Geometry with Applications and Proofs

Advanced Geometry for Senior High School, Student Text and Background Information

Aad Goddijn, Martin Kindt and Wolfgang Reuter

SensePublishers

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Student Text and Background Information

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Contents

Geometry between application and proof, a general introduction

Aad Goddijn

About this book

The main parts of this book (I–III) make up a course in geometry for secondary education, specially designed for students in the Nature & Technology strand, which prepares students in their last three years of secondary education for studying the sciences or technology at university level.
The course was used in an experimental setting by schools in the so-called Profi project. Its goal was basically to reintroduce 'proof' in a substantial way in the Dutch secondary education curriculum in the 1990s.
This book contains the greater part of the geometry course. To restrict the size of the book, some parts were not included; we restricted ourselves to chapters, subjects and tasks which characterise the specific approach to geometry well.
Readers and educators who really love mathematics and teaching may immediately try their hand at part I–III, but some background information about the materials may be useful nonetheless. That is why we include this short general introduction, an explanation about the combination of application and proof in the educational approach and some stories from the classroom about learning to prove.

Geometry in Dutch education

Geometry in the first years of Dutch secondary education is often strongly related to realistic experiences, and is in a way highly intuition-based. Exploring spatial objects and shapes, relating different types of images of objects and situations, calculating with proportions on similar figures, some Pythagoras, computation with angles, most of the time in concrete situations, are what the focus is on. Discussion is elicited, but remains in general situation-dependent and the abstraction level is quite low. Understanding the physical world with the help of some basic mathematical tools is the main goal. It is a geometry for daily life, also

preparing well (or at least not too badly) for vocational schooling and practical jobs.
This junior high school geometry curriculum was developed in the early nineties, as a counterbalance for the then prevailing boring geometry tasks. Currently there is a tendency to move part of this intuition-based (and intuition-stimulating) type of geometry to its proper place in primary education; an outline of possible goals and learning-teaching trajectories has been published recently by the Freudenthal Institute. An English translation is available.[1]
In the same period many teachers, especially at schools of the traditional 'gymnasium'-type (similar to 'grammar schools' in the UK, streaming for university), liked to include more and more thought-provoking and proof-related problems in their geometry teaching, in the certainty that their students can do more intellectually than the curriculum seems to allow.
The present geometry course for upper secondary education in this book fits very well in this environment: in the course, a good orientation base of intuitive insight in geometry will help in becoming familiar with the more formal demands of mathematical proofs. We tried to link the intuitive and formal approaches without mixing them up, including clarifying for the students what the differences are. In this respect the first chapter (*Geometry, classical topics & new applications,* by Martin Kindt) is instructive. Important moments in the course around this theme can be found in chapter 2 of part I, *Reasoning with distances*, and chapter 1 of part II, *Using what you Know*.

Mathematical contents of the course

Part II (*Thinking in circles and lines*) is the closest approach the course offers to *The Elements* of Euclid; the title can be read as a reference to the well-known 'ruler and compass'. But no attempt has been made to cover the first six books of *The Elements*, where the traditional secondary education geometry subjects have their origin. The course focuses strongly on distance and angle related subjects; proportion, similarity and area share a relatively low exposure in the text of this part.
This is clear from the very beginning of part I, (*Distances, Edges & Regions*) where a famous – rather modern – division of a plane area is introduced. The division is natural in situations where there a finite number of points in the area and comparing distances to this points is important in an application. The main idea is the so called *Voronoi diagram*. Voronoi diagrams are used in many sciences today, from archeology to astronomy and medicine. Basic geometrical

1. Marja van den Heuvel-Panhuizen and Kees Buys (Eds.) . *Young children learn measurement and geometry.* Sense Publishers, 2008.

ideas like perpendicular bisectors, distances, circles spring up here almost by themselves.

Other distance-related subjects can be found in part I too, for instance the so-called iso-distance lines around regions. An example is the famous 200 mile fishery exclusion zone around Iceland. Distance optimisation of routes in diverse situations, with some attention to the Fermat principle, concludes part I.

In part I as a whole one may see a gradual road from application-oriented problems to more pure mathematical thinking. But in this part, the choice of problems is not yet guided by systematic mathematical deduction.

That changes in part II, by far the most ‘pure’ part of the course. Several ideas which originated in the distance geometry of part I, are taken up again in a systematic way. Circles and angles (midpoint and peripheral) play a part in determining Voronoi diagrams, as do special lines like perpendicular and angular bisectors. In part II they will be placed in their proper mathematical environment, an environment ruled by clarifying descriptions and organized argumentation, where one is supposed to use only certain statements in the process of argumentation. This typically mathematical way of handling figures and their relations has its own form of expression: the style of definitions and proofs.

A proof should not be a virtuoso performance by a gifted teacher or student on the blackboard in front of the silent class. It should ideally be found and formulated by the students themselves. This is a heavy demand, which is why, in part II, we pay a lot of attention to the problem of finding and writing down a proof. This also requires some reflection by the students on their own thinking behavior. In the current Dutch situation, 16 and 17 year old students are involved; with them, such an approach can be realized a lot easier than with the students who traditionally read Euclid already at 12 or 13 in the not so remote past.

Important in part II is the stimulating role of Dynamic Geometry Software. We will spend a separate paragraph on it.

Part III (*Conflict lines and Reflections*) is also connected with part I. A conflict line[2] of two separate regions A and B is the line consisting of the points which have equal distances to the regions A and B. It is in a way a generalization of the Voronoi diagram concept of part I and of the perpendicular bisector of two points. The division of the North Sea between Norway, Germany, Holland and Great Britain is a starting example; an important one, because of the oil deposits in the bottom of the North Sea. Later we specialize for simple regions like points, lines and circles. The conflict lines turn out to be good old ellipse, parabola and hyperbola, as characterized by Apollonius of Perga around 200 B.C. Their

2. Or conflict *set*. In the course all of them have the character of lines, so there was not much reason to use the term ‘set’.

properties are studied with distance-based arguments and again with DGS: tangents, directrices, foci and reflection properties.
One of the deep wonders of mathematics is that, as soon as you have clarified your concepts, say from conflict lines in the North Sea to mathematical ones like ellipses and parabolas, those newly constructed ideal objects start to generate new applications by themselves. In part III several old and new acoustic and optical applications of conics are taken in.
In the current 2014 edition of this book an extra chapter focuses on examples of analytic geometry related to distance geometry. This part is specially aimed at connecting analytic and synthetic methods in this field. It is not an independent analytic geometry course. The chapter was not included in the Dutch curriculum in the 1990s, as analytic geometry was put 'on the back burner' for about twenty years.

A short note on axioms and deduction

Euclid started *The Elements* with:

Definition 1: A point is that which has no parts.

We do not. We start in the middle, where the problems are. Therefore the practice of the Voronoi diagram is used to start building arguments in two directions; downward, looking for basic facts to support the properties of found figures, and upward, by constructing new structures with them. The two directions are called, in the words of Pappos, *analysis* and *synthesis*. Euclid on the other hand, and many of his educationally oriented followers, presented mathematics as building upward only. This course, especially in chapter 2 of part I, indicates that we see axioms also as objects to discover or to construct, not as dictated by the old and unknown bearded Gods of mathematics. A anecdote from the classroom may clarify this:

> *At a certain moment, triangle inequality was introduced as a basic underpinning of the distance concept. It expresses the shortest-route idea very well. "Let us agree about the triangle inequality, we will use that as a sure base for our arguments!" But no, this was not accepted by everybody in the class. A group of three students asked: "Why the triangle inequality and not something else?"*
> *I countered: "Well, it is just a proposal. By the way, if I propose something else, you will again probably have objections too, don't you think?" They agreed with that. So I asked: "What would be your choice?" After a few moments they decided on 'Pythagoras' as a basic tool to argue about distances. I said, "That's okay, but there is a problem here: can you base the triangle inequality on 'Pythagoras'?"*

Ten minutes later they called me again. Yes, they could, and they showed a proof!

The main point of this little story is not the debate over what is an axiom and what is not. It is that theses students were actively involved in building up the (or 'a'?) system itself.
It took the mathematical community over two thousand years (from before Euclid to after Hilbert) to build a safe underpinning for geometry in a fully axiomatic-deductive way. It is an illusion to think we can *teach* students such a system in a few lessons. But we can make them help to become part of the thinking process. Probably we will get no further than local organization of some theorems and results by this approach in secondary education. But we did reach a cornerstone of mathematics in the anecdote above: building by arguments, actively done by yourself.

Dynamic geometry software

Part I of the course includes a computer practical; a program which can draw Voronoi diagrams for a given set of point is used. Such a program allows us to explore properties of this special subject in a handy way. The program that was used in the experiments in the nineties is difficult to run on modern computers running Windows Vista, 7 and 8. It had the advantage that prepared point configurations with some hundreds of points could be read in from a file. On the internet several modern and (a little bit) easier to handle applets are available; maybe you will not find one which can read point sets from file but in the text of this edition we supply drawings of some larger examples to work with on paper.
In part II, Dynamic Geometry Software is used with a different purpose in mind.[3]
One of the initial exercises was the following. On the screen, draw a circle and a triangle ABC with its vertices on the circle. Construct the perpendiculars from A and B to BC and AC and call the intersection H. Now move C over the circle. The lines BC and AC, the perpendicular and H will join the movement. Let H leave a trace on the screen during the movement. Big surprise: the trace looks like a circle with the same size as the original one. The student sitting beside me in the computer lab, after lazily performing this construction act: "I suppose we have to prove this?" "Yes, yes, that's exactly the point. You have to prove your own conjectures, which you found yourself while working with the program!"
After finding conjectures on the screen, the question arises again: how to find proofs. A DGS-program does not give any direct hints, it only supplies a adaptable drawing

3. For this 2014 edition we adapted tasks requiring DGS from the original *Cabri Géomètre II* to *Geogebra*.

and offers measuring tools. But especially the changeability and animation of the figures supply all kinds of clues to the careful observer. In the above case you may see for instance that point C is always above H (if you made AB horizontal) and that CH looks constant. So if you can *prove* that, you are not very far from your goal. Looking for constant elements in an animated construction turned out to be a very good heuristic in finding proofs.

The aftermath of the Profi project

The Profi project was performed in close collaboration by teachers and a group of designers at the Freudenthal Institute and overseen by a committee of university researchers. Experimental textbooks were designed, tried out in class and improved. In a later phase, elaborate textbooks were produced by commercial editors, which is the usual approach in the Netherlands. Many aspects of the experimental textbooks illustrated the underlying ideas (which are in a way an upper secondary education elaboration of the theory of Realistic Mathematics Education) much more clearly than the commercial books do, but on the other hand – it should be said also – the commercial books are often better geared to the daily organizational problems faced by the common teacher and student.

Other activities in the Profi project were the design of the so called ‘project tasks’. These are intended for individual or team-use by students, to help them do some independent mathematical research, related to a real life or purely mathematical problem situation. In many cases, students themselves chose DGS as a tool in those tasks.

Each year the Mathematics B-day is organized in the Netherlands, for this group of students. It is a nationwide (recently international) team competition on one day. Students attack (in teams of four) such a problem situation, send in their results, hoping for the honour to be one of the best teams and getting a small prize. The enthusiasm is overwhelming and the number of teams involved is still growing. Students themselves commented that working in-depth on one problem for a longer period is very stimulating for them. Their sound view is not supported by current antididactical trends in education, where subjects are often split up in small digestible bits and mathematics as the activity of building structures disappears almost totally out of sight.

Shortly after the Profi project ended, a major organizational change was introduced in upper secondary education in the Netherlands, the much debated ‘New Second Phase’. Students were supposed to become overnight independent learners, teachers should lay down their supposed superior role of educator in front of the classroom and become counsellors; regular classroom situations were diminished heavily in time.

The contribution 'Circle and Butterfly' in this introductory part of the book is a informal report about a regular (but small) class, wrestling with the notion of proof. Our viewpoint there is very clear: learning to prove goes very well by communicating arguments in a debate in the traditional classroom, based on provoking problems; independent student work may be part of the process. The teacher is also a sparring partner in the debate and a guide to help students get some order in their arguments.

The authors

Wolfgang Reuter was one of the teachers involved in the Profi project. At important moments, he put the other designers of the course with their feet on the ground where the students are. His contribution is visible in the careful working out of some task sequences in part I and III. A year after the project ended, Wolfgang died suddenly. Almost all his students came to the funeral. Some of them told moving stories about the way he worked with them.
Martin Kindt did not only design parts of the geometry course for the project, he also gave shape to the calculus course of the project, which breathes the same air. Part of this course is available in a Swiss edition.[4]
Aad Goddijn has been involved in many curriculum development projects in mathematical education in both lower and upper secondary education.

4. '*Differenzieren - Do it Yourself*' (ISBN 3-280-04020-5; Orell Füssli Verlag, Zurich). The translated title is in line with what is said above, but beware: the book is in German.

Geometry, classical topics and new applications

Martin Kindt

This chapter was previously published in: Developments in School Mathematics Education Around the World, Proceedings of the 4th UCSMP International Conference on Mathematics Education, Chicago 1999.

Geometry is a Greek invention, without which modern science would be impossible. (Bertrand Russell)

Modeling, abstracting, reasoning

The meaning of the word 'classical' depends on the context. The classic interpretation of 'classical geometry' is 'Greek geometry', as described by Euclid. In his work *History of Western Philosophy*, Bertrand Russell expressed his admiration for the phenomenal achievements of the scientific culture in Greece. Maybe the above quotation gives a sufficient reason why some part of Euclidean geometry should be taught today and in the future and in the future of the future... To be honest, I must say that Russell is also critical about the Greek approach; he considers it to be one-sided. The Greeks were principally interested in *logical deduction* and they hardly had an eye for *empirical induction.*

Lately I found a lovely booklet, *Excursions in Geometry* by C. Stanley Ogilvy. The first sentence of the first page says: *What is Geometry? One young lady, when asked this question, answered without hesitation, "Oh, that is the subject in which we proved things". When pressed to give an example of one of the 'things' proved, she was unable to do so. Why it was a good idea to prove things also eluded her.* The book was written in 1969. If it were written in 1996 a young lady in my country, confronted with Stanley Ogilvy's question, would perhaps answer: "Oh, that's the subject in which my daddy told me that he had to prove things."
Stanley Ogilvy very rightly observes that the traditional method of geometry education failed. The things to prove were too obvious to inspire students, the system was too formal, too cold, too bare. In the late sixties, when he published his geometrical essays, the Dutch curriculum more or less skipped the Euclidean approach. Alas, the alternatives such as 'transformation geometry' and 'vector

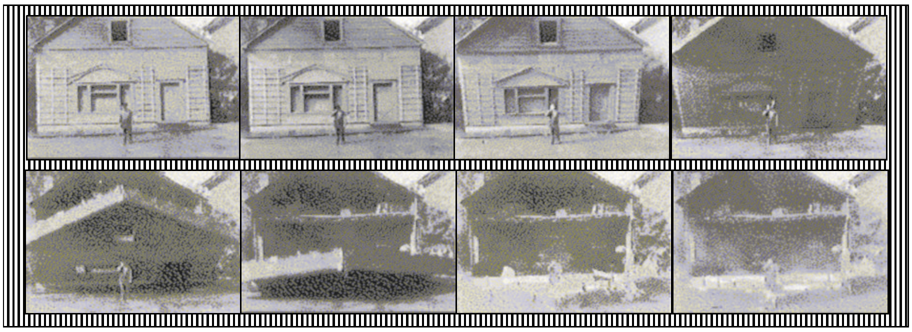

fig. 1. Buster Keaton in Steamboat Bill Jr.

geometry' did not fulfil the high expectations. Proofs disappeared gradually, the system (if there was one) was not clear for the students.

Back to the classics in a wider sense. The movies of Buster Keaton may undoubtedly be considered as classics.
I remember the famous scene in which he is standing, with his back turned, in front of a house just when the front is falling over, see fig. 1, or better: YouTube. It was a miracle; Buster was standing in the right place, where the open window of the roof landed. The brave actor didn't use a stand-in. Could it be because Buster had an absolute confidence in geometry? Indeed, with geometry you can exactly determine the safe position!
Make a side view as in fig. 2. The segment *AB* represents the rectangle on the ground where the falling window will end up.

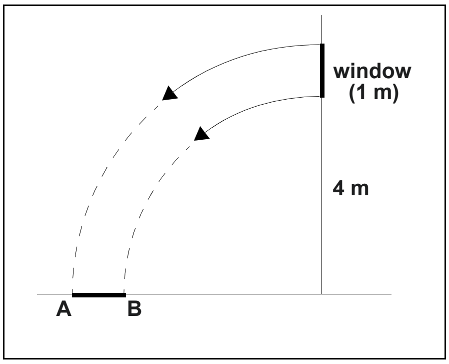

fig. 2. Side view.

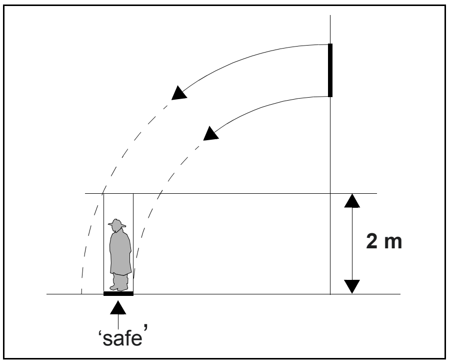

fig. 3. Safe area.

But is the whole of the rectangle a safe area? Of course not! A man has three dimensions and you have also to take height into account. Fig. 3 shows the side view of the safe area. The Buster Keaton problem gives a good exercise in *geometrical modeling* for young students:

– they have to translate ‘fall over’ as ‘rotate’;
– they have to interpret a side view;
– they have to be aware of the three dimensions of the person (especially of his height);
– they have to combine things, to reason, but... there is no need for formal proof.

An interesting question to follow this is: ‘Could the scene be made with a giant?’

As a second example of geometrical modeling I will take the story of a fishery conflict between England and Iceland (in the 1970s). England had a big problem with the extension of the Icelandic fishery zone from a width of 50 miles to a width of 200 miles. In the newspaper we found the picture in fig. 4.

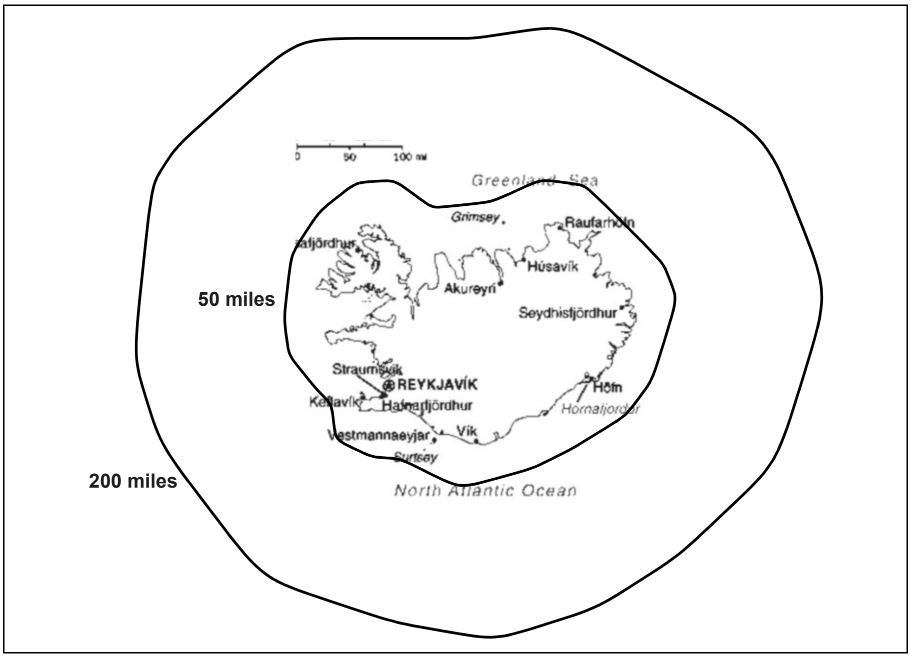

fig. 4. Iceland fishery zones.

The picture is not only provoking in a political sense, but also geometrically! For instance one can wonder:

– how to measure distances to an island from a position at sea (or vice versa)?
– how to draw the so called *iso-distance-curves*?
– why is the shape of the boundary of both zones rather smooth compared with the fractal-like coast of Iceland?
– moreover; why is the 200 miles curve more smooth than the 50 miles one?

These are typical geometrical questions to investigate. I restrict myself now to the first two questions.
How to determine on a map the distance between an island and an exterior point? You should give this as a, preferably open, question to students (of age 15 for instance).

At some point they will feel the need for a definition. Let them formulate their own definition! After a discussion the class will reach an agreement.
For instance: *the distance from an exterior point to the island is the length of the shortest route from that point to the coast.*
This definition is descriptive, not constructive. It does not say how to find the distance, how to determine the shortest route. A primitive way is to measure some routes departing from a given point *P*. In most of the cases you can quickly make a rather good estimation of the nearest point, without measuring all the distances (if... you don't have too bad an eye for measurements).
More sophisticated is the method using circles. The 'wave front' around *P* touches the island once; the smallest circle around *P* which has at least one common point with the island determines the distance, as shown in fig. 5.

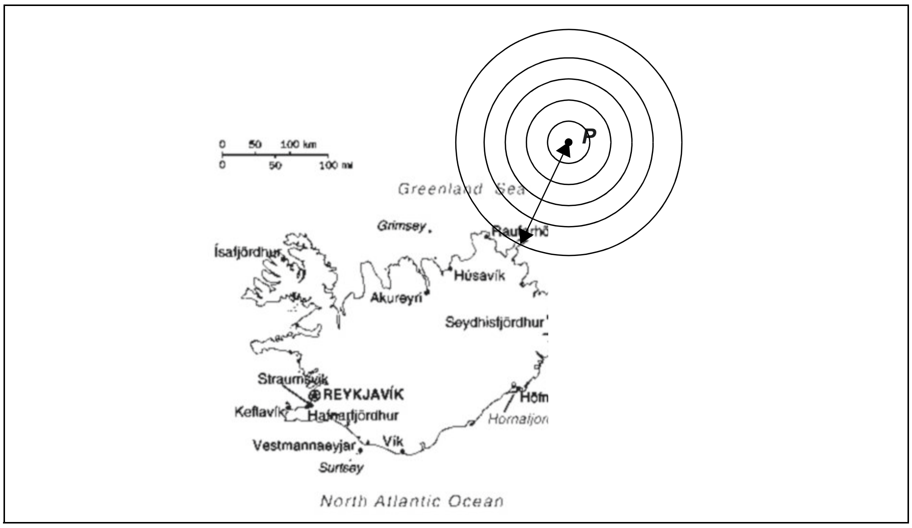

fig. 5. Smallest touching circle.

From this idea, the step to the strategy of drawing an iso-distance curve by means of a rolling circle is not a big one. See fig. 6.
Remark: there is an interesting alternative approach of drawing the iso-distance line. The curve arises also as the envelope of the circles with a fixed radius and their *centers* on the boundary of the island.

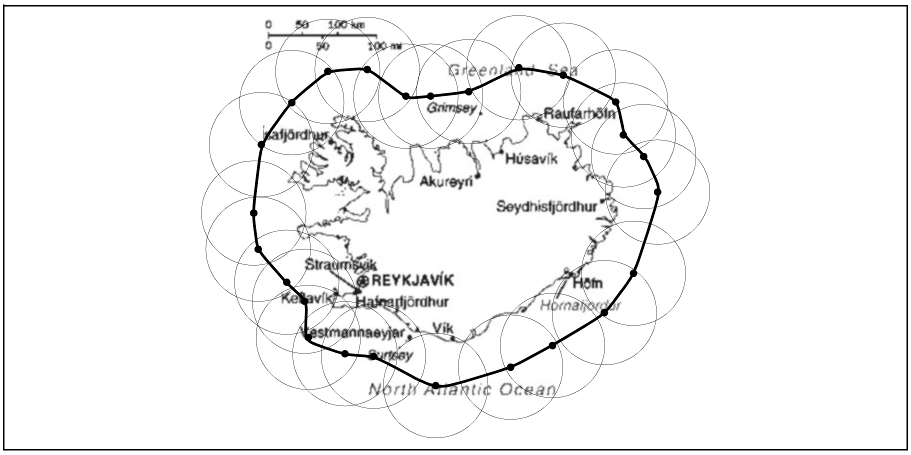

fig. 6. Circle rolling around Iceland.

Did the Greek geometers have no eye for the aspect of geometrical modeling? They certainly did; Euclid for instance wrote a book about optics ('vision geometry'). But they differentiated between pure mathematics (the geometry of the philosopher) and practical mathematics (the geometry of the architect). There is an interesting dialogue in Plato between Socrates and Protarchos about the two types of mathematics.

One of the characteristics of the philosophy of Hans Freudenthal is a complete integration between mathematics of real life and so-called pure mathematics. Mathematizing is an activity within mathematics.

In the Iceland case, the fishery conflict can be a good starting point to develop a theory about iso-distance curves of simple geometrical shapes like a quadrangle, to study the difference between convex and not-convex shapes and to make local deductions. There are also opportunities to link this subject with calculus. For instance, it is easy to understand geometrically that in the case of an island with a 'differentiable boundary', the shortest route from an exterior point to the island has to be perpendicular to the boundary; see fig. 7.

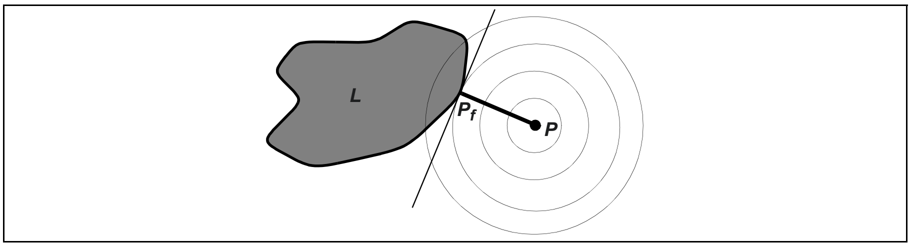

fig. 7. Shortest route is perpendicular to boundary.

Indeed, the circle which determines the distance has a common tangent line with the boundary of the island *L* and the tangent of the circle is perpendicular to the line segment PP_f.
In our standards for the math curriculum on pre-university level, the following three important aspects are mentioned:[1]
- *Modeling*: the student will get insight in the coherence between a mathematical model and its realistic source;
- *Abstracting*: the student will learn to see that a mathematical model may lead to an autonomous mathematical theory in which the realistic source disappears to the background;
- *Reasoning*: the students will learn to reason logically from given premises and in certain situations, will learn to give a mathematical proof.

The Dutch geometry curriculum

The Buster Keaton problem fits very well in the curriculum for the age group 12-15 ('geometry for all'), it is an example of 'localization', which is one of the four strands:
1. *Geometry of vision* (about vision lines and vision angles, shadows and projections, side views and perspective drawings);
2. *Shapes* (two- and three-dimensional);
3. *Localization* (different types of coordinates, elementary loci);
4. *Calculations in geometry* (proportions, distances, areas, volumes, theorem of Pythagoras).

As characteristics of this 'realistic geometry' I will mention:
- An intuitive and informal approach;
- A strong relationship with reality;
- No distinction between plane and solid geometry, everything is directed at 'grasping space'.

The Iceland problem can be extended to a rich field of geometry which I will call here 'geometry of territories'.
It fits very well within the new geometry curriculum envisaged in the nature and technology profile of pre-university level (age 16–18).
From 1998 we distinguish four profiles in the Dutch curriculum:
- *Culture and Society*;
- *Economy and Society*;

1. Freudenthal respectively used the terms 'horizontal mathematization, 'vertical mathematization' and 'local deduction'.

– *Nature and Health*;
– *Nature and Technology.*

In each of the four profiles mathematics is a compulsory subject, but only in the fourth profile is geometry a substantial part of the curriculum (along with probability and calculus). Over the last three years, we developed (and experimented with) a new program for the 'Nature profiles', with attention to:
– the relationship between mathematics and the subjects of the profile (physics, chemistry, biology);
– the mathematical language (how specific should it be?);
– the role of history (mathematics was and is a human activity);
– the use of technology (graphic calculator, software such as Derive or Geogebra);
– the ideas of horizontal and vertical mathematization, local deduction.

I will focus on the geometry part here. We chose the following three strands:
1. *Classical metric plane geometry (especially: loci based on distance and angle)*;
2. *Conic sections (synthetic approach)*;
3. *Analytic geometry (elementary equations of loci).*

The most important contextual sources in the new program are:
– *Territories (conflict lines and iso distance curves)*;
– *Mirrors (focus, normal, tangent)*;
– *Optimization (shortest path, minimal angle).*

Involving as main activities:
– *Exploring (using computers)*;
– *Modeling*;
– *Proving (local deduction).*

Geometry of Territories

The North Sea is divided in national territories. A point in the sea which is equidistant from England (GB) and the Netherlands (NL) is called a *conflict point* of both countries. All possible conflict points form a *conflict line* (or conflict curve or set).
The boundaries of the national territories at sea are parts of conflict lines. Studying a map, it is noticeable that there are 'three nation points'. For instance, there is one point which is equidistant from GB, NL and DK (Denmark). Students can reason why: the intersection of the conflict lines (GB, NL) and (NL, DK) is a point which is on the one hand equidistant from GB and NL, and on the other equidistant from

NL and DK. Conclusion: the point is equidistant from GB and DK (following the first common notion of Euclid: *things which are equal to the same thing are also equal to one another*). So the intersection point has to be a point of the conflict line of GB and DK. This is a well-known scheme of reasoning; Polya speaks about *the pattern of two loci.*

To simplify things I will study the territories of five small islands (say points) in the ocean, as in fig. 8.

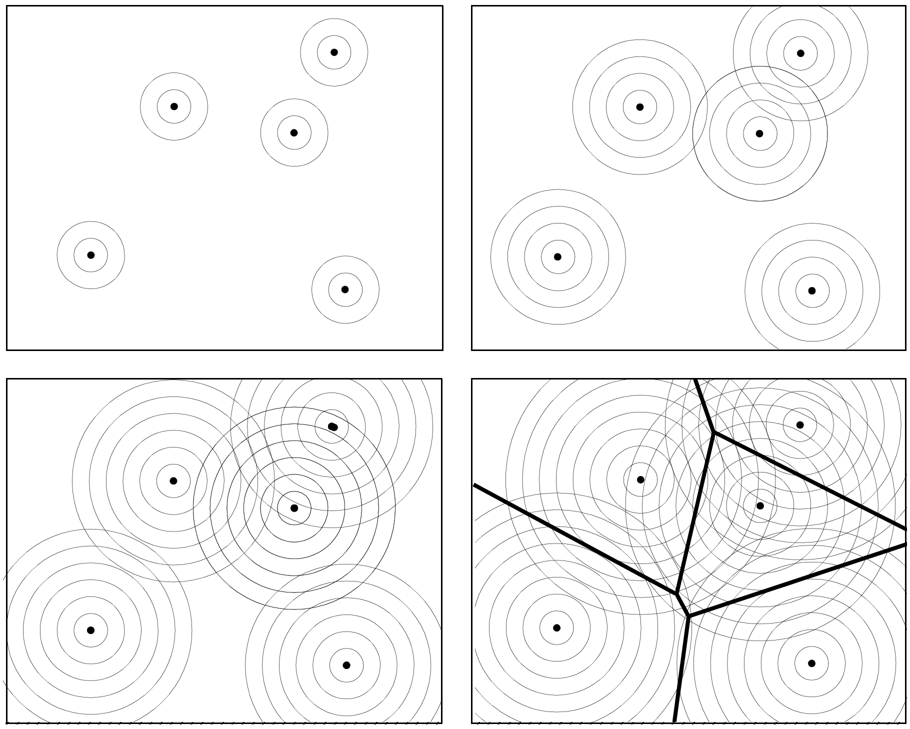

fig. 8. Five small islands and meeting wave fronts.

Where the 'wave fronts' around two islands meet each other, we have a conflict line. In this case the conflict lines are *perpendicular bisectors*. The fourth picture, without the circles is called a Voronoi diagram. The territories are called Voronoi cells. A boundary between two adjacent cells is called an *edge*. Three edges can meet in one point ('3 nations point'), such a point is called a *vertex*. The 'islands' are the *centers* of the diagram.

Voronoi diagrams (also called 'Thiessen polygons') are applied in a lot of disciplines: archeology, geography, informatics, robotics, etc. There is software that makes complicated Voronoi diagrams on the computer screen and students can do a lot of explorations. After some lessons they have a rather good idea of this concept, and then we ask them typical 'sophistic' questions (fig. 9).

– *Why can we be sure that the boundaries are straight lines*?
– *Why do three boundaries sometimes meet in one point*?

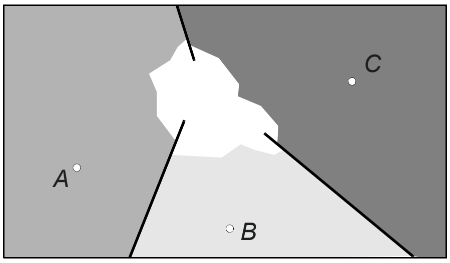

fig. 9.

Analyzing these questions we find two important reasons:
1. Every point of the perpendicular bisector has *equal* distances to the two points;
2. Every point not on the perpendicular bisector has *unequal* distances to the two points. The direction of the inequality depends on the side of the boundary.

How to prove 1 and 2?

In the past we used the congruence of triangles (the case SAS). In the age of transformation geometry we used the basic principle of reflection in a line. Our students, who have grown up with realistic geometry, proposed using the theorem of Pythagoras. It is worthwhile discussing these things. You can still ask 'why', but at a certain point you have to choose starting points; we call them *'basic rules'*. A powerful basic rule is the 'triangle inequality' from which it follows that a point on the same side as *A* of the perpendicular bisector of *A* and *B*, belongs to the territory of *A*, see fig. 10.

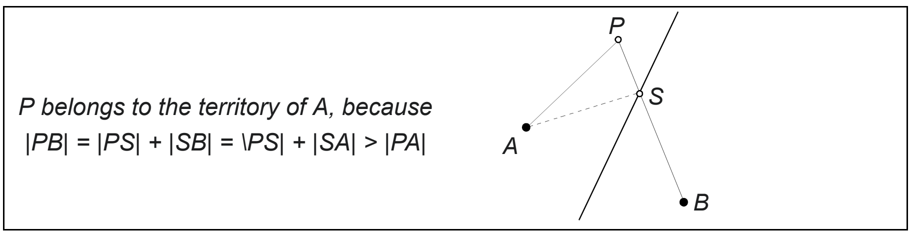

fig. 10. Why P belongs to A.

With the students we compare two directions of thinking:
In the traditional approach of geometry education we only followed the *logical path*. That was one of the big didactical mistakes. It is important to show the students (of all levels) the genesis of a piece of mathematics from time to time: *the path of exploration* (fig. 11). Often the history of mathematics is an excellent

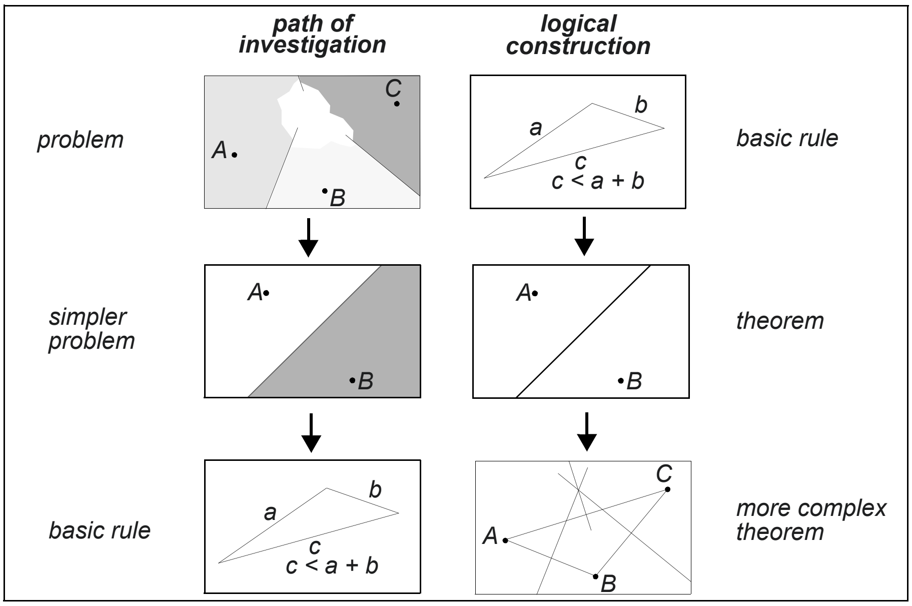

fig. 11. The two paths.

source of inspiration, but in this case I took a rather modern subject, which is a really rich one. The Voronoi theme gives rise to a lot of problems to investigate.

A few examples:
1. Given four points. Make a classification of all types of Voronoi diagrams.
2. Given four points. One point moves along an arbitrary straight line. How does the Voronoi diagram change? (see fig. 12)

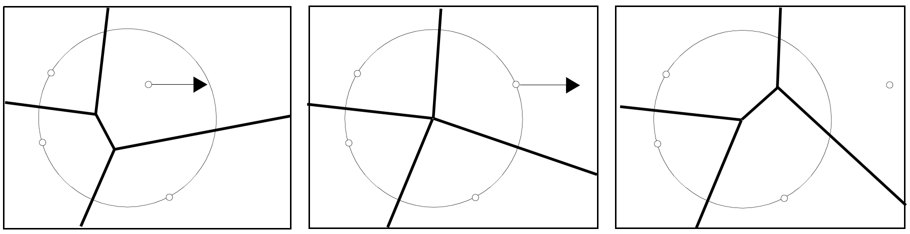

fig. 12. The influence of a moving fourth point.

3. Given three Voronoi edges, meeting in one point. Can you reconstruct the centers? How many possible solutions are there?
4. Four-nations points are very rare[2]. Can you find a criterion for a such a point'? Is

2. We looked for four-nations points in an atlas; only on the map of the USA did we find one: the common vertex of Utah, Colorado, Arizona and New Mexico.

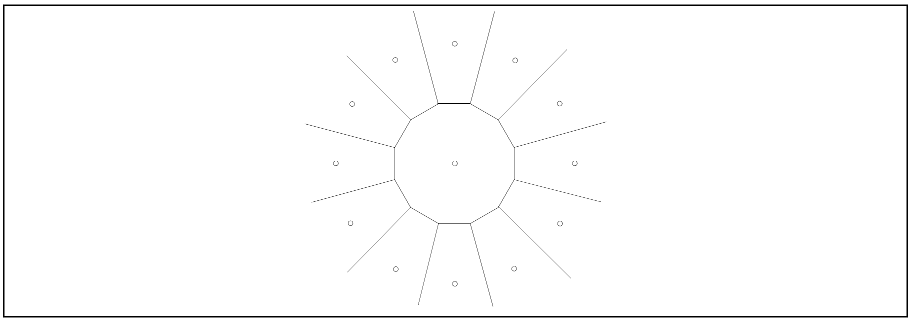

fig. 13. Circle and center.

it possible to formulate this in terms of angles?
This leads to the concept of cyclic quadrangle and the theorem of opposite angles.

5. Study the Voronoi diagrams of regular patterns. For instance: twelve points regularly lying on a circle give a star of rays. If you add a new center (the center of the circle, fig. 13) a regular polygon arises. Why?
   How does the shape of the polygon change if the center moves to the 'north'?

The discrete parabola

Do the same as in example 5 above with a row of equidistant points (fig. 14) the Voronoi diagram consists of parallel strips. If you add one new center outside the row, we get an interesting figure:

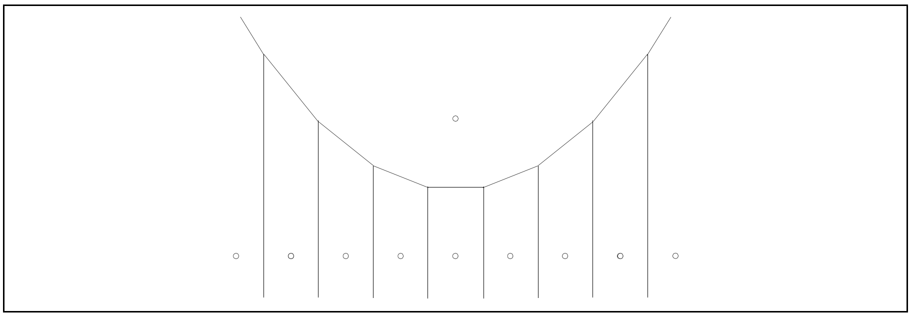

fig. 14. A special shape.

You could call a part of this pattern a 'discrete parabola. If we interpolate the row with more and more points the polygon will tend to a parabola!
In this case we get the continuous version of the Voronoi diagram if we study the conflict line between a straight line ('coast') and a point ('very small island off the coast'). Take an arbitrary point on the coast line and draw the perpendicular bisector between the island and this point. With Geogebra you can move the point

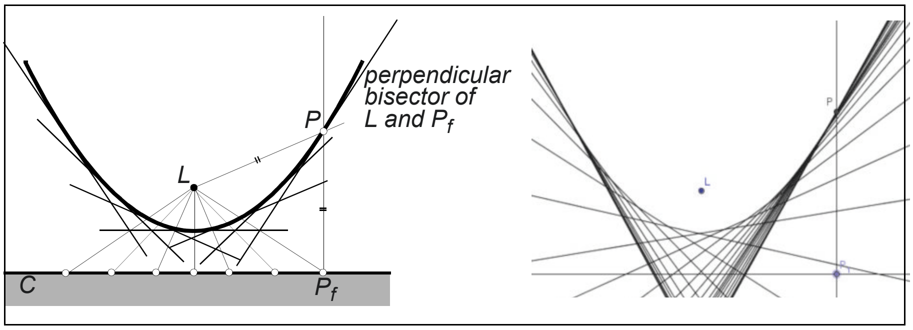

fig. 15. Left: construction. Right: screenprint Geogebra.

along the coast and see how the perpendicular bisector envelopes a curve (use the Trace-option on the object). The conflict line of island and coast is by definition a parabola and now the students spontaneously discover the property of the tangent of a parabola (see fig. 15), which leads to important technical applications (parabolic mirror, telescope).
In fig. 16 a point Q outside P is drawn on the perpendicular bisector of L and the foot of P (P_f).
If a ship is at the position Q it is clear that it is nearer to the coast than to L (for Q has equal distances a to L and P_f, and because Q is outside of P this distance is longer than the shortest route to the coast (= b)

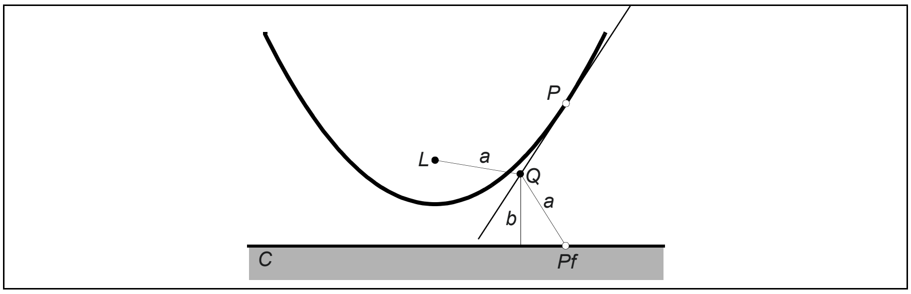

fig. 16. Proof of the tangent property.

So every point from the perpendicular bisector, except P, belongs to the territory of C and this means that you can say that the line is a tangent of the parabola. Analogously we find a hyperbola or an ellipse in the case where the coast line is a circle, see fig. 17.
You can deduce that $d(P, M) - d(P, L)$ the left picture and $d(P, M) + d(P, L)$ in the right picture are constant (namely always equal to the radius of the circle) and now you know that the conflict line is respectively (a portion) of a hyperbola and an ellipse.
For the students we used the concept of conflict line to introduce both types of

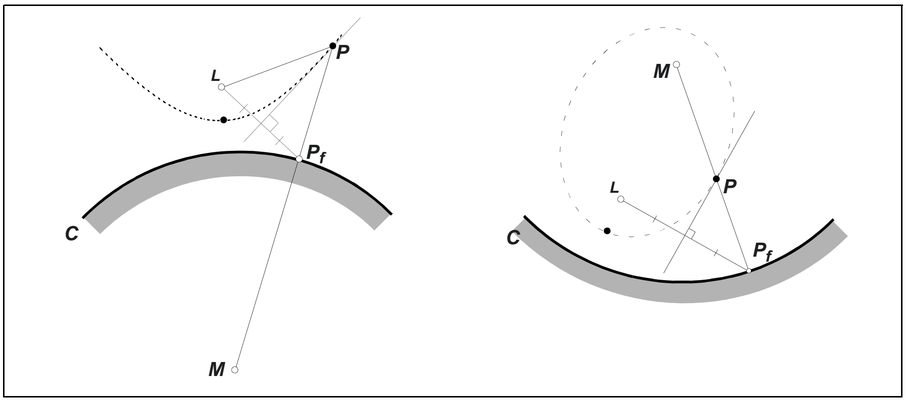

fig. 17. Circular region.

conic section and we confronted them afterwards with the classical definitions. In both cases the perpendicular bisector of L and P_f is the tangent.
If the point L is substituted by a circle (with a radius smaller than the radius of C), we get the same results.
Now there is a 'world' of nice exercises about conflict lines and parabolic, hyperbolic or elliptic mirrors.

Some conclusions

After three years of experiments with students of age 16–18, we are very hopeful that the new geometry curriculum can be realized in a motivating way.
Our experience taught that:
- the students felt challenged by classical geometrical problems, ... provided that these are either introduced by meaningful contexts or are discovered by empirical activities;
- sometimes the students are more critical with proofs than the teacher;
- the use of dynamic geometry software is really a success; students enjoy the dynamic character and they don't have difficulties with managing the program;
- students are aware of the uncertainty of a discovery by means of the computer; they experience a need to prove non-trivial results.

The geometry stuff is a lucky mixture of 'old fashioned' geometry about circles and conic sections and new applications (Voronoi diagrams). Using new technology makes things much more accessible.
While the students for whom this stuff is meant, are much more mature than the students who were confronted with classical Euclidean geometry in the past, making geometrical proofs is attainable. On the other hand, these older students

are less disciplined than the younger ones from the past and this may be a problem when presenting proofs. This last point seems to be the most difficult one. But remember the words of Stanly Ogilvy: *to avoid the catastrophe of an uninspired and uninspiring geometry course we will beg the forgiveness of the mathematicians, skip the formalities and take our chances with the rest.*

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Given: circle with butterfly
or: how do you learn proving?

Aad Goddijn

What came before?

The advanced geometry of mathematics B2-VWO explicitly contains the subject 'proving in plane geometry'. This article wants to give an impression of how this could work in a classroom. The class involved this article is a small 6 VWO-B group from the Gregorius College in Utrecht (1999); the school belongs to the group of ten schools which have been working with the experimental material of the profi-team. Advanced geometry starts with the book *Distances, edges and domains* (see bibliography). This book proceeds gradually from several applications of the concept of distance – among them Voronoi diagrams, iso-distance lines and optimization problems – towards making proving more explicit. The next part, 'Thinking in circles and lines', explains explicitly what a proof is, what you can use in one and how to write one down. The new geometrical material in this part is really geared to the previous book; the theorem of the constancy of the inscribed angle on a fixed arc and the theorem of the cyclic quadrilateral are important. Since these will play an important role in the examples used later on, let us present them in an illustration, see fig. 1.

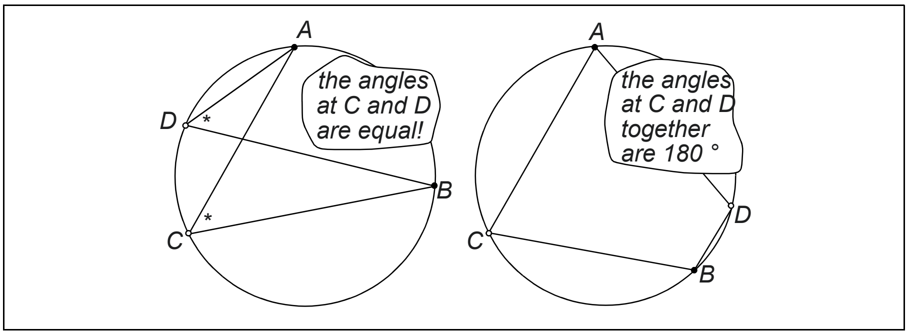


fig. 1. Inscribed angles on arc AB of cyclic quadrilateral ABCD

The constant angle theorem says that if *A*, *B*, *C* and *D* lie on one circle and *C* and *D* lie on the same side of line *AB,* then $\angle ACB$ and $\angle ADB$ are equal.

The twin of this theorem is the theorem of the cyclic quadrilateral. This says: if *A, B, C* and *D* lie in one circle and *C* and *D* lie on different sides of *AB*, then the angles at *C* and *D* are 180° together.
With such building blocks a lot can be done in numerous proofs. Next is an example of what the learning of proving could look like in this stage.
Karin, one of the students in this class, shows in her notebook, fig. 3, that there is something special about the wings of the butterfly shown in fig. 2.
In the proof, 'angles on the same arc', that is the named constant angle theorem, has been referred to twice. The idea behind the proof is good, but the execution is not yet perfect: this is called similarity rather than congruence and two pairs of equal angles is enough.
In the course of the learning process solutions and usage of terminology become more accurate and better written.

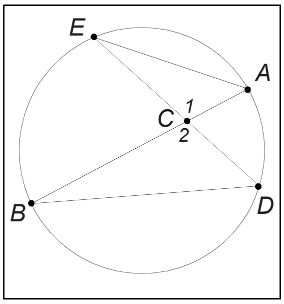
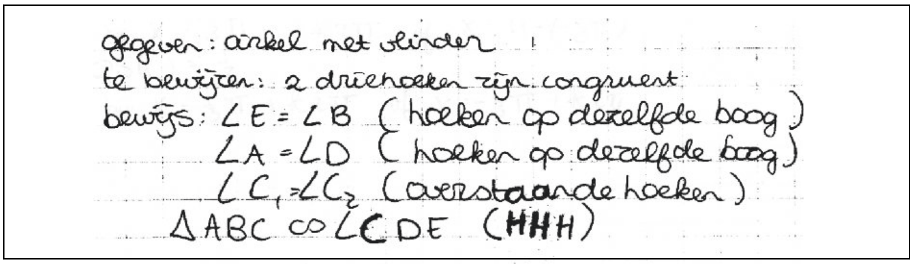


fig. 2.



gegeven: cirkel met vlinder
te bewijzen: 2 driehoeken zijn congruent
bewijs: $\angle E = \angle B$ (hoeken op dezelfde boog)
$\angle A = \angle D$ (hoeken op dezelfde boog)
$\angle C_1 = \angle C_2$ (overstaande hoeken)
$\Delta ABC \cong \angle CDE$ (HHH)

fig. 3.

This needs to be worked on in class, but this is not the key point when it comes to learning to prove. The real problem for Karin and her classmates Sigrid, Janneke, Bas, Mark, Monica, Marleen and Petra is: how do you *find* a proof in a still unfamiliar situation? And for their teacher Marcel Voorhoeve: how do I help them finding proofs by themselves? The second half of 'Thinking in circles and lines' deals especially with this search – and learning how to search – for proofs.

Form as tool

A beautiful proof is like a good sonnet: form and content support each other. Example exercise 1 explicitly asks for a proof in a certain format, which has been seen before. Part c refers back to the proof of the concurrency of the three perpendicular bisectors of a triangle *ABC*. Briefly, the proof goes like this. Let the perpendicular bisectors of *AB* and *BC* intersect in *M*. Then d*(A,M)* = d*(B,M)* holds and also d*(B,M)* = d*(C,M)*. Connect the equalities and you have that d*(A,M)* =

d*(C,M)*. From there, it also follows that *M* lies on the perpendicular bisector of *AC*.The characterization by equal distances of the perpendicular bisector is used, first twice from middle-and-perpendicular to equal distances and then after the connection step once from equal distances to middle-and-perpendicular. The students know this as the 1-1bis form. This form has been assimilated in a scheme (fig. 4) in 'Distances, edges and domains'.

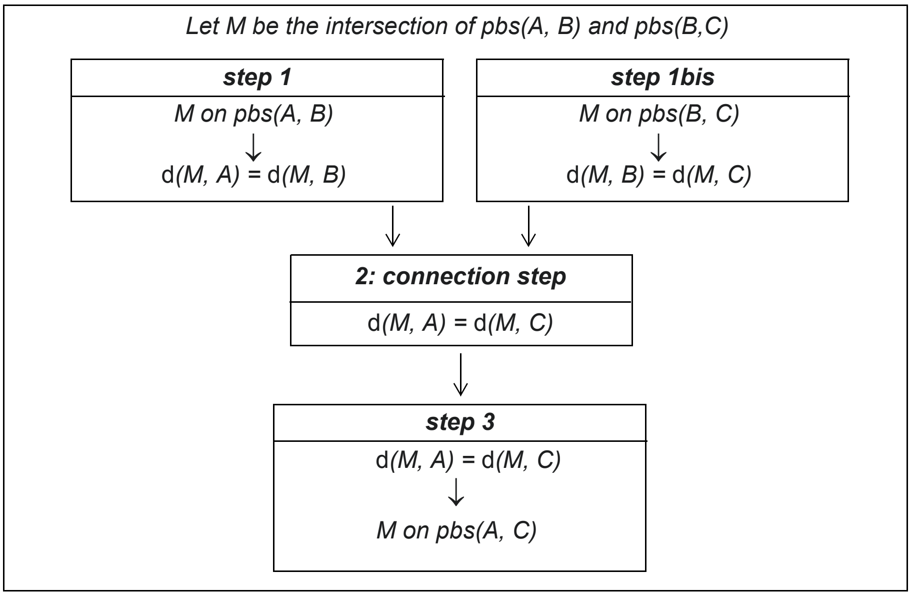
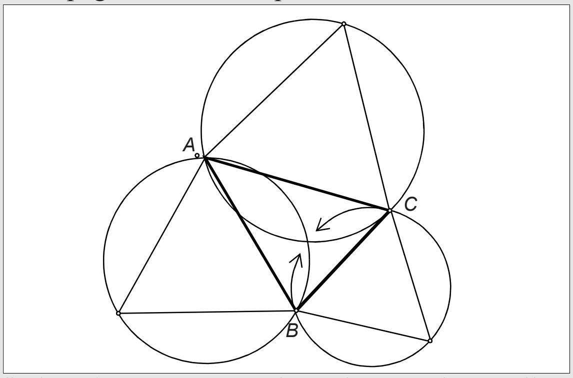


fig. 4.

In example problem 1 (fig. 5) a lot of help is offered; the reference is to the diagram in our fig. 4 and it is even made clear that you should not suppose that the circle through *B* and *C* goes through the intersection *S* of the other two circles. This explication will bear fruit; later on, in a completely different, but very difficult proof one of the students used the phrase 'You cannot assume that...'
But first Sigrid's solution (fig. 6). It is clearly structured after the model in fig. 4. The connection step is not explained, it is hardly necessary and it did not fit into the outline: the rest has been explained right above the fragment shown here, including the reverse of the cyclic quadrilateral theorem, which is used in the conclusion.
Writing down the proof in this form was obligatory. However, the student did get the chance to bring in the right ingredients, but did not have to find out in what order they should be mixed. In this phase it is actually not such a bad idea; besides, they learn a special type of proof, which is worth a high place in the repertoire of possibilities.

example exercise 1

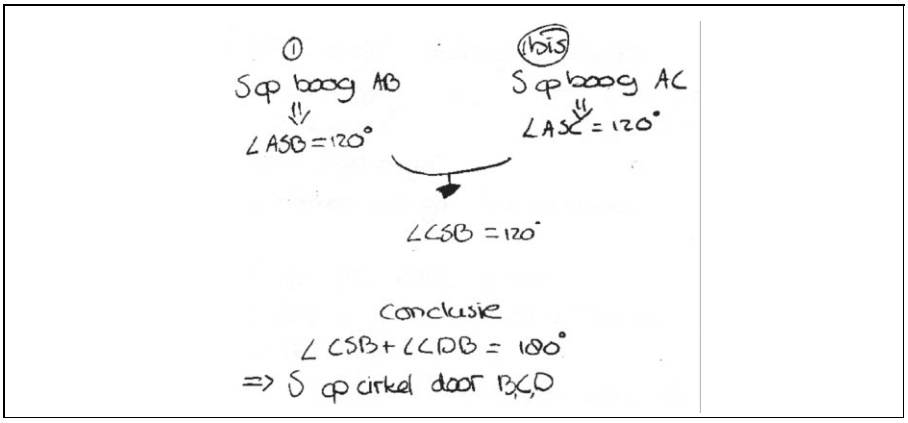
In the figure below three equilateral triangles have been put against the sides of triangle *ABC*. The circumcircle of the equilateral triangles seem to pass through one point. This needs to be proven.
Hint: Look back at page 30 and 31 of part A.



This means: find a *characterization* for points on the small arcs. Call the *intersection of those two arcs S;* show that *S* lies on the third arc.

a. What is your characterization?
b. Which theorems do you use?
c. Write down the proof in the form of page 30 in part A.

fig. 5.



① S op boog AB
⇓
∠ASB = 120°

(bis) S op boog AC
⇓
∠ASC = 120°

∠CSB = 120°

Conclusie
∠CSB + ∠CDB = 180°
⇒ S op cirkel door B,C,D

fig. 6.

Heuristics

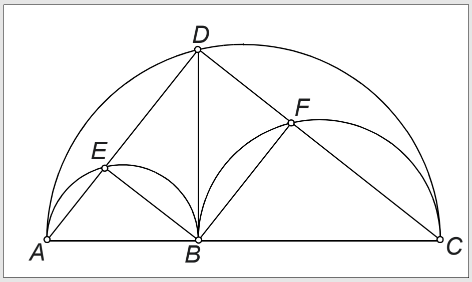
Such a format can be trained and practiced, but for me rules like *'if you need to prove that three lines or circles go through one point, then use the outline of 1-1bis'* are absolutely not done. This leads to mock results. In this fashion laws are imposed where the student needs to learn to make choices and come up with his or her own plans. Moreover, such rules lead as often to nothing as to real solutions. If one wants to help students finding (or choosing) the form of proofs, the support that is offered must have a more open character. It needs to improve oriented searching, but can never give a guaranteed solution strategy. Such guide rules are also called heuristics. In Anne van Streun's dissertation *Heuristic math education* he mentions the two just named properties. Van Streun offers a good overview of mathematicians and didactics, both having touched on this subject, and compares several approaches in this area; the mathematically-relevant target area is specified no further than 'the subject matter of 4VWO'.
Still to be recommended, especially since there are many geometric examples in there for this audience, is 'How to solve it' by George Polya. For Polya a heuristic reasoning is meant to *find* a solution; but the heuristic reasoning is certainly not meant to be the proof itself. Polya's harder founded 'Mathematical Discovery' contains a first chapter named 'The Pattern of Two Loci'. Some of the heuristics used in 'Thinking in circles and lines' can be found there.
In the remainder I will assume the view that some heuristics are very general like 'make sure you understand the problem, then come up with a plan' and others are more subject-specific, like the example of the three circles from above. I also would like to show more examples than to preach general theories. Due to the restricted size of the Nieuwe Wiskrant [see http://www.fi.uu.nl/wiskrant, the magazine where this article was published originally] not all named heuristics in 'Thinking in circles and lines' will be discussed here. I will not limit my comments on the work of students and teachers solely to heuristics. In an active process of learning a lot of things occur at the same time.

Recognizing

Nearly no one had a problem with example exercise 2, fig. 7, but it does bring some special things to light.
One of those things that novices in proving in plane geometry need to practice is recognizing several familiar configurations within new complex figures. Herein also lies an opportunity for the teacher in the classroom to revisit what is known, or at least should be known. Marcel, the teacher in our case, gratefully used this opportunity regularly. You can doubt whether the students need to know

example exercise 2

In this figure three half circles are shown. The diameters of the small circles make up the diameter of the large one. *BD* is the common tangent line of the small half circles.



You have to prove: *DEBF is a rectangle.*

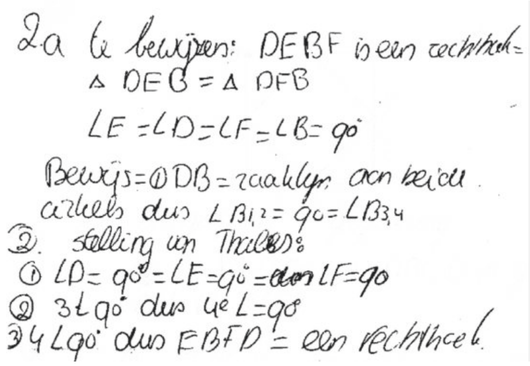
Proceed as follows:

a. Look for the main theme. It is present in the figure more than once!
b. Now write the proof down yourself in a clear, but not too detailed form.

fig. 7.

'heuristics' explicitly, but at least for the teacher it is of importance to keep a couple of heuristics in readiness as keys in a learning conversation.

Bas and Mark work together, they have recognized the theme: a right-angled triangle in a half circle, so the theorem of Thales can be used, fig. 8.



2a te bewijzen: DEBF is een rechthoek.
△ DEB = △ DFB
∠E = ∠D = ∠F = ∠B = 90°
Bewijs = ① DB = raaklijn aan beide cirkels dus ∠B1,2 = 90 = ∠B3,4
②. stelling van Thales:
① ∠D = 90° = ∠E = 90° = ~~dan~~ ∠F = 90
② 3∠90° dus 4e ∠ = 90°
3) 4∠90° dus EBFD = een rechthoek.

fig. 8.

The angles at *D*, *E* and *F* are 90°, thus the fourth angle of quadrilateral *DEBF* has to be the same. Good, but the first step of the proof, the perpendicularity of $\angle ABD$ and $\angle CBD$ now is unpleasantly useless. Mark observed that *D* could also lie somewhere else on the great half circle, then *DEBF* would still be a rectangle. Then why was the tangent *BD* to the small half circles given at all?

That was a sharp insight! Here one of the facts was redundant. Normally this is not the case in this kind of geometry and this is a good occasion to point out another general heuristic: check during your work whether you used all that was given!

Learning to note

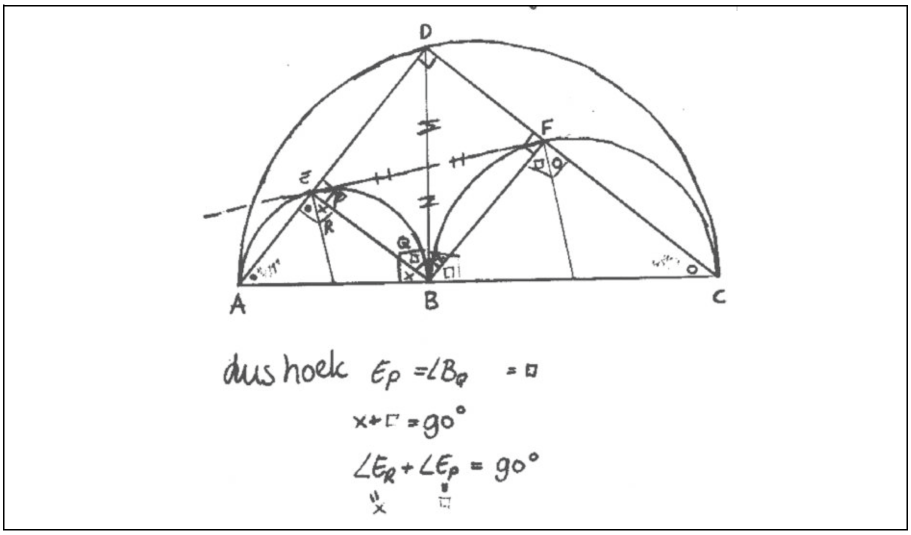


fig. 9.

The next exercise, in the figure of exercise 2 (fig. 9), was to show that *EF* is a tangent line of both small half circles. (By the way, in 1996 this question was part of the second round of the Dutch Mathematics Olympiade.) All components of the proof are shown in the figure Monica has drawn, fig. 9.

She writes down the actual proof pretty briefly: the crosses, balls, squares and other things in angles and on line segments do the actual work. Such symbols (and often a complete rainbow of felt-tip pens) come in very handy in the phase of searching for a proof. But it remains draft work, a neat form of noting must be worked on.

Initially students use several angle notations like $\angle ABD$, $\angle A_1$ and mix many symbols, also in the proofs presented to the public. The first two are, in combination with a sketch, acceptable, but the third (the crosses, balls, squares and so on) is not, since the indicated angles are not uniquely fixed, the symbols only indicate the equality in angles, not which angles they are.

There is a good traditional way to improve the correctness of the writing: let them write down a proof in detail, correct it and provide it with personal comments. It takes time, but it pays off; students often develop their own specific notations, which need comments.

Fig. 10 shows a piece of comment given by Marcel Voorhoeve on a piece of Bas' work (not shown here). Naturally Bas knew what he meant with his notes and Marcel started from there as well, but it looked as if the direction of the logic went in the opposite direction. The comment points out that the arrow is not being used correctly, and is used to introduce an explanation rather than a conclusion.

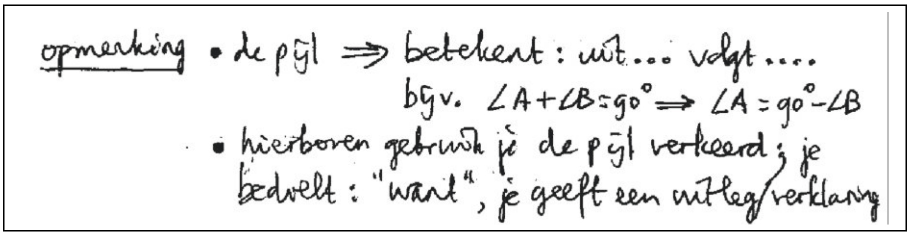

opmerking • de pijl $\Rightarrow$ betekent : uit... volgt....
bijv. $\angle A + \angle B = 90^\circ \Rightarrow \angle A = 90^\circ - \angle B$
• hierboven gebruik je de pijl verkeerd ; je bedoelt : "want", je geeft een uitleg/verklaring

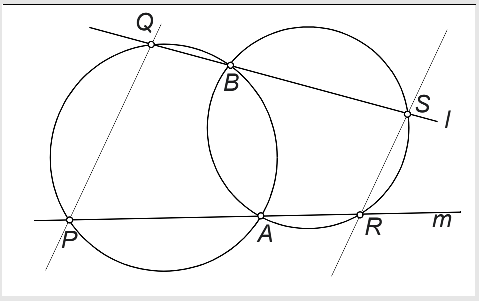
fig.10.

Find a link

A different s heuristic was introduced in example exercise 3, fig. 11 below.

example exercise 3

Here two circles are given and two lines *l* and *m*, which go through the intersections *A* and *B* of the circles.



To prove: *PQ // RS.*

Approach: consider that parallellity and equal angles often go together and look for a link. The circle and the points *A* and/or *B* of course play a big role.

fig.11.

Students are asked to present their proofs. Volunteer Mark starts his story for the class on the overhead projector after adding a few numbers with: *I am going to*

prove that $\angle Q_2 = \angle S_2$. As far as I am concerned nothing can go wrong: the very general heuristic of 'know what it is about' has been applied. Because of this the story to come has a goal and a direction. This follows from earlier class conversations; it often occurs that the story a student tells in class, is a totally dark path with lots of detours for the rest of the class. It is only a matter of time before someone – students or teacher – asks: what on earth are you talking about? These are enlightening moments, since the student in question is often able to say what it is about in one sentence!
The specific heuristic, which under the hood of 'approach' is alluded to, is really a totally different one. There is no theorem you can directly apply to show the equality of angles. Thus one needs some intermediate step, object, angle or something else. Despite the rather directive hint in the text of the assignment, which will quickly lead to sketching the help line *AB*, it will take some time before the proof can be seen clearly from the drafts. After six crossed out lines, Mark's notes look like fig. 12.

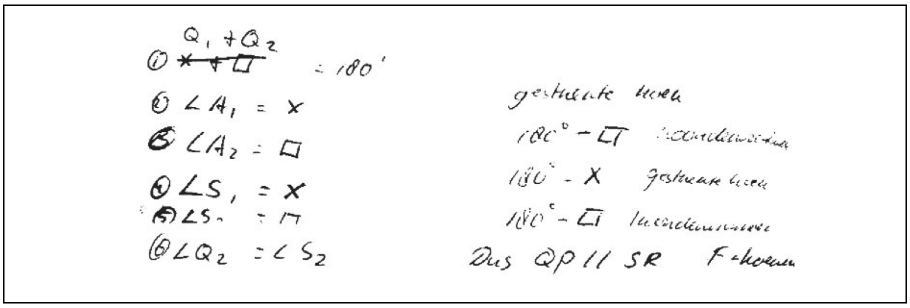


fig. 12.

From *Q* we first go to intermediate stop *A*, and from there to *S*. In the notebook the right angles get as many attention as the usage of cyclic quadrilaterals, but in the explanation the leading role is for the cyclic quadrilateral. This is based on the fact that you can use the angles at *A* as a link.
Also, there are numerous possible different variations in approach for the students. In essence they all use the same elements, but that is not seen right away. Someone who used Z-angles instead of F-angles may think that she found a different proof. In this case the class conversation is of great value; the teacher makes clear what the essential line is and what the necessary details are. Thus it came out that the proofs generally differed only in detail.
The strategy of finding links, of which later on an example in a different frame, has a very positive side effect: threshold reduction. A student who suddenly, after doing a lot of exercises, solving equations and working out brackets, is confronted with the proof question in this example sometimes is likely to sigh: well, I don't know, no idea how I should do it. 'Find a link' therefore also means: see if you are

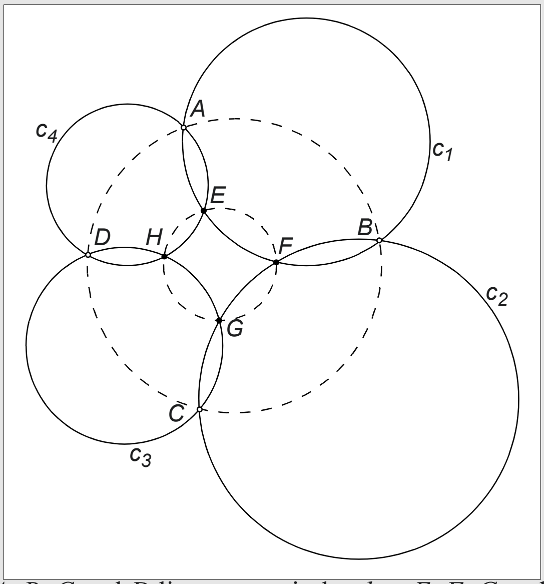
able to write something down, even if you do not know up front whether it leads to the solution. After a while you – may – ave enough pieces and even have one or two which match in order to solve your problem. The kindness of the part of geometry on which we are working is that there is so much opportunity for these learning processes, which by the way do not all evolve consciously.
The ‘link’ is as old as proving in geometry itself. Book I of Euclid’s *The Elements* contains, after a list of twenty-three definitions and five postulates, the five ‘general rules’ and the first is:

1. Things, equal to the same, are equal to each other.

In Euclid a lot of reasoning steps would need some more argumentation according to today’s mathematical standards, but this blindingly obvious platitude is called upon explicitly in crucial parts of proofs. Looked at from a logical perspective, general rule number one formulates an explicitly allowed reasoning step, but in the practice of proving in *The Elements* it is clearly a structuring tool. I consider it a heuristic.

example exercise 4

Circles c_1, c_2, c_3, and c_4 intersect as shown in A, B, C, D, E, F, G and H.



To prove: If A, B, C and D lie on one circle, *then* E, F, G and H also lie on one circle.

fig. 13.

????????? Yesssssss!!!!!

While solving a proof exercise, the keystone of the vaulting sometimes falls into your hands without prior notice. This is a nice moment, a flood of bright white light is surging trough your head, chaos alters into a pattern and suddenly all lines, angles and circles are made of sparkling crystal. And the path to the proof is wide open and lying in front of you.

Did such heavenly moments occur in the class? Yes, and not so rarely. To some extent Mark has that feeling for a moment when after six lines of bungling he starts over and the proof rolls out in a tight bow. There the watch suddenly has started ticking when the last sprocket was put in and naturally this happens more. The moment often closes a period of frustrated searching, but I do feel that the 'good feeling' for many students is more than just the relief 'Oh, am I glad this is over'. Next is such a situation, which was audible in class.

Example exercise 4, fig. 13, closes the link-section; right before, it has been said that you need more than one link.

Janneke has been working on it for a while in class. Her notes are in fig. 14.

$\angle Q + \angle E_1 = 180°$ $\cancel{\angle B + \angle E_2 = 90}$

$\angle A_1 + \angle H_1 = 180°$ $\angle A_2 + \angle F_1 = 180$

$\angle C_1 + \angle H_2 = 180°$ $\angle C_2 + \angle F_2 = 180$

$\angle H_1 + \angle H_2 + \angle H_3 = 360$ $\angle F_1 + \angle F_2 + \angle F_3 = 360°$

$180 - \angle A_1 + 180 - C_1 + \angle H_3 = 360$ $\angle H_3 + \angle F_3 = 180°$

bewys: $\angle EHG + \angle EFG = 180°$

$\angle EHG = 360 - (180 - \angle A_1) - (180 - \angle C_1)$

$\angle EFG = 360 - (180 - \angle A_2) - (180 - \angle C_2)$

$360 - (180 - \angle A_1) - (180 - \angle C_1) + 360 - (180 - \angle A_2) - (180 - \angle C_2)$

$\angle A_1 + \angle A_2 + \angle C_1 + \angle C_2 = 180°$

$360 - (180 - \angle A_1 - \angle A_2 - \angle C_1 - \angle C_2) - 180 + 360 - 180 - 180$

$360 - 0 - 180 + 360 - 180 - 180 = 180$

$\angle EHG + \angle EFG = 180$

dus $\square$

fig.14.

In the four drawn circles, she has made cyclic quadrilaterals and now she is staring at an anthill of numbered, signified and colored angle relations. She needs to show that in quadrilateral *EFGH* the two opposite angles together make 180°. But for Heaven's sake how?

Suddenly a cry: *but those also lie on a circle!*
Those are the points A, B, C and D. At this time the proof is a done deal for her, she suddenly knows for certain that you need to start with two angles of cyclic quadrilateral $ABCD$; the already found angle relations lead from outside to inside (links!) to a good sum of angles for the two opposite angles in quadrilateral $EFGH$. The breakthrough moment here is the moment where in one flash is seen that there is an unused fact after which the total plan of the proof arises. A scratch through the mess, we are starting to rewrite and the details almost fill out themselves in the computation. This will provide a second bonus: *Yeeesss, it is correct*!!!

We saw three clear stages in this event: hard work with possible frustration, breakthrough of the insight and getting the verification conclusive.
'Effort, vision, verification: aspects of doing mathematics'. This was de title of the inaugural speech of Prof. F. Oort in 1968, in Amsterdam. Beautiful to see it so clearly with VWO students!
In Janneke's notebook, fig. 14, the frustration phase is very easy to recognize in the upper part. It is clear how Janneke (in the figure in her book of course) has numbered the sub-angles. Also, there is no visible direction in the proof and no usage of the relations between the A- and C-angles. But underneath the line – the moment of insight – it goes really well, the verification is running like water. The first line is confusing for a moment; it still needs to be proven that $\angle EHG + \angle EFG = 180°$, the goal of the computation has been announced here so to speak. Look, at the end this equality returns. The fourth line (the first that starts with 360) contains the joining of the opposite angles $\angle EHG$ and $\angle EFG$; underneath this the key step has been written down in full:

$$\angle A_1 + \angle A_2 + \angle C_1 + \angle C_2 = 180°.$$

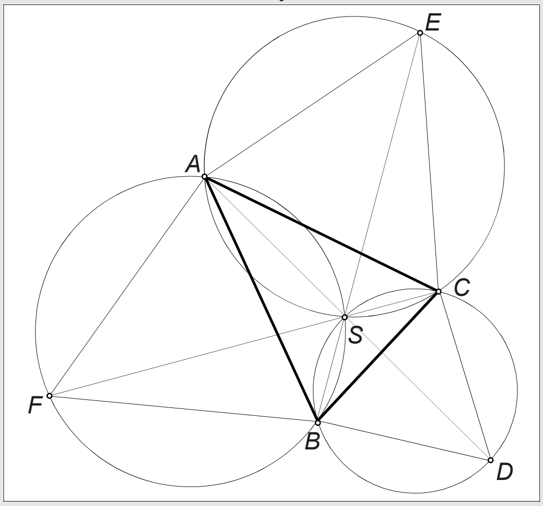
Next the deduction is continued by manoeuvering these four angles in the right positions, after which the result follows. '*thus* ' obviously means '*thus EFGH is a cyclic quadrilateral*'. Yes, yes, you need to write down after that that E, F, G and H *thus* lie on a circle, but we won't fall over a trifle right now. It is something, though, that at some point needs to be learned: that you really need to touch the finish line!

Translating

In example exercise 5, fig. 15, a by now familiar figure is shown. That the three circles have a point of intersection may be used, since this has been proven. It is useful to let something like this occur explicitly in the learning path; it shows

example exercise 5

A familiar figure. The circumcircles of the three equilateral triangles pass through one point. You know that and you can use it.



To prove: *A*, *S* and *D* lie on one line.

fig. 15.

something of the structure of the course. Students are often prepared in their schoolish kindness to again prove that the three circles intersect.

Beforehand it has been said – a heuristic – that sometimes you need to translate that which is to be proven into something else. The translation can be almost the same as the original. For example: isosceles triangles, this is the same as equiangular triangles. Or: three points lie on a line, then two line segments make an angle of 180° with each other. Things which are very close to each other, but give one handle more and some more flexibility.

In fig. 16 a fragment from Karin's work is shown; the remainder of the proof is showing that the angles at *S* are indeed 60° angles. That is simple.

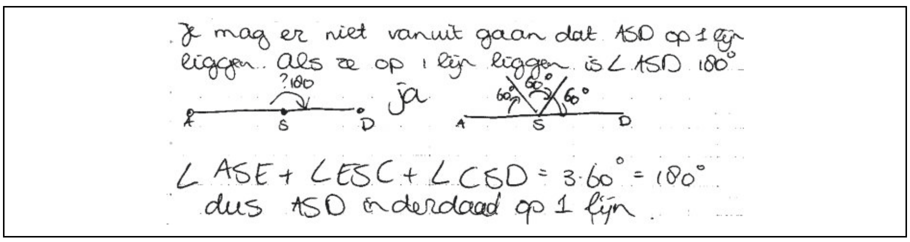


fig.16.

Transcription: You may not assume that *A*, *S*, *D* are on one line. If they are on one line, then $\angle ASD = 180°$.

The explication of what still needs to be done, namely showing that $\angle ASD = 180°$, also helps preventing to walk into the trap of already *using* that form *ASD* one line. Later this is emphasized on the blackboard by using two different colors for *AD* and *DS*, an old-fashioned neat classroom trick for the teacher.

Conjectures and Cabri (or Geogebra)

In the final chapter the students have to formulate their own conjectures while they are experimenting with Cabri[1]. These conjectures will be proven later. A special heuristic belongs to this learning method.

In the computer room I am sitting in front of the screen next to Petra and Mark. A circle has been drawn, a triangle lies on the circle with its vertices, so that the vertices can be dragged, while the circle remains fixed. The orthocentre of the triangle has been drawn. Now C is moving over the circle, and therefore H moves as well. Petra let H make a trace; Cabri has an option to do so. What happens? H also moves over a circle. The effect is spectacular when you see it happening and it immediately raises the question: why a circle? This question is a natural motive for looking for a proof.

An important heuristic in Cabri (or other dynamic geometry programs) is: look at the movements on the screen and try to find something that is moving also, but has something constant to it. If you find something like this, you may have a key, maybe a link, in your hands for the proof. This is a good working approach for Cabri and I mention it during the conversation in front of the screen.

Petra answers that $\angle AHB$ is constant and shows that she also sees the constancy of $\angle C$. The bell interrupts and all I can say is 'you can do this'. I feared that Petra looked at it a little bit differently: she may have deducted the constancy of $\angle AHB$ from the fact that H lies on a fixed circle through AB. That is assuming what you need to prove, the most deadly sin there is in mathematics.

She has worked it out on paper at home, and the result surprised me. Fig. 17, next page, is from Petra's notebook. I was wrong, or Petra changed her mind. Look at how Petra starts her argumentation: *if the path of H is a circle, then ∠AHB must be constant, so we will prove it to be.* That is the old heuristic, which Pappos named 'analysis': exploring the problem from the assumption that we have the solution. Next should be the synthesis-phase: constructing the proof from the given, the opposite direction of the analysis. Petra's synthesis starts at the fixed angle C; a long detour of almost a page in which – how else – we encounter cyclic quadrilateral *CFHE*, leads her via $\angle AHB = 180° - \angle ACB$ to the required result.

1. Note (2014): In 1999 *Cabri Géomètre I* was used. Any current Dynamic Geometry Software program will support the options used in the text.

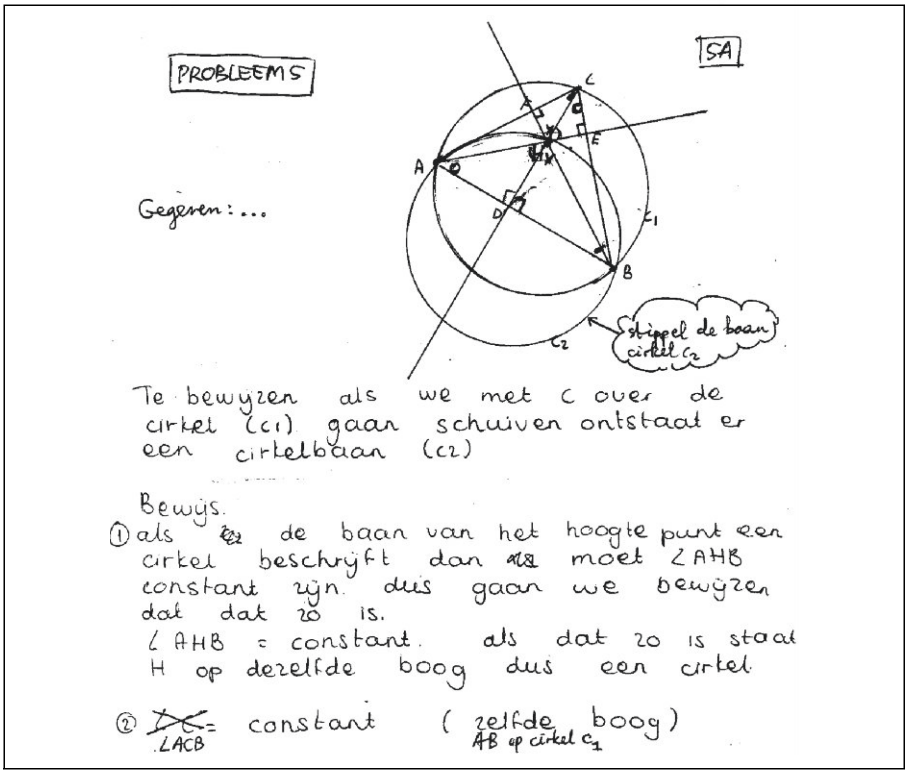


fig. 17.

