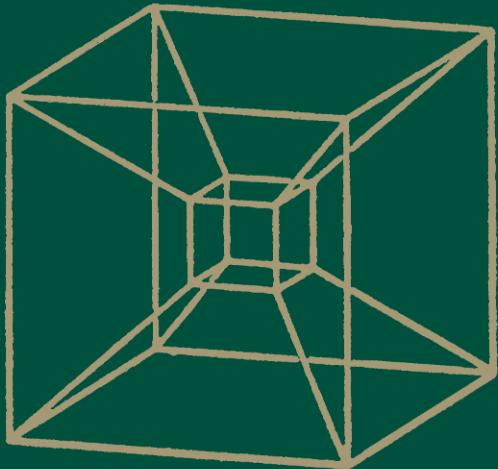


# Contributions to Geometry

Proceedings of the Geometry

Symposium in Siegen 1978

J. Tölke and J.M. Wills



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Contributions to Geometry  
Beiträge zur Geometrie

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Proceedings of the Geometry-Symposium held in Siegen  
June 28, 1978 to July 1, 1978

Edited by  
Jürgen Tölke and Jörg M. Wills  
University of Siegen



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Dedicated to  
Prof. Dr. HUGO HADWIGER

## Preface

During the time from June 28–July 1, 1978, representatives of different branches of geometry met in Siegen for discussion of and reports on current problems. In particular, the survey lectures, presented by well known geometers, gave nonspecialists the welcome opportunity to learn about the questions posed, the methods used and the results obtained in different areas of the field of geometry. The research areas represented at the meeting in Siegen are reflected in the list of participants and their contributions: Ranging from geometric convexity and related topics to differential geometry and kinematics. The foundations of geometry, an area well established in Germany, was also represented.

It is a pleasure to thank all the lecturers as well as other participants in the Geometry Symposium for their contribution to the success of the meeting. We also thank the “Minister für Wissenschaft und Forschung des Landes Nordrhein-Westfalen” and the University of Siegen for their generous support which helped make the Symposium so successful.

In order to make the contributions and results of the Symposium accessible to the general public, the publication of a proceedings volume was planned. The idea was to give a summary of a wide spectrum of research in geometry — through survey articles and original research papers.

This volume is composed of three parts: The first part contains nine broad survey articles in the general area of geometric convexity, and in particular on convex bodies, integral geometry, combinatorial geometry, discrete geometry, and geometry of numbers. These articles are complemented by an original paper and a list of 105 open problems.

The second part contains nine original papers and survey articles in the areas of differential geometry and kinematics, as well as one paper on geometric aspects of approximation theory. The volume ends with four contributions from the area of foundations of geometry.

The variety of topics and the richness of new results and open problems included in this volume reflect impressively the continuing vigour of geometric ideas and developments.

We thank all the authors for their contributions and Birkhäuser Verlag for its friendly cooperation.

Salvador and Siegen, 1979

J. Tölke and J.M. Wills

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## I. Geometric convexity

# Rolf Schneider

## Boundary structure and curvature of convex bodies

One of the fascinating features of the theory of convex bodies is the wealth of substantial results that spring from the mere assumption of convexity. The present survey is concerned with the implications that convexity of a point set has on the structure of its boundary. We have tried to collect the known results which describe, or are connected with, local geometric properties of the boundary of a convex body. Some open problems in this field will also be mentioned.

Essentially, this paper is restricted to general convex bodies in finite-dimensional Euclidean spaces. Therefore, references to corresponding results in infinite dimensional vector spaces are given only occasionally. Convex sets in finite dimensional spaces other than the Euclidean are not considered, although in recent years important work has been done, in particular, in the field of (geodesically) convex sets in Riemannian spaces. For references we refer the reader to Burago-Zalgaller [1977], Bangert [1977], [1978a], Walter [1979]. Further, since general convex bodies are the objects of interest here, we shall, as a rule, not consider results which require additional smoothness assumptions. Investigations pertaining to the differential geometry of general convex surfaces, although they may well be regarded as contributions to the local structure of convex bodies, are mentioned only as far as they are concerned with curvature properties. Comprehensive presentations are found in the books by Aleksandrov [1948], Busemann [1958], Pogorelov [1969].

In chapter I, results on different types of boundary points and faces of convex bodies are collected. Here the relevant notions are defined in terms of supporting hyperplanes or convex combinations in the boundary and may, therefore, be considered as describing first order properties of convex surfaces. Second order properties are the subject of chapter II. Here the emphasis is on a unified treatment and the applications of the area functions of convex bodies introduced by Aleksandrov [1937b] and Fenchel-Jessen [1938] and of the curvature measures of Federer [1959], restricted to convex sets. These measures replace, for general convex bodies, the differential geometric curvature functions, as they can be defined for bodies with a sufficiently smooth boundary surface. Thus they may well be considered as second order entities containing information on the local shape of a convex body.

The section headings are as follows.

- I. Boundary structure
  - §1. Singular points
  - §2. Extreme points
  - §3. Faces
- II. Curvature measures
  - §4. Second derivatives and pointwise curvatures
  - §5. Definition and properties of curvature measures
  - §6. Characterization theorems
  - §7. Integral geometric formulas
  - §8. Local properties

- §9. Uniqueness and stability theorems
- §10. Existence theorems

We collect some notation which will be used throughout the article. By  $E^d$  we denote the  $d$ -dimensional Euclidean vector space with scalar product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\|$ .

$$S^{d-1} := \{u \in E^d : \|u\| = 1\}$$

is its unit sphere,  $L^{d-1}$  denotes spherical Lebesgue measure on  $S^{d-1}$ , and  $\mathcal{H}^p$  is  $p$ -dimensional Hausdorff measure on  $E^d$ . The number  $\alpha(k) := \pi^{k/2}/\Gamma(1+k/2)$  is equal to the volume of the  $k$ -dimensional unit ball.

$\mathfrak{K}^d$  denotes the set of convex bodies (non-empty, compact, convex sets) in  $E^d$ ,  $\mathfrak{K}_0^d$  is the subset of convex bodies with interior points. As usual,  $\mathfrak{K}^d$  is equipped with the Hausdorff metric, and convergence in  $\mathfrak{K}^d$  is understood with respect to this metric.

Finally, the abbreviations int, cl, card, dim have the usual meanings, i.e., interior, closure, cardinality, dimension.

## I. Boundary structure

### §1. Singular points

The boundary point  $x$  of the convex body  $K \in \mathfrak{K}_0^d$  is called a *regular* point of  $K$  if through  $x$  there exists only one support plane of  $K$ ; otherwise it is called a *singular* point. The singular points can be classified according to the dimension of the cone of normal vectors: Let us say that  $x$  is an  $r$ -singular point if at  $x$  there exist at least  $d-r$  linearly independent normal vectors of  $K$ . Thus  $x$  is  $r$ -singular if and only if it is, in the terminology of Bonnesen-Fenchel [1934], p. 14, a  $p$ -Kantenpunkt with  $p \leq r$ . The following result is due to Anderson and Klee [1952].

(1.1) **Theorem.** *The set of  $r$ -singular points of the convex body  $K \in \mathfrak{K}_0^d$  is contained in the union of countably many compact sets of finite  $r$ -dimensional Hausdorff measure ( $r = 0, \dots, d-1$ ).*

For the proof, consider the metric projection  $\pi_K$  on to  $K$ , that is, the map which associates with each point of  $E^d$  the (unique) nearest point in  $K$  (see McMullen-Shephard [1971], pp. 30–36, for an investigation of this map, which is also called the nearest-point map). Let  $M_r$  denote the intersection of a closed ball containing  $K$  in its interior with the union of all  $r$ -dimensional affine subspaces of  $E^d$  which are spanned by points with rational coordinates. If  $x$  is an  $r$ -singular point of  $K$ , the normal cone  $\pi_K^{-1}(\{x\})$  of  $K$  at  $x$  is at least  $(d-r)$ -dimensional and hence meets  $M_r$ , thus  $x$  is in the image of  $M_r$  under  $\pi_K$ . As  $M_r$  is a union of countably many compact sets of finite  $r$ -dimensional Hausdorff measure and as  $\pi_K$  is a Lipschitz map, the assertion follows.

Note that one can similarly show that the set of  $r$ -singular points of  $K$  with a normal  $(d-r)$ -cone whose internal angle exceeds a given positive constant has finite  $r$ -dimensional Hausdorff measure.

For  $d = 3$ , a different proof of (1.1) has been given by Besicovitch [1963a]. For a

generalization to certain non-convex sets see Federer [1959], 4.15(3) on p. 447. Theorem (1.1) implies that the set of all singular points of  $K$  has  $(d-1)$ -dimensional Hausdorff measure zero. This was first proved, for  $d = 3$ , by Reidemeister [1921], see also Aleksandrov [1948], chapter V, §2 (in a dual formulation). A weaker form of Theorem (1.1) was claimed without proof by Favard [1933a], p. 228.

The above method of proof also yields immediately Fujiwara's [1916] observation that a threedimensional convex body has at most countably many straight edges, as well as the obvious higher-dimensional analogues of this result.

The above results can also be formulated in terms of differentiability properties of convex functions. Let  $f$  be a real convex function, defined on some open convex subset  $D$  of  $\mathbb{R}^{d-1}$ , and let  $x \in D$ . Then for every vector  $y \in \mathbb{R}^{d-1}$  the directional derivative  $f'(x; y)$  exists. The following properties are equivalent. (a) The partial derivatives of  $f$  at  $x$  exist; (b)  $f'(x; y)$  is a linear function of  $y$ ; (c)  $f$  is (totally) differentiable at  $x$ ; (d)  $(x, f(x))$  is a regular point of the epigraph of  $f$ . For these and further results see, e.g., Bonnesen-Fenchel [1934], section 13, Botts [1942], Rockafellar [1970], section 25, Stoer-Witzgall [1970], section 4.4, Roberts-Varberg [1973], section 44, Marti [1977], section X.3. Thus the above result implies, in particular, that every convex function of  $d-1$  real variables is almost everywhere differentiable. This is just a special case of Rademacher's theorem according to which every locally Lipschitzian function is almost everywhere differentiable (see, e.g., Federer [1969], p. 216). For different proofs in the case of convex functions, see Busemann [1958], p. 13, Rockafellar [1970], section 25, Roberts-Varberg [1973], p. 116, compare also Blaschke [1916], §20.II.

About possible infinite dimensional generalizations we mention only Mazur's [1933] density theorem for regular points (see also Marti [1977], p. 112) and point out that Asplund [1968] has studied the Banach spaces where every continuous convex function is (Fréchet, Gâteaux) differentiable on a dense  $G_\delta$  subset of its domain of continuity. In particular he proves that every continuous convex function is Gâteaux differentiable on a dense  $G_\delta$  subset of its domain of continuity if the dual ball is strictly convex. It is unknown whether or not it is enough to assume that the unit ball is smooth.

## §2. Extreme points

The point  $x$  of the convex set  $C$  is called an *extreme* point of  $C$  if  $C \setminus \{x\}$  is convex, equivalently if  $x$  is not a relatively interior point of a segment contained in  $C$ . By a theorem of Minkowski [1911], §12, every convex body in  $E^d$  is the convex hull of its set of extreme points. This result has important infinite-dimensional generalizations due to Krein-Milman and Choquet, see, e.g., Bauer [1964], Phelps [1966], Choquet [1969], Alfsen [1971]; see also the elementary introductory article by Jacobs [1971] and the comprehensive review of Šaškin [1973] which contains many references up to 1972. Of the many existing papers on extreme points of infinite-dimensional convex sets we mention only Jayne-Rogers [1977]. In the following we pick out only those results which are of interest in the finite-dimensional case.

More generally, the point  $x$  of the convex set  $K \subset E^d$  is called an  $r$ -extreme point of  $K$  if it does not lie in the relative interior of any  $(r+1)$ -dimensional convex subset of  $K$  ( $r=0, \dots, d-1$ ). The set of all the  $r$ -extreme points of  $K$  is called its  $r$ -skeleton and denoted by  $\text{skel}_r K$ . Thus  $\text{skel}_0 K = \text{ext } K$ , the set of extreme points of  $K$ , and  $\text{skel}_{d-1} K = \partial K$ , the boundary of  $K$ . The point  $x \in K$  is called an  $r$ -exposed point of  $K$  if it is contained in a supporting hyperplane of  $K$  whose intersection with  $K$  is of dimension at most  $r$  (the 0-exposed points are briefly called exposed). Thus every  $r$ -exposed point is  $r$ -extreme, and it has been proved that every  $r$ -extreme point is a limit of  $r$ -exposed points. This result is due to Straszewicz [1935] for  $r=0$  (see also Wets [1974], and for the infinite dimensional case, Klee [1958] and Bair [1976]) and to Asplund [1963] in general. Further, an  $r$ -singular point is  $r$ -exposed. The set of all  $r$ -exposed points of  $K$  is called the *exposed r-skeleton* of  $K$ .

We start with some topological questions concerning  $\text{ext } K$  and  $\text{exp } K$ , the sets of extreme points and exposed points of  $K \in \mathfrak{K}_0^d$ , respectively. It is easy to see that  $\text{ext } K$  is always a  $G_\delta$  set. (About not necessarily closed convex sets we mention the following result of H. Wegner [1974]: The set of extreme points of a convex Borel set in  $E^d$  is itself a Borel set.) If  $\dim K \leq 2$ ,  $\text{ext } K$  is closed, but for  $\dim K \geq 3$ ,  $\text{ext } K$  need not be closed (for example, in  $E^3$  let  $K$  be the convex hull of a circular disk  $D$  and a segment  $T$  orthogonal to the plane of the disk, where  $D \cap T = \{x\}$  and  $x$  is a relative boundary point of  $D$  and a relatively interior point of  $T$ ; then  $x$  is a limit of extreme points of  $K$ , but not itself an extreme point). Klee [1958] posed the following problem (for  $d=3$ ): Find a useful and simple characterization of the class  $\chi_d$  of all subsets  $X$  of the unit sphere  $S^{d-1}$  in  $E^d$  such that there is a homeomorphism of  $S^{d-1}$  onto the boundary of a convex body  $K \in \mathfrak{K}_0^d$  mapping  $X$  onto  $\text{ext } K$ . (For the situation in the infinite-dimensional case, see Klee [1955].) Clearly each closed subset of  $S^{d-1}$  with at least  $d+1$  affinely independent points is a member of  $\chi_d$ . The following partial answers to Klee's question have been given by Collier [1975].

(2.1) **Theorem.** *Let  $X$  be a subset of a compact 0-dimensional metric space  $Z$ , and let  $d \geq 3$  be an integer. There is a homeomorphism of  $Z$  into the boundary of a convex body  $K \in \mathfrak{K}_0^d$  mapping  $X$  onto  $\text{ext } K$  if and only if  $X$  is a  $G_\delta$  set with at least  $d+1$  points.*

**Corollaries:** *Any  $G_\delta$  subset of  $S^2$  with at least four points and whose closure is 0-dimensional is a member of  $\chi_3$ . Any subset of  $S^{d-1}$  with at least  $d+1$  points and whose closure is countable is a member of  $\chi_d$ .*

On the other hand, non-trivial examples of subsets of  $S^2$  which do not belong to  $\chi_3$  can be constructed by means of the following result of Collier [1976]: If  $K \in \mathfrak{K}_0^3$ , then each component of  $\text{cl}(\text{ext } K) \setminus \text{ext } K$  is a subset of a 1-dimensional face of  $K$ .

About  $\text{exp } K$  and its position in  $\text{ext } K$  the following is known. For  $\dim K = 2$ ,  $\text{exp } K$  is a  $G_\delta$  set, but Klee [1958] gives an example of a convex body  $K \in \mathfrak{K}_0^3$  for which  $\text{exp } K$  is not a  $G_\delta$  set, and Corson [1965] constructs a  $K \in \mathfrak{K}_0^3$  for which  $\text{exp } K$  is not even the union of a  $G_\delta$  set with an  $F_\sigma$  set, and contains no dense  $G_\delta$  set. He also constructs a  $K \in \mathfrak{K}_0^3$  for which  $\text{exp } K$  is the union of a countable number of closed sets each of which has no interior with respect to  $\text{exp } K$ . Choquet, Corson and Klee [1955] proved the following.

(2.2) **Theorem.** For  $K \in \mathfrak{K}_0^d$ ,  $\exp K$  is the union of a  $G_\delta$  set, an  $F_\sigma$  set, and  $d - 2$  sets each of which is the intersection of a  $G_\delta$  set with an  $F_\sigma$  set. If  $K \in \mathfrak{K}_0^3$ ,  $\exp K$  is the union of a  $G_\delta$  set and a set which is the intersection of a  $G_\delta$  set and an  $F_\sigma$  set.

The authors ask whether, for  $K \in \mathfrak{K}_0^3$ ,  $\exp K$  must be the intersection of a  $G_\delta$  set and an  $F_\sigma$  set. Klee [1958] also proves the following. If  $K \in \mathfrak{K}_0^d$  is smooth, then  $\exp K$  is a  $G_\delta$  set and  $\text{ext } K \setminus \exp K$  is a first category  $F_\sigma$  set in  $\text{ext } K$ . Further he shows that for  $d \geq 2$  there exists a smooth body  $K \in \mathfrak{K}_0^d$  such that  $\text{ext } K$  is closed and  $\text{ext } K \setminus \exp K$  is dense in  $\text{ext } K$ , and he proves the existence of a body  $K \in \mathfrak{K}_0^3$  such that  $\partial K \setminus \text{ext } K$ ,  $\text{ext } K \setminus \exp K$ , and  $\exp K$  are all dense in  $\partial K$ .

Similar questions might be asked for  $r$ -extreme and  $r$ -exposed points, where  $r > 0$ . Only a few results on the topological structure of  $\text{skel}_r K$  are known. It is not difficult to see that  $\text{skel}_r K$  is a  $G_\delta$  set (see Larman and Rogers [1973] for results in the infinite-dimensional case). Guided by some wellknown results in the theory of convex polytopes, Larman and Rogers [1970], [1971] have investigated the connectivity properties of the skeletons of convex bodies. They established the following results.

(2.3) **Theorem.** Let  $a$  and  $b$  be two distinct exposed points of a convex body  $K \in \mathfrak{K}_0^d$ . Then there are  $d$  simple arcs  $P_1, \dots, P_d$  in the  $l$ -skeleton of  $K$ , each joining  $a$  to  $b$ , such that  $P_i \cap P_j = \{a, b\}$  for  $1 \leq i < j \leq d$ .

The authors also give simple examples of a convex body in  $E^3$  whose  $l$ -skeleton is not locally connected and of a convex body in  $E^4$  whose  $l$ -skeleton is not closed. They obtain, in two refined versions, the following result on the existence of increasing paths.

(2.4) **Theorem.** Let  $L$  be a non-constant linear function on  $E^d$  and let  $K \in \mathfrak{K}_0^d$ . Then there is a continuous map  $s$  of the closed interval  $[0,1]$  to the exposed  $l$ -skeleton of  $K$  with

$$L(s(0)) = \inf_{x \in K} L(x), \quad L(s(1)) = \sup_{x \in K} L(x),$$

$$L(s(t_1)) < L(s(t_2)), \text{ when } 0 \leq t_1 < t_2 \leq 1.$$

Larman [1977] has been able to prove extensions of these results to infinite dimensional compact convex sets in a normed space. A known result on polytopes suggests the following open question.

(2.5) **Problem** (Larman). Within the  $l$ -skeleton of a  $d$ -dimensional convex body find a realization of the complete  $(d+1)$ -graph by arcs.

Next we come to some questions of a metric nature. For every convex body  $K \in \mathfrak{K}_0^d$ , and for  $r = 1, \dots, d-2$ , the inequality

$$(2.6) \quad \mathcal{H}^r(\text{skel}_r K) > (d-r+1)V_r(K)$$

has been proved by Schneider [1978b]. Here

$$V_r(K) := \frac{\binom{d}{r}}{\alpha(d-r)} W_{d-r}(K),$$

and  $W_j$  is the  $j$ -th quermassintegral.

Trivially (2.6) holds with equality for  $r = d - 1$ , and with equality admitted for  $r = 0$ , where it simply says that  $\text{card ext } K \geq d + 1$ , which is true by Minkowski's theorem. From this special case the general case is deduced by means of an integral geometric argument. The inequality (2.6) is sharp, that is, the ratio  $\mathcal{H}^r(\text{skel}_r K)/V_r(K)$  can be made arbitrarily near to the numerical factor in (2.6), e.g., by choosing for  $K$  the Cartesian product of a small  $(d-r)$ -simplex and a large  $r$ -simplex. Inequality (2.6) for polytopes and a weaker form of it for general convex bodies had been proved earlier by Firey and Schneider [1979].

From (2.6) we may deduce an inequality of isoperimetric type. Since  $W_{d-r}(K)^{d-1} \geq \alpha(d)^{d-r-1} W_1(K)^r$ , which is a special case of the Fenchel-Aleksandrov inequalities (see, e.g., Busemann [1958], pp. 49–50), and since  $d W_1(K) = \mathcal{H}^{d-1}(\partial K)$  is the surface area of  $K$ , we get

$$(2.7) \quad \mathcal{H}^r(\text{skel}_r K)^{d-1} > \beta(d, r) \mathcal{H}^{d-1}(\text{skel}_{d-1} K)^r$$

with

$$\beta(d, r) := \left[ \binom{d}{r} \frac{(d-r+1)\alpha(d)}{\alpha(d-r)} \right]^{d-1} (d\alpha(d))^{-1}$$

$(r = 1, \dots, d-2)$ . This leads to the following open question.

(2.8) **Problem.** For  $1 \leq r < s \leq d$  determine

$$c(d, r, s) := \inf_K \mathcal{H}^r(\text{skel}_r K)^{1/r} / \mathcal{H}^s(\text{skel}_s K)^{1/s},$$

where  $K$  runs through the convex bodies  $K \in \mathfrak{K}_0^d$  with  $\mathcal{H}^s(\text{skel}_s K) < \infty$ . Especially, is  $c(d, r, s) > 0$ ? Define  $c^*(d, r, s)$  analogously, but with  $K$  running only through the convex polytopes. Is  $c(d, r, s) = c^*(d, r, s)$ ?

The value  $c(d, d-1, d)$  is known from the classical isoperimetric inequality, and (2.7) gives a positive lower bound for  $c(d, r, d-1)$  and hence, by the isoperimetric inequality, for  $c(d, r, d)$ . Eggleston, Grünbaum and Klee [1964], Th. 5.5, proved that  $c^*(d, r, s) \geq 1$  if  $r$  is a divisor of  $s$ . Some further results have been obtained by Larman and Mani [1970], but it is still not known whether  $c^*(d, r, s) > 0$  in general.

Eggleston, Grünbaum and Klee [1964] also described the following “natural way of assigning such quantities as total edge-length and total area of 2-faces to an arbitrary convex body”. For a convex polytope  $P \in \mathfrak{K}_0^d$  and for  $s = 0, \dots, d-1$  let

$$\zeta_s(P) := \mathcal{H}^s(\text{skel}_s P),$$

the total  $s$ -dimensional volume of the  $s$ -faces of  $P$ . These functions are shown to be lower semicontinuous: If  $(P_j)_{j \in \mathbb{N}}$  is a sequence of polytopes in  $\mathfrak{K}_0^d$  converging to a polytope  $P$ , then

$$\liminf_{j \rightarrow \infty} \zeta_s(P_j) \geq \zeta_s(P).$$

Now for an arbitrary convex body  $K \in \mathfrak{K}^d$  let

$$\zeta_s(K) := \liminf_{\substack{P \rightarrow K \\ P \in \mathfrak{P}^d}} \zeta_s(P),$$

where  $\mathfrak{P}^d \subset \mathfrak{K}_0^d$  is the set of convex polytopes; then the extended functions are also lower semicontinuous. The authors pose the problem of finding a simple geometric characterization of the members  $K$  of  $\mathfrak{K}^d$  for which  $\zeta_s(K) < \infty$ , in the cases  $1 \leq s \leq d - 2$ . Using the integralgeometric method of Schneider [1978b] and applying Fatou's lemma one can show that

$$(2.9) \quad \int \text{card ext}(E \cap K) d\mu(E) \leq \beta_1(d,s) \zeta_s(K),$$

where the integration is over the space of all  $(d-s)$ -dimensional affine subspaces  $E$  of  $E^d$ ,  $\mu$  denotes the rigid motion invariant measure over that space, and  $\beta_1(d,s)$  is a positive constant depending on the normalization of  $\mu$ . Hence every convex body  $K \in \mathfrak{K}_0^d$  with  $\zeta_s(K) < \infty$  has the following property.

(2.10) *The function  $E \mapsto \text{card ext}(E \cap K)$  is  $\mu$ -integrable, in particular, for  $\mu$ -almost every  $(d-s)$ -plane  $E$  the set  $\text{ext}(E \cap K)$  is finite, hence  $E \cap K$  is a polytope.*

This might perhaps help to describe the bodies  $K$  with  $\zeta_s(K) < \infty$  geometrically. Apparently the first essential question to answer would be: What does (2.10) imply for the set of extreme points of  $K$ ? For instance, in the case  $d = 3$ ,  $s = 1$ , must  $\text{ext } K$  be countable? If not, is  $\text{ext } K$  of one-dimensional Hausdorff measure zero?

Another natural way to extend the functions  $\zeta_s$  from  $\mathfrak{P}^d$  to  $\mathfrak{K}^d$  would just be to define

$$\eta_s(K) := \mathcal{H}^s(\text{skel}_s K) \quad \text{for } K \in \mathfrak{K}^d.$$

Then  $\eta_s(P) = \zeta_s(P)$  for  $P \in \mathfrak{P}^d$ , and the following question arises naturally.

(2.11) **Problem.** Is  $\eta_s = \zeta_s$  on  $\mathfrak{K}^d$ ?

As long as this is unknown, it is worth remarking that the method of Schneider [1978b] also shows that

$$(2.12) \quad \int \text{card ext}(E \cap K) d\mu(E) \leq \beta_1(d,s) \eta_s(K),$$

and hence the bodies  $K$  with  $\mathcal{H}^s(\text{skel}_s K) < \infty$  also have property (2.10). One would also like to know whether  $\eta_s$  is lower semi-continuous on  $\mathfrak{K}^d$ .

Talking of semicontinuity, we remark that Price [1937], Th. 9.6, and Jerison [1954], Th. 2, have established results on the extreme points of sequences of compact convex sets which, in the special case of converging sequences in  $\mathfrak{K}^d$ , can be interpreted as a semi-continuity property of  $\text{ext}$ . This can be generalized as follows.

(2.13) **Theorem.** Let  $(K_j)_{j \in \mathbb{N}}$  be a sequence in  $\mathfrak{K}^d$  converging to the convex body  $K$ . Then, for  $r = 0, \dots, d-1$ ,

$$\liminf_{j \rightarrow r} \text{skel}_r K_j \supset \text{cl } \text{skel}_r K.$$

Here  $\liminf$  is understood in the topological sense, i.e.,  $x \in \liminf A_j$  if and only if every neighborhood of  $x$  meets  $A_j$  for almost all  $j$ . For a proof of (2.13), see §8.

### §3. Faces

A *face* or *extreme subset* of the convex set  $K \subset E^d$  is a convex subset  $F$  of  $K$  such that every closed line segment in  $K$  whose relative interior meets  $F$  is contained in  $F$ . Every intersection of  $K$  with a supporting hyperplane is a face; these special faces are called *exposed*. (Here one should observe that several authors use a different terminology. For instance, Grünbaum [1967], section 2.4, uses “face” instead of “exposed face”, and “poonem” instead of “face”; compare also his exercise 2.4.7). The family of all faces of a closed convex set, partially ordered by inclusion, is a complete lattice; the same holds for the family of exposed faces. The collection of all relative interiors of non-empty faces of the convex set  $K$  is a partition of  $K$  (see Rockafellar [1970], p. 164). A general study of faces and exposed faces is made in Bair-Fourneau [1976], chapter II. The facial structure of infinite dimensional convex sets is complicated and not yet fully explored. Some of the less obvious facial properties of finite-dimensional convex bodies are described in the following.

Collier [1976], in the course of establishing his result mentioned after Theorem (2.1), obtained the following detail of the facial structure: The family consisting of the relative interiors of all the  $(d-2)$ -dimensional faces of the convex body  $K \in \mathbb{R}_0^d$  has a quotient topology which is paracompact.

Much easier to grasp from an intuitive point of view, but not easy to prove, is the following result.

(3.1) **Theorem.** *For  $K \in \mathbb{R}_0^d$ , the union of the relative boundaries of the faces of  $K$  of dimension at least 1 has zero  $(d-1)$ -dimensional Hausdorff measure.*

In other words, the relative interiors of the maximal proper faces of  $K$  cover almost all of the boundary of  $K$ . This was conjectured by Klee and Martin [1971] (see also [1970]), who proved it for  $d \leq 3$ , and has been established in general by Larman [1971]. The question arose in connection with an investigation of the face-function of convex sets. For  $K \in \mathbb{R}_0^d$  let  $X = K$  or  $\partial K$ , and for  $x \in X$ , let  $F(x)$  denote the smallest face of  $K$  that includes  $x$ ; then  $F$  is called the *face-function* of  $X$ . Let  $X_l$  (respectively,  $X_u$ ) be the set of all points  $x$  of  $X$  at which  $F$  is lower (resp., upper) semicontinuous, which means that  $\liminf F(x_j) \supset F(x)$  (resp.,  $\limsup F(x_j) \subset F(x)$ ) for every sequence  $(x_j)_{j \in \mathbb{N}}$  in  $X$  converging to  $x$ ; here  $\liminf$  and  $\limsup$  are understood in the topological sense. Klee and Martin [1971], Th. 3.2, show that  $(\partial K)_l$  is the union of the relative interiors of all the maximal convex subsets of  $\partial K$ . Thus (3.1) shows that the face-function of  $\partial K$  is upper semicontinuous almost everywhere in the sense of measure. Klee and Martin [1971] also prove that  $(\partial K)_u$  is an  $F_{\sigma\delta}$  set, and they describe three-dimensional examples showing that  $(\partial K)_u$  may fail to be either an  $F_\sigma$  or a  $G_\delta$  set, and that it may be of the first category in  $\partial K$ . About lower semi-continuity of the face-function, Klee and Martin [1971] obtained the following result.

(3.2) **Theorem.** For  $K \in \mathfrak{K}_0^d$ , the face-function  $F$  of  $\partial K$  is lower semicontinuous almost everywhere in the sense of category, but when  $d \geq 3$  examples show that  $F$  may be lower semicontinuous almost nowhere in the sense of measure.

Lower semicontinuity of face-functions is closely connected with continuity of so-called envelopes or convex hulls of functions. For  $K \in \mathfrak{K}_0^d$  and each continuous function  $f: K \rightarrow \mathbb{R}$  let  $f_e := \sup \{g: g \text{ is an affine function on } E^d \text{ satisfying } g \leq f \text{ on } K\}$ ; then  $f_e$  is called the *envelope* of  $f$ . Let  $K_e$  denote the set of all points of  $K$  at which every envelope is continuous. Kruskal [1969] showed by an example that  $K_e \neq K$  is possible for  $d \geq 3$ . He conjectured that  $K_e = K$  for  $d = 2$ , and this was proved by Klee and Martin [1971]. Completing one of their further results, namely that  $K_e \subset K_1$ , Eisler [1977] showed:

(3.3) **Theorem.**  $K_e = K_1$  for  $K \in \mathfrak{K}_0^d$ .

The convex bodies  $K$  for which  $K_1 = K$  have been characterized by Papadopoulou [1977] as follows.

(3.4) **Theorem.** For  $K \in \mathfrak{K}_0^d$  the following assertions are equivalent:

- (a)  $K$  is stable, that is, the map  $(x,y) \mapsto \frac{1}{2}(x+y)$  from  $K \times K$  onto  $K$  is open,
- (b) the face-function of  $K$  is lower semicontinuous,
- (c) all skeletons  $\text{skel}_r K$ ,  $r = 0, 1, \dots, d$ , of  $K$  are closed (equivalently, the function  $x \mapsto \dim F(x)$  is lower semicontinuous on  $K$ ).

For  $d > 2$  the convex bodies  $K$  with properties (a), (b), (c) of (3.4) are also characterized by the fact that the metric space consisting of the proper faces of  $K$  with the Hausdorff metric is compact. Similarly, the metric space of all exposed faces is compact if and only if all the exposed  $r$ -skeletons of  $K$  are closed,  $0 \leq r \leq d-2$ . These results are due to Reiter-Stavrakas [1977].

Now we consider, more generally, arbitrary  $r$ -dimensional convex subsets in the boundary of a convex body. The question arises as to "how many" of them there can be. First we look at the case  $r = 1$ . For  $K \in \mathfrak{K}^d$ , let  $D(K) \subset S^{d-1}$  denote the set of directions of line segments in the boundary of  $K$ , i.e.,

$$D(K) := \left\{ \frac{a-b}{\|a-b\|} : a \neq b, \text{conv } \{a,b\} \subset \partial K \right\}.$$

Klee [1957] asked whether  $D(K)$  is of first category in  $S^{d-1}$  and of  $(d-1)$ -dimensional measure zero. The case  $d = 2$  is trivial,  $D$  being countable. For  $d = 3$ , McMinn [1960] gave an affirmative answer showing that  $D(K)$  is contained in the union of the ranges of countably many Lipschitz mappings from  $[-1,1]$  to  $S^2$ . In particular,  $D(K)$  is of  $\sigma$ -finite one-dimensional Hausdorff measure. A shorter proof of this fact was given by Besicovitch [1963b]. For  $d \geq 4$  Klee [1969] posed the problem of showing that  $D(K) \neq S^{d-1}$ . It was settled by the following result of Ewald, Larman and Rogers [1970].

(3.5) **Theorem.** For  $K \in \mathbb{R}^d$  the set  $D(K)$  of directions of line segments in the boundary of  $K$  has  $\sigma$ -finite  $(d-2)$ -dimensional Hausdorff measure.

Clearly, this result is best possible: If  $\partial K$  contains a  $(d-1)$ -dimensional convex set, the directions of the line segments lying in that set fill out a  $(d-2)$ -dimensional great sphere of  $S^{d-1}$ . On the other hand, Larman and Rogers [1971] proved the following: Let  $H$  be a fixed hyperplane in  $E^d$ . The set of directions of the line segments in  $\partial K$  which are parallel to  $H$  but not lying in the support-planes to  $K$  parallel to  $H$ , has zero  $(d-2)$ -dimensional Hausdorff measure.

To consider the case of  $r$ -dimensional convex subsets of  $\partial K$  for  $r > 1$ , let  $G_r^d$  denote the Grassmann manifold of all oriented  $r$ -dimensional linear subspaces of  $E^d$ . It is a compact differentiable manifold of dimension  $r(d-r)$ . Hausdorff measures in  $G_r^d$  are understood with respect to the distance function which is induced by a rotation invariant Riemannian metric on  $G_r^d$  (or by any other continuous Riemannian metric, which makes no difference in so far as finiteness,  $\sigma$ -finiteness, or vanishing of Hausdorff measures is concerned). Generalizing (3.5), Ewald, Larman and Rogers [1970] have shown the following.

(3.6) **Theorem.** For  $1 \leq r \leq d-1$  and for  $K \in \mathbb{R}^d$ , the set of all  $r$ -dimensional linear subspaces of  $E^d$  which are parallel to some  $r$ -dimensional convex subset contained in the boundary of  $K$ , has  $\sigma$ -finite  $r(d-r-1)$ -dimensional Hausdorff measure.

The result is best possible, since obviously there exist convex bodies for which the set in question has infinite  $r(d-r-1)$ -dimensional Hausdorff measure. However, the following was proved by Ewald, Larman and Rogers [1970]: If  $2 \leq r \leq d-2$ , and  $K \in \mathbb{R}^d$  has no  $(d-1)$ -dimensional faces, the set of all  $r$ -dimensional linear subspaces of  $E^d$  which are parallel to some  $r$ -dimensional convex subset in the boundary of  $K$ , has zero  $r(d-r-1)$ -dimensional Hausdorff measure.

By refining the methods which were used to prove (3.5), Zalgaller [1972] obtained a different generalization of (3.5). We say that the  $r$ -flat  $F$  ( $r$ -dimensional affine subspace of  $E^d$ ) supports  $K$  if  $F$  is contained in a supporting hyperplane of  $K$  and  $F \cap K \neq \emptyset$ . A supporting flat is called *singular* if it contains more than one point of  $K$ .

(3.7) **Theorem.** For  $1 \leq r \leq d-1$  and for  $K \in \mathbb{R}^d$ , the set of all  $r$ -dimensional linear subspaces of  $E^d$  which are parallel to some singular supporting  $r$ -flat of  $K$ , has  $\sigma$ -finite  $(r(d-r)-1)$ -dimensional Hausdorff measure.

This implies, in particular, that for almost all  $r$ -dimensional subspaces  $F$ , the shadow boundary of  $K$  in direction  $F$  is sharp (this was claimed, but not completely proved, by Ewald [1964]). Several improvements are conceivable, e.g., the following.

(3.8) **Problem** (Larman). Let  $K \in \mathbb{R}_0^d$ . For an  $r$ -dimensional subspace  $F$ , let  $K(F)$  be the shadow boundary of  $K$  in direction  $F$  and let  $\hat{K}(F)$  be the orthogonal

*projection of  $K(F)$  in direction  $F$ . Is it true that, for almost all  $r$ -dimensional subspaces  $F$ ,  $K(F) = N \cup \bigcup_{i=1}^{\infty} M_i$ , where  $N$  is a set of zero  $(d-k-1)$ -dimensional Hausdorff measure and on each  $M_i$  there is a two sided Lipschitz condition between  $M_i$  and its projection in  $\hat{K}(F)$ ?*

For an interpretation of (3.7) in terms of the existence of  $r$ -dimensional Chebyshev subspaces of  $d$ -dimensional normed vector spaces, see Klee [1969] and Zalgaller [1972].

The following is a common generalization of (3.6) and (3.7).

(3.9) **Theorem.** *For  $1 \leq s \leq r \leq d-1$  and for  $K \in \mathfrak{K}^d$ , the set of all  $r$ -dimensional linear subspaces of  $E^d$ , parallel to some supporting  $r$ -flat of  $K$  which contains an  $s$ -dimensional convex subset of  $K$ , has  $\sigma$ -finite  $(r(d-r)-s)$ -dimensional Hausdorff measure.*

This result, too, was obtained by Zalgaller [1972], but unfortunately the proof given by him is very sketchy. We remark that the case  $r = d-1$  of (3.9) can be deduced from Theorem (1.1) by applying it to a polar body of  $K$ .

The results, respectively the methods, of Ewald-Larman-Rogers and Zalgaller have been applied to other questions on convex bodies in papers by Larman-Rogers [1970], [1971], Larman [1971], Ivanov [1973], Schneider [1978a] (Lemma (5.2)), [1978c]. Ivanov [1973] proves that the union of all the lines in  $E^d$  which meet the boundary of the convex body  $K \in \mathfrak{K}^d$  in a segment, has  $\sigma$ -finite  $(d-1)$ -dimensional Hausdorff measure. This implies that almost every point  $x \in E^d \setminus K$  has the property that the shadow boundary of  $K$  under central projection from  $x$  is sharp.

Schneider [1978c] proves a result on pairs of convex bodies which can be considered as a generalization of (3.4). Let  $S0_d$  denote the rotation group of  $E^d$  equipped with the distance function which is induced by a bi-invariant Riemannian metric.

(3.10) **Theorem.** *Let  $K, K' \in \mathfrak{K}^d$  be convex bodies. The set of all rotations  $g \in S0_d$  for which  $K$  and  $gK'$  contain parallel segments lying in parallel supporting hyperplanes, has  $\sigma$ -finite  $(\frac{1}{2}d(d-1)-1)$ -dimensional Hausdorff measure.*

If  $K'$  is a segment, this is equivalent to the result (3.4) of Ewald-Larman-Rogers. The following result of Ivanov [1976] is in the spirit of, but not directly related to, the foregoing theorems.

(3.11) **Theorem.** *Let  $2 \leq r \leq d-1$ ,  $K \in \mathfrak{K}_0^d$ , and  $0 \in \text{int } K$ . The set of all  $r$ -dimensional linear subspaces of  $E^d$  containing an  $(r-1)$ -dimensional convex subset of  $\partial K$  which does not meet the relative interior of a  $(d-1)$ -face of  $K$ , has measure zero.*

Applying (3.11) to the polar of a given convex body  $K \in \mathfrak{K}_0^d$ , one obtains a result on the directions of the  $r$ -dimensional cones of normal vectors to  $K$ .

Further results on classification of boundary points of convex bodies, of types not considered here, are found in Karlin-Shapley [1953] and Waksman-Epelman [1976].

Finally we mention that recently Leichtweiss [1978] has proved some uniqueness theorems for convex bodies which are similar in spirit to some results to be explained later in §9, but make use of first order properties only.

## II. Curvature measures

### §4. Second derivatives and pointwise curvatures

For a twice continuously differentiable hypersurface the local behaviour “of the second order” is described by the curvature functions and related notions (curvatures of normal sections, principal curvatures, Dupin indicatrix, etc.). In the second part of this survey we consider those notions which replace the curvature functions in the case of a general convex hypersurface (part of the boundary of a convex body) which is not necessarily differentiable. The present section is concerned with the existence and properties of curvatures at a point, as far as they can be defined for general convex bodies. This corresponds to the investigation of second order differentiability properties of convex functions. The following sections will then treat measures defined over sets of boundary points or normal vectors which in the smooth case can be interpreted as integrals of curvature functions and thus are substitutes for differential geometric curvatures.

Since the following definitions and results concern only the neighbourhood of a point, it is convenient to represent the convex hypersurface under consideration as the graph of a convex function. Thus we shall henceforth assume that  $E^{d-1}$  is a fixed linear subspace of  $E^d$ ,  $e$  is a unit vector orthogonal to  $E^{d-1}$ ,  $U$  is an open, bounded, convex neighbourhood of 0 in  $E^{d-1}$ , and  $f: U \rightarrow \mathbb{R}$  is a convex function which satisfies  $f \geq 0$  and  $f(0) = 0$ . Further, we assume that  $E^{d-1}$  is the only support plane to the epigraph  $K_f$  of  $f$  at 0.

First we consider the case  $d = 2$ . Let  $t \in E^1$  be a unit vector. For  $\tau > 0$  (sufficiently small) let  $c(\tau)$  be the centre of the circle through 0 which is tangent to  $E^1$  and goes through the point  $p(\tau) := \tau t + f(\tau)e$ . If  $\tau_j \rightarrow 0$  for  $j \rightarrow \infty$  and  $(c(\tau_j))_{j \in \mathbb{N}}$  converges to some point  $\rho e$ , which means that

$$\rho = \lim_{j \rightarrow \infty} \frac{\tau_j^2}{2f(\tau_j)},$$

we call  $\rho$  a *radius of curvature of  $K_f$  at 0 in direction  $t$* . The set of all numbers  $\rho$  ( $\infty$  admitted) arising in this way is an interval  $[\rho_l, \rho_u]$  with  $0 \leq \rho_l \leq \rho_u \leq \infty$ . We call  $1/\rho_l$  the *upper curvature* and  $1/\rho_u$  the *lower curvature of  $K_f$  at 0 in direction  $t$* ; if both values are the same and are finite, one says that the curvature of  $K_f$  at 0 in direction  $t$  exists.

The following approach to curvatures, equivalent to that above in the  $C^2$  case, leads in general to a different notion of upper and lower curvatures. For  $\tau > 0$  let  $n$  be an interior unit normal vector to  $K_f$  at the point  $p(\tau)$ . Let  $z(\tau, n)$  be the intersection point of the normals  $\{\lambda e: \lambda \geq 0\}$  and  $\{p(\tau) + \lambda n: \lambda \geq 0\}$ . If  $\tau_j \rightarrow 0$  for  $j \rightarrow \infty$  (which implies that the corresponding chosen normal vectors  $n_j$  converge to  $e$ ) and  $(z(\tau_j, n_j))_{j \in \mathbb{N}}$  converges to some point  $r_e$ , which means that

$$r = \lim_{j \rightarrow \infty} \frac{\tau_j}{\sqrt{1 - \langle n_j, e \rangle^2}},$$

we call  $r$  a *Jessen radius of curvature of  $K_f$  at 0 in direction  $t$* . The set of all such numbers  $r$  ( $\infty$  admitted) is an interval  $[r_l, r_u]$  with  $0 \leq r_l \leq r_u \leq \infty$ . It is easy to see that  $[\rho_l, \rho_u] \subseteq [r_l, r_u]$ . Jessen [1929] shows that

$$(4.1) \quad \rho_u - \sqrt{\rho_u(\rho_u - \rho_l)} \leq r \leq \rho_u + \sqrt{\rho_u(\rho_u - \rho_l)}$$

whenever  $r$  is a Jessen radius of curvature, see also Busemann [1958], section 2 (for  $\rho_u = \infty$  the left hand side has to be replaced by  $\rho_l/2$ ). He also shows the following. Suppose that  $0 \leq a \leq b \leq \infty$  and

$$b - \sqrt{b(b-a)} \leq a' \leq a, \quad b \leq b' \leq b + \sqrt{b(b-a)}.$$

Then  $f$  can be chosen such that  $[\rho_l, \rho_u] = [a, b]$  and  $[r_l, r_u] = [a', b']$ . The essential conclusion from the inequality (4.1) is that  $\rho_l = \rho_u$  implies  $r_l = r_u$ . The relations of the foregoing to second order differentiability properties of  $f$  at 0 are discussed in detail in Busemann [1958], section 2, and we refer the reader to this for more information.

Now let  $d \geq 3$ , and let  $t \in E^{d-1}$ . By the *upper and lower curvature of  $K_f$  at 0 in direction  $t$*  we understand the corresponding values for the restriction of  $f$  to the line spanned by  $t$ . Of particular importance is the case where the curvature exists for all  $t$  and, as function of  $t$ , behaves like in the  $C^2$  case. More generally, for  $h > 0$  the hyperplane  $E^{d-1} + he$  intersects the closure of  $K_f$  in a  $(d-1)$ -dimensional convex body. Project this body orthogonally into  $E^{d-1}$  and expand it by the factor  $1/\sqrt{2h}$  to obtain  $D(h)$ . If the Hausdorff closed limit  $\lim_{h \rightarrow 0+} D(h)$  exists, this set is called the *indicatrix of  $K_f$  at 0*. Now 0 is called a *normal point* (or *Euler point*) of  $K_f$  if at 0 the indicatrix exists and is bounded by a quadric with centre 0. Thus at a normal point the curvatures in all directions exist, the principal curvatures are defined (as the reciprocal squares of the semi-axes of the indicatrix), and Euler's theorem holds.

If 0 is a normal point of  $K_f$ , then  $f$  admits at 0 a second differential. But here the existence of the second differential has to be understood in a generalized sense, since the first differential of  $f$  need not exist in a full neighbourhood of 0. The following terminology, taken from Bangert [1977], is appropriate. Let  $x \in U$ . The linear function  $L: E^{d-1} \rightarrow \mathbb{R}$  is called a *subdifferential of  $f$  at  $x$*  if

$$f(y) \geq f(x) + L(y-x) \quad \text{for } y \in U.$$

In other words,  $L$  is a subdifferential of  $f$  at  $x$  if the graph of the affine function  $y \mapsto f(x) + L(y-x)$  supports  $K_f$  at  $x + f(x)e$ . A map  $\Theta: U \rightarrow (E^{d-1})^*$  is called a *subdifferential* of  $f$  if, for each  $x \in U$ ,  $\Theta(x)$  is a subdifferential of  $f$  at  $x$ . Clearly every subdifferential of  $f$  coincides with the differential of  $f$  at each point where  $f$  is differentiable. Now the function  $f$  is called *twice differentiable* at  $x$  if the family of subdifferentials of  $f$  is uniformly differentiable at  $x$ , which means that there exists a neighbourhood  $V$  of  $x$ , a linear map  $F: E^{d-1} \rightarrow (E^{d-1})^*$  and a function  $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $\lim_{t \rightarrow 0+} \psi(t) = 0$ , such that

$$\|\Theta(y) - \Theta(x) - F(y-x)\| \leq \psi(\|y-x\|)\|y-x\|$$

for  $y \in V$  and each subdifferential  $\Theta$  of  $f$ .

(4.2) **Theorem.** *The point  $0$  is a normal point of  $K_f$  if and only if  $f$  is twice differentiable at  $0$ .*

The “if”-part is easy to see, and the other direction was obtained (in different terminology) by Aleksandrov [1939c].

So far we have considered only a single point of a convex hypersurface, and we have assumed a special representation of a neighbourhood of the point by means of a convex function. But since locally such a representation is always possible, the set of normal points of (the boundary of) a convex body  $K \in \mathbb{R}_0^d$  is well defined. Although normality of a point is apparently a strong assumption, it turns out that almost all boundary points of a convex body are normal.

(4.3) **Theorem.** *The set of all boundary points of the convex body  $K \in \mathbb{R}_0^d$  which are not normal has  $(d - 1)$ -dimensional Hausdorff measure zero.*

Clearly this is equivalent, in view of (4.2), to the following result.

(4.4) **Theorem.** *Every convex function (defined and finite on an open convex subset of  $E^{d-1}$ ) is twice differentiable almost everywhere in its domain.*

For  $d = 2$ , Theorem (4.4) is essentially an immediate consequence of the differentiability almost everywhere of a monotone function. This was first pointed out by Jessen [1929]. The assertion that the set of points where a convex function of one variable is not twice differentiable, has measure zero, is best possible: Let  $E$  be a null set in  $\mathbb{R}$  and construct a continuous, increasing function on  $\mathbb{R}$  with infinite derivative at each point of  $E$  (e.g., see Riesz-Nagy [1956], p. 4). The indefinite integral of this function yields a convex function which is not twice differentiable at each point of the given set  $E$ . For  $d = 3$ , Theorem (4.3) was proved by Busemann-Feller [1936a]. Using this result and an induction argument, Aleksandrov [1939c] obtained the general case. Other proofs of the important theorem (4.4), which are more analytic in character, have been given by Rešetnjak [1968] and Bangert [1977], [1979]. Rešetnjak’s proof uses techniques from the theory of distributions. Essential tools of Bangert’s proof are the Lipschitz property of the metric projection on to a convex body, Rademacher’s theorem on the almost everywhere differentiability of Lipschitz maps (e.g., Federer [1969], p. 216), and a generalization of a special case of Sard’s lemma (this result can be proved in an elementary way, according to a private communication of Bangert).

From Aleksandrov’s theorem (4.4), Asplund [1973] deduced that the metric projection on any closed (not necessarily convex) subset of a finite-dimensional Euclidean space is almost everywhere differentiable. This answered a question of Kruskal [1969].

For another consequence of (4.3), call the point  $x \in \partial K$  an  $\varepsilon$ -smooth point of the convex body  $K \in \mathbb{R}_0^d$  if there exists a ball  $B$  of radius  $\varepsilon$  such that  $x \in B \subset K$ . Answering a question of Sallee [1972], McMullen [1974b] proved that the set of all boundary points of the convex body  $K$  which are not  $\varepsilon$ -smooth for any  $\varepsilon > 0$ , is of  $(d - 1)$ -dimensional Hausdorff measure zero. This could also be deduced from (4.3), but the proof which McMullen gives is very short.

More results on curvatures and other differential geometric properties of general convex surfaces can be found in the papers by Busemann-Feller [1935a], [1935b], [1936a], [1936b]. A special approximation procedure of general convex bodies by very smooth bodies, which has good properties as to the convergence of radii of curvature, has been constructed and utilized by Weil [1973].

Finally we mention some differentiability and curvature properties of “typical” convex bodies. Here we say that “a typical convex body in  $E^d$  has property P” if the set of all convex bodies in  $\mathfrak{K}^d$  not having property P is a set of first category in the (complete) space  $\mathfrak{K}^d$  with the Hausdorff metric. Gruber [1977] has shown that a typical convex body has a boundary which is not a  $C^2$  hypersurface. More surprising (and stronger) is Zamfirescu’s [1979a] theorem according to which the curvatures of a typical convex body are zero wherever they exist. Further results in this spirit are found in Zamfirescu [1979b]. Schneider [1979c] shows that the boundary of a typical convex body contains a dense set such that, for each point  $x$  of this set and for each tangent direction at  $x$ , the lower curvature is 0 and the upper curvature is  $\infty$ .

## §5. Definition and properties of curvature measures

The notion of “curvature measures”, as it is used here, includes both the area functions of convex bodies introduced by Aleksandrov [1937a] and Fenchel-Jessen [1938], and the curvature measures of Federer [1959], restricted to convex bodies. Thus the curvature measures are generalizations, to arbitrary convex bodies, of (the indefinite integrals of) the elementary symmetric functions of the principle radii of curvature or the principal curvatures, as they can be defined for convex bodies with sufficiently smooth boundaries. Therefore the curvature measures play a natural rôle if one tries to generalize certain differential geometric results on closed convex surfaces satisfying differentiability assumptions to arbitrary convex surfaces. On the other hand, curvature measures may be viewed as “local” generalizations of the quermassintegrals which are so important in the Brunn-Minkowski theory of convex bodies and in integral geometry. It turns out that a number of integral geometric results on convex bodies admit natural extensions in terms of curvature measures.

In this section we shall first sketch how the curvature measures can easily be introduced and how their fundamental properties can be obtained. Then historical remarks will be made and possible extensions will be discussed.

We start with some notation. For  $K \in \mathfrak{K}^d$  and  $x \in E^d$ , let  $p(K, x)$  be the unique point in  $K$  which is nearest to  $x$  (thus  $p(K, \cdot)$  is the metric projection denoted by  $\pi_K$  in §1). By  $\mathcal{B}(X)$  we denote the  $\sigma$ -algebra of Borel subsets of  $X$ , where  $X = E^d$  or  $S^{d-1}$ . For  $K \in \mathfrak{K}^d$  and  $\beta \in \mathcal{B}(E^d)$  let  $\sigma(K, \beta)$  be the spherical image of  $K$  in  $\beta$ , that is, the set of all unit normal vectors to  $K$  at points of  $\partial K \cap \beta$ . For  $\omega \in \mathcal{B}(S^{d-1})$  we denote by  $\sigma^{-1}(K, \omega)$  the set of boundary points of  $K$  at which there exists an exterior unit normal vector which belongs to  $\omega$ .

Now for  $K \in \mathfrak{K}^d$ ,  $\beta \in \mathcal{B}(E^d)$ ,  $\omega \in \mathcal{B}(S^{d-1})$ , and  $\lambda > 0$ , define

$$A_\lambda(K, \beta) := \{x \in E^d : 0 < \|x - p(K, x)\| \leq \lambda \text{ and } p(K, x) \in \beta\},$$

$$B_\lambda(K, \omega) := \left\{ x \in E^d : 0 < \|x - p(K, x)\| \leq \lambda \text{ and } \frac{x - p(K, x)}{\|x - p(K, x)\|} \in \omega \right\}.$$

Then  $A_\lambda(K, \beta)$  and  $B_\lambda(K, \omega)$  are Borel sets, and we denote the Lebesgue measure of  $A_\lambda(K, \beta)$  by  $U_\lambda(K, \beta)$  and that of  $B_\lambda(K, \omega)$  by  $V_\lambda(K, \omega)$ . It turns out that these, as functions of  $\lambda$ , are polynomials of degree at most  $d$ , say

$$(5.1) \quad U_\lambda(K, \beta) = \frac{1}{d} \sum_{i=0}^{d-1} \binom{d}{i} \lambda^{d-i} C_i(K, \beta),$$

$$(5.2) \quad V_\lambda(K, \omega) = \frac{1}{d} \sum_{i=0}^{d-1} \binom{d}{i} \lambda^{d-i} S_i(K, \omega).$$

This defines functions  $C_i: \mathbb{R}^d \times \mathcal{B}(E^d) \rightarrow \mathbb{R}$  and  $S_i: \mathbb{R}^d \times \mathcal{B}(S^{d-1}) \rightarrow \mathbb{R}$  ( $i = 0, \dots, d-1$ ). Their fundamental properties are listed in the following theorems.

(5.3) **Theorem.** Let  $i \in \{0, \dots, d-1\}$ .

- (a) For each  $K \in \mathbb{R}^d$ ,  $C_i(K, \cdot)$  is a finite (positive) measure on  $\mathcal{B}(E^d)$ .
- (b) If  $(K_j)_{j \in \mathbb{N}}$  is a sequence in  $\mathbb{R}^d$  which converges to the convex body  $K$ , then the sequence  $(C_i(K_j, \cdot))_{j \in \mathbb{N}}$  converges weakly to  $C_i(K, \cdot)$ .
- (c) For each  $\beta \in \mathcal{B}(E^d)$  the function  $C_i(\cdot, \beta)$  is additive, that is,

$$C_i(K_1 \cup K_2, \beta) + C_i(K_1 \cap K_2, \beta) = C_i(K_1, \beta) + C_i(K_2, \beta)$$

if  $K_1, K_2, K_1 \cup K_2 \in \mathbb{R}^d$ .

- (d) For each  $\beta \in \mathcal{B}(E^d)$  the function  $C_i(\cdot, \beta): \mathbb{R}^d \rightarrow \mathbb{R}$  is measurable (with respect to the  $\sigma$ -algebra of Borel subsets of  $\mathbb{R}^d$  which is induced by the topology of the Hausdorff metric).
- (e) If  $\beta \subset E^d$  is open and if  $K_1, K_2 \in \mathbb{R}^d$  are convex bodies such that  $K_1 \cap \beta = K_2 \cap \beta$ , then  $C_i(K_1, \beta') = C_i(K_2, \beta')$  for every Borel set  $\beta' \subset \beta$ .
- (f) If  $g$  is a rigid motion of  $E^d$ , then  $C_i(gK, g\beta) = C_i(K, \beta)$  for  $K \in \mathbb{R}^d$ ,  $\beta \in \mathcal{B}(E^d)$ .
- (g) If  $K \in \mathbb{R}^d$ ,  $\beta \in \mathcal{B}(E^d)$  and  $\alpha > 0$ , then  $C_i(\alpha K, \alpha \beta) = \alpha^i C_i(K, \beta)$ .

(5.4) **Theorem.** Let  $i \in \{0, \dots, d-1\}$ .

- (a) For each  $K \in \mathbb{R}^d$ ,  $S_i(K, \cdot)$  is a finite (positive) measure on  $\mathcal{B}(S^{d-1})$ .
- (b) If  $(K_j)_{j \in \mathbb{N}}$  is a sequence in  $\mathbb{R}^d$  which converges to the convex body  $K$ , then the sequence  $(S_i(K_j, \cdot))_{j \in \mathbb{N}}$  converges weakly to  $S_i(K, \cdot)$ .
- (c) For each  $\omega \in \mathcal{B}(S^{d-1})$ , the function  $S_i(\cdot, \omega)$  is additive, that is,

$$S_i(K_1 \cup K_2, \omega) + S_i(K_1 \cap K_2, \omega) = S_i(K_1, \omega) + S_i(K_2, \omega)$$

if  $K_1, K_2, K_1 \cup K_2 \in \mathbb{R}^d$ .

- (d) For each  $\omega \in \mathcal{B}(S^{d-1})$ , the function  $S_i(\cdot, \omega): \mathbb{R}^d \rightarrow \mathbb{R}$  is measurable.
- (e) If  $\omega \in \mathcal{B}(S^{d-1})$  and if  $K_1, K_2 \in \mathbb{R}^d$  are convex bodies such that  $\sigma^{-1}(K_1, \omega) = \sigma^{-1}(K_2, \omega)$ , then  $S_i(K_1, \omega) = S_i(K_2, \omega)$ .
- (f) If  $g$  is a rigid motion of  $E^d$ , then  $S_i(gK, g\omega) = S_i(K, \omega)$ , where  $g_0$  denotes the rotation (leaving the origin fixed) which corresponds to  $g$ .

(g) If  $K \in \mathfrak{K}^d$ ,  $\omega \in \mathcal{B}(S^{d-1})$  and  $\alpha > 0$ , then  $S_i(\alpha K, \omega) = \alpha^i S_i(K, \omega)$ .

The measures  $C_i(K, \cdot)$  and  $S_i(K, \cdot)$  will be called the *curvature measures* of  $K$ ; for distinction,  $C_i(K, \cdot)$  is the  $i$ -th *curvature measure of the first kind* (or *Federer's curvature measure*), and  $S_i(K, \cdot)$  is the  $i$ -th *curvature measure of the second kind* (or *area function*). Sometimes a different normalization is convenient. We write

$$(5.5) \quad \Phi_i(K, \beta) := \frac{\binom{d}{i}}{d\alpha(d-i)} C_i(K, \beta),$$

$$(5.6) \quad \Psi_i(K, \omega) := \frac{\binom{d}{i}}{d\alpha(d-i)} S_i(K, \omega).$$

(In fact,  $\Phi_i(K, \cdot)$  rather than  $C_i(K, \cdot)$  is the measure which Federer [1959] calls the  $i$ -th curvature measure of  $K$ .) The advantage lies in the fact that  $\Phi_i(K, \cdot)$  and  $\Psi_i(K, \cdot)$  do not depend on the dimension of the space in which  $K$  is considered. The use of  $\Phi_i, \Psi_i$  leads to slightly shorter formulas in integral geometry, whereas  $C_i, S_i$  seem more natural in differential geometry.

We explain the meaning of the curvature measures in some special cases. Comparison of (5.1) and (5.2) with the classical Steiner formula (of which (5.1) and (5.2) may be viewed as "local versions") shows that

$$C_i(K, E^d) = S_i(K, S^{d-1}) = dW_{d-i}(K),$$

equivalently

$$\Phi_i(K, E^d) = \Psi_i(K, S^{d-1}) = V_i(K).$$

Next, suppose that the boundary of  $K \in \mathfrak{K}_0^d$  is a regular hypersurface of class  $C^2$ , then

$$C_i(K, \beta) = \int_{\partial K \cap \beta} H_{d-1-i} dF,$$

where  $H_k$  denotes the  $k$ -th normalized elementary symmetric function of the principal curvatures of  $\partial K$  and  $dF$  is the differential geometric surface area element. Similarly,

$$S_i(K, \omega) = \int_{\omega} R_i dL^{d-1},$$

where  $R_i$  denotes the  $i$ -th normalized elementary symmetric function of the principal radii of curvature of  $\partial K$ , considered as functions of the exterior unit normal vector, and where  $L^{d-1}$  is the spherical Lebesgue measure on  $S^{d-1}$ . If  $P \in \mathfrak{K}^d$  is a polytope, we have

$$C_i(P, \beta) = \frac{d\alpha(d-i)}{\binom{d}{i}} \sum_F \gamma(F, P) \mathcal{H}^i(F \cap \beta),$$

$$S_i(P, \omega) = \frac{d\alpha(d-i)}{\binom{d}{i}} \sum_F L^{d-1-i}(\sigma(P, F) \cap \omega) \mathcal{H}^i(F),$$

where in each case the summation extends over the  $i$ -dimensional faces  $F$  of  $P$ . Here  $\gamma(F, P)$  denotes the normalized external angle of the polytope  $P$  at its face  $F$ , and  $L^k$  denotes the  $k$ -dimensional spherical Lebesgue measure on  $k$ -dimensional unit spheres. For an arbitrary convex body  $K \in \mathbb{R}^d$  we have

$$C_{d-1}(K, \beta) = \mathcal{H}^{d-1}(\partial K \cap \beta) \text{ if } \dim K \neq d-1,$$

with an obvious modification if  $\dim K = d-1$ , and

$$C_0(K, \beta) = L^{d-1}(\sigma(K, \beta)).$$

On the other hand, we have

$$S_0(K, \omega) = L^{d-1}(\omega)$$

and

$$S_{d-1}(K, \omega) = \mathcal{H}^{d-1}(\sigma^{-1}(K, \omega)) \text{ if } \dim K \neq d-1,$$

again with an obvious modification for  $\dim K = d-1$ .

Proofs of the assertions made so far can be found in Schneider [1978a]. The curvature measures of the first kind were first introduced and used by Federer [1959], not only for convex bodies, but for the so-called sets of positive reach. In this setting, Fedérer obtained the results listed in Theorem (5.3) and others to be reviewed later. For convex bodies the proofs are much simpler, and additional results can be obtained. The curvature measures of the second kind are more than 20 years older, a fact which may have its reason in their close connection with mixed volumes. This connection can be explained as follows.

Let  $K_1, \dots, K_{d-1} \in \mathbb{R}^d$  be convex bodies, fixed for the moment. If  $h$  is the restriction to  $S^{d-1}$  of the support function of a convex body  $K \in \mathbb{R}^d$ , let  $f(h)$  be the mixed volume  $V(K, K_1, \dots, K_{d-1})$ . If  $C(S^{d-1})$ , the real vector space of continuous real functions on  $S^{d-1}$ , is equipped with the maximum norm, it turns out that  $f$  is uniformly continuous and has a unique extension to a continuous linear functional on  $C(S^{d-1})$ . An application of the Riesz representation theorem then yields the existence of a unique measure  $S(K_1, \dots, K_{d-1}; \cdot)$  on  $\mathcal{B}(S^{d-1})$  such that, in particular,

$$V(K, K_1, \dots, K_{d-1}) = \frac{1}{d} \int_{S^{d-1}} h(K, u) dS(K_1, \dots, K_{d-1}; u)$$

for  $K \in \mathbb{R}^d$ , where  $h(K, \cdot)$  denotes the support function of  $K$ . This measure  $S(K_1, \dots, K_{d-1}; \cdot)$  is called the *mixed area function* of  $K_1, \dots, K_{d-1}$ . As special cases one obtains the curvature measures of the second kind, namely from

$$S_i(K, \cdot) = S(\underbrace{K, \dots, K}_i, \underbrace{B, \dots, B}_{d-1-i}; \cdot),$$

where  $B$  denotes the unit ball.

By the above method the mixed area functions were introduced by Aleksandrov [1937a], §4, and in a similar way by Fenchel-Jessen [1938]. These two papers differ essentially in the approach by which they arrive at a geometric interpretation of area functions. Aleksandrov first defines  $S_{d-1}(K, \omega)$  as the surface area of the set  $\sigma^{-1}(K, \omega)$  (where his notion of surface area is equivalent, for the sets considered, to  $(d-1)$ -dimensional Hausdorff measure, but is introduced in a different way, using approximation by polytopes). He further shows that  $S_{d-1}(K, \cdot)$  is a measure and that

$$(5.5) \quad V(K', K, \dots, K) = \frac{1}{d} \int_{S^{d-1}} h(K', u) dS_{d-1}(K, u)$$

for  $K, K' \in \mathbb{R}^d$ . This is then used to prove that

$$S_{d-1} \left( \sum_{j=1}^m \lambda_j K_j, \omega \right) = \sum_{j_1, \dots, j_{d-1}=1}^m \lambda_{j_1} \dots \lambda_{j_{d-1}} S(K_{j_1}, \dots, K_{j_{d-1}}, \omega)$$

for  $\lambda_1, \dots, \lambda_m \geq 0$ . Later Aleksandrov [1939b] simplified part of his reasoning, essentially by working with weak convergence. Fenchel-Jessen [1938] (see also Busemann [1958], section 8) used (5.5) for the definition of  $S_{d-1}(K, \cdot)$  and at the end of their paper they obtained an intuitive interpretation by proving formula (5.2).

Generalizations of the curvature measures are possible in several directions. Comparison of (5.1), where sets  $\beta \cap \partial K$  of boundary points play a rôle, and of (5.2), where sets  $\omega$  of normal vectors occur, suggests a common generalization by considering sets of *support elements* (that is, pairs of boundary points  $x$  and normal vectors at  $x$ ). Let  $\mathcal{B}(\Omega)$  denote the  $\sigma$ -algebra of Borel sets (induced by the product topology) in  $\Omega := E^d \times S^{d-1}$ . For  $K \in \mathbb{R}^d$ ,  $\eta \in \mathcal{B}(\Omega)$ , and  $\lambda > 0$ , let  $W_\lambda(K, \eta)$  be the Lebesgue measure of the set

$$\left\{ x \in E^d : 0 < \|x - p(K, x)\| \leq \lambda \text{ and } \left( p(K, x), \frac{x - p(K, x)}{\|x - p(K, x)\|} \right) \in \eta \right\}.$$

Then a formula

$$W_\lambda(K, \eta) = \frac{1}{d} \sum_{i=0}^{d-1} \binom{d}{i} \lambda^{d-i} T_i(K, \eta)$$

is valid for  $\lambda > 0$ , and this defines finite measures  $T_0(K, \cdot), \dots, T_{d-1}(K, \cdot)$  on  $\mathcal{B}(\Omega)$ . Comparison with (5.1) and (5.2) shows that

$$C_i(K, \beta) = T_i(K, \beta \times \omega) \quad \text{if } \sigma(K, \beta) \subseteq \omega,$$

$$S_i(K, \omega) = T_i(K, \beta \times \omega) \quad \text{if } \sigma^{-1}(K, \omega) \subseteq \beta$$

( $i = 0, \dots, d-1$ ). The functions  $T_i$  have properties analogous to those of  $C_i$  and  $S_i$ , and they have proved useful in establishing connections between the  $C_i$  and the  $S_i$ . For this, see Schneider [1979].

The curvature measures can be extended to certain classes of non-convex sets. Federer [1959] defined his curvature measures for the sets of positive reach. A compact set  $A \subset E^d$  is called *of positive reach*, if there exists a number  $\rho > 0$  such that, for each  $x \in E^d$  whose distance from  $A$  is less than  $\rho$ , there is a unique point

in  $A$  which is nearest to  $x$ . Extensions to the “Konvexring”, which consists of the finite unions of convex bodies, have been considered by Schneider [1979a] (for a special case, see Schneider [1977b]). Such extensions of the curvature measures, or more generally of the functions  $T_i$  defined above, are not unique. One approach leads to signed measures, which still have the additivity property analogous to (5.3.c) and (5.4.c). A different approach, which in the case of curvature measures of the first kind is essentially due to Matheron [1975], p. 119 ff, yields positive measures. In each case, an appropriate generalization of the “Steiner formulae” (5.1) and (5.2), but for parallel sets “with multiplicity”, is the crucial device. Curvature measures for (non-convex) polyhedra have also been considered by Flaherty [1973].

Probably many of the results on curvature measures have counterparts for certain vector valued measures, which are related to the vector valued functionals treated by Hadwiger-Schneider [1971] and Schneider [1972a], [1972b] in much the same way as the curvature measures are related to the quermassintegrals. But this has not been investigated.

## §6. Characterization theorems

The fact that the classical quermassintegrals are just special values of the curvature measures leads one to expect that some of the results on quermassintegrals extend to curvature measures. The present section and the following one provide examples where such extensions or analogues are possible.

One of the most interesting and useful results on quermassintegrals is Hadwiger's (see Hadwiger [1957], section (6.1.10)) axiomatic characterization of the linear combinations of quermassintegrals by certain functional properties. Federer [1959], Remark 5.17, asked for a characterization of his curvature measures which is similar to Hadwiger's characterization of the quermassintegrals. For the case of convex bodies, the following theorem gives such a result.

(6.1) **Theorem.** Suppose that  $\varphi$  is a map from  $\mathbb{R}^d$  into the set of finite (positive) Borel measures on  $E^d$  which satisfies the following conditions (we write  $\varphi(K, \beta)$  for  $\varphi(K)(\beta)$ ):

- (a)  $\varphi$  is rigid motion invariant:  $\varphi(gK, g\beta) = \varphi(K, \beta)$  for  $K \in \mathbb{R}^d$ ,  $\beta \in \mathcal{B}(E^d)$  and every rigid motion  $g$  of  $E^d$ ,
- (b)  $\varphi$  is additive:  $\varphi(K_1 \cup K_2, \cdot) + \varphi(K_1 \cap K_2, \cdot) = \varphi(K_1, \cdot) + \varphi(K_2, \cdot)$ , if  $K_1, K_2, K_1 \cup K_2 \in \mathbb{R}^d$ ,
- (c)  $\varphi$  is weakly continuous: If  $(K_j)_{j \in \mathbb{N}}$  is a sequence in  $\mathbb{R}^d$  converging to  $K$ , then  $(\varphi(K_j, \cdot))_{j \in \mathbb{N}}$  converges weakly to  $\varphi(K, \cdot)$ ,
- (d)  $\varphi$  is defined locally: If  $\beta \in \mathcal{B}(E^d)$  is open and  $K_1 \cap \beta = K_2 \cap \beta$ , then  $\varphi(K_1, \beta') = \varphi(K_2, \beta')$  for every Borel set  $\beta' \subset \beta$ .

Then we have

$$\varphi(K, \beta) = \sum_{i=0}^d a_i C_i(K, \beta) \quad \text{for } K \in \mathbb{R}^d, \beta \in \mathcal{B}(E^d),$$

where  $C_d(K, \beta)$  is the Lebesgue measure of  $K \cap \beta$  and where  $a_0, \dots, a_d$  are nonnegative real constants.

For curvature measures of the second kind, a similar theorem is valid:

(6.2) **Theorem.** Suppose that  $\psi$  is a map from  $\mathbb{R}^d$  into the set of finite signed Borel measures on  $S^{d-1}$  which satisfies the following conditions:

- (a)  $\psi$  is rigid motion invariant:  $\psi(gK, g_0\omega) = \psi(K, \omega)$  for  $K \in \mathbb{R}^d$ ,  $\omega \in \mathcal{B}(S^{d-1})$  and every rigid motion  $g$  of  $E^d$ , where  $g_0$  denotes the rotation corresponding to  $g$ ,
- (b)  $\psi$  is additive:  $\psi(K_1 \cup K_2, \cdot) + \psi(K_1 \cap K_2, \cdot) = \psi(K_1, \cdot) + \psi(K_2, \cdot)$ , if  $K_1, K_2, K_1 \cup K_2 \in \mathbb{R}^d$ ,
- (c)  $\psi$  is weakly continuous: If  $(K_j)_{j \in \mathbb{N}}$  is a sequence in  $\mathbb{R}^d$  converging to  $K$ , then  $(\psi(K_j, \cdot))_{j \in \mathbb{N}}$  converges weakly to  $\psi(K, \cdot)$ ,
- (d)  $\psi$  is defined locally: If  $\omega \in \mathcal{B}(S^{d-1})$  and  $\sigma^{-1}(K_1, \omega) = \sigma^{-1}(K_2, \omega)$ , then  $\psi(K_1, \omega) = \psi(K_2, \omega)$ .

Then we have

$$\psi(K, \omega) = \sum_{i=0}^{d-1} a_i S_i(K, \omega) \quad \text{for } K \in \mathbb{R}^d, \omega \in \mathcal{B}(S^{d-1}),$$

where  $a_0, \dots, a_{d-1}$  are real constants.

Theorem (6.2) was proved in Schneider [1975b] and (6.1) in Schneider [1978a]. One should observe that the two characterizations are not strictly analogous to each other, in that (6.1) requires  $\varphi(K, \beta) \geq 0$  and  $\beta$  open in (d). The proofs, however, are similar, and different from the deeper proof of Hadwiger's characterization of quermassintegrals. It should be observed that none of the conditions (a) to (d) can be omitted, as can be shown by counterexamples.

A more general concept was developed by Hadwiger [1975a]. The area functions may be considered as mappings  $S_i: \mathbb{R}^d \times \mathcal{B}(S^{d-1}) \rightarrow \mathbb{R}$ . Since Borel sets may be replaced by their indicator functions, these maps can be viewed as special cases of mappings  $h: \mathbb{R}^d \times \mathcal{F} \rightarrow \mathbb{R}$ , where  $\mathcal{F}$  denotes the set of nonnegative, bounded, Borel-measurable real functions on  $S^{d-1}$ . More general examples of such maps are given by

$$W_i(K, p) := \frac{1}{d} \int_{S^{d-1}} p(u) dS_{d-i}(K, u) \quad \text{for } K \in \mathbb{R}^d, p \in \mathcal{F},$$

( $i = 1, \dots, d$ ). Hadwiger lists a number of properties which these functions share; roughly speaking, part of these properties are induced by properties of the  $S_i$ , and part of them come from the fact that the  $W_i$  are integrals. Then he shows that every function  $h: \mathbb{R}^d \times \mathcal{F} \rightarrow \mathbb{R}$  which has these properties is necessarily of the form

$$h(K, p) = \sum_{i=1}^d a_i W_i(K, p),$$

with  $a_1, \dots, a_d$  not depending on  $K$  or  $p$ . As applications he obtains integral geometric formulas involving arbitrary functions; these will be reviewed in the next section.

## §7. Integral geometric formulas

In this section we show how classical integral geometric formulas for convex bodies (as presented, e.g., in Hadwiger [1957], chapter 6) can be “localized” by replacing the quermassintegrals occurring in these formulas by curvature measures. Moreover, we review some results in a fairly recent branch of integral geometry of convex bodies, where one considers sets of convex bodies or affine subspaces which touch a given convex body. The curvature measures enter these investigations in a natural way.

The following invariant measures will be used. By  $\lambda$  we denote Lebesgue measure on  $E^d$  and by  $v$  the Haar measure on the rotation group  $SO_d$  of  $E^d$ , normalized so that  $v(SO_d) = L^{d-1}(S^{d-1})$ . If we define the map  $\gamma: E^d \times SO_d \rightarrow G_d$  into the group  $G_d$  of proper rigid motions of  $E^d$  by letting  $\gamma(t, \delta)x := \delta x + t$  ( $x \in E^d$ ), then the image measure  $\mu$  of the product measure  $\lambda \otimes v$  under  $\gamma$  is the (suitably normalized) Haar measure on  $G_d$ . For  $q \in \{0, \dots, d-1\}$  we denote by  $\mathcal{E}_q^d$  the homogeneous space of all  $q$ -flats ( $q$ -dimensional affine subspaces) of  $E^d$ . Let  $F_q \subset E^d$  be a fixed  $q$ -dimensional linear subspace of  $E^d$ ,  $F_q^\perp$  its orthogonal complement, and  $\lambda_{d-q}$  the  $(d-q)$ -dimensional Lebesgue measure on  $F_q^\perp$ . If we define the map  $\gamma_q: F_q^\perp \times SO_d \rightarrow \mathcal{E}_q^d$  by  $\gamma_q(t, \delta) := \delta(F_q + t)$ , then the image measure  $\mu_q$  of the product measure  $\lambda_{d-q} \otimes v$  under  $\gamma_q$  is the (suitably normalized) rigid motion invariant measure on  $\mathcal{E}_q^d$ .

With these definitions, the following mean value formulas are valid.

$$(7.1) \quad \int_{G_d} C_i(K \cap gK', \beta \cap g\beta') d\mu(g) \\ = \frac{\alpha(d-i)}{\alpha(i)} \sum_{k+j=d+i} \binom{d-i}{d-j} \frac{\alpha(j)\alpha(k)}{\alpha(d-j)\alpha(d-k)} C_k(K, \beta) C_j(K', \beta')$$

for  $K, K' \in \mathfrak{R}^d$ ,  $\beta, \beta' \in \mathcal{B}(E^d)$ ,  $i \in \{0, \dots, d-1\}$ , further

$$(7.2) \quad \int_{\mathcal{E}_q^d} C'_i(K \cap E_q, \beta \cap E_q) d\mu_q(E_q) = \frac{q\alpha(q)\alpha(d-q+i)}{\alpha(i)} C_{d-q+i}(K, \beta)$$

for  $K \in \mathfrak{R}^d$ ,  $\beta \in \mathcal{B}(E^d)$ ,  $0 \leq i < q \leq d-1$ ; here  $C'_i(K \cap E_q, \cdot)$  is the  $i$ -th curvature measure computed in the  $q$ -dimensional space  $E_q$ . Equalities (7.1) and (7.2) were proved for sets of positive reach by Federer [1959], Theorems 6.11 and 6.13; if one wants to compare the coefficients, one has to use the identity

$$k! \alpha(k) = 2^k \pi^{(k-1)/2} \Gamma\left(\frac{k+1}{2}\right).$$

In this general setting, the proof is fairly complicated. For convex bodies a shorter proof along the lines of Hadwiger’s method can be given by using the uniqueness theorem (6.1), see Schneider [1978a]. For  $\beta = \beta' = E^d$ , formulas (7.1) and (7.2) reduce to classical formulas involving quermassintegrals.

Other classical integral geometric formulas for quermassintegrals are generalized by the following results.

$$(7.3) \quad \int_{SO_d} S_i(K + \delta K', \omega \cap \delta \omega') d\nu(\delta) = \sum_{k=0}^i \binom{i}{k} S_k(K, \omega) S_{i-k}(K', \omega')$$

for  $K, K' \in \mathfrak{K}^d$ ,  $\omega, \omega' \in \mathcal{B}(S^{d-1})$ ,  $i \in \{0, \dots, d-1\}$ , further

$$(7.4) \quad \int_{S^{d-1}} S'_i(K | \delta E_q, \omega \cap \delta E_q) dv(\delta) = q \alpha(q) S_i(K, \omega)$$

for  $K \in \mathfrak{K}^d$ ,  $\omega \in \mathcal{B}(S^{d-1})$ ,  $0 \leq i < q \leq d-1$ . Here  $E_q$  is a linear subspace,  $K | \delta E_q$  denotes the image of  $K$  under orthogonal projection on to the linear subspace  $\delta E_q$ , and  $S'_i(K | \delta E_q, \cdot)$  is the  $i$ -th area function computed in the  $q$ -dimensional space  $\delta E_q$ . (7.3) can be deduced from the following more general formulas for mixed area functions.

$$(7.5) \quad \begin{aligned} & \int_{S^{d-1}} S(K_1, \dots, K_p, \delta K_{p+1}, \dots, \delta K_{d-1}; \omega \cap \delta \omega') dv(\delta) \\ &= S(K_1, \dots, K_p, \underbrace{B, \dots, B}_{d-1-p}; \omega) S(\underbrace{B, \dots, B}_p, K_{p+1}, \dots, K_{d-1}; \omega') \end{aligned}$$

for  $K_1, \dots, K_{d-1} \in \mathfrak{K}^d$ ,  $\omega, \omega' \in \mathcal{B}(S^{d-1})$ ,  $p \in \{0, \dots, d-1\}$ . Equalities (7.3) to (7.5) were obtained by Schneider [1975c], as an application of the uniqueness theorem (6.2). For the special case  $\omega' = S^{d-1}$ , (7.5) was proved in a different way in Schneider [1976].

Since (7.2) and (7.4) enable one to compute  $C_j, S_j$  from  $C_0, S_{q-1}$ , respectively, in  $q$ -dimensional subspaces, they may be used as recursion formulas. As such they prove useful in establishing certain properties of the curvature measures (see §8). Formulas (7.1), (7.2) can be extended to the Konvexring; similar extensions are possible for formulas (7.3), (7.4), (7.5), see Schneider [1979a], Remarks 7 and 9. In the classical kinematic formula of integral geometry (case  $i = 0$ ,  $\beta = \beta' = E^d$  of (7.1)) the integration extends, essentially, over the set of rigid motions  $g$  for which the moving body  $gK'$  intersects the fixed body  $K$  (for the other motions, the integrand vanishes). Hadwiger [1975b] has considerably extended this formula by introducing an (up to mild analytic conditions) arbitrary function of the distance between the fixed and the moving convex body. As an application of his theory developed in Hadwiger [1975a], which was mentioned in §6, he has later obtained a further generalization which involves also functions of directions (formula (7.7) below). An essential special case can be described as follows. For  $K, K' \in \mathfrak{K}^d$  (or  $K'$  a flat) let  $r[K, K']$  denote the Euclidean distance of  $K$  and  $K'$ . If  $K \cap K' = \emptyset$  and  $x \in K$ ,  $x' \in K'$  are points at distance  $r[K, K']$ , write  $u[K, K'] := x' - x / \|x' - x\|$  (which depends only on  $K, K'$ ). Further, for sets  $\omega, \omega' \in \mathcal{B}(S^{d-1})$  let

$$M_\infty(K, K'; \omega, \omega') := \{g \in G_d : K \cap gK' = \emptyset \text{ and } u[K, gK'] \in \omega \cap (-g_0 \omega')\},$$

where  $g_0$  is the rotation part of  $g$ . Then for any Borel-measurable function  $f : ]0, \infty[ \rightarrow [0, \infty[$  for which

$$M_i(f) := \int_0^\infty f(r) r^i dr < \infty \quad \text{for } i = 0, \dots, d-1,$$

the formula

$$(7.6) \quad \begin{aligned} & \int_{M_\infty(K, K', \omega, \omega')} f(r[K, gK']) d\mu(g) \\ &= \sum_{j,k=0}^{d-1} \binom{d-1}{j+k} \binom{j+k}{j} M_{d-1-j-k}(f) S_j(K, \omega) S_k(K', \omega') \end{aligned}$$

holds. Using standard arguments from integration theory, one deduces Hadwiger's formula

$$(7.7) \quad \begin{aligned} & \int_{K \cap gK' = \emptyset} f(r[K, gK']) p(u[K, gK']) p'(g_0^{-1} u[gK', K]) d\mu(g) \\ &= \sum_{j,k=0}^{d-1} \binom{d-1}{j+k} \binom{j+k}{j} M_{d-1-j-k}(f) V_j(K, p) V_k(K', p') \end{aligned}$$

with

$$V_j(K, p) := \int_{S^{d-1}} p(u) dS_j(K, u) \quad \text{for } j=0, \dots, d-1.$$

Here  $p, p'$  are nonnegative, Borel-measurable, real functions on  $S^{d-1}$ . A short proof of (7.6), which makes use of (7.3), has been given in Schneider [1977a]. Recently Weil [1979c] has further generalized (7.7) by proving that

$$(7.8) \quad \begin{aligned} & \int_{K \cap gK' = \emptyset} f(r[K, gK'], u[K, gK'], g_0^{-1} u[gK', K]) d\mu(g) \\ &= \sum_{j,k=0}^{d-1} \binom{d-1}{j+k} \binom{j+k}{j} \int_0^\infty \int_{S^{d-1}} \int_{S^{d-1}} r^{d-1-j-k} f(r, v, w) dS_j(K, v) dS_k(K', w) dr. \end{aligned}$$

Here  $f: ]0, \infty[ \times S^{d-1} \times S^{d-1} \rightarrow \mathbb{R}$  is a measurable function for which the integrals occurring in (7.8) are finite.

Considering moving  $q$ -flats instead of moving convex bodies, the extended Crofton formulas of Bokowski-Hadwiger-Wills [1976] can be generalized in a similar way. This has been done by Hadwiger [1975a], and could also be done by a method similar to that used in Schneider [1977a]. The following general result is due to Weil [1979b].

$$(7.9) \quad \begin{aligned} & \int_{K \cap E_q = \emptyset} f(r[K, E_q], u[K, E_q]) d\mu_q(E_q) \\ &= (d-q)\alpha(d-q) \sum_{j=0}^{d-1-q} \binom{d-1-q}{j} \int_0^\infty \int_{S^{d-1}} r^{d-1-q-j} f(r, v) dS_j(K, v) dr. \end{aligned}$$

Here  $f: ]0, \infty[ \times S^{d-1} \rightarrow \mathbb{R}$  is a measurable function for which the integrals in (7.9) are finite.

We mention that recently Groemer [1979] has given an interesting common generalization of the kinematic formulas involving distances due to Hadwiger [1975b] and Bokowski-Hadwiger-Wills [1976].

If in formula (7.6) we choose for  $f$  the indicator function of the interval  $]0, \varepsilon]$  and then let  $\varepsilon > 0$  tend to zero, we immediately obtain the following result.

(7.10) **Theorem.** *For  $K, K' \in \mathfrak{K}^d$ ,  $\omega, \omega' \in \mathcal{B}(S^{d-1})$ ,  $\varepsilon > 0$ , let*

$$\begin{aligned} M_\varepsilon(K, K'; \omega, \omega'): = & \{g \in G_d : 0 < r[K, gK'] \leq \varepsilon \\ & \text{and } u[K, gK'] \in \omega \cap (-g_0\omega')\}. \end{aligned}$$

Then the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \mu(M_\varepsilon(K, K'; \omega, \omega')) =: \bar{\mu}(K, K'; \omega, \omega')$$

exists and is given by

$$(7.11) \quad \bar{\mu}(K, K'; \omega, \omega') = \sum_{i=0}^{d-1} \binom{d-1}{i} S_i(K, \omega) S_{d-1-i}(K', \omega').$$

Similarly (7.9), with  $f(r, u) := 1$  if  $0 < r \leq \varepsilon$  and  $u \in \omega$ , and 0 otherwise, yields the following.

(7.12) **Theorem.** For  $K \in \mathfrak{K}^d$ ,  $\omega \in \mathcal{B}(S^{d-1})$ ,  $\varepsilon > 0$ ,  $q \in \{0, \dots, d-1\}$ , let

$$M_\varepsilon^q(K; \omega) := \{E_q \in \mathcal{E}_q^d : 0 < r[K, E_q] \leq \varepsilon \text{ and } u[K, E_q] \in \omega\}.$$

Then the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \mu_q(M_\varepsilon^q(K; \omega)) =: \bar{\mu}_q(K; \omega)$$

exists and is given by

$$(7.13) \quad \bar{\mu}_q(K; \omega) = (d-q)\alpha(d-q)S_{d-1-q}(K, \omega).$$

Both theorems are due to Firey [1972], [1974]. His proofs are different, using direct integral geometric computations in the case of polytopes and an approximation procedure. Other proofs of Firey's results have been given by Schneider [1975c], before the general formulas (7.6) and (7.9) were known. Firey's formula (7.13) gave the first integral-geometric interpretation of area functions. An analogous interpretation of Federer's curvature measures was obtained by Schneider [1978a]. The proof used the results of Schneider [1975c] and approximation by smooth bodies. A different proof could utilize the characterization theorem (6.1). As mentioned in Schneider [1979a] (Satz 4.12), the method used by Schneider [1978a] can be modified to yield an analogous result for the generalized curvature measures  $T_i$  defined in §5.

Intuitively, formula (7.13) may be interpreted by saying that  $S_{d-1-q}(K, \omega)$  measures (up to a constant factor) the  $q$ -flats which touch  $K$  in a direction of  $\omega$ . Here we say that  $E_q$  touches  $K$  in a direction  $u \in S^{d-1}$  if  $K \cap E_q \neq \emptyset$  and  $E_q$  lies in a supporting hyperplane to  $K$  with exterior normal vector  $u$ . If  $M_0^q(K; \omega)$  denotes the set of all these  $q$ -flats, the sets  $M_\varepsilon^q(K; \omega)$  with  $\varepsilon \rightarrow 0$ , may be considered as approximations to the set  $M_0^q(K; \omega)$ . Therefore it seems reasonable to consider  $\bar{\mu}_q(K; \omega)$  as a natural measure for the set  $M_0^q(K; \omega)$ . This is essentially the interpretation which Firey [1972] gives. It should, however, be observed that this approach does not lead to a measure defined on all Borel sets of  $q$ -flats which touch  $K$ . Such a measure has recently been introduced and investigated by Weil [1979b].

