\right) + 3\sin\left(2\pi(60)t\right) + 4\cos\left(2\pi(90)t\right)$$

Use the `randn` command to add noise into the signal similar to that explained in section 8.5. Then, apply the fast Fourier transform to find the three frequencies from the amplitude versus frequency plot.

9. Apply the Fourier transform to solve the first-order nonhomogeneous differential equation,

$$2y' + y = \delta(t)$$

where $\delta(t)$ is the Dirac delta function. Plot the solution of $y(t)$ and verify it by substituting into the differential equation.

10. Apply the Fourier transform to solve the first-order nonhomogeneous differential equation,

$$y' + 5y = e^{-t}\, H(t)$$

where $H(t)$ is the Heaviside function. Show detailed derivation of the solution $y(t)$. Plot the solution and verify it by substituting into the differential equation.

11. Apply the Fourier transform to solve the first-order nonhomogeneous differential equation,

$$y' - 8y = e^{-8t}\, H(t)$$

where $H(t)$ is the Heaviside function. Plot the solution of $y(t)$ for $t > 0$ and verify it by substituting into the differential equation.

12. Apply the Fourier transform to solve the second-order nonhomogeneous differential equation,

$$y'' + 4y' + 4y = \delta(t)$$

where $\delta(t)$ is the Dirac delta function. Plot the solution of $y(t)$ and verify it by substituting into the differential equation.

13. Apply the Fourier transform to solve the second-order non-homogeneous differential equation,

$$y'' + 6y' + 5y \;=\; \delta(t-3)$$

where $\delta(t)$ is the Dirac delta function. Show detailed derivation of the solution $y(t)$ and plot its variation in the interval of $3 \le t \le 7$. Note that the MATLAB command to find Fourier transform of $\delta(t-3)$ is `fourier(dirac(t-3))`.

Chapter 9

Boundary Value Problems

9.1 Introduction

Solving the boundary value problems that will learn in this chapter is the first step toward understanding how to analyze practical problems. We will solve the boundary value problems that are governed by the ordinary differential equations together with the boundary conditions. Solving the ordinary differential equations is equivalent to solve the one-dimensional problems for which their exact solutions are usually available. If the exact solutions cannot be found, we will apply the numerical methods to obtain the approximate solutions instead.

Learning how to solve the one-dimensional problems is important as the basis to continue solving two- and three-dimensional problems. For these multi-dimensional problems, the governing equations are in the form of partial differential equations which are difficult to solve. In addition, the boundary conditions are more complicated and the geometries could be complex too. In general, their exact solutions are not available and the numerical methods are the only way for finding approximate solutions. Efficient numerical methods, such as the finite element method which will be explained in the following chapter, must be applied to solve for their solutions.

In this chapter, we will learn how to find exact solutions of the one-dimensional boundary value problems that are governed by simple ordinary differential equations with boundary conditions. We will derive the exact solutions and verify them by using the `dsolve` command. If the exact solutions are not available, we will use the `bvp4c` command to find their approximate solutions. The materials in this chapter thus represent the first step toward learning how to solve the more general boundary value problems.

9.2 Two-Point Boundary Value Problems

Two approaches are normally used to solve the two-point boundary value problems. The first approach is to find the exact solutions when the governing differential equations and boundary conditions are not complicated. The second approach is to find the approximate solutions by using the numerical methods. In the latter approach, the methods for solving the initial value problems, such as the shooting, Euler and Runge-Kutta methods, are modified to include iteration process so that the boundary conditions are satisfied. MATLAB contains the `bvp4c` command to solve such problems for which we will learn how to use it in detail.

In this section, we will start by reviewing the derivation of exact solutions to the governing differential equations with boundary conditions. We will verify the derived solutions with those obtained by using the `dsolve` command. We will use the following examples to demonstrate the process.

Example Derive the exact solution of the boundary value problem governed by the second-order homogeneous differential equation,

$$\frac{d^2y}{dx^2} - 4y = 0 \qquad\qquad 0 \le x \le 1$$

with the boundary conditions of $y(0)=0$ and $y(1)=1$.

The process for deriving the exact solution of the boundary value problem is similar to the initial value problem. We first assume a general solution in the form of $e^{\lambda x}$ and substitute it into the differential equation to get,

$$\lambda^2 e^{\lambda x} - 4e^{\lambda x} = 0$$

Then, we divide it by $e^{\lambda x}$ to obtain the characteristic equation,

$$\lambda^2 - 4 = 0$$

$$(\lambda - 2)(\lambda + 2) = 0$$

i.e., $\lambda_1 = 2$ and $\lambda_2 = -2$. Thus, the general solution is,

$$y = C_1 e^{2x} + C_2 e^{-2x}$$

where C_1 and C_2 are constants that can be determined from the given boundary conditions as follows,

$$y(0) = 0; \qquad 0 = C_1 + C_2$$

$$y(1) = 1; \qquad 1 = C_1 e^2 + C_2 e^{-2}$$

By solving the two equations above, we obtain the two constants of C_1 and C_2 as,

$$C_1 = \frac{1}{e^2 - e^{-2}} \quad \text{and} \quad C_2 = -\frac{1}{e^2 - e^{-2}}$$

Hence, the exact solution is,

$$y = \frac{e^{2x}}{e^2 - e^{-2}} - \frac{e^{-2x}}{e^2 - e^{-2}}$$

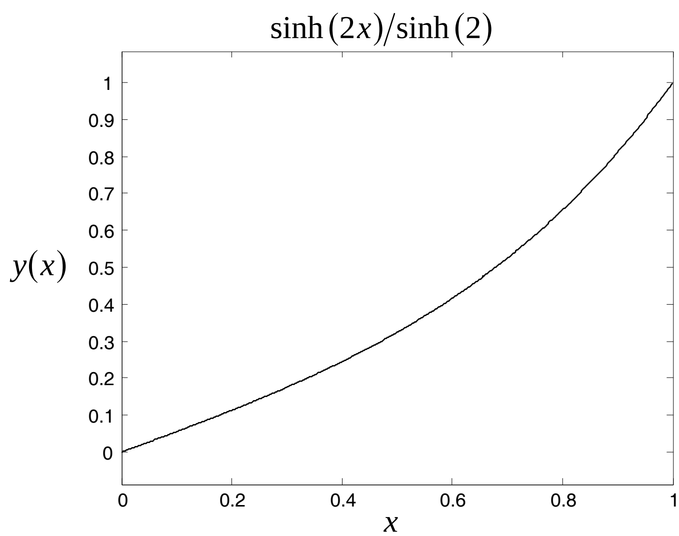
Or,
$$y = \frac{\sinh(2x)}{\sinh(2)}$$

The same exact solution is obtained by using the `dsolve` command. The solution is plotted using the `ezplot` command as shown in the figure.

```
>> syms x y
>> dsolve('D2y - 4*y = 0','y(0)=0','y(1)=1','x')

ans =

sinh(2*x)/sinh(2)

>> ezplot(ans, [0, 1])
```

dsolve

ezplot



<u>Example</u> Derive the exact solution of the boundary value problem governed by the second-order homogeneous differential equation,

$$\frac{d^2y}{dx^2} + 4y = 0 \qquad 0 \le x \le 1$$

with the boundary conditions of $y(0) = 0$ and $y(1) = 1$.

This example is identical to the previous one except the opposite sign of the zeroth-order term in the differential equation. The characteristic equation obtained from the differential equation above is,

$$\lambda^2 + 4 = 0$$

$$(\lambda - 2i)(\lambda + 2i) = 0$$

So, $\lambda_1 = 2i$ and $\lambda_2 = -2i$ where $i = \sqrt{-1}$. Thus, the general solution is,

$$y = Ae^{2ix} + Be^{-2ix}$$

where A and B are constants. By using the Euler's formula,

$$e^{i\beta x} = \cos(\beta x) + i\sin(\beta x)$$

and

$$e^{-i\beta x} = \cos(\beta x) - i\sin(\beta x)$$

the general solution above can be written in the form of sine and cosine functions as,

$$y = C_3\cos(2x) + C_4\sin(2x)$$

where C_3 and C_4 are constants that can be determined from the given boundary conditions as follows,

$$y(0) = 0; \qquad 0 = C_3 + 0$$

$$y(1) = 1; \qquad 1 = C_3\cos(2) + C_4\sin(2)$$

The two equations above give values of the two constants as $C_3 = 0$ and $C_4 = 1/\sin(2)$. Hence, the exact solution is,

$$y = \frac{\sin(2x)}{\sin(2)}$$

The same exact solution is obtained by using the `dsolve` command. The solution of y that varies with x is plotted by using the `ezplot` command as shown in the figure.

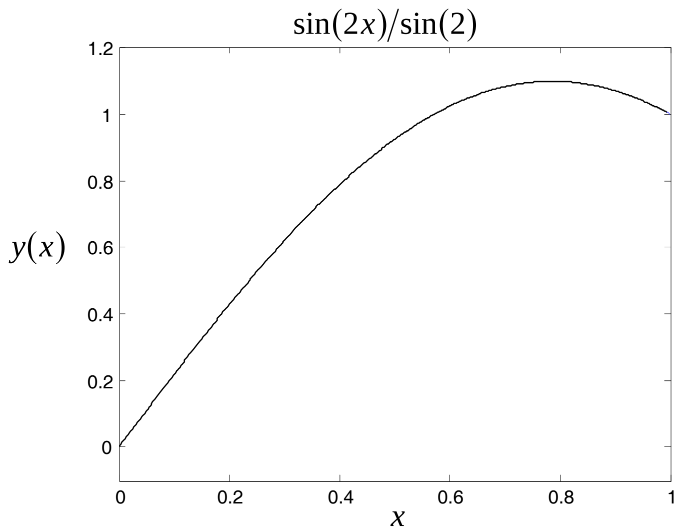
```
>> syms x y
>> dsolve('D2y + 4*y = 0','y(0)=0','y(1)=1','x')

ans =

sin(2*x)/sin(2)

>> ezplot(ans, [0, 1])
```

dsolve

ezplot

9.3 Second-Order Differential Equations

Derivation for exact solutions of the boundary value problems governed by the second-order differential equations is reviewed in the preceding section. The derived solutions were verified by using the `dsolve` command. For many second-order



differential equations, their exact solutions cannot be derived in closed-form expressions. Numerical methods must be applied to find the approximate solutions. In this section, we will use the `bvp4c` command to solve the boundary value problems. The approximate solutions will be compared with the exact solutions, if available, to measure the numerical solution accuracy.

The `bvp4c` command employs the technique similar to the commands used for solving the initial value problems. For example, to solve an initial value problem governed by the second-order differential equation in the interval of $0 \le x \le 1$, we start the solution from the initial conditions of $y(0)$ and $y'(0)$ at $x = 0$. The technique computes the next solutions with the time step of Δx until it reaches $x = 1$. The final solution of $y(1)$ at $x = 1$ thus depends to the differential equation and the two initial conditions.

If the same technique is used to solve the boundary value problem, it starts with the condition of $y(0)$ at $x = 0$ and must end with the specified condition of $y(1)$ at another end of $x = 1$. Thus, an iteration process is needed so that the computed solution agrees with the specified boundary condition at $x = 1$.

Therefore, to solve the boundary value problems by MATLAB, we need to provide the information of: (1) a main program that calls the `bvp4c` command, (2) the differential equation, and (3) the boundary conditions at both ends of the problem. We will demonstrate how to solve the boundary value problems with the `bvp4c` command by using the following examples.

<u>Example</u> Use the `bvp4c` command to solve the boundary value problem governed by the second-order homogeneous differential equation,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0 \qquad 0 \le x \le 1$$

with the boundary conditions of $y(0) = 1$ and $y(1) = 0$.

If we derive the exact solution by first assuming it in the form of $e^{\lambda x}$, we obtain two distinct roots of λ_1 and λ_2 from the characteristic equation. These two roots give the general solution in the form,

$$y = C_5 e^{-x} + C_6 e^{-2x}$$

where C_5 and C_6 are constants to be determined from the given boundary conditions of $y(0) = 1$ and $y(1) = 0$. Solving these two constants leads to the exact solution of,

$$y = \frac{e^1 - e^x}{e^{2x}(e^1 - 1)}$$

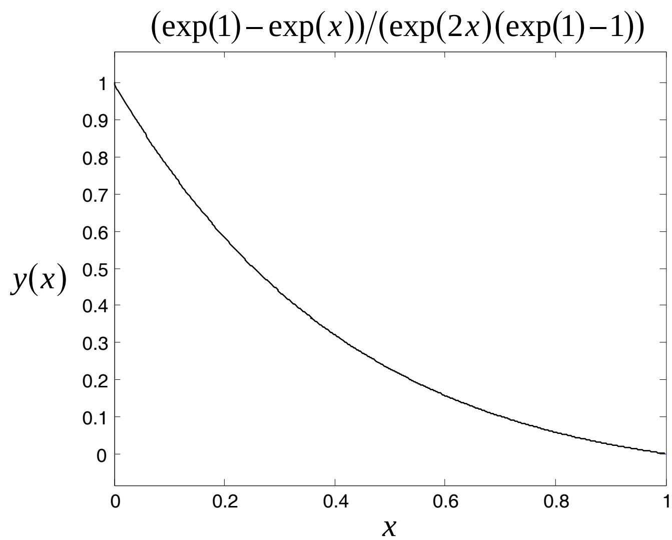
The same exact solution is obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('D2y+3*Dy+2*y=0','y(0)=1','y(1)=0','x')
ans =                                              dsolve

(exp(1) - exp(x))/(exp(2*x)*(exp(1) - 1))
```

Variation of the solution y with x is plotted using the `ezplot` command as shown in the figure.

ezplot

```
ezplot(ans, [0, 1])
```



The bvp4c command can be used to find the approximate solution of this problem. To do that, we need to create the script file, Ex1.m, containing commands as follows,

bvpinit **bvp4c** **deval**

```
%Ex1: D2y+3*Dy+2*y=0
solinit = bvpinit(linspace(0,1,5), [0 -1]);
sol = bvp4c('OdeEx1','BcEx1',solinit);
x = linspace(0,1,20);
y = deval(sol,x);
plot(x,y(1,:),'k')
xlabel('x'); ylabel('y'); hold on;
xe = 0:0.1:1;
ye = (exp(1)-exp(xe))./(exp(2.*xe)*(exp(1)-1));
plot(xe,ye,'ok')
```

plot

The MATLAB command, bvpinit, in the second line divides the interval of $0 \le x \le 1$ into five sub-intervals by further calling the linspace command. Therefore, there are four inner points and two end points in this case. The two numbers in the square bracket $y = 0$ and $y' = -1$ are the initial guess values of the four inner points at the starting of the iteration process. These initial guess values, which are provided by users, should be closed to the final solutions.

The `bvp4c` command in the third line calls the file function, `OdeEx1`, that contains the two first-order differential equations arose from the second-order differential equation,

$$\frac{d^2 y_1}{dx^2} + 3\frac{dy_1}{dx} + 2y_1 = 0$$

This second-order differential equation is separated into two first-order differential equations as,

$$\frac{dy_1}{dx} = y_2 \quad \text{and} \quad \frac{dy_2}{dx} = -3y_2 - 2y_1$$

Thus, the `OdeEx1` file function contains the details as,

function

```
function dydx = OdeEx1(x,y)
dydx = [y(2); -3*y(2)-2*y(1)];
```

The `bvp4c` command also calls the file function, `BcEx1`, that contains the initial conditions of $y(0)=1$ and $y(1)=0$ at both ends of the domain as,

```
function res = BcEx1(ya,yb)
res = [ya(1)-1; yb(1)-0];
```

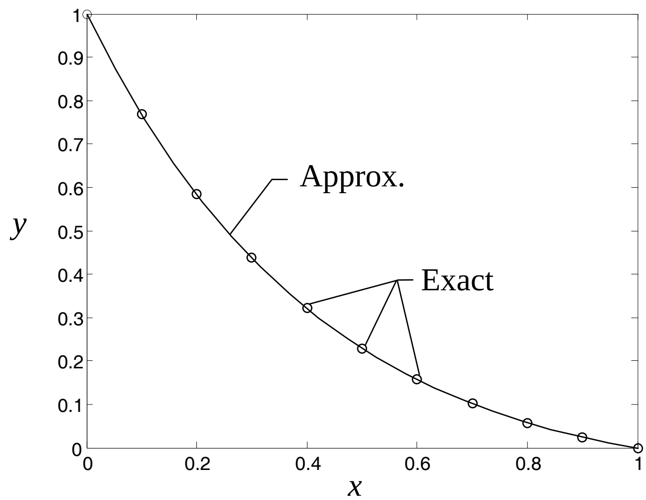
Note that the `res` statement denotes the residual values at both ends of the domain. So that the values of `ya(1)-1` and `yb(1)-0`, representing the solution errors at these two points, must be zero. The iteration process continues until these conditions are met.

The computed solutions at the four inner points are plotted as shown in the figure. The `deval` command in the 5th line of the `Ex1.m` script file is used to smooth the curve for plotting from the discrete solution data at the four inner points. As shown in the figure, the approximate solution compares very well with the exact solution.

<u>Example</u> Use the `bvp4c` command to solve the boundary value problem governed by the second-order homogeneous differential equation,

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = 0 \qquad\qquad 0 \le x \le 1$$

with the boundary conditions of $y(0) = 1$ and $y'(1) = 0$.



This example is identical to the preceding one except the boundary condition at $x = 1$ is changed from $y(1) = 0$ to $y'(1) = 0$. We can derive the exact solution by ourselves or use the `dsolve` command as follows,

dsolve

```
>> syms x y
>>
dsolve('D2y+3*Dy+2*y=0','y(0)=1','Dy(1)=0','x')

ans =

(exp(1) - 2*exp(x))/(exp(2*x)*(exp(1) - 2))
```

i.e.,
$$y = \frac{e^1 - 2e^x}{e^{2x}(e^1 - 2)}$$

The approximate solution can be obtained by creating a script file, `Ex2.m`, that employs the `bvp4c` command and calls the two file functions, `OdeEx2` and `BcEx2` as follows,

```
%Ex2: D2y+3*Dy+2*y=0
solinit = bvpinit(linspace(0,1,5), [0 -1]);
sol = bvp4c('OdeEx2','BcEx2',solinit);
x = linspace(0,1,20);
```

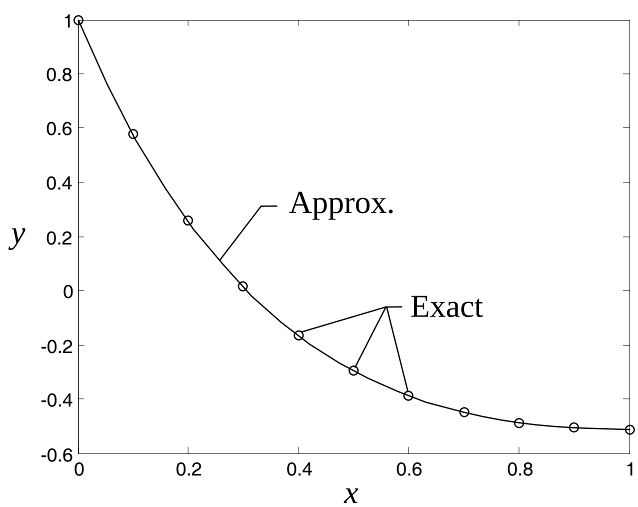
linspace

```
y = deval(sol,x);
plot(x,y(1,:),'k')
xlabel('x'); ylabel('y'); hold on;
xe = 0:0.1:1;
ye = (exp(1)-2.*exp(xe))./(exp(2.*xe)*(exp(1)-
      2));
plot(xe,ye,'ok')

function dydx = OdeEx2(x,y)
dydx = [y(2); -3*y(2)-2*y(1)];

function res = BcEx2(ya,yb)
res = [ya(1)-1; yb(2)-0];
```

xlabel

plot

The figure below shows the plot of the approximate solution as compared to the exact solution. The boundary conditions of $y(0)=1$ and $y'(1)=0$ are satisfied at the ends of the domain.



<u>Example</u> Use the `bvp4c` command to solve the boundary value problem governed by the second-order nonhomogeneous differential equation,

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y \quad = \quad 4x^3+6x^2+14x-9 \qquad\qquad 0\le x\le 1$$

with the boundary conditions of $y(0)=1$ and $y(1)=1$.

Similar to the preceding examples, we can derive the exact solution by ourselves or use the MATLAB command to find it. The `dsolve` command can be used to find the exact solution conveniently as follows,

```
>> syms x y
>> dsolve('D2y+Dy+y=4*x^3+6*x^2+14*x-9',
          'y(0)=1','y(1)=1','x')
```

dsolve

```
ans =

4*x^3 - 6*x^2 + 2*x + 1
```

i.e., the exact solution for this boundary value problem is,

$$y = 4x^3 - 6x^2 + 2x + 1$$

If we cannot find the exact solution, we can employ the `bvp4c` command to solve for the approximate solution. We first create the script file, `Ex3.m`, that calls the two function files `OdeEx3` and `BcEx3`. The script file contains the following commands including the commands for plotting as follows,

bvpinit

```
%Ex3: D2y+Dy+y=4*x^3+6*x^2+14*x-9
solinit = bvpinit(linspace(0,1,5), [0 -1]);
sol = bvp4c('OdeEx3','BcEx3',solinit);
x = linspace(0,1,20);
y = deval(sol,x);
plot(x,y(1,:),'k')
xlabel('x'); ylabel('y'); hold on;
xe = 0:0.1:1;
ye = 4.*xe.^3-6.*xe.^2+2.*xe+1;
plot(xe,ye,'ok')
```

bvp4c

deval

plot

The function file, `OdeEx3`, contains the two first-order differential equations,

```
function dydx = OdeEx3(x,y)
dydx = [y(2); -y(2)-y(1)+4*x^3+6*x^2+14*x-9];
```

which are obtained by separating the second-order differential equation as follows,

$$\frac{d^2y_1}{dx^2}+\frac{dy_1}{dx}+y_1 \;=\; 4x^3+6x^2+14x-9$$

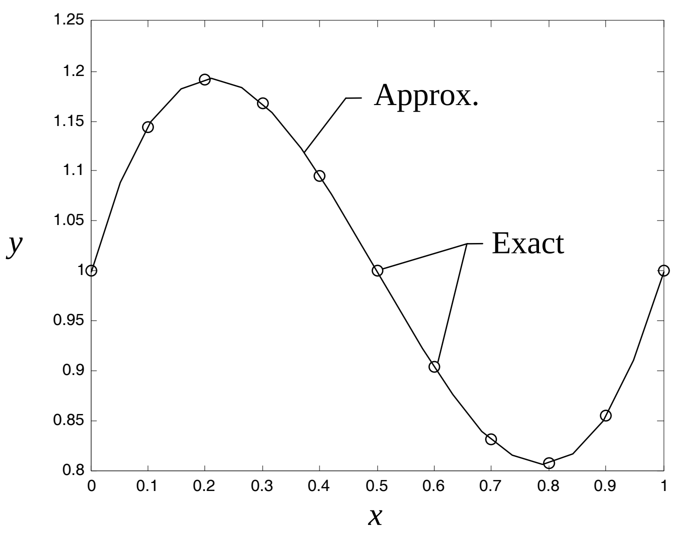
Let,
$$\frac{dy_1}{dx} \;=\; y_2$$

then,
$$\frac{dy_2}{dx} \;=\; -y_2-y_1+4x^3+6x^2+14x-9$$

The function file, `BcEx3`, contains the information of the two boundary conditions at both ends of the domain,

```
function res = BcEx3(ya,yb)
res = [ya(1)-1; yb(1)-1];
```

The approximate solution is plotted to compare with the exact solution. Again, the plot shows that the `bvp4c` command can provide high solution accuracy to the problem. The method for finding the approximate solution is thus valuable when the exact solution of the problem is not available.



9.4 Higher-Order Differential Equations

The `bvp4c` command can also be used to find approximate solutions of the boundary value problems that are governed by the higher-order differential equations as demonstrated in the examples below.

Example Use the `bvp4c` command to solve the boundary value problem governed by the third-order homogeneous differential equation,

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0 \qquad 0 \le x \le 4$$

with the boundary conditions of $y(0) = 0$, $y'(0) = 1$ and $y(4) = 0$.

This boundary value problem has an exact solution that can be found by using the `dsolve` command,

dsolve

```
>> dsolve('D3y+2*D2y-Dy-2*y=0',
   'y(0)=0','Dy(0)=1','y(4)=0','x')

ans =

-((exp(x) - 1)*(exp(2*x) - exp(4) - exp(8) +
   exp(x)))/(exp(2*x)*(exp(4) + exp(8) - 2))
```

i.e., the exact solution is,

$$y = -\frac{(e^x - 1)(e^{2x} - e^4 - e^8 + e^x)}{e^{2x}(e^4 + e^8 - 2)}$$

To employ the `bvp4c` command for finding the approximate solution, we need to break the third-order differential equation into three first-order differential equations,

$$\frac{d^3y_1}{dx^3} + 2\frac{d^2y_1}{dx^2} - \frac{dy_1}{dx} - 2y_1 = 0$$

i.e., if we let, $\frac{dy_1}{dx} = y_2$ and $\frac{dy_2}{dx} = y_3$

then, $\frac{dy_3}{dx} = -2y_3 + y_2 + 2y_1$

The boundary conditions become $y_1(0) = 0$, $y_2(0) = 1$ and $y_1(4) = 0$.

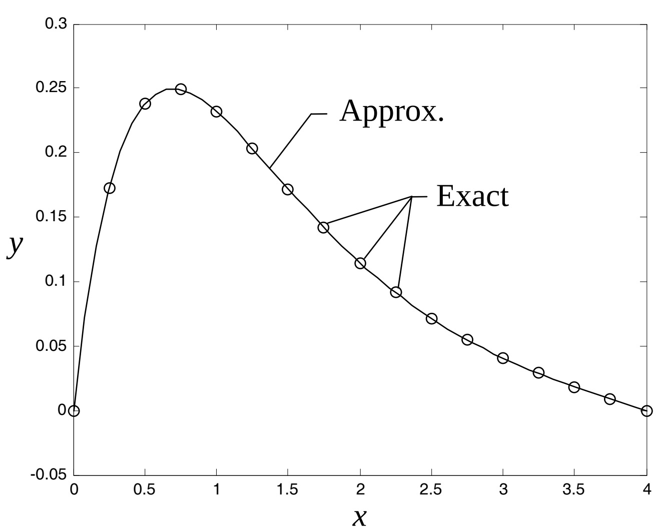
We can create the script file, `Ex4.m`, that calls the function files `OdeEx4` and `BcEx4`. The function file, `OdeEx4`, contains information of the three first-order differential equations while the function file, `BcEx4`, includes the three boundary conditions.

Details of the script file (`Ex4.m`) and the two function files (`OdeEx4` and `BcEx4`) are as follows.

bvpinit **bvp4c** **deval** **plot** **function**

```
%Ex4: D3y+2*D2y-Dy-2*y=0
solinit = bvpinit(linspace(0,4,5), [0 1 0]);
sol = bvp4c('OdeEx4','BcEx4',solinit);
x = linspace(0,4,50);
y = deval(sol,x);
plot(x,y(1,:),'k')
xlabel('x'); ylabel('y'); hold on;
xe = 0:0.25:4;
ye = -((exp(xe)-1).*(exp(2.*xe)-exp(4)-exp(8)
     +exp(xe)))./(exp(2.*xe)*(exp(4)+exp(8)- 2));
plot(xe,ye,'ok')

function dydx = OdeEx4(x,y)
dydx = [y(2); y(3); -2*y(3)+y(2)+2*y(1)];

function res = BcEx4(ya,yb)
res = [ya(1)-0; ya(2)-1; yb(1)-0];
```

The approximate solution is compared with the exact solution as shown in the figure.

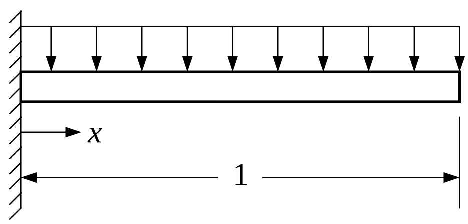


Example Use the bvp4c command to solve the boundary value problem governed by the fourth-order nonhomogeneous differential equation,

$$\frac{d^4y}{dx^4} = -1 \qquad\qquad 0 \le x \le 1$$

with the boundary conditions of $y(0)=0$, $y'(0)=0$, $y''(1)=0$ and $y'''(1)=0$.

This boundary value problem represents the deflection behavior of a cantilever beam subjected to a uniform loading as shown in the figure. The beam of a unit length is fixed at the left end ($x=0$) so that both the deflection and slope are zero. The right end ($x=1$) is free to move, therefore, both the moment and shear are zero.



The exact solution of this problem can be derived by performing integration four times and applying the boundary conditions. It can also be found conveniently by using the dsolve command as follows,

```
>> syms x y
>> dsolve('D4y=-1','y(0)=0','Dy(0)=0',
          'D2y(1)=0','D3y(1)=0','x')
```

dsolve

```
ans =

- x^4/24 + x^3/6 - x^2/4
```

i.e., $$y = -\frac{x^4}{24} + \frac{x^3}{6} - \frac{x^2}{4}$$

If we prefer to obtain the approximate solution by using the bvp4c command, we have to separate the fourth-order differential equation into four first-order differential equations. Since the given differential equation is,

$$\frac{d^4 y_1}{dx^4} = -1$$

If we let, $\dfrac{dy_1}{dx} = y_2$, $\dfrac{dy_2}{dx} = y_3$ and $\dfrac{dy_3}{dx} = y_4$

then, $$\frac{dy_4}{dx} = -1$$

The boundary conditions become $y_1(0)=0$, $y_2(0)=0$, $y_3(1)=0$ and $y_4(1)=0$.

Then, we create the script file, `Ex5.m`, which calls the two function files `OdeEx5` and `BcEx5`. The two function files `OdeEx5` and `BcEx5` contain information of the differential equations and boundary conditions, respectively. Details of these files are as follows,

```
%Ex5: D4y=-1
solinit = bvpinit(linspace(0,1,5), [0 -1 0 0]);
sol = bvp4c('OdeEx5','BcEx5',solinit);
x = linspace(0,1,50);
y = deval(sol,x);
plot(x,y(1,:),'k')
xlabel('x'); ylabel('y'); hold on;
xe = 0:0.1:1;
ye = - xe.^4/24 + xe.^3/6 - xe.^2/4;
plot(xe,ye,'ok')
```

linspace

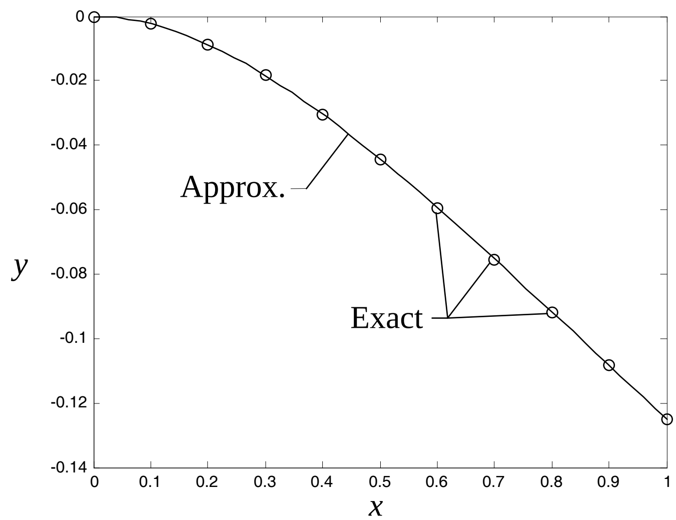
ylabel