Petra must have enjoyed this success; she closes very professionally with step 13, (fig. 18) where it becomes clear that she acknowledges the (not at any cost necessary) case distinction.

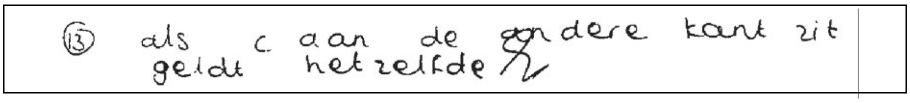


fig. 18.

Translation: if C is on the other side, it is the same.
Other side: down under *AB*, yes.

Finally: teacher and student

During this story I have pointed out several times that heuristics also belong to the conversation tool kit of the teacher. The heuristics then are an aid to finding coherence in the search process. The idea here is not about one heuristic being better than another, but about the direct effectiveness with respect to the content

of it all. The emphasis lies on stimulating the search so that one will no longer say: I do not know that now, so I cannot do it. This is achieved in this little paradise class.
Teacher Marcel Voorhoeve also takes on other roles besides the organizational-directing one: the role of co-solver and also that of critical sounding board via questions dealing for example with half-grown proof steps and sometimes students take over that last part in conversations. The task of demonstrating on the blackboard is hardly of any importance; as far as learning to find proofs goes, it does not seem very effective, and directing towards acceptable ways of noting down could also be learned through the students' work.
Sieb Kemme and Wim Groen have written in the *Nieuwe Wiskrant* (19,2) about problem solving as a trade. After their introduction, I of course started to deal with their example problem in a different manner, but also their reflections on the search process matched my approach and agree with what I have brought up here. Reflecting independently is something I see Sieb, Wim and myself however, naturally – or just because of age and professional knowledge – do more than the students in this 6 VWO class; in 6 VWO such things are more open for discussion than in say a 3 grammar school class. Here again lies a task for the teacher.

Geometrical footnotes

Sometimes to keep on solving a problem yourself and to look how you do that, remains an important exercise for those who have to teach those things. So why not add a couple of nice continuations of one of the problems presented in this article?

1. The point *S* in example exercise 1 is the first point of Fermat, F_1, although the Italians will keep it calling the point of Torricelli. Reflect the triangles also to the other side of the sides; show that the three circles then also pass through one point F_2. Use the plagiarism-heuristic: detailed copying of a proof with some small changes.
2. In example exercise 5 *AD*, *BE* and *CF* all three of course go trough the point *S* (of F_1). *But those three segments also have the same length.* This should not be difficult to prove, especially for those whose memory of a previous phase of the geometry education (transformations) is still vivid.
3. Plagiarize exercise 2 like exercise 1 plagiarizes example exercise 1.
4. In example exercise 5 it was proven that *AD*, *BE* and *CF* go through one point, for the case that the outer triangles are equilateral. Now put three isosceles, mutually uniformly, triangles with their bases on the three sides of *ABC* and now

prove also that AD, BE and CF go through one point. You need to forget about the circles! This exercise may be seen as more difficult.

5. Draw a triangle (use Geogebra for instance). Construct both Fermat points, the center of the circumcircle and the center of the nine points circle and show that these four lie on one circle. J. Lester has showed this remarkable relation in 1995. Heuristic: use someone else's work from the Internet to find a proof.

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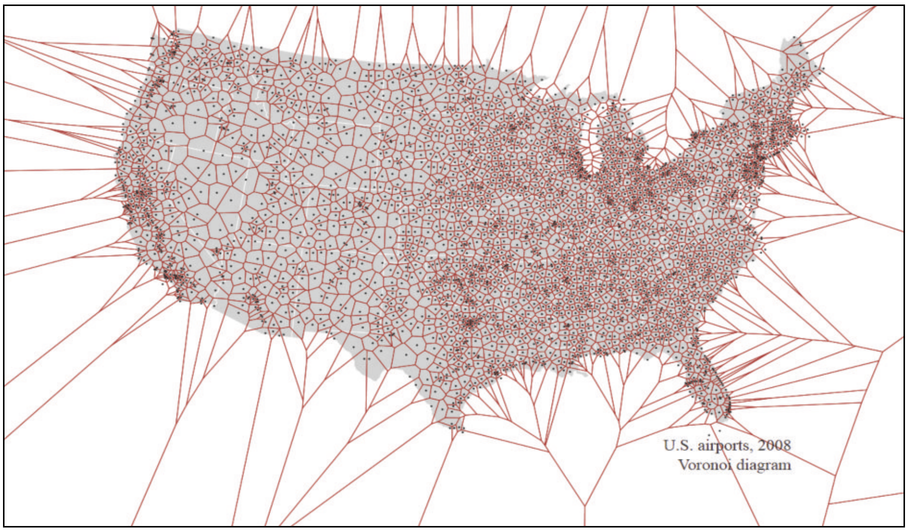
Distances, edges and domains

Advanced geometry, part I



Distances, edges and domains – Advanced geometry, part I
Project: Mathematics for the new Senior High school
Profile: Nature and Technology
Domain: Advanced Geometry
Class: VWO 5
State: Second edition (1997)
Authors: Aad Goddijn, Wolfgang Reuter
Translation: Danny Dullens, Nathalie Kuijpers

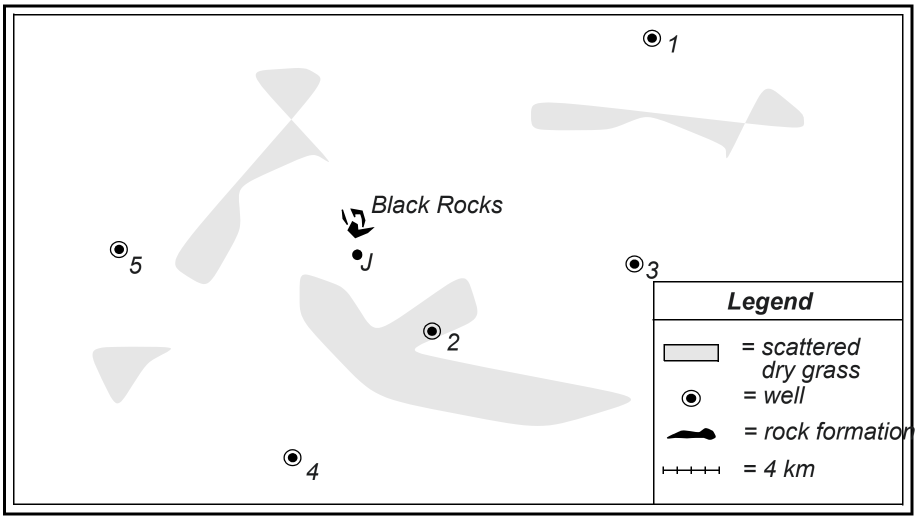
Chapter 1: Voronoi diagrams

In this first part of the geometry course you will encounter a way of partitioning an area which has many applications.
We will start with simple cases, but make sure you keep an eye on the figure on the front page. That is also an example of one of the partitions in this chapter!

You will have to sketch quite a lot. You can do that in this book. For some exercises special worksheets are included. You will usually need a protractor, ruler and compass. You may also have to make sketches and graphs in your notebook while working out the solutions.

1. *In the desert*

Below, you see part of a map of a desert. There are five wells in this area. Imagine you and your herd of sheep are standing at *J*. You are very thirsty and you only brought this map with you.



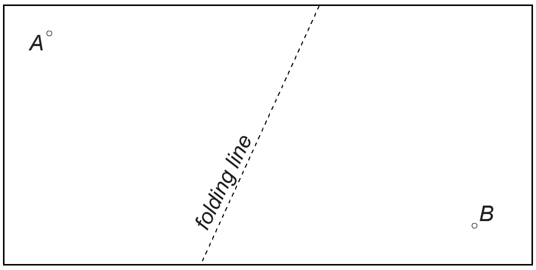
1. **a.** To which well would you go for water?
   b. Point out two other places from where you would also go to well 2. Choose them far apart from each other.
   c. Now sketch a division of the desert in five parts; each part belongs to one well. It is the domain around that particular well. Anywhere in this domain that special well must be the nearest.
   d. What can you do when you are standing exactly on the edge of two different domains?
   e. Do the domains of wells 1 and 5 adjoin? Or: try to find a point which has equal distances to wells 1 and 5 and has larger distances to all the other wells.
   f. In reality the desert is much larger than is shown on this map. If there are no other wells throughout the desert than the five on this map, do the domains of wells 3 and 4 adjoin?
   g. The edge between the domains of wells 2 and 3 crosses the line segment between wells 2 and 3 exactly in the middle. Does something similar apply to the other edges?
   h. What kind of lines are the edges? Straight? Curved?

In this exercise you just partitioned an area according to the *nearest-neighbor-principle*. Similar partitions are used in several sciences, for instance in geology,

forestry, marketing, astronomy, robotics, linguistics, crystallography, meteorology, to name but a few. We will revisit those now and then.
Next we will investigate the simple case of two wells, or actually two points, since we might not be dealing with *wells* in other applications.

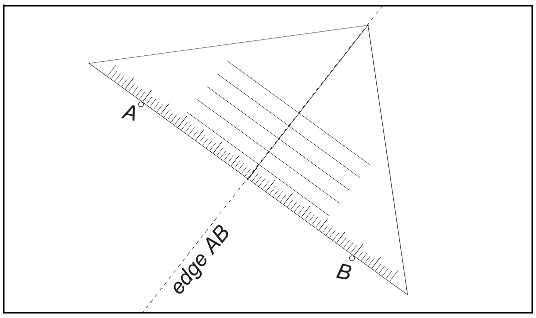
2. *The edges between two domains*

folding
A simple case with two wells is shown here. We neglect the dimensions of the wells themselves, i.e.: we pretend they have no size at all. In the figure: the *points A* and *B*.

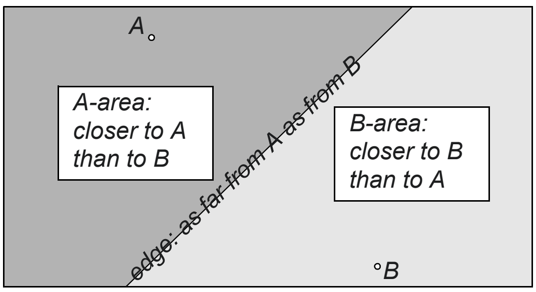


On paper the edge between the domains of *A* and *B* is easy to find, namely by folding the paper so that *A* lies on top of *B*. This folding line is the edge between the areas belonging to *A* and *B*.

protractor
There is also another method to find this edge easily and fast: with the protractor. See the figure on the right. *A* and *B* are both at the same distance from the middle of the protractor.



half plane
In this figure the areas round *A* and *B* have different colors.
To the *A*-domain applies:
  distance to A < distance to B.
To the *B*-domain applies:
  distance to A > distance to B.
Only on the edge the following applies:
  distance to A = distance to B.



Actually, you should imagine that there is more than just the sketched part: everything continues unlimited in all directions. The two domains, determined in such a way, are infinitely large and are bounded by a straight line. The name for such a domain is *half-plane*. Include the edge as part of both half-planes. In the figure the domains of A and B are therefore both half-planes. These half-planes overlap each other on the edge.

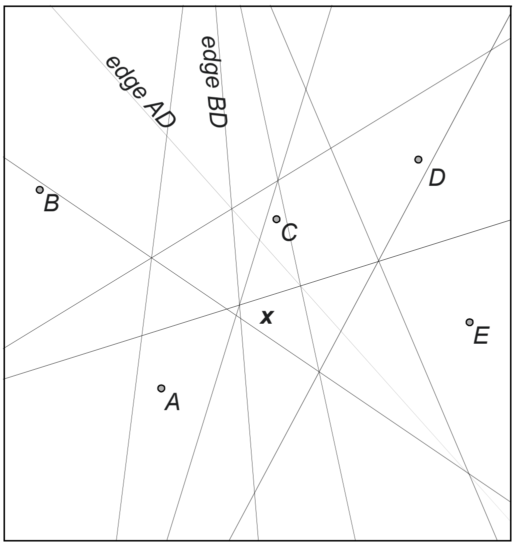
2. The edge is often called *conflict line*. A good name? Why?

3. More points, more edges

Through folding, we will now investigate a situation with four points.
Also take *worksheet A: Folding to Voronoi* (page 175).

3. For each pair of points we determine the edge by folding.
  a. First find out how many folds are necessary and then proceed with the actual folding. Try to do this as precise as possible; for instance hold the piece of paper up to the light. Use the folded lines to sketch the partition of the area.
  b. While folding, a lot of intersections of the folding lines arise. Nevertheless there are different kinds of intersections. What differences do you notice?
  c. One fold turns out to be redundant. What causes that?

excluding technique
Here, a situation with five points. The edges for all pairs of points are shown.



4. How many edges are there?
  a. Of the cross **x** near the edge BD you know for certain: it certainly does not belong to B. How can you tell?
  b. Use other lines to exclude other possible owners of **x**. In the end one remains. Which one?
  c. Try to find out for the other areas to what center they belong and with this excluding technique finish the partition using five colors.

Exact Voronoi diagram with the protractor

5. Now sketch, using the protractor method, the exact edges round the wells on the desert map in *worksheet B: Exact Voronoi diagram for the desert*, page 176.

4. Voronoi diagrams: centers, edges, cells

In this paragraph we discuss some more terminology.

nearest-neighbor- principle
In the preceding we made partitions of an area according to the '*nearest-neighbor-principle*'.

centers
The points around which everything evolves (in this example the wells) will now be called *centers*. Throughout this book we will always assume we have a finite number of centers.

Voronoi diagram
The figure of edges is called the *Voronoi diagram* belonging to the centers. Another name is *edge diagram.*

Voronoi cell
The area that belongs to a center, is called a *Voronoi cell*, or, in short, *cell* of that center.

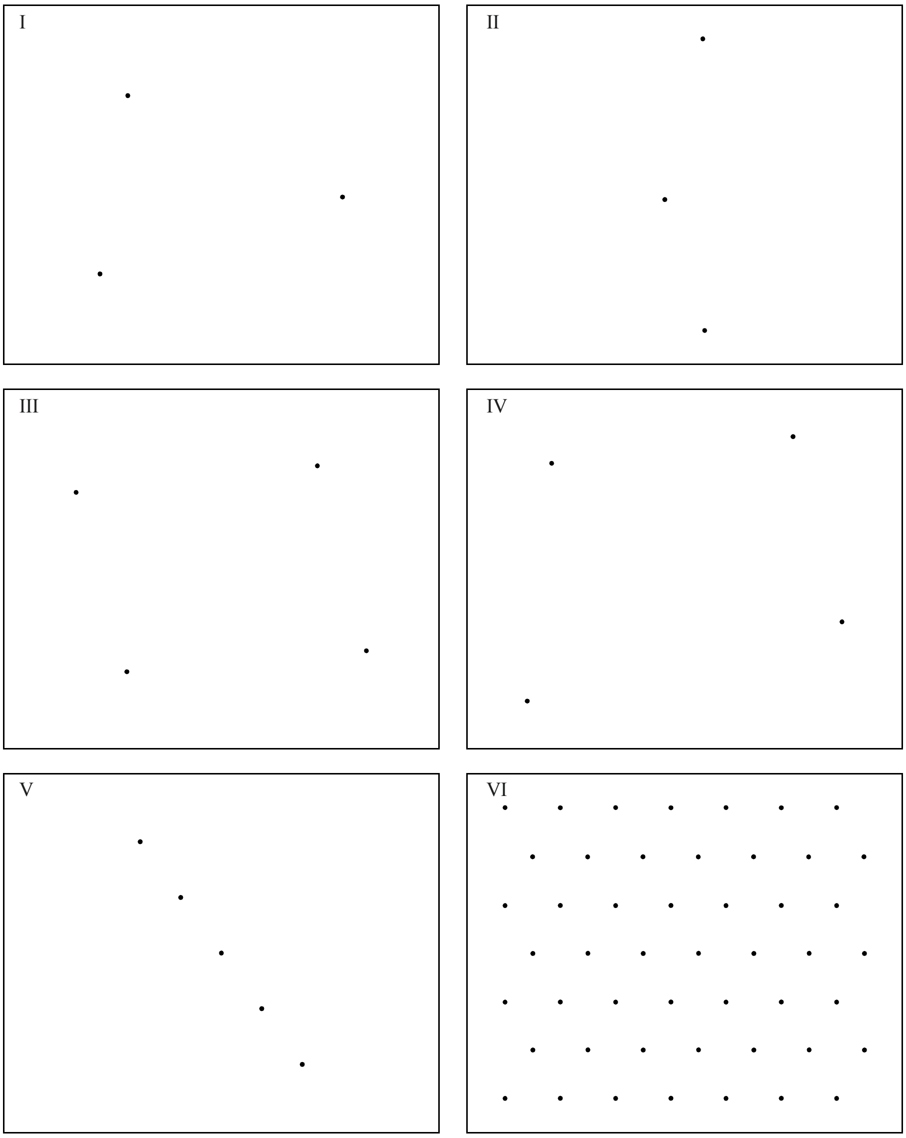
vertices
A Voronoi cell is bounded by straight lines or by segments of straight lines. The points where several lines converge, are called the *vertices* of the Voronoi diagram (singular: *vertex*).

history
Voronoi diagrams are named after the mathematician Voronoi. He (in 1908) and Dirichlet (in 1850) used these diagrams in a pure mathematical problem, the investigation of positive definite quadratic forms. In 1911, Thiessen used the same sort of diagrams while determining quantities of precipitation in an area, while only measuring at a small number of points. In meteorology, geography and archaeology the term Thiessen-polytope instead of Voronoi cell became established.

6 **a.** On the next page you can see six situations. Each dot represents a center. Sketch the edge diagrams for these situations.
b. In situation I you find one point in the middle where three edges converge. What can you say about the distances of that point to the centers?
c. Does situation II also have such a point?
d. In the situations III and IV only one center is not in the same place. However, the Voronoi diagrams differ considerably. Try to indicate the cause of that difference.
e. In situation V the centers lie on one line and thus the diagram is fairly easy to draw. What can you say about the mutual position of the edges and the shape of the Voronoi cells?
f. Situation VI has lots of centers. But thanks to the regularity, sketching of the edge diagram is again a simple affair. Once one cell is known, the rest follows automatically.
Do you know anything in nature which has this pattern as a partition?

infinitely large cells

7. Nowhere is it said that a Voronoi cell is enclosed on all sides by (segments of) lines. In fact, some cells are infinitely large, even though that is not visible in the picture.
a. How many infinite cells are there in the well example on page 43?
b. In situations with two or three wells there are only infinite cells. Now sketch two situations with four centers. In one situation all cells must be infinitely large, in the other situation not all cells are infinitely large.
c. Describe a situation with twenty centers and twenty infinite cells.
d. Where do you expect the infinite cells to occur in a Voronoi diagram?


I
II
III
IV
V
VI

5. *Three countries meeting; empty circles*

Below you see a redivision of the Netherlands as a Voronoi diagram[1]. The centers are the province capitals.

three-countries-point

On each of the vertices of the Voronoi diagram, three cells converge. This is what we call a *three-countries-point*, even in a context that does not involve countries.

8 **a.** What do you know of the distances of the 'three-countries-point' between the cities of Middelburg, Den Haag and Den Bosch to those three cities?

b. Put you compass point in that three-countries-point. Now draw a circle through those three cities with this three-countries-point as its center.

c. Now put your compass point somewhere on the edge between Zwolle and Arnhem, but not in a vertex of the diagram. Sketch a circle with this point as center, which passes through Arnhem.

---

1. Use worksheet C: map of the Netherlands, page 177, to see the official division in provinces.

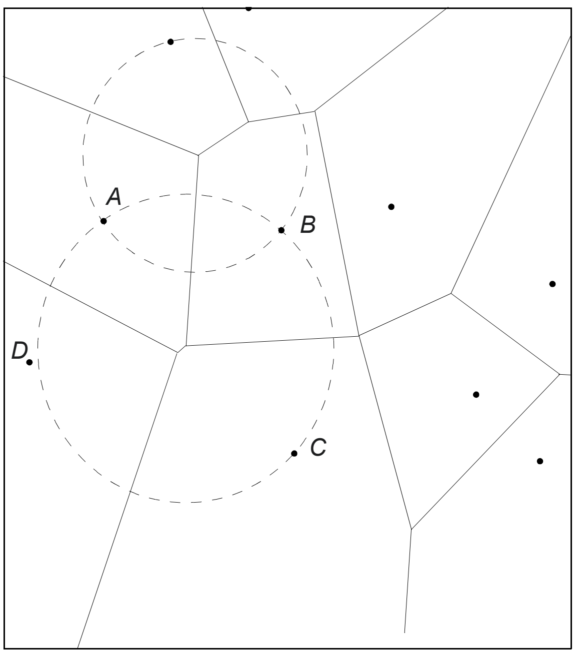
largest empty circles

What you just sketched, are examples of *largest empty circles*. A largest empty circle in a Voronoi diagram is a circle in which no centers lie and on which lies at least one center.

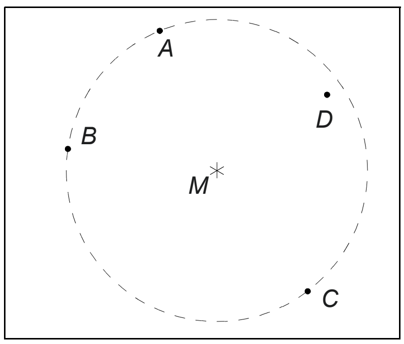
The name *largest empty circle* is chosen well: if you enlarge such a circle around its center just a tiny bit, the interior of the circle would not be empty anymore: for sure there will be one or more centers inside.

9 **a.** In this Voronoi diagram two largest empty circles are already sketched. Mark their centers.
b. Sketch several largest empty circles:
- with *four* centers on the circle,
- with *two* centers on the circle,
- with *one* center on the circle.

c. What can you say in general about the number of centers on a largest empty circle round a three-countries-point?



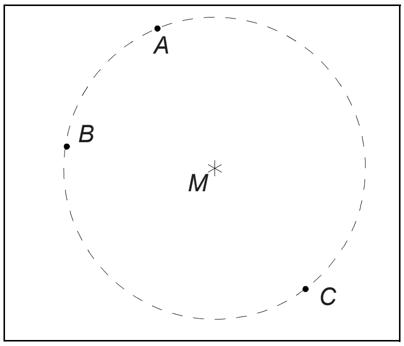
10 **a.** On the right you see a situation with four centers. The centers are the black dots. The little star at M is the center of the circle through the centers A, B and C. Can M represent the three-countries-point of *the cells around* A, B and C? Why? Or why not?



b. Here you see the same figure, only the center D is left out. Sketch the Voronoi diagram. Be careful: M is not a center itself, but you can make good use of M in some way.

c. Now add center D yourself and expand the Voronoi diagram, but do it in such a way that M becomes a four-countries-point.

d. Three-countries-points are very normal, four-countries-points are special. Explain why.



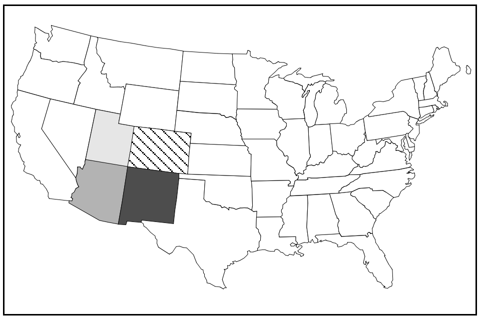
11. On the next page you will find a Voronoi diagram of which the centers lie on the coast of four islands. Parts of the Voronoi diagram have actually become edges between domains around those islands.

a. Mark those edges with a color. A partition in four domains arises.

b. You could also talk about three-countries-points in the last partition. Now sketch a couple of largest empty circles, which just touch three of the islands. Where should you place their centers?

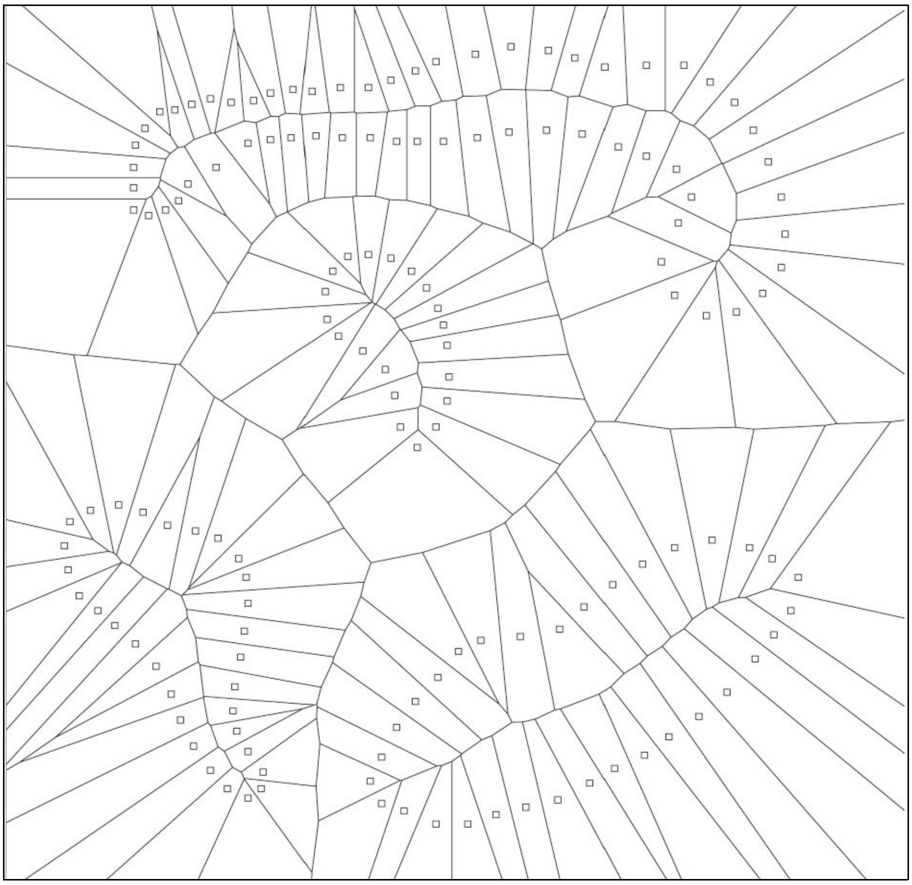
true four-countries-point

True four-countries-points occur very rarely between countries in real life. One is shown on the right, between the states Utah, Colorado, Arizona and New Mexico in the USA. If you know another one, speak up!



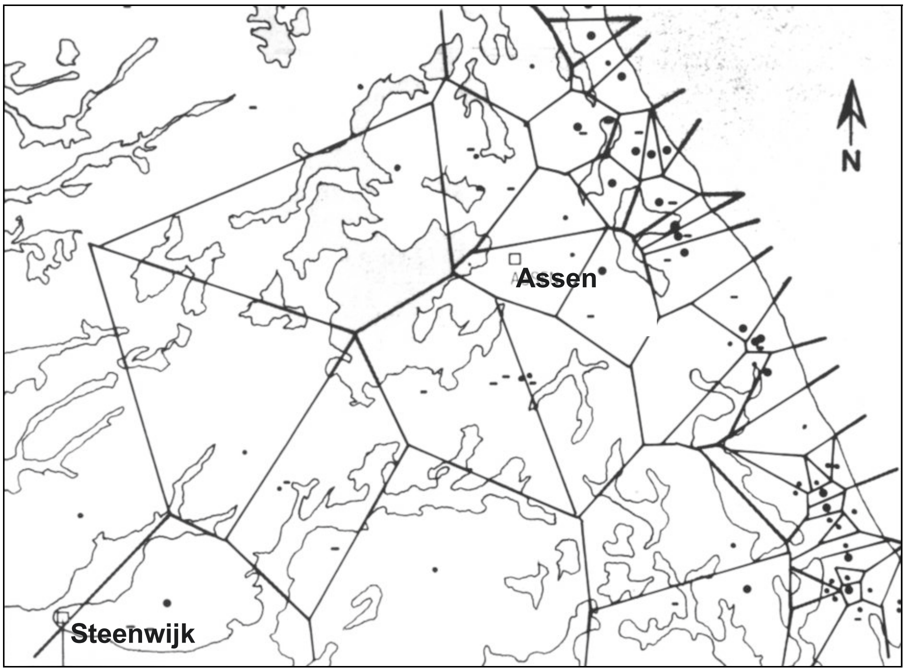
Of course this is not Voronoi diagram of the United States.

four islands



12. Sketch a situation of centers and a *Voronoi diagram* yourself, for which:
– only four-countries-points occur and no three-countries-points
– and for which there is a square cell
– and for which the edges lie in each other's extension, are parallels, or are perpendicular.

6. *Chambered tombs in Drenthe*



A partition of the eastern part of the Drents plateau in imaginary territories. Centers here represent groups of chambered tombs[2]. Some chambered tombs were used to store bones and skulls over a period of 600 years. The Voronoi diagram shows a possible partitioning of the area. Archeologists often research whether such partitions correspond to the distribution of pottery in an area. This could provide indications about the social and economical structure in former days.

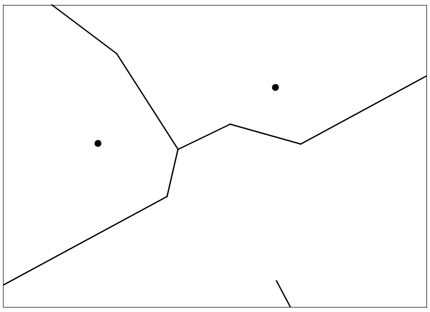
13 a. Observe: the cell in which Assen lies and the one north of it have centers which lie symmetrical in relation to the edge. Why that symmetry?

b. Do the centers lie symmetrical everywhere in relation to the borders? Is this necessarily so for a Voronoi diagram?

c. The cell southwest of Assen contains several dots. Which dot is used for making the Voronoi diagram?

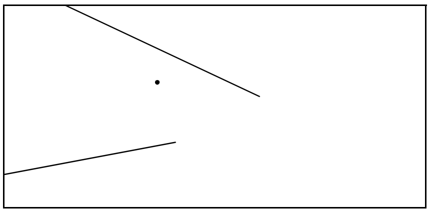
2. From about 3500 B.C. They are called 'hunebed' in Dutch.

14. Reasoning with symmetry could also help to fill up an incomplete map of centers and edges. On the right an incomplete Voronoi diagram is shown.
Complete the diagram.

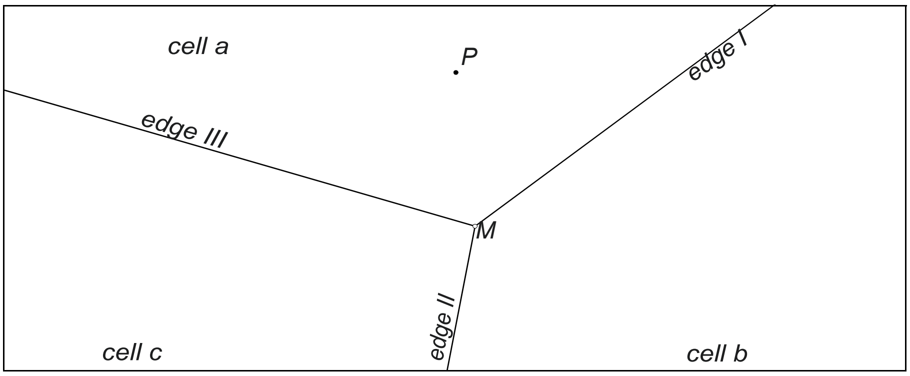


reflection
Voronoi centers of adjacent cells are always each other's mirror image in relation to their Voronoi edge. So you can recover missing centers by reflecting in an edge! We will illustrate this technique with the following two examples.

15. This figure shows only one center. Since two edges are (partially) indicated, there have to be two other centers and a third edge. Finish the sketch accurately.



16. Below, the edges of a Voronoi diagram with three centers are given. Point P lies in cell a.
Try to work as precise as possible in this exercise, or you may run into trouble. You can do the reflection exactly using your protractor. See I - 44.



a. *P is certainly not the center which belongs to cell a!*
You can verify this by reflecting P in edge I; name the reflection P_1.
Then reflect P_1 in edge II. Name the reflection P_2. Finally, reflect P_2 in edge III. Name the reflection Q.
Why is it not possible that P is the center of cell a?

b. Sketch the middle of the line *PQ* and name it *R*. Now also reflect *R* successively in the three edges; in this way point *S* arises. What do you notice about this ultimate point *S*?

c. The point *R* (or *S*) could be the center of cell *a*, but that is not necessarily true. Another option, for example, is a point that lies in the middle of *R* and the indicated three-countries-point *M*. Verify this by repeated reflection. What other points could you take to start with as center of cell a?

reconstruction problem
The final result of exercise **16** is surprising. Using the method given above, you are apparently able to reconstruct possible centers, without knowing one of them. Of course the question is: Why does this work so well? Several clues are given in the *extra exploration exercise*, on page **58**, so you can get to the bottom of this.

Summary of chapter 1

This chapter provisionally explored Voronoi diagrams, and discussed several concepts.

nearest-neighbor-principle
A Voronoi diagram arises when a number of points are given and the plane is partitioned so you can determine everywhere what the nearest point is. In this fashion a partition in subregions arises. This is called partitioning according to the *nearest-neighbor-principle*.

You will find definitions of the concepts *center, Voronoi diagram, edge diagram, Voronoi cell, cell* and *vertex* on page 48.

The Voronoi cells can be infinitely large. These infinite cells belong to centers which lie close to the side; we shall have to specify this later.

three-countries-points
In general three cells meet in a vertex. Such vertices are called three-countries-points. To have more than three cells converging in a vertex is possible, but rare.

largest empty circle
A circle in which no centers lie, but on which does lie a center, is called a largest empty circle. Such a circle cannot be enlarged from its center.

reflection, reconstructing problem
If only edges are given the centers can sometimes be recovered by reflection. For this, the fact that the centers lie *symmetrically* with respect to their edge is used.

To find the *centers for a given Voronoi diagram* is called solving the reconstruction problem.

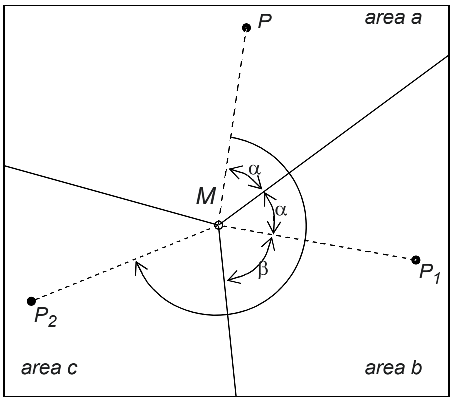
Preview

In the next chapters we will first go deeper into the mathematics, which until now we used incidentally. Doing that, we will argue more independently and draw fewer conclusions from measuring figures only.
Nevertheless one result will be very practical, namely that we will find a fast way to determine whether a center D is inside the circle through A, B, C or not, *without determining the actual circle itself.* This will make the construction of a Voronoi diagram a lot easier.
In a later chapter you will learn more about Voronoi diagrams using a computer program. For this you will need the knowledge in this chapter as well as the next.

Extra exploration exercises: Recovering the centers

This exercise combines with exercise **16**. First of all, make sure that you understand the method used to construct possible centers of a Voronoi diagram of three cells with a three-countries-point.

Exercise one
Find and describe an argumentation which proves that the method of exercise **16** always works. Several hints follow here.



- **a.** The figure on the right already shows P_1 and P_2. The next reflected point would be Q, but for now call this point P_3 and do three more reflections. You will discover something very special about P_6 if you have drawn carefully enough.
- **b.** ***If*** you were sure that indeed P_6 is always the same as P is always true, ***then*** you can conclude that the middle R of PP_3 will end up on top of itself after *three* reflections. Find out why by reflecting the line segment PP_3 three times.

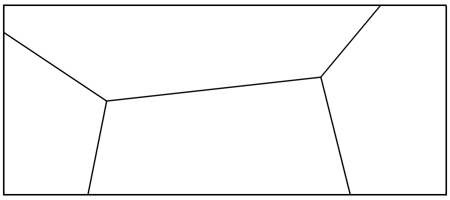
c. But *why* is P_6 equal to P? That's the main question now.
The figure shows several angles. You could also think about reflection as the rotation of the bar MP around the center of rotation M. Compare the angle over which MP has to rotate (clockwise) to arrive at MP_2 with the angle of cell b.

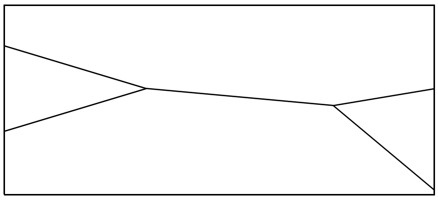
Exercise two

Once you found one possible position for the center of cell a, you also know all the other possibilities. Work this out.

Exercise three

Find a method to recover the four centers that are not given in the same type of Voronoi diagrams as shown below. Hint: Divide and rule.



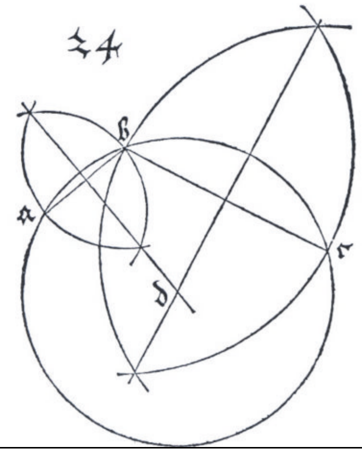


Complete your investigation as follows: Make a report of one page in which you write down your argumentations. Add clear figures.

Chapter 2: Reasoning with distances

ES ist nützlich zůwyssen/so drey punckte ungleich gestelt werden/das man sie behend so man seyn bedarff/in eyn zirckel verfassen müg diß mach also/Die drey punckten seyen.a.b.c. die zeüch mit zweyē geraden liniē zůsamē.a.b vnd.b.c.darnach thu jm zůgleicher weiß wie vorn in der.21.figur angezeigt ist/sůch die mittel beder linien.a.b.vnd.b.c. vnd laß die zwů geradē linien so die zwů für gegebnen linien. a.b.vñ.b.c.ytliche in der mitt von eynander teylen vndersich herab durch eynander streichen/ vnd so es not thut/so erlenger die bed gerad teillinien/Darnach nym ein zirckel vnnd setz den mit dem ein fuß in den punckten. d. vnnd den andern in den punckten.a. vnd reiß darauß ein gantzen zirckel ryß/so rürdt der zirckel/die drey punckten.a.b.c. wie jch das vnden hab auffgriesse.

24



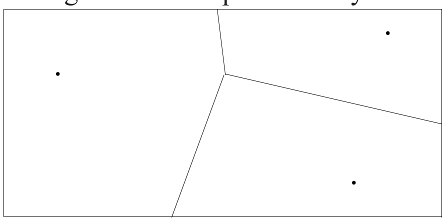
In this chapter the reasoning will be a lot more exact than in the previous chapter. In principle we deduct things from scarcely any given data. We will also consider how that kind of reasoning works and how to write it down.

The illustration with the text in gothic letters on the front page becomes a reality: UNTERWEYSUNG DER MESSUNG MIT DEM ZIRCKEL UND RICHTSCHEYT.
This is a geometry book, published in 1525 for painters and artists, by Albrecht Dürer. The figure is an illustration for what in this chapter will be *theorem 5*. That theorem says that for any triangle there is exactly one circle which passes through the three vertices of the triangle. In his figure, Dürer uses the triangle *abc*, and indicates how the center of the circle and the circle itself can be determined by ruler and compass only.

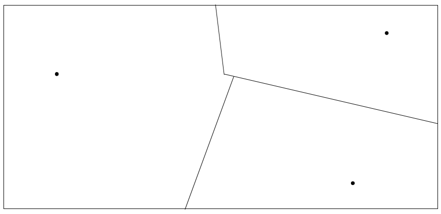
7. Introduction: reasoning in geometry

1. In the last chapter we investigated several things in relation to Voronoi diagrams. We used what could be seen in figures and sketches. However, we might have assumed things that we do not know for sure whether they are true.

problem A
Why does the Voronoi diagram of three points always looks like this?



and never like this?



problem B
The Voronoi edge of two points always appears to be a straight line. The folding technique backs up this idea. But why are folding lines always straight? We *have seen* this in many cases. We do not really know why.
In this chapter we will look further into these questions.

reasoning instead of observing
Now we want to get certainty about these questions by *reasoning in general situations,* and not by *observing a special case* in a sketched figure.
This is why this chapter will have a more theoretical character, especially towards the end, since we will start with concrete diagrams, but in the latter part *theorems* and *proofs* are discussed.
You may feel as if you are walking on egg shells. This is true, but you will get used to it. Moreover, you will start complaining if something is stated without a proof.

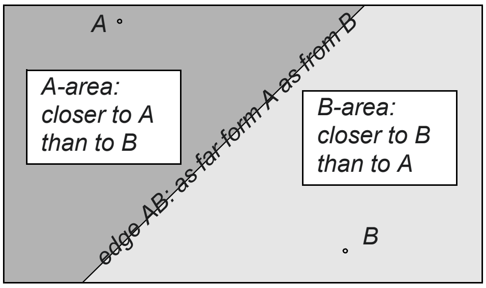
1. a. In the section '*The edges between two domains*' (page 46) more non-founded properties of the edge between two areas are stated. Which for example?
b. And how do we use those in the section '*Chambered tombs in Drenthe*' (*page* 55)?

8. An argumentation about three-countries-points and circles

First we tackle problem A above: *Why do the edges of a Voronoi diagram with three centers intersect in one point*?
You need to look at such questions with a critical eye. Hence:

2 **a.** Sketch a few three-centers situations, which don't even have three edges. Thumb through the examples of the previous chapter, if need be, to get some ideas.
b. What characterizes these situations?

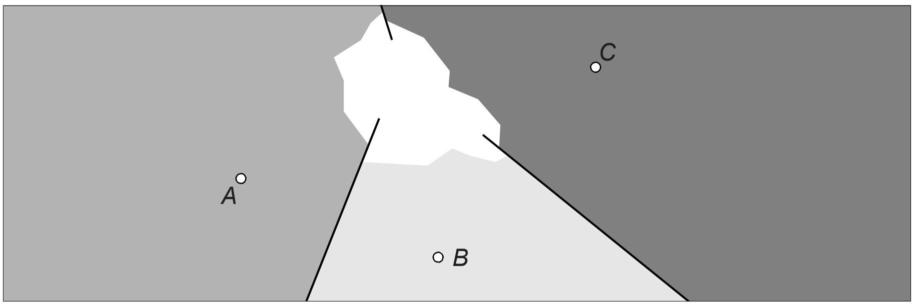
In the remainder of this section we will not reckon with this special case.
In the previous chapter the Voronoi edge played the leading role. The figure shows its characteristics.
Only the points on the edge have equal distance to A and B. Phrased differently:



property Voronoi edge

> **(distance from P to A) = (distance from P to B)**
> **applies only to the points *P* on the Voronoi edge between *A* and *B***

We will use this clear form while reasoning with three edges. Below you see a figure for which it is not known whether the three edges converge into one point.



3. This exercise will help you find an argumentation for:
the three edges converge in one point.
a. Indicate the point of intersection of Voronoi edge AB and Voronoi edge BC and call it M.

b. Write down, using the property of Voronoi edges, the two accompanying equalities and derive the third equality. Write that one down as well.
c. What does that equality say about point M? (Remember the property once more.)
d. Did you reach the goal of the argumentation?

4. The circle, which has M as its center and passes through A, also passes through B and C.
a. How do you know that for certain?
b. In the previous chapter this circle played an important role. What role was that?

critical remarks
While working on exercises **3** and **4** we solved the problem on page 59, and more, or so it seems.

5. Try to answer the following questions:
a. On what is the assumption that such an intersection point M exists based?
b. Do we not also (maybe carefully hidden, but nevertheless) use the fact that the edges are straight lines?

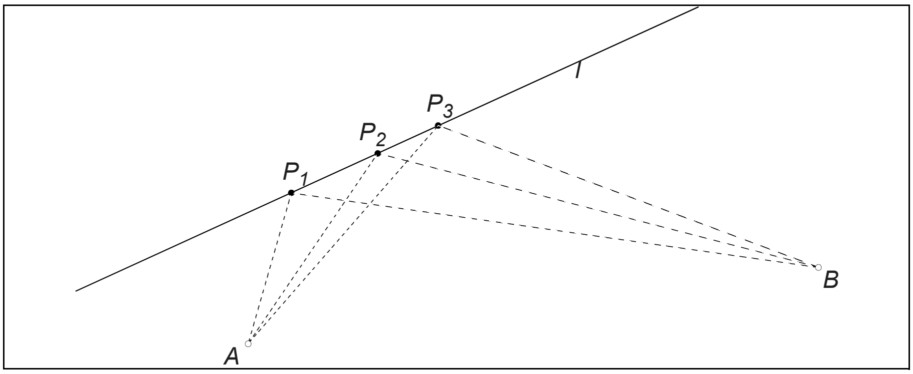
This is not as easy as it looks!
However, the argumentation of exercise **3** is beautiful, and we will hold on to it. From here on, though, we choose to look for more certainty.
We will follow this strategy:
a. Show unimpeachably that the Voronoi edge of two centers is a straight line.
b. Show that under the condition that the three centers are not on one line, both edges do intersect.

In this fashion we will look for a solid foundation in our argumentation. That has to be found in the properties of the concept of distance, because that is where it all began. In the following paragraph we will work towards this solid foundation, starting with a possibly unexpected problem.

9. Shortest paths and triangle inequality

shortest-path-principle
The shortest path from point A to point B is the straight line segment, which connects point A with B.

6. But what is the shortest route (figure on next page) from A to B if during the we also have to go *via line l* like in the figure on the next page? We will get to the bottom of this now.



a. Measure which of the three paths from A to B is the shortest.

b. We don't know whether there might be an even shorter path! Here is a pretty trick:
Reflect A in line l, name the reflection A'.
Also connect A' to the points P.
Why does the following now apply:
from A to B via P_1 is as long as from *A'* to *B* via *P_1?*

c. Now determine, using point *A'*, point *Q* on *l,* such that the path via *Q* leads to the shortest path.

d. Think of a situation where it is of importance to find a shortest path of this kind.

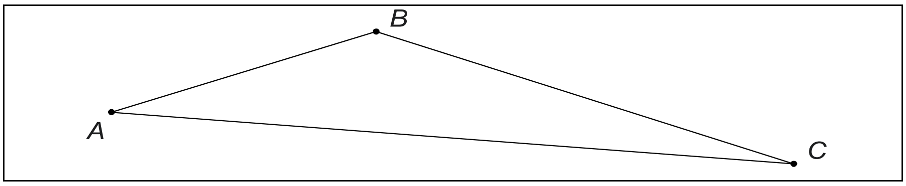
There is much more to discover about finding shortest paths in more complex situations. We will do so in chapter 6. For now we only establish that the *shortest-path-principle* is the basis of the solution. We will rephrase this principle more precisely. First, since we will be talking about *distance* all the time, we introduce a notation for the distance between two points.

distance notation
From here on we will denote the distance between two points A and B as $\mathrm{d}(A, B)$. Because we are thinking in terms of comparing distances, it does not matter whether you think of centimeters on paper or of kilometers in the landscape.
$\mathrm{d}(A, B)$ is always a nonnegative number and you can use it in expressions such as equalities and inequalities. Also expressions like $\mathrm{d}(A, B) + \mathrm{d}(C, D)$ have meaning. The *d* originates from the word *distance*.

simple properties
7 **a.** Translate in common English what is asserted here:
The following always holds for points P and Q : $d(Q, P)= d(P, Q)$

b. What can you say about points A and B if $d(A, B) = 0$?

The figure shows three points and their connections:



Next we will describe, using the d-notation, that going from A to C via B is a detour when B is not on the line segment AC. This has a name: the *triangle inequality*.

Triangle inequality

> **Triangle Inequality**
> **For each set of three points A, B and C holds: $d(A,C) \leq d(A, B) + d(B, C)$.**
> **The equality occurs only if B is on the line segment between A and C.**
> **In every other case there is a true inequality.**

The name *triangle inequality* originates from the fact that the *inequality* holds if A, B and C form a *triangle*.

8. In the figure on the previous page, expanded with A' and Q, you can apply the triangle inequality to show that A-Q-B is the shortest path.
   a. To which triangle would you apply it?
   b. Q is on $A'B$; this is drawn. What can you say about the triangle inequality in this situation?

9. On closer inspection the *triangle inequality* is somewhat more modest than the *shortest-path-principle*.
   a. Sketch a situation in which two paths are compared, and where the shortest-path-principle has some meaning, but the triangle equation has not.
   b. The argumentation of exercise **6** runs impeccably and yet only the triangle inequality is used. What causes that?

Of course we choose the simplest principles as starting-points for argumentation.

starting-point for argumentation
We therefore assume the triangle inequality as an established fact. From now on, you can refer to it in your argumentations.

10. You could legitimately ask yourself: What is the triangle inequality itself based on? If you asked yourself that question, the following exercise is for you, otherwise not.

a. Good question, difficult answer. Try to think of something yourself on which you would base the triangle inequality.

b. In that case, what would be the next question?

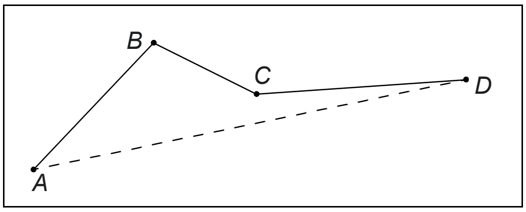
A different notation

Often the notation $|AB|$ is used for the length of a line segment with end points A and B. In this chapter, where everything breathes the air of distances, we use the notation $d(A, B)$. When we encounter figures with line segments and their lengths, you will also see $|AB|$.

extra exercise

11. Show that for every set of four points A, B, C and D holds:

$$|AD| \leq |AB| + |BC| + |CD| \ .$$



10. The concept of distance, Pythagorean Theorem

Another very important property of the concept of distance can be expressed as the well-known Pythagorean Theorem for right-angled triangles.

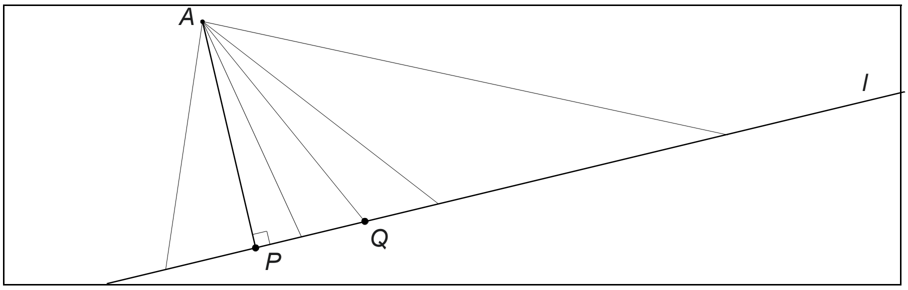
12. Phrase that theorem using the d-notation of the previous section. Your phrasing should deal with a triangle ABC, of which one angle is right.

We will now use this theorem in order to determine the shortest distance from a point to a line and also to ensure the correctness of the method.

shortest distance to a line

In this figure A is a point outside the line l. You are probably convinced of the following:

Of all possible connection line segments the line segment, which is perpendicular to l is the shortest.

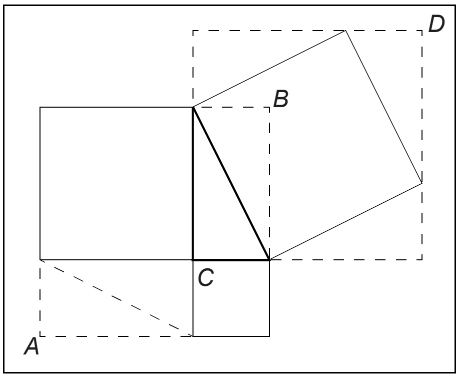


13 **a.** Write down – in *d*-notation – what holds for triangle *APQ* according to the Pythagorean Theorem.
b. How does d(*A, P*) < d(*A, Q*) result from that?

The Pythagorean Theorem is also one of the fundamentals you can use. You could also prove the Pythagorean Theorem based on more elementary things, but we will also not do this exhaustively. One possibility is outlined below as an 'extra'.

extra: a proof by Multatuli
(*This is nr. 529 from part II of the* Ideas *by Multatuli* (*1820–1887*))

I recently found a new proof for the Pythagorean theorem. Here it is. By, as shown in the adjacent figure, constructing six triangles – each equal to the given right-angled triangle – one acquires two equal squares, AB and CD. If one subtracts four triangles of each of these figures, one proves the equality of the remainder on both sides, which was to be shown.



It cannot be done any simpler, or so it seems to me. After finding this proof, I heard of the existence of an article, which discusses this topic. I purchased this little book, but it did not contain my demonstration. Furthermore I deem that none of the therein assimilated proofs is as illustrative and clear as mine.

Thus far the proud writer of the *Max Havelaar*.

Multatuli's proof leans heavily on the concept *area*. We did not exactly establish what its properties are. Moreover, it is rather easily assumed that certain parts of the figure are squares. But very well, for now we will join Multatuli.

14 a. Put some more letters in the figure and write down an argumentation which eventually leads to the equality part of the Pythagorean Theorem, expressed in areas of certain squares.

b. What would you have to show in order to conclude that the oblique 'square' is in fact a square.

extra; alternative for exercise 13 a/b

15. You can also prove without the Pythagorean Theorem that the perpendicular line from A to l provides the shortest distance. Use the following hint and your own inventiveness.
Hint: *How do you get from A to A the fastest if you have to go via line l?*

Pythagoras in a medieval monastery

The illustration below comes from a medieval manuscript. It was made in the monastery of Mont Saint Michel in Bretagne, when Robert de Torigni was the abbot, during the years 1154 through 1186. The manuscript contains figures and texts about astronomy; the abacus, bells, and of course mathematics are used in each of those. Presumably a lot is copied from Arabic manuscripts; in the Arabic world of those days a lot more attention was paid to mathematics and science than in Christian Europe.
This picture is of an application of the Pythagorean theorem in archery. You can see the arch of the bow, and the word 'sagitta' (arrow) is written at the hypotenuse and the base. You can also see close to the sides: 'filum V pedii', 'filum IIII pedii' and 'altitudo III pedii'. Translated: threads of 5 and 4 foot, a height of 3 foot. It is the well-known 3-4-5 triangle.

11. Properties of the perpendicular bisector

Next we will use the triangle inequality to prove that the
Voronoi edge of the centers A and B
is equal to the
perpendicular bisector of A and B.

important
This section is definitely the hardest of this chapter. Even if you do not catch on to all the details, you will be able to continue with the next section, but make a good effort to try to follow the reasoning. The more argumentations like this you can follow, the easier it will get later on, simply because you have had some training.

First a *definition,* which should be familiar. With definitions in mathematics we establish exactly what is meant.

definition Voronoi edge

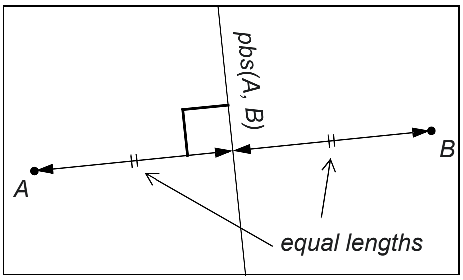
The Voronoi edge between two points *A* and *B* is the set of points *P* for which hold: d(*P, A*) = d(*P, B*).

In chapter I you got the big impression that Voronoi edge of *A* and *B* is exactly the perpendicular bisector of the line segment *AB*. We will now *prove* this. Since we want to start only from definitions and familiar things, we also have to define 'perpendicular bisector'.

definition perpendicular bisector

The perpendicular bisector of line segment *AB* is the line which is passing through the midpoint of *AB* and is perpendicular to *AB.*

pbs(A, B)
We will agree upon a notation for the 'perpendicular bisector of line segment *AB*': *pbs(A, B)*. The figure displays two characteristics of the perpendicular bisector:



– it is *perpendicular* to the line segment
– it *divides* the line segment *in two.*

What we would love to pose is the fact that the two concepts of Voronoi edge and perpendicular bisector are actually one and the same. So we state:

statement of equality

The Voronoi edge of two points A and B and the **perpendicular bisector** of the line segment AB coincide.

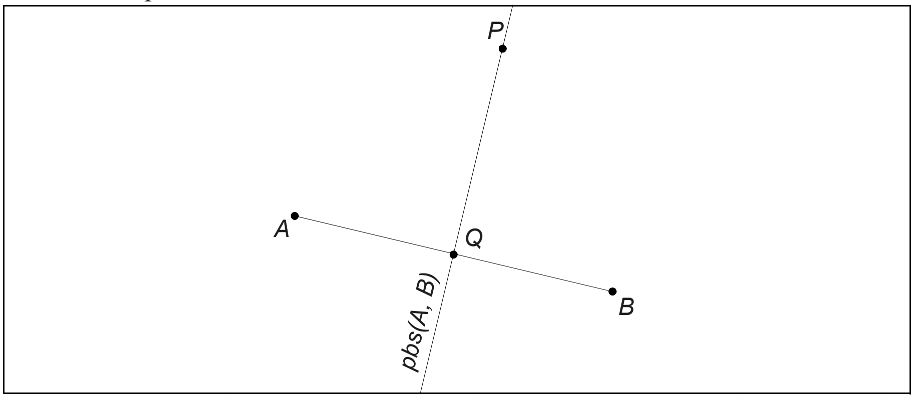
This means quite a lot: not only that all of the points of the perpendicular bisector lie on the Voronoi edge, but also that the Voronoi edge does not consist of more points. And vice versa. That is why two things need to be proven separately:

a. Every point which lies on the perpendicular bisector, is also on the Voronoi edge.

b. Every point which does *not* lie on the perpendicular bisector, is also *not* on the Voronoi edge.

We will discuss both parts separately.

16. Proof of part a:



Every point, which lies on the perpendicular bisector is also on the Voronoi edge

The figure shows line segment AB and also the *pbs(A, B)*. Q is the middle of AB. P is a point that lies on the perpendicular bisector.

a. Indicate in the figure, in green, the *two* things, which you can use now according to the definition of *pbs(A, B)*.

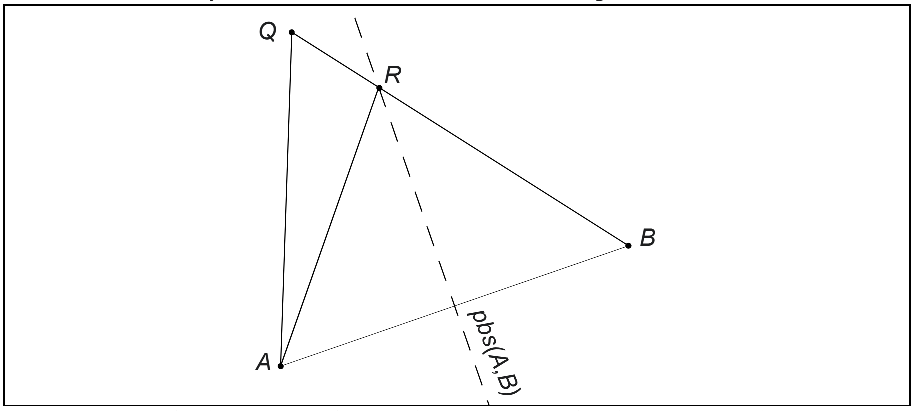
b. Color the line segments of which you have to prove that they have the same length red.

c. Write down what Pythagoras says about d(P, A) and d(P, B) and derive from that:

$$d(P, A) = d(P, B).$$

d. Did you use both of the characteristics of the perpendicular bisector? Where in the argumentation?

17. We are halfway there, but we still have to do the proof of b:



Every point, which does not lie on the perpendicular bisector, is also not on the Voronoi edge.

See the figure above. *Q* is not on *pbs(A, B)*, but on the side of *A*. *BQ* will then certainly intersect with the perpendicular bisector, call the point of intersection *R*. *R* is certainly not on line segment *AQ*. This is what you know and what you can use in your proof.

- **a.** Write down in *d*-notation what you need to prove.
- **b.** Since we have already proven part a, you do know something about point *R*. Note that as an equality.
- **c.** Also formulate an inequality, which contains *Q*, *R* and *A*.
- **d.** Combine these to obtain the wanted conclusion.

18. Does the figure of the triangle inequality remind you of a problem we encountered earlier?

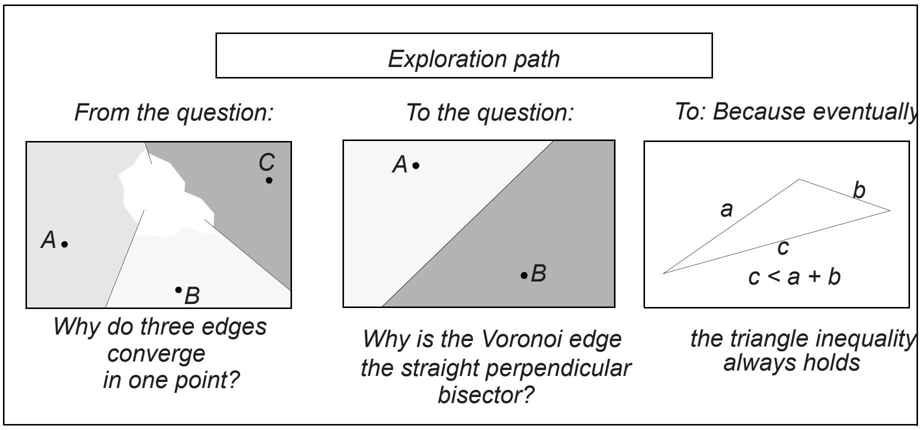
As a matter of fact, we also need to research the possibility that d(*A, Q*) > d(*B, Q*). Of course *Q* is on the other side of *B,* and this boils down the whole thing to consistently switching the letters *A* and *B*. This is no longer interesting.

12. From exploration to logical structure

Introduction to this section

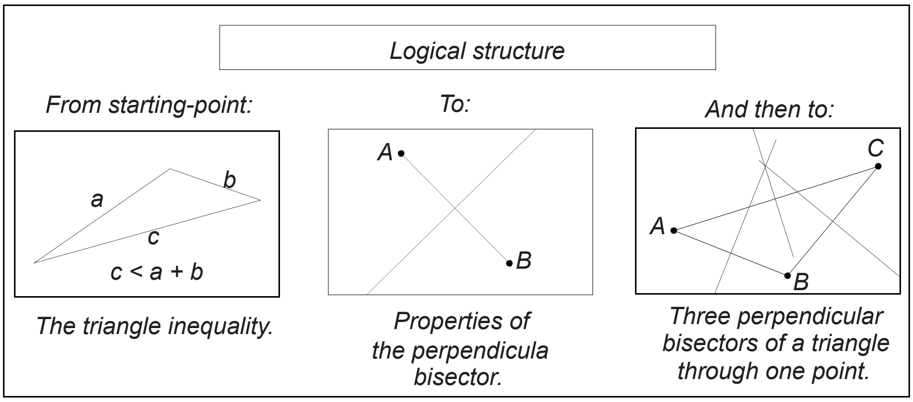
research

In the preceding we explored why the three Voronoi edges of three points (in general) meet in one point. We started off with the problem of partitioning an area and then found out that a fundamental property of the concept of distance was of importance. The exploration developed as follows:



So far the exploration phase.

logical structure
Since we are reasoning, the logical structure will be the other way around when we look at it afterwards: first the triangle inequality, and deduct from there that the *Voronoi edge* and the *perpendicular bisector* coincide and then finally derive from there the statement about the concurrency of the three edges.
In this section we will repeat the whole in that last form. We will abandon the terminology of the Voronoi diagram; we will now make our proof mathematically pure, like this:



We will now formulate the most important statements, which will be proven concisely as theorems. We will number the theorems, as this will make future reference easier. That is exactly what happens in a logical structure: what is proven before, you can use later.

Starting-points: triangle inequality and Pythagoras

The triangle inequality can be proven from other, more primitive starting-points. However, we will not do this. It will be our first theorem.

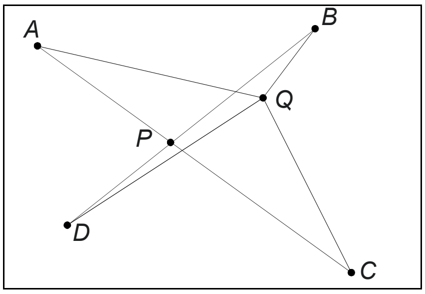
Theorem 1 (Triangle inequality)

For each set of three points A, B and C holds: $d(A,C) \leq d(A, B) + d(B, C)$.
The equal sign occurs only if B lies on the line segment AC.
In all other cases a real inequality occurs.

Next an exercise in practicing the use of the triangle inequality.

Four points are given: A, B, C and D. P is the point of intersection of the line segments AC and BD. Q is a point different from P. Note: P is special, Q is arbitrary.
You have to show that
the four distances from Q to A, B, C and D together are bigger than
the four distances from P to A, B, C and D together.

19 **a.** a. Write down the to be proven statement using the d-notation as follows:
To show: $d(P, A)$ + ... ≤ ....
b. Then start the proof with:
Proof: ... and use (one or more times) the triangle inequality.



For the sake of completeness the Pythagorean theorem is stated below. You saw a proof of it when you did the extra exercises on page 69.

Theorem 2 (Pythagoras)

If in a triangle ABC angle B is right, then this equality holds:

$$d(A, C)^2 = d(A, B)^2 + d(B, C)^2.$$

The perpendicular bisector

We gave a definition of a perpendicular bisector. We will copy it here.

definition perpendicular bisector

> **The perpendicular bisector of line segment *AB* is the line which passes through the midpoint of *AB* and is perpendicular to *AB*.**

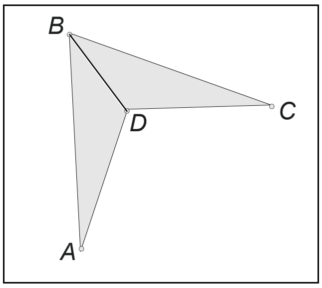
The main properties of the perpendicular bisector are mentioned in the next theorem, which we showed before, on page 72:

Theorem 3

The perpendicular bisector of line segment *AB* is the set of points *P* for which hold d(*P, A*) = d(*P, B*).
For points *P* outside of the perpendicular bisector holds:
If d(*P, A*) < d(*P, B*), then *P* lies on the *A*-side of *pbs(A, B)*.
If d(*P, A*) > d(*P, B*), then *P* lies on the *B*-side of *pbs(A, B)*.

You can apply the theorem in the following problem. Think about this: if two points of a Voronoi edge are known, you know the entire edge.

20. In this delta wing *AB* and *BC* have equal length and also *AD* and *DC* are of the same length. Show that *BD* is perpendicular to *AC*.



Perpendicular bisectors in the triangle

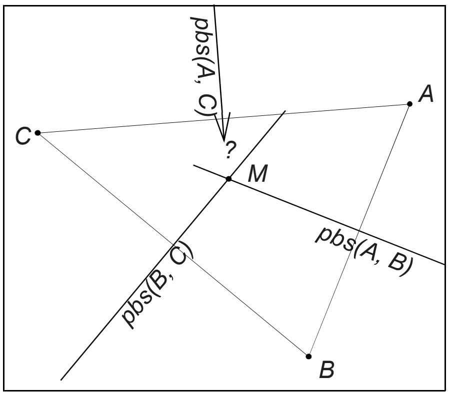
Theorem 4

In each triangle *ABC* the perpendicular bisectors of the sides meet in one point.

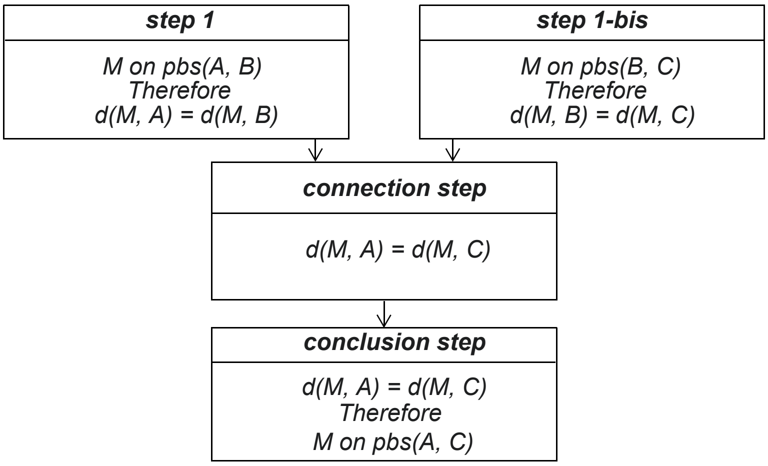
21 a. In the section '*An argumentation about three-countries-points and circles*' there was a problem: the three centers were not allowed to lie on one line. How did we get around that here? If the three points do lie on one line, what would happen to the three perpendicular bisectors? Sketch a (complete) figure.

b. How is the three points on one line situation excluded by the wording of the theorem?

Adjacent you see a figure, which illustrates the theorem. *M* is the point of intersection of the perpendicular bisectors *AB* and *BC*. On the next page you see a scheme, which represents the proof.



22. In fact, this is the proof as it was given in exercise **3**, page 60.
  a. Point out exactly how the different parts of the exercise correspond to the ones of the scheme.
  b. Step 1 and 1-bis differ from the conclusion step. In which way?



The circumscribed circle

definition of the circumcircle

> The circumcircle of a triangle *ABC* is the triangle's circumscribed circle, i.e. the circle that passes through each of the triangles' three vertices *A*, *B* and *C*.

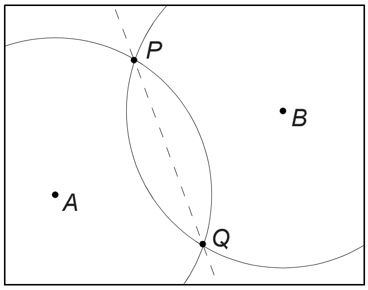
The theorem about the circumcircle is no easy to formulate.

Theorem 5

> **Each triangle *ABC* has one and only one circumcircle. The center of the circumcircle, i.e. the circumcenter, is the intersection of perpendicular bisectors of the triangle.**

The proof is simple: the point *M*, where the three perpendicular bisectors intersect, has equal distance to each of the three vertices and is the only point with that property.

We will now do some exercises with circles and perpendicular bisectors.



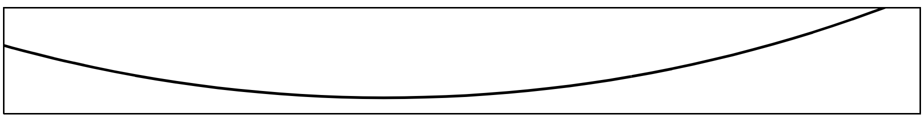
23. This figure shows two circles with equal radii. Their centers are *A* and *B*. Based on which theorem do the intersections *P* and *Q* lie on the perpendicular bisector of line segment *AB*?

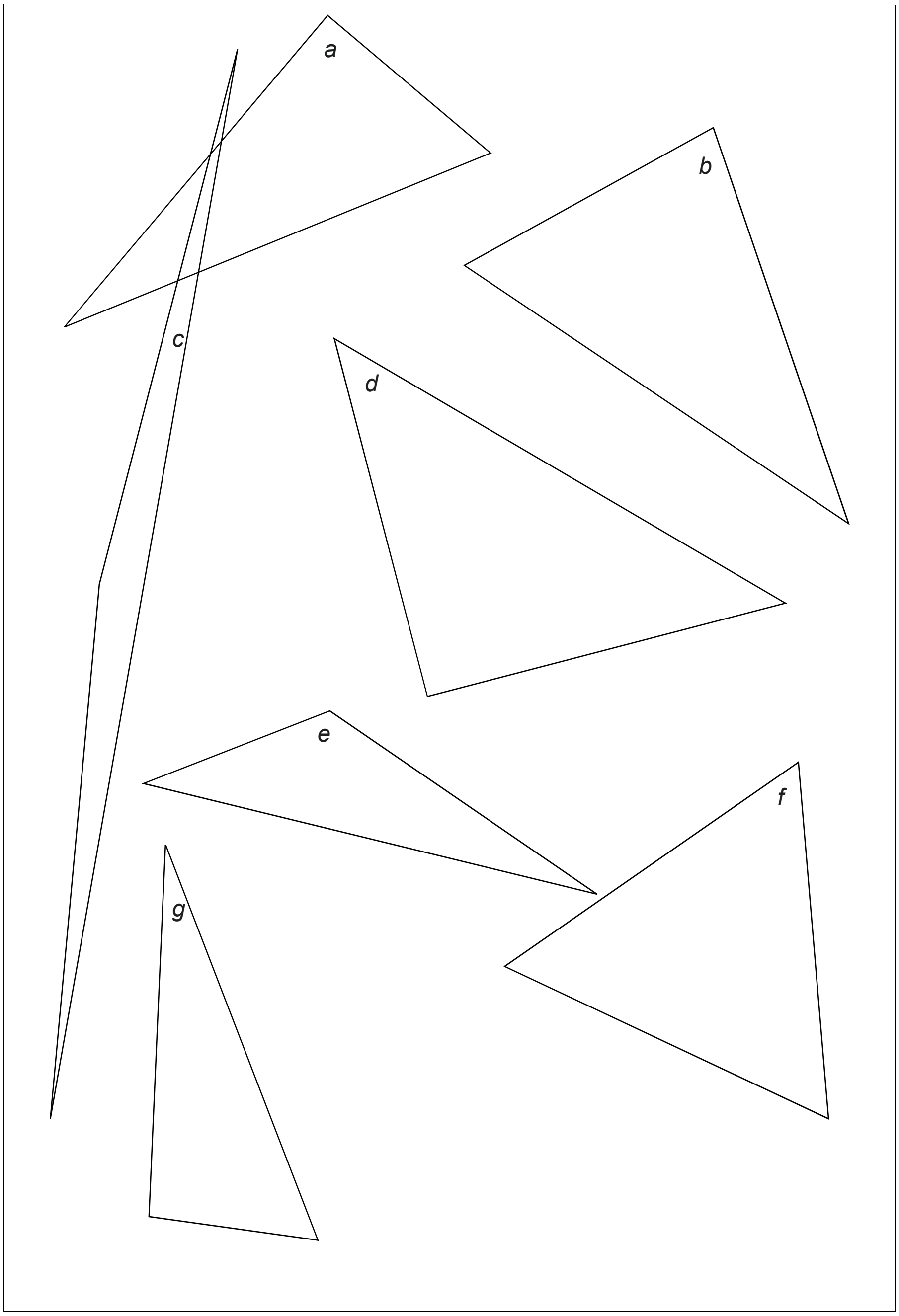
This is a recipe to construct the perpendicular bisector with compass and ruler. Do not use a protractor or the numbers on the ruler.

24. Examine how Dürer used this technique for the front page of this chapter, when finding the circumcircle about the three points *a*, *b* and *c*.

25. Pick out a few of the triangles on the next page and determine the intersection of the perpendicular bisectors and sketch the circumcircle. In at least one case, use the construction with compass and ruler.
Choose these in such a manner that one center lies *inside*, one center lies *on* and one center lies *outside* the involved triangle. How does this *inside-on-outside* link to the shape of the triangle?

26. Here a part of a circle is given. Determine, using only compass and ruler, the center of the circle.(Hint: to start, place some points on the circle)




a
b
c
d
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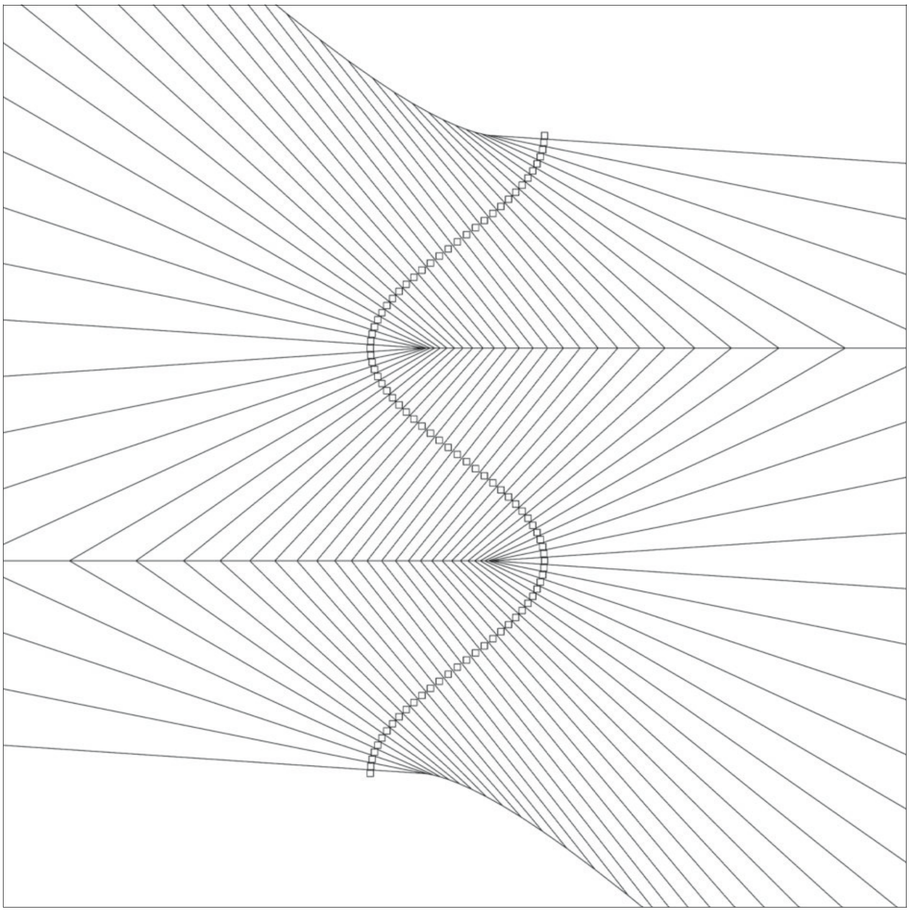
g

Chapter 3: Computer practical Voronoi diagrams



Introduction, software availability
In this chapter we will construct Voronoi diagrams using the computer. The chapter is of a practical nature: you will try a lot, but prove little to nothing.
Since we will be working with more than 5 or 6 centers rather fast and easily, we can also look at other properties than the ones in chapter 2.

Most of the tasks in this chapter can be done with the free applet VoroGlide (FernUniversität Hagen; made by Christian Icking, Rolf Klein, Peter Köllner, Lihong Ma)**.**

Starting Voroglide:
Go to http://wwwpi6.fernuni-hagen.de/GeomLab/VoroGlide/index.html
or search for VoroGlide with Google.

Basic use of VoroGlide
You can generate points by clicking your mouse.
- Already made points can be dragged.
- Remove points with right-mouseclicks or ***Clear*** under ***Edit***
- If you want several points on a line or circle, you can put a physical object (cup, ruler) on the screen to help you. Also, because you can only drag points as far as the border, you can use the borders as lines.
- Under ***Show*** you can choose three types of diagrams; combinations are possible. You will discover their meanings in this chapter.

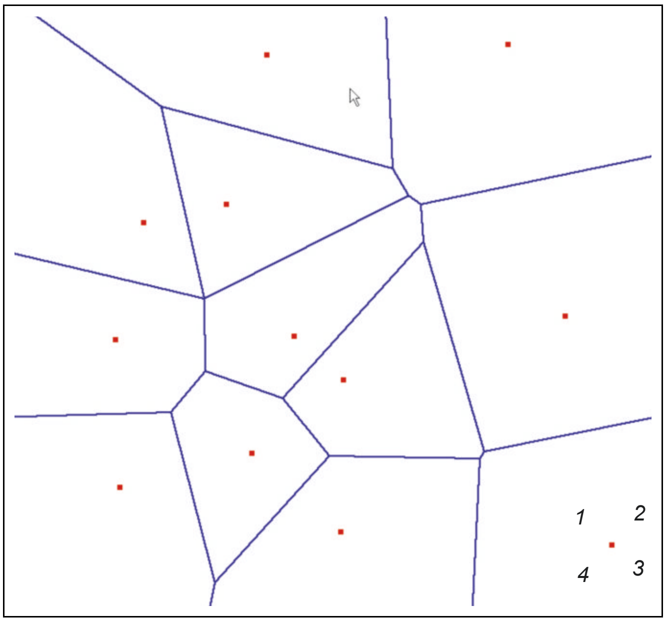
13. Introduction

You know that constructing a Voronoi diagram is based on drawing perpendicular bisectors, but finding the right line segments can be very time-consuming if there are a lot of centers. In applications of Voronoi diagrams we usually have situations with a lot of centers. We will also look for changes in the configuration when one points moves slightly. It is obvious that there is a need for computer programs that can do the time-intensive sketching. We will be working with such a program now.

start, usage
You can find information and simple directions for using the program VoroGlide on the previous page. Once the program is running, the screen will look as follows: You operate the program mainly with your mouse; you only need to click with the left button. On the left side of the screen you click points on or off. On the right side you click on the task for the program. That is all. For some assignments you will be given directions on how to use the `report`-part of the screen.

1. Start VoroGlide and explore the possibilities.

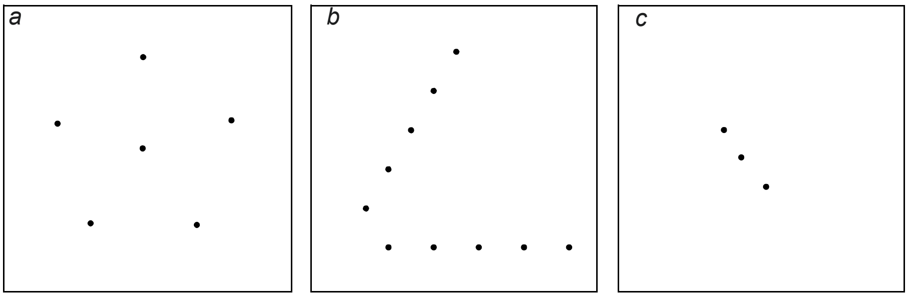
2 **a.** Make the screen look like this illustration.



b. There is a four-country point. Add a point to make it into a five-country point without drawing a circle. Indicate in the illustration above all possible places for this fifth point.

c. In the illustration there is also an ‘almost’ four-country point. In which of the four indicated direction should you drag the low-right center to get a real four-country-point. Think first, drag later!

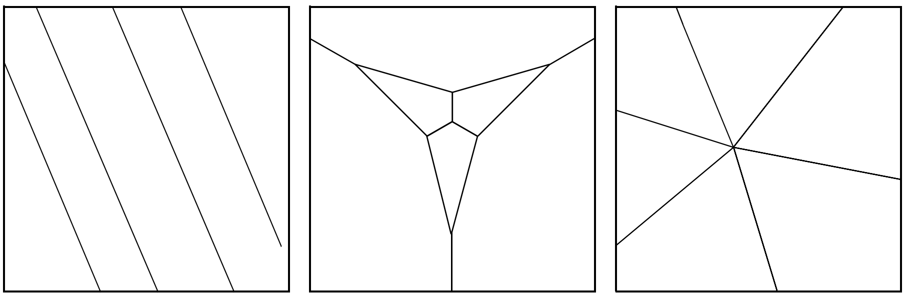
3 **a.** Create Voronoi diagrams according to the following positioning of the centers.



b. If you construct situation *b* very precisely, a number of edges will be straight behind one another. They form a line. Which line is that?