A similar intuitive interpretation can be given for the result of Theorem (7.10). The following question was suggested by Firey [1974]. Consider two convex bodies  $K, K' \in \mathbb{R}^d$  and paint subsets  $\beta, \beta'$  on the boundaries of  $K$  and  $K'$ . Now assume that  $K'$  undergoes random motion in such a way that it touches  $K$ . What is the probability that  $K$  and  $K'$  collide paint-to-paint? In order to make the question meaningful, it is necessary to define a measure for the set  $L_0(K, K'; \beta, \beta')$  of rigid motions  $g \in G_d$  for which  $gK'$  touches  $K$  and  $\beta \cap g\beta' \neq \emptyset$ , and this for "sufficiently many" sets  $\beta, \beta'$ . Suppose, in particular, that  $\beta = \sigma^{-1}(K, \omega)$ ,  $\beta' = \sigma^{-1}(K', \omega')$  and  $\omega = \sigma(K, \beta)$ ,  $\omega' = \sigma(K', \beta')$  with  $\omega, \omega' \in \mathcal{B}(S^{d-1})$ . Then the sets  $M_\epsilon(K, K'; \omega, \omega')$  defined in (7.10) may, for  $\epsilon \rightarrow 0$ , be considered as approximations to the set  $L_0(K, K'; \beta, \beta')$ . Therefore it seems reasonable to take  $\bar{\mu}(K, K'; \omega, \omega')$  as a natural measure for the set  $L_0(K, K'; \beta, \beta')$  and to consider the quotient

$$\bar{\mu}(K, K'; \omega, \omega') / \bar{\mu}(K, K'; S^{d-1}, S^{d-1}),$$

for which Theorem (7.10) gives an expression in terms of area functions, as the probability which was asked for. In order to stress the plausibility of selecting  $\bar{\mu}(K, K'; \omega, \omega')$  as a "contact measure", an axiomatic characterization of the function  $\bar{\mu}$  given by (7.11) was established by Schneider [1975b]. Such a characterization had been asked for by Firey [1975]. The approach of Firey [1974] requires one, roughly, to restrict attention to the inverse spherical images of Borel sets on  $S^{d-1}$ ; more precisely, for the touching bodies one does not prescribe sets of boundary points but rather sets of normal vectors of separating hyperplanes. If one prefers to "paint" arbitrary Borel subsets  $\beta, \beta'$  of the boundaries of  $K, K'$ , one has to use Federer's curvature measures. As a counterpart to (7.10), Schneider [1978c] proved the following.

(7.14) **Theorem.** For  $K, K' \in \mathbb{R}^d$ ,  $\beta, \beta' \in \mathcal{B}(E^d)$ ,  $\epsilon > 0$ , let  $L_\epsilon(K, K'; \beta, \beta')$  be the set of all rigid motions  $g \in G_d$  for which  $0 < r[K, gK'] \leq \epsilon$  and there exist points  $x \in \partial K \cap \beta$  and  $x' \in \partial K' \cap \beta'$  such that  $\|x - gx'\| = r[K, gK']$ . Then the limit

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \mu(L_\epsilon(K, K'; \beta, \beta')) = : \bar{v}(K, K'; \beta, \beta')$$

exists and is given by

$$\sum_{i=0}^{d-1} \binom{d-1}{i} C_i(K, \beta) C_{d-1-i}(K', \beta').$$

The proof of (7.14) given by Schneider [1978c] uses approximation by smooth bodies, so that a reduction to the approach followed by Firey (in the variant given by Schneider [1975c]) is possible. A different proof could be given which uses the characterization theorem (6.1).

The special case of the above results where  $K, K'$  are polytopes and  $\beta, \beta'$  are faces, has been investigated by McMullen [1974a]. In particular, he showed the following result and thus answered a question of Firey: The probabilities that two congruent cubes in  $E^3$  touching randomly will meet edge against edge and vertex against face are in proportion  $3\pi : 8$ .

Theorems (7.10) and (7.12) together with their interpretations in terms of contact probabilities admit a common generalization in terms of a fixed convex body

and moving congruent convex cylinders; this has been carried out by Heuser [1976].

Recently Firey [1979] has considered a variant of his contact probability problem, namely inner contact: The convex body  $K$  is assumed to roll freely inside  $K'$  and to touch the boundary of  $K'$ . If again inverse spherical images of sets  $\omega, \omega' \in \mathcal{B}(S^{d-1})$  are “painted”, the measure of all  $g$  for which  $gK$  touches  $K'$  “paint-to-paint”, is given by a formula of type (7.11), with  $(-1)^i$  occurring in the sum.

The interpretation of all these results as giving “contact probabilities” suffers from the fact that a “measure” for the relevant set of touching positions is only defined as a limit. The construction of a reasonable underlying probability space would require one to have a “natural” finite measure, defined at least on the  $\sigma$ -algebra of all Borel sets of rigid motions  $g$  for which  $gK$  touches  $K$ . Modifying the approaches of Firey and Schneider in a suitable way, Weil [1979c] has recently introduced such a measure. It appears that his theory is a natural setting for the integral geometry of touching convex sets; see his survey, Weil [1979d].

Concerning the contact measure  $\bar{\mu}$  given by (7.11), Firey [1975] asked the following. Let  $K' \in \mathfrak{K}_0^d$  be fixed. Suppose that  $K_1, K_2 \in \mathfrak{K}^d$  are convex bodies for which  $\bar{\mu}(K_1, K'; \omega, S^{d-1}) = \bar{\mu}(K_2, K'; \omega, S^{d-1})$  for each  $\omega \in \mathcal{B}(S^{d-1})$ . Is  $K_2$  a translate of  $K_1$ ? For  $d=3$  the answer is in the affirmative. For convex bodies with a twice continuously differentiable boundary this was proved by Firey [1974], and for general convex bodies by Schneider [1976]. For  $d > 3$  the question is open.

## §8. Local properties

From the definition of curvature measures, and from their geometric meaning in the special cases of sufficiently smooth convex bodies or polytopes, one should expect that there are correlations between properties of the curvature measures of a convex body and the local shape of its boundary. In this section we review the few known results which may be interpreted in this sense.

Guided by the special case of a body  $K$  with a  $C^2$ -boundary, where the formula (see §5)

$$C_i(K, \beta) = \int_{\partial K \cap \beta} H_{d-1-i} dF$$

is valid, one may ask whether, for a general convex body, it is possible to obtain the function  $H_{d-1-i}$  at the points where it is defined (in the sense of §4) from the measure  $C_i(K, \cdot)$  by means of a differentiation process. This is indeed the case. For the definition of a normal boundary point  $x$  (or Euler point) of a convex body  $K$  we refer to Aleksandrov [1939c] or to §4, and for the definition of a normal sequence (with respect to  $x$ ) of subsets of  $\partial K$ , to Aleksandrov [1939c] or Schneider [1979].

(8.1) **Theorem.** *Let  $K \in \mathfrak{K}_0^d$ ,  $x$  a normal boundary point of  $K$ , and  $(\beta_i)_{i \in \mathbb{N}}$  a (with respect to  $x$ ) normal sequence of subsets of  $\partial K$ . Then*

$$\lim_{i \rightarrow \infty} \frac{C_m(K, \beta_i)}{C_{d-1}(K, \beta_i)} = H_{d-1-m}(x)$$

for  $m = 0, \dots, d-1$ .

For  $m = 0$  this was proved by Aleksandrov [1939c] (see also Busemann-Feller [1936b] for a related investigation in  $E^3$ ). The general case has been obtained by Schneider [1979]. Aleksandrov (loc. cit. §8) used his result to deduce an analogous theorem for the mixed area function  $S(K_1, \dots, K_{d-1}; \cdot)$  of the convex bodies  $K_1, \dots, K_{d-1} \in \mathfrak{K}_0^d$  at each point of  $S^{d-1}$  where the support functions of  $K_1, \dots, K_{d-1}$  are simultaneously twice differentiable.

Observing that  $C_{d-1}(K, \beta) = \mathcal{H}^{d-1}(\beta)$  for Borel sets  $\beta \subset \partial K$  and that  $\mathcal{H}^{d-1}$ -almost all boundary points of  $K$  are normal, we deduce that

$$C_m(K, \beta) = \int_{\partial K \cap \beta} H_{d-1-m} d\mathcal{H}^{d-1} \quad \text{for } \beta \in \mathcal{B}(E^d)$$

for every convex body  $K \in \mathfrak{K}_0^d$  for which  $C_m(K, \cdot)$  is absolutely continuous with respect to  $\mathcal{H}^{d-1}$ . But it should be observed that for most convex bodies  $K$  (i.e., for all convex bodies except those of a certain set of first category) the curvature measure  $C_m(K, \cdot)$  is singular with respect to  $\mathcal{H}^{d-1}$  ( $m = 0, \dots, d-2$ ), as follows from Zamfirescu's [1979a] result mentioned in §4.

About absolute continuity (with respect to spherical Lebesgue measure) of the area functions, Weil [1973] has obtained the following result.

(8.2) **Theorem.** *Let  $K \in \mathfrak{K}_0^d$  be a convex body for which  $S_1(K, \cdot)$  is absolutely continuous and has a bounded density. Then  $S_m(K, \cdot)$  is absolutely continuous for  $m = 1, \dots, d-1$ , and the support function of  $K$  is continuously differentiable on  $E^d \setminus \{0\}$ .*

In  $E^3$  some results on the influence of the “specific curvature”  $C_0(K, \cdot)/C_2(K, \cdot)$  (defined on the Borel sets  $\beta$  with  $C_2(K, \beta) > 0$ ) on the local shape of  $\partial K$  are known.

(8.3) **Theorem.** *If the specific curvature of the convex body  $K \in \mathfrak{K}_0^3$  in a neighborhood of  $x \in \partial K$  is bounded, then either  $\partial K$  is differentiable at  $x$ , or  $x$  is a relatively interior point of a straight edge on  $\partial K$  (i.e., a segment on  $\partial K$  which lies in two distinct supporting planes).*

This theorem is due to Aleksandrov [1942b]. For consequences and further results in this spirit, see also Busemann [1958], section 5. Similar results for higher dimensions or other curvature measures seem to be unknown.

A satisfactory correlation between properties of the curvature measures of  $K$  and geometric properties of  $K$  itself is given by the following description of the supports of these measures. (The support of a Borel measure is the complement of the largest open set on which the measure vanishes.)

(8.4) **Theorem.** *For  $K \in \mathfrak{K}^d$  and  $m \in \{0, \dots, \dim K\}$ , the support of  $C_m(K, \cdot)$  is equal to the closure of the  $m$ -skeleton of  $K$  (as defined in §2).*

This was proved by Schneider [1978a]. The proof makes essential use of the integral geometric formula (7.2). By a similar method, using (7.4), the following was obtained by Schneider [1975c].

(8.5) **Theorem.** *For  $K \in \mathfrak{K}^d$  and  $m \in \{0, \dots, d-1\}$ , the support of  $S_m(K, \cdot)$  is equal to the closure of the set of all  $(d-1-m)$ -extreme unit normal vectors of  $K$ .*

The notion of *k-extreme unit normal vector* is dual to the notion of k-extreme point; see Schneider [1975c] for a formal definition. Theorem (8.5) was conjectured by Weil [1973], p. 356. It implies, in particular, that for  $m > n$  the support of  $S_n(K, \cdot)$  contains the support of  $S_m(K, \cdot)$ ; this had been asked by Firey [1975].

The argument leading to (8.5) can also be used to show that in the set  $\{S_m(K, \cdot) : K \in \mathfrak{K}_0^d\}$  the curvature measures of polytopes are characterized by their supports (see Goodey and Schneider [1979]; for  $m=1$  compare Weil [1973], Satz 4.4):

(8.6) **Theorem.** *Suppose that  $K \in \mathfrak{K}_0^d$ ,  $m \in \{1, \dots, d-1\}$ , and that the support of  $S_m(K, \cdot)$  is contained in the union of finitely many  $(d-1-m)$ -dimensional great spheres. Then  $K$  is a polytope.*

Theorem (8.4) can be used to give a short proof of theorem (2.13) (but a direct proof would also be possible). In fact, let  $r \in \{0, \dots, d-1\}$ , let  $x \in \text{skel}_r K$ , and suppose that the sequence  $(K_j)_{j \in \mathbb{N}}$  in  $\mathfrak{K}^d$  converges to  $K$ , but  $x \notin \liminf \text{skel}_r K_j$ . Then there exists an open neighborhood  $\beta$  of  $x$  such that  $\beta \cap \text{skel}_r K_{j(n)} = \emptyset$  for an infinite sequence  $(j(n))_{n \in \mathbb{N}}$ . From (8.4) it follows that  $C_r(K_{j(n)}, \beta) = 0$ , hence by (5.3.b) and since  $\beta$  is open, we have

$$C_r(K, \beta) \leq \liminf_{n \rightarrow \infty} C_r(K_{j(n)}, \beta) = 0.$$

Another application of (8.4) shows that  $\beta \cap \text{skel}_r K = \emptyset$ , a contradiction. It has been observed by Aleksandrov [1938], §4, that  $S_m(K, \beta) = 0$  for every one-pointed set  $\beta$ , if  $m \leq d-2$ . The intuitive idea that curvature measures must be small on “small” sets, can be made much more precise. Firey [1970a] proved the inequality

$$(8.7) \quad S_i(K, \omega_\alpha) \leq A D(K)^i \sin^{d-i-1} \alpha \sec \alpha$$

for  $K \in \mathfrak{K}^d$  and  $i = 1, \dots, d-1$ , where

$$\omega_\alpha := \{u \in S^{d-1} : \langle u, u_0 \rangle \geq \cos \alpha\} \quad \text{for } 0 < \alpha < \frac{\pi}{2}$$

with some fixed  $u_0 \in S^{d-1}$ . Here  $D(K)$  is the diameter of  $K$ , and  $A$  does not depend on  $u_0$ ,  $\alpha$  or  $K$ . (8.7) implies the existence of a constant  $a_i$  such that

$$(8.8) \quad S_i(K, \omega) \leq a_i D(K)^i \mathcal{H}^{d-i-1}(\omega)$$

for  $K \in \mathfrak{K}^d$  and  $\omega \in \mathcal{B}(S^{d-1})$ . A similar inequality can be obtained as follows. By (7.4),

$$S_i(K, \omega) = b_i \int_{S^{d-1}} S'_i(K | \delta E_{i+1}, \omega \cap \delta E_{i+1}) d\nu(\delta)$$

with a constant  $b_i$ . Since clearly

$$S'_i(K | \delta E_{i+1}, \{u\}) \leq D_i(K),$$

where  $D_i(K)$  denotes the maximal  $i$ -dimensional volume of any  $i$ -dimensional orthogonal projection of  $K$ , we have

$$S_i(K, \omega) \leq b_i D_i(K) \int_{S^{d-1}} \text{card}(\omega \cap \delta E_{i+1}) d\nu(\delta).$$

If we now assume that  $\omega$  is  $(\mathcal{H}^{d-i-1}, d-i-1)$  rectifiable, we can use Federer [1969], Th. 3.2.48, to obtain

$$(8.9) \quad S_i(K, \omega) \leq c_i D_i(K) \mathcal{H}^{d-i-1}(\omega)$$

with some constant  $c_i$ . Probably (8.9) is true for every Borel set  $\omega \in \mathcal{B}(S^{d-1})$ . For Federer's curvature measures we have the inequality

$$(8.10) \quad \Phi_i(K, \beta) \leq \mathcal{H}^i(\beta)$$

for  $K \in \mathbb{R}^d$  and  $\beta \in \mathcal{B}(E^d)$ . For the special case where  $\beta$  consists only of  $i$ -singular points, this was proved by Federer [1959], p. 489. He also showed, under this assumption, that  $\Phi_i(K, \beta) > 0$  if  $\mathcal{H}^i(\beta) > 0$ . Inequality (8.10) for general Borel sets  $\beta$  is proved similarly; one uses (7.2) with  $i = 0$ ,  $q = d - i$  and the integral geometric inequality

$$\mathcal{H}^i(\beta) \geq \frac{\binom{d}{i}}{d\alpha(d-i)\alpha(i)} \int \text{card}(E_{d-i} \cap \beta) d\mu_{d-i}(E_{d-i})$$

(see Federer [1969], 2.10.16 and 3.3.13).

## §9. Uniqueness and stability theorems

In global differential geometry one knows a great number of congruence theorems and of characterizations of spheres among the closed convex hypersurfaces, where one assumes some special relations between curvature functions. In this section we gather the results of this type which have been obtained for general convex bodies, that is, without any differentiability assumptions for the boundary hypersurface. Naturally, assumptions on the curvature measures have to replace the assumptions on the curvature functions which are made in the smooth case. It turns out that more results are known for curvature measures of the second kind than for those of the first kind. This is not surprising, since area functions are closely related to mixed volumes, and since the fact that they are defined on the unit sphere makes it easier and more natural to establish correspondences between the area functions of distinct convex bodies. The classical result in this field is the well-known theorem of Aleksandrov-Fenchel-Jessen:

(9.1) **Theorem.** If  $K, K' \in \mathfrak{K}^d$ ,  $i \in \{1, \dots, d-1\}$ ,  $\dim K, \dim K' \geq i+1$  and

$$S_i(K, \cdot) = S_i(K', \cdot),$$

then  $K, K'$  are translates of each other.

Proofs, both of which make essential use of the Brunn-Minkowski theory of mixed volumes, have been given by Aleksandrov [1937b] and Fenchel-Jessen [1938]. Busemann [1959] has obtained corresponding uniqueness theorems for pairs of non-closed convex hypersurfaces with identical boundaries and also, using an appropriate modification of the measures  $S_i(K, \cdot)$ , for convex caps. Applications of (9.1) to other problems in the theory of convex bodies are found, for instance, in Schneider [1970a], [1970b], Schneider-Weil [1970].

(9.2) **Problem.** Let  $K, K' \in \mathfrak{K}_0^d$  be convex bodies and assume that

$$\sum_{i=1}^{d-1} \alpha_i S_i(K, \cdot) = \sum_{i=1}^{d-1} \alpha_i S_i(K', \cdot),$$

where  $\alpha_i \geq 0$  are constants that do not all vanish. Is  $K'$  a translate of  $K$ ?

Theorem (9.1) gives an affirmative answer for the case where all but one of the  $\alpha_i$  are zero. For  $d = 3$ , an affirmative solution of (9.2) was obtained by Schneider [1976]. Even the special case of (9.2) in which  $K'$  is a ball has not been solved; more generally we propose the following problem.

(9.3) **Problem.** Determine all convex bodies  $K \in \mathfrak{K}_0^d$  that satisfy a relation

$$\sum_{i=1}^{d-1} \alpha_i S_i(K, \cdot) = 0$$

with constants  $\alpha_i$  at least two of which are unequal to zero.

In this case, without further assumptions on the signs of the  $\alpha_i$ , it should be observed that balls are not the only convex bodies that satisfy such a relation. This is shown by the following partial solution, which was obtained by Schneider [1978d].

(9.4) **Theorem.** Let  $K \in \mathfrak{K}_0^d$  be a convex body, let  $i \in \{0, \dots, d-2\}$ . Then

$$S_i(K, \cdot) = \alpha S_{d-1}(K, \cdot)$$

with some constant  $\alpha$  if and only if  $K$  is an  $i$ -tangential body of a ball.

But still other non-spherical convex bodies can satisfy linear relations of type (9.3). For instance, if  $K = k + cB$ , where  $k \in \mathfrak{K}^d$  is a body with  $\dim k \leq m-1$  and  $B$  is a unit ball, then

$$\sum_{i=0}^m \binom{m}{i} (-c)^{m-i} S_i(K, \cdot) = 0.$$

Whether the converse is true is unknown even in the simplest non-trivial case,  $d = 3$  and  $m = 2$ . Here even the following might be true.

(9.5) **Problem.** Suppose that  $K \in \mathfrak{K}_0^3$  is a convex body for which

$$S_2(K, \cdot) - 2cS_1(K, \cdot) + c^2 S_0(K, \cdot) \leq 0$$

with some constant  $c$ . Is  $K$  the sum of a (possibly degenerate) segment and a ball?

An affirmative answer would be of considerable interest since it would imply that a closed convex surface of class  $C^2$  on which the principal radii of curvature satisfy an inequality  $(r_1 - c)(r_2 - c) \leq 0$  must be a sphere. This conjecture has been the subject of a number of differential geometric investigations but has been proved only for analytic surfaces. Under the additional assumption that  $K$  admits some circular projection, problem (9.5) has been solved by Schneider [1975a].

A uniqueness theorem that involves only inequalities has been obtained by Aleksandrov [1961]:

(9.6) **Theorem.** Suppose that  $K, K' \in \mathfrak{K}_0^d$  are convex bodies for which

$$S_i(K, \cdot) \leq S_i(K', \cdot) \text{ and } S_{i+1}(K, S^{d-1}) \geq S_{i+1}(K', S^{d-1})$$

for some  $i \in \{1, \dots, d-1\}$  (where for  $i = d-1$  the second condition has to be replaced by  $\text{vol}(K) \geq \text{vol}(K')$ ). Then  $K, K'$  are translates of each other.

Sometimes it is possible to obtain quantitative improvements of uniqueness theorems in the form of stability theorems. For instance, the Aleksandrov-Fenchel-Jessen theorem says that the assumption  $S_i(K, \cdot) - S_i(K', \cdot) = 0$  implies that  $K'$  is a translate of  $K$ . Now assume that  $S_i(K, \cdot) - S_i(K', \cdot)$  is merely near to zero. Must then  $K'$  be almost a translate of  $K$ , in some definite sense? Such results are known in the cases  $i = d-1$  and  $i = 1$ . In the following theorems, the discrepancy  $\delta(K, K')$  between the convex bodies  $K, K'$  is defined as the minimum of the Hausdorff-distances between  $K$  and the translates of  $K'$ .

(9.7) **Theorem.** Let  $0 < r < R$ . There exist numbers  $\varepsilon_0 > 0$  and  $C$ , depending only on  $d, r, R$ , such that the following is true. If  $K, K' \in \mathfrak{K}_0^d$  are convex bodies which contain balls of radius  $r$  and are contained in balls of radius  $R$  and for which

$$|S_{d-1}(K, \omega) - S_{d-1}(K', \omega)| \leq \varepsilon S_{d-1}(K, S^{d-1}) \quad \text{for } \omega \in \mathcal{B}(S^{d-1})$$

with  $0 \leq \varepsilon \leq \varepsilon_0$ , then  $\delta(K, K') \leq C \varepsilon^{1/d}$ .

This result is due to Diskant [1972]. A slightly weaker result of this type had been obtained earlier by Volkov [1963]. A sketch of his proof is reproduced in Pogorelov [1969], chapter VII, §10.

(9.8) **Theorem.** There exists a number  $C$ , depending only on  $d$ , such that the following is true. If  $K, K' \in \mathfrak{K}^d$  are convex bodies for which

$$|S_1(K, \omega) - S_1(K', \omega)| \leq \varepsilon L^{d-1}(\omega) \quad \text{for } \omega \in \mathcal{B}(S^{d-1})$$

with some  $\varepsilon > 0$ , then  $\delta(K, K') \leq C\varepsilon$ .

The special case where  $d = 3$  and  $K, K'$  have twice differentiable support functions is due to Pogorelov [1969], p. 502. His argument extends to the general case as follows. According to Berg [1969], ch. 3, there exists a real function  $g_d$  such that

$$h(K, u) = \int_{S^{d-1}} g_d(\langle u, v \rangle) dS_1(K, v) + \langle s(K), u \rangle$$

for  $K \in \mathfrak{K}^d$  and  $u \in S^{d-1}$ , where  $h(K, \cdot)$  denotes the support function and  $s(K)$  is the Steiner point of  $K$ . Assuming  $s(K) = s(K')$ , we get

$$|h(K, u) - h(K', u)| \leq \int_{S^{d-1}} |g_d(\langle u, v \rangle)| d\tau(v),$$

where  $\tau$  is the total variation of the signed measure  $S_1(K, \cdot) - S_1(K', \cdot)$ . From the assumption of Th. (9.8) we deduce  $\tau(\omega) \leq 2\varepsilon L^{d-1}(\omega)$ . Since  $\int |g_d(\langle u, v \rangle)| dL^{d-1}(v) < \infty$ , the assertion follows.

A special case of theorem (9.1) says that a convex body  $K \in \mathfrak{K}_0^d$  which satisfies  $S_i(K, \cdot) = L^{d-1}$  is necessarily a ball. The following corresponding stability result is due to Diskant [1971].

(9.9) **Theorem.** *There exist numbers  $\varepsilon_0$  and  $C$ , depending only on  $d$ , such that the following is true. If  $K \in \mathfrak{K}^d$  is a convex body for which*

$$1 - \varepsilon \leq \frac{S_i(K, \omega)}{L^{d-1}(\omega)} \leq 1 + \varepsilon$$

*for some  $i \in \{1, \dots, d-1\}$  and each  $\omega \in \mathcal{B}(S^{d-1})$  with  $L^{d-1}(\omega) > 0$ , where  $0 \leq \varepsilon \leq \varepsilon_0$ , then  $K$  is contained in a ball of radius  $1 + C\varepsilon^{1/d-1}$  and contains a ball of radius  $1 - C\varepsilon^{1/d-1}$ .*

Results in the same spirit which hold for curvature measures of the first kind, are rare. Until recently only  $C_0$ , the measure of the spherical image, has been the subject of such investigations. Aleksandrov [1942a] (see also Busemann [1958], Theorem (4.11)) considered the following situation. Let  $K, K' \in \mathfrak{K}_0^d$  be convex bodies with common interior point 0. Then the radial projection  $f: \partial K \rightarrow \partial K'$  is defined by  $f(x) = \lambda(x)x$  with  $\lambda(x) > 0$  for  $x \in \partial K$ . Suppose that  $C_0(K, \beta) = C_0(K', f(\beta))$  for each Borel set  $\beta \subset \partial K$ . Aleksandrov [1942a] showed that then  $K' = \lambda K$  for some  $\lambda > 0$ . He also has related results on unbounded convex surfaces. Making use of the integral geometric formula (7.2), Schneider [1978a] proved the following.

(9.10) **Theorem.** *Let  $K, K' \in \mathfrak{K}_0^d$  be convex bodies which have the origin as common interior point. Suppose that*

$$\sum_{i=0}^{d-1} \alpha_i C_i(K, \beta) = \sum_{i=0}^{d-1} \alpha_i C_i(K', f(\beta))$$

*for each Borel set  $\beta \subset \partial K$ , where  $\alpha_0, \dots, \alpha_{d-1}$  are nonnegative constants with  $\alpha_1 + \dots + \alpha_{d-1} > 0$  and  $f: \partial K \rightarrow \partial K'$  denotes the radial projection. Then  $K = K'$ .*

In Schneider [1978a] also the following result was obtained. Let  $K \in \mathfrak{K}_0^d$  be a convex body, and let  $i \in \{0, \dots, d-1\}$ . Suppose there exists a point  $x \in \text{int } K$  such that, for every hyperplane  $H$  through  $x$ ,  $C_i(K, H^+) = C_i(K, H^-)$ , where  $H^+, H^-$  are the two open halfspaces bounded by  $H$ . Then  $K$  is centrally symmetric with respect to  $x$ . The special case  $i=d-1$  of this theorem had been proved earlier by Schneider [1970b].

As a counterpart to theorem (9.4), the following theorem was proved by Schneider [1979].

(9.11) **Theorem.** *Let  $K \in \mathfrak{K}_0^d$ ,  $i \in \{0, \dots, d-2\}$ , and  $\alpha$  a real number.*

*If*

$$C_i(K, \cdot) = \alpha C_{d-1}(K, \cdot),$$

*then  $K$  is a ball.*

This generalizes to arbitrary convex bodies the Liebmann-Süss theorem according to which a closed convex hypersurface of class  $C^2$  on which the  $(d-1-i)$ -th elementary symmetric function of the principal curvatures is constant, is necessarily a sphere. For  $i=0$ , the following stronger stability theorem is known (Diskant [1968]).

(9.12) **Theorem.** *Let  $K \in \mathfrak{K}_0^d$ ,  $0 < \varepsilon < 1/2$ , and suppose that*

$$1 - \varepsilon \leq \frac{C_0(K, \beta)}{C_{d-1}(K, \beta)} \leq 1 + \varepsilon$$

*for each  $\beta \in \mathcal{B}(E^d)$  with  $C_{d-1}(K, \beta) > 0$ . Then  $K$  is in the  $C\varepsilon$ -neighborhood of some unit ball, where  $C$  depends only on  $d$ .*

Further contributions to this field are Blind [1977] and Schneider [1977d], where a characterization of the ball is obtained, and Schneider [1979b], where convex hypersurfaces with unique projection to a hyperplane satisfying an inequality  $C_0 \geq \alpha C_{d-1}$  are considered.

## §10. Existence theorems

In this section we review the known results on the existence of a convex body for which one of the curvature measures is given in advance.

The classical result is Minkowski's existence theorem for convex polytopes. Recall that for a polytope  $P \in \mathfrak{K}^d$  the value  $S_{d-1}(P, \omega)$  is the sum of the areas of the  $(d-1)$ -faces of  $K$  which have exterior normal vectors belonging to  $\omega$ . The following theorem is, therefore, a special existence result concerning  $S_{d-1}$ .

(10.1) **Theorem.** *Let  $u_1, \dots, u_n \in S^{d-1}$  be vectors not lying in a great sphere, and let  $F_1, \dots, F_n$  be positive numbers such that*

$$\sum_{i=1}^n u_i F_i = 0.$$

Then there exists a polytope  $P \in \mathfrak{K}_0^d$  which has  $u_1, \dots, u_n$  as the exterior unit normal vectors of its  $(d-1)$ -faces and  $F_1, \dots, F_n$  as the corresponding areas.

This generalizes to arbitrary convex bodies as follows.

(10.2) **Theorem.** Let  $\mu$  be a (positive) measure on  $\mathcal{B}(S^{d-1})$  not concentrated on a great sphere, and suppose that

$$(10.3) \quad \int_{S^{d-1}} u d\mu(u) = 0.$$

Then there exists a convex body  $K \in \mathfrak{K}_0^d$  with  $S_{d-1}(K, \cdot) = \mu$ .

Condition (10.3) is clearly necessary; in fact the equalities

$$(10.4) \quad \int_{S^{d-1}} u dS_i(K, u) = 0$$

hold for  $K \in \mathfrak{K}_0^d$  and  $i = 0, 1, \dots, d-1$ . By (9.1) the body  $K$  in (10.2) is uniquely determined to within a translation.

Theorem (10.1) is due to Minkowski [1897] (for  $d = 3$ , but the proof generalizes immediately to  $d \geq 2$ ), see also Minkowski [1903], §9. A particularly elegant version of Minkowski's proof was presented by McMullen [1973], §7. Minkowski obtains the existence of the polytope in question by solving a certain extremum problem related to mixed volumes. For explanations and generalizations of this method, see Bonnesen-Fenchel [1934], p. 123, Fenchel [1938], p. 267, Aleksandrov [1950], chapter VII, §2, Firey [1970b]. Aleksandrov [1939a], see also [1950], chapter VII, §1, found a different proof, using his so-called "mapping lemma", an application of the "invariance of domain". For similar existence results for unbounded polyhedra, see Aleksandrov [1950], chapter VII, §3.

Minkowski [1903], §10, himself extended his result to convex bodies more general than polytopes, but the bodies he admits are defined in an implicit manner (see also Bonnesen-Fenchel [1934], §58). In today terminology, he considers the convex bodies  $K$  for which  $S_{d-1}(K, \cdot)$  has a continuous density with respect to spherical Lebesgue measure. The more satisfactory result (10.2) was possible only when the measure  $S_{d-1}$  was introduced, and then it could be deduced easily from Minkowski's theorem by means of approximation, using the weak continuity of  $S_{d-1}$ . This is the way followed by Fenchel-Jessen [1938] and later by Aleksandrov [1939b] to prove (10.2). Independently of Fenchel-Jessen, Aleksandrov [1938], §3, gave a proof which generalizes Minkowski's approach via an extremum problem, but in the general case the necessary variation argument is less elementary. A similar attempt, but with an insufficient variation argument (see the critique in Bonnesen-Fenchel [1934], p. 123) has been made by Süss [1931]. Related existence theorems for infinite convex surfaces are stated without proof in Aleksandrov [1950], p. 305–306; a similar existence result for convex caps has been obtained by Busemann [1959].

For  $d = 2$  the existence theorem (10.2) applies to  $S_1(K, \cdot)$ , which depends linearly on  $K$ . This fact has proved useful in treating certain decomposition problems for plane convex domains with respect to Minkowski addition. Thus Kallay [1975] characterizes the extreme convex sets  $K$  in the set of bodies in  $\mathfrak{K}^2$  with a given width function by means of a property of the Radon-Nikodym derivative of

$S_1(K, \cdot)$ . Schneider [1974] uses  $S_1$  in the investigation of asymmetry classes of convex sets in the plane.

From the existence theorem (10.2) it follows that to any two convex bodies  $K_1, K_2 \in \mathfrak{K}_0^d$  there exists a convex body  $K \in \mathfrak{K}_0^d$  for which

$$(10.5) \quad S_{d-1}(K, \cdot) = S_{d-1}(K_1, \cdot) + S_{d-1}(K_2, \cdot).$$

$K$  is unique up to a translation. If (10.5) holds, one writes  $K = K_1 \# K_2$  and calls  $K$  the *Blaschke sum* of  $K_1$  and  $K_2$ . Blaschke mentioned this process of addition briefly in [1916], p. 112, but for the case of polytopes it occurred already in the work of Minkowski [1897], p. 117. Blaschke sums have been used by Süss [1932], Kneser-Süss [1932], Firey-Grünbaum [1964], Firey [1965], [1967a], [1967c], Grünbaum [1967], chapter 15.3, Schneider [1967], Chakerian [1971], Kutateladze-Rubinov [1969], Kutateladze [1973], [1976], Goikhman [1974].

In global differential geometry one is interested in a variant of the existence result (10.2): One asks for a closed convex surface which has a given function on  $S^{d-1}$  as the product of the principal radii of curvature, considered as functions of the exterior unit normal vector. If one wants to utilize the general existence result (10.2) in this connection, one needs regularity results for convex surfaces with regular curvature functions. For such results and for more information on Minkowski's problem and related problems from the viewpoint of differential geometry, we refer the reader to the books of Pogorelov [1969], [1975] and to the survey articles of Firey [1970b] and Gluck [1975]. Further, there is a paper by Shefel' [1977].

The existence problem for  $S_i$ ,  $i < d - 1$ , in its present state seems rather inscrutable. Aleksandrov [1938], §4 (see also [1937c]) has exhibited examples of positive measures on  $\mathcal{B}(S^{d-1})$ , even with analytic densities, which satisfy the necessary condition (10.4) but are not equal to the  $i$ -th area function  $S_i(K, \cdot)$  for any convex body  $K \in \mathfrak{K}^d$  and any  $i \in \{1, \dots, d-2\}$ . These examples refuted some statements to the contrary (e.g., Bonnesen-Fenchel [1934], p. 123) and insufficient attempts (Favard [1933b], Süss [1933]) concerning the case  $i = 1$ . For this case, complete solutions have been obtained independently by Berg [1969] and Firey [1967b], [1968]. The case  $i = 1$ , however, is exceptional in that the first order area function  $S_1(K, \cdot)$  depends linearly on the support function of  $K$  and the solution can be represented explicitly by means of a Green's function. Furthermore, the necessary and sufficient conditions obtained by the authors are not easy to handle; for instance, it is not clear how other known necessary conditions for measures  $S_1(K, \cdot)$  (see, e.g., Weil [1973], Kor. 4.5) could be deduced directly from these. Also the necessary and sufficient conditions for first order area functions of polytopes found by Schneider [1977c] are apparently not obtainable from the general result.

Necessary and sufficient conditions for a measure on  $\mathcal{B}(S^{d-1})$  to be the  $i$ -th area function  $S_i(K, \cdot)$  of some convex body  $K$ , where  $1 < i < d - 1$ , are not known. One has a complete solution for the special case of sufficiently smooth bodies of revolution (Firey [1970c], a special case by Nádeník [1968]) and knows some necessary conditions (Firey [1970a], Weil [1979a], compare also Goodey and Schneider [1979]). Some of the results reported in §8 yield further necessary conditions; for instance, (8.5) implies that the support of an  $i$ -th area function is

connected if  $i < d - 1$ . Weil [1979a] has investigated the cone of all  $i$ -th area functions of convex bodies and the vector space generated by it.

As a special case of the existence problem one may ask whether, for given convex bodies  $K_1, K_2 \in \mathfrak{K}_0^d$  and given  $i \in \{2, \dots, d-2\}$ , there exists a convex body  $K \in \mathfrak{K}^d$  such that

$$(10.6) \quad S_i(K, \cdot) = S_i(K_1, \cdot) + S_i(K_2, \cdot).$$

This question was posed, more or less explicitly, by Firey [1967a], p. 99, [1970b], p. 42, [1970c], p. 389, [1975], p. 481, Chakerian [1971], p. 40. However, Goodey and Schneider [1979] showed that  $K$  satisfying (10.6) does not exist when  $K_1, K_2$  are two suitably chosen parallelepipeds.

Existence problems for the curvature measures  $C_i$  of the first kind have only been considered for  $i = 0$ . Let  $K \in \mathfrak{K}_0^d$  be a convex body which has the origin 0 as an interior point. Let  $f: S^{d-1} \rightarrow \partial K$  be defined by  $f(u) = \|f(u)\|u$ .

(10.7) **Theorem.** *Let  $\kappa$  be a positive measure on  $\mathcal{B}(S^{d-1})$ . There exists a convex body  $K \in \mathfrak{K}_0^d$  for which  $C_0(K, f(\omega)) = \kappa(\omega)$  for  $\omega \in \mathcal{B}(S^{d-1})$  if and only if the following conditions are satisfied.*

$$(1) \quad \kappa(S^{d-1}) = L^{d-1}(S^{d-1}),$$

$$(2) \quad \kappa(S^{d-1} \setminus \omega) > L^{d-1}(\omega^*) \text{ whenever } \omega \subset S^{d-1} \text{ is spherically convex and } \omega^* \subset S^{d-1} \text{ denotes the set polar to } \omega.$$

Uniqueness of  $K$  to within a dilatation follows from the result of Aleksandrov [1942a] mentioned before (9.10). Theorem (10.7) was proved by Aleksandrov [1939a], first for polytopes and then by approximation in general. The polytopal case, which Aleksandrov treated as another application of his “mapping lemma” (see also Aleksandrov [1950], chapter IX) is an existence result for polytopes with vertices on given rays through 0 and preassigned curvatures at these vertices. Similar results for unbounded convex surfaces and orthogonal projection on to a plane are obtained by Aleksandrov [1942a], [1950], chapter IX, see also Magomedov [1974].

Far-reaching generalizations of the above existence results and of some of the methods applied in their proofs have led to a geometric theory of elliptic Monge-Ampère equations, for which the reader may consult Pogorelov [1960], [1969], chapter VIII, Bakel'man [1965], chapter IV.

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## References

The abbreviations FdM, MR, Zbl refer to reviews of the papers in *Fortschritte der Mathematik*, *Mathematical Reviews*, *Zentralblatt der Mathematik*. Numbers in square brackets at the end of a reference denote the sections of this report in which that reference is quoted.

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# Wolfgang Weil

## Kinematic integral formulas for convex bodies

### §1. Introduction

In the following we describe some recent developments in integral geometry. The classical integral geometric formulas for convex bodies and the various generalizations of these formulas, for which the reader may consult the books of Hadwiger [1955], [1957] and Santaló [1976], deal with intersecting convex figures. Our aim here is to present results of a different type in two recent branches of integral geometry. In the first case, which was initiated by Hadwiger, one investigates mean value formulas for convex figures which, in contrast to the classical case, have a positive distance. In the other case, which goes back to work of Firey, one considers measures over contact positions of convex figures. Both topics are closely related. As we shall see, the search for integral formulas of the first type that are as general as possible leads one immediately to a natural definition of contact measures of convex bodies. Moreover, since the integral formulas as well as the contact measures involve curvature measures, our considerations also yield results in a third branch of integral geometry, which is concerned with local versions of the classical formulas as they have been obtained by Federer [1959], Schneider [1975], [1978a].

Besides on the results of Firey, Hadwiger, and Schneider, the following considerations are mainly based on two articles of the author (Weil [1979a], [1979b]) concerning contact measures. Here we have chosen a different approach starting from kinematic integral formulas, we include some local formulas, and moreover we prove several new results.

Since the curvature measures of convex bodies are the crucial tools in our considerations, this paper may be regarded as a supplement to the survey article of Schneider [1979b]. We therefore adopt the notations of Schneider [1979b] and especially make use of his §7. For convenience, in the following we refer to Schneider's article by [S].

The following notation is used throughout the paper:

$E^d$	Euclidean d-space
$B$	unit ball in $E^d$
$S^{d-1}$	unit sphere in $E^d$
$\alpha(d)$	volume of $B$
$\partial M$	boundary of a set $M$ w.r.t. the linear hull of $M$
$K, L$	convex bodies
$C_j(K; \cdot)$	curvature measure of the first kind (see [S], §5)
$S_j(K; \cdot)$	curvature measure of the second kind (area function) (see [S], §5)
$V_j(K)$	intrinsic j-volume (see [S], §2)

The measure spaces which underlie the following results are the d-space  $E^d$  with Lebesgue measure  $\lambda = \lambda_d$ , the group of rigid motions  $G_d$  with invariant measure  $\mu$ , the rotation group  $SO_d$  with invariant measure  $\nu$ , and the homogeneous space  $\mathcal{E}_q^d$  of q-flats with invariant measure  $\mu_q$ ,  $q = 0, \dots, d-1$ .

The measures are normalized as follows. We put  $v(SO_d) = d\alpha(d)$  and

$$(1.1) \quad \mu = \gamma \cdot (\lambda \otimes v)$$

where  $\gamma: E^d \times SO_d \rightarrow G_d$  is defined by  $\gamma(t, \delta)x = \delta x + t$  for all  $x \in E^d$ . Similarly, we define

$$(1.2) \quad \mu_q = \gamma_q \cdot (\lambda_{d-q} \otimes v).$$

Here  $\lambda_{d-q}$  is the Lebesgue measure on the orthogonal complement  $F_q^\perp$  of a fixed  $q$ -dimensional subspace  $F_q$  of  $E^d$  and  $\gamma_q: F_q^\perp \times SO_d \rightarrow \mathcal{E}_q^d$  is given by  $\gamma_q(t, \delta) = \delta(F_q + t)$ .

For a rigid motion  $g$  we denote by  $g_0$  the rotation part of  $g$ .

## §2. The classical formulas

We have four groups of classical kinematic formulas for convex bodies.

$$(2.1) \quad (\text{Principal kinematic formulas})$$

$$\int_{G_d} V_j(K \cap gL) d\mu(g) = d\alpha(d)\alpha(d-1) \sum_{k+i=d+j} \frac{\alpha(j)}{\alpha(i-1)\alpha(k-1)} V_k(K)V_i(L)$$

for  $j=0, \dots, d$ .

$$(2.2) \quad (\text{Crofton formulas})$$

$$\int_{SO_d} V_j(K \cap E_q) d\mu_q(E_q) = \frac{d\alpha(d)\alpha(d-1)\alpha(j-1)}{\alpha(q-1)\alpha(d+j-q-1)} V_{d+j-q}(K)$$

for  $j = 0, \dots, q$ ,  $q = 0, \dots, d-1$ .

$$(2.3) \quad \begin{aligned} \int_{SO_d} V_j(K + \delta L) dv(\delta) \\ = d\alpha(d)\alpha(d-1) \sum_{k=0}^j \frac{\alpha(d-j-1)}{\alpha(d-k-1)\alpha(d-j+k-1)} V_k(K)V_{j-k}(L) \end{aligned}$$

for  $j = 0, \dots, d$ .

$$(2.4) \quad \int_{SO_d} V_j(K | \delta F_q) dv(\delta) = \frac{d\alpha(d)\alpha(d-1)\alpha(q-j-1)}{\alpha(q-1)\alpha(d-j-1)} V_j(K)$$

for  $j = 0, \dots, q$ ,  $q = 0, \dots, d-1$ .

Generalizations of (2.3) and (2.4) are given by the integral theorems of Hadwiger [1955], [1957]. Local versions of (2.3) and (2.4) are due to Schneider [1975] ([S], (7.3) and (7.4)). In case of (2.1) and (2.2) we are in a similar situation. Again, there have been obtained local versions by Federer [1959] and Schneider [1978a] (see also [S], §7) and general integral formulas by Hadwiger [1955], [1957]. However, these formulas of Hadwiger do only concern functions defined on  $\{K \cap gL \neq \emptyset\}$  resp.  $\{K \cap E_q \neq \emptyset\}$ . Observe that these sets are the effective domains of integration in (2.1) and (2.2).

It was Hadwiger's idea to investigate also integrals of functions  $f(K, gL)$  and

$h(K, E_q)$  on the complementary domains  $\{K \cap gL = \emptyset\}$  and  $\{K \cap E_q = \emptyset\}$  and to ask for counterparts to (2.1) and (2.2). The first results in this direction (Hadwiger [1975a], Bokowski-Hadwiger-Wills [1976]) concern functions  $f$  and  $h$  which depend only on the distance  $r[K, gL]$  resp.  $r[K, E_q]$ . Here, the distance of two sets  $A, B \subset E^d$  is, as usual, the smallest distance between points of  $A$  and  $B$ . Apart from this, the functions are arbitrary except that they have to fulfill conditions to guarantee the existence of the integrals. The results are in terms of the intrinsic  $j$ -volumes of the bodies  $K, L$  resp.  $K$ .

Hadwiger [1975b] then generalized these results to functions  $f = f_1 f_2 f_3$  (resp.  $h = h_1 h_2$ ). Here  $f_1$  and  $h_1$  are functions of the distance  $r[K, gL]$  and  $r[K, E_q]$ ,  $f_2$  and  $h_2$  are functions of the directions  $u[K, gL]$  (from  $K$  to  $gL$ ) and  $u[K, E_q]$  (from  $K$  to  $E_q$ ) in which the distance is realized, and  $f_3$  is a function of  $g_0^{-1} u[gL, K]$ . The results are expressed by the local analogs of the intrinsic  $j$ -volumes, the curvature measures (of the second kind). (For another proof, see Schneider [1977].)

For general functions  $f, h$  of  $r[K, gL]$ ,  $u[K, gL]$ ,  $g_0^{-1} u[gL, K]$  and  $r[K, E_q]$ ,  $u[K, E_q]$ , corresponding results are proved in Weil [1979a], [1979b]. These are, up to now, the most general formulas for the integrals

$$\int_{K \cap gL = \emptyset} f(g) d\mu(g)$$

and

$$\int_{K \cap E_q = \emptyset} h(E_q) d\mu_q(E_q),$$

and include all the earlier results.

In the following, we study these integrals for functions  $f$  and  $h$  which depend in various ways on the geometry of the convex bodies  $K, L$  resp.  $K$ . We start in §3 with general integral formulas which lead us in an obvious manner to contact measures. These are studied in §4 together with their connection to curvature measures. As a consequence, the integral formulas of Weil [1979a], [1979b] are obtained. The results of §4 are used in §5 to get the rotation formulas for area functions of Schneider [1975] (these are local versions of (2.3) and (2.4)). Moreover, we find similar formulas for the curvature measures of the first kind. Conversely, the latter are used in §6 to obtain formulas corresponding to those in §4, but for functions depending on the boundary points instead of the outer normal directions. In §7 we state a few results in translative integral geometry, generalizing, thus, a formula of Groemer [1979]. Some final remarks are made in the last paragraph.

### §3. General integral formulas

Our aim in this paragraph is to express the integrals

$$\int_{K \cap gL = \emptyset} f(g) d\mu(g)$$

and

$$\int_{K \cap E_q = \emptyset} h(E_q) d\mu_q(E_q)$$

for arbitrary  $\mu$ -integrable functions  $f$  on  $G_d$  and arbitrary  $\mu_q$ -integrable functions  $h$  on  $\mathcal{E}_q^d$  in geometric terms of  $K$  and  $L$  (resp.  $K$ ).

Starting with the first integral, we observe that for  $g$  in the domain of integration, the bodies  $K$  and  $gL$  have a positive distance  $r[K, gL] > 0$ . Therefore, a natural way for a further investigation of this integral is to split up the integration over  $\{K \cap gL = \emptyset\}$  into an integration over the set  $D_r(K, L)$  of rigid motions  $g$  with  $r[K, gL] = r$ ,  $r \in (0, \infty)$ , and then an integration over  $r$ . This procedure is possible if there exists a decomposition of  $\mu$  on  $\{K \cap gL = \emptyset\}$  in the following form

$$(3.1) \quad \mu|_{\{K \cap gL = \emptyset\}} = \int_0^\infty \mu^{(r)}(K, L; \cdot) dr,$$

where  $\mu^{(r)}(K, L; \cdot)$  is a finite Borel measure concentrated on  $D_r(K, L)$ .

In order to find a decomposition (3.1), we make use of (1.1) which implies

$$(3.2) \quad \mu(A) = \int_{SO_d} \lambda(T(A, \delta)) dv(\delta)$$

for each Borel set  $A \subset G_d$ , where

$$T(A, \delta) = \{t \in E^d : \gamma(t, \delta) \in A\}.$$

Observe now that  $D_r(K, L)$  is the set of rigid motions such that  $gL$  touches  $K + rB$ . Hence we have  $\gamma(t, \delta) \in D_r(K, L)$  if and only if  $\delta L + t$  touches  $K + rB$ , which is equivalent to  $t \in \partial(K + rB + (-\delta L))$ . Therefore we obtain a decomposition of type (3.1) if we have a decomposition of  $\lambda$  in  $E^d$

$$(3.3) \quad \lambda|_{\{x: x \notin K + (-\delta L)\}} = \int_0^\infty \lambda^{(r)}(K + (-\delta L); \cdot) dr,$$

where  $\lambda^{(r)}(K + (-\delta L); \cdot)$  is a finite Borel measure on  $\{x : r[K + (-\delta L), x] = r\} = \partial(K + rB + (-\delta L))$ . Natural candidates for such measures are the  $(d-1)$ -dimensional Hausdorff measures on  $\partial(K + rB + (-\delta L))$ . These coincide with the curvature measures  $C_{d-1}(K + rB + (-\delta L); \cdot)$ . The decomposition

$$(3.4) \quad \lambda|_{\{x: x \notin K + (-\delta L)\}} = \int_0^\infty C_{d-1}(K + rB + (-\delta L); \cdot) dr$$

is a special case of a theorem of Federer [1969], (3.2.11). Another simple proof of (3.4) was given in Weil [1979a]. Since  $C_{d-1}(K + rB + (-\delta L); \cdot)$  depends weakly continuously on  $r$ , (3.4) yields the only weakly continuous solution of (3.3).

Using (3.4) in (3.2) and Fubini's theorem, we get for a Borel set  $A \subset \{K \cap gL = \emptyset\}$

$$\begin{aligned} \mu(A) &= \int_{SO_d} \lambda(T(A, \delta)) dv(\delta) \\ &= \int_{SO_d} \int_0^\infty C_{d-1}(K + rB + (-\delta L); T(A, \delta)) dr dv(\delta) \\ &= \int_0^\infty \int_{SO_d} C_{d-1}(K + rB + (-\delta L); T(A, \delta)) dv(\delta) dr. \end{aligned}$$

Therefore, if we define

$$(3.5) \quad \mu^{(r)}(K, L; A) = \int_{S_{0,d}} C_{d-1}(K + rB + (-\delta L); T(A, \delta)) d\nu(\delta),$$

for  $r \in (0, \infty)$ , we have a family of finite Borel measures fulfilling (3.1). The weak continuity of the curvature measures carries over to the measures  $\mu^{(r)}(K, L; \cdot)$ , hence (3.5) is the only weakly continuous solution of (3.1).

Consider now the transformation  $\gamma$  again. Then our result can be written in the form

$$\begin{aligned} \mu|_{\{K \cap gL = \emptyset\}} &= \gamma \circ \left[ \int_0^\infty \int_{\partial(K + rB + (-\delta L))} C_{d-1}(K + rB + (-\delta L); \cdot) dr \, d\nu(\delta) \right] \\ &= \int_0^\infty \gamma \circ \left[ \int_{\partial(K + rB + (-\delta L))} C_{d-1}(K + rB + (-\delta L); \cdot) d\nu(\delta) \right] dr \end{aligned}$$

where we have used in the brackets a shortened form to indicate measures on  $E^d \times S_{0,d}$ . Hence we get the following general integral formula.

(3.6) *Let  $f$  be a  $\mu$ -integrable function on  $G_d$ . Then*

$$\begin{aligned} &\int_{K \cap gL = \emptyset} f(g) d\mu(g) \\ &= \int_0^\infty \int_{S_{0,d}} \int_{\partial(K + rB + (-\delta L))} f(\gamma(t, \delta)) C_{d-1}(K + rB + (-\delta L); dt) d\nu(\delta) dr. \end{aligned}$$

For functions  $f$  on  $\{K \cap gL = \emptyset\}$  which depend only on  $r[K, gL]$ ,  $u[K, gL]$ , and  $g_0$ ,

$$f(g) = f_1(r[K, gL], u[K, gL], g_0),$$

we can give (3.6) a more convenient form. In that case we have

$$\begin{aligned} &\int_{K \cap gL = \emptyset} f(g) d\mu(g) \\ &= \int_0^\infty \int_{S_{0,d}} \int_{\partial(K + rB + (-\delta L))} f_1(r, u[K, \delta L + t], \delta) C_{d-1}(K + rB + (-\delta L); dt) d\nu(\delta) dr. \end{aligned}$$

Since, for  $r > 0$ ,  $K + rB + (-\delta L)$  is smooth,  $u[K, \delta L + t]$  is the unique outer normal  $U_{K+rB+(-\delta L)}(t)$  of this body at the boundary point  $t$ . But  $C_{d-1}(K + rB + (-\delta L); \cdot)$  is the image measure of  $S_{d-1}(K + rB + (-\delta L); \cdot)$  under the transformation  $U_{K+rB+(-\delta L)}$  (Schneider [1978a]). Hence we get

$$\begin{aligned} &\int_{\partial(K + rB + (-\delta L))} f_1(r, u[K, \delta L + t], \delta) C_{d-1}(K + rB + (-\delta L); dt) \\ &= \int_{S^{d-1}} f_1(r, u, \delta) S_{d-1}(K + rB + (-\delta L); du). \end{aligned}$$

By the Steiner formula for parallel bodies (see e.g. [S], §5), we obtain the following result.

(3.7) Let  $f$  be a measurable function on  $(0, \infty) \times S^{d-1} \times S0_d$ , for which the integrals in the following sum are finite. Then

$$\begin{aligned} & \int_{K \cap gL = \emptyset} f(r[K, gL], u[K, gL], g_0) d\mu(g) \\ &= \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty \int_{S0_d} \int_{S^{d-1}} r^{d-j-1} f(r, u, \delta) S_j(K + (-\delta L); du) dv(\delta) dr. \end{aligned}$$

In a similar way, the integral

$$\int_{K \cap E_q = \emptyset} h(E_q) d\mu_q(E_q)$$

can be treated. Here we use (1.2) and

$$\lambda_{d-q}|_{\{x \in \delta F_q^\perp : x \notin K \mid \delta F_q^\perp\}} = \int_0^\infty C'_{d-q-1}(K + rB | \delta F_q^\perp; \cdot) dr.$$

We define

$$(3.8) \quad \mu_q^{(r)}(K; A) = \int_{S0_d} C'_{d-q-1}(K + rB | \delta F_q^\perp; T_q(A, \delta)) dv(\delta),$$

where

$$T_q(A, \delta) = \{t \in \delta F_q^\perp : \delta F_q + t \in A\}.$$

Then (3.8) is a weakly continuous solution of

$$\mu_q|_{\{K \cap E_q = \emptyset\}} = \int_0^\infty \mu_q^{(r)}(K; \cdot) dr.$$

Equivalently, we have

$$\begin{aligned} \mu_q|_{\{K \cap E_q = \emptyset\}} &= \gamma_q \circ \left[ \int_0^\infty \int_0^\infty C'_{d-q-1}(K + rB | \delta F_q^\perp; \cdot) dr dv(\delta) \right] \\ &= \int_0^\infty \gamma_q \circ \left[ \int C'_{d-q-1}(K + rB | \delta F_q^\perp; \cdot) dv(\delta) \right] dr \end{aligned}$$

which implies the following result.

(3.9) Let  $h$  be a  $\mu_q$ -integrable function on  $\mathcal{E}_q^d$ . Then

$$\begin{aligned} & \int_{K \cap E_q = \emptyset} h(E_q) d\mu_q(E_q) \\ &= \int_0^\infty \int_{S0_d} \int_{\ell(K + rB | \delta F_q^\perp)} h(\gamma_q(t, \delta)) C'_{d-q-1}(K + rB | \delta F_q^\perp; dt) dv(\delta) dr. \end{aligned}$$

To formulate the counterpart to (3.7), we consider the set  $\mathcal{L}_q^d$  of all  $q$ -dimensional linear subspaces of  $E^d$  with the usual topology and denote by  $E_q^{(0)}$  the element of  $\mathcal{L}_q^d$  which is parallel to  $E_q \in \mathcal{E}_q^d$ . Then, the following result holds.

(3.10) Let  $h$  be a measurable function on  $(0, \infty) \times S^{d-1} \times \mathcal{L}_q^d$ , for which the integrals in the following sum are finite. Then

$$\begin{aligned} & \int_{K \cap E_q = \emptyset} h(r[K, E_q], u[K, E_q], E_q^{(0)}) d\mu_q(E_q) \\ &= \sum_{j=0}^{d-q-1} \binom{d-q-1}{j} \int_0^\infty \int_{S_{0,d}} \int_{S^{d-1}} r^{d-q-j-1} h(r, u, \delta F_q) S'_j(K | \delta F_q^\perp; du) d\nu(\delta) dr. \end{aligned}$$

Here, for simplicity, we have extended the measure  $S'_j(K | \delta F_q^\perp; \cdot)$  which originally is defined on  $S^{d-1} \cap \delta F_q^\perp$  to a measure on  $S^{d-1}$  by

$$S'_j(K | \delta F_q^\perp; A) = S'_j(K | \delta F_q^\perp; A \cap \delta F_q^\perp).$$

However, the integral formulas (3.6), (3.7), (3.9), (3.10) are slightly unsatisfactory in that they contain an integration over  $S_{0,d}$  on the right side. Of course, we can only hope to simplify these formulas, if we consider more special functions  $f$  and  $h$  (e.g. those which do not depend on  $\delta$ ), and if we are able to transform the measures occurring in these formulas. Thus, in the next paragraph we will further investigate the measures (3.4) and (3.8).

#### §4. Contact measures

The measures  $\mu^{(r)}(K, L; \cdot)$  defined by (3.5) depend weakly continuously on  $r$ . Therefore their weak limit

$$\mu(K, L; \cdot) = \text{w-lim}_{r \rightarrow 0} \mu^{(r)}(K, L; \cdot)$$

exists and obeys

$$(4.1) \quad \mu(K, L; A) = \int_{S_{0,d}} C_{d-1}(K + (-\delta L); T(A, \delta)) d\nu(\delta),$$

for all Borel sets  $A \subset G_d$ . Since  $\mu(K, L; \cdot)$  is supported by

$$D_0(K, L) = \{g : gL \text{ touches } K\}$$

and since

$$\mu(K + rB, L; \cdot) = \mu^{(r)}(K, L; \cdot),$$

$\mu(K, L; \cdot)$  may be interpreted as the natural contact measure of  $K$  and  $L$ ,  $K$  fixed,  $L$  movable. With this interpretation, the measure  $\mu(K, L; \cdot)$  was introduced in Weil [1979b] by another method, namely by a local Steiner formula in  $G_d$ . The purpose of that paper was to find a measure space as a framework for results of Firey [1974] and Schneider [1978b] (Theorems 7.10 and 7.14 in [S]). Especially,  $\mu(K, L; \cdot)$  yields the natural probability measure for problems on randomly touching convex bodies. In this connection, a first hint to the definition of  $\mu(K, L; \cdot)$  was given in McMullen [1974].

Analogously, in the case of  $q$ -flats, we get by

$$\mu_q(K; \cdot) = \text{w-lim}_{r \rightarrow 0} \mu_q^{(r)}(K; \cdot)$$

a measure on

$$D_0^q(K) = \{E_q : E_q \text{ touches } K\}$$

which obeys

$$(4.2) \quad \mu_q(K; A) = \int_{S_{0,d}} C_{d-q-1}(K | \delta F_q^\perp; T_q(A, \delta)) d\nu(\delta)$$

for all Borel sets  $A \subset \mathcal{E}_q^d$  and which is the natural contact measure of  $K$  w.r.t.  $q$ -flats. Again, this measure originally was introduced by a local Steiner formula in  $\mathcal{E}_q^d$  (Weil [1979a]) and yields the measure space in which the integral geometric interpretations of the curvature measures by Firey [1972] and Schneider [1978a] (Theorem 7.12 and subsequent remarks in [S]) find their place.

The mentioned results of Firey and Schneider give natural values for the measurement of the following subsets of  $D_0(K, L)$  resp.  $D_0^q(K)$ . For Borel sets  $\omega, \omega' \subset S^{d-1}$  and  $\beta \subset \partial K$ ,  $\beta' \subset \partial L$  we define

$$M_0^q(K; \omega) = \{E_q \in D_0^q(K) : E_q \text{ lies in a supporting hyperplane to } K \text{ with outer normal in } \omega\},$$

$$L_0^q(K; \beta) = \{E_q \in D_0^q(K) : E_q \cap \beta \neq \emptyset\},$$

$$M_0(K, L; \omega, \omega') = \{g \in D_0(K, L) : K \text{ and } gL \text{ are separated by a hyperplane with outer normal } (w.r.t. K) u \in \omega \cap (-g_0 \omega')\},$$

$$L_0(K, L; \beta, \beta') = \{g \in D_0(K, L) : \beta \cap g\beta' \neq \emptyset\}.$$

As it was recently shown by Burton [1979], the sets  $M_0^q(K; \omega)$  and  $L_0^q(K; \beta)$  need not be Borel sets in general. This is true for the other sets too. However, if we assume the measures  $\mu(K, L; \cdot)$  and  $\mu_q(K; \cdot)$  to be complete, the sets defined above are measurable (i.e. elements of the completion of the Borel  $\sigma$ -algebra) and we have the following connection with the curvature measures (Weil [1979a], [1979b])

$$(4.3) \quad \mu_q(K; M_0^q(K; \omega)) = (d - q)\alpha(d - q)S_{d-q-1}(K; \omega),$$

for all Borel sets  $\omega \subset S^{d-1}$ ,

$$(4.4) \quad \mu_q(K; L_0^q(K; \beta)) = (d - q)\alpha(d - q)C_{d-q-1}(K; \beta),$$

for all Borel sets  $\beta \subset \partial K$ ,

$$(4.5) \quad \mu(K, L; M_0(K, L; \omega, \omega')) = \sum_{i=0}^{d-1} \binom{d-1}{i} S_i(K; \omega) S_{d-i-1}(L; \omega'),$$

for all Borel sets  $\omega, \omega' \subset S^{d-1}$ , and

$$(4.6) \quad \mu(K, L; L_0(K, L; \beta, \beta')) = \sum_{i=0}^{d-1} \binom{d-1}{i} C_i(K; \beta) C_{d-i-1}(L; \beta'),$$

for all Borel sets  $\beta \subset \partial K$ ,  $\beta' \subset \partial L$ .

Since the parallel body  $K + rB$  is always smooth, the set  $M_0^q(K + rB; \omega)$  is the inverse of  $\omega$  under the map

$$V_{K+rB} : D_0^q(K + rB) \rightarrow S^{d-1}, E_q \mapsto u[K, E_q],$$

which is continuous. Then, (4.3) implies that  $(d - q)\alpha(d - q)S_{d-q-1}(K + rB; \cdot)$  is the image measure  $V_{K+rB} \circ \mu_q(K + rB; \cdot)$ . We apply this and (3.9) to the integral of a measurable function  $h$  on  $(0, \infty) \times S^{d-1}$  and get

$$\begin{aligned}
& \int_{K \cap E_q = \emptyset} h(r[K, E_q], u[K, E_q]) d\mu_q(E_q) \\
&= \int_0^\infty \int_{D_r^d(K)} h(r, u[K, E_q]) \mu_q(K + rB; dE_q) dr \\
&= (d-q)\alpha(d-q) \int_0^\infty \int_{S^{d-1}} h(r, u) S_{d-q-1}(K + rB; du) dr.
\end{aligned}$$

Hence, by the Steiner formula for parallel bodies ([S], §5), we obtain the following formula (Weil [1979a], [S], (7.9)).

(4.7) *Let  $h$  be a measurable function on  $(0, \infty) \times S^{d-1}$  for which the integrals in the following sum are finite. Then*

$$\begin{aligned}
& \int_{K \cap E_q = \emptyset} h(r[K, E_q], u[K, E_q]) d\mu_q(E_q) \\
&= (d-q)\alpha(d-q) \sum_{j=0}^{d-q-1} \binom{d-q-1}{j} \int_0^\infty \int_{S^{d-1}} r^{d-q-j-1} h(r, u) S_j(K; du) dr.
\end{aligned}$$

In the case of rigid motions we can proceed similarly. Since  $K + rB$  is smooth,  $K + rB + (-\delta L)$  has a unique outer normal in each boundary point. Hence (4.5), used for  $K + rB$  and  $L$ , implies that

$$\sum_{j=0}^{d-1} \binom{d-1}{j} S_j(K + rB; \cdot) \otimes S_{d-j-1}(L; \cdot)$$

is the image measure of  $\mu(K + rB, L; \cdot)$  under the transformation

$$W_{K+rB, L}: g \mapsto (u[K, gL], g_0^{-1}u[gL, K]).$$

Here we employ  $u[K, gL] = U_{K+rB+(-\delta L)}(t)$ , where  $r = r[K, gL]$  and  $g = \gamma(t, \delta)$ . Therefore, the integral of a function  $f$  on  $(0, \infty) \times S^{d-1} \times S^{d-1}$  can be transformed in the following way in view of (3.6)

$$\begin{aligned}
& \int_{K \cap gL = \emptyset} f(r[K, gL], u[K, gL], g_0^{-1}u[gL, K]) d\mu(g) \\
&= \int_0^\infty \int_{D_r(K, L)} f(r, u[K, gL], g_0^{-1}u[gL, K]) \mu(K + rB, L; dg) dr \\
&= \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty \int_{S^{d-1}} \int_{S^{d-1}} f(r, u, v) S_j(K + rB; du) S_{d-j-1}(L; dv) dr \\
&= \sum_{j=0}^{d-1} \sum_{i=0}^j \binom{d-1}{j} \binom{j}{i} \int_0^\infty \int_{S^{d-1}} \int_{S^{d-1}} r^{j-i} f(r, u, v) S_i(K; du) S_{d-j-1}(L; dv) dr.
\end{aligned}$$

If we put  $d-1-j=k$ , we get the following integral formula of Weil [1979b] ((7.8) in [S]).

(4.8) *Let  $f$  be a measurable function on  $(0, \infty) \times S^{d-1} \times S^{d-1}$  for which the integrals in the following sum are finite. Then*

$$\begin{aligned} & \int_{K \cap gL = \emptyset} f(r[K, gL], u[K, gL], g_0^{-1} u[gL, K]) d\mu(g) \\ &= \sum_{i,k=0}^{d-1} \binom{d-1}{i+k} \binom{i+k}{i} \int_0^\infty \int_{S^{d-1}} \int_{S^{d-1}} r^{d-i-k-1} f(r, u, v) S_i(K; du) S_k(L; dv) dr. \end{aligned}$$

For later use, we mention that, by a change in the order of summation, this formula can be written in the following way

$$(4.9) \quad \begin{aligned} & \int_{K \cap gL = \emptyset} f(r[K, gL], u[K, gL], g_0^{-1} u[gL, K]) d\mu(g) \\ &= \sum_{j=0}^{d-1} \sum_{k=0}^j \binom{d-1}{j} \binom{j}{k} \int_0^\infty \int_{S^{d-1}} \int_{S^{d-1}} r^{d-j-1} f(r, u, v) S_j(K; du) S_{j-k}(L; dv) dr. \end{aligned}$$

## §5. Local formulas

If we compare the formulas (4.7) and (4.9) with the general formulas (3.10) and (3.7) we get local formulas for the curvature measures of the second kind.

For a measurable function  $h$  on  $(0, \infty) \times S^{d-1}$ , (3.10) implies

$$\begin{aligned} & \int_{K \cap E_q = \emptyset} h(r[K, E_q], u[K, E_q]) d\mu_q(E_q) \\ &= \sum_{j=0}^{d-q-1} \binom{d-q-1}{j} \int_0^\infty \int_{S^{d-1}} r^{d-q-j-1} h(r, u) \int_{S0_d} S'_j(K | \delta F_q^\perp; du) dv(\delta) dr. \end{aligned}$$

If we compare this formula with (4.7) and take into consideration that  $h$  was arbitrary, we get the following local integral formula of Schneider [1975] ([S], (7.4)).

$$(5.1) \quad \int_{S0_d} S'_j(K | \delta F_q; \cdot) dv(\delta) = q \alpha(q) S_j(K; \cdot)$$

for  $j = 0, \dots, q-1$ ,  $q = 0, \dots, d-1$ .

For a measurable function  $f$  on  $(0, \infty) \times S^{d-1} \times S^{d-1}$ , (3.7) implies

$$(5.2) \quad \begin{aligned} & \int_{K \cap gL = \emptyset} f(r[K, gL], u[K, gL], g_0^{-1} u[gL, K]) d\mu(g) \\ &= \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty \int_{S0_d} \int_{S^{d-1}} r^{d-j-1} f(r, u, -\delta^{-1} u) S_j(K + (-\delta L); du) dv(\delta) dr. \end{aligned}$$

For  $\delta \in S0_d$  we define the continuous transformation

$$T_\delta: S^{d-1} \rightarrow S^{d-1} \times S^{d-1}$$

by  $T_\delta(u) = (u, -\delta^{-1} u)$ . Then we have

$$\begin{aligned} & \int_{S^{d-1}} f(r, u, -\delta^{-1} u) S_j(K + (-\delta L); du) \\ &= \int_{S^{d-1}} \int_{S^{d-1}} f(r, u, v) [T_\delta \circ S_j(K + (-\delta L); \cdot)] (du, dv). \end{aligned}$$

Hence, (5.2) changes to

$$\begin{aligned}
& \int_{K \cap gL = \emptyset} f(r[K, gL], u[K, gL], g_0^{-1}u[gL, K]) d\mu(g) \\
&= \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty \int_{S0_d} \int_{S^{d-1}} \int_{S^{d-1}} r^{d-j-1} f(r, u, v) [T_\delta \circ S_j(K + (-\delta L); \cdot)] \\
&\quad (du, dv) dv(\delta) dr.
\end{aligned}$$

Again, we compare this with (4.9) and notice that  $f$  was arbitrary. Hence, the following local integral formula of Schneider [1975] ([S], (7.3)) is settled.

$$(5.3) \quad \int_{S0_d} S_j(K + \delta L; \omega \cap \delta \omega') dv(\delta) = \sum_{k=0}^j \binom{j}{k} S_k(K; \omega) S_{j-k}(L; \omega')$$

for  $j=0, \dots, d-1$ , and all Borel sets  $\omega, \omega' \subset S^{d-1}$ .

Both formulas, (5.1) and (5.3) could also be obtained formally by the use of (4.3) in (4.2) and (4.5) in (4.1). This procedure is of interest since we similarly can use (4.4) in (4.2) and (4.6) in (4.1) to get local formulas for the curvature measures of the first kind. However, we then have to show that (4.1) and (4.2) are valid for measurable sets  $A$ , too. To prove this, we take a measurable set  $A$  in  $D_0(K, L)$ . Hence there are Borel sets  $A'$ ,  $N \subset D_0(K, L)$  such that the symmetric difference  $A \Delta A'$  is a subset of  $N$  and  $\mu(K, L; N) = 0$ . But

$$T(A, \delta) \Delta T(A', \delta) \subset T(N, \delta)$$

and  $C_{d-1}(K + (-\delta L); T(N, \delta)) = 0$  for  $v$ -almost all  $\delta$ . Hence  $T(A, \delta)$  is measurable for  $v$ -almost all  $\delta$  and (4.1) is valid for  $A$ . By a theorem of Schneider [1978b] we can slightly strengthen this result in the case  $A = L_0(K, L; \beta, \beta')$ . Schneider proved that for  $v$ -almost all  $\delta$  the distance  $r[K, \delta L + t]$  is realized by exactly one pair  $x_t \in K$ ,  $y_t \in \delta L + t$  for all  $t \notin K + (-\delta L)$ . This implies that for  $v$ -almost all  $\delta$  the set  $(\beta + (-\delta \beta')) \cap \partial(K + (-\delta L))$  is a Borel set, if  $\beta \subset \partial K$  and  $\beta' \subset \partial L$  are Borel sets. Hence, we can use (4.1) for  $A = L_0(K, L; \beta, \beta')$ . Because of

$$T(L_0(K, L; \beta, \beta'), \delta) = (\beta + (-\delta \beta')) \cap \partial(K + (-\delta L))$$

and (4.6) we get the following formula

$$(5.4) \quad \int_{S0_d} C_{d-1}(K + \delta L; \beta + \delta \beta') dv(\delta) = \sum_{k=0}^{d-1} \binom{d-1}{k} C_k(K; \beta) C_{d-k-1}(L; \beta').$$

for all Borel sets  $\beta \subset \partial K$ ,  $\beta' \subset \partial L$ . Especially for  $L = rB$ ,  $r > 0$ , and  $\beta' = \partial(rB)$ , (5.4) implies

$$(5.5) \quad C_{d-1}(K + rB; \beta + rB) = \sum_{k=0}^{d-1} \binom{d-1}{k} r^{d-k-1} C_k(K; \beta).$$

If in (5.5) we once replace  $r$  by  $r+s$  and then  $K$  by  $K+sB$  and  $\beta$  by  $(\beta+sB) \cap \partial(K+sB)$  we immediately get the following Steiner formula for parallel bodies (Schneider [1979a])

$$(5.6) \quad C_j(K + rB; \beta + rB) = \sum_{k=0}^j \binom{j}{k} r^{j-k} C_k(K; \beta),$$

for  $j=0, \dots, d-1$  and all Borel sets  $\beta \subset \partial K$ . Now we replace in (5.4)  $K$  by  $K+rB$

and  $\beta$  by  $(\beta + rB) \cap \partial(K + rB)$  and use (5.6). Then we get on the one hand

$$\begin{aligned} & \int_{S0_d} C_{d-1}(K + rB + \delta L; [(\beta + rB) \cap \partial(K + rB)] + \delta \beta') dv(\delta) \\ &= \sum_{k=0}^{d-1} \binom{d-1}{k} r^{d-k-1} \int_{S0_d} C_k(K + \delta L; \beta + \delta \beta') dv(\delta), \end{aligned}$$

and on the other hand

$$\begin{aligned} & \int_{S0_d} C_{d-1}(K + rB + \delta L; [(\beta + rB) \cap \partial(K + rB)] + \delta \beta') dv(\delta) \\ &= \sum_{k=0}^{d-1} \binom{d-1}{k} C_k(K + rB; \beta + rB) C_{d-k-1}(L; \beta') \\ &= \sum_{k=0}^{d-1} \sum_{j=0}^k \binom{d-1}{k} \binom{k}{j} r^{k-j} C_j(K; \beta) C_{d-k-1}(L; \beta') \\ &= \sum_{k=0}^{d-1} \sum_{j=0}^k \binom{d-1}{k} \binom{k}{j} r^{d-k-1} C_j(K; \beta) C_{k-j}(L; \beta'). \end{aligned}$$

Comparing the coefficients of  $r^{d-k-1}$  we find the following generalization of (5.4) as a counterpart to (5.3).

$$(5.7) \quad \int_{S0_d} C_j(K + \delta L; \beta + \delta \beta') dv(\delta) = \sum_{k=0}^j \binom{j}{k} C_k(K; \beta) C_{j-k}(L; \beta')$$

for  $j = 0, \dots, d-1$  and all Borel sets  $\beta \subset \partial K$ ,  $\beta' \subset \partial L$ .

Again, there are similar results for  $q$ -flats. First, (4.2) is valid for measurable sets  $A$  too and  $T_q(L_0(K; \beta), \delta)$  is a Borel set for  $v$ -almost all  $\delta$  if  $\beta \subset \partial K$  is a Borel set. Thus, we can use (4.4) in (4.2). Because of

$$T_q(L_0(K; \beta), \delta) = (\beta | \delta F_q^\perp) \cap \partial(K | \delta F_q^\perp),$$

we then get the formula

$$\int_{S0_d} C'_{d-q-1}(K | \delta F_q^\perp; \beta | \delta F_q^\perp) dv(\delta) = (d-q)\alpha(d-q) C_{d-q-1}(K; \beta)$$

for all Borel sets  $\beta \subset \partial K$ . Employing (5.6), we obtain the following counterpart to (5.1).

$$(5.8) \quad \int_{S0_d} C'_j(K | \delta F_q; \beta | \delta F_q) dv(\delta) = q\alpha(q) C_j(K; \beta)$$

for  $j = 0, \dots, q-1$ ,  $q = 0, \dots, d-1$ , and all Borel sets  $\beta \subset \partial K$ .

## §6. Boundary formulas

We have got the local integral formulas (5.1) and (5.3) by comparing the integral formulas in §4, which concern the curvature measures of the second kind, with the formulas in §3. In reverse, we can use now (5.7) and (5.8) in (3.6) and (3.9) to get formulas of type (4.7) and (4.8) where the curvature measures of the second kind are replaced by those of the first kind.

For that purpose, we consider a measurable function  $f$  on  $(0, \infty) \times \partial K \times \partial L$ . By the result of Schneider [1978b] which we used earlier in §5, for  $v$ -almost all  $\delta$  the distance  $r[K, \delta L + t]$  is realized by exactly one pair of points  $x_i \in \partial K$ ,  $y_i \in \partial(\delta L + t)$  for all  $t \notin K + (-\delta L)$ . For convex bodies  $K, L$  let  $x[K, L]$  be the point in  $\partial K$ , which fulfills  $r[x[K, L], L] = r[K, L]$ , if this point is unique. Hence the integral

$$\int_{K \cap gL = \emptyset} f(r[K, gL], x[K, gL], g^{-1}x[gL, K]) d\mu(g)$$

makes sense. If it is finite, in view of (3.5) it equals

$$\begin{aligned} & \int_0^\infty \int_{S0_d} \int_{\partial(K + rB + (-\delta L))} f(r, x[K, \delta L + t], \delta^{-1}(x[\delta L + t, K] - t)) \\ & C_{d-1}(K + rB + (-\delta L); dt) dv(\delta) dr. \end{aligned}$$

For  $v$ -almost all  $\delta$  we define the continuous transformation  $S_\delta : \partial(K + rB + (-\delta L)) \rightarrow \partial K \times \partial L$  by

$$S_\delta(t) = (x[K, \delta L + t], \delta^{-1}(x[\delta L + t, K] - t)).$$

Thus, we have

$$\begin{aligned} & \int_{K \cap gL = \emptyset} f(r[K, gL], x[K, gL], g^{-1}x[gL, K]) d\mu(g) \\ &= \int_0^\infty \int_{S0_d} \int_{\partial K} \int_{\partial L} f(r, x, y) [S_\delta \circ C_{d-1}(K + rB + (-\delta L); \cdot)] (dx, dy) dv(\delta) dr. \end{aligned}$$

Since

$$\begin{aligned} & [S_\delta \circ C_{d-1}(K + rB + (-\delta L); \cdot)] (\beta \times \beta') \\ &= C_{d-1}(K + rB + (-\delta L); \beta + rB + (-\delta \beta')) \end{aligned}$$

we can use (5.4) and (5.5) to conclude

$$\begin{aligned} & \int_{K \cap gL = \emptyset} f(r[K, gL], x[K, gL], g^{-1}x[gL, K]) d\mu(g) \\ &= \sum_{k=0}^{d-1} \sum_{j=0}^k \binom{d-1}{k} \binom{k}{j} \int_0^\infty \int_{\partial K} \int_{\partial L} r^{k-j} f(r, x, y) C_j(K; dx) C_{d-k-1}(L; dy) dr. \end{aligned}$$

This implies the following result.

(6.1) *Let  $f$  be a measurable function on  $(0, \infty) \times \partial K \times \partial L$  for which the integrals in the following sum are finite. Then*

$$\begin{aligned} & \int_{K \cap gL = \emptyset} f(r[K, gL], x[K, gL], g^{-1}x[gL, K]) d\mu(g) \\ &= \sum_{i,k=0}^{d-1} \binom{d-1}{i+k} \binom{i+k}{i} \int_0^\infty \int_{\partial K} \int_{\partial L} r^{d-i-k-1} f(r, x, y) C_i(K; dx) C_k(L; dy) dr. \end{aligned}$$

Next, we consider a measurable function  $h$  on  $(0, \infty) \times \partial K$ . For  $q$ -flats, the mentioned result of Schneider implies that for  $v$ -almost all  $\delta$  the distance  $r[K, \delta F_q + t]$  is realized by exactly one point  $x[K, \delta F_q + t] \in \partial K$  for all  $t \notin K \cap \delta F_q^\perp$ .

Hence we can investigate the integral

$$\int_{K \cap E_q = \emptyset} h(r[K, E_q], x[K, E_q]) d\mu_q(E_q)$$

and use (3.9) to get

$$(6.2) \quad \begin{aligned} & \int_{K \cap E_q = \emptyset} h(r[K, E_q], x[K, E_q]) d\mu_q(E_q) \\ &= \int_0^\infty \int_{S^0_d} \int_{\partial(K + rB | \delta F_q^\perp)} h(r, x[K, \delta F_q + t]) C'_{d-q-1}(K + rB | \delta F_q^\perp; dt) dv(\delta) dr. \end{aligned}$$

Furthermore, for  $v$ -almost all  $\delta$ , we can define a continuous transformation

$$S_\delta^{(q)} : \partial(K + rB | \delta F_q^\perp) \rightarrow \partial K$$

by

$$S_\delta^{(q)}(t) = x[K, \delta F_q + t].$$

The image measure  $S_\delta^{(q)} \circ C'_{d-q-1}(K + rB | \delta F_q^\perp; \cdot)$  obeys

$$\begin{aligned} & [S_\delta^{(q)} \circ C'_{d-q-1}(K + rB | \delta F_q^\perp; \cdot)](\beta) \\ &= C'_{d-q-1}(K + rB | \delta F_q^\perp; \beta + rB | \delta F_q^\perp) \end{aligned}$$

for all Borel sets  $\beta \subset \partial K$ . Using this, (5.5), and (5.8) in (6.2), we obtain the following result.

(6.3) *Let  $h$  be a measurable function on  $(0, \infty) \times \partial K$  for which the integrals in the following sum are finite. Then*

$$\begin{aligned} & \int_{K \cap E_q = \emptyset} h(r[K, E_q], x[K, E_q]) d\mu_q(E_q) \\ &= (d-q)\alpha(d-q) \sum_{j=0}^{d-q-1} \binom{d-q-1}{j} \int_0^\infty \int_{\partial K} r^{d-q-j-1} h(r, x) C_j(K; dx) dr. \end{aligned}$$

## §7. Translative integral formulas

Kinematic integrals with respect to the group of translations of  $E^d$  have been investigated by several authors (see e.g. Groemer [1977]). Our methods immediately apply to these integrals. In this paragraph we want to describe shortly the results.

We consider the integral

$$\int_{K \cap (L+x) = \emptyset} g(x) d\lambda(x)$$

where  $g$  is a  $\lambda$ -integrable function on  $E^d$ . Then the decomposition (3.4) leads to the following integral formula.

(7.1) *Let  $g$  be a  $\lambda$ -integrable function on  $E^d$ . Then*

$$\begin{aligned} & \int_{K \cap (L+x) = \emptyset} g(x) d\lambda(x) \\ &= \int_0^\infty \int_{\partial(K+rB+(-L))} g(x) C_{d-1}(K+rB+(-L); dx) dr. \end{aligned}$$

For functions on  $\{K \cap (L+x) = \emptyset\}$  which depend only on  $r[K, L+x]$  and  $u[K, L+x]$ , (7.1) implies

$$\begin{aligned} & \int_{K \cap (L+x) = \emptyset} g(r[K, L+x], u[K, L+x]) d\lambda(x) \\ &= \int_0^\infty \int_{S^{d-1}} g(r, u) S_{d-1}(K+rB+(-L); du) dr \\ &= \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty \int_{S^{d-1}} g(r, u) r^{d-j-1} S_j(K+(-L); du) dr. \end{aligned}$$

Using the mixed area functions (see [S], §5)

$$S(\underbrace{K, \dots, K}_i, \underbrace{-L, \dots, -L}_{j-i}, \underbrace{B, \dots, B}_{d-j-1}; \cdot)$$

for  $i=0, \dots, j$ , we get the following result.

(7.2) *Let  $g$  be a measurable function on  $(0, \infty) \times S^{d-1}$  for which the integrals in the following sum are finite. Then*

$$\begin{aligned} & \int_{K \cap (L+x) = \emptyset} g(r[K, L+x], u[K, L+x]) d\lambda(x) \\ &= \sum_{i,k=0}^{d-1} \binom{d-1}{i+k} \binom{i+k}{i} \int_0^\infty \int_{S^{d-1}} r^{d-i-k-1} g(r, u) S(\underbrace{K, \dots, K}_i, \underbrace{-L, \dots, -L}_k, \underbrace{B, \dots, B}_{d-i-k-1}; du) dr. \end{aligned}$$

If  $g$  depends only on  $r$ , this formula reduces to a formula of Groemer [1979].

## §8. Final remarks

The developments which we have described in this article are of course not yet completed, perhaps they even stand at the beginning. Further efforts will surely be made to determine contact probabilities, others than those given by (4.3) to (4.6). However, since the contact measures are integrals of curvature measures, each contact probability given in geometric terms of the bodies under consideration, implies immediately a (local) integral formula for curvature measures. Or reversely, the determination of contact probabilities will require in most cases the proof of an integral formula. We explain this shortly by two examples.

Firey [1979] recently considered an inner contact problem for convex bodies  $K, L$ . Assume that  $L$  can freely roll around in  $K$ . This means that for each  $\delta \in S0_d$  there is a convex body  $M_\delta$  such that  $K = M_\delta + \delta L$ . For special sets of rigid

motions  $g$ , for which  $gL$  touches  $K$  from inside, Firey determines natural values. The result corresponds to (4.5) except that there is an alternating sign in the sum. It is possible to find the natural contact measure  $\hat{\mu}(K,L;\cdot)$  for this inner contact problem given by

$$\hat{\mu}(K,L;A) = \int_{S0_d} C_{d-1}(M_\delta; T(A,\delta)) dv(\delta).$$

However, if we want to find the probability that the fixed body  $K$  and the movable body  $L$  touch (from inside) in prescribed boundary sets  $\beta \subset \partial K$ ,  $\beta' \subset \partial L$ , we need a result similar to (4.6). This leads us to the integral formula (compare with (5.4))

$$\int_{S0_d} C_{d-1}(M_\delta; \beta - \delta\beta') dv(\delta) = \sum_{k=0}^{d-1} (-1)^{d-k-1} \binom{d-1}{k} C_k(K; \beta) C_{d-k-1}(L; \beta')$$

which is not proved up to now.

Another natural problem is the following. Suppose that two convex bodies  $K, L$  touch a body  $M$  randomly and independently. What is the probability that they intersect? For an answer, we have to determine  $\mu(M,K;A)$  where

$$A = \{g \in D_0(M,K) : gK \cap L \neq \emptyset\}$$

and  $L$  is assumed to be a fixed body touching  $M$ . By definition

$$\mu(M,K;A) = \int_{S0_d} C_{d-1}(M + (-\delta K); T(A,\delta)) dv(\delta)$$

and

$$T(A,\delta) = \partial(M + (-\delta K)) \cap L + (-\delta K).$$

Hence, a further investigation of  $\mu(M,K;A)$  would require a formula for

$$\int_{S0_d} C_{d-1}(M + (-\delta K); L + (-\delta K)) dv(\delta).$$

((5.4) is not applicable, for  $L$  is not a subset of  $\partial M$ .)

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## Geometric theories of measure and area

Although there are a number of different methods to define the volume, or the surface area, for a convex body  $K$  in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , the values resulting from these constructions all coincide. Every reasonable definition of the volume, for example, leads to the Jordan content of  $K$ . This situation changes as soon as one starts studying the finer structure of  $K$ . Which is the natural measure for the shadow boundary of  $K$  in a given direction, or for the  $m$ -skeleton of  $K$ ?

It is the purpose of this article to give a report about various general constructions for the measure of lower dimensional subsets in certain metric spaces, and about interrelations between these constructions. In many cases, of relevance in finite dimensional convexity, the sets in question appear as the image of appropriate continuous functions, which leads us to study also the area of a function, and relate it to the measure of its image.

The origins of these investigations lie in the work of Peano [49], Lebesgue [39], de Geöcze [29], Hausdorff [37] and Caratheodory [11, 12]. Besicovitch [3–5] was the first author to give a thorough measuretheoretic examination of general onedimensional sets in the Euclidean plane  $\mathbb{R}^2$ . His methods have, at many instances, served as an inspiration for the study of higher dimensional questions. After Geöcze's pioneering work, it was especially Rado [50] and, later, Cesari [13, 14], who made important contributions to the theory of surface area, in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . A thorough account of the earlier literature can be found in the books [50] and [13]. In recent years the theories of measure and area have received a strong impetus from Federer's work. A careful and thorough introduction to his methods is given in the book [24], and also in [20]. We are here following [24] to a large extent. Those definitions and theorems whose source we don't mention explicitly, can be found in [24]. As a whole we have restricted ourselves to the aspects of the theory which have, in our opinion, a close connection to questions in finite dimensional convexity. New results and open problems are included here to the extent in which they came to our attention, but we cannot at all claim to be complete. The reader who is interested in further reaching applications, in analysis and topology, and in the calculus of variations, is referred to the corresponding chapters of [24], and also to the book [1].

### 1. m-dimensional measures

We are assuming familiarity with the basic concepts of general measure theory and with the construction of invariant measures on locally compact groups and homogenous spaces (see, for example, the initial parts of chapter 2 in [24]). All measures considered here arise from a gauge function by a process of covering and refinement, due to C. Caratheodory. Let  $X$  be a metric space and  $\mathfrak{F}$  a set of subsets of  $X$ . Denote by  $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$  the extended real number system. A gauge function over  $\mathfrak{F}$  is a mapping  $\gamma$  from  $\mathfrak{F}$  into the interval  $[0, \infty]$  of  $\bar{\mathbb{R}}$ . Given  $\mathfrak{F}$  and  $\gamma$  we define for each  $\varepsilon > 0$  a mapping  $\varphi_\varepsilon$  from the power set  $\mathfrak{P}(X)$  of  $X$  into

$[0, \infty]$ , by setting, for each  $Y \subset X$ ,  $\varphi\varepsilon(Y) := \inf \{\lambda \geq 0 : \text{there is a countable subset } \mathfrak{G} \text{ of } \mathfrak{F} \text{ with } Y \subset \cup \mathfrak{G}, \delta(G) \leq \varepsilon \text{ for all } G \in \mathfrak{G} \text{ and } \sum_{G \in \mathfrak{G}} \gamma(G) = \lambda\}$ . Here  $\delta(G) := \sup \{\rho(x, y) : \{x, y\} \subset G\}$  denotes the diameter of  $G \in \mathfrak{G}$ , with respect to the metric  $\rho$  of  $X$ . Clearly  $\varepsilon_1 < \varepsilon_2$  implies  $\varphi\varepsilon_1(Y) \geq \varphi\varepsilon_2(Y)$ , hence  $\varphi(Y) := \lim_{\varepsilon \rightarrow 0} \varphi\varepsilon(Y)$  exists in  $[0, \infty]$ .  $\varphi : \mathfrak{P}(X) \rightarrow [0, \infty]$  is a measure over  $X$ , and all open subsets of  $X$  are  $\varphi$ -measurable. One says that  $\varphi$  arises by Caratheodory's construction from  $(\mathfrak{F}, \gamma)$ . Below we give a list of the principal measures of this type. Let us, throughout, assume that  $X$  is a metric space with metric  $\rho$ , and denote by  $\delta(Y)$  the diameter of  $Y \subset X$ , with respect to  $\rho$ .