```
function dydx = OdeEx5(x,y)
dydx = [y(2); y(3); y(4); -1];
```

function

```
function res = BcEx5(ya,yb)
res = [ya(1)-0; ya(2)-0; yb(3)-0; yb(4)-0];
```

The approximate solution of the beam deflection compares very well with the exact solution as shown in the figure.

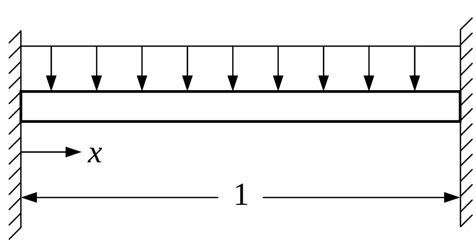


<u>Example</u> Use the `bvp4c` command to solve the boundary value problem governed by the fourth-order nonhomogeneous differential equation,

$$\frac{d^4y}{dx^4} = -1 \qquad\qquad 0 \le x \le 1$$

with the boundary conditions of $y(0) = 0$, $y'(0) = 0$, $y(1) = 0$ and $y'(1) = 0$.

This example is identical to the preceding one except the right boundary conditions at the right end of the domain. The beam is now fixed at both ends so that the deflections and slopes at ($x = 0$) and ($x = 1$) are zero as shown in the figure.



In this case, the exact solution can be obtained easily by using the `dsolve` command,

```
>> syms x y
>> dsolve('D4y=-1','y(0)=0','Dy(0)=0',
          'y(1)=0','Dy(1)=0','x')
```

dsolve

```
ans =

- x^4/24 + x^3/12 - x^2/24
```

i.e.,
$$y = -\frac{x^4}{24}+\frac{x^3}{12}-\frac{x^2}{24}$$

Here, the script file, `Ex6.m`, and the two function files `OdeEx6` and `BcEx6` are similar to those in the preceding example,

bvpinit

```
%Ex6: D4y=-1
solinit = bvpinit(linspace(0,1,5), [0 -1 0 0]);
sol = bvp4c('OdeEx6','BcEx6',solinit);
x = linspace(0,1,50);
y = deval(sol,x);
plot(x,y(1,:),'k')
xlabel('x'); ylabel('y'); hold on;
xe = 0:0.1:1;
ye = - xe.^4/24 + xe.^3/12 - xe.^2/24;
plot(xe,ye,'ok')
```

bvp4c

deval

plot

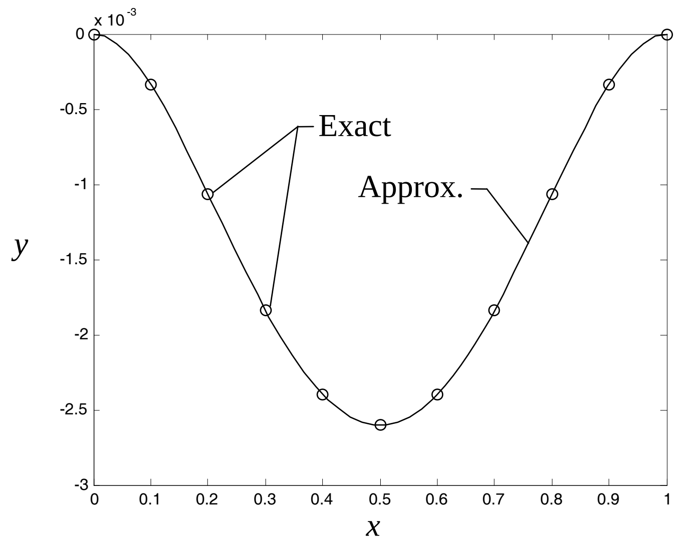
```
function dydx = OdeEx6(x,y)
dydx = [y(2); y(3); y(4); -1];

function res = BcEx6(ya,yb)
res = [ya(1)-0; ya(2)-0; yb(1)-0; yb(2)-0];
```

function

The approximate solution compares very well with the exact solution as shown in the figure.

9.5 Complicated Differential Equations

Many boundary value problems are governed by complicated differential equations. To solve them, we may start by using the `dsolve` command to find their exact solutions. If the exact solutions are not available, we can use the `bvp4c` command to find the approximate solutions instead.



Example Use the `bvp4c` command to solve the boundary value problem governed by the second-order homogeneous differential equation with variable coefficients,

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - \frac{2}{x^2}y = 0 \qquad 1 \le x \le 2$$

The boundary conditions are $y(1) = 5$ and $y(2) = 3$.

We start from employing the `dsolve` command to find the exact solution,

dsolve

```
>> syms x y
>> dsolve('D2y+(2/x)*Dy-(2/x^2)*y=0',
          'y(1)=5','y(2)=3','x')

ans =

x + 4/x^2
```

The exact solution for this boundary value problem is,

$$y = x + \frac{4}{x^2}$$

We can use the `bvp4c` command to find the approximate solution by creating the script file `Ex7.m`. The script file calls the two function files `OdeEx7` and `BcEx7` which contain information of the differential equation and boundary conditions, respectively.

linspace

```
%Ex7: D2y+(2/x)*Dy-(2/x^2)*y=0
solinit = bvpinit(linspace(1,2,5), [5 -1]);
sol = bvp4c('OdeEx7','BcEx7',solinit);
x = linspace(1,2,20);
y = deval(sol,x);
plot(x,y(1,:),'k')
xlabel('x'); ylabel('y'); hold on;
xe = 1:0.1:2;
ye = xe + 4./xe.^2;
plot(xe,ye,'ok')
```

Hold on

plot

```
function dydx = OdeEx7(x,y)
dydx = [y(2); -(2/x)*y(2)+(2/x^2)*y(1)];
```

function

```
function res = BcEx7(ya,yb)
res = [ya(1)-5; yb(1)-3];
```

Note that the function file `OdeEx7` above contains the two first-order differential equations that are separated from the given second-order differential equation,

$$\frac{d^2 y_1}{dx^2} + \frac{2}{x}\frac{dy_1}{dx} - \frac{2}{x^2} y_1 \quad = \quad 0$$

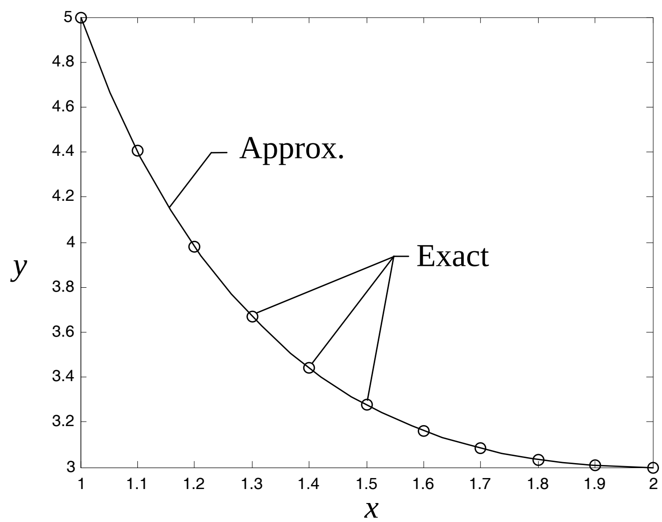
where $\dfrac{dy_1}{dx} = y_2$ then $\dfrac{dy_2}{dx} = -\dfrac{2}{x} y_2 + \dfrac{2}{x^2} y_1$

The computed approximate solution is compared with the exact solution as shown in the figure.

<u>Example</u> Use the `bvp4c` command to solve the boundary value problem governed by the second-order nonlinear differential equation,

$$y\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \quad = \quad 0 \qquad\qquad 1 \le x \le 3$$

with the boundary conditions of $y(1) = \sqrt{2}$ and $y(3) = 2$.



The `dsolve` command can provide the exact solution for the nonlinear differential equation with the boundary conditions above,

```
>> syms x y
>>
dsolve('y*D2y+Dy^2=0','y(1)=sqrt(2)','y(3)=2','x')
ans =
(x + 1)^(1/2)
```

i.e., $$y = \sqrt{x+1}$$

The approximate solution is obtained by using the `bvp4c` command. The second-order differential equation is separated into the two first-order differential equations as follows,

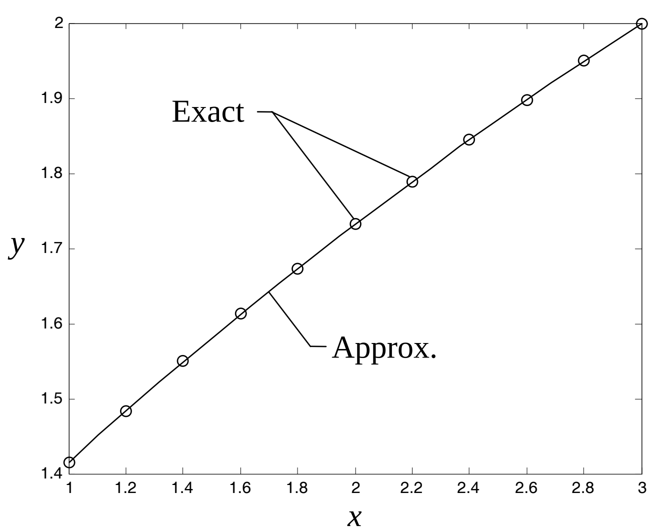
$$y_1 \frac{d^2 y_1}{dx^2} + \left(\frac{dy_1}{dx} \right)^2 = 0$$

If we let, $\dfrac{dy_1}{dx} = y_2$ then $\dfrac{dy_2}{dx} = -\dfrac{y_2^2}{y_1}$

Details of the script file, `Ex8.m`, that calls the two function files `OdeEx8` and `BcEx8` containing information of the differential equations and boundary conditions, respectively, are as follows,

```
%Ex8: y*D2y+Dy^2=0
solinit = bvpinit(linspace(1,3,5), [1 1]);
sol = bvp4c('OdeEx8','BcEx8',solinit);
x = linspace(1,3,20);
y = deval(sol,x);
plot(x,y(1,:),'k')
xlabel('x'); ylabel('y'); hold on;
xe = 1:0.2:3;
ye = sqrt(xe+1);
plot(xe,ye,'ok')

function dydx = OdeEx8(x,y)
dydx = [y(2); -y(2)^2/y(1)];

function res = BcEx8(ya,yb)
res = [ya(1)-sqrt(2); yb(1)-2];
```

The approximate solution is compared with the exact solution as shown in the figure. The figure shows that the `bvp4c` command can provide accurate approximate solution to this nonlinear boundary value problem.



<u>Example</u> Use the `bvp4c` command to solve the boundary value problem governed by the second-order nonlinear differential equation,

$$\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = 0 \qquad 0 \le x \le 1$$

with the boundary conditions of $y(0)=1$ and $y(1)=1/2$.

In this case, the `dsolve` command cannot provide the exact solution,

```
>> syms x y
>> dsolve('D2y+2*y*Dy=0','y(0)=1','y(1)=1/2','x')
Warning: possibly missing solutions [solvini]
Warning: Explicit solution could not be found.
> In dsolve at 101

ans =

[ empty sym ]
```

However, it is found that the solution, $\quad y = \dfrac{1}{1+x}$

is an exact solution because it satisfies the differential equation. We can check this by using the `diff` command,

```
>> y = 1/(1+x);
>> LHS = diff(y,x,2) + 2*y*diff(y,x)

LHS =

0
```

diff

The solution also satisfies the two boundary conditions which can be verified by using the `subs` command,

```
>> subs(y,{x},{0})

ans =

     1

>> subs(y,{x},{1})

ans =

    0.5000
```

subs

If we cannot find the exact solution of the problem, we can use the `bvp4c` command to find the approximate solution in the same fashion as explained in the preceding examples. We start from breaking the second-order differential equation into two first-order differential equations as follows,

$$\frac{d^2 y_1}{dx^2} + 2y_1 \frac{dy_1}{dx} \quad = \quad 0$$

i.e., if we let $\dfrac{dy_1}{dx} = y_2$, then $\dfrac{dy_2}{dx} = -2y_1 y_2$.

We create the script file, `Ex9.m`, which calls the two function files `OdeEx9` and `BcEx9` containing information of the differential equations and boundary conditions, respectively. Details of these file are as follows,

bvpinit

bvp4c

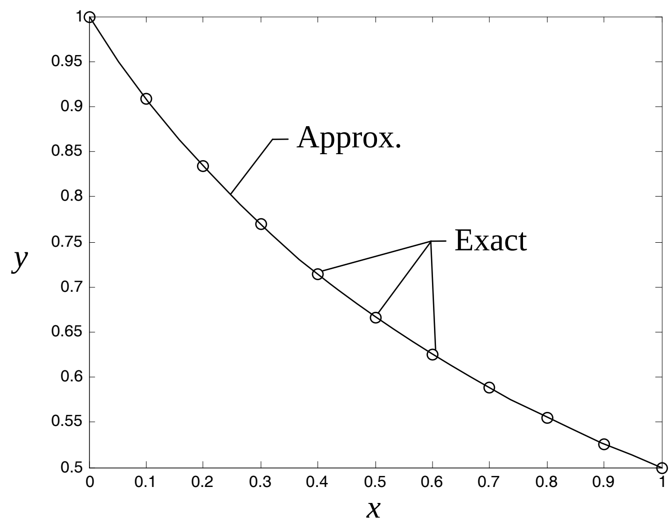
deval

plot

function

```
%Ex9: D2y+2*y*Dy=0
solinit = bvpinit(linspace(0,1,5), [1 -1]);
sol = bvp4c('OdeEx9','BcEx9',solinit);
x = linspace(0,1,20);
y = deval(sol,x);
plot(x,y(1,:),'k')
xlabel('x'); ylabel('y'); hold on;
xe = 0:0.1:1;
ye = 1./(1.+xe);
plot(xe,ye,'ok')

function dydx = OdeEx9(x,y)
dydx = [y(2); -2*y(1)*y(2)];