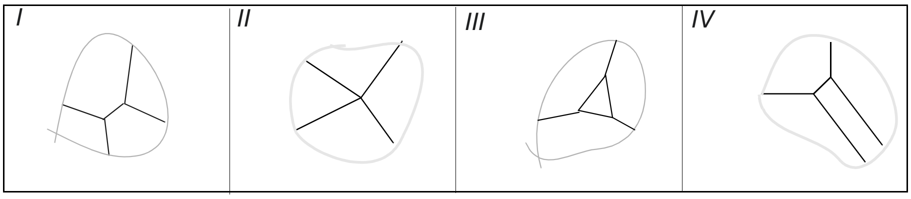
c. In situation *c* you cannot easily see a three-country-point on the screen. Now move the middle point a bit left and right. Try as best as possible to find a spot for this point which makes the diagram really not having a three-country point, also not outside the screen.

4. Construct by clever choosing the centers of the Voronoi diagrams in the figures shown below. Remember that you can also sketch circles and lines first to find out where the centers must be placed. Then indicate in the figure where the centers need to be approximately.

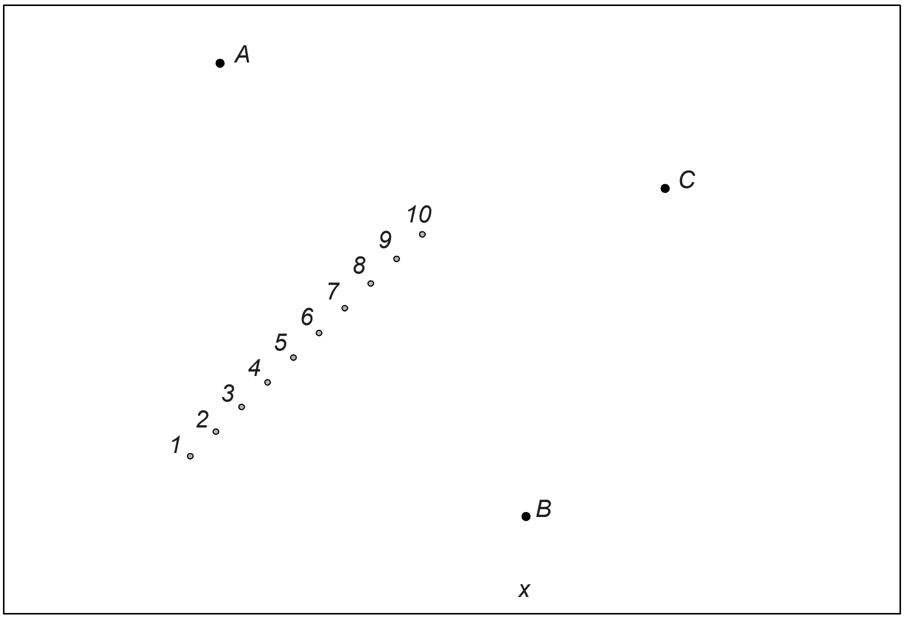


14. *The influence of the fourth point*

There are only four clearly distinctive diagrams possible for four points, namely these four:



5. Choose approximately three points like the black points *A*, *B* and *C* shown in the figure below. In this exercise we will add a fourth point (*D*) every time and



see what the effect of that point is. (The points 1 through 10 don't play a part until the next exercise.)

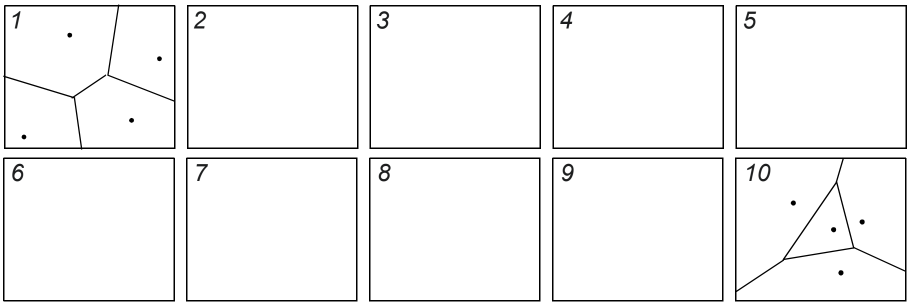
a. Indicate a fourth point in such a way that you get a type III diagram. If you move the point a little, the diagram will remain a type III.

b. Show on the screen all possible places for *D* for which the Voronoi diagram is of type II.

c. Do the same for type IV.

d. What type arises if *D* lies close to *x*?

6. If the fourth point consecutively takes the positions 1, 2, 3, ... , 10, the Voronoi diagram changes gradually. The beginning and ending of that process are given. Sketch the intermediate stages and determine the type for each state



7. From the preceding you can see that the transitions from one type to another type take place when point *D* passes through the circumscribed circle or through one of the three lines (through *A* and *B*, through *B* and *C*, through *C* and *A*).

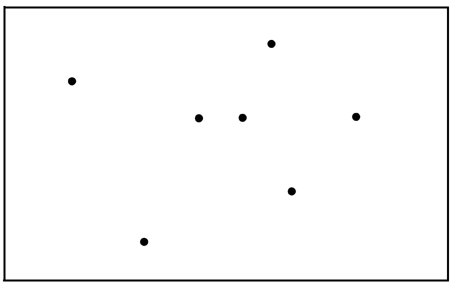
a. A type belongs to each intermediate stage. Indicate that in the figure with I, II, III or IV.

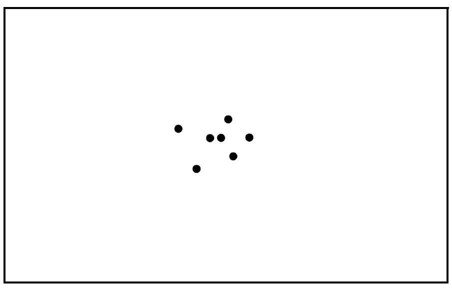
b. If you pick a point *D* ‘at random’, there is a greater chance for two of the four types, but the chance for the other two of the four types is very small. Why is that so?

15. Infinitely large cells, the convex hull

Meanwhile you have seen that there are cells which are enclosed on all sides by edges and that there are cells for which this is not the case. Now we will look especially at that last kind of cells.

8 **a.** For starters: make the Voronoi diagram for the situation at the left. There are only two finite cells.





b. Now make the situation at right; that is the same but four times smaller. What does the diagram start to look like when you reduce much more?

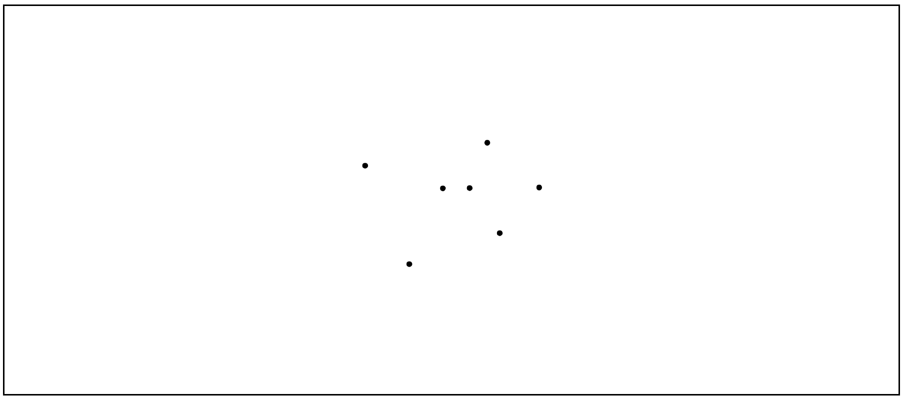
convex hull

c. Now go back to the first situation and choose *Show* >> *Convex Hull*.
A green closed line is added now. Imagine that the centers are nails in a board, which still stick out a bit. The green line looks like an elastic ribbon around all the points.

d. Add a couple of points *within* the hull. Are their any new infinite cells? Why is that so?

e. Add one new point, but do it in such a fashion that you add exactly one infinite large cell and that the other infinite large cells stay infinitely large.

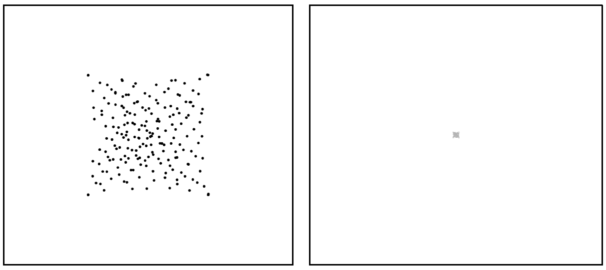
f. Here we have the situation of question **8 a** again, with some more space around. Indicate with a color the areas in which you can choose a new center in such a way that all the centers which are on the hull do not come loose from the hull.



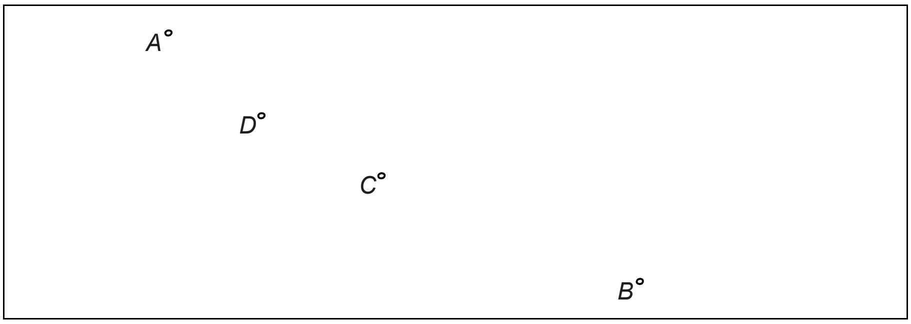
conclusion 1

The centers on the convex hull have infinitely large cells.

9. Here you see a situation where the computer cannot help you to sketch the Voronoi diagram.
This would take too long. However, you can predict how the diagram will look, if you decrease the scale a lot. Sketch that approach in the space on the right, where the little gray cloud in the middle represents the group of centers.



10 a. Construct this situation, with Voronoi diagram and hull. Verify whether the cells of *A* and *D* adjoin.



b. Sketch the circle through *A*, *D* and *B* and also the circle through *A*, *C* and *B*. Find their centers, if necessary resize the situation. Why do they need to lie on the Voronoi edge of *A* and *B* or on its extension?

c. Add within the hull of *A*, *B*, *C* and *D* a new center *E*, but in such a way that the line segment *AB* still belongs to the hull. Again, sketch the circle through *A*, *E* and *B*. Do the cells of *A* and *B* still adjoin? Why (or why not)?

We draw another temporary conclusion:

conclusion 2

Two centers which are connected with a line segment of the hull, have adjoining infinitely large cells.

11. Test with several examples whether this conclusion holds.

12. Construct an example with seven centers for which the Voronoi cells of the two furthest apart centers adjoin.

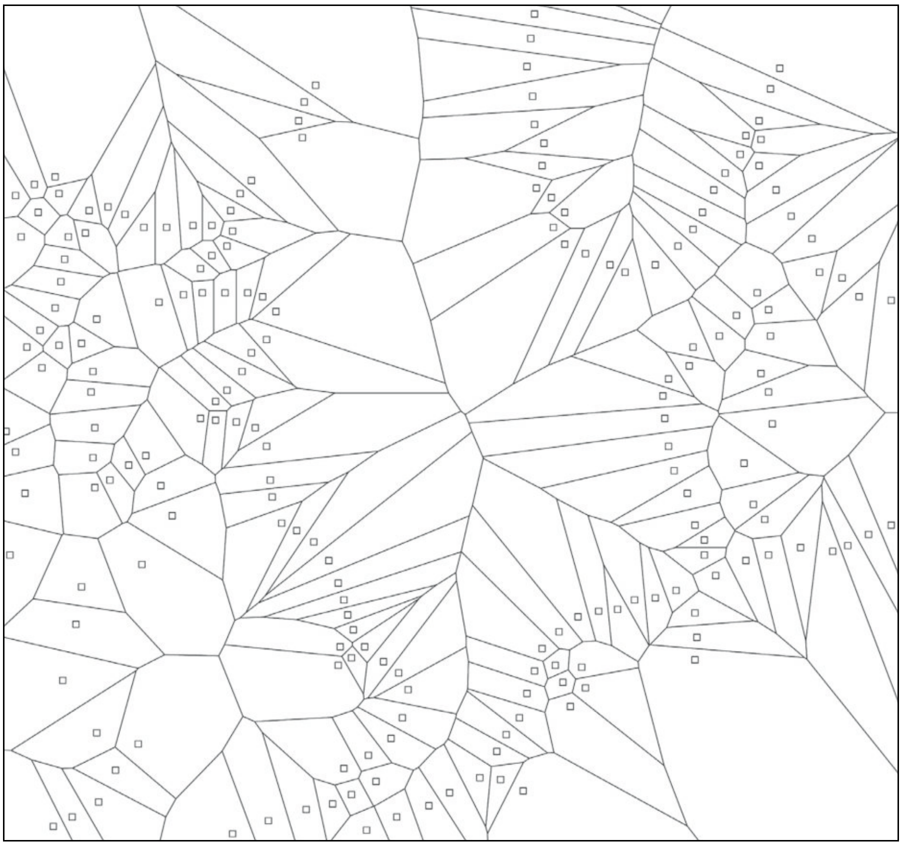
The two conclusions we drew in this paragraph look solid and reliable, but they are based only on your observations. You could prove them using the method from the previous chapter. Since it would be a lot of work and would not lead to new insights, and also since we will not build further on these conclusions, we will leave it at this.

16. Dividing the sea

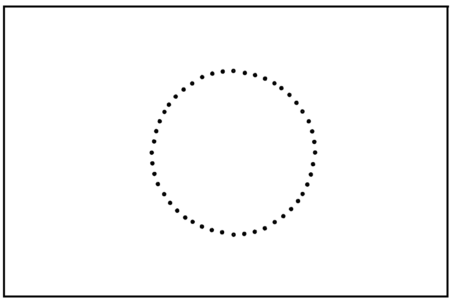
The North Sea

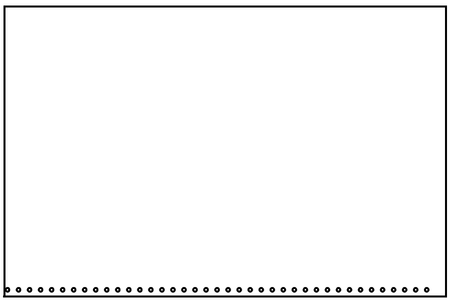
13. In the picture on the next page you see Great Britain on the left and the Netherlands, Germany, Denmark and Norway on the right; all coastlines have been sketched using only points. Using a specialised computer program, the Voronoi diagram is drawn.

- **a.** For oil exploitation purposes, the North Sea has been partitioned according to the nearest-neighbor-principle for whole countries. Draw this division in the picture.
- **b.** It is possible to draw a circle which is tangent to the coast of Great Britain and Norway. Find a possible midpoint for this circle and draw it with a compass.
- **c.** Find also a circle tangent to the coasts of Great Britain, the Netherlands and Denmark.
- **d.** Because of your circle in question c there is no circle tangent to Great Britain, the Netherlands and Norway. Is this argument okay?
- **e.** An important British oil harbor is located in the group of islands to the north-east of Great Britain (the Shetland Islands). Several years ago a mammoth tanker stranded on the rocks. Check how the North Sea would be partitioned if this group of islands belonged to a Scandinavian country. Before 1472 this was the case, but back then people did not drill for oil.



17. Exploring two more mathematical configurations





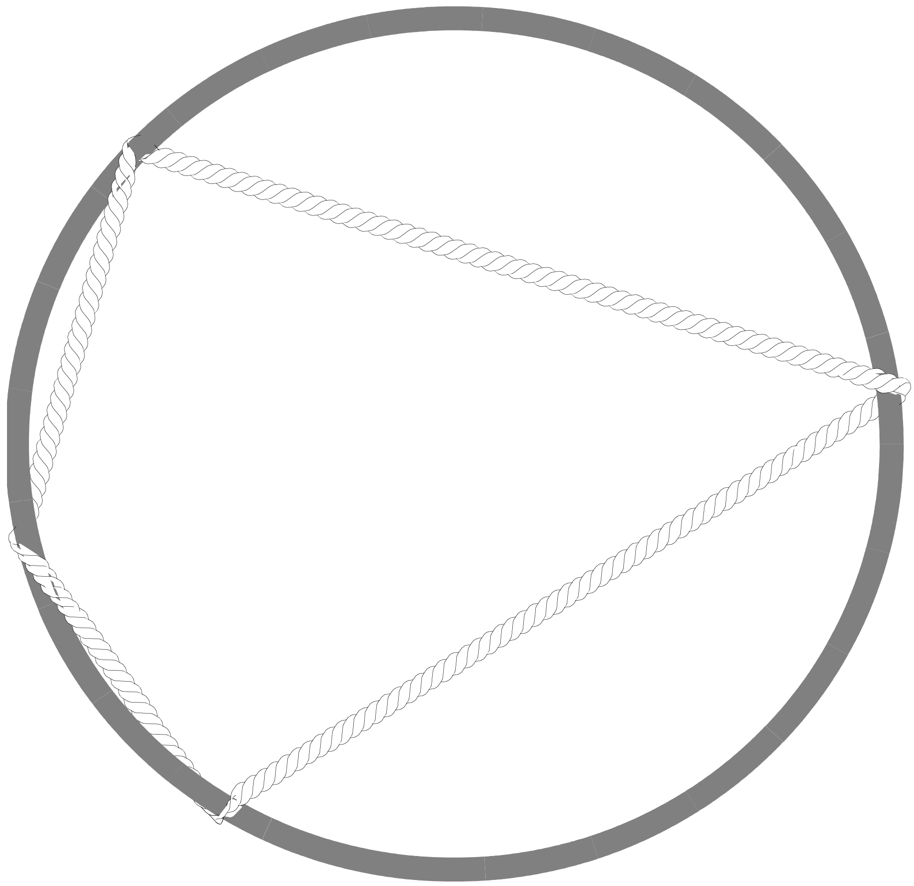
14. These two configuration can be made easily with a cardboard circle and dragging points to the bottom line.

a. Make the circle example and adjust points carefully so that all Voronoi edges meet in the center.
b. Now bring in a new center and drag that in and out of the circle. Explore what kind of cells you may get around that point. Is a new circle possible?
c. The oval shape that comes into being if the point is in the circle is an approximation of an ellipse. We will meet it again later! If the extra point is outside the circle, we find an infinite cell: it is a hyperbola, which we will also meet again later.
d. Make the points on the line. If you do well, all Voronoi edges should be parallel.
e. Now bring in a new point and explore what kind of cells you may get around that point. Does the shape looks familiar?
f. Again it is the approximation of a well known figure, the parabola. The extra point is called the focus, and the line the directrix.
g. Check this: a tangent to the parabola is the perpendicular bisector of the focus and a point on the directrix.
We will also come back to this property later!

Summary

In this computer practical you were able to try several things you have seen before, like the existence of largest empty circles. While exploring the influence of a fourth point on a diagram with three centers, it appeared that the circumscribed circle of the triangle of the three centers plays a dominant role. Furthermore, the sides of the triangle were also important. You also took a more precise look at infinite cells. Their centers turned out to lie on the convex hull. If you work with large numbers of points, you could form figures like countries on a map and explore their Voronoi borders. Using more mathematical arrangements like circles and straight lines, we discovered figures such as ellipses, hyperbolas and parabolas!

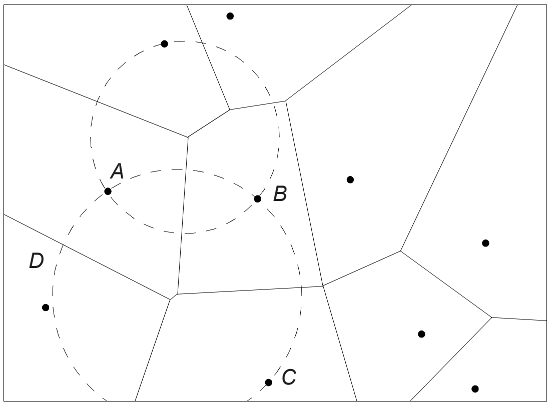
Chapter 4: A special quadrilateral



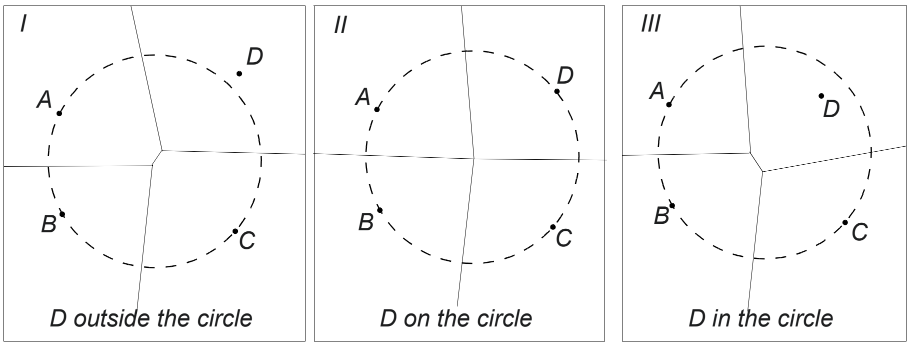
In this chapter we will continue reasoning.

In general we will deduct several things concerning distances and angles from very few given data. We will also think about the process of reasoning itself and how to write down proofs.

18. Cyclic quadrilaterals



In chapter I, when you were dealing with Voronoi diagrams, you encountered this example, in which point *D* lies just outside the circumcircle of triangle *ABC*. This circle is empty and thus the cells round *A*, *B* and *C* converge into a three-countries-point, which of course is the center of the circumcircle of triangle *ABC*. Thus *D* does not disturb the three-countries-point. For the position of the fourth point *D,* compared to the circle through the three points *A*, *B* and *C*, there are three options:



(In case III it is possible that *D* lies so close to *B* that the cell round *D* is closed.)

1. **a.** In case II the Voronoi diagram is very special: there is a four-countries-point.
Why?
b. Which vertex of the Voronoi diagram is the center of the triangle *ABC* in case I?
c. And how about case III?

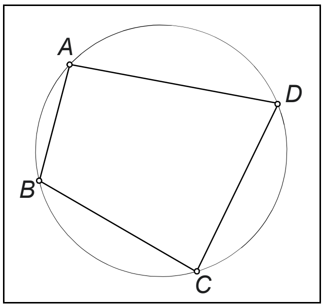
Since the sides of the quadrilateral in the special case are all *chords* of the circle, we call quadrilateral *ABCD* a cyclic quadrilateral. The **definition** is:

definition of cyclic-quadrilateral

A quadrilateral is called a **cyclic quadrilateral** if its vertices lie on one circle.

In this section we shall prove a relation between the sizes of the angles for these special quadrilaterals. Later on we shall use this relation for other purposes than Voronoi diagrams.

2. On the right you see a sketch of a cyclic quadrilateral *ABCD*. In the figure we didn't actually portray the main property of the circle: that there is a center *M* and that the line segments *MA*, *MB*, etcetera, have equal length.



a. Therefore, sketch the center and the line segments *MA, MB, MC* and *MD.*

b. The quadrilateral is now divided in four triangles. The eight angles to the vertices of the quadrilateral are equal two by two. Why?

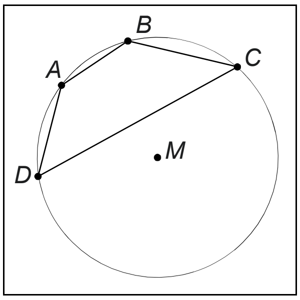
c. Indicate equal angles with the same symbol; use symbols like 1, 2, *, ∘, ′, ·. In each vertex you see a different combination of signs. But what do you notice when you compare the sum of the signs of *A* and *C* to the sum of *B* and *D*?

First a short footnote: The only correct answer to question **2b** is: because triangle *ABM* is isosceles. It is given that the sides are equal, namely d(*A, M*) = d(*B, M*). That you can conclude from this that the angles are equal, is based on an at the moment non-formulated theorem about isosceles triangles. We will not prove this theorem here, it is one of the things we accept for now, like earlier the triangle inequality. Later on, you will draw up a numbered list of these kinds of theorems, which have already been familiar to you for some time.
What you have just proven is the following *temporary* theorem:

Temporary theorem of the cyclic quadrilateral
In each cyclic quadrilateral ABCD holds: $\angle A + \angle C = \angle B + \angle D$.

We reached a fine result by smart reasoning. However, there are some problems left.

Problem A



The first question you need to ask yourself is: Is the theorem proven for every cyclic quadrilateral one can think of? For example, consider a cyclic quadrilateral as shown on the right. We again looked at just one special figure, which is not representative for all cases. For in this one, the proof above does not hold....

Problem B

The temporary theorem only deals with equality of $\angle A + \angle C$ and $\angle B + \angle D$. This is a bit meager. Maybe something can be said about the *size* of $\angle A + \angle C$ and $\angle B + \angle D$.

Problem C

Just like in the complete theorem about the perpendicular bisector, theorem 3, page 71, we need to know what happens if D lies inside or outside of the circle, since this is the most frequently occurring case!

19. Scrutinize proving

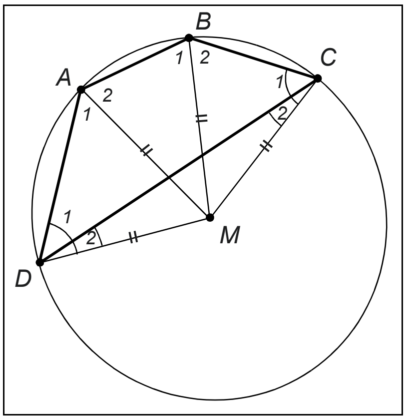
Problem A

3 **a.** What is the essential difference between this cyclic quadrilateral (last figure on previous page) and the one from exercise 2?

b. Again go over the steps of exercise 2. Where do you need to deviate from exercise **2** for this case?

It is not very difficult to find a proof for this situation. We will draw up this proof in a clearly noted form as an exercise in notation.

This is the accompanying sketch.



Now we can talk easily about all kinds of angles and parts of angles in the situation without referring to them with strange symbols.

4. In the theorem we are proving $\angle A$ plays a role. By $\angle A$ we mean the vertex angle $\angle DAB$. The sketch also shows $\angle DAB = \angle A_1 + \angle A_2$. Here we are not reasoning based on a sketch, but merely showing what we mean with all those letter notations.

a. What does $\angle D$ mean in the theorem? Write down the relation of $\angle D$ with $\angle D_1$ and $\angle D_2$.
Look carefully at how the little arches are indicated.

The complete proof could start as follows:

$\angle D_1 = \angle A_1$, $\angle A_2 = \angle B_1$, $\angle B_2 = \angle C_1$, $\angle C_2 = \angle D_2$
(since the triangles *DMA*, *AMB*, *BMC* and *CMD* are isosceles)

This is actually
a statement
with a motivation.
In the remainder of the proof we will use the data from the sketch and these four equalities to rewrite the sum of angles $\angle A + \angle C$ step by step to $\angle B + \angle D$. Between the brackets is stated why an equality holds, thus those are again

$\angle A + \angle C = \angle BAD + \angle DCB$
= (dividing angles)
$(\angle A_1 + \angle A_2) + (\angle C_1 - \angle C_2)$
= (using equal angles)
. . . .

motivations.
b. Complete this story.
The last line of this story will be
$= \angle B + \angle D$
(You could – if you get stuck – search by starting at $\angle B + \angle D$ and splitting up the angles)

For the first case (where *M* lies inside of the cyclic quadrilateral *ABCD*) you could have written down the proof in the same fashion.

5. The sketch would be different, but in the proof you would need to change some details. Which?

case distinction
We are still working on the temporary theorem of the cyclic quadrilateral. We have made a careful distinction between the cases where the center of the circumcircle lies inside or outside the quadrilateral.

6 a. Is the temporary theorem of the cyclic quadrilateral now proven for áll possible cyclic quadrilaterals? In other words: are there other situations than those where *M* lies inside respectively outside of the quadrilateral?
b. If you find another case, which of the two proofs holds?

Conclusion from this part for Problem A

The temporary theorem of the cyclic quadrilateral was put under some pressure, but is eventually saved by adjusting the proof for the other case. While doing that we also practiced how to write down a proof clearly. We distinguished statements and motivations. Noting angles with indices was handy for keeping the relation between proof and sketch. On to problem B now.

Problem B

That was:

The temporary theorem only talks about the equality of $\angle A + \angle C$ *and* $\angle B + \angle D$. *This is a bit meager. Maybe something more can be said about the size of* $\angle A + \angle C$ *and* $\angle B + \angle D$ *themselves.*

7 **a.** a. Find out in exercises **2** and **4** what $\angle A + \angle C$ and $\angle B + \angle D$ are, using a protractor. The four results are not very different!

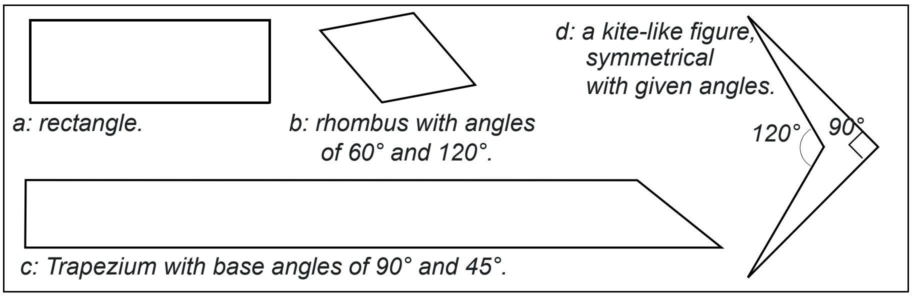
b. What is your conjecture (still to be proved!) about the sum of opposite angles in a cyclic quadrilateral?

If your statement is right, you can also say something about $\angle A + \angle C + \angle B + \angle D$ in such a quadrilateral. Before we will prove that the sum of the four vertex angles in a *cyclic quadrilateral* is 360°, we check whether we really need to restrict ourselves to cyclic quadrilaterals.

Namely, we switched from $\angle A + \angle C = \angle B + \angle D = 180°$ to $\angle A + \angle C + \angle B + \angle D = 360°$, but the last thing also holds if we have for instance $\angle A + \angle C = 140°$ and $\angle B + \angle D = 220°$. In other words: for the total sum of angles of 360° we maybe should not restrict ourselves to cyclic quadrilaterals.

We will first try to find out how general $\angle A + \angle C + \angle B + \angle D = 360°$ can be true.

8. Determine the total internal sum of angles for these examples. These are a few special cases for which it is easy to compute and determine angles.



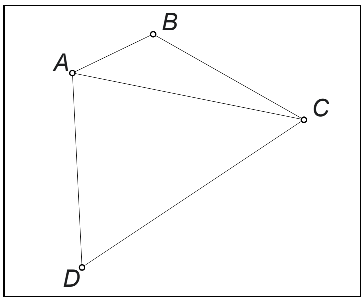
(Note that in ***d*** the angle of 120° is not an inside-angle of the quadrilateral.)

Let's formulate the statement first as a theorem, the proof will follow.

Theorem 8
In each quadrilateral the sum of angles is equal to 360°.

You will give the proof in the form which just has been shown. For the kernel of the proof you will of course need to know where to look, but you do already know that the sum of angles in a triangle is 180°. This is something you learnt earlier which you can use now. In the numbered list at the end of this chapter we will assimilate this fact as a theorem. In short: split the quadrilateral into two triangles!

9 **a.** This is a sketch which goes with theorem **8**. However, note that it is sometimes possible that the connection AC does not lie inside the quadrilateral. Sketch such a case and divide that quadrilateral in two internal triangles with a connection line. As long as we do not use anything except the properties of triangles, everything will work out just fine after this case distinction.



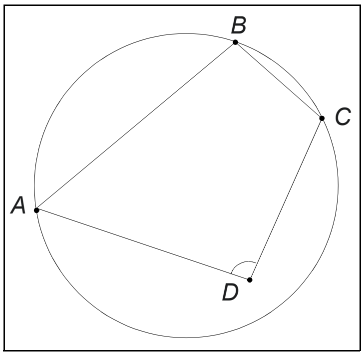
b. Now add the necessary numbers and arches in the figure and write down the proof using the format of exercise **4**.

The proof of theorem 8 is now complete. It is now safe to use the theorem to improve the temporary theorem of the cyclic quadrilateral to:

> Temporary theorem of the cyclic quadrilateral, improved version
>
> For each cyclic quadrilateral ABCD holds $\angle A + \angle C = \angle B + \angle D = 180°$

Problem C

Now we will look at the situation where D lies within the circle through A, B and C.
The next assumption almost goes without saying: $\angle B + \angle D > 180°$. Imagine that you are standing somewhere on the circle, opposite B. If you starting walking forwards, you need to widen your view to left and right to be able to still see A and C.
The proof of $\angle B + \angle D > 180°$ shall be based on this idea: we will compare the situation in point D to one with a point on the circle.



10. For that point we do not choose a completely new point, but a point, which has a relation to the other points.

a. Choose a point on the extension of AD and call it E. Make sure that the quadrilateral $ABCE$ is fully drawn.

b. Now show, using a familiar property of triangles, that $\angle ADC = \angle DCE + \angle CED$.

c. Which inequality does now apply to $\angle ADC$ and $\angle CED$?

d. Complete the proof of $\angle B + \angle D > 180°$ by applying the temporary theorem of the cyclic quadrilateral to $ABCE$ and combining that with the result of **c**.

The proof has not been put in a strictly organized form, but this is not always necessary.
Now the following has been proven:

I. If in a quadrilateral $ABCD$ point D lies **INSIDE** the circumcircle of triangle ABC, then $\angle B + \angle D > 180°$ applies.

We already knew:

II. If in a quadrilateral $ABCD$ point D lies **ON** the circumcircle of triangle ABC, then $\angle B + \angle D = 180°$ applies.

And of course we also expect that:

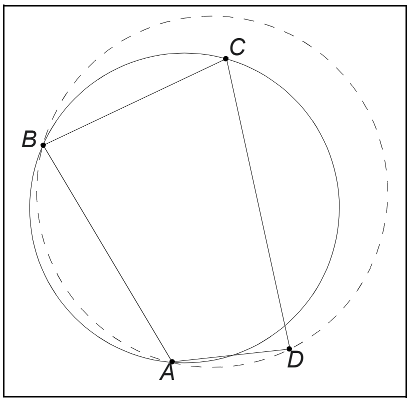
III. If in a quadrilateral $ABCD$ point D lies **OUTSIDE** the circumcircle of triangle ABC then $\angle B + \angle D < 180°$ applies.

You could try prove III almost similar to I in exercise **10**, but you will have to check several cases. It is possible that AD and both intersect the circle, or one of them or none.
It is better to use the proven cases I and II in a smart way and deducting case III from there. We will do this in this extra exercise.

extra

11. First of all the sketch. D lies outside the circle through A, B and C. The dotted circle goes through A, B and D. It looks like the complete arch from A via D to B lies outside the circle through A, B and C.

a. Argue that, using theorem 5, page 78. Actually, you need to show that both circles cannot have a third point X in common, because what would then be the circumcircle of AXB?



b. Thus C lies inside the circle through A, B and D. The proven part I of above now leads to an equality in where $\angle C$ occurs. Write it down.

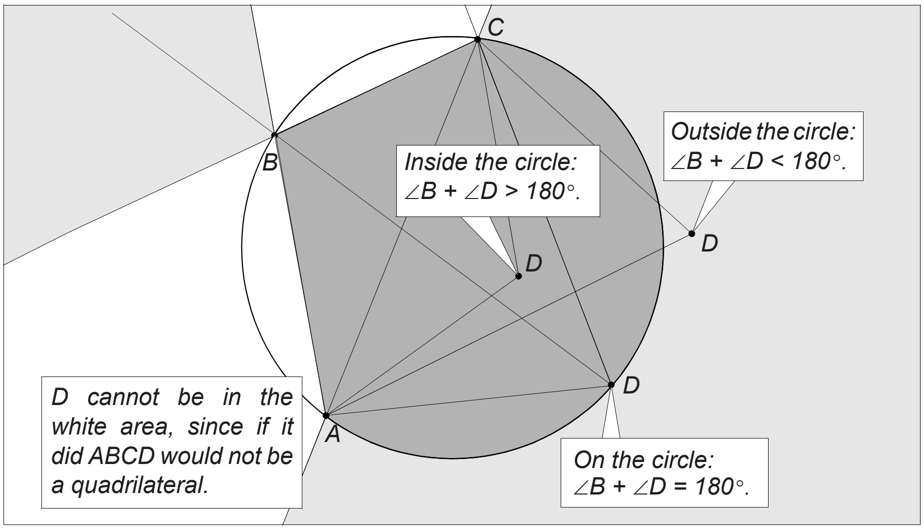
c. Now deduct, using also theorem 8, the desired statement $\angle B + \angle D < 180°$.

Part III is proved now and we will summarize the results of this section in one theorem.

Theorem 9 (properties of the cyclic quadrilateral)

If *ABCD* is a cyclic quadrilateral, then $\angle A + \angle C = \angle B + \angle D = 180°$.
If *D* lies inside the circumcircle of *A*, *B* and *C*, then $\angle B + \angle D > 180°$.
If *D* lies outside the circumcircle of *A*, *B* and *C*, then $\angle B + \angle D < 180$.

This deserves a survey illustration.



Remark: Since the three cases of the theorem exclude each other, you can immediately draw conclusions like:

If in a quadrilateral $\angle B + \angle D = 180°$, then that quadrilateral is a cyclic quadrilateral.

After all, if this equality holds, then the last two statements of the theorem ensure the fact that D neither lies inside nor outside the circle. Remains: on the circle.

extra: finding a good definition

12. It is said that D cannot lie in the white area, since then $ABCD$ would not be a quadrilateral. This is rather vague as long as we not have agreed upon what a quadrilateral is. Find out what is going on and give a definition of 'quadrilateral', which exactly excludes these cases.

extra: a warning!

In mathematics it can, and will, happen that you have a proof which looks right and clear, but later on somebody finds a small error in it. It does not always mean that the proof is totally wrong; it can be fixed in most cases. You are now in such a situation.

13. Check again the position of point E in 10a above. It could be on arc BC! In that case you cannot work with $ABCE$ as a quadrilateral.

a. How to fix this hole in the proof?

20. Using cyclic quadrilaterals

In this section we return to constructing Voronoi diagrams. We use what we know of cyclic quadrilaterals, so mostly theorem 9. Since that theorem talks about angles, you need to measure angles very precisely several times.

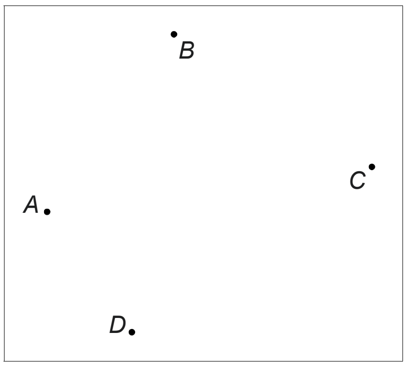
14. Given are four centers.

a. Find out whether in this Voronoi diagram of these four points the cells round A and C, or the cells round B and D adjoin.

b. Sketch all connection lines of centers which have adjoining cells with a color.

c. Finish the Voronoi diagram by sketching perpendicular bisectors.

d. Unlike exercise **4**, page 45, now you did not sketch too many perpendicular bisectors. Why?



15. Is this also true?

If a Voronoi cell is a quadrilateral, then that quadrilateral is a cyclic quadrilateral.

If necessary, give a counterexample.

16. Sketch a situation with six centers, where you have two four-countries-points in the Voronoi diagram. (Avoid the flat example that the centers which have a four-countries-point form a square or a rectangle.)

Summary of chapter 4

reasoning
In this chapter we got results through reasoning. The direction of concluding things was from former knowledge to new all the time.

Writing down proofs
You have learned that you can write down proofs in a neat way. Two aids were:

a. Indicating angles with indices: A_1, B_2, etcetera. In the sketch you can also indicate angles with symbols like $*$, $\circ$, $\times$, and $\bullet$. However, it looks kind of weird if you start your proof with: * = *. A good compromise is: indicate in the sketch equal angles with the same color or symbols and use unambiguous denominations.

b. Note the statement and motivation in this format:

a statement

with a motivation.

motivations
As motivations the following are allowed:
– references to definitions
– basic unproven known facts we agreed about (like the triangle inequality)
– statements that have been proven earlier.

theorems
Important things which we know to be true and which we will use again, are laid down in the form of a theorem. Several theorems mentioned below have not been proven in this book. Those are the first two of the following overview.

overview of theorems in this chapter

Theorem 6 (Isosceles triangle)

If in triangle *ABC*

$d(A, C) = d(B, C)$,

then also

$\angle CAB = \angle CBA$.

The reverse is also true:

If in triangle *ABC*

$\angle CAB = \angle CBA$,

then also

$d(A, C) = d(B, C)$.

Theorem 7 (Sum of angles in a triangle)

In each triangle the sum of the angles is equal to 180°.

Theorem 8 (Sum of angles in a quadrilateral)

In each quadrilateral the sum of the angles is equal to 360°.

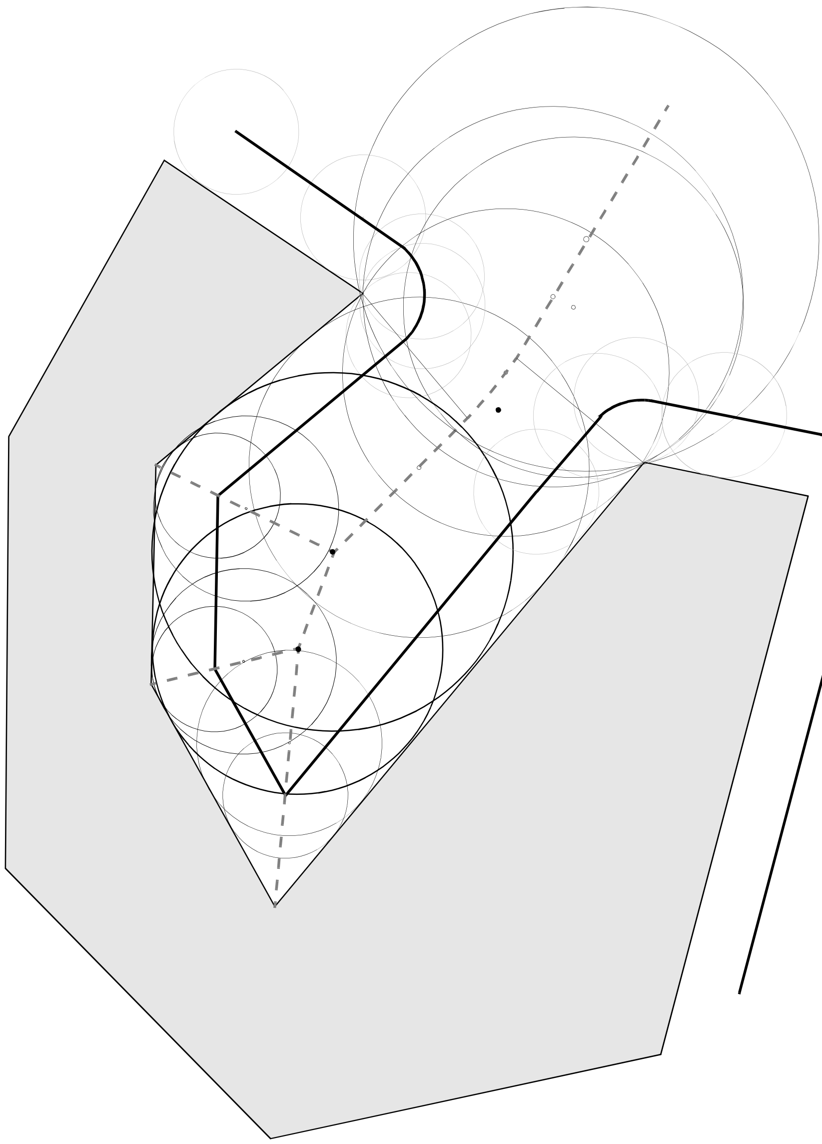
Theorem 9 (properties of the cyclic quadrilateral)

If *ABCD* is a cyclic quadrilateral, then $\angle A + \angle C = \angle B + \angle D = 180°$.

If *D* lies inside the circumcircle of *A*, *B* and *C*, then $\angle B + \angle D > 180°$.

If *D* lies outside the circumcircle of *A*, *B* and *C*, then $\angle B + \angle D < 180$.

Chapter 5: Exploring isodistance lines



When dividing fishing grounds between different countries the concept of the isodistance line plays an important role. This is a line which lies at sea at a fixed distance from the different shores.
Studying these lines puts us on the track of several geometric relations, where tangent circles and bisectors play a major role.

21. Isodistance lines, distance to areas

The small island of Mururoa in the South Pacific was frequently in the news in 1995. An international fleet, lead by two Greenpeace ships, wanted to protest on the spot against nuclear activities which would take place on the island. When rubber boats and the helicopter from the Rainbow Warrior operated within the 12-mile-zone round Mururoa, the French government saw this as an occasion to board this ship. On the one hand this lead to dramatic TV, but on the other it was a sensitive blow to Greenpeace's further plans.

The 12-mile-zone is seen as part of a country's territory. Unauthorized passing of the imaginary edge of this zone is a violation of international treaties.

isodistance line

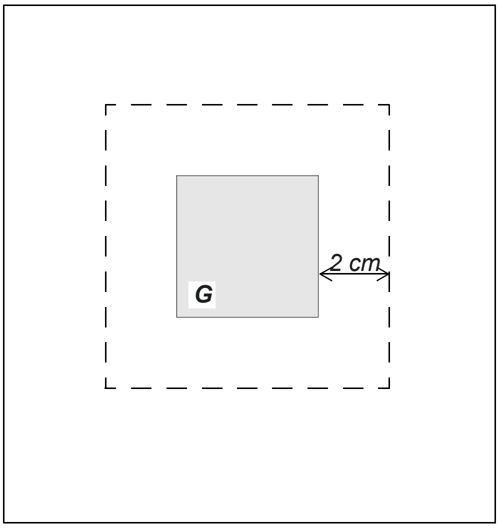
The edge of this 12-mile-zone is an example of an isodistance line: each point of this line has a distance of 12 miles to the coast. More exact: it is the iso-12 mile-line. But how do you know where this imaginary line lies? To what point on the shore is the distance 12 miles? What determines the shape of such a line? We will deal with these questions in this chapter.

1. Mururoa is an atoll. The diameter of this small island is a couple of hundred meters, very little compared to 12 mile (1 mile = 1,61 km).

   a. First consider Mururoa as a point-like area.
   Which shape does the edge of the 12-mile-zone have?

   b. Consider Mururoa as a circular area with a radius of 680 meter. What then is the iso-12-mile-line?

2. On the right a square – reduced – with a side of 4 cm is sketched. That is the area G which we look at. Around the first square a second square with a side of 8 cm is sketched. The edge of this square is *not* the iso-2 cm-line of G.
You can find a full-size sketch in the *worksheet D: isodistance lines round a square)* on page 178.



   a. Indicate on the edge of the larger square five points that do belong to the iso-2 cm-line of G and also five that do not.

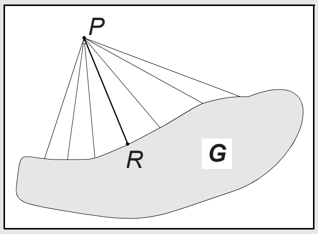
b. Now sketch the iso-2 cm-line of *G*.
Also sketch the iso-1 cm-line, the iso-3 cm-line and the iso-4 cm-line of *G*.
c. Is the iso-300 cm-line of *G* a circle?
Explain your answer.

Before we start sketching isodistance lines for more complicated areas, we will first describe what we mean by the distance from a point *P* to an area *G*. The description uses the familiar distance concept for two points.

description

Determine, for all points on the edge of an area *G*, the distance to *P*.
For one (or more) of these points, say *R*, is d(*P*, *R*) the smallest.

We then call:
- d(*P*, *R*) de distance from *P* to *G*
- *R* a footpoint of *P*
We denote the distance from *P* to *G* by d(*P*, *G*).

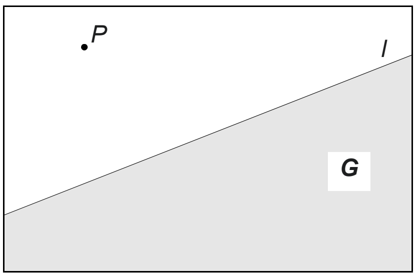


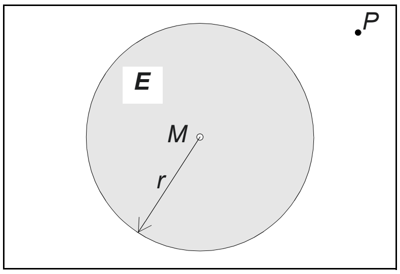
In the description a lot of undefined words are used: area, edge. Also, without further introduction it is simply assumed that such a smallest distance exists.
In this chapter we will use the terms *area* and *edge*, just as if they have a solid mathematical meaning, and we will accept also that the minimum distance exists.
We will agree on one other thing: *the edge belongs to the area*.
Furthermore: in situations where misunderstandings are not possible, from now on we will leave out the unit for distances.

simple areas

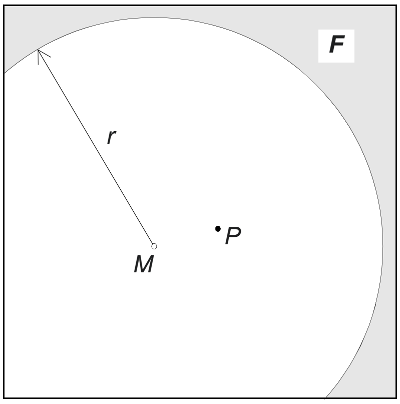
In some cases it is easy to determine *footpoint* and *distance*. We are also able to give a full proof in those cases. We will do so in the following exercises.

3. Here the area *G* is a half-plane; the edge of *A* is a line, say *l*.
a. How can you determine the footpoint *R* of *P* and the distance d(*P, G*) immediately?
b. In what way(s) has been proven that this is true?





4. A circular island E with center M and radius r. Again a point P outside of E is indicated.
   a. Of course you *expect* the footpoint of P on E to be on the connection line PM. Sketch the line and name the point R.
   b. Take a point Q on the edge of E, not equal to R, and *prove* that indeed $d(P, Q) > d(P, R)$ applies.
   (Aside from the triangle inequality, you also need to use your knowledge of circles.)
   c. Express, using the d-notation, the distance from P to E in the distance from P to M and the radius r.



5. Here the area F is the outside area of a circle. M again is the center of the circle. We need to find the footpoint R for the given point P.
   a. Again sketch the points R and a point Q on the circle, (almost) like in the previous exercise.
   b. Also here the triangle inequality helps while *proving* that $d(P, Q) > d(P, R)$. Now you definitely need
   $d(M, Q) = d(M, R)$.
   c. Again express, using the d-notation, the distance from P to F in the distance from P to M and the radius r.

Now we know how to find footpoints of given points for line-shaped and circular edges of areas, it is also clear that footpoints can be found for areas which are restricted by several pieces of lines and circles. With that, for these types of areas, the existence of a minimum distance, which is assumed in the description on page 110, is made secure.

Because the concept distance to an area is known, you can also define what an isodistance line is.

definition isodistance line

For a given positive distance *a*, the isodistance line at distance *a* from an area *G* is the set of all point to which apply: d(*G*,*P*) = *a*.
The isodistance line at distance *a* is indicated by ‘iso-*a*-line’.

6 **a.** For exercises **3**, **4** and **5**, sketch the isodistance line at distances 1, 2 and 3; also sketch the isodistance line which crosses point P.

b. From what does it follow immediately that these lines are indeed circles in exercises **4** and **5** ?

7. Below a triangular area G is sketched. (a larger picture is on *worksheet D: isodistance lines round a square)*, page 179.)

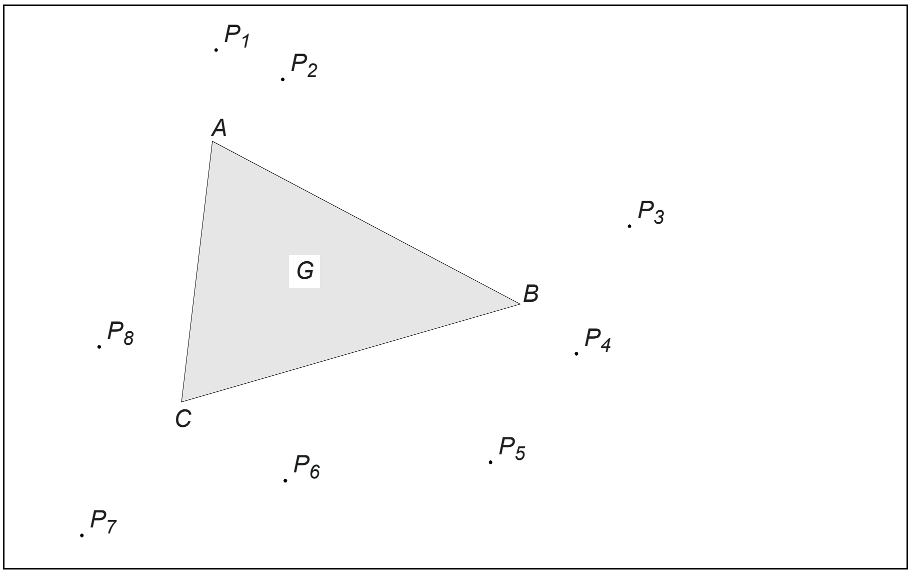
a. Indicate for each of the points P_i the footpoint R_i on the edge of G.

b. Vertices A, B and C of the area play an important role. In the figure, sketch the 'zone' for which point A is the closest point for points in the area G. For instance P_1 is in this zone. Also sketch these zones for points B and C.

c. The outside area of G is now divided into six zones.
For each zone, describe the shape of the isodistance lines.

d. Sketch the isodistance lines for $d = 2$ cm, $d = 4$ cm and $d = 6$ cm. What stands out?

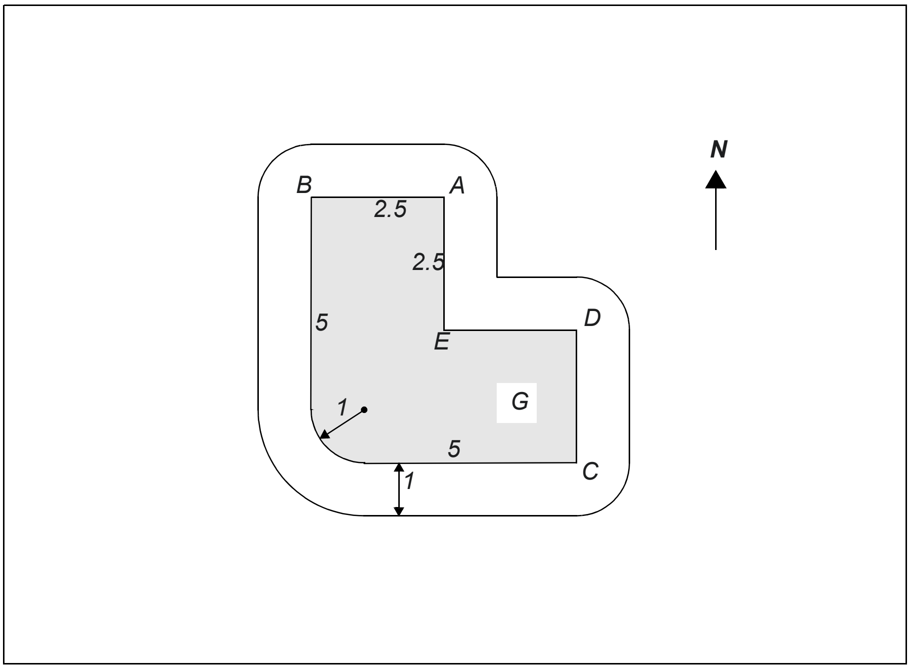
e. The isodistance lines are longer than the circumference of area G. How much longer?



cape

Points A, B and C are a kind of *cape*. According to the dictionary a cape is *a strip of land projecting into a body of water*. Such a projecting point is the footpoint for a large number of points, all of which lie in a certain sector.

8 **a.** Explain why all the isodistance lines in the neighborhood of a *cape* are parts of concentric circles.



b. How do you find the lines which confine the sector of such a cape?

9. Here you see an L-shaped area *G* with iso-1-line. *A*, *B*, *C*, and *D* are capes.

a. Now make a very exact sketch of the northeast part of the isodistance lines for the distances 0.5, 2, 3 and 4.

Near the capes the arcs of the circle run smoothly into the straight pieces, but this is somewhat different for *E*. There, the isodistance lines show a kink.

b. On which line do these kinks lie?

c. The iso-1-line makes a kink of 90^o. How large is the kink for the other four isodistance lines?

d. Up to which value of *d* does the de iso-*d*-line make a kink of 90^o? Explain.

e. As the distance increases, the angle between the two arcs decreases. Will the kink completely disappear at a great distance? Give a clear reasoning.

Areas with bays

For the simple areas that we looked at up to now, the isodistance lines were built from arcs of circles and straight line segments and could be sketched precisely. You can say from the L-shaped area of exercise **9** that the northeast has a bay. Near this bay the isodistance lines showed kinks. In the next section we will go further into bays with straight-lined shores.

The next two exercises deal with bays with a small passage to the open sea.

10. The bay above has a passage to the open sea with a minimal width of 2 kilometer. The scale of the map is therefore determined by: in reality 1 cm on the map is 1 kilometer.

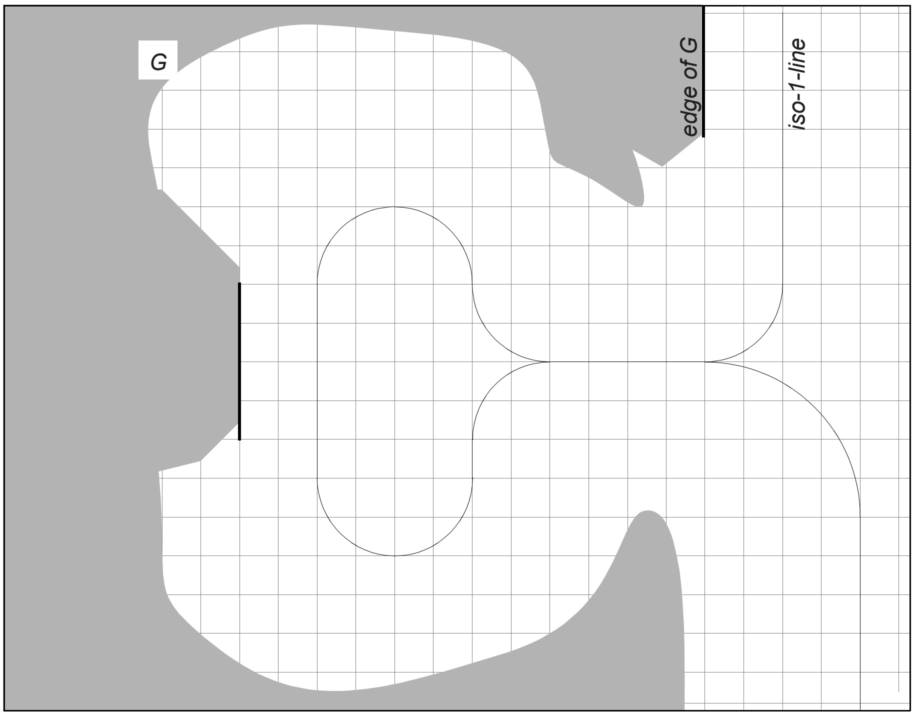
a. Successively sketch in different colors the iso-0.2-, iso-0.5-, iso-1-, iso-1.50- and iso-2-lines. You do not need to measure very precisely, but the important characteristics of the isodistance lines should clearly come out.

b. What, according to you, is the most important difference between the shapes of the iso-0.2-line and the iso-1.5-line?

c. Indicate all points on the iso-1-line, which have two footpoints.

d. Indicate with an extra color on the iso-2-line those pieces of which the footpoints lie on the coast of offshoot *B*.

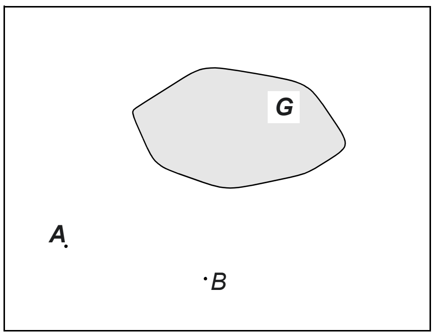
You can see from this example that the name isodistance *line* is not perfect: a line does not split up in two pieces halfway through. No problems occur if you know what you mean by the total concept *isodistance line*, namely the *set* of points which are ... etcetera.



11. In the top figure on the next page an iso-1-line is shown, as well as two short pieces of the edge of the involved area. The isodistance line consists entirely of quarter circles, half circles and straight line segments. *Finish the edge of the area G*.
You can use the grid lines.

extra

12. In the proofs at the beginning of this section (the half plane and the circular island) the triangle inequality played an important role.
The triangle inequality applies to three arbitrary *points*. Now the concept of distance is extended with distances to areas. Would the triangle inequality still be true if you were to replace one or more points by areas? In short:



a. Does $d(A,B) \leq d(A,G) + d(B,G)$ apply to the sketched situation?
b. Move B in such a way that this 'triangle inequality' is no longer true.

22. Angle bisectors

In this section we investigate isodistance lines in bays which are bounded by straight-lined shores. Bisectors play an important role here.

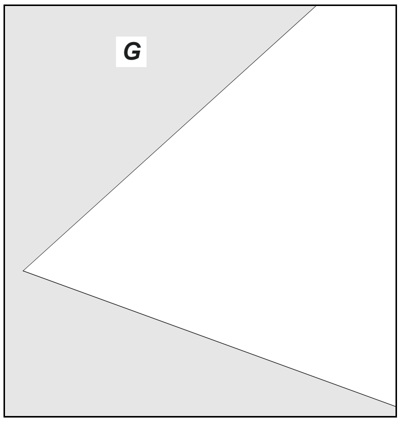
13. You only see a part of the area *G*. Suppose that the straight edges of the bay continue indefinitely. The arrows should represent this.

a. Sketch accurately in the bay
the iso-1-line
the iso-2-line
the iso-3-line
the iso-4-line.

b. All these lines have a kink. Which figure do these kinks form?

c. How many footpoints does the 'kink point' of an isodistance line have?

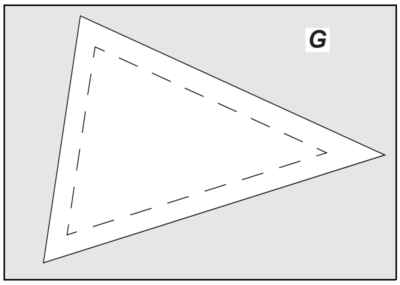
d. Sketch the footpoints of the 'kink point' of the iso-3-line.



14. Here you also see part of an area. The white triangle does not belong to the area and the isodistance lines lie inside the triangle.

a. The iso-0.5-line is sketched. Also sketch the distance lines at the distances 1, 1.5, 2, 2.5, et cetera.

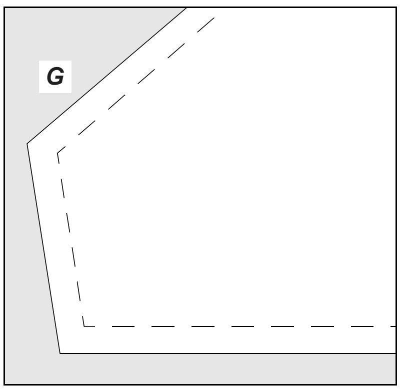
b. Describe what you notice.



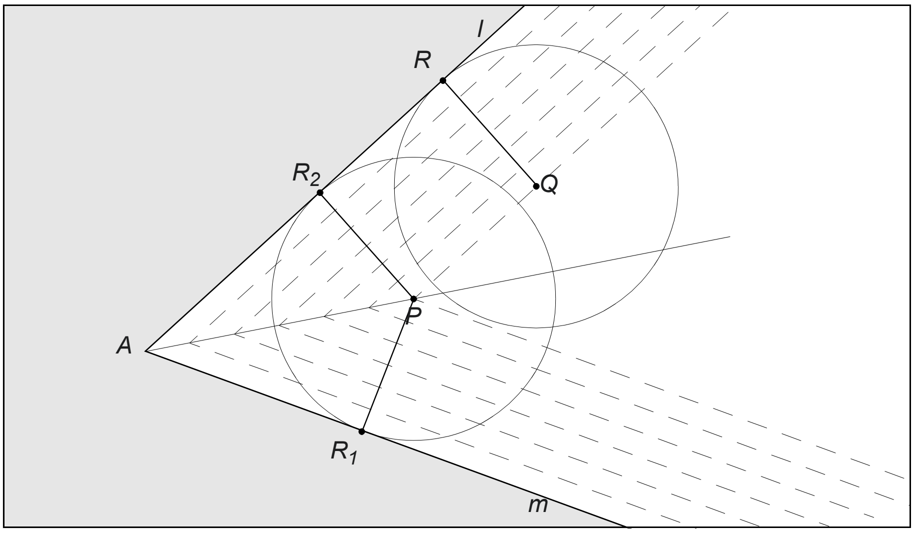
15. This area has a bay with two kinks. Also, the iso-0.5-line shows two kinks.

a. Do all isodistance lines have *two* kinks?

b. Find a point *P* in this bay that has three footpoints on the area *G*. Show how you found *P*.
Also sketch the circle with center *P*, which runs through the three footpoints.



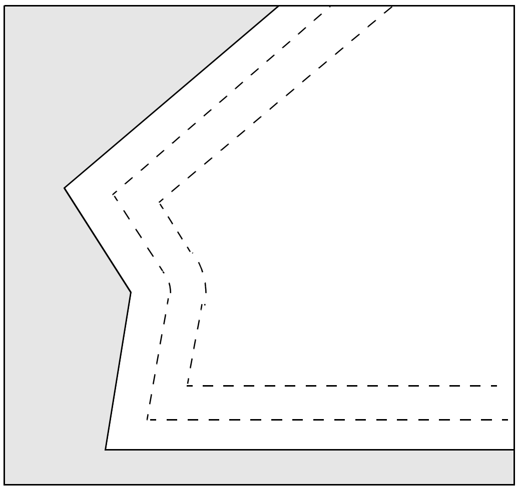
In a bay which is bounded by straight line segments the isodistance lines show kinks. Those kinks lie on a straight line. Each kink has a footpoint on both edges of the bay, in contrast to all the other points on an isodistance line.



For the two lines which bound the bay, a 'kink line' plays a similar role as the Voronoi edge for two points.
d(P, A) = d(P,B) applies to the points on the Voronoi edge of centers A and B.
d(P, l) = d(P,m) applies to the points on the 'kink line' for l and m.

In the next section this analogy will be looked into further.

16. Also, for the bay on the right we have a lot of isodistance lines with two kinks, but not all. Sketch an area which has bay(s) with straight-lined shores and for which *all* isodistance lines have two kinks.



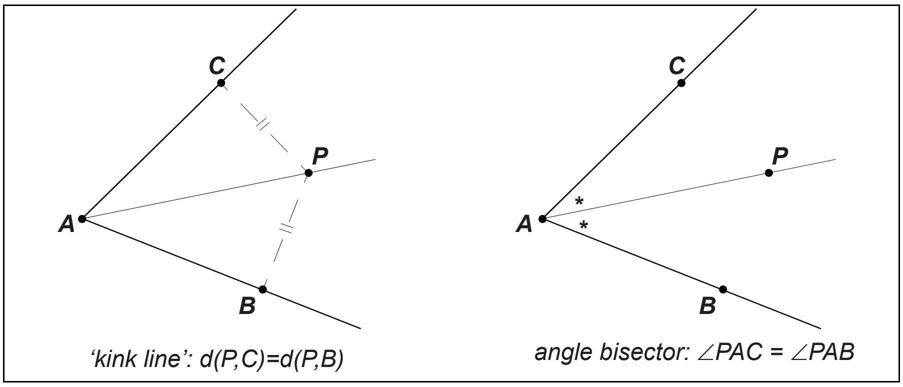
23. Theorems about angle bisectors

Here is a definition of *bisector of an angle.*

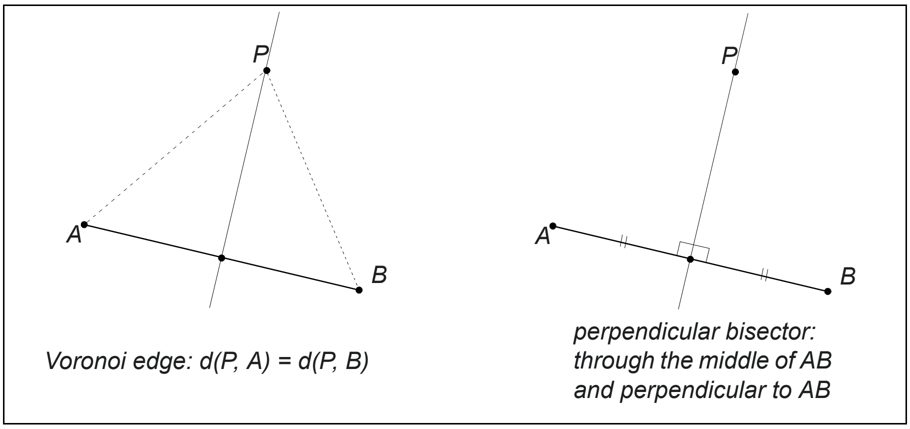
definition angle bisector

The angle bisector of an angle CAB is the line which makes equal angles with the legs CA and BA of that angle.

To a point P on the bisector applies: $\angle PAC = \angle PAB$.
In the previous section we found that to points on the 'kink line' of a bay which is bounded by straight line segments, applies: $d(P, C) = d(P, B)$.
There, B and C are the footpoints of P on the legs of the angle.
The figure strengthens the idea that the 'kink line' and the angle bisector coincide.



It all looks a lot like the equality of the Voronoi edge and the perpendicular bisector.



In section 11 of chapter 2 (page 71) we have proven this last equality. The proof consisted of two parts.

17. Now prove only that each point on the 'kink line' also lies on the bisector.

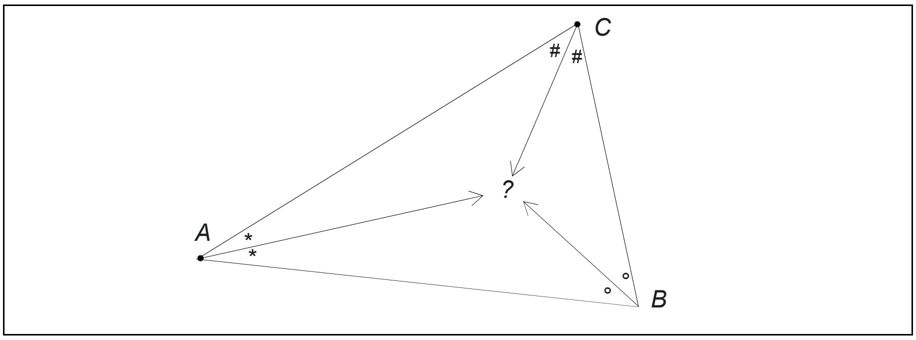
On page 76 we have formulated the following theorem about the properties of the perpendicular bisector.

Theorem 10 Properties of the perpendicular bisector
The perpendicular bisector of line segment *AB* is the set of points *P* for which hold d(*P, A*) = d(*P, B*).
For points *P* outside of the perpendicular bisector holds:
If d(*P, A*) < d(*P, B*), then *P* lies on the *A*-side of pbs(*A, B*).
If d(*P, A*) > d(*P, B*), then *P* lies on the B-side of pbs(*A, B*).

18. Formulate an analogue theorem about the angle bisector. It starts like:

Theorem 11 Properties of the angle bisector
....
....

This theorem can be proven just as fundamentally as theorem 3 in chapter 2; we will not do this in detail. In exercise **17** you have already done a part.
There also is an analogue for theorem 4, see page 76.
The figure below already suggests it.



Theorem 12
The three angle bisectors of triangle *ABC* meet in one point.

In the proof of the theorem of the perpendicular bisector we used the concept of distance. Since theorem 11 associates bisectors with distances, we can do the same here.

19. Give a proof of Theorem 12; use the structure of page 77.

We will carry on with the analogy in a minute.The *circum*scribed circle of the triangle belongs to the intersection of the perpendicular bisectors. Here we have an *in*scribed circle.

definition inscribed circle

> An inscribed circle of a triangle is a circle which lies in the triangle and which has one, and only one, point in common with each of the three sides.

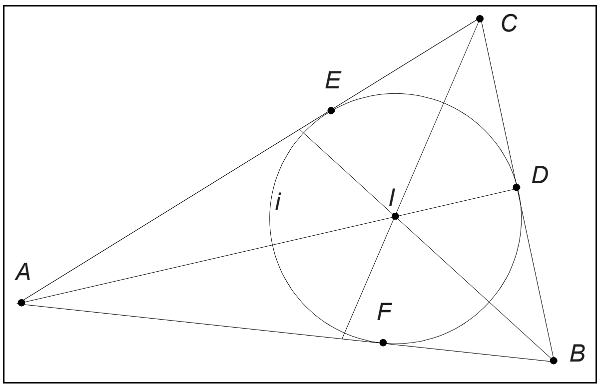
This could be the theorem that looks like theorem 5 in Chapter 2.

Theorem 13

Each triangle has an inscribed circle (=incircle). The center (=incenter) of the incircle is the point of concurrence of the triangle's angle bisectors.

glance forward

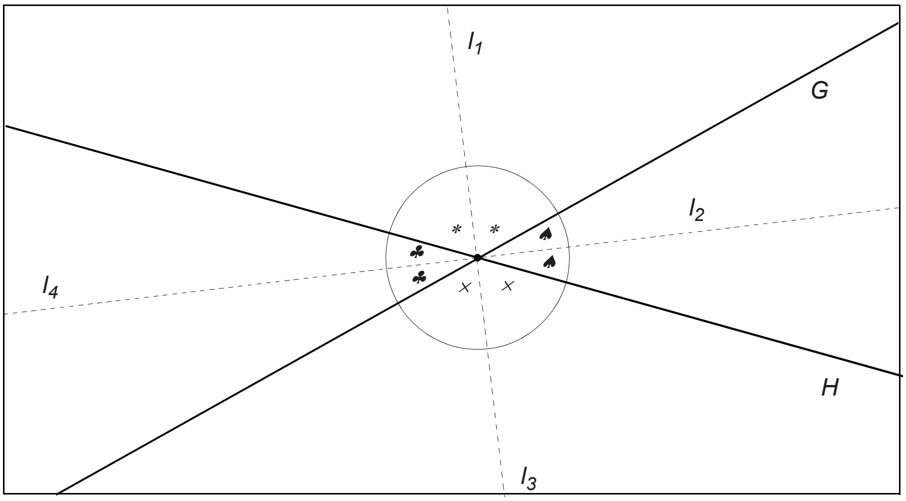
The figure on the right shows a triangle with its incircle. It is tempting to say that the sides are tangents to the circle. This is not wrong, but then we should underpin the word 'tangent'. We will elaborate on this later.



As the theorem was formulated here, everything is fine; the circle lies completely on the right side of all three lines, so *within* the triangle.

24. There are more angle bisectors!

For preparation purposes on a theorem about angle bisectors, which does *not* have an analogue in perpendicular bisectors, we investigate converging angles and their bisectors.
Here two line shaped areas G and H are indicated. (Therefore, the intersection of the lines belongs to both areas, but this is not very important). The two areas enclose four bays. For each of the bays the bisector is sketched. Equal angles are indicated with the same symbol.



According to the definition on page 118 we should talk about *four* angle bisectors. Up to now we only had *half* angle bisectors. We will rectify this in a moment. Of course you know that these four half bisectors complete each other into two lines. You can also prove this.

20. Prove that l_2 and l_4 do indeed form a straight line. Do the same for l_1 and l_3
Find out which of the little signs indicate the same angles.
Then prove that l_1 is perpendicular to l_4 and also to l_2.
Use the facts that H and G are 'whole' lines and that a straight angle is 180°.

agreement
From now on, when we refer to bisector, we mean a *whole* line, which passes through the intersection of two lines and makes equal angles with those lines.

The result of exercise **20** can now be formulated in the following theorem.

Theorem 14

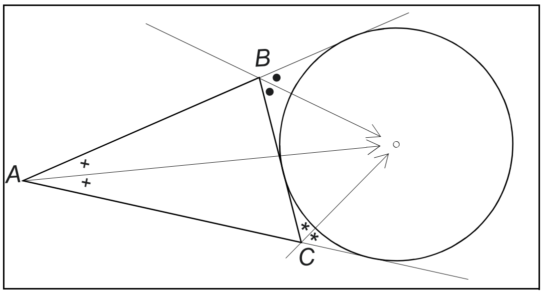
The two angle bisectors of two crossing lines are perpendicular to each other.

internal and external angle bisectors

In a triangle *ABC* we now have six bisectors:
- three, which coincide in one point inside the triangle; these are called the *internal angle bisectors*;
- three others, which, besides a vertex they cross, lie outside the triangle; they are called the *external angle bisectors*.

21. In this triangle the internal angle bisector through *A* and the external angle bisector through *B* and *C* are sketched.

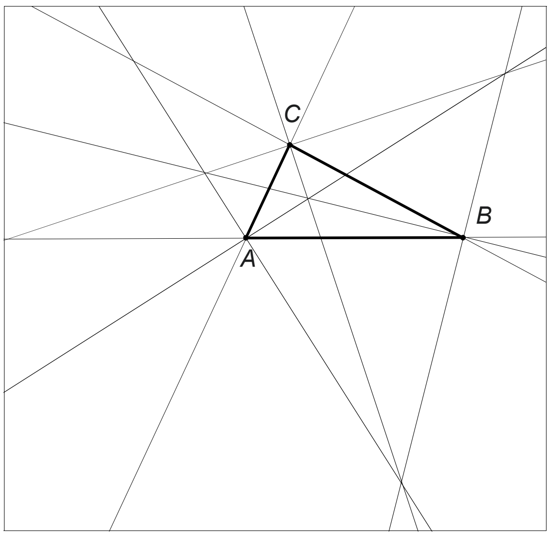
Prove that these three bisectors also coincide in one point and that this point is the center of a circle, which only has one point in common with each of the three lines.



escribed circle

Such a circle is called an *escribed* circle or *excircle* of triangle *ABC*. Each triangle has three excircles.

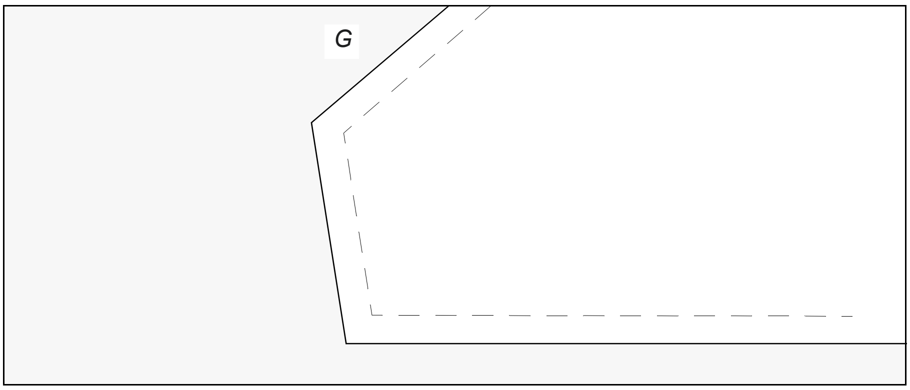
22. The following figure shows a triangle *ABC* and all its internal and external angle bisectors. Sketch the *incircle* and the three *excircles*. Also indicate equal and right angles in the figure.



The following theorem does not state anything new, it is a summary of the preceding. It even repeats the previous theorem. There is nothing against that in such a summary. For now, we are keeping in the undefined concept of '*tangent*', and you can think about 'tangent' as just having one point in common and for the other points as lying on one side of the lines.

Theorem 15 In a triangle the three internal angle bisectors coincide in one point. That point is the incenter of the incircle.
If one chooses the external angle bisector at two vertices and the internal angle bisector at the third vertex, then those three lines also concur in one point. This point is the excenter of an excircle of the triangle.

23. In exercise **15** you have sketched a point *P* in a bay, which has a footpoint on each of the three shores. You can now think of that point as the center of an excircle of some triangle.
But then *P* needs to be the intersection of two external angle bisectors and one internal angle bisector of this triangle.
Sketch those three angle bisectors of the triangle.



extra

24. Do the incircle and excircle have the same point in common with a side in a triangle? Find out for which triangles this is true.
You do not have to prove your statement.

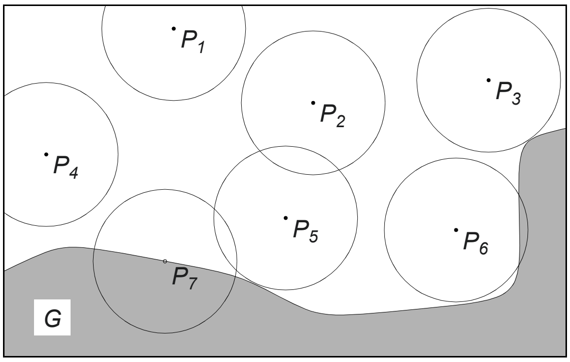
25. Tangent circles

In this section we discuss an easy way to draw isodistance lines of capricious areas. We will meet tangent circles in a natural way.

bumping circle

We call a circle round a point P outside G a *bumping circle* for G if there is one (or more) point of G **on** the circle and there are no points of G **inside** the circle.

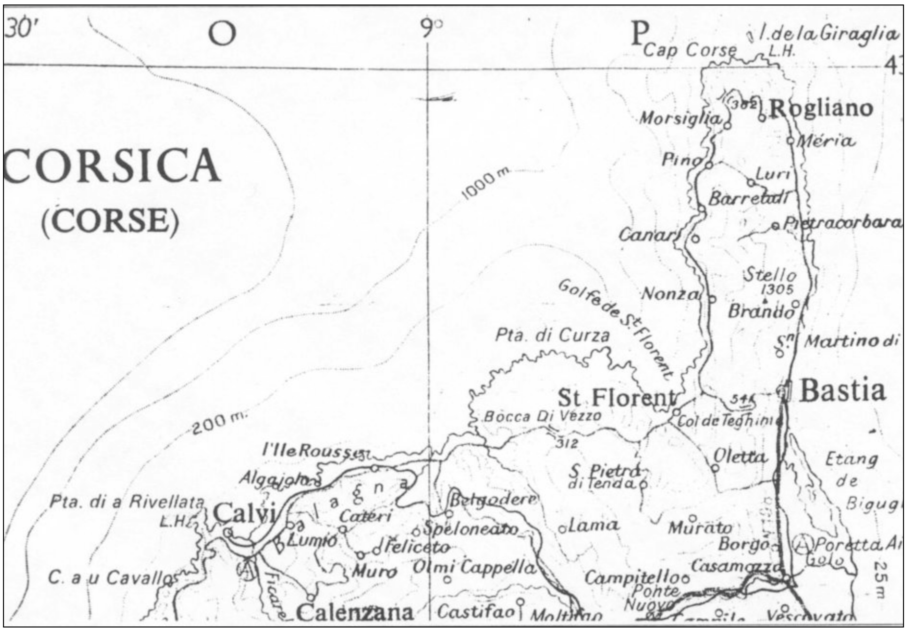
25 a. Here an area G and a number of circles with radius 1 with centers $P_1, \ldots , P_7$ is sketched. Indicate which points P_i belong to the iso-1-line.



b. If a point Q lies on the iso-1-line, is it certain that there *is* a bumping circle with radius 1 and center Q?