### 1. Hausdorff measures

Let  $h : [0, \infty] \rightarrow \mathbb{R}_+$  be a function, with  $h(0) = 0$ . Let  $\mathfrak{F} = \mathfrak{P}(X)$  be the power set of  $X$  and set, for  $Y \in \mathfrak{F}$ ,  $\gamma(Y) := h(\delta(Y))$ . The measure which arises by Caratheodory's construction from  $(\mathfrak{F}, \gamma)$ , is called the  $(\rho, h)$ -Hausdorff measure over  $X$ , and denoted by  $\mathcal{H}_{(\rho, h)}$ , or by  $\mathcal{H}_h$ . A good introduction to general Hausdorff measures can be found in C.A. Rogers' book [51]. The most interesting case for our purpose arises when we set  $h_m(t) := (\alpha(m)/(2^m))t^m$ , where  $\alpha(m) := (\Gamma(1/2))^m/\Gamma(m+2)/2$  is the volume of the  $m$ -dimensional Euclidean unit ball and  $\Gamma$  stands for Euler's Gamma function. The resulting measure  $\mathcal{H}_{(\rho, h_m)}$  will be denoted by  $\mathcal{H}_\rho^m$ , or by  $\mathcal{H}^m$ , and is called the  $m$ -dimensional Hausdorff measure over  $X$ .

### 2. Spherical measures

$F \subset X$  is called a closed ball, if there is an element  $x \in X$  and a number  $\varepsilon > 0$  in  $\mathbb{R}$  such that  $F = \{y \in X : \rho(y, x) \leq \varepsilon\}$ . Set  $\mathfrak{F} := \{F \subset X : F \text{ is a closed ball}\}$  and, for  $F \in \mathfrak{F}$ ,  $\gamma(F) := h_m(\delta(F))$ , where  $h_m(t) = (\alpha(m)/(2^m))t^m$  is as in the preceding definition. We call the measure which arises by Caratheodory's construction from  $(\mathfrak{F}, \gamma)$  the  $m$ -dimensional spherical measure over  $X$ , and denote it by  $\mathcal{S}_\rho^m$ , or by  $\mathcal{S}^m$ .

### 3. Cross measures

These measures are only defined for  $X = \mathbb{R}^n$ . Let  $\mathfrak{F}$  be the set of all nonempty subsets of  $\mathbb{R}^n$ , and set, for  $F \in \mathfrak{F}$ ,  $\gamma_m(F) := (\alpha(m)/(2^m)) \sup \{I(L_1 + \dots + L_m) : L_i \text{ is a line segment whose endpoints lie in } F\}$ , where  $I(P)$  denotes the  $m$ -dimensional elementary volume of a polytope  $P$ , of dimension  $\dim(P)$  at most  $m$ . The measure  $\mathcal{T}^m$  which arises by Caratheodory's construction from  $(\mathfrak{F}, \gamma_m)$ , shall be called the  $m$ -dimensional cross measure over  $\mathbb{R}^n$ . Notice that  $\gamma_m(F)$  gives an estimate for the size of the largest cross whose  $m$  arms are parallel to line segments with endpoints in  $F$ . Theorem 2.10.32 in [24] gives an inequality between the functional  $\gamma_n$  and the volume, for convex bodies in  $\mathbb{R}^n$ , from which the isodiametric inequality can easily be derived. Compare also [35]. It would be interesting to know whether there are other nontrivial relations between the functionals  $\gamma_m$ ,  $1 \leq m \leq n$ , and the various Quermassintegrals.

The gauge functions from which the following measures are derived, are all based on projection properties of subsets of the space  $X$ ; we therefore assume  $X = \mathbb{R}^n$ . Instead of projection properties we could study intersection properties of  $Y \subset X$  with "a moving subspace" of  $X$ , a concept which becomes meaningful

when  $X$  is, for example, a manifold on which a Lie group acts. Such general situations have been investigated by Brothers [8–10]. Following Federer we denote, for integers  $n$  and  $m$ , with  $1 \leq m \leq n$ , by  $0^*(n,m)$  the set of all orthogonal projections from  $\mathbb{R}^n$  onto  $\mathbb{R}^m$ .  $0^*(n,m)$  thus consists of all maps  $\varphi = \pi \circ \rho$ , where  $\rho: \mathbb{R}^n \rightarrow \mathbb{R}^n$  belongs to the orthogonal group  $O(n)$  and  $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is the canonical projection,  $\pi(x_1, \dots, x_n) := (x_1, \dots, x_m)$ .  $\alpha: O(n) \times 0^*(n,m) \rightarrow 0^*(n,m)$  defined by  $\alpha(\rho, \varphi) := \varphi \circ \rho^{-1}$ , is a left action of  $O(n)$  onto  $0^*(n,m)$ . Denote by  $\theta_{n,m}^*$  the unique  $\alpha$ -invariant measure on  $0^*(n,m)$  for which  $\theta_{n,m}^*(0^*(n,m)) = 1$ . Let  $\mathcal{L}^m$  be the Lebesgue measure in  $\mathbb{R}^m$ .

#### 4. Integralgeometric measures, Gillespie measures

Let  $n$  and  $m \leq n$  be positive integers. For  $t \geq 1$  in  $\bar{\mathbb{R}}$ , set  $\beta_t(n,m) := [\pi_{j=m+1}^n ((\Gamma(j/2)\Gamma((t+j-m)/2))/(\Gamma((j-m)/2)\Gamma((t+h)/2))]^{1/t}$ . Set  $\beta_\infty(n,m) := 1$ . If  $S \subset \mathbb{R}^n$  is a Borel set, we define, with the notation introduced above,

$$\gamma_t(S) := (1/\beta_t(n,m)) \left[ \int_{0^*(n,m)} (\mathcal{L}^m(\varphi(S)))^t d\theta_{n,m}^*(\varphi) \right]^{(1/t)}$$

Denote by  $\gamma_\infty(S)$  the essential supremum of the function  $1: 0^*(n,m) \rightarrow \bar{\mathbb{R}}$  defined by  $1(\varphi) := \mathcal{L}^m(\varphi(S))$ . Notice that, for a convex body  $S \subset \mathbb{R}^n$ ,  $\gamma_1(S)$  is the  $(n-m)$ -th Quermassintegral. Let  $\mathfrak{F}$  be the set of all Borel sets in  $\mathbb{R}^n$ . The measure which arises by Caratheodory's construction from  $(\mathfrak{F}, \gamma_t)$ , is called the  $m$ -dimensional integralgeometric measure with exponent  $t$  over  $\mathbb{R}^n$ , and is denoted by  $\mathcal{J}_t^m$ . If we restrict  $\mathfrak{F}$  to the set of all open convex subsets of  $\mathbb{R}^n$ , the measure arising from  $(\mathfrak{F}, \gamma_t)$  is denoted by  $\mathcal{Q}_t^m$ .  $\mathcal{Q}_1^m$  is called the  $m$ -dimensional Gillespie measure over  $\mathbb{R}^n$ .

#### 5. Gross measures, Caratheodory measures

They arise if we replace the essential supremum by the supremum, in the preceding section. If  $n$  and  $m \leq n$  are again positive integers we set, for every Borel set  $S \subset \mathbb{R}^n$ ,  $\gamma(S) := \sup \{ \mathcal{L}^m(\varphi(S)) : \varphi \in 0^*(n,m) \}$ . If  $\mathfrak{F}$  is the set of all Borel subsets of  $\mathbb{R}^n$ , the measure  $\mathcal{G}^m$  which arises by Caratheodory's construction from  $(\mathfrak{F}, \gamma)$ , is called the  $m$ -dimensional Gross measure over  $\mathbb{R}^n$ . If we take for  $\mathfrak{F}$  the set of all open convex subsets of  $\mathbb{R}^n$ , the measure obtained from  $(\mathfrak{F}, \gamma)$  is the  $m$ -dimensional Caratheodory measure over  $\mathbb{R}^n$ , denoted by  $\mathcal{C}^m$ . Various classical geometric results have been used to study the relationships between the above measures. In the case  $n=m$  they all coincide with the Lebesgue measure  $\mathcal{L}^n$  over  $\mathbb{R}^n$ . In general we have, for any  $t \in [1, \infty)$ ,

$$\begin{aligned} \mathcal{S}^m &\geq \mathcal{H}^m \geq \mathcal{T}^m \geq \mathcal{C}^m \geq \mathcal{Q}_\infty^m \geq \beta_t(n,m) \mathcal{Q}_t^m \\ &\leq \leq \leq \\ \mathcal{G}^m &\geq \mathcal{J}_\infty^m \geq \beta_t(n,m) \mathcal{J}_t^m. \end{aligned}$$

Furthermore,  $\beta_t(n,m) \mathcal{Q}_t^m$  and  $\beta_t(n,m) \mathcal{J}_t^m$  are nondecreasing, with respect to  $t$ , and  $A \mapsto \mathcal{Q}_t^m(A)$  depends continuously on  $t$ , for every subset  $A$  of  $\mathbb{R}^n$ . The ratio between any two measures in  $\{\mathcal{S}^m, \mathcal{H}^m, \mathcal{T}^m, \mathcal{Q}_\infty^m, \mathcal{Q}_t^m\}$  is bounded, whereas the measures  $\mathcal{C}^m, \mathcal{G}^m, \mathcal{J}^m$  do not have bounded ratios. Furthermore it is known that

$\mathcal{S}^1 \neq \mathcal{H}^1 \neq \mathcal{G}^1$ ,  $\mathcal{H}^2 \neq \mathcal{T}^2 \neq \mathcal{C}^2$ , and that the ratio between  $\mathcal{G}^1$  and  $\mathcal{Q}_t^1$ ,  $t \in [1, \infty]$ , is not bounded. Perhaps the most interesting open problem is the following.

**Question 1.** Do we have, in every Euclidean space  $\mathbb{R}^n$ ,  $\mathcal{J}_{\infty}^m = \mathcal{J}_t^m$ , for every  $m$  and every  $t \in [1, \infty)$ ?

For more references, see section 2.10.6 in [24], as well as [15] and [16].

## 2. Tame sets

There is a large class of sets for which all the measures constructed above yield the same values. These sets are built up from images of Lipschitzian maps. Recall that a map  $f: X \rightarrow Y$ , where  $X$  and  $Y$  are metric spaces with metrics  $\rho$  and  $\sigma$ , is called Lipschitzian, if there is a number  $c \in \mathbb{R}$  such that  $\sigma(fx, fy) \leq c\rho(x, y)$ . The smallest of these numbers  $c$  is the Lipschitz constant  $\text{Lip}(f)$  of  $f$ . Let us first recall three basic results about differentiability properties of Lipschitzian maps.

**Theorem 1. (Kirschbaum's theorem)** Let  $S$  be a subset of  $\mathbb{R}^m$  and  $f: S \rightarrow \mathbb{R}^n$  a Lipschitzian map. There is an extension  $g: \mathbb{R}^m \rightarrow \mathbb{R}^n$  of  $f$ , with  $\text{Lip}(g) = \text{Lip}(f)$ .

A corresponding result does not always hold, if we replace the Euclidian metric on  $\mathbb{R}^n$  and  $\mathbb{R}^m$  by one arising from an arbitrary Minkowskian norm.

Recall the following definitions. If  $f$  is a map from  $A \subset \mathbb{R}^m$  into  $\mathbb{R}^n$  and  $p$  is an interior point of  $A$ , we say that  $f$  is differentiable at  $p$ , if there exists a linear map  $g: \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that  $\lim_{x \rightarrow p} (\|f(x) - f(p) - g(x - p)\| / \|x - p\|) = 0$ . We call  $g$  the derivative of  $f$  at  $p$ , and write  $g = Df(p)$ . If  $A$  is open and the map  $Df$ , from  $A$  into the space of all linear maps  $g: \mathbb{R}^m \rightarrow \mathbb{R}^n$ , is continuous, then we say that  $f$  is a  $C^1$ -map. Assume  $m \leq n$ , and let  $M$  be the  $(n, m)$ -matrix whose  $i$ -th column vector is  $Df(p)[e_i] \in \mathbb{R}^n$ , where  $(e_1, \dots, e_m)$  denotes the standard linear base of  $\mathbb{R}^m$ . The Jacobian  $Jf(p)$  is defined by  $Jf(p) := (\sum_{N \in S} (\text{Det}(N))^2)^{1/2}$ , where  $S$  is the set of all  $(m, m)$ -submatrices of  $M$ . Alternatively  $Jf(p)$  is the  $m$ -dimensional volume of the polytope  $Df(p)[W]$ ,  $W = \{x \in \mathbb{R}^m : 0 \leq \langle x, e_i \rangle \leq 1\}$  being the  $m$ -dimensional unit cube.

**Theorem 2. (Rademacher's theorem)** If  $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a Lipschitzian map, then  $f$  is differentiable at  $\mathcal{L}^m$  almost all points of  $\mathbb{R}^m$ .

**Theorem 3.** If  $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a Lipschitzian map, then for every  $\varepsilon > 0$  there is a  $C^1$ -map  $g: \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that  $\mathcal{L}^m\{x \in \mathbb{R}^m : f(x) \neq gx\} < \varepsilon$ .

There is a classical formula for the area of a  $C^1$ -map by means of an integral over the Jacobian  $J(f)$ . Theorem 3 above indicates that a similar formula might hold for Lipschitzian maps. Indeed, let us associate to any map  $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$  and every subset  $A$  of  $\mathbb{R}^m$  the multiplicity function  $N(f, A): \mathbb{R}^n \rightarrow [0, \infty]$  by setting, for  $y \in \mathbb{R}^n$ ,  $N(f, A)y := \text{card } ((f|A)^{-1}[y]) \in \mathbb{N}$ , if  $(f|A)^{-1}[y]$  is a finite set, and  $N(f, A)[y] := \infty$ , otherwise.

**Theorem 4.** Let  $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a Lipschitzian map, with  $m \leq n$ , and let  $A \subset \mathbb{R}^m$  be an  $\mathcal{L}^m$ -measurable set. Then  $\int_A Jf(x) d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} N(f, A)[y] d\mathcal{H}^m(y)$ .

Notice that, by theorem 2,  $Jf(x)$  is defined almost everywhere on  $A$ . It has to be shown, however, that  $Jf$  is integrable with respect to the restriction of the Lebesgue measure  $\mathcal{L}^m$  to  $A$ , and that  $N(f,A)$  is integrable with respect to  $\mathcal{H}^m$ . Theorem 4 becomes especially simple when  $f$  is injective. We then find  $\int_A Jf(x) d\mathcal{L}^m(x) = \mathcal{H}^m(f[A])$ . As we shall see later, similar formulae hold for the other  $m$ -dimensional measures over  $\mathbb{R}^n$ . There is an interesting analogue to theorem 4 for the case  $m > n$ , expressing the coarea of a Lipschitzian map as an integral over its  $n$ -dimensional Jacobian. As an application of this coarea formula one can, for example, verify the geometric meaning of the constants  $\alpha(m)$  and  $\beta_i(n,m)$ , introduced in the first paragraph, compare 3.2.13 in [24]. Let  $X$  be a metric space,  $E$  a subset of  $X$ ,  $\varphi$  a measure over  $X$  and  $m$  a positive integer. We say that  $E$  is  $m$ -rectifiable, if there is a bounded subset  $B$  of  $\mathbb{R}^m$  and a Lipschitzian map from  $B$  onto  $E$ . We say that  $E$  is  $(\varphi, m)$ -rectifiable, or  $(\varphi, m)$ -tame, if there exists a countable family  $(F_i)_{i \in \mathbb{N}}$  of subsets of  $E$ , such that  $\varphi(E) \in \mathbb{R}$ ,  $E = \bigcup_{i=1}^{\infty} F_i$ ,  $\varphi(F_1) = 0$ , and  $F_i$  is  $m$ -rectifiable, for every  $i \geq 2$ . The  $(\varphi, m)$ -rectifiable subsets of  $X$  are in many respects the best behaved sets for geometric measure theory. The next result, which describes the structure of such sets in  $\mathbb{R}^n$ , is based on the foregoing theorems about Lipschitzian maps between Euclidean spaces. Let us first define, following Besicovitch, the density of a measure, and the tangent cone of a set, at a point  $p$ .

**Definition 1.** Let  $X$  be a subset of  $\mathbb{R}^n$ ,  $p$  a point of  $X$  and  $\varphi$  a measure over  $X$ . For each integer  $m \geq 0$ , set  $\Theta^{m*}(\varphi, p) := \limsup_{\rho \rightarrow 0+} (\varphi(B\rho(p, X)) / \alpha(m)\rho^m)$ ,  $\Theta_*^m(\varphi, p) := \liminf_{\rho \rightarrow 0+} (\varphi(B\rho(p, X)) / \alpha(m)\rho^m)$ . If  $\Theta^{m*}(\varphi, p)$  and  $\Theta_*^m(\varphi, p)$  coincide, denote their common value by  $\Theta^m(\varphi, p)$ .

Here  $B\rho(p, X) := \{q \in X : \|q - p\| \leq \rho\}$  is the ball with radius  $\rho$  and centre  $p$  in  $X$ .  $\Theta^{m*}(\varphi, p)$  and  $\Theta_*^m(\varphi, p)$  are the  $m$ -dimensional upper and lower densities of  $\varphi$  at  $p$ .  $\Theta^m(\varphi, p)$  is called the  $m$ -dimensional density of  $\varphi$  at  $p$ .

**Definition 2.** Let  $X$  be a subset of  $\mathbb{R}^n$  and  $p$  a point of  $X$ . Set  $\text{Tan}(X, p) := \{u \in \mathbb{R}^n : \text{for every } \varepsilon > 0 \text{ there is a point } q \in X \text{ and a number } \rho > 0 \text{ in } \mathbb{R} \text{ such that } \|q - p\| < \varepsilon \text{ and } \|\rho(q - p) - u\| < \varepsilon\}$ .

In other words, a ray  $S \subset \mathbb{R}^n$ , issuing from  $p$ , is contained in  $\text{Tan}(X, p)$  if and only if for every neighbourhood  $U$  of  $S$  and every neighbourhood  $V$  of  $p$  there is a ray  $T \in U$  issuing from  $p$  and meeting  $V - \{p\}$ .

**Theorem 5.** Let  $X$  be a  $(\mathcal{H}^m, m)$ -rectifiable and  $\mathcal{H}^m$ -measurable subset of  $\mathbb{R}^n$ . Then for  $\mathcal{H}^m$  almost all  $p$  in  $X$ ,  $\Theta^m(\mathcal{H}^m \llcorner X, p) = 1$  and  $\text{Tan}(X, p)$  is an  $m$ -dimensional linear subspace of  $\mathbb{R}^n$ .

$\mathcal{H}^m \llcorner X$  stands for the restriction of  $\mathcal{H}^m$  to  $X$ . Thus, in many respects, a  $(\mathcal{H}^m, m)$ -rectifiable subset of  $\mathbb{R}^n$  is similar to an  $m$ -dimensional submanifold of  $\mathbb{R}^n$ . This can be made more precise.

**Theorem 6.** Let  $X$  be a  $(\mathcal{H}^m, m)$ -rectifiable subset of  $\mathbb{R}^n$ . Then  $\mathcal{H}^m$  almost all of  $X$  is contained in the union of some countable family of  $C^1$ -submanifolds of  $\mathbb{R}^n$ .

The rectifiable subsets of  $\mathbb{R}^n$  have an especially nice measuretheoretic behaviour. The following three theorems may illustrate this.

**Theorem 7.** *Let  $X$  be a  $(\mathcal{H}^m, m)$ -rectifiable subset of  $\mathbb{R}^n$ . Then  $\mathcal{S}^m(X) = \mathcal{H}^m(X) = \mathcal{T}^m(X) = \mathcal{C}^m(X) = \mathcal{G}^m(X) = \mathcal{Q}_t^m(X) = \mathcal{J}_t^m(X)$ ,  $t \in [1, \infty]$  arbitrarily.*

Thus all geometric measures defined in the preceding paragraph coincide on  $X$ . It can be shown that for a closed  $m$ -rectifiable subset  $X$  of  $\mathbb{R}^n$  the above measures also coincide with the  $m$ -dimensional Minkowski content  $\mathcal{M}^m(X)$ , which is often used in the theory of convex bodies. On the other hand there is a  $(\mathcal{H}^2, 2)$ -rectifiable subset  $X$  of  $\mathbb{R}^3$ , actually a countable union of Euclidean triangles, such that  $\infty = \mathcal{M}^2(X) > \mathcal{H}^2(X) > 0$ , see [38]. Many questions concerning  $\mathcal{M}^m$  and its relations to the  $m$ -dimensional measures, are still open. The next theorem is an integralgeometric formula of Crofton type, related also to the principal kinematic formula of Blaschke-Santalò and Hadwiger, see [35]. Let  $I(n)$  be the group of all isometries of  $\mathbb{R}^n$ . There is an essentially unique invariant measure  $\delta_n$  on  $I(n)$ .

**Theorem 8.** *Let  $k$  and  $l$  be nonnegative integers and  $n$  a positive integer, with  $n \leq k + l$ . There is a constant  $c$ , such that for all  $(\mathcal{H}^k, k)$ -rectifiable Borel sets  $A$  in  $\mathbb{R}^n$  and all  $(\mathcal{H}^l, l)$ -rectifiable Borel sets  $B$  in  $\mathbb{R}^n$ ,  $\int_{I(n)} \mathcal{H}^{k+l-n}(A \cap \rho B) d\delta_n(\rho) = c \mathcal{H}^k(A) \mathcal{H}^l(B)$ .*

For a proof see [21]. Analogues of this result, and further properties of tame sets in general homogenous manifolds, have been studied in [8], [9] and [10]. The cartesian product of two rectifiable sets is again rectifiable. Actually we have a stronger statement.

**Theorem 9.** *Let  $A$  be a  $k$ -rectifiable Borel set in  $\mathbb{R}^n$  and  $B$  an  $(\mathcal{H}^l, l)$ -rectifiable Borel set in  $\mathbb{R}^m$ . Then  $A \times B$  is an  $(\mathcal{H}^{k+l}, k+l)$ -rectifiable subset of  $\mathbb{R}^{n+m}$ , and  $\mathcal{H}^{k+l} \llcorner (A \times B) = (\mathcal{H}^k \llcorner A) \times (\mathcal{H}^l \llcorner B)$ .*

Again,  $\varphi \llcorner X$  denotes the restriction of the measure  $\varphi$  to  $X$ . Notice that the requirements about  $A$  and  $B$  are not the same. Theorem 9 implies, of course,  $\mathcal{H}^{k+l} \llcorner (A \times B) = \mathcal{H}^k(A) \mathcal{H}^l(B)$ . In general, our geometric measures don't have such a simple behaviour with respect to the formation of products. There is an early example by Besicovitch and Moran [6] of a linear set  $A \subset \mathbb{R}^2$  such that  $\mathcal{H}^2(A + [0, h e_3]) > h \mathcal{H}^1(A)$ , for some  $h > 0$ . Their investigations have been continued and extended to other geometric measures in [17], [25], [26], [36] and [57].

### 3. Structure theory

Our main question in this section is: To which extent can a set of finite  $m$ -dimensional measure differ from a rectifiable set? In order to study the rectifiability of an  $m$ -dimensional subset  $X$  of  $\mathbb{R}^n$ , we consider its intersections with certain cones. Let  $p$  be a point in  $\mathbb{R}^n$ ,  $V \subset \mathbb{R}^n$  an  $(n-m)$ -dimensional linear subspace,  $r$  a positive real number and  $s$  a number in the interval  $(0, 1)$ . Set  $X(p, r, V, s) := \{x \in \mathbb{R}^n - \{p\} : \|x - p\| \leq r, \sin(\alpha_x) < s\}$ , where  $\alpha_x$  is the angle between  $x$

$-p$  and  $\pi_V(x - p)$ , and  $\pi_V: \mathbb{R}^n \rightarrow V$  denotes the orthogonal projection of  $\mathbb{R}^n$  onto  $V$ . Thus  $X(p, r, V, s)$  is the intersection of the cone with centre  $p$ , axis  $p + V$  and radius  $s$ , and the Euclidean ball with centre  $p$  and radius  $r$ . Denote by  $\gamma_{n,k}$  the unique rotationinvariant probability measure on the space  $G(n,k)$  of all  $k$ -dimensional linear subspaces of  $\mathbb{R}^n$ . Let  $\varphi$  be a Borel measure over  $\mathbb{R}^n$ , consider a Borel set  $A \subset \mathbb{R}^n$  and a point  $p \in A$ .

**Theorem 10.** *With the above notation, for each integer  $m \in [0, n]$ ,  $\gamma_{n,n-m}$  almost all flats  $V$  in  $G(n, n-m)$  satisfy one of the following conditions:*

(1) *There is  $\delta > 0$  such that*

$$\lim_{s \rightarrow 0+} \sup_{0 < r < \delta} (\varphi[A \cap X(p, r, V, s)] / r^m s^m) = 0$$

(2) *For all  $\delta > 0$ ,*

$$\limsup_{s \rightarrow 0+} \sup_{0 < r < \delta} (\varphi[A \cap X(p, r, V, s)] / r^m s^m) = \infty$$

(3)  $p \in \text{cl}\{x \in A - \{p\} : (x - p) \in V\}$ .

$\text{cl}(Y)$  denotes the closure of  $Y \subset \mathbb{R}^n$ . Roughly speaking, if (1) is fulfilled for  $V$ , then some cone around  $p + V$ , with apex  $p$ , is almost free of  $A$ , in a neighbourhood of  $p$ . If (1) holds for many  $(V, p)$  in  $G(n, n-m) \times A$ , then  $A$  is close to being rectifiable. On the other hand, if  $\varphi(A)$  is finite and (2) holds for many  $(V, p)$ , then the  $m$ -dimensional Lebesgue measure  $\mathcal{L}^m(\pi_V(A))$  is almost always equal to zero. Here  $\pi_V: \mathbb{R}^n \rightarrow V^\perp$  denotes the projection along  $V$  onto the orthogonal complement of  $V$ . In the case  $\varphi = \mathcal{J}_\infty^m$ , the set  $A$  can be split into subsets  $A_1, A_2$ , such that the first possibility holds for  $A_1$  and the second for  $A_2$ . Thus one obtains the following result.

**Theorem 11.** *Let  $A \subset \mathbb{R}^n$  be a Borel set with  $\mathcal{J}_\infty^m(A) < \infty$ . Then  $A$  is  $(\mathcal{J}_\infty^m, m)$ -rectifiable.*

The following question still seems open.

**Question 2.** *Let  $n$  and  $m$  be integers with  $1 \leq m < n$ , and  $t \geq 1$  a real number. Is every Borel set  $A \subset \mathbb{R}^n$ , for which  $\mathcal{J}_t^m(A) < \infty$ ,  $(\mathcal{J}_t^m, m)$ -rectifiable?*

Let us call a set  $B \subset \mathbb{R}^n$  purely  $(\mathcal{H}^m, m)$ -unrectifiable, if there exists no  $(\mathcal{H}^m, m)$ -rectifiable subset  $C$  of  $B$  with  $\mathcal{H}^m(C) > 0$ .

**Theorem 12.** *Let  $A \subset \mathbb{R}^n$  be a Borel set with  $\mathcal{H}^m(A) < \infty$ . Then  $A$  is the disjoint union of two Borel subsets  $A_1, A_2$  such that  $A_1$  is the union of a countable family of  $m$ -rectifiable sets,  $A_2$  is purely  $(\mathcal{H}^m, m)$ -unrectifiable, and  $\mathcal{J}_\infty^m(A_2) = 0$ .*

One might be tempted to believe that appropriate conditions on the topological nature of  $A$  would ensure that  $\mathcal{H}^m(A_2) = 0$ . Indeed, if  $A$  is a Jordancurve in  $\mathbb{R}^2$ , then  $\mathcal{H}^1(A) < \infty$  implies the  $(\mathcal{H}^1, 1)$ -rectifiability of  $A$ . However there is, already in  $\mathbb{R}^3$ , a topological 2-sphere  $A$ , of finite  $\mathcal{H}^2$ -measure, which is not  $(\mathcal{H}^2, 2)$ -rectifiable (see 3.3.22 and 4.2.25 in [24]).  $\mathcal{J}_\infty^m(A_2) = 0$  means that almost all

projections of  $A_2$  onto  $m$ -dimensional subspaces have zero  $m$ -dimensional Lebesgue measure, hence also  $\mathcal{J}_t^m(A_2)=0$ , for all  $t\in[0,\infty]$ . The analogue of theorem 12 holds for all the other  $m$ -dimensional measures which we have been studying. There are generalizations of these results to homogenous manifolds, see [9]. The case  $n=2$ ,  $m=1$  of theorem 12 is one part of the pioneering work [3], by Besicovitch. He characterized the regular set  $A_1$  and the irregular set  $A_2$  by density properties: The density  $\Theta^1(\mathcal{H}^1 \llcorner A, p)$  is, for almost all  $p \in A_1$ , equal to one. The question whether similar characterizations hold for the other measures, in higher dimensions, is still not completely settled. We have the following important result.

**Theorem 13.** *Let  $A \subset \mathbb{R}^n$  be a Borel set with  $\mathcal{H}^m(A) < \infty$  and  $\Theta^m(\mathcal{H}^m \llcorner A, p) = 1$ , for almost all  $p \in A$ . Then  $A$  is  $(\mathcal{H}^m, m)$ -rectifiable.*

This has been proved for  $n=2$ ,  $m=1$  in [3], for  $n=3$ ,  $m=2$  in [40] and for general  $(n,m)$  in [41]. A much further reaching conjecture is still unsettled.

**Question 3.** *Let  $\varphi$  be a Borel regular measure over  $\mathbb{R}^n$ , and  $m$  a positive integer. If the density  $\Theta^m(\varphi, p)$  is positive and finite, for  $\varphi$  almost all  $p \in \mathbb{R}^n$ , is then  $\mathbb{R}^n$  countably  $(\varphi, m)$ -rectifiable?*

In other words, does there exist a countable family  $(A_i)_{i \in \mathbb{N}}$  of subsets  $A_i \subset \mathbb{R}^n$  such that  $\mathbb{R}^n = \bigcup_{i=1}^{\infty} A_i$ ,  $\varphi(A_1) = 0$ , and each  $A_i$ ,  $i \geq 2$ , is  $m$ -rectifiable?

#### 4. Areas of continuous functions

There is a considerable number of definitions for the area of a continuous map. We have chosen to present three which seem to be of major importance. For further references, the reader may wish to consult [50], concerning the situations in  $\mathbb{R}^3$ , and [20], as well as [48], for higher dimensions.

##### 1. Lebesgue area

The basic notions about simplicial complexes and maps are taken from [52]. A compact topological space  $X$  is called triangulable, if there are a finite simplicial complex  $C$  and a homeomorphism  $\varphi$  from the space  $|C|$  of  $C$  onto  $X$ . Let  $\mathfrak{C}_n(X)$  be the Banach space of all continuous mappings  $f: X \rightarrow \mathbb{R}^n$ , with the norm of uniform convergence on  $X$ ,  $\|f\| := \sup \{ \|f(x)\| : x \in X \}$ .  $f \in \mathfrak{C}_n(X)$  is called polyhedral if there are a finite simplicial complex  $C$  and a homeomorphism  $\varphi: |C| \rightarrow X$  such that  $f \circ \varphi: |C| \rightarrow \mathbb{R}^n$  is piecewise linear over  $C$ , that is, the restriction  $r$  of  $f \circ \varphi$  to the space  $[s]$  of any simplex  $s$  of  $C$ , satisfies  $r(\alpha p + (1-\alpha)q) = \alpha r(p) + (1-\alpha)r(q)$ , for every  $p, q$  in  $s$  and  $\alpha \in [0,1]$ . Let  $\mathfrak{P}_n(X)$  be the space of all polyhedral maps in  $\mathfrak{C}_n(X)$ . Clearly  $\mathfrak{P}_n(X)$  is dense in  $\mathfrak{C}_n(X)$ . Denote by  $I: \mathfrak{P}_n(X) \rightarrow \mathbb{R}_+$  the elementary area. That is, given  $f \in \mathfrak{P}_n(X)$  and a triangulation  $(C, \varphi)$  of  $X$  such that  $f \circ \varphi$  is piecewise linear over  $C$ , denote, for every maximal simplex  $s$  of  $C$ , by  $\alpha(f, s)$  the  $k$ -dimensional volume of the convex polytope  $f \circ \varphi[S]$  in  $\mathbb{R}^n$ , where  $k$  is the dimension of  $C$ . Set  $I(f) := \sum \alpha(f, s)$ , where the summation is extended over all maximal simplexes  $S$  of  $C$ .  $I(f)$  depends on  $f$  alone, and not on the specific

triangulation  $(C, \varphi)$ . To see this, use, for example the remark 6.4 on page 314 of [22], where  $I(f)$  is expressed independently of  $(C, \varphi)$ , by means of an integral of the multiplicity function of  $f$ . Notice that  $I$  is lower semicontinuous over  $\mathfrak{P}_n(X)$ . The largest lower semicontinuous extension of  $I$  to  $\mathfrak{C}_n(X)$  is called the Lebesgue area over  $X$ , and denoted by  $L_X$ . Thus, for  $f \in \mathfrak{C}_n(X)$  we have  $L_X(f) := \inf \{\lambda \in \mathbb{R} : \text{there is a sequence } (g_i)_{i \in \mathbb{N}} \text{ in } \mathfrak{P}_n(X) \text{ with } \|f - g_i\| \rightarrow 0 \text{ (} i \rightarrow 0 \text{) and } I(g_i) \rightarrow \lambda \text{ (} i \rightarrow \infty \text{)}\}$ . Much more is known about the Lebesgue area over a 2-dimensional space  $X$  (see [13]) than in the case of higher dimensions (see [20] and [22] for an introduction). Part of the reason for this lies in the fact that for sufficiently high dimensional spaces  $X$  one can find several classes of essentially different triangulations, see [42]. Thus the following two basic theorems about Lebesgue area require some rather involved topological arguments. Given a triangulation  $(C, \varphi)$  of  $X$ , denote by  $\mathfrak{P}_n(X, C, \varphi)$  the set of all maps  $f \in \mathfrak{C}_n(X)$  for which there exists a subdivision  $(C', \varphi')$  of  $(C, \varphi)$  such that  $f \circ \varphi' : |C'| \rightarrow \mathbb{R}^n$  is piecewise linear over  $C'$ , and by  $L_{(X, C, \varphi)} : \mathfrak{C}_n(X) \rightarrow \bar{\mathbb{R}}$  the largest lower semicontinuous extension of  $I|_{\mathfrak{P}_n(X, C, \varphi)}$  to  $\mathfrak{C}_n(X)$ . Notice that  $\mathfrak{P}_n(X, C, \varphi)$  is dense in  $\mathfrak{C}_n(X)$ .  $L_{(X, C, \varphi)}$  is similar to the Lebesgue area, but for its definition we use only functions which are piecewise linear over some subdivision of a fixed triangulation of  $X$ .

**Theorem 14.** *For every triangulation  $(C, \varphi)$  of the triangulable space  $X$  we have  $L_X = L_{(X, C, \varphi)}$ .*

**Theorem 15.** *Let  $X$  and  $X'$  be triangulable spaces, with  $X' \subset X$ . For every  $f \in \mathfrak{C}_n(X)$  we have  $L_{X'}(f|X') \leq L_X(f)$ .*

Both theorems are proved in [22].

## 2. Peano area, Geöcze area

If  $C \subset \mathbb{R}^2$  is a Jordan curve, the winding number  $o(p, C)$  tells us whether a point  $p \in \mathbb{R}^2 \setminus C$  belongs to the bounded region enclosed by  $C$ . Thus the area of this region can be determined by integrating the function  $p \mapsto o(p, C)$  over  $\mathbb{R}^2$ . The notion of Peano area is based on a construction related to this observation. Let  $M$  be a  $k$ -dimensional topological manifold and  $X \subset M$  a triangulable subspace. For  $n \geq k$ , denote by  $\mathfrak{C}_n(X)$  the Banach space of all continuous mappings  $f : X \rightarrow \mathbb{R}^n$ , with the supremum norm. We assume first  $n = k$ , and define the combinatorial multiplicity function  $M(f) : \mathbb{R}^k \rightarrow \mathbb{N} \cup \{\infty\}$ ,  $f \in \mathfrak{C}_k(X)$ , in analogy to the winding number. Let  $U$  be a bounded, open, connected subset of  $\mathbb{R}^k$ , and  $V$  a component of  $f^{-1}(U)$ . The  $k$ -th Čech cohomology group with integer coefficients of the pair  $(\text{cl}(U), \text{bd}(U))$ ,  $H^k(\text{cl}(U), \text{bd}(U))$ , is infinite cyclic, while  $H^k(\text{cl}(V), \text{bd}(V))$  is isomorphic to  $\mathbb{Z}$ , or to  $\mathbb{Z}_2$ , or to  $\{0\}$ . Consider generators  $x$  of  $H^k(\text{cl}(U), \text{bd}(U))$  and  $y$  of  $H^k(\text{cl}(V), \text{bd}(V))$ . Denoting by  $f^* : H^k(\text{cl}(U), \text{bd}(U)) \rightarrow H^k(\text{cl}(V), \text{bd}(V))$  the homomorphism induced by  $f$ , we let  $D(f, U, V)$  be the smallest nonnegative integer  $n$  for which  $ny \in \{f^*x, -f^*x\}$ . Set  $D(f, U) := \sum_V D(f, U, V)$ , where the summation is extended over all components  $V$  of  $f^{-1}(U)$ . The combinatorial multiplicity  $M(f)[y]$  of  $f$  at a point  $y \in \mathbb{R}^k$  is defined by  $M(f)[y] := \sup \{D(f, U) : U \subset \mathbb{R}^k \text{ is open, bounded and connected, with } y \in U\}$ . As shown on page 612 of [18], the mapping  $M(f) : \mathbb{R}^k \rightarrow \mathbb{N} \cup \{\infty\}$  is lower semicontinuous, and thus the integral  $P_X(f)$ :

$= \int_{\mathbb{R}^k} M(f) d\mathcal{H}^k$  exists.  $P_X(f)$  shall be called the Peano area of  $f$  over  $X$ , compare [19], where  $P_X(f)$  is denoted by  $M^*(f)$ . In the case  $k < n$  we take the average over all projections, as follows. Let  $0^*(n,k)$  be, as in the definition of the integralgeometric measures in paragraph 1, the set of all orthogonal projections  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^k$ , and  $\theta_{n,k}^*$  the  $0^*(n)$ -invariant probability measure on  $0^*(n,k)$ . Let  $\beta_1(n,k)$  be the integralgeometric constant introduced there, and set

$$P_X(f) := (1/\beta_1(n,k)) \int_{0^*(n,k)} P_X(\varphi \circ f) d\theta_{n,k}^*(\varphi),$$

where the Peano area of the map  $\varphi \circ f$  from  $X$  to  $\mathbb{R}^k$  has been defined above. We say that  $P_X(f)$  is the Peano area of  $f$  over  $X$ . Notice that this definition, due to Federer, differs slightly from the classical one [14], where  $X$  is supposed to lie in the plane  $\mathbb{R}^2$  and  $f$  in  $C_3(X)$ , homology, rather than cohomology is used for the construction of the multiplicity function  $M(f)$ , and a subdivision process for  $X$  is also applied. However, as one can see by formula (13) on page 93 of [19] and theorem (6,i) on page 181 of [14], the outcome is the same, in this case. One could, of course, imagine different averaging procedures for the areas  $P_X(\varphi \circ f)$ , just as we had, in definition 4 of paragraph 1, integralgeometric measures with different exponents. One of these procedures leads to the notion of Geöcze area of  $f$  over  $X$ , see [50] and [14] for the lower dimensional situations and [48] for the general case.

### 3. Stable area

As before, consider a  $k$ -dimensional manifold  $M$ , a triangulable subspace  $X$  of  $M$  and the space  $C_n(X)$  of mappings into  $\mathbb{R}^n$ ,  $n \geq k$ . Assuming at first  $k = n$ , we say that  $y \in \mathbb{R}^k$  is a stable value for  $f \in C_k(X)$ , if there exists a neighbourhood  $U$  of  $f$  in  $C_k(X)$  such that, given any mapping  $g \in U$ , we find a point  $q \in X$  with  $g(q) = y$ . Notice that if  $f$  is differentiable at  $p$  and if the derivative  $Df(p)$  of  $f$  at  $p$  is nonsingular, then  $f(p)$  is a stable value for  $f$ . We define a multiplicity function  $T(f): \mathbb{R}^k \rightarrow \mathbb{N} \cup \{\infty\}$ , as follows. Given  $y \in \mathbb{R}^k$ , denote by  $T(f)[y]$  the supremum (possibly 0 or  $\infty$ ) of the set of all nonnegative integers  $m$  for which there are bounded open sets  $U_1, \dots, U_m$  in  $X$  such that  $y$  is a stable value of each function  $f|U_i \in C_k(U_i)$ , and such that  $\text{cl}(U_i) \cap \text{cl}(U_j) = \emptyset$ ,  $i \neq j$ , as well as  $\text{cl}(U_i) \subset \text{int}(X)$ , for all  $i$ .  $T(f): \mathbb{R}^k \rightarrow \mathbb{N} \cup \{\infty\}$  is lower semicontinuous, and thus the stable area  $S_X(f)$  of  $f$  over  $X$  can be defined as the integral  $S_X(f) := \int_{\mathbb{R}^k} T(f) d\mathcal{H}^k$ , compare page 91 of [19]. In the case  $k < n$  we set, as in the previous section,  $S_X(f) := (1/\beta_1(n,k)) \int_{0^*(n,k)} S_X(\varphi \circ f) d\theta_{n,k}^*(\varphi)$ .

It is not hard to see that for every triangulable subspace  $X$  of a  $k$ -dimensional manifold and every  $f \in C_n(X)$ ,  $n \geq k$ , we have  $L_X(f) \geq P_X(f) \geq S_X(f)$ . Notice that  $L_X$  is the largest lower semicontinuous extension of the elementary content of polyhedral mappings. Furthermore, if  $k = n$ , then  $L_X(f) = P_X(f)$ . (See (2) and (3) on page 92 of [19] as well as [18].) This last equation is quite important. For example  $P_X(f)$ , being expressed as an integral, has the usual additivity properties, which are in no way obvious for  $L_X(f)$ . For further relations between the above area functions and their connections to certain geometric measures, see [19].

## 5. Smooth functions

In analogy to the situation described in paragraph 2, our area functions all associate the same value to a Lipschitzian map.

**Theorem 16.** *Let  $X$  be a triangulable subset of  $\mathbb{R}^k$  and  $f$  a Lipschitzian map from  $X$  into  $\mathbb{R}^n$ . Then  $L_X(f) = P_X(f) = S_X(f)$ .*

For a proof, see formula (9) in [19]. At the same place it is shown that the above areas agree with the  $k$ -dimensional Hausdorff measure, counted with appropriate multiplicities, of  $\text{Im}(f)$ . We use the following notation.

**Definition 3.** *Let  $f: X \rightarrow Y$  be any map. Define  $N(f): Y \rightarrow \mathbb{N} \cup \{\infty\}$  by setting  $N(f)[y] := \text{card } \{x \in X : f(x) = y\}$ , if this latter set is finite, and  $N(f)[y] := \infty$ , otherwise.*

**Theorem 17.** *Under the assumptions of theorem 16 we have, in addition,  $L_X(f) = \int_{\mathbb{R}^n} N(f) d\mathcal{H}^k$ .*

Similar, but weaker, statements can be made about maps whose range is  $(\mathcal{H}^k, k)$ -rectifiable. There are several other classes of maps with a regular behaviour towards the above area functions: maps whose domain is lowdimensional, or whose domain and range have the same dimension, or nonparametric surfaces, as defined below.

**Definition 4.** *Let  $X \subset \mathbb{R}^n$  be a (rectilinear)  $n$ -dimensional polyhedron. A continuous map  $g: X \rightarrow \mathbb{R}^{n+k}$ ,  $k \geq 1$ , is called a nonparametric  $n$ -manifold in  $\mathbb{R}^{n+k}$ , if  $\pi \circ g(p) = p$ , for all  $p \in X$ , where  $\pi: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$  is the natural projection,  $\pi(x_1, \dots, x_{n+k}) = (x_1, \dots, x_k)$ . In the case  $k = 1$  we call  $g$  a nonparametric surface in  $\mathbb{R}^{n+1}$ .*

**Theorem 18.** *Let  $g: X \rightarrow \mathbb{R}^3$  be a nonparametric surface in  $\mathbb{R}^3$ . Then  $L_X(g) = P_X(g) = S_X(g) = \mathcal{H}^2(\text{Im}(g))$ .*

See [19], relation (18) on page 93.

**Theorem 19.** *Let  $g: X \rightarrow \mathbb{R}^n$ ,  $n \geq 2$ , be a nonparametric surface in  $\mathbb{R}^n$ . Then  $L_X(g) = \mathcal{H}^{n-1}(\text{Im}(g)) = \mathcal{J}_1^{n-1}(\text{Im}(g))$ , and if this number is finite, then  $\text{Im}(g)$  is  $(\mathcal{H}^{n-1}, n-1)$ -rectifiable.*

This has been proved in [23], with the use of [53]. In view of the lower semicontinuity of the Lebesgue area, it would be interesting for geometric applications to know the answer to the following question.

**Question 4.** *Assume  $2 \leq k \leq n-2$ , and let  $g: X \rightarrow \mathbb{R}^n$  be a nonparametric  $k$ -manifold in  $\mathbb{R}^n$ . Is it true that  $L_X(g) = \mathcal{H}^{n-1}(\text{Im}(g))$ ?*

## 6. Structure theory

The finiteness of the Lebesgue area of a map  $f$  does not imply the regularity of  $\text{Im}(f)$  in the way in which one might hope at first. For example de Geöcze

constructed a map  $f$  from a planar region  $X \subset \mathbb{R}^2$  onto a cube in  $\mathbb{R}^3$ , such that  $L_X(f) = 0$ . A similar phenomenon for injective maps has been found by Besicovitch (see [50], page 551). On the other hand one can obtain, at least in low dimensions, certain regularity properties after a reparametrization.

**Definition 5.** Let  $X$  be a triangulable space. Two elements  $f, g$  in  $\mathfrak{C}_n(X)$  are called Fréchet equivalent, if there is a sequence  $(T_i)_{i \in \mathbb{N}}$  of homeomorphisms  $T_i : X \rightarrow X$  such that  $(f \circ T_i)_{i \in \mathbb{N}}$  converges to  $g$ , uniformly on  $X$ .

Fréchet equivalence can be defined in more general contexts, see page 348 in [20]. The following result is due to Morrey [44, 45], compare also [20] and [14].

**Theorem 20.** Let  $X \in \mathbb{R}^2$  be a 2-cell, and  $f \in \mathfrak{C}_3(X)$  a map with  $L_X(f) < \infty$ . There is a map  $g$  in  $\mathfrak{C}_3(X)$ , Fréchet equivalent to  $f$ , such that  $g$  is differentiable  $\mathcal{L}^2$  almost everywhere in  $X$ , and such that  $L_X(g) = L_X(f) = \int_X J(g) d\mathcal{L}^2$ .

Here  $J(g)$  stands for the Jacobian of  $g$ , as introduced in paragraph 2. Even in this simple case we cannot expect, though, that  $\text{Im}(f)$  will be  $(\mathcal{H}^2, 2)$ -rectifiable. The most important case known to us where such a rectifiability statement holds, is that of nonparametric surfaces, mentioned in theorem 19. Although the definition of the Lebesgue area  $L_X(f)$  by means of polyhedral approximation appears as a quite natural concept, its actual calculation is, in general, not easy. Thus it is interesting to obtain estimates for  $L_X(f)$  by means of integrals over some generalized Jacobian  $J'(f)$ , compare [14] and [50], where this question is discussed thoroughly for the classical case. In general one would expect  $L_X(f) \geq \int_X J'(f)$ . That such an inequality need not always hold, is shown in [30] and [31]. For further studies in this direction, also involving the Hausdorff measure of  $\text{Im}(f)$ , with appropriate multiplicities, and including certain classes of noncontinuous measurable functions, see [7], [33], [34], and [58]. In this context of computing  $L_X(f)$ , a long standing conjecture by Göcze asks whether  $L_X(f)$  can be obtained by just considering polyhedral functions inscribed to  $f$ . This conjecture has been solved in the affirmative by Mulholland [40] for nonparametric surfaces in  $\mathbb{R}^3$  and by Toralballa [54–56], for general surfaces in  $\mathbb{R}^3$ . To the best of our knowledge the corresponding conjecture for  $\mathbb{R}^n$ ,  $n \geq 4$ , has not been settled yet. To be more precise, let us introduce the following terminology.

**Definition 6.** Let  $X \subset \mathbb{R}^k$  be a rectilinear polyhedron and  $f$  an element of  $\mathfrak{C}_n(X)$ . We set  $\mathfrak{P}_n(X, f) := \{g \in \mathfrak{C}_n(X) : \text{there is a simplicial subdivision } C \text{ of } X \text{ such that } g|_s \text{ is affine for every simplex } s \in C, \text{ and } g(p) = f(p) \text{ for every vertex } p \text{ of } C\}$ . Set  $L'_X(f) := \inf \{\lambda \in \mathbb{R} : \text{there is a sequence } (g_i)_{i \in \mathbb{N}} \text{ of elements } g_i \text{ in } \mathfrak{P}_n(X, f) \text{ such that } g_i \rightarrow f \text{ and } I(g_i) \rightarrow \lambda \text{ (} i \rightarrow \infty \text{)}\}$ .

Here  $I(g_i)$  denotes the elementary content of the polyhedral map  $g_i$ . By a simplicial subdivision of  $X$  we understand a finite set  $C$  of Euclidean simplexes, with the usual closedness — and intersection properties, such that  $\cup C = X$ .

**Question 5.** Do we have  $L_X(f) = L'_X(f)$ , for every  $k, n$ , every polyhedron  $X \subset \mathbb{R}^k$  and every  $f \in \mathfrak{C}_n(X)$ ?

It does not seem likely that the answer is affirmative in all cases. It might be affirmative for  $k = 2$ , and, in view of theorem 19, for all nonparametric surfaces  $f \in \mathfrak{C}_{k+1}(X)$ ,  $X \subset \mathbb{R}^k$ . For nonparametric surfaces of class  $C^1$  it is known to be affirmative [2]. Further investigations involving the Lebesgue area of nonparametric surfaces are [32] and [43]. In higher dimensions the relations between the different areas of a continuous function, and between the various geometric measures of its range, are not fully understood yet. An important contribution has been made by Federer in [22], where he established the following result.

**Theorem 21.** Let  $X$  be a triangulable subset of  $\mathbb{R}^2$  and  $f$  a function in  $\mathfrak{C}_n(X)$ . Then  $L_X(f) \leq \sum L_X(\pi_E f)$ , where the summation is extended over all 2-dimensional coordinate planes  $E = \text{lin}\{e_i, e_j\}$ ,  $1 \leq i \neq j \leq n$ , and  $\pi_E: \mathbb{R}^n \rightarrow E$  denotes the orthogonal projection onto  $E$ .

From this (see theorem 7.16 in [22] and the relations (11) and (13) on page 93 of [19]) we can derive, furthermore, the following result.

**Theorem 22.** Let  $X \subset \mathbb{R}^2$  be a topological disc. For every  $n$  and every  $f \in \mathfrak{C}_n(X)$ , we have  $L_X(f) = P_X(f)$ .

There are some interesting extensions of this result, but a complete answer is, to the best of our knowledge, still missing, for higher dimensional  $X$ . Compare [27] and [28].

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## Transforms, Diagrams and Representations

### 1. Introduction

#### 1A. Foreword

A number of the more striking advances in the combinatorial theory of convexity in recent years have come from the various kinds of diagram techniques. With benefit of hindsight, we can trace the origins of the theory to Whitney [1935], a paper which also initiated the fruitful study of matroids (see also 8A below). However, the idea as presently envisaged first appears in a recognizable form in Gale [1956], though related ideas appear in Motzkin [1951] and Davis [1954]. The present development of the subject is due largely to Perles (whose work is described in Grünbaum [1967]) and, later, to a series of papers by McMullen and Shephard (references will be given later, in the appropriate places).

The basic idea of a Gale diagram is to represent a  $d$ -polytope  $P$  with  $n$  vertices by an (ordered) set of  $n$  points in  $E^{n-d-1}$ , whose combinatorial properties reflect those of  $P$ . Later variants of this technique represent centrally symmetric polytopes, zonotopes, positive bases and classes of polyhedral sets whose facets have the same outer normal vectors.

While, in what follows, we hope to give enough of the theory to enable the interested reader to fill in missing details for himself, it would be completely impossible to do more than merely mention most of the applications in the space available. However, we have attempted to give a complete survey of the results obtained by means of diagram techniques, and, in particular, to give a full bibliography of the area. To save space, we have not given any references to papers which do not employ diagrams; such references may be culled from the literature cited.

While writing this, the author has remained extremely conscious of the fact that a survey article of this kind is likely to standardize matters of terminology, in an area where writers (notably including the author) have often changed their terminology from paper to paper. Here we have largely returned to the original usages, particularly in preferring the term "transform" to "representation", since it seems to us that the latter suggests the idea of a linear or affine mapping (that is, homomorphism in some context) rather more than the former.

If, then, we have tended to be conservative in matters of terminology, our approach to the theory is better characterized as radical. We have followed up hints of a categorical approach in a couple of papers, and dealt with our diagrams in general linear or affine spaces, rather than in coordinatized Euclidean spaces. But the reader should not become alarmed; this approach, in fact, makes more clear the salient features of the subject.

With some reluctance, and in spite of several requests, we have decided not to include any figures in the text. One reason is that, even without figures, the article is already rather long. But figures are readily available in all the main

articles and books cited, which our text is, after all, designed to complement rather than replace.

### 1B. Conventions and notations

While we shall construct our theory in real finite dimensional linear or affine spaces, the reader may conveniently think of the spaces as euclidean. To reinforce this aspect, we shall denote the value taken by the linear functional  $v^*$  on the vector  $v$  by  $\langle v, v^* \rangle$ , using the notation of the inner product; and similarly for affine functionals.

The space  $V^*$  of linear functionals on a linear space  $V$  is again a linear space, called the (*linear*) *dual* of  $V$ . If  $\Phi: V \rightarrow U$  is a linear mapping between linear spaces, then  $\Phi$  induces a linear mapping  $\Phi^*: U^* \rightarrow V^*$ , called its *dual*, defined by  $\langle v, u^* \Phi^* \rangle = \langle v\Phi, u^* \rangle$  for all  $u^* \in U^*$  and  $v \in V$ . If  $B = (b_1, \dots, b_n)$  is a (linear) basis of  $V$ , then the dual basis  $B^* = (b_1^*, \dots, b_n^*)$  of  $V^*$  satisfies  $\langle b_i, b_j^* \rangle = \delta_{ij}$ , the Kronecker delta. Note that (essentially)  $V^{**} = V$ .

If  $V$  is an affine space, the space  $V^\ddagger$  of affine functionals on  $V$  is a linear space, called the (*affine*) *dual* of  $V$ . An affine mapping  $\Phi: V \rightarrow U$  induces a dual linear mapping  $\Phi^\ddagger: U^\ddagger \rightarrow V^\ddagger$ , defined as for linear duals. An affine basis  $B$  of  $V$  again has a dual linear basis  $B^\ddagger$  of  $V$ . We do not now have  $V^{\ddagger\ddagger} = V$ ; for one thing,  $\dim V^\ddagger = \dim V + 1$ . However, in the context, we shall be able to identify  $V$  with an affine subspace (flat) of  $V^{\ddagger*}$  (see 2B).

As far as coordinate vectors are concerned, our convention is that they are always row vectors; accordingly, linear and affine mappings act on the right. The term *rank* will mean the dimension of the linear or affine hull, as appropriate. Finally, the remaining notations, such as lin, aff and so on, are those usually employed by Grünbaum [1967] or McMullen & Shephard [1971].

## 2. Linear and affine transforms

### 2A. Linear transforms

Let  $X = (x_1, \dots, x_n)$  be an ordered set of points in the real linear space  $U$ . The investigation of the linear structure of  $X$ , and of the facial structure of the cone  $\text{pos}X$  generated by  $X$  with apex the origin  $o$  is facilitated by using a linear transform of  $X$ . This may be constructed as follows.  $X$  is the image under some linear mapping  $\Phi$  of an ordered basis  $B = (b_1, \dots, b_n)$  of an  $n$ -dimensional space  $V$ . Let  $W$  be another linear space, and  $\Psi: W \rightarrow V$  a linear mapping, such that the sequence of linear spaces and mappings

$$W \xrightarrow{\Psi} V \xrightarrow{\Phi} U$$

is exact at  $V$ ; that is,  $\text{im} \Psi = \ker \Phi$ . Let

$$U^* \xrightarrow{\Phi^*} V^* \xrightarrow{\Psi^*} W^*$$

be the dual sequence of spaces and mappings; this sequence is exact at  $V^*$ , since the annihilator of  $\text{im} \Psi$  is

$$\begin{aligned}
(\text{im } \Psi)^\perp &= \{v^* \in V^* | \langle v, v^* \rangle = 0 \text{ for all } v \in \text{im } \Psi\} \\
&= \{v^* \in V^* | \langle w, v^* \Psi^* \rangle = 0 \text{ for all } w \in W\} \\
&= \ker \Psi^*,
\end{aligned}$$

and similarly  $(\ker \Phi)^\perp = \text{im } \Phi^*$ . Let  $B^* = (b_1^*, \dots, b_n^*)$  be the basis of  $V^*$  dual to  $B$ . Then the image  $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n)$  of  $B^*$  under  $\Psi^*$  is called a *linear transform* of  $X$ . The obvious symmetry between  $X$  and  $\bar{X}$  shows:

2A1 *If  $\bar{X}$  is a linear transform of  $X$ , then  $X$  is a linear transform of  $\bar{X}$ .*

Since  $\dim \ker \Phi + \dim \text{im } \Phi = \dim V = \dim \ker \Psi + \dim (\ker \Psi)^\perp$ , we have:

2A2  $\text{rank } X + \text{rank } \bar{X} = \text{card } X$ .

We observe that linear transforms are only determined up to linear equivalence, and that linearly equivalent sets have the same linear transforms. Clearly, also, if we replace  $x_i$  by  $\lambda x_i$  ( $\lambda \neq 0$ ), we replace  $\bar{x}_i$  by  $\lambda^{-1} \bar{x}_i$ .

A vector  $(\alpha_1, \dots, \alpha_n)$  is called a *linear dependence* of  $X$  if  $\sum_{i=1}^n \alpha_i x_i = 0$ . The set  $L(X)$  of linear dependences of  $X$  clearly forms a subspace of  $E^n$ . We have:

2A3  $(\alpha_1, \dots, \alpha_n) \in L(X)$  if and only if  $\alpha_i = \langle \bar{x}_i, a \rangle$  ( $i = 1, \dots, n$ ), for some  $a \in W$ .

For,  $(\alpha_1, \dots, \alpha_n) \in L(X)$  if and only if  $\sum_{i=1}^n \alpha_i b_i \in \ker \Phi = \text{im } \Psi$ . Equivalently,  $\sum_{i=1}^n \alpha_i b_i = a \Psi$ , for some  $a \in W$ , and since  $\alpha_i = \langle b_i^*, a \Psi \rangle = \langle b_i^* \Psi^*, a \rangle = \langle \bar{x}_i, a \rangle$ , we have the result.

Note, in particular:

2A4  $x_i \notin \text{lin}(X \setminus \{x_i\})$  if and only if  $\bar{x}_i = 0$ .

If  $Y \subseteq X$ , we write  $\tilde{Y} = \{\bar{x}_i \in \bar{X} | x_i \notin Y\}$ . As a consequence of 2A3, we have:

2A5  $Y$  is linearly independent if and only if  $\text{lin } \tilde{Y} = \text{lin } \bar{X}$ .

For, observe that there is a natural embedding  $L(Y) \subseteq L(X)$  (setting  $\alpha_i = 0$  for  $x_i \notin Y$ ). Now  $Y$  is linearly independent if and only if  $L(Y) = \{0\}$ , which, by 2A3, states that every linear functional  $a \in W$  on  $W^*$  which vanishes on  $\tilde{Y}$  vanishes on  $\bar{X}$ . This, in turn, is equivalent to  $\text{lin } \tilde{Y} = \text{lin } \bar{X}$ .

As a particular case:

2A6  $Y$  is a basis of  $\text{lin } X$  if and only if  $\tilde{Y}$  is a basis of  $\text{lin } \bar{X}$ .

We next have:

2A7 *Let  $X, Z$  be ordered  $n$ -tuples, and  $\bar{X}, \bar{Z}$  their linear transforms. If  $X$  is the image of  $Z$  under some linear mapping  $\Theta$ , then  $\bar{Z}$  is the image of  $\bar{X}$  under some linear mapping  $\bar{\Theta}$ .*

For, if  $X, Z$  are the images of  $B$  under the mappings  $\Phi$  and  $\Omega$ , then  $\Theta$  induces an embedding of  $\ker \Omega$  in  $\ker \Phi$ ; conversely, such an embedding induces a corresponding linear mapping  $\Theta$ . Taking annihilators, we have an embedding of  $\text{im } \Phi^*$  in  $\text{im } \Omega^*$ , and using the exactness of the defining sequences for  $\bar{X}$  and  $\bar{Z}$  (with these subspaces as kernels), we obtain the required linear mapping  $\bar{\Theta}$  of  $\bar{X}$  onto  $\bar{Z}$ .

In particular:

2A8 *To each linear symmetry  $\Theta$  of  $X$  corresponds a linear symmetry  $\bar{\Theta}$  of  $\bar{X}$ .*

Note that some of these symmetries (of  $X$  or  $\bar{X}$ ) may be *intrinsic*; that is (for example),  $\Theta$  permutes coincident points of  $X$ , while the induced linear mapping of  $\text{lin } X$  is the identity.

If  $Y \subseteq X$ , and in 2A7 we choose  $Z$  so that  $z_i = x_i$  if  $x_i \in Y$ , and otherwise the points of  $Z$  are linearly independent of  $\text{lin } Y$  and each other, we obtain:

2A9 *If  $Y \subseteq X$ , a linear transform  $\bar{Y}$  of  $Y$  is the image of  $\bar{X} \setminus \bar{Y}$  under a linear mapping with kernel  $\text{lin } \bar{Y}$ .*

For, from 2A4 we see that  $\bar{Z}$  consists of  $\bar{Y}$ , together with a point  $o$  corresponding to each point  $z_i \notin Y$ . The result then follows from 2A7.

Counting ranks and cardinalities of sets in 2A9 then leads to  $\text{rank } Y = \text{rank } \bar{X} - \text{card } Y + \text{rank } Y$ .

This relationship is more succinctly expressed in terms of the *deficiency*  $\text{def } Y = \text{rank } X - \text{rank } Y$  of  $Y$  in  $X$ , and the *excess*  $\text{exc } Y = \text{card } Y - \text{rank } Y$  of  $Y$ , and then becomes:

2A10 *If  $Y \subseteq X$ , then  $\text{def } \bar{Y} = \text{exc } Y$  and  $\text{exc } \bar{Y} = \text{def } Y$ .*

Of course, the relations are equivalent.

We now consider results on the facial structure of the cone  $\text{pos } X$ . We call  $Y \subseteq X$  a *facial set* if  $Y = H \cap X$  for some support hyperplane  $H$  of  $\text{pos } X$ . We call  $\bar{Y} \subseteq \bar{X}$  a *cofacial set* if  $o \in \text{relint pos } \bar{Y}$ . Then:

2A11  *$Y \subseteq X$  is a facial set if and only if  $\bar{Y} \subseteq \bar{X}$  is a cofacial set.*

For,  $Y$  is a facial set if and only if there is some  $a^* \in U^*$  with

$$\langle x_i, a^* \rangle = \alpha_i \begin{cases} = 0 & \text{for } x_i \in Y, \\ > 0 & \text{for } x_i \notin Y. \end{cases}$$

By 2A3 (and using 2A1), this is equivalent to  $\sum_{i=0}^n \alpha_i \bar{x}_i = o$ , or  $o \in \text{relint pos } Y$ . There is a unique subset  $A$  of  $X$  which is both facial and cofacial.  $A$  is called the *apical set* of  $X$ , since  $\text{pos } A$  is the face of apices of the cone  $\text{pos } X$ . Thus:

2A12 *If  $A$  is the apical set of  $X$ , then  $\bar{A}$  is the apical set of  $\bar{X}$ .*

A special case of this is:

2A13  *$\text{pos } X = \text{lin } X$  if and only if  $o \notin \text{conv } \bar{X}$ .*

For, note that, if  $\bar{Y} \neq \emptyset$ , then  $o \in \text{relint pos } \bar{Y}$  if and only if  $o \in \text{relint conv } \bar{Y}$ ; further,  $o \in \text{conv } \bar{X}$  if and only if  $o \in \text{relint conv } \bar{Y}$  for some  $\bar{Y} \subseteq \bar{X}$ . The condition then states that  $\emptyset$  is the apical set of  $\bar{X}$ .

The last results of this section illustrate the meaning of linear transforms in ordinary euclidean spaces. The definition implies that the kernels of the defining linear mappings  $\Phi: E^n \rightarrow E^k$  and  $\Psi^*: E^m \rightarrow E^m$  are mutual annihilators, so that  $\langle y, y^* \rangle = 0$  for all  $y \in \ker \Phi$  and  $y^* \in \ker \Psi^*$ . If we let  $M$  be the  $(k+m) \times n$  matrix with rows  $(x_i, \bar{x}_i)$  ( $i = 1, \dots, n$ ), we then see:

2A14  *$\bar{X} = (\bar{x}_1, \dots, \bar{x}_n) \subseteq E^m$  is a linear transform of  $X = (x_1, \dots, x_n) \subseteq E^k$  if and only if  $\text{rank } M = n$ , and each of the last  $m$  columns of  $M$  is orthogonal to each of the first  $k$ .*

The last  $m$  columns of  $M$  span  $L(X)$ , of course, so 2A14 gives us a method of constructing  $\bar{X}$  if we are given  $X$ . For, it suffices to pick any spanning set  $a_j = (\alpha_{1j}, \dots, \alpha_{nj})$  ( $j = 1, \dots, m$ ) of  $L(X)$ , and let  $\bar{X}$  be the rows of the  $m \times n$  matrix whose columns are  $a_1, \dots, a_m$ . In particular, if  $X$  is so ordered that  $\{x_1, \dots, x_d\}$  is a basis of  $\text{lin } X$ , we can express each of  $x_{d+1}, \dots, x_n$  as a (unique) linear combination of  $x_1, \dots, x_d$ , and these linear combinations at once yield a basis of  $L(X)$ . We conclude:

2A15 *Let  $\text{rank } X = d$ , and suppose  $X$  (and  $\bar{X}$ ) recorded, if necessary, so that  $Y = \{x_1, \dots, x_d\}$  is a linear basis of  $\text{lin } X$ . If the linear coordinates of  $X$  with respect to  $Y$  are the rows of the matrix  $\begin{bmatrix} I_d \\ A \end{bmatrix}$ , then the linear coordinates of  $\bar{X}$  with respect to  $\tilde{Y}$  are the rows of the matrix  $\begin{bmatrix} -A^T \\ I_{n-d} \end{bmatrix}$ .*

Finally, a definition we shall use later. If  $X = (\pm x_1, \dots, \pm x_n)$  is an ordered centrally symmetric set, and  $(\bar{x}_1, \dots, \bar{x}_n)$  is a linear transform of  $(x_1, \dots, x_n)$ , we call  $\bar{X} = (\pm \bar{x}_1, \dots, \pm \bar{x}_n)$  a *central transform* of  $X$ .

As we observed in the introduction, the idea of linear transforms goes back to Whitney [1935]. The seeds of the present approach were sown by Ewald & Voss [1973], though there they worked with matrices in Euclidean spaces. The treatment by Bair & Fourneau [1975] also uses coordinates. In convexity, linear transforms were introduced by Shephard [1971b], to whom 2A11–13 are due, although they were implicitly defined in McMullen & Shephard [1968].

## 2B. Affine transforms

Affine transforms perform the function analogous to linear transforms in studying the affine properties of an ordered set  $X$  in an affine space  $U$ . We now take  $X$  to be the image under an affine mapping  $\Phi$  of an ordered affine basis  $B$  of an  $(n-1)$ -dimensional affine space  $V$ . We let

$$W \xrightarrow{\Psi} V \xrightarrow{\Phi} U$$

be a sequence of affine spaces and mappings, which is exact at  $V$  in the sense that  $\text{im } \Psi = W\Psi$  is the inverse image  $\{a\}\Phi^{-1}$  of some point  $a \in U$ . The induced affine dual sequence

$$U^\ddagger \xrightarrow{\Phi^\ddagger} V^\ddagger \xrightarrow{\Psi^\ddagger} W^\ddagger$$

of linear spaces and mappings is exact at  $V^\ddagger$ . The image  $\bar{X}$  of the basis  $B^\ddagger$  of  $V^\ddagger$  dual to  $B$  under  $\Psi^\ddagger$  is then called an *affine transform* of  $X$ . We first note that:

$$2B1 \quad \text{rank } X + \text{rank } \bar{X} = \text{card } X - 1.$$

The affine properties of  $X$  are then reflected in the linear properties of  $\bar{X}$ . An *affine dependence* of  $X$  is a vector  $(\alpha_1, \dots, \alpha_n)$ , such that  $\sum_{i=1}^n \alpha_i x_i = 0$  and  $\sum_{i=1}^n \alpha_i = 0$ . The linear subspace of affine dependences of  $X$  is denoted  $A(X)$ . We then have:

- 2B2 (i)  $(\alpha_1, \dots, \alpha_n) \in A(X)$  if and only if  $\alpha_i = \langle \bar{x}_i, a \rangle$  for some  $a \in W$ ;  
(ii)  $(\beta_1, \dots, \beta_n) \in L(\bar{X})$  if and only if  $\beta_i = \langle x_i, b^* \rangle$  for some  $b^* \in U^*$ .

Note, in particular:

- 2B3  $x_i \notin \text{aff}(X \setminus \{x_i\})$  if and only if  $\bar{x}_i = o$ .

We also observe that the existence of the constant affine functional 1 gives us one half of:

- 2B4  $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n)$  is an affine transform of some set  $X$  if and only if  $\sum_{i=1}^n \bar{x}_i = o$ .

We shall prove the other half below. We remark here that affinely equivalent sets have the same affine transforms, which are defined up to linear equivalence. Much of the theory of affine transforms is elucidated by finding connexions between them and linear transforms. First, we embed  $U$  as a hyperplane in an affine space  $\hat{U}$ , and make  $\hat{U}$  into a linear space by specifying some point not on  $U$  as origin. Then  $U = \{x \in \hat{U} \mid \langle x, f^* \rangle = 1\}$  for some  $f^* \in \hat{U}^*$ . Moreover, there is a natural isomorphism  $u^* \rightarrow u^*$  between  $U^*$  and  $\hat{U}^*$ , since the null-space of  $u^* \in U^*$  is a hyperplane in  $U$ , which spans a linear hyperplane in  $\hat{U}$ . It is now easy to choose the correct multiple  $u^*$  of a linear functional with this hyperplane as null-space, so that  $u^*$  coincides with  $u^*$  on  $U$ . Finally, every affine dependence of our given set  $X$  is clearly a linear dependence (in  $\hat{U}$ ); the converse also holds, since, if  $\sum_{i=1}^n \alpha_i x_i = o$ , applying  $f^*$  (as defined above) shows that  $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \alpha_i \langle x_i, f^* \rangle = 0$ .

The required connexion is now clear:

- 2B5 Let  $X$  lie in a hyperplane not containing  $o$  of a linear space. Then linear and affine transforms of  $X$  coincide.

For, the linear and affine transforms are determined (up to linear equivalence) by the subspaces  $L(X) = A(X)$  of linear and affine dependences.

Further, the proof of 2B4 is easily completed. For, the points of a linear transform  $X$  of  $\bar{X}$  lie in a hyperplane not containing  $o$ , since the relation  $\sum_{i=1}^n \bar{x}_i = o$  implies that there is some linear functional  $f^*$  with  $\langle x_i, f^* \rangle = 1$  ( $i = 1, \dots, n$ ). Thus  $\bar{X}$  is an affine transform of  $X$ .

As a further remark to elucidate the geometry, notice that the construction described above implies that we regard the affine basis  $B$  whose image under  $\Phi$  is  $X$  as a linear basis in a space of one more dimension.

2B5 now enables us to find the appropriate analogues of results on linear transforms. Firstly:

- 2B6 Let  $X, Z$  be ordered  $n$ -tuples, and let  $\bar{X}, \bar{Z}$  be their affine transforms. Then  $X$  is the image of  $Z$  under some affine mapping  $\Theta$  if and only if  $\bar{Z}$  is the image of  $\bar{X}$  under some linear mapping  $\bar{\Theta}$ .

In particular:

- 2B7 To each affine symmetry of  $X$  corresponds a linear symmetry of its affine transform  $\bar{X}$ , and conversely.

- 2B8 An affine transform of a subset  $Y$  of  $X$  is the image of  $\bar{X} \setminus \bar{Y}$  under a linear mapping with kernel  $\text{lin } \bar{Y}$ .

Note that the image of  $\bar{X} \setminus \bar{Y}$  automatically has centroid  $o$ . We shall see in 3A7 what will correspond to suitable subsets of  $\bar{X}$ .

The *deficiency*  $\text{def } Y$  of  $Y \subseteq X$  is defined as for linear spaces (with the affine definition of rank), but since an affine basis contains one more point than the dimension of its affine hull, it is appropriate here to define the *excess* of  $Y$  by  $\text{exc } Y = \text{card } Y - \text{rank } Y - 1$ . We then have, as for linear transforms:

2B9 *If  $Y \subseteq X$ , then  $\text{def } \bar{Y} = \text{exc } Y$  and  $\text{exc } \bar{Y} = \text{def } Y$ .*

Deficiency and excess in  $\bar{X}$  are, of course, the linear terms. In particular, we have a result we could have proved much earlier:

2B10 (i)  $\text{aff } Y = \text{aff } X$  if and only if  $\bar{Y}$  is linearly independent;  
(ii)  $Y$  is affinely independent if and only if  $\text{lin } \bar{Y} = \text{lin } \bar{X}$ .

There is another connexion between linear and affine transforms.

2B11 *Let  $X = (x_1, \dots, x_n)$  be a subset of a linear space, let  $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n)$  be a linear transform of  $X$ , and write  $\bar{x}_0 = \sum_{i=1}^n \bar{x}_i$ . Then the image of  $\bar{X}$  under a linear mapping  $\Theta$  with kernel  $\text{lin } \bar{x}_0$  is an affine transform of  $X$ .*

For,  $A(X)$  is the subspace of  $L(X)$  consisting of those  $(\alpha_1, \dots, \alpha_n) \in L(X)$  with  $\sum_{i=1}^n \alpha_i = 0$ . Hence (compare 2A7), in constructing the affine transform as a linear image of the dual basis  $(b_1^*, \dots, b_n^*)$  of  $V^*$ , to obtain the new kernel, we must adjoin to the kernel of the mapping giving the linear transform the vector  $\sum_{i=1}^n b_i^*$ . The result now follows.

From this, we see:

2B12 *Let  $\bar{X}$  be an affine transform of  $X$ . Then  $X$  is affinely equivalent to an affine transform of  $\bar{X}$ .*

For, let  $X' = (x'_1, \dots, x'_n)$  be a linear transform of  $\bar{X}$ , so that  $X'$  is affinely equivalent to  $X$ , by 2B4 and 2B5. By 2B11, an affine transform of  $\bar{X}$  is obtained from  $X'$ , by taking the image of  $X'$  under a linear mapping with kernel  $\text{lin } \{\sum_{i=1}^n x'_i\}$ . But this mapping is one-to-one on  $X'$ , and we thus have the required result.

Let our affine space  $U$  be embedded as before as a hyperplane in a linear space  $\hat{U}$ . To each point of  $U$  corresponds the line of  $\hat{U}$  through  $o$  to which it belongs; and conversely, except for lines through  $o$  parallel to  $U$ , and hence lying in the parallel hyperplane to  $U$ :  $U_0 = \{u \in \hat{U} | \langle u, f^* \rangle = 0\}$ , which corresponds to the “hyperplane at infinity” of  $U$ . A *projectivity* of  $U$  is then a mapping  $\Theta$  induced by a non-singular linear mapping  $\hat{\Theta}$  of  $\hat{U}$ . If  $X \subseteq U$ , then  $X\hat{\Theta} \subseteq U\hat{\Theta}$ , so to  $x_i \in X$  corresponds  $x_i\hat{\Theta} = \langle x_i\hat{\Theta}, f^* \rangle^{-1} x_i\hat{\Theta}$ . Thus  $\Theta$  is not *permissible* for  $X$  if, for some  $i$ ,  $x_i\hat{\Theta} \in U_0$ . If  $\Theta$  is permissible for  $X$ , then, writing  $g^* = f^*\hat{\Theta}^*$ , using 2B5, and bearing in mind that linearly equivalent sets have the same linear transforms, we see that an affine transform  $\bar{X}' = (\bar{x}'_1, \dots, \bar{x}'_n)$  of  $X\Theta$  has  $\bar{x}'_i = \langle x_i, g^* \rangle \bar{x}_i$  ( $i = 1, \dots, n$ ). Note that  $\Theta$  is permissible for  $\text{conv } X$  if and only if the  $\langle x_i, g^* \rangle$  all have the same sign. We shall call a mapping of the form  $\bar{x}_i \rightarrow \lambda_i \bar{x}_i \Lambda$ , where  $\lambda_i > 0$  ( $i = 1, \dots, n$ ) and  $\Lambda$  is a non-singular linear mapping an *equivalence*. The argument can be extended without difficulty to projective mappings which are not necessarily non-singular, and we obtain:

2B13 *Let  $X, Z$  be ordered  $n$ -tuples, and let  $\bar{X}, \bar{Z}$  be their affine transforms. Then  $X$*

*is the image of  $Z$  under some projective mapping permissible for  $Z$  if and only if  $\bar{Z}$  is the image of  $\bar{X}$  under some mapping of the form  $\bar{z}_i = \lambda_i \bar{x}_i \Lambda$ , where  $\lambda_i \neq 0$  ( $i = 1, \dots, n$ ) and  $\Lambda$  is a linear mapping. The projective mapping is permissible for  $\text{conv} Z$  if and only if the  $\lambda_i$  have the same sign.*

To complete the proof, again we need only discuss the case where  $\Lambda$  is non-singular. Since  $\sum_{i=1}^n \lambda_i \bar{x}_i = 0$ , we have  $\lambda_i = \langle x_i, g^* \rangle$  for some  $g^* \in \bar{U}^*$ . We now define a non-singular  $\bar{\Theta}$  by  $g^* = f^* \bar{\Theta}^*$ ; then  $Z$  is (affinely equivalent to) the image of  $X$  under the induced projectivity  $\Theta$ , as above.

We note a particular case:

**2B14** *To each projective symmetry of  $X$  permissible for  $\text{conv} X$  corresponds a self-equivalence of its affine transform  $\bar{X}$ , and conversely.*

2B13 shows us that equivalence in affine transforms reflects permissible (for the convex hull) projective equivalence of the original sets. This suggests the following definition. A *projective transform* of a set  $X$  is any set  $\bar{X}$  equivalent to an affine transform of  $X$ . This concept was introduced by McMullen & Shephard [1970b], p. 7, as a linear transform of a set equivalent to  $X$  (embedded, as above, in some hyperplane). We shall not, in fact, use the term; for one thing, there seems no obvious reason why one should only admit projectivities permissible for the convex hull, and so the term might conceivably be open to misinterpretation.

The affine transform has useful continuity properties.

**2B15** *Let  $X, X_k$  ( $k = 1, 2, \dots$ ) be ordered  $n$ -tuples, with  $\text{rank } X_k = \text{rank } X$  for all  $k$ , and  $\lim X_k = X$ . Then affine transforms  $\bar{X}, \bar{X}_k$  ( $k = 1, 2, \dots$ ) can be chosen so that  $\lim \bar{X}_k = \bar{X}$ .*

For, let  $\Phi, \Phi_k$  map the basis  $B$  of  $V$  onto  $X, X_k$ . If  $Y \subseteq X$  is an affine basis of  $\text{aff} X$ , then the corresponding subset  $Y_k \subseteq X_k$  will be an affine basis of  $\text{aff} X_k$ , for all large enough  $k$ . We can now suppose  $\tilde{Y}_k$  chosen so that  $\tilde{Y}_k = \tilde{Y}$  for all such  $k$  (in fact,  $\tilde{Y}_k \rightarrow \tilde{Y}$  would do). Since  $\ker \Phi_k \subseteq V$  and  $\ker \Psi_k^\dagger \subseteq V^\dagger$  are mutual annihilators, we conclude at once that  $\bar{X}_k \rightarrow \bar{X}$ .

The last result of this section concerns coordinates.

**2B16** *Let  $x, \bar{x}$  denote coordinate vectors in Euclidean spaces  $E^k, E^m$ . Let  $M$  be the matrix with rows  $(x_i, 1, \bar{x}_i)$  ( $i = 1, \dots, n$ ). Then  $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n)$  is an affine transform of  $X = (x_1, \dots, x_n)$  if and only if  $\text{rank } M = n$ , and each of the last  $m$  columns of  $M$  is orthogonal to each of the first  $k+1$  columns.*

We observe that 2B16 allows us to construct  $\bar{X}$  if we are given  $X$ , exactly like 2A14 for linear transforms. In the same way, 2A15 extends immediately to affine transforms, if we interpret the rows of the first matrix as giving the affine coordinates of  $X$  with respect to  $Y$  (and, of course, we must change  $d$  to  $d+1$ , if we retain  $\text{rank } X = d$ ).

We have not forgotten to discuss the facial structure of  $\text{conv} X$ ; the results can, of course, be deduced from those of 2A and 2B5. But we shall consider this topic in 3A.

The presentation of the material in this section is mostly new, though the actual results are not. For earlier descriptions, largely using coordinates, the reader

should consult Grünbaum [1967], §5.4, McMullen & Shephard [1971], Chapter 3, the three survey articles Grünbaum & Shephard [1969], Grünbaum [1970] and McMullen [1977], or the lecture notes McMullen & Shephard [1970b] and Doignon & Valette [1975]. The initial paper Gale [1956] naturally uses coordinates. Stoer & Witzgall [1970], §2.15, adopt a fairly abstract approach in their brief description of Gale diagrams. 2B7 is due to McMullen & Shephard [1970a], and 2B15 to Kömhoff & Shephard [1974].

### 3. Gale diagrams

#### 3A. Gale diagrams

We shall define Gale diagrams in several ways, which we shall show to be essentially equivalent. Our treatment here will roughly follow the historical development of the subject, although, of course, we shall try to benefit from hindsight.

We begin with an arbitrary ordered set  $X = (x_1, \dots, x_n)$  in an affine space  $U$ . Let  $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n) \subseteq W^\ddagger$  be an affine transform of  $X$ . We call  $Y \subseteq X$  a *facial set* if  $Y = H \cap X$  for some support hyperplane  $H$  of  $X$  (or of  $\text{conv } X$ ), and we call  $\tilde{Y} \subseteq \bar{X}$  a *cofacial set* if  $o \in \text{relint pos } X$  (or, what amounts to the same thing,  $o \in \text{relint conv } X$ ). Our first result gives the *Gale diagram relationship*:

3A1  $Y \subseteq X$  is a facial set if and only if  $\tilde{Y} \subseteq \bar{X}$  is a cofacial set.

For,  $Y$  is a facial set if and only if there is some  $u^\ddagger \in U^\ddagger$ , such that

$$\langle x_i, u^\ddagger \rangle = \alpha_i \begin{cases} = 0, & x_i \in Y, \\ > 0, & x_i \notin Y. \end{cases}$$

By 2B2(ii), this is equivalent to  $\sum_{i=1}^n \alpha_i \bar{x}_i = o$ ; that is, that  $\tilde{Y}$  is a cofacial set. We say that two ordered subsets are *facially (cofacially) isomorphic* if there is a one-to-one correspondence between them, such that corresponding subsets are facial (cofacial) or not together. Clearly:

3A2 Two ordered sets are facially isomorphic if and only if their affine transforms are cofacially isomorphic.

Where no ambiguity is likely, we usually drop the qualifiers “facially” and “cofacially” and talk about sets being isomorphic.

A particularly important case is where  $X = \text{vert } P$ , the set of vertices of some polytope  $P$ , taken in some order. Any set  $\bar{X}$  isomorphic to an affine transform of  $X$  will be called a *Gale diagram* of  $P$ . Note that  $\bar{X}$  will not, in general, be an affine transform; but some set equivalent to  $\bar{X}$  will be, since we can replace  $\bar{x}_i \in \bar{X}$  by  $\lambda_i \bar{x}_i$ , where the  $\lambda_i > 0$  are such that  $\sum_{i=1}^n \lambda_i \bar{x}_i = o$ . Since each singleton subset of  $X$  is facial, we obtain the *Gale diagram condition*:

3A3 An ordered set  $\bar{X}$  is a Gale diagram of some polytope if and only if every open linear half-space of  $\text{lin } \bar{X}$  contains at least two points of  $\bar{X}$ .

By a *linear* half-space, we mean a half-space bounded by a hyperplane through  $o$ . Among many special results, let us note a consequence of 2B3:

3A4  $P$  is a pyramid with apex  $x_i$  if and only if  $\bar{x}_i = 0$ .

From 2B10(ii) also follows:

3A5  $P$  is simplicial if and only if every cofacial set in  $\bar{X}$  spans  $\text{lin } \bar{X}$ .

We note that, if  $P = \text{conv } X$ , but  $X \neq \text{vert } P$ , we can construct a Gale diagram of  $P$  from an affine transform of  $X$  (or any set isomorphic to it). For, if  $P = \text{conv}(X \setminus \{x_i\})$ , then  $\bar{X} \setminus \{\bar{x}_i\}$  is not cofacial (and so the Gale diagram condition 3A3 is violated). But an affine transform of  $X \setminus \{x_i\}$  is obtained from  $\bar{X}$  by taking the image of  $\bar{X} \setminus \{\bar{x}_i\}$  under a linear mapping with kernel  $\text{lin } \{\bar{x}_i\}$ , by 2B8. Performing this operation as often as necessary then reduces  $\bar{X}$  to a Gale diagram of  $P$ .

2B8 also shows:

3A6 Let  $F$  be a face of  $P$ , corresponding to the cofacial set  $Y$ . Then the image of  $\bar{X} \setminus \bar{Y}$  under a linear mapping with kernel  $\text{lin } \bar{Y}$  is a Gale diagram of  $F$ .

If  $F$  is a face of  $P$ , then the lattice of faces  $G$  of  $P$  with  $F \subseteq G \subseteq P$  is isomorphic to the lattice of faces of some polytope of dimension  $\dim P - \dim F - 1$ , called the *face-figure* of  $P$  at  $F$ , and denoted  $P/F$ . Then:

3A7 A Gale diagram of  $P/F$  is obtained by reducing the cofacial set  $\bar{Y}$  corresponding to  $F$ .

To see this, we use 2A9 and 2B5. For, as in 2B5, embed the space  $U$  containing  $X = \text{vert } P$  as a hyperplane in  $\hat{U}$ . The image of  $X \setminus Y$  under a linear mapping with kernel  $\text{lin } Y$  gives a set of points  $Z$  whose positive hull is (essentially) the cone spanned by  $P/F$ . 2A9 shows that  $\bar{Y}$  is a linear transform of  $Z$ , and hence equivalent to an affine transform of a set  $Z'$  with  $\text{conv } Z' = P/F$ , by 2B5 again. Since equivalent Gale diagrams are isomorphic, we see that the non-zero points of a Gale diagram in  $E^k$  can always be chosen to lie on the unit sphere  $S^{k-1}$ . Such a Gale diagram is called *standard*. Gale diagrams in  $E^2$  admit yet more regularization, which we shall discuss in 3B.

We now consider the geometric formulation of Gale diagrams. We begin with a general result, which we shall apply here and elsewhere. Let  $Q$  be a polytope in the linear space  $V$ , with  $o \in \text{int } Q$ . The *polar set*  $Q^0$  of  $Q$  is defined by

$$Q^0 = \{v^* \in V^* \mid \langle v, v^* \rangle \leq 1 \text{ for all } v \in Q\}.$$

Then  $Q^0$  is again a polytope, with  $o \in \text{int } Q^0$ .  $Q^0$  is actually *dual* to  $Q$ , in the sense that there is a one-to-one inclusion reversing correspondence  $G \leftrightarrow \hat{G}$  between the faces  $G$  of  $Q$  and  $\hat{G}$  of  $Q^0$ , which is given by

$$\hat{G} = \{v^* \in Q^0 \mid \langle v, v^* \rangle = 1 \text{ for all } v \in G\}.$$

Now let  $\Phi: V \rightarrow U$  be a linear mapping of  $V$  onto  $U$ . Then  $Q\Phi$  is again a polytope with  $o \in \text{int}(Q\Phi)$ , and its polar set in  $U$  is:

3A8  $(Q\Phi)^0 = Q^0(\Phi^*)^{-1}$ .

For,

$$(Q\Phi)^0 = \{u^* \in U^* \mid \langle u, u^* \rangle \leq 1 \text{ for all } u \in Q\Phi\}$$

$$= \{u^* \in U^* \mid \langle v, u^* \Phi^* \rangle \leq 1 \text{ for all } v \in Q\} \\ = Q^0(\Phi^*)^{-1},$$

as claimed.

Now suppose, as usual, that

$$W \xrightarrow{\Psi} V \xrightarrow{\Phi} U$$

is a sequence of linear spaces and mappings exact at  $V$ . Then we have:

3A9 *The face  $G$  of  $Q$  is the inverse image under  $\Phi$  of some face of  $Q\Phi$  if and only if  $o \in \text{relint}(\hat{G}\Psi^*)$ .*

For, we can choose the corresponding supporting hyperplane of  $Q\Phi$  to be  $\{u \in U \mid \langle u, u^* \rangle = 1\}$ , for some  $u^* \in U^*$ . Then the corresponding supporting hyperplane of  $Q$  is  $\{v \in V \mid \langle v, u^* \Phi^* \rangle = 1\}$ . Since this meets  $Q$  exactly in  $G$ , we have  $u^* \Phi^* \in \text{relint}(\hat{G})$ . Applying  $\Psi^*$  yields  $o \in \text{relint}(\hat{G}\Psi^*)$ . The argument is reversible, and so we have the result.