function res = BcEx9(ya,yb)
res = [ya(1)-1; yb(1)-0.5];
```

The approximate solution is compared with the exact solution as shown in the figure. Again, the figure shows that the `bvp4c` command can provide accurate approximate solution to this nonlinear boundary value problem.



9.6 Concluding Remarks

Methods for solving the boundary value problems governed by the ordinary differential equations and boundary conditions were presented. For the problems with simple differential equations, their exact solutions can be derived. The process for deriving exact solutions is similar to that for the initial value problems except the application of the boundary conditions instead of the initial conditions. MATLAB contains the `dsolve` command that can be used to find exact solutions for many types of the differential equations.

For the boundary value problems that are governed by more complicated differential equations, such as those with variable coefficients and in nonlinear form, their exact solutions may not be available. In this case, the numerical methods must be applied to obtain approximate solutions. MATLAB contains the `bvp4c` command that can provide accurate numerical solutions. Users need to break the given higher-order differential equation into many first-order differential equations before using the command.

The boundary value problems solved in this chapter are all in one-dimensional domain. The problems are governed by the ordinary differential equations with simple boundary conditions. For two- and three-dimensional problems, they are governed by the partial differential equations. The boundary conditions are more complicated and their domains could be arbitrary. In general, their exact solutions cannot be found. Finding the approximate solutions by using the numerical methods is the only way to solve the problems. We will learn a popular method, so called the finite element method, for solving these problems in the next chapter.

Exercises

1. Derive the exact solutions of the following boundary value problems,

   (a) $y'' + 4y = 0 \qquad 0 \le x \le 1$
   $y(0) = 0, \; y(1) = 5$

   (b) $y'' + 9y = 0 \qquad 0 \le x \le 2$
   $y(0) = 4, \; y(2) = 1$

   (c) $y'' + y = 0 \qquad 0 \le x \le \pi/2$
   $y(0) = -1, \; y(\pi/2) = 1$

   (d) $y'' + y' + 2y = 0 \qquad 0 \le x \le 1$
   $y(0) = 0, \; y(1) = 1$

   (e) $y'' + 2y' - 3y = 0 \qquad 0 \le x \le 1$
   $y(0) = 1, \; y(1) = 2$

   Verify the solutions with those obtained by using the `dsolve` command. In each sub-problem, employ the `ezplot` command to plot the solution of y that varies with x within the given domain.

2. Derive the exact solutions of the following boundary value problems,

   (a) $y'' - y = x^2 \qquad 0 \le x \le 1$
   $y(0) = 0\,,\ \ y(1) = 0$

   (b) $y'' + 4y = \cos x \qquad 0 \le x \le 1$
   $y(0) = 0\,,\ \ y(1) = 0$

   (c) $y'' + 2y' + y = 5x \qquad 0 \le x \le 1$
   $y(0) = 0\,,\ \ y(1) = 0$

   (d) $y'' - 2y' + y = x^2 - 1 \qquad 0 \le x \le 1$
   $y(0) = 5\,,\ \ y(1) = 10$

   (e) $y'' - 4y' + 4y = (x+1)e^{2x} \qquad 0 \le x \le 1$
   $y(0) = 3\,,\ \ y(1) = 0$

   Verify the solutions with those obtained by using the `dsolve` command. In each sub-problem, employ the `ezplot` command to plot the solution of y that varies with x within the given domain.

3. Use the `dsolve` command to solve the following boundary value problems,

   (a) $x^2 y'' + 3xy' + 3y = 0 \qquad 1 \le x \le 2$
   $y(1) = 5\,,\ \ y(2) = 0$

   (b) $x^2 y'' - 2xy' + 2y = 0 \qquad 1 \le x \le 2$
   $y(1) = -2\,,\ \ y(2) = 2$

   (c) $x^2 y'' + 3xy' + y = x^2 \qquad 1 \le x \le 2$
   $y(1) = 0\,,\ \ y(2) = 0$

   (d) $y'' + \dfrac{1}{x} y' = 0 \qquad 1 \le x \le 4$
   $y(1) = 50\,,\ \ y(4) = 100$

   (e) $x^2 y'' - xy' + y = \ln x \qquad 1 \le x \le 2$
   $y(1) = 0\,,\ \ y(2) = -2$

   In each sub-problem, verify the solution by substituting it into the differential equation and boundary conditions.

4. Use the `diff` command to show that,

$$y = 32\left[\frac{\cos(5)-1}{\sin(5)}\sin\left(\frac{x}{2}\right)-\cos\left(\frac{x}{2}\right)+1\right]$$

is the exact solution of the boundary value problem governed by the second-order differential equation,

$$y'' + \frac{1}{4}y = 8 \qquad 0 \le x \le 10$$

with the boundary conditions of $y(0)=0$ and $y(10)=0$. Then, use the `dsolve` command to check whether MATLAB can provide the above solution.

5. Use the `diff` command to show that,

$$y = x\sin x$$

is the exact solution of the boundary value problem governed by the second-order nonhomogeneous differential equation,

$$y'' + 2xy' - y = 2(1+x^2)\cos x \qquad 0 \le x \le \frac{\pi}{2}$$

with the boundary conditions of $y(0)=0$ and $y(\pi/2)=\pi/2$. Then, use the `dsolve` command to check whether MATLAB can provide the above solution.

6. Use the `diff` command to show that,

$$y = \tan\left(x-\frac{\pi}{4}\right)$$

is the exact solution of the boundary value problem governed by the second-order nonlinear differential equation,

$$y'' - 2yy' = 0 \qquad 0 \le x \le \frac{\pi}{2}$$

with the boundary conditions of $y(0)=-1$ and $y(\pi/2)=1$. Then, use the `dsolve` command to check whether MATLAB can provide the above solution.

7. Use the `diff` command to show that,

$$y = \sqrt{1+x}$$

is the exact solution of the boundary value problem governed by the second-order nonlinear differential equation,

$$y y'' + (y')^2 = 0 \qquad 1 \leq x \leq 3$$

with the boundary conditions of $y(1) = \sqrt{2}$ and $y(3) = 2$. Then, use the `dsolve` command to check whether MATLAB can provide the above solution.

8. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order homogeneous differential equation,

$$y'' + y = 0 \qquad 0 \leq x \leq 1$$

with the boundary conditions of $y(0) = 1$ and $y(1) = -1$. Plot to compare the approximate solution with the exact solution obtained by using the `dsolve` command.

9. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order nonhomogeneous differential equation,

$$y'' + 2y' + y = x^2 \qquad 0 \leq x \leq 1$$

with the boundary conditions of $y(0) = 5$ and $y(1) = 2$. Plot to compare the approximate solution with the exact solution obtained by using the `dsolve` command.

10. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order homogeneous differential equation with variable coefficients,

$$x^2 y'' + xy' + x^2 y = 0 \qquad 1 \leq x \leq 8$$

The boundary conditions are $y(1) = 1$ and $y(8) = 0$. Plot to compare the approximate solution with the exact solution obtained by using the `dsolve` command.

11. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order homogeneous differential equation with variable coefficients,

$$x^2 y'' - xy' + y = 0 \qquad 1 \le x \le 3$$

The boundary conditions are $y(1)=1$ and $y(3)=4$. Plot to compare the approximate solution with the exact solution obtained by using the `dsolve` command.

12. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order homogeneous differential equation with variable coefficients,

$$6x^2 y'' + xy' + y = 0 \qquad 1 \le x \le 64$$

The boundary conditions are $y(1)=2$ and $y(64)=12$. Plot to compare the approximate solution with the exact solution obtained by using the `dsolve` command.

13. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order nonhomogeneous differential equation with variable coefficients,

$$xy'' - y' = x^5 \qquad 1 \le x \le 2$$

The boundary conditions are $y(1)=1/2$ and $y(2)=4$. Plot to compare the approximate solution with the exact solution obtained by using the `dsolve` command.

14. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order nonhomogeneous differential equation with variable coefficients,

$$x^2 y'' + xy' - 3y = 3x \qquad 1 \le x \le 2$$

The boundary conditions are $y(1)=1$ and $y(2)=5$. Then, use the `dsolve` command to check whether MATLAB can provide and exact solution. If it can, plot to compare the approximate solution with the exact solution. If it cannot, plot to show only the approximate solution.

15. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order nonhomogeneous differential equation with variable coefficients,

$$y'' + xy' - x^2 y \quad = \quad 2x^3 \qquad\qquad 0 \le x \le 1$$

The boundary conditions are $y(0)=1$ and $y(1)=-1$. Then, use the `dsolve` command to check whether MATLAB can provide and exact solution. If it can, plot to compare the approximate solution with the exact solution. If it cannot, plot to show only the approximate solution.

16. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order nonhomogeneous differential equation with variable coefficients,

$$y'' - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y \quad = \quad x^2+1 \qquad\qquad 0 \le x \le 1$$

The boundary conditions are $y(0)=2$ and $y(1)=5/3$. Plot to compare the approximate solution with the exact solution of,

$$y \quad = \quad \frac{x^4}{6} - \frac{3x^2}{2} + x + 2$$

17. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order nonhomogeneous differential equation with variable coefficients,

$$y'' - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y \quad = \quad x^2+1 \qquad\qquad 0 \le x \le 1$$

The boundary conditions are $y(0)+y'(0)=0$ and $y(1)-y'(1)$ $=-3$. Plot to compare the approximate solution with the exact solution of,

$$y \quad = \quad \frac{x^4}{6} + \frac{3x^2}{2} + x - 1$$

18. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order homogeneous nonlinear differential equation,

$$y'' + y' - y^2 = 0 \qquad 0 \le x \le 1$$

with the boundary conditions of $y(0) = 1$ and $y(1) = 2$. Then, use the `dsolve` command to check whether MATLAB can provide and exact solution. If it can, plot to compare the approximate solution with the exact solution. If it cannot, plot to show only the approximate solution.

19. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order homogeneous nonlinear differential equation,

$$y'' - 2yy' = 0 \qquad 1 \le x \le 2$$

with the boundary conditions of $y(1) = 1$ and $y(2) = 1/2$. Then, use the `dsolve` command to check whether MATLAB can provide and exact solution. If it can, plot to compare the approximate solution with the exact solution. If it cannot, plot to show only the approximate solution.

20. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order homogeneous nonlinear differential equation,

$$yy'' + (y')^2 = 0 \qquad 0 \le x \le 1$$

with the boundary conditions of $y(0) = 1$ and $y(1) = 2$. Plot to compare the approximate solution with the exact solution of,

$$y = \sqrt{3x + 1}$$

21. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order homogeneous nonlinear differential equation,

$$y'' + 2yy' = 0 \qquad 0 \le x \le 1$$

with the boundary conditions of $y(0)=1$ and $y(1)=1/2$. Plot to compare the approximate solution with the exact solution of,

$$y = 1/(x+1)$$

22. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the second-order homogeneous nonlinear differential equation,

$$y'' - x(y')^2 = 0 \qquad 0 \le x \le 2$$

with the boundary conditions of $y(0)=\pi/2$ and $y(2)=\pi/4$. Plot to compare the approximate solution with the exact solution of,

$$y = \cot^{-1}(x/2)$$

23. Derive the exact solutions of the following boundary value problems,

(a) $y''' + 2y'' + 3y' + 2y = -2(x+2)\sin x \qquad 0 \le x \le \pi/2$
$y(0)=0$, $y'(0)=1$, $y(\pi/2)=0$

(b) $y''' - 3y'' - 10y' + 24y = 24x^2 - 20x + 18 \qquad 0 \le x \le 1$
$y(0)=1$, $y'(0)=0$, $y(1)=2$

(c) $y''' - 3y' + 2y = -16e^{-3x} \qquad 0 \le x \le 2$
$y(0)=2$, $y(2)=e^{-6}-e^{2}$, $y'(2)=-3e^{-6}-e^{2}$

(d) $y''' - y'' - 8y' + 12y = 4\sin x - 36\cos x + 12$
$y(0)=0$, $y(\pi)=0$, $y'(\pi)=-4 \qquad 0 \le x \le \pi$

Verify the solutions with those obtained from using the `dsolve` command. In each sub-problem, employ the `ezplot` command to plot the solution of y that varies with x within the given domain.

24. Derive the exact solutions of the following boundary value problems,

(a) $y^{IV} - 2y'' + y \;=\; 9e^{2x} + 2 \qquad 0 \le x \le 1$
$y(0) = 3,\;\; y'(0) = 2,\;\; y(1) = e^2 + 2,\;\; y'(1) = 2e^2$

(b) $y^{IV} + 10y'' + 16y \;=\; 7\sin x + 48 \qquad 0 \le x \le \pi$
$y(0) = 3,\;\; y'(0) = 1,\;\; y(\pi) = 3,\;\; y'(\pi) = -1$

(c) $y^{IV} - y \;=\; -1 \qquad 0 \le x \le 5$
$y(0) = 2,\;\; y'(0) = 1,\;\; y(5) = e^5 + 1,\;\; y'(5) = e^5$

(d) $y^{IV} - 8y''' + 24y'' - 32y' + 16y \;=\; 16x^2 - 48x + 32$
$y(0) = 1,\;\; y'(0) = 1,\;\; y(2) = 7,\;\; y'(2) = 5 \qquad 0 \le x \le 2$

Verify the solutions with those obtained from using the `dsolve` command. In each sub-problem, employ the `ezplot` command to plot the solution of *y* that varies with *x* within the given domain.

25. Use the `dsolve` command to find the exact solution of the boundary value problem governed by the third-order homogeneous differential equation,

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y \;=\; 0 \qquad 0 \le x \le 1$$

with the boundary conditions of $y(0) = 0$, $y'(0) = 0$ and $y(1) = 1$. Plot the solution of *y* that varies with *x* by using the `ezplot` command. Repeat the problem but by employing the `bvp4c` command to solve for the approximate solution. Plot to compare the approximate solution with the exact solution.

26. If the differential equation in preceding problem becomes nonlinear,

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + y^2 \;=\; 0 \qquad 0 \le x \le 1$$

with the same boundary conditions of $y(0)=0$, $y'(0)=0$ and $y(1)=1$, use the `dsolve` command to check whether it can find and exact solution. If it cannot, employ the `bvp4c` command to find the approximate solution and plot to show its variation.

27. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the third-order nonhomogeneous nonlinear differential equation,

$$y\frac{d^3y}{dx^3}-\frac{dy}{dx} \quad = \quad -2x \qquad\qquad 0 \le x \le 1$$

with the boundary conditions of $y(0)=1$, $y'(0)=0$ and $y(1)=2$. Plot to compare the approximate solution with the exact solution of,

$$y \quad = \quad x^2+1$$

28. The velocity profile of a laminar boundary layer flow over a flat plate can be found by solving the third-order homogeneous nonlinear differential equation,

$$2\frac{d^3y}{dx^3}+y\frac{d^2y}{dx^2} \quad = \quad 0 \qquad\qquad 0 \le x \le 1$$

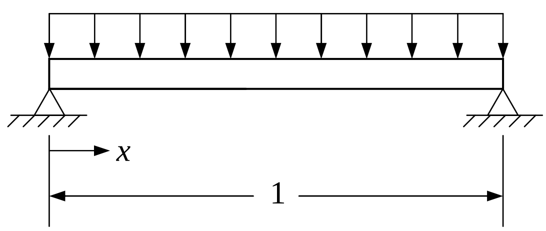
with the boundary conditions of $y(0)=0$, $y'(0)=0$ and $y(1)=1$. Employ the `bvp4c` command to find the approximate velocity profile *y* that varies with the depth *x* in the interval of $0 \le x \le 1$.

29. A simply-support beam of a unit length is subjected to a uniform loading as shown in the figure. The deflection *y*(*x*) along the beam length in the *x*-direction can be found by solving the fourth-order nonhomogeneous differential equation,

$$\frac{d^4y}{dx^4} \quad = \quad -1 \qquad\qquad 0 \le x \le 1$$

with the boundary conditions of $y(0) = y''(0) = y(1) = y''(1) = 0$. Employ the `bvp4c` command to find the approximate solution of y that varies with x. Plot to compare the approximate solution with the exact solution of,

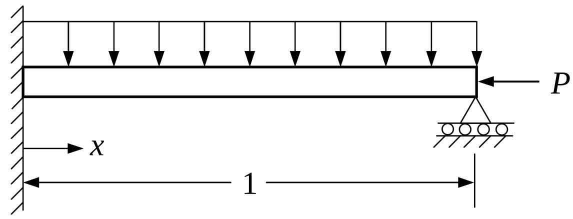
$$y = -\frac{x^4}{24} + \frac{x^3}{12} - \frac{x}{24}$$



30. The deflection y of a cantilever beam subjected to an axial force P as shown in the figure is governed by the fourth-order nonhomogeneous differential equation,

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = -1 \qquad 0 \le x \le 1$$

with the boundary conditions of $y(0) = y'(0) = y(1) = y''(1) = 0$. Employ the `bvp4c` command to find the approximate deflection of y that varies with x along the beam length. Plot to compare the approximate solution with the exact solution that can be found by using the `dsolve` command.



31. Employ the `bvp4c` command to find the approximate solution of the boundary value problem governed by the fourth-order nonhomogeneous nonlinear differential equation,

$$y\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} + y = x^3 - 6x - 1 \qquad 0 \le x \le 1$$

with the boundary conditions of $y(0) = -1$, $y'(0) = 0$, $y(1) = 0$ and $y'(1) = 3$. Plot to compare the approximate solution with the exact solution of,

$$y = x^3 - 1$$

Then, check whether the `dsolve` command can find the exact solution above. If it cannot, verify the exact solution by substituting it into the differential equation and boundary conditions via the `diff` and `subs` commands, respectively.

Chapter 10

Partial Differential Equations

10.1 Introduction

Most scientific and engineering problems are governed by partial differential equations for which the dependent variable u varies with the three coordinates of x, y, z, and time t. Solving the partial differential equations is more difficult than the ordinary differential equations learned earlier in the preceding chapters. Their exact solutions are not available, in general, and numerical methods are used to obtain the approximate solutions.

The widely used numerical methods are the finite difference, finite element and finite volume methods. The finite element method is the popular one because it can handle complicated boundary conditions and domain geometries effectively. MATLAB contains a toolbox that uses the finite element method to solve for approximate solutions of the partial differential equations in two dimensions.

This chapter begins with the classification of the partial differential equations. The MATLAB toolbox for solving these differential equations is explained. The toolbox is then used to solve the elliptic, parabolic and hyperbolic partial differential equations, respectively. Exact solutions for simple domain geometries are derived so that the finite element solutions can be compared to measure their accuracy. More complicated examples, for which their exact solutions are not available, are then used to demonstrate the efficiency of the finite element method implemented in the toolbox.

10.2 Classification of Partial Differential Equations

The Partial Differential Equation toolbox (PDE toolbox) is used to solve the problems in two dimensions. The unknown dependent variable u is function of x- and y- coordinates and may be time t. The partial differential equations are classified into three types:

(a) Elliptic equation,

$$-\left(\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right)+au = f$$

(b) Parabolic equation,

$$d\frac{\partial u}{\partial t}-\left(\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right)+au = f$$

and (c) Hyperbolic equation,

$$d\frac{\partial^2 u}{\partial t^2}-\left(\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right)+au = f$$

where d, c, a and f are constants or may be function of x, y and u.

The boundary conditions consist of,

(a) Specifying value of the dependent variable u along the boundary (Dirichlet condition) ,

$$hu \quad = \quad r$$

where h and r are constants or function of x, y and u.

(b) Specifying gradient value of the dependent variable u along the boundary (Neumann condition) ,

$$c\frac{\partial u}{\partial n} + qu \quad = \quad g$$

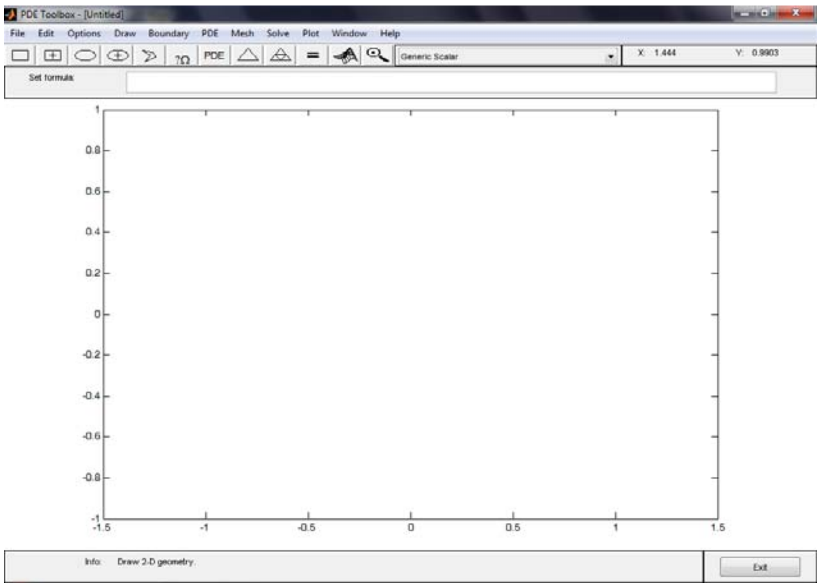
where c, q and g are constants or function of x, y and u. The letter n denotes a unit vector normal to the boundary.

Solving the elliptic equation does not need the initial condition while the parabolic equation needs an initial condition of $u(x, y, 0)$ at time $t = 0$. The hyperbolic equation needs two initial conditions of $u(x, y, 0)$ and $\partial u(x, y, 0)/\partial t$ at time $t = 0$.

10.3 The Finite Element Toolbox

The finite element method is used to solve the partial differential equations. The main advantage of the method is its flexibility for modelling arbitrary geometry easily. The problem domain is first divided into small elements connected at nodes where the unknowns are located. The finite element equations for each element are derived from the governing differential equation of the problems. These finite element equations are assembled together to form a set of simultaneous algebraic equations. The boundary conditions are then imposed prior to solving for solutions of the unknowns at nodes.

Understanding the finite element procedure above leads to the development of the PDE toolbox which contains commands under the menus, tool bars, etc. via the Graphical User Interface (GUI). The PDE toolbox can be initialized by simply typing `>> pdetool` on the Command Window, a graphic interface as shown in the figure is displayed.



On the graphic interface, the menu commands are:

File	New, Open…, Save, Save As…, Print…, Exit.
Edit	Undo, Cut, Copy, Paste…, Clear, Select All.
Options	Grid, Grid Spacing…, Snap, Axes Limits…, Axes Equal, Zoom, etc.
Draw	Draw Mode, Rectangle/square, Ellipse/circle, Polygon, etc.
Boundary	Boundary Mode, Specify Boundary Conditions, etc.
PDE	PDE Mode, PDE Specification…, etc.
Mesh	Mesh Mode, Initialize Mesh, Refine Mesh, Show Node Labels, Show Triangle Labels, etc.
Solve	Solve PDE, Parameters…, etc.
Plot	Plot Solution, Parameters…, Export Movie….

Icons in the tool bar under the menu bar perform the same tasks through the graphic interface. We will be familiar with this graphic interface by solving examples in the following sections.

10.4 Elliptic Equations

The elliptic partial differential equation is in the form,

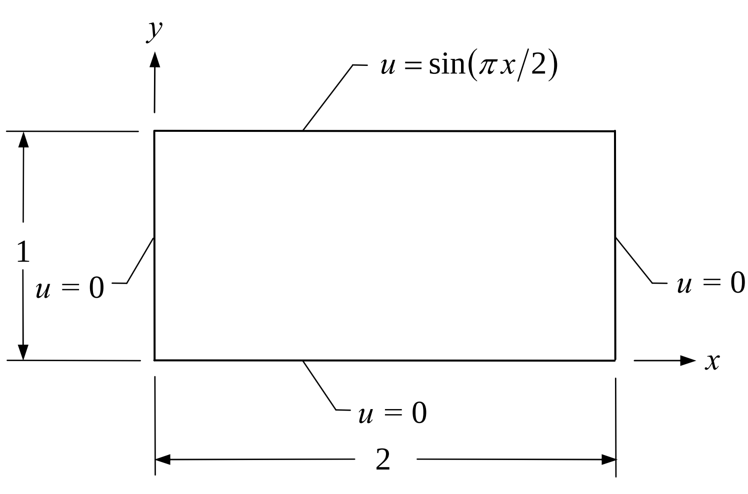
$$-\left(\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right)+au \quad = \quad f$$

where c, a and f are constants or function of x, y and u. The partial differential equations in the elliptic form above arise in many applications. These include steady-state heat transfer, potential flow, cross-sectional stress in a bar under torsion, electric potential, underground seepage flow, etc. We will use the PDE toolbox to solve for the solution of $u(x,y)$ from the elliptic differential equation with the specified boundary conditions.

Example Use the PDE toolbox to solve the elliptic partial differential equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \quad = \quad 0$$

for the solution of $u(x,y)$ in a rectangular domain with the boundary conditions as shown in the figure.



It is noted that the exact solution for this problem which is derived in the next example is,

$$u(x,y) \quad = \quad \frac{\sin(\pi x/2)\sinh(\pi y/2)}{\sinh(\pi/2)}$$

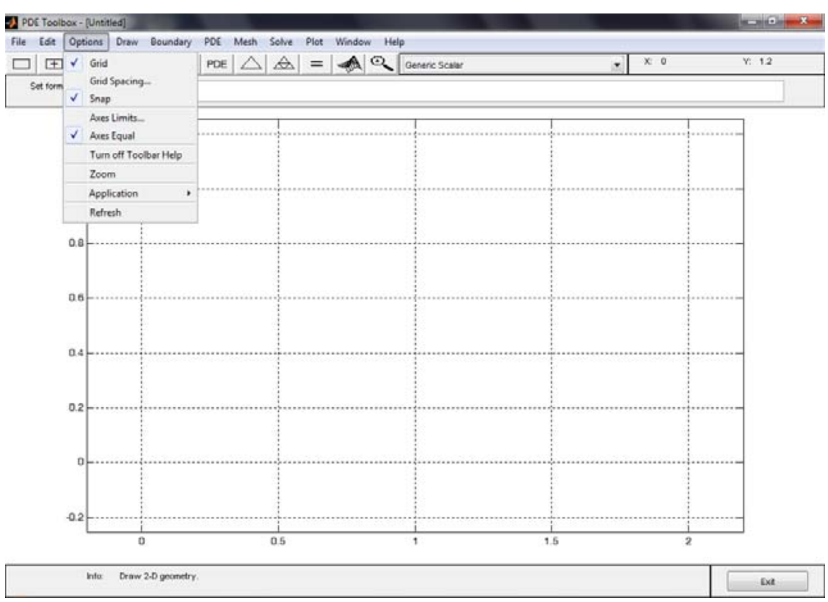
By comparing coefficients of the governing differential equation with the standard form of the elliptic differential equation, we find that $c = 1$, $a = 0$ and $f = 0$. This problem is equivalent to a steady-state heat conduction in a rectangular plate with the size of 2×1. Zero temperature ($u = 0$) is specified along the left, right and bottom edges while the temperature is $u = \sin(\pi x/2)$ along the top edge.

To use the PDE toolbox, we start by typing,

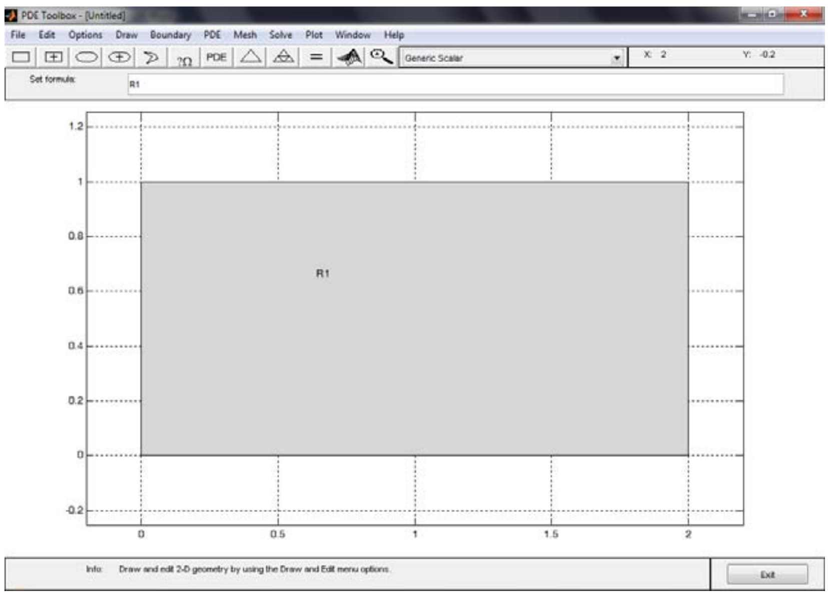
```
>> pdetool
```

pdetool

on the Command Window. We select the menu **Options**, choose the sub-menu **Grid** and click at the **Snap** option so that the points we create will be located at the round off coordinates. We select the **Axes Limits** as [-0.2 2.2] and [-0.2 1.2] for x- and y-axis, respectively. Then, we click **Apply** and **Axes Equal** options so that the model has equal scalings in both x- and y-directions. Note that if the background grid overflows the screen area, the screen resolution should be changed to 1024×768.



To create the rectangular domain of 2×1, we select the menu **Draw** and sub-menu **Rectangle/square**. Then, move the cursor from the coordinate of (0,1) to the coordinate of (2,0) and click at the mouse, a gray rectangle will appear with the letter of R1 as shown in the figure.

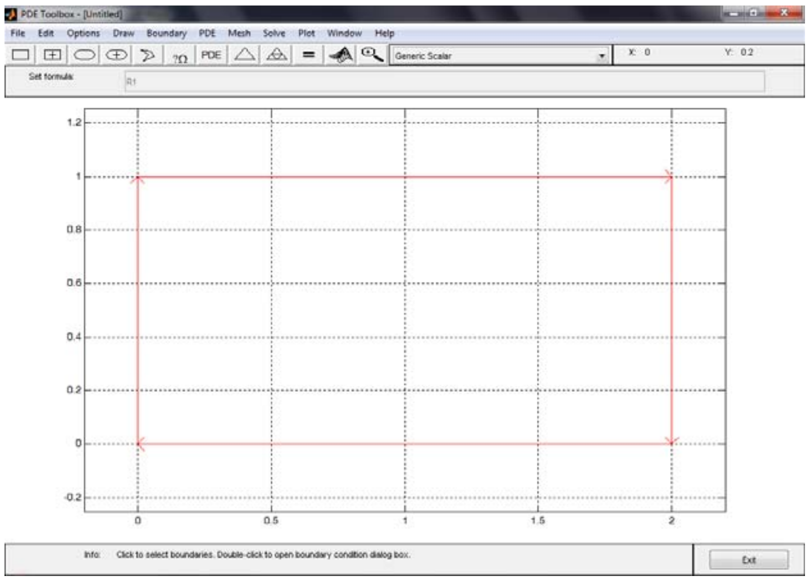


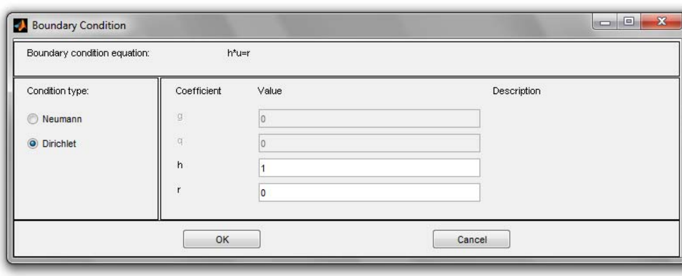
Next, to apply the boundary conditions, we select the menu **Boundary** and sub-menu **Boundary Mode**. The gray rectangle disappears while the rectangle edges become red arrows as shown in the figure.

To specify the boundary condition along the left edge with the value of $u = 0$, we double click at that edge, the Boundary Condition dialog box with the equation,

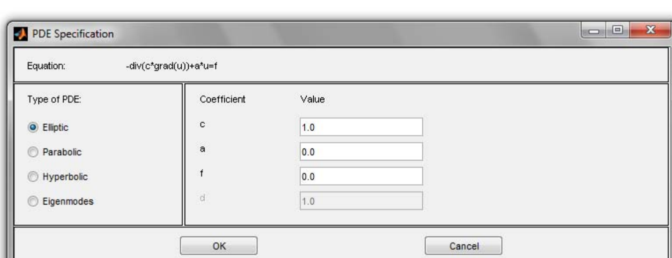
$$h * u \quad = \quad r$$

appears. We enter $h = 1$ and $r = 0$ which mean u is zero along this edge and click **OK**. We repeat applying the same boundary conditions of $u = 0$ for the right and bottom edges. For the top edge, we enter $h = 1$ and $r = \sin(pi * x./2)$, then click **OK**. Note that the letter x in the r expression must follow by a period due to the expression format requirement in MATLAB.

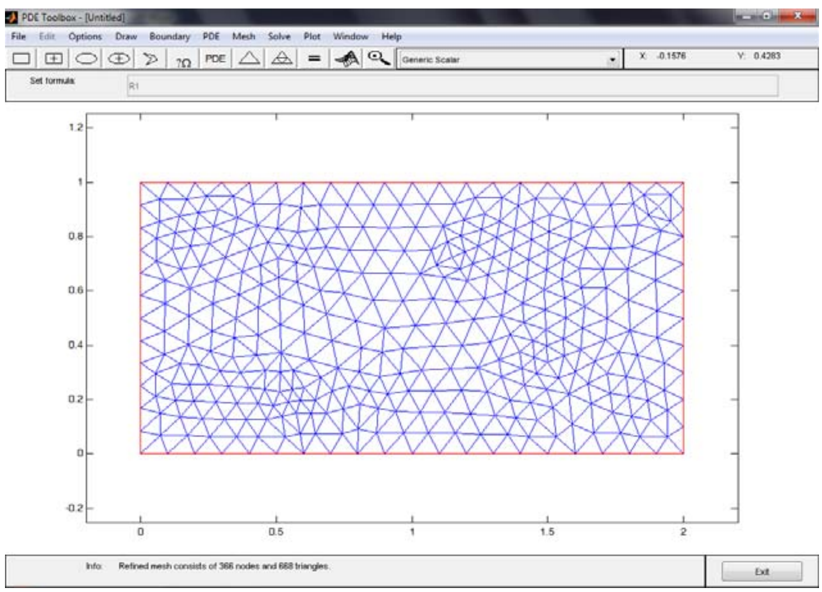




The next step is to choose the type of the differential equation. We select the menu **PDE** and sub-menu **PDE Specification**, the PDE Specification dialog box appears. After selecting Type of PDE as **Elliptic**, we enter $c = 1$, $a = 0$ and $f = 0$, and click **OK**.

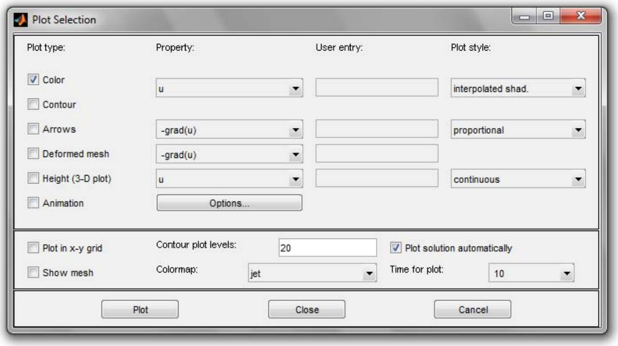


A finite element mesh can now be constructed. After selecting the menu **Mesh** and sub-menu **Initialize Mesh**, a mesh with triangular elements is created. The mesh can be refined by clicking at the sub-menu **Refine Mesh**. The mesh without the background grid is shown in the figure.

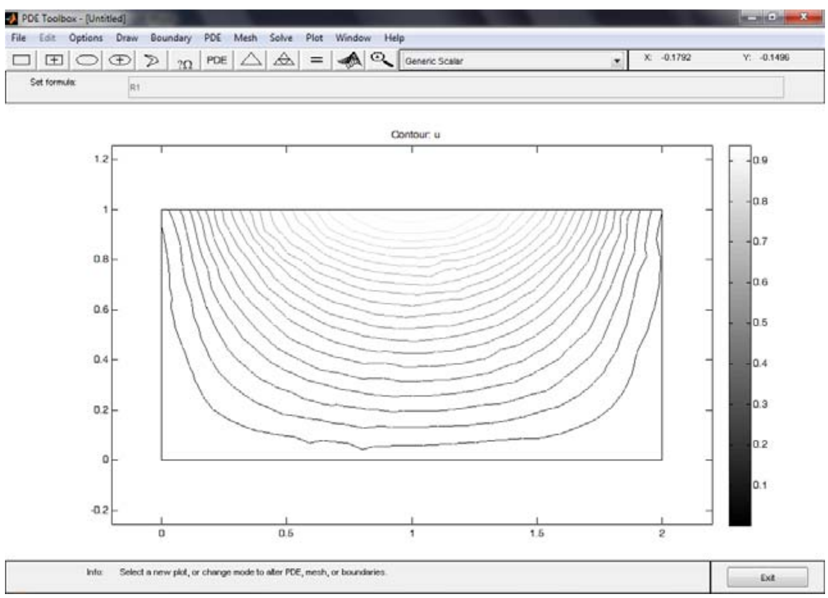


Then, we can solve for solution of the problem. We select the menu **Solve** and sub-menu **Solve PDE** so that the problem is executed. The computed solution is plotted by selecting the menu **Plot** and sub-menu **Plot Solution**. To display the colors similar to those in commercial finite element software, the Colormap which is under the sub-menu **Parameter** in the Plot Selection dialog box should be selected as **jet** as shown in the figure.

We can display the computed solution in the form of color fringe plot, line contour plot, deformed plot, plot with arrows, as well as generating an animation. Displaying solution in the form of animation is useful to show the solution behaviors of the parabolic and hyperbolic problems. Most of the solutions shown herein are in form of the line contour plots for clarity. Distribution of the



computed solution in the form of contour lines for this example is shown in the figure.



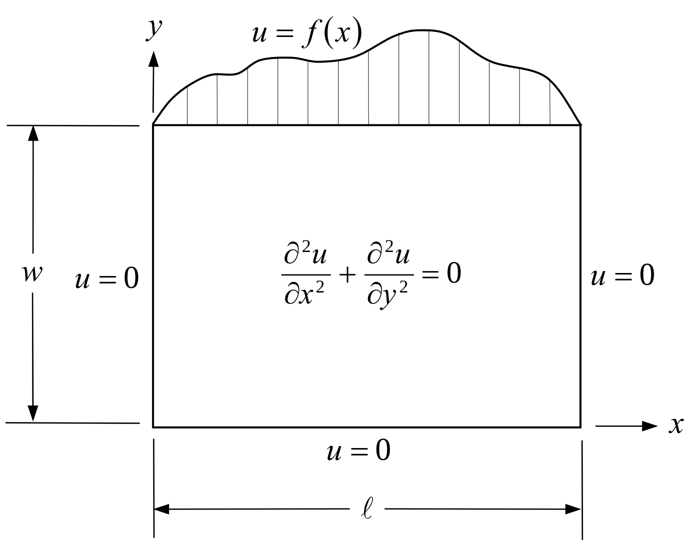
The exact solution in the preceding example can be derived by using the method of separation of variable. Detailed derivation of the exact solution is presented in the following example.

Example Derive the exact solution of the elliptic differential equation which is in form of the Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for a rectangular domain with dimensions of $\ell \times w$ as shown in the figure. The boundary conditions along the four edges are,

$$u(0,y) = 0\,, \qquad u(\ell,y) = 0$$
$$u(x,0) = 0\,, \qquad u(x,w) = f(x)$$



The method of separation of variables is applied by first assuming that the solution is in the product form of the functions $X(x)$ and $Y(y)$,

$$u(x,y) = X(x)\,Y(y)$$

By substituting the assumed solution into the differential equation, we obtain,

$$X''Y + XY'' = 0$$

where the prime symbol ($'$) denotes the derivative order. The equation above can be written as,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$$

Since X is only function of x while Y is only function of y, then both sides of this equation must be equal to a constant,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda^2$$

Thus, the given partial differential equation becomes two ordinary differential equations which should be solved easier.

If we consider the first ordinary differential equation,

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

Its general solution is in the form of sine and cosine functions,

$$X(x) = A\sin\lambda x + B\cos\lambda x$$

where A and B are constants. These two constants are determined as follows. Since the left edge boundary condition is $u(0,y)=0$, then $X(0)\,Y(y)=0$, or $X(0)=0$, so that $B=0$. Similarly, the right edge boundary condition is $u(\ell,y)=0$, then $X(\ell)\,Y(y)=0$, or $X(\ell)=0$. Then, $X(x)$ above becomes,

$$X(\ell) = 0 = A\sin\lambda\ell$$

But A cannot be zero, thus,

$$\lambda\ell = n\pi \qquad \text{and} \qquad \lambda = \frac{n\pi}{\ell}$$

where n is an integer. Hence, in general,

$$\lambda_n = \frac{n\pi}{\ell} \qquad\qquad n = 1, 2, 3, \ldots$$

which are called the *eigenvalues*.

Similarly, if we consider the second ordinary differential equation,

$$\frac{d^2Y}{dy^2} - \lambda^2 Y = 0$$

which has the general solution of,

$$Y(y) = E\sinh\lambda y + F\cosh\lambda y$$

where E and F are constants. The constant F is determined from the bottom edge boundary condition of $u(x,0)=0$, i.e., $X(x)\,Y(0)=0$. Then $Y(0)=0$, which leads to $F=0$. Thus, the general solution of the partial differential equation is,

$$u(x,y) = X(x)\,Y(y) = (A\sin\lambda x)(E\sinh\lambda y)$$

Or,
$$u(x,y) = C\sin\lambda x\sinh\lambda y$$

where $C = AE$ denotes a new constant. Since there are n values of λ, i.e, λ_1, λ_2, λ_3, ..., λ_n, thus, the general solution can be written in the summation form for n values of λ as,

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\lambda_n x\sinh\lambda_n y$$

The constants C_n are determined from the top edge boundary condition at $y = w$ as,

$$u(x,w) = f(x) = \sum_{n=1}^{\infty} (C_n \sinh\lambda_n w)\sin\lambda_n x$$

Or,
$$\sum_{n=1}^{\infty} D_n \sin\lambda_n x = f(x)$$

where $D_n = C_n \sinh\lambda_n w$. This means, once we obtain D_n from the above equation, we can find C_n and also the exact solution of $u(x,y)$.

To solve for D_n, we multiply both sides of the equations by $\sin(m\pi x/\ell)$ where m denotes an integer, and perform integration from $-\ell$ to ℓ as follows,

$$\sum_{n=1}^{\infty} D_n \int_{-\ell}^{\ell} \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx = \int_{-\ell}^{\ell} f(x)\sin\frac{m\pi x}{\ell}\,dx$$

But from the integration formula,

$$\int_{-\ell}^{\ell} \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx \quad = \quad \begin{cases} \ell, & m = n \\ 0, & m \neq n \end{cases}$$

which is known as the *orthogonal properties*.

Then, the equation above reduces to,

$$D_n\,\ell \quad = \quad \int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Or, $$D_n \quad = \quad \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx \quad = \quad \frac{2}{\ell}\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

But $D_n = C_n \sinh\lambda_n w = C_n \sinh(n\pi w/\ell)$, therefore,

$$C_n \quad = \quad \frac{2}{\ell\sinh\dfrac{n\pi w}{\ell}}\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Hence, the exact solution of this problem is,

$$u(x,y) \quad = \quad \frac{2}{\ell}\sum_{n=1}^{\infty}\frac{\sinh\dfrac{n\pi y}{\ell}}{\sinh\dfrac{n\pi w}{\ell}}\sin\frac{n\pi x}{\ell}\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

For the special case in the preceding example, if

$$f(x) = \quad \sin\frac{\pi x}{2}$$

with the domain sizes of $\ell = 2$ and $w = 1$, the integral in the exact solution reduces to,

$$\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx \quad = \quad \int_{0}^{2} \sin\frac{\pi x}{2}\sin\frac{n\pi x}{2}\,dx \quad = \quad \begin{cases} 1, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

Thus, the exact solution of the preceding example is,

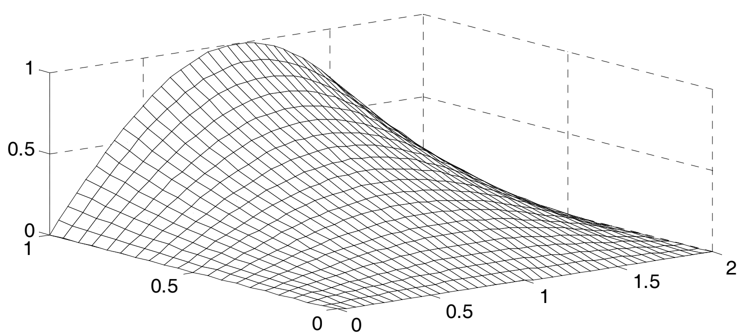
$$u(x,y) \quad = \quad \frac{2}{2}\,\frac{\sinh\dfrac{\pi y}{2}}{\sinh\dfrac{\pi}{2}}\sin\frac{\pi x}{2}$$

Or,
$$u(x,y) = \frac{\sin(\pi x/2)\sinh(\pi y/2)}{\sinh(\pi/2)}$$

This exact solution can be plotted to show the distribution of $u(x,y)$ by using the `meshgrid` and `mesh` commands as follows,

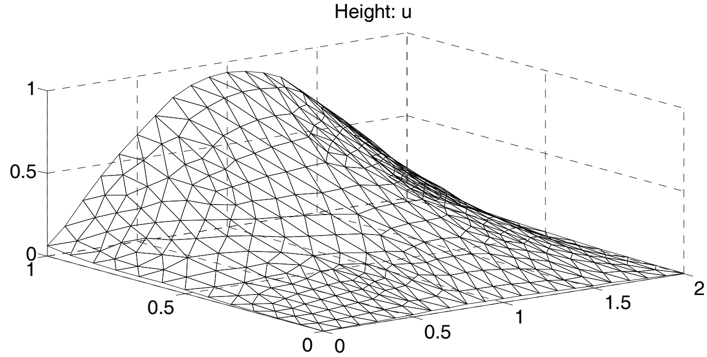
```
xx = 0:.05:2;  yy = 0:.05:1;
[x,y] = meshgrid(xx,yy);
u = sin(pi.*x/2).*sinh(pi.*y/2)/sinh(pi/2);
mesh(x,y,u);  view(-37.5,30);
```

Distribution of the exact solution $u(x,y)$ is shown in the figure.



Exact solution

The finite element solution obtained from the preceding example can be plotted in the same fashion as shown in the figure.

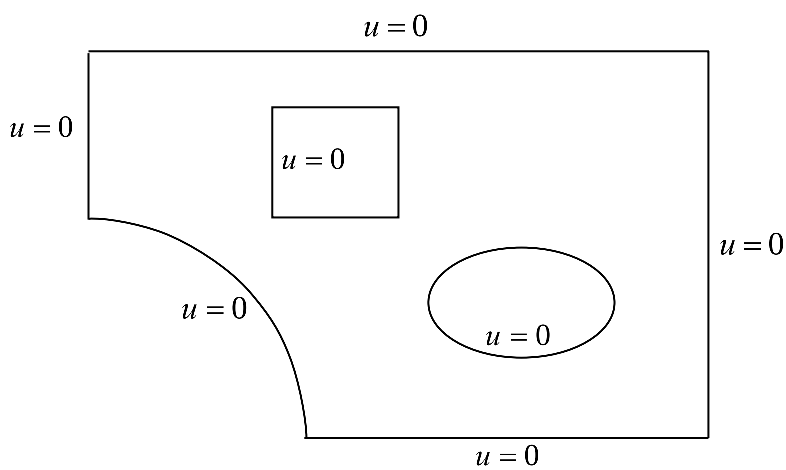


Approximate solution

Example Use the PDE toolbox to solve the elliptic partial differential equation,

$$-3\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+5u \quad = \quad 8$$

for an arbitrary two-dimensional domain as shown in the figure. The domain has the specified boundary condition of $u=0$ for all the edges. The problem statement is equivalent to the two-dimensional steady-state heat transfer in a plate. The plate material has its thermal conductivity coefficient of 3 units and is subjected to surface convection with the convection coefficient of 5 units. The plate is also subjected to a surface heating with the magnitude of 8 units.



We begin solving this problem by typing,

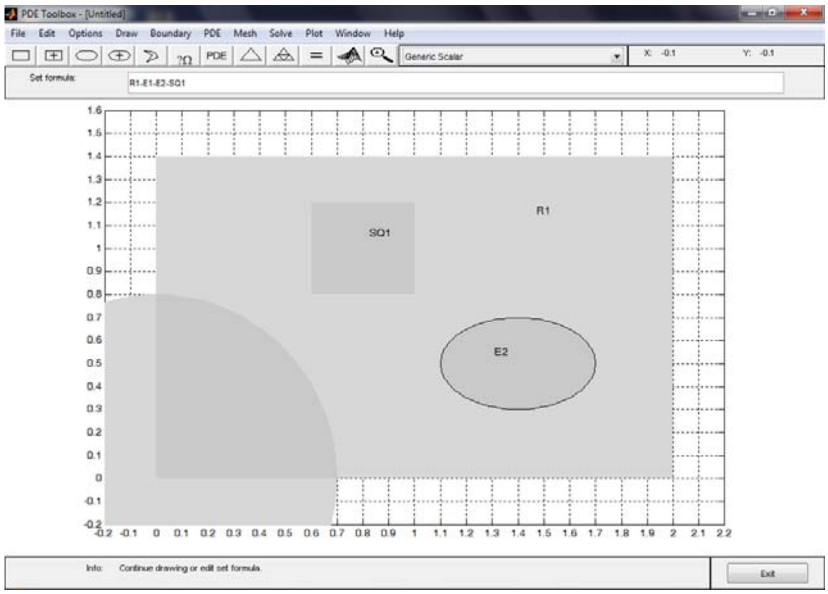
```
>> pdetool
```

pdetool

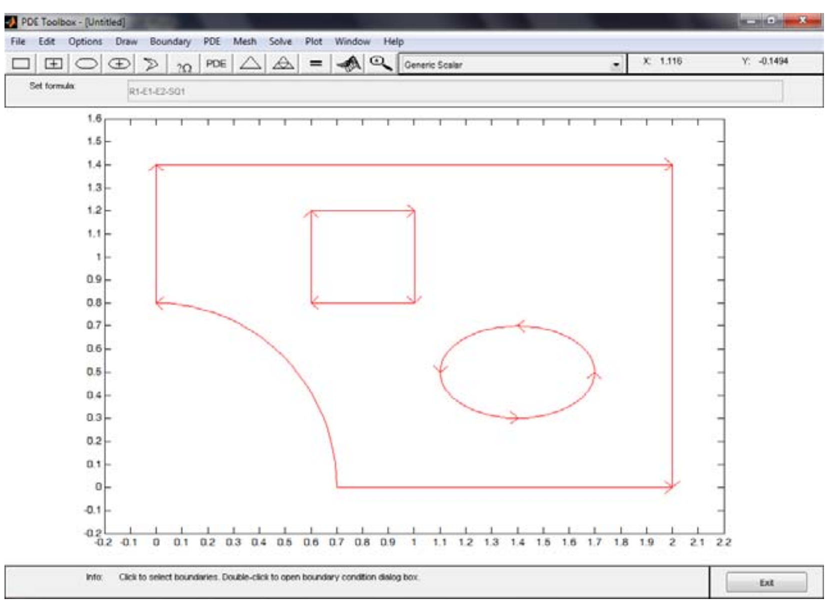
on the Command Window similar to the preceding example. We select the menu **Axes Limits** as [-0.2 2.2] and [-0.2 1.6] for *x*- and *y*-axis, respectively, then click the sub-menu **Grid Spacing** to enter -0.2:0.1:2.2 and -0.2:0.1:1.6 into the **X-axis** and **Y-axis** boxes, respectively, and click **Apply**.

Next, we draw the geometry of the domain by creating the rectangle R1, the circle E1 (lower left of the figure), the ellipse E2 and the square SQ1. Then, we modify the command in the **Set**

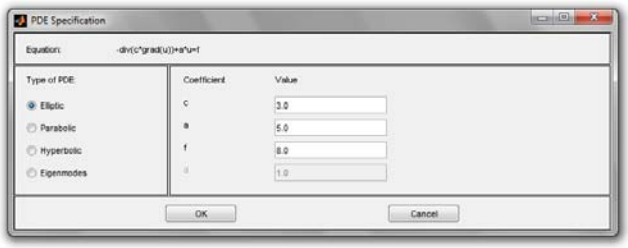
formula box to R1-E1-E2-SQ1 which means we subtract the areas of E1, E2 and SQ1 from R1 as shown in the figure.



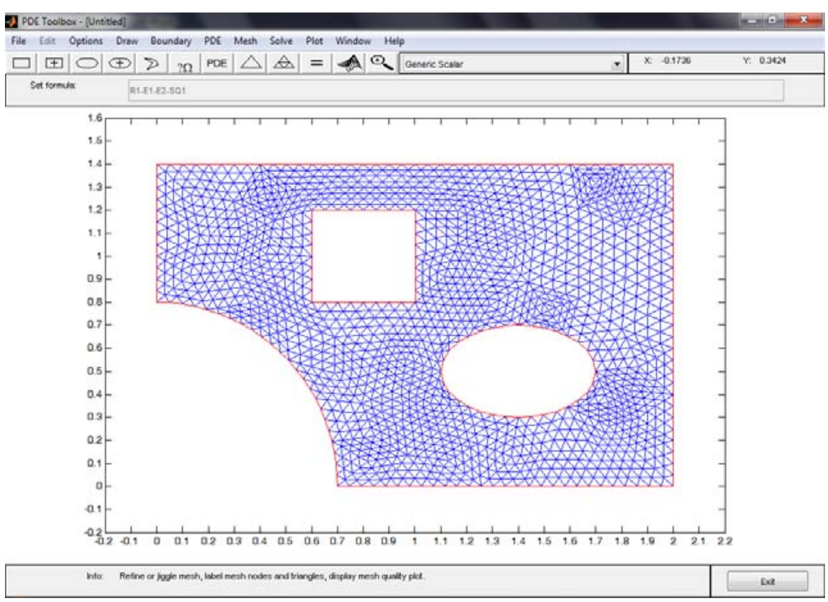
After that, we select the menu **Boundary** followed by the sub-menu **Boundary Mode**, the red arrows representing the domain boundaries appear as shown in the figure.



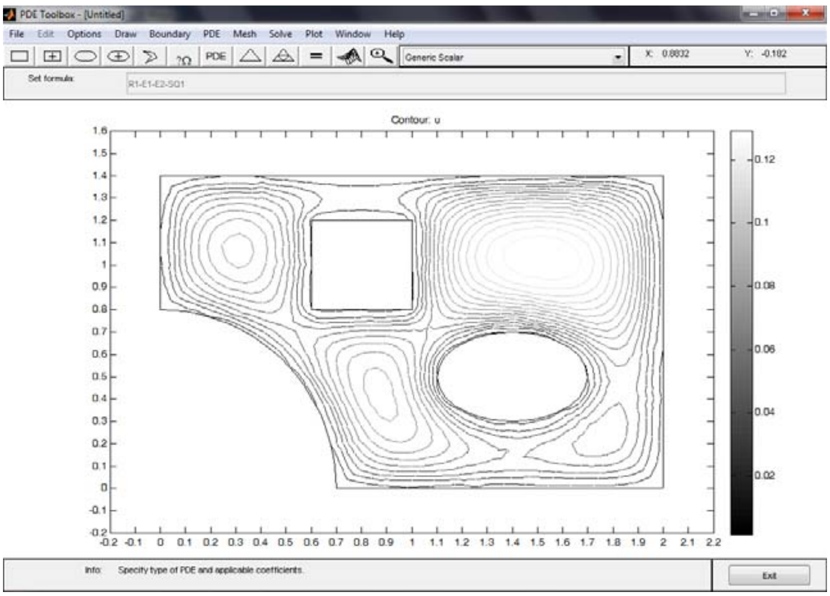
Next, we select the type of the differential equation that governs the problem. Here, we select the menu **PDE** and sub-menu **PDE Specification**, then click on **Elliptic** in PDE Specification dialog box and enter $c = 3$, $a = 5$ and $f = 8$ as shown in the figure, and click **OK**.



A mesh can now be constructed by selecting the menu **Mesh** and sub-menu **Initialize Mesh**. If the element sizes are too large, we click at the sub-menu **Refine Mesh** so that smaller elements are generated as shown in the figure.



The final step is to perform the analysis to find solutions of u at nodes. We select the menu **Solve** and sub-menu **Solve PDE**, the color fringe plot of the computed solution appears on the screen. If other plotting style is preferred, we select the menu **Plot** followed by sub-menu **Parameters** and choose different options there. The line contour plot representing the distribution of the computed solution is shown in the figure.



10.5 Parabolic Equations

The parabolic partial differential equation is in the form,

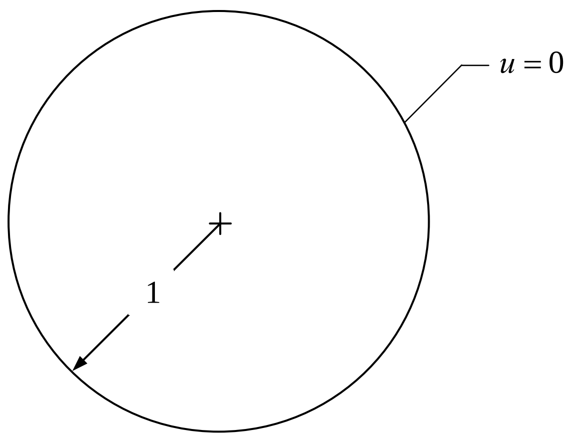
$$d\frac{\partial u}{\partial t}-\left(\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right)+au \quad = \quad f$$

where d, c, a and f are constants or may be function of x, y and u. The parabolic equation is more complicated than the elliptic equation because of an additional independent variable of t. Finding the exact solution is more difficult, especially for domain with arbitrary geometry. The PDE toolbox using the finite element method can alleviate such difficulty by finding the approximate solution instead.

Example Use the PDE toolbox to solve the parabolic partial differential equation in the form,

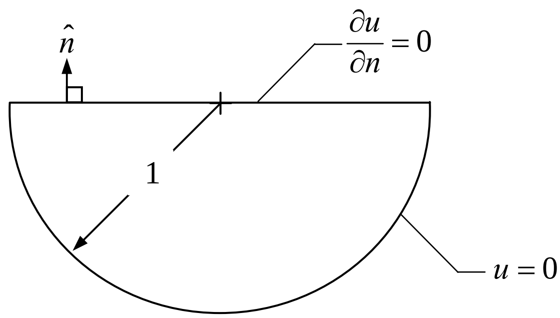
$$2\frac{\partial u}{\partial t} - 3\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2}\right) = 8$$

for a circular domain with a unit radius. The domain has the specified boundary condition of $u = 0$ along the edge as shown in the figure and the initial condition of $u(x, y, 0) = 0$ at time $t = 0$.



The problem statement is equivalent to transient heat conduction in a circular plate with specified zero temperature along the edge and zero initial temperature. The plate material has the thermal conductivity coefficient of 3 units and the specific heat of 2 units with a unit of material density. The plate is also subjected to a specified surface heating of 8 units.

We can create a finite element model for the entire plate but, due to symmetry of the solution, only the lower half of the plate can be modelled. As shown in the figure, the boundary condition of $u = 0$ is applied along the outer edge. Along the center line, the gradient of u normal to the edge must be equal to zero representing the insulated boundary condition.



Again, to analyze this problem by using the PDE toolbox, we begin by entering,

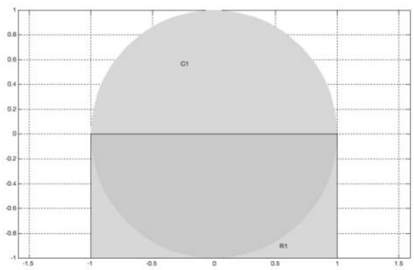
```
>> pdetool
```

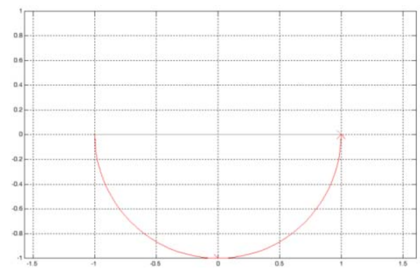
pdetool

on the Command Window. We select the menu **Option** followed by sub-menu **Grid** to show the grid background. We choose the sub-menu **Snap** so that the constructed points are located at the round off coordinates. Then, we select the sub-menu **Axes Equal** for equal scalings in both *x*- and *y*-directions.

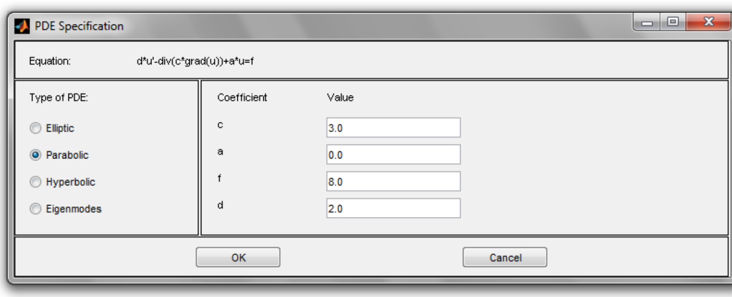
To construct the domain with half circle geometry, we start by clicking at the **ellipse** icon in the tool bar. We place the cursor arrow head at the coordinates of (0,0) and move it to the coordinates of (1,-1), click the mouse, then a circle with the letter C1 appears. We follow the same procedure by choosing the **rectangle** icon in the tool bar. We place the cursor arrow head at the coordinates of (-1,0), drag it to the point at the coordinates of (1,-1), click the mouse, a rectangle with the letter R1 appears as shown in the figure. Then, we modify the expression in the **Set formula** box to C1∗R1, click the icon **?Ω**, a half circle domain indicated by the red arrows appears as shown in the figure.

The next step is to apply the boundary conditions. We double click at the circle centerline, the **Boundary condition** dialog box appears, then choose the **Neumann** condition type followed by **OK**. Note that the default boundary condition of $u = 0$ is automatically applied along the outer edge.

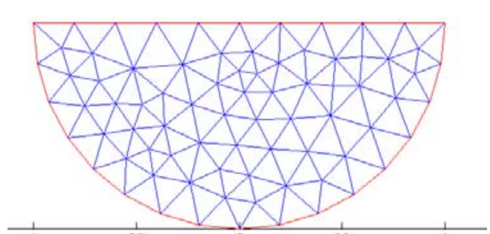


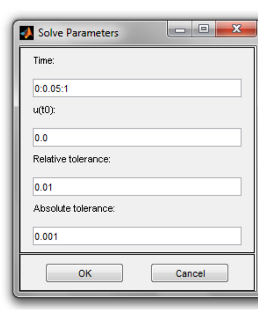


We now can select the type of the differential equation to be solved. We click the menu **PDE** and sub-menu **PDE Specification**. We choose the **Parabolic** Type of PDE in the PDE Specification dialog box, enter $c = 3$, $a = 0$, $f = 8$ and $d = 2$ as shown in the figure, then click **OK**.

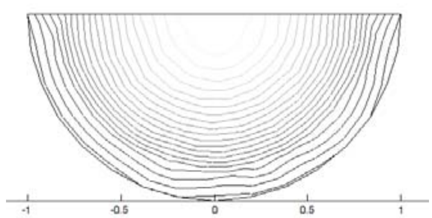


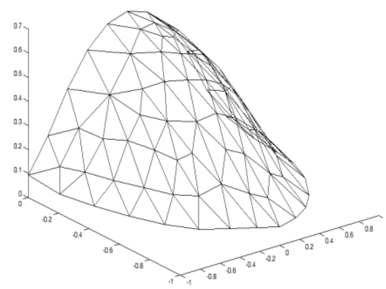
A finite element mesh consisting of triangular elements can now be constructed. We select the menu **Mesh** and sub-menu **Initialize Mesh**, the mesh as shown in the figure is generated. To solve for the transient solution, we choose the menu **Solve** and click at the **Parameters** icon. In the Solve Parameters dialog box as shown in the figure, we enter the values of 0:0.05:1 in the **Time** box which means we perform the calculation for the interval of $0 \le t \le 1$ and save the solution at every $t = 0.05$.





By clicking the = icon in the tool bar, the computed solutions at different times are obtained. These solutions can be plotted as contour lines in two dimensions or the carpet plots in three dimensions as shown in the figure. The transient solution which varies with time t can be animated by clicking at the **Animation** icon in the sub-menu **Plot Selection** under the menu **Plot** $\rightarrow$ **Parameters**. Users may try different commands in this sub-menu to display solutions in various ways.



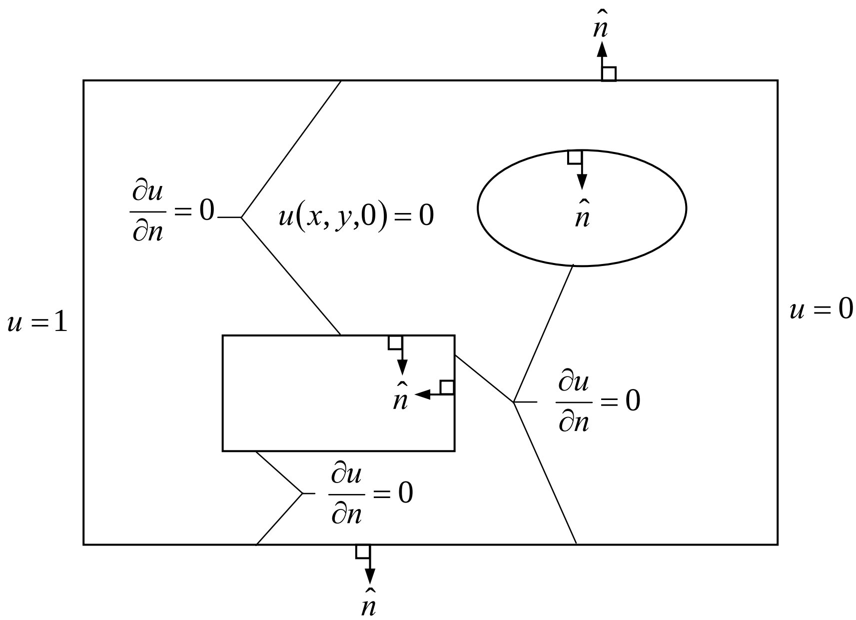


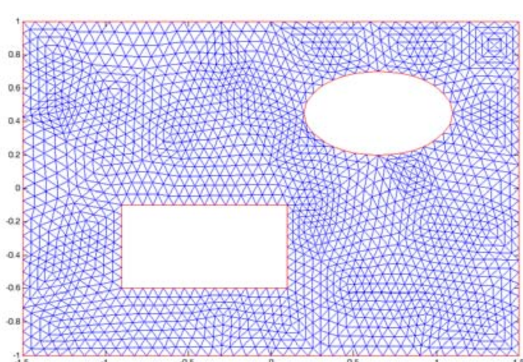
Example Use the PDE tool to solve the parabolic partial differential equation,

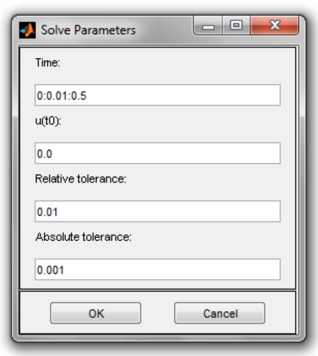
$$\frac{\partial u}{\partial t} - 2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2}\right) + 3u \quad = \quad 4$$

on the 3×2 rectangular domain with small ellipse and rectangular holes inside. The initial condition is $u(x, y, 0) = 0$ while the boundary conditions are shown in the figure.

The procedure for solving this problem is similar to that explained the preceding example. The domain geometry, however, is more complicated with the boundary condition along the left edge changes abruptly from $u = 0$ to $u = 1$ at $t = 0$. A finite element mesh can be constructed as shown in the figure. We select **Parabolic** as the type of PDE to be solved in the PDE Specification dialog box under the **PDE** menu with the parameters of $c = 2$, $a = 3$, $f = 4$ and $d = 1$. To obtain the transient solution, we enter 0:0.01:0.5 in the **Time** box of the **Solve Parameters** dialog box as shown in the figure.

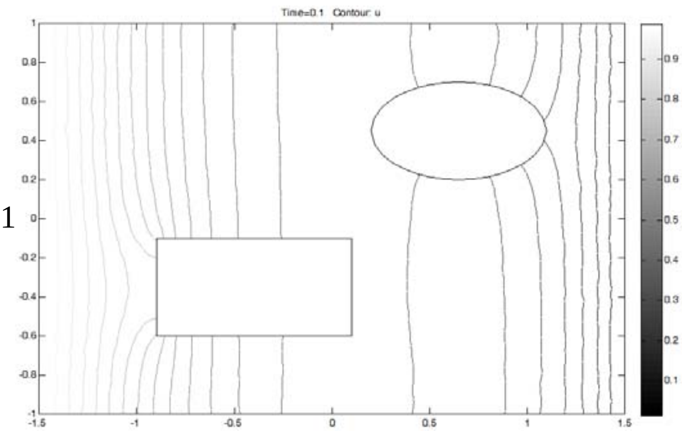




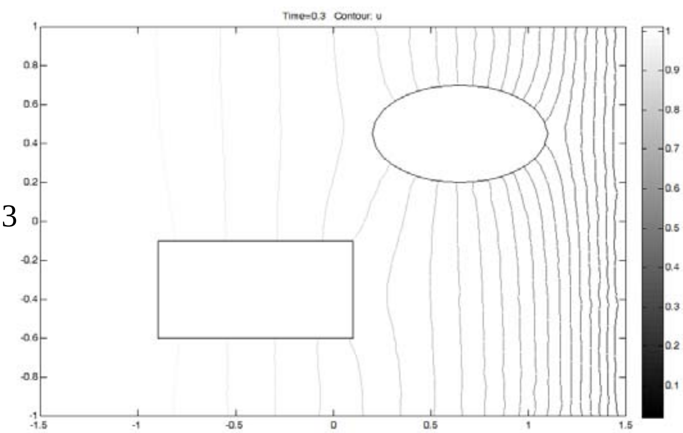


The computed solutions of $u(x,y,t)$ at $t=0.1$ and 0.3 are plotted using contour lines as shown in the figures.

$t = 0.1$

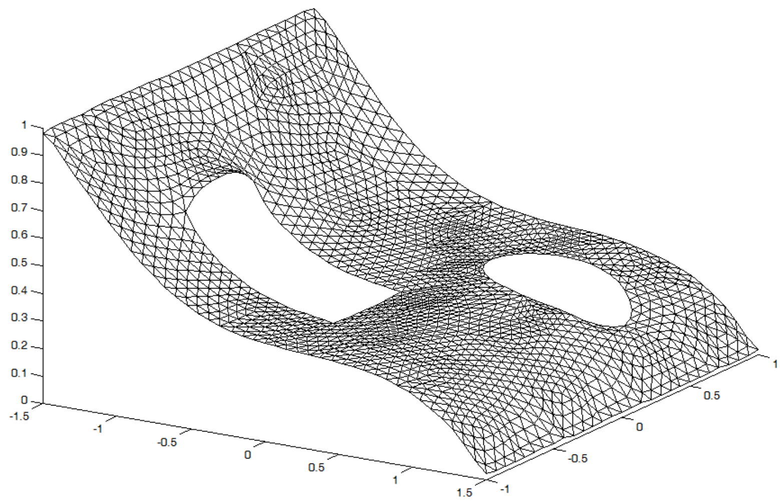


$t = 0.3$



Animation of the transient solution $u(x, y, t)$ can help us to understand the solution behavior clearly. A sample of the transient solution $u(x, y)$ at time $t = 0.1$ is shown as a carpet plot in the figure.

This example demonstrates that the finite element method can provide approximate solution to the differential equation with complicated boundary conditions and geometry effectively. The next example shows detailed derivation of the exact solution for the parabolic differential equation with simple boundary conditions and domain geometry. The example will show that the exact solution is not easy to derive even for a simple problem. These two examples thus highlight benefits of using the finite element method to obtain approximate solutions for complex problems.



Example Derive the exact solution of the parabolic partial differential equation,

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} \right) \;=\; 0$$

for the $\ell \times w$ rectangular domain as shown in the figure. The boundary conditions are,

$$\begin{aligned} u(x,0,t) &= 0, & u(x,w,t) &= 0, & 0 \le x \le \ell \\ u(0,y,t) &= 0, & u(\ell,y,t) &= 0, & 0 \le y \le w \end{aligned}$$

and the initial condition is,

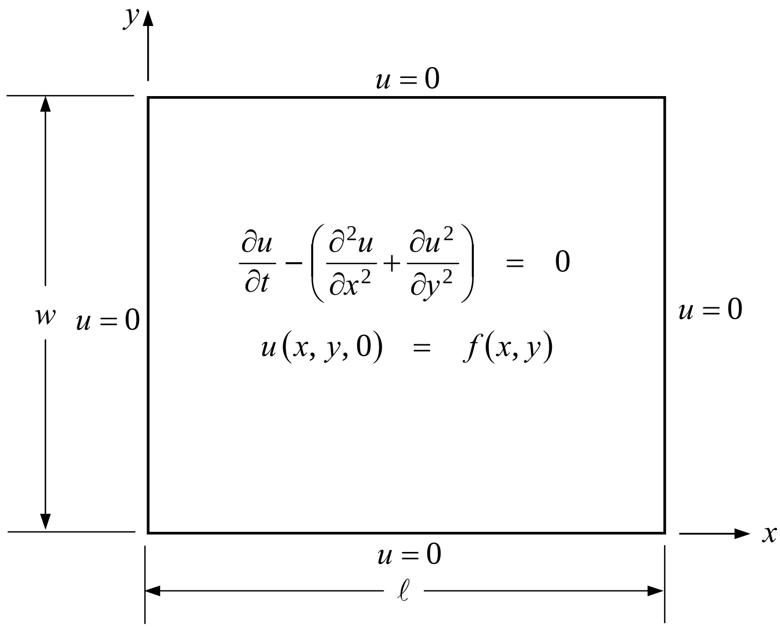
$$u(x,y,0) \;=\; f(x,y) \qquad\qquad t > 0$$

The method of separation of variables is used to solve for the exact solution. We start by assuming the solution $u(x,y,t)$ as the product of the three functions $X(x)$, $Y(y)$ and $T(t)$,

$$u(x,y,t) \;=\; X(x)\,Y(y)\,T(t)$$

By substituting it into the differential equation, we obtain,

$$XYT' - (X''YT + XY''T) \;=\; 0$$



where the prime symbol ($'$) denotes the derivative order. We divide the equation through by XYT and move some terms to get,

$$\frac{T'}{T} - \frac{Y''}{Y} = \frac{X''}{X}$$

Since the terms on the left-hand-side of the equation are only functions of t and y while the term on the right-hand-side of the equation is only function of x, then they must be equal to a constant,

$$\frac{T'}{T} - \frac{Y''}{Y} = \frac{X''}{X} = -\lambda^2$$

Similarly,

$$\frac{T'}{T} + \lambda^2 = \frac{Y''}{Y} = -\mu^2$$

Hence, the method of separation of variables changes the given partial differential equation into three ordinary differential equations as,

$$X'' + \lambda^2 X = 0, \qquad Y'' + \mu^2 Y = 0 \quad \text{and} \quad T' + (\lambda^2 + \mu^2)T = 0$$

At the same time, the given boundary conditions become,

$$X(0) = 0, \quad X(\ell) = 0, \quad Y(0) = 0 \quad \text{and} \quad Y(w) = 0$$

By using the same procedure as shown in the preceding example of the elliptic differential equation, the *eigenvalues* and *eigenvectors* are obtained,

$$\lambda_m = \frac{m\pi}{\ell}, \qquad X_m(x) = \sin\left(\frac{m\pi x}{\ell}\right)$$

and

$$\mu_n = \frac{n\pi}{w}, \qquad Y_n(y) = \sin\left(\frac{n\pi y}{w}\right)$$

The ordinary differential equation related to the time t is,

$$T' + (\lambda^2 + \mu^2)T = 0$$

Its general solution is in the form,

$$T = A_{mn} e^{-\alpha_{mn}^2 t}$$

where

$$\alpha_{mn}^2 = \lambda_m^2 + \mu_n^2 = \left(\frac{m\pi}{\ell}\right)^2 + \left(\frac{n\pi}{w}\right)^2$$

Thus, the general solution is,

$$\begin{aligned} u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_m Y_n T \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi y}{w}\right) e^{-\alpha_{mn}^2 t} \end{aligned}$$

where the constants A_{mn} are determined from the initial condition of $u(x,y,0) = f(x,y)$ by using the orthogonal properties,

$$A_{mn} = \frac{4}{\ell w}\int_0^{\ell}\int_0^{w} f(x,y)\, \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi y}{w}\right) dx\, dy$$

As an example, if the initial condition is,

$$u(x,y,0) = f(x,y) = x(\ell - x^2)\, y(w - y)$$

then, the constants A_{mn} are,

$$
\begin{aligned}
A_{mn} &= \frac{4}{\ell w}\int_0^{\ell}\int_0^{w} x(\ell - x^2)\, y(w-y)\, \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi y}{w}\right) dx\, dy \\
&= \frac{48\,\ell^2 w^2}{(mn\pi^2)^3}(-1)^m\left((-1)^n - 1\right)
\end{aligned}
$$

Hence, the exact solution for this particular case is,

$$
\begin{aligned}
u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{48\,\ell^2 w^2}{(mn\pi^2)^3}(-1)^m\left((-1)^n - 1\right)\times \\
&\qquad \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi y}{w}\right) e^{-\left(m^2/\ell^2 + n^2/w^2\right)\pi^2 t}
\end{aligned}
$$

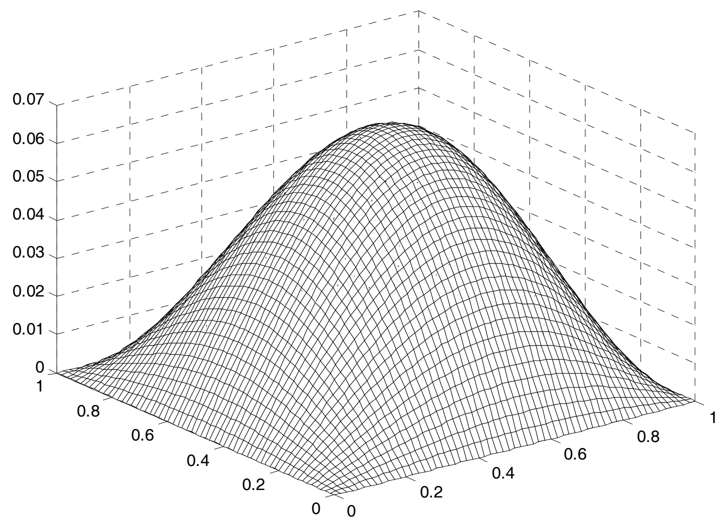
The exact solution above is in the form of infinite series. We can create a MATLAB script file to compute the solution. For example, the script file for determining the solution of $u(x,y,t)$ at $t = 0.02$ for a unit square domain $(\ell = w = 1)$ consists of the following statements,