With one bumping circle, moving around the area, and a small hole in the middle, you could quickly sketch an isodistance line. Let's try!

26. Below you see part of a map of Corsica; the coast of Corsica is very irregular. Make a cardboard circle with radius 2 and a hole in the middle. With a pencil, which sticks through the hole, you can quickly sketch the iso-2-line. The scale of the map is 1 : 600000; so you are actually sketching the iso-12 km-line.

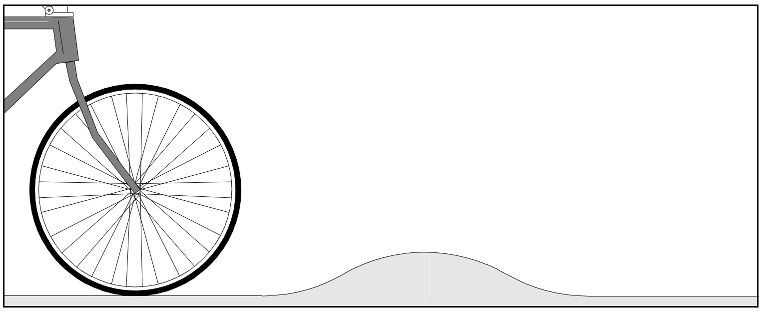


Note: On the map some lines with ‘200 m’ and ‘1000 m’ are visible. They are not iso-distance lines. What are those lines?

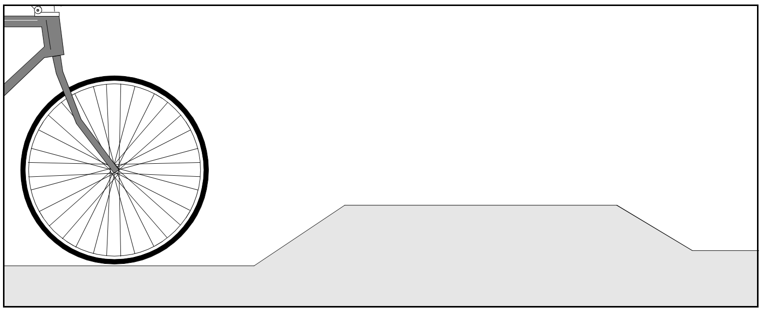
27. Consider a turning cycle wheel as a moving bumping circle. In the following three figures you see different obstacles.

a. For each case sketch the path that the middle of the wheel follows.

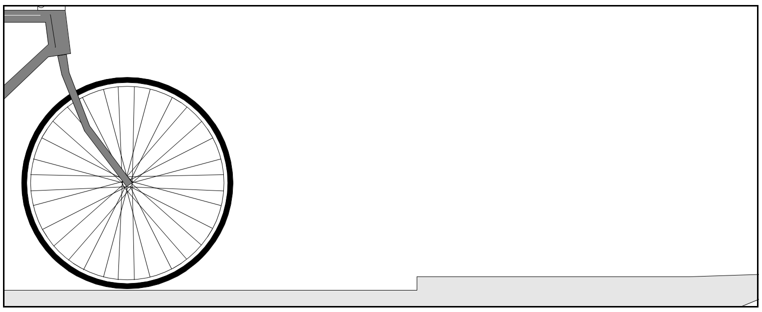
1. round bump



2. speed ramp



3. low pavement



If the path of the middle of the wheel makes a kink, the cyclist will feel a blow. The blow is heavier when the angle between the directions before and after the kink is larger.

b. How many blows does the cyclist feel at the speed ramp?
c. In which case does the cyclist feel the severest blow: at the ramp or at the low pavement?

Theorem 16 The theorem of the bumping circle.
The centers of all bumping circles with radius *a* of an area *A* form the iso-*a*-line of that area.

28. The statement has two directions:
– each center of a tangent circle with radius a lies on the iso-a-line
– each point of the iso-a-line is the center of a tangent circle with radius a.
Did we prove this theorem? Where and how?

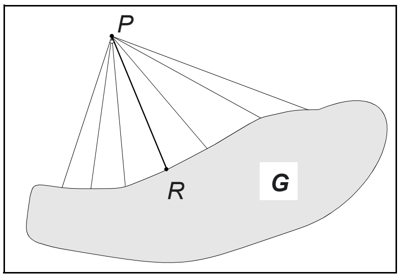
29. The next statement is *not* true in general:
- *If a bumping circle with radius a touches two or points of an area, then the iso-a-line has a kink.*
Find an example and counterexample for this statement in section 21, Isodistance lines, distance to areas

Summary of chapter 5

area, edge

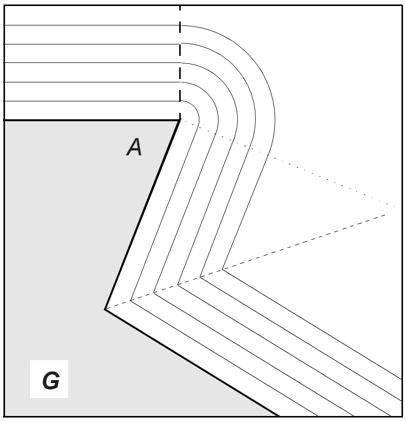
We did not say what an area and an edge are in general. We could, but it would go too far for this book. No difficulties will occur if we stick to the following:
– An area in the plane is nothing else but a piece of the plane we work in. It could be bounded or unbounded. A piece of straight line is also an area.
– The edge of an area always belongs to the area.

distance from a point to an area, footpoint



The concept of distance between points has been expanded in this chapter to distances from points to areas. The distance from a point P to an area is the smallest possible distance from P to the edge points of the area. Such an edge point is a footpoint of P. P can have more footpoints.
When we deal with areas, bounded by several arcs and line segments, we can be certain of the fact that each point outside the area has one or more well defined footpoints on the edge of the area.

isodistance line
The iso-a-line of an area G consists of all points which have distance a to the area G.



In the sector which belongs to cape A, the isodistance lines are pieces of concentric circles with center A.
If the footpoints lie on a straight line, then the isodistance lines will also be straight-lined.
For bays, which are bounded by straight line segments, the isodistance lines consist of parallel line segments, which are connected with a kink. The 'kink points' have two footpoints. They lie on the bisector of the angle the two straight edge segments make.

angle bisector
An angle bisector is a line which makes equal angles with the legs of the angle.

similarities between angle bisectors and perpendicular bisectors
Angle bisectors divide angles in half, perpendicular bisectors divide line segments in half. We have found several theorems about angle bisectors, partly analogous to the theorems of the perpendicular bisector.
Each triangle has three perpendicular bisectors, their intersection is the center of the circumcircle. Each triangle has three internal angle bisectors and three external angle bisectors. There are four points where three of those lines intersect. Those are the centers of the incircle and the three excircles.

theorems about angle bisectors
We proved in more or less formal way a lot of theorems; they are easy to find in the text. For completeness we formulate theorem 11 here.

Theorem 11 Properties of the angle bisector)

> **The two angle bisectors of an intersecting pair of lines l and m form the set of points P to which applies d(*P, l*) = d(*P, m*).**
> **To points *P* within triangle *ASB* applies: (Here *A* and *B* are points on *a* and *b* respectively, not equal to *S*)**
> **If d(*P, a*) < d(*P, b*), then ∠*ASP* < ∠*BSP***
> **If d(*P, a*) = d(*P, b*), then ∠*ASP* = ∠*BSP***
> **If d(*P, a*) > d(*P, b*), then ∠*ASP* > ∠*BSP***

Chapter 6: Shortest paths

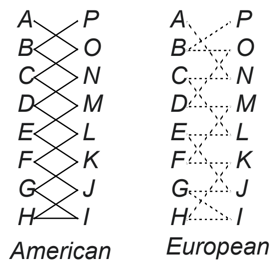
In this chapter a different type of geometry problems comes up: finding shortest paths.
We start by finding economic ways to tie your shoelaces.
Reflection and working with angles will be used many times.
Thus you will often need your protractor.

The character of this chapter is mostly very practical. In many exercises sketching is required.

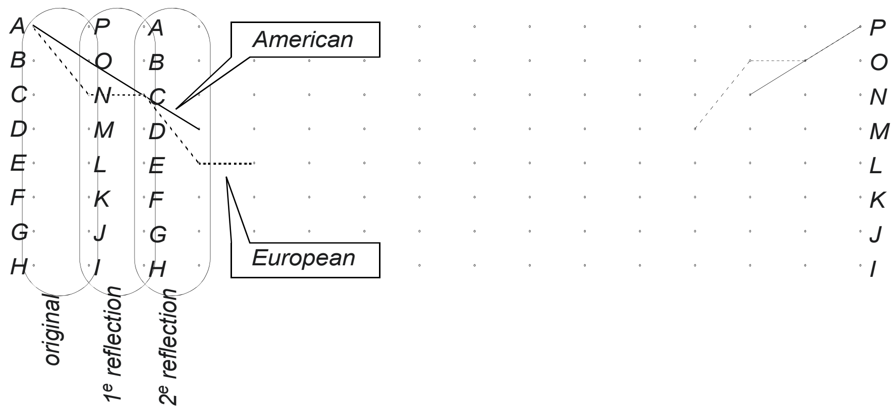
26. From shoelaces to shortest paths

The pictures on the first page of this chapter show two different ways to tie your shoelaces. The dressy shoes are tied European style, the cowboy boots American style. We will compare the styles on efficiency of use of shoelace and work with a method which gives rise to an interesting geometrical investigation: optimizing through reflection in mirrors.

1. The two methods are applied to the same training shoe. Alongside you see the schemes. According to you, which method uses the shortest length of shoelace between *A* and *P*? (Base your answer on the schemes and restrict yourself to a first estimate.)



2. Below, the start and finish of the shoelace patterns are represented transformed; reflections were used, so that the crossings no longer appear See the shoelace going from *A* to *O*, to *C*: the American style.

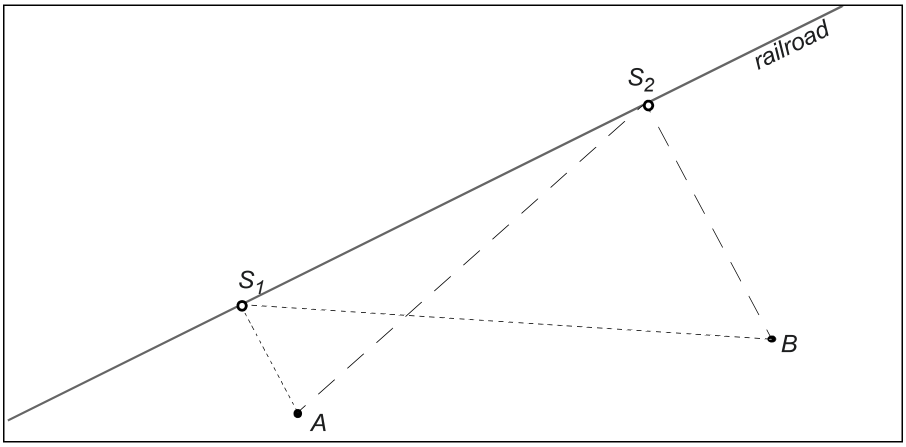


a. First finish both patterns in this fold-out representation.
b. lengths are not changed in this new representation, compared to the earlier scheme. Why is that so?
c. Now decide again which pattern uses the shortest length of shoelace. Explain your choice.

shortest road length

In the shoelace problem you saw that it is sometimes easier to determine a shortest path by reflection if the path needs to meet certain conditions of reflection. We have seen an earlier situation in which this method was applicable.

an old problem
The straight line below represents a railroad line; A and B are cities, which lie at some distance from the railroad. One station needs to be built and of course roads from both cities to the station need to be constructed.



The separate city councils of the cities A and B suggested S_1 and S_2. The roads necessary for both propositions are indicated with dotted lines.

3. After deliberation the cities decide in favor of the overall cheapest solution:

position S of the station needs to be located in such a way that the total length of constructed roads $|AS| + |SB|$ is minimal.[1]

a. Determine the exact point S, which meets this condition.
(You have done this before! But again a hint: if the cities lie on different sides of the railway, it would not be difficult to decide where S needs to be located. In case of emergency consult page 66.)
b. Now also show that for the point S holds: $\angle S_1SA = \angle S_2SB$.

think ahead for a moment
Strictly speaking, you have not learned why that last thing is true in this book! You need some knowledge of angles, but not all that much. In the next geometry book

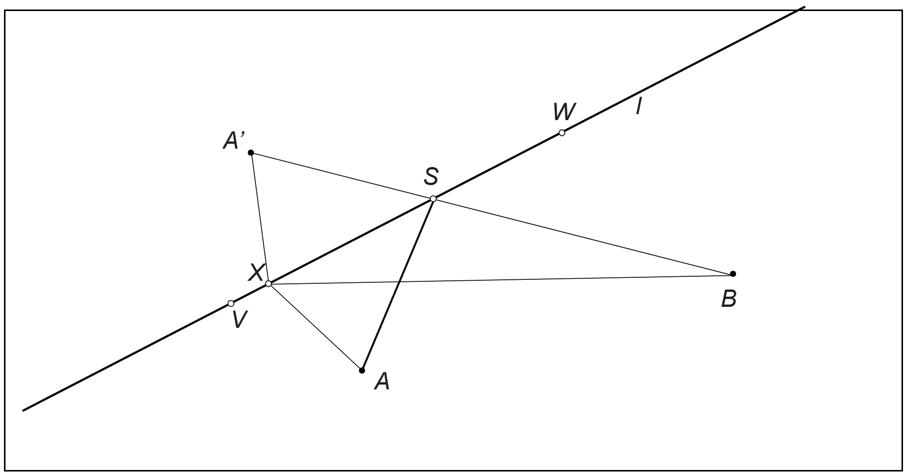
1. Here the notation $|XY|$ is used for the length of the line segment XY.

you will work with angles more, and then proving something like this would be a piece of cake. In this chapter you will gain some experience in using angles. Simply use what you already know:
– opposite angles of intersecting lines (as was used here)
– *F*- and *Z*-angles
– the total sum of angles in a triangle is 180 degrees.

We summarize what we found in the form of a theorem.

Theorem 17 (Theorem of the shortest route)
If a line *l* is given and two points *A* and *B* on the same side of the line, then there is exactly one point *S* on the line *l* for which $|AS| + |SB|$ is minimal. For that point *S* holds
- *S* is on the connection line of *B* and the *reflection of A* in *l*,
- *AS* and *BS* make equal angles with *l*.

extra
If you move in the figure below a point *X* over line *l* from *V* to *W*, then the total length $|XA| + |XB|$ would vary.



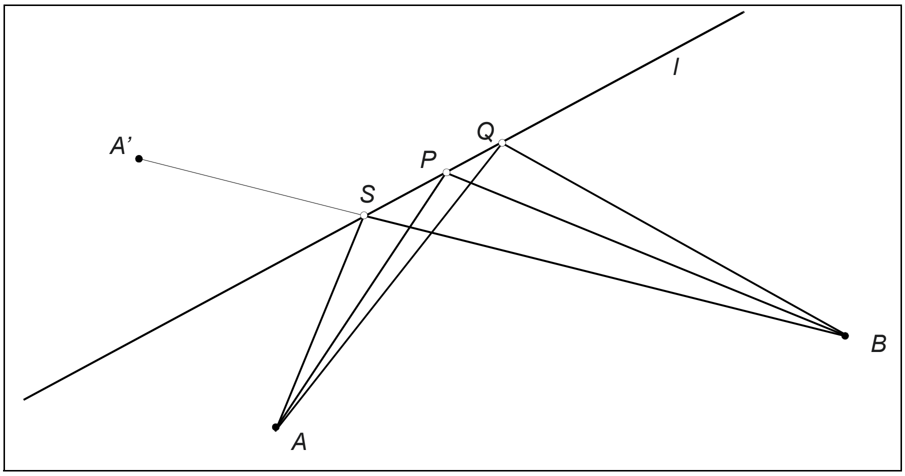
From theorem 17 you only know that this quantity would be minimal if $X = S$, on the straight line *AB*. But it ís also true that the length decreases continuously if you move *X* from *V* to *S* and after that increases continuously if you move on to *W*. Said plainly: if you make a bigger detour, it will be longer.

Thus we would like to add to theorem 17 something like this:

> *Furthermore: if P is point which moves away from S along l,*
> *then |AP| + |PB| constantly increases.*

elementary, but subtle
The proof of this statement does fit within the framework of proving with the *triangle inequality*. But it is not easy!

In the figure above you should show that:



$|AQ| + |QB| > |AP| + |PB|$.

Unpleasant as it is, it does seem that $|AQ| > |AP|$, but also $|QB| < |PB|$; so even when we could prove those inequalities, it will not help adding them.

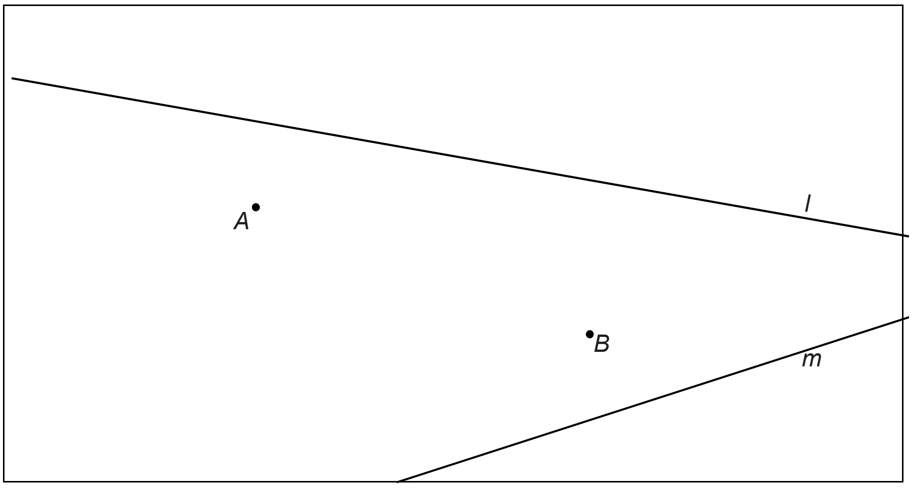
4. You still need to find a proof!
The easy part: draw *A'P* and *A'Q*. Now go for $|A'Q| + |QB| > |A'P| + |PB|$.

Hint: Bring a third route from *A'* to *B* into the game! First, walk from *A'* to *P* and go straight on until you hit line *QB*, at new point *R*. Then follow *RB*. Compare lengths of this route with both other routes. [2]

2.Archimedes used our addition to theorem 17 in his work about measuring lengths of curves, but he did not prove it.

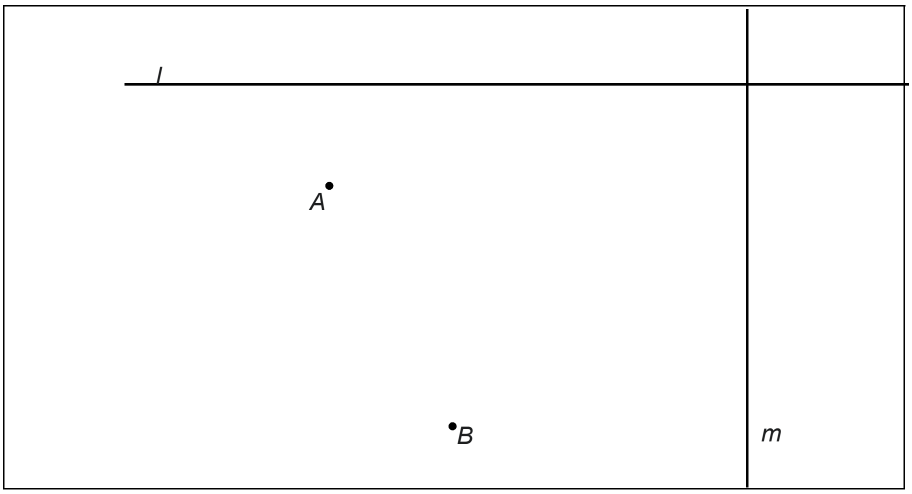
meeting more lines on shortest routes

5. Here you see two lines l and m and two points A and B.



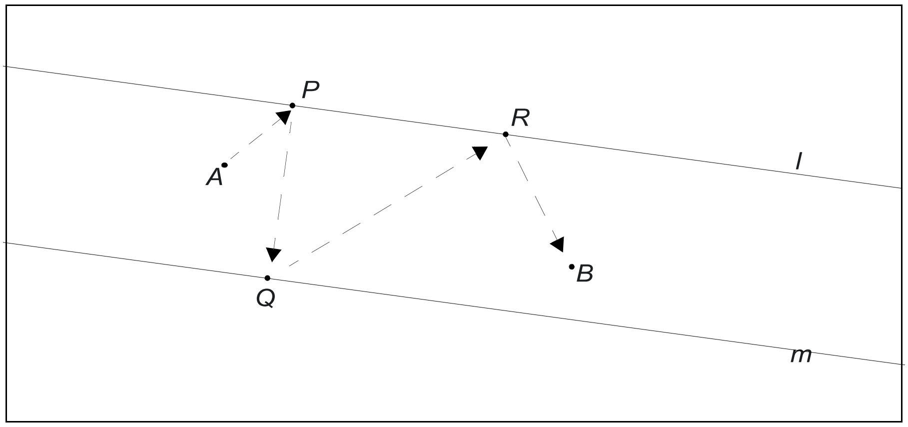
a. Determine the shortest path from A via l and m to B, in this order; first make a sketch and then think about how you can make use of reflection(s).
b. Also construct the road from A via m and then via l to B.
c. Finally: the shortest path from A via l and m back to A.

6. In this next situation the lines l and m are perpendicular.



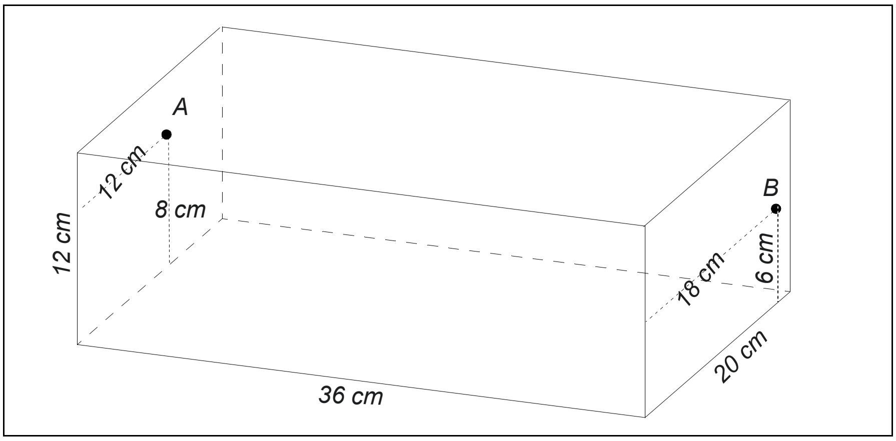
a. Again construct the minimal route $APQB$ with P on l and Q on m.
b. What do you notice about the directions of AP and QB?

7. In the situation below l and m are parallel.
A route has been sketched from A to l, to m, again to l and only then to B. The consecutive contact points with l and m are P, Q and R, but this is not the shortest route.



a. Suppose Q has already been found. How to find R to make Q-R-B as short as possible?
b. What about the angles of PQ and QR with the line m?
c. Think back to the shoelaces; now construct the shortest path precisely.
d. Also indicate why, when solution is finished, AP // QR and also PQ // BR.

8. Shortest route on a shoebox!
There is an ant on the back of this (shoe)box of 12 by 20 by 36 cm at point A.



Determine the fastest route from A to B for this ant along the outside of the box. First try to find a way yourself, before you use the following hint.

hint
On *worksheet F: an ant on a shoebox* (page 180) you find a possible net for part of the box. Now it looks like earlier problems. It is just an idea, but this net might not help you to find the shortest path. Think about adapting it concerning the way the faces of de box are linked. There is some freedom left.

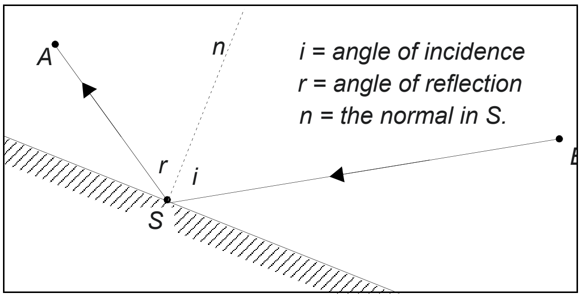
The principle of Fermat

The reflection principle applied to shortest paths via a straight line, and then you found equal angles. There are also situations where you *know* already that there are equal angles, so you could work in the opposite direction.

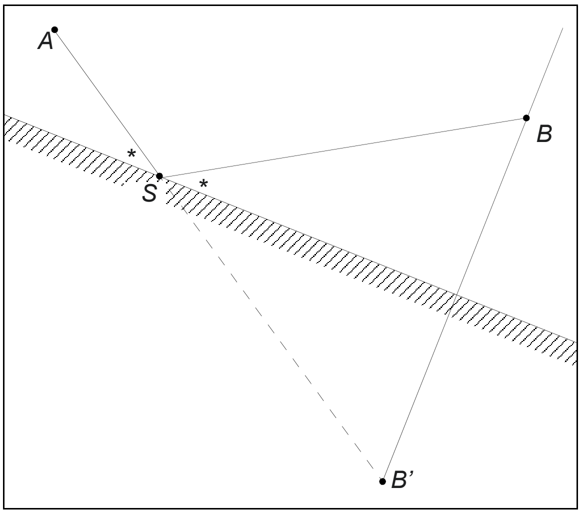
You know that for the reflection of a ray of light this principle holds:

> *the angle of incidence is equal angle to the angle of reflection*

See the illustration. The *normal n* is the line perpendicular to the mirror.



Our figures looked a lot more like the figure on the right, where it is also indicated how you can find the reflection line from *A* to *B*.
We looked at the angles *AS* and *SB* make with the *mirror*, not at the ones with the *normal* on the *mirror*.



You can see also that

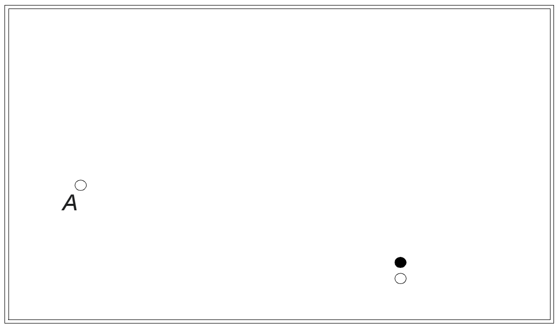
> *the light ray chooses the shortest path to go from B to A via the mirror.*

This is the so-called principle of Fermat.

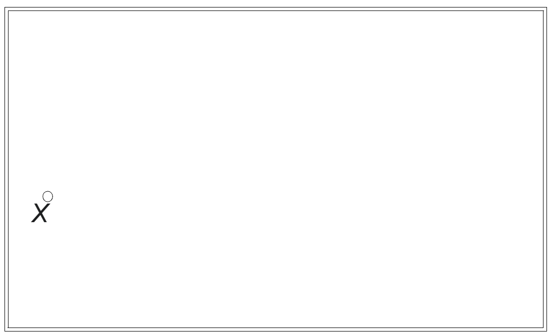
billiards problems

In billiards the balls bump against the so-called *cushions* (the edges round the green cloth) and go on from there. In this part we assume that this happens according to the reflection principle. According to experienced billiards players, this is not true, especially if some *effect* (spinning of the ball) is added to the shot or effect arises when touching the cushion. If you think this is becoming a bit theoretical, you might also think of a room with mirror walls instead of billiards, and then you'd change from ball routes to light paths.

9. Here a billiard table is sketched in top view. Your cue-ball is *A*. It is a three-cushions billiards game, i.e. the shooter's cue ball must contact the cushions of the table at least three times before first touching the *third* object ball of his shot. You can also hit the same cushion twice, as long as the cue-ball touches a cushion three times. When you hit the *second* ball is irrelevant; in this case ball two and three lie so close together that it is very likely you will hit the three cushions first before touching the other balls practically at the same time. Now construct *two* different possible routes for ball *A*.



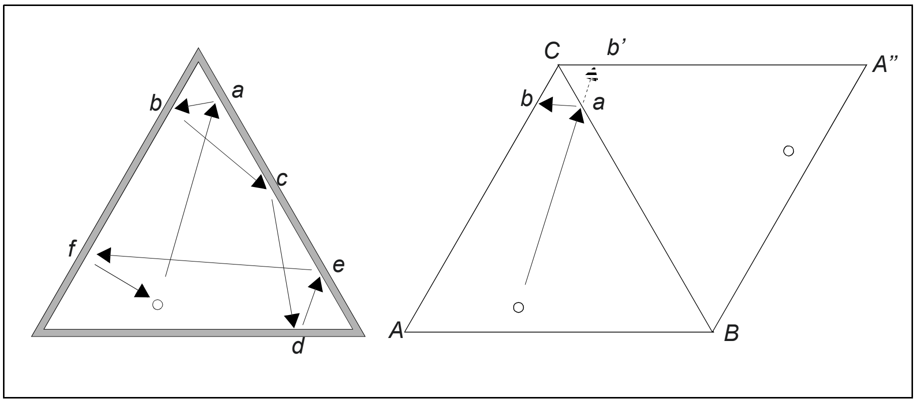
10. Play billiards with one ball! Show that you can shoot the ball in such a way that after contacting all four cushions, it returns to the same spot and continues in the same direction.



Hint:
reflect the billiard table repeatedly to all sides. Set *A, B, C, D* on the vertices and reflect these letters at the same time. It will be easier to indicate the reflected *X*-positions

extra: a triangular billiard

11. We want to draw the exact six-cushion path in this triangular billiards. On the right part of a solution is suggested. Complete it, if necessary use *worksheet G: triangles and mirrors* (page 181).



Summary of chapter 6

shortest connection
In this chapter you used the fact that the shortest connection between two points is a straight line.

principle of Fermat
You noticed that you could find the shortest path via one or more lines by applying the reflection principle. This principle is known as the *principle of Fermat.*

reasoning with angles
Therefore you needed to reason with angles in various situations. It is time we pick up the how and why of that reasoning. This will happen in part two of Advanced Geometry: *Thinking in circles and lines*.

Example solutions

General remarks

– Different solutions are possible for many exercises. Thus your answer will often differ from the solution given here, without it being wrong. In most cases, the example solutions will be sufficient to determine whether you answer is acceptable.

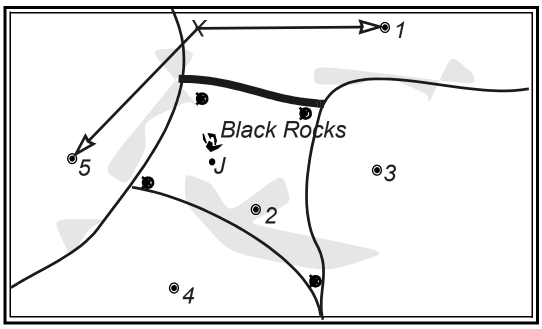
– Some comments with these solutions will give a deeper explanation. You will see this when you encounter them.

– The original drawings from the books were used in these solutions. That is why the solutions look prettier than is expected of you. You yourself will have to copy figures and then only sketch the necessary things. Do maintain clarity in your sketches.

– A few times you will not find an answer here, for example, when you were asked to make a figure with the computer. Sometimes you will find a printed screen. It will help you to check whether you are heading in the right direction, but of course you do not need to reproduce such pictures in your notebook, unless absolutely necessary.

Chapter 1: Voronoi diagrams

1: In the desert

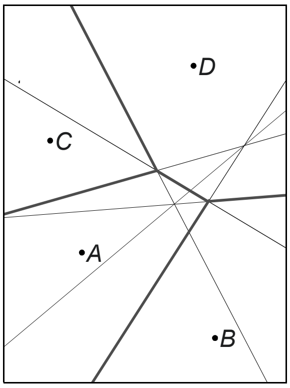


1. **a.** 2
   b.
   c. See figure (it does not need to be exact, that is for later on).
   d. Choose to which well you go.
   e. Yes. See point X. The arrows are of equal length and the distances to well 2,3 and 4 are larger.
   f. Yes. Very far to the southeast.
   g. No. Not for 3 and 4. (It does hold for the extensions of the edge.)
   h. Straight. (This will be investigated thoroughly later.)

2: The edge between two areas

2. That is a good name. If we were dealing with the partitioning of for example oil fields, you definitely would get conflicts on the edges, because the points on the edges belong to both fields. By the way: Geologist can partly explain why most of the oil under the North Sea is in the middle, at equal distances from the coasts.

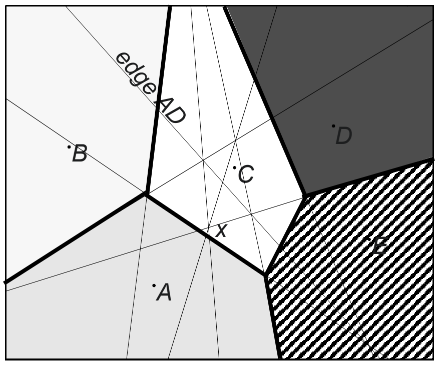
3: More points, more edges

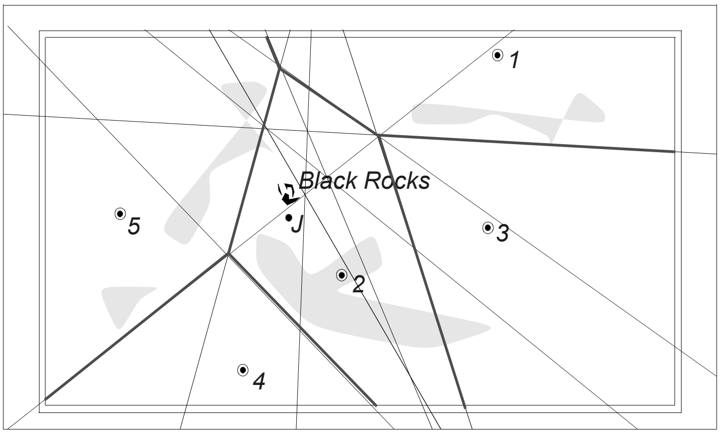


3.
- **a.** 6. For every pair one. Thus 3 + 2 + 1 = 6.
- **b.** Sometimes 3 folding lines intersect. For the other intersections 2.
- **c.** The one between *B* and *E.* Not the fold between *B* and *D*!

4
- **a.** 4 + 3 + 2 + 1 = 10.
- **b.** It is closer to *D*; you can tell by the edge *BD*.
- **c.** It does not belong to *A* (edge *AC*). Also not to *D* (edge *AD*) and not to *E* (edge *AE*). Remains *C*. Be careful: Not belonging to *B* because of edge *BD* only means: the distance to *D* is smaller than to *B*. It does not mean that the distance to *D* is the smallest.
- **d.**

5.





4: *Voronoi diagrams, centers, edges*

6 a.

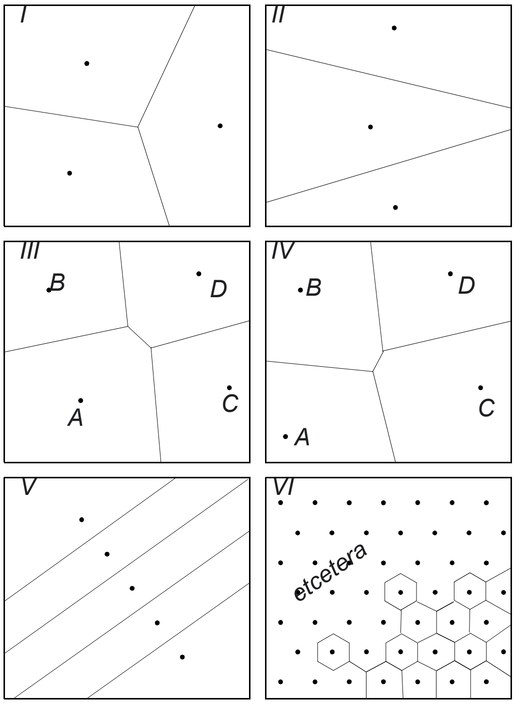
b. That point lies on equal distances from the centers.

c. Yes, but that lies outside the figure.

d. For IV point *A* is further away. Therefore the cells round *B* and *C* adjoin instead of those round *A* and *D*.

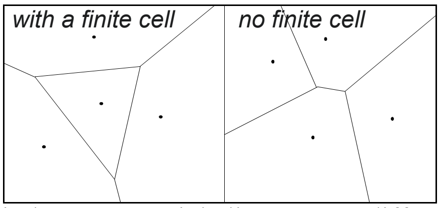
e. The edges are parallel. Ribbon villages in Noord-Holland have such fields.

f. The honeycomb. Basalt (solidified lava) often shows these hexagons. Famous examples are the Giants' Causeway in Northern Ireland and the island Staffa in Scotland.



7 **a.** 4

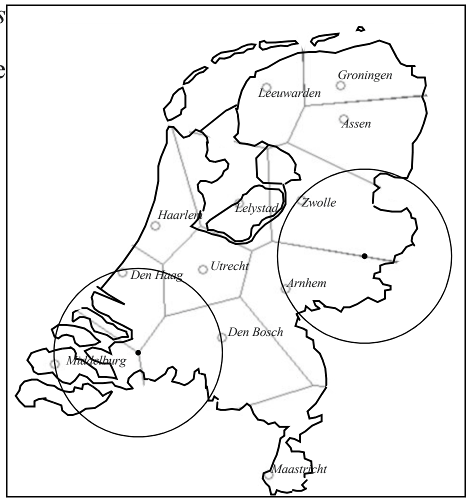
b.



c. 20 points on a circle or on a straight line. Or on a different figure, which does not have bays, for example an oval. It does not matter how they lie on that circle, line or curved figure.

d. On the edges. (We will investigate this later on.)

5: Three-countries-points, empty circles

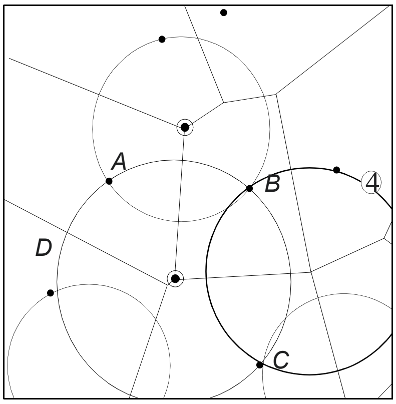


8 **a.** It has equal distances to the three cities.

b.

c.

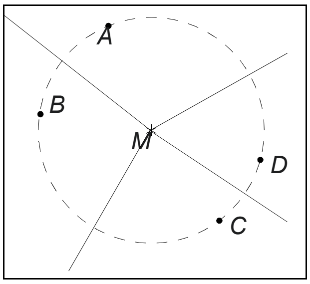
9 **a.** See



b. See figure.

c. There are always three.

10 **a.** *D* lies closer to *M*. Thus the circle through *A*, *B* and *C* is not a largest empty circle.

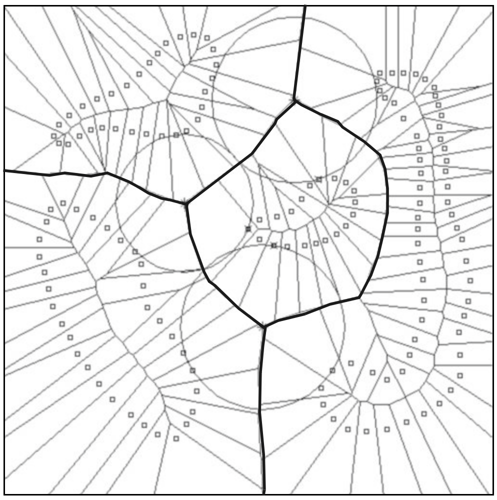


b.

c.

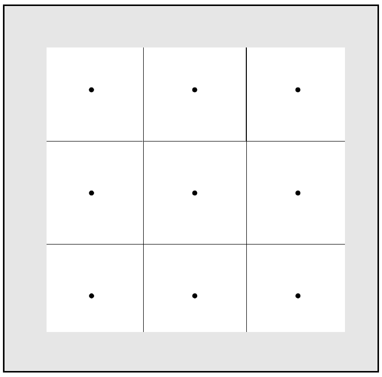
d. If there are three centers, there always is a three-countries-point (except if the centers lie on one line).
If a four-countries-point needs to occur, the fourth center needs to lie on the circle through the three other points. This happens by coincidence. Thus it is something special.

11 **a.**



b. Where the three colored edges converge.

12.



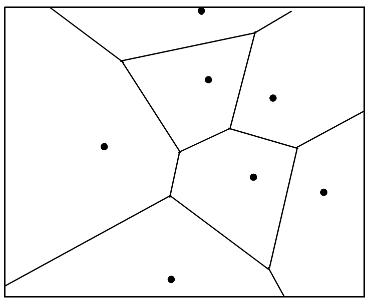
6: Chambered tombs in Drente; reflecting

13 **a.** The centers all lie at the same distance from the edge and the edge is perpendicular to the connection line.

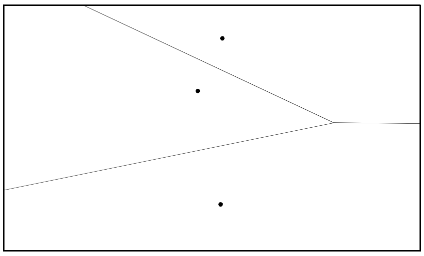
b. It is not completely accurate, but it should be so.

c. The round dot closest to the east edge.

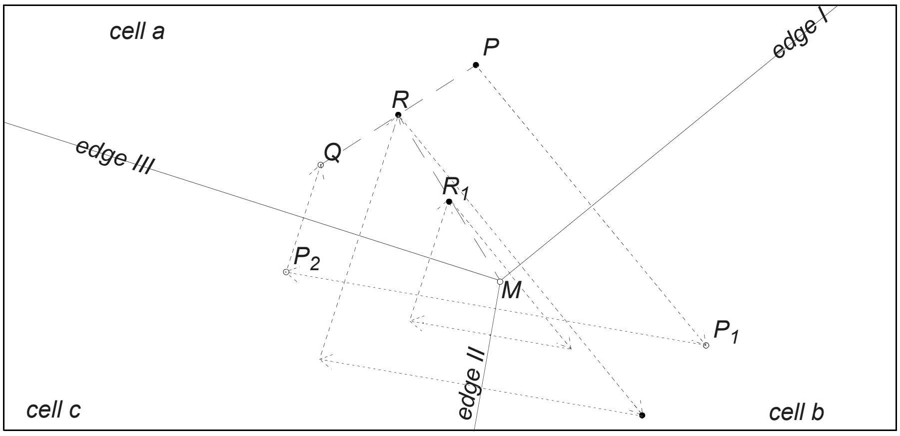
14.



15.



16.



a. P and Q are different. If P was the center of cell a, then P_2 should be the center of cell c. This is not possible.
b. S is equal to R.
c. See R_1.

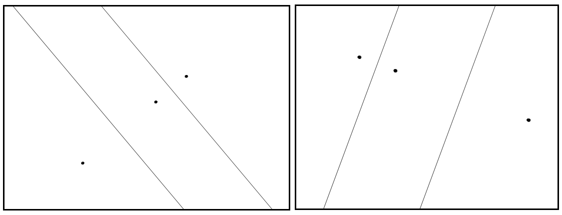
Chapter 2: Reasoning with distances and angles

7: Introduction: reasoning in geometry

1. **a.** That it is a line and not part of an area, that it is a straight line, that the points lie symmetrical in relation to the line (There probably is more!)
b. The symmetry was used to find the missing centers.

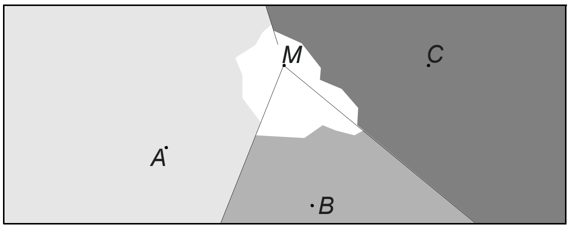
8: An argumentation about three-countries-points and circles

2
a.



b. The three centers lie on one line.

3
a.



b. (distance M to A) = (distance M to B)
(distance M to B) = (distance M to C)
From this directly follows:
(distance M to A) = (distance M to C).

c. Thus M lies on the Voronoi edge of A and C.

d. Yes, all of the three edges pass through M.

4.
a. The distances from M to A, B and C are all three equal. Thus the circle with center M, which passes through A, also passes through B and C.

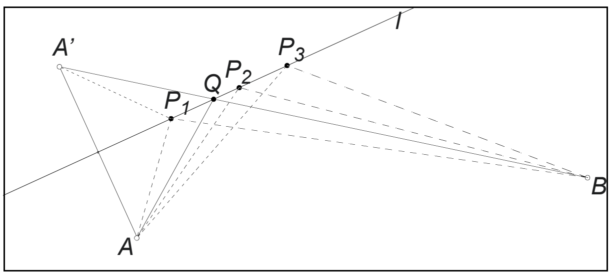
b. That was the largest empty circle round the three-countries-point M.

5.
a. That the two edges indeed have a point of intersection.

b. No, apparently not. If the intersection of the AB-edge and the BC-edge *exists*, then it also lies on the AC-edge. The shape of the edge is irrelevant for the argumentation, only the distance equalities matter. (That the intersection exists, follows from rectilinearity and non-parallelism.)

9: shortest paths and triangle inequality

6.



a. Via P_2. But it differs very little.

b. Since $d(A, P_1) = d(A', P_1)$.

c. Draw the straight line A'B. The intersection with l is Q.

d. *Example one*: Getting water from river l and bringing it to village B.
Example two: l is a railroad line: the cities A and B get one station together.

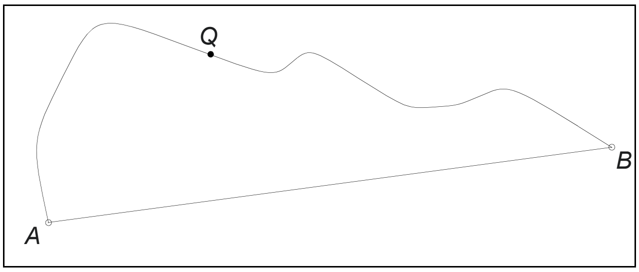
7.

a. From P to Q is as far as from Q to P.

8.

a. For example on $A'P_1B$.

b. $d(A', Q) + d(Q, B) = d(A', B)$.



9.

a. Something with a route that does not consist of straight pieces.

b. It only deals with distances.

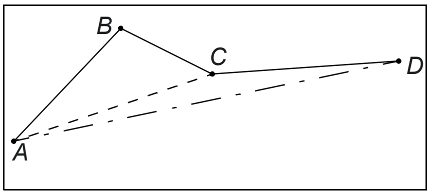
10.

a. You could think of the Pythagorean theorem.

b. What is this Pythagorean theorem based on?
But also: how does the triangle inequality follow from the Pythagorean theorem?

11. Draw AC and apply the triangle inequality twice: to AD with a detour via C and to AC with a detour via B:

$$|AD| \leq |AC| + |CD| \leq |AB| + |BC| + |CD|$$



10: The concept of distance, Pythagorean theorem

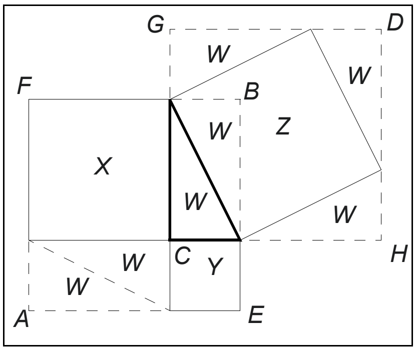
12. If in triangle ABC angle B is right, then: $d(A, C)^2 = d(A, B)^2 + d(B, C)^2$.

13 a. $d(A, Q)^2 = d(A, P)^2 + d(P, Q)^2$.

b. Since $d(P, Q)^2 > 0$, $d(A, Q)^2 > d(A, P)^2$ and since distances are non-negative: $d(A, Q) > d(A, P)$.

14 a. There are many ways to do this. Also put letters in the areas of the squares and the triangular areas.
Area(AEBF) = Area(CHDG)
Thus*:*
$X + Y + 4 \times W = Z + 4 \times W$
Thus
$X + Y = Z$

b. That the sides are equal and perpendicular to each other.



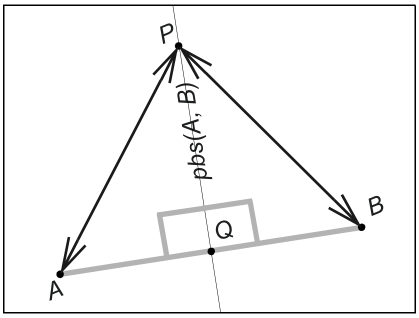
15. You do the same as in exercise **6**, for the case that *A* and *B* coincide.

11: Properties of the perpendicular bisector

16.

a. = green. Equal line segments and angles of 90°.

b. $\longleftrightarrow$ = red. Equality of the two line segments needs to be proven.



c. Since $\angle PQA = 90°$:
$d(P, A)^2 = d(P, Q)^2 + d(Q, A)^2$
Since $\angle PQB = 90°$:
$d(P, B)^2 = d(P, Q)^2 + d(Q, B)^2$
Since we can use that $d(Q, A) = d(Q, B)$, we conclude:
$d(P, A) = d(P, B)$.

d. Yes, perpendicularity is used in Pythagoras, and the equality of the lengths *AQ* and *BQ* is used in the deduction.

17.

a. To prove: $d(Q, A) \neq d(Q, B)$.
But we actually prove: $d(Q, A) < d(Q, B)$.

b. *R* is on the perpendicular bisector, thus: $d(R, A) = d(R, B)$.

c. Triangle inequality for *Q*, *R*, *A*. *R* is not on *QA*, thus we have a true inequality:
$d(Q, A) < d(Q, R) + d(R, A)$.

d. $d(Q, A) < d(Q, R) + d(R, A) = d(Q, R) + d(R, B) = d(Q, B)$.
the last equality holds, since *R* does lie on *QB*.

18. Finding the shortest route via a line. Here you could go from *A* to *Q* via the perpendicular bisector of *AB*.

12: From exploration to logical structure

19.

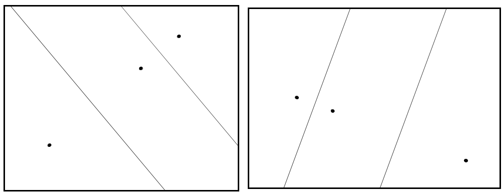
a. *To prove*
$d(P, A) + d(P, B) + d(P, C) + d(P, D) < d(Q, A) + d(Q, B) + d(Q, C) + d(Q, D)$

b. *Proof*:
Use the triangle inequality in triangles *ACQ* and *DBQ*.
$d(P, A) + d(P, C) = d(A, C) \leq d(Q, A) + d(Q, C)$.
and
$d(P, B) + d(P, D) = d(B, D) \leq d(Q, B) + d(Q, D)$.
In at least one of the cases we have an inequality, because else *Q* would be equal to *P*.
By adding, what needed to be proven follows immediately.

20. *Idea*: Use that *B* and *D* lie ***on*** the 'Voronoi edge' of *A* and *C*.
Proof:
$d(A, B) = d(B, C)$ (this is given). Thus *B* lies on *pbs(A, C)* according to theorem 3.
$d(A, D) = d(D, C)$ (this is given). Thus *D* lies on *pbs(A, C)* according to theorem 3.
Combine: the line through *B* and *D* *is* the *pbs(A, C)*. Thus *BD* is perpendicular to *AC*.

21 **a.** By demanding that *A*, *B* and *C* form a triangle.

b. The perpendicular bisectors are perpendicular to that line and are thus parallel. Re-using an old figure.



c. Then the perpendicular bisectors are not parallel and thus do intersect.

22.

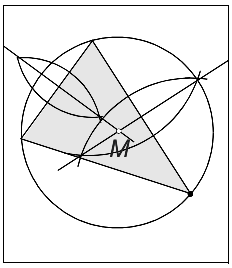
a. 3b, writing down the equalities in distance <> step 1 and step 1bis
3b, conclude third equality<>connection step
3c conclusion that *M* lies on the third Voronoi edge<>conclusion step.

b. step 1 and 1bis: from *on perpendicular bisector* to *equality of distances*,
The conclusion step is the other way around: from *equality of distances* to *on perpendicular bisector*.

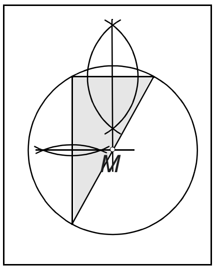
23. *P* and *Q* lie at equal distances from *A* and *B* and thus on the perpendicular bisector of *A* and *B*. Theorem 3.

24. With the techniques of the two circles, Dürer constructs the perpendicular bisectors of *a* and *b*, and also those of *b* and *c*. He calls the intersection *d*. That intersection is the center of the circle through *a*, *b* and *c*.

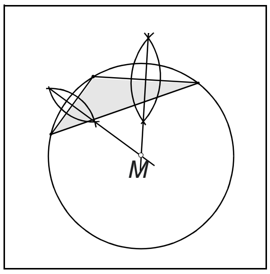
25. Three examples of the construction: (circles with equal letters have the same size).



Acute triangle: center circumcircle inside the triangle.

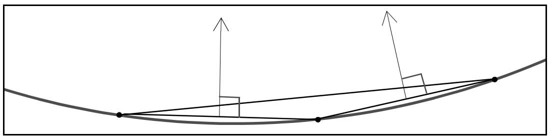


Right triangle: center circumcircle l on the triangle



Obtuse triangle: center circumcircle outside the triangle

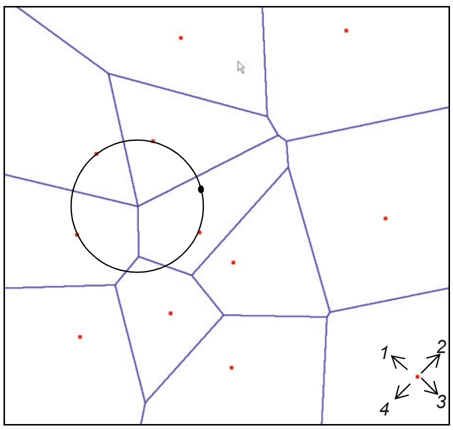
26. Choose three points on the circle. Construct the circumcircle of that triangle. You already have the circle, but the construction also gives the center!



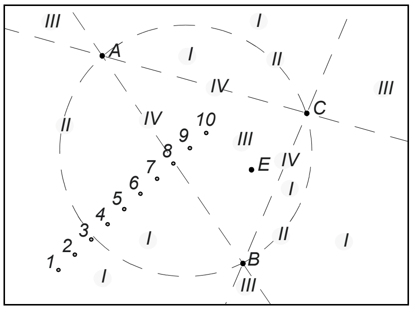
Chapter 3: Computer practical Voronoi diagrams

13: Introduction

1.
2 a.
b.
c. Direction 3.
3 a.
b. the angle bisector of the angle of the two lines through the points.
c.
4.



14: The influence of the fourth point

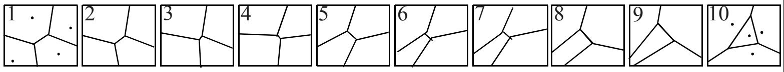


5.

a. For example point *E* shown above on the right.
b. The button `circle(A,  B,  C)` does exactly that.
c. Then there are two perpendicular edges. Thus three points need to lie on one line. Thus the three lines *AB*, *CB* and *AC*. All of them!
d. Type III.

6 a.

Voronoi:



Type:	I	I	I	I	I	I	I	IV	III	III

b. When *D* passes through the circle.

7.

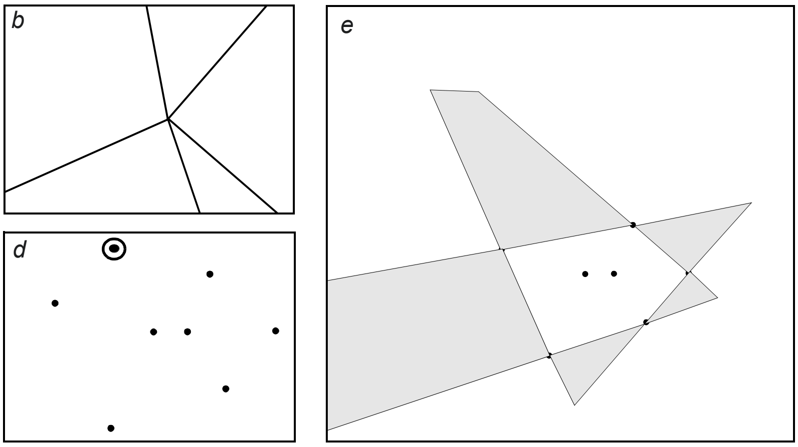
a. See above.
b. Type II and IV only occur when *D* lies on certain lines lies. This is rare.

15: Infinitely large cells, the convex hull

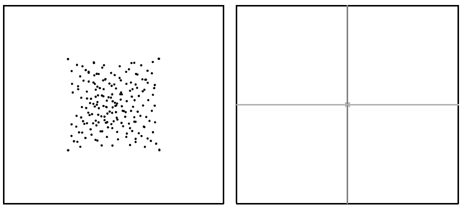
8 a.

b. Five rays from 'one' point.
c. A rubber band.
d. No new infinitely large cells! Think about two centers which are connected by a segment of the hull. There are always circles through those centers, with no other center inside this circle, if you have a midpoint of the circle far away enough.
e. For example:

f.



9.

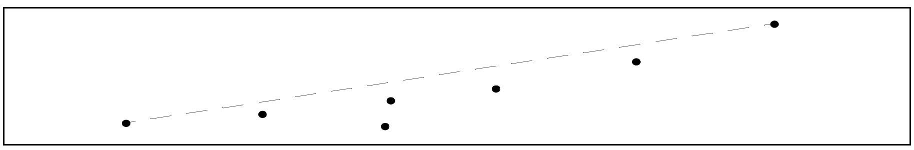


10 **a.** They adjoin.

b. Those centers lie at equal distances from *A* and of *B*.

c. Those cells still adjoin. The perpendicular bisector of AB continues past the center of the circle through *A*, *B* and *E*. There it is the Voronoi edge of *A* and *B*. If you take more points in the half-plane of the line *AB* on the side of *C* and *D*, it still holds that far on the perpendicular bisector of *AB* lie points for which the closest centers are *A* and *B*.

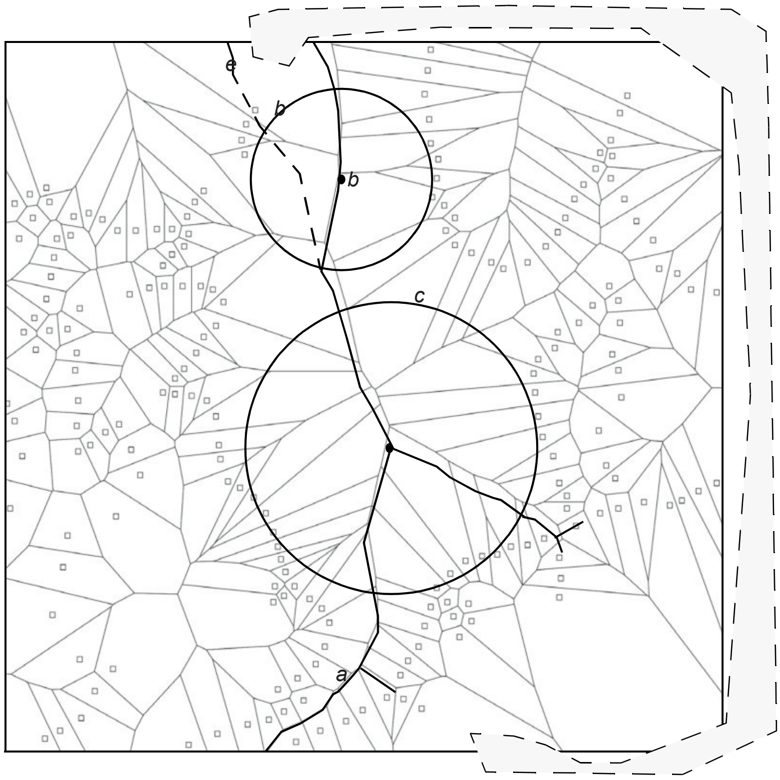
11.



12.

16: Dividing the sea

13.



a.
b.
c.
d. No! If the countries behaved like simple centers in a Voronoi diagram, it should be okay. But imagine the Netherlands having a small strip of land starting in the southwest, which goes around the pictured area, north over Norway and comes in at the top of the picture.
e.

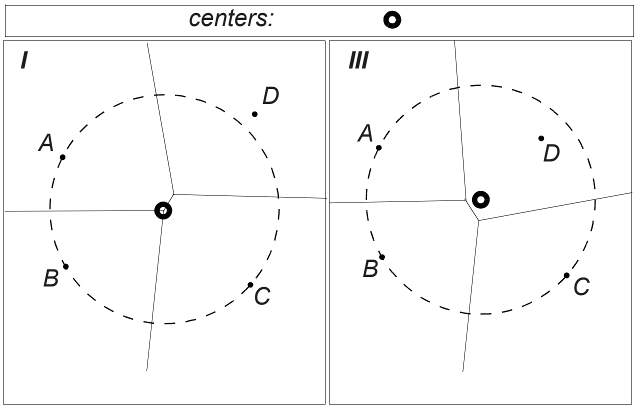
17. Exploring two more mathematical configurations

14.

a.
b. Yes. The new center must be the midpoint of the circle.
c.

Chapter 4: A special quadrilateral

17: Cyclic quadrilaterals



D outside the circle through ABC

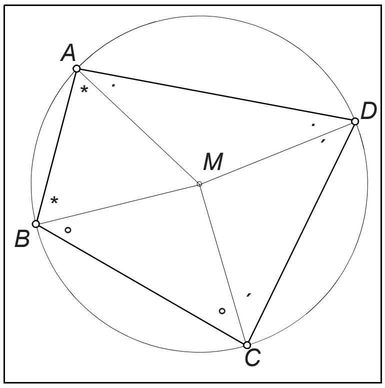
D inside the circle through ABC

1. a. The center of the circle lies on equal distances from *A*, *B*, *C* and *D*. That point belongs to all four cells.

b. ◎

c. *D* lies inside the circle through *A*, *B* and *C*; therefore the center of that circle lies in the cell of *D*.

2.



a.

b. the four triangles are isosceles.

c. The same symbols can be found at *A* and *C* together as at *B* and *D* together. Thus $\angle A + \angle C = \angle B + \angle D$.

18: Scrutinize proving

3 **a.** Here *M* does not lie inside the quadrilateral.

b. When comparing the symbols in the angles. This is simply impossible.

4.

a. $\angle ADC$. For the size of the angles holds: $\angle ADC = \angle D_1 - \angle D_2$.

$\angle D_1 = \angle A_1$, $\angle A_2 = \angle B_1$, $\varsigma B_2 = \angle C_1$, $\angle C_2 = \angle D_2$
(since the triangles *DMA*, *AMB*, *BMC* and *CMD* are isosceles

$\angle A + \angle C$
= (refinement angles)
$\angle BAD + \angle DCB$
= (subdividing angles)
$(\angle A_1 + \angle A_2) + (\angle C_1 - \angle C_2)$
= (use equal angles, working towards either *A* and *C* or *B* and *D*)
$(\angle D_1 + \angle B_1) + (\angle B_2 - \angle D_2)$
= (rearrange to lead to *B* and *D*)
$(\angle D_1 - \angle D_2) + (\angle B_2 + \angle B_1)$
= (use subdivision)
$\angle ADC + \angle ABC$
= (back to the needed angles)
$\angle D + \angle B$
= (calculation step: $x+y = y+x$)
$\angle B + \angle D$.

(Such motivations as in the last two steps need not to be indicated all the time!)

5. Replacing the minus signs by plus signs is enough.

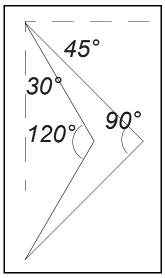
6 **a.** *M can still be on* the quadrilateral.

b. Take as example *M* on *DC*.
In that case both proofs work.
(That is actually because then $\angle C_2$ and $\angle D_2$ are equal to 0. Then is does not matter whether it has plus or minus signs.)

7.

a. Results are 180° or lie real close to that.

b. A reasonable statement is: the sum of two opposite angles in a quadrilateral is 180°.



8. All four 360°:

a. $4 \times 90° = 360°$

b. $60° + 120° + 60° + 120° = 360°$

c. $2 \times 90° + 45° + 135° = 360°$

d. the acute angles are (see figure) $90° - 30° - 45° = 15°$.
$90° + 2 \times 15° + (360° - 120°) = 360°$

9 a.

b.

Proof ONE

$\angle A + \angle B + \angle C + \angle D$

= (subdivision of the angles)

$\angle A_1 + \angle A_2 + \angle B + \angle C_1 + \angle C_2 + \angle D$

= (rearranging to triangles)

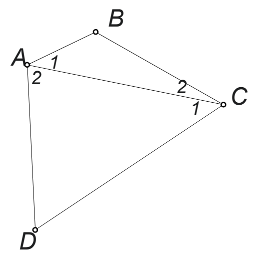
$\angle A_1 + \angle B + \angle C_2 + \angle C_1 + \angle D + \angle A_2$

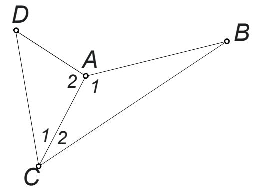
= (sum of angles in triangle is 180×, twice)

180° + 180°

= 360°

for 9 a and 9 a





Proof TWO

$\angle A_1 + \angle B + \angle C_2 = 180°$ (sum of angles in triangle *ABC* is 180°)

$\angle C_1 + \angle D + \angle A_2 = 180°$ (σum of angles in triangle *ADC* is 180°)

$\angle A_1 + \angle B + \angle C_2 + \angle C_1 + \angle D + \angle A_2 = 180° + 180°$ (combine)

$\angle A + \angle B + \angle C + \angle D = 360°$ (rearrange)

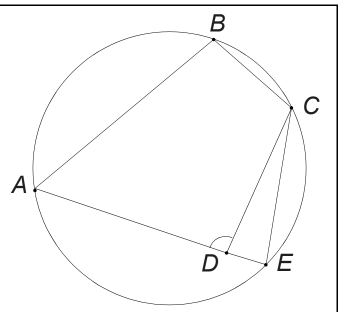
10.

a.

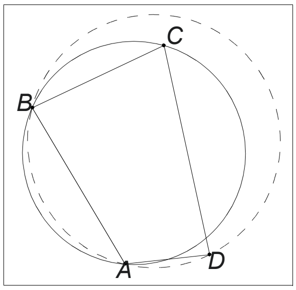
b. $\angle DCE + \angle CED + \angle EDC = 180°$.
But also: $\angle ADC + \angle EDC = 180°$.
Thus: $\angle ADC = \angle DCE + \angle CED$

c. $\angle ADC > \angle CED$.

d. Since $\angle B + \angle E = 180°$ according to the temporary theorem of the cyclic quadrilateral, follows from the inequality in **b**: $\angle B + \angle D > 180°$.



11.

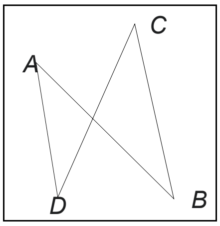


a. No other points but A and B can lie on the dotted circle as well as on the sketched circle, since then the circles would coincide according to theorem 5. Thus if D lies outside circle ABC, the whole arch BAD lies outside circle ABC and C lies inside circle ABD.

b. $\angle A + \angle C > 180°$.

c. Since the four angles together are 360°, this needs to be true: $\angle B + \angle D < 180°$.

12. CD then passes through AB; for example like this:

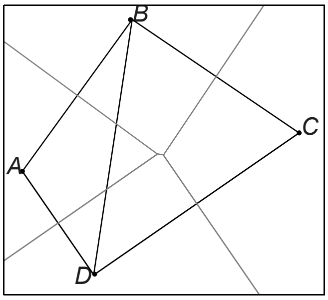


A **definition** of a useful kind of quadrilateral here could be:

A quadrilateral ABCD consists of four points, and the four line segments AB, BC, CD and DA, for which no three points lie on one line, the line segments AB and CD do not intersect and the line segments AD and BC do not intersect.

13 a. In this case the angle at $\angle ADC$ on its own is already over 180°.

19: Using cyclic quadrilaterals



14.

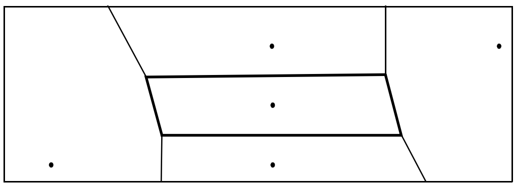
a. according to the protractor is $\angle B + \angle D = 183°$. Thus D lies inside the circle through A, B and C. Cells B and D adjoin.

b.

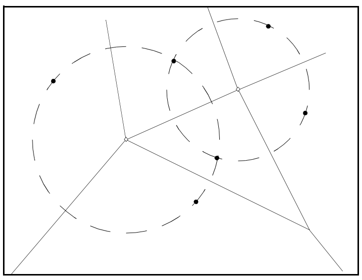
c.

d. You already knew which cells adjoined.

15. That is a mistake and here is a counter-example.



16.



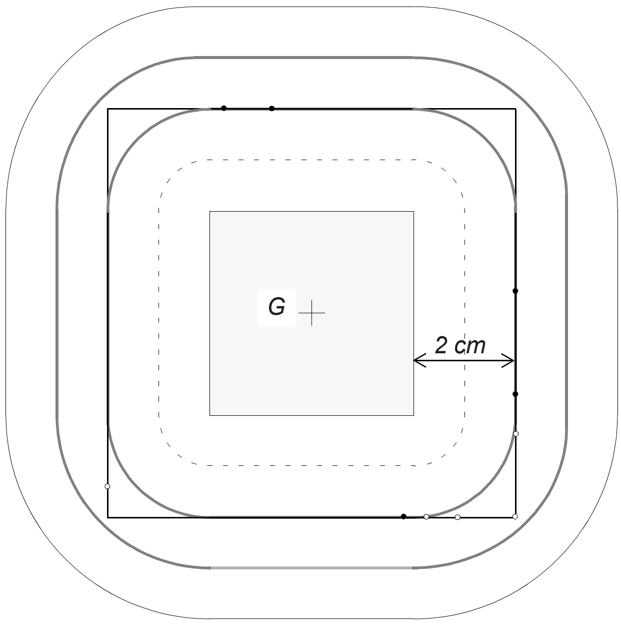
Chapter 5: Exploring isodistance lines

20: isodistance lines, distance to areas

1.

a. A circle with a radius of 12 miles.
b. A circle with a radius of 12 miles + 680 meter.

2.

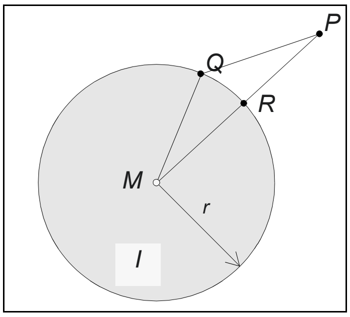


a. does: • does not: ○
b. iso-1 cm-line - - - - - - - - - -; iso-2 cm-line ———;
iso-3 cm-line ———; iso-4 cm-line ——— .
c. *the iso-300 metre-line is not a circle since it has four straight segments, each of 4 cm.*

3.

a. Sketch the line through *P* perpendicular to *l*. In intersects *l* in *R*. *d(P, G)* = *d(P, R)*.
b. With the triangle inequality and with Pythagoras.

4.

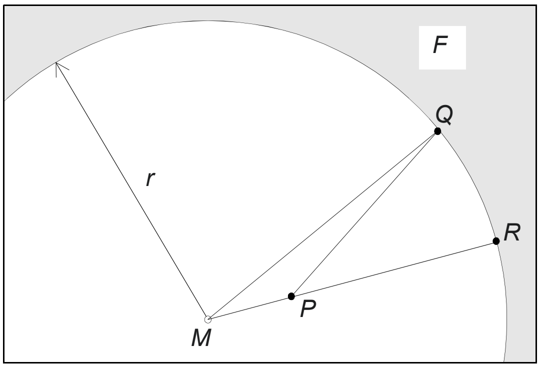


a. on the intersection of *MP* with the circle.

b. *d(P, Q) + d(M, Q) > d(P, M)*.
Since *Q* and *R* lie on the circle, holds*:*
d(R, M) = d(M, Q).
d(P, M) = d(P, R) + d(M, R) since *R* lies on *PM*.
Filling out gives *d(P, Q) + d(M, R) > d(P, R) + d(M, R)*.
Thus*: d(P, Q) > d(P, R)*.

c. Since *d(P, I) = d(P, R)*, holds:
d(P, I) = d(P, M) − r.

5.

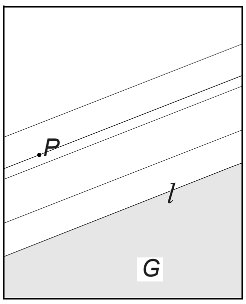


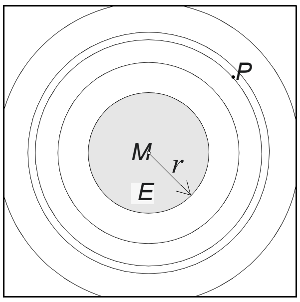
a.

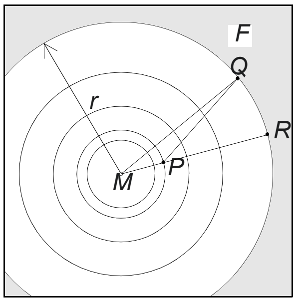
b. the triangle inequality in triangle *MPQ* says*:*
d(M, R) = d(M, Q) < d(M, P) + d(P, Q).
Since
d(M, R) = d(M, P) + d(P, R) also holds,
d(P, Q) > d(P, R) needs to hold as well.

c. *d(P, F) =r − d(P, M)*.

6 **a.** Reduced, scale 1 : 2.







b. From the expressions for the distance in the **c**-questions.
From those formulas, it follows that the points lie on a fixed distance to the circle and also lie on a fixed distance to the center *M*.
Elaborating on, for example, exercise 4:

(1) *d(P, E) = d(P, M) − r* *(exercise 4 c)*

(2)If *P* lies on the iso-*a*-line:

d(P, E) = a *(definition iso-a-line)*

(3) *d(P, E) = d(P, R)(according to exercise 4a)*

From 1, 2 and 3 follow:

d(P, M) = d(P, R) + d(R, M) = a + r

a + *r* is constant, Thus:

then *P* lies on a circle with radius *a* + *r* and center *M*.

Now we also need to show: *each point on the circle with radius a + r and center M, lies on the iso-a-line of E.*

That goes like this (written down in shorter form than the first half of the proof):

If *d(P, M) = = a + r,*

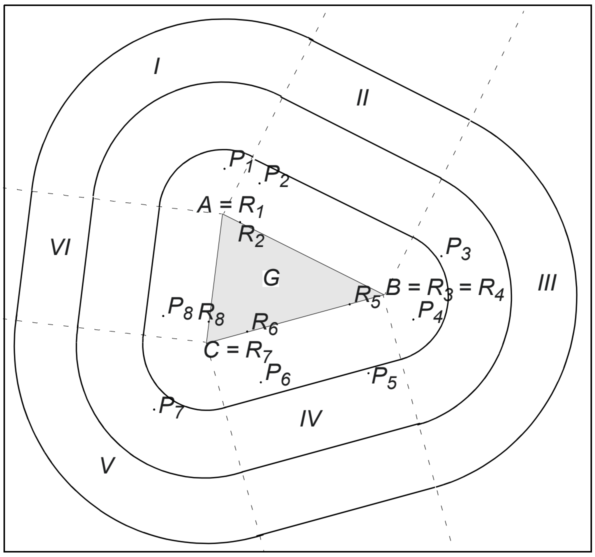
and *R* is the point on the circle on the line segment *PM*, then also:

d(P, E) = d(P, R) = d(P, M) – d(R, M) = a + r – r = a.

Thus *P* on the iso-*a*-line of *E*.

7. *Scale 1 : 2.*

a.



b. The dotted lines are the edges of the zones in this drawing.

c. Zone I, III and V (the zones at the vertices): segments of the circle around the vertex. Zone II, IV and IV (the zones of the line segments): pieces of straight line parallel to the sides.

d. That the straight pieces have the same length. That the circles run smoothly into the straight pieces.

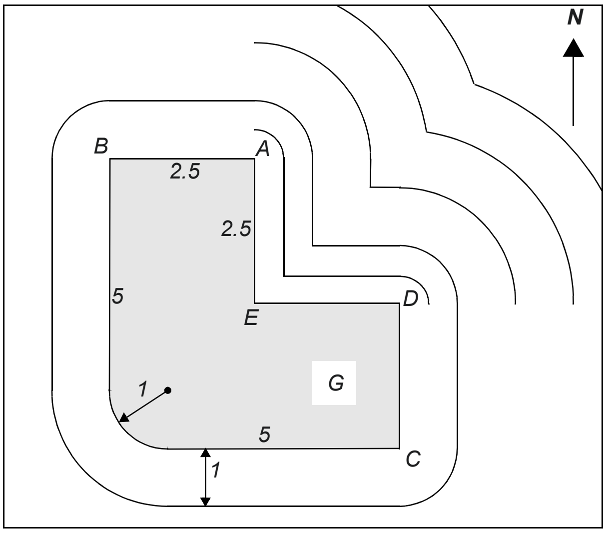
e. Only the arcs are extra compared to the triangle itself. These three arcs of each isodistance line can be put together as a circle. Thus the extra lengths are: 4π, 8π, 16π.

8 **a.** The points on the isodistance lines in the neighborhood of a cape all lie at the same distance from the cape. Thus the cape is the center of the circle on which that piece of isodistance line lies.

b. These lines are perpendicular to the pieces of edges, which lie next to the cape.

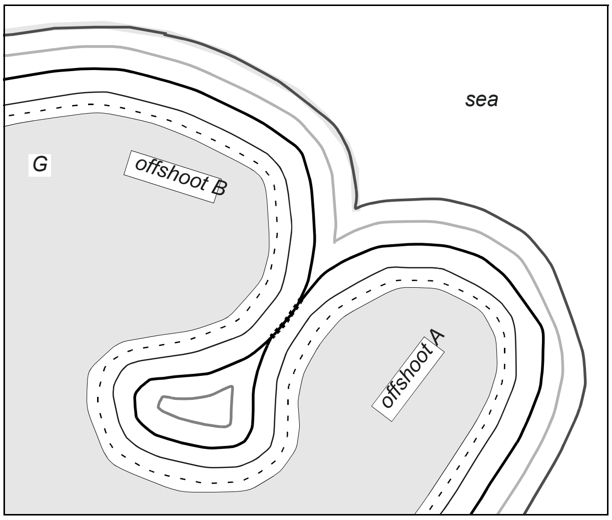
9.

- ***a.***
- **b.** On the angle bisector of angle *AED*
- **c.** The iso-0.5-line and the iso-2-line show a kink of 90^{o}. The iso-3 and iso-4-line show kinks of approximately 108^{o} and 128^{o}.
- **d.** Till 2.5. Because up to there the kink is formed by two straight line segments, which are perpendicular.
- **e.** Since the kink is formed by the intersection of two circles and the isodistance line switches from one circle to the other in the kink, there always will be a kink.

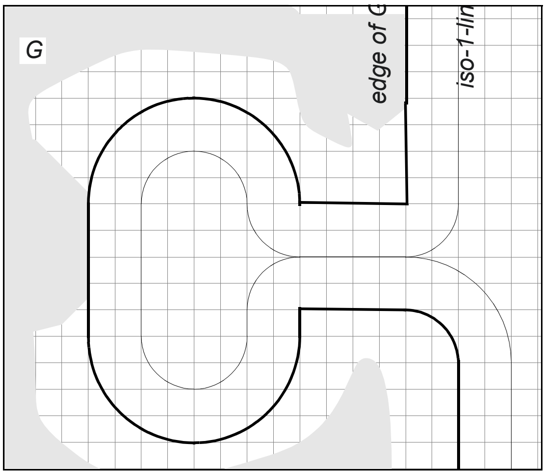


10.

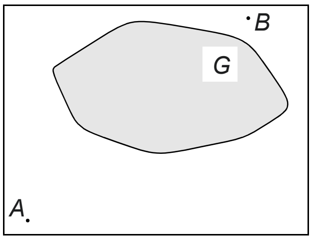
- **a.** Something like the figure.
- **b.** the iso-1.5-line falls apart into two pieces; the iso-0.2-line does not.
- **c.**
- **d.**



11.



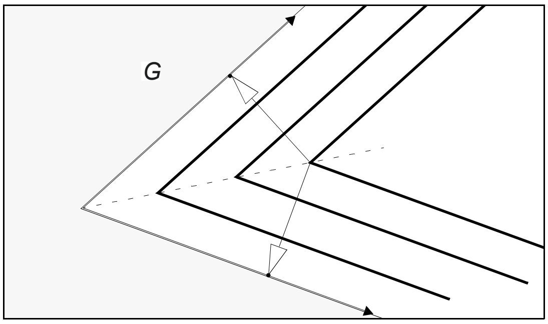
12.



a. Yes.
b.

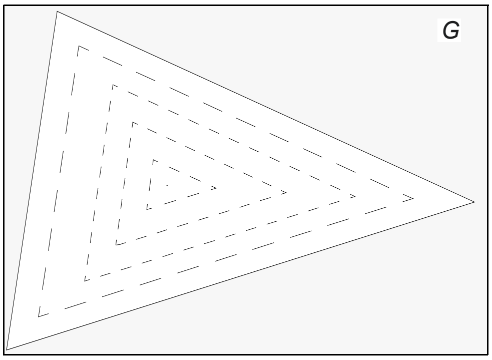
21: Angle bisectors

13. (scale 1:2)



a.
b. the angle bisector of the angle.
c. In this case 2.

14.

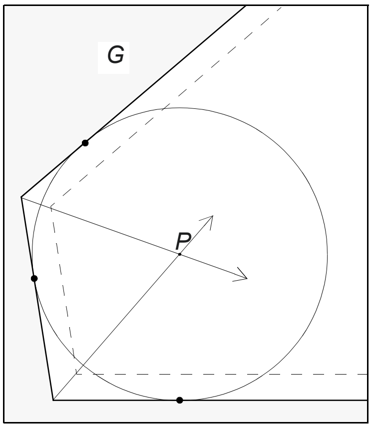


a.
b. They all are triangles with the same shape. The last triangle becomes one point.