We first apply these results by taking  $Q$  to be an  $(n-1)$ -simplex, with vertices  $q_1, \dots, q_n$  whose centroid lies at  $o$ . Let the polar simplex  $Q^0$  have vertices  $q_1^*, \dots, q_n^*$ , so that  $\langle q_i, q_j^* \rangle = 1$  ( $i \neq j$ ); the centroid of the vertices of  $Q^0$  also lies at  $o$ . If  $b_j$  is the affine functional  $b_j = n^{-1}(1 - q_j^*)$ , then  $(b_1, \dots, b_n)$  is the basis of  $V$  dual to  $(q_1, \dots, q_n)$ . Writing  $u^\ddagger = (u^*, \lambda)$  to mean  $\langle x, u^\ddagger \rangle = \langle x, u^* \rangle + \lambda$ , we easily check that  $(u^*, \lambda)\Phi^\ddagger = (u^*\Phi^*, \lambda)$ , so that the constant affine functionals in  $V^\ddagger$  belong to  $\ker \Psi^\ddagger$ , and so  $b_j\Psi^\ddagger = -n^{-1}q_j^*\Psi^*$ . Thus  $(q_1^*, \dots, q_n^*)\Psi^*$  is linearly equivalent to  $(b_1, \dots, b_n)\Psi^\ddagger$ , and so:

3A10  *$(q_1^*, \dots, q_n^*)\Psi^*$  is an affine transform of  $(\text{vert } Q)\Phi$ .*

In this case, 3A9 reduces to the Gale diagram relationship 3A1, since  $\text{vert } \hat{G} = \{q_i^* \mid q_i \notin \text{vert } G\}$ . Note also that 3A10 provides an alternative proof of 2B12. However, the fact that  $o = \sum_{i=1}^n q_i$  is not really relevant to the geometry of the construction; all that is needed is  $o \in \text{int } Q$ . If we recall that polar sets of projectively equivalent polytopes are projectively equivalent, and that the permissible projective mappings of a simplex  $\text{conv}\{a_1, \dots, a_n\}$  into itself, leaving each vertex  $a_i$  fixed, are of the form

$$\sum_{i=1}^n \tau_i a_i \rightarrow \left( \sum_{i=1}^n \lambda_i \tau_i \right)^{-1} \sum_{i=1}^n \lambda_i \tau_i a_i,$$

where  $\lambda_i > 0$  ( $i = 1, \dots, n$ ), and the coordinates  $\tau_i$  satisfy  $\tau_i \geq 0$  and  $\sum_{i=1}^n \tau_i = 1$ , we should readily believe that different choices of origin in  $Q$  lead to equivalent sets in  $W^*$ . For fuller details, we refer the reader to McMullen [1968], Chapter 5, or McMullen & Shephard [1971], Chapter 3. (Compare also McMullen & Shephard [1970b], §I.1.)

The following observation, which gives the inverse of the construction described above, further clarifies the geometry.

3A11 *Let  $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n)$  be a Gale diagram of a polytope  $P$ , and let  $(a_1, \dots, a_n)$  be the vertices of an  $(n-1)$ -simplex. Then*

$$\left\{ \sum_{i=1}^n \tau_i a_i \mid \sum_{i=1}^n \tau_i \bar{x}_i = 0, \tau_i \geq 0 \ (i=1,\dots,n), \sum_{i=1}^n \tau_i = 1 \right\}$$

is a polytope dual to P.

This idea, in a different context, is due to Davis [1954] and Klee [1966].

The following observation allows us to calculate Gale diagrams.

- 3A12 Let P be a polytope in  $E^d$ , with vertices  $x_1, \dots, x_n$  and facets  $F_1, \dots, F_m$ . Let  $\alpha_{ij}$  be the euclidean distance from  $x_i$  to  $\text{aff } F_j$ , and let A be the  $n \times m$  matrix with entries  $\alpha_{ij}$ . Then a linear transform of the set of row vectors of A is a Gale diagram of P, and a linear transform of the column vectors of A is a Gale diagram of a dual  $P^0$  of P.

Suppose, without loss of generality, that  $\dim P = d$ . Embed  $E^d$  as a hyperplane not through o in the linear space  $\hat{U}$ . Let  $C = \text{pos } P$  be the cone spanned by P, and let

$$C^0 = \{u^* \in U^* \mid \langle u, u^* \rangle \geq 0 \text{ for all } u \in C\}$$

be (the negative of) the cone polar to C.  $C^0$  is met by some hyperplane in  $U^*$  in a polytope polar to P. Let the vertices of this polytope  $P^0$  be  $y_1^*, \dots, y_m^*$ , corresponding to the facets of P.

Now,  $\dim P^0 = d$  also, so the  $y_j^*$  span  $U^*$ . We have  $\langle x_i, y_j^* \rangle = \lambda_i \alpha_{ij}$  for some  $\lambda_i > 0$ , or  $\alpha_{ij} = \langle x_i, \lambda_j^{-1} y_j^* \rangle$ . Thus the rows of A are the images of  $x_1, \dots, x_n$  under a one-to-one linear mapping, and the first conclusion follows. By the same token, the columns of A are equivalent to the images of  $y_1^*, \dots, y_m^*$  under a one-to-one linear mapping, and we have the second result.

Gale diagrams have been discussed by many authors. Apart from the references given before 3A11, useful sources are Grünbaum [1967], §5.4, Grünbaum & Shephard [1969], §2.5, Stoer & Witzgall [1970], §2.15, Bair & Fourneau [1975], §VI.1, and Doignon & Valette [1975], Chapter 6. A rather different, abstract approach is due to Marcus [1979c], by whom 3A12 was also pointed out (private communication).

### 3B. Applications

Since the technique of Gale diagrams is the oldest of those under discussion, it is perhaps not surprising that Gale diagrams have been the most comprehensively exploited. We shall survey the various applications of Gale diagrams here.

Our first result uses affine transforms, rather than Gale diagrams. A *Radon partition* of our finite set X consists of disjoint subsets A and B of X, satisfying  $\text{conv } A \cap \text{conv } B \neq \emptyset$ . Then we have the classical *Radon's theorem*:

- 3B1 If  $\text{exc } X > 0$  (that is, if  $\text{card } X > \text{rank } X$ ), then X admits a Radon partition.

For, an affine transform  $\bar{X}$  of X has rank  $\bar{X} = \text{def } \emptyset \geq 1$ , and  $\sum_{i=1}^n \bar{x}_i = o$ . Let H be any hyperplane in  $\text{lin } \bar{X}$  through o. Then both open half-spaces bounded by H contain points of  $\bar{X}$  corresponding to the (disjoint) subsets A and B of X. A non-zero linear functional with null-space H then gives rise to an affine dependence of X, the coefficients of which are (say) positive, negative and zero on A, B and  $X \setminus (A \cup B)$ , respectively. The result follows at once.

Moreover, every Radon partition (essentially) arises in this way.

This connexion between Gale diagrams and Radon partitions was first explicitly noted by Reay [1968]. The connexion has been further explored by Shephard [1969], Doignon & Valette [1975], Chapter 7, Eckhoff [1974; 1975] and Doignon [1979].

We say  $X$  is  $k$ -neighbourly if every  $k$  of its points form a facial set. Thus, every subset obtained by deleting  $k$  points of  $\bar{X}$  is cofacial, by 3A1. In other words (Grünbaum [1967], Exercise 7.3.7):

3B2  *$X$  is  $k$ -neighbourly if and only if every open linear half-space of  $\text{lin } \bar{X}$  contains at least  $k + 1$  points of  $\bar{X}$ .*

If  $\text{rank } \bar{X} \geq 1$ , a hyperplane bounding such a half-space may contain  $\text{rank } \bar{X} - 1$  points of  $\bar{X}$ , or more. We thus conclude that  $\text{card } \bar{X} \geq \text{rank } \bar{X} - 1 + 2(k + 1)$ , or, since  $\text{rank } \bar{X} = \text{card } X - \text{rank } X - 1$ :

3B3 *If  $X$  is not the set of vertices of a simplex, then  $X$  is at most  $[\frac{1}{2}\text{rank } X]$ -neighbourly.*

By considering images of  $\bar{X}$  under linear mappings, whose kernels are spanned hyperplanes (spanned, that is, by points of  $\bar{X}$ ), we see:

3B4 *Let  $\text{rank } X = d$ . Then  $X$  is  $k$ -neighbourly if and only if every subset of  $d + 2$  points of  $X$  is  $k$ -neighbourly.*

3B4 comes from Grünbaum [1967], Exercise 7.3.5(i). Shephard [1969] and Poláková [1973] discuss restrictions on Radon partitions imposed by neighbourliness; for example, from 3B2 we clearly deduce:

3B5 *If  $X$  is  $k$ -neighbourly, and  $(A, B)$  is a Radon partition of  $X$ , then  $\text{card } A \geq k + 1$ .*

It should be recalled that the pioneering paper Gale [1956] was concerned with the construction of neighbourly polytopes.

If  $X$  is  $k$ -neighbourly, then, by 3B3, every  $k$  points of  $X$  are the vertices of a  $(k - 1)$ -simplex, which is a face of  $\text{conv } X$ . Hence, by 3B2, an affine transform  $\bar{X}$  of  $X$  is a *positive*  $(k + 1)$ -spanning set, in the sense that every subset  $\bar{Y}$  of  $n - k$  points of  $\bar{X}$  satisfies  $\text{pos } \bar{Y} = \text{lin } \bar{X}$ . In particular, 3A3 says that the Gale diagram of every polytope is a positive 2-spanning set. We then have the following relevant result of Marcus [1979a]:

3B6 *Let  $Z$  be a minimal (with respect to inclusion) positive 2-spanning set, with  $\text{rank } Z = m$  and  $\text{card } Z = n$ . Then  $n \leq 4m$  if  $m \leq 5$ , and  $n \leq \frac{1}{2}m(m + 1) + 5$  if  $m \geq 5$ .*

The bound  $n \leq 4m$  is best possible if  $m \leq 4$ , and is conjectured to be so generally. A polytope  $P$  whose Gale diagram is a minimal positive 2-spanning set may be called *unneighbourly*, since no vertex of  $P$  is joined to every other one by an edge. If generally true, 3B6 would imply that an unneighbourly  $d$ -polytope must have at least  $\frac{4}{3}(d + 1)$  vertices. (See also 3B18 below.)

Our next topic is the enumeration of combinatorial types. If the number of vertices of a polytope  $P$  does not greatly exceed its dimension, then a Gale

diagram of  $P$  has small rank, and so is comparatively easy to investigate. We shall use this fact here and elsewhere.

We write  $c(n,d)$  for the number of combinatorial types of  $d$ -polytopes with  $n$  vertices, and  $c_s(n,d)$  for the corresponding number of simplicial polytopes. Obviously,  $c(d+1,d) = 1 = c_s(d+1,d)$ . If  $n=d+2$ , a standard Gale diagram of  $P$  on the real line  $R$  consists of the point 1 with multiplicity  $r \geq 2$ , the point  $-1$  with multiplicity  $s \geq 2$  (and we can suppose  $r \geq s$ ), and the point 0 with multiplicity  $t = d+2-r-s \geq 0$ , where  $t=0$  if  $P$  is simplicial. Thus (see McMullen & Shephard [1971], Chapter 3, Theorem 9):

$$3B7 \quad c_s(d+2,d) = [\frac{1}{2}d]; \quad c(d+2,d) = [\frac{1}{4}d^2].$$

Perles (see Grünbaum [1967], 6.3.2) used Gale diagrams to prove:

$$3B8 \quad c_s(d+3,d) = 2^{[d/2]} - [\frac{1}{2}(d+4)] + \sum_{h|d+3, h \text{ odd}} \varphi(h) 2^{(d+3)/h} \quad \text{where } \varphi \text{ is Euler's totient function.}$$

Perles actually used *distended* standard Gale diagrams, which have  $d+3$  equally spaced diameters; the *contracted* diagrams, in which the points lie on as few diameters as possible, are perhaps slightly easier to use. Lloyd [1970] also used Gale diagrams to evaluate  $c(d+3,d)$ , but his expression is too complicated to reproduce here.

Altshuler & McMullen [1973] enumerated the simplicial  $[\frac{1}{2}d]$ -neighbourly  $d$ -polytopes with  $d+3$  vertices, while McMullen [1974] solved the corresponding problem for general polytopes. Gale diagrams have also been used to enumerate combinatorial types of polytopes with certain symmetry groups, but we postpone discussion of this subject to 6C.

The *f-vector* of a  $d$ -polytope  $P$  is the sequence  $(f_0, \dots, f_{d-1})$  of the numbers  $f_j = f_j(P)$  of  $j$ -faces of  $P$ . McMullen [1971a] has used Gale diagrams to verify his so-called *g-conjecture* concerning the possible *f*-vectors of simplicial  $d$ -polytopes, in case  $f_0 \leq d+3$ . The *g-conjecture*, whose involved statement we shall not reproduce here, encompasses the upper bound theorem, a proof of which in case  $f_0 \leq d+3$  was earlier given by McMullen [1968], Chapter 5, using Gale diagrams (see also McMullen [1970] and McMullen & Shephard [1971], §3.4), as well as the generalized lower bound conjecture of McMullen & Walkup [1971], where it was remarked that certain cases for small values of  $f_0$  follow from a stronger version of the next result.

3B9 *Let  $P$  be a polytope, and  $\bar{X}$  equivalent to an affine transform of  $\text{vert } P$ . If  $\bar{x}_0 \in \text{lin } \bar{X}$  is in general position with respect to  $\bar{X}$ , the cofacial sets of  $\bar{X} \cup \{\bar{x}_0\}$  containing  $\bar{x}_0$  correspond to the simplices of a triangulation of  $P$ .*

For,  $\bar{X} \cup \{\bar{x}_0\}$  is equivalent to an affine transform of the set of vertices of a polytope  $Q$ , whose vertex-figure at its vertex  $z_0$  corresponding to  $\bar{x}_0$  is (projectively equivalent to)  $P$ . Moreover, the faces of  $Q$  not containing  $z_0$  are all simplices, and the result follows by radial projection of these simplices from  $z_0$  into  $P$ .

We see, in fact, that it is only necessary to place  $\bar{x}_0$ , so that all the cofacial sets containing  $\bar{x}_0$  span  $\text{lin } \bar{X}$ .

Finally, on *f*-vectors, Klee [1974] uses Gale diagrams to calculate  $f_{d-1}(P)$ , when  $P$  is a simplicial  $d$ -polytope with  $d+3$  vertices and  $\binom{d+3}{2}-1$  edges, and

Grünbaum [1970], §3.2, lists the Gale diagrams of all 4-polytopes with 7 vertices, and their f-vectors (three examples are also given by Barnette & Reay [1973]).

A polytope  $P$  is called *projectively unique* if every polytope combinatorially isomorphic to  $P$  is projectively equivalent to  $P$ . There is a corresponding condition on Gale diagrams:

**3B10**  $P$  is projectively unique if and only if every two Gale diagrams of  $P$  are equivalent.

Thus Perles (see Grünbaum [1967], Exercise 6.5.5) has classified the projectively unique  $d$ -polytopes with  $d+3$  vertices; clearly,  $d$ -polytopes with  $d+1$  or  $d+2$  vertices are always projectively unique. Perles & Shephard [1974] showed how to construct projectively unique  $d$ -polytopes with about  $3^{d/3}$  vertices, and McMullen [1976] (see also [1968], Chapter 6) investigated other inductive constructions for projectively unique polytopes.

One of the projectively unique polytopes found by Perles (see Grünbaum [1967], 5.5.4) settles a problem of Klee in the negative:

**3B11** There is an 8-polytope  $P$  with 12 vertices, such that no polytope combinatorially isomorphic to  $P$  can have all its vertices at points with rational coordinates.

Two more examples of Perles (see Grünbaum [1967], §11.1), this time of projectively unique 9-polytopes  $P$  and  $Q$ , which have facets  $F$  and  $G$  which are combinatorially isomorphic but not projectively equivalent, and so cannot be glued together along  $F$  and  $G$ , show:

**3B12** There is a cell complex which cannot be realized geometrically with convex cells in any dimension.

Concerning realizability of complexes, Shephard [1971a] has used Gale diagrams to find necessary and sufficient conditions for a spherical cell complex to be obtainable from the boundary complex of a polytope by radial projection from the centre. Kleinschmidt [1976] has shown:

**3B13** Every  $(d-1)$ -dimensional spherical cell complex with  $d+3$  vertices is isomorphic to the boundary complex of some  $d$ -polytope.

Mani [1972] solved the problem for spherical simplicial complexes using the closely related star diagrams, for which see Grünbaum [1967], §6.2, and also 6.3.4, which provides the connexion with Gale diagrams.

In contrast to 3B11, we have the following result, due to Perles (see Grünbaum [1967], Exercise 6.5.3):

**3B14** Let  $P$  be a  $d$ -polytope with at most  $d+3$  vertices, and let  $\varepsilon > 0$ . Then there is a polytope combinatorially isomorphic to  $P$ , each of whose vertices has rational coordinates, and is at distance at most  $\varepsilon$  from the corresponding vertex of  $P$ .

Concerning approximation problems, Kömhoff & Shephard [1974] have investigated the following situation. Let  $[P]$  denote the combinatorial isomor-

phism class of the polytope  $P$ , write  $[P] \rightarrow [Q]$  if the polytope  $Q$  can be approximated by polytopes in  $[P]$ , and  $[P] \rightarrow [Q]$  if every polytope in  $[Q]$  can be so approximated. Then (among other results) they proved:

- 3B15 (i) *If  $P$  and  $Q$  are  $d$ -polytopes, and each have at most  $d+3$  vertices or at most  $d+3$  facets, then  $[P] \rightarrow [Q]$  implies  $[P] \rightarrow [Q]$ .*  
(ii) *There exist 6-polytopes  $P$  and  $Q$ , each with 10 vertices, such that  $[P] \rightarrow [Q]$  does not imply  $[P] \rightarrow [Q]$ .*

2B7 and the existence of contracted standard Gale diagrams provide the proof of another result of Perles (see Grünbaum [1967], Exercise 6.5.7):

- 3B16 *If  $n \leq d+3$ , then for each  $d$ -polytope  $P$  with  $n$  vertices, there is a polytope  $P' \subseteq E^d$  combinatorially isomorphic to  $P$ , such that every combinatorial automorphism of  $P'$  is realized by an isometry of  $E^d$ .*

If the complex  $\mathcal{K}_2$  is a refinement of  $\mathcal{K}_1$ , a vertex of  $\mathcal{K}_2$  is said to be *principal* if it corresponds (in the refinement) to a vertex of  $\mathcal{K}_1$ . Lockeberg [1977] has shown:

- 3B17 *For each  $d \geq 6$ , there is a  $d$ -polytope  $P$  with  $d+3$  facets, such that some two vertices of  $P$  cannot be principal in any expression of the boundary complex of  $P$  as a refinement of that of the  $d$ -simplex.*

For  $d \geq 4$ , Lockeberg has shown (without using Gale diagrams) that there is a simple such  $d$ -polytope with  $d+4$  facets. These results are best possible.

Let  $k(d)$  be the largest number  $k$ , such that, given any  $d$ -polytope  $P$  and any  $k$  pairs  $(a_i, b_i)$  ( $i = 1, \dots, k$ ) of vertices of  $P$ , there are disjoint paths in the graph of  $P$  joining  $a_i$  and  $b_i$  for each  $i$ . Then Gallivan [1974; 1979] has shown:

- 3B18  $k(d) \leq [(2d+3)/5]$ .

We construct examples for which  $k+1$  pairs of vertices cannot be so joined, as follows. Let  $m = \frac{1}{2}(k+1)$  or  $\frac{1}{2}k+1$  as  $k$  is odd or even. Let  $\{\bar{a}_1, \dots, \bar{a}_{k+1}\}$  be a positive basis of  $E^m$  (see the next chapter for more details about these), let  $\bar{b}_i = \bar{a}_i$  ( $i = 1, \dots, k+1$ ), and let  $\bar{c}_1 = \dots = \bar{c}_p = 0$ , where  $p = k-2$ ,  $k-1$  or  $k$  if  $k$  is odd, or  $p = k-1$  or  $k$  if  $k$  is even. Then  $\bar{X} = (\bar{a}_1, \dots, \bar{a}_{k+1}, \bar{b}_1, \dots, \bar{b}_{k+1}, \bar{c}_1, \dots, \bar{c}_p)$  satisfies the Gale diagram condition 3A3, and so is the Gale diagram of some polytope  $P$  of dimension  $d = 2k + p - m + 1$  (which takes the values  $\frac{1}{2}(5k + \{-3, -1, 1\})$  if  $k$  is odd, and  $\frac{1}{2}(5k + \{-2, 0\})$  if  $k$  is even), and the corresponding pairs of vertices  $(a_i, b_i)$  ( $i = 1, \dots, k+1$ ) of  $P$  cannot be joined as required.

There is a fairly close connexion between this problem and 3B6, since it is not hard to see that, to determine  $k(d)$ , we need only look at (several fold) pyramids, whose bases are unneighbourly polytopes.

While it is more appropriate to discuss sections of polytopes by hyperplanes, and the related ideas of projections and shadow boundaries, in 5B, one recent result has been obtained by use of Gale diagrams alone. Let  $\delta(P)$  be the largest number  $k$ , such that the polytope  $P$  has a section (by some  $k$ -flat) which is a  $k$ -simplex. Marcus [1979b] shows:

- 3B19  $\delta(P)$  is the largest  $k$ , such that  $P$  has vertices  $v_0, \dots, v_k$  and facets  $F_0, \dots, F_k$ , with  $v_i \in F_j$  if and only if  $i \neq j$ .

As a somewhat surprising consequence:

3B20  $\delta(P) = \delta(P^0)$ , where  $P^0$  is dual to  $P$ .

Finally, we consider the problem of determining the largest  $n = n(d)$ , such that for every set  $X$  of  $n$  points in general position in  $E^d$ , there exists a projective mapping  $\Theta$  permissible for  $X$ , so that  $X\Theta$  is the set of vertices of a convex polytope. Larman [1972] has shown:

3B21  $n(d) \geq 2d + 1$ .

In Gale diagram terms, this is equivalent to:

3B22 Every set  $\bar{X}$  of at least  $2k+3$  points in general position in  $E^k$  admits a partition into disjoint subsets  $\bar{A}$  and  $\bar{B}$ , such that, for each  $\bar{x} \in \bar{X}$ ,  $(\bar{A} \setminus \{\bar{x}\}, \bar{B} \setminus \{\bar{x}\})$  is a Radon partition.

That  $2k+3$  is best possible is an open conjecture.

#### 4. Positive diagrams

##### 4A. Positive diagrams

Let  $X$  be any finite (ordered) set of vectors in a real linear space  $U$ , such that  $\text{pos } X = \text{lin } X$ . Then  $X$  is equivalent to an affine transform of some set  $\bar{X}$  in a space  $W^*$ , by 2B4. If  $X$  is a *positive basis* of its linear hull, that is,  $\text{pos } X = \text{lin } X$ , but  $\text{pos}(X \setminus \{x\}) \neq \text{lin } X$  for each  $x \in X$ , then any set facially isomorphic to  $\bar{X}$  is called a *positive diagram* of  $X$ .

It is often convenient to discuss the structure of  $X$  in terms of the *diagram polytope*  $\bar{P} = \text{conv } \bar{X}$ . For, the structure of  $X$  as a positive basis is its cofacial structure, and so corresponds to the facial structure of  $\bar{X}$ , and hence of  $\bar{P}$ . However, we cannot recover  $X$  directly from  $\bar{P}$ ; we need to know, for each face  $\bar{F}$  of  $\bar{P}$ , which points of  $\bar{X}$  lie in  $\bar{F}$ , or, what amounts to the same thing in the end, in  $\text{relint } \bar{F}$ . So, to each face  $\bar{F}$  of  $\bar{P}$  is attached a *label*, which is the number of points of  $\bar{X}$  in  $\text{relint } \bar{F}$ ; thus  $\bar{P}$  is a *labelled polytope*. It is clear that the isomorphism class of  $\bar{X}$  (and hence of  $X$ ) depends only on the labelling, and not on the actual position of the points of  $\bar{X}$  in the respective faces.

Now  $X$  is a positive basis if and only if no subset  $X \setminus \{x\}$  of  $X$  is cofacial. Thus we obtain the *positive diagram condition*:

4A1 A labelled polytope  $\bar{P}$  is a diagram polytope of some positive basis if and only if each of its vertices carries a label of at least 2.

Since positive diagrams and Gale diagrams are, in a certain sense, inverses of each other, most of the properties of positive diagrams follow from the results of 2B and 3A. Thus our discussion here will be fairly brief. The fact that  $X$  is a positive basis, rather than merely positively spanning its linear hull, is also not crucial in what follows.

If  $\text{pos } Y = \text{lin } Y$ , clearly  $\text{card } Y \geq \text{rank } Y + 1$ ; if we have equality, we call  $Y$  a *minimal positive basis* (for its linear hull). From 2B9 and 3A1, we thus have:

4A2  $Y \subseteq X$  is a minimal positive basis if and only if  $\bar{F} = \text{conv } \tilde{Y}$  is a facet of  $\bar{P}$ , with  $\tilde{Y} = \bar{F} \cap \bar{X}$ .

In particular, since no point of  $\bar{P}$  lies in every facet, we have:

**4A3** *Every point  $x \in X$  lies in some minimal positive basis.*

Positive diagrams were first defined by Shephard [1971b], from where the above is taken; some of the basic ideas are implicit in Davis [1954].

#### 4B. Applications

Our first applications give new proofs of classical results. If  $X$  is a subset of an affine space, and  $a \in \text{conv } X$ , we can take  $a$  as origin, so that some subset of  $X$  positively spans its linear hull. 4A3 then yields *Carathéodory's theorem*:

**4B1** *Let  $a \in \text{conv } X$ . Then  $a \in \text{conv } Y$  for some affinely independent subset  $Y$  of  $X$ . Moreover,  $Y$  can be chosen to contain any given point  $x \in X$ .*

Next, we have *Steinitz's theorem*:

**4B2** *Let  $X$  affinely span the space  $U$ , and let  $a \in \text{int conv } X$ . Then  $a \in \text{int conv } Z$  for some subset  $Z$  of  $X$ , with  $\text{card } Z \leq 2 \text{ rank } X$ .*

For, we can suppose  $X$  finite, and choose  $a$  as origin, so that  $\text{pos } X = U$ . Then  $X$  contains a positive basis  $Z$  of  $U$ . Let  $\bar{Z}$  be a positive diagram of  $Z$ , and  $\bar{Q} = \text{conv } Z$ , so that  $\bar{Q}$  is a polytope of dimension  $\dim \bar{Q} = \text{card } Z - \text{rank } Z - 1$ . Since  $\bar{Q}$  is at least a simplex, it has at least  $\dim \bar{Q} + 1$  vertices, and, by 4A1, each vertex carries a label of at least 2. Thus  $\text{card } Z \geq 2(\dim \bar{Q} + 1) = 2(\text{card } Z - \text{rank } Z)$ , and since  $\text{rank } Z = \text{rank } X$ , we have the result.

We next have a result due originally to Reay (see Shephard [1971b]):

**4B3** *Let  $X$  be a positive basis of  $E^d$  with  $n$  points. Then  $X$  admits a partition  $X = X_1 \cup \dots \cup X_{n-d}$ , such that*  
 (i)  $\text{card } X_i \geq \text{card } X_{i+1} \geq 2$  ( $i = 1, \dots, n-d-1$ ), and  
 (ii)  $\text{pos}(X_1 \cup \dots \cup X_j)$  is a linear subspace of  $E^d$  of dimension  $\text{card}(X_1 \cup \dots \cup X_j) - j$ , for  $j = 1, \dots, n-d$ .

To show this, we successively choose  $\bar{F}_{i+1}$  to be a facet of  $\bar{F}_i$  (with  $\bar{F}_0 = \bar{P}$ ), which contains as many points of  $\bar{X}$  as possible. Then  $X_1 \cup \dots \cup X_i$  is the cofacial set corresponding to  $\bar{F}_i$ .

Let  $p(n,d)$  denote the number of isomorphism classes of positive bases of  $E^d$  with  $n$  points. Enumeration of the corresponding (labelled) diagram polytopes yields:

- 4B4**
- (i)  $p(d+1,d) = 1$ ;
  - (ii)  $p(d+2,d) = [\frac{1}{4}d^2]$ ;
  - (iii)  $p(2d-1,d) = d - 1$ ;
  - (iv)  $p(2d,d) = 1$ .

It is not hard, in principle, to evaluate  $p(d+3,d)$  and  $p(2d-2,d)$  also.

Shephard [1971b] also uses diagram polytopes to find (best possible) upper and lower bounds for the number of minimal positive bases which a positive basis of  $E^d$  containing  $n$  points can have. This same paper reproves several other results of Reay, including some about minimal representations of points as non-negative linear combinations of the points of a positive basis.

All the results of this section, except 4B4 which is new, appear in Shephard [1971b].

## 5. Representation of polyhedral sets

### 5A. Representations

Let  $X$  be a fixed finite ordered set of vectors in the linear space  $U$ . In many situations (for examples of which see 5B below), we are interested in the family  $\mathcal{P}(X)$  of polyhedral sets in the dual space  $U^*$  of the form  $P = \{u^* \in U^* \mid \langle u^*, x_i \rangle \leq \eta_i \ (i=1, \dots, n)\}$ . We shall not, for the moment, impose any conditions on  $X$ . (At this point, we pause to warn the reader who may be acquainted with McMullen [1973], from which the material of this section is taken, that the notation here is chosen to harmonize with the rest of this article, and so is quite different from that of the paper.)

Often in such situations, we do not usually wish to distinguish between  $P$  and one of its translates  $P + t^*$ ,  $t^* \in U^*$ . Now  $P$  is determined by the vector  $y = (\eta_1, \dots, \eta_n)$ , which we interpret as a coordinate vector with respect to the basis  $B^* = (b_1^*, \dots, b_n^*)$  of  $V^*$ , dual to the basis  $B = (b_1, \dots, b_n)$  of  $V$ . (The notation is, of course, that of 2A.) For the moment, therefore, our attention is on the ordered family of closed half-spaces  $\{u^* \in U^* \mid \langle u^*, x_i \rangle \leq \eta_i\}$  ( $x_i \in X$ ), which determines  $P$ . The vector in  $V^*$  which corresponds to  $P + t^*$  is  $(\eta_1 + \langle t^*, x_1 \rangle, \dots, \eta_n + \langle t^*, x_n \rangle)$ , so the natural way to identify  $P$  and  $P + t^*$  is by means of a linear map  $\Psi^*$  on  $V^*$ , with kernel

$$\ker \Psi^* = \{(\langle t^*, x_1 \rangle, \dots, \langle t^*, x_n \rangle) \mid t^* \in U^*\}.$$

But, comparing with 2A, we see at once that  $\bar{X} = B^* \Psi^*$  is a linear transform of  $X$ . To  $P$  then naturally corresponds the point  $p = y \Psi^* = \sum_{i=1}^n \eta_i \bar{x}_i$ . We shall say that  $\Psi^*$  is a *representation associated with  $\mathcal{P}(X)$* , and that  $p$  is *associated with  $P$* . Note that, by definition,  $p$  is associated with every translate of  $P$ .

A number of results follow directly from the definition.

- 5A1 (i)  $P \neq \emptyset$  if and only if  $p \in \text{pos } \bar{X}$ ;
- (ii)  $\text{int } P \neq \emptyset$  if and only if  $p \in \text{relint pos } \bar{X}$ .

For,  $P \neq \emptyset$  if and only if, up to translation,  $o \in P$ , so that each  $\eta_i \geq 0$ . With  $\text{int } P \neq \emptyset$ , we have  $\eta_i > 0$ .

For  $j=1, \dots, n$ , we write  $F_j = \{u^* \in P \mid \langle u^*, x_j \rangle = \eta_j\}$ . If  $Y \subseteq X$ , we write  $F(Y) = \cap \{F_j \mid x_j \in Y\}$ . Every face  $F$  of  $P$  is the intersection of certain of the  $F_j$ ; if  $F = F(Y)$ , we call  $Y$  a *partial subset* of  $X$  for  $F$ , and if, in addition,  $F \subseteq F_j$  only if  $x_j \in Y$ , we call  $Y$  the *complete subset* of  $X$  for  $F$ . Then:

- 5A2  $Y \subseteq X$  is a complete (partial) subset for some non-empty face  $F$  of  $P$  if and only if  $p \in \text{relint pos } \tilde{Y}$  ( $p \in \text{pos } \tilde{Y}$ ).

For, we may suppose  $o \in \text{relint } F$ . If  $F \subseteq F_j$ , then  $\eta_j = 0$ ; if  $F \not\subseteq F_j$ , then  $\eta_j > 0$ . The result follows.

If  $Y$  is the complete subset for  $F$ , then  $\dim F = \dim U - \text{rank } Y$ . Hence:

- 5A3 If  $\text{lin } X = U$ , then  $\dim F = \text{exc } \tilde{Y}$ .

Supposing  $o \in \text{relint } F$ , we have  $F \subseteq (\text{lin } Y)^\perp$ . We can then use 2A9 (or, rather, its application to  $\bar{Y} \subseteq \bar{X}$ ) to show:

5A4 *With  $F$  is associated the point  $p \in \text{pos } \bar{Y}$ .*

As a particular case of 5A2,  $\{x_j\}$  is a complete subset of  $X$ , of course for the face  $F_j$ , if and only if  $p \in \text{relint pos}(\bar{X} \setminus \{\bar{x}_j\})$ . In this case,  $F_j$  must be a facet of  $P$ , which is full dimensional. We call

$$\text{ir } \bar{X} = \cap \{\text{relint pos}(\bar{X} \setminus \{\bar{x}_j\}) \mid j = 1, \dots, n\}$$

the *inner region* of  $\text{pos } \bar{X}$ ; points in  $\text{ir } \bar{X}$  are thus associated with those  $P \in \mathcal{P}(X)$  for which each  $x_j$  is irredundant (so that  $P \notin \mathcal{P}(X \setminus \{x_j\})$ ). In general, if  $P \in \mathcal{P}(X)$  is associated with the point  $p = \sum_{i=1}^n \eta_i \bar{x}_i \in \text{pos } \bar{X} \setminus \text{ir } \bar{X}$ , then some  $x_j$  is redundant for  $P$ ; for each such  $x_j$ , we may replace  $\eta_j$  by any  $\eta'_j > \eta_j$ , to obtain another point associated with  $P$ .

However, it is clear that every (non-empty)  $P \in \mathcal{P}(X)$  is associated with a unique point  $p = \sum_{i=1}^n \eta_i \bar{x}_i$  in the *closed inner region* of  $\text{pos } \bar{X}$ ,

$$\text{clir } \bar{X} = \cap \{\text{pos}(\bar{X} \setminus \{\bar{x}_j\}) \mid j = 1, \dots, n\},$$

which is  $\text{cl } \text{ir } \bar{X}$  if  $\text{ir } \bar{X} \neq \emptyset$ ; here,  $\eta_i = \sup \{\langle u^*, x_i \rangle \mid u^* \in P\}$ . The mapping  $\psi: \mathcal{P}(X) \rightarrow \text{clir } \bar{X}$ , defined by  $p = P\psi$  in this way, is called a *representation* of  $\mathcal{P}(X)$ , and  $p$  is the *representative* of  $P$ . Thus:

5A5 *Two (non-empty) polyhedral sets in  $\mathcal{P}(X)$  have the same representative in  $\text{clir } \bar{X}$  if and only if one is a translate of the other.*

We can deal with redundancy as follows (compare 2A9 again):

5A6 *If  $x_j$  is redundant for  $P$ , and we reduce  $\bar{X}$ , by taking the image of  $\bar{X} \setminus \{\bar{x}_j\}$  under a linear mapping  $\Theta$  with kernel  $\text{lin } \{\bar{x}_j\}$ , then  $p\Theta$  is associated with  $P$  in  $\mathcal{P}(X \setminus \{x_j\})$ .*

The mapping  $\psi$  is obviously continuous on  $\mathcal{P}(X)$ . Moreover, if  $p \in \text{clir } \bar{X}$ , then  $p\psi^{-1} \in \mathcal{P}_T(X)$ , the set of translation classes in  $\mathcal{P}(X)$ . Indeed,  $(\Psi^*)^{-1}$  induces a mapping  $\psi^*: \text{pos } \bar{X} \rightarrow \mathcal{P}_T(X)$ , which extends  $\psi^{-1}$ . In an obvious sense,  $\psi^{-1}$  and  $\psi^*$  are continuous.

The above results show that  $\text{clir } \bar{X}$  is partitioned into the relative interiors of convex cones (with apex  $o$ ), which are intersections of sets of the form  $\text{relint pos } \bar{Y}$ , with  $\bar{Y} \subseteq \bar{X}$ . We call these sets *type-cones*, since they correspond to the different combinatorial types of polyhedral sets in  $\mathcal{P}(X)$ . The combinatorial isomorphism is here in the *strong* sense, that corresponding faces have parallel supporting hyperplanes. The type-cone containing  $p \in \text{clir } \bar{X}$  is the intersection of all sets of the form  $\text{relint pos } \bar{Y}$  which contain  $p$ ; in fact, we can restrict the intersection to linearly independent such  $\bar{Y}$ , since the combinatorial type of  $P$  is determined by which faces  $F_j$  contain each face of  $P$ .

If  $P$  and  $Q$  are polyhedral sets, and  $\lambda, \mu \geq 0$ , then  $\lambda P + \mu Q$  is a polyhedral set. (In this context, it is appropriate to interpret  $0P$  as corresponding to the case  $\eta_i = 0$  for each  $i$ .) In general,  $P, Q \in \mathcal{P}(X)$  does not imply  $\lambda P + \mu Q \in \mathcal{P}(X)$ . However, if  $P$  and  $Q$  are of the same strong combinatorial type, then so is  $\lambda P + \mu Q$ , at least if at least one of  $\lambda$  or  $\mu$  is positive. In this case, if  $P, Q$  are represented by  $p, q$ , in the

type-cone  $\mathcal{K}$  in  $\text{clir}\bar{X}$ , then  $\lambda P + \mu Q$  is represented by  $\lambda p + \mu q$ , so the representation  $\psi$  is non-negative linear.

In an obvious sense, we can talk about the linearity of  $\psi^{-1}$ , since the non-negative linear combination  $\lambda[P] + \mu[Q]$  of the translation classes  $[P], [Q] \in \mathcal{P}_T(X)$  is well-defined as  $[\lambda P + \mu Q]$ . In view of the above discussion, it is not surprising to find:

5A7  *$\mathcal{K} \subseteq \text{clir}\bar{X}$  is a type-cone if and only if  $\mathcal{K}$  is a maximal relatively open convex subset of  $\text{clir}\bar{X}$  on which  $\psi^{-1}$  is non-negative linear.*

By continuity, if  $\mathcal{K}$  is a type-cone, then  $\psi^{-1}$  is linear on  $\text{cl}\mathcal{K}$ . We then deduce:

5A8 *Let  $\mathcal{K}$  be a type-cone in  $\text{clir}\bar{X}$ , and let  $G$  be a face of  $\text{cl}\mathcal{K}$ . Then  $\text{relint } G$  is a type-cone. The closures of the type cones in  $\text{clir}\bar{X}$  form a polyhedral complex.*

If  $X$  positively spans  $U$ , then  $\mathcal{P}(X)$  consists of polytopes, and if  $P \in \mathcal{P}(X)$  has  $\text{int } P \neq \emptyset$ , we can suppose  $o \in \text{int } P$ , and then construct a Gale diagram of the (polar) dual  $P^0$  of  $P$  from the representation of  $P$ . For, with  $P$  in its usual form, we have  $\eta_i > 0$  ( $i = 1, \dots, n$ ), and then, assuming irredundancy, the vertices of  $P^0$  are  $(\eta_1^{-1}x_1, \dots, \eta_n^{-1}x_n)$ . A linear transform of this is  $(\eta_1\bar{x}_1, \dots, \eta_n\bar{x}_n)$ , and an affine transform is the image of this under a linear mapping whose kernel is spanned by  $\sum_{i=1}^n \eta_i \bar{x}_i = p$ , by 2B11. We thus have:

5A9 *If  $\text{pos } X = U$  and  $p \in \text{irr}\bar{X}$ , then the image of  $\bar{X}$  under a linear mapping with kernel  $\text{lin } p$  is a Gale diagram of the dual  $P^0$  of  $P$ .*

The process can be reversed, and a representation of  $P$  constructed from a Gale diagram of  $P^0$ .

If  $Y$  is any subset of  $X$ , then the polyhedral set

$$Q = \{u^* \in U^* \mid \langle u^*, x_i \rangle \leq \eta_i \ (x_i \in Y)\},$$

obtained from  $P$  by deleting those of its determining half-spaces corresponding to the vectors  $x_i \in X \setminus Y$ , is associated with the point  $q = p\Theta \in \text{pos}(\bar{X} \setminus \bar{Y})$ , where  $\Theta$  is a linear mapping with kernel  $\text{lin } \bar{Y}$  (compare 5A6). In particular, if  $Y$  is the complete subset of  $X$  for some face  $F$  of  $P$ , then  $Q$  is the cone generated by  $P$  with apex any point of  $\text{relint } F$ . The structure of  $Q$  then just mirrors that of the face-figure  $P/F$ . If we take  $o$  to be an apex of  $Q$ , then  $\text{pos } Y$  is the polar cone of  $Q$ . Comparing 3A7 then yields:

5A10 *If  $Y$  is the complete subset of  $X$  for some face  $F$  of  $P$ , then a Gale diagram of the dual of the face figure  $P/F$  is obtained by reducing the image of  $\bar{X} \setminus \bar{Y}$  under a linear mapping with kernel  $\text{lin } \bar{Y}$ .*

Note that such a mapping automatically sends  $p$  to  $o$ .

If  $\mathcal{P}(X)$  consists of polytopes, then  $\text{pos } X = U$  and (by 2A13)  $o \notin \text{conv } \bar{X}$ . If we are only interested in the combinatorial properties of the  $P \in \mathcal{P}(X)$ , we can modify the picture of 5A9 a little. There is a hyperplane which supports  $\text{pos } \bar{X}$  in  $o$  alone; let  $H$  be a parallel hyperplane meeting  $\text{relint } \text{pos } \bar{X}$ , write  $\hat{x}_i, \hat{p}$  for the intersections of  $H$  with the lines joining  $o$  and  $\bar{x}_i, p$ , and let  $\hat{X} = (\hat{x}_1, \dots, \hat{x}_n)$ . Then the position of  $\hat{p}$  in  $\text{conv } \hat{X}$  gives us the combinatorial type of  $P$ . Indeed, we say  $Q$  is *homothetic* to  $P$  if  $Q = \lambda P + t^*$  for some  $\lambda > 0$  and  $t^* \in U^*$ ; then  $\hat{p}$  represents the *homothety class* of  $P$ .

Comparing 2B11 again, we see that, taking  $\hat{p}$  as origin,  $\bar{X}$  (reduced if necessary) will be a Gale diagram of  $P^0$ .

The material in this section is mostly taken from McMullen [1973], where a geometric formulation of the technique can also be found. Rather earlier, Shephard [1971c] had developed a closely related diagram technique for polyhedral sets; however, his point of view was quite different, since his diagrams were essentially Gale diagrams of certain polar sets of the polyhedral sets, and did not obviously correspond to translation classes. Similarly, Shephard [1971b] discusses those polytopes whose vertices form a positive basis of  $E^d$ , noting that their Gale diagrams can be obtained by choosing a varying origin in the relative interior of the diagram polytope. This ties up, of course, with the previous paragraph, which comes from McMullen, Schneider & Shephard [1974]. We also remark that Bair & Fourneau [1976] describe the representation technique, with a considerably different notation.

### 5B. Applications

As already mentioned, the representation technique has an advantage over that of Gale diagrams, in that it enables us to study linear systems of polytopes. For example, we say a polytope  $P$  is *decomposable* if we can write  $P = P_1 + P_2$ , where neither  $P_i$  is homothetic to  $P$ ; we call the  $P_i$  *summands* of  $P$ . Otherwise,  $P$  is *indecomposable*. Clearly,  $P \in \mathcal{P}(X)$  is indecomposable if and only if the type-cone containing its representative in  $\text{clir} \bar{X}$  has dimension 1. More generally, we have (see McMullen [1973]):

5B1 *Let the polytope  $P \in \mathcal{P}(X)$  be represented by the point  $p$  in the type-cone  $\mathcal{K}$  in  $\text{clir} \bar{X}$ . Then  $P$  is the sum of at most  $\dim \mathcal{K}$  indecomposable polytopes.*

For,  $p$  is the sum of points on at most  $\dim \mathcal{K}$  extreme rays of  $\text{cl} \mathcal{K}$ ; using 5A8, these points correspond to indecomposable polytopes.

In particular:

5B2 *A d-polytope  $P$  with  $n$  facets is a sum of at most  $n-d$  indecomposable polytopes, and  $n-d$  are needed only if  $P$  is simple.*

5B3 *Every simple polytope, except a simplex, is decomposable.*

Fourneau [1979] uses representations to impose on polytopes a metric related to decomposability.

Decomposability also plays a rôle in the following situation. If  $P$  is a d-polytope in  $E^d$  (say), we write  $N(P)$  for the set of (unit) outer normal vectors to the facets of  $P$ ; thus  $N(P) \subseteq X$  if  $P \in \mathcal{P}(X)$ . We define the following four classes of polytopes; in each case, the condition concerns a polytope  $Q$  with  $N(Q) = N(P)$ .

$P \in \mathcal{P}_M$ , and is called *monotypic*, if  $Q$  is strongly combinatorially isomorphic to  $P$ .  $P \in \mathcal{P}_S$ , and is called *strongly monotypic*, if the euclidean arrangements of facet hyperplanes of  $P$  and  $Q$  are isomorphic. (We shall discuss arrangements of hyperplanes in 7D.)

$P \in \mathcal{P}_A$  if  $Q$  is affinely equivalent to  $P$ .

$P \in \mathcal{P}_H$  if  $Q$  is homothetic to  $P$ .

Then we have (McMullen, Schneider & Shephard [1974]):

- 5B4 (i)  $P \in \mathcal{P}_M$  if and only if every non-empty intersection  $P_t = P \cap (P - t)$  is homothetic to a summand of  $P$ .  
(ii) If  $P \in \mathcal{P}_S$ , then  $P_t$  is a summand of  $P$ .  
(iii)  $P \in \mathcal{P}_A$  if and only if  $P_t$  is an affine image of  $P$ .  
(iv)  $P \in \mathcal{P}_H$  if and only if  $P_t$  is homothetic to  $P$  (or a point).

We shall only comment on 5B4 here. Let  $X = N(P)$ . Then  $P \in \mathcal{P}_M$  if and only if  $\text{ir } \bar{X}$  is a single type-cone. The converse to (ii) is, as yet, only conjectured.  $\mathcal{P}_A$  comprises the direct sums (products) of simplices.  $\mathcal{P}_H$  consists of simplices; thus  $P \in \mathcal{P}_H$  if and only if  $\text{pos } \bar{X}$  is one dimensional.

There are some relationships between these classes, and the class  $\mathcal{P}_P$  of primitive polytopes:  $P \in \mathcal{P}_P$  if  $N(P)$  forms a positive basis of  $E^d$ . This is equivalent to  $\text{ir } \bar{X} = \text{relint pos } \bar{X}$  (compare 4A1). We have:

$$5B5 \quad \mathcal{P}_H \subset \mathcal{P}_A \subset \mathcal{P}_S \subset \mathcal{P}_M; \quad \mathcal{P}_A \subset \mathcal{P}_P; \quad \mathcal{P}_A = \mathcal{P}_M \cap \mathcal{P}_P.$$

Our next applications, again from McMullen [1973], involve volume and surface area. Here we suppose  $X$  to consist of distinct unit vectors which positively span  $E^d$ . Since volume is translation invariant, we can write  $V(p) = V(P)$  for the volume of  $P \in \mathcal{P}(X)$ , where  $p \in \text{pos } \bar{X}$  is any point associated with  $P$ . (The freedom to have redundant normals is useful here.) Using our earlier notation, let  $A_j(p) = A_j(P)$  be the area of the face  $F_j$  (note that  $F_j$  may be empty). As is well known (from translation invariance)  $\sum_{i=1}^n A_i(p)x_i = 0$ , so that there is some vector  $A(p)$  with  $A_j(p) = \langle A(p), \bar{x}_j \rangle$  ( $j = 1, \dots, n$ ). If we suppose  $\bar{X} \subseteq E^{n-d}$ , we can identify  $A(p)$  with the gradient  $\nabla V(p)$ , and from the definition follows  $V(p) = d^{-1} \langle p, A(p) \rangle$ .

From the Brunn-Minkowski theorem, we obtain:

- 5B6 Let  $\alpha > 0$ . Then  $K(\alpha) = \{p \in \text{pos } \bar{X} | V(p) \geq \alpha\}$  is a smooth closed convex set, which is strictly convex at points of  $\text{ir } \bar{X}$ .

Thence it is a short step to proving Minkowski's theorem:

- 5B7 Let  $x_1, \dots, x_n$  be distinct unit vectors which positively span  $E^d$ , and let  $A_1, \dots, A_n$  be positive numbers such that  $\sum_{i=1}^n A_i x_i = 0$ . Then there is a  $d$ -polytope  $P$ , unique up to translation, whose  $n$  facets have outer normal vectors  $x_i$  and corresponding areas  $A_i$ .

For, some hyperplane with (inner) normal  $A$ , such that  $A_j = \langle A, \bar{x}_j \rangle$ , supports  $K(1)$  in a point  $q$ , which lies in  $\text{ir } \bar{X}$ , and so is unique, since all  $A_i > 0$ . An appropriate positive multiple of  $q$  then represents the required polytope  $P$ .

The proof closely follows that of Minkowski, but the use of representations emphasizes the salient features. This is also true of the proof of the Aleksandrov-Fenchel inequalities for mixed volumes. We note further that the possibility of adding area vectors, together with 5B7, gives another addition of polytopes; the Blaschke sum  $P \# Q$  of two polytopes  $P$  and  $Q$  (assumed to be  $d$ -polytopes in  $\mathcal{P}(X)$ ) is defined by  $A(P \# Q) = A(P) + A(Q)$ . An easy counterexample of a 3-polytope with 5 facets which is not a Blaschke sum of primitive 3-polytopes shows that one obvious analogue of 5B1 is false. However, we do have the

following. Let  $\mathcal{S}_r^d$  denote the class of Blaschke sums of at most  $r$   $d$ -simplices. Then:

**5B8** *A  $d$ -polytope  $Q$  with  $n$  facets can be approximated by polytopes in  $\mathcal{S}_{n-d}^d$ .*

For, let  $P$  be a  $d$ -polytope with  $n$  facets, whose normals are in linearly general position, approximating to  $Q$ . Let  $p$  represent  $P$  in  $\text{ir } \bar{X}$ , and let  $A(p)$  be its area vector. Now  $A(p)$  lies in the interior of the cone  $\bar{X}^0 = (\text{pos } \bar{X})^0$ , and is the sum of at most  $n-d$  vectors on extreme rays of  $\bar{X}^0$ . These extreme rays are outer normals to facets of  $\text{pos } \bar{X}$ , and so by 4A2 and 5B7 give area vectors of polytopes in  $\mathcal{P}(X)$  whose outer normals form a minimal positive basis, which necessarily spans  $E^d$ . Thus  $P$  is a Blaschke sum of at most  $n-d$   $d$ -simplices, and the result follows. We now come to sections and projections of polytopes. Although originally described by Shephard [1972] as a variant of Gale diagrams, the topic probably fits in better here. Let  $X = (x_1, \dots, x_n)$  as usual, let  $x_0$  be a general (non-zero) vector, and write  $Z = (x_0, x_1, \dots, x_n)$ . Then the sections of a polyhedral set  $P \in \mathcal{P}(X)$  by hyperplanes with normal  $x_0$ , and the halves into which such hyperplanes divide  $P$ , are related to the linear transform  $\bar{Z} = (\bar{z}_0, \bar{z}_1, \dots, \bar{z}_n)$  of  $Z$ . For, the image of  $\bar{z}_i$  under a linear mapping  $\Theta$  with kernel  $\text{lin } \{\bar{z}_0\}$  is  $\bar{x}_i$  ( $i = 1, \dots, n$ ). Moreover, if  $p \in \text{pos } \bar{X}$  is associated with  $P$ , then each point of the line  $p\Theta^{-1}$  is associated with one of the sections. We can thus determine the combinatorial properties of the varying sections, and of the halves of  $P$  (corresponding to normals  $x_0$  and  $-x_0$ , and so utilizing  $\bar{z}_0$  or  $-\bar{z}_0$  as appropriate).

Now, as we saw in 3A8, dual to a section of a polytope  $P$  is a projection (parallel or from a point) of its dual  $P^0$ . The inverse image of the boundary of the projection of  $P^0$  is its *shadow boundary*, and this shadow boundary is *sharp* if the projection is one-to-one on it. If  $\dim P^0 = d$ , then a sharp shadow boundary is homeomorphic to a  $(d-2)$ -sphere. If every subcomplex of the boundary complex of  $P^0$  which is homeomorphic to a  $(d-2)$ -sphere corresponds to a shadow boundary of some polytope combinatorially isomorphic to  $P^0$ , then we say that  $P^0$  has the *universal shadow boundary property*. Then Shephard [1972] has shown:

**5B9** (i) *Every  $d$ -polytope with at most  $d+2$  vertices has the universal shadow boundary property.*  
(ii) *For each  $d \geq 4$ , there is a  $d$ -polytope with  $d+3$  vertices which does not have the universal shadow boundary property.*

## 6. Central diagrams

### 6A. Central diagrams

If  $P$  is a centrally symmetric polytope, we can suppose the centre of  $P$  to lie at the origin  $o$ . More generally, let  $X = (\pm x_1, \dots, \pm x_n)$  be any centrally symmetric set in our linear space  $U$ . We shall relate the facial structure of  $X$  to properties of a central transform  $\bar{X} = (\pm \bar{x}_1, \dots, \pm \bar{x}_n)$  of  $X$ , as defined after 2A15.

We shall find it convenient to introduce here some notation which we shall use throughout the last three chapters. Write  $N = \{1, \dots, n\}$ . A *signed subset* of  $N$  is a function  $J: N \rightarrow \{0, 1, -1\}$ ; the set *underlying*  $J$  is  $J = \{i \in N | J(i) \neq 0\}$ . We write  $X(J)$

$=\{J(i)x_i|i \in J\}$ , and  $C(J)=x_J + \sum_{i \notin J} S_i$ , where  $x_J=\sum_{i \in J} J(i)x_i$  and  $S_i=\text{conv}\{-x_i, x_i\}$  ( $i=1, \dots, n$ );  $\bar{X}(J)$  and  $\bar{C}(J)$  are derived similarly from  $\bar{X}$ . In the present context, it is appropriate to call  $\bar{C}(J)$  a *cofacial set* if  $o \in \text{relint } \bar{C}(J)$ ; facial sets have their usual meaning.

Since antipodal points  $\pm x_i$  cannot both belong to a single proper face of  $\text{conv } X$ , a facial set is of the form  $X(J)$  for some signed subset  $J$  of  $N$ . If  $X(J)$  is a facial subset, let  $u^* \in U^*$  be such that the hyperplane  $\{u \in U | \langle u, u^* \rangle = 1\}$  supports  $X$ , and meets  $X$  in  $X(J)$ . (We can suppose the support parameter to be 1, by multiplying  $u^*$  by a suitable positive factor.) Then we have  $|\langle x_i, u^* \rangle| < 1$  for  $i \notin J$ . Since  $u^*$  gives rise to a linear dependence of  $\bar{X}_+ = (\bar{x}_1, \dots, \bar{x}_n)$ , we deduce at once the *central diagram relationship*:

6A1  $X(J)$  is a facial set if and only if  $\bar{C}(J)$  is a cofacial set.

The relationship 6A1 leads to an isomorphism between central transforms, analogous to that for Gale and positive diagrams, and we define a *central diagram* of a centrally symmetric polytope  $P$  to be any centrally symmetric set cofacially isomorphic to a central transform of  $\text{vert } P$ .

We easily deduce the *central diagram condition*:

6A2  $\bar{X}=(\pm \bar{x}_1, \dots, \pm \bar{x}_n)$  is a central diagram of some polytope if and only if, for each  $i=1, \dots, n$ ,  $\bar{x}_i \in \text{relint}(\sum_{j \neq i} \bar{S}_j)$ , where  $\bar{S}_j=\text{conv}\{-\bar{x}_j, \bar{x}_j\}$  ( $j=1, \dots, n$ ).

We shall construct an affine transform of  $X$  from a central transform in 6C5; however, another connexion follows from 2B8, 2B11 and the definition of central transforms.

6A3 An affine transform of the subset  $X(J)$  of  $X$  is the image of  $\bar{X}(J)$  under a linear mapping with kernel  $\text{lin}(\{\bar{x}_j\} \cup \{\bar{x}_i | i \notin J\})$ .

As a particularly interesting case:

6A4 If  $F=\text{conv } X(J)$  is a face of the centrally symmetric polytope  $P$ , with  $\text{vert } P=X$ , then the image of  $\bar{X}(J)$  under a linear mapping with kernel  $\text{lin}\{\bar{x}_i | i \notin J\}$  is a Gale diagram of  $F$ .

For, note that 6A1 ensures that  $\bar{x}_i$  is also sent to  $o$  by the mapping.

As with Gale diagrams, there is a geometric formulation of central diagrams. Let  $B=(b_1, \dots, b_n)$  be our linear basis of  $V$ , and let  $B^*=(b_1^*, \dots, b_n^*)$  be the dual basis of  $V^*$ . Write  $B_j^*=\text{conv}\{-b_j^*, b_j^*\}$  ( $j=1, \dots, n$ ). In 3A8 and 3A9, we now take  $Q$  to be the cross-polytope  $\text{conv}\{\pm b_1, \dots, \pm b_n\}$ , so that its dual is the cube  $Q^0=B_1^*+\dots+B_n^*$ . To the face  $G=\text{conv } B(J)$  (with the obvious notation) of  $Q$  corresponds the face  $\hat{G}=\sum_{i \in J} J(i)b_i^* + \sum_{i \notin J} B_i^*$ . The relationship 6A1 can be deduced immediately. Central diagrams were first formulated by McMullen & Shephard [1968], from which the above (with considerable change of notation) is taken.

## 6B. Applications

The central diagram relationship 6A1 is clearly less easy to work with than the Gale diagram or zonal diagram (see 7A1) relationships. For this reason alone, there are few applications of central diagrams. For example, in contrast to the

two other situations just mentioned, even the first non-trivial enumeration problem, that of counting the combinatorial types of centrally symmetric d-polytopes with  $2(d+1)$  vertices, remains unsolved.

Indeed, our only application in this section concerns neighbourliness. Here, we say that a centrally symmetric set  $X$  is *k-neighbourly* if every  $k$  points of  $X$  which do not contain an antipodal pair  $x_i$  and  $-x_i$  form a facial set. Let  $k(d,m)$  denote the maximal possible neighbourliness of such sets  $X$  of  $2(d+m)$  points with rank  $d$  (or, equivalently, of centrally symmetric d-polytopes with  $2(d+m)$  vertices). Ignoring the trivial case  $m=0$ , we have:

- 6B1 (i)  $k(d,1) = [\frac{1}{2}d]$ ;  
(ii)  $k(d,2) = [\frac{1}{3}(d+1)]$ .

6B2 For  $m \geq 2$ ,  $\liminf_{d \rightarrow \infty} \frac{k(d,m)}{d+m} \geq \beta(m)$ , where

$$\beta(m) = \frac{2\Gamma(\frac{1}{2}m)}{\sqrt{\pi}\Gamma(\frac{1}{2}(m-1))} \int_0^\varphi \sin^{m-2} \theta d\theta,$$

and  $\sin^{m-1} \varphi = \frac{1}{2}$ , with  $0 \leq \varphi \leq \frac{1}{2}\pi$ .

The results of 6B1 appear in McMullen & Shephard [1968]; part (i) is straightforward, but part (ii) involves grouping together the points of the central diagram in “cyclic order”, and the argument cannot be extended to higher dimensional diagrams in any obvious way. McMullen & Shephard made a general conjecture about  $k(d,m)$ , but this was refuted by Halsey [1972]. Halsey’s results were much improved by Schneider [1975], to whom 6B2 is due. It may be conjectured that the bound of 6B2 is exact. In Schneider’s paper, it is shown that  $\beta(m)$  decreases with  $m$ , and that

$$\lim_{m \rightarrow \infty} \beta(m) = 1 - \operatorname{erf}(\sqrt{\log 2}) \approx 0.2390.$$

The value  $\beta(2) = \frac{1}{3}$  agrees, naturally, with 6B1(ii); we also have  $\beta(3) = 1 - 2^{-\frac{1}{2}}$ . It may be remarked that an easy direct argument shows that  $k(d,m) = 1$  for each fixed  $d$  and all large enough  $m$ .

### 6C. Polytopes with prescribed symmetries

The topics of this section should, perhaps, occur in Chapter 3; we consider them here for historical reasons, as well as expository convenience.

We have already remarked in 2B7 that to an affine symmetry  $\Theta$  of a polytope  $P$  corresponds a linear symmetry  $\bar{\Theta}$  of its (suitably chosen) Gale diagram. If  $\Theta$  is involutory, and  $o$  is a fixed point, then  $U$  has complementary subspaces  $A$  (the *axis*) invariant under  $\Theta$ , and  $C$  (the *coaxis*) reversed by  $\Theta$ . There are similar such subspaces  $\bar{A}$  and  $\bar{C}$  for the Gale diagram. We then have:

- 6C1 If the vertices of  $P$  are  $x_1, \dots, x_r$  on  $A$ , and  $y_1, y'_1, \dots, y_s, y'_s$  paired about  $A$  (so that  $y_i - y'_i \in C$ ), then the affine transform  $(\bar{x}_1, \dots, \bar{x}_r, \bar{y}_1, \bar{y}'_1, \dots, \bar{y}_s, \bar{y}'_s)$  of vert  $P$  is such that  $(\bar{x}_1, \dots, \bar{x}_r, \bar{y}_1 + \bar{y}'_1, \dots, \bar{y}_s + \bar{y}'_s)$  in  $\bar{A}$  is an affine transform of  $(x_1, \dots, x_r, \frac{1}{2}(y_1 + y'_1), \dots, \frac{1}{2}(y_s + y'_s))$  in  $A$ , and  $(\bar{y}_1 - \bar{y}'_1, \dots, y_s - \bar{y}'_s)$  in  $\bar{C}$  is a linear

transform of  $(y_1 - y'_1, \dots, y_s - y'_s)$  in C. Further,  $\dim A + \dim \bar{A} = r + s - 1$ .

It can also be seen that  $s \geq \max(\dim A + 1 - r, d - \dim A)$ . If  $\dim A = a$  is near  $\frac{1}{2}d$ ,  $r = 0$  and  $s$  is near minimal, then the Gale diagram has small rank. This enables us to enumerate  $c^*(v,d,a)$ , where  $v$  is the number of vertices of the axisymmetric  $d$ -polytope, none on the axis  $A$  of dimension  $a$ , in case  $d$  is near  $2a + 1$ . Thus McMullen & Shephard [1970a] showed:

- 6C2 (i)  $c^*(2(a+1), 2a+1, a) = 1$ ;  
(ii)  $c^*(2(a+1), 2a, a) = a$ ;  
(iii)  $c^*(2(a+1), 2a-1, a) = \sum_{r=1}^{a+1} \frac{1}{2r} \sum_{s|r} 2^s \varphi\left(\frac{r}{s}\right) - 3a - \frac{9}{2}$   
 $+ \begin{cases} 7 \cdot 2^{(a-2)/2}, & a \text{ even,} \\ 5 \cdot 2^{(a-1)/2}, & a \text{ odd;} \end{cases}$   
(iv)  $c^*(2(a+2), 2a+2, a) = [\frac{1}{4}(a+2)^2]$ ;  
(v)  $c^*(2(a+3), 2a+3, a) = c(a+3, a) + \frac{1}{2} \binom{a+5}{5} + \frac{5}{8} \binom{a+1}{3} + \frac{1}{2} - \frac{1}{8}(a-3)[\frac{1}{2}(3a+1)]$ .

There are corresponding calculations for  $c_s^*(v,d,a)$ ; the subscript stands for “simplicial”.

- 6C3 (i)  $c_s^*(2(a+1), d, a) = 0$  if  $a > \frac{1}{2}d$ ;  
(ii)  $c_s^*(2(d-a), d, a) = \sum_{v \leq d-a} c_s(v, a)$  if  $a \leq \frac{1}{2}(d-1)$ .

3B7, and Perles's result 3B8, give the values for small values of  $d - 2a$ . The proof of 6C3(ii) does not use Gale diagrams, but, of course, the actual enumerations do.

Ewald & Voss [1973] and Ewald, Kleinschmidt & Schulz [1976] have investigated polytopes with symmetries of larger orders than 2. Let  $c^*(v,d;G,a)$  denote the number of  $d$ -polytopes, with a group of symmetries  $G$  having an  $a$ -dimensional axis  $A$  of fixed points, and  $v$  vertices, none on  $A$ . As before, the subscript  $s$  refers to simplicial polytopes. Then Ewald & Voss [1973] showed:

- 6C3 Let  $Z_p$  be the cyclic group of prime order  $p > 2$ . Then:  
(i)  $c^*(pk, (p-1)k+1; Z_p, 1) = [\frac{1}{4}k^2]$ ;  
(ii)  $c^*(p(a+1), p(a+1)-3; Z_p, a) = c(a+1, a-2) + f(a) + 1 - \delta_{p,3} + g(a)$ ,  
where  $f(a) = \begin{cases} 0, & \text{if } a = 0 \text{ or } 1, \\ \frac{1}{2} \binom{a+3}{5} + \frac{5}{8} \binom{a-1}{3} + \frac{1}{2} - \frac{1}{8}(a-5)[\frac{1}{2}(3a-5)] & \text{otherwise,} \end{cases}$   
 $g(a) = \frac{1}{6}[\frac{1}{2}(a+1)][\frac{1}{2}(a+3)][6[\frac{1}{2}a] - 2[\frac{1}{2}(a+1)] + 5]$ .  
(iii)  $c_s^*(p(a+1), p(a+1)-3; Z_p, a) = c_s(a+1, a-2) + [\frac{1}{2}a] + 1 - \delta(p, a)$ ,  
where  $\delta(p, a) = 1$  if  $a = 0$  and  $p = 3$ , and  $= 0$  otherwise.  
(iv)  $c_s^*(p(a+3), p(a+3)-3; Z_p, a) = c_s(a+3, a) + [\frac{1}{2}a] + 1$ .

Some of the enumerations above were made by Schulte-Ladbeck [1972]. Many of the expressions obtained by Ewald, Kleinschmidt & Schultz [1976] are even more complicated; their results involve, for example, groups of order  $2 \cdot p^n$ ,

with  $p$  an odd prime. On the other hand, a number of their results, involving groups  $G$  of general order  $g$ , are the same as, or near to, the results of 6C3; for example, 6C3(i) holds with any odd  $g$  instead of  $p$ . We conclude with one of their more distinctive results.

**6C4** *Let  $p, q$  be odd primes, and  $k, m$  natural numbers, with  $m < p < q$ , and let  $G$  be a group of order  $pq$ . Then*

$$c^*(pk + qm, pk + qm - 2; G, k + m - 2) = \frac{1}{8}(k^2(m+1) + m^2(k+1) + (km + k + m) - f(k, m)),$$

where  $f(k, m) = 0$ ,  $k + 2$ ,  $m + 2$  or  $k + m + 3$  as  $(k, m)$  is (even, even), (even, odd), (odd, even) or (odd, odd).

The expression 6C1 enables us to relate Gale and central diagrams. The case where  $r = 0$  and  $A = \{o\}$  shows:

**6C5** *Let  $X = (\pm x_1, \dots, \pm x_n)$  be a centrally symmetric set, and  $\bar{X} = (\pm \bar{x}_1, \dots, \pm \bar{x}_n) \subseteq W^*$  a central transform of  $X$ . Let  $(\bar{z}_1, \dots, \bar{z}_n)$  be a minimal positive basis of some other space  $W'$  with  $\sum_{i=0}^n \bar{z}_i = o$ . Then  $(\bar{z}_1 \pm \bar{x}_1, \dots, \bar{z}_n \pm \bar{x}_n) \subseteq W' \times W^*$  is an affine transform of  $X$ .*

## 7. Zonal diagrams

### 7A. Zonal diagrams

A *zonotope* is a vector sum  $Z = S_1 + \dots + S_n$  of line segments  $S_i$  ( $i = 1, \dots, n$ ), which are called *components* of  $Z$ . There is no loss of generality in supposing each  $S_i$ , and hence  $Z$ , to be centred at the origin  $o$ . Thus we shall write  $S_i = \text{conv}\{-x_i, x_i\}$  ( $i = 1, \dots, n$ ). It is often convenient, and sometimes necessary, to allow two or more of the component segments  $S_i$  to be parallel, or to let some  $S_i$  be of zero length, so that the corresponding  $x_i = o$ .

Write  $X = (\pm x_1, \dots, \pm x_n)$  for the ordered centrally symmetric set of end-points of the segments  $S_i$ , and let  $\bar{X} = (\pm \bar{x}_1, \dots, \pm \bar{x}_n)$  be a central transform of  $X$ . We shall obtain a combinatorial relationship between  $Z$  and  $\bar{X}$ .

We use the notation introduced in 6A. We shall call a set of the form  $C(J)$ , where  $J$  is a signed subset of  $N = \{1, \dots, n\}$ , a *cell*. A face  $F$  of  $Z$  is a cell  $C(J)$  for some  $J$ . For, if  $u^* \in U^*$  is an outer normal to a hyperplane which supports  $Z$  in  $F$ ,  $J$  is determined by  $\langle J(i)x_i, u^* \rangle > 0$  if  $i \in J$ , and  $\langle x_i, u^* \rangle = 0$  if  $i \notin J$ . Notice that each zero segment  $S_i$  is automatically a component of the zonotope  $C(J)$ . If  $C(J)$  is a face of  $Z$  in this way, we call  $C(J)$  a *facial set*. Cofacial sets in  $\bar{X}$  are defined as for Gale diagrams:  $\bar{X}(J)$  is *cofacial* if  $o \in \text{relint pos } \bar{X}(J)$ . From the above description of  $C(J)$ , we at once deduce the *zonal diagram relationship*:

**7A1**  *$C(J)$  is a facial set if and only if  $\bar{X}(J)$  is a cofacial set.*

This should be compared with 6A1. As with the other techniques we have discussed, 7A1 leads to an isomorphism between central transforms (different, of course, from that of 6A). If  $x_i \neq o$ , the faces of  $Z$  with  $S_i$  as a component form a *zone*, and their normal vectors lie in a hyperplane in  $U^*$ . Of course,  $Z$  will have  $n$  distinct zones provided no  $x_i$  is  $o$  or a multiple of some other  $x_j$ . If  $Z$  is a

zonotope with  $n$  zones, then any centrally symmetric set  $\bar{X}$  cofacially isomorphic to a central transform of the set  $X$  of end-points of the  $n$  components of  $Z$  is called a *zonal diagram* of  $Z$ .

It is straightforward to translate the above condition into a property of  $\bar{X}$ , to obtain the *zonal diagram condition*:

7A2  $\bar{X} = (\pm \bar{x}_1, \dots, \pm \bar{x}_n)$  is a zonal diagram of some zonotope if and only if every open linear half-space of  $\text{lin } \bar{X}$  contains at least 3 points of  $\bar{X}$ .

As with Gale diagrams, we can reduce a centrally symmetric set  $\bar{X}$  which violates the zonal diagram condition 7A2, to obtain a zonal diagram of a zonotope which is a sum of segments, some of which may be parallel or of zero length. In this context, it is useful to note that equivalent centrally symmetric sets  $\bar{X}$  and  $\bar{X}'$  (by which we mean  $\bar{x}'_i = \lambda_i \bar{x}_i \Lambda$ , for some  $\lambda_i > 0$  ( $i = 1, \dots, n$ ) and some non-singular linear mapping  $\Lambda$ ) are isomorphic, and correspond to *equivalent* zonotopes, obtained from one another by varying the lengths of the non-zero components, and applying non-singular linear mappings.

We say the zonotope  $Z$  is a *prism* with *upright*  $S_j$ , if the  $S_i$  with  $i \neq j$  are all parallel to some hyperplane in  $\text{lin } X$ . We then have the analogue of 3A4:

7A3  $Z$  is a prism with upright  $S_j$  if and only if  $\bar{x}_j = 0$ .

Analogous to 3A7, and proved by the same method, is:

7A4 If  $C(J)$  is a facial set of  $Z$ , then the reduced diagram of  $\bar{X}(J)$  is a Gale diagram of the face-figure  $Z/C(J)$ .

As a consequence, we have:

7A5 Every polytope  $P$  is a vertex-figure of some zonotope.

For, let  $\bar{X}$  be an affine transform of  $\text{vert } P$ , and let  $Z$  be the zonotope whose zonal diagram is  $\bar{X} \cup (-\bar{X})$ .

Of course, we obtain a zonal diagram of a face of  $Z$  by applying 2A9. We note that, since  $\dim C(J) = \text{rank } \{x_i | i \notin J\}$ , we have:

7A6  $\dim C(J) = \dim Z - \text{exc } \bar{X}(J)$ .

As with the other diagram techniques, we have a geometric approach. In 3A8 and 3A9, we now take  $Q$  to be the cube  $B_1 + \dots + B_n$ , where  $B_i = \text{conv} \{ -b_i, b_i \}$  ( $i = 1, \dots, n$ ), so that  $Q^0$  is the polar cross-polytope  $\text{conv} \{ \pm b_1^*, \dots, \pm b_n^* \}$ . The faces  $G$  of  $Q$  and  $\hat{G}$  of  $Q^0$  are then exactly as those for central diagrams in 6A, with the rôles of the  $b_i$  and  $b_i^*$  reversed. As before, the zonal diagram relationship follows at once.

Most of this section is taken from McMullen [1971b], which should be consulted for further details; the notation used here is, however, quite different. The paper Shephard [1971b] also contains much relevant material; the structure of a zonotope around one of its faces is obviously closely connected with that of the convex cone generated by the zonotope, with a relatively interior point of the face as apex.

### 7B. Associated zonotopes

Let  $Z = S_1 + \dots + S_n$ ,  $S_i = \text{conv}\{-x_i, x_i\}$  ( $i = 1, \dots, n$ ), be a zonotope, and let  $\bar{X} = (\pm \bar{x}_1, \dots, \pm \bar{x}_n)$  be isomorphic to a central transform of  $X = (\pm x_1, \dots, \pm x_n)$  (and so a zonal diagram of  $Z$  if  $Z$  has  $n$  zones). Let us write  $\bar{S}_i = \text{conv}\{-\bar{x}_i, \bar{x}_i\}$  ( $i = 1, \dots, n$ ). Then the zonotope  $\bar{Z} = \bar{S}_1 + \dots + \bar{S}_n$  is said to be *associated with Z*. (In McMullen [1971b], where this concept was introduced, the term “derived from” was used. The present term is due to Shephard [1974a].) Note that the definition of  $\bar{Z}$  depends upon the particular expression of  $Z$  as a sum of line segments. By symmetry,  $Z$  is associated with  $\bar{Z}$ . The definition implies that the association is combinatorial in nature, and this we shall now verify. The relationship, in fact, follows from:

7B1 *Let  $J$  be a signed subset of  $N$ , with  $J = N$ . Let  $J_1, J_2$  be the restrictions of  $J$  to complementary subsets of  $N$ . Then  $C(J_1)$  is facial, with  $x_j \in \text{relint } C(J_1)$  if and only if  $\bar{C}(J_2)$  is facial, with  $\bar{x}_j \in \text{relint } \bar{C}(J_2)$ .*

In the above statement, we mean  $J_2 = N \setminus J_1$ , with  $J_i$  agreeing with  $J$  on  $J_i$  ( $i = 1, 2$ ). To prove this result, we need only remark that the condition  $x_j \in \text{relint } C(J_1)$  is equivalent to  $0 \in \text{relint pos } X(J_2)$ .

We now partition the points  $x_j$ , with  $J = N$ , into three classes. We say  $x_j$  is a *vertex* if some hyperplane supports  $Z$  in  $x_j$  alone (so that no  $x_i = 0$ ), an *interior point* if  $x_j \in \text{relint } Z$ , and otherwise a *boundary point*. Then 7B1 implies:

7B2  *$x_j$  is a vertex (boundary point, interior point) of  $Z$  if and only if  $\bar{x}_j$  is an interior point (boundary point, vertex) of  $\bar{Z}$ .*

The combinatorial relationship now follows from the observation that the facets of  $Z$  are the cells  $C(I)$ , which are maximal with respect to inclusion, and are such that every point  $x_j$  with  $J$  agreeing with  $I$  on  $I$  is a vertex or boundary point. Let  $Z^*(n, d)$  denote the class of  $d$ -zonotopes with  $n$  zones, with the additional property that no facet contains translates of all but one or two of the component line segments; this condition is just that of 7A2, applied to  $X$  rather than  $\bar{X}$ . Let  $z^*(n, d)$  denote the number of combinatorial types of zonotopes in  $Z^*(n, d)$ . Then 7B2 and the subsequent remark yield:

7B3 *There is a one-to-one correspondence between the combinatorial types of zonotopes in  $Z^*(n, d)$  and in  $Z^*(n, n - d)$ . In particular,  $z^*(n, d) = z^*(n, n - d)$ .*

A special case is worth mentioning. If  $Z$  is *cubical*, that is, if the facets of  $Z$  are all cubes, then  $X_+$  is in linearly general position; the same is then true of  $\bar{X}_+$ , so that  $\bar{Z}$  is also a cubical zonotope. Denoting by  $z_c(n, d)$  the number of combinatorial types of cubical  $d$ -zonotopes with  $n$  zones, we then have:

7B4 *For  $n - 2 \geq d \geq 2$ ,  $z_c(n, d) = z_c(n, n - d)$ .*

Associated zonotopes have an obvious geometric formulation. For, using the notation of 6A and 7A,  $Z$  is the image of the  $n$ -cube  $B_1 + \dots + B_n$  under  $\Phi$ , while  $\bar{Z}$  is the image of  $B_1^* + \dots + B_n^*$  under  $\Psi^*$ . If we take  $V$  to be Euclidean space  $E^n$ , and identify  $V^*$  with  $V$ , then we can regard  $Z$  and  $\bar{Z}$  as images of the same cube under orthogonal projection on to orthogonal complementary subspaces.

### 7C. Applications

The zonal diagram relationship is, if anything, easier to use than the Gale diagram relationship. So, we can solve certain enumeration problems, where the diagrams have small dimension. We denote by  $z(n,d)$  the number of  $d$ -zonotopes with  $n$  zones;  $z_c(n,d)$  is as in 7B. Then we have (McMullen [1971b]):

- 7C1 (i) For  $d \geq 1$ ,  $z(d,d) = 1 = z_c(d,d)$ .  
(ii) For  $d \geq 2$ ,  $z(d+1,d) = d-1$ ;  $z_c(d+1,d) = 1$ .  
(iii) For  $d \geq 2$ ,  $z_c(d+2,d) = 1$ ;

$$z(d+2,d) = \sum_{r=1}^{d+2} \frac{1}{2r} \left\{ \sum_{s|r} 2^s \varphi\left(\frac{r}{s}\right) \right\} - 5d - \frac{13}{2} + \begin{cases} 5.2^{d/2}, & d \text{ even,} \\ 7.2^{(d-1)/2}, & d \text{ odd.} \end{cases}$$

For part (iii), we note that, as with Gale diagrams, we have *standard* zonal diagrams, with points on the unit sphere or at 0. Further, if rank  $\bar{X} = 2$ , we can assume the diameters of the circle carrying points of  $\bar{X}$  to be evenly spaced. From a result of Hansen on arrangements of hyperplanes (see 7D), it can be shown that every  $d$ -zonotope has faces which are  $j$ -cubes, for each  $j \leq [\frac{1}{2}(d+1)]$ . For  $n \geq [3d/2]$  zones, this is best possible, but an application of Steinitz's theorem 5B2 to the zonal diagram yields the following, which is an improvement for smaller  $n$  (McMullen [1971b]).

- 7C2 A  $d$ -zonotope with  $n$  zones has  $j$ -cube faces for each  $j \leq 2d-n$ .

The remainder of our applications (in this section) concern associated zonotopes. Shephard [1974a] has found a number of interesting combinatorial properties. First, we can extend 7B2, as follows. Let  $I$  be a signed subset of  $N$ , and let  $J$  run over those signed subsets with  $J=N$  which agree with  $I$  on  $\underline{I}$ . We call  $C(I)$  a *vertex cell* if at least one  $x_i$  is a vertex, but none are interior points of  $Z$ , an *interior cell* if at least one  $x_i$  is an interior point, but none are vertices, and otherwise a *boundary cell*. Then:

- 7C3  $C(I)$  is a vertex (boundary, interior) cell if and only if  $\bar{C}(I)$  is an interior (boundary, vertex) cell.

More interesting, however, is the following connexion between cells of  $Z$  and  $\bar{Z}$ . If  $C=C(J)$  is a cell of  $Z$ , write  $[C] = \{C(I)|\underline{I}=J\}$ . Also, let  $K$  be any signed subset with  $K=N\setminus J$ , and write  $\bar{C}=\bar{C}(K)$ . We define the deficiency and excess of  $C$  by  $\text{def}C=\text{def}X(K)$ ,  $\text{exc}C=\text{exc}X(K)$ , and extend the definition to  $\text{def}[C]$  and  $\text{exc}[C]$  in the obvious way. From 2A10, we have at once:

- 7C4  $\text{def}[C] = \text{exc}[\bar{C}]$ ;  $\text{exc}[C] = \text{def}[\bar{C}]$ .

An easy counting argument also gives:

$$7C5 \quad \sum_{[C]} (-1)^{\text{def}[C]+\text{exc}[C]} = 0.$$

Denote by  $v(Z)$ ,  $b(Z)$  and  $i(Z)$  the number of points  $x_i$  which are vertices, boundary points or interior points of  $Z$ , respectively. Then:

$$7C6 \quad v(Z) = \sum_{[C]} (-1)^{\text{exc}[C]}, \quad b(Z) = 2 \sum_{[C]} (1 - (-1)^{\text{def}[C]+\text{exc}[C]}), \\ i(Z) = \sum_{[C]} (-1)^{\text{def}[C]}.$$

Let  $\alpha_{ij}$  be the number of families  $[C]$  with  $\text{def}[C]=i$  and  $\text{exc}[C]=j$ . The matrix  $A(Z)=(\alpha_{ij})$  is called the *deficiency-excess matrix* of  $Z$ . We immediately have:

$$7C7 \quad A(\bar{Z}) = A(Z)^T, \text{ where } {}^T \text{ denotes transpose.}$$

7C4 and 7C5 can clearly be rephrased in terms of  $A(Z)$ . Shephard [1974a] also finds other relationships involving  $A(Z)$ .

A *cubical dissection* of a  $d$ -zonotope is a dissection of  $Z$  into  $d$ -cubes, each of which is a cell; the *order* of the dissection is the number of such cubes. Our final result from Shephard [1974a] is:

$$7C8 \quad \text{Every cubical dissection of a zonotope } Z \text{ has order } \alpha_{00}. \text{ Thus every cubical dissection of the associated zonotope } \bar{Z} \text{ has the same order.}$$

Finally, let  $Z$  be a  $d$ -zonotope in  $E^d$  with  $n$  zones; we may take its associated zonotope  $\bar{Z}$  to lie in  $E^{n-d}$ . We say  $Z$  *tiles*  $E^d$  if there is a family  $T$  of translation vectors in  $E^d$ , such that the translates  $Z+t$  ( $t \in T$ ) cover  $E^d$ , with the intersection of two different translates being empty or a face of each. Shephard [1974b] formulated a number of conditions on  $d$ -zonotopes, which he showed to be equivalent for  $d \leq 4$ , and conjectured to be equivalent generally. This conjecture was established by McMullen [1975]. We list some of the more striking conditions here.

7C9 *The following conditions on our  $d$ -zonotope  $Z$  are equivalent.*

- (i)  $Z$  tiles  $E^d$ .
- (ii) Every linear  $(d-2)$ -space spanned by points of  $X$  lies in two or three such spanned hyperplanes.
- (iii) There is some set  $X'$  equivalent to  $X$ , such that for each spanned hyperplane  $H$  of  $X'$ ,  $X'$  is contained in  $H$  and two hyperplanes parallel to  $H$ .
- (iv) Some zonotope  $Z'$  equivalent to  $Z$  is the Voronoi polytope of some lattice.
- (v)  $\bar{Z}$  tiles  $E^{n-d}$ .

Of course, from our present point of view, the equivalence of (i) and (v) is the most interesting.

#### 7D. Arrangements of hyperplanes

A  $k$ -*arrangement* is a finite set of hyperplanes of  $k$ -dimensional projective space  $P^k$ , which we shall suppose do not contain a common point. The components of the complement of the hyperplanes are the interiors of convex  $k$ -polytopes. These polytopes, together with their faces, are the *faces* of the arrangement. Two arrangements are *combinatorially isomorphic* if there is a one-to-one inclusion preserving correspondence between their faces.

There is a one-to-two correspondence between points of  $P^k$  and pairs of antipodal points of the  $k$ -sphere  $S^k$  in  $E^{k+1}$ , so an arrangement corresponds to a set of hyperplanes through the origin of  $E^{k+1}$  (which cut out a set of great spheres of  $S^k$ ), and so to the zonotope which is the vector sum of the diameters of  $S^k$  orthogonal to these hyperplanes. In fact, the dual of the zonotope is isomorphic to the complex cut out on  $S^k$  by the hyperplanes.

All this means that there is a natural combinatorial correspondence between

arrangements of hyperplanes in  $P^k$  and  $(k+1)$ -zonotopes, and so results on zonotopes can be transferred to give results on *associated* arrangements. For example, as a consequence of 7B4, we have:

- 7D1 *There is a one-to-one correspondence between the combinatorial types of simple k-arrangements of n hyperplanes and simple  $(n-k-2)$ -arrangements of n hyperplanes.*

Here, a k-arrangement is *simple* if no point lies in more than k hyperplanes. Canham [1971], Chapters 4–7, applies 7B2 (and its consequence 7B3) to investigate the combinatorial relationship between associated arrangements, and, in particular, between their faces. Halsey [1972] generalizes this association to one between simple arrangements of pseudohyperplanes; we shall return to this topic in 8A. Shannon [1979] finds a correspondence between simplicial faces of associated arrangements, which are *excellent* in the sense that no point lies in all but one or two of the hyperplanes; this condition is again equivalent to 7A2, applied to the set X. Shannon's result, which also follows from Theorem 7 of Canham [1971], Chapter 4, is:

- 7D2 *Let  $\mathcal{H}$  be an excellent arrangement of n hyperplanes in  $P^k$ , and let  $\tilde{\mathcal{H}}$  be the associated arrangement in  $P^{n-k-2}$ . Then there is a one-to-one correspondence between the simplicial k-faces of  $\mathcal{H}$  and the simplicial  $(n-k-2)$ -faces of  $\tilde{\mathcal{H}}$ . Such a k-face of  $\mathcal{H}$  has a facet in the hyperplane  $H_i$  if and only if the corresponding  $(n-k-2)$ -face of  $\tilde{\mathcal{H}}$  does not have a facet in the hyperplane  $\tilde{H}_i$  corresponding to  $H_i$ .*

There is a close connexion between euclidean arrangements of hyperplanes, and the representation theory of Chapter 5. For, we were there representing polyhedral sets, which are intersections of closed half-spaces of the form  $H_j^- = \{u^* \in E^d \mid \langle u^*, x_j \rangle \leq \eta_j\}$ . If we replace each half-space  $H_j^-$ , for  $j$  in some subset J of  $\{1, \dots, n\}$ , by the opposite half-space  $H_j^+$ , this is equivalent to replacing  $x_j$  by  $-x_j$  and  $\eta_j$  by  $-\eta_j$  for  $j \in J$ . The corresponding polyhedral set in  $\mathcal{P}(X_J)$  (say) is still associated with the point p associated with the original polyhedral set, since  $(-\eta_j)(-\bar{x}_j) = \eta_j \bar{x}_j$ . We thus see:

- 7D3 *The position of p in  $\text{lin} \bar{X}$  relative to  $\bar{X} \cup (-\bar{X})$  determines the combinatorial type of the euclidean arrangement determined by the facet hyperplanes of the associated polyhedral set P.*

Of course,  $-p$  gives the same arrangement (with  $p \in \text{relint pos } \bar{Y}$  replaced by  $-p \in \text{relint pos}(-\bar{Y})$ , for  $\bar{Y} \subseteq \bar{X} \cup (-\bar{X})$ ). It is then easy to see:

- 7D4 *Let  $\mathcal{H} = (H_0, H_1, \dots, H_n)$  be a projective arrangement of  $n+1$  hyperplanes, and let  $(\pm \bar{x}_0, \pm \bar{x}_1, \dots, \pm \bar{x}_n)$  be a zonal diagram of the corresponding zonotope. Then  $\bar{x}_0$  represents in  $\text{lin}\{\pm \bar{x}_1, \dots, \pm \bar{x}_n\}$  the euclidean arrangement of hyperplanes obtained from  $\mathcal{H}$  by setting  $H_0$  as the hyperplane at infinity. Conversely, every euclidean arrangement embeds in a projective arrangement in this way.*

7D1 comes from McMullen [1971b], 7D3 is implicit in McMullen, Schneider & Shephard [1974], while 7D4 has not been explicitly noted before.

## 8. Conclusion

### 8A. Oriented matroids

Just as matroids provide the abstract framework in which to study properties of linear and affine dependence, so oriented matroids provide an appropriate abstract context for diagram theory.

There are three different, but equivalent ways of defining matroids (see Whitney [1935]); in terms of their independent (and dependent) subsets, in terms of their bases, which are maximal independent subsets, and in terms of their circuits, which are minimal dependent subsets. In defining oriented matroids, in which the order properties of the underlying field are also to be reflected, it is the third concept which it is appropriate to generalize.

We shall discuss matroids on the set  $N = \{1, \dots, n\}$ . As in 6A, a *signed subset* of  $N$  is a function  $J: N \rightarrow \{0, 1, -1\}$ ; the *set underlying*  $J$  is denoted  $\underline{J} = \{i \in N | J(i) \neq 0\}$ , and the *positive* and *negative subsets* of  $J$  are  $J^+ = \{i \in N | J(i) = 1\}$  and  $J^- = \{i \in N | J(i) = -1\}$ . The signed subset  $-J$  defined by  $(-J)(i) = -J(i)$  ( $i \in N$ ) is the *opposite* of  $J$ , so that  $(-J)^+ = J^-$ ,  $(-J)^- = J^+$ . If  $I = J$  or  $I = -J$ , we write  $I = \pm J$ . If  $J = \emptyset$ , we write  $J = \emptyset$ .

An *oriented matroid* on  $N$  is a collection  $\mathcal{C}$  of signed subsets of  $N$ , called *(signed) circuits*, such that:

- (i) for all  $J \in \mathcal{C}$ ,  $J \neq \emptyset$  and  $-J \in \mathcal{C}$ ; and for all  $I, J \in \mathcal{C}$ , if  $I \subseteq \underline{J}$  then  $I = \pm J$ ; and
- (ii) for all  $I, J \in \mathcal{C}$ ,  $i \in (I^+ \cap J^-) \cup (I^- \cap J^+)$  and  $j \in (I^+ \setminus J^-) \cup (I^- \setminus J^+)$ , there exists  $K \in \mathcal{C}$  such that  $K^+ \subseteq (I^+ \cup J^+) \setminus \{i\}$ ,  $K^- \subseteq (I^- \cup J^-) \setminus \{j\}$ , and  $j \in K$ .

The collection  $\mathcal{C}$  of underlying sets  $J$  corresponding to the signed circuits  $J$  of an oriented matroid  $(N, \mathcal{C})$  form the circuits of an ordinary matroid.

The appropriate connexion with ordered point-sets in a real linear space is:

- 8A1 Let  $X = (x_1, \dots, x_n)$  be an ordered set, and for a signed subset  $J$  of  $N$ , write  $X(J) = \{J(i)x_i | i \in \underline{J}\}$ . Then those  $J$  corresponding to minimal positive bases  $X(J)$  in  $\text{lin } X$  form the signed circuits of an oriented matroid on  $N$ .