```
xx = 0:1/60:1;  yy = 0:1/60:1;
[x,y] = meshgrid(xx,yy);
t = 0.02;  u = 0;
for m = 1:50;
  for n = 1:50;
    a = (-1)^m;  b = (-1)^n - 1;
    c = sin(m*pi.*x);  d = sin(n*pi.*y);
    ee= (m*m+n*n)*pi*pi*t;  e = exp(-ee);
    fac = 48/(m^3*n^3*pi^6);
    u = u + fac*a*b.*c.*d*e;
  end
end
mesh(x,y,u);  view(-37.5,30);
```

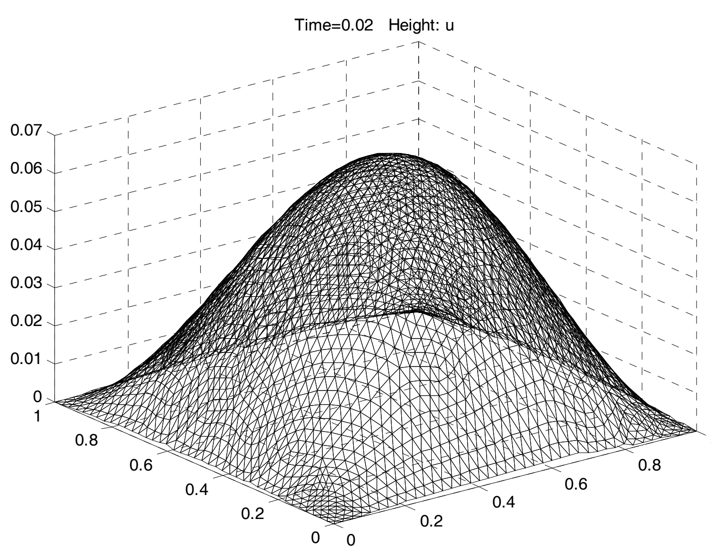
meshgrid

mesh

The same problem is solved by using the PDE toolbox for the approximate solution. The exact and approximate solutions are compared in form of the carpet plots as shown in the figures.



Exact solution



Approximate solution

10.6 Hyperbolic Equations

The hyperbolic partial differential equation is in the form,

$$d\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right) + au = f$$

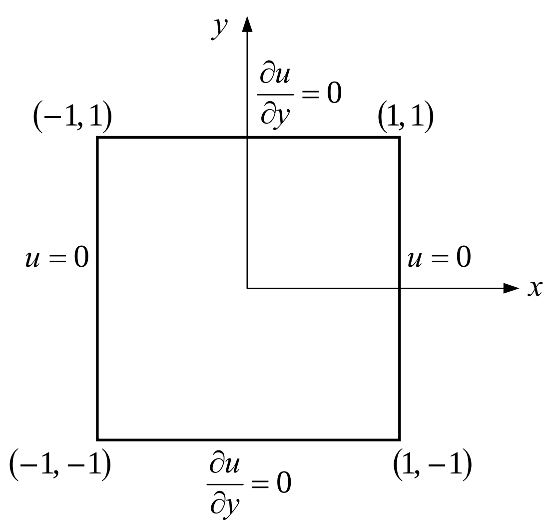
where d, c, a and f are constants or function of x, y and u. The hyperbolic equation is more complicated than the elliptic and parabolic equations. The dependent variable u is function of the independent variables x, y and t. Two initial conditions of $u(x, y, 0)$ and $\partial u(x, y, 0)/\partial t$ are needed for solving the problem.

In this section, we will use the PDE toolbox to solve the hyperbolic partial differential equation for both simple and complex domains.

Example Use the PDE toolbox to solve the hyperbolic partial differential equation in the form,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

for a 2×2 unit square domain. The boundary conditions are $u = 0$ on the left and right edges while $\partial u/\partial y = 0$ on the top and bottom edges as shown in the figure.



The initial conditions are,

$$u(x,y,0) = \tan^{-1}(\cos(\pi x/2))$$

$$\partial u(x,y,0)/\partial t = 4\sin(\pi x)e^{\sin(\pi y/2)}$$

To use the PDE toolbox, we start by typing,

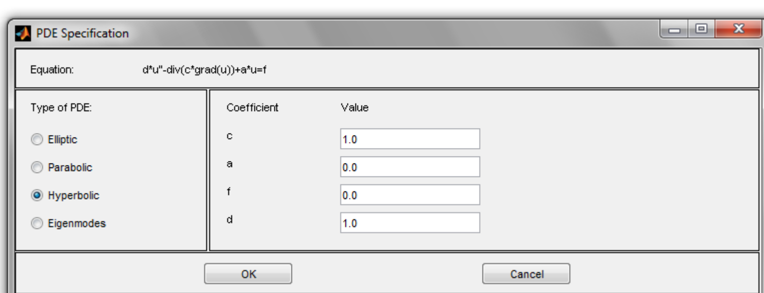
```
>> pdetool
```

pdetool

on the Command Window. We select the menu **Options** and click the **Grid** and **Snap** options to show the background grids and to place the points exactly at the round off coordinates. We also select **Axes Equal** option to have equal scalings for both x- and y-axes.

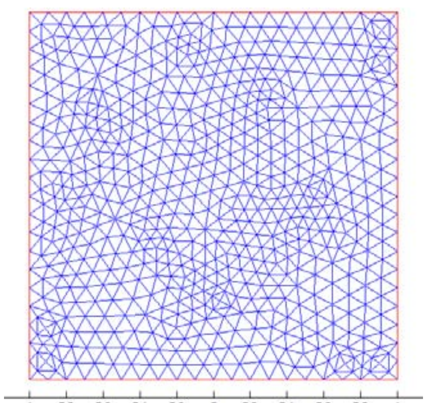
Next, we create a square domain with the dimensions of 2×2 units. We click at the rectangle icon in the tool bar, place the cursor arrow head at the coordinate of (-1,1), drag the mouse to move the cursor arrow head to the coordinate of (1,-1), then click the mouse again, a square domain with the letter SQ1 appears. Then, we apply the boundary conditions for the four edges, one at a time. We select the Dirichlet condition for the left and right edges, and the Neumann condition for the top and bottom edges.

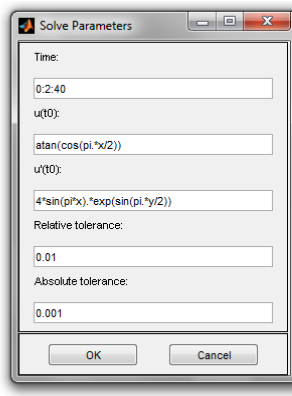
To specify the type of the differential equation, we select the menu **PDE** and sub-menu **PDE Specification**. We choose the type of PDE as **Hyperbolic**, and enter the values of $c=1$, $a=0$, $f=0$ and $d=1$, and click **OK** as shown in the figure.



A finite element mesh can now be constructed. We select the menu **Mesh** followed by sub-menu **Initialize Mesh**, a mesh with triangular elements appears. The initial mesh may be too crude, we can click the sub-menu **Refine Mesh** for couple times so that elements become smaller as shown in the figure.

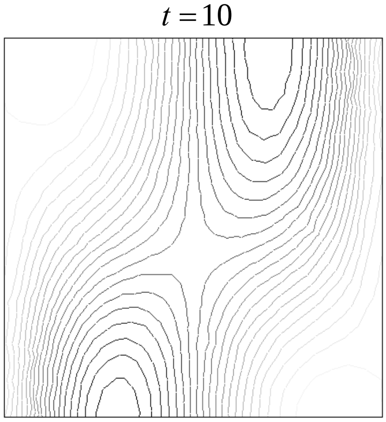
To solve the problem, we choose the menu **Solve** and sub-menu **Parameters**, and enter values in the **Time** box as 0:2:40. We also need to provide the initial conditions of $u(x, y, 0)$ and $\partial u(x, y, 0)/\partial t$ in the next two small boxes as shown in the figure.



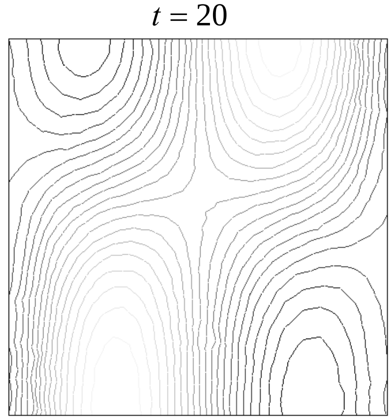


Then, we execute the problem by choosing the menu **Solve** and the sub-menu **Solve PDE**. The computed solutions of $u(x, y, 10)$ and $u(x, y, 20)$ at time $t = 10$ and 20 are shown as the contour line plots in the figures.

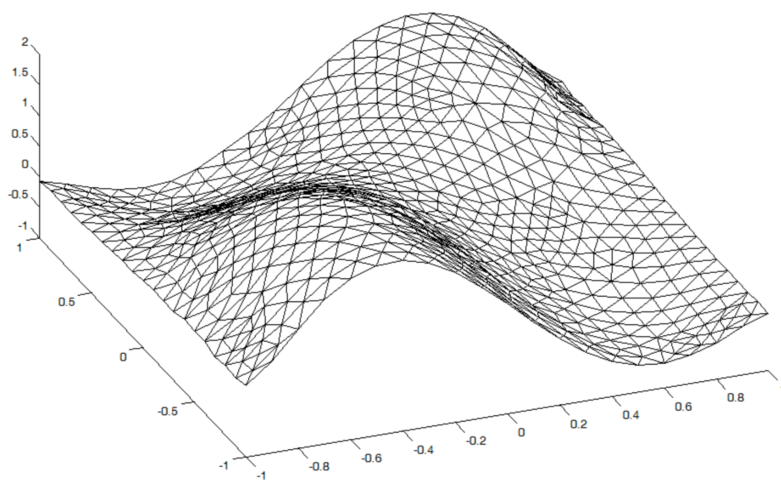
$t = 10$



$t = 20$



To display the solution that changes with time, we can select the **Animation** icon. Herein, a typical solution at $t = 40$ is shown in the figure in form of a carpet plot.

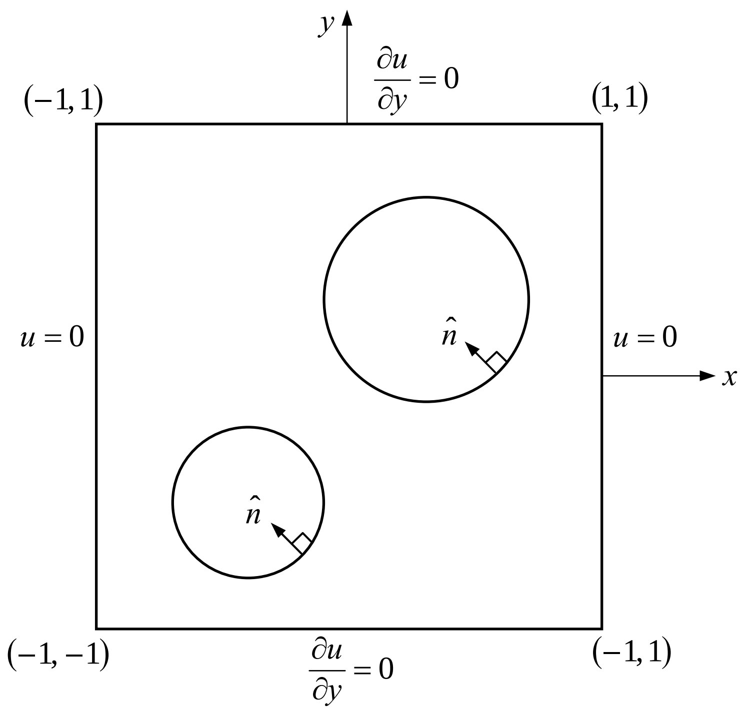


<u>Example</u> Use the PDE toolbox to solve the hyperbolic partial differential equation,

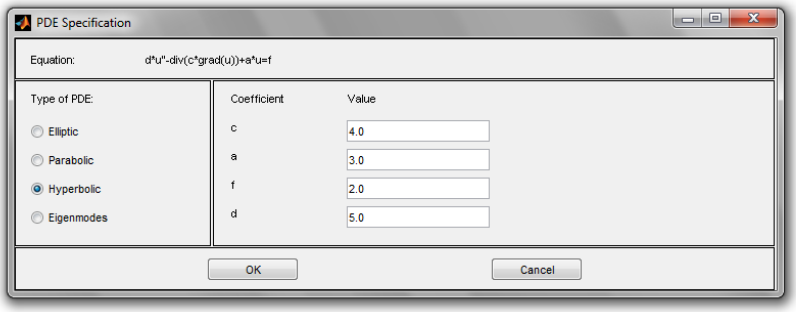
$$5\frac{\partial^2 u}{\partial t^2} - 4\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 3u \quad = \quad 2$$

for a unit square domain with two small holes inside as shown in the figure. The boundary conditions are $u = 0$ on the left and right edges, and $\partial u/\partial y = 0$ on the top and bottom edges while $\partial u/\partial n = 0$ along inner edges of the holes. The two initial conditions are,

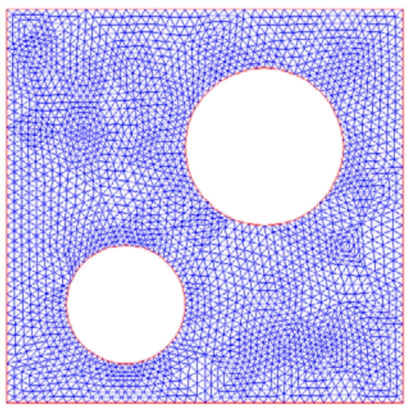
$$\begin{aligned} u(x, y, 0) \quad &= \quad \tan^{-1}(\cos(\pi x/2)) \\ \partial u(x, y, 0)/\partial t \quad &= \quad 4\sin(\pi x)e^{\sin(\pi y/2)} \end{aligned}$$

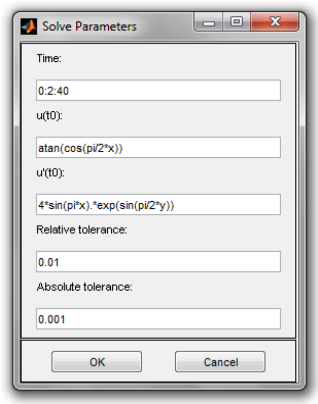


Since this example is similar to the preceding one except the differential equation and the two holes inside the domain, we can follow the same procedure to solve the problem. To provide information of the differential equation, we select the menu **PDE** and sub-menu **PDE Specification**, choose the Type of PDE as **Hyperbolic** and enter the values of $c = 4$, $a = 3$, $f = 2$ and $d = 5$, then click **OK**.



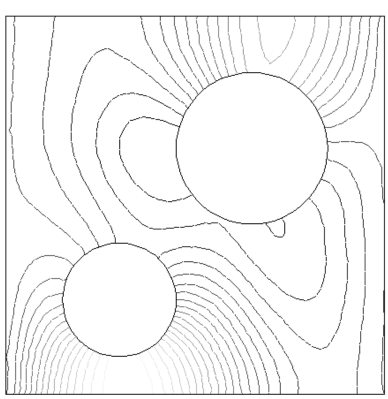
A finite element mesh can be constructed by using the same procedure as explained in the preceding example. The **Solve Parameter** dialog box remains the same as shown in the figure.



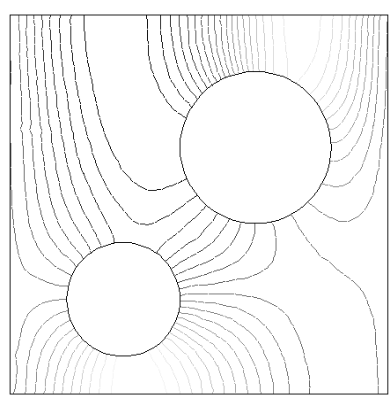


Typical solutions of $u(x, y, t)$ at times $t = 10$ and 40 are plotted in the form of contour lines as shown in the figures.

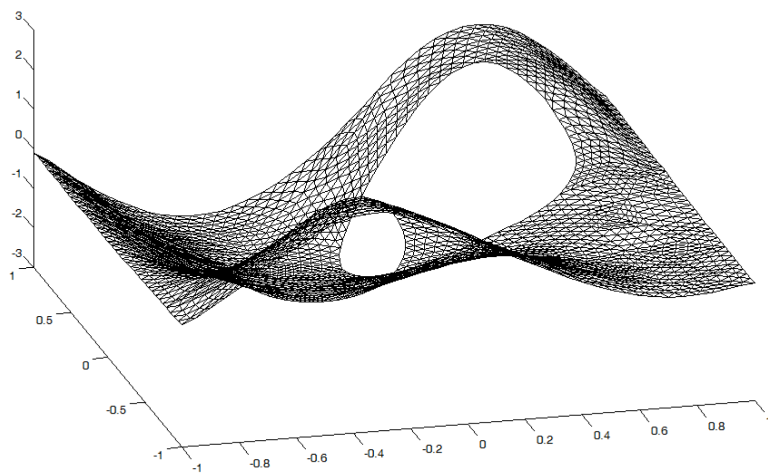
$t = 10$



$t = 40$



The solution can be displayed as a carpet plot in three dimensions. A typical solution of $u(x, y, t)$ at time $t = 40$ in form of the carpet plot is shown in the figure.



To highlight the advantage of using the finite element method to solve for approximate solution of the hyperbolic problem, we will use the example below to show the derivation of exact solution for a very simple problem. We will see that the derivation is rather lengthy and complicated.

Example Derive the exact solution of the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} \right) \quad = \quad 0$$

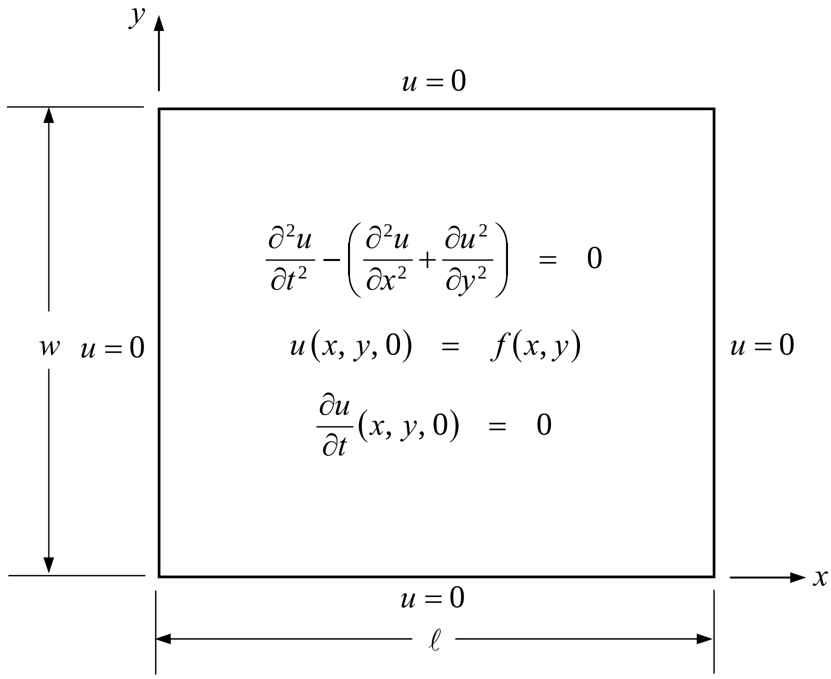
for a rectangular domain with the dimensions of $\ell \times w$ as shown in the figure. The boundary conditions are,

$$\begin{array}{llllllll} u(x,0,t) & = & 0\,, & u(x,w,t) & = & 0\,, & & 0 \le x \le \ell \\ u(0,y,t) & = & 0\,, & u(\ell,y,t) & = & 0\,, & & 0 \le y \le w \end{array}$$

and the initial conditions are,

$$u(x,y,0) \quad = \quad f(x,y), \qquad \frac{\partial u}{\partial t}(x,y,0) \quad = \quad 0 \qquad\qquad t > 0$$

The method of separation of variables is again used to derive the exact solution. We first assume the exact solution $u(x,y,t)$ in form of the product of the three functions $X(x)$, $Y(y)$ and $T(t)$ as,



$$u(x, y, t) = X(x)\,Y(y)\,T(t)$$

By substituting it into the differential equation, we obtain,

$$X\,Y\,T'' - (X''\,Y\,T + X\,Y''\,T) = 0$$

where the prime symbol ($'$) denotes the derivative order. We divide the equation through by $X\,Y\,T$ and move some terms to get,

$$\frac{T''}{T} - \frac{Y''}{Y} = \frac{X''}{X}$$

Since the terms on the left-hand-side of the equation are only functions of *t* and *y* while the term on the right-hand-side of the equation is only function of *x*, then they must be equal to a constant,

$$\frac{T''}{T} - \frac{Y''}{Y} = \frac{X''}{X} = -\lambda^2$$

which leads to,

$$\frac{T''}{T} + \lambda^2 = \frac{Y''}{Y} = -\mu^2$$

Hence, the method of separation of variables changes the partial differential equation into three ordinary differential equations as,

$$X'' + \lambda^2 X = 0, \quad Y'' + \mu^2 Y = 0 \quad \text{and} \quad T'' + (\lambda^2 + \mu^2)T = 0$$

while the boundary conditions become,

$$X(0) = 0, \quad X(\ell) = 0, \quad Y(0) = 0 \quad \text{and} \quad Y(w) = 0$$

The process leads to the *eigenvalues* and the *eigenvectors* similar to those explained in the elliptic and parabolic problems,

$$\lambda_m = \frac{m\pi}{\ell}, \qquad X_m(x) = \sin\left(\frac{m\pi x}{\ell}\right)$$

and

$$\mu_n = \frac{n\pi}{w}, \qquad Y_n(y) = \sin\left(\frac{n\pi y}{w}\right)$$

The ordinary differential equation for the time t is,

$$T'' + (\lambda^2 + \mu^2)T = 0$$

The general solution of this differential equation is in the form of sine and cosine functions. But after applying the initial condition of $\partial u(x, y, 0)/\partial t = 0$, the solution is only in the form of cosine function as,

$$T(t) = A_{mn}\cos(\alpha_{mn} t)$$

where

$$\alpha_{mn}^2 = \lambda_m^2 + \mu_n^2 = \left(\frac{m\pi}{\ell}\right)^2 + \left(\frac{n\pi}{w}\right)^2$$

Thus, the general solution of the partial differential equation is,

$$\begin{aligned} u(x, y, t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_m Y_n T \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi y}{w}\right)\cos(\alpha_{mn} t) \end{aligned}$$

where the constant A_{mn} can be determined from the initial condition of $u(x, y, 0) = f(x, y)$ by using the orthogonal properties as explained in the preceding examples. These constants A_{mn} are,

$$A_{mn} \;=\; \frac{4}{\ell w}\int_0^\ell\int_0^w f(x,y)\,\sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi y}{w}\right)dx\,dy$$

As an example, if the initial condition is,

$$u(x,y,0) \;=\; f(x,y) \;=\; x(\ell-x)\,y(w-y)$$

then, the constants A_{mn} are,

$$\begin{aligned} A_{mn} &= \frac{4}{\ell w}\int_0^\ell\int_0^w x(\ell-x)\,y(w-y)\,\sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi y}{w}\right)dx\,dy \\ &= \frac{16\,\ell^2 w^2}{(mn\pi^2)^3}\left((-1)^m-1\right)\left((-1)^n-1\right) \end{aligned}$$

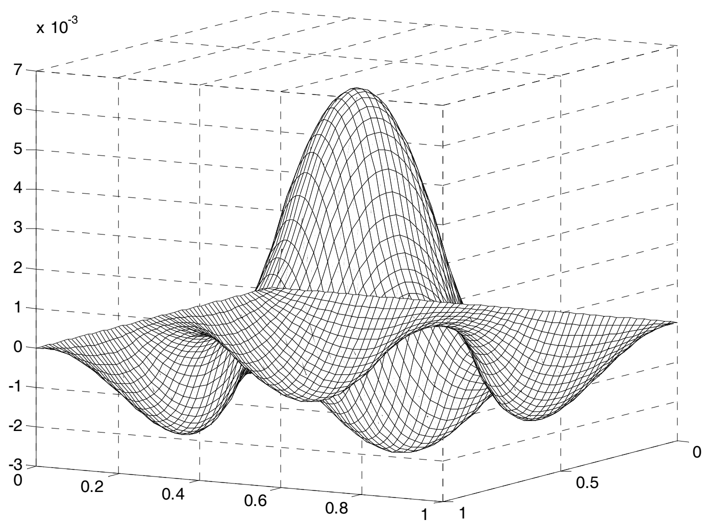
So that the exact solution for this case is,

$$\begin{aligned} u(x,y,t) \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{16\,\ell^2 w^2}{(mn\pi^2)^3}\left((-1)^m-1\right)\left((-1)^n-1\right)\times \\ \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi y}{w}\right)\cos(\alpha_{mn}t) \end{aligned}$$

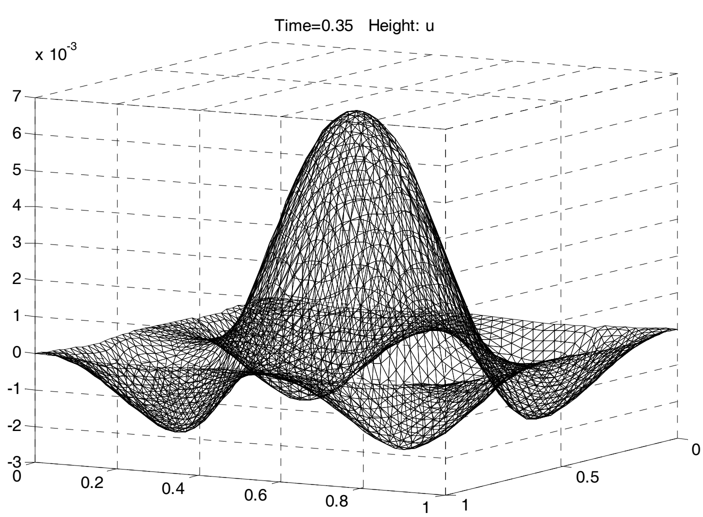
The exact solution above is in the form of infinite series. To solve for the solution of $u(x,y,t)$ at $t=0.35$ for a unit square domain $(\ell=w=1)$, a MATLAB script file can be created with details as follows,