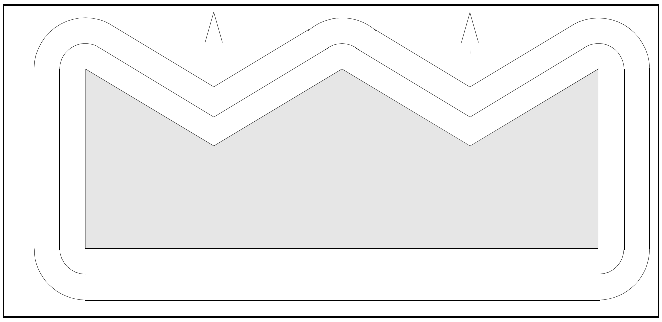
15.

a. No. For larger distances there is only one kink.

b. Sketch the angle bisectors from both angles. If a point lies on one angle bisector, it has footpoints on two sides. If the point lies on both angle bisectors, it has footpoints on all three of the sides.

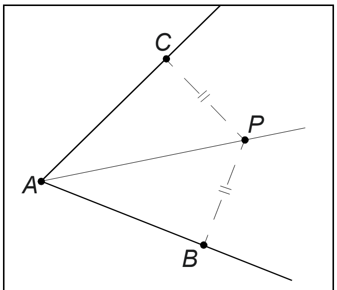


16.



22: Theorems about angle bisectors

17. *Given:* P lies on the kink line of $\angle CAB$.
To prove: the line AP cuts $\angle CAB$ *in half.*
Proof: Since P lies on the kink line, like in the left figure, where C and B are footpoints of P, the triangles APC and APB are right. Since the hypotenuses are equal, and PC and PB also have the same length, the triangles are completely equal. Thus $\angle CAP = \angle BAP$.

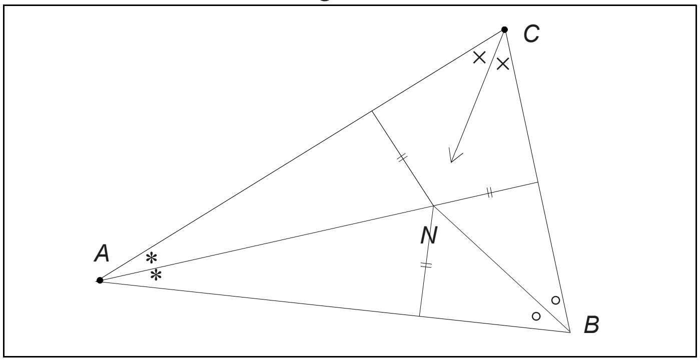


'kink line': d(P, C) = d(P, B)

18. If two half lines a and b converge in an end point S and make an angle of less than 180°, the set of points for which holds:
$d(P, a) = d(P, b)$
is a half line c from S, which makes equal angles with a and b.

For points P inside the angle ASB hold: (Here A and B are points on a and b respectively, not equal to S)
if $d(P, a) < d(P, b)$, then $\angle ASP < \angle BSP$
if $d(P, a) = d(P, b)$, then $\angle ASP = \angle BSP$
if $d(P, a) > d(P, b)$, then $\angle ASP > \angle BSP$.

19. Let N be the intersection of the angle bisectors from A and B.



The proof that N also lies on the angle bisector from C, fits exactly into a scheme.

step 1

N on angle bisector of $\angle CAB$
Thus
$d(N, CA) = d(N, BA)$

step 1bis

N on angle bisector of $\angle CBA$
Thus
$d(N, CB) = d(N, AB)$

↓ ↓

connecting step

Thus

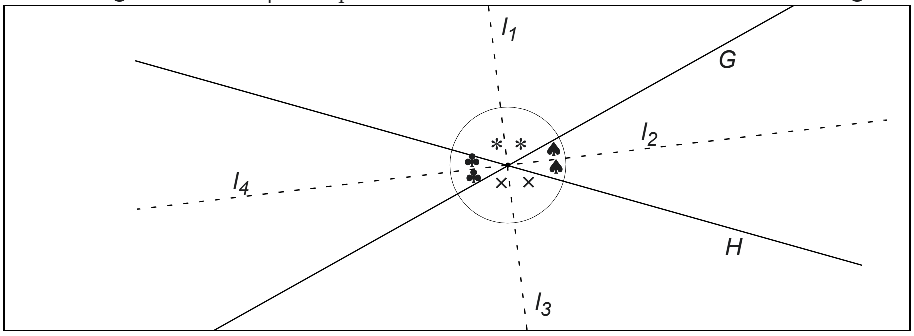
$d(N, CA)= d(N, CB)$

↓

conclusion step

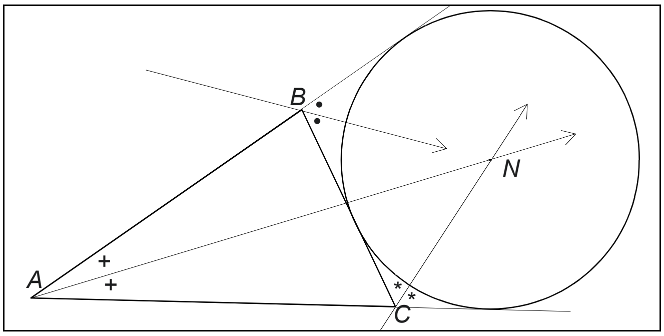
$d(N, CA)= d(N, CB)$
Thus
N on angle bisector of $\angle ACB$

20. the angle between l_4 and l_1 is ♣ + ∗. Since ♣ + ♣ + ∗+ ∗ = 180°, that angle is



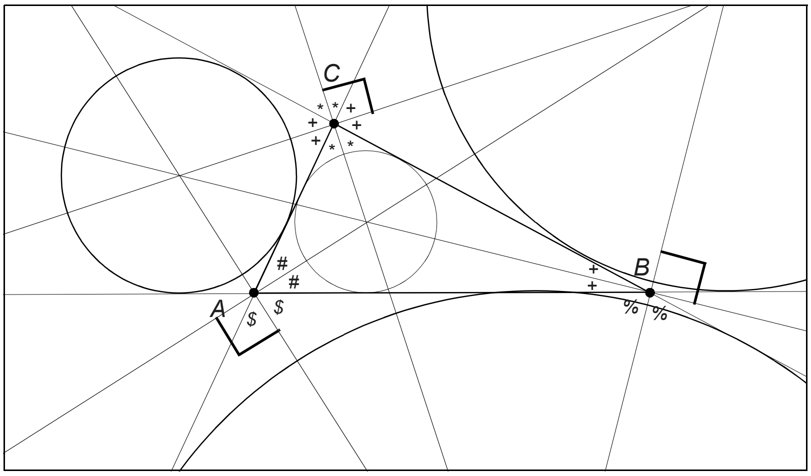
90°. Likewise, the angle between l_1 and l_2 is 90°. Thus in total is the angle between l_4 and l_2 180°. Thus the two halve l_4 and l_2 lines form one straight line.

21. Let N be the intersection of the external angle bisectors from C and the internal angle bisector from A.
Then again this applies:

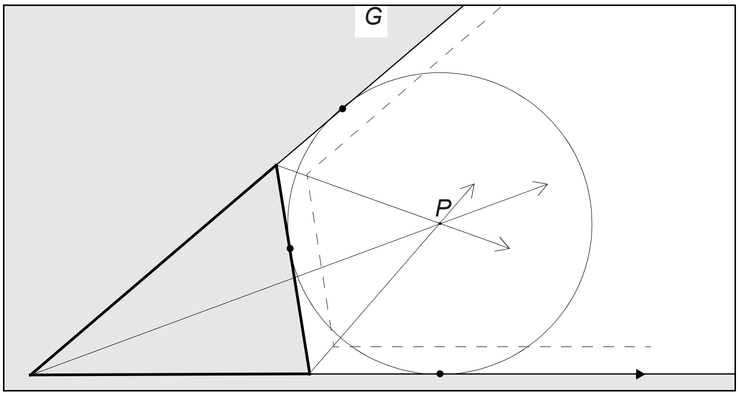


d(N, line AC) = d(N, line BC) and also:*d(N, line AC) = d(N, line AB)*
Thus: *d(N, line BC) = d(N, line AB)*. Thus *N* also lies on the external angle bisector from *B*. (After all, *N* lies outside the triangle).

22.



23.



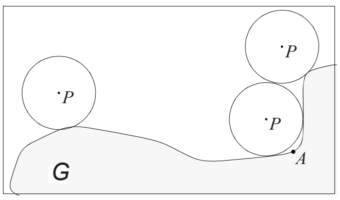
24. Mostly not. For isosceles triangles the angle bisector between the equal sides does go through the tangent point on the third side. Thus for equilateral triangles it happens three times.

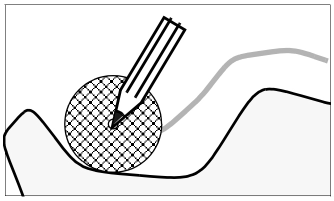
23: Bumping circles

25 a.

b. yes.

26.

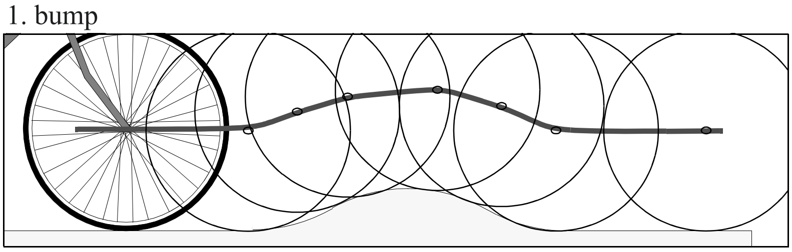




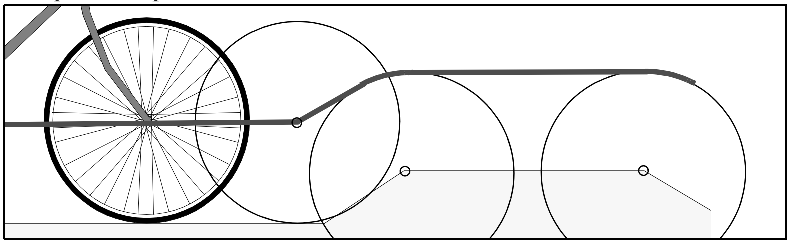
27.

a.

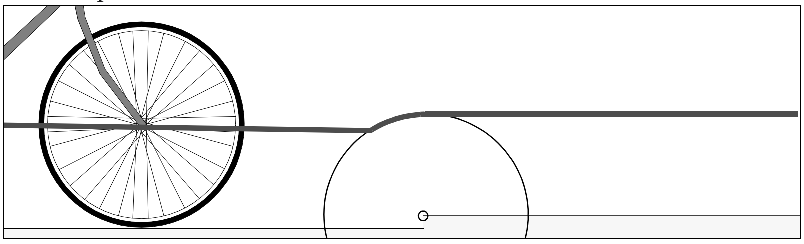
1. bump



2. speed ramp



3. low pavement



b. Two. At the beginning and at the end.

c. It does not matter very much when you look at the kink angles. If the pavement was a little lower, the blow on the speed ramp would be bigger.

28. Yes, in exercise **25**. But not very formal.

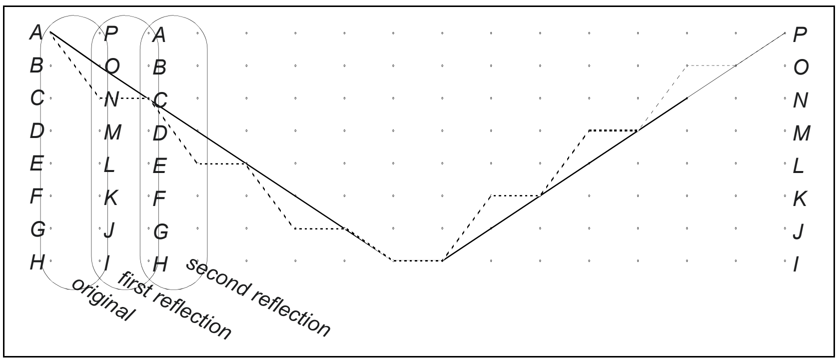
29. It does not apply to the two bays with narrow entrances. Exercise **10** and **11** have no kinks in de isodistance lines, while two examples in exercise 27 have kinks.

Chapter 6: shortest paths

24: From shoelaces to shortest paths

1.

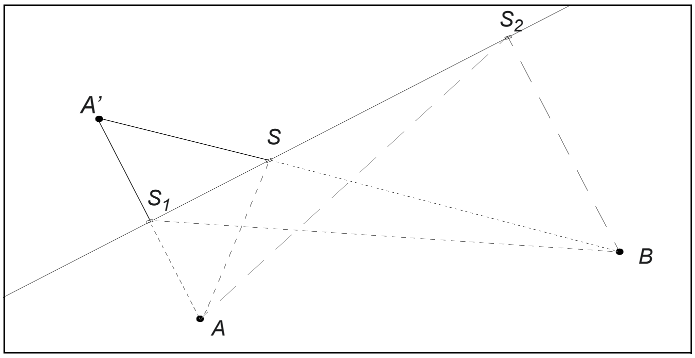
2. a.



a. The lengths stay constant when reflecting.

b. The American. Since it uses the shortest path from *A* to the lowest point *I* and also back up from *H* to *P*.

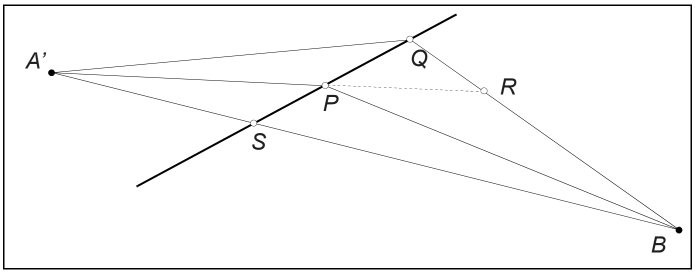
3.



a.

b. $\angle A'SS_1 = \angle S_2SB$ *(Opposite angles are equal for intersecting lines).*
$\angle A'SS_1 = \angle ASS_1$ *(Since the triangles are equal, due to the reflecting).*
Thus $\angle ASS_1 = \angle S_2SB$.

4. All that is needed is mentioned in this figure.



Given: *P* lies between *S* and *Q*.
To proof: *d(A', P) + d(P, B) < d(A', Q) + d(Q, B)*

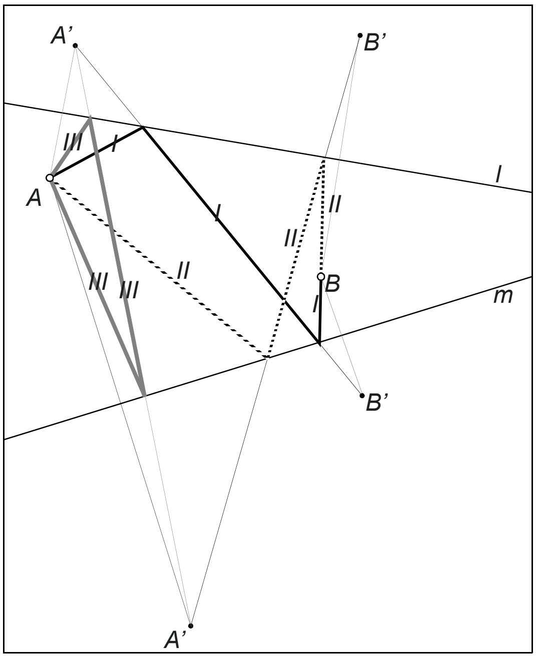
Proof:
Compare the *P*- route with the new route via *R*:
d(*A', P*) + d(*P, B*) < d(*A', P*) + d(*P, R*) + d(*R, B*) = d(*A', R*) + d(*R, B*)
Compare the new route via *R* with the route via *Q*:
d(*A', R*) + d(*R, B*) < d(*A', Q*) + d(*Q, R)* + d(*R, B*) = d(*A', Q*) + d(*Q, B*)
Combine!

5.

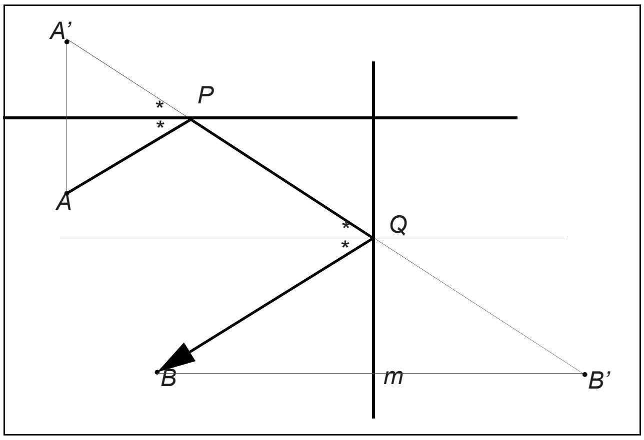
a. route I - I - I.

b. route II - II - II

route III - III - III.



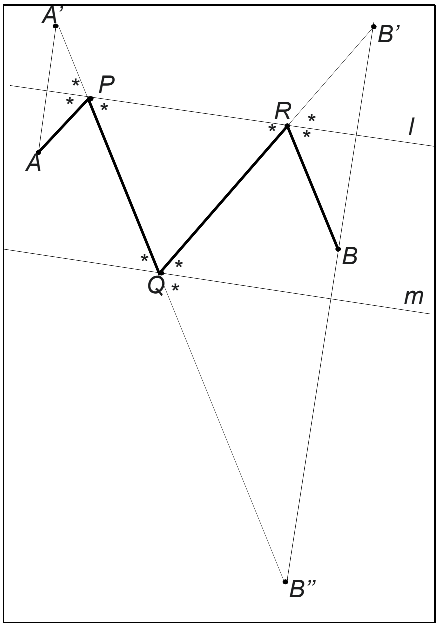
6.



a.

b. *AP* and *QB* are parallel. In the figure all angles with * are equal.

7.

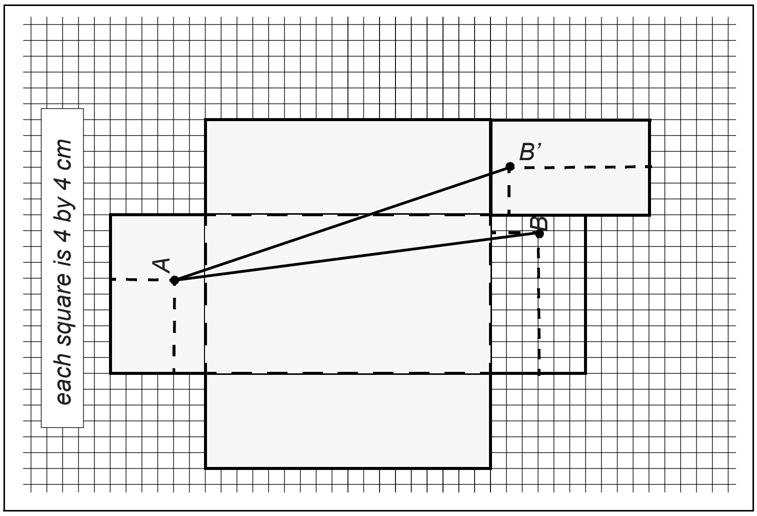


a. On the reflection of *B* in *l*. Call that *B'*.

b. On the reflection of *B'* in *m*. Call that *B''*.

c. *A'* is the reflection of *A* in *l*. *A'B''* is sketched. You find *P* and *Q*. Then draw *QB'*. You find *R*.

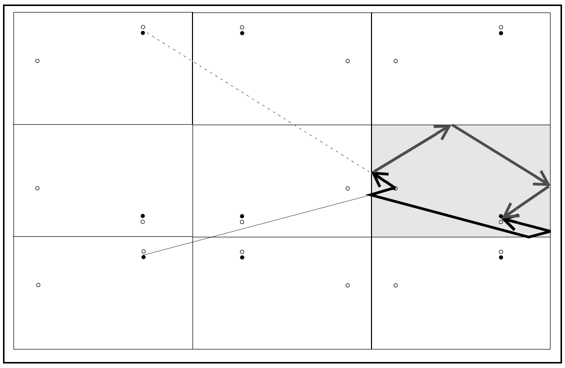
d. Since all angles with a∗ are equal. There are many *Z*-angles!



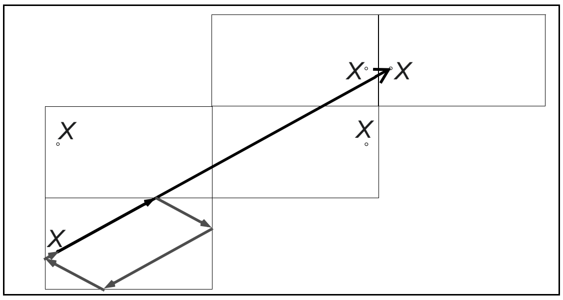
8. On the net shortest routes can be sketched as straight lines. However, there are many ways to make a net. In the figure connecting the square with ant B to the long side yields minor profit.

$$|AB] = \sqrt{46^2 + 6^2} \; = \; 46{,}389 \; |AB'| = \sqrt{42^2 + 14^2} \; = \; 44{,}271$$

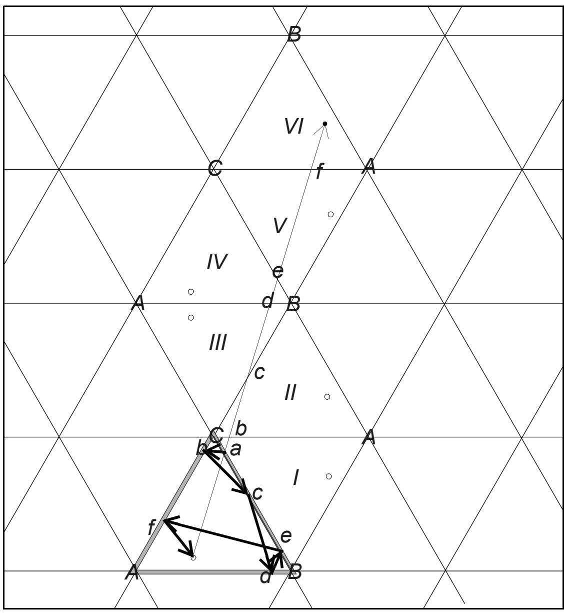
9. The figure shows the billiards reflected, reflected, reflected, reflected. Each line from *A* to a reflection of the goal, which is reached via three cushions is correct. One example is shown.



10. The solution to this question is practically the same as the last one. Of course there are other possibilities, which you can find by making a different mirror-mirror-mirror-mirror-image of *X*.



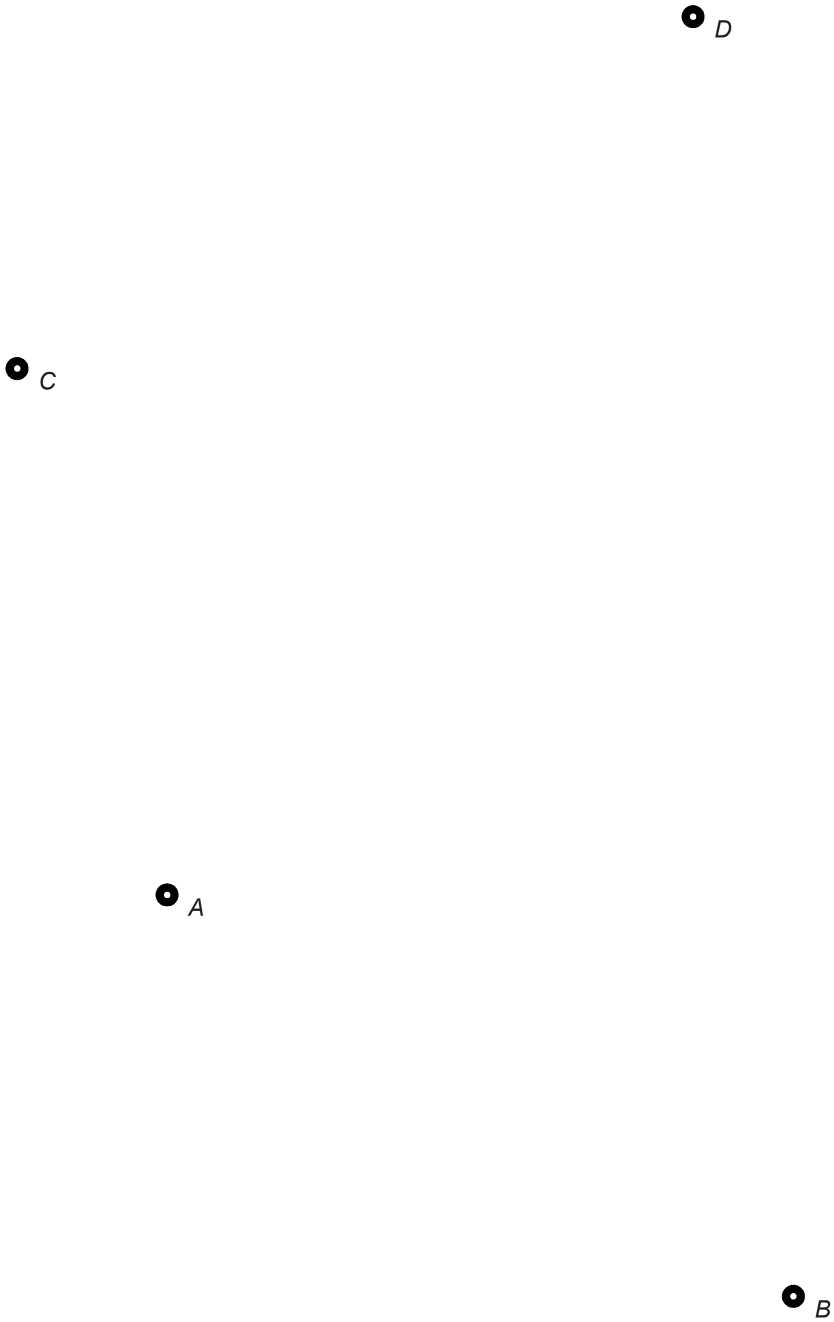
11. When using *worksheet G: triangles and mirrors*, page 181, it is not so hard. Putting letters on the angles of the reflections of the triangle will help you in the right direction. The reflections are numbered I, II, III etcetera. From the long line to the broken line: measure piece-wise along the cushions of this billiards.



Worksheets part I

Distances, edges and domains

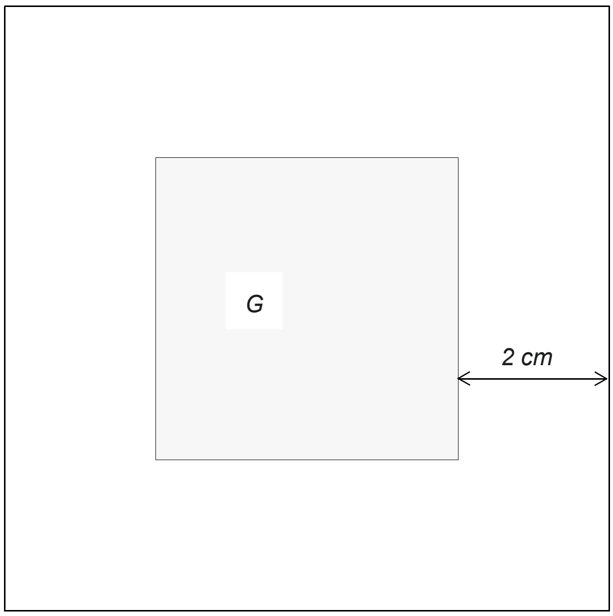
worksheet A: Folding to Voronoi



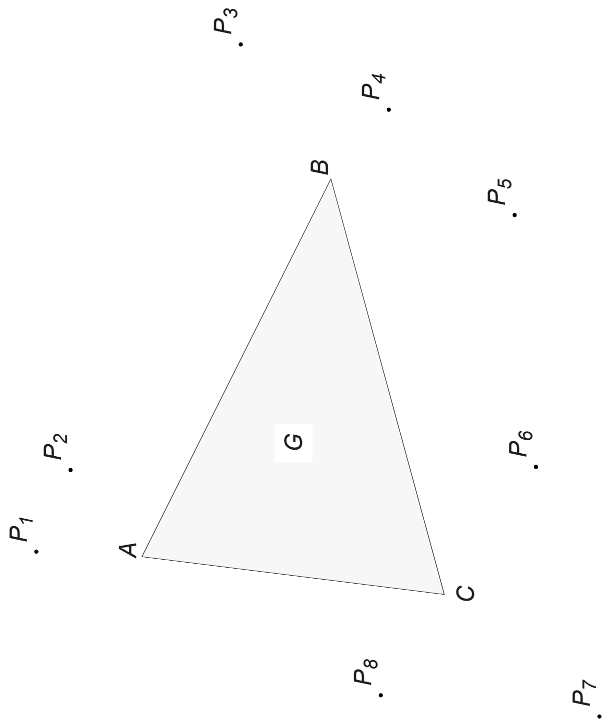
worksheet B: Exact Voronoi diagram for the desert

worksheet C: map of the Netherlands

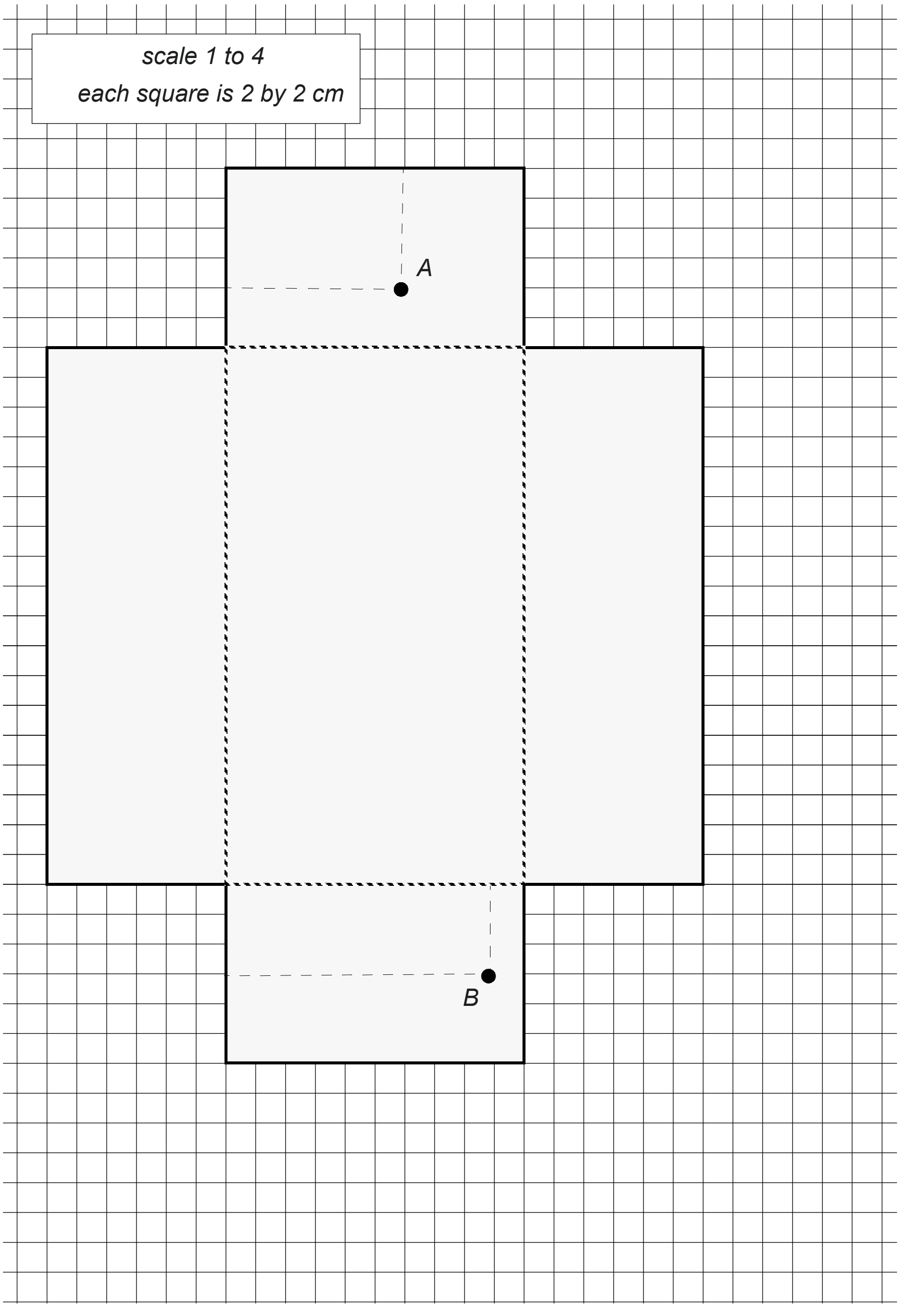
worksheet D: isodistance lines round a square)



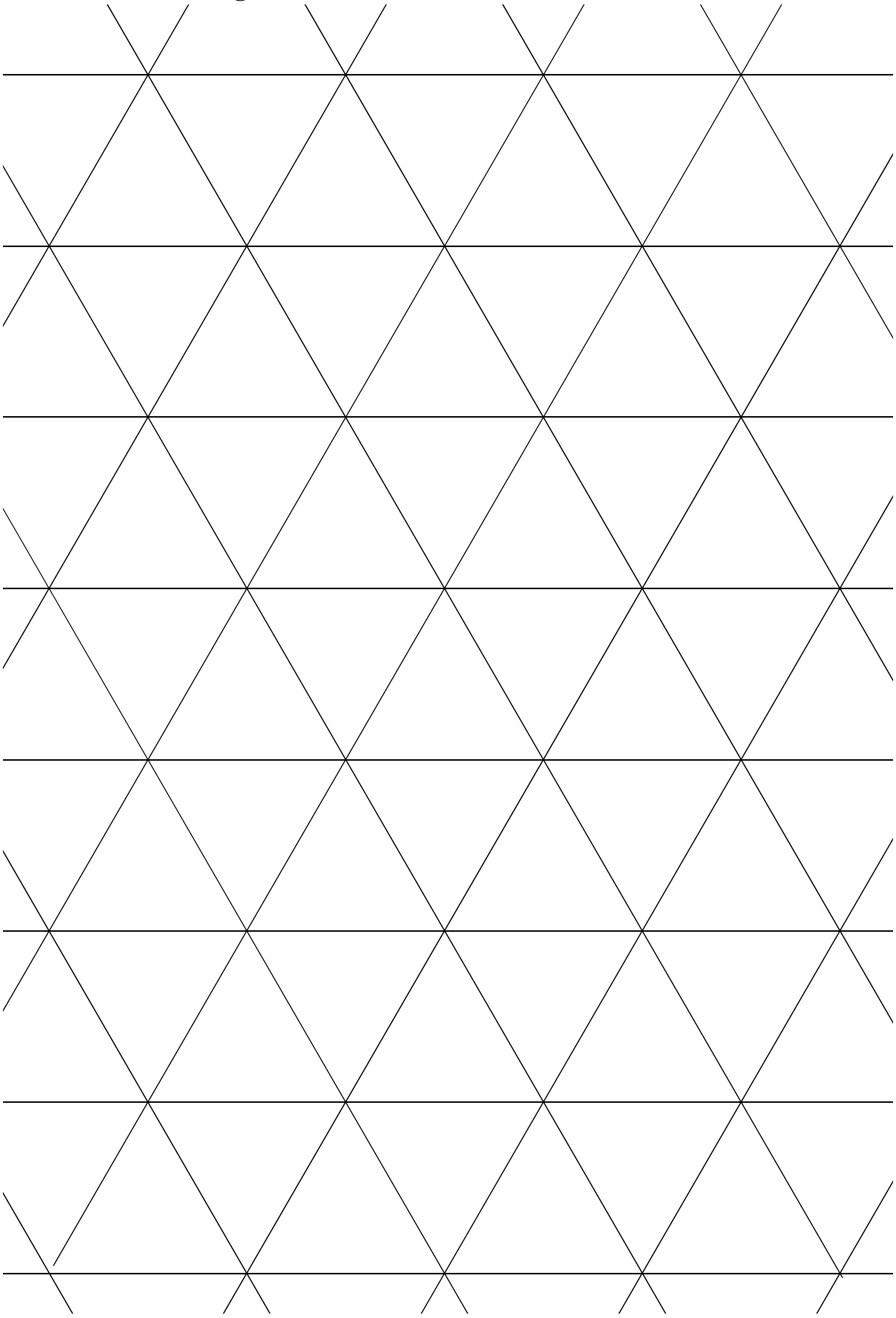
worksheet E: triangle, feet, sectors



worksheet F: an ant on a shoebox



worksheet G: triangles and mirrors



Thinking in circles and lines

Advanced geometry, part II

The illustration on the front of this section is a fragment of 'The school of Athens', a fresco by Rafaello Sanzio, 1483–1520.
On this fresco several scientists from ancient times can be found. The fragment shows the Greek mathematician Euclid, surrounded by enthusiastic students. Much of the geometry in this book was already described by Euclid. Euclid lived around 300 B.C. For the portrait of Euclid, Rafael used the face of his friend Bramante, architect of the dome of the St. Peter in Rome. Not a bad choice: an architect must know a great deal about geometry.
On the next page the whole fresco is shown; the original is seven meters wide.
The two figures in the centre are Plato and Aristotle, important philosophers of Ancient Greece. More to the left, Socrates is shown. He is counting his arguments on his fingers. Argumentation, reasoning, that is what this book is about.

Thinking in circles and lines – Advanced geometry, part II
Project: Mathematics for the second phase.
Profile: N&T
Domain: Advanced Geometry
Class: VWO 5
State: Revised version, January 1998
Design: Aad Goddijn, Wolfgang Reuter
Translation: Danny Dullens, Nathalie Kuijpers

Chapter 1: Using what you know

In advance

review
In the previous part, ‘Distances, edges & regions’, dividing regions based on the nearest neighbor principle was discussed. Gradually we entered the field of pure mathematical matter. There, definitions were given and theorems were proven. We found several new things through reasoning.

In this volume we continue to build op argumentations in geometry. An argument in mathematics that is built up in a strict way is called a proof.

what you know
In a proof you can only refer to simple geometrical theorems and definitions you know for sure, for example about:
– isosceles triangles
– intersecting of parallel lines with a third line (F- and Z-angles)
– circle, parallelogram

You may also use the new theorems from ‘Distances, edges and domains.’ But you should never write in your proof: this is so-and-so because you can see it in this drawing already. Drawings may help you to find a proof, but they are not proofs themselves.

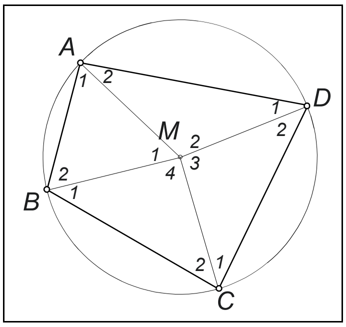
1. Finding arguments and writing down proofs

Introduction about notations

Earlier on you have already seen a couple of tools to indicate certain relations *in sketches*. This section deals with *writing down* reasonings and again there are tools for that. You already know most of them, but they are listed here as well.

equal lengths and distances
The distance between two points A and B is indicated by d(A, B).
The length of the line segment AB is indicated by $|AB|$.
The d-notation can also be used for the distance between a point P and a domain R: d(P, R).

indicating angles
First way (can be used without a sketch: the angle at point B with legs BA and BC is indicated by $\angle ABC$. If there is a sketch you can sometimes use a shorter notation: just the letter and an index, usually a number. So, in the sketch on the right, you might observe: $\angle A_1 = \angle B_2$, since AMB is an isosceles triangle.
If it starts to become unclear, you can always add: $\angle A_1 = \angle MAB$.



the symbols $//$ and $\perp$
Parallelism and perpendicularity will often be used; that is why there are two special symbols:
If you want to say that line DE is parallel to line GH, you write: $DE \ // \ GH$.
If you want to say that line PQ is perpendicular to line QR, you write: $PQ \perp QR$.

motivations, given

Above you saw a small example of an argumentation
$\angle A_1 = \angle B_2$, since AMB is an isosceles triangle.

This is an *assertion*, namely $\angle A_1 = \angle B_2$, which is *motivated* by referencing a known theorem about isosceles triangles. Mentioning the key words *isosceles triangle* is enough here.

But why is *AMB* an isosceles triangle? In this case .... since $|AM| = |BM|$. But why *should* $|AM| = |BM|$ be true here? An obvious answer is: since A and B lie on a circle with center M. This should have been said in advance!

given
The fact that A and B lie on a circle with center M actually follows from nothing, it is what is *given* in this situation. In al problems you will solve, there one of more givens.

1. In this part you have seen several types of motivation for the reasoning steps. Which are they?

The definite form of the reasoning from *given* to the *end result* $\angle A_1 = \angle B_2$ can be written down as follows:

	argumentation step	motivation
1	A and B lie on a circle with center M	given
2	$\|AM\| = \|BM\|$	definition circle
3	$\angle A_1 = \angle B_2$	isosceles triangle *AMB*.

2 **a.** There has been a change in the order of this story compared to the introduction. Explain what this change is, and why it was made.
b. We have seen such a change of order before! Were?

In the motivations you can thus refer to:
– givens
– definitions
– familiar theorems
but also
– to *what has been shown already in your proof*; for this, it can be wise to number the lines of the proof, or just number some of them.
– to *what has been shown earlier* (this is often a theorem; sometimes it is an earlier exercise).
From now on we will of course need to know exactly what is given in a problem.

proof
We shall call a correct reasoning, set up in this fashion, a *proof*. Thus in a proof you only find reasoning steps, which are motivated according to the above.

clarity!
That you need to motivate all reasoning steps in proofs, is something you have seen before in *Distances, Edges and Regions.* The new thing in this chapter is that you now know exactly which motivations you can use. This seems a restriction, but on the other hand, you do now have a whole list of possible motivations at your disposal that you can use as a resource while finding proofs.
We will practice this in the next section.

2. examples of proofs

In this section we will prove a couple of new geometrical facts. We use the method of working and notation from the previous section.

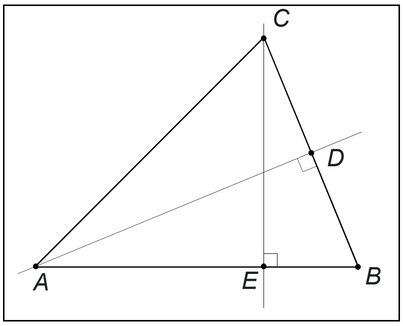
example one: two altitudes

Since we work with altitudes in this example, we first *define* this concept.

definition altitude

> An altitude from the vertex of a triangle is the perpendicular line from this point to the opposite side.

Now the actual problem:
Given here is a triangle ABC with the altitudes from A on BC and from C on AB. The altitude from A intersects BC in D, the altitude from C intersects AB in E. When measuring in different cases you will soon notice: $\angle DAB = \angle ECB$. This needs to be proven.
Thus, to prove: $\angle DAB = \angle ECB$.



3. First try to find an argument for why these angles are equal yourself and then see whether you can shape it into a proof which meets the conditions of the previous section.

Well done if you succeeded, no harm done if you did not. In any case, read on.

4. First look at $\angle DAB$. It is often a good idea to take an angle to be an *angle of a triangle, because then you can possibly use known theorems about triangles*.

a. You have two choices, but in one of them all vertices already have names. This looks the easiest. Which triangle do we take?
b. With $\angle ECB$ you can do the same thing. Which triangle do you use?
c. Which characteristics do these triangles have in common?

5. After these observations, you have the route of the proof lying in front of you in broad outlines: $\angle DAB$ and $\angle ECB$ can be expressed in $\angle B$ and from that it must become clear whether they are equal.

Now the issue is to write down the proof exactly. Especially 'expressing in'. Start as follows, fill the holes with the right motivations and finish the proof. Remember that you can refer to the previous lines.

	argumentation step	motivation
1:	$\angle DAB + \angle ADB + \angle B = 180°$	sum angles in triangle ....
2:	$\angle ADB =$	given
3:	$\angle DAB =$	from step 1 and 2, simple algebra.
4		
5		

In this proof you have by now reached an expression for $\angle DAB$; you reached step 3 by some simple algebra. Probably you found: $\angle DAB = 90° - \angle B$.
You should do almost exactly the same for $\angle ECB$; you will find in step 6 of course $\angle ECB = 90° - \angle B$ and the proof is rounded of with $\angle DAB = \angle ECB$, motivated by step 3 and 6.

[After some experience you will not write the same story twice with only names changed. You will refer to the earlier part, for instance as: "same steps as ...., in triangle ...". We will come back to this foreshortening strategy later.]

*addendum: case distinction*In the figure with the two altitudes the intersection of the two altitudes lies *within* the triangle. That is why the proof was easy: the two triangles have an overlapping angle. Still, you have to be careful: it does not say in the data that the triangle is acute. In such situations you must also deal with the other case (here the obtuse triangle). In short: case distinction.

6 **a.** Sketch a triangle *ABC* with an obtuse angle at *B* and again sketch the altitudes from *A* and from *C*. These now lie outside the triangle.
b. Is the proof is still correct? If necessary, add a step (with motivation).

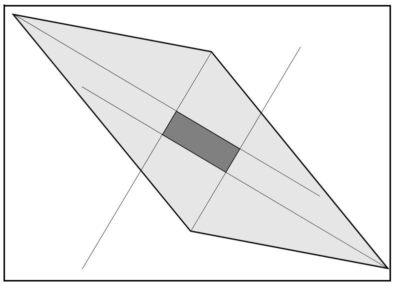
example two: the four bisectors of a parallelogram

In this section you prove two properties of parallelograms.

7. The first property is: *Consecutive angles of a parallelogram add up to 180°*. Follow the steps in this exercise.
a. Make a sketch.
b. Note what is given and what is to prove.
c. Give a (short) proof.

In the next exercise you can use the result of exercise **7** in the proof. Motivate for example with 'according to exercise **7**'.

8. Here a parallelogram is sketched with four bisectors of the angles.
a. Sketch two other parallelograms and their four angle bisectors.
b. The figure which is enclosed by the bisectors, appears to be a special kind of quadrangle. What kind of quadrangle is it?



This last observation is just an idea which you think is true, but you have no proof by now. It is called a conjecture. It still has to be proven, to know that it is true for *all* quadrilaterals.

c. We continue in the spirit of the previous problems. Start with writing down the *given* facts, what is *to prove* and make a sketch where all givens is visualized with symbols.
d. Again find a reasoning to ground the conjecture.

A couple of hints:
- Name the vertices of the parallelogram: A, B, C and D. And also those of the to be investigated quadrangle: P, Q, R and S.
- Now some relation has to be found between an angle of $PQRS$ and the angles of $ABCD$. Choose an angle of $PQRS$ to work with. It does not matter which one.
- The bisectors also play a part. Find a triangle where ... etc.
- ....
- etcetera ....

e. If you found a reasoning, then write it down in the form of a correct proof.

summary, three step plan

In these sections you have worked on a couple of proofs yourself. Looking back at the exercises, you can see that a *three-step plan* was used every time:

step one: sketch and exploration phase
the making of a sketch where you indicate what is given and where you can see clearly what needs to be shown.
For this, you make a sketch and find relations that you indicate with signs, for instance to denote equal angles.

step two: the reason and search phase
Finding a reasoning for what is to prove. In this step you will often add new things to the figure, and you may also get good use of your eraser! At the end of this phase you may indeed have found a path towards what needs to be shown, but you do not yet have a correctly written down proof.

step three: the finishing phase
writing down a correct proof.
You wrote down the whole in the form of a threefold:

given:
....

to prove:
....

proof:
....

Under *given* it usually does not say much, but what is there, is of vital importance. It only says what is given for *this* situation. So no temporary conclusions, no other things you know, no etcetera. This needs to be very succinct.
In *to prove* only the final conclusion is stated, which you are going for.
In the *proof* you show the *proof steps* and you give the *motivations* too.

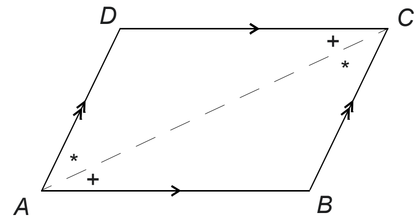
This is the most extensive part. The motivations can refer to:
– givens
– definitions
– earlier proof steps
– earlier proven assertions and familiar theorems
– several fundamentals (like only one straight line goes trough two points)

The final line of the proof should state what you set out to prove. Traditionally people add ***QED***.
This is short for ***Quod Erat Demonstrandum***, or in the original Greek ὅπερ ἔδει δεῖξαι or in English: *That which was to be demonstrated.*

Given:

a quadrangle *ABCD;*
AB parallel to *CD* and *BC* parallel to *AD*.

Figure:



To prove:

$\angle DAB = \angle BCD$

Proof:

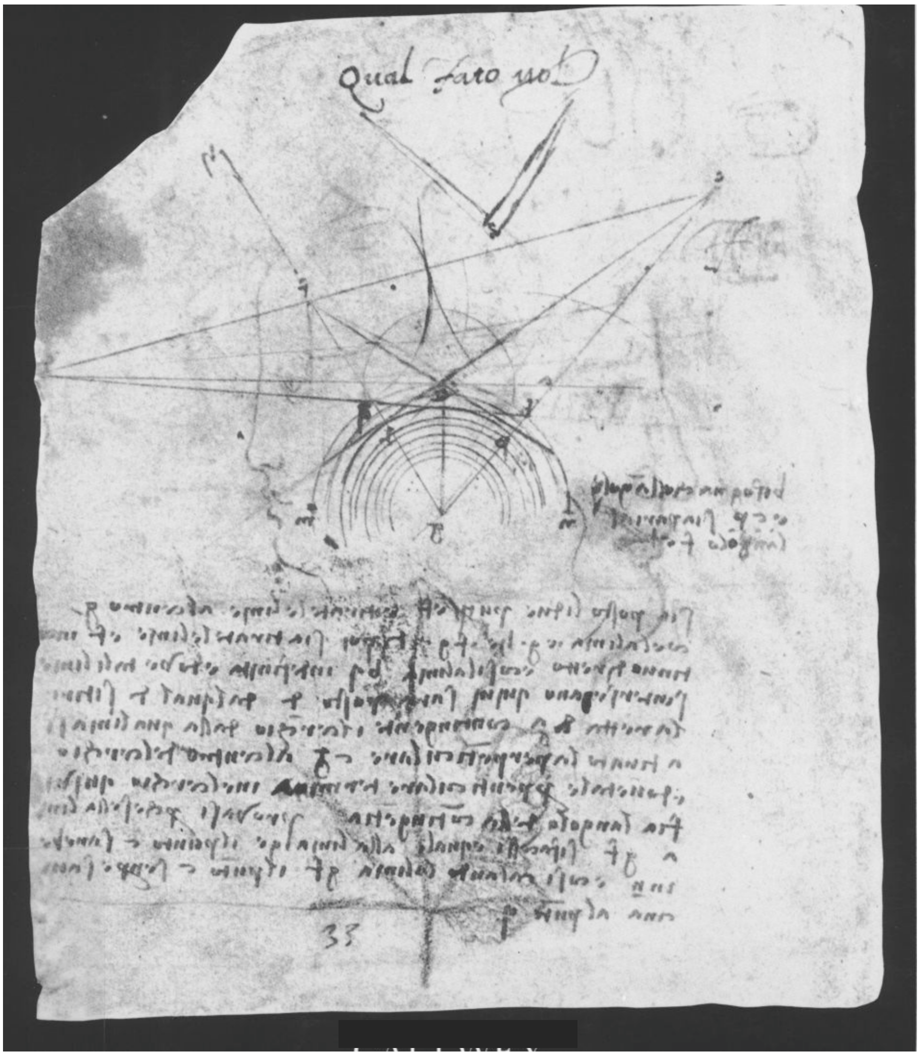
Proof Step	**motivation**
Draw the diagonal *AC*. Then this holds:	
$\angle DAC = \angle ACB$	*Z-angles, given AD // BC*
And also:	
$\angle CAB = \angle ACD$	*Z-angles, given AB // CD*
Therefore	
$\angle DAB = \angle DCB$	*From lines 1 and 2. Adding of angles.*
This was to be proven.	

To end this summary, a neatly written proof. The assertion:
If a quadrangle is a parallelogram, then the opposite angles of the quadrangle are pair-wise equal.
Under *'Given:'* you can see what comes after 'if'.
Under *'To prove:*' you will find the sentence behind 'then'.
You may find the proof itself easy, but this example concerns only how a proof is written down.
The proof is not complicated. Still, there is one problem: how do you hit on the idea to draw the smart help line *AC*? If you do not see something like this at once, do not give up, there are often many ways.

preview

Step two of the step plan is especially important: the search for a path. You will have noticed that not just one fact can lead to the right connection; you sometimes need to link several things together. Further on in this book, we will search extensively for this.

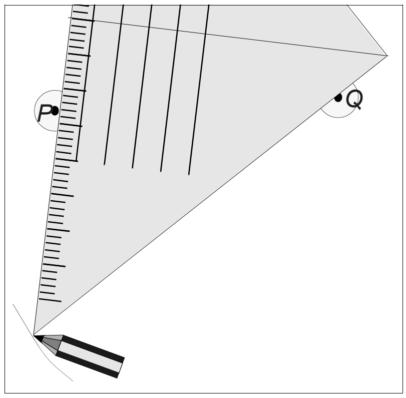
Chapter 2: The circle scrutinized



In this chapter you will encounter several theorems about circles, where the concept of the angle is important. We will compare angles from the midpoint of the circle and angles from points on the circle itself, and we will prove theorems about them.

3. between two thumbtacks

an experiment



We start this chapter with a small experiment.

1. Pin, from the *back* of this page, two thumbtacks through the points P and Q. Now, carefully, put your protractor in between, as shown. You can still move point X, while the protractor slides along both thumbtacks. This way X describes a *path*.

a. Sketch this path with a pencil.

b. Do the same with the *right* angle of the protractor and also with a piece of cardboard which you cut in the shape of an acute angle of approximately 20 degrees.

conjecture

c. You probably have a *conjecture* about what the shape of this path is. Note your conjecture.

2. Let's scrutinize the case of the right angle first. We want to prove the conjecture and use the step plan from page 192.

a. Make a *sketch* of the situation. Sketch $\angle PQX$ right-angled.
You now know: $\angle PQX = 90°$. You can *use* this.
You suspect: X lies on the circle with diameter PQ. This you need to *prove*.
Thus also sketch the middle of PQ.

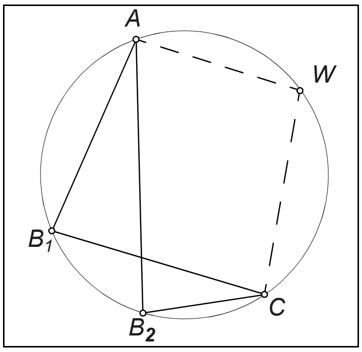
b. Now the search for the *reasoning*. The sure thing is that right angle; so that's *given*. When searching for a proof you will most definitely use this. Two *associations* are possible:
– Pythagoras. But this has no relation to the middle of PQ.
– Rectangles. However, these cannot be seen, but try whether you can look at PXQ as half of a rectangle. We do know some stuff about rectangles.

c. Now write down a *correct proof* for the following theorem.

Theorem 18 (Theorem of Thales)
If angle C of triangle ABC is right, then C lies on the circle with diameter AB.

4. the constant angle theorem

In the experiment of the previous section we worked with a constant angle, the one of the protractor or the piece of cardboard. It looked as if a circle resulted from that. If the constant angle is not right, we have not yet proven anything. We only have our conjecture that circles play a part. What we will do now, is prove something new about circles that fits the situation of the constant angle. After that, we will see how to continue.

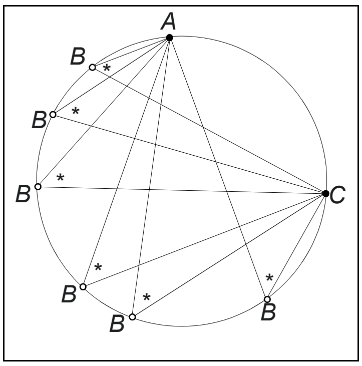


3. In this sketch you see two cyclic quadrilaterals, which partly coincide. Argument that the angles at B_1 and B_2 must be equal to each other.

(You do not have to give a complete proof, just an explanation. Later on, we will write a complete proof.)
Point *W* has an unusual role in this figure. In order to write down that the angles at B_1 and B_2 are equal, *W* is redundant, you just note:
$\angle AB_1C = \angle AB_2C$.
However, $\angle W$ is used to argue that the angles at B_1 and B_2 are equal. Here point *W* plays the role of help point, which is used in the proof. (The special thing here is that you may freely choose the help point on the arc of the circle!)

The result of exercise **3** can be written down as a theorem without mentioning the help point.

Theorem 19 (Constant angle theorem)
If *A*, *B* and *C* are on a circle, point *B* moves over one of the arcs between *A* and *C*, the size of $\angle ABC$ does not change.



The size of ∠ABC apparently only depends on the mutual positioning of A and C on the circle and does not depend on the position of B. The figure shows exactly what is said in theorem: *B* can be set anywhere on the indicated arc, it does not matter for $\angle ABC$.

4 a. Can you *explain* the origin of the arc in exercise **1** with this theorem?
b. But has it really been *proven* that the sketched path is part of a circle? Why or why not?

5. the inverse theorems

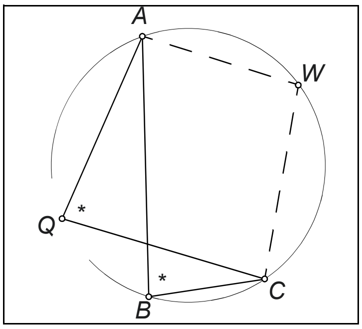
We need an inverse of the theorem of Thales as well as the inverse of the constant angle theorem. First the inverse of Thales. This has been formulated as follows:

Theorem 20 (Inverse of the theorem of Thales)
If AC is the diameter of a circle, and B lies on this circle, then $\angle ABC$ is right.

5. We are going to prove this in two ways. Be complete for one of them, limit yourself to a short proof outline for the other. Make a sketch in both proofs. The ideas are:
  a. Extend triangle ABC to a parallelogram $ABCD$ and show that this parallelogram is a rectangle.
  b. Use the already proven constant angle theorem. Choose a point P wisely on the arc ABC, for which it is easy to prove that $\angle APC$ is right.

Now the inverse of the constant angle theorem. Here there is something annoying the matter, which does not occur in Thales. For Thales you are allowed to place point B anywhere on the whole circle with diameter AC, $\angle ABC$ will always be right. For the other theorem the mobile point has to be on one of the arcs of the circle. That is why the inverse of the constant angle theorem demands a little more attention.

Theorem 21 (Inverse of the constant angle theorem)
If point Q lies on the same side of AC as point B and $\angle AQC = \angle ABC$, then Q lies on the arc ABC.



For the proof we use a figure, which is as like as two peas in a pod to the one in exercise **3**. But in this figure is shown:
- $\angle AQC = \angle ABC$
- that you *cannot* use that Q lies on the circle through A, B and C, because this is what you need to prove.

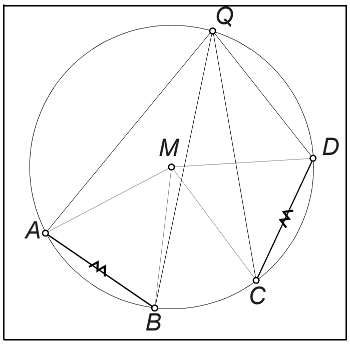
6. Prove theorem 21. Be inspired by exercise **3**, but do not *use* that $AQCW$ is a cyclic quadrilateral; *prove* this by calling upon the appropriate theorems. Of course you can use that $ABCW$ is a cyclic quadrilateral. Write down the complete proof.

7. A last look at the experiment at the beginning of the previous section. With which theorem(s) can you explain:

a. that point X moves over a fixed arc.
b. that the center of this circle lies on PQ if angle PXQ is right. (The circle is already present here.)

6. *bisectors, half angle theorem*

equal chords, equal arcs, equal angles

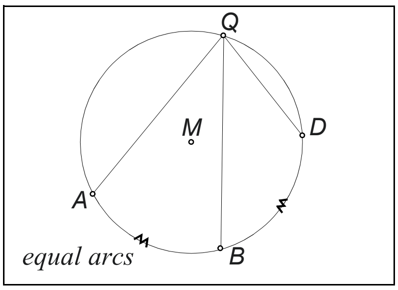


8. In the figure on the right the chords AB and CD have the same length.
a. Now indicate which angles at the center M must also be equal.
b. Now explain why the angles $\angle AQB$ and $\angle CQD$ must be equal. [Hint: rotate one of the triangles.]
c. Write down concisely what has been proven here; use the *if ...then* ... pattern.
d. Formulate and prove the inverse of this theorem completely. Then it is *given* that the angles at Q are equal and it needs to be *proven* that the chords AB and CD are equal.
e. At Q it is possible to find two more equal angles. Which angles?

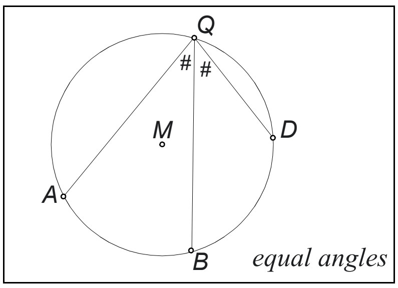
Theorem 22 (Theorem of the equal arcs)
To equal arcs on a circle belong equal inscribed angles and vise versa.

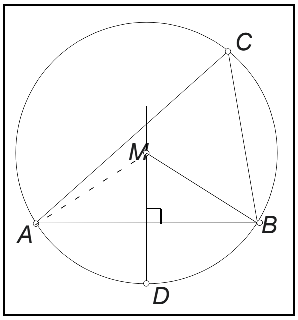
an application with angle bisector and perpendicular bisector

A special situation occurs in the figure shown above if B and C coincide. Then the arc AD through C (or B) is divided in two equal parts. The situation can be expressed in pictures as follows. So QB is the angle bisector of $\angle AQD$.



$\Leftrightarrow$





You know both directions from **8c** and **d**. Here is another application of a halved arc.

9. In this figure the perpendicular bisector of *AB*, which intersects the circumcircle in *D,* has been sketched. *M* is the center of the circumcircle.
a. Show: $\angle DMB = \angle ACB$.
[Hint: extend *AM* to *C** on the circle].

From this you may directly deduce an important relation between angles from a midpoint and angles from the circumference on the same arc of a circle: the midpoint angle $\angle DMB$ is double the angle form the circumference $\angle ACB$. Like the results in exercise **8c** and **d**, this will be stated again in the theorem summary below.

Theorem 23 (Double angle theorem)
If A, B and C lie on a circle with midpoint M, then ∠AMB = 2∠ACB. (The angle ∠AMB must be chosen as the angle on the arc Ab not containing C)

7. iso-angle-lines

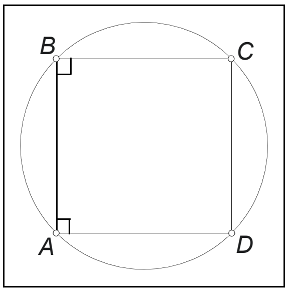
The main task of this section is:

Given is a line segment AB and a fixed angle γ. Now draw exactly the set of the points X, for which holds ∠AXB = γ.

You could, analogous to the iso-distance-lines, say:

Determine the ***iso-angle–line*** *with angle γ of line segment AB.*

If $\gamma = 90°$, it is easy: find the midpoint *M* of *AB* and sketch the circle around *M*, which goes through *A* and *B*.
In other cases, an initial point *X* must first be determined with $\angle AXB = \gamma$. Then the center of the circumcircle of *AXB* can be found, and we are done. The constant angle theorem now states that any point *Y* on arc *AXB* fulfils the condition $\angle AYB = \gamma$. On the other side of *AB* there is another such arc; together these arcs form the *iso-angle-line*. The question is: how do we find such an initial point *X*? We handle a special case first ($\gamma = 45°$), after that the general one.



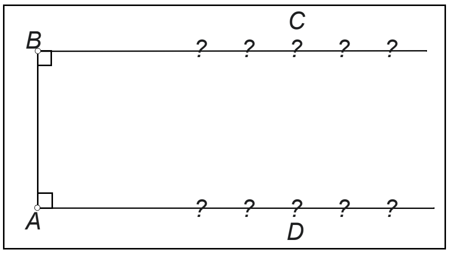
10. First we take $\gamma = 45°$.

a. Show that X must lie on the circumcircle of a square of which AB is a side.

b. Now it is easy to indicate how such an X can be found: X can be point C.

11. In the general case, thus with a different angle γ, we can still make good use of the lines BC and AD, but $ABCD$ will no longer be a square.

a. What kind of figure is $ABCD$?



For the construction in this exercise take an angle γ between 0° and 90° as desired.

b. Point C must be found, so we cannot work from C yet.
But if you know γ, you know $\angle CAD$. Why?

c. Now you can also construct AC *with the protractor*.
Finish the construction by determining the center of the circumcircle of ACB. Remember: this is a special type of triangle.

8. summary of theorems, practice exercises

To start with, you will find a list of the proven theorems up to now. These form the basis for the activities in the following chapters. Later on we will prove new results based on these theorems. There it will be somewhat harder to find the proofs than for the upcoming exercises. Hence *practice exercises.*

theorem summary

18: (Theorem of Thales)
If angle B of triangle ABC is right, then B lies on the circle with diameter AC.

19: (Constant angle theorem)
If point B moves over one of the arcs between A and C, the size of ∠ABC does not change.

20: (Inverse of the theorem of Thales)
If AC is the diameter of a circle, and B lies on this circle, then ∠ABC is right.

21: (Inverse of the constant angle-theorem)
If point Q lies on the same side of AC as point B and $\angle AQC = \angle ABC$, then Q lies on the arc ABC.

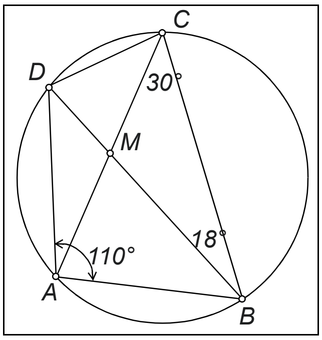
22: (Theorem of the equal arcs)
To equal arcs on a circle belong equal inscribed angles and vise versa.

23: (Double angle theorem)
If A, B and C lie on a circle with midpoint M, then $\angle AMB = 2\angle ACB$. (The angle $\angle AMB$ must be chosen as the angle on the arc Ab not containing C)

9. practice exercises

The main thing is practising what has been investigated in this chapter. Only when you are asked explicitly do you need to write a complete proof. In the other cases it says ‘show’ or ‘explain’ or ‘determine’. For ‘show’ you do need to mention the used theorems.

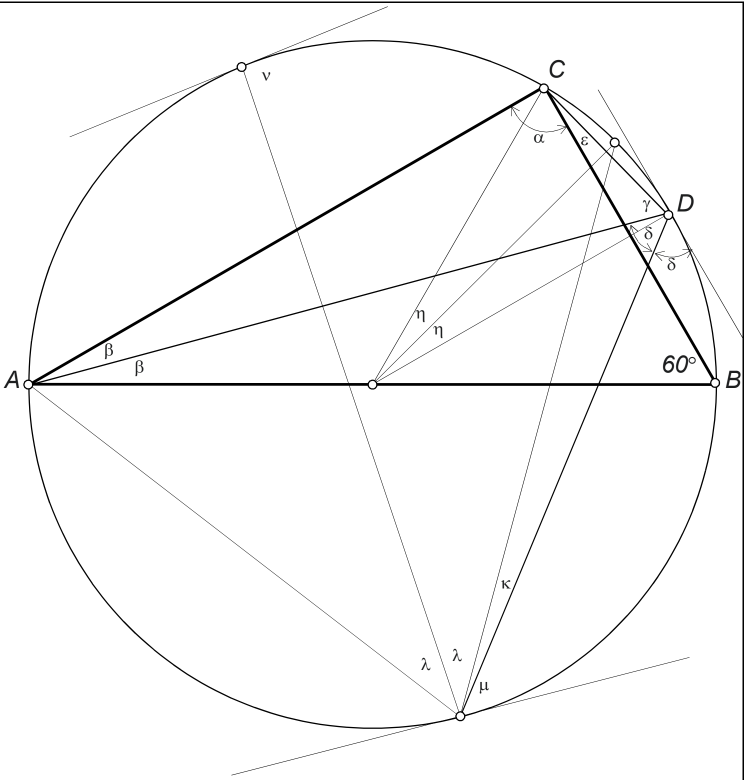
12. Determine all not yet indicated angles in this figure (it does not have to be in this order!):
- **a.** $\angle DAC$
- **b.** $\angle CDB$
- **c.** $\angle DCA$
- **d.** $\angle ABC$
- **e.** $\angle ABD$
- **f.** $\angle DAC$
- **g.** $\angle AMB$



13. Given is triangle *ABC* with right angle *C*; $\angle B = 60°$. Σεε νεξτ παγε.
As an exercise, you have to compute all angles marked with Greek symbols. Equal symbols mean (of course) equal angles. So you find some angle bisectors.
- **a.** There are also some tangent lines to the circle, for instance at *D*. You have to compute the angle between the tangent at *D* and line *AD* to find $\delta + \delta$ and δ itself. Will the arc below *AD* be split up in equal halves by the angle bisector?

b. Fill in all angles in the table.

$\alpha = 90°$
$\beta =$
$\gamma =$
$\delta =$
$\varepsilon =$
$\eta =$
$\kappa =$
$\lambda =$
$\mu =$
$\nu =$



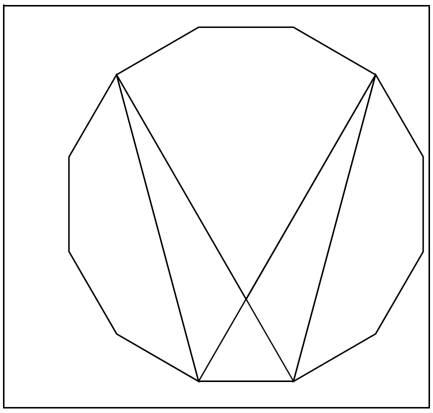
a puzzling division

14. In this figure a regular dodecagon has been divided in six parts.

a. Determine *all* angles of the six parts.
Hint: Clocks are circular.

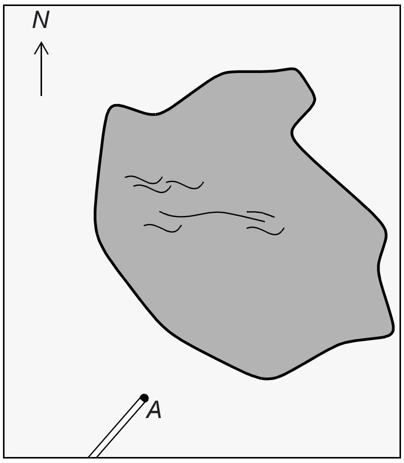
b. Cut the pieces from the worksheet on page 249 and make them into a square.

Hint: Try to make angles of 90° in all cases. Show that angles and sides that have been put together match exactly.



A geodetics application

15. A surveyor needs to make a circular path round a lake. The center of the circle is out of reach for him. The path needs to start and end at point *A*. The surveyor has an instrument at his disposal, with which he can measure angles in the field. The surveyor starts the job by placing a leveling staff (a clearly visible red-white pole) in point *A*. Where should he place a second leveling staff and how would he complete the job (on his own)?

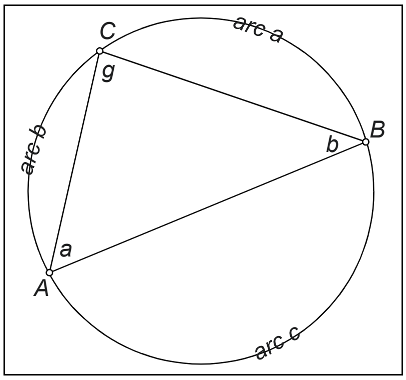


Given arc proportions

16. A triangle and its circumcircle contain three sides and three arcs. See the figure.

a. Why is there no triangle of which the sides are proportioned as 1 : 2 : 3?

b. Now investigate the case when the *arcs* are proportioned as 1 : 2 : 3. What are the sizes of the angles of this triangle?
[Hint: Remember theorem 22, page 200.]



17. Can you find for *any* arc proportion $a : b : c$ a triangle with that arc proportion? What are the angles?

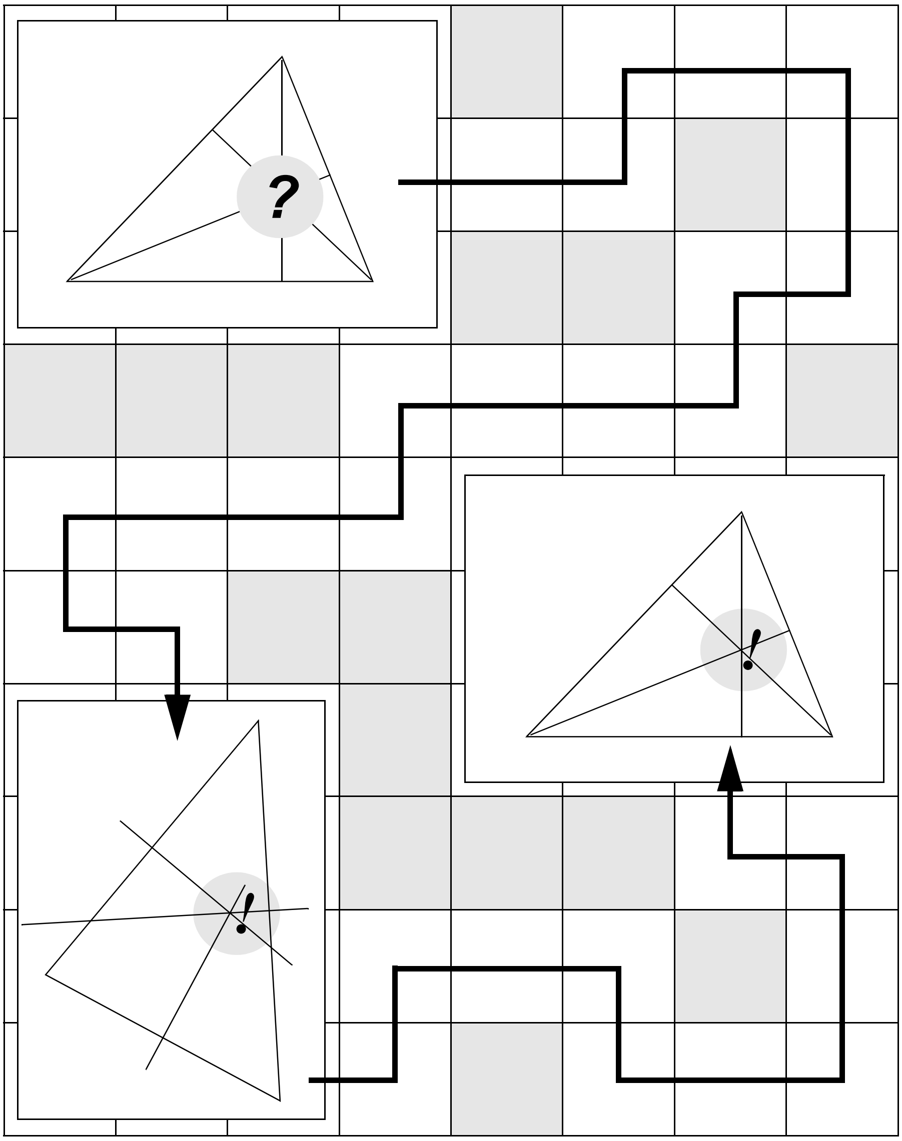
18. Such questions are possible also for pentagons of which the vertices lie on a circle.

a. Sketch a pentagon with five vertices on a circle with arc proportion 1 : 2 : 3 : 4 : 5. What are the angles of the pentagon itself if the arcs lie on the circle in the order 1-2-3-4-5 ?

b. Sketch an example where the order of the arcs is different and where the pentagon has *two* equal angles. Also determine the angles of the pentagon.

c. Also sketch an example where the order of the arcs is again different and where the pentagon has *two pairs* of equal angles. Also determine the angles.

Chapter 3: Finding proofs



10. Chapter introduction

In previous chapters the proofs of the theorems were presented in little lumps. You didn't need to choose your own path. You learned:

– What proving is.
– That a proof can sometimes be written down schematically, but that you can also write down things in great detail.
– That there always is the three step form *Given - To prove - Proof*.
– That finding a proof starts with searching for an *idea*, after that you fill out the *details* and finally you write it down *properly*. You have done the last two steps elaborately.

procedure in this chapter
It is obvious what is going to happen next: practising in finding a proof. This will not be done like in the previous chapter, because then you would have not enough opportunity for your own research. We choose a different approach. Later, several search methods will be explained and sometimes be clarified with an example. After that you will be given a problem to work on. Thus the tasks you carry out are somewhat more extensive.

clues
What you need to try is: find the proofs yourself as much as possible, using the indicated search method. If you get stuck, you can glance through the *clues*, which start on page 231. But do not read these clues completely at once. Try again and again to make progress yourself – preferably with only one clue.

proof plan is the main thing
Finding a good proof plan is always the point. Your solution gives a clear proof story. It does not necessarily need to be written down in the two-columns-form of previous chapters.

methods to find proofs
In this chapter we start with using theorems about circles in small proofs. The paragraph is named: *know your theorems!*
Then there follows a short section about a pattern you have seen before. Because it is very important, it should be mentioned in our toolbox. The paragraph is named: *an important proof-pattern.*

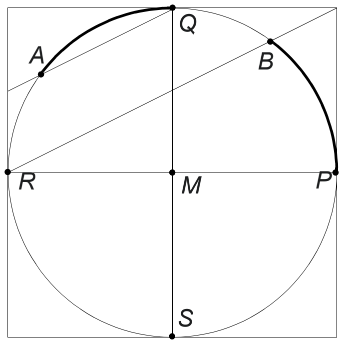
After that we will use three special methods to search for proofs. We will call them *links, split, plagiarism*

These are also the titles of the following sections. Thus the structure of this chapter is not built up mathematically, but around these search methods.

11. know your theorems!

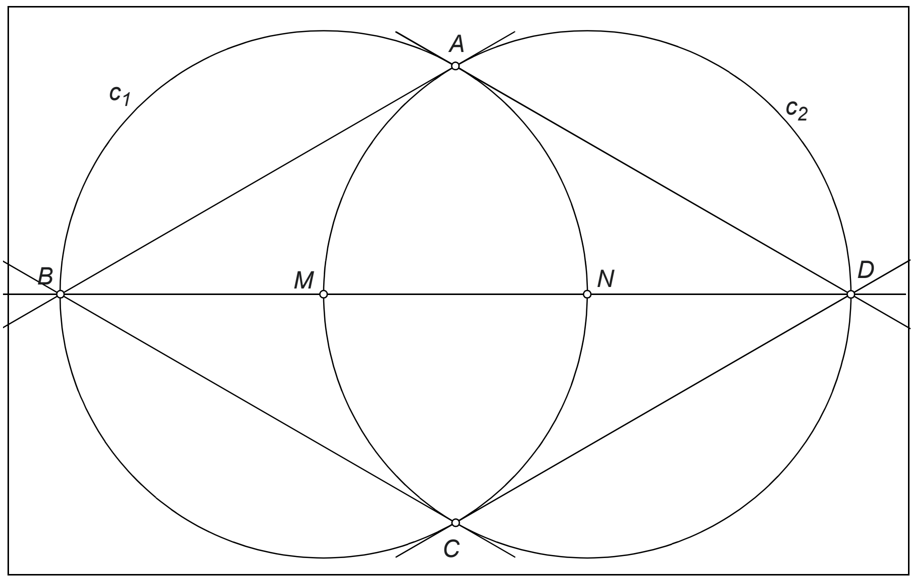
square and circle

1. A square, tangent to a circle. Given is also that *RB* and *AQ* are parallel.
a. Prove that the bold sketched arcs have the same size and that $\angle ARB$ is 45°.



Two circles: writing down proofs

2. Make a precise sketch of the figure below with compass and ruler.



M and *N* are the centers of the circles c_1 and c_2. The rest is obvious from the figure. prove the following two assertions and write your proof with full motivations.
a. *ABCD* is a rhombus.
b. The line *AB* is tangent to the circle c_2 in *A*.

Of course there are three other tangent lines: *BC* to c_2, *AD* and *CD* to c_1.

12. Reviewing an important proof-pattern

3. In the next figure three equilateral triangles have been set against the sides of triangle *ABC*. The circumcircles of the equilateral triangles seem to pass through one point. This needs to be proven!

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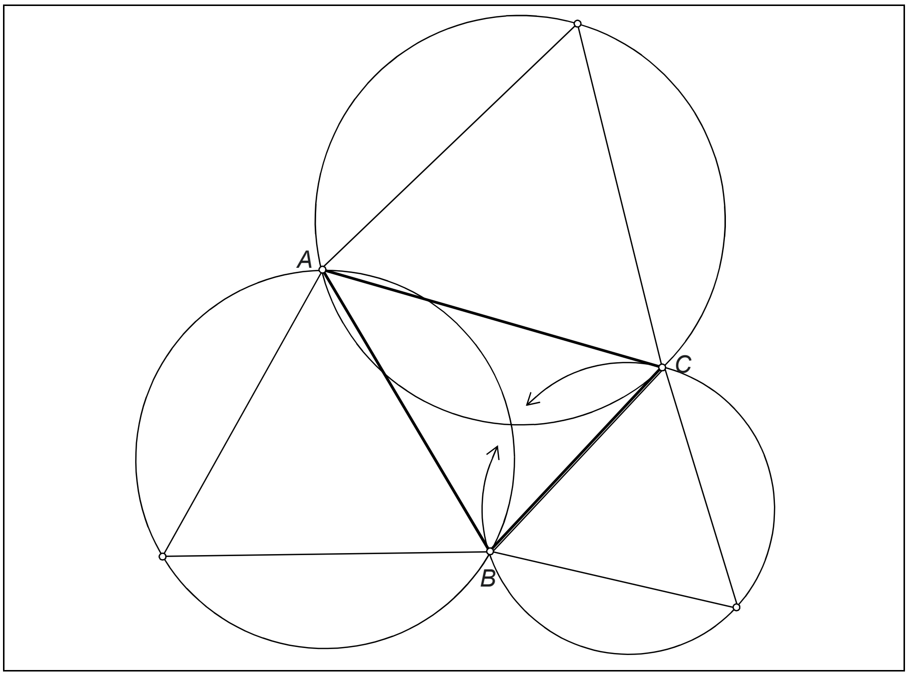


The proof pattern is:

a. Find a *characterization* for points on the small arcs.
b. Name the *intersection of two of these arcs S*
c. Show that *S* also lies on the third arc.

You used this pattern several times before!

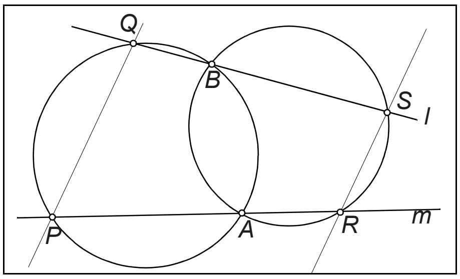
a. What is your characterization?
b. Which theorems do you use?
c. Write down the proof in the way you used earlier with the perpendicular bisector theorem in '*Distances, edges & domains*', page 76.