The condition (i) is easy to check. For (ii), observe that the set  $(X(I) \cup X(J)) \setminus \{-x_i, x_i\}$  will positively span its linear hull, and so contains a minimal positive basis, which by 4A3 can be chosen to contain any given point  $I(j)x_j$ .

An oriented matroid arising from such an ordered set (in a linear space over some ordered field) is said to be *coordinatizable*.

If  $M = (N, \mathcal{C})$  is a matroid with circuits  $\mathcal{C}$ , there is a *dual* matroid  $\bar{M} = (N, \bar{\mathcal{C}})$ ; the duality is most easily described in terms of bases, since a base in  $\bar{M}$  is just the complement (in  $N$ ) of a base in  $M$ . But one structure can be derived from the other, since a circuit is a minimal subset contained in no base, while a base is a maximal subset containing no circuit.

Let  $M = (N, \mathcal{C})$  be a matroid. If  $(N, \mathcal{O})$  is an oriented matroid with  $\mathcal{O} = \mathcal{C}$ , then  $\mathcal{O}$  is called an *orientation* of  $M$ . Then we have:

- 8A2 Let  $M, \bar{M}$  be dual matroids on  $N$ . Then  $\mathcal{O}$  is an orientation of  $M$  if and only if there is an orientation  $\bar{\mathcal{O}}$  of  $\bar{M}$ , such that for all  $I \in \mathcal{O}$  and  $J \in \bar{\mathcal{O}}$ , if  $I \cap J \neq \emptyset$ , then  $(I^+ \cap J^+) \cup (I^- \cap J^-) \neq \emptyset$  and  $(I^+ \cap J^-) \cup (I^- \cap J^+) \neq \emptyset$ . Moreover, such an orientation  $\bar{\mathcal{O}}$  is uniquely determined by  $\mathcal{O}$ .

We shall use 8A2 as a definition of the *dual* oriented matroid. The connexion with coordinatization is given by:

8A3 *If the oriented matroid  $(N, \mathcal{O})$  corresponds to the ordered set  $X$ , then the dual oriented matroid  $(\bar{N}, \bar{\mathcal{O}})$  corresponds to a linear transform  $\bar{X}$  of  $X$ . Thus,  $(N, \mathcal{O})$  is coordinatizable if and only if  $(\bar{N}, \bar{\mathcal{O}})$  is coordinatizable.*

For,  $X(I)$  is cofacial, so  $\bar{X}(I)$  lies strictly to one side of a hyperplane  $H$  in  $\text{lin } \bar{X}$  containing (and actually spanned by) the set  $\{\bar{x}_i \in \bar{X} | i \in N \setminus I\}$ . Since  $I \cap J \neq \emptyset$  (so  $J \not\subseteq N \setminus I$ ), and  $\bar{X}(J)$  is cofacial, we see that  $\bar{X}(J) \cap \bar{X}(I) \neq \emptyset$  and  $\bar{X}(J) \cap \bar{X}(-I) \neq \emptyset$ . This is just the required conclusion.

It would be inappropriate for us to pursue this topic much further here, as the theory of oriented matroids seems, so far at least, not to have contributed anything strikingly new to diagram theory. The above description of oriented matroids is adapted from Bland & Las Vergnas [1978]. Oriented matroids were implicitly introduced by Rockafellar [1969], and formalized, independently, by Bland [1974], Las Vergnas [1975] and Lawrence [1975]. Other references the interested reader may wish to consult are Bland [1977] (concerning an abstraction of linear programming, including a rule which specializes to the usual simplex algorithm, with a different pivot rule), Las Vergnas [1977; 1978; 1979], Zaslavsky [1975] and, particularly, Folkman & Lawrence [1978].

However, there is one class of non-coordinatizable oriented matroids related to a topic we have previously mentioned. An *r-pseudoflat* in projective space  $P^k$  corresponds to a centrally symmetric (but not necessarily great) *r-sphere* in  $S^k$ . A simple *k-pseudoarrangement* is a finite collection of pseudohyperplanes in  $P^k$ , such that, for each  $m = 1, \dots, k+1$ , every  $m$  of them meet in a  $(k-m)$ -pseudoflat. As mentioned in 7D, Halsey [1972] develops a combinatorial association between simple pseudoarrangements of  $n$  hyperplanes in  $P^k$  and  $P^{n-k-2}$ , which extends that for ordinary arrangements.

Halsey also forms a correspondence between a simple *k-pseudoarrangement*  $\mathcal{H}$  of  $n$  hyperplanes and a centrally symmetric *k-sphere*  $\mathcal{S}$  composed of *k-faces* of an  $n$ -cube  $C^n$ . In case of ordinary arrangements, these spheres project onto the boundaries of the corresponding  $(k+1)$ -zonotopes. There is a unique  $(n-k-2)$ -sphere  $\bar{\mathcal{S}}$  in  $C^n$ , such that each vertex of  $C^n$  belongs to just one of  $\mathcal{S}$  or  $\bar{\mathcal{S}}$ .  $\bar{\mathcal{S}}$  then corresponds to the associated  $(n-k-2)$ -pseudoarrangement.

If we now take the vertices of  $C^n$  to be all  $(\varepsilon_1, \dots, \varepsilon_n)$  with  $\varepsilon_i = \pm 1$  ( $i = 1, \dots, n$ ), each *k-cube* in  $\mathcal{S}$  uniquely corresponds to its centroid, which in turn gives a signed subset of  $N$ . To  $\mathcal{S}$  then corresponds an oriented matroid  $M$  (which would actually be associated with the zonal diagram of the corresponding zonotope, were the arrangement ordinary), and to  $\bar{\mathcal{S}}$  corresponds its dual matroid  $\bar{M}$ .

Now if  $\mathcal{S}$  corresponds to a pseudoarrangement which is not realizable as an arrangement of ordinary hyperplanes, then  $M$  is non-coordinatizable, and conversely. Since such arrangements exist (and Halsey gives examples), we have further examples of non-coordinatizable oriented matroids to add to the list of Bland & Las Vergnas [1978].

## 8B. Afterword

We have tried here to present a complete survey of the theory of diagrams, and of their various applications. It is almost certain that there are inadvertent

omissions; it is hoped that the reader will treat these with indulgence. A number of colleagues, too many to mention by name, have helped fill in what would otherwise have been gaps, by drawing the author's attention to work he had overlooked; the author's grateful thanks go to them. However, the final responsibility for the presentation here remains, of course, the author's own.

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## Neuere Entwicklungen in der kombinatorischen Konvexgeometrie

### I. Einführung

#### 1. Allgemeines

Das geometrische Studium dreidimensionaler konvexer Polytope hat eine gute und reiche Tradition, vor allem im 19. und beginnenden 20. Jahrhundert. Ihr Höhepunkt kann in dem Satz von Steinitz erblickt werden, der diejenigen planaren Graphen bzw. zweidimensionalen Komplexe kennzeichnet, die zu Kantengraphen von dreidimensionalen Polytopen isomorph sind. Höherdimensionale Komplexe gewannen parallel hierzu immer mehr Bedeutung in der Entwicklung der Topologie, vor allem derjenigen der zwanziger und dreißiger Jahre. Hier waren konvexe Polytope Bausteine für die Approximation von Mannigfaltigkeiten, also eher Hilfsmittel als Objekte eigener Untersuchungen. Erst in den beiden letzten Jahrzehnten hat man entdeckt, daß die kombinatorische Theorie höherdimensionaler konvexer Polytope eine eigene Welt interessanter und tiefgründiger Probleme verbirgt. Wesentliche Anstöße hierzu kamen von Anwendungsfragen, die inzwischen als Theorie der linearen Optimierung große Bedeutung gewonnen haben. Merkwürdigerweise stößt man gerade im Umkreis der numerischen Methode des Simplexalgorithmus auf die Frage der rein kombinatorischen Struktur von Randkomplexen konvexer Polytope. Auch an die Tradition der Graphentheorie wird hierbei mit angeknüpft.

So hat sich sehr stürmisch ein umfassendes Gebiet entwickelt, zunächst vorwiegend in den USA, in England und Israel, neuerdings mehr und mehr auch in Deutschland und in der Schweiz. B. Grünbaum hat mit seinem 1967 erschienenen Buch «Convex Polytopes» nicht nur die erste Phase dieser Entwicklung systematisch dargestellt, sondern mit zahlreichen Ideen und Problemen eine nächste Phase stimuliert [43]. Bereits 1970 hat Grünbaum in einem Beitrag «Polytopes, graphs, and complexes» [45] mehr als hundert neue Titel referiert.

Im folgenden werden wir uns vorwiegend auf nach 1970 erschienene Arbeiten konzentrieren und Schwerpunkte setzen. Die meisten Arbeiten in der kombinatorischen Theorie der Polytope (d.h. immer der konvexen Polytope) lassen sich einem der folgenden Teilgebiete zuordnen:

- a) Abzählung von Klassen kombinatorisch äquivalenter Polytope.
- b) Extremalprobleme bezüglich der Seitenzahl von Polytopen.
- c) Charakterisierung der f-Vektoren von Polytopen (Seitenzahlen als Komponenten).

d) Innere Kennzeichnung der Randkomplexe von Polytopen (Steinitz-Problem).

e) Struktur der Randkomplexe von Polytopen, insbesondere eingebettete Graphen und Mannigfaltigkeiten.

Wir beschränken uns hier auf die Punkte d) und e). Zu a) erscheint ein eigener Beitrag von McMullen («Representations and Diagrams»). Es ist zu hoffen, daß vor allem die eindrucksvollen Fortschritte hinsichtlich b), aber auch c) an anderer Stelle ausgiebig referiert werden.

Hat auch die kombinatorische Konvexgeometrie sich zu einer eigenständigen Disziplin entwickelt, so ist nicht ausgeschlossen, daß sie auf andere Gebiete stärker als bisher angenommen ausstrahlen wird, und zwar nicht nur auf Fragen der linearen Optimierung. So erscheint eines der aus der älteren Topologie überhängenden ungelösten Probleme, das sogenannte Schönlies-Problem, durch folgende, von U. Pachner (s. u. III. 1.) angegebene Charakterisierung in neuem Licht: Die Schönlies-Vermutung (d.h. jede  $(d-1)$ -Sphäre in einer  $d$ -Sphäre ist unverknotet) ist genau dann richtig, wenn in den einfachen Polytopen, die man aus dem 5-Simplex durch sukzessives «Abschneiden» von Seiten erhält, jede drei dimensionale Subsphäre im 3-Skelett unverknotet ist. Neuartige Beziehungen zur Algebra hat Stanley [90] geknüpft, indem er die Lösung der sogenannten «upper bound conjecture» (durch McMullen) mit Hilfe der Theorie der Macauley-Ringe auf beliebige kombinatorische Sphären ausgedehnt hat. Umgekehrt hat Kleinschmidt [63] eine Deutung gewisser Ring-Invarianten von Macauley-Ringen mit Hilfe geometrischer Zellkomplexe gefunden. In [55] werden diese Untersuchungen von Kind und Kleinschmidt fortgesetzt. (Hier nicht näher behandelt.)

Es sei noch angemerkt, daß einige der im folgenden referierten Arbeiten infolge ihres großen Umfangs nicht publiziert ([24], [76], [92]) und somit hier dokumentiert werden sollen.

## 2. Begriffe und Bezeichnungen

Im Zusammenhang mit Polytopen übernehmen wir die Terminologie von Grünbaum [43]. Dabei finden folgende Begriffe Verwendung:  $d$ -Polytop, Ecke, Kante, Facette, Seite,  $d$ -Simplex ( $T^d$ ), simpliziales Polytop, einfaches Polytop, zyklisches Polytop,  $\mathcal{B}(P)$  (Randkomplex des Polytops  $P$ ),  $\text{conv } M$  (Konvexe Hülle der Menge  $M$ ),  $k$ -Skelett, duales Polytop. Einige weitere allgemeine Begriffe seien hier zusammengestellt; speziellere Definitionen werden in den einzelnen Abschnitten angegeben.

Wir gehen von folgendem Komplexbegriff aus:

Eine endliche Menge  $\mathcal{C}$  von Polytopen, die sämtlich in einem euklidischen Raum  $E^d$  liegen, heißt ein (eingebetteter, geometrischer) *Komplex*, wenn folgendes gilt:

(a) Ist  $A \in \mathcal{C}$  und ist  $B$  Seite von  $A$ , dann gilt  $B \in \mathcal{C}$ .

(b) Aus  $A \in \mathcal{C}$  und  $B \in \mathcal{C}$  folgt, daß  $A \cap B$  Seite von  $A$  und  $B$  ist.

Die Elemente von  $\mathcal{C}$  heißen *Zellen*. Sind alle Zellen von  $\mathcal{C}$  Simplizes, so heißt  $\mathcal{C}$  ein *Simplizialkomplex*. Ein  $n$ -Komplex ist ein Komplex, dessen Zellen maximaler Dimension  $n$ -Polytope sind. Ein *Graph* ist ein 1-Komplex. Der *Graph eines Polytopes* ist dessen Kantengraph, also dessen 1-Skelett.

Im Zusammenhang mit Komplexen treten folgende Begriffe auf: Teil- oder Subkomplex,  $st(C, \mathcal{C})$  («Stern» von  $C$  in  $\mathcal{C}$ ),  $ast(C, \mathcal{C})$  («Antistern» von  $C$  in  $\mathcal{C}$ ),  $link(C, \mathcal{C})$  («Verkettungskomplex» von  $C$  in  $\mathcal{C}$ ),  $|\mathcal{C}|$  («Trägermenge» oder «Körper» von  $\mathcal{C}$ ),  $\mathcal{C} \cdot \mathcal{D}$  («Verbindungskomplex» der Komplexe  $\mathcal{C}$  und  $\mathcal{D}$ , vgl. [40]).

Zwei Komplexe  $\mathcal{C}$  und  $\mathcal{C}'$  heißen *kombinatorisch äquivalent* ( $\mathcal{C} \approx \mathcal{C}'$ ) oder *isomorph*, wenn es eine bijektive Abbildung  $\varphi$  von  $\mathcal{C}$  auf  $\mathcal{C}'$  gibt, die inklusionserhaltend ist, d.h.: Aus  $C$  Seite von  $D$  in  $\mathcal{C}$  folgt  $\varphi(C)$  Seite von  $\varphi(D)$  in  $\mathcal{C}'$ . Gilt  $\mathcal{C} = \mathcal{B}(P)$  und  $\mathcal{C}' = \mathcal{B}(P')$  für zwei Polytope  $P, P'$ , dann nennen wir im Falle  $\mathcal{C} \approx \mathcal{C}'$  auch  $P$  und  $P'$  *kombinatorisch äquivalent*.

Sei  $\mathcal{C}$  ein Simplizialkomplex. Den Übergang von  $\mathcal{C}$  zu  $\sigma(\mathcal{C}) := [\mathcal{C} : st(C, \mathcal{C})] \cup [\{c\} : \mathcal{B}(C) : link(C, \mathcal{C})]$  bezeichnen wir als *stellare Unterteilung in der Zelle C*, den Übergang von  $\sigma(\mathcal{C})$  zu  $\mathcal{C}$  als *inverse stellare Unterteilung*. Eine *stellare Operation* ist eine stellare Unterteilung oder eine inverse stellare Unterteilung oder ein Isomorphismus. Dabei ist  $c$  ein innerer Punkt der Zelle  $C$ . Gibt es eine Kette vonstellaren Operationen, die den Komplex  $\mathcal{C}$  in den Komplex  $\mathcal{D}$  überführen, so heißen  $\mathcal{C}$  und  $\mathcal{D}$  *stellar äquivalent*. Die stellare Äquivalenz von Komplexen ist bekannterweise äquivalent zu der stückweise-linearen Homöomorphie ihrer Trägermengen (vgl. etwa [40]) im Sinne der stückweise-linearen Topologie.

Ist ein Komplex  $\mathcal{C}$  stellar äquivalent zum  $d$ -Simplex oder dessen Randkomplex, so heißt  $\mathcal{C}$  eine (stückweise-lineare)  *$d$ -Kugel* bzw.  *$(d-1)$ -Sphäre*. Gilt für einen simplizialen  $d$ -Komplex  $\mathcal{C}$  und jede  $0$ -Zelle  $v$  von  $\mathcal{C}$ , daß  $link(v, \mathcal{C})$  eine  $(d-1)$ -Kugel oder  $(d-1)$ -Sphäre ist, so heißt  $\mathcal{C}$  eine (stückweise-lineare)  *$d$ -Mannigfaltigkeit*. Sind speziell alle Verkettungskomplexe von Ecken  $(d-1)$ -Sphären, so ist  $\mathcal{C}$  eine *geschlossene  $d$ -Mannigfaltigkeit*. Ein simplizialer  $d$ -Komplex  $\mathcal{C}$  heißt eine *(geschlossene)  $d$ -Pseudomannigfaltigkeit*, falls gilt:

- (a) Jede  $(d-1)$ -Zelle von  $\mathcal{C}$  liegt in genau 2  $d$ -Zellen.
- (b)  $\mathcal{C}$  ist stark zusammenhängend, d.h. zu je 2  $d$ -Zellen  $C$  und  $D$  von  $\mathcal{C}$  existiert eine Folge  $C =: C_1, \dots, C_n := D$  von  $d$ -Zellen, derart, daß je 2 aufeinanderfolgende  $d$ -Zellen eine  $(d-1)$ -Zelle gemein haben.

Die obigen Bedingungen sind unabhängig von einer speziellen Triangulierung der Polyeder. Häufig reden wir deshalb im folgenden auch von  $d$ -Sphären, Kugeln, Mannigfaltigkeiten und Pseudomannigfaltigkeiten, wenn wir keinen speziellen Komplex, sondern ein Polyeder vorliegen haben, das eine Triangulierung besitzt, die obigen Eigenschaften genügt. Damit können wir etwa auch von nichtsimplicialen Mannigfaltigkeiten reden. Es wird sich stets aus dem Zusammenhang ergeben, welchen Mannigfaltigkeitsbegriff wir verwenden.

## II. Die Frage der inneren Kennzeichnung von Randkomplexen konvexer Polytope (Steinitz-Problem)

### 1. 3-polytopale Sphären und allgemeine Problemstellung

Steinitz [91] gab eine vollständige kombinatorische Charakterisierung derjenigen Zellkomplexe an, die zu Randkomplexen von 3-Polytopen isomorph sind. Für höherdimensionale Zellkomplexe ist eine entsprechende Kennzeichnung noch nicht bekannt. Die Lösung dieses Problems — Steinitz-

Problem genannt — stößt auf erhebliche Schwierigkeiten. Nur wenige Spezialfälle sind befriedigend gelöst. Wir stellen diese in den Abschnitten 2 und 3 dar. In Abschnitt 4 skizzieren wir einen Algorithmus zur Entscheidung darüber, ob eine gegebene kombinatorische Sphäre polytopal ist. In Abschnitt 5 betrachten wir Schälbarkeitsfragen, die neben ihrer Bedeutung für das Steinitz-Problem auch von eigenem Interesse sind.

Ein  $(d-1)$ -Komplex heißt *polytopal*, wenn er isomorph zum Randkomplex eines  $d$ -Polytops ist. Für  $d=2$  sind die polytopalen Komplexe genau die kombinatorischen 1-Sphären, also Kreisgraphen.

Ein Graph heißt *n-zusammenhängend*, wenn je zwei seiner Ecken durch  $n$  verschiedene Kantenwege verbunden werden können, die paarweise nur ihre beiden Endpunkte gemeinsam haben.

Ein altes Ergebnis besagt, daß der Kantengraph eines  $d$ -Polytops stets  $d$ -zusammenhängend ist (vgl. [43]). Das Steinitzsche Ergebnis besagt:

- (1) Ein Graph  $G$  ist genau dann zum Kantengraph eines 3-Polytops isomorph, wenn er planar und 3-zusammenhängend ist.

Kombinatorische Probleme für 3-Polytope können also mit Hilfe von (1) auf graphentheoretische Probleme zurückgeführt werden. Steinitz formulierte (1) in der Terminologie von 2-Komplexen und nannte seinen Satz den «Fundamentalsatz konvexer Bereiche». Wie weit der Satz wirklich reicht, ist erst durch Grünbaums Buch [43] deutlich geworden, in dem eine Fülle von Anwendungen und verwandte Sätze angegeben werden. Zwei Beispiele zu letzterem:

- (2) Ein Graph  $G$  ist genau dann isomorph zum Kantengraph eines zentral-symmetrischen 3-Polytops, wenn  $G$  planar und 3-zusammenhängend ist und wenn es eine Involution  $\varphi$  von  $G$  gibt derart, daß jede Ecke  $v$  von  $G$  von  $\varphi(v)$  durch einen Kreis getrennt wird.
- (3) Ein Graph  $G$  ist genau dann isomorph zum Kantengraph eines 3-Polytops mit Symmetrieebene, wenn  $G$  planar und 3-zusammenhängend ist und wenn es eine Involution von  $G$  gibt, die die Orientierung der Seiten umkehrt.

Die Beweise von (1) bis (3) sind in [43] ausführlich beschrieben. Sie benutzten verschiedene Typen von «Reduktionen» planarer Graphen. Dabei werden Kanten und Ecken von  $G$  geeignet entfernt bzw. hinzugefügt derart, daß der neue Graph planar und 3-zusammenhängend bleibt, wodurch ein Induktionsbeweis über die Kantenzahl von  $G$  möglich wird.

Eine andere Beweismethode benutzt den Prozeß des «facet-splitting», bei dem gezeigt wird, daß  $G$  aus dem Kantengraphen des Tetraeders hervorgeht, indem 2-Seiten durch Hinzufügen neuer Kanten in mehrere 2-Seiten aufgespalten werden [91].

Weitere interessante Korollare von (1) ergeben sich aufgrund der folgenden Definition:

Eine endliche Familie  $\mathcal{D} = \{D_0\} \cup \mathcal{C}$  von Polytopen im  $E^d$  heißt ein  $d$ -Diagramm, wenn gilt:

- (a)  $\mathcal{C}$  ist ein Komplex.

- (b)  $D_0$  ist ein d-Polytop mit  $D_0 = |\mathcal{C}|$  und jede echte Seite von  $D_0$  ist eine Zelle von  $\mathcal{C}$ .
- (c)  $C \cap \partial D_0 \in \mathcal{C}$  sofern  $C \in \mathcal{C}$ .

$D_0$  heißt *Basis* von  $\mathcal{D}$ . Beispiele von d-Diagrammen liefern die *Schlegel-Diagramme* von  $(d+1)$ -Polytopen:

Sei  $P$  ein  $(d+1)$ -Polytop mit der Facette  $D_0$  und  $x_0 \notin P$  ein Punkt, so daß von allen affinen Hüllen von Facetten von  $P$  nur aff  $D_0$   $x_0$  und  $P$  trennt. Sei  $\mathcal{C} := \mathcal{B}(P) \setminus \{D_0\}$ . Projiziert man  $\mathcal{C}$  auf  $D_0$  vermöge Strahlen, die von  $x_0$  ausgehen, so erhält man einen Komplex  $\mathcal{C}'$ .

Die Familie  $S := \{D_0\} \cup \mathcal{C}'$  heißt dann Schlegel-Diagramm von  $P$  (vgl. Bild 1).

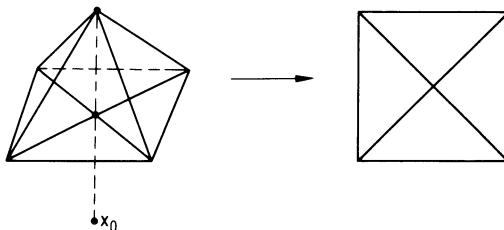


Bild 1

Aus (1) folgt dann sofort:

- (4) Jedes 2-Diagramm ist isomorph zum Schlegel-Diagramm eines 3-Polytops [43].

Außerdem folgt aus (1):

- (5) Jede 2-Sphäre ist polytopal. [43]

Da die Beweismethoden zu (1) auch im rationalen 3-dimensionalen Raum übernommen werden können, folgt noch:

- (6) Jede 2-Sphäre ist isomorph zum Randkomplex eines rationalen 3-Polytops. [43]

Weitere Folgerungen konnten mit Hilfe der Beweismethode zu (1) in [10] und [14] gewonnen werden:

- (7) Sei  $P$  ein 3-Polytop mit dem n-Eck  $F$  als Facette und  $F'$  ein beliebig gewähltes zu  $F$  isomorphes Polytop. Dann gilt: Es gibt ein zu  $P$  kombinatorisch äquivalentes Polytop  $P'$  mit der Facette  $F'$ , so daß  $F'$  das Bild von  $F$  unter dem Isomorphismus von  $P$  und  $P'$  ist.
- (8) Seien  $P$  ein 3-Polytop und  $S$  eine 1-Sphäre in  $\mathcal{B}(P)$ . Dann ist  $S$  eine *Schattengrenze* von  $P$ , d.h. es gibt ein zu  $P$  kombinatorisch äquivalentes Polytop  $P'$  und einen Punkt  $x \notin P'$  derart, daß  $S'$  (das Bild von  $S$  unter dem Isomorphismus von  $P$  und  $P'$ ) der Rand des Teils von  $\mathcal{B}(P')$  ist, der von  $x$  aus "sichtbar" ist.

Es stellt sich nun die Frage, ob auch für  $d > 3$  der Satz von Steinitz und die übrigen genannten Ergebnisse Analoga besitzen. Da schon der Beweis von (1)

einen erheblichen Aufwand erfordert, ist eine vollständige und genügend einfache Lösung des Steinitz-Problems kaum zu erwarten. Dies wird auch dadurch gestützt, daß sich fast alle direkten Analoga zu (1)–(8) als falsch erwiesen haben. Insbesondere wurde von Brückner [27] in Analogie zu (4) stillschweigend angenommen, daß jedes d-Diagramm isomorph zu einem Schlegel-Diagramm eines  $(d+1)$ -Polytops sei. Zur Bestimmung der Anzahl kombinatorischer Klassen von einfachen 4-Polytopen mit 8 Facetten zählte Brückner verschiedene Typen von 3-Diagrammen ab. In [49] und [43] wurde jedoch nachgewiesen, daß eines der dort betrachteten 3-Diagramme keinem Schlegel-Diagramm entsprechen kann; dual formuliert:

- (9) Es gibt ein (simpliziales) 3-Diagramm mit 8 Ecken, das nicht isomorph zum Schlegel-Diagramm eines 4-Polytops ist.

Weitere Beispiele solcher Diagramme (auch nichtsimpliziale) werden in [9], [21] und [60] angegeben. Nach diesen Beispielen gibt es kein Analogon zu (6), d.h. nicht jede kombinatorische 3-Sphäre ist polytopal.

## 2. *Invertierbare und dualisierbare Sphären*

Die obigen 3-Diagramme haben die Eigenschaft, nicht *invertierbar* zu sein, d.h. man kann nicht jede ihrer 3-Zellen als Basis verwenden. Schlegel-Diagramme sind natürlich invertierbar.

Die in [43] geäußerte Vermutung, daß invertierbare Diagramme stets isomorph zu einem Schlegel-Diagramm sind, wurde jedoch in [14] widerlegt:

- (10) Es gibt ein 3-Diagramm mit 14 Ecken, das invertierbar, aber nicht isomorph zu einem Schlegel-Diagramm ist.

Ein polytopaler Komplex  $\mathcal{C}$  ist stets *dualisierbar*, d.h. es gibt einen (geometrischen) Komplex  $\mathcal{C}'$  und eine bijektive Abbildung  $\varphi$  von  $\mathcal{C}$  auf  $\mathcal{C}'$ , die inklusionsumkehrend ist. In [19] und [60] wird gezeigt, daß die obigen 3-Komplexe sämtlich nicht dualisierbar sind, und es ist unbekannt, ob es dualisierbare nicht-polytopale Komplexe gibt. Die Dualisierbarkeit scheint von besonderer Bedeutung für das Steinitz-Problem zu sein, denn es gilt [43]:

- (11) Ein simplizialer Komplex ist genau dann polytopal, wenn er dualisierbar ist.

Dieses und die oben genannten Ergebnisse aus [19] und [60] legen die Vermutung nahe, daß (11) auch für nicht-simpliziale Komplexe gilt. Die Dualisierbarkeit ist natürlich am konkreten Beispiel im allgemeinen sehr schwer nachprüfbar. Ein brauchbares Kriterium für Dualisierbarkeit wird jedoch in [97] nachgewiesen:

Ein  $(d-1)$  Komplex heißt einfach, wenn jede seiner  $k$ -Zellen,  $0 \leq k \leq d-1$ , in genau  $d-k$   $(d-1)$ -Zellen liegt.

- (12) Ein einfacher abstrakter  $(d-1)$ -Komplex  $\mathcal{C}$  ist genau dann geometrisch im  $E^d$  realisierbar, wenn jede seiner 3-Zellen separat so realisierbar ist, daß die 2-Zellen von verschiedenen 3-Zellen, die in  $\mathcal{C}$  einander entsprechen, paarweise affin äquivalent sind.

Die Dualisierbarkeit ist damit auf ein 3-dimensionales Problem zurückgeführt.

*Problem:* Gibt es (nicht-simpliziale) dualisierbare Komplexe, die nicht-polytopal sind? Gibt es im Falle einer positiven Antwort, dualisierbare nicht-polytopale Komplexe, die isomorph zu einem invertierbaren Diagramm sind?

Eine dem Steinitz-Problem verwandte Fragestellung wird in [87] behandelt: Sei  $P$  ein  $d$ -Polytop im  $E^d$  und  $0 \in \text{int } P$ . Die radiale Projektion von  $\mathcal{B}(P)$  auf die Einheitssphäre  $S^{d-1}$  des  $E^d$  liefert dann einen «sphärischen» Komplex auf  $S^{d-1}$ . Die Frage, wann umgekehrt ein Komplex  $\mathcal{C}$  auf  $S^{d-1}$ , dessen Zellen sphärische Polytope sind, d.h. endliche Durchschnitte von Hemisphären auf  $S^{d-1}$ , radiale Projektion eines Polytops ist, wird in [87] beantwortet, indem analog zu der in [43] beschriebenen Methode der Gale-Diagramme der Eckenmenge  $\{v_1, \dots, v_s\}$  von  $\mathcal{C}$  ein Gale-Diagramm  $\{\bar{v}_1, \dots, \bar{v}_s\}$  zugeordnet wird. Ist  $\mathcal{Z}_i$  das relative Innere der konvexen Hülle der Koseite (vgl. [43]) einer  $(d-1)$ -Zelle von  $\mathcal{C}$  und  $\mathcal{Z}$  die Menge aller  $\mathcal{Z}_i$ , so gilt:

- (13) Ein sphärischer Komplex  $\mathcal{C}$  ist genau dann die radiale Projektion des Randkomplexes eines konvexen Polytops  $P$ , wenn gilt:  
 $\cap \mathcal{Z} \neq \emptyset$ .

### 3. Sphären mit wenigen Ecken

Da es gemäß (9) 3-Sphären mit 8 Ecken gibt, die nicht polytopal sind, und da der Randkomplex eines Polytops stets eine Sphäre ist, stellt sich die Frage, für welche Eckenzahlen Sphären noch stets polytopal sind. Dies wurde von Mani [64] für simpliziale Sphären vollständig beantwortet:

- (14) Besitzt eine simpliziale  $(d-1)$ -Sphäre höchstens  $d+3$  Ecken, so ist sie polytopal. Für jedes  $d > 3$  und jedes  $v > d+3$  gibt es simpliziale  $(d-1)$ -Sphären, die nicht polytopal sind.

Es wird im Beweis von (14) gezeigt, daß für eine  $(d-1)$ -Sphäre  $S$  mit  $d+3$  Ecken der Antistern  $\text{ast}(v, S)$  einer Ecke  $v$  isomorph in den Randkomplex eines  $d$ -Polytops  $P'$  mit  $d+2$  Ecken einbettbar ist. Mit Hilfe der Stern-Diagramm-Technik (vgl. [43]) folgt, daß der Rand dieses Komplexes eine Schattengrenze von  $P'$  darstellt, woraus sich ergibt, daß der Komplex  $[\{x\} \cdot \partial \text{ast}(v, S)] \cup \text{ast}(v, S)$  polytopal und isomorph zu  $S$  ist.

Die negative Aussage benutzt eine induktive Konstruktion, basierend auf dem von Barnette [9] gefundenen nicht-polytopalen 3-Diagramm mit 8 Ecken. In [58] wurde (12) auf nicht-simpliziale Komplexe erweitert:

- (15) Besitzt eine  $(d-1)$ -Sphäre  $S$  höchstens  $d+3$  Ecken, so ist sie polytopal.

Wie im Beweis von (14) wird dabei gezeigt, daß der Antistern  $\text{ast}(v, S)$  einer Ecke von  $S$  isomorph in den Randkomplex eines  $d$ -Polytops  $P_1$  mit  $d+2$  Ecken einbettbar ist. Für  $d$ -Polytope mit  $d+2$  Ecken hat Shephard in [88] bewiesen, daß jede  $(d-2)$ -Sphäre in ihrem Randkomplex eine Schattengrenze ist, woraus folgt, daß jede  $(d-2)$ -Sphäre in einem solchen Polytop zwei  $(d-1)$ -

Kugeln  $Q_1$  und  $Q_2$  berandet, derart daß für geeignete Punkte  $x$  und  $y$  die Sphären  $[\{x\} \cdot \partial Q_1] \cup Q_1$  und  $[\{y\} \cdot \partial Q_2] \cup Q_2$  polytopal sind. Die Tatsache, daß Shephards Satz für Polytope mit  $d+3$  Ecken für  $d > 3$  falsch ist, bildet ein wesentliches Hindernis bei der Lösung des Steinitz-Problems. Für 3-Polytope gibt es diese Schwierigkeit wegen (8) nicht. Wendet man Shephards Satz über Schattengrenzen auf  $P_1$  an und berücksichtigt man die Einbettbarkeit von  $\text{ast}(v, S)$  in  $\mathcal{B}(P_1)$ , so ergibt sich, daß die Sphäre  $[\{x\} \cdot \partial \text{ast}(v, S)] \cup \text{ast}(v, S)$  polytopal ist. Zum Beweis von (15) ist also bereits ein Polytop  $P_2$  konstruiert, mit  $\mathcal{B}(P_2) \approx [\{x\} \cdot \partial(\text{ast } v, S)] \cup \text{ast}(v, S)$ , d.h. in  $P_2$  besitzt eine Ecke  $v$  den gleichen Antistern wie die Ecke  $v$  in  $S$ . Für nichtsimpliciales  $S$  gilt aber noch nicht  $\mathcal{B}(P_2) \approx S$ , denn  $\text{st}(v', \mathcal{B}(P_2))$  und  $\text{st}(v, S)$  sind dann nicht isomorph. Der Stern von  $v'$  in  $\mathcal{B}(P_2)$  ist aber eine Unterteilung von  $\text{st}(v, S)$ . Zellen von  $\text{st}(v, S)$  treten also in  $\text{st}(v', \mathcal{B}(P_2))$  eventuell als «geknickte» Zellen auf. Zum Beweis von (13) wird  $\mathcal{B}(P_2)$  so abgeändert, daß schrittweise Zellen von  $\text{st}(v, S)$  im Stern von  $v'$  realisiert werden. Dabei werden die «Knicke», die diese Zellen in  $\text{st}(v', \mathcal{B}(P_2))$  noch aufweisen, entfernt. Dies geschieht durch Transformationen des *Gale-Diagrammes* von  $P_2$ , an dem die konvexe Realisierbarkeit leicht abzulesen ist (vgl. [43]). Dadurch wird schrittweise ein Polytop  $P$  mit  $\mathcal{B}(P) \approx S$  konstruiert. Aufgrund der Ergebnisse in [70] kann leicht der zu (15) duale Satz hergeleitet werden:

- (16) Sei  $S$  eine  $(d-1)$ -Sphäre mit höchstens  $d+3$   $(d-1)$ -Zellen, dann ist  $S$  polytopal.

Die Annahme, daß  $S$  eine (stückweise-lineare) Sphäre ist, kann noch erheblich abgeschwächt werden. Nach (16) genügt es anzunehmen, daß  $S$  eine Homologie-Sphäre ist. Dies kann jedoch auch direkt mit den Methoden des Beweises in [58] hergeleitet werden.

Für zentrale symmetrische Polytope mit «wenigen» Ecken werden in [59] Analoga zu (2) und (12) bewiesen, die Vermutungen von Grünbaum in [44] positiv beantworten.

Eine  $(d-1)$ -Sphäre  $S$  heißt in (kombinatorischer) Analogie zu Randkomplexen zentrale symmetrischer Polytope *symmetrisch*, wenn es eine inklusionserhaltende bijektive Involution  $\varphi$  von  $S$  gibt mit:

$$\varphi(F) \cap F = \emptyset \text{ für jede Zelle } F \text{ von } S.$$

Dann gilt:

- (17) Jede symmetrische  $(d-1)$ -Sphäre mit  $2d$  Ecken ist polytopal.  
 (18) Für jedes  $d \geq 4$  und jedes gerade  $v \geq 2d+2$  gibt es symmetrische  $(d-1)$ -Sphären mit  $v$  Ecken, die nicht polytopal sind.

Es ist hier überraschend, daß schon für sämtliche Eckenzahlen von symmetrischen Sphären, die die Eckenzahl des einfachsten zentrale symmetrischen Polytops, des Kreuzpolytops, übersteigen, die konvexe Realisierbarkeit nicht mehr gewährleistet ist.

*Problem:* Gibt es Analoga zu (17) und (18) für Sphären mit anderen Symmetrien? Welches ist die kleinste Eckenzahl von nichtpolytopalen Sphären, die eine (abstrakte) Hyperebene als Symmetriechse besitzen?

Ein enger Zusammenhang zwischen dem Steinitz-Problem und höherdimensionalen Analoga von (7) ergibt sich aus den Ergebnissen in [57]:

Die Form einer Facette  $F$  eines Polytops  $P$  heißt *beliebig vorschreibbar*, sofern es zu jedem Polytop  $F' \approx F$  ein Polytop  $P' \approx P$  gibt, so daß  $F'$  das Bild von  $F$  unter dem Isomorphismus zwischen  $P$  und  $P'$  ist. In Beantwortung von Vermutungen aus [43] und [17] wird in [57] bewiesen:

- (19) Sei  $P$  ein  $d$ -Polytop mit  $d+3$  Ecken. Dann ist die Form einer jeden Facette von  $P$  beliebig vorschreibbar. Es gibt jedoch ein 4-Polytop  $Q$  mit 8 Ecken und einer Facette, deren Form nicht vorgeschrieben werden kann.

Wir beschreiben die Facetten von  $Q$  durch die Angabe ihrer Ecken in Tabelle 1.

Tabelle 1

235678	1248
1237	2348
1347	1568
1467	1458
4567	1268
1267	3458
3457	1234
1456	

Nach [57] kann die Form des Oktaeders 235678 in  $Q$  nicht vorgeschrieben werden. Die 3 Diagonalen dieses Oktaeders dürfen sich nämlich nicht in einem Punkt schneiden.

Durch Bildung von  $k$ -fachen Pyramiden über  $Q$  ergibt sich, daß (19) bestmöglich ist.

Unterteilt man das Oktaeder ohne zusätzliche Ecken geeignet in 4 Tetraeder, so ist die davon induzierte Unterteilung von  $\mathcal{B}(Q)$  isomorph zu der nicht-polytopalen Sphäre von Barnette in [9].

*Problem:*

- Gilt die folgende Aussage: Ist die Form der Facette  $F$  eines Polytops  $P$  nicht beliebig vorschreibbar, so gibt es eine Unterteilung ohne neue Ecken von  $F$ , derart daß die induzierte Unterteilung von  $\mathcal{B}(P)$  nicht-polytopal ist?
- Ist  $Q$  aus (19) projektiv eindeutig, d.h. ist jedes  $Q' \approx Q$  projektiv äquivalent zu  $Q$ ?
- Ist  $Q$  in  $\mathbb{Q}^4$  konvex realisierbar?

#### 4. Komplexe mit speziellen kombinatorischen Eigenschaften

Barnette bewies in [13], daß unter allen  $d$ -Polytopen mit fest gegebener Eckenanzahl die  $d$ -Stapelpolytope die kleinste Anzahl von  $j$ -dimensionalen Seiten haben,  $0 \leq j \leq d-1$ . Ein  $d$ -Polytop  $P$  heißt ein *d-Stapelpolytop*, wenn es eine Folge  $T^d = P_1, \dots, P_n = P$  von  $d$ -Polytopen gibt, derart daß  $\mathcal{B}(P_{i+1})$  isomorph zu einer stellaren Unterteilung von  $\mathcal{B}(P_i)$  in einer Facette von  $P_i$  ist,  $i = 1, \dots, n-1$ . Das  $d$ -Simplex  $T^d$  sei ebenfalls ein Stapelpolytop.

Für diese Klasse von Polytopen wird in [56] eine vollständige graphentheoretische Charakterisierung angegeben, die eine Zurückführung sämtlicher kombinatorischer Probleme bei Stapelpolytopen auf graphentheoretische Methoden erlaubt. Die wichtigste der charakterisierenden Eigenschaften des Kantengraphen  $G$  eines  $d$ -Stapelpolytops lautet:

- (20) Zu je zwei nicht benachbarten Ecken  $x$  und  $y$  von  $G$  existiert ein vollständiger Teilgraph von  $G$  mit  $d$  Ecken, der  $x$  und  $y$  trennt.

Aus (20) kann man eine Folgerung herleiten, die für die Problematik in [89] von Teilstapelpolytopen von Stapelpolytopen von Bedeutung ist:

- (21) Der Graph eines 3-Stapelpolytops  $Q$  enthält keinen Teilgraphen, der eine Verfeinerung des Kantengraphen eines Oktaeders  $\mathcal{O}$  ist.

Gäbe es eine solche Verfeinerung, so wären nach (20) die antipodalen Ecken des Kantengraphen von  $\mathcal{O}$  paarweise durch eine Kante verbunden. Der Graph von  $Q$  enthielte dann eine Verfeinerung eines vollständigen Graphen mit 6 Ecken, was für 3-Polytope nicht möglich ist.

Eine weitere Klasse von simplizialen  $d$ -Polytopen wird in [75] kombinatorisch gekennzeichnet. Ein Polytop  $P$  mit  $\mathcal{B}(P) \approx \mathcal{B}(T_0) \cup \dots \cup \mathcal{B}(T_k)$ , wobei die  $T_i$  Simplizes sind, heißt *Simplexoid*. Anders als die Stapelpolytope sind die Simplexeide nicht durch ihren Graphen eindeutig bestimmt. Zur Charakterisierung unter den simplizialen geschlossenen  $(d-1)$ -Pseudomannigfaltigkeiten  $\mathcal{C}$  wird der Graph  $\mathcal{E}(\mathcal{C})$  hinzugezogen, der als Kanten genau die Verbindungen der jeweils zwei Ecken des Verkettungskomplexes der  $(d-2)$ -Zellen von  $\mathcal{C}$  hat.

- (22) Eine simpliziale geschlossene  $(d-1)$ -Pseudomannigfaltigkeit mit  $d+r$  Ecken ist genau dann isomorph zum Randkomplex eines  $d$ -Simplexoids, wenn  $\mathcal{E}(\mathcal{C})$  disjunkte Vereinigung vollständiger Graphen ist.

In [75] wird ferner gezeigt, wie man die zu den Simplexoiden dualen Polytope durch spezielle Unterraumschnitte eines Simplexes konstruieren kann, und diese Unterraumschnitte werden durch Eckenpartitionen des Simplex rein graphentheoretisch gekennzeichnet. Diese Betrachtungen sind auch insofern von Interesse, als sie mögliche Ansatzpunkte für einen Algorithmus zur Konstruktion aller einfachen Polytope liefern. Bei den Unterraumschnitten des Simplex wird nämlich ein Schnittbegriff verwendet, der dual genau den scharfen Schattengrenzen entspricht und daher genau die einfachen Polytope als Schnitte liefert (vgl. auch Abschnitt II.5).

## 5. Algorithmische Lösung des Steinitz-Problems

Angesichts der großen Schwierigkeiten, notwendige und hinreichende «innere», rein kombinatorische Eigenschaften dafür zu finden, daß ein Komplex polytopal ist, stellte Grünbaum [43] die Frage, ob es überhaupt einen Algorithmus gibt, mit dem man von einer beliebigen, durch ein kombinatorisches Schema gegebenen Sphäre entscheiden kann, ob sie polytopal ist. Die Antwort ist positiv, wobei allgemeine Sätze der Logik herangezogen werden. Allerdings gibt der Nachweis der Existenz eines derartigen Algorithmus noch kein Rechenverfahren

in die Hand, die auftretenden Ungleichungen zu lösen. In der Arbeit [38] wird eine Alternative zu Grünbaum wie folgt angegeben:

Bekanntlich [43] läßt sich jedes Polytop  $P$  mit  $n+1$  Ecken darstellen als Parallelprojektion  $\varphi$  eines geeigneten  $n$ -Simplex  $T^n \subset E^n$  auf einen Unterraum von  $E^n$ . Das Urbild von  $\varphi(P)$  unter  $\varphi$  ist eine sogenannte *Schattengrenze* von  $T^n$ . Ist  $\mathcal{C}$  eine gegebene kombinatorische  $(k-1)$ -Sphäre mit  $n+1$  Ecken, dann ordne man  $\mathcal{C}$  zunächst eine simpliziale Sphäre  $S(\mathcal{C})$  zu, die die gleichen Ecken besitzt wie  $\mathcal{C}$  und deren  $(k-1)$ -Zellen durch Zerschneiden der  $(k-1)$ -Zellen von  $\mathcal{C}$  entstehen.  $S(\mathcal{C})$  kann man nun als Subkomplex  $\mathcal{C}'$  in  $\mathcal{B}(T^n)$  einbetten, indem man die Ecken von  $S(\mathcal{C})$  in beliebiger Reihenfolge bijektiv den Ecken von  $T^n$  zuordnet. Die Frage ist dann, ob man eine Projektionsrichtung  $X = x_1 \Lambda \cdots \Lambda x_{n-k}$  so angeben kann, daß bei der Projection  $\varphi$  von  $T^n$  in Richtung  $X$  auf einen  $k$ -dimensionalen Unterraum  $U \subset E^n$  das Bild  $\varphi(\mathcal{C}')$  von  $\mathcal{C}'$  nach Zusammenfügen gewisser Simplexe zu neuen Zellen einen zu  $\mathcal{C}$  isomorphen Randkomplex eines  $k$ -Polytops ergibt. Dies führt zur Frage, ob ein System von maximal  $f_0(\mathcal{C}) \cdot f_{k-1}(\mathcal{C})$  Determinantenungleichungen und -gleichungen in den Koordinaten der  $x_j$  eine gemeinsame Lösung besitzen.

Der konkrete Rechenaufwand ist erwartungsgemäß recht hoch. Jedoch erscheint es möglich, mit einem verbesserten Algorithmus die von Altshuler und Steinberg [6] durchgeführte konkrete Untersuchung von 4-Sphären mit wenig Ecken fortzuführen.

## 6. Schälbarkeit und stellare Operationen

Eine in der stückweise-linearen Topologie häufig benutzte Technik ist die *Schälung* von Komplexen:

Eine Teilschälung eines  $d$ -Komplexes  $\mathcal{C}$ , der  $n$   $d$ -Zellen besitzt, ist eine Folge  $(C_1, \dots, C_k)$  von  $d$ -Zellen von  $\mathcal{C}$  mit der Eigenschaft, daß die Trägermenge von  $\bigcup_{i=1}^{m-1} C_i$  für  $1 \leq m \leq k$  homöomorph zu einer  $d$ -Kugel ist.

Nur falls  $m = k = n$  ist, sei auch erlaubt, daß diese Vereinigung eine  $d$ -Sphäre ist. Nur in diesem Fall ist  $\mathcal{C}$  eine  $d$ -Sphäre, anderenfalls eine  $d$ -Kugel. Eine Teilschälung ist ein *Schälung* von  $\mathcal{C}$ , wenn  $k = n$  gilt. Besitzt der Komplex  $\mathcal{C}$  eine Schälung, so heißt er *schälbar*. Kann jede Teilschälung von  $\mathcal{C}$  zu einer Schälung fortgesetzt werden, so nennen wir  $\mathcal{C}$  *fortsetzbar schälbar*.

Bruggesser und Mani bewiesen in [28], daß der Randkomplex eines  $d$ -Polytops stets schälbar ist, insbesondere wurde dort gezeigt:

- (23) Seien  $F_1$  und  $F_2$  beliebig gewählte Facetten des  $d$ -Polytops  $P$ . Dann gibt es eine Schälung von  $\mathcal{B}(P)$  mit  $F_1$  als erster und  $F_2$  als letzter Facette.

Die Schälbarkeit der Randkomplexe von Polytopen wurde wesentlich in McMullens Beweis [65] der «upper-bound-conjecture» benutzt, und ist wohl als eine der wesentlichsten kombinatorischen Eigenschaften von Polytopen zu bezeichnen.

Danaraj und Klee äußern in [30] die Vermutung, daß es einen engen Zusammenhang zwischen dem Steinitz-Problem und der Schälbarkeit gibt, derart, daß aus der Existenz einer großen Anzahl verschiedener Schälungen eines Komplexes folgt, daß er polytopal ist.

In den letzten Jahren sind von verschiedenen Autoren Beiträge zur Schälbarkeit

von Komplexen geleistet worden, wobei insbesondere Beziehungen zwischen derstellaren Theorie und Schälbarkeit nachgewiesen wurden. Wir werden im folgenden besonders diejenigen Aspekte der Schälbarkeit aufführen, die sich im Zusammenhang mit Polytopen ergeben.

Als hilfreiches Instrument für diesen Problemkreis hat sich der Begriff der *p-stellaren Operation* erwiesen (vgl. [62]):

$\mathcal{C}$  sei eine d-Sphäre mit einer 0-Zelle  $p$ .  $F$  sei eine d-Zelle ( $p \notin F$ ) von  $\mathcal{C}$  und  $F \cap \text{link}(p, \mathcal{C})$  eine  $(d-1)$ -Kugel  $B$ .  $\bar{B}$  sei die von den  $(d-1)$ -Zellen von  $F$ , die nicht in  $B$  liegen, erzeugte  $(d-1)$ -Kugel in  $\mathcal{B}(F)$ .

Der Komplex  $\mathcal{C}'$  sei wie folgt definiert:

$$\mathcal{C}' := [\mathcal{C} \setminus \{(F \cup \mathcal{B}(F)) \cup (p \cdot B)\}] \cup p \cdot \bar{B} \quad (\text{vgl. Bild 2}).$$

Damit  $\mathcal{C}'$  wohldefiniert ist, sei es erlaubt, auf  $\mathcal{C}$  vorher einen geeigneten Isomorphismus anzuwenden.

Der Übergang von  $\mathcal{C}$  zu  $\mathcal{C}'$  heiße *p-stellare Operation* (im simplizialen Fall in [37] *p-stellare Abänderung* genannt).

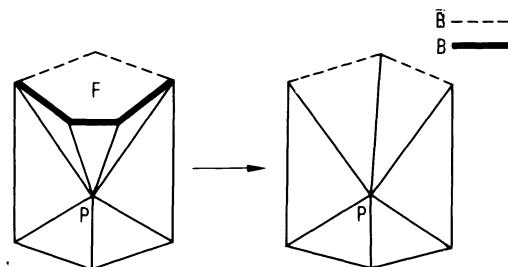


Bild 2

Mit einem Prozeß, der ein modifiziertes Verfahren der von Bruggesser und Mani zum Beweis von (23) verwandten Methoden darstellt, lässt sich zeigen:

- (24)  $P$  sei ein d-Polytop,  $p$  eine fest gegebene Ecke und  $F$  eine Facette von  $P$ , die  $p$  nicht enthält. Dann gibt es eine Kette von *p-stellaren Operationen*  $\sigma_1, \dots, \sigma_n$  mit:  
 $\sigma_n \cdots \sigma_1(\mathcal{B}(P))$  ist isomorph zu  $\text{conv}(p \cup F)$ , wobei  $\sigma_i \cdots \sigma_1(\mathcal{B}(P))$  ein konvexes Polytop ist für  $1 \leq i \leq n$ .

Für simpliziale Polytope ist (24) von Ewald in [37] gezeigt worden. Damit folgte für simpliziale Polytope das folgende, auch schon von Ewald-Shephard in [39] bewiesene Resultat:

- (25) Die Randkomplexe zweier simplizialer d-Polytope können durch stellare Operationen derart ineinander überführt werden, daß bei sämtlichen Zwischenschritten d-Polytope auftreten.

Damit konnte zu dem in der stückweise-linearen Topologie bekannten Ergebnis derstellaren Äquivalenz von (stückweise-linearen) Sphären (vgl. [40]) für Polytope ein stärkeres Analogon gefunden werden. Ewalds Ergebnis aus [37] führt auch zu einer Oberabschätzung der benötigten Schritte bei einemstellaren Prozeß vom Simplex zu einem Polytop:

- (26) Sei  $P$  ein simpliziales  $d$ -Polytop und sei  $T^d$  ein  $d$ -Simplex.  $v_{\max}(P)$  sei die maximale Anzahl von Facetten, die in einer Ecke von  $P$  zusammenstoßen. Dann gilt für die Minimalzahl  $k$  der benötigten stellaren Operationen, die  $\mathcal{B}(T^d)$  in  $\mathcal{B}(P)$  überführen:

$$k \leq 2(f_{d-1}(P) - v_{\max}(P)) + d - 1 - f_0(P),$$

wobei mit  $f_i(P)$  die Anzahl der  $i$ -Seiten von  $P$  bezeichnet wird.

*Problem:* Man bestimme eine schärfere Abschätzung für  $k$ ! Lassen sich auch die Anzahlen der stellaren Operationen einer festen Dimension günstig abschätzen (dies wäre evtl. von Bedeutung für die sogenannte g-Vermutung [66])?

Als Korollar von (24) erhält man (23) und insbesondere induzieren die  $p$ -stellaren Operationen eine Schälung des Antisterns von  $p$ :

- (27) Sei  $p$  Ecke des  $d$ -Polytops  $P$ . Dann ist  $\text{ast}(p, P)$  schälbar.

Aus (27) und wenigen geometrischen Überlegungen ergibt sich zusammen mit einem induktiven Argument, daß Polytope stets sehr starken Schälbarkeitsbedingungen genügen. Dieser Satz wurde schon von Danaraj und Klee in [30] bewiesen, die den zur Schälbarkeit dualen Prozeß betrachteten. Eine  $d$ -Sphäre  $S$  heiße *sternförmig schälbar*, wenn für jede Folge  $F_0, \dots, F_d$  von Zellen aus  $S$  mit  $\dim F_i = i$ ,  $0 \leq i \leq d$  und  $F_0 \subset F_1 \subset \dots \subset F_d$  gilt:

Es gibt eine Schälung von  $S$ , bei der die  $d$ -Zellen in der folgenden Reihenfolge auftreten:

$F_d, \mathcal{F}_{d-1}, \dots, \mathcal{F}_0, \text{ast}(F_0, S)$ , wobei

$$\mathcal{F}_i := \Delta^d(\text{st}(F_i, S)) \setminus \Delta^d(\text{st}(F_{i+1}, S)), \quad 0 \leq i \leq d-1 \quad (\Delta^d(\mathcal{C}) = \{F \in \mathcal{C} / \dim F = d\}).$$

Es gilt dann:

- (28) Der Randkomplex eines  $d$ -Polytops  $P$  ist derart sternförmig schälbar, daß die zuletzt zu schälende Facette  $F$  von  $\mathcal{F}_i$  beliebig wählbar ist. Ist  $P$  simplizial, so können diejenigen Facetten von  $\mathcal{F}_i$ , die mit  $F$  eine  $(d-2)$ -Seite gemeinsam haben, in beliebiger Reihenfolge direkt vor  $F$  in der Schälung gewählt werden.

(28) wird in [61] direkt geometrisch ohne die in [30] verwandte duale Theorie bewiesen. (28) ist die stärkste bekannte Bedingung für Schälungen von Polytopen. Daß (28) nicht für beliebige Sphären gilt, folgt aus der Tatsache, daß nichtschälbare 3-Kugeln bekannt sind.

Rudin konstruierte in [81] eine nichtschälbare Triangulierung von  $T^3$  mit 14 Ecken. Grünbaum [46] konstruierte eine nichtschälbare 3-Kugel (wir nennen sie  $B_G$ ) mit 14 Ecken mit weit weniger Facetten, die wir in Tabelle 2 durch die Ecken ihrer Tetraeder angeben:

Sei  $p$  ein Punkt in  $E^4$  und  $B_G$  eingebettet in einen dreidimensionalen Unterraum von  $E^4$ , der  $p$  nicht enthält, dann ist die 3-Sphäre  $S := \{p\} \cdot \partial B_G \cup B_G$  natürlich nicht sternförmig schälbar, da  $\text{ast}(p, S)$  nicht schälbar ist. Es gilt also:

- (29) Es gibt eine 3-Sphäre mit 15 Ecken, die nicht sternförmig schälbar ist (vgl. [62]).

Tabelle 2

AA'BD	ADFG
AA'B'D'	A'D'F'G'
AA'DD'	A'D'F'G'
ABCD	BCDE
A'B'C'D'	B'C'D'E'
AB'D'E'	CC'GG'
A'BDE	CDEG
ACC'G	C'D'E'G'
A'CC'G'	CF'GG'
ACDG	C'FGG'
A'C'FG	DD'EG
A'CF'G'	DD'E'G'
ADD'E'	DD'GG'
A'DD'E	DFGG'
	D'F'GG'

Versucht man diejenigen Sphären zu bestimmen, die polytopal sind, so kann man also wegen (28) und (29) gewisse Kandidaten für polytopale Komplexe ausscheiden, nämlich, die nicht sternförmig schälbaren. Diese Methode liefert jedoch nicht alle Polytope, da auch nichtpolytopale Komplexe wie die Barnette-Sphäre (s.o.(9)) sternförmig schälbar sind.

Für algorithmische Tests auf Schälbarkeit, wie sie in [31] und [6] mit Rechenprogrammen durchgeführt werden, ist die oben definierte fortsetzbare Schälbarkeit nützlich, denn fortsetzbar schälbare Komplexe erzeugen keine «Irrwege» in solchen Programmen.

Eine äquivalente, aber meist leichter überprüfbarere Definition für fortsetzbare Schälbarkeit wird in [62] gegeben:

Eine d-Sphäre  $S$  ist genau dann fortsetzbar schälbar, wenn mit jeder schälbaren d-dimensionalen Teilkugel von  $S$  auch ihre bezüglich  $S$  komplementäre Kugel schälbar ist.

Danaraj, Klee und Tverberg fragten (vgl. [32]), ob polytopale Komplexe stets fortsetzbar schälbar sind. Für «wenige» Ecken wurde dies unter Ausnutzung von (14) und der letzteren Definition der fortsetzbaren Schälbarkeit in [62] bewiesen:

(30) Jede d-Sphäre mit  $d + 3$  Ecken ist fortsetzbar schälbar.

Mit Hilfe einer Konstruktion, die auf der oben angegebenen Kugel  $B_G$  von Grünbaum basiert, wird dort jedoch auch gezeigt:

(31) Es gibt eine nicht-polytopale 3-Sphäre, die sternförmig schälbar, aber nicht fortsetzbar schälbar ist. Sie besitzt 14 Ecken.

(31) ist in gewisser Hinsicht minimal, da Rudin [81] gezeigt hat, daß Triangulierungen von  $T^3$  mit weniger als 14 Ecken stets schälbar sind.

Es wäre interessant, die Lücke zwischen (30) und (31) zu schließen oder zumindest zu verkleinern.

*Problem:* Welches ist das kleinste  $n$ , derart, daß jede d-Sphäre mit  $d + n$  Ecken fortsetzbar schälbar ist?

Die Randkomplexe von 3-Polytopen sind stets sowohl sternförmig als auch fortsetzbar schälbar. Die Versuche, fortsetzbare Schälbarkeit für Randkomplexe von Polytopen zu zeigen, scheitern hauptsächlich daran, daß man keinen metrischen Prozeß analog dem von Bruggesser und Mani verwandten benutzen kann, denn es gibt auch bei Polytopen Teilkugeln, deren Rand keine Schattengrenze ist (vgl. [88]).

Barnette bestimmte in [11] alle kombinatorischen Typen von simplizialen 3-Sphären mit 8 Ecken, von denen die polytopalen Sphären bereits in [49] charakterisiert wurden. Für 3-Sphären mit 9 Ecken bestimmten Altshuler und Steinberg in [6] alle Typen. Sie konnten jedoch für einige kombinatorische Klassen nicht entscheiden, ob sie polytopal sind. Es konnte für alle diese Sphären mit Hilfe eines Rechnerprogramms nachgewiesen werden, daß sie im Sinne von (23) schälbar sind. Dabei trat innerhalb der Rechnung niemals eine «Sackgasse» auf, d.h. eine begonnene Schälung konnte stets fortgesetzt werden. Es bleibt weiterhin unbekannt, ob es (stückweise-lineare) Sphären gibt, die nicht schälbar sind. Es gibt allerdings stets eine Unterteilung, die schälbar ist [82].

*Problem:* Gibt es stückweise-lineare Sphären, die nicht schälbar sind?

In [34] wird mit Hilfe einer doppelten Einhängung einer 3-Homologie-Sphäre eine 5-Sphäre konstruiert, die nicht stückweise-linear und daher nicht schälbar ist.

Weitere Zusammenhänge zwischen Schälbarkeit undstellaren Operationen werden in [72] betrachtet. Sei  $\mathcal{C}$  ein simplizialer d-Komplex,  $A \in \mathcal{C}$  eine k-Zelle ( $0 \leq k \leq d$ ) und es gelte:

- (a)  $\text{link}(A; \mathcal{C}) = \mathcal{B}(B)$ , wobei  $B$  ein  $(d - k)$ -Simplex ist.
- (b)  $B \notin \mathcal{C}$

Dann heißt  $\chi_A \mathcal{C} := (\mathcal{C} \setminus A \cdot \mathcal{B}(B)) \cup \mathcal{B}(A) \cdot B$  eine *bistellare k-Operation* (vgl. Bild 3 für  $d = 2$ ). Sie unterscheidet sich von der p-stellaren Operation bzw. p-stellaren Abänderung (s.o.) nur dadurch, daß keine Ecke  $p$  vorgegeben ist. Die Eckenzahl der Komplexe ändert sich genau für  $k = 0, d$ . Bezeichnet  $\sigma_A, \sigma_B$  diestellaren Unterteilungen in  $A$  bzw.  $B$ , so läßt sich auch  $\chi_A = \sigma_B^{-1} \sigma_A$  schreiben. Bistellare Operationen sind im Vergleich zu allgemeinenstellaren Operationen insofern besser überschaubar, als die Verkettungskomplexe der Zellen  $A$  und  $B$ , in denen stellar unterteilt wird, Randkomplexe von Simplizes sind, also die denkbar einfachste kombinatorische Struktur haben.

Die p-stellaren Operationen in Simplizialkomplexen sind «lokalierte» bistellare

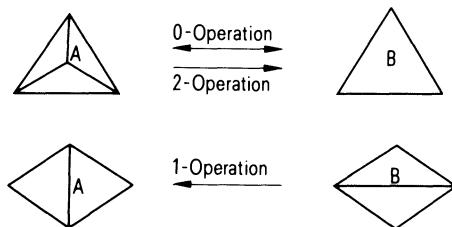


Bild 3

Operationen. Sie unterscheiden sich nur dadurch, daß eine Ecke  $p \in \mathcal{C}$  ausgezeichnet und  $p \in B$  verlangt wird.

Zwei Simplizialkomplexe heißen *bistellar äquivalent*, wenn sie durch eine Kette bistellarer Operationen ineinander überführt werden können. Die Randkomplexe zweier d-Polytope sind wegen (24) stets bistellar äquivalent. U.a. unter Verwendung des Satzes von Steinitz zeigte Pachner in [72] folgende teilweise Verallgemeinerung:

- (32) Zwei simpliziale Mannigfaltigkeiten  $M_1, M_2$  mit  $\dim M_i \leq 4$  sind genau dann stellar äquivalent (also ihre Trägermengen p.l. homöomorph), wenn sie bistellar äquivalent sind.

Sätze von diesem Typ sind häufig beweistechnisch interessant, da sie eine einfache Berechnung von topologischen Invarianten gestatten. So hat sich etwa der Satz von Alexander [1], nach dem man bei stellaren Äquivalenzen stets mit Operationen an Kanten auskommt, als sehr nützlich erwiesen.

Für Polytope wurde (24) folgendermaßen verschärft:

- (33) (a) Zwei polytopale simpliziale d-Sphären mit gleicher Eckenzahl lassen sich stets durch bistellare k-Operationen mit  $0 < k < n$ , also ohne daß sich zwischendurch die Eckenzahl ändert, ineinander überführen.  
Insbesondere gilt also:  
(b) Jede polytopale simpliziale d-Sphäre läßt sich durch bistellare k-Operationen mit  $0 < k < n$  in den Randkomplex eines  $(d+1)$ -Stapelpolytops gleicher Eckenzahl überführen.

Sätze von diesem Typ lassen sich zur Konstruktion kombinatorischer Typen von Polytopen und Sphären benutzen und könnten evtl. auch für Abzählungen solcher Typen von Bedeutung sein.

(33b) besagt, daß man bei der bistellaren Äquivalenz eines  $(d+1)$ -Polytops zum Simplex ohne d-Operationen auskommt und die 0-Operationen als letzte ausgeführt werden können.

In diesem Zusammenhang ergibt sich das folgende

*Problem:* Läßt sich jede d-Sphäre,  $d \geq 4$ , durch bistellare Operationen in  $\mathcal{B}(T^{d+1})$  überführen und ist dies für  $d \geq 3$  stets ohne bistellare d-Operationen möglich?

Die Existenz einer bistellaren Äquivalenz überhaupt ist natürlich eine notwendige Voraussetzung für die Schälbarkeit. Eine d-Sphäre, bei der bei jeder bistellaren Äquivalenz zu  $\mathcal{B}(T^{d+1})$  mindestens eine bzw. mindestens zwei bistellare d-Operationen vorkommen, wäre zugleich ein Beispiel für eine Sphäre, die nicht sternförmig bzgl. irgendeiner Ecke schälbar ist bzw. die überhaupt nicht schälbar ist.

### III. Zellzerlegungen von Mannigfaltigkeiten und deren Einbettung in Randkomplexe von Polytopen

#### 1. Allgemeines

Ebenso wie Kreise als 1-dimensionale Submannigfaltigkeiten von Polytopen sind als höherdimensionales Analogon Zellzerlegungen von Mannigfaltigkeiten

und deren Einbettung als Teilkomplex konvexer Polytope unter verschiedenen Aspekten behandelt worden. Zum einen ist das große Interesse in den Objekten selbst begründet, wobei einmal die kombinatorische Topologie der Mannigfaltigkeiten im Vordergrund steht und das andere Mal die Struktur der Polytope erhellt werden soll. Zum anderen haben sich die Ergebnisse auch für weitere Problemkreise als fruchtbar erwiesen und sind aus anderen Gebieten und Fragestellungen erwachsen.

So könnte z.B. die Existenz bzw. Nichtexistenz von gewissen Submannigfaltigkeiten für das Steinitz-Problem eine Rolle spielen.

Ein schönes Beispiel dafür liegt in [21] vor, wo im konkreten Fall aufgrund der Existenz eines speziellen Möbiusbandes als Subkomplex ein nicht-polytopales 3-Diagramm konstruiert wird. Als weiteres Anwendungsbeispiel sei der Problemkreis der Existenz kombinatorischer Typen von Polytopen genannt, z.B. solcher  $(d+1)$ -Polytope, deren Facetten alle zueinander isomorph sind (vgl. [15] und [77]). Ein  $d$ -Polytop, das nicht als Facette eines derartigen  $(d+1)$ -Polytops auftreten kann, wird als «nonfacet» bezeichnet. Für  $d=2$  sind genau die  $n$ -Ecke mit  $n \geq 6$  nonfacets. Dieses ergibt sich direkt aus der Euler-Relation. Für höhere Dimensionen ist die Frage weitaus schwieriger. Perles und Shephard [77] zeigten mit Hilfe von Winkelsummen, daß die Eigenschaft eines  $d$ -Polytops, nonfacet zu sein oder nicht, stark mit der Existenz «größer»  $(d-2)$ -Subsphären im  $(d-2)$ -Skelett zusammenhängt:

- (34) Enthält jede  $(d-2)$ -Sphäre im  $(d-2)$ -Skelett eines  $d$ -Polytops  $P$  für  $0 \leq k \leq d-2$  höchstens das  $\left(\frac{d-1-k}{d+1-k}\right)$ -fache der Anzahl aller  $k$ -Seiten von  $P$ , so ist  $P$  nonfacet.

Die folgende Verallgemeinerung von Barnette [15] ist besonders im Hinblick auf die Tatsache von Interesse, daß es unbekannt ist, ob es nicht schälbare zellzerlegte Sphären gibt (vgl. Abschnitt II.6). Der Beweis wird rein kombinatorisch geführt.

- (35) Jede schälbare  $d$ -Sphäre hat eine Facette  $F$  mit einer  $(d-2)$ -Sphäre in  $\text{skel}_{d-2}(F)$ , die mehr als ein Drittel aller  $k$ -Seiten,  $0 \leq k \leq d-2$ , von  $F$  enthält.

Die nun folgenden Ergebnisse haben wir in vier Bereiche eingeteilt, die sich natürlich teilweise überschneiden.

## 2. Hamiltonsche Mannigfaltigkeiten

Bei der Betrachtung hamiltonscher Mannigfaltigkeit steht hier die kombinatorische Struktur der Polytope im Mittelpunkt des Interesses.

Unter einer  $k$ -hamiltonschen  $m$ -Mannigfaltigkeit in einer zellzerlegten Mannigfaltigkeit  $\mathcal{C}$  verstehen wir eine orientierbare, geschlossene, zellzerlegte  $m$ -Mannigfaltigkeit im  $m$ -Skelett von  $\mathcal{C}$ , die das ganze  $k$ -Skelett von  $\mathcal{C}$  enthält. [Beispiel: Im vierdimensionalen Kubus gibt es einen 1-hamiltonschen 2-Torus (vgl. Bild 4)].

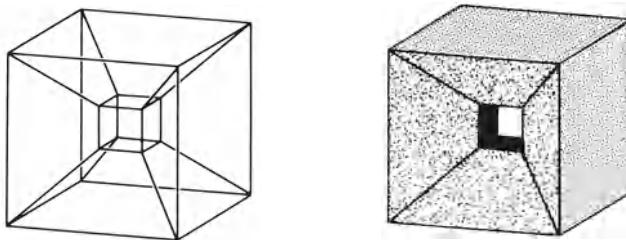


Bild 4

Es liegt nahe, zuerst Hamiltonflächen zu studieren, da deren kombinatorische Struktur noch recht gut zu erfassen ist. In [2] gibt Altshuler eine weitgehende Charakterisierung der Existenz von 1-hamiltonischen Flächen in Stapelpolytopen. Daneben beweist er:

- (36) Zu jedem  $g \geq 1$  existiert ein simpliziales 4-Polytop (welches kein Stapelpolytop ist) mit genau  $3g+4$  Ecken, das eine 1-hamiltonische Fläche vom Geschlecht  $g$  besitzt.

Dieses Ergebnis ist von besonderer Bedeutung für das schwierige Problem, für Flächen minimale Zellzerlegungen zu finden und diese in einem Raum möglichst niedriger Dimension zu realisieren (in diesem Falle sind die Flächen sogar vermöge des Schlegel-Diagramms des 4-Polytops im  $E^3$  realisierbar).

Diese Problematik, auf die wir später noch zurückkommen, steht auch im Vordergrund der Untersuchungen Altshulers [3] über Hamiltonflächen in zyklischen 4-Polytopen, wo u.a. gezeigt wird:

- (37) Die zyklischen 4-Polytope enthalten als geschlossene Flächen nur Sphären und Tori in ihrem 2-Skelett. Die zyklischen 4-Polytope mit mindestens 7 Ecken enthalten alle einen Torus und alle diese Tori sind 0-hamiltonsch. Nur das zyklische 4-Polytop mit 7 Ecken enthält eine 1-hamiltonische Fläche.

In [83] sind von Schulz diese Untersuchungen für benachbarte Polytope fortgesetzt worden. Ein Polytop  $P$  heißt  $k$ -benachbart, wenn je  $k$  seiner Ecken eine Seite von  $P$  bilden.

- (38) (a) Ein 3-benachbartes Polytop mit  $v$  Ecken besitzt genau dann eine 1-hamiltonische Fläche vom Geschlecht  $g$ , wenn gilt:

$$g = \frac{(v - 3)(v - 4)}{12}.$$

- (b) Unter den zyklischen 5-Polytopen ist das mit 7 Ecken das einzige mit einer 1-hamiltonischen Fläche.

- (c) Existiert in einem 2-benachbarten 4-Polytop  $P$  mit  $v$  Ecken (bei geeigneter geometrischer Realisierung) eine Teilmenge der Eckenmenge mit mehr als  $\frac{v+7}{2}$  Elementen, deren konvexe Hülle ein zyklisches 4-Polytop ist, so besitzt  $P$  keine 1-hamiltonische Fläche.

*Problem:* Gibt es 2-benachbare 4-Polytope mit mehr als 7 Ecken, die (38c) nicht erfüllen?

In [83] und [84] untersucht Schulz auch nichtsimpiziale 4-Polytope, nämlich Prismen und Pyramiden, auf Hamiltonflächen:

- (39) Vierdimensionale Pyramiden enthalten keine 1-hamiltonschen Flächen. Für jede ganze Zahl  $g \geq 1$  gibt es eine vier dimensionale Bipyramide mit  $3g + 5$  Ecken, die eine 1-hamiltonsche (nichtsimpiziale) Fläche vom Geschlecht  $g$  zuläßt; diese Eckenzahl ist für Bipyramiden minimal.
- (40) Sei  $P$  das Prisma über einem einfachen 3-Polytop  $Q$ , welches  $p_k$   $k$ -Ecke als Facetten hat. Dann gilt:
  - (a) Ist  $p_{2k+1} = 0$  für alle  $k$ , so hat  $P$  genau drei verschiedene 1-hamiltonsche Flächen.
  - (b) Im Fall  $\sum_{k \geq 1} p_{2k+1} = 2$  hat  $P$  genau eine 1-hamiltonsche Fläche.
  - (c) In den übrigen Fällen existiert höchstens eine, für  $\sum_{k \geq 1} (2k+1)p_{2k+1} > \sum_{k \geq 1} kp_k$  keine hamiltonsche Fläche.

Für höherdimensionale simpliziale Mannigfaltigkeiten bewies Ewald [35] mit Hilfe der «pulling method» (siehe auch IV):

- (41) Für eine geschlossene simpliziale  $d$ -Mannigfaltigkeit  $\mathcal{C}$  gilt:
  - (a) Hat jede  $(d-1)$ -Zelle von  $\mathcal{C}$  in  $\mathcal{C}$  mindestens eine  $(d+2)$ -valente Ecke, so besitzt  $\mathcal{C}$  für  $m = 1, \dots, d-1$  eine 0-hamiltonsche  $m$ -Sphäre.
  - (b) Es besitze jede  $(d-1)$ -Zelle von  $\mathcal{C}$  mindestens 3 Ecken mit Valenz höchstens  $d+3$ . Enthält dann  $\mathcal{C}$  eine beliebige geschlossene  $m$ -Mannigfaltigkeit als Subkomplex, so auch eine 0-hamiltonsche  $m$ -Mannigfaltigkeit vom gleichen Homöomorphietyp. Insbesondere enthält ein derartiger Komplex  $\mathcal{C}$  also auch stets eine 0-hamiltonsche  $m$ -Sphäre für  $m = 1, \dots, d-1$ .

Recht gut in den Griff zu bekommen sind auch Submannigfaltigkeiten der Kodimension 1 in einfachen Sphären aufgrund des folgenden Satzes von Pachner [71] und [74].

- (42) Im Randkomplex eines einfachen Polytops ist jede geschlossene  $(d-2)$ -Pseudomannigfaltigkeit eine Mannigfaltigkeit.

Pachner beweist (42) noch in allgemeinerer Form für Komplexe, deren Zellen nicht mehr Polytope, sondern zellzerlegte Kugeln sind.

Technisch noch handlicher ist eine rein graphtheoretische Kennzeichnung im dualen Polytop, die wir im Abschnitt III.5 behandeln. Mit dieser Technik wird in [70] bewiesen:

- (43) Enthält das einfache  $d$ -Polytop  $P$  eine  $k$ -hamiltonsche  $(d-2)$ -Sphäre, so auch jedes  $d$ -Polytop, das man aus  $P$  durch sukzessives Abschneiden von Ecken mittels Hyperebenen erhält. Insbesondere enthalten die dualen Polytope zu  $d$ -Stapelpolytopen stets eine  $\left\lceil \frac{d-4}{2} \right\rceil$ -hamiltonsche  $(d-2)$ -Sphäre.

Man kann zeigen, daß in den dualen Polytopen zu d-Stapelpolytopen als geschlossene  $(d-2)$ -Pseudomannigfaltigkeiten im  $(d-2)$ -Skelett nur Sphären enthalten sind, die außerdem stets *unverknotet* sind, d.h. sie zerlegen den Rand des Polytops in 2 Kugeln. Ergebnisse dieser Art sind im Hinblick auf die (für stückweise-lineare Sphären) ungelöste Schönlies-Vermutung von Interesse, insbesondere wegen der folgenden Reduktion dieses Problems durch Pachner [71]:

- (44) Die Schönlies-Vermutung (d.h. jede  $(d-1)$ -Sphäre in einer d-Sphäre ist unverknotet) ist genau dann richtig, wenn in den einfachen Polytopen, die man aus dem 5-Simplex durch sukzessives Abschneiden von Seiten (beliebiger Dimension) erhält, jede dreidimensionale Subsphäre im 3-Skelett unverknotet ist.

Es sei bemerkt, daß das Abschneiden von Seiten bei einfachen Polytopen kombinatorisch im Dualen genau dem stellaren Unterteilen entspricht und von daher kombinatorisch gut beherrschbar ist.

Ebenfalls mit Hilfe von (42) bewies Schulz in [83]:

- (45) (a) Besitzt das einfache d-Polytop  $Q$  eine  $k$ -hamiltonsche  $(d-2)$ -Mannigfaltigkeit, so besitzt das  $r$ -fache Prisma über  $Q$  eine  $\left(k + \left[ \frac{r}{2} \right] \right)$ -hamiltonsche  $(d+r-2)$ -Mannigfaltigkeit.  
 (b) Jedes  $r$ -fache Prisma über einem einfachen d-Polytop enthält eine  $\left[ \frac{r-1}{2} \right]$ -hamiltonsche  $(d+r-2)$ -Mannigfaltigkeit.

### 3. Überdeckungen und Zerlegungen der Skelette von Sphären mit Mannigfaltigkeiten

Eng verwandt mit der Frage nach hamiltonschen Mannigfaltigkeiten ist die Frage nach der Minimalzahl von  $m$ -Mannigfaltigkeiten im  $m$ -Skelett einer Sphäre, deren Vereinigung das ganze  $m$ -Skelett enthält. Eine enge Beziehung besteht auch zu den dualen Problemen, die wir im letzten Abschnitt behandeln werden.

Pachner zeigt in [73]:

- (46) (a) Zur Überdeckung des  $(d-1)$ -Skeletts einer d-Sphäre benötigt man mindestens  $\lceil \log_2(d+2) \rceil$   $(d-1)$ -Subsphären. Diese Zahl ist für  $\mathcal{B}(T^{d+1})$  scharf.  
 (b) Zur Überdeckung des  $(d-1)$ -Skeletts einer d-Sphäre benötigt man mindestens  $\lceil \log_2(d+1) \rceil$  geschlossene  $(d-1)$ -Mannigfaltigkeiten. Diese Zahl ist für den Rand des  $(d+1)$ -Würfels scharf.

Die Überdeckungszahlen bzgl. Überdeckungen mit geschlossenen Pseudomannigfaltigkeiten wachsen erstaunlicherweise nicht notwendig unbegrenzt mit der Dimension. In [73] zeigt Pachner, daß man beim Prisma über dem d-Kreuzpolytop stets mit drei Pseudomannigfaltigkeiten auskommt. Die endgültige Lösung fand Paus [76]:

- (47) Zur Überdeckung des  $(d-1)$ -Skeletts einer  $d$ -Sphäre benötigt man mindestens zwei geschlossene  $(d-1)$ -Pseudomannigfaltigkeiten, im Randkomplex des  $(d+1)$ -dimensionalen Kreuzpolytops genügen stets 2 Pseudomannigfaltigkeiten.

Es sei bemerkt, daß zur Überdeckung des  $(d-1)$ -Skeletts einfacher  $(d+1)$ -Polytope mindestens  $\lceil \log_2(d+1) \rceil$  geschlossene  $(d-1)$ -Pseudomannigfaltigkeiten benötigt werden. Dies ergibt sich als unmittelbare Folgerung aus (42) und (46).

Eine Mittelstellung zwischen Mannigfaltigkeiten und Pseudomannigfaltigkeiten nehmen die l.st.z. Pseudomannigfaltigkeiten (l.st.z.=lokal stark zusammenhängend) ein, das sind solche Pseudomannigfaltigkeiten, bei denen der Stern einer jeden Zelle (bei beliebiger Zellzerlegung) stark zusammenhängend, also ebenfalls eine Pseudomannigfaltigkeit entsprechend niedriger Dimension ist. Diese induktive Eigenschaft läßt die l.st.z. Pseudomannigfaltigkeiten als Verallgemeinerung von Mannigfaltigkeiten eigentlich natürlicher erscheinen als die Pseudomannigfaltigkeiten.

Es ist nicht bekannt, ob bei geschlossenen l.st.z. Pseudomannigfaltigkeiten die Überdeckungszahlen unbeschränkt mit der Dimension wachsen, man weiß lediglich [73]:

- (48) Für jedes  $d$  gibt es  $(d+1)$ -Polytope, deren  $(d-1)$ -Skelett mit  $\left\lceil \log_2 \frac{4}{5} (d+2) \right\rceil$  geschlossenen l.st.z.  $(d-1)$ -Pseudomannigfaltigkeiten überdeckt werden kann.

Ebenso wie bei den zuvor angegebenen Lösungen ist das (48) betreffende Problem in erster Linie topologischer Natur. Es geht dabei nämlich um die Frage der Einbettbarkeit von geschlossenen l.st.z. Pseudomannigfaltigkeiten von bestimmtem topologischen Typ in einem Raum möglichst niedriger Dimension. Dieses Problem wird im nächsten Abschnitt noch einmal angeschnitten.

Eine  $(k,r)$ -Zerlegung eines Polytops  $P$  ist ein System von  $k$ -Mannigfaltigkeiten  $M_1, \dots, M_r$  (nicht notwendig orientierbar und geschlossen), die das  $k$ -Skelett von  $P$  überdecken und für die zusätzlich gilt, daß die  $M_i$ 's paarweise keine  $k$ -Seiten gemein haben. Ein erstes Ergebnis in dieser Richtung stammt von Ringel [79]:

- (49) Für ungerade Dimension  $d \geq 3$  hat der  $d$ -Würfel eine  $\binom{d-1}{2}$ -Zerlegung in geschlossene orientierbare Flächen.

Für 4-Polytope existiert trivialerweise eine  $(2,r)$ -Zerlegung für  $r = f_2(P)$ . Bezeichnet man mit  $z(P)$  das minimale  $r$  für eine  $(2,r)$ -Zerlegung von  $P$ , so ist das Maximum von  $z(P)$ , genommen über alle 4-Polytope, unbekannt.

Betke, Schulz und Wills [23] zeigten:

- (50) Für 4-Polytope gilt stets  $z(P) < \frac{1}{2} f_2(P)$ , und es gibt unendlich viele Polytope mit  $z(P) \geq \frac{1}{4} f_2(P) - \frac{1}{2}$ .

In [22] behandeln Betke, Schulz und Wills (2,2)-Zerlegungen von 4-Polytopen. Die Autoren zeigen u.a. die Äquivalenz der Existenz einer Zerlegung zum Vierfarbensatz. Da dieser inzwischen bewiesen ist [7], können wir hier das Ergebnis ohne die entsprechende Zusatzvoraussetzung notieren.

- (51) (a) Prismen über einfachen 3-Polytopen sind (2,2)-zerlegbar. Die Flächen solcher Zerlegungen sind stets berandet. Ferner wird gezeigt:
  - (b) Prismen über (2,2)-zerlegbaren einfachen 4-Polytopen sind (2,2)-zerlegbar. Hierbei erhält man Zerlegungen in geschlossenen Flächen.
  - (c) Die (2,2)-zerlegbaren simplizialen 4-Polytope sind genau das 4-Simplex, das zyklische 4-Polytop mit 6 Ecken und die Bipyramide über dem Tetraeder und dem Oktaeder. Das 5-Simplex ist das einzige (2,2)-zerlegbare simpliziale 5-Polytop, 6-Polytope sind nie (2,2)-zerlegbar.
- Als einige Beispiele seien angeführt:
- (d)  $T^4$  hat eine Zerlegung in zwei Möbiusbänder, das Prisma über  $T^4$  in zwei Kleinsche Flaschen und die Pyramide über dem Oktaeder in zwei gelochte Tori.

#### 4. Realisierungen zellzerlegter Mannigfaltigkeiten

Die ersten fundamentalen Sätze über (geradlinige) Einbettungen von Simplizialkomplexen in euklidische Räume gehen auf van Kampen [95] aus dem Jahre 1932 zurück und finden immer wieder in neueren Sätzen Anwendungen.

- (52) (a) Jeder  $n$ -dimensionale simpliziale Komplex ist in den  $E^{2n+1}$  einbettbar.
- (b) Jede simpliziale  $n$ -Pseudomannigfaltigkeit ist in den  $E^{2n}$  einbettbar.
- (c)  $\text{skel}_n(T^{2n+2})$  ist nicht in den  $E^{2n}$  einbettbar.

Seitdem hat es viele Bemühungen gegeben, diese Aussagen, insbesondere (52b) zu verbessern. In diesem Zusammenhang interessiert auch die Einbettung von Mannigfaltigkeiten als Subkomplexe in Randkomplexe von  $(d+1)$ -Polytopen. Eine solche Realisierung vermittelt durch ein Schlegeldiagramm stets auch eine Realisierung im  $E^d$ . Eine Realisierung im Sinne von (52a) und eine problemlose Angabe der Koordinaten der Einbettung erhält man z.B. durch Einbettung in ein zyklisches  $(2n+2)$ -Polytop.

Daß die Beschränkung auf Einbettungen in Polytope — wenigstens topologisch — keine Einschränkung bedeutet, zeigt der nachstehende Satz von Pachner [71]. Dazu bezeichne  $P_{st}$  die Familie der simplizialen  $(d+1)$ -Polytope, deren Randkomplex man kombinatorisch aus  $\mathcal{B}(T^{d+1})$  durch eine Kette stellarer Unterteilungen (ohne Inverse!) erhält.

- (53) Jedes Polyeder  $R \subset E^d$  hat eine  $P_{st}$ -realisierbare Zellzerlegung also eine solche, die isomorph zu einem Subkomplex eines Polytops aus  $P_{st}$  ist. Zu jeder vorgegebenen Zellzerlegung  $\mathcal{R}$  von  $R$  existiert eine  $P_{st}$ -realisierbare Unterteilung.

Für Mannigfaltigkeiten wird in derselben Arbeit bewiesen:

- (54) Sei  $M \subset E^d$  eine geschlossene  $(d-1)$ -Pseudomannigfaltigkeit. Genau dann existiert eine  $P_{st}^*$ -realisierbare Zellzerlegung von  $M$  ( $P_{st}^*$  = Klasse der

zu  $P_{st}$  dualen Polytope), wenn die abgeschlossene Hülle der beschränkten Zusammenhangskomponente von  $E^d \setminus M$  eine Mannigfaltigkeit ist, d.h. wenn  $M$  eine flach im  $E^d$  einbettbare Mannigfaltigkeit ist.

Bevor wir zu den Flächen übergehen, erwähnen wir noch das Hauptergebnis aus Altshulers Arbeit [5] über l.st.z. 3-Pseudomannigfaltigkeiten.

- (55) Zu jedem endlichen System von topologischen Typen von geschlossenen Flächen gibt es eine simpliziale 3-Pseudomannigfaltigkeit, so daß der Verkettungskomplex einer jeden Ecke eine Fläche vom vorgegebenen Typ ist und jeder der Typen auch als Verkettungskomplex auftritt. Insbesondere existieren also z.B. solche 3-Pseudomannigfaltigkeiten, wo der Verkettungskomplex jeder Ecke ein Torus ist oder eine projektive Ebene usw.

Die von Altshuler konstruierten Pseudomannigfaltigkeiten sind nicht in Randkomplexe von 5-Polytopen einbettbar, und es ist ein ungelöstes Problem, ob es geschlossene l.st.z. 3-Pseudomannigfaltigkeiten gibt, die in den  $E^4$  einbettbar und in jeder Ecke singulär sind.

Eine Vermutung von Grünbaum lautet, daß jede triangulierte, orientierbare geschlossene Fläche in den  $E^3$  einbettbar ist. Für Sphären (auch für nicht simplizial zerlegte) folgt dies unmittelbar aus dem Satz von Steinitz, für andere topologische Typen, so z.B. schon für Tori, ist dieses Problem noch völlig offen. An Zellzerlegungen von Flächen, die minimal sind bzgl. ihrer Eckenzahl bzw. Kanten- oder Seitenzahl besteht neben der kombinatorischen Bedeutung einmal ein Interesse im Zusammenhang mit der obigen Vermutung von Grünbaum. Für Triangulierungen orientierbarer geschlossener Flächen vom Geschlecht  $g$  erhält man nämlich aus der Euler-Formel  $f_0 \geq \frac{1}{2}(7 + \sqrt{1 + 48g})$ , mit Gleichheit, wenn der Graph der Triangulierung vollständig ist. In [80] werden von Ringel abstrakte und damit in den  $E^5$  einbettbare Triangulierungen für den Gleichheitsfall  $f_0 \equiv 0, 3, 4, 7 \pmod{12}$  angegeben. Für  $g = 20$  erhält man  $f_0 = 19$ , d.h. die Anzahl der Henkel ist größer als die Eckenzahl, und es wäre sehr überraschend, wenn hier Realisierungen im  $E^3$  möglich wären. Für Realisierungen in Randkomplexen von 4-Polytopen kämen wegen der Vollständigkeit des 1-Skeletts nur die 2-benachbarten Polytope in Frage. Diese sind aber, soweit untersucht, arm an eingebetteten Flächen [siehe (37) und (38)], was auch gegen eine mögliche Realisierung im  $E^3$  spricht.

Wir geben jetzt eine kurze Aufzählung von bekannten Ergebnissen über Realisierungen von Flächen.

- (56) Der simpliziale Torus mit minimaler Eckenzahl hat den f-Vektor  $f = (f_0, f_1, f_2) = (7, 21, 14)$  und ist in den  $E^3$  einbettbar.

Császár [29] hat ohne nähere Konstruktionshinweise die Koordinaten einer Realisierung im  $E^3$  angegeben. Altshuler [3] zeigt die Realisierbarkeit des Császárschen Torus im zyklischen 4-Polytop mit 7 Ecken und charakterisiert alle in zyklische 4-Polytope einbettbare Tori:

- (57) Eine Triangulation  $M$  des Torus mit  $v \geq 7$  Ecken ist genau dann in ein zyklisches Polytop einbettbar, wenn  $M$  einen Kreis enthält, der von jedem Dreieck von  $M$  mindestens eine Kante enthält.