```
xx = 0:1/60:1;  yy = 0:1/60:1;
[x,y] = meshgrid(xx,yy);                        meshgrid
t = 0.35;  u = 0;
for m = 1:50;
  for n = 1:50;
    a = (-1)^m - 1;  b = (-1)^n - 1;
    c = sin(m*pi.*x);  d = sin(n*pi.*y);
    al= sqrt(m^2*pi^2+n^2*pi^2);  e = cos(al*t);
    fac = 16/(m^3*n^3*pi^6);
    u = u + fac*a*b.*c.*d*e;
  end
end
mesh(x,y,u);  view(-30,-10);                    mesh
```

The same problem is solved by using the PDE toolbox. The approximate solution obtained from the PDE toolbox is compared with the exact solution in the form of carpet plot as shown in the figures. The comparison indicates that the finite element method in the PDE toolbox can provide accurate approximate solution by agreeing very well with the exact solution.



Exact solution



Approximate solution

10.7 Concluding Remarks

In this chapter, we learned three types of the partial differential equations which are in the elliptic, parabolic and hyperbolic forms. We solved these partial differential equations for their solutions in two dimensions. Exact solutions can be derived only for simple differential equations with plain boundary conditions and geometry. For more complicated problems, we have to employ the numerical methods to solve for approximate solutions.

MATLAB contains the PDE toolbox that uses the finite element method to solve these problems for approximate solutions. The toolbox solves the partial differential equations that are in different forms. The boundary conditions may be complicated and the geometry could be arbitrary. The process starts from discretizing the problem domain into a number of small triangular elements. These elements are connected at nodes where the unknowns are located. The finite element equations are derived for each element and assemble together to form up a set of simultaneous equations. The boundary conditions are then imposed on the set of equations before solving them for the solutions at nodes.

Several examples are used to show derivation of exact solutions for simple problems and to find approximate solutions for more complicated ones. The results have demonstrated that the PDE toolbox can provide approximate solutions with high accuracy if the finite element meshes are refined with small elements. These results highlight the advantages of using the finite element method for solving partial differential equations with complicated boundary conditions and domain geometries.

Exercises

1. Use the PDE toolbox to solve the elliptic partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for a unit square domain of $0 \le x \le 1$ and $0 \le y \le 1$ with the boundary conditions of,

$$u(x,0) = 0, \quad u(x,1) = 1 \qquad 0 \le x \le 1$$
$$u(0,y) = 0, \quad u(1,y) = 0 \qquad 0 \le y \le 1$$

Plot to compare the solution with the exact solution of,

$$u(x,y) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin(n\pi x)\sinh(n\pi y)}{n \sinh(n\pi)}$$

2. If the boundary condition along the right edge of the square domain in Problem 1 is changed to,

$$u(1,y) = 1 \qquad 0 \le y \le 1$$

use the PDE toolbox to solve the problem again. Plot to compare the solution with the exact solution if it can be derived.

3. Use the PDE toolbox to solve the elliptic partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1$$

for a unit square domain of $0 \le x \le 1$ and $0 \le y \le 1$. The four edges have the specified boundary conditions of $u = 0$. Plot to compare the solution with the exact solution of,

$$u(x,y) = \frac{16}{\pi^4} \sum_{i,j=1,3,5,\ldots}^{\infty} \frac{\sin(i\pi x)\,\sin(j\pi y)}{i^3 j^2 + i^2 j^3}$$

4. If there is a circular hole with radius of 0.2 at the center of the square domain in Problem 3, use the PDE toolbox to solve the problem again when the hole edge has the: (a) Dirichlet boundary condition of $u = 0$ and (b) Neumann boundary condition of $\partial u / \partial n = 0$. Plot to compare the two solutions in form of the carpet plot.

5. Use the PDE toolbox to solve the elliptic partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for a unit square domain of $0 \le x \le 1$ and $0 \le y \le 1$ with the boundary conditions of,

$$\begin{aligned} u(x,0) &= 0, \quad u(x,1) = x \qquad 0 \le x \le 1 \\ u(0,y) &= 0, \quad u(1,y) = y \qquad 0 \le y \le 1 \end{aligned}$$

Plot to compare the solution with the exact solution of $u(x, y) = xy$.

6. Use the PDE toolbox to solve Problem 5 again if the differential equation is changed to,

$$-5\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 7u = 10$$

Explain the physical meaning of the solution which is different from that obtained in Problem 5.

7. Use the PDE toolbox to solve the elliptic partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for a unit square domain of $0 \le x \le 1$ and $0 \le y \le 1$ with the boundary conditions of,

$$u(x,0) = 1, \qquad \frac{\partial u}{\partial y}(x,1) + u(x,1) = 2 \qquad 0 \le x \le 1$$

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(1,y) = 0 \qquad 0 \le y \le 1$$

Plot to compare the solution with the exact solution of,

$$u(x,y) = 1 + \frac{y}{2}$$

8. Use the PDE toolbox to solve Problem 7 again if the differential equation is changed to,

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 5u = 9$$

Plot the solution $u(x,y)$ in form of the carpet plot.

9. Use the PDE toolbox to solve the elliptic partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(x^2 + y^2\right) e^{xy}$$

for a rectangular domain with the size of 2×1 units, i.e., $0 \le x \le 2$ and $0 \le y \le 1$. The boundary conditions are,

$$u(x,0) = 1, \qquad u(x,1) = e^x, \qquad 0 \le x \le 2$$

$$u(0,y) = 1, \qquad u(2,y) = e^{2y}, \qquad 0 \le y \le 1$$

Plot to compare the solution with the exact solution of,

$$u(x,y) = e^{xy}$$

10. Use the PDE toolbox to solve the elliptic partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2x\left(x^3 - 6xy + 6xy^2 - 1\right)$$

for a unit square domain of $0 \le x \le 1$ and $0 \le y \le 1$. The four edges have the specified boundary conditions of $u = 0$. Plot to compare the solution with the exact solution of,

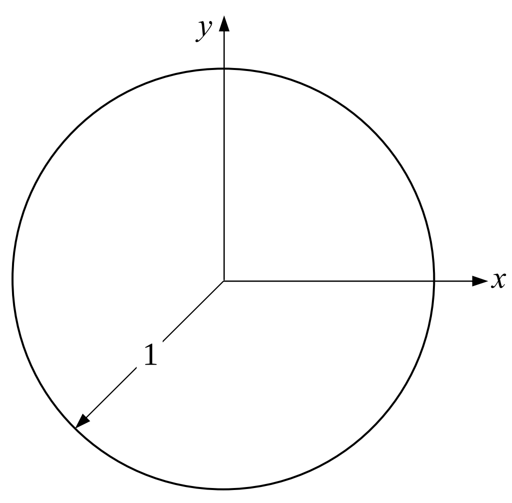
$$u(x,y) = \left(x - x^4\right)\left(y - y^2\right)$$

11. Use the PDE toolbox to solve the elliptic partial differential equation,

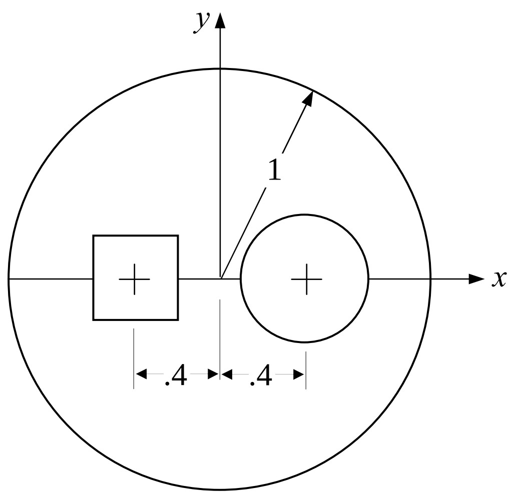
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -20$$

for a circular domain with a unit radius. The boundary condition along the edge is $u = 0$. Compare the solution with the exact solution of,

$$u(x,y) = 5\left(1 - x^2 - y^2\right)$$

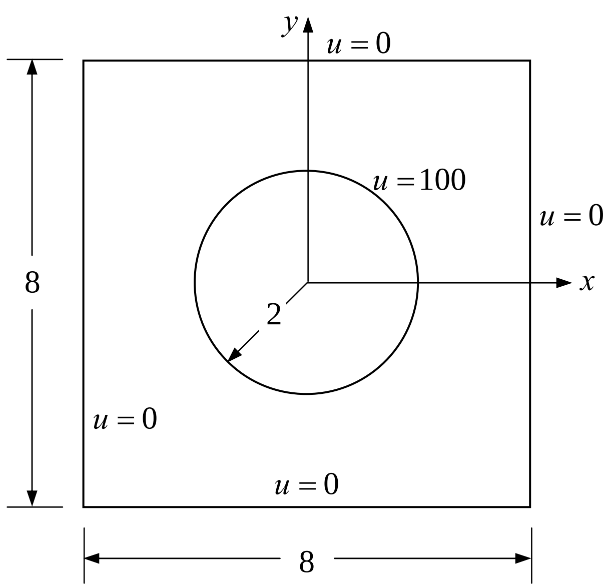


12. Use the PDE toolbox to solve Problem 11 again if the circular domain contains a small circular hole with radius of 0.3 unit and a square hole with dimension of 0.2×0.2 units as shown in the figure. The boundary conditions for the edges of the circular hole and square hole are $u = 0$ and $\partial u / \partial n = 0$, respectively. Plot the solution $u(x,y)$ in form of the carpet plot.



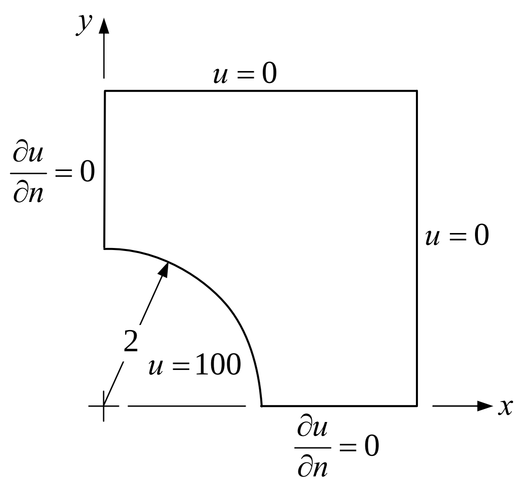
13. Use the PDE toolbox to solve the elliptic partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



for a square domain with the size of 8×8 units. The domain contains a circular hole with the radius of 2 units at its center as shown in the figure. If the boundary conditions of the hole and the outer boundary of the domain are $u = 100$ and $u = 0$, respectively, solve for the solution of $u(x, y)$. Then, plot the solution in form of the carpet plot.

14. Due to symmetry of the solution in Problem 13, only the upper right quarter of the domain as shown in the figure can be used for the analysis. Solve the problem again with the boundary conditions as shown in the figure. Plot the solution in form of the carpet plot.



15. Use the PDE toolbox to solve the elliptic partial differential equation which is in form of the Helmholtz equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a(x, y)u \quad = \quad f(x, y)$$

for a unit square domain of $0 \le x \le 1$ and $0 \le y \le 1$. The four edges have the specified boundary conditions of $u = 0$. Find the solution if $a = -2$ and

$$f(x, y) \quad = \quad xy\left[(x^2 - 7)(1 - y^2) + (1 - x^2)(y^2 - 7)\right]$$

Plot to compare the solution with the exact solution of,

$$u(x, y) \quad = \quad (x - x^3)(y - y^3)$$

16. Use the PDE toolbox to solve the parabolic partial differential equation,

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

for a rectangular domain with the dimensions of $\pi \times 0.3$ units, i.e., $0 \leq x \leq \pi$ and $0 \leq y \leq 0.3$. The boundary conditions along the four edges are $\partial u / \partial n = 0$ and the initial condition is,

$$u(x, y, 0) = \cos x$$

Plot to compare the solution with the exact solution of,

$$u(x, y, t) = e^{-t} \cos x$$

when $t = 0.2$.

17. Use the PDE toolbox to solve the parabolic partial differential equation,

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 2$$

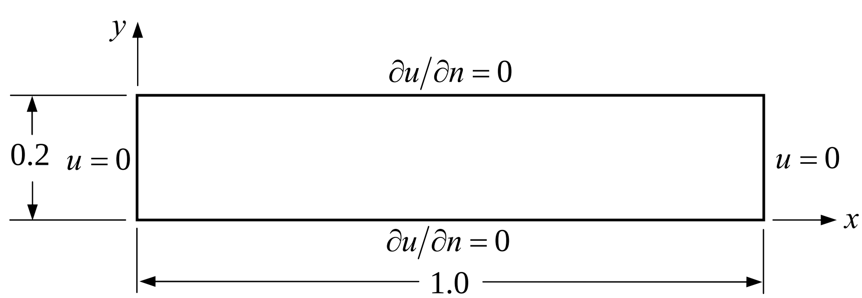
for the rectangular domain of $0 \leq x \leq 1$ and $0 \leq y \leq 0.2$. The boundary conditions are shown in the figure. The initial condition is,

$$u(x, y, 0) = \sin(\pi x) + x(1 - x)$$

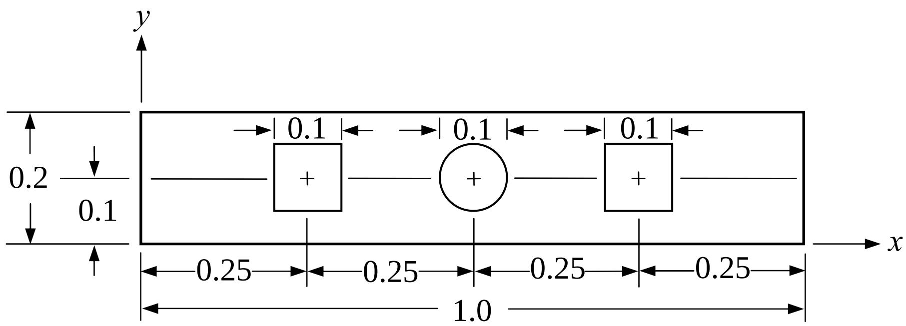
Plot to compare the solution with the exact solution of,

$$u(x, y, t) = e^{-\pi^2 t} \sin(\pi x) + x(1 - x)$$

when $t = 0.1$. Study the solution behavior from animation in the form of carpet plot.



18. Solve Problem 17 again when the domain contains two square holes and a circular hole as shown in the figure. The boundary conditions along the edges of the holes are $\partial u/\partial n = 0$. Provide comments on the solution behavior which is different from that obtained in Problem 17.



19. Use the PDE toolbox to solve the parabolic partial differential equation,

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

for the rectangular domain of $0 \le x \le 1$ and $0 \le y \le 0.2$ with the boundary conditions of,

$$\frac{\partial u}{\partial y}(x,0,t) = 0, \qquad \frac{\partial u}{\partial y}(x,0.2,t) = 0, \qquad 0 \le x \le 1$$

$$u(0,y,t) = 1, \qquad u(1,y,t) = 0, \qquad 0 \le y \le 0.2$$

and the initial condition of,

$$u(x,y,0) = 1 - x - \frac{1}{\pi}\sin(2\pi x)$$

Plot to compare the solution with the exact solution of,

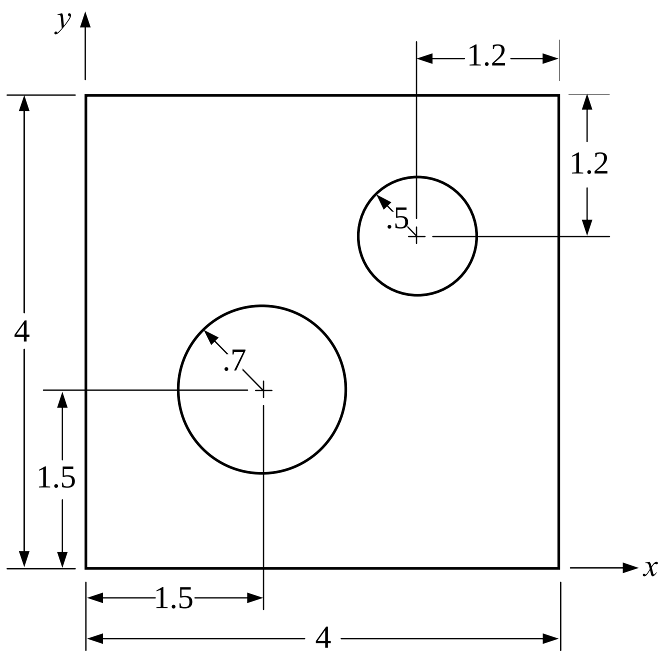
$$u(x,y,t) = 1 - x - \frac{1}{\pi}e^{-4\pi^2 t}\sin(2\pi x)$$

when $t = 0.01$, 0.02 and 0.05.

20. Use the PDE toolbox to solve the parabolic partial differential equation,

$$7\frac{\partial u}{\partial t} - 4\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 3u \quad = \quad 12$$

for the 4×4 unit square domain with two holes as shown in the figure. The boundary conditions along the outer four edges are $u = 0$. The initial condition is given by $u(x, y, 0) = 0$. Find the solutions at $t = 0.2$, 0.5 and 1.0 when the boundary conditions along the hole edges are: (a) $u = 0$ and (b) $\partial u / \partial n = 0$. Plot the solutions in form of the carpet plot.



21. Use the PDE toolbox to solve the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad = \quad 0$$

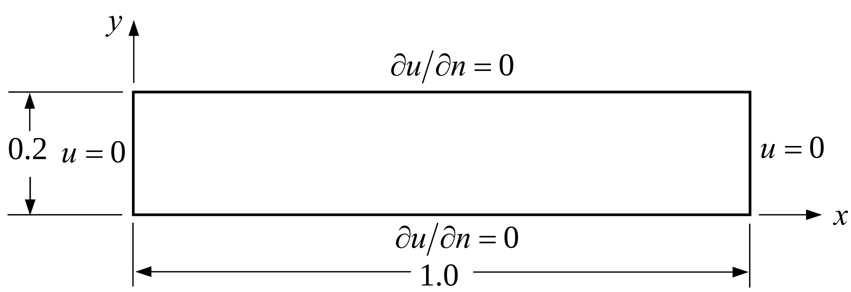
for the domain of $0 \le x \le 1$ and $0 \le y \le 0.2$ with the boundary conditions as shown in the figure. The initial conditions are,

$$u(x, y, 0) \quad = \quad \sin(\pi x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, y, 0) \quad = \quad 0$$

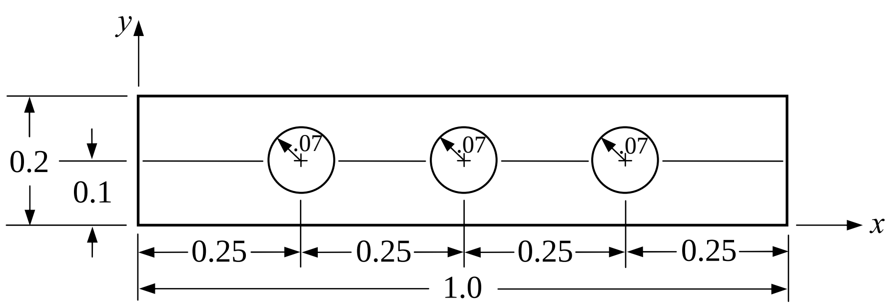
Plot to compare the solution with the exact solution of,

$$u(x, y, t) \quad = \quad \sin(\pi x)\cos(\pi t)$$

when $t = 1.0$ and 2.0.



22. Solve Problem 21 again if the rectangular domain contains three small circular holes as shown in the figure. The boundary conditions along the hole edges are $\partial u / \partial n = 0$. Compare the solution with that obtained in Problem 21 by plotting it in form of the carpet plot.



23. Use the PDE toolbox to solve the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 2e^{-t}\sin x$$

for a rectangular domain of $\pi \times 0.3$ units, i.e., $0 \le x \le \pi$ and $0 \le y \le 0.3$ with the boundary conditions of,

$$\frac{\partial u}{\partial y}(x,0,t) = 0, \qquad \frac{\partial u}{\partial y}(x,0.3,t) = 0 \qquad 0 \le x \le \pi$$

$$u(0,y,t) = 0, \qquad u(\pi,y,t) = 0 \qquad 0 \le y \le 0.3$$

and the initial conditions of,

$$u(x,y,0) = \sin x \quad \text{and} \quad \frac{\partial u}{\partial t}(x,y,0) = -\sin x$$

Plot to compare the solution with the exact solution of,

$$u(x,y,t) = e^{-t}\sin x$$

when $t = 0.2$ and 0.5.

24. Use the PDE toolbox to solve the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

for a rectangular domain of 1×0.2 units, i.e., $0 \le x \le 1$ and $0 \le y \le 0.2$ with the boundary conditions of,

$$\frac{\partial u}{\partial y}(x,0,t) = 0, \qquad \frac{\partial u}{\partial y}(x,0.2,t) = 0 \qquad 0 \le x \le 1$$

$$u(0,y,t) = 0, \qquad u(1,y,t) = 0 \qquad 0 \le y \le 0.2$$

and the initial conditions of,

$$u(x,y,0) = \sin(2\pi x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,y,0) = 2\pi\sin(2\pi x)$$

Plot to compare the solution with the exact solution of,

$$u(x,y,t) = \sin(2\pi x)[\sin(2\pi t) + \cos(2\pi t)]$$

when $t = 0.3$ and 0.6.

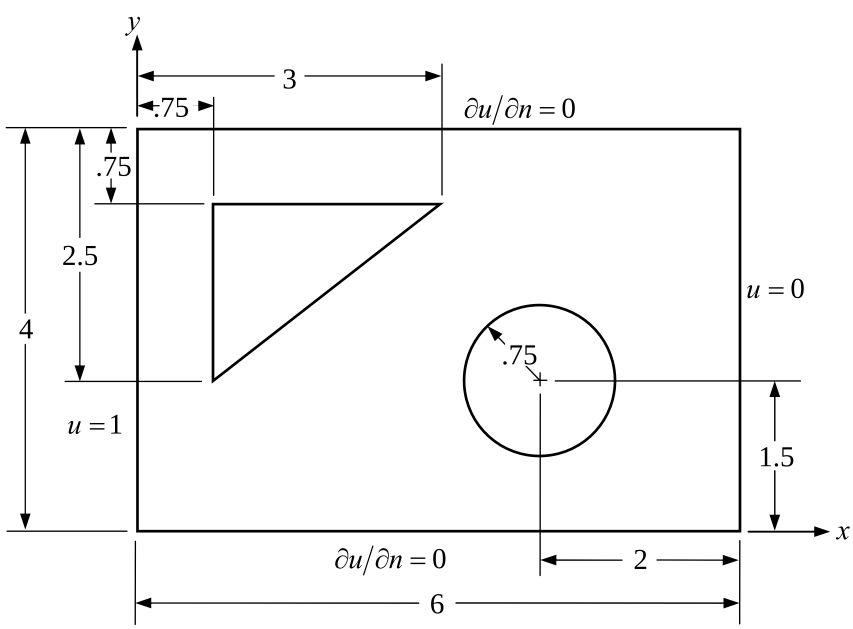
25. Use the PDE toolbox to solve the hyperbolic partial differential equation,

$$3\frac{\partial^2 u}{\partial t^2} - 5\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2u \quad = \quad 4$$

for a rectangular domain of 6×4 units with the boundary conditions along the outer edges as shown in the figure. The boundary conditions along the hole edges are $\partial u/\partial n = 0$. The initial conditions are given by,

$$u(x, y, 0) \quad = \quad 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, y, 0) \quad = \quad 0$$

Find the solutions for the interval of $0 \le t \le 1$. Display the solutions at $t = 0.3, 0.5$ and 1.0 in form of the carpet plot.



Chapter 11

Special Functions

11.1 Introduction

Special functions often occur while solving mathematical problems in science and engineering. These special functions include the error functions, Gamma functions, Beta functions, Bessel functions, Airy functions and Legendre functions. These functions are in various forms of the infinite series as well as in the integral forms. In the past, their values could not be determined conveniently, so many textbooks have to provide them as tables in appendices.

MATLAB contains commands for determining these functions easily. The commands can be implemented in a program to work together with other numerical methods for solving the entire problem. This chapter begins with the definitions of special functions that are generally encountered in mathematics. The MATLAB commands for determining these functions are presented by using examples. The examples demonstrate high efficiency of these commands for determining values of the special functions.

11.2 Error Functions

The error function occurs during solving many forms of differential equations in scientific and engineering problems. The definition of the error function is,

$$erf(x) \;=\; \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

As an example, the exact solution of the initial value problem governed by the first-order ordinary differential equation and the initial condition,

$$\frac{dy}{dx} - 2xy \;=\; 2 \qquad ; \qquad y(0) \;=\; 1$$

is,
$$y(x) \;=\; e^{x^2}\left[1 + \sqrt{\pi}\; erf(x)\right]$$

where $erf(x)$ denotes the error function evaluated at x.

The exact solution above can be obtained by using the `dsolve` command,