13. Links

finding links

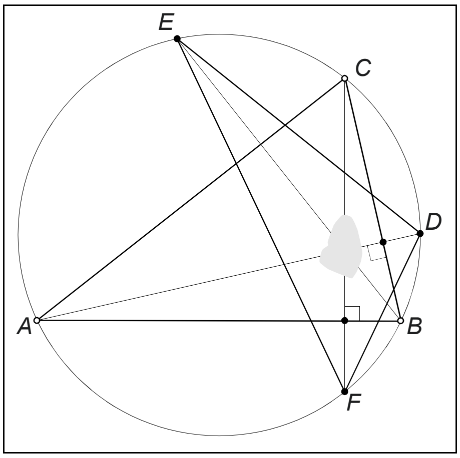
Often you encounter the situation that you need to prove an equality like $\angle X = \angle Y$ and it is not seen *directly* why these angles are equal.
If there is another angle, say $\angle Z$, of which you can prove both $\angle X = \angle Z$ and $\angle Z = \angle Y$, then you are done.
Or you may find that both $\angle X = 180° - \angle Z$ and $\angle Y = 180° - \angle Z$. Then you also hit the jackpot.
$\angle Z$ forms a *link* to get from one angle to the other. You prove the equality of $\angle X$ and $\angle Y$ *via* angle $\angle Z$. A *link* in a proof can be anything: not just one angle or one point. Sometimes you need more links, or two links at the same time.

4. Here two circles and two lines l and m through the intersections A and B of the circles are given.
To prove: $PQ \parallel RS$.
Approach: assume the idea that you need to show parallelism through indicating equal angles and search for a link. The circle and the points A and/or B of course play a role.



The next exercise leads to a familiar theorem in a new way. Thus this theorem should not be used in this proof!

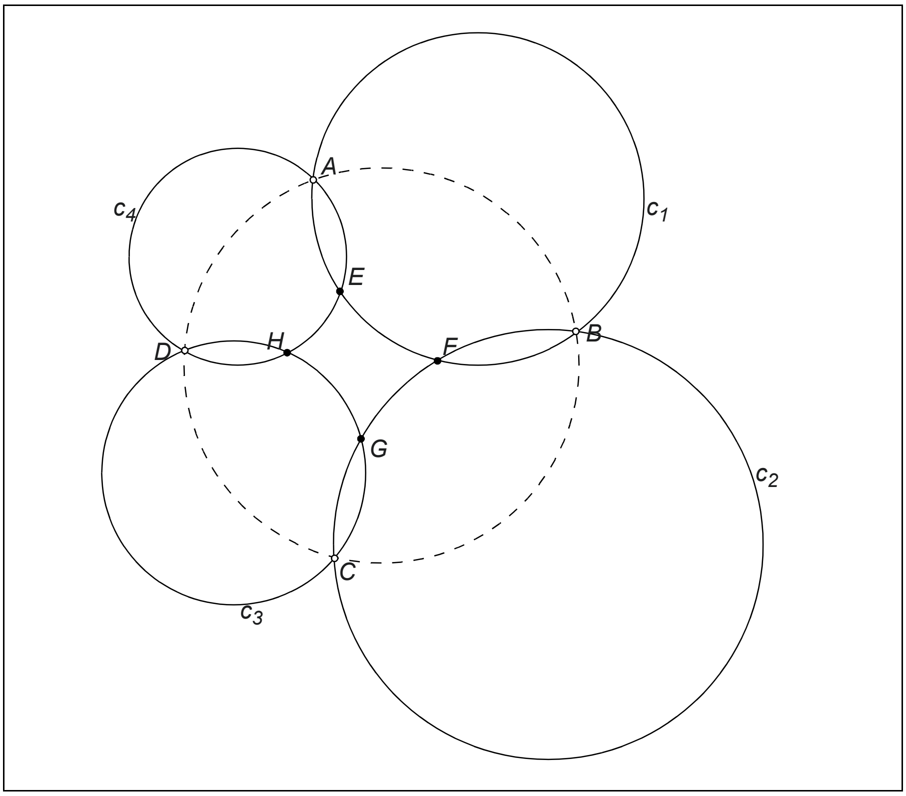
5. Given an acute triangle ABC with its circumcircle, the three altitudes intersect this circle in D, E and F.
To prove: the altitudes of triangle ABC are the bisectors of triangle DEF.
Approach:
Line EB should be the bisector of $\angle DEF$.
Try to find *both* angles, which should be equal to each other, somewhere else in the figure based on familiar theorems. Can these two new angles be linked via another angle?



6. How does the theorem of the altitudes follow from this? Is the proof general?

Next an example where the links will look alike, since the figure is built up from almost identical elements. You do need a substantial amount of links, but they are all of the same kind; if you operate in an orderly way, it will not be so bad.

7. The circles c_1, c_2, c_3, and c_4 intersect in the indicated manner in *A*, *B*, *C*, *D, E*, *F*, *G* and *H*. The statement you have to proof is surprising!



Prove: *If A*, *B*, *C* and *D* lie on one circle, *then E*, *F*, *G* and *H* also lie on one circle.

Clue: *EFGD* should be a cyclic quadrilateral. Connect its angles via links with angles of quadrilateral *ABCD*. Use a known theorem in the circles you know.

links in short

Often it has to be proved that two things, say *A* and *B*, are equal, while there is no theorem immediately applicable to the situation. Sometimes an extra step can be

found, a *C* of which you know that *A* and *C* are equal and that *C* and *B* are equal. Then *C* is a link. A link can be another angle, a line segment, several angles together.
When searching for links from one angle to another you have seen
- that the one angle sometimes is an angle of a cyclic quadrilateral; the opposite angle of the cyclic quadrilateral can also be a link.
- that a link angle can be on the same arc as the first one
- that link angle can be a neighbor angle of an isosceles triangle

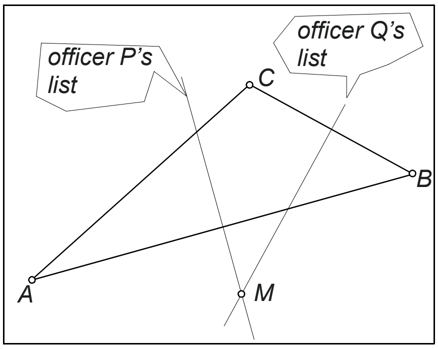
You can also say in general:
- you often find links if you indicate equal things with symbols in the figure. At some point you will find a link or a series of links from *A* to *B*.

14. Splitting up conditions

bad plan
Imagine: the police know that a pickpocket, who is taller than 1.87 m and has red hair, is in a certain street. The investigation is split up over two officers as follows: On one street corner officer *P* collects all people who are taller than 1.87 m and officer *Q* collects all redheads on the other corner. The pickpocket will be the one who is standing on both corners.

... does work great in mathematics
A bad plan, clearly, in this case. Still: in geometry it often is the right plan and we have applied it many times.

a familiar example
Think back to how to find the center *M* of the circumcircle of triangle *ABC*. What was wanted was a point *M* to which applied:



$$d(M, A) = d(M, B) = d(M, C).$$

We sent, so to speak, two officers: Officer *P* searched all points with:

$$d(M, A) = d(M, B)$$

and officer *Q* searched all points with:

$$d(M, B) = d(M, C).$$

Or said in another way:

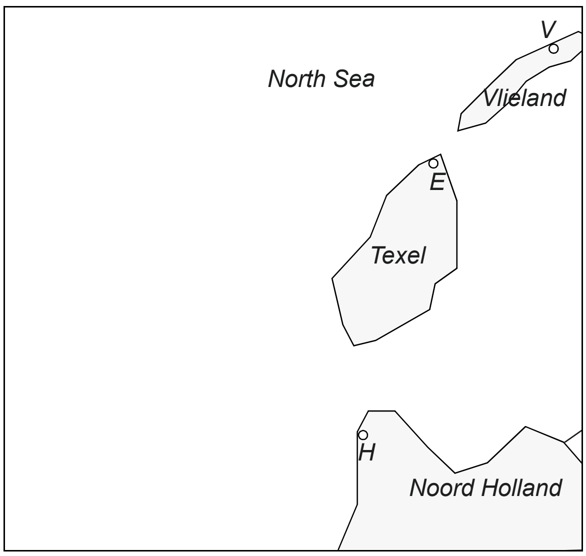
> *the condition d(M, A) = d(M, B) = d(M, C) is split up in two conditions, namely d(M, A) = d(M, B) and d(M, B)= d(M, C).*

For each of the two, all possibilities were investigated. That resulted in two lines. The intersection of the lines met both conditions and thus is the wanted point. Next, several examples. The first one is not particularly difficult, but does illustrate the core of the method very well.

8. Sketch very accurately a triangle with sides of 12, 9 and 17 cm. You can use ruler and compass.
  a. First sketch the side AB of length 12 cm. Point C now meets two conditions. Which?
  b. Now sketch – in the terminology of a minute ago – the lists of both officers.
  c. You now find two possibilities. Are these triangles different?

9. Wanted a triangle ABC with a triangle of which you know:
$|AB| = 10$, $\angle CAB = 45°$ and $|BC| = 8$.
  a. Sketch AB and again find C using the split method.
  b. Also here there are more possibilities for C. Still the situation is different than in the previous exercise. Explain this.

10. Apply the splitting method to the following *navigation* problem. A damaged ship is (still) afloat on the North Sea. The compass no longer works. To determine their location to ask for help, the crew looks at the lighthouses of Huisduinen (H, at Den Helder), het Eierlandse Gat (E on the north point of Texel), and of Vlieland (V on the northeast side of Vlieland). The position of the ship is called S. The angles $\angle HSV$ and $\angle HSE$ can be measured easily from the ship. They are: 90° and 45°. Now show how position S of the ship on the map can be determined.



'splitting conditions' in short

Sometimes *one* point with *two* properties needs to be found. Then it is often easy
– to find the figure of all points with one property;
– and also to find the figure of all points with the other property.
A point at the cross-section of the two figures then has both properties.
In a manner of speaking: the point is the one on the lists of both officers.

Example: finding a point that has two given distances to other points.

Another example: The familiar method of finding the center of the circumcircle (the so-called 1-1-bis manner) also works with separating the two conditions: lie on the *pbs(A, B)* and lie on *pbs(B, C)*.

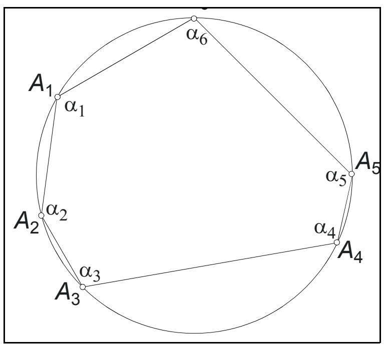
You see the split method is not only suitable for finding a certain point, but also for certain proofs. In a lot of cases it is the 1-1-bis manner, if the conditions, which are separated, are of the same type.

15. Plagiarism

If you are writing a book and then copy pieces from another writer (or from the internet), maybe slightly altered, but without acknowledgment, then you are plagiarizing. Even if you take something slightly modified so it won't show, it remains plagiarism. Also if you copy someone's ideas without acknowledgement, you plagiarize.

In this section you practice mathematical plagiarism in proving. Do not worry, the way we do it here, is a honorable cause. It works like this: You have a problem in front of you and it reminds you of something you did before. You look whether the proof can be used in the new situation, of course adapted to it.
Here are several problems where you can apply the method of plagiarism.

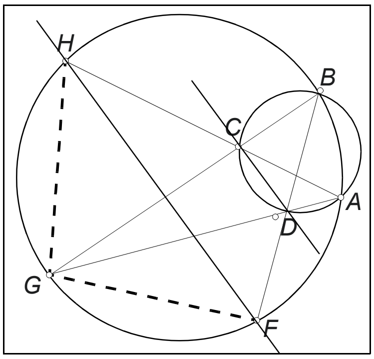
11. The theorem of the *cyclic hexagon*:
If the vertices of hexagon $A_1A_2A_3A_4A_5A_6$ lie on a circle and α_1, α_2, α_3, α_4, α_5 and α_6 are the angles, the following applies:
$\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6$
Prove this by plagiarizing the proof of theorem 9 of the cyclic quadrilateral as mentioned in *Distances, Edges and Domains*, page 105.



a. Prove this by dividing the hexagon in two cyclic quadrilaterals and using the theorem of the cyclic quadrilateral.
b. One of these two proofs is complete, for the other a case distinction should be made. What is up with that?
c. Formulate a *theorem of the cyclic thousand-angle*. You do not need to be very original, plagiarize.
d. Does such a theorem also exist for a *cyclic pentangle* or a *thousand and one* sided figure?

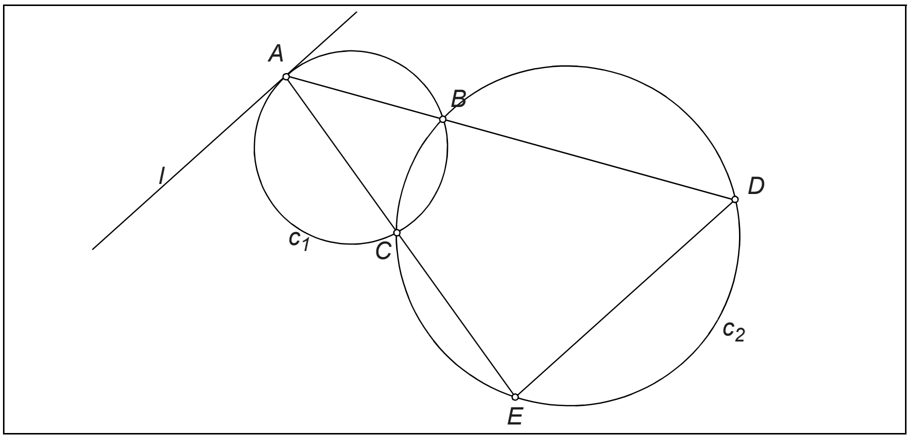
The next two problems are variations on exercises from this chapter.

12. Given: two intersecting circles with points as shown in the figure.
a. Show: $HF \parallel CD$.
(The position of the points was slightly different from the original exercise, but the differences are not very big.)
b. Extra for this figure, independent from question a: show that $|FG| = |GH|$.



In the next exercise the figure is a little simpler than the original on page 212. The exercises have the parallelism and the two circles in common.

13. In the figure below l is the tangent line in A to circle c_1.
B and C are the intersections of circles c_1 and c_2.
D and E are the other intersections of AB and AC with circle c_2.



Prove that l is parallel to DE.

16. Review on this chapter

finding proofs

In this chapter you have seen three different methods to find a proof.

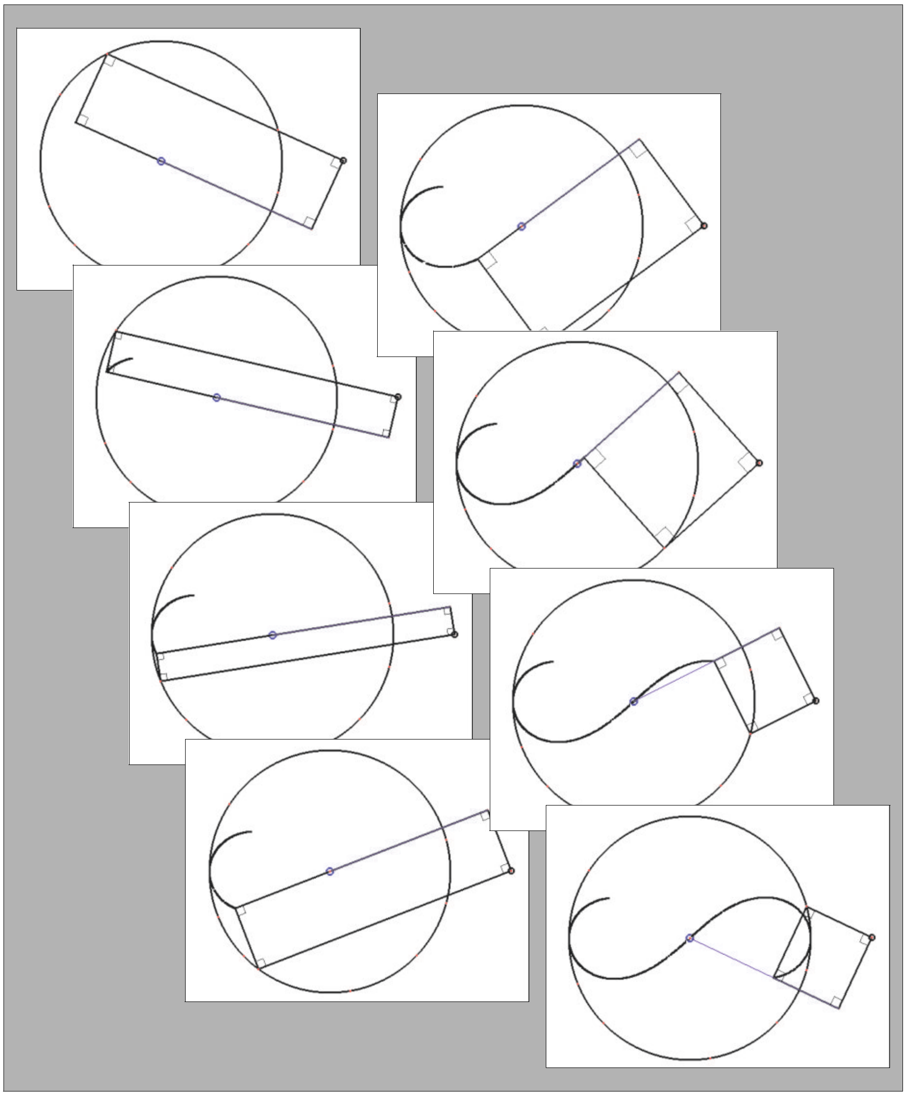
three search methods
Sometimes you need to look for appropriate ***links***. You use these if you need to prove that two things are equal to each other, and there is no theorem that shows this relation immediately. Sometimes you need more than one link.
The method ***split of conditions*** can be used if a point has multiple properties, or if you need to prove that three figures (lines or circles) concur.
The method ***plagiarism*** was nothing new. The most important is that you get used to reusing elements from other proofs.

success never guaranteed
It is not the case that you will immediately find a proof by applying one of the methods. It also is not true that a proof cannot be found in several ways. The methods do not exclude each other, they supplement each other.

preview

In the next chapter you will explore several figures using a computer program. This leads to all kinds of conjectures, which are unproven assertions of which you have a strong impression that they are true, without having proved them. In the chapter after that you apply the techniques of this chapter to prove these conjectures.

Chapter 4: Conjectures on screen



On the front of this chapter

The 'movie' on the front page of this chapter can be made with the computer program GeoGebra that you will be using in this chapter. You see snapshots of a moving and distorted rectangle. One vertex leaves a trace: the path of that point.
With GEOGEBRA it is easy to make moving images and you are able to show paths of moving points.
In diverse situations the paths seem to be unexpected simple figures, like circles or lines. In such cases you then need to try to prove that the path really is like that. You cannot just assume it is true.

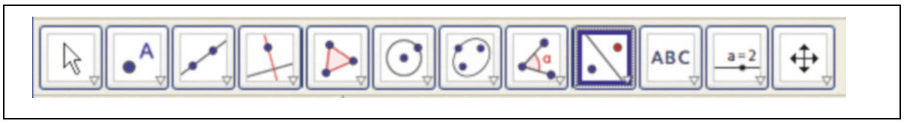
17. Introduction: getting to know GeoGebra

This chapter is a practical with the computer program GeoGebra. With GeoGebra you sketch geometrical figures on the screen. You can constantly change and move them. The software is free and you can easily find it online.

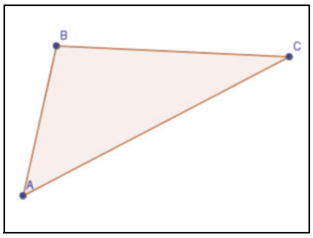
This practical consists of three parts:
- a few initial exercises to get to know the controls (this section)
- making drawings of (almost) familiar cases (see the next section)
- investigating new problems, where a lot of movement will take place on screen (see page 225)

controls, initial exercises
You will learn the controls in a minute by making a familiar figure. As an exercise we draw a triangle with two perpendicular bisectors and the circumcircle of the triangle.

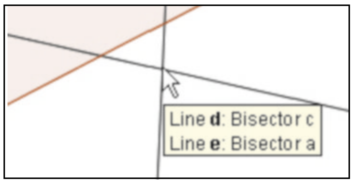
tool bar, buttons
When you start, an empty screen appears with at the top a **tool bar** with buttons. Some buttons represent types of objects you can make, for instances points, lines, polygons (triangles among them), circles. If you click on the little triangle on a button, you see a list of further options to choose from.



Just explore the buttons a bit to get an impression.
No we start the initial exercise.



1. Choose the option **Polygon** (under button 5 from left)
   a. Now click three points of your choice on the drawing screen and close of with clicking the first point again.
   And your triangle is done, including labels A, B and C!
2. Perpendicular Bisectors.
   a. Choose the option **Perpendicular bisector** (button 4). Click in the drawing screen in on an *edge* of the triangle.



b. Draw another perpendicular bisector.

c. Choose *Intersect* under the point button (second on the left) the create the intersection point of the perpendicular bisectors.

d. Finally draw the circumcircle of the triangle.Use *Circle with center through point* (find the option yourself) and click first on the intersection of the perpendicular bisectors, then on one of the vertices of the triangle.

3. Now we shall investigate the figure that has been drawn.

a. Choose the very important option **Move** in the leftmost button. We investigate what can or cannot be *dragged*.

b. By dragging the vertices, you distort the triangle. Try this and establish: perpendicular bisectors and circle adjust to the new triangle.

c. Also try to drag: (it will not work, but try it anyway!)
– the perpendicular bisectors
– the center of the circle
– the circle
– a side of the triangle
– the whole triangle (by a point of the interior.

dependent, independent

The explanation for all this is important for the useful continuation of this practical.

You could drag the vertices of the triangle in full freedom. These are so-called INDEPENDENT OBJECTS.

You made the perpendicular bisectors and the circle in relation to the first points. They are called DEPENDENT OBJECTS, because they need other objects created before.

Dependent objects move or distort in general, if you drag the objects on which they depend. Independent objects can be dragged freely. You can only make them with the DRAW-buttons.

4. A short closing exercise (do not clean the screen, it is not necessary).

a. Sketch a new *independent* circle.

b. Sketch with the option **Point on object** a point on this circle.

c. Try how you can drag this point. (Click **Move** before you drag!)

point on object

Such a point is a dependent point that can move over the circle, but stays on it. Call it *a point moving over the circle*.

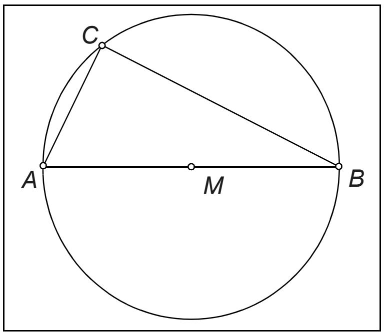
5. Under **Edit** you will find an Undo-option as usual in computer programs; Ctrl+Z works also. You will use this option regularly.

18. Constructing with GeoGebra: old acquaintances

dragging or constructing: an important distinction

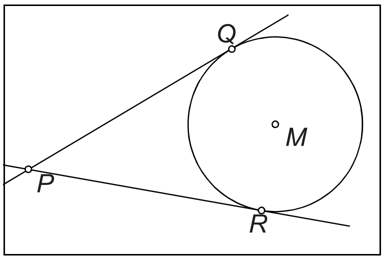
6. On your screen you can still see the triangle with circumcircle.
 a. By dragging vertex *C* you can move the center of the circumcircle to the side *AB*.

The situation you have created now is a drawn illustration for the theorem of Thales. But as soon as you move a vertex, it no longer is! It is just a freehand drawing, you are not sure that the midpoint of the circle really is on the segment. Let's create the figure in another way, so that by dragging *A* or *B* the structure stays intact. That is called *constructing exactly*.



7. Clear the screen completely for the next exercise. (Edit, Select All, Delete).
 A and *B* as the independent points, which you start with. You can make them point by point, but it can be done faster.
 a. Draw a line segment *AB* (Find your option under the line options.)
 b. Construct the midpoint of *AB*. (Find your option under the point options.).
 c. Rename the midpoint to *M*. ***Right-click*** on the point and select **Rename**, or doubleclick the name in the drawing screen directly.
 d. Now create the circle with diameter *AB*. Use the same circle-option as above.
 e. Create a new point on the circle.
 f. The angle at the new point should be 90°. So says Thales!
 Check this, by using the option **Angle** and indicate the angle by the *three points method*. If you don't get the 90° and the special sign for right angles, change the order of choosing the three points.
 Positive angels are imagined rotating counterclockwise, so in the above figure you should choose *A*-*C*-*B*.
 g. Now move *C* over the circle and watch for the angle at *C*. There maybe some a small surprise but it can be explained by the information under point **f** above.

constructing tangent lines to a circle



You could make the figure on the right by first drawing the circle, point P, and then shifting and dragging the tangent lines to it. Not pretty, we want to *construct* the tangent lines.

8. Clear the screen and draw a *circle* with center M and a *point P* outside it. These are the independent objects. Let's *construct* the tangent lines from P to the circle exactly. You can construct tangent points Q via the *split method*, since you now know *two* things about point Q:
– *one*: Q lies on the circle with center M.
– *two*: $\angle PQM$ is perpendicular.
The figure for condition one already exists. That is the circle itself.
a. Construct the figure for the points satisfying the second condition:
the figure consisting of all possible points W, for which $\angle PWM$ is right angle.
You now what figure this is and how to construct it!
b. Now create intersection ***points*** Q and R with the **Intersect** option.
c. Draw the tangent lines and test by dragging M and P and enlarging the circle whether the construction works. What happens if you place P inside the circle?

Dress up

If you want to – sometimes it is worth the effort for ease of survey – you can 'dress up' your drawing. The next exercise shows some possibilities.
The main tool in Geogebra is to right click on an object. Some tools are already visible; for coloring, thickness of lines, choose **Object properties**

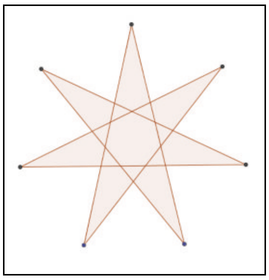
9. Practice a bit with the drawing you made.
a. Show the name of the new circle which you used in the construction.
b. Hide this circle. Hiding does not remove the circle. It still in the so-called algebra list. Right click on it there.
c. You may also choose a lighter color for this circle, or a thinner line. Find out how.

Exercises: incircle and regular stars

10. Now you are going to construct the *incircle* of a triangle.
a. Draw a triangle ABC, the *angle bisectors* of $\angle A$ and $\angle B$ and their intersection I.
There is an option to do this under the second line-button.
Point out the to be divided angle with the *three points method*. Thus for the bisector of $\angle B$ you point out A, B and C.

b. In order to construct the incircle, you need the *exact* point where the circle is tangent to one of the three sides. You can find such a point by constructing the perpendicular line through I on AB. Construct this perpendicular line and the incircle.

c. Again test your construction by dragging the vertices of the triangle.



Extra exercise: regular star

11. Draw this septagonal star. Use **Regular polygon** and then use the option **Polygon**; in this last option you are free to choose the vertices of the polygon. Remember to close by clicking the first point again.

19. Conjectures, movement, traces, the locus

thinking up conjectures and investigating

You now know enough to go on an exploratory computer expedition for special relations in geometrical figures.

conjectures

The result of such an investigation is fundamentally no more than a (strong) conjecture. But that is also quite a lot: Geogebra helps finding new conjectures.You will need to prove such a conjectures. Only then is the conjecture promoted to a theorem. Thus make sure you write down enough information.

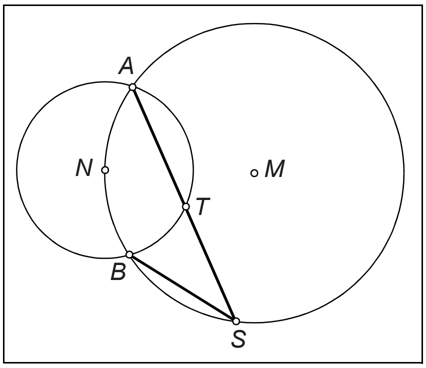
Here are some strategic hints, which do not differ much from the ones in chapter 2.

strategic tips

First play a little with the initial drawing. Write down what you notice. Then look at special cases: for example symmetrical situations, situations with a special angle or extreme situations. Make a sketch in your notebook. Possibly add information to your initial drawing: measure the size of angles or the length of line segments; it may be handy to work with colors, or drawing an extra line (segment) might help you make progress.

problem 1: a special triangle

12. Draw two points M and N. Draw the circle around M through N. Choose a point A on this circle and draw the circle around N through A. The two circles intersect in a second point B. Choose on the circle around M a point S and draw the lines SA and SB.
The line SA intersects the circle around N in another point T. Your drawing should contain the same elements as the figure on the right. Triangle STB has a special property.



If you want more assurance, measure the lengths of BS and TS with the appropriate option under the angle-button.
Formulate a conjecture with regard to this.

problem 2: the four bisectors of a quadrilateral

13. Draw an arbitrary quadrilateral $ABCD$. Also draw the four interior angle bisectors.
The angle bisectors enclose a quadrilateral $PQRS$. Label these points on the screen.
At first sight this bisectors-quadrilateral does not have any special properties.
So we first investigate a few special cases.

a. Shift the points A, B, C and D so that quadrilateral $ABCD$ (practically) becomes a parallelogram. Formulate a conjecture for this situation about $PQRS$.

b. Also investigate which shape quadrilateral $PQRS$ has if $ABCD$ has the shape of a rectangle, a rhombus, a trapezium.

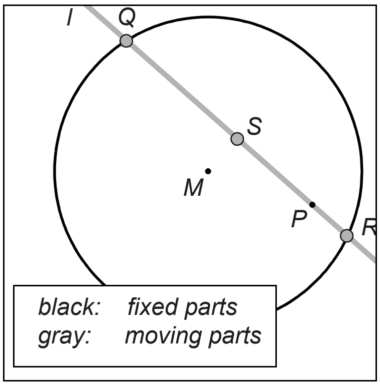
c. In the general case quadrilateral $PQRS$ also has a special property.
Maybe you can already formulate a conjecture yourself.
If not, here comes a hint: measure the angles of quadrilateral $PQRS$.

movement with left traces

In the next problems you can see something special if you move the figure (automatically or not). GeoGebra has special tools to show movements.
The tools you will use have names: Animation, Trace, Locus.

problem 3: middle of rotating chord, *Animation, Trace, Locus*
Here is a simple problem that you can draw with CABRI. After the problem you get some tips on how to investigate the situation.

14. In this figure you can see a circle with center *M* and a point *P*. *Q* is a moving point on the circle. A line through *Q* and *P*, which intersects the circle in *R*, is drawn. The middle of *QR* is the point *S* that also lies on the line. We will investigate how *S* moves if *Q* passes through the circle.



a. Make the construction. Start with the circle and point *P*.
b. You *must* create *Q* by the option **Point on object**. Create the line, point *R* and point *S*.
c. Move Q around; the line and point *S* will move also.

Probably you already have a conjecture over the path of *S*. But you should be able to use the nice tools Geogebra offers to help here. They are explained below

15. Animation, Trace, Locus
a. Next we will let the computer do the moving. So ***Animate*** point *Q*. (Right-click *Q*, and choose Animation *on*. It is a bit easier to see the movement of *S* now.
b. Next step: show the ***Trace*** point *S* would leave, if it had a leaking paint pot.
c. Finale step: draw the path of *S* in one go, by using the ***Locus*** option.

Animation, leaving a Trace, the locus-option: Instructions

Animation in Geogebra

Right click the point *Q* that should move and choose *Animation on.*

Note: *Q* is a point-on-object and drives the rest of the construction

Leaving a Trace

Right click the object (*S*) which should leave a trace choose *Trace on.*

Drag (or animate) the driving point (*Q*)

Drawing a Locus

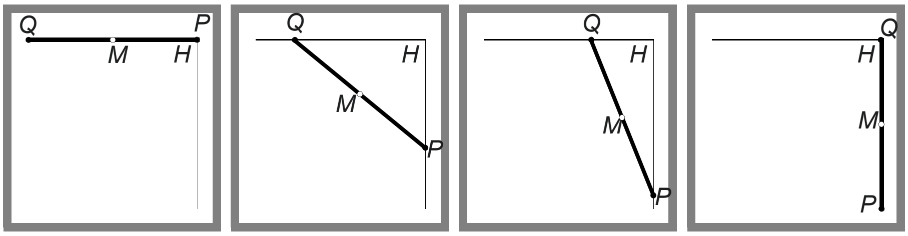
Choose Locus under the fourth button.

Click the point (*S*) which should show its trajectory and then the point that should move over an object (*Q*).

problem 4: midpoint between two hikers

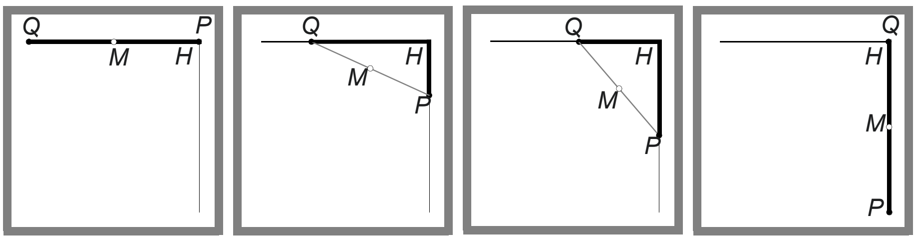
Below you see some images belonging to a simple movement. There are two people, represented by the points *P* and *Q*, who walk around an angle *H* one after the other. *P* walks in front all the time and *Q* follows. They are connected though a line of which we will trace the middle *M*. The film starts when *P* is at the angle and ends when *Q* arrives at the angle.

first case: the real distance of P and Q remains constant



Here the dark line segment, the direct connection, has a fixed length and thus *P* and *Q* cannot travel at the same speed.

second case: the distance of P and Q along the road remains constant



Here the dark line segment that goes around the corner has a fixed length and thus the direct connection line, on which *M* lies, is elastic.

16. Simulate both situations in GEOGEBRA and also investigate for both cases the path of *M.* Write down conjectures about both paths.

Clues:

- First draw the piece of road where *P* is walking with the option **Segment** and choose a moving point on that segment.
- Now draw the line where *Q* walks on with CONSTRUCT1>PERPENDICULAR LINE. Then determine *Q*, dependent on *P*. In the two cases this must be done differently.
- Handy options are **Compass** and **Circle with center and radius** under the Circle button.

problem 5: the path of the orthocenter

This next problem closely associates with exercise **5** in chapter 3, page **212**.

17. Draw a circle with a triangle *ABC* on it. In this order, because later on we want to let point *C* move over the circle.

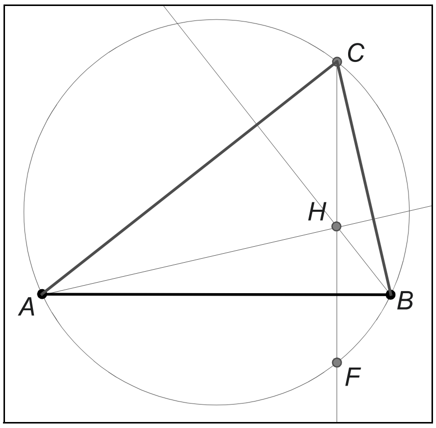
a. Draw the altitudes of triangle *ABC*.

b. Mark the orthocenter via **Intersect two Objects.**

c. If you move point *C* over the circle, *H* also moves. What is your impression of the *path* which *H* describes?

d. Investigate what the path of *H* is using the *Locus* option.
Also investigate what changes occur in the path if you:
– enlarge the circle
– move the points *A* and *B* over the circle.

e. Also try to phrase a conjecture about the relation between the position of points *F* and *H* in the picture.

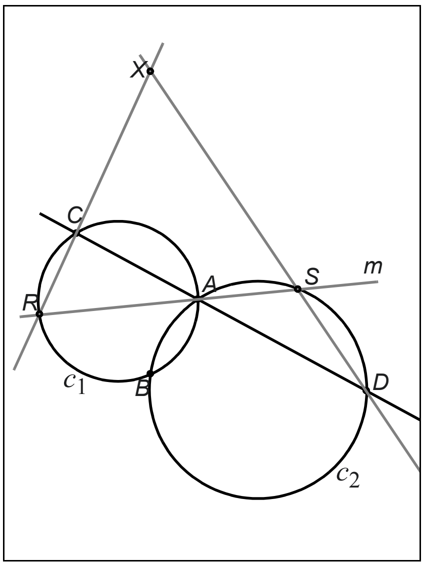


problem 6: path of a special intersection

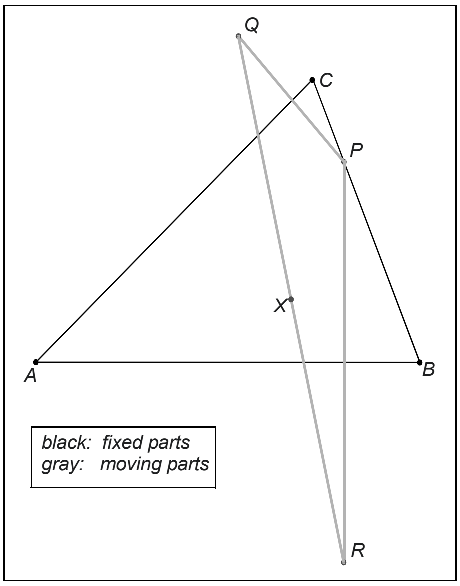
18. In this figure (a screen print) you see:
– two circles initial drawn c_1 and c_2
– their intersections *A* and *B*
– a line through *A*
– intersection points *C* and *D*.
These are the fixed parts of the drawing, which again are colored black.

A point *R* moves over c_1.
These also depend on *R:*
the line *m* through *R* and *A*
– the intersection *S* of *m* with c_2
– The lines *CR* and *DS*
– The intersection *X* of *CR* and *DS*
These are all gray moving parts.

a. Construct this all and investigate the path of *X* if *R* passes through the circle c_1.



problem 7: the middle of two reflections



19. Draw a triangle *ABC*. The point *P* lies on side *BC*.

a. Draw the reflections of *P* in *AB* and *AC* and call them *Q* and *R*.

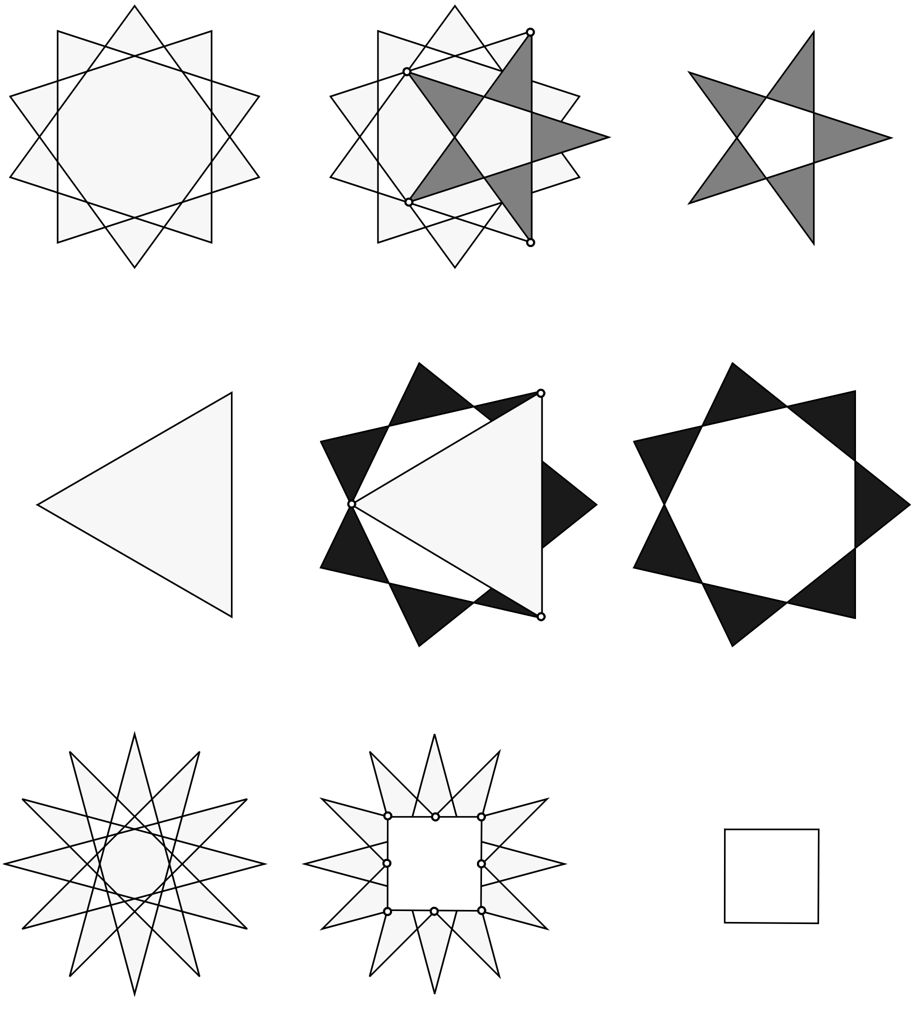
(Use the option **Reflect about line** under the transformations-button.)
The midpoint of *QR* is called *X*.

b. If *P* moves, *X* must also move. What is the path of *X*, if *P* passes through the side *BC*? Formulate a conjecture about this situation.

research tips

- *P* moves from *B* to *C*. What are the accompanying extreme points for *X*?
- If *P* moves away from *C*, *X* moves in the opposite direction. Measure $\angle PAC$ and $\angle XAB$. Measure other angles at *A*, for example also $\angle QAX$.
- Investigate special cases. You can do this by distorting the triangle. For example to isosceles and equilateral.

Chapter 5: Proving conjectures



Note about the illustration on the front page of this chapter

The figures in each row on the front page of this chapter are related to each other. The middle one is made by combining the left and right one in special ways.
On the first row the pentagonal star fits perfectly – or seems to – on four special points of the ten-pointed star. On the second row the equilateral triangle fits perfectly – or seems to – on three special points of the seven-pointed starlike figure. On the third row the square fits perfectly – or seems to – on eight special points of the twelve-pointed starlike figure.

So it appears.

But one of the pictures is certainly a fraud: on closer inspection deviations of the regularities in the middle figure will come to light. Maybe you can find out by measuring which one is certainly a fraud. But even then you cannot be sure about the other ones. Measuring will not help.

You need proofs!

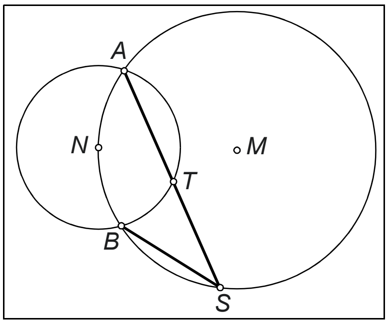
20. Introduction and procedure

In the practical with GEOGEBRA you have seen a number of things on the computer screen. In the problems of section 19*: Conjectures, movement, traces, the locus* you have formulated conjectures about what is going on. Now the final step: investigating whether and why the conjectures are true. In other words, you need to *prove* what you discovered.
A proof shows how your discovery is connected with what you already know, moreover: how your discovery is a logic result of what has been proven before.

procedure
The problems from the last chapter will be repeated briefly. They come with a conjecture, probably the same one you came up with.
You will need to prove this conjecture.
If you have observed more than the conjecture itself, for example by drawing extra lines, this will probably work in your advantage when finding a proof, so have your notes at hand.
What has been said in chapter 3 about *Clues* also goes for this chapter.
Use the *Clues* (they start on page 243) sparingly. They give you some more hints, so that hopefully you can continue if you get stuck.

21. Proving conjectures

problem 1: a special triangle



Sketch two points M and N. Sketch the circle around M through N. Choose point A on this circle and draw the circle around N through A. The two circles intersect in a second point B. Choose a point S on the circle around M and draw the lines SA and SB. The line SA intersects the circle around N in another point T. Triangle BST has a special property. Which? Formulate a conjecture.

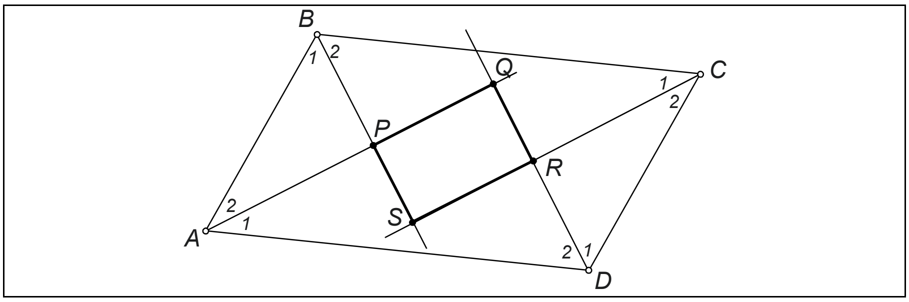
conjecture: Triangle *BST* is isosceles

1. Find a proof for this conjecture, for example use strategies you have seen before.
   a. Write the proof in the form *Given - To prove - Proof.*
   b. *SB* intersects the circle in *R*. Indicate another isosceles triangle in the figure.

problem 2: the bisectors-quadrilateral

Draw an arbitrary quadrilateral *ABCD*. Also draw the four interior angle bisectors. The bisectors enclose a quadrilateral *PQRS*.

conjecture I: If *ABCD* is a parallelogram, then the interior angle bisectors enclose a rectangle *PQRS*.



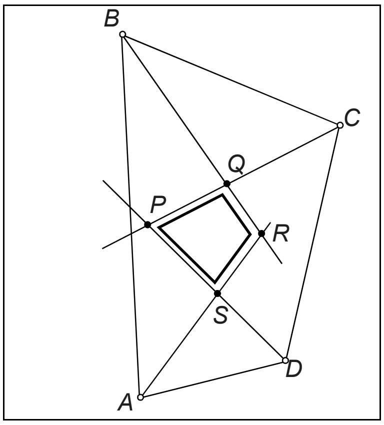
This first conjecture may look familiar, since it already came up in chapter 1. Despite this, it is not a bad idea to write down the proof. It will probably give you some ideas, which you can use in the next two exercises.

2. Thus, if you want to prove this conjecture, you are allowed to use the properties of parallelogram, but you can also use properties of bisectors.
   a. Now prove this conjecture and make sure that you connect the steps with the properties you use.
   b. Specification of the previous: if *P* is the intersection of the bisectors from *A* and *B*, which parallelism is used to show that angle *P* is right?

3. Now investigate other special cases; in all three cases give an outline of the proof. Use, if possible, things you already have proved.
   a. If *ABCD* is a rectangle, then *PQRS* is a square.
   b. If *ABCD* is a rhombus, then *PQRS* degenerates in one point.
   c. If *ABCD* is a trapezium, then *PQRS* is a quadrilateral with two opposite right angles.

The general conjecture, that is in case you know nothing about *ABCD*, needs to contain all the preceding. It probably looks most like the case of the trapezium. There *PQRS* was a quadrilateral with two opposite right angles. You know that *PQRS* then is a cyclic quadrilateral.

If we change the trapezium shaped quadrilateral *ABCD*, then the right-angled part also drowns. The last thing that can keep its head above water, is: *PQRS* is a cyclic quadrilateral. Hence:

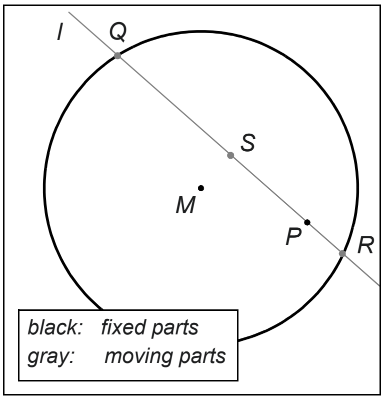


conjecture: If ABCD is a quadrilateral is, then the interior bisectors of the angles A, B, C and D enclose a cyclic quadrilateral.

4. Prove this conjecture.

problem 3: midpoint of rotating chord

In this figure you see you the circle with center *M* and point *P*. *Q* is a moving point on the circle. A line through *Q* and *P*, which intersects the circle in *R*, is drawn. The middle of QR is the point S that also lies on the line. Investigate how *S* moves if *Q* passes through the circle.
Draw up a conjecture about the path of *S*.



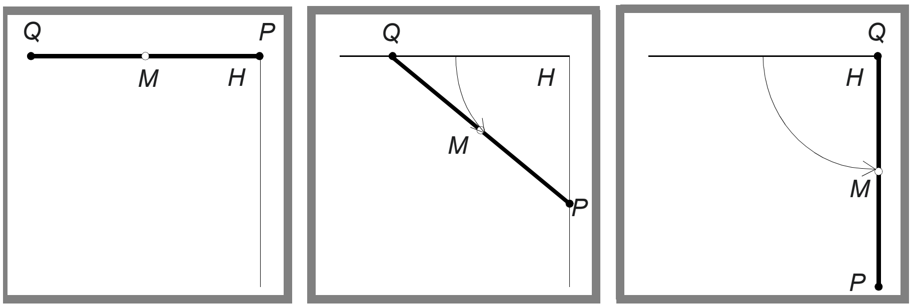
conjecture: *S* moves over the circle, which has *PM* as diameter.

5. Prove this conjecture.
Note that you actually need to prove two things:
a. Such a point *S* lies on the circle with diameter *MP*.
b. Each point *T* of the circle with diameter *MP* is the middle of the chord (of the large circle), which goes through *P* and *T*.

problem 4: middle between two hikers

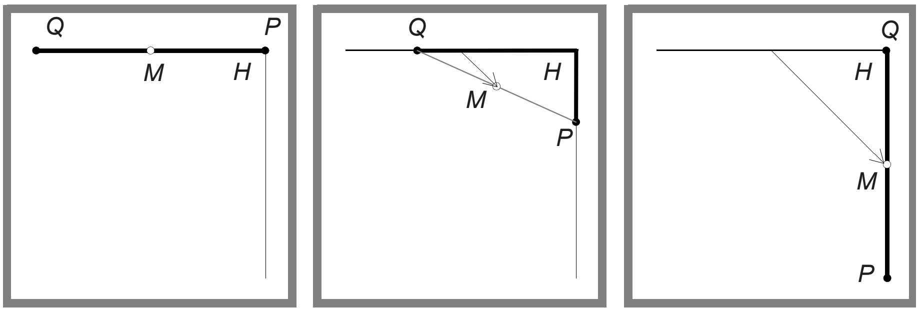
Below you can see some images belonging to the simple movement. There are two people, represented by the points *P* and *Q*, who walk around an angle *H* one after the other. *P* walks in front all the time and *Q* follows. They are connected though a line of which we will trace the middle *M*.
The film starts when *P* is at the angle and ends if *Q* arrived at the angle.

first case: the real mutual distance of *P* and *Q* remains constant



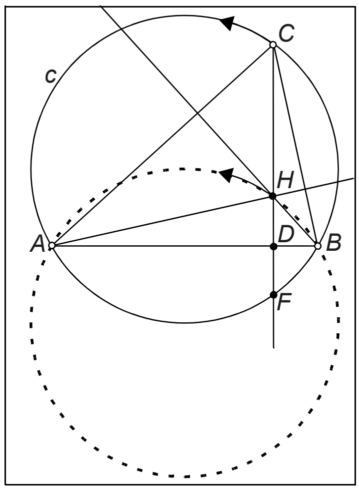
conjecture: *M* moves over a quarter circle. The radius is half of the distance between *P* and *Q* and the center is the vertex.

second case: the distance of *P* and *Q* along the road remains constant



conjecture: *M* moves over the connection line between the middles of the line segments where *P* and *Q* walk.

6. Prove both conjectures.



problem 5: the path of the orthocenter

Given is a circle *c* with three points *A*, *B* and *C*. If *C* moves over the circle and *A* and B *lie* idle, then the orthocenter *H* of triangle *ABC* describes a path.
If you saw this happen on the computer screen, you cannot escape the following:

conjecture: Point *H* moves over a circle through *A* and *B*, which has the same size as the original circle and which also goes through *A* and *B*.

7. The odds are that you have seen more, or something equivalent to this conjecture. This is why we first look into the coherence of some things. We will also use point F, the other intersection of the altitude from C with the circle, and the foot D of this altitude. F and D also move, if C moves.

a. *If* the conjecture is true, what will then be the relation between $|DF|$ and $|HD|$?
What does that mean for the circle where H lies, if you compare its position with the position of c?

b. If the conjecture is true, then $|CH|$ must have a fixed size. Why? And how do you translate this into a relation between both circles?

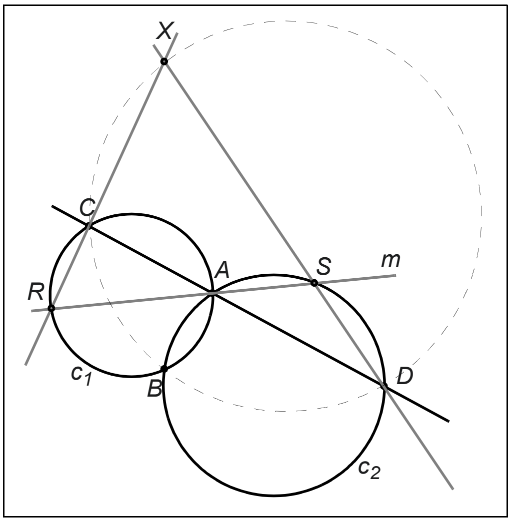
c. What is the relation between $\angle ACB$ and $\angle AHB$?

8. Now prove the conjecture.
You can let the exploration in the previous exercise inspire you. The relations connected to the conjecture could also serve as a step in the proof. This is not necessary; first find your own path! Maybe you do not need everything.

problem 6: path of a special intersection

In this figure you see:
- two circles, c_1 and c_2,
- their intersections A and B
- a line through A, which still intersects the circles respectively in C and D.

A point R moves over c_1. On R depend furthermore:
- the line m through R and A
- the intersection of m with c_2: the point S,
- the lines CR and DS
- the intersection X of CR and DS

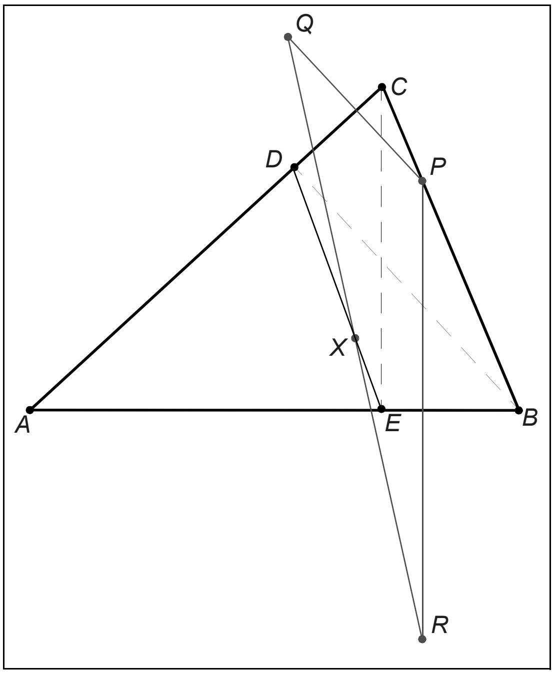


Investigate the path of X if R passes through the circle c_1.

conjecture: *X* moves over the circle, which goes through *C*, *D* and *B*.

9. Prove this conjecture.
In the clues you can look for certain familiar figures, but of course you were already doing that!

problem 7: the middle of two reflections



Sketch a triangle ABC. The point P lies on side BC.
Sketch the reflections of P in AB and AC and call them Q and R.
The middle of QR is called X.

If P moves, X also moves. What is the path of X if P passes through the side BC? Formulate a conjecture about this.

conjecture: If D and E are the feet of the altitudes from B and C, then X moves over line segment DE if P moves over CB.

X lies on E if P lies on C, this is no so hard: in this position Q coincides with P and C together and PR is the doubled altitude from C.

In the practical assignment several suggestions for investigation were given. This surely led to conjectures about some angles:

conjectures about the angles at A:

a: $\angle CAP = \angle BAX$

b: $\angle QAX = \angle CAB$

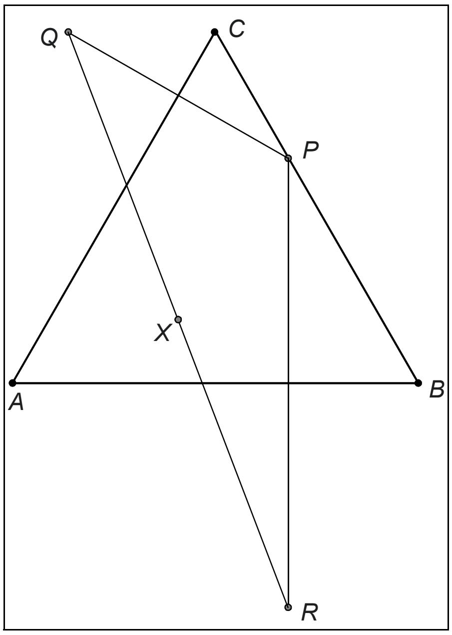
10. Draw the necessary lines in the figure and prove the two conjectures about the angles. Choose the order you want to do it in yourself.

The general proof that X moves over the indicated line segment DE is not so simple; but in the case that triangle ABC is special, it is not so bad. You need to find this proof now.

11. The case that ABC is equilateral, has been sketched again on the right. Of course you can use the equalities of angles you have proven. First a further exploration of this situation.

a. Fill up the figure with the points D and E. Indicate in metaphorical language as many equalities and familiar angles in the figure as you know. Also use the proven assertions of the previous exercise.

b. Also use what you know of triangles with angles of 90°, 60° and 30° in order to show that $|AX| = \frac{1}{2}|AP|$.



The strong impression arises that X lies on the midparallel of BC and point A. This also adds up with the general conjecture.

Unfortunately, this does not follow directly from what has been proven in **b**, since point X is *not* the middle of AP. We need an extra link.

c. Add a point in the sketch – call it P'–, so that X is the middle of AP'.

d. Now prove this assertion:

if P lies on side BC of the equilateral triangle ABC, then the point X, which is the middle of the connection line of the reflections of P in AB and in AC, lies on the midparallel of BC and A.

extra (and still more difficult!): the general case

12. For the proof of the general case you could plagiarize the preceding, but then you would need to adjust two things in your story:

– the conclusion via the midparallel principal changes slightly; this is due to the fact that factor $\frac{1}{2}$ needs to be replaced by another and strictly seen that conclusion falls just outside our geometry domain.

– you need to make an appropriate choice for P'. This has not only to lie on the line AX, but also must hold $|AP'| = |AP|$. Then P' does not (always) lie on BC, but on another line: the reflection of BC in the bisector from A.

Accept the challenge: prove the general conjecture.

Clues for chapter 3 and 5

Clues for chapter 3 Finding proofs

1. ***Clue 1***:
Equal arcs, equal angels from the midpoint. What about $\angle QMP$ and $\angle AMB$?

Clue 2:
You need 45, not 90 degrees. There was a theorem about midpoint and circumference-point with a factor 2 in it!

2.

3. ***Clue 1***:
For points R on arc AB (the smaller one) you can know the size of angle $\angle ARB$. *That's a good characterization of those points.*

Clue 2:
Take over the whole scheme for the proof from page 77 and fill it in box by box

4. ***Clue 1:***
Proving parallelism probably goes via *F*- or *Z*-angles. Thus you need to associate the angles at *P* and/or *Q* with angles at *S* and/or *R*.
The link will probably be an *angle* (or more) at *A* and/or *B*.

Clue 2:
Sketch an extra line, which makes angles at *A* and/or *B*: line *AB*. You now have at *A* a left angle and a right angle.

Clue 3:
You can see two cyclic quadrilaterals. You know something about the sum of opposite angles in cyclic quadrilaterals.

Clue 4:
Use the two angles at *A* as a link.

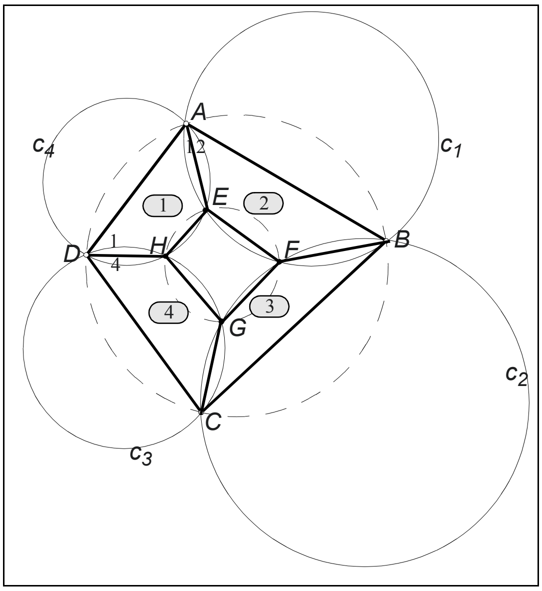
5. ***Clue 1:*** Put a symbol $*$ in $\angle DEB$. This angle stands on arc *DB*. Are there any more angles on this arc?

Clue 2: Put a symbol # at the other angle at *E*. Where else can you put this symbol?

Clue 3: The two new angles ∗ and # should be equal. Take each angle to be an angle of an already visible triangle.

6. The three altitudes of a triangle concur.

7. ***Clue 1***:
Start with cyclic quadrilateral *ABCD* and find links to the angles of quadrilateral *EFGH*.



Clue 2: Cyclic quadrilateral *ADHE* helps relating a partial angle at A to one at *H*.

Clue 3: You want to prove that *E*, *F*, *G* and *H* lie on a circle, or *EFGH* is a cyclic quadrilateral. Then the angles at *E* and *G* should together ...
Thus express the correct angles at *E* and *G* in other angles, with which you can work in the exterior.

Clue 4: number all sub angles; use numbers which correspond with the small cyclic quadrilaterals.

8.

9.

10. The two conditions (looking under 90° and under 45°) belong to special figures.

11. e:
There is a split in even- and odd-numbered angles. With 5 or 1001 or any odd numbers you will be in trouble: two odd-numbered angles are adjacent.

12. As two peas in a pod, which lie somewhat more on top of each other.

13. ***Clue 1****:* Plagiarism.

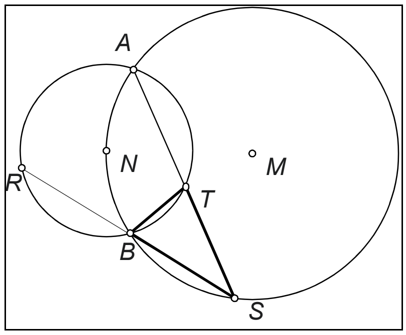
Clue 2: You also can think of tangent in *A* like: *A* actually consists of two in-

tersection points of l with the circle, which now coincide.
Clue 3: In this chapter, exercise 4, page 202.
Partly you could follow the proof of the original, but at a certain point in time you will need a theorem about tangent lines.

Clues for chapter 5 Proving conjectures

special triangle

1. ***a. Clue 1:***
Angle *SBT* and *STB* should be equal. It is annoying that *S*, *T* and *B* do not lie on one of the circles. But there are more points on the *large* circle in the game than *B* and *S*.



Clue 2:
See if you can find equal angles in the figure by using the large circle.

Clue 3:
By the look of it, arc *AN* is equal to *NB*. If this is really true you can – if need be by drawing extra lines – find a couple of equal angles.

Clue 4:
Find associations for 'isosceles'. A very general one is: symmetry. Which line should be the symmetrical axis of triangle *BST*? Sketch it. Can you now find a pair of *presumably* equal angles of which you can *prove* the equality?

b.
c. Also *SRA* is isosceles.

bisectors-quadrilateral

2. ***Clue 1***: Make a sketch. Use little signs for the equal angles or indicate subangles with numbers.

Clue 2: $\angle QPS$ should be 90°. In this figure it is easier to determine $\angle APB$.

Clue 3: In triangle *APB* the vertices *A* and *B* are also vertices of the parallelogram. Which *associations* do you have with 'neighboring angles in parallelograms'?

Clue 4: Determine the sum of the angles *A* and *B* in triangle *ABP,* and then *P* itself.

3. Make a new sketch for each case.

a. ***Clue 1****: ABCD* is a rectangle. Thus you know more than what you need just for the parallelogram. What else do you know now of triangle *APB*?

Clue 2: *Associations* for 'square'. How else can you recognize a square if you do not know directly that all sides have the same length? Think of the diagonals. The same length? That is not enough!

Clue 3: maybe another way: In order to find a proof, at first you can use the symmetrical axes of rectangle *ABCD*. What do they mean for 'rectangle' *PQRS*, according to the sketch?

Clue 4a: Show that triangle *APB* is isosceles. This way you actually use, without saying out loud, the symmetrical axis. From this, it follows that *P* lies on the perpendicular bisector of *AB*.

Clue 4b: Involve *R*. How does *RP* run? *RP* has two roles: it is diagonal in *PQRS*, but in this case also .....

b. ***Clue***: What other properties do the bisectors in a rhombus have besides being just bisectors?

c. ***Clue:*** *ABCD* is a trapezium; then holds for example $AD \parallel BC$. Now you can surely use something from the preceding.

4. ***Clue 1:*** Make a sketch and ask yourself: what is the most important property of cyclic quadrilaterals?

Clue 2: Express as before the angle at *P* in the angles at *A* and *B*. If you want to do something with opposite angles, you need to something similar for *R*.

middle of rotating chord

5. ***Clue***: It apparently revolves around the circle with diameter *MP*. There is a theorem about circles with diameter!

midpoint between two hikers

6. ***Clue 1 for both cases:*** There is a rectangle and a line segment is divided in half. If QP was diagonal of ..., then you could probably do something with it.

   Clue 2 for both cases: Fill up QHP to a rectangle.***Clue 3 for case two:*** The new point X of the rectangle also describes a path!

the path of the orthocenter

7. ***Clue*:** You could think of the two circles as each other's reflection in AB, but you could also look at the second circle as a translation of the first.
8. ***Clue 1*:** Again indicate as many equal angles in the sketch as you can. At first you might limit yourself to special positions of C (to acute triangles), but that is not so bad for the time being. This will solve itself later on.
   ***Clue 2*:** If $|HD| = |DF|$ is true, then things look really nice for the reflection! But if this is true, then $\angle HAD = \angle DAF$ must also hold. Maybe first show the equality of these angles? Can you link these angles through another angle?

   Clue 2a: (supplement for clue 2.) Have you already used the fact that there is a circle, the one round ABC with F on it? You can prove the equality of angles with it, if need be, sketch an extra chord.

   ***Clue 3*:** If $|CH|$ is constant, then it looks beautiful for a translation approach. It would come in handy to find a special one of C, wherein CH can easily be located and compared with the other positions.
   ***Clue 3a*:** (supplement for clue 3.) Can H coincide with B? Where does C lie then?

   Clue 4: Since all kinds of equalities can be found, in the long run you do not remember whether you have proven it or not. Remedy: write on a draft what you know for sure: triangle, orthocenter, position of F and D. Now only write on the draft what you know for sure within your reasoning, so not a variety of conjectures.

   ***Clue 5*:** Make sure that you have the given figure all thought out before you look at other positions. Check whether a new proof is necessary for other positions of C. If this is the case, use the plagiarism technique, as has been practiced before: adjusting a proof through almost word for word translation to the new situation.

path of a special intersection

9. *Clue*: First assume the situation in the sketch and prove that X goes through CBD on the circle.

Clue: If you want to get rid of $\angle CBD + \angle CXD$ you do need to come up with another point A.

Clue: Split $\angle CBD$ and consider the parts separately. Can they also be found somewhere else?

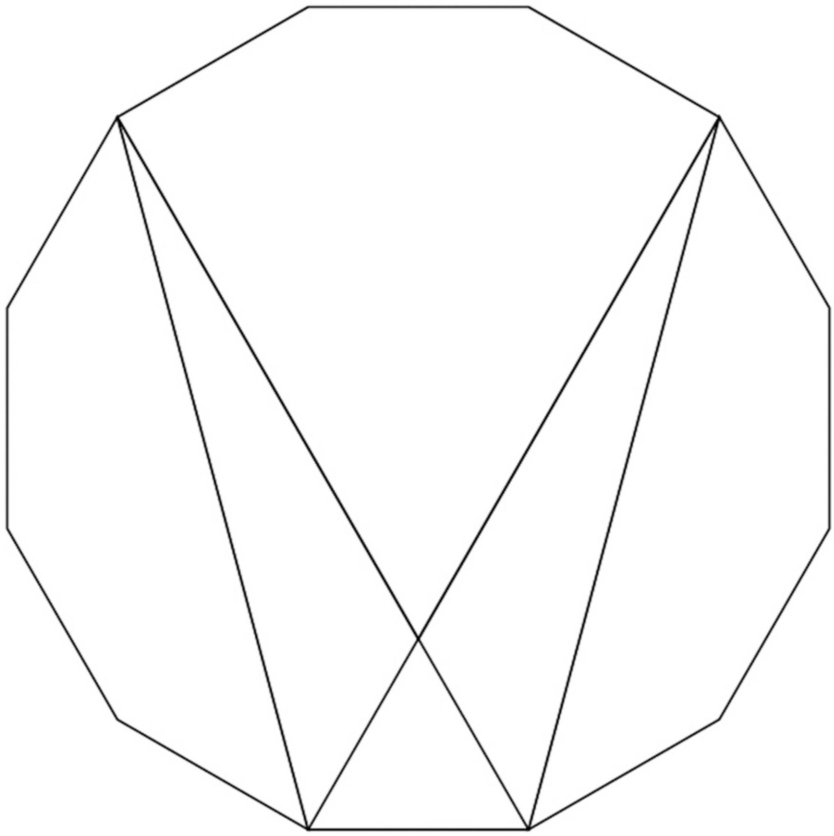
Clue: 180° is of importance for cyclic quadrilaterals, but also in several other theorems. Associate!

10. This is by far the hardest part of the section.

11. In the chapter, clues have been assimilated in the text, also disguised as help exercises.

Worksheet

Cutting figure for exercise 14, page 204: a puzzling division



Conflict lines and reflections

Advanced geometry, part III

Conflict lines and reflections – Advanced geometry, part III

Project: Mathematics for senior high school
Profile: Nature and Technology
Class: VWO 6
Authors: Wolfgang Reuter, Martin Kindt, Aad Goddijn
Translation: Danny Dullens, Nathalie Kuijpers

Preface

edge and conflict

Throughout history differences of opinions about the exact course of a border often led to conflicts and even wars. In the first two sections of chapter one we will take a look at several division and border problems:

– To whom do the oil and gas which are found at the bottom of the North Sea belong?
– How can you divide ‘new land’ among old neighbors?
– How do you determine the ‘middle’ of a border river, if that is exactly where the edge should be?

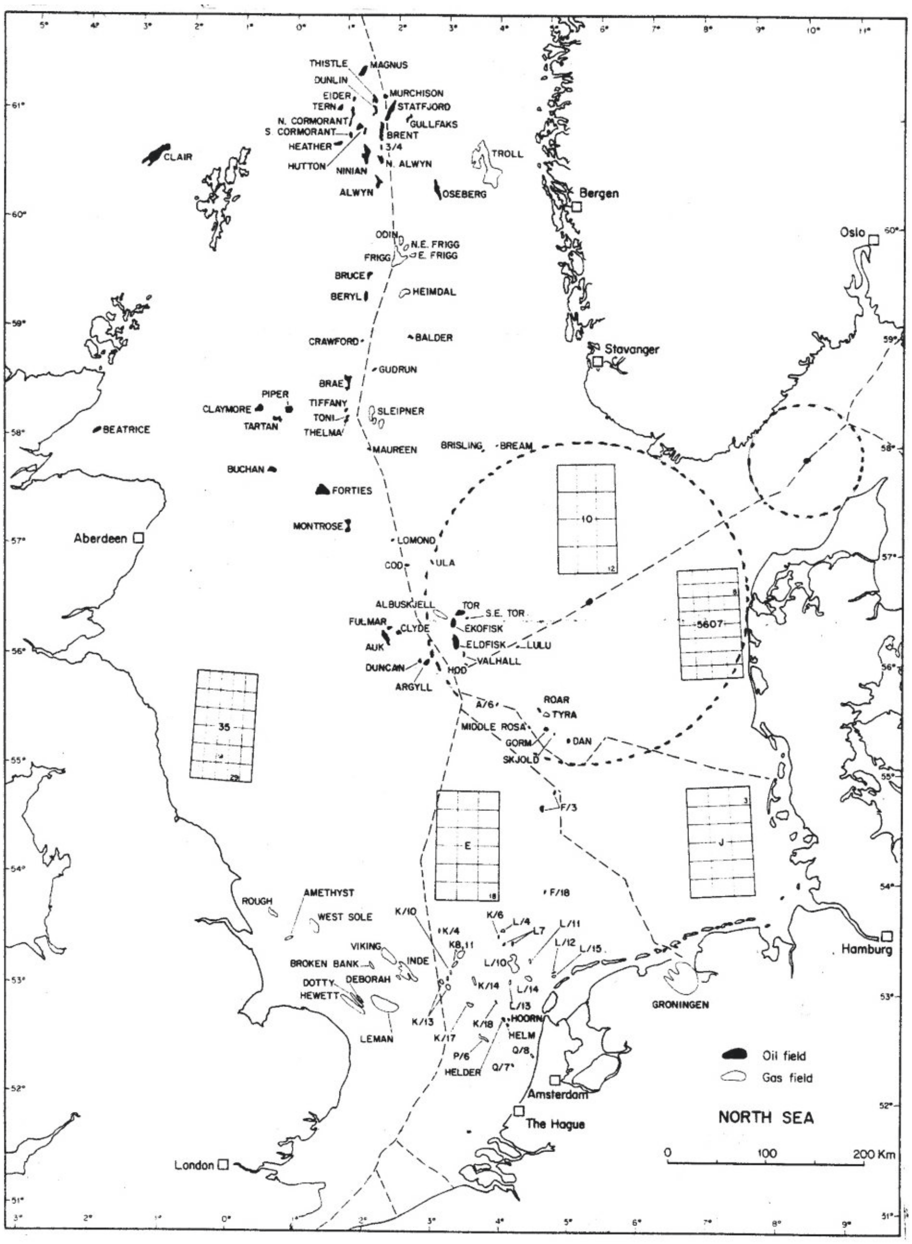
We are looking for peaceful mathematical solutions, of course.

parabola, ellipse, hyperbola

A systematic research of a special type of border lines (which we will call conflict lines) brings us to classic curves, namely *parabolas, ellipses* and *hyperbolas*. In this research we will again use ***Geogebra***. The special properties of these curves will be used in many applications. That will be dealt with in chapter 2. This knowledge originates, like much of the rest of the geometry you have done this year, in ancient Greece. The modern terms parabola, ellipse and hyperbola are derived from the work of Apollonius of Perga (200 before Christ). This is also true for the word *asymptote*. Asymptotes play a role in hyperbolas.

Chapter 1: Edge and conflict

Greenpeace activists at the Brent Spar oil platform, 1995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GULLFAKS
BRENT
HEATHER
3/4
CLAIR
TROLL
HUTTON
NINIAN
N. ALWYN
ALWYN
OSEBERG
Bergen
ODIN
N.E. FRIGG
FRIGG
E. FRIGG
Oslo
BRUCE
BERYL
HEIMDAL
CRAWFORD
BALDER
Stavanger
GUDRUN
BRAE
PIPER
TIFFANY
CLAYMORE
SLEIPNER
TONI
TARTAN
THELMA
BEATRICE
MAUREEN
BRISLING
BREAM
BUCHAN
FORTIES
MONTROSE
Aberdeen
LOMOND
COD
ULA
ALBUSKJELL
TOR
S.E. TOR
FULMAR
CLYDE
EKOFISK
5607
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ELDFISK
LULU
VALHALL
DUNCAN
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TYRA
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DAN
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J
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ROUGH
F/18
WEST SOLE
K/10
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L/4
L/11
K/4
L7
VIKING
K8, 11
L/12
L/15
BROKEN BANK
INDE
L/10
Hamburg
DEBORAH
DOTTY
K/14
L/14
HEWETT
L/13
GRONINGEN
K/13
K/18
HOORN
LEMAN
K/17
HELM
P/6
Q/8
Oil field
HELDER
Q/7
Gas field
Amsterdam
The Hague
NORTH SEA
0
100
200 Km
London

1. *Borders under water*

The border between the Netherlands and Germany does not end at the waters of the Dollard, but it continues on the seabed.
On the map on the left you can see that the Netherlands also have a long common border with Great Britain, and that we are very close to being direct neighbors with the Norwegians. You can also see why these borders are so important: the proprietary rights of the oil- and natural gas fields must be arranged appropiately. These borders were determined in 1945. For example, on the Danish-Norwegian border all points are equidistant from the coasts of both countries.

1. **a.** A circle is sketched around two points on this border. What do these circles mean?
b. ***If*** P lies on the Norwegian-Danish border, ***then*** d(P, *Norway*) = d(P, *Denmark*).
Is the reverse also true:
if d(P, *Norway*) = d(P, *Denmark*), ***then*** P on the Norwegian-Danish border?

distance, point-regions
In chapter 5 of DISTANCES, EDGES & DOMAINS we identified the distance from a point to the area like this.

> The distance from a point P to an area A is equal to the radius of the smallest circle round P, which has at least one point in common with the edge of A.

In this chapter distances will again play an important role, so we will recall some earlier knowledge. You may have all this knowledge at hand; if not, consult chapter 5 of DISTANCES, EDGES & DOMAINS.

2 **a.** What do you understand by a foot (point)?
b. Can one point have different feet on the edge of a region?
c. What is an iso-distance line?
d. How can you sketch an iso-distance line for a complex region?

3. The Norwegian-Danish border ends in a three-countries-point.
a. Which is the third country in question?
b. Show that this point truly has equal distances to the three countries by drawing the right circle.
c. South of this point there are two other three-countries-points. Indicate for each point which three countries are involved.

The circles in exercise **1. a** can be called *largest empty circles* again, as was done when working with Voronoi diagrams.

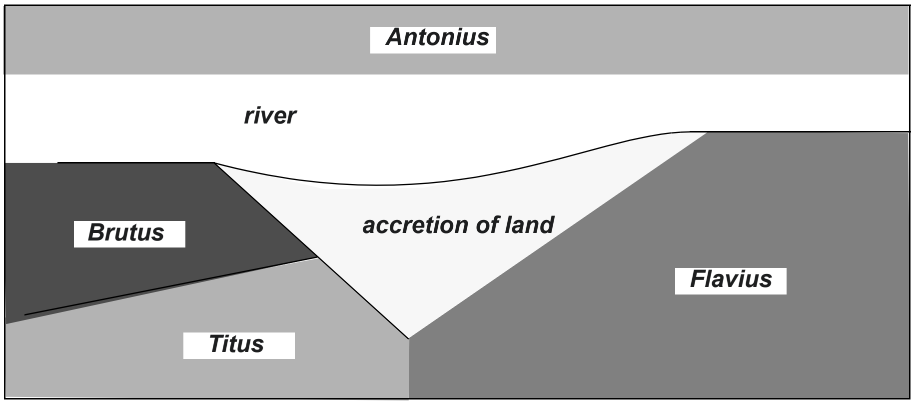
If you sketch circles around the two points mentioned in exercise 3 which touch the concerned countries, you will see that there are no feet along the German coast. There is a historic reason for this. During the division in 1945 people were not concerned with German interests. When later on, the international community did want to involve Germany in international conventions, the Netherlands and Denmark had both started drilling. For the demarcation of the German sector some political compromises were made.

4. Sketch the original course of the Dutch-Danish under water border.

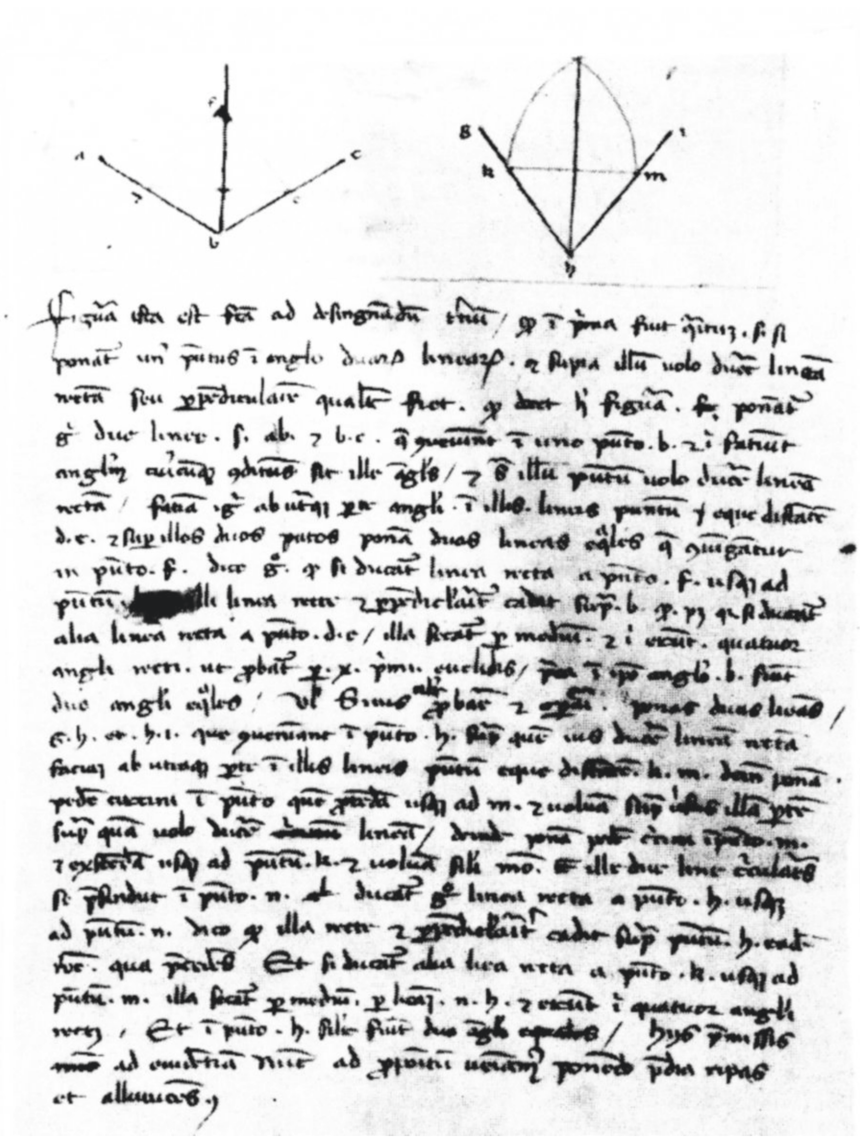
5. The borderline between the two regions is also called *conflict line*. A good name?

2. Division problems

In this section we move to a river in the Roman Empire. Once the river had a big bay on the south side. Over the years the bay gradually became silted up. This new land is very fertile, and each of the neighbors wants to add as large a piece as possible to his domain. But according to what rules must the alluvium by divided between the neighbors Brutus, Titus and Flavius? And can the opposite neighbor Antonius maybe also claim a part of this land?