Altshuler [3] gibt noch ein weiteres komplizierteres Kriterium für die Realisierbarkeit einer Triangulation des Torus im  $E^3$ , aber auch dieses Kriterium ist nicht notwendig, wie durch ein Beispiel gezeigt wird.

Für weitere Realisierungen orientierbarer geschlossener Flächen verweisen wir noch auf die Ergebnisse (36) und (39) von Altshuler bzw. Schulz.

Für nichtorientierbare Flächen ist folgendes lange bekannt:

- (58) Die Triangulation der projektiven Ebene mit minimaler Eckenzahl hat 6 Ecken. Man erhält sie aus dem Ikosaeder durch Identifizierung antipodaler Ecken und sie ist in  $\mathcal{B}(T^5)$  also auch in den  $E^4$  einbettbar.

Möbiusbänder sind von Betke, Schulz und Wills in [21] untersucht worden. Wie beim Torus, wo man außer dem Czásárschen Torus andere Minimalmodelle, z.B. bzgl. der Kantenzahl, findet, gibt es auch hier verschiedene Minimalmodelle.

- (59) Für das Möbiusband gibt es Minimalmodelle mit den f-Vektoren (5, 10, 5) und (6, 10, 4). Das erste ist minimal bzgl.  $f_0, f_1$ , das zweite bzgl.  $f_1, f_2$ , beide sind minimal bzgl. der Norm  $\|f\| = f_0 + f_1 + f_2$ . Der erste Typ ist in ein 4-Polytop einbettbar, der zweite ist in kein Polytop irgendeiner Dimension, wohl aber in den  $E^3$  einbettbar.

Mit Hilfe des Möbiusbandes mit f-Vektor (6, 10, 4) wird in [21] ein nicht-polytopales Diagramm konstruiert.

In [86] studieren Schulz und Wills gelochte Kleinsche Flaschen.

- (60) Es gibt genau 4 gelochte Kleinsche Flaschen, die bzgl.  $\|f\| = f_0 + f_1 + f_2$  minimal sind, und zwar eine mit f-Vektor (7, 14, 6) und drei verschiedene mit f-Vektor (8, 14, 5). Bei (7, 14, 6) sind  $f_0, f_1$ , bei (8, 14, 5) zugleich  $f_1, f_2$  minimal. Alle vier Typen sind nicht in den Randkomplex eines Polytops irgendeiner Dimension einbettbar.

## 5. Schnitte von Graphen und Polytopen

In [69] beschäftigt sich O’Neil mit einem konkreten Problem aus dem Bereich «Erkennung von Mustern und Strukturen»: Wieviel Hyperebenen benötigt man mindestens, um alle Kanten des d-Würfels  $C^d$  zu schneiden, wobei keine der Hyperebenen eine Ecke von  $C^d$  enthält? Das Problem ist immer noch ungelöst, O’Neil konnte lediglich zeigen:

- (61) Man benötigt für  $C^d$  mindestens  

$$d \cdot 2^{d-1} / [\frac{1}{2}(d+1)] \cdot \binom{d}{[\frac{1}{2}d]} \text{ Hyperebenen.}$$

In [47] dehnt Grünbaum die Frage allgemein auf Polytope aus und betrachtet verschiedene Verallgemeinerungen von Hyperebenenschnitten. In [12] beweist Barnette eine Vermutung von Grünbaum:

- (62) Man benötigt mindestens  $\lceil \log_2(d+1) \rceil$  Hyperebenen, um alle Kanten eines d-Polytops zu schneiden. (Hier, wie im folgenden, ist stets vorausgesetzt, daß die Hyperebenen keine Ecke treffen.)

Barnette führt den Beweis dual mit Hilfe von Überdeckungen des  $(d-2)$ -

Skeletts eines Polytops mit scharfen Schattengrenzen. Der Satz läßt sich daher auch sofort als Spezialfall des Satzes (46a) von Pachner ableiten.

In [48] zeigt Grünbaum:

- (63) Für zentrale symmetrische 3-Polytope benötigt man mindestens 3 Hyperebenen, um alle Kanten zu schneiden.

Unbewiesen ist für  $d \geq 4$  die Vermutung Grünbaums, daß man für jedes zentrale symmetrische  $d$ -Polytop mindestens  $\lceil 1 + \log_2 d \rceil$  Hyperebenenschnitte benötigt.

In [54] gibt Katchalski eine genaue kombinatorische Kennzeichnung der 3-Polytope, deren Kanten man (bei geeigneter Realisierung) mit genau 2 Hyperebenen schneiden kann. Sie sind alle nicht zentrale symmetrisch. Aus dieser Charakterisierung erhält man daher auch das Ergebnis (63) von Grünbaum.

Betrachtet man Hyperebenenschnitte vom Graphen des Polytops her, so kann man als größte Verallgemeinerung *Schnitte* (coboundary bei Harary [51]) von zusammenhängenden Graphen betrachten. Hierunter versteht man jede Kantenmenge eines zusammenhängenden Graphen  $G$  vom Typ  $N = \{x_1 \cdot x_2 \in G : x_i \in X_i\}$ , wo  $X_1, X_2$  eine beliebige Partition der Ecken von  $G$  ist. Die maximalen Subgraphen von  $G$  mit Eckenmengen  $X_1$  bzw.  $X_2$  heißen die *Schnittkomponenten* von  $N$ .

Bezeichnet  $m(G)$  die Minimalzahl von Schnitten, die benötigt werden, um alle Kanten von  $G$  zu überdecken und  $r(G)$  die Färbungszahl von  $G$  bzgl. Eckenfärbung, so gilt (Pachner [70]):

$$(64) \quad m(G) = \lceil \log_2 r(G) \rceil.$$

Aus dem Vierfarbensatz folgt daraus sofort:

- (65) Die Kanten eines jeden polyedrischen (planar, 3-zusammenhängend) Graphen sind durch zwei Schnitte überdeckbar.

Sind die Schnittkomponenten eines Schnittes  $N$  zusammenhängend, so heißt  $N$  ein *g-Schnitt* (*g-cut* in [47], *cocycle* in [51]). Äquivalent kann man einen *g-Schnitt* als minimale separierende Kantenmengen definieren. Jeder Schnitt ist Vereinigung paarweise disjunkter *g-Schnitte*. Bezeichne  $m_g(G)$  die Minimalzahl von *g-Schnitten*, die nötig ist, um alle Kanten eines Graphen  $G$  zu überdecken und  $m_g(k)$  sei das Minimum der  $m_g(G)$  genommen über alle  $k$ -zusammenhängenden Graphen. In [47] zeigte Grünbaum  $m_g(k) = 2$  für  $k = 2, 3, 4, 5$ . In [73] verallgemeinert Pachner dies mit Hilfe der Doppelpyramide über  $C^{k-1}$ :

- (66)  $m_g(k) = 2$  für alle  $k$ . Für jede Dimension gibt es Polytope, deren Kantenmenge durch 2 *g-Schnitte* überdeckbar ist.

Ist  $N$  *g-Schnitt* des Graphen  $\text{skel}_1(M)$  einer Mannigfaltigkeit, so ist für jede Zelle  $S$  von  $M$ , die Kanten aus  $N$  enthält,  $N(S) := N \cap \text{skel}_1(S)$  trivialerweise ein Schnitt von  $\text{skel}_1(S)$ . Verlangt man zusätzlich, daß  $N(S)$  stets ein *g-Schnitt* ist, so heißt  $N$  ein *p-Schnitt* von  $M$ . Bei diesem Schnittbegriff geht also nicht nur der Graph, sondern die gesamte Komplexstruktur von  $M$  ein und man nähert sich begrifflich weiter den Hyperebenenschnitten bei Polytopen.

In [74] wird gezeigt:

- (67) In geschlossenen Mannigfaltigkeiten  $M$  mit Dimension  $d \geq 3$  ist genau dann jeder  $g$ -Schnitt ein  $p$ -Schnitt, wenn die Facetten von  $M$  2-benachbart sind.

Über minimale Schnittzahlen mit  $p$ -Schnitten ist bisher nur bekannt [73]:

- (68) Für simpliziale  $d$ -Sphären benötigt man mindestens  $\lceil \log_2(d+1) \rceil$   $p$ -Schnitte ( $g$ -Schnitte) um alle Kanten zu überdecken. Für jedes  $d$  gibt es ein  $(d+1)$ -Polytop, dessen Kantenmenge durch  $\lceil \log_2 \frac{4}{5}(d+2) \rceil$   $p$ -Schnitte überdeckt werden kann.

Wie schon nach (62) erwähnt und wie auch beim Vergleich der Ergebnisse (68) und (48) erkennbar, besteht ein enger Zusammenhang zwischen Schnitten und Submannigfaltigkeiten, der durch den Übergang zu dualen Sphären bzw. Polytopen vermittelt wird. Ist  $M$  eine geschlossene  $(d-1)$ -Pseudomannigfaltigkeit im  $(d-1)$ -Skelett einer  $d$ -Sphäre  $\mathcal{C}$ , so entsprechen den Facetten von  $M$  mittels der dualen Abbildung in der zu  $\mathcal{C}$  dualen Sphäre  $\mathcal{C}^*$  eine gewisse Kantenmenge, die man als Kopseudomannigfaltigkeit von  $\mathcal{C}^*$  bezeichnet. Jede Kopseudomannigfaltigkeit ist ein  $g$ -Schnitt, aber nicht umgekehrt. Folgende Beziehungen sind bekannt (vgl. [88], [74], [70]):

- (69) Die Hyperebenenschnitte von  $d$ -Polytopen entsprechen in dualen Polytopen genau den scharfen  $(d-2)$ -Schattengrenzen.
- (70) Die  $p$ -Schnitte von Sphären sind genau die l.st.z. Kopseudomannigfaltigkeiten.
- (71) Für eine  $n$ -Sphäre  $\mathcal{C}$  ist jeder  $p$ -Schnitt eine Komannigfaltigkeit, wenn für jede Facette  $F$  von  $\mathcal{C}$  eine der folgenden Bedingungen gilt:
- $\dim F \leq 3$ .
  - $F$  hat höchstens  $n+2$  Facetten.
  - $F$  ist ein Stapelpolytop.
  - $F$  ist ein Pyramidoid,  $F = F_1 \cdots F_s$ , wo  $F_1, \dots, F_s$  vom Typ (a), (b) oder (c) sind.

Insbesondere gilt also:

- (72) Für 3-Sphären ist jeder  $p$ -Schnitt eine Komannigfaltigkeit.
- (73) Für simpliziale Sphären ist jeder  $g$ -Schnitt eine Komannigfaltigkeit.

Diese Sätze gestatten es in vielen Fällen, Aussagen über Mannigfaltigkeiten, z.B. hamiltonsche Mannigfaltigkeiten, rein graphentheoretisch zu gewinnen.

Wir wollen noch kurz das unter (67) genannte Einbettungsproblem aufgreifen. Nach (66) und (48) liegt die Schnittzahl für 5-Polytope bzgl.  $p$ -Schnitten bei 2 oder 3.

*Problem:* Man finde ein 5-Polytop, dessen Kanten durch 2  $p$ -Schnitte überdeckt werden können und damit wegen (70) eine geschlossene l.st.z. 3-Pseudomannigfaltigkeit, die in jeder Ecke singulär und in den  $E^4$  einbettbar ist.

#### IV. Hamiltonkreise und Längenexponenten

Ergebnisse und Methoden der Graphentheorie sind wegen ihrer vielen Anwendungsmöglichkeiten sehr vielfältig. Zu den Anwendungsgebieten zählen Probleme aus der linearen Programmierung, der Kodierungstheorie, der kombinatorischen Topologie, der Theorie der konvexen Polytope sowie Fragen im Zusammenhang mit der Nomenklatur organischer Verbindungen, Färbungsprobleme und vieles andere mehr. Wir beschränken uns hier auf die klassische Frage der Graphentheorie nach längsten Kreisen in gewissen Graphenfamilien. Insbesondere betrachten wir *polyedrische Graphen*, d.h. Kantographen von 3-Polytopen, außerdem Kantographen von zellzerlegten geschlossenen Flächen. Die Untersuchungen über polyedrische Graphen werden erheblich erleichtert durch den Satz von Steinitz [vgl. (1)], der es gestattet, sämtliche Probleme auf 3-zusammenhängende planare Graphen zurückzuhören. Eine Graphenfamilie heißt *hamiltonsch*, wenn jeder Graph  $G$  der Familie einen *Hamiltonkreis* besitzt, das ist ein Subgraph von  $G$ , dessen Trägermenge eine geschlossene Jordankurve ist, die alle Ecken von  $G$  enthält.

Tutte [94] bewies 1946 für polyedrische Graphen den folgenden Satz:

- (74) Die Familie der 4-zusammenhängenden planaren Graphen ist hamiltonsch.

In jüngerer Zeit hat es besondere Fortschritte bei polyedrischen Graphen mit beschränkter Eckenzahl bzw. mit beschränkter Eckenzahl der Facetten der zugehörigen 3-Polytope gegeben.

Mit  $\mathcal{G}(q,r)$  bezeichnen wir die Familie der polyedrischen Graphen, deren Seiten höchstens  $q$  Ecken haben und deren Ecken höchstens  $r$ -valent sind. Analog definieren wir die Menge der Kantographen von zellzerlegten geschlossenen Flächen als  $\mathcal{U}(q,r)$ . Setzen wir in diesen Definitionen z.B. voraus, daß alle Ecken genau  $r$ -valent sein sollen, so drücken wir das in der Bezeichnung durch  $\mathcal{G}(q,r)$  bzw.  $\mathcal{U}(q,r)$  usw. aus.

Neben dem Satz von Steinitz wird in einigen Arbeiten über Hamiltonkreise, deren Ergebnisse wir im folgenden beschreiben, die von Ewald in [35] geschilderte «pulling method» verwandt. Diese besteht darin, daß man einen Kreis, der die Kante  $a_1a_2$  eines Dreiecks  $a_1a_2a_3$ , nicht aber die Ecke  $a_3$  enthält, verlängert, indem man  $a_1a_2$  entfernt und dafür dem Kreis die Kanten  $a_1a_3$  und  $a_2a_3$  hinzufügt. Natürlich ist es von Fall zu Fall mehr oder weniger kompliziert zu zeigen, daß man auf diese Weise einen Hamiltonkreis erhalten kann. Oft ist es z.B. notwendig, zwischendurch einen Kreis «zurückzuziehen», d.h. den obigen Prozeß rückgängig zu machen.

Ewald beweist in [35]:

- (75) (a)  $\mathcal{U}(3, 6)$  ist hamiltonsch.

Ferner sind Graphen simplizialer geschlossener Flächen hamiltonsch, wenn eine der beiden folgenden Bedingungen erfüllt sind:

- (b) Von 2 benachbarten Ecken hat stets eine höchstens die Valenz 4.
- (c) Von 2 benachbarten Ecken hat stets eine höchstens die Valenz 5, die andere höchstens die Valenz 8.

$\mathcal{G}(3, 8)$  ist bereits nicht mehr hamiltonsch, wie der Kantograph des Kleetops

über dem Doppeltetraeder, d.i. das Polytop, das entsteht, indem man auf jede Dreiecksseite eines Doppeltetraeders ein Tetraeder setzt, zeigt. Die Lücke zum Ergebnis (75a) ist von Streubel [92] mit ähnlichen Methoden teilweise geschlossen worden:

(76)  $\mathcal{G}(3, 7)$  ist hamiltonsch.

*Problem:* Ist  $\mathcal{U}(3, 7)$  hamiltonsch?

Das duale Analogon zu den zuvor betrachteten Dreiecksgraphen sind die einfachen (3-valenten) Graphen.

Tait [93] vermutete 1890, daß alle einfachen Graphen hamiltonsch sind, womit das Vierfarbenproblem gelöst gewesen wäre. Erst 1946 wurde von Tutte [94] ein Gegenbeispiel mit 46 Ecken gefunden.

Inzwischen sind Gegenbeispiele mit 38 Ecken bekannt [45]. Einfache Polytope mit höchstens 26 Ecken sind jedoch hamiltonsch [20].

$\mathcal{G}(5, 3)$  ist endlich und hamiltonsch. Ob  $\mathcal{G}(6, 3)$  hamiltonsch ist, ist ein offenes Problem.

Es ist jedoch bekannt, daß der Längenexponent von  $\mathcal{G}(6, 3)$  gleich 1, ist, d.h. man kann stets einen gewissen Prozentsatz der Ecken eines solchen Graphen mit einem Kreis treffen. Goodey zeigte in [41] bzw. [42], daß diejenigen Graphen aus  $\mathcal{G}(6, 3)$  hamiltonsch sind, deren Facetten sämtlich 4- oder 6-Ecke bzw. 3- oder 6-Ecke sind.

Kürzlich konstruierte Zaks (private Mitteilung) einen Graphen aus  $\mathcal{G}(7, 3)$ , der nicht hamiltonsch ist.

In [98] beschreibt Zaks einen nicht-hamiltonschen Graphen aus  $\mathcal{G}(8, 3)$ , dessen Facetten nur 5- und 8-Ecke sind.

Läßt man sowohl höhere Eckenzahl als auch höhere Eckenzahl der Seiten zu, so ist bereits  $\mathcal{G}(4, 4)$  nicht mehr hamiltonsch [78]. Immerhin konnte Bielig [24] mit erheblichem Aufwand zeigen:

(77)  $\mathcal{G}(4,=4)$  ist hamiltonsch. Jeder  $\mathcal{G}(4, 4)$ -Graph enthält einen Kreis, der höchstens 2 Ecken des Graphen ausläßt.

Der zweite Teil löst eine Vermutung von Grünbaum.

Zu Graphen von Flächen sind noch die Ergebnisse von Altshuler [4] über Graphen von Tori zu erwähnen. Die entsprechende Graphenfamilie sei mit  $\mathcal{U}_1$  bezeichnet. Altshuler beweist u.a.:

- (78) (a)  $\mathcal{U}_1(=4,=4)$  ist hamiltonsch.
- (b) Jeder Graph aus  $\mathcal{U}_1(=6,=3)$  mit gerader Anzahl von 6-Ecken ist hamiltonsch.
- (c) Jeder 6-zusammenhängende Graph aus  $\mathcal{U}_1(\infty,\infty)$  ist hamiltonsch.

Über Graphen höherdimensionaler Polytope liegen weit weniger Ergebnisse bezüglich der Existenz von Hamiltonkreisen vor. Für simpliziale Mannigfaltigkeiten gibt es ein Resultat von Ewald [35] vom ähnlichen Typ wie (75). Es ist als Spezialfall in einem Satz über hamiltonsche Mannigfaltigkeiten enthalten, den wir in 2. aufgeführt haben.

Ungelöst ist noch die Vermutung von Barnette, daß die Kantengraphen einfacher 4-Polytope hamiltonsch sind. Da jedoch inzwischen der

Vierfarbensatz von Appel und Haken [7] gelöst wurde, gilt nach Barnette und Rosenfeld [18]:

(79) Kantengraphen einfacher 4-Prismen sind hamiltonsch.

In der gleichen Arbeit wird ferner bewiesen:

(80) Kantengraphen von  $k$ -fachen Prismen über 3-Polytopen sind für  $k \geq 2$  stets hamiltonsch.

Wie bereits erwähnt, sind einfache Graphen nicht notwendig hamiltonsch und Grünbaum und Walther [50] zeigten, daß man für dreivalente polyedrische Graphen nicht einmal eine positive untere Schranke für das Verhältnis zwischen der Länge  $h(G)$  des längsten Kreises und der Eckenanzahl  $v(G)$  solcher Graphen  $G$  finden kann. Um dennoch ein vernünftiges Maß für maximale Kreise in nicht hamiltonschen Graphenfamilien zu haben, hat man den *Längenexponenten*  $\sigma(\mathcal{G}) := \liminf \frac{\log h(G)}{\log v(G)}$  eingeführt. Mit ähnlichen Mitteln wie in [35] bewies Ewald [36]:

$$(81) \quad \sigma(\mathcal{U}(4, 4)) = \sigma(\mathcal{U}(3, 6)) = \sigma(\mathcal{U}(3, 7)) = 1.$$

(Das letztgenannte Ergebnis ist durch den Satz (37) von Streubel inzwischen überholt.)

Es ist nicht bekannt, ob der Längenexponent aller polyedrischen Graphen überhaupt positiv ist. Die kleinste bekannte Oberabschätzung stammt von Moon und Moser [68]:

$$(82) \quad \sigma(\mathcal{G}(3, \infty)) \leq \log 2 / \log 3 = 0,63093 \dots$$

Grünbaum und Walther [50] vermuten:  $\sigma(\mathcal{G}(\infty, \infty)) = \frac{\log 2}{\log 3}$

Sie zeigen dort weiterhin u.a.:

$$(83) \quad \begin{aligned} & (a) \quad \sigma(\mathcal{G}(3, 12)), \sigma(\mathcal{G}(6, 6)) \leq \log 5 / \log 7 = 0,82826 \dots \\ & (b) \quad \sigma(\mathcal{G}(15, 3)) \leq \log 22 / \log 23 = 0,98583 \dots \end{aligned}$$

Der Beweis wird durch induktive Konstruktion geführt. Mit ähnlichen Methoden und Argumenten hat Bielig [25] kleinere Oberabschätzungen für Graphenfamilien mit beschränkter Eckenzahl und Eckenanzahl der Seiten gefunden, die gegen  $\log 2 / \log 3$  konvergieren:

$$(84) \quad \sigma(\mathcal{G}(6, 5)), \sigma(\mathcal{G}(3, 15)), \sigma(\mathcal{G}(=5, 30)) \leq \frac{\log 8}{\log 13} = 0,810714,$$

$$(85) \quad \sigma(\mathcal{G}(3, 9 \cdot 2^m)), \sigma(\mathcal{G}(=4, 6 \cdot 2^m)), \sigma(\mathcal{G}(6, 3 \cdot 2^m)) \leq \frac{\log(4 \cdot 2^m - 1)}{\log(4 \cdot 3^m - 1)}.$$

Weitere Ergebnisse kann man aus bekannten Resultaten, z.B. aufgrund der Tatsache gewinnen, daß beim Übergang zu den Medialgraphen einfacher Graphenfamilien und den Radialgraphen simplizialer Graphenfamilien sich der Längenexponent nicht ändert.

Hieraus folgt:  $\sigma(\mathcal{G}(q, =4)) = \sigma(\mathcal{G}(q, 3))$  und  $\sigma(\mathcal{G}(=4, v)) = \sigma(\mathcal{G}(3, q))$ , was z.B. folgende Ergebnisse aus [25] und [50] liefert:

- (86)  $\sigma(\mathcal{G}(12,=4)), \sigma(\mathcal{G}(=4, 12)) \leq \log 5 / \log 7.$   
 (87) (a)  $\sigma(\mathcal{G}(13,=4)) \leq \log 24 / \log 25.$   
 (b)  $\sigma(\mathcal{G}(15,=4)) \leq \log 22 / \log 23.$

Zum Abschluß wollen wir eine von Bielig und Schulz [26] gefundene Beziehung zwischen hamiltonschen Graphenfamilien und solchen Graphenfamilien mit Längenexponent 1 angeben, die insbesondere im Hinblick auf die Vermutung Barnettes, daß jeder einfache Graph mit geradzahliger Eckenzahl der Seiten hamiltonsch ist, interessiert. Zur leichteren Formulierung bezeichne  $\mathcal{G}'(2q,r)$  die Graphen von  $\mathcal{G}(2q,r)$ , deren Seiten alle gerade Eckenzahl haben. Dann gilt:

- (88) Aus  $\sigma(\mathcal{G}'(4q, 3)) = 1$  folgt, daß  $\mathcal{G}'(2q, 3)$  hamiltonsch ist.

Für weitere Ergebnisse über Hamiltonkreise und Längenexponenten verweisen wir noch auf [50], [96] und [45].

Für die Kantengraphen höherdimensionaler Polytope liegen noch recht wenige Aussagen über nicht hamiltonsche Familien vor.

Für die Graphenfamilie  $\Gamma_d(\alpha)$  von Graphen simplizialer d-Polytope mit maximaler Eckenzahl  $\alpha$  zeigten Illing und Walther [53]:

$$(89) \quad \begin{aligned} \text{(a)} \quad & \sigma(\Gamma_4(488)) \leq \frac{\log 8}{\log 16} = 0,75 \ 00 \dots \\ \text{(b)} \quad & \sigma(\Gamma_5(6830)) \leq \frac{\log 10}{\log 25} = 0,71 \ 53 \dots \end{aligned}$$

Moon und Moser zeigten in [68]:

$$(90) \quad \sigma(\Gamma_4(\infty)) \leq \frac{1}{2}.$$

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## Radon's theorem revisited

Radon's theorem is one of the cornerstones of combinatorial geometry. It asserts that each set of  $d + 2$  points in  $\mathbb{R}^d$  can be expressed as the union of two disjoint subsets whose convex hulls have a common point. Moreover, the number  $d + 2$  is the smallest which has the stated property.

The theorem was discovered by Radon [1921] and used by him to obtain a short proof of Helly's theorem (see Danzer, Grünbaum, and Klee [1963]). Radon's own proof of the result is algebraic in nature and probably the simplest possible. Geometric proofs have been given by Markoff [1939], Rado [1952], Peterson [1972], and others. We mention also the papers by Bryant and Webster [1969], Reay [1972], and Cantwell [1974], which place the theorem in an axiomatic setting, or use a generalized definition of convexity. For applications of Radon's theorem to the phase analysis of multicomponent systems, to problems in approximation theory and in the theory of convex polytopes, see Beck [1945], Pepper [1949], Brosowski [1965], Šaškin [1967], Grünbaum [1967], Eckhoff [1976], and Kind and Kleinschmidt [1976].

The first comprehensive survey of Radon's theorem, its relationship to the theorems of Helly and Carathéodory as well as its generalizations and applications, was given by Danzer, Grünbaum, and Klee [1963]. (An updated Russian translation of their paper has been published in *Biblioteka Sbornika Matematika*, Moscow 1968.) However, in the fifteen years since the appearance of the survey a large body of refinements and far-reaching generalizations of Radon's theorem has been assembled. In the process, two main lines of research have emerged. The first is concerned with the geometrical and combinatorial properties of Radon partitions, while the second deals with developments in the spirit of Tverberg's [1966] famous theorem. This will be explained in more detail later. The two lines are intimately connected, and both can be followed also in an abstract setting.

The purpose of this article is to present an up-to-date survey of Radon's theorem and its relatives. We have attempted to summarize the known results in the area, to state the more important unsolved problems, and to give appropriate references. It should be remarked that some of the material treated here has already appeared in an earlier report on the same topic (see Doignon and Valette [1975]).

The paper is organized into six sections which emphasize the different aspects of the subject. The section headings are as follows.

1. Radon partitions.
2. The number of Radon partitions.
3. Radon vectors.
4.  $(m,k)$ -divisible sets.
5. Generalizations.
6. Abstract convexity.

With the exception of the last section the setting of the survey is the  $d$ -

dimensional affine space  $R^d$ . A  $k$ -flat in  $R^d$  is a  $k$ -dimensional subspace of  $R^d$ . A set  $X \subset R^d$  is in general position in  $R^d$  if any  $d+1$  or fewer points of  $X$  are affinely independent. The standard abbreviations  $\text{aff } X$ ,  $\text{conv } X$ ,  $\text{bd } X$ ,  $\dim X$ , and  $\text{card } X$  are used to denote the affine hull, convex hull, boundary, dimension, and cardinality of  $X$ .

The following special terminology will also be used. A partition of  $X$  is a finite set of pairwise disjoint subsets of  $X$  whose union is  $X$ . Notice that some of the subsets may be empty. A partition in  $X$  is a partition of some subset of  $X$ . If  $\{X_1, \dots, X_m\}$ ,  $\{Y_1, \dots, Y_m\}$  are two partitions in  $X$ , then  $\{X_1, \dots, X_m\}$  is said to extend  $\{Y_1, \dots, Y_m\}$  provided, after a suitable renumbering,  $X_i \supseteq Y_i$ ,  $i = 1, \dots, m$ .

## 1. Radon partitions

Let  $X$  be a set of points in  $R^d$ , and let  $\{X_1, X_2\}$  be a partition in  $X$ . Then  $\{X_1, X_2\}$  is called a *Radon partition* in  $X$ , and  $X_1, X_2$  are called the *Radon components* of  $\{X_1, X_2\}$ , provided  $\text{conv } X_1 \cap \text{conv } X_2 \neq \emptyset$ . If  $X = X_1 \cup X_2$ , then we say that  $\{X_1, X_2\}$  is a Radon partition of  $X$ .

Radon's theorem asserts that if  $X$  is a set of at least  $d+2$  points in  $R^d$ , then  $X$  admits a Radon partition. More generally, a Radon partition of  $X$  exists if, and only if,  $X$  is affinely dependent. This can be viewed as the case  $Y = \emptyset$  of the following theorem.

- (1.1) **Theorem.** *A set  $Y \subset X$  is contained in a component of some Radon partition of  $X$  if, and only if,*
- (i)  $\text{conv } Y \cap \text{aff}(X \setminus Y) \neq \emptyset$ , or
  - (ii)  $X \setminus Y$  is affinely dependent.

For finite sets in general position in  $R^d$  this was first proved by Hare and Kenelly [1971]. The above generalization has been given by Petty [1975] and Doignon and Valette [1975]. Note that condition (ii) is satisfied when  $\text{card}(X \setminus Y) \geq d+2$ , and can be dropped when  $X$  is in general position in  $\text{aff } X$ .

A special case of Theorem (1.1) is worth singling out. Let  $X$  be a set of  $d+2$  points in general position in  $R^d$ . Then two points of  $X$  lie in the same component of some Radon partition of  $X$  if, and only if, they are separated by the hyperplane through the remaining  $d$  points. Furthermore, the Radon partition of  $X$  is unique. This result has been discovered independently by many authors (see Khudekov [1941], Beck [1945], Pepper [1949], Proskuryakov [1959], and Kosmák [1963]). The paper by Hare and Kenelly [1971] contains a similar characterization of the unique partition in which the separating hyperplanes are replaced by spheres.

- (1.2) **Theorem.** *Let  $X$  be a set of  $d+2$  points in general position in  $R^d$ , not all on a common  $(d-1)$ -sphere. Two points of  $X$  lie in the same Radon component of  $X$  if, and only if, they are both inside (or both outside) the respective  $(d-1)$ -sphere through the other  $d+1$  points.*

For a geometric proof of Theorem (1.2) and of the result preceding it, see Peterson [1972].

In order to study the Radon partitions in a given set more closely we define the notion of a *primitive Radon partition in X*. This is a Radon partition in X which is minimal in the sense that it does not extend any other Radon partition in X. The concept was introduced by Hare and Kenelly [1971] and has proved useful in various investigations.

The following theorem collects the basic properties of primitive Radon partitions. It is essentially due to Hare and Kenelly [1971] (but see also Breen [1972], Peterson [1972], Petty [1975], Doignon and Valette [1975]).

(1.3) **Theorem.** *Every Radon partition in X extends a primitive Radon partition in X. If  $\{Y_1, Y_2\}$  is a primitive Radon partition in X, then*

- (i)  $Y_1 \cup Y_2$  is in general position in  $\text{aff}(Y_1 \cup Y_2)$ ,
- (ii)  $\text{card}(Y_1 \cup Y_2) = \dim(Y_1 \cup Y_2) + 2 \leq d + 2$ ,
- (iii)  $\dim(\text{conv } Y_1 \cap \text{conv } Y_2) = 0$ .

Thus X admits a unique Radon partition if, and only if, X is in general position in  $\text{aff } X$  and  $\text{card } X = \dim X + 2$ .

For connections between primitive Radon partitions in X and *poonems* of X, see Petty [1975].

The collection of all Radon partitions of a given set describes, in a sense, the ‘Radon type’ of the set. This is made precise in the following definition.

Two sets X,Y in  $R^d$  are said to be *Radon equivalent* provided there exists a bijection  $\sigma: X \rightarrow Y$  which preserves Radon partitions (i.e.  $\{X_1, X_2\}$  is a Radon partition of X iff  $\{\sigma X_1, \sigma X_2\}$  is a Radon partition of Y). Clearly, Radon equivalence is an equivalence relation; its representatives are called *Radon types* (see Eckhoff [1974], [1975]).

There is a slightly different way of looking at Radon equivalence. It is obviously possible to determine the Radon partitions in X, and in particular the primitive ones, if the Radon partitions of X are given. Hence, in view of (1.3), two sets X,Y are Radon equivalent provided there is a bijection  $\sigma: X \rightarrow Y$  which preserves primitive Radon partitions. For finite sets in  $R^d$ , this is related to the definition of “dependence structures” given implicitly in Brylawski [1976]. Let  $\mathcal{R}(X)$  denote the matroid whose independent sets are the affinely independent subsets of X (for general information about matroids, see Welsh [1976]). Then the circuits (i.e. minimal dependent sets) of  $\mathcal{R}(X)$  correspond to the primitive Radon partitions in X. Hence if X and Y are Radon equivalent, then  $\mathcal{R}(X)$  and  $\mathcal{R}(Y)$  are isomorphic. The converse statement is false.

The main (but rather hopeless) problem about Radon types is the following.

(1.4) **Problem.** *Characterize the Radon types of finite sets of points in  $R^d$ .*

Very little is known about the Radon types even of sets in general position in  $R^d$ . Using (1.8) below Breen [1973] has described the types which arise from the vertex sets of cyclic polytopes in  $R^d$  (for general information about convex polytopes, see Grünbaum [1976]). Eckhoff [1969] has done the same for certain sets in general position in  $R^d$  which are not the vertex sets of convex polytopes. The Radon types of sets of  $d + 3$  points in general position in  $R^d$  can be classified by means of standard (distended) Gale diagrams (see Eckhoff [1975]). Let  $t(d, d + 3)$  denote the number of such Radon types. Then we have

$$(1.5) \quad t(d,d+3) = 2^{\lceil \frac{d}{2} \rceil} - 1 + \frac{1}{4(d+3)} \sum \varphi(h) 2^{\frac{d+3}{h}},$$

where  $\varphi(h)$  is Euler's  $\varphi$ -function and the summation extends over all odd divisors  $h$  of  $d+3$ . The proof of (1.5) is based on Perles' formula for the number of simplicial  $d$ -polytopes with  $d+3$  vertices (see Grünbaum [1967]). The corresponding number  $t(d,d+2)$  for sets of  $d+2$  points is  $\lceil \frac{d}{2} \rceil + 1$ .

In passing we remark that Eckhoff [1976] has characterized the sets  $X$  in  $\mathbb{R}^d$  with the following property: If  $\{X_1, X_2\}$  is any Radon partition of  $X$ , then  $\dim(\text{conv } X_1 \cap \text{conv } X_2) = 0$ . It turns out that such a set can have at most  $2d+1$  points. Moreover, the intersection  $\text{conv } X_1 \cap \text{conv } X_2$  does not depend on  $\{X_1, X_2\}$ . The corresponding number of Radon types (of sets affinely spanning  $\mathbb{R}^d$ ) equals

$$2p(1) + \cdots + 2p(d) - d + 1,$$

where  $p(k)$  is the number of partitions of  $k$ .

We turn now to a description of the relations between the combinatorial type of a convex polytope and the Radon type of its vertex set. This relationship was first investigated by Breen [1970], [1972] and later by Eckhoff [1975] but seems far from being fully explored.

The main result of Breen [1972] is as follows (for a simpler proof, see Doignon and Valette [1975]).

(1.6) **Theorem.** *Let  $X$  be the vertex set of a convex polytope in  $\mathbb{R}^d$ , and suppose  $Y \subset X$ . Then the following assertions are equivalent:*

- (i)  $\text{conv } Y$  is a face of  $\text{conv } X$ ,
- (ii) if  $\{X_1, X_2\}$  is a primitive Radon partition in  $X$ , then  $X_1 \subset Y$  implies  $X_2 \subset Y$ .

The case  $Y = X$  is included, meaning that  $\text{conv } X$  is a face of itself. Theorem (1.6) enables one to read off the facial structure of  $\text{conv } X$  from the set of Radon partitions of  $X$ . Thus we have the following corollary.

(1.7) **Corollary.** *Two convex polytopes in  $\mathbb{R}^d$  are combinatorially equivalent if their vertex sets are Radon equivalent.*

Simple examples show that the converse of (1.7) is false. The condition of Radon equivalence in (1.7) can be considerably weakened in that only relatively few Radon partitions have to be checked. Breen [1972] has proved that the combinatorial type of  $\text{conv } X$  is determined, for any given  $x \in X$ , by the primitive Radon partitions  $\{Y_1, Y_2\}$  in  $X$  with either  $x \in Y_1 \cup Y_2$ , or  $\text{card}(Y_1 \cup Y_2) \leq d+1$ .

For simplicial polytopes Theorem (1.6) takes the following simpler form.

(1.8) **Theorem.** *Let  $X$  be the vertex set of a simplicial convex polytope in  $\mathbb{R}^d$ , and suppose  $Y \subset X$ . Then  $\text{conv } Y$  is a face of  $\text{conv } X$  if, and only if, no subset of  $Y$  is a Radon component of  $X$ .*

For any  $x \in X$ , the combinatorial type of  $\text{conv } X$  is determined by the primitive

Radon partitions in  $X$  which contain  $x$  in one component. Again the combinatorial type does not determine the Radon type, even if  $X$  is in general position in  $\mathbb{R}^d$ . (However, this is true when  $X$  consists of at most  $d+3$  points in general position; see Eckhoff [1975].) Breen [1970] has shown that if  $X$  is in general position in  $\mathbb{R}^d$ , then the combinatorial type of  $\text{conv } X$  can be ‘reconstructed’ from an appropriate selection of  $\text{card } X-d-1$  primitive Radon partitions in  $X$ , and this number cannot be replaced by a smaller number.

## 2. The number of Radon partitions

Throughout this section,  $X$  will be a set of  $s$  points in  $\mathbb{R}^d$ . By  $N(X)$  we denote the number of distinct Radon partitions of  $X$ . Hence  $N(X) \geq 1$ , if  $s \geq d+2$ , and  $N(X) = 1$  if, and only if,  $s = \dim X + 2$  and  $X$  is in general position in  $\text{aff } X$ . More generally, letting

$$N(r,s) := \sum_{i=0}^{s-r} \binom{s-1}{i},$$

we have the following theorem.

(2.1) **Theorem.** *If  $X$  is a set of  $s$  points in  $\mathbb{R}^d$ , then  $N(X) \geq N(d+2,s)$ . Moreover, equality holds if, and only if,  $X$  is in general position in  $\mathbb{R}^d$ .*

Note that  $N(r,s)=0$  when  $s < r$ .

Theorem (2.1) was first obtained in Eckhoff [1969]. The proof uses a result on partitions of space by hyperplanes which has appeared many times in the literature (for references and a unified treatment of the underlying theory, see Zaslavsky [1975]). A direct geometrical proof is also possible. In a remarkable paper by Sierksma and Boland [1974] the first part of (2.1) has recently been generalized to arbitrary “convexity spaces” (see Section 6).

Brylawski [1976] has studied the number  $N(X)$  from a matroid-theoretic point of view. He showed that  $N(X)$  is an invariant of the matroid  $\mathcal{R}(X)$  defined in Section 1 and gave a new proof of Theorem (2.1). His paper contains several interesting representations for  $N(X)$ , two of which will be described here. Note that Brylawski’s “Radon number”  $R(X)$  is equal to  $2(2^{s-1} - N(X))$ .

The first result is as follows.

(2.2) **Theorem.**  $N(X) = 2^{s-1} - \frac{1}{2} \sum_z |\mu(0,z)|$ .

Here  $\mu(0,z)$  is the Möbius function computed in the lattice of those flats in  $\mathbb{R}^d$  which are spanned by subsets of  $X$ , and the sum is over all such flats (see Welsh [1976] for a definition of these terms). The formula is essentially due to Zaslavsky [1975].

The second result is the dualized version of a theorem of Winder [1966].

(2.3) **Theorem.**  *$N(X)$  equals the number of subsets  $Y$  of  $X$  for which  $\dim Y + \text{card } Y$  is even.*

It would be interesting to find a ‘natural’ bijection between the set of Radon

partitions of  $X$  and the set of subsets of  $X$  whose sum of dimension and cardinality is even.

Theorems (2.1) and (2.3) can be combined to yield the following formula which facilitates the actual computation of  $N(X)$ .

$$(2.4) \quad N(X) = N(d+2,s) + \sum_{\substack{i,j \neq d \\ i+j \leq d}} N_{ij} + \sum_{\substack{i,j \equiv d \\ j \leq d}} N_{ij} - \sum_{\substack{i \neq j, d \\ j > d}} N_{ij}.$$

Here  $N_{ij}$  denotes the number of subsets  $Y$  of  $X$  satisfying  $\dim Y = i$  and  $\text{card } Y = j$ , and the congruences are to be taken modulo 2. For  $d=2$ , the formula reduces to (8.24) in Eckhoff [1969].

We turn now to the problem of establishing upper bounds for  $N(X)$ . When no restriction is placed on  $X$ , then  $N(X) \leq 2^{s-1} - s$ , and equality holds if, and only if,  $X$  is a set of collinear points. However, this bound can be considerably lowered when attention is restricted to sets of maximal dimension.

(2.5) **Theorem.** *Let  $X$  be a set of  $s$  points in  $R^d$ , and suppose  $\dim X = d$ . Then*

$$N(X) \leq 2^{s-1} - 2^{d-1}(s-d+1),$$

*with equality if, and only if,  $X$  consists of  $s-d+1$  collinear points and  $d-1$  affinely independent points which together span  $R^d$ .*

Theorem (2.5) was proved in a somewhat different context by Dowling and Wilson [1974]. For a geometric proof (and references to other proofs), see Brylawski [1976].

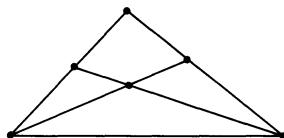
If  $X$  is *non-pyramidal* in  $R^d$ , i.e.  $\dim(X \setminus \{x\}) = d$  for each  $x \in X$ , then the term  $2^{d-1}(s-d+1)$  above can be approximately doubled. This was shown in another paper by Brylawski [1977] (where  $2(2^{s-1} - N(X))$  is denoted by  $\gamma(X)$ ). In geometrical language the result reads as follows.

(2.6) **Theorem.** *Let  $d \geq 2$ ,  $s \geq d+3$ ,  $(d,s) \neq (3,7)$ , and let  $X$  be a non-pyramidal set of  $s$  points in  $R^d$ . Then*

$$N(X) \leq 2^{s-1} - (2^d - 1)(s-d),$$

*with equality if  $X$  consists of  $s-d$  collinear points and  $d$  points spanning a hyperplane through one of the collinear points.*

In the example below we have  $d=2$ ,  $s=6$ , and  $N(X)=20$ . Thus the sets described in Theorem (2.6) do not exhaust the class of all sets for which the upper bound is achieved. A characterization of this class is not known.



Strangely enough the best upper bound in the case  $(d,s)=(3,7)$  is one higher than would be expected from (2.6).

(2.7) **Theorem.** *Let  $X$  be a non-pyramidal set of 7 points in  $\mathbb{R}^3$ . Then  $N(X) \leq 37$ , with equality if, and only if,  $X$  is the union of three sets of 3 collinear points, each two of which intersect in at most one point, and which together span  $\mathbb{R}^3$ .*

The number of Radon types of such sets is 11.

It should be observed that Theorems (2.5), (2.6), and (2.7) are geometric instances of more general results which apply to all simple (respectively connected simple) matroids of dimension  $d$  and cardinality  $s$ . Since  $2(2^{s-1} - N(X))$  turns out to be the sum of the “Whitney numbers of the first kind” associated to  $\mathcal{R}(X)$  (see Brylawski [1977]), it may be asked whether these numbers (and the corresponding “reduced” Whitney numbers studied by Brylawski) have a geometrical meaning.

In conclusion we pose the following problem.

(2.8) **Problem.** *Determine the smallest possible number of primitive Radon partitions in a set of  $s$  points in  $\mathbb{R}^d$ . For which sets is this number attained?*

It is easily seen that the corresponding largest possible number (for sets of dimension  $d$ ) is  $\binom{s}{d+2}$ ; it is attained if, and only if, the set is in general position in  $\mathbb{R}^d$  (see Welsh [1976]). For recent partial results on Problem (2.8), see Doignon [1979].

### 3. Radon vectors

Though some of the results in this section remain valid for arbitrary sets we assume, for convenience, that all sets considered are in general position in  $\mathbb{R}^d$ . Let  $\mathcal{M}_s^d$  denote the class of such sets which consist of  $s$  points. Throughout, the abbreviations  $n = \lceil \frac{d}{2} \rceil$  and  $t = \lceil \frac{s}{2} \rceil$  will be used.

When studying the Radon partitions of a set one is not only interested in their number but also in the size of their components. Thus the following definitions seem appropriate. A partition  $\{X_1, X_2\}$  in  $X$  is of type  $\{s_1, s_2\}$  provided  $\text{card } X_1 = s_1$  and  $\text{card } X_2 = s_2$ . Suppose  $X \in \mathcal{M}_s^d$ , and denote by  $r_i = r_i(X)$ , for  $i = 1, \dots, t$ , the number of Radon partitions of  $X$  of type  $\{i, s-i\}$ . Similarly, for  $i = 1, \dots, n+1$ , denote by  $p_i = p_i(X)$  the number of primitive Radon partitions in  $X$  of type  $\{i, d+2-i\}$ . Then the vectors

$$(r_1, \dots, r_t) \text{ and } (p_1, \dots, p_{n+1})$$

are called respectively the *Radon vector* and the *primitive Radon vector* of  $X$ . Clearly, both vectors depend only on the Radon type of  $X$ . The general problem is to describe the sets of all possible Radon vectors and primitive Radon vectors, but unfortunately our present knowledge about these sets is rather limited.

(3.1) **Problem.** *Find an intrinsic characterization of those vectors  $(r_1, \dots, r_t)$  and  $(p_1, \dots, p_{n+1})$  which occur respectively as the Radon vector and the primitive Radon vector of some set  $X \in \mathcal{M}_s^d$ .*

It appears that Problem (3.1) is largely inaccessible, and no conjecture as to the form of the answer has, so far, been proposed. Even the “upper bound problem” of determining the numbers

$$r_{\max}(d,s,i) := \max[r_i(X) | X \in \mathcal{M}_s^d], i = 1, \dots, t,$$

$$p_{\max}(d,s,i) := \max[p_i(X) | X \in \mathcal{M}_s^d], i = 1, \dots, n+1,$$

has hardly been touched (see Eckhoff [1969]). The same applies to the corresponding “lower bound problem”; however, this is of interest only when  $i \geq n+1$  (see (3.4)).

Before stating the known results we write down the following equations which hold for each set  $X \in \mathcal{M}_s^d$ . Note that (3.2) is a reformulation of (2.1), while (3.3) is trivial.

$$(3.2) \quad \sum_{i=1}^t r_i = N(d+2,s).$$

$$(3.3) \quad \sum_{i=1}^{n+1} p_i = \binom{s}{d+2}.$$

Let us first remark on the special case  $s=d+2$  where the two types of Radon vectors coincide. The sequence  $(r_1, \dots, r_{n+1})$  is the Radon vector of some set  $X \in \mathcal{M}_{d+2}^d$  if, and only if, for some  $k \in \{1, \dots, n+1\}$ ,

$$r_i = \delta_{ik}, i = 1, \dots, n+1.$$

Here  $\delta_{ik}$  is the Kronecker delta. Hence there are  $n+1$  different Radon vectors. Now if  $s \geq d+3$  the behaviour of Radon partitions changes completely as becomes apparent from the following theorem.

(3.4) **Theorem.** *Let  $s \geq d+3$  and  $X \in \mathcal{M}_s^d$ . Then, for some  $k \in \{0, \dots, n\}$ ,*

- (i)  $r_i = p_i = 0, i = 1, \dots, k,$
- (ii)  $r_i > 0, i = k+1, \dots, t,$
- (iii)  $p_i > 0, i = k+1, \dots, n+1.$

*Conversely, for any  $k \in \{1, \dots, n\}$ , the vertex set of a  $k$ -neighbourly but not  $(k+1)$ -neighbourly convex polytope satisfies (i), (ii), and (iii).*

For a definition of  $k$ -neighbourliness, see Grünbaum [1967]. It is understood that if  $k=0$ , all  $r_i$  and  $p_i$  are positive. This occurs when some point of  $X$  lies in the convex hull of the remaining points.

Theorem (3.4) was proved by Shephard [1969] for Radon vectors and by Eckhoff [1974] (employing Shephard's method) for primitive Radon vectors. Both proofs make use of the *Gale transform technique* developed in Grünbaum [1967] which can be successfully applied to establishing various generalizations of Radon's theorem (see Reay [1968], Eckhoff [1975], Doignon and Valette [1975], Doignon [1978], [1979]).

The following is an easy deduction from (3.4).

(3.5) **Theorem.** *Every linear relation among  $p_1, \dots, p_{n+1}$  which is satisfied by the primitive Radon vectors of all sets  $X \in \mathcal{M}_s^d$ , is a multiple of (3.3).*

For a proof of (3.5), see Eckhoff [1974]. The analogous result for Radon vectors is false since, for instance,

$$\sum_{i=1}^{n+2} (-1)^i r_i = 0$$

when  $s=d+3$  and  $d=2n+1$  (see (3.2), (3.3), and (3.7) below).

We turn now to the case  $s=d+3$  where the problem of characterizing all possible Radon vectors and primitive Radon vectors can be completely solved. This is due to the fact that the Gale transforms of a set  $X \in \mathcal{M}_{d+3}^d$  have dimension two.

(3.6) **Theorem.** *The sequence  $(p_1, \dots, p_{n+1})$  is the primitive Radon vector of some set  $X \in \mathcal{M}_{d+3}^d$  if, and only if, in addition to (3.3) and the existence of an index  $k$  as described in (3.4), the components  $p_1, \dots, p_n$  are even.*

This was proved in Eckhoff [1974]. From (3.6) it follows that the number of primitive Radon vectors in the case  $s=d+3$  is  $2^{n+1}-1$ . The corresponding characterization of Radon vectors was obtained in Eckhoff [1975]. It can easily be derived from (3.6) once the following formulae have been established. For a proof of these, see Eckhoff [1974].

(3.7) **Theorem.** *Let  $X \in \mathcal{M}_{d+3}^d$ . Then, for  $i=1, \dots, [\frac{d+1}{2}]$ , the following relations between the Radon vector and the primitive Radon vector of  $X$  hold:*

- (i)  $r_i = \frac{1}{2}(p_i + p_{i-1})$ ,
- (ii)  $p_i = 2 \sum_{j=0}^i (-1)^j r_{i-j}$ .

Here we put  $r_0 = p_0 = 0$ . In other words, the Radon vector of  $X$  completely determines the primitive Radon vector (and *vice versa*). As examples with  $(d,s)=(4,8)$  show, this is no longer true when  $s \geq d+4$ . In fact, the formulae expressing the primitive Radon vector of a set  $X \in \mathcal{M}_{d+4}^d$  in terms of the Radon vector depend on the f-vector of the simplicial polytope  $\text{conv } X$  (for a definition of f-vectors, see Grünbaum [1967]). This result is due to Kramer [1975].

(3.8) **Theorem.** *Let  $s=d+4$  and  $X \in \mathcal{M}_s^d$ . Then, for  $i=1, \dots, n+1$ ,*

$$r_i = p_{i-1} - \sum_{j=0}^i (-1)^{i-j} \binom{s-j}{i-j} f_{j-1},$$

where  $p_0 = 0$ ,  $f_{-1} = 1$ , and  $(f_0, \dots, f_{d-1})$  is the f-vector of  $\text{conv } X$ .

Notice that (3.2), (3.3), and (3.8) together imply, for  $i=n+2$ , the additional relation

$$r_i = p_{i-1} - \sum_{j=0}^{i-1} (-1)^{i-j} \binom{s-j-1}{i-j-1} f_{j-1}.$$

It can be shown that the equations in (3.8) also hold when  $s=d+3$ , and are then equivalent to those in (3.7) (see Eckhoff [1975]). They certainly do not hold when  $s$  is large compared to  $d$ . However, we conjecture that Theorem (3.8) remains true for all  $s \geq d+3$  provided  $X$  is the vertex set of a convex polytope. This would imply that the primitive Radon vector of such a set  $X$  is determined

by the Radon vector of  $X$  and the  $f$ -vector of  $\text{conv } X$ . (Kramer [1975] has proved that if  $X \in \mathcal{U}^3$  is the vertex set of a convex polytope, then the Radon vector of  $X$  depends only on  $s$ .) Furthermore, the Dehn-Sommerville equations for  $\text{conv } X$  (see Grünbaum [1967]) would yield the linear relation

$$(3.9) \quad \sum_{i=1}^{n+1} a_i(r_i - p_{i-1}) = a_0,$$

where

$$a_i := \binom{s-n-i-2}{s-d-2} + \delta_{d,2n} \binom{s-n-i-1}{s-d-2}.$$

Returning to the general case, we remark that Kramer [1975] has also obtained the equation

$$(3.10) \quad \sum_{i=1}^{t-1} b_{is} r_i = \sum_{i=1}^n b_{i,d+2} p_i + c_{ds},$$

which holds for all  $s \geq d+2$  and all sets  $X \in \mathcal{U}_s^d$ . The coefficients  $b_{ij}$  are given by

$$b_{ij} := \left( i - \left[ \frac{j}{2} \right] \right) \left( i - \left[ \frac{j+1}{2} \right] \right),$$

and the constant  $c_{ds}$  can be determined by considering, say, the vertex set of the cyclic polytope  $C(s,d)$  (see Grünbaum [1967]). The (rather tedious) calculation uses the result of Breen [1973] described in Section 6 and is omitted. If  $s=d+4$ , then (3.10) reduces to (3.9), and consequently  $c_{d,d+4} = b_{0,d+4}$ . It would be interesting to have a complete system of linear relations (including (3.2), (3.3), and (3.10)) which are satisfied by the Radon vector and the primitive Radon vector of every set  $X \in \mathcal{U}^d$ .

#### 4. $(m,k)$ -divisible sets

According to Reay [1968] a set  $X \subset R^d$  is said to be  $(m,k)$ -divisible<sup>1</sup> if it can be partitioned into  $m$  subsets whose convex hulls intersect in a set of dimension at least  $k$ . (We always assume  $m \geq 2$  and  $0 \leq k \leq d$ .) An  $(m,0)$ -divisible set is also called  $m$ -divisible (see Tverberg [1966], Doignon and Valette [1975], [1977]), but this is not to be confused with the notion of “divisible set” considered by Larman [1972] (see Section 5).

Let  $\gamma(d,m,k)$  denote the smallest integer  $\gamma$  with the property that each set of at least  $\gamma$  points in  $R^d$  is  $(m,k)$ -divisible. If no such integer exists, put  $\gamma(d,m,k) = \infty$ . The function  $\gamma(d,m,0)$  was first studied by Rado [1952] and Birch [1959] (see Danzer, Grünbaum, and Klee [1963] for a discussion of their results). In particular, Birch conjectured the following theorem which was eventually proved by Tverberg [1966]. It is regarded by now as a “classical” result in combinatorial geometry.

<sup>1</sup> Here we deviate from the generally adopted notation  $(r,k)$ -divisible in order to reserve the letter  $r$  for the Radon number of a convexity space (see Section 6).

(4.1) **Theorem.**  $\gamma(d, m, 0) = (m - 1)(d + 1) + 1$ .

When  $m = 2$ , this is Radon's theorem.

As an immediate consequence of Theorem (4.1) one obtains a special case of a theorem of Rado (see Tverberg [1966]): For each set of  $(m - 1)(d + 1) + 1$  points in  $\mathbb{R}^d$  there exists a point  $x \in \mathbb{R}^d$  such that any closed halfspace containing  $x$  contains at least  $m$  of the points. For a combinatorial application of (4.1), see Tverberg [1971]. For an application to the theory of convex polytopes, see McMullen and Walkup [1971].

Tverberg's original proof of Theorem (4.1) can be somewhat simplified (though it is still by no means easy). For example, the notion of "algebraic independence" which he employed can be replaced by the weaker (and more geometric) notion of "strong independence" introduced by Reay [1968]. A set  $X \subset \mathbb{R}^d$  is called *strongly independent* provided each finite collection  $\{X_1, \dots, X_t\}$  of pairwise disjoint subsets of  $X$  has the following property: if  $\text{card } X_i = d_i + 1 \leq d + 1$ , then

$$\dim \left( \bigcap_{i=1}^t \text{aff } X_i \right) = \max \left\{ -1, d - \sum_{i=1}^t (d - d_i) \right\}.$$

Since  $X$  is easily seen to be in general position in  $\mathbb{R}^d$ , the condition states that if the flats  $\text{aff } X_i$  have a nonempty intersection, then their codimensions add up to the codimension of the intersection. This keeps the flats from forming pencils of lines, planes, etc.

A further simplification of Tverberg's proof has recently been obtained by Doignon and Valette [1975], [1977] who showed, moreover, that Theorem (4.1) remains valid in any affine space over an ordered division ring. This is not obvious because several steps in Tverberg's proof rely heavily on the properties of real numbers.

Doignon and Valette work with a notion of *full independence* which is defined in the same way as strong independence except that in the condition above  $\text{aff } X_i$  is replaced by  $\text{proj } X_i$  (i.e. the projective hull of  $X_i$  in the projective extension of  $\mathbb{R}^d$ ). Neither independence condition is implied by the other; in fact, a strongly independent set is maximal for inclusion if, and only if, it is not fully independent (see Doignon and Valette [1977]).

Theorem (4.1) is best possible in the sense that no strongly (or fully) independent set of  $s$  points is  $m$ -divisible if  $s < \gamma(d, m, 0)$ . The collection of such sets is open and dense in the space of all sets of  $s$  points in  $\mathbb{R}^d$  (see Doignon and Valette [1975]). The following generalization of Theorem (4.1) was obtained in Tverberg [1968].

(4.2) **Theorem.** Let  $m \geq 2$ ,  $0 \leq c \leq d$ ,  $s = (m - 1)(c + 1) + 1$ , and let  $F_1, \dots, F_s$  be flats of codimension  $c$  in  $\mathbb{R}^d$ . Then it is possible to split the set  $\{1, \dots, s\}$  into  $m$  pairwise disjoint subsets  $A_1, \dots, A_m$  such that the intersection

$$\bigcap_{i=1}^m \left( \text{conv} \left( \bigcup_{j \in A_i} F_j \right) \right)$$

contains a flat of codimension  $c$ .

When  $c = d$ , this reduces to (4.1). The case  $m = 2$  was proved earlier by De Santis [1957]. It can be shown that the number  $(m - 1)(c + 1) + 1$  is always best possible.

We return now to the discussion of  $\gamma(d,m,k)$ . Clearly, when  $k > 1$  no set of collinear points can be  $(m,k)$ -divisible. Hence  $\gamma(d,m,k) = \infty$  for  $k > 1$ , and so it remains to consider the case  $k = 1$ . The following conjecture has been proposed by Reay [1978].

(4.3) **Conjecture.**  $\gamma(d,m,1) = 2d(m - 1) + 2$ .

The conjecture is trivially true when  $d = 1$ ; it was proved for  $m = 2$  by Eckhoff [1976] and for  $d = 2$  by Reay [1978]. That the right-hand side in (4.3) may not be replaced by a smaller number is demonstrated by the following example. Take  $d$  lines which intersect in a point  $x$  and together span  $R^d$ , and take  $m - 1$  points on each of the open halflines emanating from  $x$ . The resulting set of  $2d(m - 1) + 1$  points (including  $x$ ) is not  $(m,1)$ -divisible because each point  $y \neq x$  lies on the boundary of a closed halfspace which contains at most  $m - 1$  of the points. On the other hand, Reay [1978] has shown that  $\gamma(d,m,1) \leq 2(d + 1)(m - 1) + 1$ . His paper contains several interesting results which strongly support the conjecture.

For  $k > 1$ , one has to impose independence conditions (for example, strong independence) in order to ensure that the sets considered be  $(m,k)$ -divisible. This was carried out by Reay [1968]. Let  $\delta(d,m,k)$  denote the smallest integer  $\delta$  such that each strongly independent set of  $\delta$  points in  $R^d$  is  $(m,k)$ -divisible. Reay's result is as follows.

(4.4) **Theorem.**  $\delta(d,m,k) = (m - 1)(d + 1) + k + 1$ .

Theorem (4.4) is a direct consequence of the following stronger result (also due to Reay [1968]).

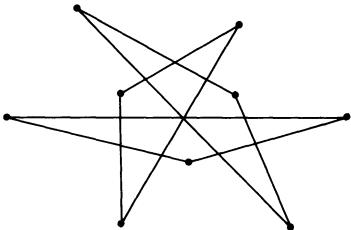
(4.5) **Theorem.** *Let  $X$  be a strongly independent set of  $(m - 1)(d + 1) + k + 1$  points in  $R^d$ , and suppose that  $X$  is partitioned into  $m$  subsets  $X_1, \dots, X_m$  satisfying  $\text{card } X_i \leq d + 1$ ,  $i = 1, \dots, m$ . Then if the intersection*

$$\bigcap_{i=1}^m \text{conv } X_i$$

*is nonempty, it is of dimension  $k$ .*

We remark that no strongly independent set of less than  $\delta(d,m,k)$  points in  $R^d$  is  $(m,k)$ -divisible. This follows at once from the definitions. Both (4.4) and (4.5) remain valid if strong independence is replaced throughout by full independence. But while Theorem (4.5) fails (at least for  $m > 2$ ) if strong independence is weakened to general position, it seems very likely that every set of  $(m - 1)(d + 1) + k + 1$  points in general position in  $R^d$  is  $(m,k)$ -divisible. For example, the set of 9 points in  $R^2$  shown below is  $(3,2)$ -divisible although the indicated partition refutes the conclusion of Theorem (4.5). (The example is taken from Reay [1968].)

The conjecture stated above was proved for  $d = 2$  and for  $m = 2$  in Reay [1968] (see Peterson [1972] for a more geometric proof in the latter case), and for  $d = m = 3$  in Reay [1979]. For  $k = d$ , it takes the following appealing form.<sup>2</sup>



(4.6) **Conjecture.** Any  $m(d+1)$  points in general position in  $R^d$  can be organized into the vertex sets of  $m$  nondegenerate simplices which have a common interior point.

Recently Doignon [1978] has used Gale transforms to obtain a generalization of one of the foregoing theorems. Call a set  $k$ -independent if any  $k+1$  or fewer of its points are affinely independent. Doignon's results imply that each  $k$ -independent set of  $k(d-k+2)+2$  points in  $R^d$  is  $(2,k)$ -divisible provided  $k > 1$ . The number  $k(d-k+2)+2$  is best possible when  $k=d$ , but unfortunately not in all cases. So the question arises whether the result can be improved and how it can be extended to  $(m,k)$ -divisible sets.

In conclusion we mention two problems which deserve a place in the context of this section.<sup>2</sup>

1. (J.R. Reay) When are Tverberg partitions unique? When does every Tverberg partition give a unique single point as the intersection of the corresponding convex hulls? (A *Tverberg partition* of a set is a partition into subsets whose convex hulls have a common point.)
2. (J.-P. Doignon) Does every set  $X$  of  $2d+1$  points in  $R^d$  admit a Radon partition  $\{X_1, X_2\}$  such that  $\dim(\text{conv } X_1 \cap \text{conv } X_2) \geq \min \{\dim X_1, \dim X_2\}$ ? Examples show that the number  $2d+1$  may not be replaced by  $2d$ .

## 5. Generalizations

This section is devoted to some recent generalizations and other relatives of Tverberg's Theorem (4.1). In contrast to the preceding section no independence condition upon the sets considered will be assumed.

We begin with a problem which was proposed by Bajmóczy and Bárány in 1977 and communicated by them to Tverberg.<sup>2</sup> Let  $P$  be a convex polytope with interior points in  $R^{(m-1)(d+1)}$ , and let  $f: \partial P \rightarrow R^d$  be a continuous function. Is it possible to find  $m$  pairwise disjoint faces  $F_1, \dots, F_m$  of  $P$  such that

$$\bigcap_{i=1}^m f(F_i) \neq \emptyset?$$

An affirmative answer to Tverberg's question would imply Theorem (4.1). This is clear from the fact that any set of  $(m-1)(d+1)+1$  points in  $R^d$  may be regarded, in many different ways, as the parallel projection of the vertex set of a (nondegenerate) simplex in  $R^{(m-1)(d+1)}$ . Bajmóczy and Bárány [1977] have

<sup>2</sup> Compare the list "Problems in Geometric Convexity", these Proceedings, pp. 255–278.

proved that the answer is indeed positive when  $m=2$ , thus establishing a continuous analogue of Radon's theorem. For  $P$  a simplex and  $m$  an odd prime, an affirmative solution was obtained by Bárány, Shlosman, and Szücs [1978] with the aid of algebraic topology.

Yet another generalization of Tverberg's theorem has been considered by Reay [1978]. Suppose  $m \geq t \geq 1$ , and let  $\tau(d,m,t)$  denote the smallest integer  $\tau$  for which the following is true: Every set of  $\tau$  points in  $R^d$  admits a partition into  $m$  subsets such that the convex hulls of any  $t$  of these have a common point.

Of course,  $\tau(d,m,1)=m$  and  $\tau(d,m,t) \leq (m-1)(d+1)+1$ . Reay has given some results to support his conjecture that always  $\tau(d,m,t)=(m-1)(d+1)+1$  provided  $t > 1$ . Thus  $\tau(d,m,t)$  would not depend on  $t$  for  $t > 1$ . For example, he showed that  $\tau(2,m,2)=3m-2$  and  $\tau(3,3,2)=9$ . The conjectured equality reduces to (4.1) when  $t=m$ , and follows from Helly's theorem (see Danzer, Grünbaum, and Klee [1963]) when  $t \geq d+1$ . Reay's results imply  $\tau(d,d+1,d) \geq d(d+1)$  and in particular,  $12 \leq \tau(3,4,3) \leq 13$ , but the exact value of  $\tau(3,4,3)$  is unknown.

Now define  $\lambda(d,v)$  to be the smallest integer  $\lambda$  with the following property: If  $X$  is a set of  $\lambda$  points in  $R^d$ , then  $X$  admits a Radon partition  $\{X_1, X_2\}$  such that, for any set  $Y \subset X$  containing at most  $v$  points,  $\{X_1 \setminus Y, X_2 \setminus Y\}$  is a Radon partition in  $X$ .

Hence if  $X$  is a set of  $\lambda(d,v)$  points in  $R^d$ , then some Radon partition of  $X$  will induce a Radon partition on every subset which arises by deleting at most  $v$  points of  $X$ .

Clearly,  $\lambda(d,0)=d+2$  by Radon's theorem.

The problem of determining  $\lambda(d,v)$  was first considered, in the special case  $v=1$ , by Larman [1972] (who writes  $\lambda(d)$  instead of  $\lambda(d,1)$  and calls a set *divisible* if it has the property stated above). He obtained the following result which is surprisingly hard to prove.

(5.1) **Theorem.** *For all  $d$ ,  $\lambda(d,1) \leq 2d+3$ , with equality when  $d \leq 3$ .*

It is conjectured that always  $\lambda(d,1)=2d+3$ , but so far all attempts to construct the necessary counter-examples have failed.<sup>2</sup> (By Theorem (2.1), a set of  $2d+2$  points in  $R^d$  admits at least  $2^{2d}$  Radon partitions!)

The paper of Larman contains two interesting reformulations of the problem. In fact, the formulation above is derived, via Gale diagrams, from the original formulation due to McMullen. For  $d > 1$ , the first part of Theorem (5.1) is equivalent to the assertion that each set of  $2d+1$  points in general position in  $R^d$  can be mapped, by a permissible projective transformation, onto the vertices of a convex polytope in  $R^d$ . It is conjectured that  $2d+1$  cannot be replaced by a larger number.

As for the general case we remark that Theorem (4.1) easily gives  $\lambda(d,v) \leq (2v+1)(d+1)+1$ . This bound can be approximately halved as was proved by Stangeland [1978].

(5.2) **Theorem.**  $\lambda(d,v) \leq (v+1)(d+1)+1$ .

Again it seems likely that equality holds in (5.2) for all  $d$  and  $v$ . This is a special case of Conjecture (5.3) below.

We turn now to a far-reaching generalization of both Tverberg's Theorem (4.1) and Larman's problem considered above. It consists of replacing the desired point in the intersection of certain convex hulls by an affine subspace of given dimension which meets all the convex hulls and passes through a given subspace of lower dimension. This leads to the following definition.

Let  $d, m, p, q, v$  be integers satisfying  $d \geq q > p \geq -1$ ,  $m \geq q - p$ ,  $v \geq 0$ , and let  $\rho(d, m, p, q, v)$  denote the smallest integer  $\rho$  for which the following is true: Given a set of at least  $\rho$  points in  $R^d$  and a  $p$ -flat  $F \subset R^d$ , there exist a partition  $\{X_1, \dots, X_m\}$  of  $X$  and a  $q$ -flat  $G \subset R^d$  containing  $F$  such that, for any subset  $Y \subset X$  with at most  $v$  points,

$$G \cap \text{conv}(X_i \setminus Y) \neq \emptyset, \quad i = 1, \dots, m.$$

The function  $\rho(d, m, p, q, v)$  has been introduced by Stangeland [1978] (though his notation is different from ours) and studied by him for special values of its arguments. Some earlier results of Iversland [1969] can also be formulated in terms of  $\rho(d, m, p, q, 0)$ . Notice that  $\rho(d, m, -1, 0, 0)$  is Tverberg's function  $\gamma(d, m, 0)$ , while  $\rho(d, 2, -1, 0, v)$  is Larman's (extended) function  $\lambda(d, v)$ .

It seems that no general expression for  $\rho(d, m, p, q, v)$  has, so far, been conjectured but Stangeland ventures the following.

(5.3) **Conjecture.**  $\rho(d, m, -1, 0, v) = m(d(v + 1) + 1) - (d(v + 1) - v)$ .

This reduces to the conjecture about  $\lambda(d, v)$  when  $m = 2$ . As in that case Theorem (4.1) can be used to obtain an upper bound for  $\rho(d, m, -1, 0, v)$ , however, this bound is probably very rough.

(5.4) **Conjecture.**  $\rho(d, m, p, q, 0) = m(d - q + 1) - (q - p)(d - q)$ .

Conjecture (5.4) is trivially true when  $q = d$  or  $m = q - p$ . Stangeland [1978] proved it for  $q = d - 1$ , for  $m = q - p + 1$ , and for  $p = -1$ . Furthermore, he showed that the right-hand side in (5.4) cannot be replaced by a smaller number. He also obtained the following upper bound, but again this seems to be far from the truth.

(5.5) **Theorem.**  $\rho(d, m, p, q, 0) \leq m(d - q + p + 2) - (q - p)(d - q + p + 1)$ .

Let us write down the special case  $p = -1$  in a separate theorem, parts of which were proved earlier by Iversland [1969].

(5.6) **Theorem.** Every set of  $m(d - q + 1) - (q + 1)(d - q)$  points in  $R^d$  can be divided into  $m$  pairwise disjoint subsets whose convex hulls admit a "common transversal" of dimension  $q$ , i.e. a  $q$ -flat intersecting them all. Moreover, the integer above is the smallest which has the stated property.

The next result is also due to Iversland [1969].

(5.7) **Theorem.** If  $X$  is a set of  $(m - 1)(d - q + 1) + 1$  points in  $R^d$ , and  $G \subset R^d$  is a  $q$ -flat, then there exist a partition  $\{X_1, \dots, X_m\}$  of  $X$  and a translate  $G'$  of  $G$  such that

$$G \cap \text{conv } X_i \neq \emptyset, i = 1, \dots, m.$$

When  $q=0$ , this reduces to Tverberg's theorem. Is the number  $(m-1)(d-q+1)+1$  always best possible?

By regarding the translates of a given  $q$ -flat as passing through a common  $(q-1)$ -flat "at infinity" one could interpret (5.7) in terms of the suitably generalized function  $\rho(d,m,p,q,0)$ . In fact, it seems likely that the function will not be affected if the flat  $F$  in its definition is allowed to lie "at infinity".

We end the section by stating two problems of Iversland [1969] which do not fit into the context above but are closely related in spirit.

The first problem is concerned with partitions of a set into  $m$  subsets with the property that through any given  $p$ -flat (which may even be "at infinity") there passes a  $q$ -flat which intersects each of the corresponding convex hulls. Note that the partition is chosen first whereas in the definition of  $\rho(d,m,p,q,0)$  it is chosen according to the given  $p$ -flat. Iversland proved that if  $q=p+1$  and  $X$  is a set of at least

$$\begin{cases} (m-1)(d+1)+1, & m \leq d-q+1, \\ \tau(d,m,d-q+1), & m \geq d-q+1, \end{cases}$$

points in  $R^d$ , then  $X$  has the desired property. Here  $\tau(d,m,d-q+1)$  is Reay's function considered earlier in this section. In view of what was said there it may be conjectured that the number  $(m-1)(d+1)+1$  is always best possible (provided  $q < d$ ). This would again generalize Theorem (4.1) which is the special case  $q=0$ .

For the second problem, suppose  $m > t \geq 1$ .

(5.8) **Problem.** *Can every set of  $(m-1)d+t+1$  points in  $R^d$  be partitioned into  $m$  subsets in such a way that among the convex hulls of these sets there are  $t$ , each of which intersects all  $m$  convex hulls?*

Iversland has constructed a set of  $2(m-1)+t$  points in  $R^2$  which cannot be partitioned in the way described above. When  $t=m-1$ , an affirmative answer is given by Tverberg's theorem.

## 6. Abstract convexity

This final section is concerned with Radon-type theorems in an abstract setting. During the last fifteen years there has been an increasing interest in what might be called the axiomatic theory of convexity. For a discussion of some of the underlying ideas, see Danzer, Grünbaum, and Klee [1963]. Here we restrict ourselves to a review of those problems and results which have a direct connection with Radon's theorem.

Let  $S$  be a nonempty set, and let  $\mathcal{C}$  be a family of subsets of  $S$  such that  $\emptyset \in \mathcal{C}$ ,  $S \in \mathcal{C}$ , and  $\mathcal{C}$  is closed under intersection. Then the pair  $(S, \mathcal{C})$  is called a *convexity space*, and the members of  $\mathcal{C}$  are called *convex*. For any  $X \subset S$ , define

$$\text{conv } X := \bigcap \{Y \in \mathcal{C} \mid Y \supset X\}$$

to be the *convex hull* of  $X$ .

Many of the definitions used in the preceding sections carry over to arbitrary convexity spaces. In particular, this applies to the notion of a *Radon partition*. By  $r(S, \mathcal{C})$  we denote the *Radon number* of  $(S, \mathcal{C})$ , that is, the smallest number  $r$  with the property that each set of  $r$  points in  $S$  admits a Radon partition. If no such number exists, we put  $r(S, \mathcal{C}) = \infty$ .

The classical example of a convexity space is  $(R^d, \mathcal{C}^d)$ , where  $\mathcal{C}^d$  denotes the family of ordinary convex sets in  $R^d$ . Its Radon number is  $d+2$ . Some efforts have been made to determine the Radon numbers of various other convexity spaces, several of which are listed below. The following notation will be used.

Let  $\dot{R}^d := R^d \setminus \{o\}$  and  $\dot{\mathcal{C}}^d := \{X \setminus \{o\} | X \in \mathcal{C}^d\}$ , where  $o$  is the origin in  $R^d$ , and let  $\mathcal{C}_k^d := \{X \subset R^d | X \in \mathcal{C}^d \vee \text{card } X \leq k\}$ . Denote by  $\mathcal{P}^d$  the family of parallelotopes in  $R^d$  with edges parallel to the coordinate axes, by  $L^d$  a fixed  $d$ -dimensional lattice in  $R^d$ , and by  $S^d$  the unit sphere in  $R^{d+1}$ . Finally, let  $\mathcal{L}^d := \{X \cap L^d | X \in \mathcal{C}^d\}$ , and denote by  $\mathcal{S}^d$  (resp.  $\mathcal{R}^d$ ) the family of *strongly convex* (resp. *Robinson-convex*) subsets of  $S^d$  including  $S^d$ . For a definition of these sets, see Danzer, Grünbaum, and Klee [1963].

The following is known.

$$(6.1) \quad r(\dot{R}^d, \dot{\mathcal{C}}^d) = 2d + 1.$$

$$(6.2) \quad r(R^d, \mathcal{C}_k^d) = \max(d+3, 2k+2), \quad d, k > 1.$$

$$(6.3) \quad r(R^d, \mathcal{P}^d) = \min \left[ r \left| \binom{r}{\left[\frac{r}{2}\right]} > 2d \right. \right].$$

$$(6.4) \quad r(L^d, \mathcal{L}^d) \geq 2^d + 2^{d-2} + 1, \quad d > 1.$$

$$(6.5) \quad r(S^d, \mathcal{S}^d) = d+3.$$

$$(6.6) \quad r(S^d, \mathcal{R}^d) = 2d+3.$$

Theorem (6.1) was proved in Eckhoff [1976] and (6.3) in Eckhoff [1969], while Theorem (6.2) is due to Kramer.<sup>3</sup> Theorem (6.4) is taken from Doignon's thesis [1974/75] which contains some information about the Radon numbers of "generalized affine spaces". The lower bound in (6.4) is best possible when  $d=2$ , and equality is conjectured for all  $d$ .<sup>2</sup> Theorem (6.5) is an easy consequence of Radon's theorem. Finally, Theorem (6.6) was obtained in Hansen and Klee [1969] and independently in Šaraburova and Šaškin [1975]; for a variant, see Doignon and Valette [1975].

The Radon number of a convexity space is of interest not only in its own right, but also in its relation to other numbers such as, for example, the *Helly number*  $h$  and the *Carathéodory number*  $c$  of the space. (For a definition of these numbers, see Kay and Womble [1971].) In fact, the close relationship between the classical theorems of Radon, Helly, and Carathéodory (see Danzer, Grünbaum, and Klee [1963]) has stimulated much of the work in abstract convexity theory. The following result is best possible in the sense that no other type of relation between  $r, h$ , and  $c$  exists in arbitrary convexity spaces.

<sup>3</sup> Oral communication.

(6.7) **Theorem.** If  $r, h, c$  are respectively the Radon, Helly, and Carathéodory number of a convexity space, then

$$h + 1 \leq r \leq c(h - 1) + 2.$$

The lower bound in (6.7) is due to Levi [1951], and is attained by a large number of convexity spaces. The upper bound has been obtained by several authors (for a proof, see Sierksma [1977]). It improves an earlier bound of Kay and Womble [1971], but whether it can be replaced by an even smaller number is open. We remark that (6.7) may be used to show that the Radon number in (6.4) is finite. For an investigation of  $r, h$ , and  $c$  under additional axiomatic hypotheses, see Kay and Womble [1971], Calder [1971], Bean [1974], Sierksma [1977], and Hammer [1977].

Now let  $X$  be a finite subset of  $S$ , where  $(S, \mathcal{C})$  is a convexity space. Extending an earlier definition we denote by  $N(X)$  the number of distinct Radon partitions of  $X$ . The following result was proved by Sierksma and Boland [1974].

(6.8) **Theorem.** If  $r$  is the Radon number of the convexity space  $(S, \mathcal{C})$  and  $X$  is a set of  $s$  points in  $S$ , then  $N(X) \geq N(r, s)$  (where  $N(r, s)$  is defined in Section 2).

In view of Theorem (2.1) this reveals a remarkable extremal property of the space of ordinary convex sets. The proof of Theorem (6.8) can be reduced to the following combinatorial lemma which is of interest in itself.

Suppose  $s \geq r \geq 1$ , and let  $X$  be a set of  $s$  elements. Further, let  $\varphi$  be a map which assigns to each set  $Y \subset X$  containing  $r$  points a partition  $\{Y_1, Y_2\}$  of  $Y$ . Denote by  $N(r, s; \varphi)$  the number of partitions of  $X$  which extend, for at least one set  $Y$ , the given partition  $\varphi(Y)$ . Then the lemma states that

$$(6.9) \quad \min_{\varphi} N(r, s; \varphi) = N(r, s).$$

The minimum is attained, for example, by the following map  $\hat{\varphi}$ : Suppose that  $X$  is linearly ordered, and let  $\hat{\varphi}$  assign to each  $Y$  the partition  $\{Y_1, Y_2\}$ , where the elements of  $Y_1$  and  $Y_2$  alternate in the ordering of  $Y$  inherited from  $X$ . Every subset of  $X$  decomposes into maximal blocks of consecutive elements. A partition  $\{X_1, X_2\}$  of  $X$  extends some partition of the form  $\hat{\varphi}(Y)$  if, and only if, either  $X_1$  or  $X_2$  contains more than  $\lceil \frac{r}{2} \rceil$  blocks, or (in case  $r$  is even) both  $X_1$  and  $X_2$  contain  $\frac{r}{2}$  blocks. Hence  $N(r, s; \hat{\varphi}) = N(r, s)$ .

The application to convexity spaces is now immediate because if  $\{X_1, X_2\}$  extends  $\{Y_1, Y_2\}$ , and  $\{Y_1, Y_2\}$  is a Radon partition in  $X$ , then  $\{X_1, X_2\}$  is a Radon partition of  $X$ . In the case of ordinary convex sets the example  $\hat{\varphi}$  above describes the Radon partitions of the vertex sets of cyclic polytopes. This is essentially a result of Breen [1973].

It should be mentioned that Theorem (6.8) has been used to solve a long-standing problem in abstract convexity theory. Suppose  $(S_1, \mathcal{C}_1)$  and  $(S_2, \mathcal{C}_2)$  are convexity spaces with Radon numbers  $r_1$  and  $r_2$ , and let  $r$  be the Radon number of the *product convexity space*  $(S_1 \times S_2, \mathcal{C}_1 \otimes \mathcal{C}_2)$  defined in Eckhoff [1968]. While it is easy to see that  $\max(r_1, r_2) \leq r \leq r_1 + r_2 - 1$  (see Eckhoff [1968]), it was only with the help of (6.8) that the following stronger result could be proved (see Sierksma and Boland [1974], Sierksma [1976]).

(6.10) **Theorem.**  $\max(r_1, r_2) \leq r \leq r_1 + r_2 - 2$ .

Examples in Eckhoff [1968], [1969] show that the bounds in (6.10) are best possible for all  $r_1$  and  $r_2$ . On the other hand, Kramer<sup>3</sup> has given various examples in which  $r$  lies in the interior of the above interval. For the Radon numbers of certain product spaces with more than two factors, see Eckhoff [1969].

Returning now to the space  $(S, \mathcal{C})$  we denote by  $r_m(S, \mathcal{C})$ , for each  $m \geq 2$ , the smallest number  $r_m$  such that any set of  $r_m$  points in  $S$  is  $m$ -divisible. (The definition of  $m$ -divisibility in Section 4 carries over to convexity spaces.) This number (if it exists) is called the  $m$ -th *Tverberg number* of  $(S, \mathcal{C})$ . In particular,  $r_2$  is the Radon number.

The determination of Tverberg numbers appears to be rather difficult and has not been achieved, for example, for any of the spaces in (6.1) to (6.6). (The Tverberg numbers of product convexity spaces were studied by Hare and Thompson [1975].) It can be shown that if  $r_2$  is finite, then  $r_m$  is finite for all  $m$ .<sup>4</sup> But what is the best upper bound for  $r_m$  in terms of  $m$  and  $r_2$ ? For the ordinary convexity space we have, by Theorem (4.1),

$$r_m = (m-1)(r_2-1) + 1,$$

and the same would hold for the space in (6.1) if Conjecture (4.3) were true. One is tempted to speculate whether the relation

$$r_m \leq (m-1)(r_2-1) + 1$$

might hold for any  $m$  and any convexity space with finite Radon number  $r_2$ . Some results in this direction have been obtained by Calder [1971], Bean [1974], Reay and Bean [1974], and Jamison [1979]. Of course, to prove such a result would mean to give a purely combinatorial proof of Tverberg's theorem.

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## Geometry of numbers

### 1. Introduction

The geometry of numbers can be traced back at least to Lagrange [1773], who proved important results about quadratic forms in two variables. The proofs as well as the formulations of results were purely arithmetic. Reviewing a book of Seeber [1831B] on ternary quadratic forms, Gauß [1831] introduced for the first time geometric methods. Geometric methods were predominant in the work of Dirichlet [1850]. On the other hand Hermite [1850] and Korkine and Zolotareff [1872], [1873], [1877] gave arithmetic proofs for their results on quadratic forms in more than three variables. Finally Minkowski [1891] noticed that a simple geometric argument which he used to give a new proof of a theorem of Hermite could be adapted to much more general situations. Then Minkowski [1896B], [7B], [11B] started a systematic study of geometric methods in number theory and called this new branch of number theory *geometry of numbers*. Many results and most concepts of modern geometry of numbers have their origin in the work of Minkowski. After Minkowski many eminent mathematicians made contributions to this field. In order to avoid controversies I will not mention any of them. Geometry of numbers is closely related to other branches of number theory such as algebraic number theory and Diophantine approximation. A flourishing offspring is discrete geometry, developed mainly by Fejes Tóth and his school.

The geometry of numbers is a problem-oriented branch of mathematics with a small number of major problems. We mention the following:

- (1) Determination of lattice constants.
- (2) Refinements of the theorem of Hlawka-Minkowski.
- (3) Mordell's problem concerning Minkowski's theorem on linear forms.
- (4) Voronoi's problem on parallelohedra.
- (5) Determination of the densest lattice packing of unit balls.
- (6) Determination of the thinnest lattice covering of unit balls.
- (7) Minkowski's conjecture on the product of non-homogeneous linear forms.

From a more systematic point of view one can distinguish in geometry of numbers the homogeneous or packing problem and the inhomogeneous or covering problem.

Detailed information about geometry of numbers may be obtained from the collected works of Minkowski [11B] and Davenport [77B] and the books of Minkowski [1896B], [7B], Keller [54B], Cassels [59B] and Lekkerkerker [69B]. For further information we refer the reader to the books and survey articles listed in the references.

In the following we will survey part of the work on geometry of numbers that has been done after the publication of the books of Cassels [59B] and Lekkerkerker [69B]. In some instances it has been necessary to include also earlier references. From our considerations results belonging primarily to discrete geometry, lattice point problems in the sense of Landau and

Diophantine approximation will be excluded. Russian papers were considered only in those cases where translations were available.

For recent results in geometry of numbers over fields other than  $\mathbb{R}$  or in spaces other than  $\mathbb{R}^n$  we refer to papers of Armitage [69], Spence [70], Savov [71], Cook [75], Dubois [76], Karian [76], Luthar and Vinygard [77] and Johnston [78].

## 2. Preliminaries

A *lattice*  $L$  in  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is the set of integral linear combinations of  $n$  linearly independent points  $b_1, \dots, b_n \in \mathbb{R}^n$ .  $\{b_1, \dots, b_n\}$  is called a *basis* of  $L$ . Any two bases of  $L$  are related to each other by an integral unimodular transformation.  $|\det\{b_1, \dots, b_n\}|$  is called the *determinant* of  $L$  and is denoted by  $d(L)$ . Obviously it is independent of the special choice of a basis. Bases are important in many contexts, in particular in reduction theory. (For this and some other important results about lattices see Cassels [59B] and Lekkerkerker [69B].) Let  $L_0$  denote the lattice of all points with integer coordinates.  $L_0$  is called the *fundamental lattice*. The unit vectors of  $\mathbb{R}^n$  form a basis of  $L_0$ . Schmidt [69] proved that in each sublattice of  $L_0$  (i.e. a subset of  $L_0$  which is a lattice itself) there is a finite set consisting of vectors with non-negative coordinates such that each vector of the sublattice with non-negative coordinates is a non-negative integer linear combination of  $n$  of these vectors. For other results consult Foster [64] and Ozeki [76]. Every lattice  $L$  can be represented in the form  $L = BL_0$  where  $B$  is real non-singular  $n \times n$ -matrix the columns of which form a basis of  $L$ .

The set of all lattices of  $\mathbb{R}^n$  can be topologized by the following definition: A sequence of lattices  $L^{(1)}, L^{(2)}, \dots$  converges to a lattice  $L$  if there are bases  $\{b_1^{(1)}, \dots, b_n^{(1)}\}, \dots$  and  $\{b_1, \dots, b_n\}$  of  $L^{(1)}, \dots$  and  $L$  respectively, such that  $b_1^{(1)}, b_1^{(2)}, \dots \rightarrow b_1, \dots, b_n^{(1)}, b_n^{(2)}, \dots \rightarrow b_n$ . Woods [66] and Schmidt [71] proved that certain (small) sets of lattices are dense in the space of all lattices of  $\mathbb{R}^n$  of determinant 1.

A lattice is called *admissible* for a subset  $K$  of  $\mathbb{R}^n$  if it contains no interior point of  $K$  except possibly the origin 0 of  $\mathbb{R}^n$ . An important tool in the geometry of numbers is the following *selection theorem of Mahler* [46]:

*Let a sequence of lattices in  $\mathbb{R}^n$  be given. If the corresponding sequence of the determinants of the lattices is bounded and if there is a ball with centre 0 such that every lattice of the sequence is admissible for it, then the sequence contains a convergent subsequence.*

This theorem implies in many instances the existence of lattices with certain extremal properties. An interesting proof of it, based on Blaschke's selection theorem and the concept of Dirichlet-cells was given by Groemer [71]. For generalizations of this see the paper of Hammer and Dwyer [76].

On the set of all lattices of determinant 1 Siegel [45] introduced a particular *measure* such that the whole space has measure 1. For more information see 3.2 below.

A *convex body* in  $\mathbb{R}^n$  is a compact convex subset of  $\mathbb{R}^n$  with non-empty interior. A *symmetric* convex body is a convex body symmetric in 0. A *star body* in  $\mathbb{R}^n$  is a

closed subset of  $\mathbb{R}^n$  such that 0 is an interior point and such that each ray from 0 meets its boundary in at most one point.

The Euclidean norm in  $\mathbb{R}^n$  will be denoted by  $| \cdot |$ , the inner product by  $\langle \cdot, \cdot \rangle$  and Lebesgue measure by  $v$ . Let  $\mathbb{Z}$  denote the ring of rational integers.

### 3. The homogeneous problem

#### 3.1 The fundamental theorems of Minkowski and Blichfeldt

The following theorem of Minkowski (see [1896B], [7B], [11B]) turned out to be of fundamental importance in number theory:

*Let  $K$  be a symmetric convex body and  $L$  a lattice in  $\mathbb{R}^n$ . If*

$$v(K) \geq 2^n d(L),$$

*then  $K$  contains at least one pair  $\pm a \neq 0$  of lattice points of  $L$ .*

This result has been refined in various ways. Among the older refinements we should like to mention theorems of Siegel [35], van der Corput [35] and van der Corput and Davenport [46]. Siegel made use of Parseval's theorem to show the following result:

*Let  $K$  and  $L$  be as in Minkowski's theorem and suppose that  $K$  does not contain a lattice point  $\neq 0$  of  $L$  in its interior. Then*

$$v(K) + 4^n v(K)^{-1} \sum_{b \in L^* \setminus \{0\}} \left| \int_K e^{-2\pi i \langle b, x \rangle} dx \right|^2 = 2^n d(L).$$

*Here  $L^*$  denotes the polar lattice of  $L$  ( $:= \{b \in \mathbb{R}^n \mid \langle a, b \rangle \in \mathbb{Z} \text{ for all } a \in L\}$ ).*

Van der Corput proved,

*if  $K$  and  $L$  are as in Minkowski's theorem, then*

$$v(K) \geq 2^n k d(L)$$

*implies that  $K$  contains at least  $k$  pairs of lattice points  $\neq 0$  of  $L$ .*

Van der Corput and Davenport make use of the curvature of  $K$ . More recent refinements and generalizations concerning primarily the case  $n = 2$  are reviewed in the booklet of Hammer [77B]. Most of them suffer from the defect that they depend on metric invariants. Interesting exceptions are the papers by Groemer [59] (affine length), and Ehrhart [64] (centroid), to mention some of them. Unions of convex bodies have been considered by Narzullaev and Skubenko [70a] and polar bodies by Mahler [74], [76]. White [63] indicated a more precise form of van der Corput's theorem.