```
>> syms x y
>> dsolve('Dy - 2*x*y = 2', 'y(0) = 1', 'x')
ans =

exp(x^2)*(pi^(1/2)*erf(x) + 1)
```

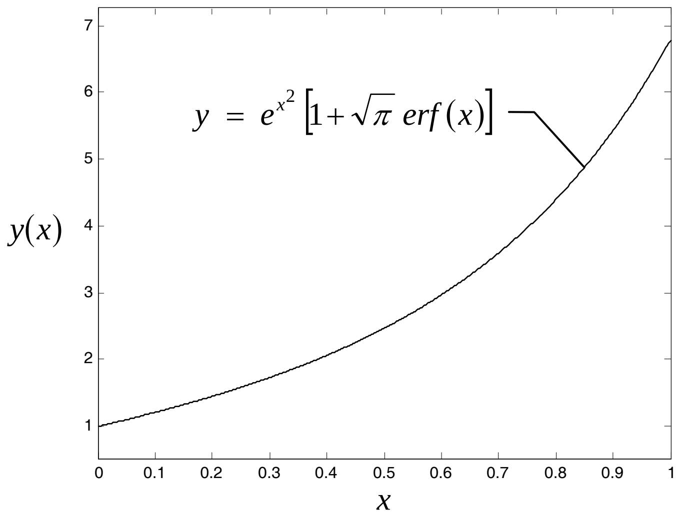
dsolve

The solution of y that varies with x can be plotted by using the `ezplot` command for the interval of $0 \le x \le 1$ as shown in the figure.

```
>> ezplot(ans, [0 1])
```

ezplot

The error function is determined automatically at any x in the commands above to provide the solution for plotting. To determine the error function, e.g. for $x = 1$, we use the `erf` command by entering,




```
>> erf(1)
ans =
    0.8427
```

erf

Note that the error function changes moderately from -1 to 1 in the interval of $-3 \le x \le 3$ as shown in the figure. Values of the error function approach -1 and 1 for $x < -3$ and $x > 3$, respectively.

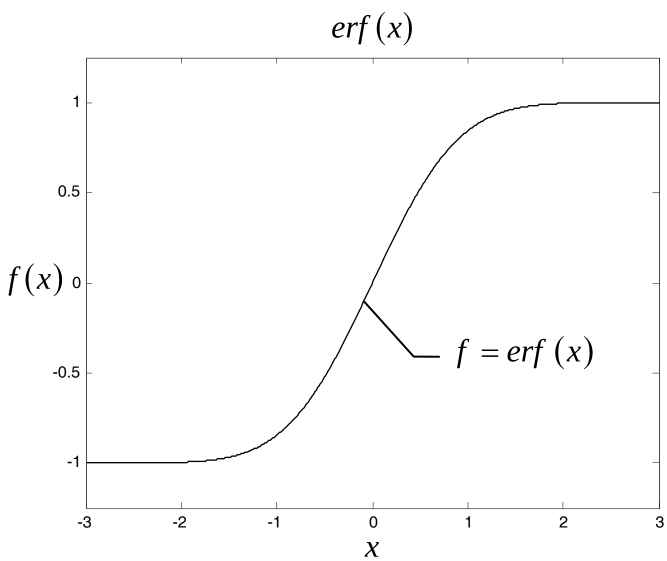
The figure is plotted by using the commands,

```
 >> syms x f
>> f = erf(x);
>> ezplot(f, [-3 3])
```

erf

MATLAB also contains the `erfc` command to determine the complementary error function with the definition of,

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$
$$= 1 - erf(x)$$



As an example, the complementary error function of 1 is,

```
>> erfc(1)
ans =
    0.1573
```

erfc

Variation of the complementary error function from -3 to 3 is shown in the figure.

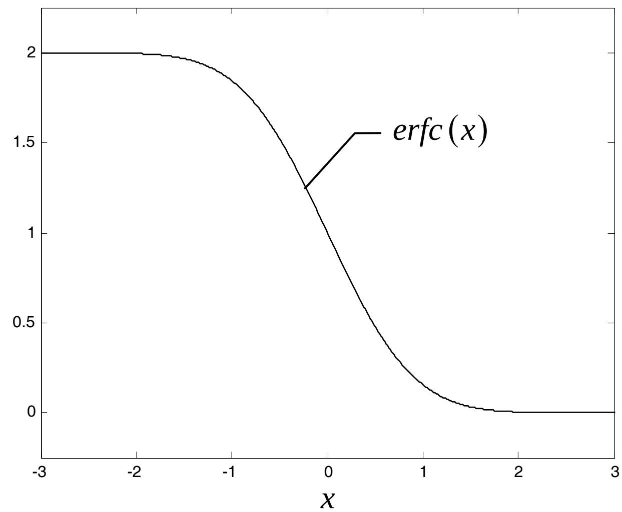
11.3 Gamma Functions

The Gamma function is another function often occurs during solving mathematical problems. The definition of the Gamma function is,

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$$

As an example, the Gamma function when $x = 1$ is,

$1 - erf\,(x)$



$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t}\, t^{1-1}\, dt &&= \int_0^\infty e^{-t}\, dt \\ &= -e^{-t}\Big|_0^\infty &&= -0+1 \;=\; 1\end{aligned}$$

The Gamma function has the special property of,

$$\Gamma(x+1) \;=\; x\Gamma(x)$$

which can be verified by performing the integrations by parts as follows,

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x\, e^{-t}\, dt \;=\; \left[t^x(-e^{-t})\right]_0^\infty - \int_0^\infty x t^{x-1}(-1)e^{-t}\, dt \\ &= x\int_0^\infty t^{x-1}\, e^{-t}\, dt \;=\; x\Gamma(x)\end{aligned}$$

Furthermore, if x is a positive integer, then,

$$\Gamma(x+1) \;=\; x! \qquad\qquad x = 1, 2, 3, \ldots$$

This relation helps determining their values easily. For examples,

$$x = 0; \qquad \Gamma(0+1) = \Gamma(1) = 0! = 1$$
$$x = 1; \qquad \Gamma(1+1) = \Gamma(2) = 1! = 1$$
$$x = 2; \qquad \Gamma(2+1) = \Gamma(3) = 2! = 2$$

MATLAB contains the `gamma` command for finding values of the Gamma function conveniently. For examples,

```
>> gamma(1)
ans =
     1
>> gamma(5)
ans =
    24
```

gamma

The latter result can be verified by $\Gamma(5) = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

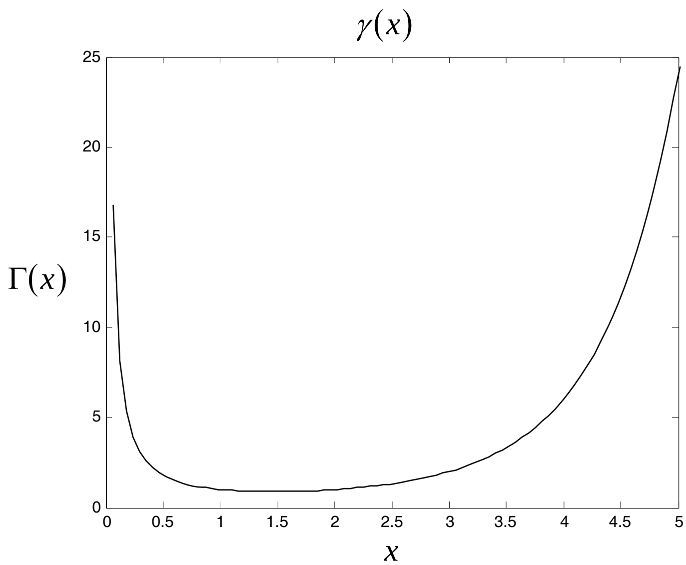
If x is not an integer, the `gamma` command can still be used. For examples,

```
>> gamma(0.71)
ans =
   1.2825
>> gamma(4.38)
ans =
   9.8639
```

gamma

We can employ the `ezplot` command to display variation of the Gamma function for $0 \le x \le 5$ as shown in the figure.

```
>> syms x
>> ezplot('gamma(x)', [0 5 0 25])
```



If x is a negative integer, the Gamma function has the value of infinity. For examples,

```
>> gamma(-1)
ans =
   Inf
>> gamma(-5)
ans =
   Inf
```

gamma

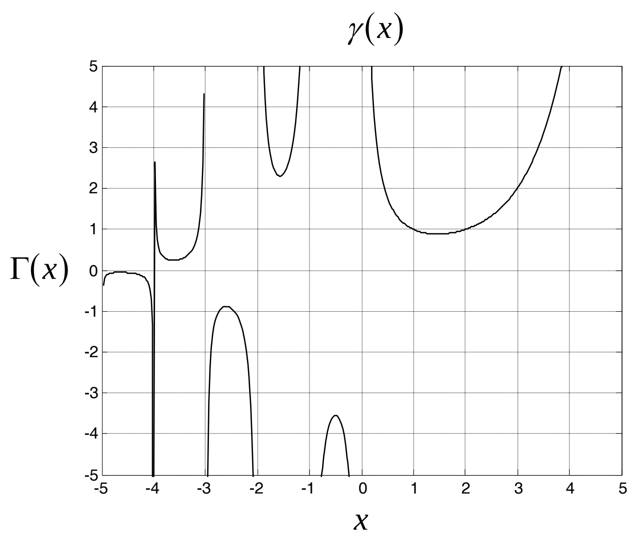
If x is not an integer but a negative value, the Gamma function has a finite value. For examples,

```
>> gamma(-0.71)
ans =
   -4.3664
>> gamma(-4.38)
ans =
   -0.0782
```

We can use the `ezplot` command together with the `grid on` command to display variation of the Gamma function in the interval of $-5 \le x \le 5$ as shown in the figure.