According to Roman law, accretion of land was considered as *accessio*, increase: the increase to something which is my property is in itself also my property. According to this principle the owner of a cow also becomes the owner of the calf and the owner of a tree also becomes the owner of the fruits the tree bears. If you take the principle literally, the land must be divided among Brutus, Titus and Flavius, since the land has ‘grown’ onto their land. But how do the borders progress?

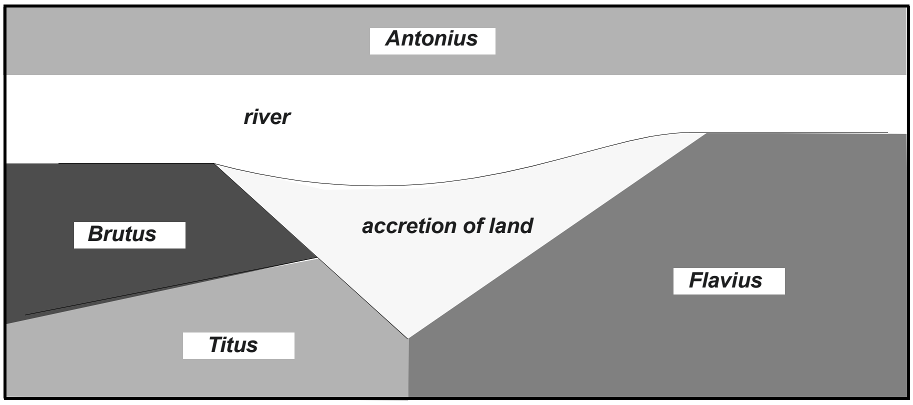


A page from Bartolus' book showing the construction of the bisector of an angle. The original manuscript is in the Vatican Library.

6. Brutus suggests extending the borders in the direction they run.
   a. Construct a possible division of the accretion of land according to this idea.
   b. Titus does not agree with this division. Which arguments could he use?

The Roman jurist Gaius (second century after Christ) already looked into this kind of problems in his *Institutiones*. But the Italian jurist Bartolus the Saxoferrato (1313–1357) was the first to realize that the problem of accretion of land was basically a mathematical one. Inspired by the conflict which he experienced during a vacation by the Tiber, he wrote a treatise under the title *Tractatus the fluminibus* (*flume*n is the Latin word for river) dealing with the problem of accretion of land. In this work he determined the 'conflict lines' with geometrical instruments for a large number of situations. He used the same criterion as we will in this chapter, namely: *the smallest distance to the old shores determines who will be the owner.*

7. Divide the accretion of land among the neighbors Brutus, Titus and Flavius according to that principle.

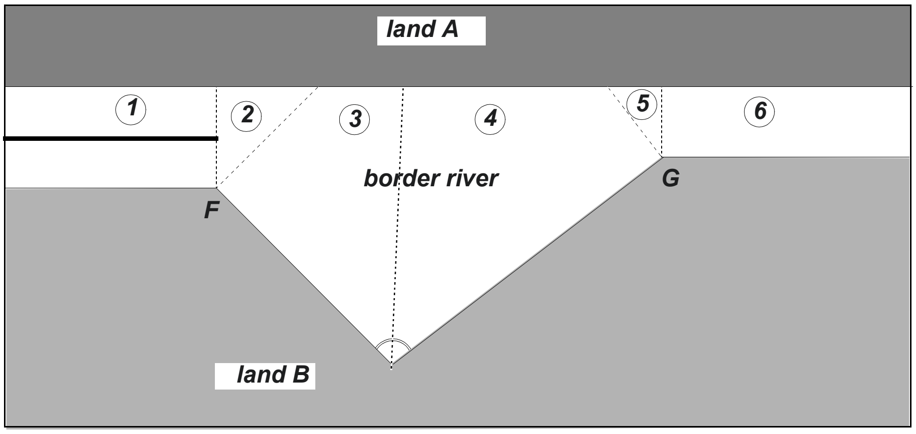


Antonius (other side of the river!) does not want to accept that he has no role in the division. He secured an expert's support and challenged the decision in a court of law. According to the expert, Antonius is also entitled to a small part of the alluvium. Since it is hard for him to cultivate that part, he is willing to sell it to one of his opposite neighbors.

8. Find out whether Antonius does have the right to part of the alluvium.

bisector
In the solution of division problems bisectors (whether or whether not of a straight angle) play an important role. This is also true for the following problem.

9. The border between the countries A and B must lie exactly in the 'middle' of the border river. To find this border the river is divided into six sectors through four perpendicular lines and one bisector (in the figure those five lines are dotted).



midparallel

- **a.** In sector 1 the borderline between A and B has been drawn. The line there is the midparallel of the two parallel banks.
  For which other sector is the borderline also a midparallel?
- **b.** In two sectors, the border lies on a bisector.
  Sketch these parts of the borderline.
- **c.** In the two remaining sectors the borderline is curved. Try to sketch those borders as accurately as possible.
- **d.** Why are these curved lines *not* circle arcs?
- **e.** Which role do the points F and G play for the 'edge' of B ?
  (Think about problems round iso-distance lines)

conflict line, conflict point

All three problems revolved round a set of P with the property:

$$\mathrm{d}(P, \textit{region } X) = \mathrm{d}(P, \textit{region } Y).$$

From here on we will call such a point a *conflict point*. The set of all conflict points is called *conflict line*. Thus:

The conflict line of two areas A_1 and A_2 is the set van all points P for which hold: $\mathrm{d}(P, A_1) = \mathrm{d}(P, A_2)$.

Examples:
- if the edges of the areas are two parallel lines, then the conflict points lie on the midparallel of these lines;
- if the edges of the areas are non-parallel lines, then the conflict points lie on the bisector of the angle, which (after extension of the edges is necessary) is enclosed by these lines.

If the edges of the areas have a whimsical shape, then the conflict line can still be fairly straight as was seen in the Norwegian-Danish border on the bottom of the North Sea.
A conflict line can also be curved; that was the case in exercise **9d**.
The demand for the conflict points there was as follows:
d(*point, point*) = d(*point, straight line*).
This and similar cases will be looked into in the next sections.

3. Construction of conflict points

In this section we start with a systematic research for the conflict lines between two simple areas. This research will be continued in section 4 with a computer practical and will be finished in following chapter. We investigate all cases in this table.

conflict line between ... and ...	point	straight line	circle
point	perpendicular bisector		
straight line		pair of angle bisectors, midparallel	
circle			

Two cells have been filled. Remark: in the table the (straight) lines and circles are not meant as shores of regions, but they are (abstract) areas themselves. That is why you find a *pair* of bisectors in the middle cell. Also a point is considered to be an area. The well known theorem of the perpendicular bisector between two points allies beautifully with the definition of a conflict line.

perpendicular bisector

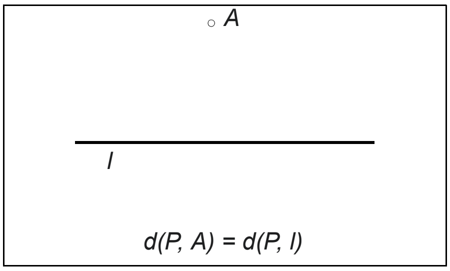
The perpendicular bisector van two points A and B is the set of all points P for which hold: $d(P, A) = d(P, B)$.

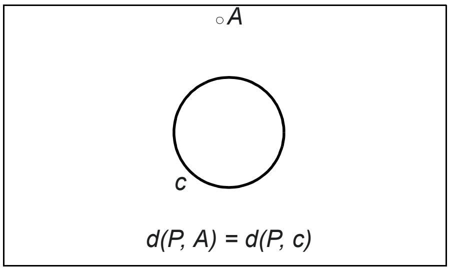
The midparallel and the bisector came up in the previous section.

10. Explain why the set of all conflict points of two intersecting lines is a bisector pair.

Due to the symmetry four cases are left in the table. Below you see two sketches for the cases ‘point – straight line’ and ‘point – circle’.

11 a. Try to sketch the conflict line in the lefthand figure. First consider:



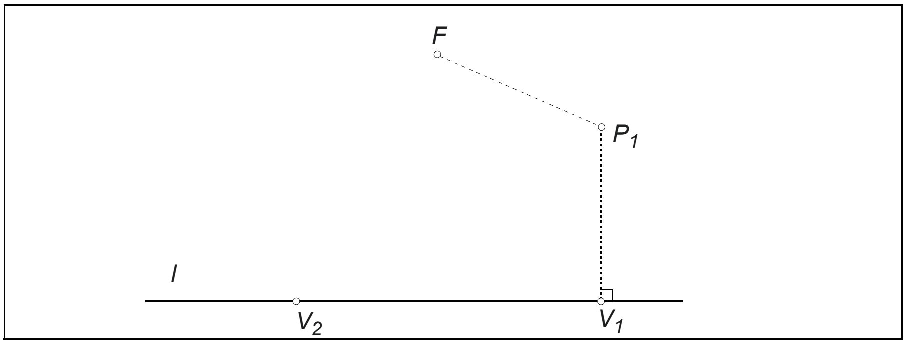


– for which conflict points can you indicate the exact location easily?
– how you find the foot of an arbitrary conflict point on line l?
– which symmetry the conflict line has;
– whether the conflict line is infinitely long or closed.

b. Sketch, after the same considerations, the conflict line in the righthand figure.

the case point-line

In a Geogebra-practical we want to study these cases extensively. But to get these conflict lines on the screen we have to come up with a construction method. We will first scrutinize the case point-line.



12. In the sketch you see at the point F and the line l a conflict point P_1 and its foot V_1.

a. How can you verify that $d(P_1, F) = d(P_1, V_1)$?

b. Is P_1 the only conflict point that has V_1 as a foot? Explain why/why not.

13. In the sketch another point V_2 is indicated on line l. Starting with V_2 we execute the construction again, but now we incidentally use some appropriate terminology

a. Sketch all points of which V_2 is the foot point on line l. What is this collection of points?

b. Draw all points equidistant from V_2 and F. What is this set of points?

c. Now mark the wanted point P_2: the conflict point of F and l with foot V_2 on l.

14. Construct a conflict point P for *each* point V on l.

a. Why does the intersecting of the two lines (13a and 13b) never go wrong?

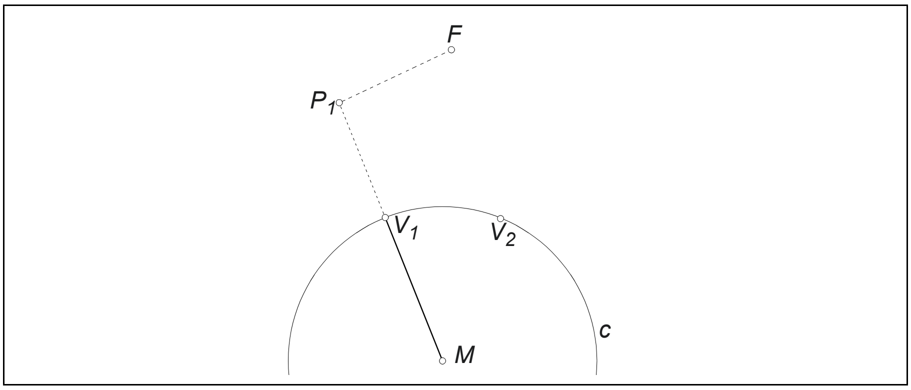
b. Summarize the method in a construction scheme, which we will execute later with GEOGEBRA. Complete this scheme.

How do you construct a point on the conflict line between a point and a line?

Draw the point F. Draw the line l. Draw a point V on l.	preparation phase
Draw .... Draw .... Mark the intersection and call this point P.	construction phase

With this scheme you can construct any conflict point. You show this construction in GEOGEBRA once very precisely and after that you let CABRI do the construction for all the other points on the line *l*, using the Locus-option

the case point-circle



15. Now the case point-circle. The figure shown below looks a lot like the figure in the case point-line. Point P_1 lies on the conflict line since d(P_1, F) = d(P_1, V_1).

a. V_2 is the foot of another different conflict point P_2. Demonstrate that construction of P_2 can be executed in practically the same way as was done for the case point-line.

b. Complete the schedule shown under **a**.

c. Do you find a conflict point P for *each* point V on the circle this way?

d. Do you find *all* points of the conflict line this way?

In applications interpreting the circle *c* as the edge of a region goes without saying. In exercise **15** you are so to speak looking for the conflict line between a large island and a tiny, but independent island, which lies a few miles offshore. However, point F can also lie in the interior of circle c. In that case you could think of an inland lake, wherein a 'very small' independent island is located.

How do you construct a point on the conflict line between a point and a circle?	
Draw the point *F*. Draw the circle *c*. Draw a point *V* on *c*.	preparation phase
Draw .... Draw	construction phase

16 a. Draw a circle with a point F in the *interior* area of the circle and sketch the conflict line between F and the circle.

b. Check whether or not you construct all points of this conflict line with the construction scheme of exercise **15b**. If not, make a appropriate scheme for it.

c. Suppose F happens to be the center of circle c. What is the conflict line in this case?

circle-line and circle-circle come later on
Both here and in the next GEOGEBRA-practical we do not yet pay attention to the two cases 'circle-straight line' and 'circle-circle'. Later on it will become clear that these cases can be reduced fairly easily to the familiar cases 'point-straight line' and 'point-circle'.

4. *Conflict lines with GEOGEBRA*

In this paragraph we construct the conflict lines with GeoGebra. GeoGebra was used in chapter 4 of part II of this book; the Locus and Trace options are very important here.

the case point-line

17. We first investigate the case 'point-straight line'. We found the steps of the preparation phase and the construction phase in the previous section. The final three steps are to clarify the drawing as much as is possible

How do you construct a point on the conflict line between a point and a line?	
1. Draw a point F.	
2. Draw a line l.	(preparation)
3. Draw a ***new*** point V on l.	
4. Draw the perpendicular line in V on l.	(construction)
5. Draw the perpendicular bisector of V and F.	
6. Mark the intersection of the two lines.	
7. Put names at the points F, V and P.	(embellishment
8. Draw the line segments FP and VP and color them	)

a. Execute this construction in GeoGebra.

b. Now drag away point V and look what happens with point P. This point now walks on the conflict line. Watch triangle FPV carefully. What does and what does not change?

c. Up to now you have seen point P moving, but you have not seen the conflict line itself.
Make this curve visible with the option LOCUS.
Choose this option. **First** click **on *P* and then on *V*.** The computer now sketches the conflict line.

locus = place
'Locus' is the Latin word for 'place'. With the option LOCUS you sketch the place of all points P when V moves along line l.You may also speak more modernly of the orbit of P, when moved by V.

18. Now investigate how the shape of this conflict line changes when you enlarge or decrease the distance from F to line l. You can also turn l and see what happens.

19. When you drag point V, the perpendicular bisector of FV turns.

a. Which role does this perpendicular bisector appear to play for the conflict line?

b. With the TRACE-option you are also able to sketch many positions of this perpendicular bisector in one go. Think which actions you need to take and let the computer sketch this locus.

parabola
The conflict line between a point F and a straight line l is called a *parabola*. In the next chapter we will give an exact geometrical definition of the parabola.

20. Write down in the survey what you already know about parabolas.

A parabola is the conflict line between ....
A parabola looks symmetrical in ....
The vertex of the parabola is the middle of..
The parabola looks narrower/wider as the distance between focus and directrix decreases.
The .... appear to be tangent lines to the parabola

the case point-circle, part 1

We now investigate the case point-circle.

We first investigate the situation where point F lies outside the circle.

21 a. Clean the drawing screen.

b. Adjust two lines in the construction scheme for the parabola, now you already have the construction scheme for the new conflict line.
In line 4 use the option **Ray through two points** to draw a 'half line', which begins in point M. First click on M and then on V, or the ray will go in the wrong direction.

c. Now execute the construction.

d. Do the same investigation steps as in exercise thus
– draw the conflict line with the LOCUS-option
– draw the perpendicular bisectors $pbs(F, V)$ with the TRACE-option.

For the parabola it looked as if the perpendicular bisectors filled the whole exterior of the parabola. In this case it looks as if the perpendicular bisectors leave a whole area free, which resembles the interior region of the conflict line!

22 a. Now look again what happens when you drag V fully around the circle. The perpendicular bisector and the half line do not always intersect.
Now adjust line 4 in the construction scheme in such a way that the computer does not sketch the *half* line, but the *whole* line through M and V.

b. Remove the half line from the screen and execute the new construction. If you now sketch the set of points P with the LOCUS-option, you notice that the sketched figure consists of two parts, i.e. has two *branches*. These two branches are each separately mirror symmetrical, but they also look like each other's images. Which two lines are the symmetrical axes?

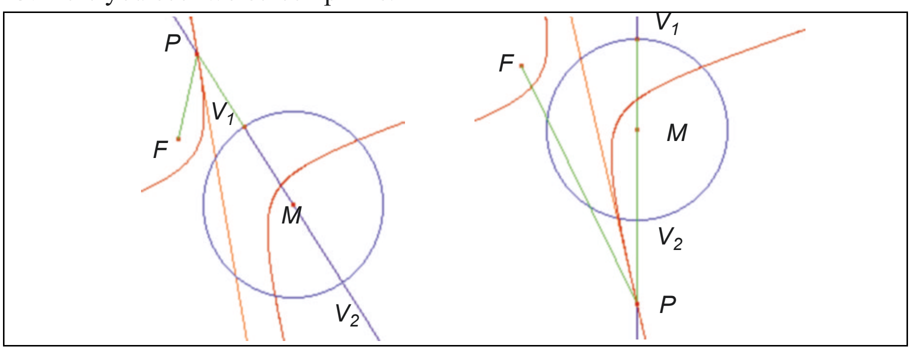
c. There are two points V on the circle for which the construction does not result in a point P on either of the branches. Try to find these two places on the circle.

d. Which shape does the triangle FVM have in these cases?

e. Which role does the perpendicular bisector $pbs(F, V)$ play for the conflict line in these cases?

f. And which role does the perpendicular bisector $pbs(F, V)$ seem to play if V lies elsewhere on the circle?

23. Here you see two screen prints.



In the lefthand figure you see P, a point of the branch, which lies completely outside the circle. In the righthand figure, P lies on the branch which intersects the circle. The line through P and M intersects the circle in points V_1 and V_2 in both figures. Answer these questions for each of the figures.

a. Which of the points V_1 and V_2 is, *according to the figure,* the foot of P on the circle?

b. And which of the two points is, *according to the construction,* the foot of P?

c. To which of these points does $d(F, P) = d(V, P)$ apply?

d. What can you say about these two distances for the other point?

hyperbola

The conflict line between a circle and a point outside the circle is one of the two branches of a *hyperbola*. In the next section we will give an exact definition of the hyperbola.

24. Insert your results in the following summary.

A hyperbola has two branches. The conflict line between a circle and a point F outside the circle is .... A hyperbola looks symmetrical in .... The hyperbola looks narrower/wider as the distance between point F and the circle decreases. The .... appear to be tangent lines to the hyperbola. For two points V on the circle the construction does not give a point P of the hyperbola. In these cases – triangle VFM is .... – $pbs(V, F)$ is .... of the hyperbola.

the case point-circle, part 2

25. Now drag point F to the interior region of the circle. If necessary, first increase the circle using the pointer and pull. Also execute all investigation steps for this situation, thus:

a. first drag V by hand, try to understand what happens

b. then sketch the conflict line with the LOCUS-option; change the place of A within the circle and investigate the consequences for the shape of the conflict line

c. sketch the perpendicular bisectors $pbs(A, V)$ with the LOCUS-option.

ellipse

The conflict line between a circle and a point A within the circle is an *ellipse*. In the next section we will give an exact definition of the ellipse.

26 **a.** An ellipse appears to have two symmetrical axes. Which two lines are these?

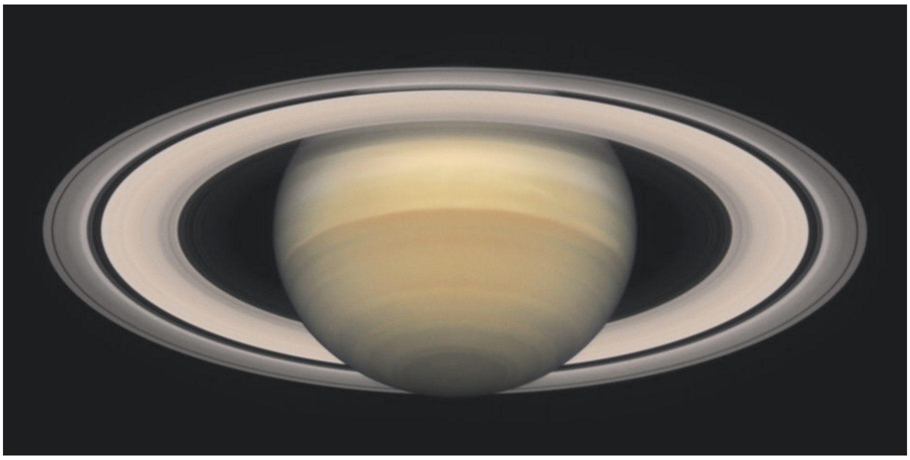
b. Draw these two lines with GEOGEBRA. Change the shape of the ellipse and look whether these two lines remain the symmetrical axes of the ellipse.

c. Now drag point F in such a way that it coincides with the center of the circle. What happens to the ellipse? And what can you say about symmetrical axes in this case?

27. Now summarize the most important results for the ellipse.

An ellipse is the conflict line between .... An ellipse looks symmetrical in... The ellipse looks narrower/wider as the distance between point F and the circle decreases. The .... appear to be tangent lines to the ellipse. ....

Chapter 2: Parabola, ellipse and hyperbola



Saturn, Lord of the rings. Photographed with the Hubble Space Telescope

The planet Saturn revolves around the sun in 29 years at a distance of 1.429.400.000 km and has a diameter of 121.000 km. Round its equator, a gigantic storm rages, big enough to make the whole earth disappear. See the white spot on the photo.

Saturn has a number of rings, which can be seen clearly on the photo. These rings are circles, but since you are looking at them from an angle, they are distorted.

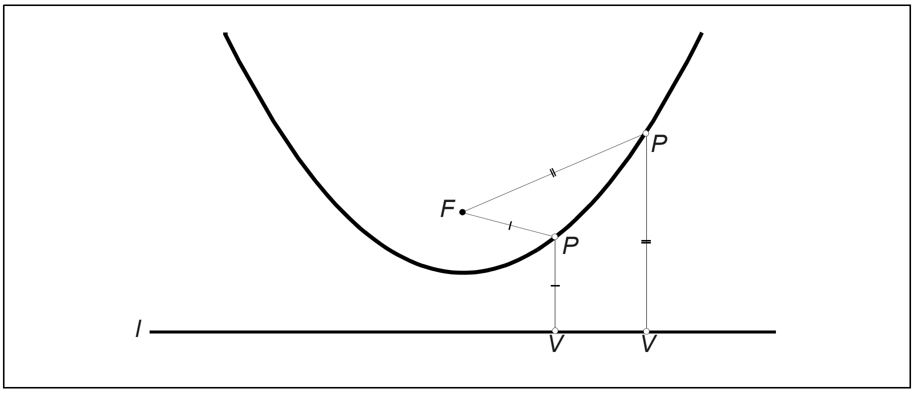
The Hubble Space Telescope rotates around the earth in a slight elliptic orbit at a height of approximately 600 km. Due to the good atmospheric circumstances at that height – no atmosphere, to be precise – it is possible to take very detailed photos. This telescope does not use lenses, but a curved mirror. The main mirror of the system is a parabolic mirror with a diameter of 2.5 meter. The way the parabolic mirror works is explained in the geometry of this chapter.

5. *Definitions and properties of parabola, ellipse and hyperbola*

what is a parabola?

definition parabola

Let F be a point and l a straight line, which does not go through F.
The set of all points P for which hold: $d(P, F) = d(P, l)$ is called a *parabola.*
F is called the *focus* of the parabola, l is called the *directrix* of the parabola.

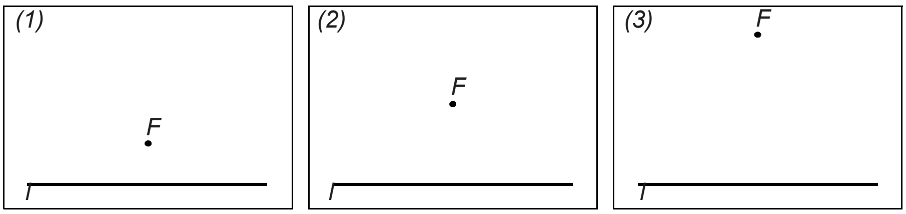


That F is called a focus originates from a beautiful application: the parabolic mirror. This will be explained in section 26 of this chapter.

1. Is the definition much different from the description given earlier on?

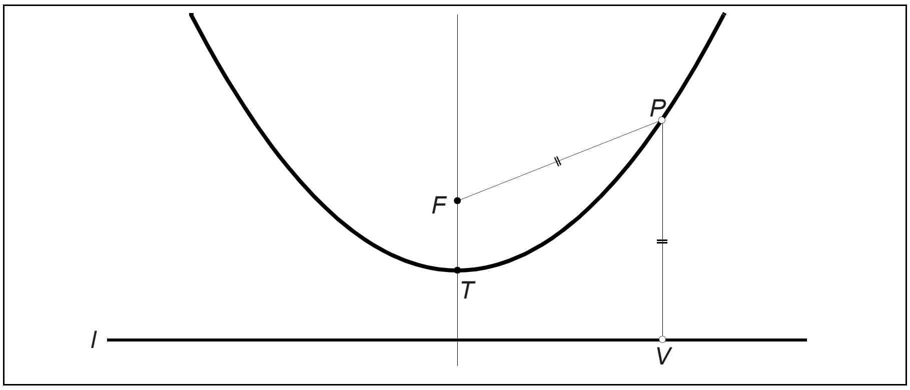
symmetry, vertex
A parabola is thus completely determined by a point F and a straight line l (not through F). The figure formed by F and l is symmetrical, and therefore so is the parabola. The point of the parabola, which lies on the axis of symmetry, is called the *vertex* of the parabola. It is the point of the parabola which has the smallest distance to the directrix and focus.

2. If you have the vertex and a couple of points (like the ones in exercise **11 a**, page 243), you can already make a quite reasonable sketch of the parabola. Do this in the three situations shown below.



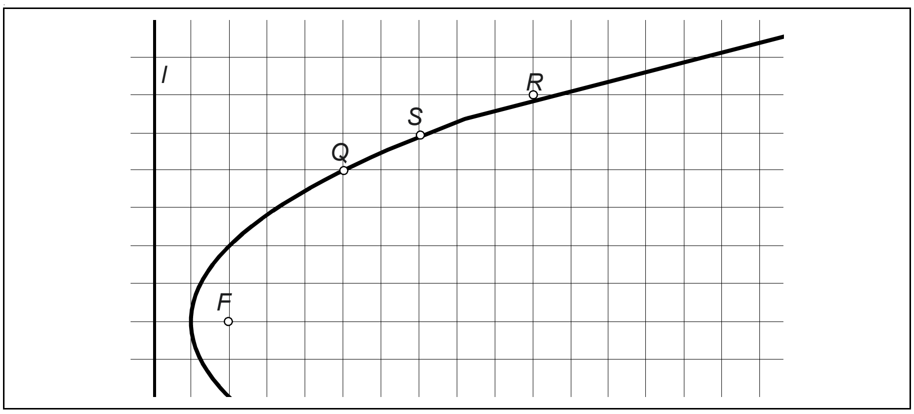
all parabolas are similar
It looks as if you can make a distinction between 'narrow' and 'wide' parabolas, but that depends! When you enlarge the parabola of (1) so that the distance of *F* to *l* is equal to the distance in (3), then you get the parabola of (3). By zooming in or out you can make any two parabolas equal to each other. Therefore we say that all parabolas *are similar.*

3. Here the axis of symmetry is indicated in the figure.



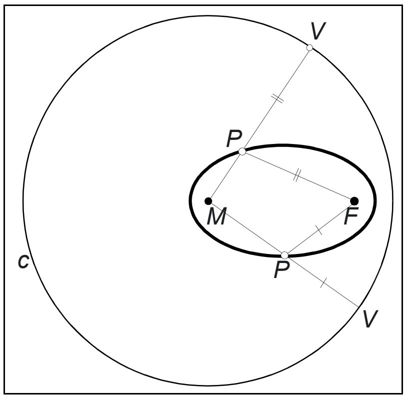
a. Check whether the figure is drawn correctly by sketching a circle with center *P,* which is tangent to *l*. What should be correct now?
b. Show that there is another parabola with the same directrix and axis of symmetry, which also goes trough *P*; determine the focus of this parabola precisely.

4. Here a parabola is sketched with a square grid in the background. Focus and directrix lie exactly on this grid



Show through computation that the grid points *Q* and *R* lie on the parabola and *S* does not according to the definition of the parabola.

what is an ellipse?



We described the ellipse as the conflict line between a circle c and a point F within the circle. Just like the parabola, you can immediately make a definition from the description. It would probably look like this:

> *Let c be a circle and F a point in the interior region of c.*
> *The set of all points P for which hold: $d(P, F) = d(P, c)$ is an ellipse.*

But this is not very pretty due to the following.
An ellipse is completely determined by a circle c and a point F (within c). The figure formed by F and c has one axis of symmetry (the line which connects F with the center M of c). This line is therefore also a axis of symmetry of the ellipse. The ellipse turns out to have a second axis of symmetry, namely the perpendicular bisector of MF. This cannot be seen directly from looking at the temporary definition, but will become clear in the next exercise.

5. Look at the figure on page 275. Circle c has radius r.
  a. Explain that the condition $\mathrm{d}(P, F) = \mathrm{d}(P, c)$ is equivalent to:
  $\mathrm{d}(P, M) + \mathrm{d}(P, F) = r$.
  b. Explain why the perpendicular bisector of MF is an axis of symmetry of the ellipse.

In the second condition of exercise **5a** the points F and M do no longer play different roles; this was the case originally. In order to emphasize the equivalence, from now on we will call these points F_1 and F_2. So we have finally:

standard definition of the ellipse

> F_1 and F_2 are two different points.
> The set of all points P for which the *sum* of the distances to F_1 and F_2 is constant, is called an *ellipse*.
> F_1 and F_2 are called the *foci* of the ellipse.

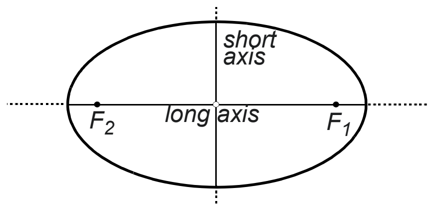
Please note: the constant must be bigger than the distance between the foci.
We will return to the term *foci* later.

6. Here is a way to create an elliptical flowerbed in the garden.

Put two poles in the ground at two meters from each other. Knot the ends of a nine meter long rope together so you get a loop. Put the loop around the poles and pull the rope taut with a stick. Move the stick around and scratch an oval in the ground with the point.

a. Why does a flowerbed made in this way have the form of an ellipse?
b. How long and how wide will this ellipse be?
c. How do the length and width of the ellipse change if you put the two poles closer than two meters from each other in the ground?
d. It is also possible to make a three meter wide ellipse with the rope. At what distance from each other should you put the poles in the ground?

long and short axis, vertices
You have seen that an ellipse has two symmetrical axes:
– the line that connects the foci;
– the perpendicular bisector of the two foci.



The segments of the symmetrical axes which lie inside the ellipse are called: *long axis* and *short axis*. Both foci lie on the long axis. The four intersections of the axes with the ellipse are called the *vertices* of the ellipse.

7. Of a point P it is given that it lies on an ellipse with foci F_1 and F_2 and that $d(P, F_1) = 7$ and $d(P, F_2) = 3$ hold. Additionally $d(F_1, F_2) = 8$ holds.
How long are both axes (long axis and short axis) of the ellipse?

8 **a.** Two different ellipses do not have to be similar (like two parabolas). That is, (most of the time) it is not possible by zooming in or out to make two ellipses equal to each other. This follows for example from the temporary definition where an ellipse is determined by a circle c and a point F. Explain this.
b. What kind of shape does the ellipse have when F coincides with the center of c? And what do you then know about the long axis and the short axis?
c. Suppose you have two similar ellipses. Which property should axes of these ellipses have?

9. Given two points F_1 and F_2 and a number $r > d(F_1, F_2)$. c_1 is the circle with center F_1 and radius r; c_2 is the circle with center F_2 and radius r.

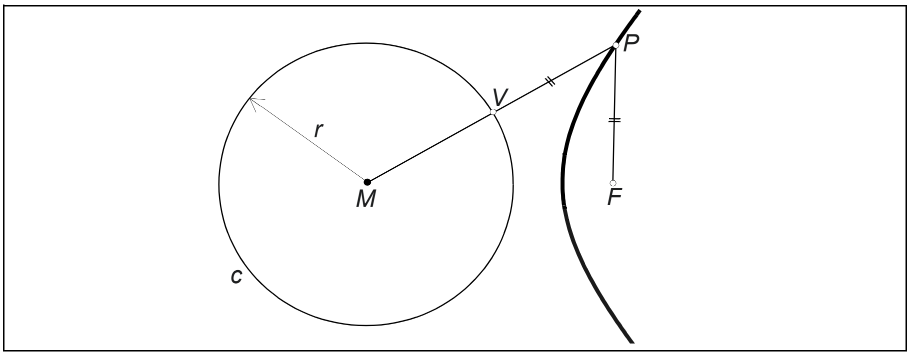
Show that the conflict line of F_1 and c_2 coincides with the conflict line of F_2 and c_1.

director circle

In the standard definition of the ellipse the circle *c* does no longer play a role. When constructing an ellipse and in several reasonings, it will come in very handy. That is why this circle has a special name: it is a *director circle* of the ellipse. It is a circle with radius, the constant from the definition, and center, one of the foci. Thus there are *two* circles directrices.

what is a hyperbola?

The conflict line of a circle and a point outside that circle was just one branch of a hyperbola. This is why we let the standard definition of the hyperbola differ even more from the conflict line description than was done for the ellipse. In the figure below, you see one hyperbola branch, sketch according to the conflict line description.



Such a hyperbola branch is completely determined by a circle *c* and a point *F* outside *c*. The figure formed by *F* and *c* has one axis of symmetry (the line, which connects *F* with the center *M* of *c*). This line is therefore also an axis of symmetry of the hyperbola branch. The point of the hyperbola branch, which lies on the axis of symmetry, is the vertex of the hyperbola branch.

10. Explain that the condition: $d(P, F) = d(P, c)$ is equivalent to:

$$d(P, M) - d(P, F) = r.$$

With this last condition we are close to the definition of a hyperbola. But the points *M* and *F* here are not, as for the ellipse, interchangeable. For the sum of two distances the order is not important, for a difference it is.

11 a. Verify that the set of points P which satisfy $d(P, F) - d(P, M) = r$ are in fact a new hyperbola branch, namely the mirror image of the hyperbola branch which belongs to the condition $d(P, M) - d(P, F) = r$.

b. For which circle and which point is the new hyperbola branch the conflict line?

absolute difference

The hyperbola branch meant in **11 a** forms a complete hyperbola together with the other branch. Or: the point P lies on the hyperbola when $d(P, M) - d(P, F) = r$ *or* $d(P, F) - d(P, M) = r$.
In brief: P lies on the hyperbola when:

$$d(P, M) - d(P, F) = \pm r$$

Or, using the *absolute value:*

$$| d(P, M) - d(P, F) | = r$$

Say:

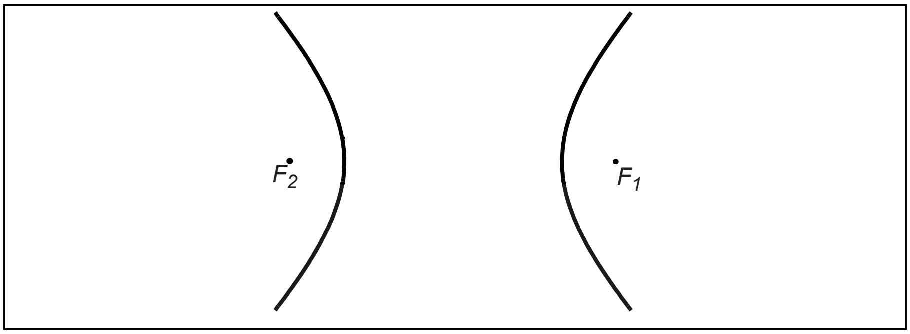
for all points P the absolute difference of d(P, M) and d(P, F) is equal to r.

To emphasize that the points M and F are equivalent in this condition, we call them F_1 and F_2. The definition of the hyperbola, as can be found in most books, finally comes out as:

standard definition of the hyperbola

> F_1 and F_2 are two different points.
> The set of all points P for which holds:
> $| d(F_1, P) - d(F_2, P) | = r$
> is called a *hyperbola.*
> F_1 and F_2 are called the *foci* of the hyperbola.

12. Here a hyperbola with foci F_1 and F_2 is sketched.



a. Determine what the used value of r is by measuring.
b. Choose a point P on the branch at F_1 and a point Q on the other branch, so that the quadrangle F_1PF_2Q is a parallelogram. Which role does the middle of PQ play?
c. Now so that F_1PF_2Q is a rectangle; if you use your protractor this should not be difficult.
d. Can you also make F_1PF_2Q square?

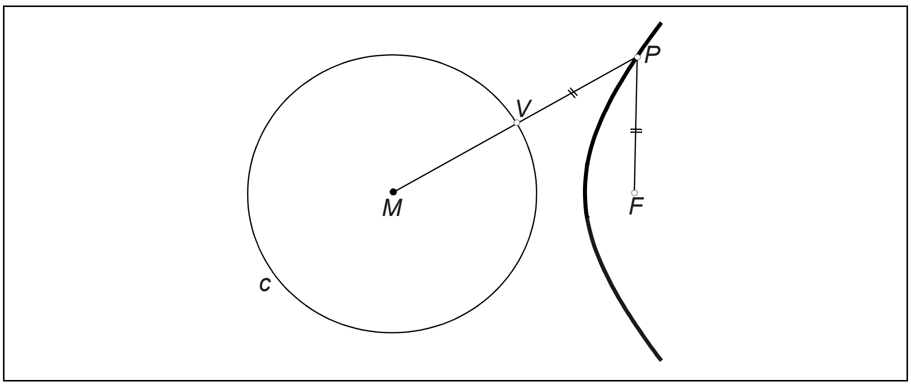
13. When speaking about a hyperbola it is somewhat off to talk about long and short axes, but it is reasonable beyond a doubt to talk about axes. Why?

14. Describe what could be meant with the concept *director circle* of a *hyperbola*.

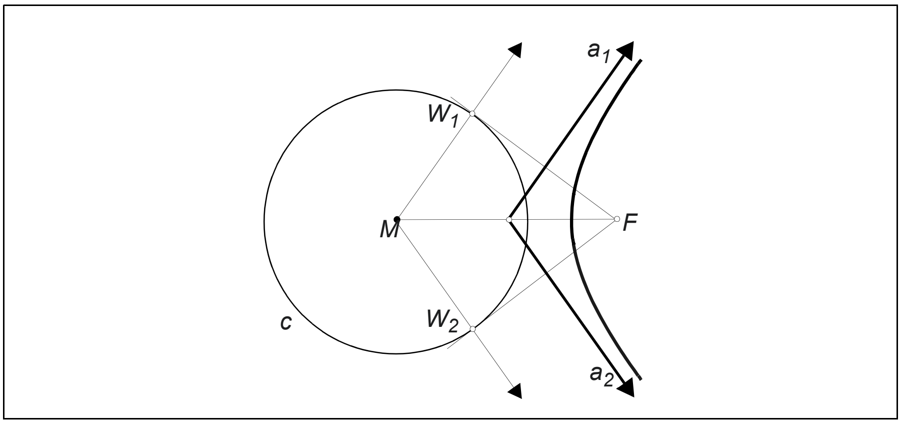
15. All parabolas are similar. Not all ellipses are similar. What about hyperbolas?

the asymptotes of the hyperbola

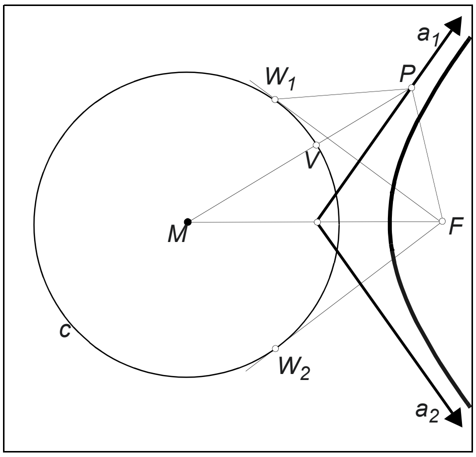
A specific detail, which you discovered earlier for the hyperbola branch as conflict line, is that the feet of the conflictpoints P only lie on a part of the circle. Just look at the figure which belongs to the conflict description.



In the next figure, W_1 and W_2 are just the edge points where V must stay between. W_1 and W_2 are the tangent points of the tangent lines from F to the circle. The line MW_1 is parallel to the perpendicular bisector of FW_1 and thus does not yield a P. The same can be said about MW_2 and the perpendicular bisector of FW_2.



The perpendicular bisectors of FW_1 and FW_2 are called the *asymptotes* (say a_1 and a_2) of the hyperbola branch. When you move the foot V over the arc W_1W_2, then point P will move along the hyperbola branch. When V comes close to W_1 or W_2, then P moves very far away; then the distance to a_1 (of a_2) becomes very small. This distance approaches 0 as V approaches W_1 (of W_2). We will look into this in an extra exercise (number **17**).



16. It looks as if the hyperbola branch is completely formed *within* the angle made by the half lines a_1 and a_2. But you are not able to see what happens when V lies very close to one of the edge points.

a. To prove that the hyperbola branch does stay within the angle, you take an arbitrary point P on one of the asymptotes. For that, it can be proven that it belongs to the sphere of influence of c, in other words that:

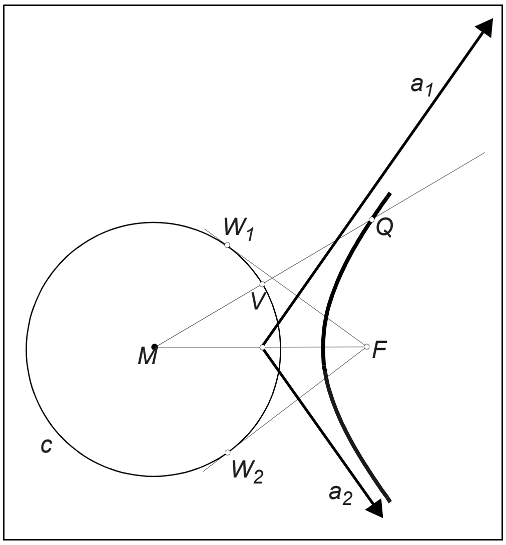
$$d(P, V) < d(P,F)$$

Prove the latter.

b. Point S lies on the same side from a_1 as W_1. Show that S certainly does not lie on the indicated branch of the hyperbola.

For a complete hyperbola – with both branches – the asymptotes are whole lines, it should be obvious.

extra, investigation asymptote



17. That the hyperbola comes 'infinitely close' to a_1 is somewhat more difficult to see. Here is an idea for a proof

a. Add the perpendicular bisector of VF. It intersects W_1F in K.

b. What do you know about $\angle QKW_1$?

c. Q looks closer to a_1 than K. Why?

d. What happens with K when V approaches W?

e. Now finish the proof that Q can be as close to a_1 as you want.

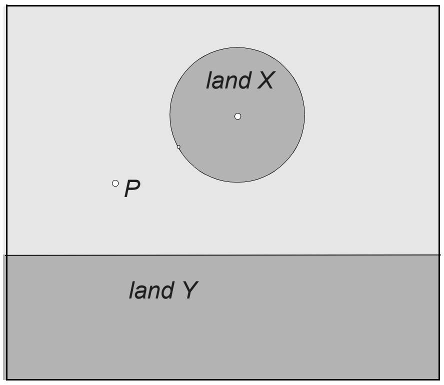
6. Reduce and split Conflict Lines

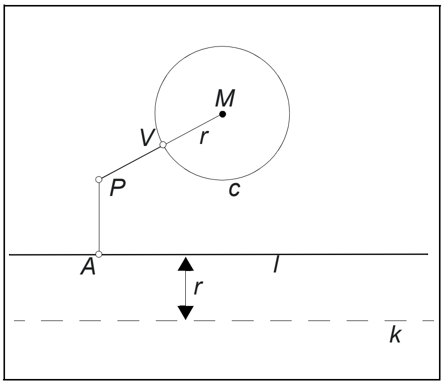
The table in section 3, page 249, has now been filled out for the most part. In this section you will learn how you can reduce the two missing cases to familiar situations.

conflict line between ... and ...	point	straight line	circle
point	perpendicular bisector	parabola	ellipse, hyperbola branch
straight line		pair of bisectors, midparallel	

conflict line between ... and ...	point	straight line	circle
circle			

the case line-circle





18 a. In the figure on the left a circular island lies opposite the straight shore of a neighboring country. Sketch the conflict line in this figure.

b. In the figure on the right, there is, besides line *l* and circle *c,* also a line *k,* which runs parallel to *l* at distance *r*. For each point *P* on the conflict line, the following holds: $d(P, c) = d(P, l)$ thus $d(P, V) = d(P, A)$.
Explain that $d(P, M) = d(P, k)$ also applies to *P*.

c. So, what shape does the wanted conflict line have?

Reduction

In the last exercise we applied an important technique. We reduced the circle, so to speak, to its center. After all, we were no longer looking for the conflict line between a *circle* and a ... , but for a conflict line between a *point* and a .... As a result of this we had to replace the distance $d(P, c)$ with $d(P, M) - r$. The equation $d(P, c) = d(P, l)$ became $d(P, M) = d(P, l) + r$.
We could think of $d(P, l) + r$ as the distance from *P* to a line *k*, which runs parallel to *l* at distance *r*.
Thus: the condition $d(P, c) = d(P, l)$ is equivalent to $d(P, M) = d(P,k)$.
In this way we reduced the problem to a familiar situation. We will apply this technique also to exercises **20** and **21**.

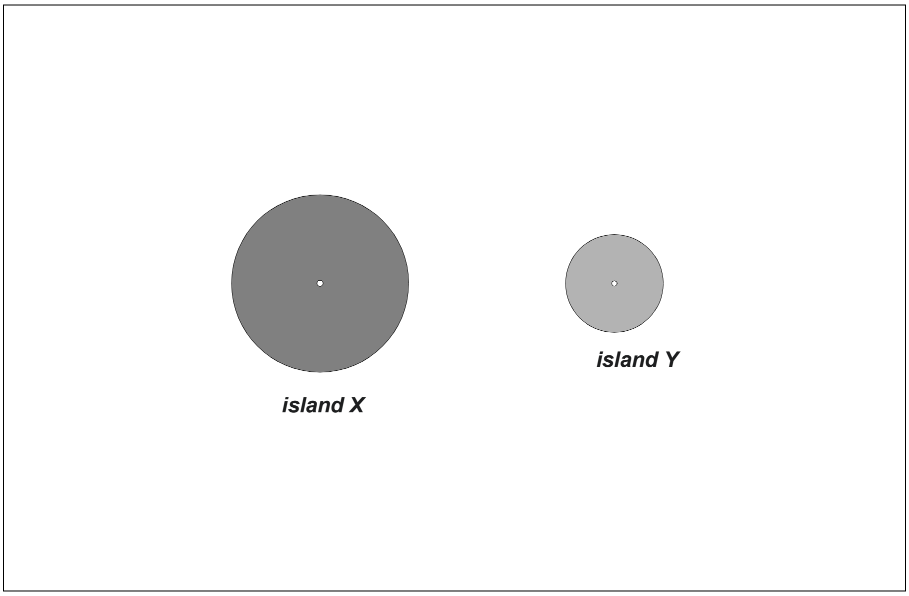
case circle – circle

In the case circle – circle we limit ourselves to the situations where the circles do not have common points. We look at two special cases first.

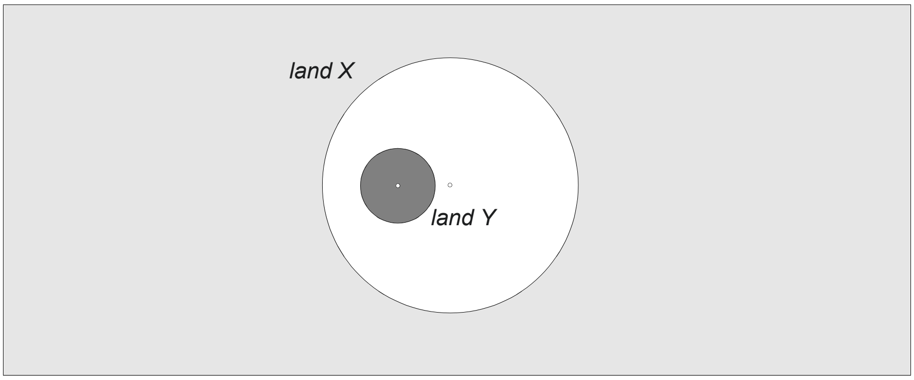
19 a. Which shape does the conflict line between two circles of the same size, which lie outside each other, have?

b. Which shape does the conflict line between two circles with different radius, but with the same center, have?

20. Now determine the conflict line between these two circular islands.



A good approach is:
– first draw a couple of points of the conflict line
– formulate an assumption: which familiar conflict line could it be?
– make a new sketch with relevant data
– reduce the problem to a familiar situation.



21. The circular island Y lies in a circular inland lake of land X.
  a. Argue which shape the conflict line of these two lands has.
  b. Draw several points of the conflict line and then sketch the conflict line.

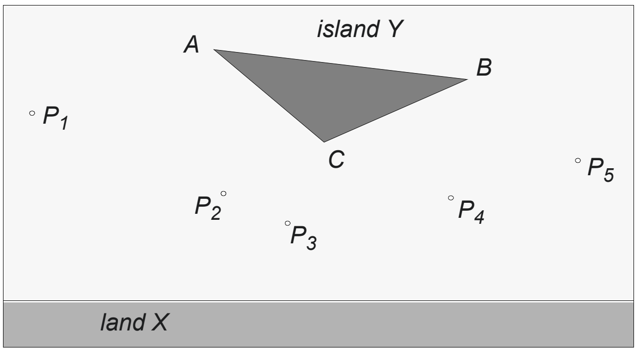
22. How does the shape of this conflict line change if
  a. the radius of island Y increases/decreases?
  b. the distance between the centers of the two circles increases/decreases?

23. Process the results of exercises **20** and **21** into the survey scheme.

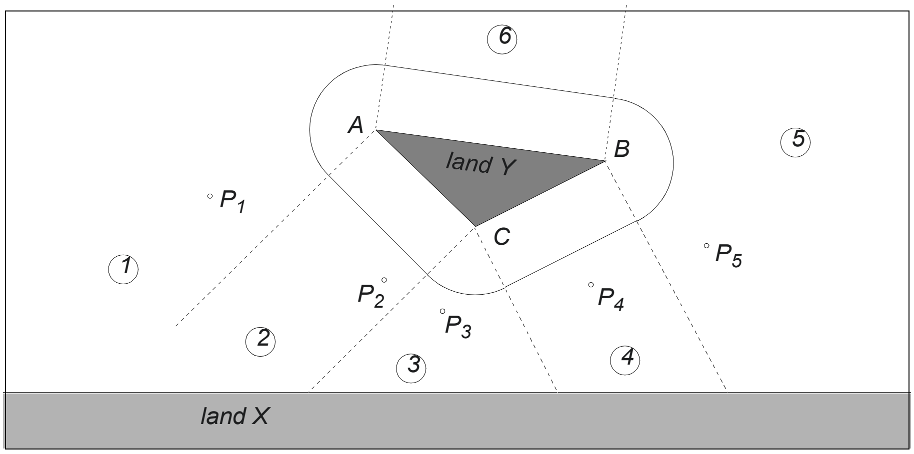
splitting up

We now look at a few situations where we need to split up the to be divided region in sectors. You also encountered such problems in chapter 5 of DISTANCES, EDGES AND REGIONS for iso-distance lines. Then capes played an important role.

24. A triangular island Y lies opposite the straight-lined shores of land X. In the region between the two lands five points are indicated.
  a. For each of these five points, draw the feet on the edge of land X and also on the edge of island Y.
  b. Investigate for each of the points whether it lies closer to land X or to land Y.



  c. Sketch how the conflict line runs approximately between the two lands. Make sure that the points P_1 trough P_5 lie on the right side of the conflict line.



25. In this figure an iso-distance line for the island has been sketched. This line consists of circle arcs and straight line segments. The segments of the iso-distance line join to each other on the dotted lines. These dotted lines are of great importance for the conflict line.

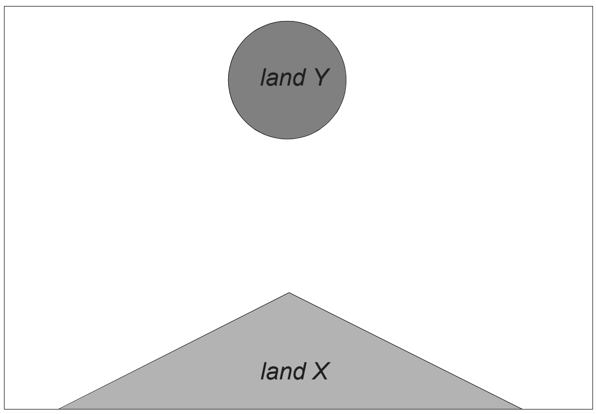
a. The dotted lines divide the region round the island in six sectors. Indicate for the sectors 1 through 5 which shape the conflict line has in that segment. First think about which case from the survey scheme you are dealing with.

b. Sector 6 lies 'behind' the island. Investigate whether the conflict line also continues in this sector.

c. Draw an improved version (compared to exercise **24**) of the conflict line in the figure.

extra exercise

26. In this situation the conflict line between the two lands consists of three parts. Sketch this conflict line. Use the techniques you have seen in this section for the exact description of the conflict line. Add a clear argumentation.



7. *The tangent line property of the parabola*

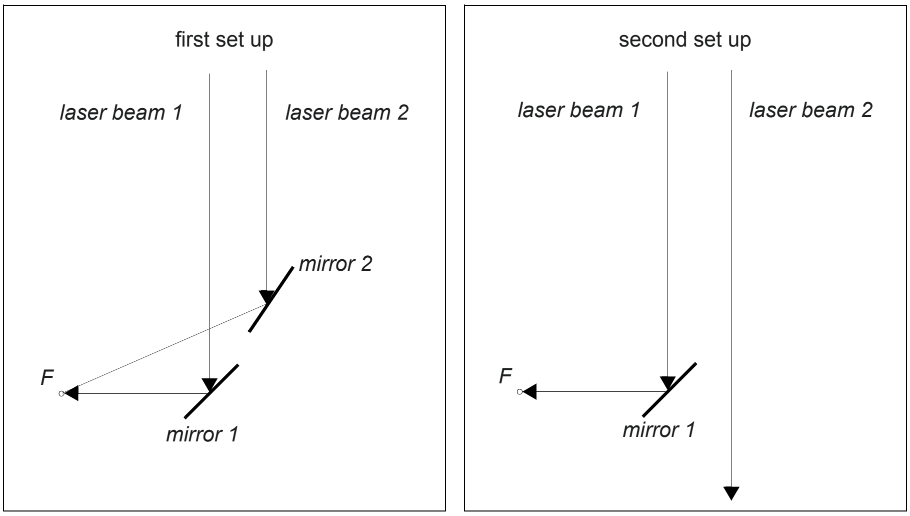
In practical applications of parabolas, ellipses and hyperbolas the special *tangent line property* of these curves is often being used. You have *seen* that the perpendicular bisectors, which appeared in the constructions were tangent lines to the parabola, the ellipse and the hyperbola. In this section we will prove this and then directly use the properties of the tangent lines in important applications of the three figures. The most important application has to do with reflection. **angle of incidence = angle of reflection**

law of reflection
As we know, the following *law of reflection* applies to plane mirrors:

angle of incidence = angle of reflection

If you have a plane mirror and you shine a parallel beam on it, a parallel reflected beam will reflect. This is not spectacular.
Our first goal is to design a mirror, which converts a parallel beam into a converging beam, i.e. into a beam which goes through one point.

In the figure on the left you can see two mirrors and two laser beams. Laser beam 1 (2) is reflected via mirror 1 (2). Both rays reach point *F*.

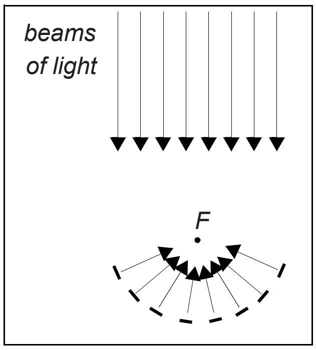


27 a. In the figure on the left, how can you check with your protractor that the reflection rays indeed go through *F*?

b. Draw in the right figure also a mirror 2 so that laser beam 2 is reflected on point *F* (on the place of the point of the arrow of laser beam 2).

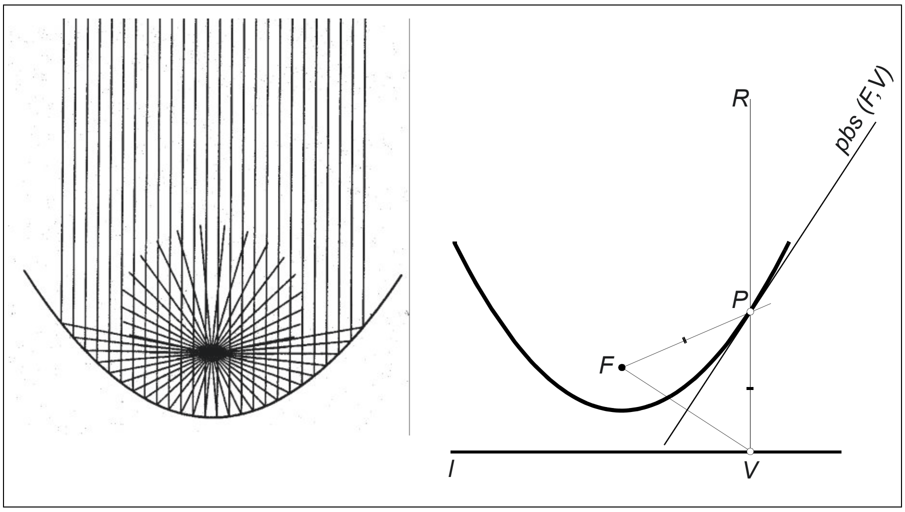
c. What do you notice when you compare the two positionings of mirror 2?

curved mirror



In principle you could reflect parallel laser beams using a hundred plane mirrors to one specific point. This is not very practical, for example the mirrors should not be in each other's way. With a *beam* of rays this is impossible. Then you would need infinitely many very small mirrors, which all together form a curved mirror. These need to ground in a certain shape. Also, for a curved mirror the law of reflection stands. With angle of incidence and angle of reflection we then mean the angles with the *tangent line* to the curve!

In the figure on the left below you see a mirror with the desired curved shape: after reflection, all rays of light of the parallel beam converge to one point. Next to that you see a sketch of a parabola as you have seen before.



A strong impression exists that the parabola is the desired figure. A ray of light would approach from R, parallel to the axis of the parabola; the ray would hit the parabola at P and end up at F after reflection. Since this would be true for all rays parallel to the axis, the beam does converge in F after reflection.

First two things need to be checked:
I : the line $pbs(F, V)$ is indeed the tangent line to the parabola
II : the lines RP and PF behave in relation to this perpendicular bisector according to the law of reflection: *angle of incidence = angle of reflection.*

28. The second point is the easiest to prove. Do that yourself first.
Next you are going to prove that the perpendicular bisector of *F* and *V* lies completely outside the parabola except for point *P*.

Given: *P* is a point of the parabola with focus *F* and directrix *l*. *V* is the foot of *P* on *l*. To prove: *pbs*(*F*, *V*) lies outside the parabola except for *P*.

29. Choose an arbitrary point *Q* on *pbs* (*F*, *V*), other than *P*.
Now prove that *Q* lies outside the parabola, i.e. that d(*Q*, *F*) > *d*(*Q*, *l*).
Hint: draw the line segment that realizes the distance d(*Q*, *l*).

a consideration in the margin
From this proof it follows that *pbs*(*F*, *V*) lies outside the parabola except for *P*. Is it therefore also the tangent line? is the question now. Since the parabola is a smooth figure, it is hard to avoid that impression. But you actually should prove that there exists one and only one such line, which lies outside the parabola except for point *P*. Proving that this is the case would go too far at this moment. We will revisit this later on, when it will be shown that:
– the graph of $y = x^2$ is legitimately indicated as 'parabola' and thus that you can find a focus *F* and a directrix *l*;
– the graph of a linear approximation of $y = x^2$ in a point *P coincides with the perpendicular bisector pbs*(*F*, *V*), where *V* again is the foot is of the perpendicular line from *P* on *l*.
Or put another way: the tangent line as found in this chapter is the same as the one in differential calculus.

You can assume from now on that *pbs*(*F*, *V*) is a tangent line to the parabola. I.e.: you can use the assertions made in I and II on the previous page from now on.

Everything about the *tangent line property of a parabola* is summarized in the next theorem:

tangent line property of the parabola

***P* is a point on the parabola with focus *F* and directrix *l*.** **The tangent line in *P* to the parabola makes equal angles with line *PF* and the perpendicular line through *P* on *l*.**

The physical meaning of the preceding is:

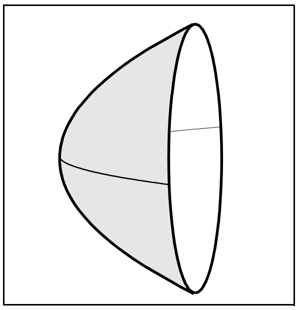
All rays of light, which approach a parabolic mirror parallel to the axis, will be reflected in the direction of the focus of the parabola.

30. Fill in:

If rays ***go out*** *from the focus F, then the rays reflected by the parabolic mirror form a ....*

paraboloid

The headlight of a bike has the shape of a *paraboloid*. A paraboloid is a spatial shape which arises from rotating a parabola about its axis of symmetry.



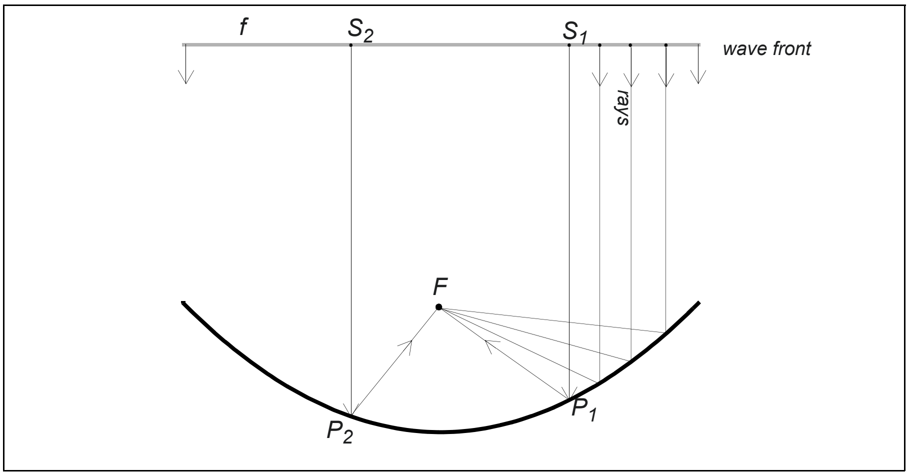
31. What can you say about the beam of light from such a bike lamp:

a. if the filament of the light is exactly located in the focus?

b. if the filament of the light is located a bit behind the focus?

parallel wave fronts

Dish aerials and radio telescopes also have the shape of a paraboloid (see the picture on the initial page of part III of this book). When receiving radio or tv signals, it is of importance that all 'rays' reach the focal point at the same moment; only then is optimal reception possible. You could imagine that a wave front (line *f* in the figure below) consists of points, which all move at the same speed in the same direction, namely the direction of the arrows.



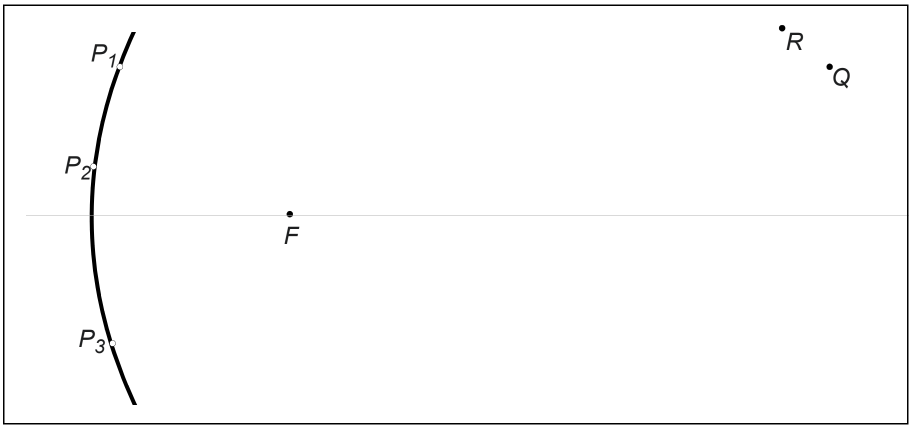
32. Explain that for a parabolic antenna all rays of light travel the same distance and that therefore the condition is met. Or: show that the paths S_1P_1F and S_2P_2F have the same length.

Nowadays you can see as many parabolic antennas as you want: satellite antennas are attached to many houses. They are aimed at satellites that have a fixed position above the equator. That means that in our regions the satellite antennas point south. In the city you do not need a compass to determine your orientation.

33. Can you name a few other applications of parabolic antennas?

construct oblique rays

34. Below, you can see a parabolic mirror; the axis of symmetry is indicated.



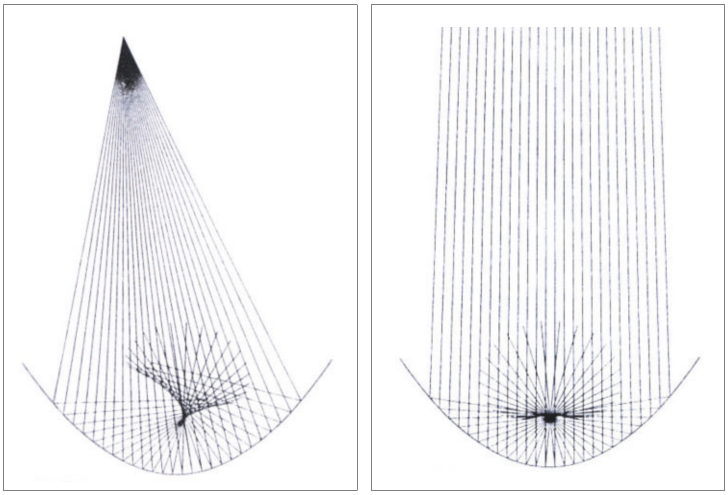
a. A ray, which falls from Q to P_1, is parallel to the axis. Draw the ray and its reflected continuation.

b. The ray, which falls from R on P_1, also answers to the law of reflection. Draw the ray and its reflected continuation. Now you need to sketch the tangent line in P_1 or its perpendicular line, i.e. the bisector of QP_1F. Use the protractor.

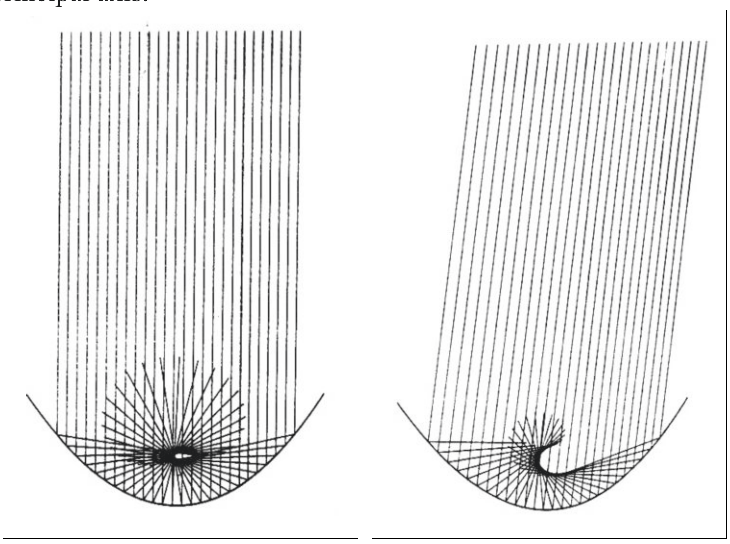
c. Also draw rays approaching from R to P_2 to P_3, and their reflected continuations.

d. What do you notice about the rays approaching from R, and their reflected continuation?

From the preceding you could draw the conclusion that non-parallel rays do not converge. In the figures below this can be seen again, but the right figure also shows: if the incident rays come from a point very far away, then convergence does occur in a good approximation because there is parallelism in good approximation.



35. Below you see two cases where the beam is parallel, but not in the direction of the principal axis.



Give nuanced comments.

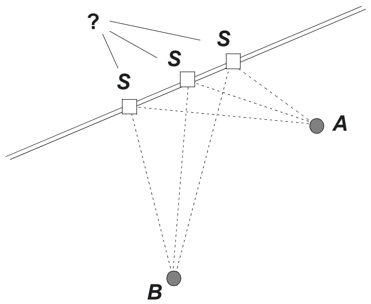
8. *The tangent line properties of ellipse and hyperbola*

the case of the ellipse

While investigating the tangent line property of the ellipse we can unexpectedly reuse an old problem.

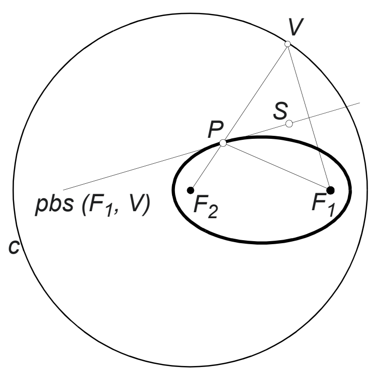
the reflection principle
Maybe you recognize the following optimization problem.

Cities A and B lie on the same side of a railroad. One station needs to be built for both cities. The bus company, which needs to bring the people from A and B either to or from this station, wants the position of the station to be chosen so that distance to the sum of the distances to A and B is as small as possible.



36. Solve this problem again by applying the reflection principle.

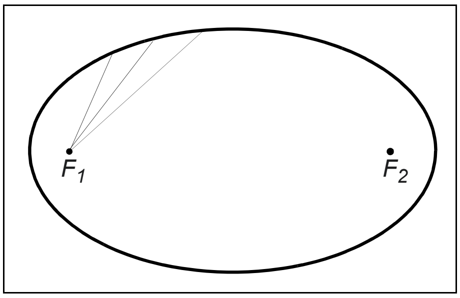
The sketch on the right contains the same elements. A and B are now called F_1 and F_2. The optimal place of the station (point P in the figure) is the point where the line l (in the figure *pbs*(F_1, V)) touches the ellipse with the foci F_1 and F_2. r is here the radius of the director circle c.



37 a. Prove that each other point S of the line l lies outside the sketched ellipse.
b. How does it follow from this that P is the desired point?

38. Now prove the theorem about the tangent line property of ellipses. (Here a hole in the proof will remain unplugged. Make the same assumption that we did for the parabola.)

P is a point on the ellipse with foci F_1 and F_2.
The tangent line in P to the ellipse makes equal angles with the lines PF_1 and P



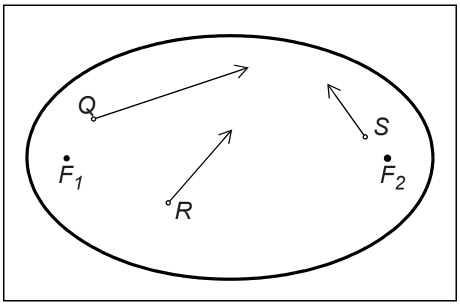
39. Now apply the law of reflection to a concave elliptic mirror. What happens to the rays of light which depart from focus F_1 and are reflected by the mirror?

40. *At the dentist*. Over the patient's head there is a bright lamp. The light of the lamp is reflected in a mirror and aimed at the oral cavity of the patient.

a. The dentist must perform his actions in a very small area, approximately 1 cm^2. Suppose that he wants the light concentrated here as much as possible, what shape should the mirror have?

b. Too large differences in light intensity between the work area and the direct surroundings are tiring for the eye. Also, too much heat would be concentrated on an already ill molar. That is why the dentist would like a larger area to be lighted. How can that be achieved?

41. In the Taj Mahal in India there is a so-called *hall of private audience*. Bridal couples who visited the Taj Mahal in the past had to stand on two special places in the hall, fifteen meters apart. The groom whispered the vows of eternal love, and his words were only heard by the bride. Give your comments, on love and ellipses.



42. The case of a ray of light within a reflecting ellipse that does not start in a focal point is also interesting.

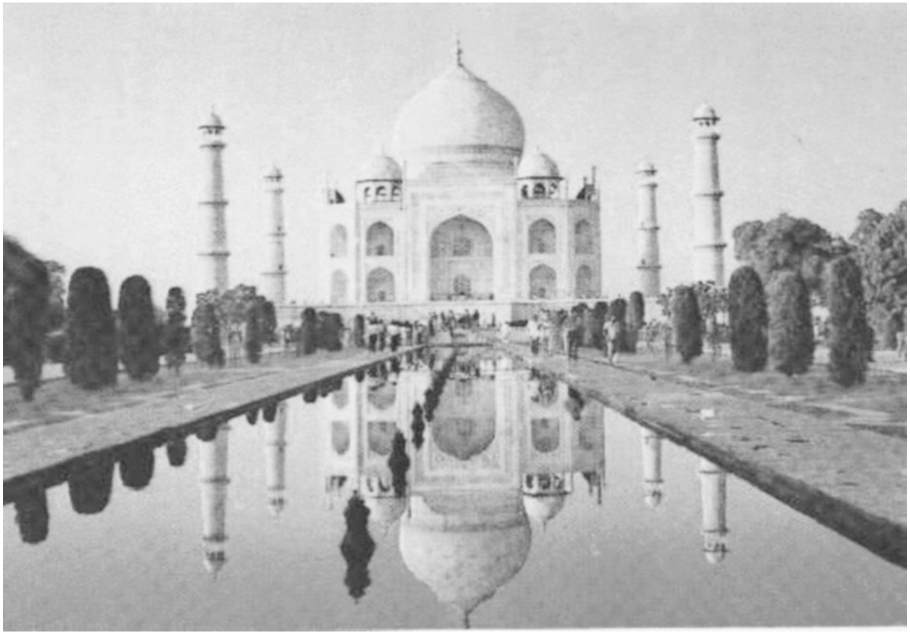
a. Continue with the ray starting at Q for a few reflections. Use the technique you have seen for the parabola in a modified way.

b. The ray from Q and all its reflections will never cross between the foci. Show this.

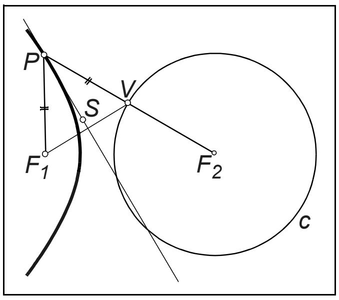
c. How is this for the rays starting at R and S?

the case of the hyperbola

Also in the figure of the hyperbola you see the same elements as for the parabola and ellipse. The hyperbola branch lies, except for point P, again completely on one side of *pbs* (F_1, V).



43. For that must be shown:
$d(F_2, S) - d(F_1, S) < r$,
where r is the radius of the circle. Give this proof by applying the triangle inequality to $d(F_2, S)$. Find an appropriate triangle.



Thus here also applies:
The perpendicular bisector *pbs* (F_1, V) is the tangent line in P to the hyperbola branch.
And the tangent line property also applies:

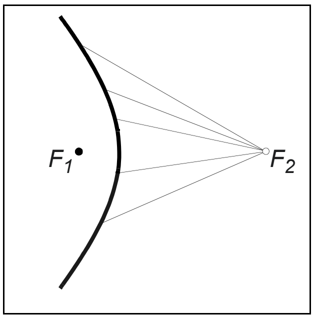
tangent line property hyperbola

Let P be a point on the hyperbola with foci F_1 and F_2.
The tangent line in P to the hyperbola makes equal angles with the lines PF_1 and PF_2.

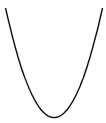
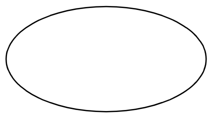
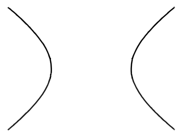
44 **a.** On the right you see a *convex* hyperbolic mirror. F_1 and F_2 are the foci of the hyperbola. Find out in which direction the rays of light that depart from F_2 are being reflected.

b. Suppose, you have a *concave* hyperbolic mirror with a light source in F_1. Where do all rays of light appear to come from?

c. If, in one way or another, you have rays which are all aimed at F_2, how will their reflection rays behave?



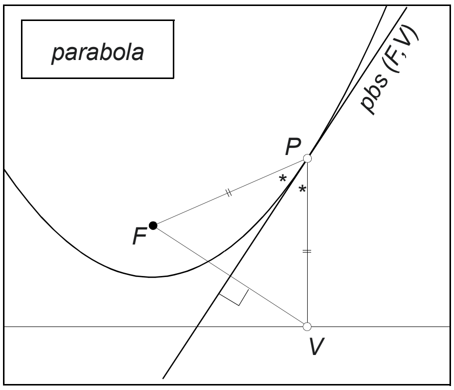
9. *Survey parabola, ellipse and hyperbola*

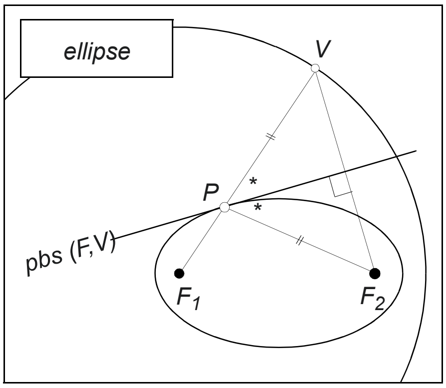
	parabola	ellipse	hyperbola
data	Focus F, directrix l.	Foci F_1 and F_2, constant r ; $r > d(F_1, F_2)$.	Foci F_1 and F_2, constant r ; $r < d(F_1, F_2)$.
condition for point P in the figure	$d(P, F) = d(P, l)$.	$d(P, F_1) + d(P, F_2) = r$.	$\mid d(P, F_1) - d(P, F_2) \mid = r$.
shape	One branch, open on one side. 	Closed figure. 	Two open branches 
number of vertices	1	4	2
symmetrical axes	1	2	2
directrices, circle directrices	One director line.	Two director circles.	Two director circles.
details	All parabolas are similar.	The segments of the axes which lie inside the circle are called short and long axis.	One hyperbola has two asymptotes.
tangent line property	The tangent line in P to the parabola makes equal angles with line PF and the perpendicular line through P on l.	The tangent line in P to the ellipse makes equal angles with the lines PF_1 and PF_2.	The tangent line in P to the hyperbola makes equal angles with the lines PF_1 and PF_2

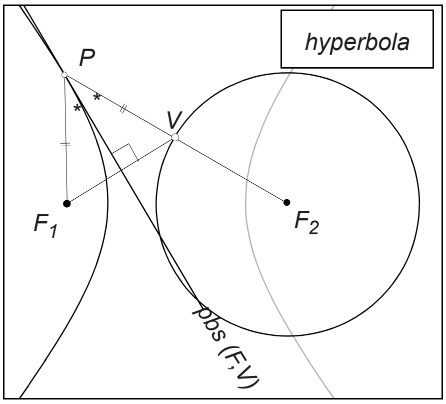
10. Summarizing illustrations to the tangent line properties

The naming in the following figures is the same everywhere:
- *P* is still a point on the figure (parabola, ellipse, hyperbola)
- *F* is the focus, or F_1 and F_2 are the foci.
- *V* is the foot point of *P* on the directrix or the director circle (if there are two, on the other focus).

  In all three cases the tangent line in P is the bisector of ∠ FPV and at the same time pbs(F, V)







11. Extra: The folded path of light in a telescope

You require from a strong telephoto lens, which you photograph stars, sports or birds with, that:
- beams of parallel approaching light converge in one point

– that two beams, which hardly differ in direction (for instance two stars that can barely be told apart with the eye) still have convergence points which lie at a reasonable distance from each other.

If you want to accomplish this with ordinary lenses, you will end up with heavy lenses with long focal distances. On the right you see a so called *catadioptric lens*. This is fairly compact and still has the desired properties. Inside it contains two curved mirrors; the illustration on the next page shows a cross-section of the lens. A parabolic mirror can be seen; actually, only part of it and it has a hole in the middle; its focus is indicated with F_1. The ray of light approaching from S_1 runs parallel to the axis and thus reflects back in the direction of F_1. But a small curved mirror has been placed in front of F_1. This reflects the ray of light eventually to F_2, where the photographic film is located. (The shutter mechanism, which determines the exposure time, is not shown).

45. We further investigate the beam of rays.

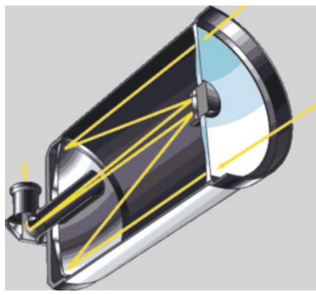
a. Finish the path of the rays approaching from S_2 and S_3. (These are parallel to the ray from S_1.) What shape should the little mirror have to make sure that these rays also end up in F_1?

b. The ray of light which approached from R_1, does not run parallel to the axis. Check that the ray does end up on point Q of the photographic film according to the law of reflection.

c. Construct the paths of the rays, which approach from R_2 and R_3 and are parallel to the one from R_1. Do all these arrive exactly in Q?