Recent applications of the above theorems have been given by Davenport [71], Mordell [72], [72a], Schmidt [76] and Dumir and Hans-Gill [77].

If  $K$  is a subset of  $\mathbb{R}^n$  and  $L$  a lattice then the *homogeneous minimum* or *first successive minimum* of  $K$  with respect to  $L$  is defined by

$$\lambda_1 := \inf \{ \lambda > 0 \mid \lambda K \cap L \text{ contains a point } \neq 0 \}.$$

Now let  $K$  be a convex body, symmetric in 0. Then the fundamental theorem can be written as follows:

$$\lambda_1^n v(K) \leq 2^n d(L).$$

The *successive minima*  $\lambda_1, \dots, \lambda_n$  of a set  $K$  with respect to a lattice  $L$  are defined by

$$\lambda_i := \inf \{ \lambda > 0 \mid \lambda K \cap L \text{ contains } i \text{ linearly independent points} \}.$$

Obviously  $\lambda_1 \leq \dots \leq \lambda_n$ . There exists a vast literature on successive minima (see e.g. Cassels [59B] and Lekkerkerker [69B]). The central result is the following *second fundamental theorem of Minkowski* [1896B]:

*Let K be a symmetric convex body and L a lattice in  $\mathbb{R}^n$ . Then*

$$2^n/n! \leq \lambda_1 \cdots \lambda_n v(K) \leq 2^n d(L).$$

The first really satisfactory proofs were given by Bambah, Woods and Zassenhaus [65]. Their proofs follow the spirit of classical proofs. A proof based on Dirichlet's pigeon hole principle was given by Danicic [69]. Refinements of this theorem which are similar to those of van der Corput and Davenport [46] for the fundamental theorem are contained in an (almost forgotten) paper of Jarník [48]. Another refinement has been given by Woods [66a]. Theorems about successive minima of polar bodies  $K^*$  ( $:= \{y \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1 \text{ for all } x \in K\}$ ) and of so-called compound bodies, introduced by Mahler, have been given by Mahler [74], [76] (see also Mahler [71S]) and Cook [74]. Deep applications of these results on successive minima are due to Schmidt [69a], [70], [70B].

Now define the *lattice constant* of a set K by

$$\Delta(K) := \inf \{d(L) \mid L \text{ is an admissible lattice for } K\}.$$

If 0 is an interior point of K and  $\Delta(K) < \infty$  then Mahler's selection theorem implies that there are admissible lattices of K with  $d(L) = \Delta(K)$ . These lattices are called *critical*. The homogeneous problem is essentially the problem of finding lattice constants and critical lattices of particular sets or classes of sets. If K is a symmetric convex body then the fundamental theorem can be written in the form

$$2^{-n}v(K) \leq \Delta(K).$$

This shows that the following conjecture is a sharpened form of the second fundamental theorem:

$$\lambda_1 \cdots \lambda_n \Delta(K) \leq d(L)$$

if K and L are as above. This conjecture is true if K is an ellipsoid and also if  $n \leq 3$  (Woods [56]).

One may conjecture that other refinements can be given, depending on the differences of the  $\lambda_i$ 's.

For other results on successive minima we refer to Chalk and Rogers [68], Bantegnie [70] and Delone, Galilulin, Dolbilin, Zalgaller and Štogrin [73].

Using successive minima, Mahler [66] gave a new proof of Kronecker's approximation theorem.

A generalization of a different sort is the following *theorem of Blichfeldt* [14]: *Let K be a measurable subset and L a lattice of  $\mathbb{R}^n$ . If*

$$v(K) > d(L),$$

*then K contains (at least) two different points, the difference of which is contained in L.*

A modern version of this is the following: Let M be a measure-space and m a

positive measure on  $M$  such that  $m(M)=1$ . If  $f$  is an integrable function such that  $m(f) > 1$ , then there exists at least one point  $a \in M$  such that  $f(a) > 1$ . Several generalizations of Blichfeldt's theorem to special types of topological groups are of this sort.

Minkowski's fundamental theorem is an easy consequence of Blichfeldt's theorem.

There are several generalizations of Blichfeldt's theorem analogous to generalizations of Minkowski's theorem (see e.g. Lekkerkerker [69B]). Bombieri's generalization [62] is analogous to Siegel's generalization of Minkowski's fundamental theorem. Other generalizations have been given by Woods [58a], White [63] and Gruber [67a].

Many of the older applications of Blichfeldt's theorem are stated in the books of Cassels [59B] and Lekkerkerker [69B]. Woods [58a], Bombieri [62], [63] and Gruber [67a] applied their results to the product of non-homogeneous linear forms (see 5.3 below).

### 3.2 The Hlawka-Minkowski theorem and measure theory in the geometry of numbers

The following theorem was conjectured by Minkowski [11B] and proved by Hlawka [44]:

*Let  $K$  be a Jordan measurable subset of  $\mathbb{R}^n$ . If  $v(K) < 1$ , then there exists a lattice  $L$  with  $d(L) = 1$  admissible for  $K$ . If  $K$  is a Jordan measurable star set then  $v(K) < 1$  can be replaced by  $v(K) < \zeta(n)$  and if additionally  $K$  is symmetric in 0 we may assume  $v(K) < 2\zeta(n)$ .*

(Here  $\zeta$  denotes the Riemann  $\zeta$ -function.) Making use of Minkowski's reduction theory of quadratic forms, Siegel [45] constructed in a natural way a measure  $\mu$  on the space of all lattices of  $\mathbb{R}^n$  of determinant 1, i.e. on a fundamental region of the group of all  $n \times n$ -matrices of determinant 1 with respect to the subgroup of all integer matrices. The whole space has measure 1. A central result is *Siegel's mean value theorem*, of which Hlawka's theorem is an immediate consequence:

*Let  $f \geq 0$  be a Riemann-integrable function on  $\mathbb{R}^n$ . Then*

$$\int \left( \sum_{a \in L \setminus \{0\}} f(a) \right) d\mu(L) = \int_{\mathbb{R}^n} f(x) dx. \quad (1)$$

A lattice point  $a \neq 0$  is *primitive* if on the segment  $[0, a]$  there is no lattice point besides the endpoints. Siegel proved that (1) can be replaced by

$$\int \left( \sum_{a \in L, \text{ prim.}} f(a) \right) d\mu(L) = \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f(x) dx.$$

This was generalized by Macbeath and Rogers [58] and Rogers [55]. If  $K$  is Jordan-measurable, then Siegel's mean value theorem implies that the measure of the set of admissible lattices of determinant 1 for  $K$  is

$$\geq 1 - v(K).$$

This has been improved by Rogers and Schmidt. The best result in this direction is a result of Schmidt [59] showing that this measure is equal to

$$e^{-v(K)}(1-R) \quad (2)$$

where

$$|R| < v(K)^{n-1} e^{v(K)+n}(1+\varepsilon) + \varepsilon \quad \text{if } v(K) \leq n-1, n \geq n(\varepsilon).$$

Hlawka's and Siegel's results imply immediately that

$$\Delta(K) \leq v(K)$$

for general  $K$  and

$$\Delta(K) \leq \frac{1}{2\zeta(n)} v(K)$$

if  $K$  is a star set symmetric in 0. Improvements of this can be obtained from (2). The best result was given by Schmidt [63] and shows that for general  $K$

$$\Delta(K) \leq \frac{15}{16} v(K) \quad \text{if } n = 2$$

and

$$\Delta(K) \leq \frac{1}{n\alpha - \beta} v(K) \text{ with } \alpha = 0.3465\dots, \beta \in \mathbb{R} \text{ if } n \text{ is sufficiently large.}$$

If  $K$  is symmetric in 0, the right hand sides can be replaced by  $1/2$  of them. Schwald [65] considered  $k$ -tuples of lattice points instead of single lattice points as above. Skubenko [65] indicated a new approach to find the volume of the fundamental region considered by Siegel [45] (and earlier by Minkowski [11B]). An application of Siegel's mean value theorem was given by Randol [70]. More complete accounts on measure theory in geometry of numbers have been given by Rogers [64B], Schmidt [66S] and Lekkerkerker [69B].

The fundamental theorem of Minkowski and the theorem of Hlawka-Minkowski and its refinements give lower and upper bounds for lattice constants. One may conjecture that at least for symmetric convex bodies the best upper bounds should be considerably smaller than the ones given above, but so far there is no hope of achieving even minor improvements.

### 3.3 Mordell's problem on Minkowski's linear form theorem

A simple application of Minkowski's fundamental theorem yields the following *linear form theorem of Minkowski* [7B]:

*Let  $L$  be a lattice with  $d(L)=1$  and  $c_1, \dots, c_n > 0$  such that  $c_1 \cdots c_n = 1$ . Then there is a lattice point  $\neq 0$  of  $L$  in the box*

$$|x_1| \leq c_1, \dots, |x_n| \leq c_n. \quad (1)$$

The question, which lattices  $L$  with  $d(L)=1$  are admissible for the box (1) is difficult. It was solved by Hajós [42], confirming a conjecture of Minkowski. An application of the linear form theorem has been given by Fujiwara [75]. Mordell [36] posed the following problem which is a sort of *converse to the linear form theorem*:

Find the largest possible  $\kappa(n) > 0$ , such that for each lattice  $L$  of  $\mathbb{R}^n$  with  $d(L) = 1$  there exists a box of the form (1) with  $c_1 \cdots c_n \geq \kappa(n)$  for which  $L$  is admissible. For  $n = 2$  Szekeres [36] proved

$$\kappa(2) = \frac{1}{2} + \frac{1}{2\sqrt{5}} = 0.7236\dots$$

Alternative proofs and refinements were given by Szűsz [56], Surányi [71] and Gruber [71]. The latter pointed out the relation of this problem to the homogeneous minimum

$$\lambda = \inf\{|x_1 x_2| | x \in L \setminus \{0\}\}.$$

For  $n = 3$  Szekeres [36a] and Ko [36] showed

$$\kappa(3) > \frac{1}{4} = 0.2500\dots$$

A result of Ramharter [79] indicates that there is again a close relation to an analogous homogeneous minimum and supports the conjecture that in his estimate

$$\kappa(3) \leq \frac{8}{7} \cos^2 \frac{\pi}{7} \cos \frac{2\pi}{7} = 0.5784\dots$$

equality holds. For  $n = 4$  a good lower bound for  $\kappa(4)$  may be obtained from a result of Bambah and Woods [74]. For  $n \geq 5$  lower bounds for  $\kappa(n)$  are contained in a paper by Hlawka [50]. Some recent results have been given by Gruber and Ramharter [80].

### 3.4 Lattice constants

Let  $K$  be a *symmetric convex body*. There are some general results available for finding  $\Delta(K)$  and the critical lattices of  $K$ . Minkowski [7B], Reinhardt [34] and Mahler [47] proved results of this type for  $n = 2$ , Minkowski [11B] for  $n = 3$  and Brunngraber [44] and Wolff [54] for  $n = 4$ . Hans [69] showed that the result of Reinhardt and Mahler has no analogue if  $n \geq 3$ . Upper bounds for the number of lattice points on the boundary of  $K$  of an admissible lattice were given by Minkowski [7B] for  $K$  strictly convex and  $K$  not strictly convex. An interesting result of Swinnerton-Dyer [53] says that each critical lattice of  $K$  contains at least  $n(n+1)/2$  pairs  $\pm a$  of boundary points of  $K$ . Among them are  $n$  linearly independent points. A simple application of Minkowski's fundamental theorem shows that their determinant is  $\leq n! d(L)$ . This estimate can be improved. Some recent improvements were given by Bantegnie [65], [67] and Delone, Galiulin et al. [73]. For related results and many references see Groemer [70]. Reinhardt [34] and Mahler [47] considered a sort of smoothed octagon  $K$  for which

$$v(K) = 3.6096\dots \Delta(K).$$

It is conjectured that  $3.6096\dots$  is the minimal such number for all symmetric

convex bodies in the plane. Improving upon earlier results it was shown that

$$\begin{aligned} v(K) &\geq 3.5252 \dots \Delta(K) & \text{Ennola [61],} \\ v(K) &\geq 3.5706 \dots \Delta(K) & \text{Tammela [70].} \end{aligned}$$

Minkowski [7B] stated a conjecture about the lattice constant and the critical lattices of the set

$$|x_1|^p + |x_2|^p \leq 1 \quad (p \geq 1).$$

This conjecture (in a slightly modified form) has been proved for many values of  $p$ . In recent years the following values of  $p$  have been considered:

$$\begin{aligned} 1.3, 1.4, 1.5, 1.6, 1.7, 2.2 &\quad \left\{ \begin{array}{l} \text{Kuharev [66], [68], [71]} \\ \text{2.3, 3, 4, 5, [1.3, 1.301]} \end{array} \right. \\ [6, +\infty[, [5, 6] &\quad \left\{ \begin{array}{l} \text{Malyšev and Voronetskii [75]} \\ \text{Grišanovskaja, Malyšev,} \\ \text{Pačev and Fridarova [77].} \end{array} \right. \end{aligned}$$

Ohnari [62] proved that for  $k \equiv 0 \pmod{6}$  the lattice constant of the regular  $k$ -gon inscribed in the unit circle is equal to

$$\frac{\sqrt{3}}{2} \cos^2 \frac{\pi}{n}.$$

Mahler [54] conjectured the following: Let  $K(t) := \{x \in K \mid |x_n| \leq t\}$  for  $t > 0$ . Then

$$\frac{\Delta(K(t))}{t}$$
 is decreasing.

He proved this for  $n = 2$ . Woods [72a], [73] proved the conjecture for  $n = 3$  and showed that for  $n \geq 3$

$$\frac{\Delta(K(t))}{t^{n-1}}$$
 is decreasing.

If  $K$  is a symmetric convex body of  $\mathbb{R}^2$  then it follows from a result of Rogers [51] that

$$\Delta(K) = \Delta(K') \tag{1}$$

where

$$K' := \{(x, y) \in \mathbb{R}^{2+1} \mid x \in K, y \in [-1, 1]\}. \tag{2}$$

The so called *conjecture of cylinders* says that (2) implies (1) in other cases too (see 3.5). If  $K$  is not convex, then (2) need not imply (1) as can be seen from an example of Davenport and Rogers [50]. Since for convex symmetric bodies the problem to find the lattice constant and the problem to find the maximal lattice packing density are equivalent, some results which are actually results about lattice constants and critical lattices will be mentioned in section 3.5 below. For surveys of older results we refer to Keller [54] and Lekkerkerker [69B]. Lattice constants for *star bodies*, critical lattices and so-called isolation problems

have been considered quite frequently during recent years. Some of these results actually belong to the more general setting of Minkowski-Hlawka packings (see 3.7). We mention the following sets:

$$\begin{aligned}
 n=2: \quad & |x_1 x_2| \leq 1 \quad \left\{ \begin{array}{l} \text{v. Wolff [64], Hightower [70], Hall [71],} \\ \text{Jackson [72], Gruber [67]. (See also} \\ \text{Malyšev [77S].)} \end{array} \right. \\
 & 0 \leq x_1 x_2 \leq 1 \quad \text{Dumir [70], Jackson [68], Lindgren [75]} \\
 & 2|x_2|\sqrt{3}|x_1| - |y_1| \leq \quad \text{or } |x_2| \leq \sqrt{3}|x_1| \quad \text{or } x_2^2 - 3x_1^2 \leq 1 \\
 & \quad \text{Rehman [67]} \\
 & |x_2(3x_1^2 - x_2^2)| \leq 1 \quad \text{Dauenhauer [68]} \\
 n=3: \quad & -1 \leq x_1^2 - x_2^2 - x_3^2 \leq t \quad \text{Worley [67]} \\
 & 0 \leq x_1^2 - x_2^2 - x_3^2 \leq 1 \quad \text{Worley [71]} \\
 & |x_1 x_2 x_3| \leq 1 \quad \text{Swinnerton-Dyer [71]} \\
 n=4: \quad & \left. \begin{array}{l} 0 \leq x_1^2 + x_2^2 - x_3^2 - x_4^2 \leq 1, \\ 0 \leq x_1^2 + x_2^2 - x_3^2 - x_4^2 \leq 1, \\ 0 \leq x_1^2 - x_2^2 - x_3^2 - x_4^2 \leq 1 \end{array} \right\} \quad \text{Jackson [68]} \\
 & |x_1 x_2 x_3 x_4| \leq 1 \quad \text{Noordzij [67]} \\
 n=5: \quad & \left. \begin{array}{l} |x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2| \leq 1, \\ |x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2| \leq 1, \\ |x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2| \leq 1, \\ |x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2| \leq 1 \end{array} \right\} \quad \text{Raka [78]} \\
 n \text{ arb.:} \quad & \left. \begin{array}{l} |x_1^n + \dots + x_n^n + kx_1 \dots x_n| \leq 1 \quad (k \in \mathbb{R}), \\ |x_1^m + \dots + x_n^m| \leq 1 \quad (m \in \mathbb{N}) \end{array} \right\} \quad \text{Golser [75]}
 \end{aligned}$$

Keller [54B], Cassels [59B] and Lekkerkerker [69B] give complete surveys of older results about lattice constants of star bodies.

A method (called *Mordell's method*) to find lattice constants of star bodies and convex bodies is to reduce the  $n$ -dimensional problem to one of an  $(n-1)$ -dimensional cross-section of the body. In particular this method is applicable to bodies with large automorphism groups. The first traces of this method go back to Gauß [1831] and Hermite [1850]. A clear application of it was given by Korkine and Zolotareff [1872]. More modern applications of it are summed up in Lekkerkerker's book [69B]. Recently Malyšev [73], [73a] gave several interesting generalizations of results of Lekkerkerker [61] and Mullender [64]. The article of Malyšev [73a] contains a good survey of Mordell's method.

A star body or convex body  $K$  is called *reducible* if there exists a star body  $M \subsetneq K$  with the same lattice constant, *irreducible* otherwise. Questions of reducibility have been investigated in great detail in the 1940's and 1950's (see Cassels [59B] and Lekkerkerker [69B]). More recently Hans [71a] proved results of this type.

### 3.5 Lattice packings

A set  $\{K_i | i \in I\}$  of subsets of  $\mathbb{R}^n$  will be called a *packing* if any two of them have disjoint interior. In the following we will consider only packings

$$\{K + a_i | i \in I\}$$

of translates of a given bounded Jordan-measurable set  $K$ . If the vectors  $a_i$  are distributed in a sufficiently regular way then the limit

$$\lim_{\rho \rightarrow +\infty} \frac{v(\bigcup \{K + a_i | i \in I\} \cap \rho B)}{v(\rho B)}$$

exists and is called the *density* of the packing (1). (Here  $B$  denotes the unit ball of  $\mathbb{R}^n$ .) The supremum of the densities of all packings of translates of  $K$  is called the *maximal packing density* (by translates) of  $K$  and is denoted by  $\delta(K)$ . A packing of the form (1) where the  $a_i$ 's are the points of a lattice  $L$  is called a *lattice packing* and  $L$  is called a *packing lattice* for  $K$ . The density always exists and is equal to  $v(K)/d(L)$ . The supremum of these densities  $\delta_L(K)$  is called the *maximal lattice packing density* of  $K$ . If  $K$  is a convex body or a star body and  $\delta_L(K) < +\infty$ , Mahler's selection theorem implies that there are always packing lattices for  $K$  such that the density is equal to  $\delta_L(K)$ . Such lattices are called *maximal density packing lattices* for  $K$ . Obviously

$$\delta_L(K) \leq \delta(K) \leq 1.$$

There exists a vast literature about packings. For excellent accounts see e.g. the books of Fejes Tóth [53B], Rogers [64B] and the article of Baranovskii [69S]. Let  $K$  be a *convex symmetric body*. Since a lattice is a packing lattice for  $K$  if and only if it is admissible for  $2K$ , it is easy to see that

$$\delta_L(K) = \frac{v(K)}{2^n \Delta(K)}$$

and that the maximal density packing lattices for  $K$  are precisely the critical lattices for  $2K$ . Hence all that was said in section 3.4 for convex symmetric bodies, applies to the problem of densest lattice packings too. It is not known for which  $K$   $\delta_L(K)$  is minimal even if  $n=2$  (but see 3.4). For arbitrary  $n$  the best lower bounds for  $\delta_L(K)$  are those of section 3.2:

$$\delta_L(K) \geq 2(n\alpha - \beta)2^{-n} \text{ with } \alpha = 0.3465\dots \text{ and } \beta \in \mathbb{R}. \quad (1)$$

Let  $K$  be a *convex body*, not necessarily symmetric. If  $n=2$  it has been proved by Rogers [51], that

$$\delta_L(K) = \delta(K). \quad (2)$$

There are far-reaching refinements of this result, for which we refer the reader to papers and books on discrete geometry. Fary [50] proved that for  $n=2$

$$\delta_L(K) \geq \frac{2}{3}.$$

Equality holds if and only if  $K$  is a triangle. A different proof of this was given by Courant [65]. For arbitrary  $n$  the best lower bounds of  $\delta_L$  are derived from (1): We have

$$\delta_L(K) = \frac{v(K)}{v(\frac{1}{2}(K-K))} \delta_L(\frac{1}{2}(K-K)).$$

Now note that  $(K - K)/2 = \{x - y / 2 | x, y \in K\}$  is a convex symmetric body. Hence (1) holds if  $K$  is substituted by  $(K - K)/2$ . Furthermore  $v(K)/v((K - K)/2) \geq 2^n(n!)^2/(2n)!$  (see Rogers [64B]). If

$$K' := \{(x, y) \in \mathbb{R}^{n+1} | x \in K, y \in [-1, 1]\},$$

then (2) implies that

$$\delta_L(K') = \delta(K') = \delta_L(K). \quad (3)$$

This leads to the so-called *conjecture of cylinders*: The maximal (lattice) packing density of a cylinder with a convex base is equal to the maximal (lattice) packing density of the base. We have just mentioned that the conjecture is true if the base is 2-dimensional. If the base is a 3-dimensional ball it was proved by Woods [58]. If  $K$  is not convex, (3) does not hold necessarily. More recent results about the conjecture of cylinders were given by Bantegnie [74].

For a 3-dimensional simplex  $S$  Groemer [62] proved  $\delta_L(S) \geq 18/49 = 0.3673\dots$  and conjectured that equality holds. This conjecture has been confirmed by Hoyzman [70].

The minimal density of so-called connected lattice packings has been investigated by Groemer [66] and Fejes Tóth [73].

A connection between packings and coverings of convex bodies has been considered by Butler [72].

There are only a few results about packings of (solid) *star bodies*. Stein [72], [74] produced examples of star bodies  $K \subset \mathbb{R}^5$  for which

$$\delta_L(K) < \delta(K). \quad (4)$$

Using these examples it is easy to construct symmetric star bodies in  $\mathbb{R}^n$  for each  $n \geq 10$  for which (4) holds.

$\{K + a_i | i \in I\}$  is called a *k-fold packing* of translates of  $K$  if each point of  $\mathbb{R}^d$  is an interior point of at most  $k$  of the sets  $K + a_i$ . It is obvious how to define the concepts of *k-fold lattice packings* and of *maximal k-fold (lattice) packing density*  $\delta_L^{(k)}(K), \delta^{(k)}(K)$ . For  $n=2$  Dumir and Hans-Gill [72a] proved that

$$\delta_L^{(2)}(K) = 2\delta_L(K)$$

for each convex symmetric body  $K$ . They produced an example of a star body  $S \subset \mathbb{R}^2$  for which

$$\delta_L^{(2)}(S) > 2\delta_L(S).$$

(Here the interesting case is the case of k-fold lattice packing.)

### 3.6 Lattice tilings

A packing  $\{K_i | i \in I\}$  is called a *tiling* or a *tessellation* if every point of  $\mathbb{R}^n$  belongs to at least one of the sets  $K_i$ . A lattice packing is called a *lattice tiling* if it is a tiling. A convex body is called a *parallelohedron* if it admits a lattice tiling. A parallelohedron is necessarily a convex symmetric polytope. There exists a huge literature about tilings for which we refer the reader to books on discrete geometry and to the forthcoming book of Grünbaum and Shephard on tilings. In the

following we mention only some aspects which are of relevance to the geometry of numbers.

In his 18<sup>th</sup> problem Hilbert asks whether there exist polyhedra which do not appear as fundamental regions of groups of motions of  $\mathbb{R}^n$ , by means of which nevertheless by a suitable juxtaposition of congruent copies a tiling of  $\mathbb{R}^n$  is possible. For information about the general problem see Milnor [76S]. A convex body which admits a tiling by translates is necessarily symmetric. If  $n=2$  it is easy to show that each convex body that admits a tiling by translates also admits a lattice tiling. For  $n=3$  Delone [33] sketched a proof of the corresponding result. A proof for all  $n$  was given by McMullen [80]. For star bodies it has been proved by Stein [72], [74] that for  $n=5$  there exist star bodies admitting tilings by translates but no lattice tilings. Using this example it is easy to construct a symmetric star body in  $\mathbb{R}^n$  for  $n \geq 10$  which admits a tiling by translates but not as a lattice.

A *zonotope* is a polytope of the form  $S_1 + \dots + S_k (= \{x_1 + \dots + x_k | x_i \in S_i\})$  with line segments  $S_1, \dots, S_k$ . The problem which zonotopes give lattice tilings of  $\mathbb{R}^n$  has been considered by Coxeter [62] who also stated several interesting conjectures, Shephard [74] and McMullen [75]. The latter gave a complete solution to this problem. Interesting results about multiple tilings by zonotopes are given by Groemer [79].

Let  $L$  be a lattice and  $f$  a positive definite quadratic form on  $\mathbb{R}^n$ . Then

$$D = D(f, L) := \{x | f(x) \leq f(x - a) \text{ for all } a \in L\}$$

is called the *honeycomb* of  $L$  and  $f$ . (This is an affine image of a so called *Dirichlet* or *Voronoi cell*.) It is necessarily a convex symmetric polyhedron with at most  $2^n - 1$  pairs of centrally symmetric facets. It is a parallelohedron. It is no restriction of generality, if one assumes that  $L = L_0$ . Dirichlet [1850] and in particular Voronoi [8], [8a], [9] thoroughly investigated the properties of honeycombs. A parallelohedron  $K$  is called *primitive*, if a lattice tiling  $\{K + a | a \in L\}$  of  $K$  has the following properties: Each two different translates  $K + a, K + b$  of  $K$  with non empty intersection meet in entire facets and at each vertex of  $K$  there meet exactly  $n+1$  of the translates of  $K$  (including  $K$ ). Voronoi proved that *each primitive parallelhedron is a honeycomb*.

It is an open problem of Voronoi whether each parallelohedron is a honeycomb. For a discussion of this see McMullen [80]. See also Delone [29] and Žitomirskii [29].

The following table indicates the number of *combinatorially non-isomorphic types of parallelohedra* in  $\mathbb{R}^n$

$n$	primitive	non primitive
2	1 Fedorov [1885B]	1 Fedorov [1885B]
3	1 Voronoi [8a], [9]	4 Voronoi [8a], [9]
4	3 Delone [29] – [37]	49 Delone [29] – [37],
5	221 Baranovskii and Ryškov [73], [75]	Štogrin [75B]

Voronoi and also Delone and Sandakova [61] gave algorithms for finding parallelohedra. These algorithms lead to great computational difficulties. The work of Baranovskii and Ryškov [73], [75] makes use of results of Baranovskii

[71] and Ryškov [73c] on so called C-types of parallelohedra. Besides the enumeration of the topological types of primitive and non primitive parallelohedra it is of interest to give a precise description of their combinatorial structure and to find the quadratic forms (in case  $L = L_0$ ) which correspond to each type of primitive parallelohedra. Contributions to these problems are contained in the papers mentioned above, in papers on  $L$ -partitions and in some papers on lattice coverings by balls.

Each honeycomb  $D = D(f, L)$  give rise to a tiling of space, called an  $L$ -partition which is dual to the tiling by translates of  $D$ : For each vertex  $v$  of one of the polytopes  $D + a | a \in L$  consider all lattice points  $a_1, \dots, a_k$  such that  $v$  is a vertex of each of the polytopes  $a_1 + D, \dots, a_k + D$ . Now let

$$V = V(f, L, v) := \text{conv} \{a_1, \dots, a_k\}.$$

The set of all such polytopes is a tiling of  $\mathbb{R}^n$ . If  $D$  is primitive then each  $V$  is a simplex. Since the investigations of tilings with honeycombs and of  $L$ -partitions are equivalent, we refer to the articles mentioned above. Accounts on the theory of  $L$ -partitions (for discrete point sets instead of lattices) are contained in Rogers [64B] and Baranovskii [69S]. Some further results are contained in the papers of Baranovskii [71], [73], [73a] and Golubev [73]. An interesting account on multiple tilings is contained in a paper of Robinson [79].

### 3.7 Minkowski-Hlawka packings

Let  $K$  be a convex body or a star body. A set of translates  $\{K + a_i | i \in I\}$  of  $K$  is called a *Minkowski-Hlawka packing* of  $K$  if for each  $i$  none of the points  $a_j, j \in I \setminus \{i\}$  is an interior point of  $K + a_i$ . If  $L$  is a lattice such that  $\{K + a | a \in L\}$  is a Minkowski-Hlawka packing,  $\{K + a | a \in L\}$  is called a *Minkowski-Hlawka lattice packing*. For all sufficiently regular Minkowski-Hlawka packings the limit

$$\lim_{\rho \rightarrow +\infty} \frac{\#\{i | a_i \in \rho B\}}{v(\rho B)}$$

exists. The supremum of these limits is denoted by  $\delta_{MH}(K)$  and called (*Minkowski-Hlawka*) *packing density* of  $K$ . The (*Minkowski-Hlawka*) *lattice packing density*  $\delta_{MHL}(K)$  is defined correspondingly. Obviously

$$\delta_{MHL}(K) \leq \delta_{MH}(K).$$

If  $K$  is convex and symmetric, then

$$v(K)\delta_{MH}(K) = 2^n \delta(K) \text{ and } v(K)\delta_{MHL}(K) = 2^n \delta_L(K).$$

Therefore all the results mentioned in 3.5 for convex symmetric bodies hold true for Minkowski-Hlawka packings.

v.Wolff [62] gave on example of a 2-dimensional star body  $K$  with

$$\delta_{MHL}(K) < \delta_{MH}(K).$$

Groemer [64] showed that for each  $\alpha > 1$  there exists a 2-dimensional star body for which

$$\alpha \delta_{MHL}(K) < \delta_{MH}(K).$$

On the other hand, for the star bodies

$$\begin{aligned} |x_1 x_2| &\leq 1, & \text{v. Wolff [64]} \\ |(3x_1^2 - x_2^2)x_2| &\leq 1, & \text{Dauenauer [68]} \end{aligned}$$

it was shown that the Minkowski-Hlawka packing density and lattice packing density coincide. For interesting surveys see Zassenhaus [61S] and Fejes Tóth [67S]. Since Minkowski-Hlawka packings belong to discrete geometry rather than to geometry of numbers see articles on discrete geometry for more recent results.

### 3.8 Lattice packings of balls

The problem of the densest lattice packing of the Euclidean unit ball  $B$  has been solved for  $n \leq 8$ . We have

$n$	$\delta_L(B)$	
2	$\frac{\pi}{2\sqrt{3}} 0.9069\dots$	Lagrange [1773]
3	$\frac{\pi}{3\sqrt{2}} 0.7404\dots$	Gauß [1831]
4	$\frac{\pi^2}{16} = 0.6168\dots$	
5	$\frac{\pi^2}{15\sqrt{2}} = 0.4652\dots$	Korkine and Zolotareff [1872], [1877]
6	$\frac{\pi^3}{48\sqrt{3}} = 0.3729\dots$	
7	$\frac{\pi^3}{105} = 0.2952\dots$	Blichfeldt [34]
8	$\frac{\pi^4}{384} = 0.2536\dots$	

(The calculations of Blichfeldt have been verified by Watson [66].) For  $n \geq 9$  there exist lower and upper bounds for  $\delta_L(B)$  (with large gaps between them), many explicitly constructed dense lattice packings and results on so called extreme forms.

An old result of Minkowski, later generalized by Hlawka [44] (see 3.2) implies that

$$\delta_L(B) \geq \frac{\zeta(n)}{2^{n-1}}.$$

Rogers [47] and Davenport and Rogers [47] found slightly larger lower bounds. All these bounds are not essentially better than the bounds derived from

the theorem of Minkowski-Hlawka. Among upper bounds we mention the following ones:

$$\begin{aligned}
 \delta_L(B) &\leq \frac{n+1}{2} 2^{-n/2} & \text{Blichfeldt [14]} \\
 \delta_L(B) &\leq \frac{n}{\alpha_n} 2^{-n/2}, \alpha_n \rightarrow e = 2.7182\dots & \text{Rogers [58a], Baranovskii [64]} \\
 \delta_L(B) &\leq 2^{-(0.509 + o(1))n} & \text{Sidel'nikov [73], [74]} \\
 \delta_L(B) &\leq 2^{-(0.5237 + o(1))n} & \text{Levenštejn [75]}
 \end{aligned}$$

The last two estimates are based on results in *information theory*. Rogers [64B] and Lekkerkerker [69B] give excellent surveys of results on ball packings of the type just described. Also Milnor [76S] gives a useful survey of ball packings. See also Kabat'janski and Levenštejn [78].

Since the above results do not permit to give explicitly dense packings of spheres, it is of interest to have explicit descriptions of dense (lattice) packings of balls available. Explicit descriptions of dense (lattice) packings of balls have been given by, among others, Leech [64], [67], [67a], [69], Ignat'ev [64], [68], Leech and Sloane [70], [70a], [71], Sloane [72], [77] and Vetčinkin [74]. Some of these results make use of error correcting codes. Surveys are contained in the articles of Sloane [77S] and Milnor [76S].

So far we have been concerned with the absolute maximum of the densities of lattice packings of equal balls. There exists an extensive literature on local maxima of the densities of lattice packings of equal balls. For more information about this see Lekkerkerker [69B].

In connection with ball packings several special problems have been considered. Regular dense non-lattice packings of balls have been considered by various authors. So far no example of a non lattice packing of balls (and also of arbitrary symmetric convex bodies) is known which is denser than the densest lattice packing. Coxeter [63] considers upper bounds for the number of balls that can touch another of the same size. See also Leech [64]. Ryškov and Horvath [75] give an estimate of the radius of a circular cylinder of infinite length which can be embedded in every lattice packing of unit balls. Ryškov [74], Dolbilin [76] and Delone, Dolbilin, Štočkin and Galiulin [76] consider so-called  $(r, R)$ -systems which form a bridge between packings and coverings.

### 3.9 The Epstein zeta function

Let  $L$  be a lattice of determinant 1. Then the *Epstein zeta function* is defined by

$$\zeta = \zeta(L, s) := \sum_{a \in L \setminus \{0\}} \frac{1}{|a|^{2s}} \quad \text{for } s > \frac{n}{2}.$$

Equivalently, let  $f$  be a positive definite quadratic form, then

$$\zeta(f, s) := \sum_{a \in L_0 \setminus \{0\}} \frac{1}{f(a)^s}.$$

Many papers have been written about analytic properties of the Epstein  $\zeta$ -function. From the point of view of geometry of numbers the study of the Epstein zeta function began with a paper of Rankin [53]. If one denotes the ball packing problem the problem of “hard” balls, the problem to find the minimum of  $\zeta(L,s)$  for fixed  $s$  can be considered as a packing problem for “soft” balls. A question of Sobolev on numerical integration led to the interest of the Russian school of geometry of numbers in the Epstein  $\zeta$ -function.

A lattice  $L$  of determinant 1 is called  $\zeta$ -optimal for some value of  $s$ , if  $\zeta(L,s)$  is minimal among all lattices of determinant 1.  $L$  is called  $\zeta$ -stationary for  $s$  if  $\zeta(L,s)$  is a local minimum.

For  $n=3$  it has been shown by Rankin [53] ( $s \geq 1.035$ ), Cassels [59], Diananda [64], Ennola [64] ( $s > 1$ ) and Delone, Sandakova and Ryškov [5] ( $s > 2$ ) that the lattice which gives the densest packing of equal circular disks is  $\zeta$ -optimal. Ryškov [69] proved that up to rotations this lattice is the only  $\zeta$ -stationary lattice for  $s \geq 3$ . For  $n=3$  Ennola [64a] ( $s > 3/2$ ) and Sandakova [67] ( $s \geq 2$ ) prove that the lattice which gives the densest packing of equal balls is  $\zeta$ -stationary. Ryškov [73] shows that for  $n=2, 3, \dots, 8$  the lattices providing the densest lattice packing of equal balls are  $\zeta$ -optimal for all sufficiently large  $s$ . General results of this type are contained in papers of Delone and Ryškov [67] and Ryškov [[3]].

### 3.10 Multiple packings of balls

For  $n=2$  Heppes [59], Blundon [64], G. Fejes Tóth and Florian [75] and G. Fejes Tóth [76] gave various bounds for the density of  $k$ -fold lattice and non-lattice packings of equal circles. For  $n=3$  Few and Kanagasabapathy [69] show that

$$\delta_L^{(3)}(B) = \frac{8\pi}{9\sqrt{3}} = 1.6122\dots > 2\delta_1(B) = \frac{2\pi}{3\sqrt{2}} = 1.4809\dots$$

An estimate for  $\delta_L^{(3)}(B)$  was given by Purdy [73]. For other results see Few [67] and Baranovskii [69S].

## 4. Reduction theory of positive definite quadratic forms

One can describe the reduction theory of positive definite quadratic forms either geometrically, using lattices and ellipsoids or arithmetically. For the geometric approach see e.g. Keller [54B], Lekkerkerker [69B] and Aitchison [72]. If  $L = BL_0$  is a lattice in  $\mathbb{R}^n$  one can associate with it the positive quadratic form  $f = x^t B^t B x = x^t A x = \sum a_{ij} x_i x_j$  ( $a_{ij} = a_{ji}$  for all  $i, j \in \{1, \dots, n\}$ ). If on the other hand  $f = x^t A x$  ( $A = (a_{ij})$ ,  $a_{ij} = a_{ji}$ ) is a given quadratic form, then there is a lattice  $L$  which is unique up to rotation such that  $f$  corresponds to  $L$ . This connects the arithmetic and geometric description of reduction theory.

Each positive definite  $n$ -ary quadratic form  $f = \sum a_{ij} x_i x_j$  ( $a_{ij} = a_{ji}$ ) can be represented as a point in  $N (= n(n+1)/2)$ -dimensional coefficient space  $\mathbb{R}^N$ . The set of all such forms corresponds to an open convex cone  $\mathcal{K}$  in  $\mathbb{R}^N$  with apex 0. We will identify forms with points in  $\mathbb{R}^N$ .

Two positive definite n-ary quadratic forms  $f = x^T A x$ ,  $g$  are called *equivalent*, if there exists an integral unimodular transformation  $G$  such that  $g = x^T G^T A G x = f(Gx)$ . The main problems of a reduction theory are the following:

- (1) To describe a set of forms (i.e. a subset of  $\mathcal{H}$ ) which contains at most a finite number of forms of each equivalence class of forms, so-called *reduced forms*. Such sets are in general convex cones or finite unions of convex cones. They are called *fundamental regions* if each two forms in the interior are non-equivalent.
- (2) To find out which transformations transform a reduced form into all other equivalent reduced forms.
- (3) To find an algorithm which transforms an arbitrary form into a reduced form.

Denote by  $f$  an arbitrary positive definite n-ary quadratic form.  $f$  is called *Hermite-reduced* if the function

$$h(t) := a_{11}t^{n-1} + a_{22}t^{n-2} + \cdots + a_{nn}$$

is minimal as  $t \rightarrow \infty$  among all functions defined in this way for all forms equivalent to  $f$ . Let  $\mathcal{H}^* \subset \mathcal{H}$  denote the set of Hermite-reduced forms. For each permutation  $\pi$  of  $\{1, \dots, n\}$  define  $\mathcal{H}_\pi^*$  as the set of all forms for which

$$h_\pi(t) = a_{\pi(1)\pi(1)}t^{n-1} + \cdots + a_{\pi(n)\pi(n)}$$

is minimal as  $t \rightarrow +\infty$  among all functions defined in this way for all forms equivalent to  $f$ . Let

$$\mathcal{H} := \bigcup \mathcal{H}_\pi^*.$$

$f$  is *Minkowski-reduced* if

$$f(u_1, \dots, u_n) \geq a_{ii} \text{ for each } i \in \{1, \dots, n\} \text{ and all integer } n\text{-tuples } (u_1, \dots, u_n) \text{ with g.c.d. } (u_1, \dots, u_n) = 1.$$

Denote the set of these forms by  $\mathcal{M}^*$ . (An additional condition for reduced forms considered by Minkowski is omitted, as is common.) Now define  $\mathcal{M}$  analogously to  $\mathcal{H}$ . Minkowski [11B] proved the following two *finiteness theorems* for reduction:

*There is a finite set of integer n-tuples  $U = U_1 \cup \cdots \cup U_n$  such that  $f$  is Minkowski-reduced if and only if*

$$f(u_1, \dots, u_n) \geq a_{ii} \text{ for each } i \in \{1, \dots, n\} \text{ and } (u_1, \dots, u_n) \in U_i. \quad (1)$$

*There is a finite set of integral unimodular transformations  $\{G_1, \dots, G_m\}$  such that for each reduced form the reduced equivalent forms are  $f(G_1 x), \dots, f(G_m x)$ .*

For  $n \leq 6$  sets  $U$  have been described explicitly by Minkowski [11B], but he proved for  $n \leq 4$  only that conditions (1) imply that  $f$  is reduced. For  $n = 5, 6$  proofs were given by Ryškov [72a] and Tammela [73a], [73b]. Thus  $f$  is Minkowski-reduced if

$$a_{ii} \geq a_{i+1, i+1}$$

and  $f$  satisfies the inequalities

$$f(u_1, \dots, u_n) \geq a_{ii}$$

where  $i \in \{1, \dots, n\}$  ( $n \leq 6$ ) and  $(u_1, \dots, u_n)$  is an arbitrary permutation of one of the rows up to index  $n$  of the following table:

2	$1 \pm 1$	0	0	0	0
3	$1 \pm 1 \pm 1$	0	0	0	0
4	$1 \pm 1 \pm 1 \pm 1$	0	0	0	0
5	$1 \pm 1 \pm 1 \pm 1 \pm 1$	0	0	0	0
6	$1 \pm 1 \pm 1 \pm 1 \pm 2$	0	0	0	0
	$1 \pm 1 \pm 1 \pm 1 \pm 1 \pm 1$				
	$1 \pm 1 \pm 1 \pm 1 \pm 1 \pm 2$				
	$1 \pm 1 \pm 1 \pm 1 \pm 2 \pm 2$				
	$1 \pm 1 \pm 1 \pm 1 \pm 2 \pm 3$				

Furthermore Tammela proved that for  $n \leq 6$  the transformations which transform a given Minkowski-reduced form  $f$  into all equivalent reduced forms are the unimodular integral transformations  $G = (g_{ij})$  such that for  $i \in \{1, \dots, n\}$   $(g_{i1}, \dots, g_{in})$  is a permutation of a row up to index  $n$  of the following table:

2	$\pm 1$	0	0	0	0
3	$\pm 1 \pm 1$	0	0	0	0
4	$\pm 1 \pm 1 \pm 1$	0	0	0	0
5	$\pm 1 \pm 1 \pm 1 \pm 1$	0	0	0	0
6	$\pm 1 \pm 1 \pm 1 \pm 1 \pm 1$	0	0	0	0
	$\pm 1 \pm 1 \pm 1 \pm 1 \pm 1 \pm 1$				
	$\pm 1 \pm 1 \pm 1 \pm 1 \pm 1 \pm 2$				
	$\pm 1 \pm 1 \pm 1 \pm 1 \pm 1 \pm 3$				
	$\pm 1 \pm 1 \pm 1 \pm 1 \pm 2 \pm 2$				
	$\pm 1 \pm 1 \pm 1 \pm 1 \pm 2 \pm 3$				
	$\pm 1 \pm 1 \pm 1 \pm 1 \pm 2 \pm 2 \pm 2$				
	$\pm 1 \pm 1 \pm 1 \pm 1 \pm 2 \pm 2 \pm 3$				
	$\pm 1 \pm 1 \pm 1 \pm 1 \pm 2 \pm 2 \pm 4$				
	$\pm 1 \pm 1 \pm 1 \pm 2 \pm 2 \pm 2 \pm 3$				

Based on this result Tammela describes an algorithm to find for each  $f$  an equivalent reduced form. Finally the honeycomb

$$D = D(f, L_0) = \{x \mid f(x) \leq f(x - u) \text{ for all } u \in L_0\}$$

for Minkowski-reduced  $f$  is equal to the set

$$\{x \mid f(x) \leq f(x - u)\}$$

where  $u^t = (u_1, \dots, u_n)$  is a permutation of an arbitrary row of the last table above up to index  $n$  or of the following table ( $n = 5, 6$ ):

$$\begin{array}{r|ccccc|c} 5 & \pm 1 & \pm 1 & \pm 1 & \pm 2 & \pm 3 & 0 \\ \hline & \pm 1 & \pm 1 & \pm 1 & \pm 2 & \pm 3 & \pm 3 \\ & \pm 1 & \pm 1 & \pm 1 & \pm 2 & \pm 3 & \pm 4 \\ & \pm 1 & \pm 1 & \pm 2 & \pm 2 & \pm 3 & \pm 4 \\ 6 & \pm 1 & \pm 2 & \pm 2 & \pm 2 & \pm 3 & \pm 3 \end{array}$$

Results for  $n = 7$  are contained in Tammela [77].

Let  $g = \sum b_{ij}x_i x_j \in K$  be given. We consider the set of all  $f = \sum a_{ij}x_i x_j \in K$  with  $\det a_{ij} = \det b_{ij}$ . This is a smooth unbounded convex surface contained in  $\mathcal{K}$ , called the *discriminant surface*. All  $g_k \in \mathcal{K}$  equivalent to  $g$  are on this surface. The tangent hyperplanes of the discriminant surface at all  $g_k$  define an unbounded convex polyhedral set. Each  $(N - 1)$ -dimensional facet of this set is a convex polytope (with finitely many vertices). Each two of these facets are equivalent. Now a quadratic form  $f$  is called  *$g$ -Venkov-reduced*, if the ray starting at the origin and containing  $f$  meets the facet which contains  $g$ . Denote the set of these forms  $f$  by  $\mathcal{V}_g$ . In case  $g = x_1^2 + \dots + x_n^2$  we write  $\mathcal{V}$  instead of  $\mathcal{V}_g$  and we have

$$\mathcal{V} = \{f \in \mathcal{K} \mid a_{11} + \dots + a_{nn} \leq f(g_{11}, \dots, g_{1n}) + f(g_{21}, \dots, g_{2n}) + \dots + f(g_{n1}, \dots, g_{nn})\},$$

where  $G = (g_{ij})$  is an arbitrary integral unimodular transformation.

Let

$$\mathcal{V}^* := \{f \in \mathcal{V} \mid a_{11} \leq a_{22} \leq \dots \leq a_{nn}\}.$$

We consider the boundary points of  $\mathcal{K}$  of the form  $(q_1^2, \dots, q_n^2, q_1 q_2, q_1 q_3, \dots, q_{n-1} q_n)^\top$  where  $(q_1, \dots, q_n)^\top \in L_0$  with  $\gcd(q_1, \dots, q_n) = 1$ . Their convex hull is a closed unbounded polyhedral set. Denote it by  $\Pi$ .  $\Pi$  is called the complete *Voronoi polyhedron*. Each  $((N - 1)$ -dimensional) facet of  $\Pi$  is a convex polytope (with finitely many vertices). Consider those facets whose relative interiors are contained in  $\mathcal{K}$ . It is possible to select from this set of facets a maximal finite subset of non-equivalent facets. Now  $f$  is called *Voronoi-reduced* if the ray starting at the origin and containing  $f$  meets one of these facets.

There are several connections between  $\mathcal{H}, \mathcal{M}, \mathcal{V}$  and  $\mathcal{H}^*, \mathcal{M}^*, \mathcal{V}^*$  respectively. Ryškov [73a] showed that

$$\mathcal{H} \subset \mathcal{M}, \mathcal{H}^* \subset \mathcal{M}^*.$$

Furthermore  $\mathcal{H}, \mathcal{M}, \mathcal{V}$  (and also  $\mathcal{H}^*, \mathcal{M}^*, \mathcal{V}^*$ ) are equal to each other if any two of them are equal. Ryškov [73a] and Tammela [73a] proved that

$$\mathcal{H} = \mathcal{M} = \mathcal{V}, \mathcal{H}^* = \mathcal{M}^* = \mathcal{V}^* \quad \text{for } n \leq 6$$

$$\mathcal{H} \neq \mathcal{M} \neq \mathcal{V} \neq \mathcal{H}, \mathcal{H}^* \neq \mathcal{M}^* \neq \mathcal{V}^* \neq \mathcal{H}^* \quad \text{for } n \geq 7.$$

A number of other interesting results about reduction of positive quadratic forms can be found in recent papers of Delone, Ryškov and Štogrin [67], Chandra and Prakash [72], Dickson [72], Štogrin [72], Ryškov [73b], Barnes and Cohn [76], [76a], the booklet of Baranovskii Moldavanskii and Ceremisin [74B] and others. Several papers mentioned above are collected in Malyšev [73B].

Aitchison [72] proved that it is not possible to extend for  $n \geq 3$  the two finiteness theorems of Minkowski mentioned above from ellipsoids to general convex symmetric bodies. A proof that the first finiteness theorem cannot be extended was also given by Bruneder [31] in a Vienna thesis.

## 5. The inhomogeneous problem

### 5.1 Lattice coverings

A set  $\{K_i | i \in I\}$  of subsets of  $\mathbb{R}^n$  is called a *covering* of  $\mathbb{R}^n$  if their union is equal to  $\mathbb{R}^n$ . We will consider only coverings

$$\{K + a_i | i \in I\} \quad (1)$$

of translates of a convex body or a star body  $K$ . If the  $a_i$  are distributed sufficiently regularly, then

$$\lim_{\rho \rightarrow +\infty} \frac{\sum_i v((K + a_i) \cap \rho B)}{v(\rho B)}$$

exists. It is then called the *density* of the covering (1). The infimum of the densities of all coverings of translates of  $K$  is called the *minimal covering density* (by translates) of  $K$ . Denote it by  $\vartheta(K)$ . A covering of the form (1) is called a *lattice covering* of  $K$  if the  $a_i$ 's are the points of a lattice  $L$ .  $L$  is called a *covering lattice* for  $K$ . The density of a lattice covering always exists and is equal to  $v(K)/d(L)$ . The infimum of these densities is denoted by  $\vartheta_L(K)$  and we call it the *minimal lattice covering density*. If  $\vartheta_L(K) < +\infty$  and  $K$  is a convex body or a star body, then there exist always covering lattices for  $K$  such that the density is equal to  $\vartheta_L(K)$ . They are called *minimal density covering lattices* of  $K$  (see Hlawka [54]). Obviously

$$1 \leq \vartheta(K) \leq \vartheta_L(K).$$

Let  $K$  be a *symmetric convex body*. Suppose first  $n = 2$ . A result of Fejes Tóth [46] says that

$$\vartheta_L(K) \leq \frac{2\pi}{3\sqrt{3}} = 1.2091\dots$$

with equality if and only if  $K$  is an ellipse.

Fejes Tóth [50] proved

$$\vartheta_L(K) = \vartheta(K) = \frac{v(K)}{v(H)} \quad (1)$$

where  $H$  is the largest symmetric hexagon inscribed in  $K$ . Alternative proofs of (1) are contained in papers of Bambah and Rogers [52], Bambah, Rogers and Zassenhaus [64] and Bambah and Woods [68a]. (1) implies that for

$$K' := \{(x, y) \in \mathbb{R}^{2+1} | x \in K, y \in [-1, 1]\} \quad (2)$$

we have

$$\vartheta_L(K') = \vartheta_L(K). \quad (3)$$

This corresponds to the packing case for general planar convex bodies (see 3.5). For general  $n$  Rogers [58], [59] gave surprisingly small upper bounds for  $\vartheta_L(K)$ ,  $\vartheta(K)$ :

$$\begin{aligned}\vartheta_L(K) &\leq n^2 \log \log n + \text{const}, \\ \vartheta(K) &\leq n \log n + n \log \log n + 5n.\end{aligned}$$

These results improve many previous results of the same type. For further information see Rogers [64B], [74S] and Lekkerkerker [69B]. Related results are due to Erdős and Rogers [64] and Butler [72].

Let  $K$  be an arbitrary *convex body*. For  $n=2$  Fejes Tóth [50] proved that

$$\vartheta_L(K) \geq \vartheta(K) \geq \frac{v(K)}{v(H)}$$

and Bambah, Rogers and Zassenhaus [64] proved

$$\vartheta_L(K) \geq \vartheta(K) \geq \frac{v(K)}{2v(T)},$$

which in some cases is stronger than the result of Fejes Tóth. Here  $H, T$  denote the largest symmetric hexagon and the largest triangle respectively, inscribed in  $K$ . It is an open question whether

$$\vartheta_L(K) = \vartheta(K)$$

holds for each convex body. Fary [50] gave the following upper bound for  $\vartheta_L(K)$ :

$$\vartheta_L(K) \leq \frac{3}{2}.$$

Here equality holds if and only if  $K$  is a triangle.

We will consider *star bodies*. For  $n=2$  Bambah, Dumir and Hans-Gill [77] gave an example of a bounded, symmetric star body  $K$  for which

$$\vartheta_L(K) > \vartheta(K). \quad (4)$$

They also show that for star bodies (2) does not necessarily imply (3). A result of Stein [72], [74] also implies the existence of bounded star bodies  $K$  in  $\mathbb{R}^5$  for which (4) holds. By the use of these examples it is easy to construct in each dimension  $\geq 10$  bounded symmetric star bodies for which (4) holds. Hans [70] and Hans-Gill [70a] found  $\vartheta_L(K)$  for some star bodies in  $\mathbb{R}^2$ .

There exists a number of *covering criterions* for convex (symmetric) bodies. The earlier result in most cases make use of the homogeneous minimum or of successive minima. They are called *transference theorems*. More recent results make use of other geometric properties of a convex body. Chalk [64], [67] gave the following interesting result:

*Let  $K$  be a convex symmetric body which contains  $n+1$  non-coplanar points of a lattice  $L$ . Then  $L$  is a covering lattice for*

$$\frac{n+1}{2} K \text{ (n odd), } n \frac{n+2}{2(n+1)} K \text{ (n even)}$$

*If  $n$  is odd,  $(n+1)/2$  is best possible.*

Generalizing a theorem of Bender [62] for  $n=2$ , Wills [68] and Hadwiger [70] proved the following:

Let  $K$  be a convex body of measure  $v(K)$  and surface area  $a(K)$ . If

$$v(K) - \frac{1}{2}a(K) > 0,$$

then  $L_0$  is a covering lattice for  $K$ .

More generally Bokowski, Hadwiger and Wills [72] showed that

$$\#(L_0 \cap K) > v(K) - \frac{1}{2}a(K).$$

These two theorems are central results of a new branch of lattice point problems for which we refer to a survey article in this volume. Covering criterions of different types mainly for  $n=2$  are reviewed (implicitly) by Hammer [77B]. Reducibility questions which are dual to analogous problems for packings or for lattice constants (see 3.4) have been considered in recent years by Mullender [68], Hans [71], [71a] and Dumir and Hans-Gill [72b] where there are references to related papers.

Since tilings are also coverings, much that was said in section 3.6 is of interest for the present section too.

$\{K + a_i | i \in I\}$  is called a  $k$ -fold covering of translates of  $K$  if each point of  $\mathbb{R}^n$  belongs to at least  $k$  different sets  $K + a_i$ . We will not give the obvious definitions of  $k$ -fold lattice coverings and of minimal  $k$ -fold (lattice) covering densities  $\vartheta_L^{(k)}(K), \vartheta^{(k)}(K)$  (see 3.5 above). Dumir and Hans-Gill [72] proved that for each convex symmetric body  $K$  in  $\mathbb{R}^2$

$$\vartheta_L^{(2)}(K) = 2\vartheta_L^{(1)}(K).$$

They gave an example of a compact symmetric star-body  $S$  in  $\mathbb{R}^2$  for which

$$\vartheta_L^{(2)}(S) < 2\vartheta_L^{(1)}(S).$$

For general  $n$  Cohn [76] modified a result of Rogers [59] (referred to above) in order to obtain a high multiplicity of lattice coverings for convex symmetric bodies without increasing the upper bound for the density essentially.

## 5.2 Lattice coverings by unit balls

A minimal density covering lattice of the Euclidean unit ball  $B$  is called *optimal*. A covering lattice for  $B$  is called *extreme* if the density is a local minimum among the densities for all covering lattices.

For  $n=2$  Kershner [39] proved

$$\vartheta_L^{(B)} = \frac{2\pi}{3\sqrt{3}} = 1.2091\dots$$

(Up to rotations) there is only one extreme (and thus optimal) lattice. For  $n=3$  Bambah [54] showed

$$\vartheta_L^{(B)} = \frac{5\sqrt{5}\pi}{24} = 1.4635\dots$$

Alternative proofs were given by Few [56] and Barnes [56]. The latter and Baranovskii [63] proved that there is only one extreme lattice. For  $n=4$  Delone and Ryškov [63], using results of Gameckii [62] indicated

$$\vartheta_L^{(B)} = \frac{2\sqrt{5}\pi^2}{25} = 1.7655\dots$$

For  $n=4$  there are 3 types of lattices in the sense of Voronoi (see 3.5). Gameckii found among all covering lattices of the first type the one with minimal density (which is unique up to rotation). Delone and Ryškov found the corresponding lattices among all lattices of the second and third type. Other extreme lattices do not exist, as was shown by Baranovskii [65], [66] and Dickson [67]. For  $n=5$  the lattice coverings of least density were studied by Ryškov and Baranovskii [75]. They showed that

$$\vartheta_L^{(B)} = \frac{5^{3/2}7^{5/2}\pi^2}{2^43^{11/2}} = 2.1242\dots$$

Bleicher [62], Gameckii [63] and some other authors investigated a certain lattice  $L_n$  and showed that it is extreme for each  $n$ . It corresponds to the quadratic form  $n \sum x_i^2 - \sum_{i \neq j} x_i x_j$  (see 4). It is optimal for  $n=2,3,4,5$ . For large  $n$  it cannot be optimal, since the corresponding density is larger than known upper bounds for  $\vartheta_L(B)$  for  $n$  sufficiently large. Another type of extreme lattices was found by Barnes and Trenerry [72]. For further results one may consult the papers of Barnes and Dickson [67], Ryškov [67], Baranovskii [69], Delone, Dolbilin, Ryškov and Štogrin [70] and Baranovskii's survey article [69S].

For general  $n$  Rogers [59] gave the estimate

$$\vartheta_L(B) \leq \text{const } n (\log n)^\alpha, \quad \alpha = \frac{1}{2} \log_2 2\pi e = 1.4189\dots$$

On the other hand a result of Coxeter, Few and Rogers [59] shows that

$$\vartheta_L(B) \geq \alpha_n n \quad \text{with } \alpha_n \rightarrow \frac{1}{e\sqrt{e}} = 0.2313\dots$$

It is remarkable how close these two estimates are.

For recent results about *multiple coverings by unit balls* we refer to the papers of Few [67], [71], G. Fejes Tóth and Florian [75] and G. Fejes Tóth [76].

### 5.3 The product of inhomogeneous linear forms

Let  $K \subset \mathbb{R}^n$ . The lattice covering problem for  $K$  is essentially the problem of finding

$$\sup \{d(L) | L \text{ is a covering lattice for } K\}$$

Another type of inhomogeneous problem is to find

$$\inf \{d(L) | L \text{ is not a covering lattice for } K\}.$$

Inhomogeneous problems of this sort are called by Bambah [71S] *inhomogeneous problems of the first type*. Unlike the problem of finding the least density lattice coverings, the general results known in this case are rather weak (see Lekkerkerker [69B]).

In the following we consider only one example of an inhomogeneous problem of first type, the so-called *conjecture of Minkowski on the product of inhomogeneous linear forms*. It can be formulated as follows:

Each lattice  $L$  of  $\mathbb{R}^n$  with  $d(L) = 1$  is a covering lattice for the star body

$$S : |x_1 \cdots x_n| \leq \frac{1}{2^n}.$$

If  $L$  is not of the form  $L = DL_0$  where  $D$  is a diagonal matrix, then  $L$  is a covering lattice for  $\lambda S$  for a suitable  $\lambda < 1$ .

So far this has been proved for  $n \leq 5$ . For  $n = 2$  there exist many proofs and refinements of various types (compare Gruber [67], Grover [78]). The first proof was given by Minkowski [7B]. For  $n = 3$  there are proofs of Remak [23], later simplified by Davenport [39], Birch and Swinnerton-Dyer [56] and Narzullaev [68]–[75]. Refinements for special types of lattices were given by several authors. We mention Godwin [65], who also gives references to earlier work in this area. Gruber [70a] proved a refinement employing the homogeneous minimum

$$\lambda(L) := \inf \{|x_1 x_2 x_3| \mid x \in L \setminus \{0\}\}. \quad (1)$$

Asymmetric results were given by Grover [78] and Woods [81]. For  $n = 4, 5$  Dyson [48] and Skubenko [72], [73] gave rather complicated proofs. The proof of Skubenko depends on a result of Woods [65a]. A simpler version of the proof for  $n = 5$  will be published soon by Skubenko. With the exception of the proofs of Birch and Swinnerton-Dyer and Narzullaev all proofs consist of the following propositions: (i) For each lattice  $L$  in  $\mathbb{R}^n$  there exists an ellipsoid  $E: a_1 x_1^2 + \dots + a_n x_n^2 \leq 1$  such that  $L$  is admissible for  $E$  and on the boundary of  $E$  there are  $n$  linearly independent points of  $L$ . (ii) Let  $L$  and  $E$  be as in 1. Then  $L$  is a covering lattice for  $(\sqrt{n}/2)E$ .

For  $n = 3$  a result analogous to (i) for parallelepipeds  $E$  was given by Ko [36] (see also 3.3). For  $n = 4$  a more general version of (i) was proved by Bambah and Woods [74]. For  $n = 4$  alternative proofs of (ii) were given by Hofreiter [33] and Cleaver [65a]. For  $n = 5, 6$  proposition (ii) was proved by Woods [65a], [72]. A qualitative result similar to (ii) was given by Dolbilin [74].

For  $n \geq 6$  various methods to prove the conjecture on the product of inhomogeneous linear forms have been considered:

Several authors show that the conjecture is true if  $L$  is a lattice of a specific type. It is easy to show that the conjecture holds, if  $L$  has a basis consisting of vectors with rational coordinates. For some older results of this type see Lekkerkerker [69B]. Birch and Swinnerton-Dyer [56] show that the conjecture is true in  $n+1$  dimensions for all lattices for which a certain homogeneous minimum (see (1) for  $n = 3$ ) is 0, provided the conjecture is true in  $n$  dimensions. Gruber [67] proved that for a set of lattices (of determinant 1) of measure  $\geq 1 - e^{-0.279n}$  the conjecture is true, even if one replaces  $2^n$  by (approximately)  $e^n$ . Furthermore, the conjecture is true in  $n+1$  dimensions for almost all lattices if it is true in  $n$  dimensions. A  $n \times n$ -Matrix  $A$  is called a DOTU-matrix, if  $A = DOTU$  where  $D$  is a diagonal matrix,  $O$  an orthogonal matrix,  $T$  an upper triangular matrix with 1's in the main diagonal and  $U$  an integral unimodular matrix (see Macbeath [61]). If  $L = AL_0$  where  $A$  is a DOTU-matrix, then it is easy to show that Minkowski's conjecture holds for  $L$ . Macbeath [61] proved that each  $2 \times 2$ -matrix is a DOTU-matrix. For  $n = 3$  this was indicated by Narzullaev [68]–[75]. Macbeath [61] showed that the set of DOTU-matrices is a dense open subset of

the set of all  $n \times n$ -matrices, containing the set of rational matrices. Gruber [76] and Ahmedov [77] indicated constructions of  $n \times n$ -matrices for sufficiently large  $n$  which are not DOTU-matrices.

Each lattice of determinant 1 is a covering lattice for the star body

$$|x_1 \dots x_n| \leq \frac{1}{c}$$

if  $c$  is chosen suitably: It may be taken to be

$$2^{n/2} \quad \text{Tschebotareff [34]}$$

$$2^{n/2} \alpha_n, \alpha_n \rightarrow 2e - 1 = 4.4365\dots \quad \text{Davenport [46]}$$

$$2^{n/2} \beta_n, \beta_n \rightarrow 2(2e - 1) = 8.8731\dots \quad \text{consequence of a result of Woods [58a]}$$

$$2^{n/2} \gamma_n, \gamma_n \rightarrow 3.0001 (2e - 1) = 13.3101\dots \quad \text{Bombieri [63]}$$

$$2^{n/2} \delta_n, \delta_n \rightarrow 3(2e - 1) = 13.3096\dots \quad \text{Gruber [67a]}$$

$$2^{n/2} \left( \frac{n}{\log^2 n} \right)^{1/3} \varepsilon_n, \varepsilon_n \rightarrow e^{-2} = 0.1353\dots \quad \text{Skubenko [77]}$$

Still another line of attack has been considered by Bombieri [62], Gruber [67] and Narzullaev [74]. Estimates for  $c$  depending on  $\lambda(L)$  (see (1) for the case  $n=3$ ) have been given by Gruber [70a], Mordell [72] and Bakjev, Pen and Skubenko [78].

The following results are related to Minkowski's conjecture:

*Each lattice of determinant 1 is a covering lattice for the sets*

$$\begin{aligned} |x_1 \dots x_n| &\leq 1, x_1, \dots, x_n > 0, & \text{Chalk [47]} \\ |x_1 \dots x_n| &\leq \frac{1}{2}, x_2, \dots, x_n > 0. & \text{Cole [52]} \end{aligned}$$

This leads to the conjecture that each lattice of determinant 1 is a covering lattice for the set

$$|x_1 \dots x_n| \leq \frac{1}{2^k}, x_{k+1}, \dots, x_n > 0.$$

Results in this direction have been given recently by Bambah and Woods [77] and Woods [80].

Additional results related to Minkowski's conjecture are contained in papers by Blanksby [68], Varnavides [70], [70a] and Mordell [72].

Inhomogeneous problems of the first type for sets related to indefinite ternary and quaternary quadratic forms have been considered by Dumir [67]–[70]. Chander and Hans [65] considered more general sets. For earlier results in this area consult the books of Cassels [59B] and Lekkerkerker [69B].

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## Stetige und diskrete Funktionale konvexer Körper

Die vorliegende Arbeit gibt einen Überblick über ein Bindeglied zwischen Konvexgeometrie, diskreter Geometrie und Geometrie der Zahlen. Ist  $E^d$  der  $d$ -dimensionale euklidische Raum,  $\mathfrak{K}^d = \{K \subset E^d / K \text{ kompakt und konvex}\}$  die Menge der konvexen Körper,  $\mathfrak{P}^d \subset \mathfrak{K}^d$  die Menge der konvexen Gitterpolytope (d.h. konvexe Hülle von Punkten aus  $\mathbb{Z}^d$ ), so interessieren Eigenschaften diskreter Funktionale auf  $\mathfrak{K}^d$  (bzw.  $\mathfrak{P}^d$ ) wie z.B. Gitterpunktanzahl  $G(K) = \text{card}(K \cap \mathbb{Z}^d)$  sowie ihre Beziehungen zu stetigen Funktionalen wie Volumen  $V$  und Oberfläche  $F$ , aber auch Beziehungen diskreter Funktionale zueinander. Dies steht in Analogie zur Theorie der konvexen Körper, wo u.a. Eigenschaften stetiger Funktionale (z.B. Satz von Brunn-Minkowski) oder Beziehungen stetiger Funktionale zueinander (z.B. Isoperimetrie) untersucht werden.

In der analytischen Zahlentheorie interessiert für spezielle  $K \in \mathfrak{K}^d$  (z.B. Kugeln, Ellipsoide, aber auch Sternkörper) die asymptotische Abweichung von  $V$  und  $G$  bei Dilatation (z.B. Gaußsches Kreisproblem).

In der klassischen Minkowskischen Geometrie der Zahlen interessieren für zentrale symmetrische  $K \in \mathfrak{K}^d$  Zusammenhänge zwischen  $V$  und  $G$  (z.B. Minkowskis Fundamentalsatz).

Während bei diesen beiden Gebieten die betrachteten  $K \in \mathfrak{K}^d$  wegen der Anwendung auf Zahlentheorie sehr speziell sind, werden in der vorliegenden Arbeit allgemeinere Klassen konvexer Körper betrachtet, um möglichst viel über die diskreten Funktionale zu erfahren. Auch in der kombinatorischen Geometrie betrachtet man diskrete Funktionale (z.B. Eckenanzahl) auf konvexen Polytopen und Relationen zueinander (z.B. Eulergleichung). Wegen der rein kombinatorischen Eigenschaften bestehen keine Beziehungen zu den metrisch definierten Funktionalen wie  $V$  und  $F$ . Dagegen gibt es bei Beschränkung auf Gitterpolytope durchaus Beziehungen zwischen Eckenanzahl und Volumen, wie wir sehen werden.

Man kann die zu untersuchenden Probleme grob in zwei Typen aufteilen: in Gleichungen und Ungleichungen. Diese Unterteilung ist nicht nur äußerlich: Die Gleichungen oder Identitäten spiegeln die kombinatorischen und algebraischen Eigenschaften der Gitterpolytope, und die Konvexität spielt eher eine untergeordnete Rolle. Es gibt sehr viele Identitäten zwischen stetigen und diskreten Funktionalen bei Gitterpolytopen. Die Theorie ist durch Arbeiten von MacDonald, Ehrhart, McMullen, Hadwiger u.a. weit entwickelt und gliedert sich gut in die allgemeine Theorie der konvexen Körper ein.

Bei den Ungleichungen ist im allgemeinen die Konvexität wesentlich; es liegen relativ wenige Ergebnisse vor; die Probleme sind, wenn man von den diversen und oft einfachen 2-dimensionalen Problemen absieht, schwer angreifbar. Von einer Theorie kann hier nicht die Rede sein, und so wollen wir uns auf einige (unserer Meinung nach) relevante in allen Dimensionen auftretende Probleme konzentrieren. Zuvor ein Überblick über die Identitäten.

### A. Identitäten der Gitterfunktionale

Zu  $K \in \mathfrak{K}^d$  sei  $G^0(K) = \text{card}(\text{relint } K \cap \mathbb{Z}^d)$ ,  $\hat{G}(K) = \text{card}(\text{bd } K \cap \mathbb{Z}^d)$  und speziell für  $P \in \mathfrak{P}^d$  noch  $\hat{G}(P) = \sum \alpha(g, P)$ , wobei  $g \in P \cap \mathbb{Z}^d$  und  $\alpha(g, P)$  der Winkel in  $g$  bezüglich  $P$  ist.  $\hat{G}$  heißt die gewogene Gitterpunktanzahl. Zwischen diesen Funktionalen bestehen diverse Relationen und Analogien. Sie gestatten z.B. nach Ehrhart [14a] eine Polynomentwicklung:

$$(1) \quad G^*(nP) = \sum_{i=0}^d n^i G_i^*(P),$$

wobei  $G^*$  für  $G, \hat{G}, G^0, \hat{G}$  steht und die  $G_i^*$  für die entsprechenden Koeffizienten. (1) folgt aus der für Simplices  $S = (x_0, \dots, x_d)$  expliziten Formel:

$$(2) \quad G(nS) = \binom{d+n}{d} + a_1 \binom{d+n-1}{d} + \cdots + a_d \binom{n}{d}.$$

Dabei ist

$$a_i = \text{card} \{x \in \mathbb{Z}^d \mid x = \sum \lambda_i (x_i - x_0), 0 \leq \lambda_i < 1, \sum \lambda_i = i\}.$$

Über die  $G_i^*$  ist noch wenig bekannt. Es ist

$$\begin{aligned} G_d(P) &= G_d^0(P) = \hat{G}_d(P) = V(P), \hat{G}_d(P) = 0 \\ G_{d-1}(P) &= -G_{d-1}^0(P) = 1/2 G_{d-1}(P) \\ &= 1/2 \sum_{\dim f=d-1} \frac{V^{d-1}(f)}{\det \text{aff}(f)}, \hat{G}_{d-1} = 0 \\ G_0(P) &= (-1)^d G_0^0(P) = \chi(P), \hat{G}_0 = 0, (1 - (-1)^d) G_0(P) = \hat{G}_0(P). \end{aligned}$$

Dabei ist  $\chi(P)$  die Eulercharakteristik von  $P$ .

Weitere Eigenschaften der  $G_i^*$  werden am Ende dieser Arbeit besprochen (s. auch [4] und [44g]).

Die Identitäten sind von Macdonald [26a], der sie benutzt, um durch Inversion von (1) das Volumen durch die Gitterpunktanzahl von Vielfachen von  $P$  auszudrücken:

$$(3) \quad 1/2(d-1)d!V(P) = \sum_{j=0}^{d-1} \binom{d-1}{j} (-1)^j (G((d-1-j)P) - 1/2\hat{G}((d-1-j)P))$$

für reine Gitterpolyeder.

$$(4) \quad d! V(P) = \sum_{j=0}^d \binom{d}{j} (-1)^j G((d-j)P)$$

$$(5) \quad 1/2(d-1) d! V(P) = \sum_{j=0}^{d-2} \binom{d-1}{j} (-1)^j \hat{G}(d-1-j)P)$$

für reine Gitterpolyeder.

Dabei ist in (3) und (4)  $G(0P) = \chi(P)$ ,  $\hat{G}(0P) = \chi(P)$  gesetzt.

(3) und (5) sind schon vorher für  $d=3$  von Reeve [35a] [35b] gezeigt worden.

Für  $d=2$ ,  $P$  Polygon ist (3) gerade die bekannte Picksche Identität [34]. Von Ehrhart [14e] stammt die Bemerkung, daß zur Berechnung von  $V(P)$  bereits die Kenntnis von  $G(iP)$ ,  $\hat{G}(iP)$ ,  $i=0, \dots, [\frac{d+1}{2}]$  ausreicht. Die Koeffizienten  $G_i, \hat{G}_i, G_i^0, \hat{G}_i^0$  sind nicht voneinander unabhängig. Die Beziehung zwischen  $\hat{G}_i, G_i, G_i^0$  wurde von Ehrhart [14a] vermutet und von Macdonald [26b] durch Untersuchung der kombinatorischen Struktur des Randkomplexes des Polyeders gezeigt:

Ist  $P$  topologisch eine Mannigfaltigkeit, dann gilt:

$$(6) \quad G_i(P) = (-1)^{d-i} G_i^0(P); \quad \hat{G}_i = (1 - (-1)^{d-i}) G_i(P).$$

Der Zusammenhang zwischen  $G$  und  $\hat{G}$  ist in einer wesentlich umfassenderen Arbeit von McMullen [27b] enthalten:

Es ist

$$\hat{G}(P) = \sum_f (-1)^{\dim P - \dim f} \beta(f, P) G(f)$$

und

$$G(P) = \sum_f \gamma(f, P) \hat{G}(f),$$

wo  $\hat{G}(f)$  bezüglich der affinen Hülle von  $f$  bestimmt ist. Durch Entwicklung beider Seiten ergeben sich daraus die Beziehungen zwischen den  $\hat{G}_i$  und den  $G_i$ . Die  $G_i^*$  erfüllen nach der zitierten Arbeit von McMullen die Eulertyprelation

$$(-1)^i G_i^*(P) = \sum_f (-1)^{\dim f} G_i^*(f).$$

Hieraus folgt insbesondere für  $\hat{G}$  wegen  $\hat{G}(f)=0$  für  $\dim f < d$   $\hat{G}_i(P)=0$  für  $d-i=2k+1$ , was vorher auf anderem Weg von MacDonald [26a] gefunden wurde. Von McMullen [27b] wird weiter gezeigt, daß es für eine weite Klasse von Funktionalen, zu denen auch die Gitterpunktlfunktionale gehören, eine Theorie der «gemischten Volumina» gibt. Es ist:

$$G^*(n_1 P_1 + \dots + n_j P_j) = \sum n_1^{r_1} \dots n_j^{r_j} G^*(P_1, r_1, \dots, P_j, r_j).$$

Dabei ist der Grad des nicht homogenen Polynoms auf der rechten Seite  $d$ . Dies wurde unabhängig von Bernstein [3] speziell für  $G$  durch eine geeignete Überdeckung von  $n_1 P_1 + \dots + n_j P_j$  gefunden. Es zeigt sich jedoch, daß seine Argumente allgemeiner für die von McMullen untersuchte Klasse von Funktionalen gelten. Bernstein gibt als Anwendung seiner Polynomentwicklung folgende Darstellung des gemischten Volumens von Gitterpolytopen durch Gitterpunktanzahlen:

$$d! V(P_1, \dots, P_d) = (-1)^d (1 - \sum G(P_i) + \sum G(P_i + P_j) + \dots + (-1)^d G(P_1 + \dots + P_d)).$$

Hieraus ergibt sich (4) wenn  $P_1 = P_2 = \dots = P_d$  gesetzt wird. Die Rolle der  $G_i$  in der Theorie der translationsinvarianten, additiven Funktionale auf Gitterpolytopen zeigt folgender Satz von Betke [4]:

Ist  $\varphi$  ein unimodular invariantes Funktional auf dem Raum der Gitterpolytope, das dem Inklusion- Exklusionprinzip genügt, dann gilt mit passenden Konstanten  $\alpha_i$

$$\varphi(P) = \sum \alpha_i G_i(P).$$

Es ist noch offen, ob das Inklusion-Exklusion-Prinzip durch die schwächere Forderung der Additivität ersetzt werden kann. Von Ehrhart [14f] ist die Theorie der Gitterpunktfunctionale auf Polytope übertragen worden, deren Ecken rationale Koordinaten haben. In diesem Fall gilt:

$$G(nP) = \sum n^i G_i(P, r),$$

wo die  $G_i$  außer von  $P$  von der Restklasse von  $n \bmod r$  abhängen. Dabei ist  $r$  das kleinste gemeinsame Vielfache der Nenner der Ecken von  $P$ . McMullen zeigte, daß sich die angegebene Theorie weitgehend auf diesen Fall übertragen läßt [27c].

## B. Ungleichungen für Gitterfunktionale

Da keine Theorie vorliegt, gliedern wir in 5 Problemkreise:

1. Untere Schranken für  $G$ .
2. Obere Schranken für  $G$ .
3. Obere Schranken für  $\hat{G}$ .
4. Ungleichungen für die  $G_i$ .
5. Schranken für  $G, \hat{G}, G^0$  bei Nebenbedingungen.

### 1. Untere Schranken für $G$

In der klassischen Minkowskischen Geometrie der Zahlen interessiert für zentrale symmetrische  $K \in \mathbb{R}_s^d$  (kurz:  $K \in \mathbb{R}_s^d$ ), wann sie einen bzw. mehrere linear unabhängige vom Ursprung verschiedene Gitterpunkte enthalten. Zentral ist neben dem Satz über sukzessive Minima der Fundamentalsatz von Minkowski ([24] oder [29])

Aus  $K \in \mathbb{R}_s^d$  und  $V(K) \geq 2^d$  folgt  $G(K) > 1$

der in der Verallgemeinerung von van der Corput ([24] oder [29]) lautet:

$$\text{Für } K \in \mathbb{R}_s^d \text{ gilt } 2 \left\lceil \frac{V(K)}{2^d} \right\rceil + 1 \leq G(K).$$

Minkowskis Satz ist elementar und vielseitig anwendbar. Er gilt jedoch nur für  $K \in \mathbb{R}_s^d$  und wird wegen  $G_d = V$  für große  $K$  sehr unscharf.

Ähnliches besagt der Mittelwertsatz für Translationen ( $C^d$  Einheitswürfel):

$$\text{Für } K \in \mathbb{R}^d \text{ ist } \int_{C^d} G(K + t) dt = V(K).$$

Eine in diesem Sinne schärfere untere Schranke und dazu noch für beliebige konvexe Körper liefert Hadwigers Gitterpunktsatz [17b], [7].

$$\text{Für } K \in \mathbb{R}^d \text{ ist } V(K) - \frac{1}{2} F(K) \leq G(K),$$

der auch unter Hinzunahme weiterer Quermaßintegrale keine wesentliche Verbesserung erlaubt.

Da die Extremfälle stark von Gitterpolytopen abweichen, wäre allenfalls eine Verschärfung auf Gitterpolytope von Interesse. In [44e] wurde dazu die Vermutung:

Für  $P \in \mathfrak{P}^d$  gilt  $V(P) - (\frac{1}{2} - \frac{1}{d}) F(P) + c(d) \leq G(P)$   
aufgestellt, die bisher nur für  $d = 2$  bewiesen ist.

## 2. Obere Schranken für $G$

Diese Frage ist noch weitgehend ungelöst. Dabei hat gerade dieses Problem wesentliche Schritte zum Verständnis der Zusammenhänge zwischen stetigen Funktionalen sowie den diskreten Funktionalen gebracht. Ein klassischer einfacher Satz, der  $G$  für eine relativ große Klasse konvexer Körper nach oben nur durch  $V$  abschätzt, ist der Satz von Blichfeldt (s. z.B. [24]):

Für  $K \in \mathfrak{K}^d$  mit  $\text{aff}(K \cap \mathbb{Z}^d) = E^d$  gilt  $G(K) \leq d!V(K) + d$ .

Die Struktur des Standardgitters  $\mathbb{Z}^d$  wird in folgenden trivialen Abschätzungen ausgenutzt:

Ist  $C \in \mathfrak{K}^d$  der Einheitswürfel, so gilt für  $K \in \mathfrak{K}^d$

$$G(K) \leq V(K + C)$$

und Gleichheit genau dann, wenn  $K \in Q^d$ . Entsprechend gilt für ein Fundamentalparallelepiped  $P$  des Gitters und beliebige  $K \in \mathfrak{K}^d$ :

$$G(K) \leq V(K + P)$$

und Gleichheit entsprechend. Beide Ungleichungen gelten sogar für beliebige Mengen  $K$ , wenn nur die rechte Seite erklärt ist. In diesem Zusammenhang interessiert eine Charakterisierung derjenigen  $K' \in \mathfrak{K}^d$ , so daß für alle  $K \in \mathfrak{K}^d$  gilt:  $G(K) \leq V(K + K')$ ; jedoch scheint dies hoffnungslos schwierig zu sein. Notwendig ist offenbar, daß die Projektion von  $K'$  in Richtung eines Teilgitters mit Gitterdeterminante  $\Delta$  den Inhalt  $\geq \frac{1}{\Delta}$  hat, und für  $K' \in \mathfrak{K}_s^d$  ist das im  $E^2$  auch hinreichend. Daß die Schranken für die Projektionen allein schon im  $E^2$  nicht genügen, zeigt das Beispiel

$$K' = \text{conv}\{(0,0), (\sqrt{2},0), (0,\sqrt{2})\}, \quad K = \text{conv}\{(0,0), (q,0), (0,q)\} \quad q \in \mathbb{N},$$

also  $G(K) - V(K + K') = (\frac{3}{2} - \sqrt{2})q > 0$ . Mehr dazu in [4].

Wegen  $C \subset \frac{1}{2}\sqrt{d} S$  ( $S$ : Einheitskugel) ist  $G(K) \leq V(K + \frac{1}{2}\sqrt{d} S)$  ein triviales Beispiel zu diesem Problem.

Interessanter ist Bokowskis [6b] für  $d \leq 5$  bewiesenes

$$(8) \quad G(K) \leq V(K + \omega_d^{-\frac{1}{d}} S)$$

( $\omega_d$ : Volumen der  $d$ -dimensionalen Einheitskugel).

Wegen  $V(\omega_d^{-\frac{1}{d}} S) = 1$  ist der Faktor  $\omega_d^{-\frac{1}{d}}$  kleinstmöglich.

Überraschenderweise gilt diese Ungleichung nicht für hohe Dimensionen, wie kürzlich gezeigt wurde [20].

Ausgehend von (7) und (8) und der Steinerschen Formel für Parallelkörper stellt sich die Frage nach oberen Schranken für  $G$  durch Linearkombinationen der Quermaßintegrale  $W_i$  bzw. der normierten Quermaßintegrale oder inneren

Volumina  $V_i = \binom{d}{i} \frac{W_{d-i}}{\omega_{d-i}}$  (vgl. [27a]). Wenn für alle  $K \in \mathfrak{K}^d$  gilt:

$$(9) \quad G(K) \leq \sum_{i=0}^d \lambda_i V_i(K),$$

so ist notwendig [44a]:  $\lambda_i \geq 1; i = 0, 1, \dots, d$  und wegen (7) hinreichend  $\lambda_i \leq \frac{1}{2} \sqrt{d^{d-i}}$ . Die Vermutung  $\lambda_i = 1; i = 0, 1, \dots, d$  [44a], die für  $d \leq 3$  [33], Rotationskörper mit  $d \leq 20$  [18b] und Gitterzonotope [4] gilt, wurde für  $d \geq 441$  durch Hadwiger [17c] widerlegt. Hadwigers Gegenbeispiel ist ein Simplex und es scheint, daß für hohe Dimensionen Simplex und Kreuzpolytop, also nicht etwa der Würfel, extremes Verhalten bei Abschätzungen in (9) zeigen. Wegen (8) gilt (9) für  $d \leq 5$  mit  $\lambda_i = \omega_d^{\frac{i}{d}-1}$ .

Was für  $d > 5$  gilt, ist ebenso offen wie die Frage nach den besten  $\lambda_i$  mit  $i \leq d-2$ , wenn  $\lambda_d = \lambda_{d-1} = 1$  ist.

Setzt man in (9) alle  $\lambda_i = 1$  und ersetzt zugleich für  $i = 0, 1, \dots, d-2$   $K$  durch das kleinste  $Q \in Q^d$  mit  $K \subset Q$ , so erhält man die Vermutung von Ehrhart [14f]: Für  $K \in \mathbb{R}^d$  ist

$$0 \leq G(K) - V(K) - \frac{1}{2} F(K) \leq \sum_{i=0}^{d-2} V_i(Q)$$

(die linke Ungleichung ist Hadwigers Gitterpunktsatz). Auch hier gilt Gleichheit für  $K = Q \in Q^d$ . Die Vermutung ist bisher nur für  $d \leq 3$  bewiesen. Ist  $a_i$  die Länge der Orthogonalprojektion von  $Q$  auf der  $x_i$ -Achse, so ist  $V_i(Q)$  gerade die  $i$ -te elementar-symmetrische Funktion der  $a_i$ . In dieser Form hatte Ehrhart seine Vermutung im  $E^3$  formuliert [14c].

Durch ganz andere Funktionale wird  $G$  im Satz von Davenport [12] abgeschätzt:

Für  $K \in \mathbb{R}^d$  ist

$$G(K) \leq \sum_{v=0}^d \tilde{V}_v(K).$$

Dabei ist  $\tilde{V}_v$  die Summe der Volumina der Projektionen auf die  $v$ -dimensionalen achsenorientierten Unterräume von  $\mathbb{Z}^d$ . Auch hier gilt Gleichheit für  $K \in Q^d$ . Die  $\tilde{V}_v$  sind additiv, translationsinvariant, stetig, homogen vom Grad  $v$ , aber natürlich nicht bewegungsinvariant. Weiter ist  $\tilde{V}_0 = V_0 = 1$  und  $\tilde{V}_d = V_d = V$ . Einfache Überlegungen zeigen weiter:  $V_v(K) \leq \tilde{V}_v(K) \leq V_v(Q)$   $v = 1, \dots, d-1$  und Gleichheit für  $v = 1$  und sonst genau für  $K = Q$ .

Also ist Ehrharts Vermutung im ersten Fehlerglied  $V_{d-1}$  schärfer, während Davenports Satz in den übrigen Fehlergliedern schärfer ist.

Interessant ist (vgl. [4]), daß die  $\tilde{V}_v(K)$  gerade die gemischten Volumina zwischen  $K$  und dem Einheitswürfel  $C$  sind, d.h. der Satz von Davenport ist für  $K \in \mathbb{R}^d$  ein Korollar von  $G(K) \leq V(K+C)$ . (Davenport bewies diesen Satz mit anderen Methoden.)

Ungleichungen vom Typ

$$G(K) \leq (\lambda f + 1)^d,$$

wobei für  $f$  der Durchmesser  $D$ , mittlere Breite  $B$ , Inkugelradius  $r$  oder Umkugelradius  $R$  stehen, sind eher von peripherem Interesse. Immerhin folgt aus Bokowskis Ungleichung noch [6b]

$$G(K) \leq \left( \omega_d^{-\frac{1}{d}} \frac{B(K)}{2} + 1 \right)^d \quad B: \text{mittlere Breite } (d \leq 5).$$

### 3. Obere Schranken für $\hat{G}$

Da es beliebig große Gittersimplices gibt, bei denen die Ecken die einzigen Gitterpunkte auf dem Rande sind, ist die Frage nach unteren Schranken uninteressant. Die Frage nach oberen Schranken ist dagegen ein interessantes Problem, zu dem einige Ergebnisse vorliegen: z.B. gilt, falls  $\text{aff}(K \cap \mathbb{Z}^d) = E^d$  im  $E^2$ :

$$\hat{G}(K) \leq U(K) \quad (U: \text{Umfang})$$

und im  $E^3$ :  $\hat{G}(K) \leq F(K) + 2$  (Ehrhart [14f]).

Beides kann auf  $G(K) - G\left(K - \frac{1}{\sqrt{2}}S\right) \leq U(K)$  und

$$G(K) - G\left(K - \frac{1}{\sqrt{3}}S\right) \leq F(K) + 2$$

verschärft werden [28] (links steht die Gitterpunktanzahl einer  $\frac{1}{\sqrt{2}}$  bzw.  $\frac{1}{\sqrt{3}}$  dicken Randschicht).

Die im Anschluß hieran naheliegende Vermutung

$$K \in \Re^d, \text{aff}(K \cap \mathbb{Z}^d) = E^d \Rightarrow \hat{G}(K) \leq 2 \sum_{\substack{i=0 \\ i \equiv d-1(2)}}^{d-1} V_i(K)$$

gilt nicht für  $d \geq 6$ , wie Gritzmann [15] zeigt.

Dagegen gilt in Analogie zum Satz von Blichfeldt das einfache  $K \in \Re^d$ ,  $\text{aff}(K \cap \mathbb{Z}^d) = E^d \Rightarrow \hat{G}(K) \leq (d-1)!F(K)$ .