```
>> ezplot('gamma(x)', [-5 5 -5 5]), grid on
```

`grid on`



The Gamma function also has other interesting properties,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(x+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\ldots(2x-1)\pi}{2^x}, \qquad x = 1, 2, 3, \ldots$$

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin(x\pi)}$$

$$2^{2x-1}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}\;\Gamma(2x)$$

which can be verified by the `gamma` command. For example, the first property above,

```
>> gamma(1/2)
ans =
    1.7725
>> sqrt(pi)
ans =
    1.7725
```

gamma

MATLAB also contains the `gammainc` command to find the incomplete Gamma function. The definition of the incomplete Gamma function is,

$$g(x,n) \;=\; \frac{1}{\Gamma(n)}\int_0^x e^{-t}\, t^{n-1}\, dt$$

where x and n are positive values. As an example, when $x = 0.5$ and $n = 1.5$,

```
>> gammainc(0.5,1.5)
ans =
    0.1987
```

gammainc

Variations of the incomplete Gamma function when $n = 1, 2$ and 3 are shown in the figure.

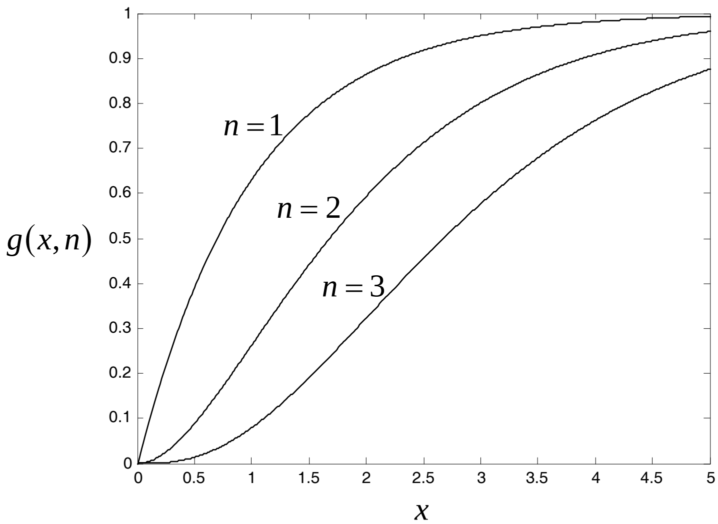
Note that if both x and n are very small, the incomplete Gamma function may be approximate by,

$$g(x,n) \;\approx\; x^n$$

For example, when $x = 0.002$ and $n = 0.001$, then,

```
>> x = 0.002; n = 0.001; g = x^n
g =
    0.9938
>> gammainc(0.002,0.001)
ans =
    0.9944
```

gammainc



11.4 Beta Functions

Definition of the Beta function is given by,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

where $x > 0$ and $y > 0$. For examples,

$$B(1,1) = \int_0^1 t^{1-1}(1-t)^{1-1}\, dt = \int_0^1 dt = 1$$

$$B(2,1) = \int_0^1 t^{2-1}(1-t)^{1-1}\, dt = \int_0^1 t\, dt = \frac{1}{2}$$

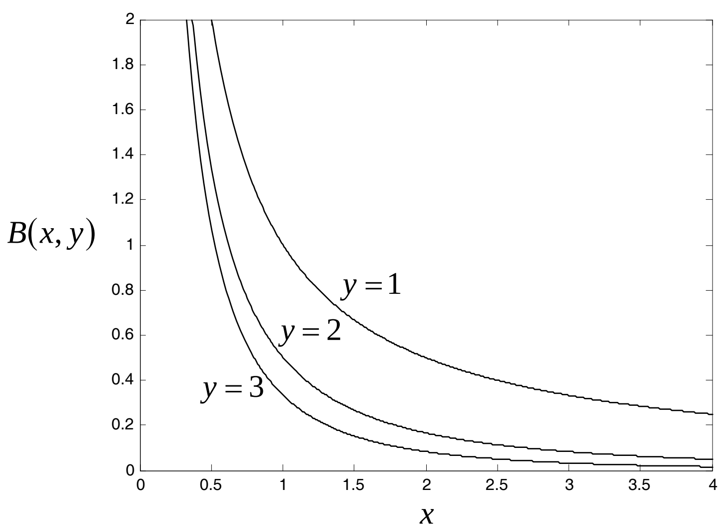
$$B(1,2) = \int_0^1 t^{1-1}(1-t)^{2-1}\, dt = \int_0^1 (1-t)\, dt = \frac{1}{2}$$

MATLAB contains the `beta` command that can be used to determine values of the Beta function conveniently. For examples, the three functions above are obtained by entering,

```
>> beta(1,1)
ans =
     1
>> beta(2,1)
ans =
    0.5000
>> beta(1,2)
ans =
    0.5000
```

beta

Variations of the Beta function with x for $y = 1, 2$ and 3 are shown in the figure.



From the three examples shown above, we found a special property of the Beta function,

$$B(x, y) \quad = \quad B(y, x)$$

which can be verified by starting from the definition of the Beta function,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

If we substitute t by $1-s$, we obtain,

$$\begin{aligned} B(x,y) &= \int_1^0 (1-s)^{x-1} s^{y-1}(-ds) \\ &= \int_0^1 s^{y-1}(1-s)^{x-1}\,ds = B(y,x) \end{aligned}$$

In addition, the Beta function can be determined from the Gamma function,

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

We can also verify this relation by substituting value of *x* and *y*. As an example, when $x = 1.5$ and $y = 2.5$, then,

$$B(1.5,2.5) = \frac{\Gamma(1.5)\,\Gamma(2.5)}{\Gamma(1.5+2.5)}$$

beta

```
>> LHS = beta(1.5,2.5)

LHS =

    0.1963

>> RHS = gamma(1.5)*gamma(2.5)/gamma(1.5+2.5)

RHS =

    0.1963
```

The Beta function also represents the integral value of the product between sine and cosine functions in the form,

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\,d\theta$$

As an example, if we want to find value of the integral,

$$I \;=\; 2\int_0^{\pi/2} \cos^2\theta \sin^3\theta \, d\theta$$

which means when $x = 3/2$ and $y = 2$, we can determine it from,

$$B(1.5,2) \qquad \text{or} \qquad \frac{\Gamma(1.5)\,\Gamma(2)}{\Gamma(3.5)}$$

as follows,

```
>> beta(1.5,2)

ans =

    0.2667

>> gamma(1.5)*gamma(2)/gamma(3.5)

ans =

    0.2667
```

beta

The solution above is confirmed by using the `int` command directly,

```
>> syms theta F
>> F = cos(theta)^2*sin(theta)^3;
>> I = 2*int(F,theta,0,pi/2);
>> double(I)

ans =

    0.2667
```

double

As for another example, if we want to find the solution of the integral,

$$I \;=\; 2\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$$

We can rewrite this integral in the form,

$$I \;=\; 2\int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}} \, d\theta \;=\; 2\int_0^{\pi/2} \cos^{-1/2}\theta \sin^{1/2}\theta \, d\theta$$

which means $x = 1/4$ and $y = 3/4$. The solution is obtained from,

$$B(1/4,3/4) \qquad \text{or} \qquad \frac{\Gamma(1/4)\,\Gamma(3/4)}{\Gamma(1)}$$

```
>> beta(1/4,3/4)

ans =

    4.4429

>> gamma(1/4)*gamma(3/4)/gamma(1)

ans =

    4.4429
```

gamma

Again, the solution can be verified by using the `int` command,

```
>> F = (tan(theta))^(1/2);
>> I = 2*int(F,theta,0,pi/2);
>> double(I)

ans =

    4.4429
```

int

11.5 Bessel Functions

Solutions of some differential equations are in the form of Bessel functions. The standard form of the differential equation, so called the Bessel differential equation, that yields solution in the form of Bessel functions is,

$$x^2 y'' + x y' + \left(x^2 - n^2\right) y = 0$$

where $n \geq 0$. The general solution is,

$$y(x) = C_1 J_n(x) + C_2 Y_n(x)$$

where C_1 and C_2 are constants that can be determined from the initial conditions. The function $J_n(x)$ is called the Bessel function of the first kind of order n while the function $Y_n(x)$ is called the Bessel function of the second kind of order n.

The Bessel function of the first kind of order n is expressed in the form of infinite series containing the Gamma function as,

$$J_n(x) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\,\Gamma(n+k+1)}$$

MATLAB has the `besselj(n,x)` command that can determine the Bessel function of the first kind of order of n at the value of x easily. As an example, The Bessel function of the first kind of order zero at $x=1$ is obtained by typing,

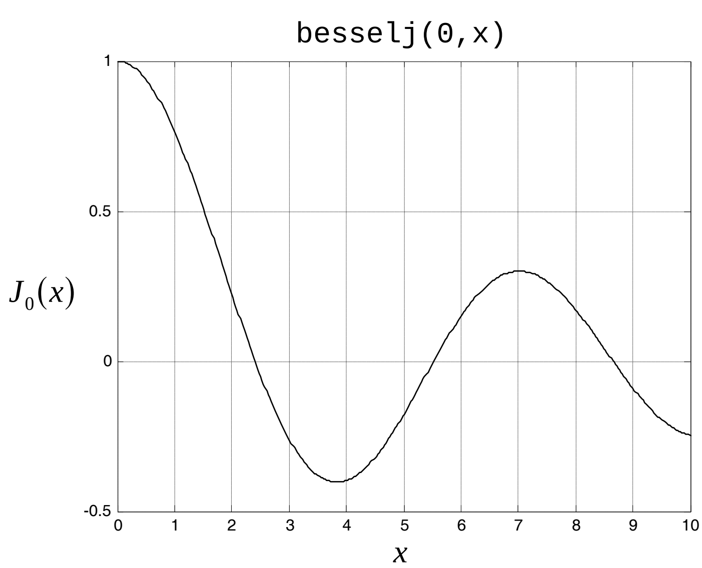
besselj

```
>> besselj(0,1)
ans =
    0.7652
```

i.e., $$J_0(1) \;=\; 0.7652$$

Variation of the Bessel function of the first kind of order zero is plotted with x by using the `ezplot` command as shown in the figure.

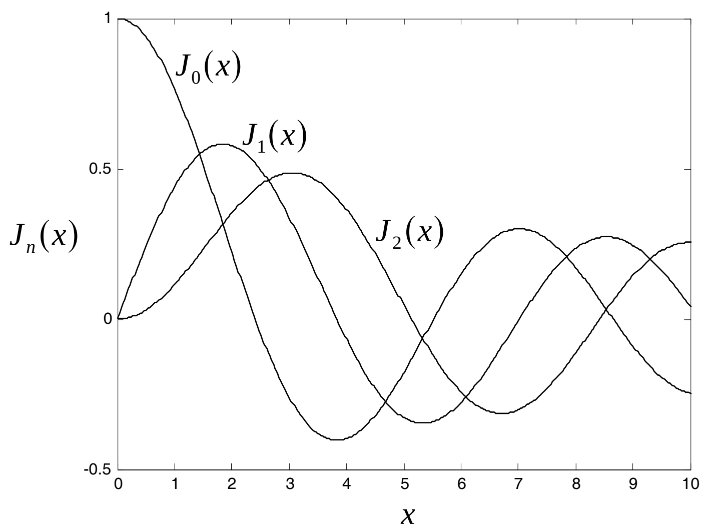
```
>> syms x
>> ezplot('besselj(0,x)', [0 10 -.5 1]), grid on
```



The Bessel functions of the first kind of order zero, one and two that vary with x are plotted together by using the `ezplot` and `hold on` commands as shown in the figure.

```
>> ezplot('besselj(0,x)', [0 10 -.5 1]), hold on
>> ezplot('besselj(1,x)', [0 10 -.5 1]), hold on
>> ezplot('besselj(2,x)', [0 10 -.5 1])
```

ezplot



The Bessel function of the first kind of order $-n$ can be determined from the Bessel function of the first kind of order n by using the relation,

$$J_{-n}(x) \;=\; (-1)^n \, J_n(x)$$

As an example, for $n=1$ and $x=2$, the value on the left-hand-side of the equation above is,

```
>> LHS = besselj(-1,2)
```

besselj

```
LHS =

   -0.5767
```

which is equal to the value on the left-hand-side of the equation,

```
>> RHS = (-1)^1*besselj(1,2)

RHS =

   -0.5767
```

MATLAB can find the derivative of the Bessel function symbolically by using the `diff` command. For example,

$$\frac{d}{dx}(J_n(x)) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

we enter the commands as follows,

```
>> syms n x
>> diff(besselj(n,x))
```

besselj

```
ans =

 (n*besselj(n, x))/x - besselj(n + 1, x)
```

Similarly, MATLAB can also perform integration of the Bessel function symbolically by using the `int` command. For example,

$$\int x^n J_{n-1}(x)\, dx = x^n J_n(x)$$

```
>> int(x^n*besselj(n-1,x))
ans =
x^n*besselj(n, x)
```

The Bessel function of the second kind of order n has the definition of,

$$Y_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}$$

The Bessel function of the second kind of order n at any x can be determined from the `bessely(n,x)` command. As an example, the Bessel function of the second kind of order zero at $x = 2$ is obtained by entering,

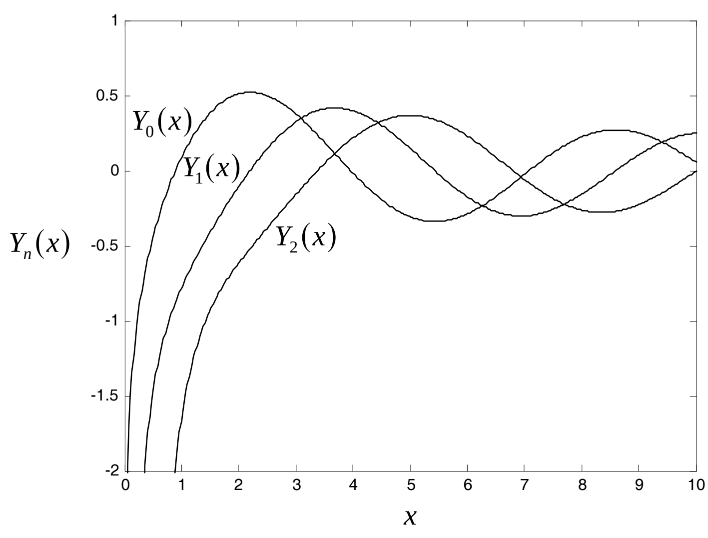
bessely

```
>> bessely(0,2)

ans =

   0.5104
```

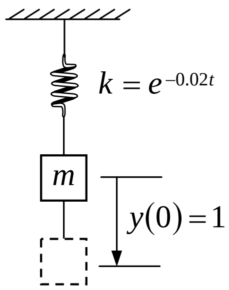
i.e., $$Y_0(2) = 0.5104$$

The Bessel functions of the second kind of order zero, one and two that vary with x are plotted together by using the `ezplot` and `hold on` commands as shown in the figure.

```
>> syms x
>> ezplot('bessely(0,x)', [0 10 -2 1]), hold on
>> ezplot('bessely(1,x)', [0 10 -2 1]), hold on
>> ezplot('bessely(2,x)', [0 10 -2 1])
```



To understand the behavior of the Bessel functions, we study the oscillation of the mass-spring system as shown in the figure.



A mass with $m = 1$ is attached to a spring for which its stiffness k varies with time t in the form,

$$k = e^{-0.02t}$$

The equilibrium condition during oscillation leads to the governing differential equation in the form,

$$\frac{d^2y}{dt^2} + e^{-0.02t} y = 0$$

where y is the unknown displacement that varies with time t. The initial conditions are given by a unit displacement and zero velocity at time $t = 0$ as,

$$y(0) = 1 \qquad \text{and} \qquad \frac{dy}{dt}(0) = 0$$

To solve for the exact solution in form of the Bessel functions, we change the form of the differential equation by letting,

$$x = 100e^{-0.01t}$$

so that,

$$\frac{dx}{dt} = -e^{-0.01t} = -0.01x$$

From the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -0.01x\frac{dy}{dx}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{dx}{dt} = \frac{d}{dx}\left(-0.01x\frac{dy}{dx}\right)(-0.01x)$$

$$= 0.0001x\frac{d}{dx}\left(x\frac{dy}{dx}\right)$$

Since

$$x^2 = 10000e^{-0.02t}$$

then,

$$e^{-0.02t} = 0.0001x^2$$

Thus, the differential equation becomes,

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right)+x^2y \;=\; 0$$

or,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x^2y \;=\; 0$$

which is in form of the Bessel differential equation of order zero. The general solution of this differential equation is,

$$y(x) \;=\; C_1J_0(x)+C_2Y_0(x)$$

where C_1 and C_2 are constants that can be determined from the two initial conditions as follows,

$$y(t=0) \;=\; 1 \qquad \text{or} \qquad y(x=100) \;=\; 1$$

and

$$\frac{dy}{dt}(t=0) \;=\; 0 \qquad \text{or} \qquad \frac{dy}{dx}(x=100) \;=\; 0$$

The two conditions above lead to a set of two simultaneous equations,

$$J_0(100)\,C_1+Y_0(100)\,C_2 \;=\; 1$$

and

$$J_0'(100)\,C_1+Y_0'(100)\,C_2 \;=\; 0$$

After solving for the two constants of C_1 and C_2, we substitute them into the general solution to obtain the exact solution of,

$$y(x) \;=\; 50\pi\left[J_1(100)Y_0(x)-Y_1(100)J_0(x)\right]$$

Then, by substituting $x=100e^{-0.01t}$, the exact solution of y can be written in form of time t as,

$$y(t) \;=\; 50\pi\left[J_1(100)Y_0\left(100e^{-0.01t}\right)-Y_1(100)J_0\left(100e^{-0.01t}\right)\right]$$

The solution can be determined for the interval of $0\le t\le 100$ by creating a script file that contains the following commands,

```
>> t = 0:.01:100;
>> ex = exp(-.01.*t);
>> ee = 100.*ex;
>> J1 = besselj(1,100);
>> Y0 = bessely(0,ee);
```

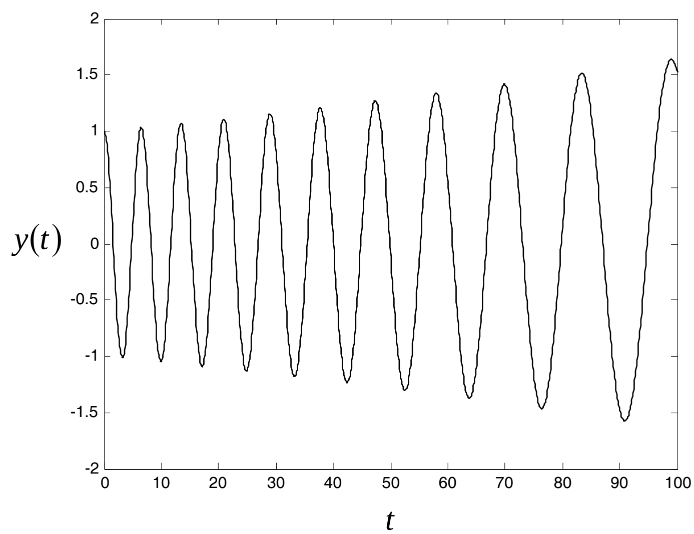
besselj

bessely

```
>> Y1 = bessely(1,100);
>> J0 = besselj(0,ee);
>> y = 50*pi*(J1.*Y0 - Y1.*J0);
>> plot(t,y), axis([0 100 -2 2])
```

plot

The file generates a plot of the mass movement that varies with time as shown in the figure. The figure shows that the magnitude of the mass movement increases with time. Such behavior agrees with the fact that the spring stiffness weakens with time.



If the sign in front of the third term of the Bessel differential equation changes from positive to negative, the equation is called the modified Bessel differential equation,

$$x^2 y'' + xy' - \left(x^2 + n^2\right)y \;=\; 0$$

The general solution of the modified Bessel differential equation is in the form,

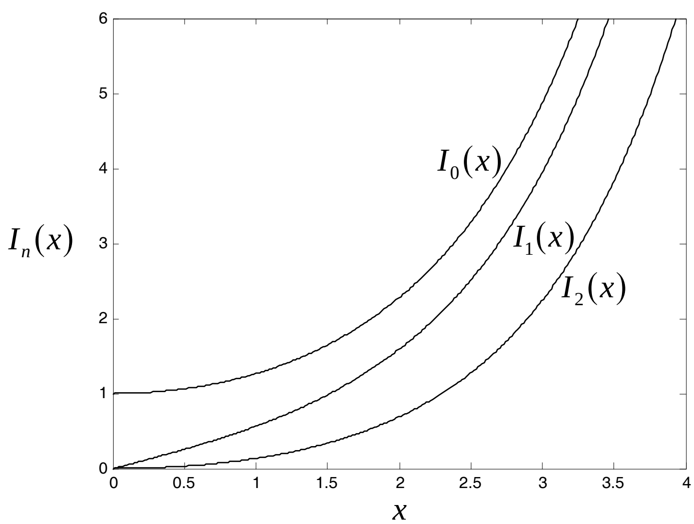
$$y(x) \;=\; C_1 I_n(x) + C_2 K_n(x)$$

where C_1 and C_2 are constants that can be determined from the initial conditions. The function $I_n(x)$ is called the modified Bessel function of the first kind of order n, while the function $K_n(x)$ is called the modified Bessel function of the second kind of order n.

MATLAB contains the `besseli(n,x)` command to determine the modified Bessel function of the first kind of order n at a value of x. The modified Bessel functions of the first kind of order zero, one and two that vary with x are plotted together by using the `ezplot` and `hold on` commands as shown in the figure.

```
>> syms x
>> ezplot('besseli(0,x)', [0 4 0 6]), hold on
>> ezplot('besseli(1,x)', [0 4 0 6]), hold on
>> ezplot('besseli(2,x)', [0 4 0 6])
```

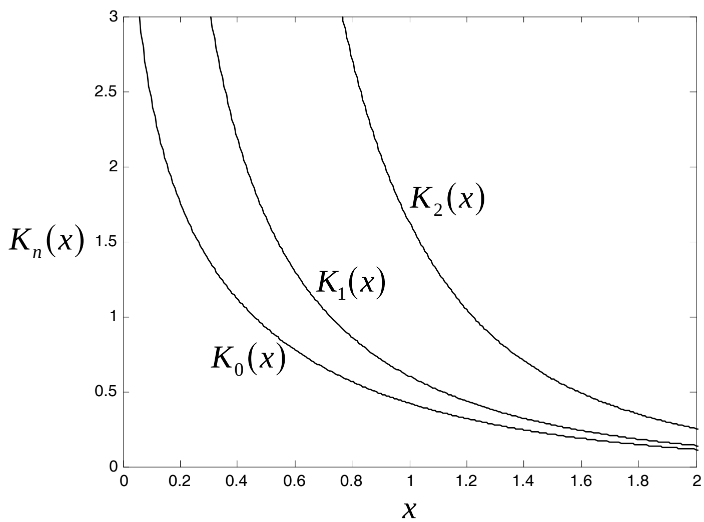
besseli



Similarly, the modified Bessel functions of the second kind of order zero, one and two that vary with x can be plotted together by using the `ezplot` and `hold on` commands as shown in the figure.

```
>> ezplot('besselk(0,x)', [0 2 0 3]), hold on
>> ezplot('besselk(1,x)', [0 2 0 3]), hold on
>> ezplot('besselk(2,x)', [0 2 0 3])
```

besselk



11.6 Airy Functions

Airy functions are special functions that arise from solving the Airy differential equation,

$$\frac{d^2y}{dx^2} - xy = 0$$

The differential equation in the form above occurs while solving some problems in physics. The form looks quite clean but a simple solution is not available. The general solution of the Airy differential equation is,

$$y(x) = C_1 Ai(x) + C_2 Bi(x)$$

where C_1 and C_2 are constants. The $Ai(x)$ and $Bi(x)$ denote the Airy and Bairy functions, respectively. These two functions are,

$$Ai(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(xt + \frac{1}{3}t^3\right)dt$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right)dt + \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right)dt$$

MATLAB contains the `airy(k,x)` command for determining the Airy and Bairy functions at any x when $k = 0$, 1, 2 and 3,

$k = 0$;	`airy(0,x)`	means	$Ai(x)$
$k = 1$;	`airy(1,x)`	means	$Ai'(x)$
$k = 2$;	`airy(2,x)`	means	$Bi(x)$
$k = 3$;	`airy(3,x)`	means	$Bi'(x)$

where the prime symbol (′) denotes the derivative order of the function.

For examples, if we want to determine $Ai(0)$, we type,

airy

```
>> airy(0,0)

ans =

    0.3550
```

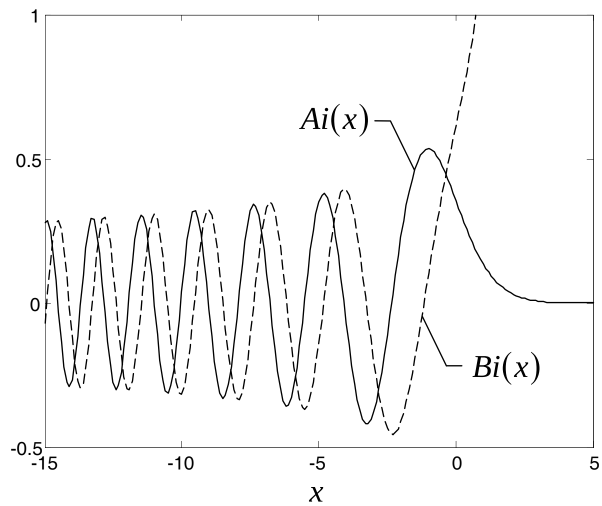
Or, to determine $Bi'(-1)$, we enter,

```
>> airy(3,-1)

ans =

    0.5924
```

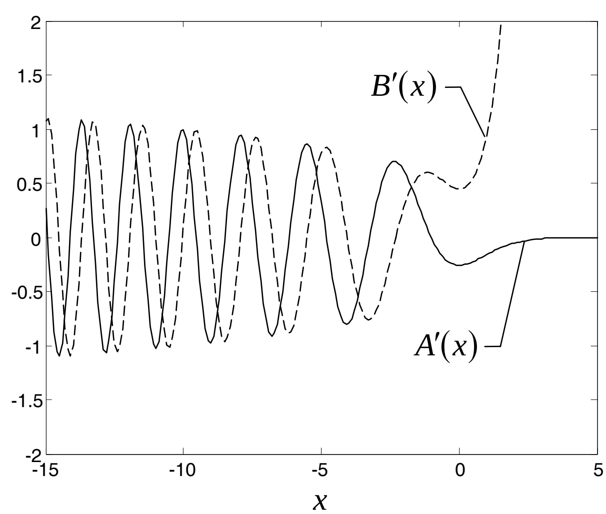
Variations of the Airy and Bairy functions can be plotted with x as shown in the figure by creating commands as follows,

plot

```
>> x = [-15:.1:5];
>> ai = airy(0,x);
>> bi = airy(2,x);
>> plot(x,ai,'-k'), axis([-15 5 -0.5 1])
>> hold on
>> plot(x,bi,'--k')
```



Similarly, variations for the derivatives of the Airy and Bairy functions can also be plotted as shown in the figure.



Now, we can solve for the solution of the Airy differential equation,

$$\frac{d^2y}{dx^2} - x\,y \;=\; 0 \qquad\qquad 0 \le x \le 1$$

with the initial conditions of $y(0) = \alpha$ and $y'(0) = \beta$.

The general solution of the differential equation is,

$$y(x) = C_1\,Ai(x) + C_2\,Bi(x)$$

where C_1 and C_2 are constants that can be determined from the two initial conditions as follows,

$$\begin{aligned} C_1\,Ai(0) + C_2\,Bi(0) &= \alpha \\ C_1\,Ai'(0) + C_2\,Bi'(0) &= \beta \end{aligned}$$

Results of C_1 and C_2 are in form of the Airy function at $x = 0$ which can be found by using the `airy(k,0)` command. Results of the C_1 and C_2 can also be written in form of the Gamma functions as,

$$\begin{aligned} C_1 &= \pi\left[\frac{3^{1/6}\alpha}{\Gamma(1/3)} - \frac{\beta}{3^{1/6}\Gamma(2/3)}\right] \\ C_2 &= \pi\left[\frac{\alpha}{3^{1/3}\Gamma(1/3)} + \frac{\beta}{3^{2/3}\Gamma(2/3)}\right] \end{aligned}$$

where the values of the Gamma functions are obtained from the `gamma` command.

If the initial conditions are given by,

$$y(0) = \alpha = 1 \qquad \text{and} \qquad y'(0) = \beta = 0$$

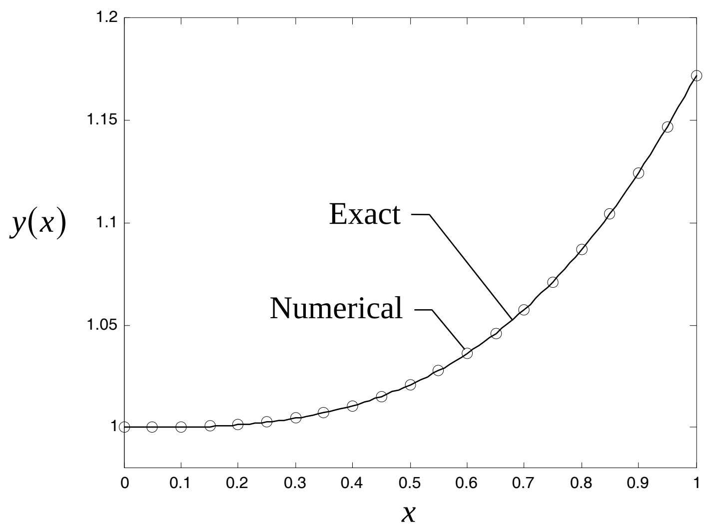
then, C_1 and C_2 become,

$$C_1 = \frac{\pi 3^{1/6}}{\Gamma(1/3)} \qquad \text{and} \qquad C_2 = \frac{\pi}{3^{1/3}\Gamma(1/3)}$$

Thus, the exact solution $y(x)$ of the initial value problem governed by the Airy differential equation is,

$$y(x) = \frac{\pi 3^{1/6}}{\Gamma(1/3)}\,Ai(x) + \frac{\pi}{3^{1/3}\Gamma(1/3)}\,Bi(x)$$

The exact solution is plotted and compared with the approximate solution obtained from using the numerical method via the `ode45` command as shown in the figure.



11.7 Legendre Functions

Many problems in the fields of applied mathematics, physics and chemistry are governed by the differential equations in form of the Legendre equation. The Legendre differential equation is,

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y \quad = \quad 0 \qquad\qquad 0<x<1$$

where n = 0, 1, 2, … are positive integers. General solution of the Legendre differential equation consists of the Legendre functions of order n,

$$y(x) \quad = \quad C_1 P_n(x)+C_2 Q_n(x)$$

where C_1 and C_2 are constants. The $P_n(x)$ and $Q_n(x)$ are called the Legendre functions of the first and second kind, respectively. These functions are given by,

$$P_n(x) \quad = \quad \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

and $$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x}$$

For example, when $n=1$,

$$P_1(x) = \frac{1}{2^1 1!}\frac{d}{dx}(x^2-1) = x$$

$$Q_1(x) = \frac{1}{2}P_1(x)\ln\frac{1+x}{1-x} = \frac{x}{2}\ln\frac{1+x}{1-x}$$

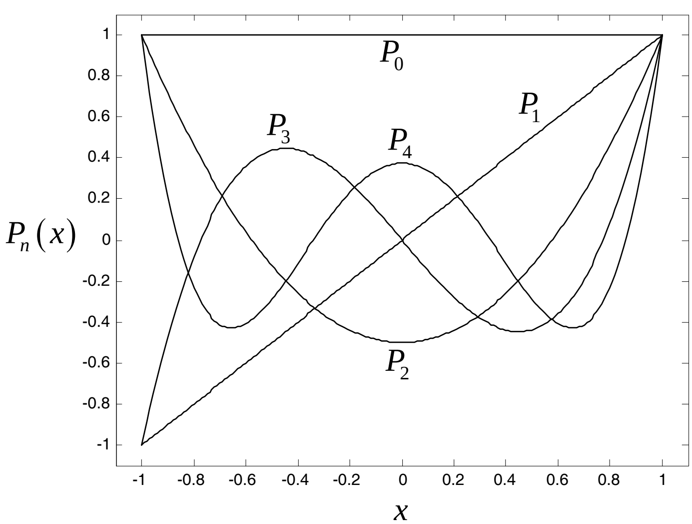
Thus, for $n=0$ to 5, these functions are,

$$P_0(x) = 1 \quad ; \quad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2-1\right) \quad ; \quad P_3(x) = \frac{1}{2}\left(5x^3-3x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4-30x^2+3\right) \quad ; \quad P_5(x) = \frac{1}{8}\left(63x^5-70x^3+15x\right)$$

Variations of Legendre functions of the first kind, P_0 to P_4 are shown in the figure.



Similarly, the Legendre functions of the second kind when $n=0$ to 4 are,

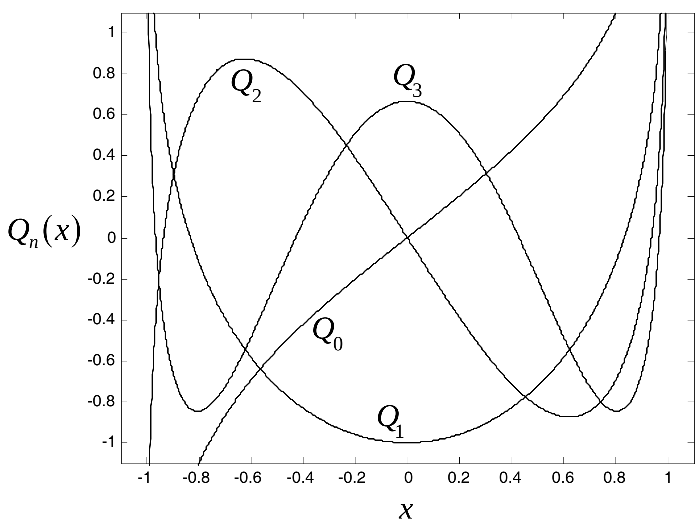
$$Q_0(x) \;=\; \frac{1}{2}\ln\frac{1+x}{1-x} \qquad ; \qquad Q_1(x) \;=\; \frac{x}{2}\ln\frac{1+x}{1-x}-1$$

$$Q_2(x) \;=\; \frac{1}{4}\left(3x^2-1\right)\ln\frac{1+x}{1-x}-\frac{3}{2}x$$

$$Q_3(x) \;=\; \frac{1}{4}\left(5x^3-3x\right)\ln\frac{1+x}{1-x}-\frac{5}{2}x^2+\frac{2}{3}$$

$$Q_4(x) \;=\; \frac{1}{16}\left(35x^4-30x^2+3\right)\ln\frac{1+x}{1-x}-\frac{35}{8}x^3+\frac{55}{24}x$$

Variations of Legendre functions of the second kind, Q_0 to Q_3 are shown in the figure.



The Legendre functions $P_n(x)$ and $Q_n(x)$ above are solutions of the Legendre differential equations. For example, when $n=1$, the Legendre differential equation reduces to,

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y \;=\; 0$$

If the boundary conditions are given such that both constants C_1 and C_2 are equal to one, then the exact solution to this boundary value problem is,

$$y(x) = P_1(x) + Q_1(x)$$

or, $$y(x) = x + \frac{x}{2}\ln\frac{1+x}{1-x} - 1$$

We can verify the validity of this solution by substituting it into the differential equation. This can be easily done by using the symbolic mathematics capability in MATLAB as follows,