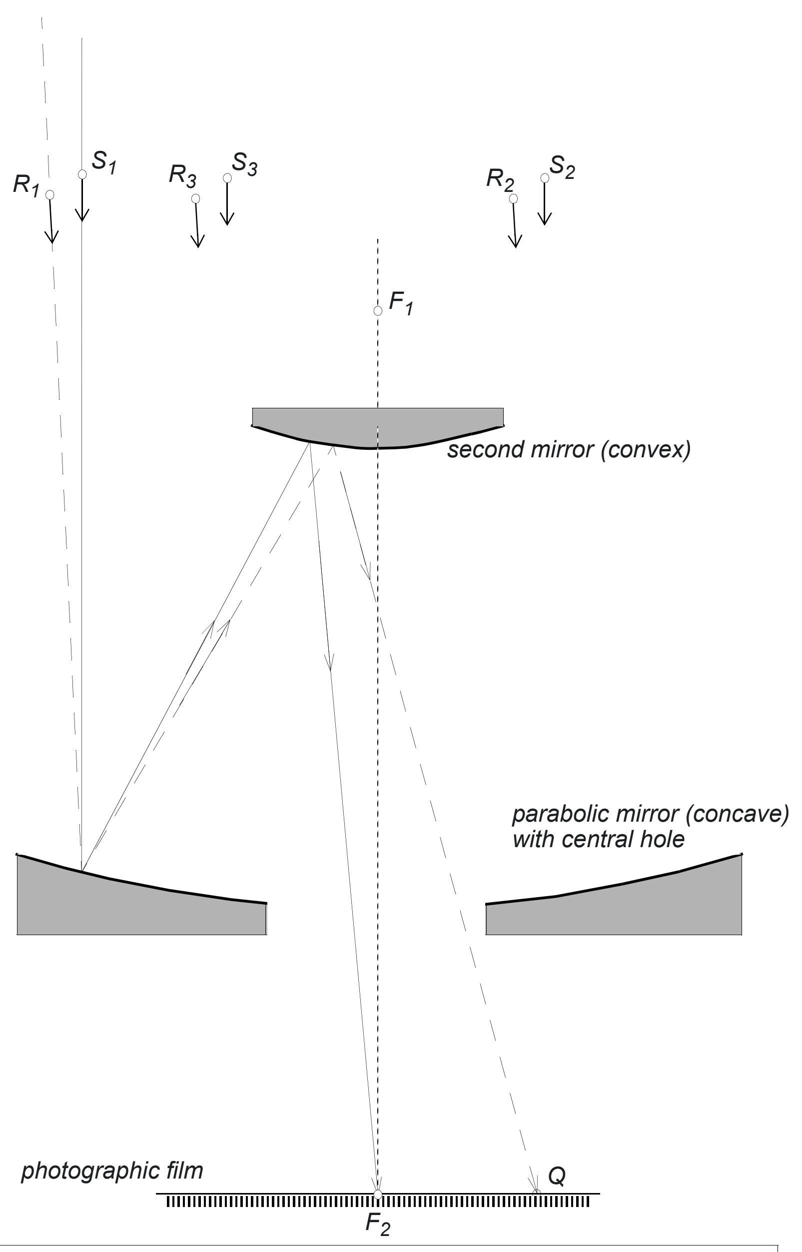
d. If the beam of the *S*- rays comes from one star and the one of the *R*-rays from another star close by, what happens then on the photographic plate?

e. Does the system meet the requirements?



The *R*-beam does not completely converge on the film. There are two solutions for this problem: make sure that the angle that the *R*-beam makes with the *S*-beam stays small

– construct a more complicated system, where a specially curved correction lens is used as well, as for example in the Schmidt-Cassegrain system, that among others is distributed by Cestron International. See the figure on the right.



beam of rays in catadioptric lens

Chapter 3: Analytic geometry

LA

GEOMETRIE.

LIVRE PREMIER.

Des probleſmes qu'on peut conſtruire ſans y employer que des cercles & des lignes droites.

TOus les Probleſmes de Geometrie ſe peuuent facilement reduire a tels termes, qu'il n'eſt beſoin par aprés que de connoiſtre la longeur de quelques lignes droites, pour les conſtruire.

Et comme toute l'Arithmetique n'eſt compoſée, que de quatre ou cinq operations, qui ſont l'Addition, la Souſtraction, la Multiplication, la Diuiſion, & l'Extraction des racines, qu'on peut prendre pour vne eſpece de Diuiſion : Ainſi n'a-t'on autre choſe a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a eſtre connuës, que leur en adiouſter d'autres, ou en oſter, Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement eſtre priſe a diſcretion, puis en ayant encore deux autres, en trouuer vne quatrieſme, qui ſoit à l'vne de ces deux, comme l'autre eſt a l'vnité, ce qui eſt le meſme que la Multiplication ; oubien en trouuer vne quatrieſme, qui ſoit a l'vne de ces deux, comme l'vnité

Commẽt le calcul d'Arithmetique ſe rapporte aux operations de Geometrie.

P p eſt

Facsimile of the first page of 'La Geometrie' by René Descartes (1637)

12. Cartesian thinking

Cartesius

The title of this paragraph is a tribute to philosopher and mathematician René Descartes (Cartesius, 1596–1650), who has had a great influence on both philosophy and mathematics in his own day and in the centuries after. Descartes' philosophy started with absolute doubt: nothing should be assumed to be true. He formulated: *'I think, therefore I am*,' and so found at least one certainty to work with.[1] In Descartes' philosophy the human body and soul were completely independent of each other, and only connected through the pineal gland. Nowadays this *dualism* has come under fire from more modern psychology.

method

Around 1630 Descartes had already formulated rules for 'the direction of thinking.' Simply put, these rules came down to the following:

- **a.** every question or problem in the world can be reduced to a *geometric* question;
- **b.** every geometric problem must be reduced to an *algebraic* problem;
- **c.** every algebraic problem must be reduced to *solving* an *equation* with one unknown.

Because mathematics offered certainty, it made it possible to reach certainty on other subjects. As a thinker, Descartes is an extreme *rationalist*: only thinking will lead humanity to truth (and for instance sensory perception or belief will not).

algebra and geometry

Descartes published the mathematics of part *b* and *c* of this textbook in 1637 in the final essay of his *'Discours de la Méthode pour bien conduire sa raison et chercher la vérité dans les sciences'*[2] which has as its subtitle *'La Geometrie*'. Ultimately every geometry problem comes down to determining the length of line segments, Descartes stated in the opening sentence of *'La Geometrie*' (see the first page of this chapter). He invented the following very general method:

– name all line segments in the figure (using letters), both known and unknown segments;

1. Better known in Latin: *Cogito ergo sum*.
2. Discourse on the method of rightly conducting one's reason and of seeking truth in the sciences.

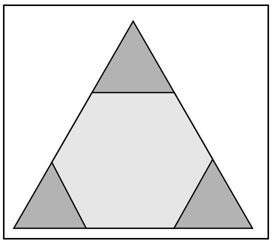
- try to express one quantity in two different ways using the named line segments;
- the expressions are equal, that gives an *equation*;
- solve the equation for the unknown. Now, everything in the figure is known, and the problem has been solved.

In this way, Descartes did much both to stimulate the still-young letter algebra and to supply one of its most important applications.
We will follow him in this chapter in a closer examination of the figures parabola, ellipse and hyperbola. We use items **b** and **c** of his programme; item **a** (that *every* problem can be translated into mathematics) is a philosophical assumption, on which opinions do of course differ.

13. Example of the algebraic method

Here is a simple problem that we will translate to a geometric problem, which we will then solve with algebra.

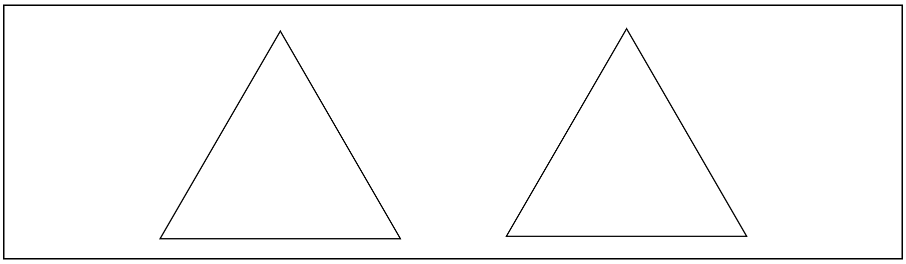
1. A business's logo should look like a triangle, as shown to the right; the points in the corners of the triangle are green, the hexagon in the centre is yellow.
The combined circumferences of the small green triangles should be exactly the same as that of the yellow hexagon; this requirement has a special meaning within the 'corporate identity' of this firm that we won't go into.



For our geometric translation we assume that the whole triangle is equilateral and that all three green points are the same size and are equilateral.
Now the algebraization. Descartes takes you by the hand:

a. Put an a along the (known) side of the entire triangle, and an x along a line segment that has to be determined. Where possible place other expressions along the sides of the triangles and the hexagon.
b. Expressed the combined circumferences of the yellow triangles in a and/or x.
c. Do the same for the hexagon.
d. As stated in the problem, the expressions of b and c should be the same. Write down the equation that follows from that, and solve it.

e. Descartes (and the firm involved...) requires that the solution that has been found must also be constructed in the figure, since it is a geometric problem. Do this on the left below, where the choice $a = 5$ has been made.

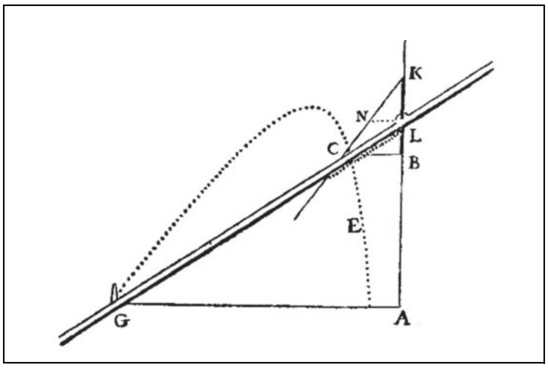


2. The design is criticized: the yellow is too dominant. This can be improved by requiring that the surface area of the green and yellow is the same. Do the calculations for that plan as well, and draw the result in the second triangle.

14. Studying coordinates

Coordinates with Descartes
Descartes was especially interested in observing and comparing curvilinear figures. This example is from *La Geometrie*. It's about the dotted curve, which is made with a simple mechanism.
GA is a fixed line segment. Point *L* moves on the vertical line through *A.* The fixed form triangle *KLN* will move along, following the vertical line, when *L* moves. Point *C* describes the dotted curve, it is the intersection of the line *GL* with the line through *K* and *N*.

Descartes' text says that point *C* can be described using the lengths of the segments *CB* and *BA*, which he calls *y* and *x*.



a la deſcrire eſt appliqué, ie tire de ce point C la ligne C B parallele a G A, & pourceque C B & B A ſont deux quantités indeterminées & inconnuës , ie les nomme l'vne *y* & l'autre *x*. mais affin de trouuer le rapport de

pliant la premiere par la derniere. & ainſi l'equation qu'il falloit trouuer eſt .

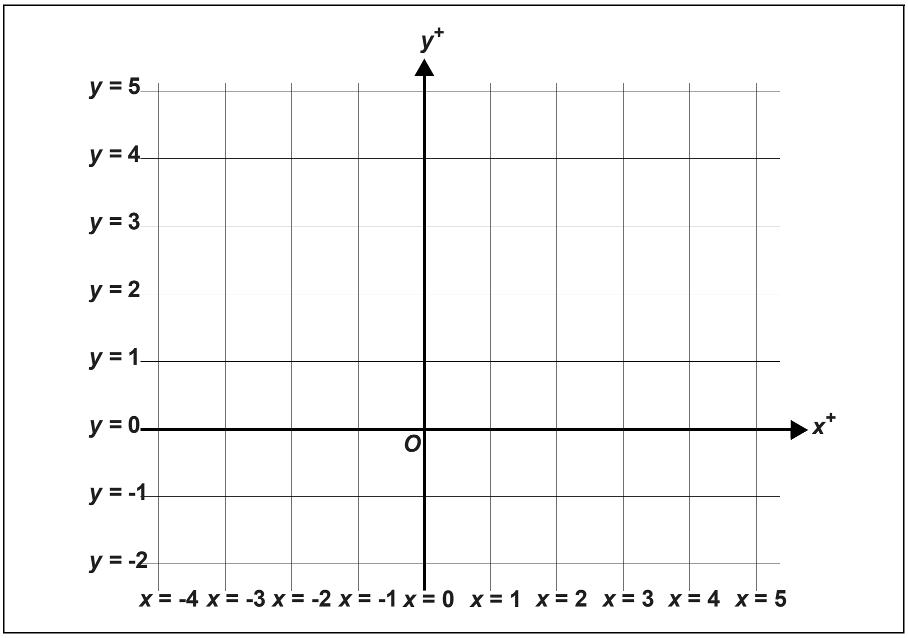
$$yy \infty cy - \frac{cx}{b}y + ay - ac.$$

de laquelle on connoiſt que la ligne E C eſt du premier genre , comme en effect elle n'eſt autre qu'vne Hyperbole.

He also uses *GA* = *a*, *Kl* = *b*, *NL* = *c*. So *a*, *b* and *c* are known quantities.
After some algebraic calculations, Descartes finds an equation showing the connection between *x* and *y* (Descartes uses his own sign for 'equals'). His conclusion is that the curve is a *hyperbola*;

Modern coordinates
The key for using algebra in geometry as Descartes did is to indicate the position of a point with two distances to fixed lines. Or, to put it in a modern way: with coordinates. In the figure above these are the distances *CB* and *CA* to the fixed lines *AK* and *GA*. Descartes calls the distances *x* and *y*, the other given line segments are given letter names from the beginning of the alphabet: *a*, *b*, *c*, etc.
That is almost exactly what we are used to doing in an *xy*-coordinate system such as the one that follows. Our *x* and *y* can be negative, though, since they are used to mark *positions* on the axes, and are no longer interpreted as *lengths*.
The image below shows a so-called *Cartesian coordinate system Oxy*. This type of coordinate system meets the following requirements:

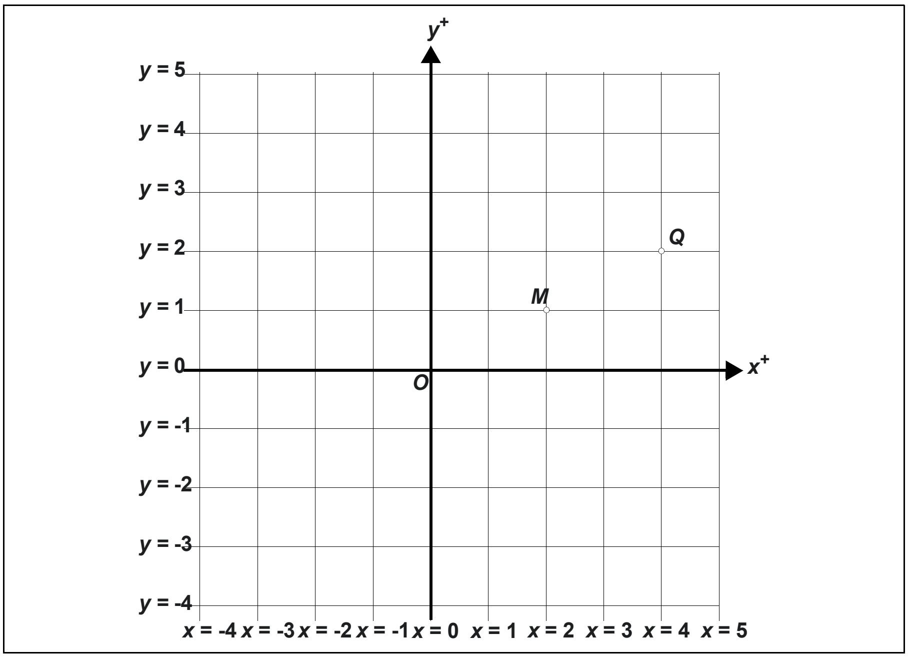
– *the two coordinate axes are perpendicular to each other* (not surprising),
– *the length units on both axes are the same* (this type of coordinate system is also called a square coordinate system),
– *the orientation of the coordinate system is positive* (i.e. a positive 90^o rotation over O will move the x^+-axis to the y^+-axis).



analytic representation
Geometric concepts such as ‘point’, ‘straight line’, ‘distance’, ‘perpendicular’, ‘circle’, ‘square area’, ‘hyperbola’, ..., acquire a so-called *analytic representation* in such a coordinate system. Such a representation may in turn be an equation, a formula, an inequality, a parametric representation or a combination of these forms. In this chapter you will learn about analytic representations and using them.

15. Distances, the circle equation

gridlines, distances, Pythagoras
In the coordinate system the so-called *gridlines* have been drawn. They can help in calculating *distances* in a coordinate system. Descartes indicates that the *Pythagorean theorem* is indispensable here.



3 **a.** Explain that in the example figure the following is true: $d(Q,M) = \sqrt{5}$. Mark the right-angled triangle that you used for ‘Pythagoras’.
b. Now let P be a point with the as yet undetermined coordinates (x, y). Express the distance from P to M in x and y: $d(P,M) =$
Hint: think of the sides of the right-angled triangle you want to use.

c. If we want P to lie on the circle with centre M that goes through Q, we know hat:

$$d(P,M) = \sqrt{5}$$

Descartes states: taking these two lengths as equal gives the equation for the circle. Set up that equation.

d. All the distances are of course positive; so squaring left and right allows you to get safely rid of the square roots. Draw the intended circle in the figure, and write down your equation alongside.

To determine the distance from M to Q, you made use of the *horizontal* and *vertical* distance in the grid, that form the sides adjacent to the right angle of the triangle. Those distance were 2 and 1. In calculating the distance from $P(x, y)$ to $M(2, 1)$ the *horizontal* and *vertical* distance are equal to the *absolute values* of the coordinate differences $|x - 2|$ and $|y - 1|$.
So your equation for the distance may have been the following:

$$d(P,M) = \sqrt{(|x-2|)^2 + (y-1)^2}$$

Because of the squares, the absolute-bars are unnecessary. The final equation after squaring for the circle could therefore have been:

$$(x-2)^2 + (y-1)^2 = 5$$

In this equation the structure of the Pythagorean theorem and the horizontal and vertical distance are still visible. Expanding the brackets in the equation is an option, but it is not always an improvement.

general circle equations

4. Now let M be the point (a, b) and let r be the radius of the circle that has M as its centre. Write down the equation for this circle.

5 **a.** Write down the equation for the circle with centre $M(-5,2)$ and radius 13.
b. Decide, by filling in the equation, whether $P(6, 9)$ and $Q(7, 7)$ are on this circle.
c. Decide by computation only, without drawing or filling in whether $I(2, -9)$ is inside or outside the circle.

6 **a.** Formulate the equation for the circle with centre $M(-2, 3)$ that goes through $(1, 7)$.
b. How many points with integer coordinates are on this circle?

from equation to figure

an equation represents a figure!

You know what is *represented* by the equation $y = 2x + 1$. A straight line, specifically one that goes through the points (0, 1) and (1,3).

That means that all points (x, y) that satisfy the equation together make up that line. You can also ask the question: what does it represent? for other equations, especially if the equation doesn't look as you expected.

7 **a.** Show that this equation represents a circle:

$$x^2 - 4x + y^2 + 6y = 51$$

b. What are the centre and the radius?

8. What is the circle $x^2 + y^2 = 1$?

9 **a.** Show that for each a the circle $(x - a)^2 + (y - a)^2 = a^2$ is tangent to the axes.

b. Four of these circles are tangent to the circle $x^2 + y^2 = 2$. Determine the two positive values of a where this happens. Make a sketch to find the point of contact.

10. Set up the equation of the circle with centre $(a, 0)$ that is tangent to the line $x = y$.

16. *Finding straight lines*

11. Which lines are represented by:

a. $x = 5$
b. $x = y$
c. $x = -y$
d. $y + 2 = 0$

straight line through two given points

We want to find the equation for a line that goes through two given points. To do this, we follow a few steps:

– First we take the origin (0, 0) as one of the points, and give the line the right slope.
– Then we investigate what must be done to move a line.

12. Lines through the origin.
Equations of form $y = ax$ represent lines through the origin (0, 0).
What is a when the line goes through the point (13, 5)?

13. Moving a line from the origin to another point.
We can represent the line through (0, 0) and (2, 5) as follows:

$$\boldsymbol{y = \frac{5}{2}x}$$

We want to move the line, so that the point (0, 0) is moved to (4, 1). This means

*If (x, y) is on the **new** line, then (x – 4, y – 1) is on the **old** line, and vice versa.*

a. Why does

$$\boldsymbol{y - 1 = \frac{5}{2}(x - 4)}$$

give the wanted equation?

This equation is called the ***point-slope*** form of the equation of the line.
The next exercise uses slope-point form to reach point-point form.

14. So a good way to set up the equation for a line through two given points is the following:
(1)first determine the slope (Δy divided by Δx)
(2)set up the equation $y - y_0 = m(x - x_0)$; here m is the slope you found and x_0 and y_0 are the coordinates of one of the two points.

a. Apply the method to find the equation for the line through (-3, 2) and (5, 6).
b. Now do the same for a line through the points (*a, b*) and (*c, d*).
c. In which situation does it go wrong? And how can you still immediately come up with an equation for the line?

15. Investigate whether the three points (5, 8), (8, 13) and (13, 21) are in one straight line.

point-point form of a line
For completeness we state the traditional point-point form for a line going through (x_1, y_1) and $(x_2, y2)$.

$$\boldsymbol{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)}$$

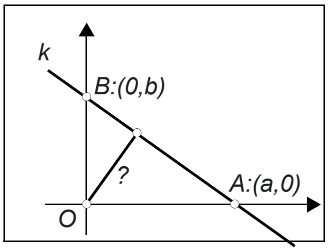
It is a small step to a form which is generally correct, also if $x_1 = x_2$:

$$\boldsymbol{(y - y_1) \cdot (x_2 - x_1) = (y_2 - y_1) \cdot (x - x_1)}$$

the intercept form equation for a straight line

16. This equation also represents a line: $2x + 5y = 1$.
a. What are the intersections with the axes?
b. Set up the equation for the line that goes through the points (1/3, 0) and (0, 1/7).
c. Set up the equation for the line that goes through the points (2, 0) and (0, 3).

17. Let k be a straight line that doesn't go through O and that intersects the x-axis in $(a , 0)$ and the y-axis in $(0 , b)$.



a. Show that an equation of that line is:

$$\frac{x}{a} + \frac{y}{b} = 1$$

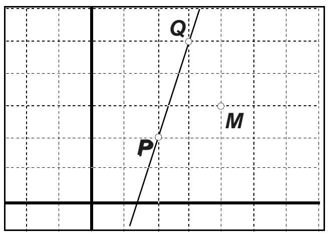
This equation is also known as the '***intercept form equation***' of the line. This form can only be used for lines that do not intersect the origin.
b. Claim: the distance from the origin O to k is equal to:

$$\frac{|ab|}{\sqrt{a^2 + b^2}}$$

Test this formula for $a = 1$ and $b = 1$.
c. Another test: when a and b are multiplied by the same positive factor, that should also be done with the distance. Check if this is the case.
d. You might prove this by setting up an equation for the line through O that is perpendicular to k and then calculate the coordinates of the footpoint V of O. Finally you will find $|OV|$. The calculations are slightly complicated. If you want to take up the challenge you can try.
A simpler way is to look at the area of triangle OAB. You can determine this area in two ways: first with $|OA|$ as the base and height $|OB|$, second with base $|AB|$ and height $|OV|$. Now you can find $|OV|$!

Perpendicular lines

18. Given the points P: (2, 2) and Q: (3, 5).

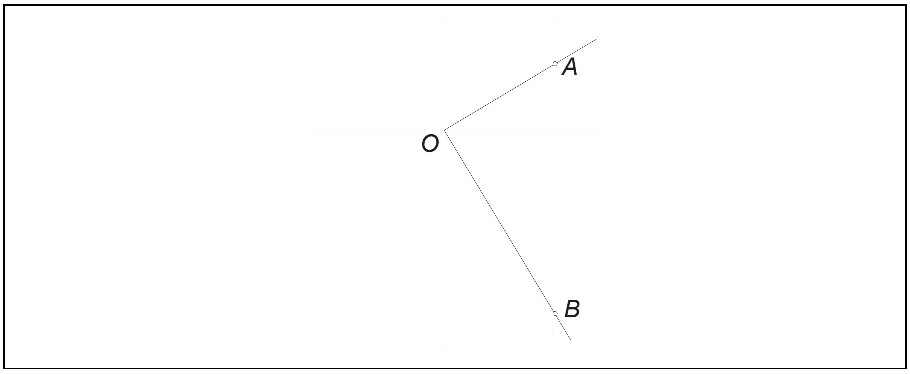


a. Give an equation for the line PQ.
b. There is exactly one line through $M(4, 3)$ that is perpendicular to PQ. It is easy to indicate another point on that perpendicular line. Now give the equation for that perpendicular line.

perpendicular lines

Take another look at problem **18**. The slopes of PQ and MN are respectively 3 and $-\frac{1}{3}$. The second is the *opposite of the reverse* of the first. Or, which comes down to the same, the product of the two slopes is equal to –1. As can be seen in the next problem, this is always the case for two lines that are perpendicular to each other and that are not parallel to the coordinate axes.

19. In a square coordinate system, the points $A = (1, a)$ and $B = (1, b)$ are given, so that the lines OA and OB are perpendicular to each other.



a. In the figure a and b are respectively positive and negative. Is it possible that a and b are both positive (or both negative)?
b. What is the slope of OA? And of OB?
c. Express $|OA|$, $|OB|$ and $|AB|$ in a and b.
d. Because triangle AOB is right-angled in O, you can apply Pythagoras' theorem. Use your results from **c** and show that the law for the slopes of perpendicular lines that is stated above is correct.

The theorem below has now been proved:

theorem

If the lines l_1 and l_2 with slopes h_1 and h_2 (both not 0) are perpendicular to each other, the following is true:

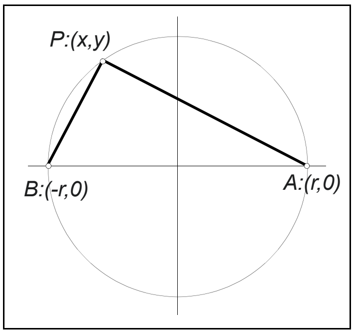
$$h_1 h_2 = -1$$

Reversed:
If the product of the slopes of two lines equals –1, those lines are perpendicular to each other.

20. It follows from Pythagoras' theorem in a simple way that the circle with centre O and radius r can be analytically represented by:

$$\boldsymbol{x^2 + y^2 = r^2}$$

There is yet another way to find this equation, and that is to use Thales' theorem.
Let A: (r, 0) and B: ($-r$, 0).
The points P: (x, y) for which $\angle APB = 90°$ is true, together with A and B form the circle with centre O and radius r. That is what Thales' theorem says.



a. You can express the slopes of PA and PB in x, y and r. How?
b. Perpendicular means: the product of the slopes equals -1. Apply this to the result of **a** and derive the circle equation again.
c. Call V the footpoint of P on AB. The following is true:

$$\boldsymbol{|PV|^2 = |AV| \times |BV|}.$$

Look at your calculations for **b** again to see if you can recognise this answer (translated to algebra).

the perpendicular bisector of two points

You can construct the perpendicular bisector of two points by:
– determining the line through the two points,
– determining the middle of the two points,
– determining the perpendicular on the first line that goes through the middle.

That's doable, but there's another interesting way!

21. We use the fact that the perpendicular bisector of two points is the conflict line (or Voronoi border) of these two points. Let the points be A: (3, 4) and B: (5, 2). P: (x, y) is a conflict point, if (and only if) d(P, A) = d(P, B).
Analytically:

$$\sqrt{(x-3)^2 + (y-4)^2} = \sqrt{(x-5)^2 + (y-2)^2}$$

But there is not that much to see directly from this equation, except for some distances being equal. Therefore you should rework it to a recognizable form.

a. The first step you might take is to leave out the square root signs. Why is that allowed?
b. Rework the resulting equation to an equation for a straight line.

c. You will have noticed that the squares on the left and right cancelled. You could have predicted that. How?

d. Check that the line goes through the middle of A and B and is also perpendicular to AB.

22. Set up an equation for the perpendicular bisector of the points (–5, 7) and (3, –3) by treating the perpendicular bisector as a conflict line.

17. Equation for the parabola

Now you will apply the method to find equations of conflict lines to the parabola. In the next paragraph it will be the turn of the ellipse and the hyperbola.

Where will you place the axes?

An important question to start with is: where will you place the axes in relation to the parabola? Previous experience will have shown that symmetry leads to simple calculations. Therefore: one axis of the coordinate system will become the symmetry axis of the parabola; take the y-axis for example. Also, just as with lines, it is handy if the parabola goes through (0, 0), as that is an easy point to fill in.

23. You still have to assign where focal point F is placed. Let that be (0, 1).

a. You now know the directrix as well. Make a sketch.

b. Go through the standard plan with the equation d(....) = d(.....) to find the equation for the conflict line!

24. Afterwards you will find that you did not find the most simple parabolic equation. How large should the distance between the top and the focal point be to find $y = x^2$ as the equation for the conflict line?

parabolic equation

It is possible to state what you found in problem **23** in a more general way.

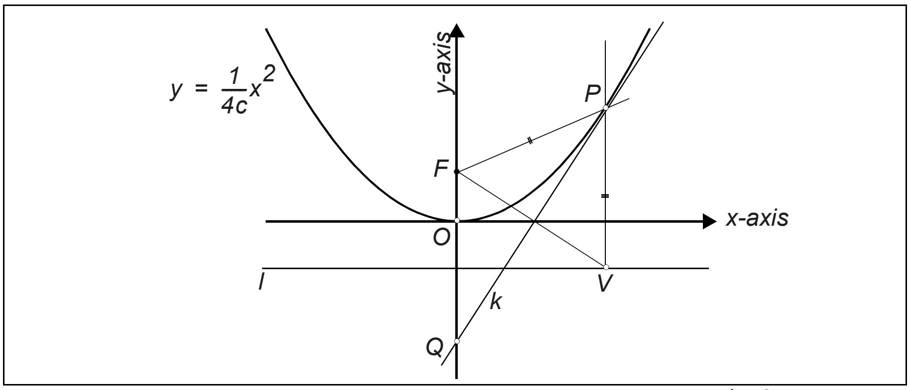
Let the y-axis be the symmetry axis for a parabola and the origin the top. If the focal point F has the coordinates $(0, c)$, the analytic representation of the parabola is:

$$y = \frac{1}{4c}x^2$$

25. Find an equation for the parabola with focal point (4, 0) and directrix $x = -4$. Use the method you prefer.

tangent to a parabola

In chapter 2 you learned that the tangent in a point P of the parabola is the perpendicular bisector of F and V, with V the footpoint of P on the directrix. In problem **26** you will check whether this is the same tangent as the one you encountered in differential and integral calculus.



26. Let the point P (x_P, y_P) lie on the parabola with equation $y = \frac{1}{4c}x^2$. Remember F has coordinates (0, c).

a. Formulate the equation for the tangent-from-differential-calculus in point P. Call that line k.

b. It intersects the y-axis in Q. The y-coordinates of P and Q are each other's opposites. Show this by intersecting the tangent with the y-axis to find the coordinates of Q.

c. Now prove that line segments FQ and PV are the same length.

d. It follows from **b** that k is indeed the perpendicular bisector of FV. How? (Hint: pay attention to quadrangle $FQVP$).

27. There is yet another, instructive, way to show analytically that the perpendicular bisector of VF is a tangent to the parabola. Read ***a***, ***b*** and ***c*** before you start solving the problems. It will help you to understand the underlying plan before!

a. Set up the equation for that perpendicular bisector of VF; you will find something like y = *line equation*.

b. We will intersect that *line* with the *parabola*, but we will do so analytically. That is to say that we will also look at the parabolic equation y =*parabolic equation.* Now we have to solve x from

line equation = *parabolic equation.*

Follow this plan and see if you can follow the argument to end up with:

the perpendicular bisector (V, F) has exactly one point in common with the parabola.

c. Is this sufficient proof that line and parabola are tangent?

18. Equations for the ellipse

The method to find conflict lines analytically, can be used to find analytic representations of ellipse and hyperbola. The calculations needed to find a 'pretty' equation, take a bit more stamina than the examples in the previous paragraph.

ellipse

You have seen a definition of the ellipse as the conflict line of a point and a circle. You also know an ellipse has two symmetry axes. The coordinate system has more or less chosen itself already with that information!

28. Now take the x-axis along the long axis and the y-axis along the short one. Let the distance from a focal point F_1 to the origin be equal to 4, say F_1: (4, 0). The radius of the circle around the other focal point (F_2) may not be smaller than 8! Let that radius be 10 for example.

a. How long are both axes of the ellipse?

b. $P(x,\ y)$ is a conflict point of c and F_1 will then mean the same as:

$$\mathbf{10} - \sqrt{(\boldsymbol{x}+\mathbf{4})^2+\boldsymbol{y}^2} \;=\; \sqrt{(\boldsymbol{x}-\mathbf{4})^2+\boldsymbol{y}^2}$$

Explain.

c. Expanding this to a simple equation is not easy. If you square both left and right to get rid of the roots, you will still have one root sign left. Write down the equation that results from doing this.

d. If you now 'clean up' and bring the square root to one side an the other terms to the other side, square again and clean up again, you end up with the following equation:

$$\mathbf{9}\boldsymbol{x}^2+\mathbf{25}\boldsymbol{y}^2 \;=\; \mathbf{225}$$

which can also be written as:

$$\frac{\boldsymbol{x}^2}{\mathbf{25}}+\frac{\boldsymbol{y}^2}{\mathbf{9}} \;=\; \mathbf{1}$$

Check all this.

e. There are a few things you can immediately check from this equation, like the symmetry and the length of the axes. How?

ellipse equation

Generally, it can be proved that an ellipse where the x- and y-axes are the symmetry axes, has an equation of the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

We assume that $a > 0$ and $b > 0$.

This is called the 'standard form of the equation' of the ellipse.

29 a. How long are the axes of the ellipse in this equation?

b. What does this equation represent if $a = b$?

30. Imagine a point $P = (x_P,y_P)$ going over an ellipse with standard form equation.

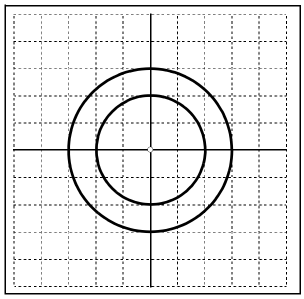
a. Let e and f be positive numbers. Show that point $Q = (e{\cdot}x_P, f{\cdot}y_P)$ also describes an ellipse by making an equation for it.

b. How would you describe the new ellipse in relation with the first?

c. You can say that each ellipse is an elongated circle. Explain this.

31. On the right you see an illustration of the circles represented by:

$$\frac{x^2}{9}+\frac{y^2}{9} = 1 \text{ and } \frac{x^2}{4}+\frac{y^2}{4} = 1 .$$



a. Copy the figure and add a sketch of the ellipse:

$$\frac{x^2}{9}+\frac{y^2}{4} = 1$$

b. Also add the ellipse:

$$\frac{x^2}{4}+\frac{y^2}{9} = 1$$

c. The four intersections of these two ellipses are symmetric around the origin, so they lie on an other circle. Such an intersection (x, y) will satisfy *both* equations, so also the equations you will find by adding the two equations:.

$$\frac{x^2}{9}+\frac{y^2}{4}+\frac{x^2}{4}+\frac{y^2}{9} = 1+1$$

Now determine how far the four intersections are away from the origin.

***abc* of the ellipse**

32. Assume that $a > b$ in the ellipse equation $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

The long axis will lie along the x-axis and therefore the focal points will lie along it as well.

a. What are the coordinates of the four tops of the ellipse?

b. Let the distance from the focal points to the origin be c.
Prove that the following is valid:

$$a^2 = b^2 + c^2$$

Hint: Remember the gardeners method in exercise 6 of chapter 2, page 276.

33. Calculate the coordinates of the focal points of the ellipses in problem **31**.

34. Of an ellipse (0, 0) is the centre, (2,0) is a top and $(\sqrt{3}, 0)$ is a focal point. Give an equation for this ellipse.

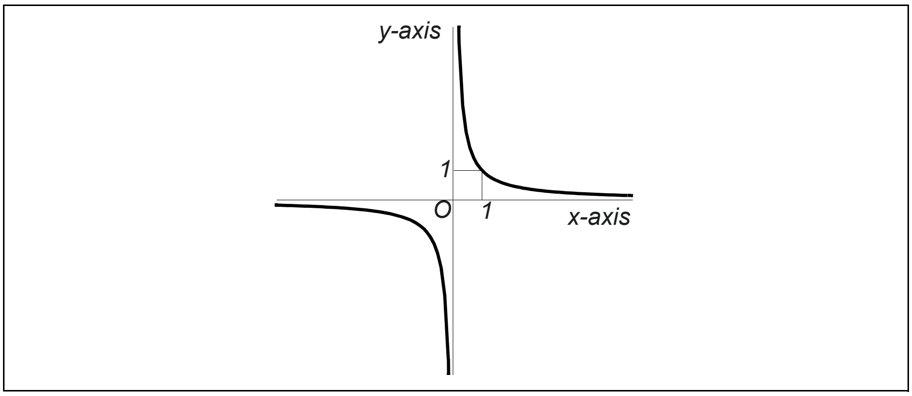
35. Use the graphic calculator (square system of coordinates) to look at the movement according to the equations:

$$\begin{cases} x = 13\cos t \\ y = 5\sin t \end{cases}$$

a. The path is an ellipse. How can you be certain?
b. Which points are the focal points?

36. The circles c_1: $(x+1)^2 + y^2 = 9$ and c_2: $(x-1)^2 + y^2 = 25$ are given.
a. Draw both circles in a system of coordinates.
b. What type of graph is the conflict line of both circles?
c. Find an equation for that conflict line.

19. Equation orthogonal hyperbola



A hyperbola that you met before meets the equation: $y = \frac{1}{x}$ or $xy = 1$.
In more general form the equation is: $xy = c$, with c a constant unequal to 0.
You will also have encountered this graph and a similar equation in physics for Boyle's law (at a given temperature, *pressure* × *volume =constant).*

The asymptotes of this kind of hyperbola are perpendicular, which is why it is called an *orthogonal hyperbola* (orthogonal means perpendicular).

Should a graph with equation $xy = c$ really be called a hyperbola? Or in other words, does the graph meet the definition from the previous chapter? We will take a closer look in the next problem.

37. Is it really a hyperbola?
  a. The coordinate axes are the asymptotes of a hyperbola. Which lines are the symmetry axes?
  b. Assume that there are focal points and that the coordinates of focal point F_1 are (2, 2). What are then the coordinates of the other focal point (F_2)?
  c. We now limit ourselves to the branch in the area $x > 0$ and $y > 0$. You can look at it as the conflict line of the point (2, 2) and a circle with centre F_2. What should be the radius of that circle? (Consult chapter 2 if necessary).
  d. Now translate the conflict condition to an x, y-equation and expand it. If you make no errors, you should end up with an equation of the form $xy = c$!

Now you are certain that the hyperbola you encountered before is the same as the hyperbola (with mutually perpendicular asymptotes) of the standard definition.

The form $xy = c$ immediately shows us a special property of the orthogonal hyperbola:

> *If you draw two perpendicular lines from a point P of the orthogonal hyperbola to the asymptotes, together with the asymptotes, these will create a rectangle of which the area is independent from the place of P on the hyperbola.*

38. That is a complicated statement. Look at it carefully and try to find out whether or not you agree with it.

20. Standard form equation of the hyperbola

Just as for the ellipse, the favorite equation for the hyperbola is the one where the coordinate axes are the symmetry axes. You can draw up such an equation in the same way as you did for the ellipse. Select your focal points (on the x-axis), formulate the conflict condition, expand equations. You will end up with an equation that looks a lot like that for the ellipse, with one important difference: on the left, instead of a plus sign, there is a minus sign. That is to say:

Again, just as for the ellipse, there is a relation between the distance of the focal points to the centre and this a and b.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

For the sake of completeness we will formulate that as follows, just as with the ellipse:

***abc* of the hyperbola**
If c is the distance from the middle of the hyperbola to a focal point, then the following is true:

$$a^2 + b^2 = c^2$$

39. Given the hyperbola with equation: $4x^2 - y^2 = 4$.
  a. What are the coordinates of the tops and of the focal points?
  b. What are the equations for the asymptotes?

40. Using the graphic calculator (square coordinate system), look at the movement according to the equations:

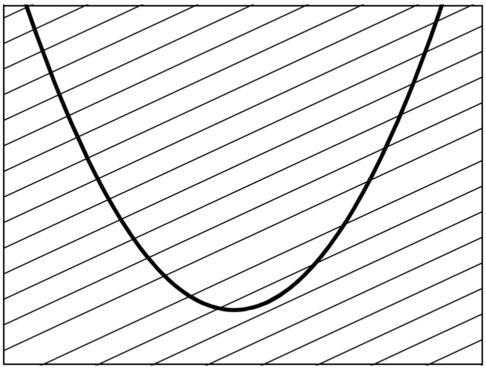
$$\begin{cases} x = \dfrac{3}{\cos t} \\ y = 4\tan t \end{cases}$$

  a. The path is a hyperbola. Explain this by deriving the equation in x and y.
  b. What are the equations for the asymptotes?
  c. What are the coordinates of the focal points?

21. Parametrization as a method; four different examples

Example 1: parabola with a sheaf of lines

41. A parabola, intersected by a sheaf of parallel lines.



a. A special phenomenon occurs when you look at the middles of the segments of the lines within the parabola. Draw the middles and formulate a conjecture.

We will prove the conjecture using the analytic method.
We take as simple a case as possible for the parabola, $y = x^2$. The lines have a specific slope; let's call it b.
Now, you could set up a general equation for the lines with a specific slope, but you would not be making clever use of the parabola. You would probably end up with complicated calculations, as you would have to find two intersections for the intersection lines, and you might end up with square roots.
It would be smarter to include one intersection right from the start by looking at a line with slope b and going *through a point of the parabola.* The trick of 'parametrization' is to write down the point in such a way that you know in advance that it lies on the parabola.
Points of the parabola are $(1, 1^2)$, $(3,3^2)$, etcetera. Generally: (a, a^2).
We say that $P_a = (a, a^2)$ is a parametrization. If a runs through all numbers, P_a will go through all the points parabola. Now the rest of the plan.

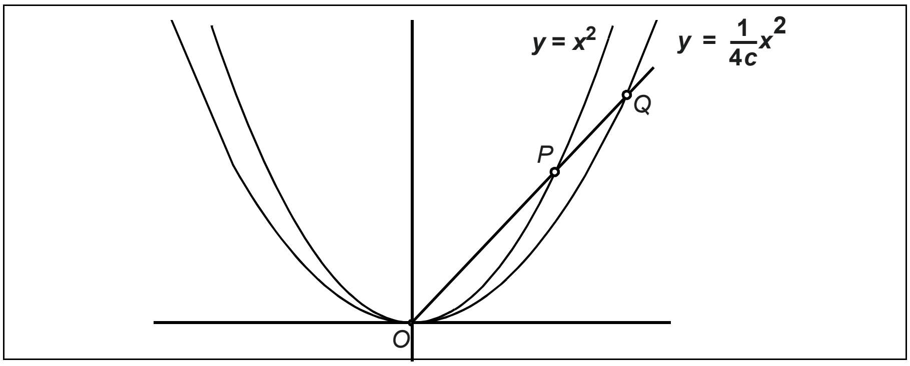
b. Formulate the equation of the line with the slope equal to b, that goes through $P_a = (a, a^2)$. (Consider problem 13 of this chapter, page 310).
c. We have to intersect this line with the parabola $y = x^2$. So fill in x^2 in your line equations and solve to x. It will initially result in a quadratic equation. *But you already know a solution for that equation, namely $x = a$.* Why?
d. Now bring the equation (if still necessary) in the form ..... = 0
e. Solve the equation by bringing a factor $(x -a)$ outside the brackets. It must be possible!
f. It should not take you long to find the x-coordinate of the second intersection. Determine the middle of the x-coordinates of this intersection and the known intersection P_a.
g. Complete the argument that will lead to a proof of the conjecture about the midpoints of parallel chords in a parabola.

Example 2: All parabolas are similar

To prove that all parabolas are similar, it is enough to prove that for every value of c the parabolas $\boldsymbol{y = \frac{1}{4c}x^2}$ and $\boldsymbol{y = x^2}$ are similar.

The two parabolas have the top and the symmetry axis in common. Can we multiply one of the parabolas from (0,0) in such a way that we end up with the other one?

42. Here is a figure.



The multiplication factor should be $|OQ| / |OP|$. It should be independent from P, but of course dependent on c.

a. Use the known parametrization for P, and calculate the coordinates of Q.
b. We have to calculate $|OQ| / |OP|$ and we hope that quotient is constant.
Is it necessary to calculate the distances themselves, or are the x-coordinates enough in calculating $|OQ| / |OP|$?
c. Complete the proof.

Example 3: There is more than $3^2 + 4^2 = 5^2$

You are probably familiar with the number relation: $3^2 + 4^2 = 5^2$ and perhaps also $12^2 + 5^2 = 13^2$ and $21^2 + 20^2 = 29^2$.
Such triples of whole numbers, seen as lengths, form a right-angled triangle, and are therefore called *pythagorean triples.* There are more: $335517^2 + 717956^2 = 792485^2$
and also: $13444655170416^2 + 27975811156937^2 = 31038762258505^2$

You would think that it took centuries to find the last one. Not quite; you yourself can find one like it after the next problems; larger ones as well. And all with the aid of a smidgen of analytic geometry!

formulating a plan
The basic idea is this: if $a^2 + b^2 = c^2$, then the point (a/c, b/c) is on the circle $x^2 + y^2 = 1$.
So we are looking for points on that circle that have common fractions as their coordinates.
Looking for points with the parametrization (t, $\sqrt{1-t^2}$) will not work, as we do not want square roots! We need to look for a better parametrization, without square roots.
We have seen finding the second intersecting of a line with a parabola is simple if we already know the first intersection point. We use the idea again.

the trick
For the well known point on the circle we take the point (-1, 0). Our lines will be the lines going through (-1, 0). The second intersection well be found without square roots in the computation. If we use a common fraction for the slope of the line, the second intersection will be have common fractions as coordinates.

43. Realization!

a. Formulate the equation for the line with slope = t that goes through the point

(-1, 0), in the form y = ...

b. Intersect with $x^2 + y^2 = 1$; so fill in y = ... in this circle equation.
c. You have a quadratic equation in x, for which you know that $x = -1$ is a solution.
d. Handle the equation like in exercise 41, page 321.
e. Now you find x = ..., an expression in t and quickly y = ... as well, another expression in t.
f. Fill in $t = 1/2$ to find a point on the circle.
g. Turn your solution into a pythagorean triple by multiplying with the common denominator
h. Construct a pythagorean triple with six-digit numbers by starting with more a value of t with a large denominator.

looking back
You solved a problem about finding special number relations using a combination of geometry and algebra. That is quite special, but it happens a lot in mathematics that you use methods from one area in very different areas as well!

Example 4: The folium of Descartes

The following equation in x and y was invented by Descartes.

$$x^3 + y^3 = 6xy$$

You will probably think that it doesn't represent a circle, parabola, ellipse or hyperbola, since those all have quadratic equations. But what then is it?

44. Some very basic properties.

a. Why is the figure for the equation symmetric in the line $x = y$?

b. .There is at least one easy to find point on the curve, (0, 0). Find another point with rational coordinates.

We will find more about the shape of the curve in two steps. First we construct a parametrization. Second we feed the parametrization into a graphic calculator (or other graphing tool) and enjoy the view.

45. Building the parametrization.

Since the point (0, 0) is really very simple, we will use it in the same way that we used (-1, 0) in the previous problem. In short: we will look for the other intersections of the curve with the line $y = tx$ for the given direction t.

a. Fill in this y in the equation and solve x. You have now got hold of a fairly simple expression for x in t and you can also find one for y.

b. Point (270/152, 450/152) is also on the curve. Using your equations, give another point with common fractions as coordinates.

46. Now you can also easily visualize the curve on the graphic calculator.

a. Select MODE **Par** (for parametrization).

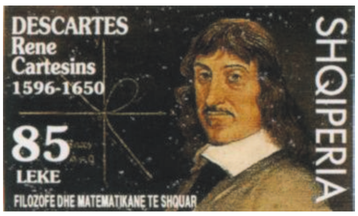
b. Select button **Y**=. You can now fill in your formulas for x and y after $\mathbf{X_{1T}}$= and $\mathbf{Y_{1T}}$=. (The calculator may insist on capital T).

c. Now the graphic calculator has all the information needed to perform the command **Graph**, but you need to make some changes under **Window**. Let T run from for instance -10 to +10 and take 0.1 as the step size.

d. Sketch your graph.

47. While calculating the expression in T for x you lost *two* factors x from the equation (see **45**). You found the not so complicated formula $x(t) = 6t/(1 + t^3)$. Perhaps we already had *two well known* points on the curve beside the knew intersection?

The curve with the nice loop is called the *Folium of Descartes*. It is difficult, without the parametrization given here, to draw the curve with the graphic calculator.

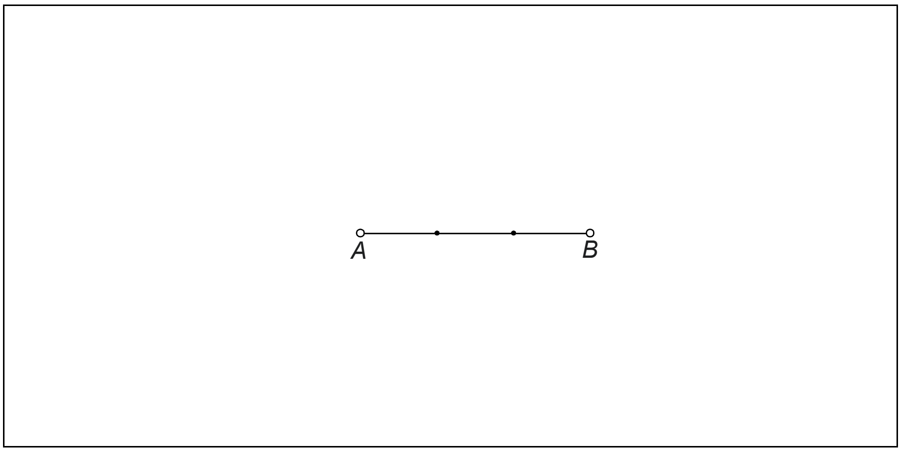


The above methods to find numeric solutions that match this kind of equation were already

used by Diophantos of Alexandria in the third century C.E.
French mathematicians of the calibre of Descartes and Fermat were familiar with Diophantos, and maybe Descartes used a similar method to draw the curve.

22. Weighted distances, other conflicts

Imagine two provincial capitals, A and B, that are 120 km apart in a straight line. City A has twice as many inhabitants as B. To determine a new border between the two provinces, the following is agreed on: cities and towns that are in A's province can be twice as far from the capital as places in B's province. Or to put it another way:
point P is a '2-*conflict point*' of A and B, if (and only if) $d(P, A) = 2 \times d(P, B)$.



48. The question of course is what the border will look like and where exactly it is.
- **a.** On the straight line between A and B you can immediately indicate a 2-conflict point P_1. What point is it?
- **b.** Someone proposes that surely the perpendicular line through P_1 on AB will be the conflict line. Why is this not true?
- **c.** You can indicate another 2-conflict point beyond B on the line AB!
- **d.** Try to locate a few points P, satisfying $d(P, A) = 2 \times d(P, B)$.
- **e.** Make a sketch of what you expect the conflict line to be.

To be able to make the transition to analytic geometry, you must select a system of axes. That can be done in many ways. It is important, though, that you do not do it haphazardly, but select the axes in such a way that the calculations will go reasonably well, and the results are in a reasonably simple form. Often you can achieve much (even before doing any calculations!), by paying attention to special

points and to *symmetry* in the figure. In this case: use the x-axis for the symmetry. And make one of the points A and B really simple.
From here on we will take A = (0, 0) and B = (6, 0)

49. Now, we continue with the analytic method, by translating

$$d(P, A) = 2 \times d(P, B)$$

into an equation in x and y.

a. Write down the equation and put it in a handy form; you will see that the quadratic terms do *not* cancel out.

b. Which type of figure does the equation represent?

The conflict line you just found is known as the *circle of Apollonius*. Apollonius was a Greek mathematician whom you might call a real expert on parabolas, ellipses and hyperbolas. However, Greek mathematics did not use analytic methods, so Apollonius probably found the circle that was named after him by traditional geometric means.

50. In the end the algebraic exercise itself was not all that much of a shock, and we did find a surprising result.
Think about an answer for the next questions and statements, because it is good to think for a moment a bit philosophical about using coordinates in geometry.

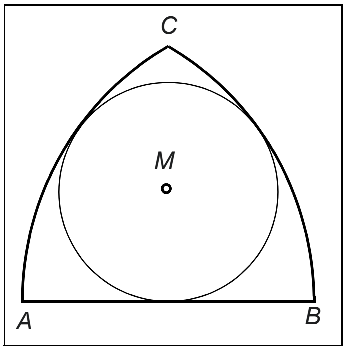
a. By now we do *know* that the 2-conflict line is a circle. But did you expect that beforehand? Would you be inclined to say: I *understand* why it is a circle?

b. *Statement 1*: regular geometry leads to endless searching for a solution in this kind of problem. The analytic approach offers a safe route. It is much more efficient.

c. *Statement 2:* analytic geometry is geometry for all. Hard work solves everything, you don not need to be as smart as in regular geometry without algebra.
But it is, to say the least, a different way to understand a problem.

23. Making a gothic window with the aid of a parabola

One of the standard shapes of a gothic window consists of a symmetric 'triangle' *ABC*, of which the upright sides are not straight lines, but arcs with an inscribed circles. You can see this type of window in the illustration on the right. On the next page are three of those windows from The royal basilica of Saint-Denis in Paris.



In the next problem you will investigate how to construct such a figure.

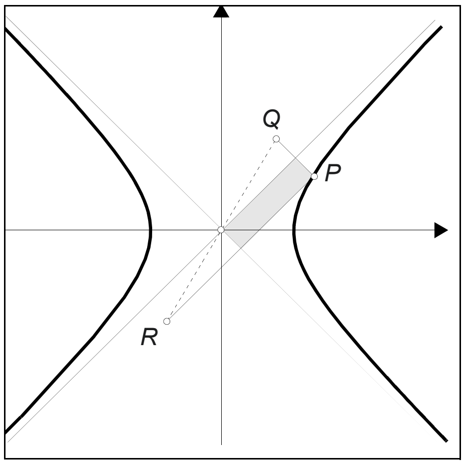
51. Analytic geometry for Gothic architects.
Let d(A, B) = r.
Arc *BC* is part of the circle with centre *A* and radius *r*. Arc *AC* should be obvious now.
- **a.** Construct a gothic triangle yourself using a pair of compasses.
- **b.** To be able to construct the inscribed circle you need to know what the height of the centre above the base *AB* is. You can use the analytic method for this, so the first thing you need to do is place the figure in a system of coordinates. Make your own choice for the system of coordinates.
- **c.** What now are the coordinates of *A* expressed in *r*? and of *B*?
- **d.** The centre *M* of the inscribed circle is the same distance from line segment *AB* as from arc *BC*. Calculate the coordinates of the point *M*, expressed in *r*.

52. The figure in the problem looks like a triangle with an inscribed circle. As you know, the centre of an inscribed circle is the intersection of the three bisectors.

The bisector from an angle is (also) the line of points having similar distance to the sides converging in that angle.
a. What shape do the 'bisectors' of the gothic window have?
b. Provide an equation for each of these 'bisectors'.
c. The 'bisector' from A intersects the line AB again. Where?

24. Stretching the hyperbola; asymptotes

Just as the case $a = b$ is a special kind of ellipse (a circle), so $a = b$ results in a special hyperbola as well, namely the orthogonal hyperbola. By making use of the 'rectangle with constant surface' (see 37, page 319) we can prove this second statement with very little work.

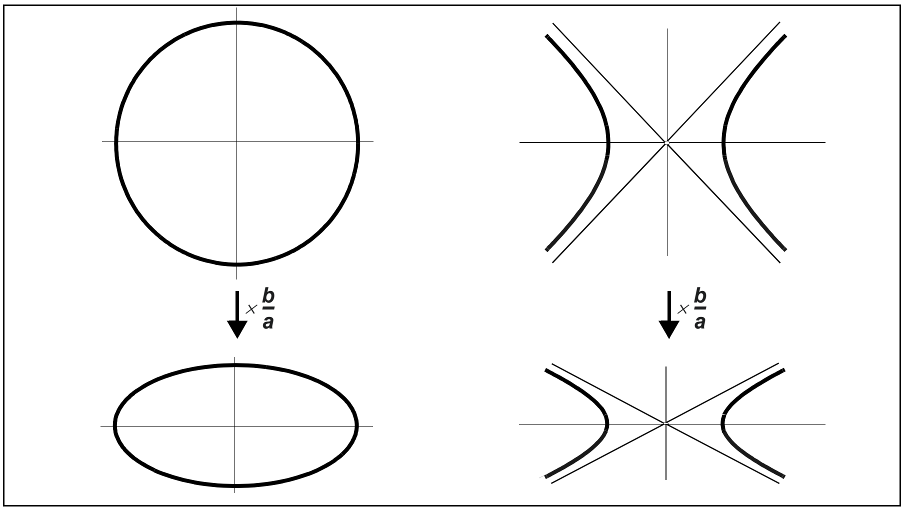


53. In the figure you can see the hyperbola that results from rotating the known orthogonal hyperbola $xy = 1$ over an angle of $-45°$. The area of the grey triangle = 1.
a. What are the equations for the asymptotes?
b. An arbitrary point P: (x, y) is reflected against both asymptotes, resulting in points Q and R.
What are the coordinates of Q and R?
c. The area of the grey rectangle is exactly half of the area of right-angled triangle PQR. Explain.
d. Using the distance formula, you can calculate the right-angle sides PQ and PR.
Check that these are equal to $|x - y|\sqrt{2}$ and $|x + y|\sqrt{2}$.
e. Explain from this that the equation of the hyperbola is $x^2 - y^2 = 2$.

oblique hyperbola
Just as you can create an arbitrary ellipse by stretching/shrinking from the circle $x^2 + y^2 = a^2$, (as you showed in exercise 30, page 317) you can create an arbitrary hyperbola by stretching/shrinking from the orthogonal hyperbola $x^2 - y^2 = a^2$. Look at the image below, where you see the stretching geometrically in the top part of the picture and algebraically in the bottom part. The asymptotes no longer form a right angle and this is called an *oblique hyperbola*.

54 a. What are the coordinates of the tops of the hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$?

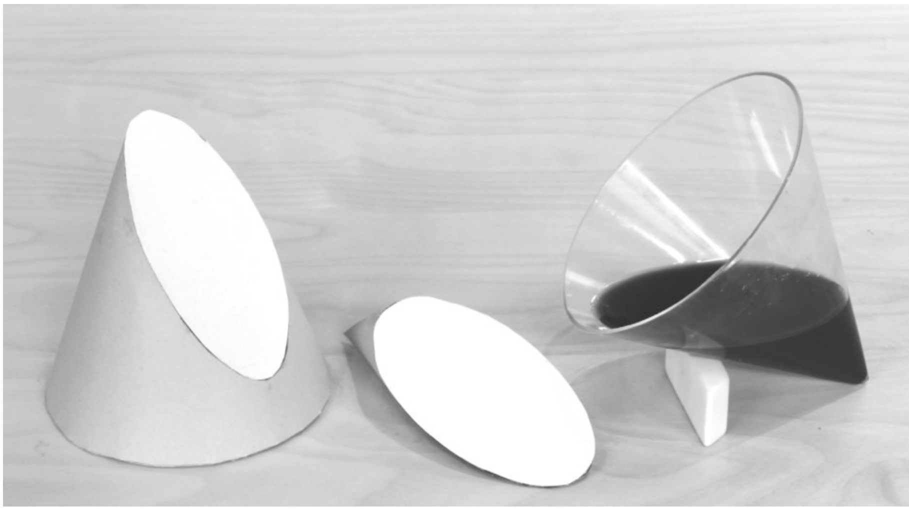
b. The lines with equation $\frac{x}{a} - \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} = 0$ are the asymptotes of the hyperbola. Explain this from the figure.



circle $x^2 + y^2 = a^2$ → *stretch/shrink* → ***ellipse*** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

orthogonal hyperbola $x^2 - y^2 = a^2$ → *stretch/shrink* → ***oblique hyperbola*** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Chapter 4: Conic sections



Introduction

A plexiglass cone has been partly filled with colored water. A massive cone has been cut along a plane. The edge of the surface and of the cut have an elongated oval shape; they look as if they might be ellipses. Could it be, or are they another type of closed curve?

The last option is the most likely one; after all, an ellipse has two symmetry axes, and looking at this photo, we expect only one, in the longitudinal direction. But for once the unlikely is true!

The mathematical family name of the figures parabola, ellipse, hyperbola and circle is: *conic sections*.
We do have to prove one thing now, because in the previous chapters we worked only on a plane, and the figures mentioned above had all been defined using distance relations within the plane. Not a cone in sight!

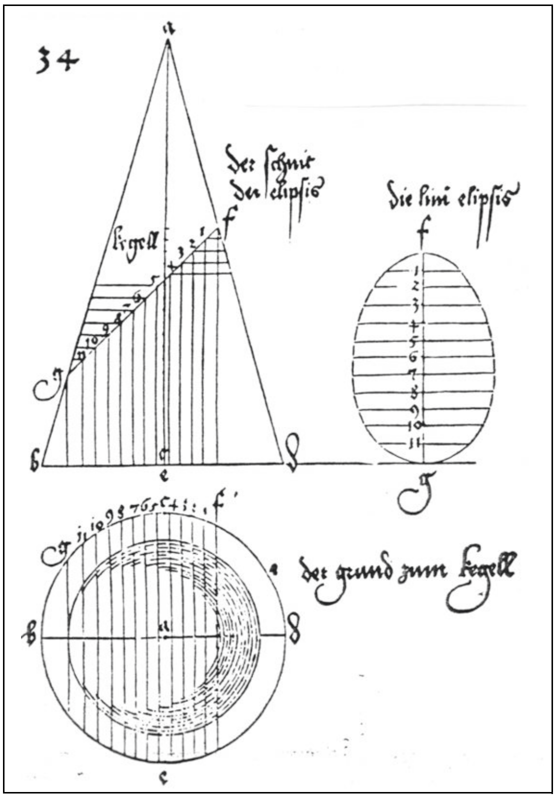
We have to show now that these cross sections of a cone with a plane are indeed our old friends parabola, ellipse, hyperbola and circle. We do have to keep a cone that extends very far in mind.

Description of a cone

For a mathematical cone you need a circle and a point on the circle's axis (but not on the plane of the circle). From that point you add lines to all points on the circle. These lines are called the *generating lines* of the cone; together they form the cone. The single point that all lines go through is called the top of the cone.

Such a cone is therefore infinitely large and not closed off by a plane like the cone on the first page of this chapter. Such a cone has two halves which meet on the sharp point: so it's more like some kind of diabolo. However, in many cases we draw only one half, and only a small part of that, bordered by a circle that we would like to see as the base of the cone. You will learn to recognise the images.

25. An ellipse according to Dürer

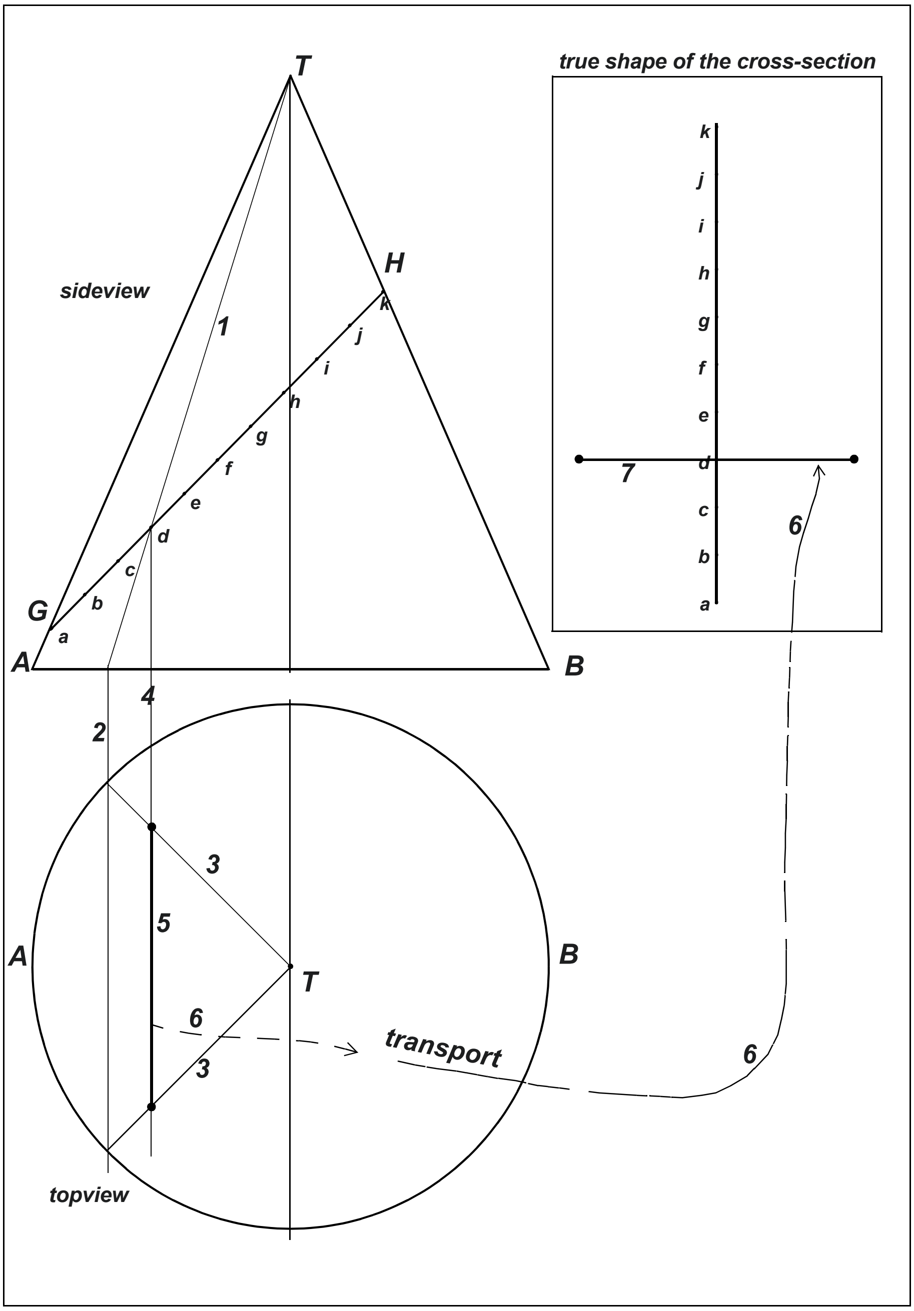


The painter and illustrator Albrecht Dürer (1471–1528) studied the figures that can result from the intersection of a plane and a cone.
Here you see one of his illustrations from a geometry book for painters.

1. You can see a cone from the top and from the side. The actual cross-section has been highlighted and constructed alongside. The numbers show the relation between the two. Somehow Dürer has been able to find out the width of the cross-section at the various positions of the numbers by using the top and side views.
   - **a.** Examine how that was done.
   - **b.** The captions can be read. Dürer writes: *Schnit der Ellipsi*s. What do you think of the result, the figure in the top right corner, where it says: *Die lini elipsis?*

Either the cross-section of the cone is an egg-shape, and NOT the ellipse as we know it, or the drawing is a bit so-so.

2. Using the precise drawing on the next page, we will re-do Dürer's work in a slightly different way.
   - **a.** Here, the order of the numbers represents the method to find the real shape from the top and side views. Follow the method using the numbers, demonstrating the construction, for point *d*.
   - **b.** Once you are certain that you understand the steps, repeat the construction so that you can decide on whether it is an ellipse or an egg.
   - **c.** State your view on this.



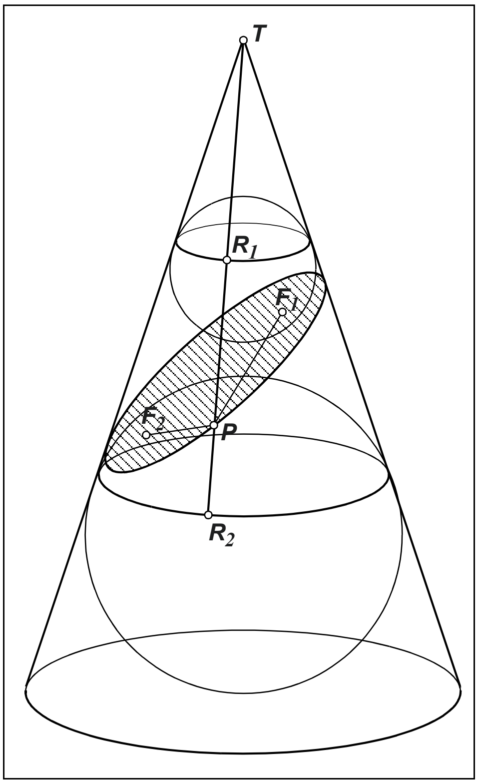
true shape of the cross-section
k
j
i
h
g
f
e
d
c
b
a
7
6
T
H
sideview
1
k
j
i
h
g
f
e
d
c
b
G
a
A
B
4
2
3
5
A
B
T
6
transport
6
3
topview

26. *Dandelin's spheres*

The case of the ellipse

First we will show that the slanted (closed) cross-section of the cone is definitely a real ellipse. This will establish without a doubt that Dürer's illustration is inaccurate|.

The proof that follows here is by Germinal Pierre Dandelin (1794–1847). You aren't expected to think of this yourself; it's quite an extraordinary find, and more or less the only thing that is remembered of Dandelin. It will require some work from your power of three-dimensional imagination, but is in fact fairly simple.

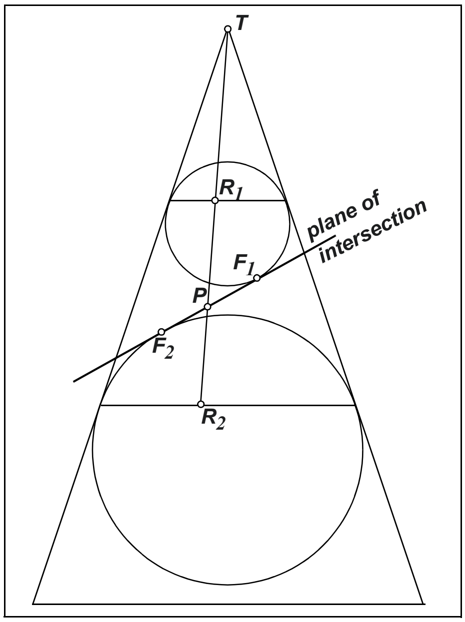


3. The shaded part of the plane is the cross-section we are looking at.
   a. Within the cone you can see two spheres. They touch the cross-section and fit exactly within the cone. So keep in mind that the spheres are determined entirely by the cone and the cross-section.
   b. In the figure, look for the circles that are the tangents of the spheres and the cone.

P is a point on the cross-section. T is the top of the cone. The points of tangency of the cross-section and the spheres are the points F_1 and F_2. The line PT has been drawn; it is a straight line on the cone, a generating line. It intersects the contact circles in R_1 and R_2.

   c. The most important key to the proof is: $|PF_1| = |PR_1|$. Why does that equivalence apply?
   Keep in mind: this involves two tangent lines from one point to a sphere.
   Of course this is also true: $|PF_2| = |PR_2|$.
   d. From this follows $|PF_1| + |PF_2| = \mathit{constant}.$
   Explain.
   e. So why is the cross-section an ellipse?

In the case of the ellipse the three-dimensional representation is easy to take in.
The illustration on this page again shows you the situation with the ellipse, but now in side view. Because of the previous problem, you now know how to interpret the various elements.
The bold line is the cross-section from the side. The circles are tangent to both cone and cross-section. The R-points are on the tangent circles, P, R_1, R_2 and T are on one generating line. F_1 and F_2 are the tangents of the cross-section and the two spheres.



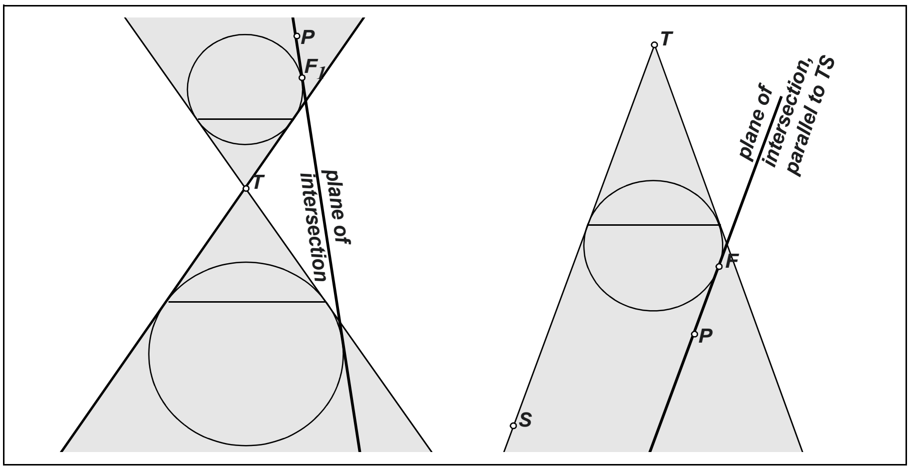
4. You can check the proof in this illustration.
It's good preparation before you tackle the analogue proof for the hyperbola and the parabola.
a. Go through the proof in the illustration step by step.

The first main step was that $|PR_1| = |PF_1|$.
In the drawing it's difficult to ***see*** that these line sections are equal. You can ***know*** it though! Make use of the tangency with the sphere.

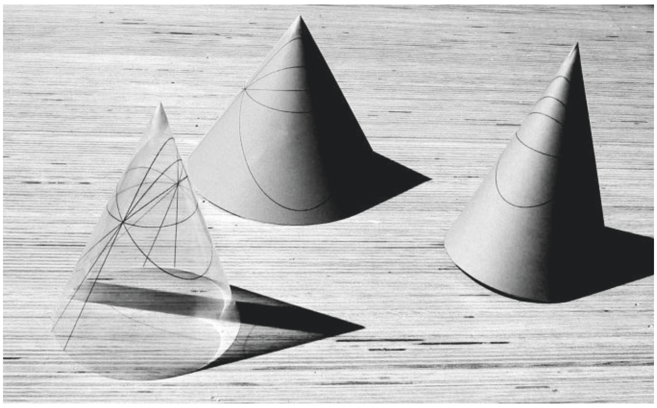
b. Adding $|PR_1|$ and $|PR_2|$ to $|R_1R_2|$ was the second step. Why was that possible?
It is not always the case that if three points A, B, and C are in one line, that $|AB| + |AC| = |BC|$? What is the extra condition?

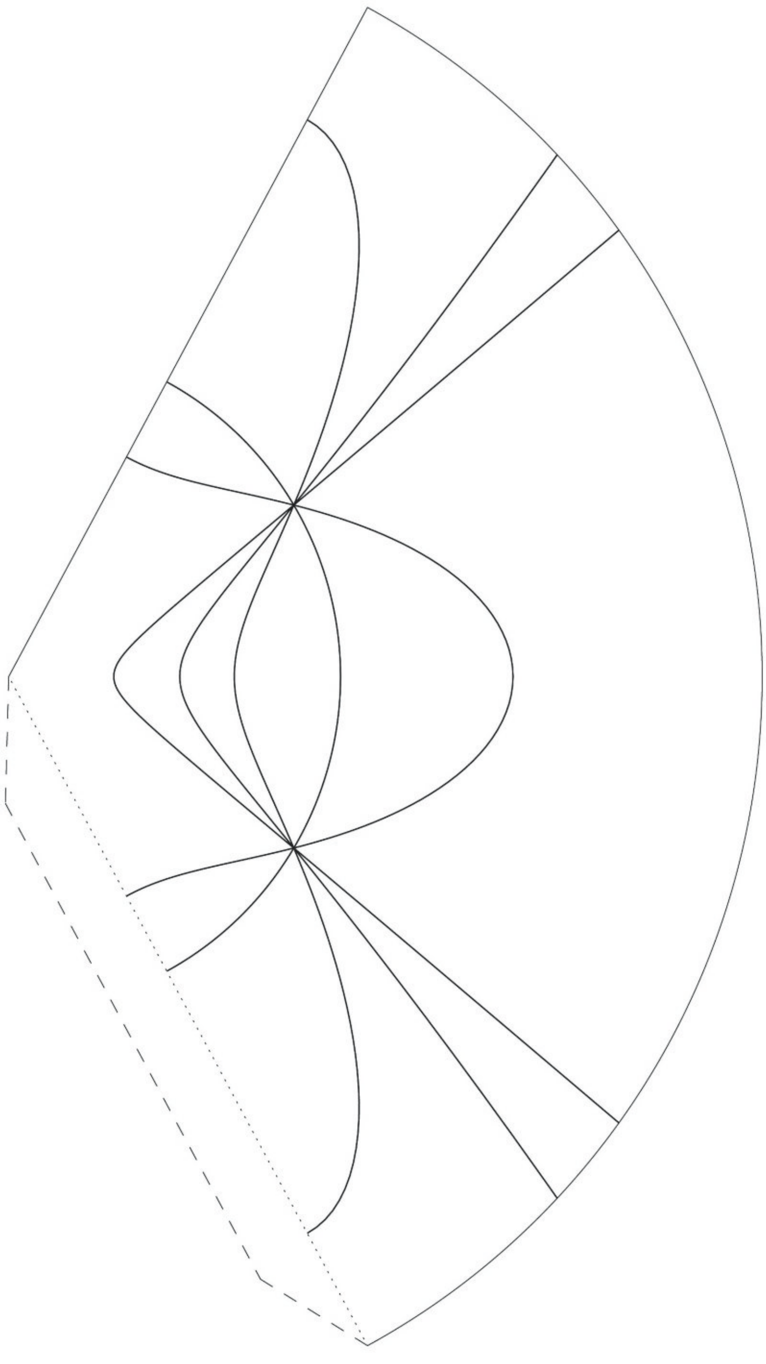
27. Dandelin's spheres for hyperbola and parabola

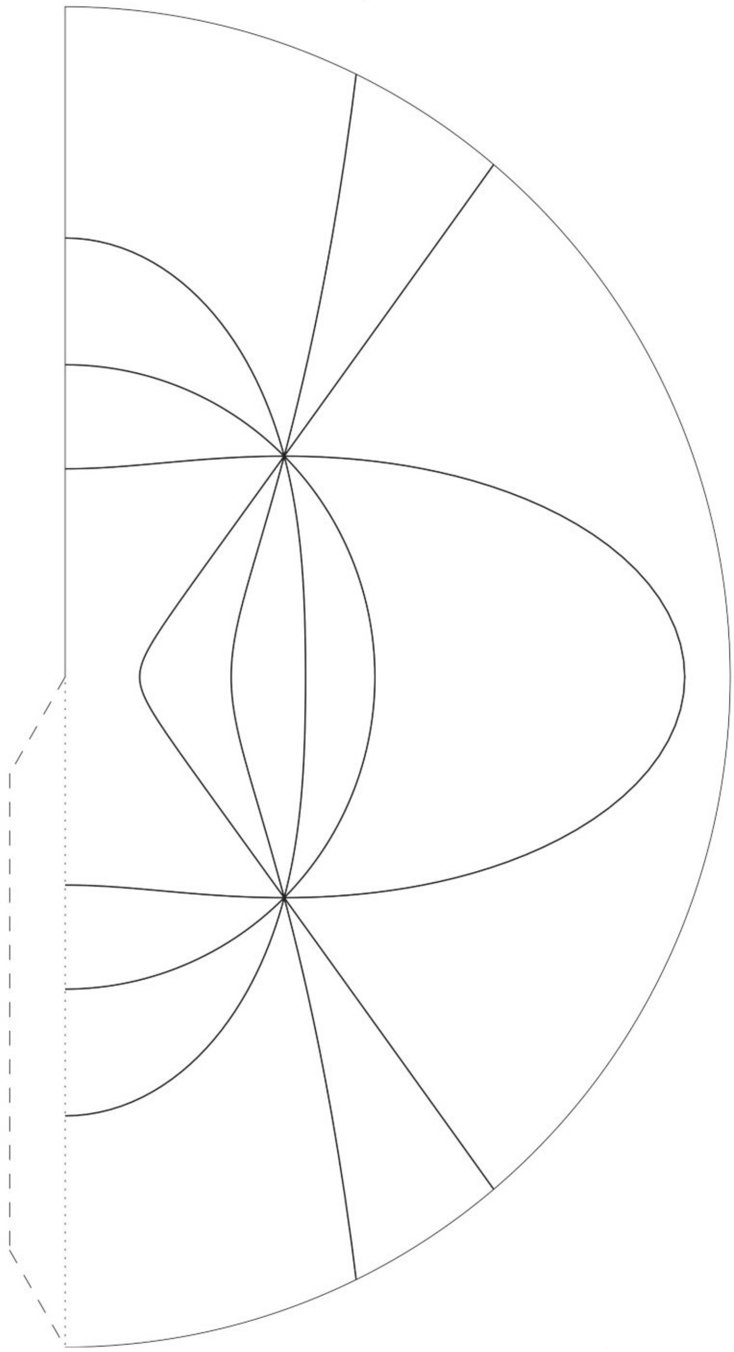
5. On the following page you can see the two-sided (grey) cone from the side.For the hyperbola we can only perform our two tasks if we make use of both sides of the cone. We will prove that the intersection, even if you cannot see it, is a hyperbola.
a. Points P, T and F_1 have already been marked. Add F_2 and the R-points.
b. From here, you should be able to complete the proof that the plane of intersection is a hyperbola!



6. The parabola is a borderline case between ellipse and hyperbola: the cross-section with the cone now has to be parallel with a generating line. There is only one Dandelin sphere.
   a. You can only add one focal point *F* and one *R*-point. Do so and conclude again *PF* = *PR*.
   b. The directrix of the parabola will be the intersection line *q* of the plane of the cross-section and the plane through the tangent circle. In the illustration *q* looks like a point. The footpoint of *P* on *q* is *V*; mark *V* in the figure.
   c. *PV* should be equal to *PF*. Use *PV* // *TS*. Draw *P'V'* equal to *PV* on *TS*, with *V'* on the tangent circle. Now finish the proof that the cross-section is a parabola in the sense of the earlier definition.
7. You can easily make these paper cones using the cutouts on the following pages. Parabola, ellipse, circle and hyperbola are all included, even if it looks quite different on the cutout. Copy (enlarge if possible), cut, glue.







Sources of some of the illustrations

Page 10: Stills from *Steamboat Bill Jr. with Buster Keaton, 1927.* http://www.pdcomedy.com/Movies/BusterKeaton/SteamboatBillJr. Public domain.

Page 41: Voronoi diagram for the U.S. airports, 2008. From http://bl.ocks.org/mbostock, a collection of computational geometry illustrations by Mike Bostock.

Page 61, 334: Unterweysung der Messung mit dem Zirckel und Richtscheyt, Albrecht Dürer, 1525.

Page 70: From a 12th century manuscript in the library of the medieval monastery of Mont Saint Michel. Copied from a postcard.

Page 183, 185: The school of Athens, fresco by Rafaello Sanzio, 1483–1520. Stanza della Segnatura, Vatican city.

Page 251: From http://pd2alx.nl/dwingeloo.html, a web page about the Dwingeloo Radio Telescope.

Page 259: The Brent Spar. From http://www.greenpeace.org/international/en/about/history/the-brent-spar.

Page 271: Saturn as seen from the Hubble space telescope. From http://apod.nasa.gov/apod/astropix.html; NASA.

Page 324: Albanian postal stamp, 1996. Found on http://curvebank.calstatela.edu/descartes/descartes.htm.

Page 327: Stained-glass windows in the Saint Denis cathedral, Paris. From traveltoeat.com/Saint-denis.

The majority of the other drawings and photographs were produced in the Profi-project, between 1995 and 2004; see introduction. For some of the other illustration used in that project the sources could not be retrieved during the 2014 revision.