```
>> syms x
>> P1 = x;
>> Q1 = (x/2)*log((1+x)/(1-x)) - 1;
>> y = P1 + Q1;
>> dy = diff(y,x);
>> d2y = diff(dy,x);
>> LHS = (1-x^2)*d2y - 2*x*dy + 2*y;
>> RHS = simple(LHS)

RHS =

0
```

`diff`

The result obtained is zero which is equal to the right-hand-side of the differential equation. This means the solution containing the Legendre functions $P_1(x)$ and $Q_1(x)$ is the solution to the differential equation above.

A more complicated Legendre differential equation is the associated Legendre differential equation in the form of,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0 \qquad 0 < x < 1$$

where n and m are positive integers. If $m = 0$, the associated Legendre differential equation reduces to the Legendre differential equation. The general solution of the associated Legendre differential equation is,

$$y(x) = C_1 P_n^m(x) + C_2 Q_n^m(x)$$

where C_1 and C_2 are constants which are determined from the given boundary conditions. The $P_n^m(x)$ and $Q_n^m(x)$ are the associated Legendre functions of the first and second kind, respectively. These functions are in the forms,

$$P_n^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

and

$$Q_n^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} Q_n(x)$$

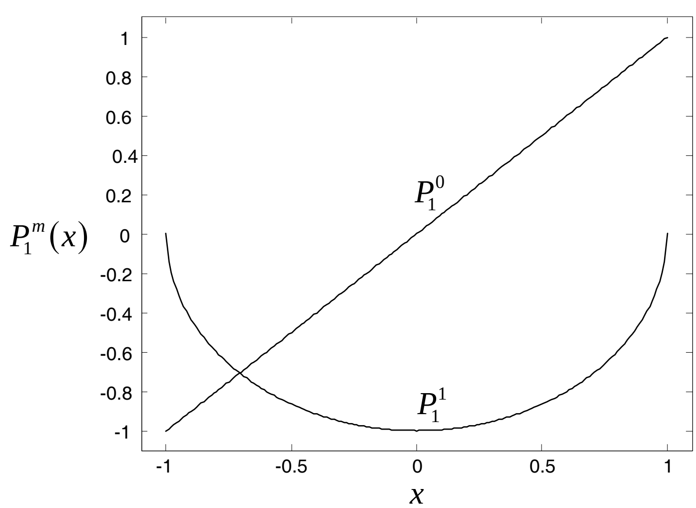
The associated Legendre function of the first kind has the property of,

$$P_n^m(x) = 0 \qquad \text{if} \qquad m > n$$

For example, when $n=1$ and $m=0,1$, the associated Legendre functions are,

$$P_1^0 = x \qquad \text{and} \qquad P_1^1 = -\left(1-x^2\right)^{1/2}$$

Variations of P_1^0 and P_1^1 are plotted as shown in the figure.



The associated Legendre function of order n at any x is obtained by using the `legendre(n,x)` command. As an example, the associated Legendre functions of order $n=1$ at

$x = -1.0, -0.5, 0.0, 0.5$ and 1.0 are obtained from the following commands,

```
>> x = [-1:0.5:1];
>> legendre(1,x)
```

legendre

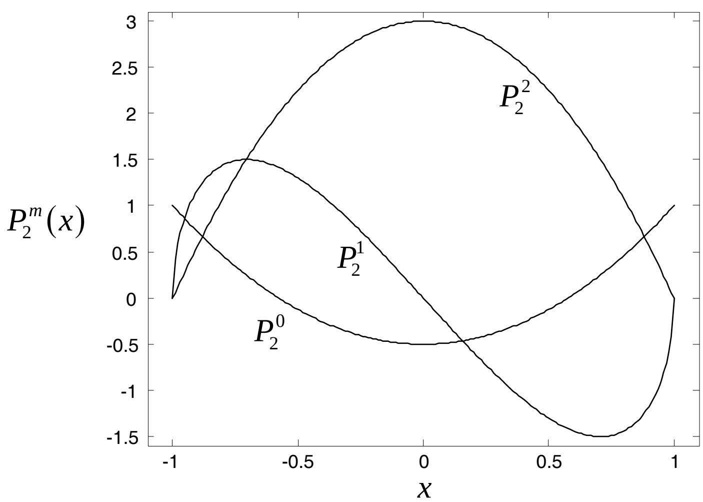
```
ans =

   -1.0000   -0.5000         0    0.5000    1.0000
         0   -0.8660   -1.0000   -0.8660         0
```

The values in the first line are P_1^0 while the results in the second line are P_1^1. These values agree with those shown in the figure.

Similarly, the associated Legendre functions of order $n = 2$ when $m = 0, 1$ and 2 are,

$$P_2^0 = \frac{1}{2}\left(3x^2 - 1\right) \quad ; \quad P_2^1 = -3x\left(1 - x^2\right)^{1/2} \quad ; \quad P_2^2 = 3\left(1 - x^2\right)$$

Variations of these functions are plotted as shown in the figure.



In this case, the associated Legendre function of order $n = 2$ and $x = -1.0, -0.5, 0.0, 0.5$ and 1.0 are obtained by entering the commands,

```
>> x = [-1:0.5:1];
>> legendre(2,x)
```

legendre

```
ans =

   1.0000   -0.1250   -0.5000   -0.1250    1.0000
        0    1.2990         0   -1.2990         0
        0    2.2500    3.0000    2.2500         0
```

The values in the first, second and third lines are P_2^0, P_2^1 and P_2^2, respectively, at the five *x* locations. These values agree with those shown in the figure. The two examples presented above highlight the `legendre(n,x)` command that can be used to find values of the associated Legendre functions of order *n* at any *x* conveniently.

11.8 Special Integrals

MATLAB contains commands for determining several special integrals. Different types of special integrals can be found by typing,

```
>> help mfunlist
```

mfunlist

In this section, some special integrals normally encountered while solving mathematical problems are presented.

The *Dawson integral* is a special integral that occurs in conduction heat transfer and theory of electric oscillation. The integral is in the form,

$$F(x) = e^{-x^2} \int_0^x e^{t^2}\, dt$$

The function $F(x)$ above is the exact solution of the differential equation,

$$\frac{dF}{dx} + 2xF = 1$$

with the initial condition of $F(0)=0$. We can verify that the function $F(x)$ above satisfies the differential equation by substituting it into the left-hand-side of the equation as follows,

int

```
>> syms x t
>> F = exp(-x^2)*int(exp(t^2),t,0,x);
>> diff(F,x) + 2*x*F

ans =

1
```

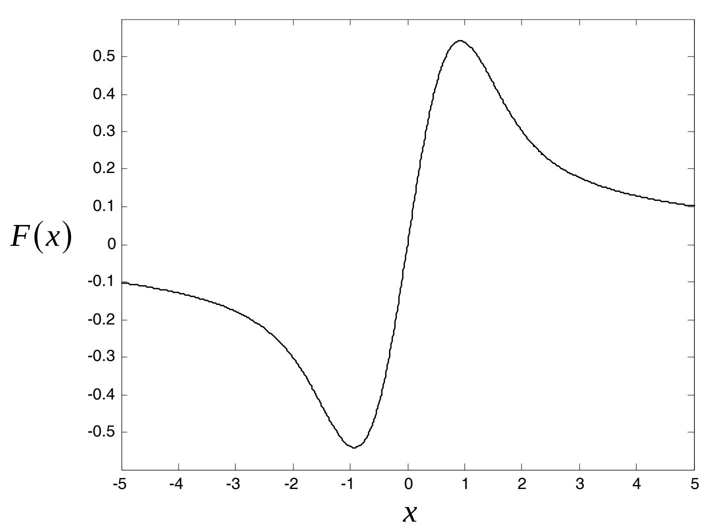
The result is one which is equal to the value on the right-hand-side of the equation.

To find value of the Dawson integral, $F(x)$, MATLAB uses the `mfun` function to call the `Dawson` command. For example, we can find values and plot variation of the Dawson integral in the interval of $-5 \le x \le 5$ by entering the commands,

Dawson

```
>> x = [-5:.01:5];
>> d = mfun('Dawson', x);
>> plot(x,d,'k'); axis([-5 5 -0.6 0.6]);
```

which lead to the plot of $F(x)$ as shown in the figure.



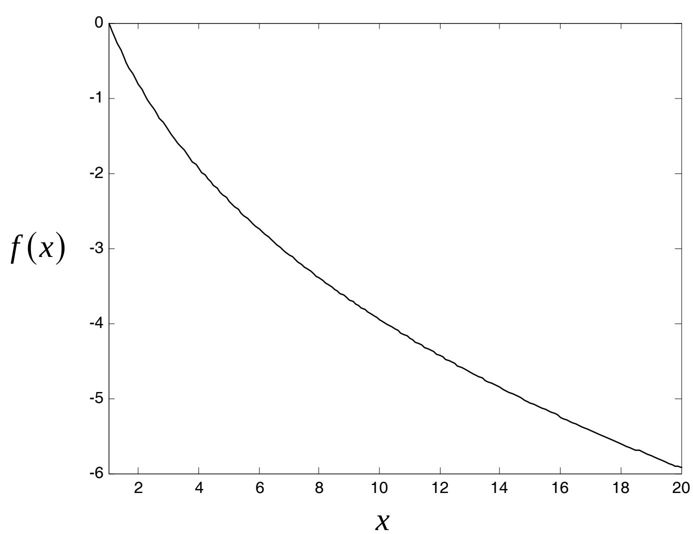
The *dilogarithm integral* is another special integral normally occurs while solving some mathematical problems. The integral is in a specific form of,

$$f(x) \; = \; \int_1^x \frac{\ln(t)}{1-t}\,dt \qquad\qquad x > 1$$

We can employ the `dilog` command via the `mfun` function to determine the dilogarithm integral $f(x)$ at a given *x*. For example, values of the dilogarithm integral $f(x)$ in the interval of $1 \le x \le 20$ can be plotted as shown in the figure by using the commands,

```
>> x = [1:.1:20];
>> di = mfun('dilog', x);
>> plot(x,di,'k'); axis([1 20 -6 0]);
```

`dilog`



The *exponential integral* is another special integral that arises while solving some mathematical problems. The integral is in the form,

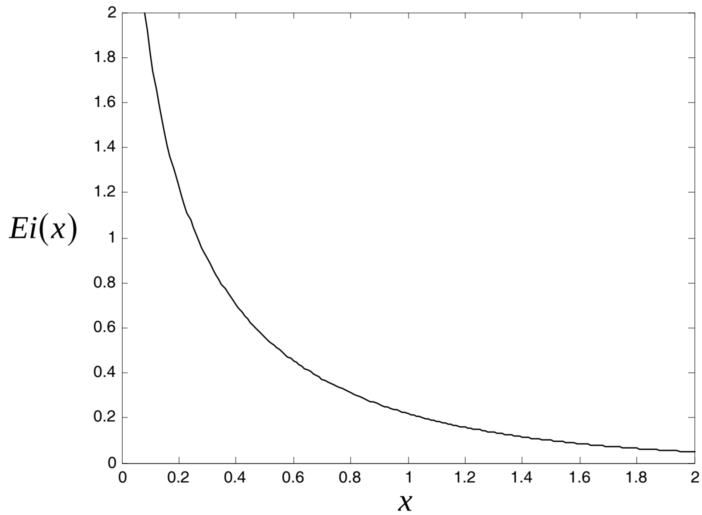
$$Ei(x) \; = \; \int_x^\infty \frac{e^{-t}}{t}\,dt$$

MATLAB contains the `expint(x)` command to determine this special integral at a given *x*. As an example, the exponential integral at $x = 0.5$ is obtained by entering,

```
>> expint(0.5)
ans =

    0.5598
```

expint

Variation of the exponential integral for $0 < x < 2$ is plotted as shown in the figure.



The *Fresnel integrals* are special integrals that contain sine and cosine functions in the form,

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt$$

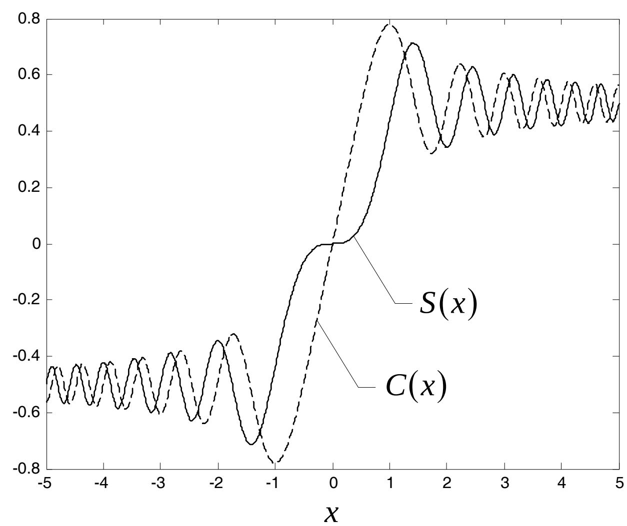
and
$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt$$

Values of the Fresnel integrals, $S(x)$ and $C(x)$, are obtained by using the `FresnelS` and `FresnelC` commands via calling the `mfun` function. For example, these two special integrals are determined in the interval of $-5 \le x \le 5$ and plotted as shown in the figure by using the commands,

```
>> x = [-5:.01:5];
>> s = mfun('FresnelS', x);
>> plot(x,s,'k');
>> axis([-5 5 -0.8 0.8]);
>> hold on
>> c = mfun('FresnelC', x);
>> plot(x,c,'--k');
```

FresnelS

FresnelC



It can be seen from the figure that,

$$S(-\infty) = C(-\infty) = -\frac{1}{2}$$

$$S(0) = C(0) = 0$$

and

$$S(\infty) = C(\infty) = \frac{1}{2}$$

11.9 Concluding Remarks

In this chapter, details of special functions and integrals normally arise while solving mathematical problems are presented. The special functions include the error and complementary error functions, the Gamma and incomplete Gamma functions, the Beta functions, the Bessel and modified Bessel functions, the Airy and Bairy functions, and the Legendre and associated Legendre functions. The special integrals presented herein are the Dawson integral, the dilogarithm integral, the exponential integral and the Fresnel integrals. Because these special functions and integrals are in complicated forms, they are tabulated as values and provided in appendices in many mathematical textbooks.

MATLAB contains commands for determining values of these special functions and integrals conveniently. Several examples are presented to demonstrate how to use these commands. Variations of the special functions and integrals are plotted to display their physical meanings. These commands can be included in a computer program to alleviate difficulty for solving the complete problem. The ability of these commands thus helps us to solve mathematical problems more effectively.

Exercises

1. Develop a computer program to find values of the error and complementary error functions by using the `erf` and `erfc` commands. Show these values in form of a table for $-3 \leq x \leq 3$ with the increment of $\Delta x = 0.05$.

2. For a small value of *x*, the error function may be determined from the series,

$$erf\ (x) \;\cong\; \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \ldots\ldots\right)$$

Compute the percentages of error by comparing the series solutions with those obtained by using the `erf` command for $x = 0.01$, 0.1 and 0.5, respectively.

3. If x is large, the error function may be determined from the series,

$$erf\ (x) \cong 1-\frac{e^{-x^2}}{\sqrt{\pi}\, x}\left(1-\frac{1}{2x^2}+\frac{1\cdot 3}{\left(2x^2\right)^2}-\frac{1\cdot 3\cdot 5}{\left(2x^2\right)^3}+........\right)$$

Compute the percentages of error by comparing the series solutions with those obtained by using the `erf` command for $x=1.5$, 2 and 3, respectively.

4. The error function can be written in form of the Maclaurin series as,

$$\begin{aligned} erf\ (x) &= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)} \\ &= \frac{2}{\sqrt{\pi}}\left(x-\frac{x^3}{3}+\frac{x^5}{10}-\frac{x^7}{42}+\frac{x^9}{216}-......\right) \end{aligned}$$

Develop a computer program to determine the error function from the series above. Compare the solutions with those obtained by using the `erf` command for $x=0.5$, 1 and 2, respectively.

5. Use the `gamma` command to find the values of,

(a) $\Gamma(8.37)$	(b) $\Gamma(1.5)\,\Gamma(2.3)$
(c) $\Gamma(3.64)/\Gamma(1.18)$	(d) $\Gamma(-1002)$
(e) $\log(\Gamma(0.5))$	(f) $e^{-\Gamma(1.6)}$

6. If x is a negative non-integer value, use the `gamma` command to show that,

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

Then, plot the variations of $\Gamma(x)$ and $\Gamma(x+1)$ for the interval of $-2<x<-1$.

7. Develop a computer program to prove that,

(a) $\Gamma\left(m+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5...(2m-1)}{2^m}\sqrt{\pi}$

(b) $\Gamma\left(-m+\frac{1}{2}\right) = \frac{(-1)^m\, 2^m\sqrt{\pi}}{1\cdot 3\cdot 5...(2m-1)}$

(c) $(m+1)(m+2)(m+3) = \frac{\Gamma(3+m+1)}{\Gamma(m+1)}$

where m is any positive integer. Show results for two cases when $m=2$ and 4.

8. Plot the variations of $\Gamma(x)$ and $1/\Gamma(x)$ functions for $-5\le x\le 5$ on the same graph. Explain the relation between the two functions.

9. Plot the Beta function in two dimensions, e.g. for $1\le x\le 4$ and $1\le y\le 4$, to show that $B(x,y)=B(y,x)$. The commands given below may help to verify such property. Additional commands, such as the figure orientation, may be needed to improve the plotting clarity.

```
>> x = 1:.1:4;
>> y = 1:.1:4;
>> [X,Y] = meshgrid(x,y);
>> Z = beta(X,Y);
>> mesh(X,Y,Z)
```

10. Employ the Beta function to determine the following integrals,

(a) $\int_0^1 t\sqrt{1-t}\; dt$ (b) $\int_0^1 \sqrt{t(1-t)}\; dt$

(c) $\int_0^1 \frac{t}{\sqrt{1-t}}\, dt$ (d) $\int_0^1 \sqrt{\frac{1+t}{1-t}}\, dt$

(e) $\int_0^{\pi/2} \sin^2\theta\cos\theta\; d\theta$ (f) $\int_0^{\pi/2} \sin\theta\cos^3\theta\; d\theta$

Verify the results by comparing with those obtained from numerical integration using the `int` command.

11. Employ the `beta` command to show that,

(a) $B(x,x) = (2)^{1-2x} B\left(x, \frac{1}{2}\right), \qquad x > 0$

(b) $B(n,n) B\left(n + \frac{1}{2}, n + \frac{1}{2}\right) = \frac{\pi}{(2)^{4n-1} n}, \qquad n = 1, 2, 3, \ldots$

12. For $m = 1$ and $n = 2$, show that,

$$B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} \, dt$$

The value on the left-hand-side of the equation is obtained by using the `beta` command. The value on the right-hand-side of the equation is determined from numerical integration using the `int` command. Repeat the problem when $m = 2$ and $n = 3$.

13. Develop a computer program to determine results of the following series for x = 0.5, 1, 2 and 3. Verify the results by using the `besselj` and `besseli` commands

(a) $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots.$

(b) $J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \ldots$

(c) $I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$

(d) $I_1(x) = \frac{x}{2} + \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \ldots$

14. Use the `besselj` command for the Bessel function of the first kind of order n to show that,

(a) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$

(b) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$

(c) $J_{3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\sin x}{x} - \cos x\right)$

(d) $J_{-3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\cos x}{x} + \sin x\right)$

(e) $J_{5/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left[\left(\dfrac{3}{x^2} - 1\right)\sin x - \dfrac{3}{x}\cos x\right]$

(f) $J_{-5/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left[\dfrac{3}{x}\sin x + \left(\dfrac{3}{x^2} - 1\right)\cos x\right]$

15. Use the `besseli` command for the modified Bessel function of the first kind of order n to show that,

(a) $I_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sinh x$

(b) $I_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cosh x$

(c) $I_{3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\cosh x - \dfrac{\sinh x}{x}\right)$

(d) $I_{-3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\sinh x - \dfrac{\cosh x}{x}\right)$

(e) $I_{5/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left[\left(\dfrac{3}{x^2} + 1\right)\sinh x - \dfrac{3}{x}\cosh x\right]$

(f) $I_{-5/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left[\left(\dfrac{3}{x^2} + 1\right)\cosh x - \dfrac{3}{x}\sinh x\right]$

16. Use the `besselj` and `besseli` commands for the Bessel and modified Bessel functions of the first kind of order n to show that,

    (a) $J_{n+1}(x) = \dfrac{2n}{x} J_n(x) - J_{n-1}(x)$

    (b) $I_{n+1}(x) = I_{n-1}(x) - \dfrac{2n}{x} I_n(x)$

    when n = 1, 2, 3 and x = 0.5, 1, 2, 3.

17. Develop a computer program by using the Bessel function with `besselj` command and the modified Bessel function with `besseli` command to show that,

    (a) $\sin x = 2[J_1(x) - J_3(x) + J_5(x) - ...]$

    (b) $\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - ...$

    (c) $\sinh x = 2[I_1(x) + I_3(x) + I_5(x) + ...]$

    (d) $\cosh x = I_0(x) + 2[I_2(x) + I_4(x) + I_6(x) + ...]$

18. Use the Bessel functions with `besselj`, `bessely` and `besseli` commands to show that, when x is very large $(x >> 0)$, the Bessel functions below can be approximate by,

    (a) $J_n(x) \approx \sqrt{\dfrac{2}{\pi x}} \cos\left(x - \dfrac{n\pi}{2} - \dfrac{\pi}{4}\right)$

    (b) $Y_n(x) \approx \sqrt{\dfrac{2}{\pi x}} \sin\left(x - \dfrac{n\pi}{2} - \dfrac{\pi}{4}\right)$

    (c) $I_n(x) \approx \dfrac{e^x}{\sqrt{2\pi x}}$

19. Use the symbolic mathematics `diff` command to find derivatives of the following Bessel functions,

    (a) $J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

(b) $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$

(c) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

(d) $\dfrac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

(e) $\dfrac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

(f) $J_n''(x) = \dfrac{1}{4}\left[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\right]$

(g) $J_n'''(x) = \dfrac{1}{8}\left[J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)\right]$

20. Use the symbolic mathematics `diff` command to find derivatives of the Bessel functions for validating the following relations,

(a) $I_n'(x) = \dfrac{1}{2}\left[I_{n-1}(x) + I_{n+1}(x)\right]$

(b) $x I_n'(x) = x I_{n-1}(x) - n I_n(x)$

(c) $x I_n'(x) = x I_{n+1}(x) + n I_n(x)$

(d) $\dfrac{d}{dx}\left[x^n I_n(x)\right] = x^n I_{n-1}(x)$

(e) $\dfrac{d}{dx}\left[x^{-n} I_n(x)\right] = x^{-n} I_{n+1}(x)$

21. Use the `int` command to symbolically integrate,

(a) $\int x J_0(x)\, dx$ (b) $\int x J_0^2(x)\, dx$

(c) $\int_0^1 J_0\left(a\sqrt{x}\right) dx$ (d) $\int_0^\infty J_0(3x)\, dx$

(e) $\int_0^\infty \dfrac{J_2(3x)}{x}\, dx$ (f) $\int_0^\infty e^{-x} J_0\left(2\sqrt{x}\right) dx$

where a is a constant.

22. Use the symbolic mathematics `int` command together with the Bessel function `besselj`, `bessely` and `besseli` commands to show that,

(a) $$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, d\theta$$

(b) $$Y_0(x) = -\frac{2}{\pi}\int_0^{\infty} \cos(x\cosh\theta)\, d\theta$$

(c) $$I_0(x) = \frac{1}{\pi}\int_0^{\pi} \cosh(x\sin\theta)\, d\theta$$

when $x = 0.5$, 1, 2 and 3.

23. In a mass-spring system, the mass and spring stiffness are,

$$m = 1 \qquad \text{and} \qquad k = e^{0.02t}$$

The governing differential equation representing the mass motion $y(t)$ is,

$$\frac{d^2y}{dt^2} + e^{0.02t}y = 0$$

If the initial conditions are,

$$y(0) = 1 \qquad \text{and} \qquad \frac{dy}{dt}(0) = 0$$

solve the initial value problem above for solution in form of the Bessel functions. Plot the solution of the displacement $y(t)$ that varies with time t. Explain the physical meanings of the solution.

24. Use the `airy` command to show that, as $x \to -\infty$, then,

$$Ai(x) \cong \pi^{-1/2}(-x)^{-1/4}\sin\left[\frac{2}{3}(-x)^{3/2} + \frac{1}{4}\pi\right]$$

$$Bi(x) \cong \pi^{-1/2}(-x)^{-1/4}\cos\left[\frac{2}{3}(-x)^{3/2} + \frac{1}{4}\pi\right]$$

In the opposite way, as $x \to \infty$, then,

$$Ai(x) \cong \frac{1}{2}\pi^{-1/2}x^{-1/4}e^{-2x^{3/2}/3}$$

$$Bi(x) \cong \pi^{-1/2}x^{-1/4}e^{2x^{3/2}/3}$$

25. Solve the Airy differential equation,

$$\frac{d^2y}{dx^2} - xy = 0 \qquad\qquad 0 \le x \le 2$$

with the boundary conditions of $y(0)=0$ and $y'(0)=1$. Plot to compare the exact solution $y(x)$ with the approximate solution obtained from solving the same boundary value problem by the numerical method. The `ode23` or `ode45` command may be used in the process for obtaining the approximate solution.

26. Employ the `legendre` and `gamma` commands to validate the relation below for the Legendre function of the first kind at $x = 0$ and $n = 1$, 2, 3.

$$P_{2n}(0) = \frac{(-1)^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}$$

27. Given the Legendre function of the first kind,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Use the symbolic mathematics capability in MATLAB to find their derivatives, $P_1'(x)$, $P_2'(x)$ and $P_3'(x)$. Then, plot their variations in the interval of $-1 < x < 1$.

28. Given the Legendre function of the second kind,

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x}$$

Use the symbolic mathematics capability in MATLAB to find the Legendre function in terms of x when n = 0, 1, 2, 3 and 4. Then, plot their variations in the interval of $-1 < x < 1$.

29. Use the symbolic mathematics capability in MATLAB to show that,

$$\int P_n(x)dx = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$$

when $n = 1,\ 2$ and 3.

30. Use the symbolic mathematics capability in MATLAB to verify the orthogonal properties of the Legendre functions,

$$\int_{-1}^{1} P_\ell(x) P_n(x)\, dx = 0 \qquad (\ell \neq n)$$

and

$$\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1} \qquad (\ell = n)$$

when $\ell, n = 1,\ 2$ and 3.

31. Develop a computer program to show that the Fresnel integrals can be expressed in form of the infinite series,

$$\begin{aligned} S(x) &= \int_0^x \sin\left(\frac{\pi}{2} t^2\right) dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n+1} x^{4n+3}}{(2n+1)!\,(4n+3)} \\ C(x) &= \int_0^x \cos\left(\frac{\pi}{2} t^2\right) dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n} x^{4n+1}}{(2n)!\,(4n+1)} \end{aligned}$$

Verify the computer program when $x = -4$, -2, 0, 1, 3 by comparing the computed results with those obtained from using the `FresnelS` and `FresnelC` commands. Also compare the results with the numerical integration solutions by using the Lobatto integration method with the `quadl` command.

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