Ob es ein Analogon zu Ehrharts Vermutung gibt, bleibt ebenso offen wie eine Charakterisierung der  $K'$  mit  $\hat{G}(K) \leq V(K + K')$  für alle  $K \in \Re^d$ . Die Frage nach der Gitterpunktanzahl auf dem  $i$ -Skelett von  $K$ ,  $i \leq d-2$  ist offenbar weniger reizvoll; einfache Überlegungen zeigen, daß es keine  $\lambda_v(d) \geq 0$  so gibt, daß gilt:

$$G(i\text{-Skelett } K) \leq \sum_{v=0}^{d-i} \lambda_v V_v(K)$$

### 4. Ungleichungen für die $G_i$

Bei Zentralsymmetrie gelten Minkowski-Typ-Sätze:

$$P \in \mathfrak{P}_s^d, G^0(P) = 1 \Rightarrow G_i(P) \leq c_i(d).$$

Verschiedene Eigenschaften der  $G_i$  (und der  $\hat{G}_i$ ) werden in [44g] und [4] aufgezählt.

Es gibt Ungleichungen vom Typ

$$G_i \leq \alpha_i G_d \quad i = 0, 1, \dots, d-1$$

und

$$G_i \leq \beta_i G_{d-1} \quad i \equiv d-1 \pmod{2},$$

insbesondere ist  $G_{d-1} \leq \binom{d+1}{2} G_d$ .

Die  $G_i$  sind wie die  $V_i$  additiv, translationsinvariant und homogen vom Grad  $i$ . Daher gelten für sie analoge Identitäten wie für die  $V_i$ . Sie verhalten sich jedoch sonst völlig anders als die  $V_i$ ; sie sind unstetig, nicht bewegungs invariant, nicht monoton und nicht positiv definit. Daher gelten nur wenige Ungleichungen zwischen den  $G_i$  und den  $V_i$ , z.B.

$$G_{d-1} \leq V_{d-1}, \quad G_{d-2} - \hat{G}_{d-2} \leq V_{d-2}, \quad \text{aber i.a. nicht } G_i \leq V_i.$$

### 5. Schranken für $G, \hat{G}, G^0$ bei Nebenbedingungen

Bei strikt konvexen Körpern lässt sich  $\hat{G}$  nach oben allein durch  $V$  abschätzen. Die konvexe Hülle der Gitterpunkte enthält nur Facetten mit wenigen Gitterpunkten und  $\hat{G}$  kann nicht zu groß werden, ohne daß  $V$  groß wird. Es gibt etliche 2-dimensionale Ergebnisse [13a], [13b] und einen allgemeinen Satz von Andrews [1a], [1b], der von Chaix [9] wiederentdeckt wurde:

$$\hat{G}(K) \leq c_1(d) V^{\frac{d-1}{d+1}}(K)$$

bzw.  $\hat{G}(K) \leq c_2(d) F^{\frac{d}{d+1}}(K)$ , wobei  $c_1(d) > 0, c_2(d) > 0$  nur von  $d$  abhängen. Den Unterschied zwischen beliebigen  $K \in \mathbb{R}^d$  und strikt konvexen  $K \in \mathbb{R}^d$  machen auch zwei Sätze von Minkowski [29] im zentral-symmetrischen Fall deutlich:

a) Aus  $K \in \mathbb{R}_s^d, G^0(K) = 1$  folgt  $G(K) \leq 3^d - 1$ .

b) Aus  $K \in \mathbb{R}_s^d, K$  strikt konvex,  $G^0(K) = 1$  folgt  $G(K) \leq 2^{d+1} - 2$ .

Beide Sätze ähneln formal Minkowskis Fundamentalsatz, Verallgemeinerungen im Sinne van der Corput sind unbekannt. Interessanter ist im Anschluß hieran die Verallgemeinerung Minkowskischer Sätze unter Abschwächung der Zentralsymmetrie. Einen ersten Schritt in diese Richtung machte Ehrhart: Da für  $K \in \mathbb{R}_s^d$  der Ursprung zugleich Schwerpunkt von  $K$  ist, lässt sich die Zentralsymmetrie durch die Forderung ersetzen, daß der Schwerpunkt von  $K$  aus  $\mathbb{Z}^d$  sei. Für diesen Fall bewies Ehrhart [14a] im  $E^2$ : Aus  $V(K) \geq \frac{9}{2}$  folgt  $G(K) > 1$  und vermutete für den  $E^d$ :

$$\text{Aus } V(K) \geq \frac{(d+1)^d}{d!} \text{ folgt } G(K) > 1.$$

Eine andere Verallgemeinerung geht davon aus, daß für  $K \in \mathbb{R}_s^d$  der Ursprung innerer Punkt von  $K$  ist; genauer: Bei den Extremfällen ist er innerer Punkt der konvexen Hülle der Gitterpunkte. Dies liefert sinnvolle Analoga zu Minkowskis Sätzen: Zu  $K \in \mathbb{R}^d$  sei  $P(K) = \text{conv}(K \cap \mathbb{Z}^d)$ . Weiter sei  $G^0(P(K)) > 0$ .

Dann gibt es lineare Ungleichungen vom Typ

- a)  $\hat{G}(K) \leq \alpha_1 G^0(K) + \beta_1 (?)$
- b)  $\alpha_2 V(K) + \beta_2 \leq G^0(K) (?)$
- c)  $G(K) \leq \alpha_3 V(K) + \beta_3.$
- d)  $\hat{G}(K) \leq \alpha_4 F(K) + \beta_4.$

Die Existenz von Ungleichungen a) und b) ist noch offen, die Existenz von c) und d) ist nach dem Satz von Blichfeldt klar. Man beachte, daß für  $G^0(P(K)) = 0$  keine Ungleichungen a) und b) existieren. Zu a) gibt es Teilergebnisse:

Für  $d = 2$  zeigte Scott [39f]  $\hat{G}(P) \leq 2G^0(P) + 7$  (d.h.  $G(P) \leq 3G^0(P) + 7$ ), dies ist bestmöglich. Ist

$$P_k = \left\{ x \in E^d \mid \sum_{i=1}^{d-1} x_i 2^{-i} + x_d 2^{-d} (k+1) \leq 1, x_i \geq 0 \text{ for } i=1, \dots, d \right\},$$

so ist

$$G^0(P_k) = k, \quad V(P_k) = \frac{1}{d!} 2^{\binom{d+1}{2}} (k+1) \quad \text{und} \quad G(P_k) > V(P_k).$$

Dieses im wesentlichen von Zaks [45] stammende Simplex zeigt, daß der Verzicht auf Zentralsymmetrie ganz andere Ergebnisse als Minkowskis Sätze liefert, wo vor allem der Würfel den Extremfall bildet.

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## Neuere Entwicklungen über reguläre Polyeder

Man kennt heute eine Reihe von Extremaleigenschaften, welche einige oder alle regulären Polyeder des dreidimensionalen euklidischen Raumes  $E^3$  innerhalb bestimmter Polyederklassen charakterisieren. Dabei nimmt jeweils eine gewisse, mit den Polyedern verbundene Größe wie Volumen, Oberfläche, Kantenkrümmung oder Kantenlängensumme für einige oder alle regulären Polyeder einen extremalen Wert an. Im Zusammenhang mit Verteilungsproblemen von Punkten auf der Kugeloberfläche treten auch einige halbreguläre Polyeder als Extremalkörper hervor. Viele Ergebnisse in dieser Richtung findet man in den bekannten Büchern [11] und [12] von Fejes Tóth. Es soll hier ein Überblick über einige neuere Resultate auf diesem Gebiet gegeben werden.

Innerhalb der Theorie der konvexen Polyeder des gewöhnlichen Raumes spielen bekanntlich die vier fundamentalen Maßzahlen eines Polyeders, nämlich Volumen  $V$ , Oberfläche  $F$ , Kantenkrümmung  $M$  und Eckenkrümmung  $C = 4\pi$  eine fundamentale Rolle (s. hiezu das exzellente Buch [20] von Hadwiger). Diese Größen stehen auch im Mittelpunkt des vorliegenden Artikels.

Im folgenden beziehen wir uns öfters auf die Klasse  $\mathcal{P}(e,f)$  der eigentlichen konvexen Polyeder mit  $e$  Ecken und  $f$  Flächen (die Kantenzahl  $k$  folgt dann aus der Eulerschen Polyederformel  $k = e + f - 2$ ). Eine Serie von Ungleichungen und damit verbundene Extremaleigenschaften erhält man, indem man die drei nichttrivialen fundamentalen Maßzahlen eines beliebigen Polyeders  $P \in \mathcal{P}(e,f)$  mit Hilfe seines Umkugelradius  $R$  bzw. Inkugelradius  $r$  nach oben bzw. unten in passender Weise abschätzt. Es ist dabei zweckmäßig, die mittlere Seitenzahl der Flächen  $p = \frac{2k}{f}$  und die mittlere Kantenzahl der Ecken des Polyeders  $q = \frac{2k}{e}$  einzuführen.

### §1

Für das Volumen  $V$  eines Polyeders  $P \in \mathcal{P}(e,f)$  gilt

$$\begin{aligned} \frac{k}{3} \sin \frac{2\pi}{p} \left( \operatorname{tg}^2 \frac{\pi}{p} \operatorname{tg}^2 \frac{\pi}{q} - 1 \right) r^3 &\leq V \leq \\ \frac{2k}{3} \cos^2 \frac{\pi}{p} \operatorname{ctg} \frac{\pi}{q} \left( 1 - \operatorname{ctg}^2 \frac{\pi}{p} \operatorname{ctg}^2 \frac{\pi}{q} \right) R^3 \end{aligned} \quad (1)$$

mit Gleichheit in beiden Fällen nur für die fünf regulären Polyeder.

Die Ungleichung links samt der Verschärfung, wonach  $V$  durch das Volumen des Durchschnittes des Polyeders mit der zur Inkugel konzentrischen Kugel vom Radius  $r \operatorname{tg} \frac{\pi}{p} \operatorname{tg} \frac{\pi}{q}$  ersetzt werden kann, stammt von Fejes Tóth [6]. Der Beweis beruht auf einer allgemeinen Ungleichung, die sich auf ein beliebiges endliches System konvexer sphärischer Polygone bezieht und verschiedenartige Anwendungen gestattet. Im einfachsten Fall lautet dieses sog. *Momentenlemma* [10]: Auf der Einheitssphäre sei  $C(a)$  eine Kappe mit dem Mittelpunkt 0 und

dem Radius  $a$ .  $P$  sei ein konvexes sphärisches  $n$ -Eck und  $\bar{P}$  das zu  $P$  flächengleiche reguläre  $n$ -Eck vom Mittelpunkt  $O$ . Dann gilt, wenn  $|B|$  den Flächeninhalt der Menge  $B$  bezeichnet,

$$|P \cap C(a)| \leq |\bar{P} \cap C(a)|. \quad (2)$$

Weitere Beweise dieses auch in der euklidischen und in der hyperbolischen Ebene gültigen Satzes findet man in [22], [5] und [18].

Die Ungleichung rechts in (1) wurde in [14] bewiesen; siehe auch [15]. Eine Verschärfung enthalten die Arbeiten [9] und [19].

Die Schranken von  $V$  in (1) haben folgende geometrische Bedeutung: Man betrachte das Tetraeder  $T_1 = OABC$  (Fig. 1), wobei  $OA$  normal auf der Ebene des Dreiecks  $ABC$  steht, das die Winkel  $\angle A = \frac{\pi}{p}$  und  $\angle B = \frac{\pi}{2}$  hat; der räumliche Winkel von  $T_1$  bei  $O$  ist  $\angle O = \frac{\pi}{k}$ , und es ist  $OC = R$ . Daneben betrachten wir das Tetraeder  $T_2$ , das aus  $T_1$  dadurch hervorgeht, daß man unter Beibehaltung der übrigen Größen  $OC = R$  durch  $OA = r$  ersetzt. Dann lautet (1), wenn  $|M|$  das Volumen von  $M$  bedeutet,

$$4k|T_2| \leq V \leq 4k|T_1|. \quad (1')$$

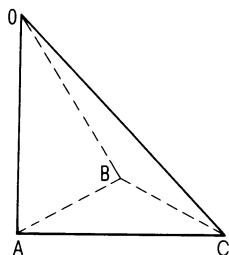


Fig. 1

Die linke Abschätzungsformel wurde von Fejes Tóth [8] auf Polyeder des dreidimensionalen sphärischen und hyperbolischen Raumes erweitert. Wie im euklidischen Raum läßt sich dieses Problem auf ein zweidimensionales zurückführen. Hingegen wurde die rechte Ungleichung noch nicht auf nichteuklidische Räume ausgedehnt.

Aus (1) folgt noch, daß für jedes konvexe Polyeder die Ungleichung

$$\frac{R}{r} \geq \operatorname{tg} \frac{\pi}{p} \operatorname{tg} \frac{\pi}{q}$$

besteht, in der Gleichheit wieder nur von den regulären Körpern beansprucht wird.

Wenden wir uns nun den analogen Problemen für die beiden anderen fundamentalen Maßzahlen zu.

## §2

Für die Oberfläche  $F$  eines Polyeders der Klasse  $\mathcal{P}(e,f)$  gilt [6]

$$k \sin \frac{2\pi}{p} \left( \operatorname{tg}^2 \frac{\pi}{p} \operatorname{tg}^2 \frac{\pi}{q} - 1 \right) r^2 \leq F \quad (3)$$

mit Gleichheit nur für die regulären Polyeder. Leider konnte die entsprechende obere Abschätzung

$$F \leq k \sin \frac{2\pi}{p} \left( 1 - \operatorname{ctg}^2 \frac{\pi}{p} \operatorname{ctg}^2 \frac{\pi}{q} \right) R^2 \quad (4)$$

bisher nur unter den folgenden einschränkenden Bedingungen bewiesen werden: Die Fußpunkte der Normalen aus dem Umkugelmittelpunkt auf die Flächenebenen bzw. Kantengeraden des Polyeders liegen in den betreffenden Flächen bzw. Kanten. Wir werden darauf als «Fußpunktbedingung» bezüglich der Flächen bzw. Kanten verweisen. Es wäre sehr wünschenswert, die Gültigkeit der Abschätzung (4) von dieser höchst wahrscheinlich überflüssigen Einschränkung zu befreien.

Die Schranken in (3) und (4) stellen die mit  $4k$  multiplizierten Flächeninhalte der Basisdreiecke ABC der in §1 beschriebenen Tetraeder  $T_2$  und  $T_1$  dar.

Auch die Ungleichung (3) wurde auf nichteuklidische Räume ausgedehnt [30].

### §3

Die Kantenkrümmung  $M$  eines konvexen Polyeders im  $E^3$  ist durch

$$M = \frac{1}{2} \sum_i l_i \alpha_i$$

definiert. Hierbei ist die Summe über alle Kanten  $L_i$  (der Länge  $l_i$ ) zu erstrecken, und  $\alpha_i$  bezeichnet den Winkel zwischen den nach außen weisenden Normalenvektoren der bei  $L_i$  anstoßenden Randflächen des Polyeders.

Für die Kantenkrümmung eines beliebigen Polyeders aus  $\mathcal{P}(e,f)$  bestehen die beiden Relationen

$$\begin{aligned} 2k \sin \frac{\pi}{p} \left( \operatorname{tg}^2 \frac{\pi}{p} \operatorname{tg}^2 \frac{\pi}{q} - 1 \right)^{1/2} \arccos \left( \cos \frac{\pi}{q} \operatorname{cosec} \frac{\pi}{p} \right) r &\leq \\ \leq M &\leq 2k \sin \frac{\pi}{p} \left( 1 - \operatorname{ctg}^2 \frac{\pi}{p} \operatorname{ctg}^2 \frac{\pi}{q} \right)^{1/2} \arccos \left( \cos \frac{\pi}{q} \operatorname{cosec} \frac{\pi}{p} \right) R, \end{aligned} \quad (5)$$

in denen Gleichheit wieder nur für die fünf regulären Polyeder eintritt. Diese Ungleichungen wurden zuerst unter Annahme der «Fußpunktbedingung» hinsichtlich der Kanten *und* Flächen bewiesen, wobei die Normalen durch den Inkugel- bzw. Umkugelmittelpunkt zu nehmen sind. Den Beweis und die Vermutung ihrer allgemeinen Gültigkeit findet man in [15]. Die Schranken in (5) haben eine einfache geometrische Bedeutung: Sei  $\delta$  der innere Diederwinkel an der Kante BC des Tetraeders  $T_2$  bzw.  $T_1$ . Dann sind diese Schranken gleich  $2k \cdot BC \cdot (\frac{\pi}{2} - \delta)$ .

Später zeigte Florian [16], daß man in der ersten Ungleichung (5) auf die Fußpunktbedingung bezüglich der Kanten verzichten kann; jedoch gelang es trotz verschiedenen Bemühungen vorerst nicht, auch die entsprechende Bedingung bezüglich der Flächen fallenzulassen.

Im vergangenen Jahr gab Linhart [29] einen überraschend kurzen Beweis der rechten Ungleichung (5), der ohne jede Zusatzbedingung auskommt. Die Idee dabei war, neben dem Polyeder  $P$  das bezüglich seiner Umkugel polar konjugierte Polyeder  $P^*$  zu betrachten. Damit gelang es, das räumliche Problem auf ein zweidimensionales zurückzuführen, das die Anwendung der Ungleichung von Fejes Tóth, ähnlich wie beim Beweis der linken Ungleichung (1), gestattet.

Ausgehend von der Linhartschen Beweisidee zeigte Florian kürzlich die linke Ungleichung in (5) ohne jede zusätzliche Voraussetzung. Der Beweis soll ganz kurz skizziert werden, ohne auf die komplizierten Details, die hauptsächlich analytischer Art sind, einzugehen.

Ohne Einschränkung kann  $r=1$  angenommen werden. Als Ursprung  $Z$  werde der Mittelpunkt der Inkugel  $K$  des Polyeders gewählt. Es sei  $u$  ein Einheitsvektor; dann ist der Wert der *Stützfunktion*  $t$  an der Stelle  $u$  durch

$$t(u) = \max_{x \in P} \langle x, u \rangle$$

definiert. Darin bezeichnet  $x$  den in  $Z$  angreifenden Ortsvektor des variablen Punktes des Polyeders  $P$ , und die Winkelklammer zeigt das Skalarprodukt an. Nach einer bekannten Integralformel ([20], S. 48) ist

$$M = \int_{S^2} t(u) du, \quad (6)$$

wobei sich die Integration über die Einheitssphäre  $S^2$  erstreckt und  $du$  das Flächendifferential auf  $S^2$  bedeutet.

Der zum Polyeder  $P$  bezüglich  $K$  polar konjugierte Körper ist ein in  $K$  enthaltenes konvexes Polyeder  $P^*$ , das den Punkt  $Z$  im Innern enthält. Seine  $f$  Ecken und  $e$  Randflächenebenen sind zu den Randflächenebenen bzw. Ecken von  $P$  polar konjugiert bezüglich  $K$ . Der im Ursprung  $Z$  angreifende Richtungsvektor  $u$  schneidet den Rand von  $P^*$  im Punkt  $\frac{1}{t(u)} u$ .

Wir bezeichnen die Randflächen von  $P^*$  mit  $F_i$ , ihre Projektionen aus  $Z$  auf den Rand von  $K$  mit  $U_i$  und die Abstände der Randflächenebenen vom Kugelmittelpunkt mit  $h_i$  ( $i = 1, \dots, e$ ).  $u_i$  ( $i = 1, \dots, e$ ) seien die Einheitsvektoren normal zu den Randflächenebenen von  $P^*$ . Dann folgt aus (6)

$$\begin{aligned} M &= \sum_{i=1}^e \int_{U_i} t(u) du = \sum_{i=1}^e \frac{1}{h_i} \int_{U_i} \langle u_i, u \rangle du \\ &= \sum_{i=1}^e \frac{1}{h_i} \int_{U_i} \cos \vartheta_i du, \end{aligned}$$

wobei  $\vartheta_i$  ( $0 \leq \vartheta_i < \pi/2$ ) den sphärischen Abstand der Punkte  $u_i$  und  $u$  bedeutet. Die einzelnen Summanden sind bei gegebenem Flächeninhalt von  $U_i$  und gegebener Seitenzahl von  $F_i$  nach unten abzuschätzen. Die darauf folgende Anwendung der Jensenschen Ungleichung auf eine gewisse konvexe Funktion von zwei Variablen führt schließlich zum gewünschten Ergebnis.

## §4

Weitere Probleme entstehen, wenn man die Kantenlängensumme  $L$  eines Polyeders in Betracht zieht. Besicovitch und Eggleston [3] zeigten, daß jedes konvexe Polyeder der Ungleichung

$$L \geq 24r \quad (7)$$

genügt mit Gleichheit nur für den Würfel. Damit wurde eine Vermutung von Fejes Tóth [7] bestätigt. Von ihm stammt auch die Vermutung, daß für konvexe, nur von Dreiecken begrenzte Polyeder sogar

$$L \geq 12\sqrt{6}r = 29,39\dots r \quad (8)$$

gilt mit Gleichheit nur für die regulären Tetraeder und Oktaeder.

Es ist bemerkenswert, daß durch (7) bzw. (8) gerade der Würfel bzw. zwei reguläre Polyeder in der Klasse aller konvexen Polyeder bzw. Dreieckpolyeder als extreme Körper charakterisiert sind.

Die schwächere Ungleichung  $L > 28r$  war von Fejes Tóth gezeigt worden. Kürzlich gelang es Linhart [27] durch geschickte Kombination elementar-geometrischer und analytischer Überlegungen, die lange offen gebliebene Vermutung (8) zu beweisen.

Für einige bestimmte Inkugelradien wurde das analoge Problem im sphärischen und hyperbolischen Raum von Coxeter und Fejes Tóth gelöst [4]. Im sphärischen Raum haben unter allen konvexen Dreieckpolyedern vom Inkugelradius  $\arcsin \frac{1}{4} = 0,252680\dots$  das Dieder und das Tetraeder die kleinstmögliche Kantenlängensumme; im hyperbolischen Raum treten bei dem Radius  $0,364054\dots$  das Oktaeder und bei dem Radius  $0,828375\dots$  das Ikosaeder als Dreieckpolyeder kleinster Kantenlängensumme auf.

In einer Reihe von Arbeiten werden Beschränkungen der Kantenlängensumme mittels der fundamentalen Maßzahlen oder des Umkugelradius des Polyeders untersucht. Wir erwähnen die für alle konvexen Polyeder gültige interessante Ungleichung von Aberth [1]

$$6\pi F < L^2.$$

Beschränkt man sich auf Dreieckpolyeder, so gilt nach Kömhoff [23]

$$12\sqrt{3}F \leq L^2,$$

mit Gleichheit nur für das reguläre Tetraeder. Weiters sind hier Aufsätze von Aberth [2], Linhart [28] und Lillington [24–26], zu nennen. Aus einem allgemeinen schönen Ergebnis von Firey und Schneider [13] folgt als sehr spezieller Fall die Ungleichung

$$L \geq \frac{3}{\pi} M.$$

In diesen Abschätzungen wird Gleichheit z.T. bei entarteten Polyedern erreicht.

## §5

Um einen weiteren Gesichtspunkt zu gewinnen, legen wir uns die folgende Frage vor: Wie groß kann das Volumen des Durchschnittes eines Polyeders aus einer vorgegebenen Klasse mit einer Kugel vom gegebenen Radius  $a$  höchstens sein? Betrachten wir etwa ein konkavexes Polyeder  $P$  mit  $e$  Ecken und dem Volumen  $V$ .  $K(a)$  bezeichne die Kugel mit einem festen Mittelpunkt  $O$  und gegebenem Radius  $a$ . Wie Florian gezeigt hat [17], besteht die Ungleichung

$$|P \cap K(a)| \leq 12(e-2)|T_3 \cap K(a)|. \quad (9)$$

Hierin bezeichnet, ähnlich wie  $T_1$  und  $T_2$  in §1,  $T_3$  das Tetraeder  $OABC$  mit den Größen  $\angle A = \frac{\pi}{3}$ ,  $\angle B = \frac{\pi}{2}$ ,  $\angle O = 4\pi/12(e-2)$  und  $|T_3| = V/12(e-2)$ . Das Gleichheitszeichen wird beansprucht, wenn  $e=4, 6$  oder  $12$  und  $P$  das entsprechende reguläre Dreieckpolyeder mit dem Mittelpunkt  $O$  ist. Wenn  $OA \leq a \leq OC$  ist, tritt nur in diesen Fällen Gleichheit ein. (9) stellt ein Analogon des Momentenlemmas (2) für Polyeder mit vorgegebener Eckenzahl dar. Das entsprechende Problem bei fester Flächenzahl dürfte außerordentlich schwierig sein.

Die Beweismethode der Ungleichung (9) ergibt allgemeiner eine Abschätzung nach oben für das mittlere Volumen der Durchschnitte endlich vieler konkavexer Polyeder gegebener Volumina und gegebener Eckenzahlen mit festen Kugeln  $K_v(a)$  vom Radius  $a$ : Seien  $P_v$  ( $v=1, \dots, N$ ) konvexe Polyeder mit durchschnittlicher Eckenzahl  $\bar{e}$  und dem durchschnittlichen Volumen  $\bar{V}$ , so gilt

$$\frac{1}{N} \sum_{v=1}^N |P_v \cap K_v(a)| \leq 12(\bar{e}-2)|\bar{T}_3 \cap K(a)|.$$

Dabei entsteht  $\bar{T}_3$  aus  $T_3$ , indem man  $V$  durch  $\bar{V}$  und  $e$  durch  $\bar{e}$  ersetzt.  $K(a)$  bedeutet wie oben die Kugel mit Mittelpunkt  $O$  und Radius  $a$ .

Wir lassen eine Skizze des Beweises von (9) folgen, da sich ähnliche Methoden auch bei anderen Problemen anwenden lassen. Ohne Einschränkung der Allgemeinheit können wir den Mittelpunkt  $O$  von  $K(a)$  im Polyeder  $P$  liegend annehmen; andernfalls braucht man nur  $P$  an  $O$  heranzuschieben, wodurch die Größe  $|P \cap K(a)|$  offenbar nicht verkleinert wird. Wir zerlegen die mehr als dreiseitigen Randflächen von  $P$  durch einander nicht kreuzende Diagonalen in Dreiecke, deren Anzahl nach der Eulerschen Polyederformel  $2(e-2)$  ist.  $P$  wird durch die Verbindung von  $O$  mit den Ecken in dreiseitige Pyramiden zerlegt, deren Volumina  $v$  und deren räumliche Winkel an der Spitze  $O$   $\tau_v$  ( $v=1, \dots, 2e-4$ ) seien.

$S(v, \tau)$  sei eine dieser Pyramiden und  $\bar{S}$  jene gerade dreiseitige Pyramide mit der Spitze  $O$  und regulärer Basis, die zu den gleichen Zahlen  $v$  und  $\tau$  gehört. Der schwierigste Teil des Beweises besteht in dem Nachweis der Ungleichung

$$|S \cap K(a)| \leq |\bar{S} \cap K(a)|. \quad (10)$$

Dazu hat man, wenn man von gewissen trivialen Fällen absieht, zu zeigen, daß sich die linke Seite von (10) durch passende infinitesimale Änderungen von  $S$  vergrößern läßt, falls  $S \neq \bar{S}$  ist. Im zweiten Beweisschritt hat man nachzuweisen, daß die rechte Seite von (10), als Funktion der beiden Variablen  $v$  und  $\tau$

betrachtet, konkav ist. Schreibt man dann die Ungleichung (10) für die  $2e-4$  Pyramiden  $S_v$  an, addiert und macht von der Jensenschen Ungleichung Gebrauch, so folgt die gewünschte Abschätzung (9).

Für die Eckenzahlen  $e=4, 6$  bzw.  $12$  und gegebenes Volumen erreicht das Volumen des Durchschnittes eines Polyeders  $P$  mit einer *beliebigen* Kugel seinen maximalen Wert, wenn  $P$  ein mit der Kugel konzentrisches reguläres Dreieckpolyeder ist. Es ist nicht bekannt, ob es auch zu anderen Eckenzahlen Polyeder mit dieser Eigenschaft gibt.

Den folgenden (noch nicht publizierten) Satz hat gleichfalls Florian bewiesen.  $P$  sei ein konvexes Polyeder der Klasse  $\mathcal{P}(e,f)$ , das in der Kugel um den festen Punkt  $O$  vom Radius  $R$  enthalten ist. Bedeutet  $K(a)$  die Kugel um  $O$  vom Radius  $a$ , so gilt

$$|P \cap K(a)| \leq 4k |T_1 \cap K(a)|, \quad (11)$$

wobei  $T_1 = OABC$  das Tetraeder aus §1 ist. Gleichheit besteht für die fünf  $K(R)$  einbeschriebenen regulären Polyeder und, falls  $a \geq \overline{OA}$ , nur für diese. Dieser Satz enthält mit  $a=R$  die Ungleichung rechts in (1) als Spezialfall.

Es sei  $P$  ein Polyeder aus  $\mathcal{P}(e,f)$  mit gegebenem Volumen  $V$ . Der Beweis der nachstehenden Ungleichung würde in einem gewissen Sinn den Abschluß der vorangegangenen Betrachtungen bilden:

$$|P \cap K(A)| \leq 4k |T_4 \cap K(a)|. \quad (12)$$

Darin ist  $T_4 = OABC$  ein Tetraeder analog  $T_1$ , jedoch mit den Größen  $\angle A = \frac{\pi}{p}$ ,  $\angle B = \frac{\pi}{2}$ ,  $\angle O = \frac{\pi}{k}$  und  $|T_4| = V/4k$ . Das Gleichheitszeichen tritt für die regulären Polyeder mit Mittelpunkt  $O$  in Kraft und, falls  $\overline{OA} \leq a \leq \overline{OC}$ , nur für diese.

Die Ungleichung (12) würde das dreidimensionale Analogon des Momentenlemmas (2) für Polyeder gegebener Ecken- und Flächenzahl darstellen. Leider kann (12) nur als Vermutung angesehen werden. Sie wird allerdings dadurch unterstützt, daß gewisse Spezialfälle und Folgerungen aus (12) bereits bewiesen wurden. Es sind dies die beiden Ungleichungen (1') sowie (9) und (11). Um dies etwa für die Ungleichung rechts in (1') einzusehen, setzen wir in (12)  $a = R$ . Auf der linken Seite steht dann  $V$ ; da  $|T_4| = V/4k$  ist, muß daher  $R \geq \overline{OC}$  sein. Deshalb und mit Rücksicht auf die Definition von  $T_1$  und  $T_4$  haben wir

$$T_4 \subset T_1 \quad (13)$$

und daher

$$|T_4| \leq |T_1|,$$

in Übereinstimmung mit dem zweiten Teil von (1'). Gleichheit tritt nur dann ein, wenn  $R = \overline{OC}$  und daher  $P$  ein reguläres Polyeder mit Mittelpunkt  $O$  und Umkugelradius  $R$  ist. Analog zeigt man den ersten Teil von (1').

Aus (12) und (13) folgt (11). Zur Unterscheidung setzen wir jetzt  $T_i = OA_iB_iC_i$ . Ist  $T_4 \neq T_1$ , so ist  $\overline{OA}_4 < \overline{OA}_1$ , und aus  $a \geq \overline{OA}_1$  folgt dann

$$|T_4 \cap K(a)| < |T_1 \cap K(a)|.$$

Somit muß, wenn in (11) für ein  $a \geq \overline{OA}_1$  Gleichheit eintritt,  $T_4 = T_1$  und daher

$P$  ein reguläres Polyeder mit dem Mittelpunkt  $O$  und dem Umkugelradius  $R$  sein.

Die Ungleichung (11) läßt sich in einem gewissen Sinn dualisieren. Enthält ein Polyeder  $P$  aus  $\mathcal{P}(e,f)$  eine Kugel vom Radius  $r$  und ist  $K(a)$  eine dazu konzentrische Kugel vom Radius  $a$ , so gilt

$$|P \cap K(a)| \geq 4k |T_2 \cap K(a)|, \quad (14)$$

wobei  $T_2 = OA_2B_2C_2$  das in §1 eingeführte (dort mit OABC bezeichnete) Tetraeder ist. Gleichheit besteht für die fünf regulären Polyeder vom Inkugelradius  $r$  und im Fall  $a \geq OB_2$  nur für diese. Im Unterschied zu (11) ist diese Feststellung sehr leicht zu beweisen, u.z. mit Hilfe des Momentenlemmas (2), angewendet auf das sphärische Netz von  $P$ . (14) umfaßt den ersten Teil von (1') samt der im Nachsatz zu (1) angegebenen Verschärfung.

Wie schon bemerkt, hat das Momentenlemma (2) außer auf der Kugelfläche auch in der euklidischen und in der hyperbolischen Ebene Gültigkeit. Es ist möglich, daß das räumliche Analogon (12) auch im sphärischen und im hyperbolischen Raum bestehen bleibt. Allerdings ist der vorläufig einzige Anhaltspunkt dafür die schon erwähnte Tatsache, daß der erste Teil von (1') allgemeiner in Räumen konstanter Krümmung gilt.

## §6

Das isoperimetrische Problem für konvexe Körper hat in der Literatur ausführliche Darstellungen gefunden, z.B. in [20] und [21]; siehe auch [11], S.282ff., und [12], S.133ff. Andererseits steht das isoperimetrische Problem für konvexe Polyeder in enger Beziehung zu manchen der vorhin besprochenen Ergebnisse und hat zum Teil zu diesen Untersuchungen Anlaß gegeben. Daher schließen wir den Aufsatz mit einigen diesbezüglichen Bemerkungen.

Die Tatsache, daß im  $E^3$  unter allen Tetraedern gleicher Oberfläche das reguläre Tetraeder das größte Volumen hat, war bereits Lhuilier bekannt. Auch der entsprechende allgemeinere Satz für Simplices im  $E^n$  ( $n \geq 2$ ) läßt sich mittels Steinerscher Symmetrisierung beweisen (s. [21], S.273). Steiner zeigte mit Hilfe seiner Symmetrisierung, daß das reguläre Oktaeder unter allen konvexen Polyedern vom selben kombinatorischen Typ den kleinsten isoperimetrischen Quotienten  $F^3/V^2$  besitzt, und formulierte die Vermutung für die übrigen regulären Polyeder.

Nach dem Theorem von Lindelöf-Minkowski hat unter allen konvexen Polytopen im  $E^n$ , die mit einem vorgegebenen konvexen Polytop  $P$  in den Richtungen der Seitenflächen übereinstimmen, das den Bedingungen genügende *Tangentialpolytop* einer Kugel (d.i. ein Polytop, dessen Seitenflächen die Kugel berühren) den kleinsten isoperimetrischen Quotienten  $F^n/V^{n-1}$  (für einen Beweis s. [21], S.276ff.). Für ein Tangentialpolyeder der Einheitskugel im  $E^3$  wird aber  $V = \frac{1}{3} F$  und daher  $F^3/V^2 = 9F$ . Damit ist das isoperimetrische Problem bei Polyedern vorgegebener Flächenzahl  $f$  auf die Bestimmung des der Einheitskugel umbeschriebenen Polyeders kleinster Oberfläche zurückgeführt. Mit Rücksicht auf (3) gilt daher

$$\frac{F^3}{V^2} \geq 54(f-2) \operatorname{tg} \frac{\pi f}{6(f-2)} \left( 4 \sin^2 \frac{\pi f}{6(f-2)} - 1 \right) \quad (15)$$

mit Gleichheit nur für die regulären Dreikantpolyeder Tetraeder, Würfel und Dodekaeder. Der erste strenge Beweis dieser von Goldberg entdeckten Ungleichung wurde von Fejes Tóth [6] gegeben. Dadurch war Steiners Vermutung für den Fall des Würfels und des Dodekaeders unter viel allgemeineren Bedingungen bewiesen.

Hingegen ist der Fall des Ikosaeders noch nicht geklärt. Nach einer Vermutung von Fejes Tóth gilt aber für ein konvexes Polyeder von  $e$  Ecken die (15) entsprechende Beziehung

$$\frac{F^3}{V^2} \geq \frac{27\sqrt{3}}{2}(e-2) \left( 3 \operatorname{tg}^2 \frac{\pi e}{6(e-2)} - 1 \right) \quad (16)$$

mit Gleichheit nur für die regulären Dreieckpolyeder Tetraeder, Oktaeder und Ikosaeder. Wahrscheinlich ist sogar die (15) und (16) enthaltende Ungleichung

$$\frac{F^3}{V^2} \geq 9k \sin \frac{2\pi}{p} \left( \operatorname{tg}^2 \frac{\pi}{p} \operatorname{tg}^2 \frac{\pi}{q} - 1 \right) \quad (17)$$

erfüllt, in der das Gleichheitszeichen nur für die regulären Polyeder in Anspruch genommen wird.

Mit Hilfe des in §5 definierten Polyeders  $T_4$  lässt sich (17) in einfacher Weise anschreiben. Bedeutet nämlich  $G$  den Flächeninhalt der Grundfläche  $A_4B_4C_4$  von  $T_4$ , so lautet (17)

$$F \geq 4k G. \quad (18)$$

Die isoperimetrische Eigenschaft des regulären Tetraeders im dreidimensionalen hyperbolischen Raum wurde von Fejes Tóth festgestellt [10]. Florian [18] hat etwas allgemeiner die Gültigkeit von (18) für Tangentialpolyeder im hyperbolischen Raum gezeigt. Im euklidischen Raum ist (18) die schon bewiesene Ungleichung (3).

Dagegen ist der Beweis der isoperimetrischen Eigenschaft des regulären Tetraeders im sphärischen Raum ein noch ungeklärtes Problem.

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## Konvexe Polytope mit regulären Facetten im $\mathbb{R}^n$ ( $n \geq 4$ )

1. Das Problem, im  $\mathbb{R}^3$  alle konvexen Polyeder mit regulären Seiten zu bestimmen, wurde in [6] gelöst: Außer den  $\infty$  vielen Prismen und Antiprismen gibt es genau 110 solche Polyeder.

Auf die analoge Fragestellung in höheren Dimensionen wird in [4], S.414, hingewiesen. Der Spezialfall von konvexen  $n$ -Polytopen im  $\mathbb{R}^n$  ( $n \geq 4$ ) mit regulären und zueinander kongruenten Facetten wurde in [1] behandelt. Im folgenden werden nun allgemein im  $\mathbb{R}^n$  ( $n \geq 4$ ) die konvexen  $n$ -Polytope mit regulären Facetten charakterisiert. Zu diesen Polytopen gehören trivialerweise die semiregulären (vgl. [4], S.413, und [3]), die Pyramide über dem  $(n-1)$ -Kreuzpolytop, die Bipyramide über dem  $(n-1)$ -Simplex; weiter für  $n=4$  die Pyramide über dem Ikosaeder, die Vereinigung eines semiregulären tetroktaedrischen Polytops mit einer Pyramide über einem seiner Facettenoktaeder, und alle aus dem 600-Zell durch Abschneiden paarweise nicht benachbarter Ecken entstehenden Polytope (Wenn  $\{E_i\}_{i=1}^k$  paarweise nicht benachbarte Ecken des 600-Zells sind, so sei  $H(E_i)$  der abgeschlossene Halbraum, der begrenzt wird von der Hyperebene durch die von  $E_i$  verschiedenen Endpunkte der  $E_i$  enthaltenden Kanten, und der  $E_i$  nicht enthält. Der Durchschnitt des 600-Zells mit  $\cap_{i=1}^k H(E_i)$  ist ein von Tetraedern und Ikosaedern begrenztes Polytop). Im folgenden wird gezeigt, daß dies für  $n \geq 5$  bereits alle solchen Polytope sind, und daß für  $n=4$  nur noch ausschließlich von Tetraedern begrenzte Polytope dazukommen.

Eine Seite eines Polytops  $P$  soll auch « $P$ -Seite» heißen; es werden im folgenden nur Pyramiden mit regulären Facetten betrachtet, es bezeichne  $P(B)$  die Pyramide mit der Basis  $B$ .

2. Es sei also  $P$  ein  $n$ -Polytop im  $\mathbb{R}^n$  ( $n \geq 4$ ) mit regulären Facetten. Dann sind alle  $(n-2)$ -Seiten von  $P$  zueinander kongruent. Wenn die Facetten von  $P$  nicht alle zueinander kongruent sind, so sind also die Facetten von  $P$  vom folgenden Typ:

- für  $n \geq 6$ : reg.  $(n-1)$ -Simplex oder reg.  $(n-1)$ -Kreuzpolytop,
- für  $n = 5$ : reg. 4-Simplex oder reg. 4-Kreuzpolytop oder 600-Zell,
- für  $n = 4$ : reg. 3-Simplex oder reg. 3-Kreuzpolytop oder Ikosaeder.

Wegen [1] genügt es, diesen Fall zu untersuchen.

3. Es sei  $K$  eine  $(n-3)$ -Seite von  $P$ , und  $P_j$  ( $j = 1, \dots, i; i \geq 3$ ) seien die  $K$  enthaltenden Facetten von  $P$ . Nach [5] gilt für den Innenwinkel  $\alpha_j$  von  $P_j$  bei  $K$  die Beziehung  $\sum_{j=1}^i \alpha_j < 2\pi$ ; daraus folgt durch Einsetzen der entsprechenden Werte für  $\alpha_j$  (siehe z.B. [2]), daß es für  $\{P_1, \dots, P_i\}$  höchstens die folgenden Fälle gibt ( $x := \text{reg. } (n-1)\text{-Kreuzpolytop}$ ,  $\Delta := \text{reg. } (n-1)\text{-Simplex}$ ,  $S := 600\text{-Zell}$  für  $n = 5$  bzw. Ikosaeder für  $n = 4$ ):

Tabelle 1

		$\{P_1, \dots, P_i\}$	$\{x, x, x\}$	$\{x, x, \Delta\}$	$\{x, x, S\}$	$\{x, \Delta, \Delta\}$	$\{x, \Delta, \Delta, \Delta\}$	$\{x, \Delta, S\}$
$n$								
4			+	+	+	+	+	+
5				+		+		+
6,7,8				+		+		
$\geq 9$						+		

		$\{P_1, \dots, P_i\}$	$\{S, S, \Delta\}$	$\{S, \Delta, \Delta\}$	$\{S, \Delta, \Delta, \Delta\}$	$\{\Delta, \Delta, \Delta\}$	$\{\Delta, \Delta, \Delta, \Delta\}$	$\{\Delta, \Delta, \Delta, \Delta, \Delta\}$
$n$								
4			+	+	+	+	+	+
5				+		+	+	
6,7,8						+	+	
$\geq 9$						+	+	

4. Es sei  $K^\perp$  der euklidische Raum, der zu  $K$  orthogonal ist und den Schwerpunkt von  $K$  enthält. Dann ist  $P \cap K^\perp$  ein 3-Polytop, dessen  $K \cap K^\perp$  enthaltende 2-Seiten genau die Durchschnitte von  $K^\perp$  mit den  $K$  enthaltenden Facetten von  $P$  sind, und der Innenwinkel einer 2-Seite von  $P \cap K^\perp$  bei  $K \cap K^\perp$  ist gleich dem Innenwinkel der entsprechenden Facette von  $P$  bei  $K$ . Außerdem ist der Winkel zwischen zwei  $K \cap K^\perp$  enthaltenden 2-Seiten von  $P \cap K^\perp$  gleich dem Winkel zwischen den entsprechenden Facetten von  $P$ .

Durch Übergang zu  $P \cap K^\perp$  ergibt sich daher  $\alpha_{j_0} < \sum_{i \neq j_0} \alpha_j$ , so daß für  $n=5$  der Fall  $\{S, \Delta, \Delta\}$  nicht möglich ist.

Die  $K$  enthaltenden Facetten  $P_j$  ( $j=1, \dots, i$ ) seien so nummeriert, daß  $P_j \cap P_{j+1}$  ( $n-2$ )-dimensional ist.  $\mathcal{F}$  sei eine Teilmenge der Menge aller Facetten von  $P$ , die  $P_j$  genau für  $1 \leq j \leq l$  ( $l < i$ ) enthält. Dann heißt der von  $P_i \cap P_1 \cap K^\perp$  und  $P_1 \cap P_{l+1} \cap K^\perp$  gebildete Winkel «offener Winkel an  $K$  bzgl.  $\mathcal{F}$ ».

Durch Übergang zu  $P \cap K^\perp$  läßt sich nachrechnen:

- a) Für  $l=i-1$  bestimmt der offene Winkel an  $K$  bzgl.  $\mathcal{F}$  den Typ von  $P_i$ .
- b) Es sei  $l=i-2$ . Der Innenwinkel des reg. Kreuzpolytops bzw. reg. Simplex bzw. Ikosaeders an einer  $(n-3)$ -Seite sei  $J(x)$  bzw.  $J(\Delta)$  bzw.  $J(S)$ . Wegen der Konvexität von  $P$  gibt es dann die folgenden Fälle für  $\{P_1, \dots, P_i\}$  nicht:

Tabelle 2

$n$	$\{P_1, \dots, P_i\}$	Offener Winkel an $K$ bzgl. $\mathcal{F}$
5	$\{x, \Delta, \Delta, \Delta\}$	$\leq J(x)$
	$\{x, \Delta, \Delta, \Delta\}$	$\leq J(\Delta)$
4	$\{S, \Delta, \Delta, \Delta\}$	$\leq J(x)$
	$\{\Delta, \Delta, \Delta, \Delta, \Delta\}$	$\leq J(x)$

- c) Es sei  $l=i-2$  und  $\{P_1, \dots, P_i\} = \{x, \Delta, \Delta, \Delta\}$ , wobei der offene Winkel an  $K$  bzgl.  $\mathcal{F}$  gleich  $J(x)$  sei. Für  $\mathcal{F}' := \{P_1, P_4\}$  ist dann der offene Winkel an  $K$  bzgl.  $\mathcal{F}'$  ungleich  $J(x)$ .

d) Es sei  $l=i-2$  und  $\{P_1, \dots, P_i\} = \{S, \Delta, \Delta, \Delta\}$ , wobei der offene Winkel an K bzgl.  $\mathcal{F}$  größer als  $J(S)$  sei. Für  $\mathcal{F}' := \{P_1, P_4\}$  ist dann der offene Winkel an K bzgl.  $\mathcal{F}'$  kleiner als  $J(S)$ .

e) Es sei  $l=i-2$  und  $P_{i-1}$  und  $P_i$  seien Simplizes. Ist dann  $\mathcal{F}$  auch Facettenmenge eines Poytops  $P'$  mit reg. Facetten, und ist K in genau 2  $P'$ -Facetten nicht aus  $\mathcal{F}$  enthalten, in Simplizes  $P'_{i-1}$  und  $P'_i$ , so ist  $P_{i-1} = P'_{i-1}$  und  $P_i = P'_i$ .

**5.a)** Es seien  $F_1$  und  $F_2$  Facetten von  $P$  mit  $(n-2)$ -dimensionalem Durchschnitt. Eine Bewegung  $\beta$ , die  $F_1 \cap F_2$  als Ganzes und die Schwerpunkte von  $F_1$  und  $F_2$  festlässt, lässt  $F_1$  und  $F_2$  als Ganzes fest. Daraus folgt:

a<sub>1</sub>) Alle offenen Winkel an  $(n-3)$ -Seiten  $K_k$  ( $k=1, \dots, n-1$ ) von  $F_1 \cap F_2$  bzgl.  $\{F_1, F_2\}$  sind gleich groß.

a<sub>2</sub>) Ist außerdem jedes  $K_k$  außer in  $F_1$  und  $F_2$  in genau 2 Facetten enthalten, in Simplizes mit der gemeinsamen Ecke  $E_k \notin F_1 \cap F_2$ , so permutiert  $\beta$  die  $E_k$  (wegen 4e)).

b) Wie in [1] gilt: Wenn  $\mathcal{F}$  eine echte Teilmenge der Menge aller Facetten eines Polytops ist so, daß jede Facette aus  $\mathcal{F}$  mit mindestens einer anderen Facette aus  $\mathcal{F}$  eine  $(n-2)$ -Seite gemeinsam hat, so gibt es eine  $(n-3)$ -Seite des Polytops, die in 2 Facetten aus  $\mathcal{F}$  enthalten ist und in einer Facette nicht aus  $\mathcal{F}$ .

**6.** Im folgenden werden nun für jeden der in 3 zusammengestellten Fälle (außer  $\{S, \Delta, \Delta\}$  für  $n=5$ ) alle Polytope mit reg. Facetten charakterisiert, bei denen dieser Fall an einer  $(n-3)$ -Seite auftritt.

Als erstes sei  $P$  ein  $n$ -Polytop mit reg. Facetten, bei dem an einer  $(n-3)$ -Seite K einer der Fälle aus Tabelle 3, Spalte 1 auftritt.

Tabelle 3

		$P'_1$	$P'_2$	$P'_3$
$n \geq 4, \{x, \Delta, \Delta\}$	Pyramide über x	x	$\Delta$	$\Delta$
$n=4, \{S, \Delta, \Delta\}$	Pyramide über S	S	$\Delta$	$\Delta$
$n=4, \{x, x, x\}$	24-Zell	x	x	x
$n=4, \{x, x, S\}$	semireguläres oktikosaedrisches 4-Polytop	x	x	S
$8 \geq n \geq 5, \{x, x, \Delta\}$	semireguläres, nicht reguläres n-Poytop	x	x	$\Delta$
		x	$\Delta$	x

Dann gibt es genau ein Polytop  $P'$  vom Typ der 2. Spalte von Tabelle 3 so, daß K auch Seite von  $P'$  ist, und die K enthaltenden Facetten von  $P$  und  $P'$  übereinstimmen. Es sei  $\mathcal{F}$  die Menge der K enthaltenden Facetten von  $P$  und  $P'$ . Dann gibt es nach 5b) eine  $(n-3)$ -Seite von  $P'$ , die in 2  $P'$ -Facetten  $P'_1, P'_2 \in \mathcal{F}$  enthalten ist, und in 1  $P'$ -Facette  $P'_3 \notin \mathcal{F}$ , wie in Spalte 3. Nach 4a) und b) zusammen mit 3 folgt dann, daß  $P'_3$  auch  $P$ -Facette ist. Durch successives Wiederholen der letzten Überlegungen (im nächsten Schritt spielt  $\mathcal{F} \cup \{P'_3\}$  die Rolle von  $\mathcal{F}$ ) folgt, daß die Menge aller Facetten von  $P$  und  $P'$  identisch ist.

**7.** Es sei  $P$  ein 4-Polytop mit reg. Facetten, bei dem an einer 1-Seite der Fall  $\{x, \Delta, S\}$  auftritt. O.B.d.A. tritt an keiner 1-Seite von  $P$  einer der Fälle aus Tabelle

3, Spalte 1 auf. Die durch  $\{x, \Delta, S\}$  gegebenen Facetten seien  $x_0, \Delta_0, S_0$ , und  $E$  sei eine ihrer gemeinsamen Ecken. Dann ist  $E$  in 5 1-Seiten  $K_1, \dots, K_5$  von  $S_0$  enthalten, die in natürlicher Reihenfolge nummeriert seien und so, daß  $K_1 = x_0 \cap \Delta_0 \cap S_0$ , und  $K_5 \subset x_0$ . Dann tritt an  $K_5$  ebenfalls der Fall  $\{x, \Delta, S\}$  auf. Der offene Winkel an  $K_2$  und  $K_4$  bzgl. der bisher betrachteten P-Facetten ist nach 5a<sub>1</sub>) gleich  $J(x)$ , so daß nach 4a) und b) auch an  $K_2$  und  $K_4$  der Fall  $\{x, \Delta, S\}$  auftritt. Deshalb tritt an  $K_3$  der Fall  $\{x, x, S\}$  auf, im Widerspruch zur Voraussetzung über  $P$ .

8. Es sei  $P$  ein 4-Polytop mit reg. Facetten, bei dem an einer 1-Seite der Fall  $\{x, x, \Delta\}$  auftritt. O.B.d.A. tritt an keiner 1-Seite von  $P$  einer der bisher betrachteten Fälle auf. Die durch  $\{x, x, \Delta\}$  gegebenen zwei Facettenoktaeder  $x_1, x_2$  haben eine 2-Seite  $\text{conv}(E_1, E_2, E_3)$  gemeinsam, wovon jede 1-Seite außer in  $x_1$  und  $x_2$  noch genau in einem Tetraeder enthalten ist.  $\mathcal{F}$  sei die Menge aus diesen 5 Facettenoktaedern bzw.-tetraedern von  $P$ .

Nun gibt es genau ein semireguläres tetroktaedrisches 4-Polytop  $P'$  so, daß  $\mathcal{F}$  auch Facettenmenge von  $P'$  ist. Die Ecke  $E_i$  ( $i=1,2,3$ ) ist außer in Facetten aus  $\mathcal{F}$  in genau einer weiteren  $P'$ -Facette enthalten, dem Oktaeder  $x'_i$ . Die 4 1-Seiten von  $x'_i$  durch  $E_1$  sind in jeweils 1 Oktaeder und 1 Tetraeder aus  $\mathcal{F}$  enthalten, und sie sind auch 1-Seiten von  $P$ . Deshalb ist entweder  $x'_i$  auch  $P$ -Facette, oder jede dieser 1-Seiten ist jeweils in genau 2  $P$ -Facettentetraedern nicht aus  $\mathcal{F}$  enthalten, die nach 4e) Facetten der Pyramide  $P(x'_i)$  sind; dabei ist die Spitze  $V_i$  von  $P(x'_i)$  nicht aus  $P'$ .

Im letzteren Fall sind also die vier  $E_i$  enthaltenden  $P(x'_i)$ -Facettentetraeder auch  $P$ -Facetten. Dann sei  $K_{ij}$  die  $E_i$  nicht enthaltende 1-Seite von  $x'_i \cap x_j$  ( $j=1,2$ ).  $K_{ij}$  ist als 1-Seite von  $P'$  genau in  $x_j, x'_i$  und einem Tetraeder enthalten, und als 1-Seite von  $P$  in  $x_j$  und einem  $P(x'_i)$ -Facettentetraeder. Nach 4c) ist also  $K_{ij}$  noch genau in 2 weiteren  $P$ -Facettentetraedern enthalten, von denen nach 4e) eines auch  $P(x'_i)$ -Facette ist. Damit sind 6  $P(x'_i)$ -Facettentetraeder als  $P$ -Facetten erkannt. Durch Anwendung von 4a) und b) auf geeignete  $V_i$  enthaltende 1-Seiten von  $P$  ergibt sich schließlich, daß alle 8  $P(x'_i)$ -Facettentetraeder auch  $P$ -Facetten sind. Falls also  $x'_i$  nicht Facette von  $P$  ist, kann  $P$  durch die  $x'_i$  enthaltende Hyperebene zerlegt werden in  $P(x'_i)$  und ein Polytop mit der Facette  $x'_i$ .

Es sei  $\bar{P}$  das evtl. durch Abschneiden von  $P(x'_i)$  ( $i=1,2,3$ ) aus  $P$  entstehende Polytop.  $x_1, x_2, x'_1, x'_2, x'_3$  sind jetzt paarweise verschiedene  $\bar{P}$ - und  $P'$ -Facetten. Weil  $P'$  genau 5 Facettenoktaeder besitzt, und an jeder 1-Seite von  $P'$  der Fall  $\{x, x, \Delta\}$  auftritt, ist die Menge aller Facetten von  $\bar{P}$  und  $P'$  identisch.

Nun haben 2 Facettenoktaeder von  $P'$  eine 1-Seite gemeinsam. Wären also z.B. die Facettentetraeder von  $P(x'_1)$  und  $P(x'_2)$  auch  $P$ -Facetten, so würde an einer gemeinsamen 1-Seite von  $x'_1$  und  $x'_2$  der Fall  $\{\Delta, \Delta, \Delta, \Delta, \Delta\}$  auftreten, mit  $J(x)$  als offenem Winkel bzgl. geeigneten drei  $\Delta$ , im Widerspruch zu 4b).

Es ist also  $P$  entweder das semireguläre tetroktaedrische 4-Polytop oder dessen Vereinigung mit einer Pyramide über einem seiner Facettenoktaeder.

9. Es sei  $P$  ein n-Polytop mit reg. Facetten ( $n=4,5$ ), bei dem an einer  $(n-3)$ -Seite der Fall  $\{x, \Delta, \Delta, \Delta\}$  auftritt. O.B.d.A. tritt an keiner  $(n-3)$ -Seite von  $P$  einer der schon behandelten Fälle auf. Jede  $(n-3)$ -Seite des durch  $\{x, \Delta, \Delta, \Delta\}$  gegebenen

Facettenkreuzpolytops  $x_0$  ist also außer in  $x_0$  genau in 3 Facettensimplizes enthalten. Alle diese Facettensimplizes sind dann entweder die konvexe Hülle einer  $(n-2)$ -Seite  $A_i$  von  $x_0$  und einer Ecke  $E_i \notin x_0$ , oder die konvexe Hülle einer  $(n-3)$ -Seite  $A_{i_1} \cap A_{i_2}$  und der Ecken  $E_{i_1}$  und  $E_{i_2}$  ( $i = 1, \dots, 2^{n-1}$ ).  $\mathcal{F}$  sei die Menge dieser Facettensimplizes.

Es sei nun  $A$  eine  $(n-4)$ -Seite von  $x_0$ , die o.B.d.A. in den in natürlicher Reihenfolge numerierten Seiten  $A_1, \dots, A_4$  enthalten sei. Dann ist für  $i=1, \dots, 4$   $\text{conv}(A \cup E_i)$  in 3 Facetten aus  $\mathcal{F}$  enthalten, und die  $E_i$  sind paarweise verschieden, weil der Fall  $\{\Delta, \Delta, \Delta, \Delta, \Delta, \Delta\}$  an einer  $(n-3)$ -Seite nicht auftreten kann. Nun wird der offene Winkel  $\alpha$  an  $\text{conv}(A \cup E_1)$  bzgl.  $\mathcal{F}$  eindeutig bestimmt durch den Winkel zwischen  $E_1 E_4$  und  $E_1 E_2$ . Weil dieser (gegebenfalls nach Umnumerierung von  $E_1, \dots, E_4$ )  $\leq \pi/2$  ist, ist  $\alpha \leq J(x)$ , so daß nach 4a) an  $\text{conv}(A \cup E_1)$  der Fall  $\{S, \Delta, \Delta, \Delta\}$  nicht auftreten kann, und nach 4b) nicht der Fall  $\{\Delta, \Delta, \Delta, \Delta, \Delta\}$ . An  $\text{conv}(A \cup E_1)$  tritt also einer der Fälle  $\{x, \Delta, \Delta, \Delta\}$  und  $\{\Delta, \Delta, \Delta, \Delta\}$  auf.

Angenommen, der Fall  $\{x, \Delta, \Delta, \Delta\}$  tritt auf. Das dadurch definierte Kreuzpolytop  $x_1$  enthält  $E_1$  und  $E_2$ . Es sei  $A'$  eine von  $A$  verschiedene  $(n-4)$ -Seite von  $A_1 \cap A_2$ ,  $A \cap A'$  ist dann  $(n-5)$ -dimensional. Nach 5a<sub>2</sub>) tritt an  $\text{conv}(A' \cup E_1)$  ebenfalls der Fall  $\{x, \Delta, \Delta, \Delta\}$  auf. Das dadurch definierte Kreuzpolytop  $x_2$  enthält ebenfalls  $E_1$  und  $E_2$ . An der  $(n-3)$ -Seite  $\text{conv}((A \cap A') \cup E_1 \cup E_2)$  tritt daher der Fall  $\{x, x, \Delta\}$  auf, im Widerspruch zur Voraussetzung über  $P$ .

Wenn andererseits an  $\text{conv}(A \cup E_1)$  der Fall  $\{\Delta, \Delta, \Delta, \Delta\}$  auftritt, so tritt an  $\text{conv}(A \cup E_4)$  und an  $\text{conv}(A \cup E_2)$  der Fall  $\{\Delta, \Delta, \Delta, \Delta, \Delta\}$  und an  $\text{conv}(A \cup E_3)$  wieder der Fall  $\{\Delta, \Delta, \Delta, \Delta\}$  auf. Dies ist nur für  $n=4$  möglich.

Dann seien  $A'$  und  $A''$  die von  $A$  verschiedenen Ecken von  $A_1$ , und  $A_5$  sei die 2-Seite von  $x_0$  mit  $A_5 \cap A_1 = A'A''$ . Wegen 5a<sub>2</sub>) tritt an  $\text{conv}(A' \cup E_1)$  und an  $\text{conv}(A'' \cup E_1)$  ebenfalls der Fall  $\{\Delta, \Delta, \Delta, \Delta\}$  auf. Deshalb sind  $E_4 E_2$ ,  $E_2 E_5$  und  $E_5 E_4$  1-Seiten von  $P$ .  $E_4$ ,  $E_2$ ,  $E_5$  bestimmen also ein gleichseitiges Dreieck, das wegen 5a<sub>2</sub>) in einer zu  $A_1$  parallelen Ebene liegt, und zusammen mit  $A_1$  ein Oktaeder  $x_3$  bestimmt. Dabei ist  $E_1$  die Spitze einer Pyramide  $P(x_3)$  über  $x_3$ , und 7 Facettentetraeder von  $P(x_3)$  sind auch P-Facetten. Nach 4b) sind dann alle Facettentetraeder von  $P(x_3)$  P-Facetten.

Analog ist  $E_3$  Spitze einer Pyramide  $P(x_4)$  über einem Oktaeder  $x_4$  so, daß alle Facettentetraeder von  $P(x_4)$  auch P-Facetten sind. Die  $x_3$  und  $x_4$  enthaltenden Hyperebenen zerlegen dann  $P$  in die Polytope  $P(x_3)$ ,  $P(x_4)$  und ein Polytop  $\bar{P}$  mit  $x_3$  und  $x_4$  als Facetten. Daher ist  $\bar{E}_2 \bar{E}_4$  als  $\bar{P}$ -1-Seite außer in  $x_3$  und  $x_4$  noch genau in einem Facettentetraeder, -oktaeder oder -ikosaeder enthalten. Dieses ist auch P-Facetten, zusätzlich zu je 2  $\bar{E}_2 \bar{E}_4$  enthaltenden Facettentetraedern von  $P(x_3)$  bzw.  $P(x_4)$ , was nach 3 und 4b) nicht möglich ist.

**10.** Es sei  $P$  ein 4-Polytop mit regulären Facetten, das ein Ikosaeder  $S_0$  als Facette hat. O.B.d.A. tritt an keiner  $(n-3)$ -Seite von  $P$  einer der schon behandelten Fälle auf.

Es sei nun  $E$  eine Ecke von  $S_0$ . Angenommen, es gibt ein weiteres P-Facettenikosaeder  $S_1$  mit der Ecke  $E$  so, daß  $S_1 \cap S_0$  2-dimensional ist. An den beiden  $E$  enthaltenden 1-Seiten von  $S_1 \cap S_0$  tritt dann jeweils der Fall  $\{S, S, \Delta\}$  auf, wodurch die Tetraeder  $\Delta_1$  und  $\Delta_2$  gegeben werden. Durch Betrachten der  $E$  enthaltenden 1-Seiten von  $\Delta_1$  und  $\Delta_2$  ergibt sich schließlich:  $E$  ist außer in  $S_0$ ,

$S_1, \Delta_1, \Delta_2$  genau in einem Ikosaeder und 3 Tetraedern enthalten, oder genau in 8 Tetraedern. Dabei sind die E enthaltenden P-Facetten-tetraeder und die Facettentetraeder der Pyramiden über den E enthaltenden P-Facettenikosaedern genau die E enthaltenden Facetten genau eines 600-Zells  $P'$ . Falls es aber kein Facettenikosaeder  $S_1$  mit der Ecke E so gibt, daß  $S_1 \cap S_0$  2-dimensional ist, tritt an jeder E enthaltenden 1-Seite von  $S_0$  der Fall  $\{S, \Delta, \Delta, \Delta\}$  auf. Dadurch sind 10 Facettentetraeder gegeben, von denen  $\Delta_1, \dots, \Delta_5$  einen 2-dimensionalen Durchschnitt mit  $S_0$  haben, und die restlichen 5 einen 1-dimensionalen. Nach 5a<sub>1</sub>) sind für jedes  $i \in \{1, \dots, 5\}$  die offenen Winkel an den 1-Seiten von  $\Delta_i \cap S_0$  bzgl.  $\{S_0, \Delta_i\}$  gleich groß, und durch Betrachten aller  $\Delta_i$  ergibt sich mit Hilfe von 4d), daß sie alle gleich  $J(S)$  sind. Deshalb ist E außer in  $S_0$  und den 10 betrachteten Tetraedern noch genau in einem Ikosaeder oder genau in 5 Tetraedern enthalten. Dabei sind die E enthaltenden P-Facetten-tetraeder und die Facettentetraeder der Pyramiden über den E enthaltenden P-Facettenikosaedern wieder genau die E enthaltenden Facetten genau eines 600-Zells  $P'$ .

Durch die analogen Überlegungen für jede Ecke von  $S_0$  werden alle Facetten bestimmt, die mit  $S_0$  einen  $(n-2)$ -dimensionalen Durchschnitt haben, analoges gilt für jedes Facettenikosaeder von  $P$ . Deshalb ist auch das durch die Vereinigung von  $P$  mit allen Pyramiden über P-Facettenikosaedern entstehende Polytop  $\bar{P}$  konvex.  $\bar{\mathcal{F}}$  sei die Menge der gemeinsamen Facetten von  $P'$  und  $\bar{P}$ . Angenommen,  $\bar{\mathcal{F}}$  sei nicht die Menge aller Facetten von  $\bar{P}$ . Dann gibt es Ecken  $E_1$  und  $E_2$  von  $\bar{P}$  so, daß  $E_1 E_2$  1-Seite von  $\bar{P}$  ist, und daß alle  $E_2$  enthaltenden  $\bar{P}$ -Facetten aus  $\bar{\mathcal{F}}$  sind, aber nicht alle  $E_1$  enthaltenden  $\bar{P}$ -Facetten. Die  $E_2$  enthaltenden  $\bar{P}$ -Facetten sind also  $P'$ -Facetten und somit die Facettentetraeder einer Ikosaederpyramide  $P(S_2)$  mit der Spitze  $E_2$ . Die  $S_2$  enthaltende Hyperebene zerlegt dann  $\bar{P}$  in  $P(S_2)$  und ein Polytop  $\bar{P}$  mit der Facette  $S_2$ . Dabei ist  $E_1$  Ecke des  $\bar{P}$ -Facettenikosaeders  $S_2$ .

Die Vereinigung einer Ikosaederpyramide  $P(S)$  und eines der in 6 bis 9 behandelten Polytope mit der Facette  $S$  ist nur im Fall der Ikosaederpyramide konvex. Weil aber  $\bar{P}$  keine Ikosaederpyramide ist, gelten daher für  $\bar{P}$  dieselben Voraussetzungen wie für  $P$ . Alle  $E_1$  enthaltenden  $\bar{P}$ -Facettentetraeder und die  $E_1$  enthaltenden Facettentetraeder der Ikosaederpyramide über dem  $E_1$  enthaltenden  $\bar{P}$ -Facettenikosaeder, d.h. alle  $E_1$  enthaltenden  $\bar{P}$ -Facetten, sind also genau die  $E_1$  enthaltenden Facetten eines eindeutig bestimmten 600-Zells, das mit  $P'$  zusammenfällt. Alle  $E_1$  enthaltenden  $\bar{P}$ -Facetten sind also aus  $\bar{\mathcal{F}}$ , im Widerspruch zur Definition von  $E_1$ .

Deshalb ist  $\bar{P} = P'$ , und  $P$  entsteht aus dem 600-Zell durch Abschneiden paarweise nicht benachbarter Ecken (vgl. 1).

11. Zusammen mit [1] ist damit bewiesen:

**Satz:** Im  $\mathbb{R}^n$  ( $n \geq 4$ ) sei  $P$  ein konvexes  $n$ -Polytop mit regulären Facetten. Dann ist  $P$  aus einer der folgenden Klassen:

- semireguläre Polytope,
- Pyramide über dem regulären  $(n-1)$ -Kreuzpolytop,
- Bipyramide über dem regulären  $(n-1)$ -Simplex

*und für  $n=4$  zusätzlich:*

Pyramide über dem Ikosaeder,  
 Vereinigung eines semireg. tetroktaedrischen Polytops mit einer  
 Pyramide über einem seiner Facettenoktaeder,  
 aus dem 600-Zell durch Abschneiden paarweise nicht benachbarter  
 Ecken entstehende Polytope,  
 Polytope, bei denen alle Facetten Tetraeder sind.

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# P.M. Gruber and R. Schneider Problems in Geometric Convexity

The following list consists of problems collected at the Oberwolfach conferences on Convexity in 1974, 1976, 1978. We have reproduced each of these problems, even if it has been solved in the meantime. Recent information about problems and references to solutions or partial solutions that has come to our knowledge has been added at appropriate places.

1. A family  $\{A_i | i \in I\}$  of subsets of a real linear space  $E$  is called separated if there exists a convex partition  $\{P_i | i \in I\}$  of  $E$  such that  $A_i \subset P_i$  for each  $i \in I$ . Characterize those families of convex subsets of  $E$  which are separated.

J. Bair

2. Let  $P$  be a convex polytope in  $\mathbb{R}^4$ . A set  $\{M_1, \dots, M_n\}$  of 2-dimensional subcomplexes of  $\text{skel}_2 P$  is called a decomposition of  $\text{skel}_2 P$  if it has the following properties:

- (a) set  $M_i := \cup\{F | F \in M_i\}$  is a topological 2-manifold for each  $i$ ,
- (b)  $\cup M_i = \text{skel}_2 P$ ,
- (c)  $M_i \cap M_j \subset \text{skel}_1 P$  for all  $i \neq j$ .

Let  $z(P)$  denote the smallest  $n$  for which a decomposition exists. What is the least upper bound of  $z(P)$  for all convex polytopes in  $\mathbb{R}^4$ ? ( $z(P) < f_2(P)/2$  for all convex polytopes in  $\mathbb{R}^4$  and  $z(P) > (f_2(P)/4) - (1/2)$  for infinitely many of them.)

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3. We say that a convex polytope contains a manifold if there is a subcomplex of its boundary complex homeomorphic to this manifold. *Problems:*

- (a) Does every  $d$ -polytope,  $d \geq 4$ , contain a Möbius strip with at most seven 2-faces?
- (b) Does every  $d$ -polytope,  $d \geq 4$ , except the 4-simplex contain a strip with at most six 2-faces?
- (c) Do all 4-polytopes except finitely many contain a torus with a hole?

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4. Let  $\mathfrak{G}$  be a family of graphs. The length exponent of  $\mathfrak{G}$  is defined by

$$\sigma(\mathfrak{G}) := \liminf_{G \in \mathfrak{G}} \frac{\log h(G)}{\log v(G)}.$$

Here  $v(G)$  denotes the number of knots and  $h(G)$  the number of knots on a maximal circuit of  $G$  [see Grünbaum-Walter: J. Combinat. Th. (A) 14 (1973), 364–385].

Let  $\mathfrak{G}^*(n,3)$  denote the family of all simple graphs all facets of which have an even number  $\leq n$  of knots. Is it true that  $\sigma(\mathfrak{G}^*(n,3)) = 1$  for each  $n$ ? This would give a positive answer to the following conjecture of Barnette: For each simple polytope in  $\mathbb{R}^3$ , all facets of which have an even number of vertices, there exists a Hamiltonian circuit.

G. Bielig, Chr. Schulz

**5.** Which convex body in  $\mathbb{R}^2$  with given diameter and width has minimal perimeter? It is conjectured that the extremal bodies are certain polygons inscribed in a Reuleaux triangle.

H. Bieri

**6.** Which convex body in  $\mathbb{R}^2$  with given diameter, width and perimeter has maximal (resp. minimal) area?

H. Bieri

**7.** Find a sharp lower bound for the volume of any three-dimensional convex body whose surface area  $F$  and integral of mean curvature  $M$  are given, satisfying the inequality  $2M^2 < \pi^3 F$ . (Compare Hadwiger, Altes und Neues über konvexe Körper, §28.)

H. Bieri

**8.** Reinhardt's problem. Which convex polygon of given number of vertices and given diameter has maximal area? If  $n$  is odd the solution is the regular  $n$ -gon. The solutions for  $n=4,6$  are also known.

H. Bieri

**9.** Is it possible to decompose the unit square  $Q$  into 5 congruent subsets  $B_1, \dots, B_5$  such that  $\text{int } B_i \cap \text{int } B_j = \emptyset$  for  $i \neq j$ ,  $B_i = \text{cl}(\text{int } B_i)$ ,  $\text{int } B_i$  is connected, and  $B_i$  is not a rectangle of side-lengths 1 and  $1/5$ ?

L. Danzer

**10.** Does there exist a  $d \in \mathbb{N}$  and a convex body  $K \subset \mathbb{R}^d$  such that there exists a tiling of  $\mathbb{R}^d$  by translates of  $K$  but not as a lattice?

L. Danzer

*Comment.* For  $n=2$  no such  $K$  exists. For  $n=5$  (and thus for  $n \geq 5$ ) Stein [Bull. Amer. Math. Soc. **80** (1974), 819–820] has given an example of a symmetric star body  $K$  having the required property. If the above problem has a negative solution, then a well known result of Minkowski (see, e.g., Cassels, An introduction to the geometry of numbers, IX, Th. V) implies that the following problem 11 has a positive solution. Similar questions are considered by H. Groemer [Arch. Math. **19** (1968), 445–448; Math. Ann. **154** (1964), 88–102; Math. Z. **79** (1962), 364–375; Monatsh. Math. **68** (1964), 21–32]. See also the survey article of Gruber in this volume.

**11.** *Conjecture.* Let  $K$  be a convex body in  $\mathbb{R}^d$  such that there exists a tiling of  $\mathbb{R}^d$  by translates of  $K$ . Then  $K$  has at most  $2^{d+1} - 2$  facets.

L. Danzer

**12.** What is the smallest integer  $r=r(d)$  such that every subset  $A$  of at least  $r$  points of the integer lattice  $\mathbb{Z}^d$  has a partition  $\{A_1, A_2\}$  such that

$$\text{conv } A_1 \cap \text{conv } A_2 \cap \mathbb{Z}^d \neq \emptyset?$$

$r(d)$  is called the Radon number of  $\mathbb{Z}^d$ . It exists since the Carathéodory and Helly

numbers of  $\mathbb{Z}^d$  are known to be  $d+1$  and  $2^d$ , respectively. One can prove  $r(2)=6$ .

J.-P. Doignon

**13.** The following result is due to Kramer: Consider a family of compact convex subsets of  $\mathbb{R}^2$  such that any 3 of its members have a common transversal. Then there exist 5 lines such that each member of the family is intersected by at least one of them. Is 5 best possible? [See Eckhoff: Arch. Math. **24** (1973), 196–202.]

J. Eckhoff

**14.** Let  $V, S, M$  be the volume, surface area, and integral of mean curvature of a convex body in Euclidean 3-space, which is so smooth that at each boundary point it has principal radii of curvature  $R_1, R_2$  and these radii are continuous over the unit sphere  $\Omega$  of outer normal directions  $u$  to the convex body.  
*Conjecture:*

$$\frac{3V}{M} \geq \left( \frac{1}{4\pi} \int_{\Omega} \frac{d\omega(u)}{R_1(u)R_2(u)} \right)^{-1} =: H, \quad (*)$$

where  $d\omega(u)$  is the area element at  $u$  on  $\Omega$ .

Inequality  $(*)$  is true for ellipsoids and almost spherical bodies, its 2 dimensional analogue is also true for outer parallels of Reuleaux polygons.  $H$  is the harmonic mean of  $R_1 R_2$ . The arithmetic mean of  $R_1 R_2$ ,

$$A := \frac{1}{4\pi} \int_{\Omega} R_1(u)R_2(u)d\omega(u) = \frac{S}{4\pi},$$

is a known upper bound for  $3V/M$ . Thus the conjecture asserts that between  $A \geq H$  we can insert  $3V/M$ . Note also that if  $h$  is the support function of the convex body, then

$$\frac{3V}{M} \int_{\Omega} h(u)R_1(u)R_2(u)d\omega(u) / \int_{\Omega} h(u)d\omega(u),$$

which is a specially weighted arithmetic mean of  $R_1 R_2$ . The set function  $H$  is not continuous, if we set  $H=0$  for all convex bodies whose boundary contains some segment, then  $H$  might be semi-continuous from below.

Wm.J. Firey

**15.** A.D. Aleksandrov proved that if  $K, K'$  are smooth convex bodies in Euclidean 3-space whose principal radii of curvature are  $R_1, R_2$  and  $R'_1, R'_2$ , viewed as functions of the outer unit normal direction  $u$  and if

$$f(R_1(u), R_2(u)) = (R'_1(u), R'_2(u)), \text{ for all } u,$$

where  $f$  is strictly increasing in each of its two arguments, then  $K'$  is a translate of  $K$ . Let  $\alpha, \beta, \gamma$  be positive constants, then

$$f(R_1, R_2) = \alpha + \beta(R_1 + R_2) + \gamma R_1 R_2 \quad (*)$$

satisfies the conditions of Aleksandrov's theorem. But  $(*)$  can be viewed as the Radon-Nikodym derivative of

$$\bar{\mu}(\omega) = \alpha S_0(\omega) + \beta S_1(\omega) + \gamma S_2(\omega),$$

where  $\omega$  is any Borel set of directions and  $S_0, S_1, S_2$  are the area functions of order 0,1,2 for  $K$ . Suppose  $K, K'$  are arbitrary convex sets in 3-space except that they satisfy

$$\bar{\mu}(\omega) = \bar{\mu}'(\omega)$$

where  $\bar{\mu}'$  is  $\bar{\mu}$  but calculated for  $K'$ . Does this imply  $K'$  is a translate of  $K$ ?

Wm.J. Firey

*Comment.* An affirmative answer has been given by R. Schneider [Arch. Math. 27 (1976), 99–105]. The analogous question in  $\mathbb{R}^d$  for  $d \geq 4$  is still open.

16. A convex body  $K$  supports another body  $K'$  if  $K \cap K'$  is a subset of their boundaries. Let  $\sigma$  be a subset of the boundary of  $K$ ,  $\sigma'$  a subset of the boundary of  $K'$ , suppose  $\sigma, \sigma'$  are inverse spherical images of Borel sets  $\omega, \omega'$  of unit normal directions. I say  $\sigma$  supports  $K'$  in  $\sigma'$  if  $\sigma \cap \sigma' \neq \emptyset$  and  $K$  supports  $K'$ . Write  $\mathcal{M}(\omega, \omega')$  for the set of rigid motions  $g$  such that  $g\sigma$  supports  $K'$  in  $\sigma'$ . Define  $\bar{\mu}(\mathcal{M}(\omega, \omega'))$  to be

$$\sum_{q=0}^{n-1} \binom{n-1}{q} S_q(K, \omega) S_{n-q-1}(K', \omega'),$$

where  $S_q(K, \omega)$  is the area function of order  $q$ . This rule defines, for fixed  $\omega'$ , a measure over the Borel sets  $\omega$  on the sphere and for fixed  $\omega$ , a measure over the Borel sets  $\omega'$  on the unit sphere.  $\bar{\mu}(\mathcal{M}(\omega, \omega'))$  is rigid motion invariant in the sense that if we replace  $K, K'$  by  $g_0 K, g_0 K'$  and  $\omega, \omega'$  by  $R_0 \omega, R'_0 \omega'$  where  $R_0$  means the rotation part of  $g_0$ , then this does not alter the value of  $\bar{\mu}(\mathcal{M}(\omega, \omega'))$ .

*Vague problem.* Formulate and prove a reasonable uniqueness theorem. One is to assume  $\bar{\mu}$  defined over all sets of  $g$  of the form  $\mathcal{M}(\omega, \omega')$  for all  $K, K'$  so as to be a measure in one of the arguments  $\omega, \omega'$  when the other is fixed. Assume also  $\bar{\mu}$  has rigid motion invariance and symmetry in the sense described for  $\bar{\mu}$ . I think it very unlikely that these assumptions alone will suffice to establish  $\tilde{\mu} = \bar{\mu}$ .

Wm. J. Firey

*Comment.* For information on the origin of Firey's problem see Firey [Mathematika 21 (1974), 270–281]. A uniqueness theorem of this type (but with additional assumptions, which cannot be omitted) was proved by R. Schneider [Abh. Math. Sem. Univ. Hamburg 44 (1975), 12–23]. Using this uniqueness theorem, Schneider [Math. Ann. 218 (1975), 253–267] gave a new proof of the main result of Firey's paper. A survey of subsequent developments is found in the articles by R. Schneider (§7) and W. Weil in this volume.

17. For a plane convex body  $K$ , write  $L(\omega)$  for the measure of

$$\bigcup_{u \in \omega} K \cap \Pi(u)$$

where  $\Pi(u)$  is the support line to  $K$  with outer normal direction  $u$  in the Borel set  $\omega$  of the unit circle. Let  $x$  be a boundary point of  $K$  and  $\alpha$  a closed circular arc of those directions  $u$  such that  $x$  is in  $K \cap \Pi(u)$ . Next, let  $\alpha'$  be an open circular arc of

directions which contains  $\alpha$ . Consider the sets of directions  $\omega$  satisfying  $\alpha \subseteq \omega \subseteq \alpha'$  which have Lebesgue measure (which we write as  $|\omega|$ ). For such  $\omega$  form

$$r(x, \alpha') = \inf L(\omega)/|\omega|, R(x, \alpha') = \sup L(\omega)/|\omega'|.$$

If  $\{a_j\}$  is a nested sequence of  $\alpha'$ , decreasing to  $\alpha$ , then

$$r(x) = \lim_j r(x, \alpha_j), R(x) = \lim_j R(x, \alpha_j)$$

exist and do not depend on the choice of  $\{\alpha_j\}$ . *Problem:* How do these “lower” and “upper” radii of curvature compare with those of B. Jessen [Mat. Tidsskr. B (1929), 50–62; see also H. Busemann, Convex Surfaces, ch. I]?

Wm.J. Firey

**18.** Let  $\{g\}$  signify the rigid-motion group in  $n$ -space,  $dg$  a suitable normalized kinematic density,  $\chi$  the Euler characteristic,  $\partial$  the boundary operator,  $B$  the unit ball,  $K, K'$  convex bodies, and  $W_p$  the  $p$ -th Quermassintegral. For two classes of convex bodies the following kinematic formulas hold: When  $n$  is odd

$$\int_{\{g\}} \chi(\partial K \cap g\partial K') dg = \sum_{p=1}^n \binom{n}{p} (1 - (-1)^p) W_p(K) W_{n-p}(K').$$

When  $n$  is even

$$\int_{\{g\}} \chi(\partial K \cap g\partial K') dg = 2 \sum_{p=1}^{n-1} \binom{n}{p} (1 - (-1)^p) W_p(K) W_{n-p}(K').$$

The classes of  $K, K'$  are these:

I. If  $K = K_0 + rB_1 = K_r$ ,  $K' = K'_0 + r'B = K'_r$  for some positive  $r, r'$  and convex bodies  $K_0, K'_0$ . See H. Federer [Trans. Amer. Math. Soc. 93 (1959), 473, §6,11, (4)] and observe that  $\partial K$  has positive reach precisely when  $K = K_0 + rB$ ,  $r > 0$ . For  $n$  odd the special condition on  $K'$  can be omitted.

II. If  $K$  and  $K'$  are polytopes. See H. Hadwiger, Vorlesungen über Inhalt usw., p. 243 [§6,3,6 (126)] and observe that  $\partial K$  is in the convex ring when  $K$  is a polytope.

It seems likely that these formulas can be established for convex bodies generally. This involves assigning a meaning, at least for almost all  $g$ , to the integrands and so that they are integrable over  $\{g\}$  with integrals equal to the right hand sides.

A proof of the following conjecture would contribute to this problem. If  $K_0 \cap gK'_0$  has a non-empty interior, then the sets  $gK'_r \cap \partial K_r$  and  $g\partial K'_r \cap \partial K_r$  have the same Euler characteristic for almost all  $g$  and for all  $r > 0$ . Here the Euler characteristic is in Federer's sense.

Wm.J. Firey

**19.** Let  $E$  be a Hausdorff locally convex topological space over  $\mathbb{R}$ . Characterize the maximal ideals of the lattice of all closed convex subsets of  $E$  and of the lattice of all closed bounded convex subsets of  $E$  in case  $\dim E \geq \aleph_0$ .

R. Fourneau

**20. Conjecture.** Let  $K$  be a convex body in  $\mathbb{R}^d$  ( $d \geq 3$ ) and let  $p, q \in \text{int } K$ . Then  $K$

is an ellipsoid if and only if each two parallel sections of  $K$  containing  $p, q$  are homothetic. What happens if one considers more general subsets of  $\mathbb{R}^d$ , if one omits the assumption that  $p, q$  are interior points of  $K$  and if one considers weaker relations than homothety between parallel sections?

P.M. Gruber

*Comment.* The conjecture has been proved by G.R. Burton and P. Mani (Comment. Math. Helv. **53** (1978), 485–507.)

**21.** Find all isometries of the space of convex bodies of  $\mathbb{R}^d$  with respect to the Hausdorff metric.

P.M. Gruber, R. Schneider

*Comment.* For related results see papers by R. Schneider [Colloqu. Math. **33** (1975), 219–224] and Gruber (Geom. Dedicata, to appear).

One might conjecture that every such isometry is of the form  $K \mapsto tK + L$ , where  $t$  is an isometry of  $\mathbb{E}^d$  and  $L$  is a fixed convex body.

Compare also problem 69.

**22.** Consider the space of measurable subsets of  $\mathbb{R}^d$  (or more generally of a given measure space). Given an isometry of this space with respect to the symmetric difference metric, does there exist a map of the underlying  $\mathbb{R}^d$  into itself which generates this isometry? For a result of this type for the space of convex bodies see Gruber (Mathematika **25** (1978), 270–278).

P.M. Gruber

**23.** Let  $H$  be the convex hull of  $d$  line segments in  $\mathbb{R}^d$ . Is the volume of  $H$  greater or equal to the volume of a simplex  $S$  defined by translates of these line segments having a common endpoint?

E. Heil

**24.** The following is a problem of Guggenheimer: Let  $K$  be a convex body in  $\mathbb{R}^n$  such that  $0$  is an interior point of  $K$ . Let  $K^*$  denote the polar body of  $K$ . Is it then true that the product of the volumes  $V(K)V(K^*)$  is minimal in case  $K$  is a simplex with center of gravity at  $0$ ? What about the body  $H$  (see problem 23)? This problem has been solved in the affirmative for  $d=2$  by Mahler. Guggenheimer [Israel J. Math. **14** (1973); 309–316] proved that for 0-symmetric convex bodies  $V(K)V(K^*) \geq 4^n/n!$  with equality if  $K$  is a cube (and thus  $K^*$  a cross-polytope).

E. Heil

**25.** Let  $V, F$  denote volume and surface area of a convex body  $K$ . Does  $V/F$  increase if  $K$  is centrally symmetrized? This is true if  $K$  is a body of constant width. What about the quotients of other quermassintegrals?

E. Heil

**26.** Among all convex bodies in  $\mathbb{R}^d$  ( $d \geq 3$ ) of given width, what is the body of minimal  $i$ -th quermassintegral ( $i \in \{0, 1, \dots, d-2\}$ )?

E. Heil

*Comment* (by E. Heil) (added by the editor). A general result for the volume  $V$  is given by Firey [Arch. Math. **16** (1965), 69–74]:  $V \geq 2\Delta^n/n! \sqrt{3}$  ( $\Delta$ : width). Some remarks for the special case  $d = 3$  and  $W_0 = V$ :

Meissner constructed a convex body of constant width (“Gleichdick”). Bonnesen-Fenchel, p. 136, described it and conjectured that it has minimal  $V$  among all convex bodies of given constant width. In Hilbert-Cohn-Vossen we find a picture of Meissner’s Gleichdick. Another historical example is the “Reuleaux-Gleichdick” which is generated by rotation of the Reuleaux-triangle. E. Schulte (Dortmund) showed 1978 that the regular tetrahedron can be inscribed into a “Gleichdick” of same diameter and same symmetry-group. L. Danzer conjectured that among all convex bodies of same constant width a “Gleichdick” of the symmetry group of the tetrahedron minimizes  $V$ . Steinhagen [Abh. Math. Sem. Hamburg **1** (1922), 15–26] constructed a convex body which has for given width a smaller  $V$  than the above mentioned “Gleichdicke”. With a slight modification of Steinhagen’s idea we construct a convex body  $K$  of which we think that it minimizes  $V$  (and perhaps the surface area  $A$ ) among all convex bodies of given width:

Take the regular tetrahedron of edge-length 1 and replace each edge by a circular arc of radius  $\sqrt{2}/2$  and center in the middle of the opposite edge. Take the four points of distance  $\sqrt{2}/2$  to the facets of the tetrahedron which are on the line between a vertex and the center of the opposite facet of the tetrahedron. Let  $K$  be the convex hull of these four points and the six circular arcs. (It is difficult to show that  $\Delta(K) = \sqrt{2}/2$ .) We compare  $V$  and  $A$  for the above mentioned convex bodies and for fixed  $\Delta = \sqrt{2}/2$ :

Convex body with $\Delta = \sqrt{2}/2$	Volume $V$	Surface area $A$
$K$	0,1055...	1,4653...
Regular tetrahedron	0,1179...	1,7321...
Circular cone	0,1234...	1,5703...
Meissner’s Gleichdick	0,1484...	1,4671...
Reuleaux’s Gleichdick	0,1589...	1,4967...
Ball	0,1851...	1,5708...

Problem: Does  $K$  minimize  $V$  (and  $A$ ) among all convex bodies with  $\Delta = \sqrt{2}/2$ ? (Compare: E. Heil, preprint Darmstadt 1978).

**27.** A convex body  $K \subset \mathbb{R}^d$  is called reduced if for each convex body  $K' \subseteq K$  the width of  $K'$  is less than the width of  $K$ . Is it true that each strictly convex reduced body  $K$  is of constant width?

E. Heil

**28.** Let  $P$  be a polytope and  $P^*$  its polar polytope with respect to a fixed sphere centered at 0. If  $L, L^*$  are the edge length sums of  $P, P^*$ , what is  $\inf LL^*$ , the infimum taken over all  $P$  containing 0 or over a special class of polytopes? More general: Let  $\zeta_s, \zeta_s^*$  be the sums of the contents of all  $s$ -dimensional faces.

What is  $\inf_{s \in S} \zeta_s \zeta_s^*$ ? This is not solved even for  $s=d$ ,  $s=d-1$ , where  $\zeta_s$  is the volume resp. the surface area. (It is solved for  $d=2$ ,  $s=1,2$ ;  $d=3$ ,  $s=2$ .)

E. Heil

**29.** Let  $K$  be a convex body and  $H$  a hyperplane in  $\mathbb{R}^d$  such that  $H \cap K = \emptyset$ . Let  $H'$  be a hyperplane parallel to  $H$  such that  $K$  is contained in the strip defined by  $H$  and  $H'$ . Assume that for every point  $p$  of  $H$  the projection with center  $p$  of  $K$  into  $H'$  is centrally symmetric. Is it then true that  $K$  is an ellipsoid?

J. Höbinger

*Comment.* Examples of G.R. Burton and D.G. Larman [Geom. Dedicata 5 (1976), 31–42] show that even an enumerable set of such pairs of hyperplanes in general position do not ensure that  $K$  is an ellipsoid. Another open problem concerning a characterization of ellipsoids, which Höbinger posed in his lecture, has been settled by D.G. Larman [Mathematika 21 (1974), 216–227].

**30.** Suppose  $K$  is a compact set in  $\mathbb{R}^d$  such that for each positive  $\lambda > 1$ , the contraction  $\lambda K$  is an intersection of translates of  $K$ . Must  $K$  be convex? For  $d=1$  the answer is yes.

R. Jamison

*Comment.* As noticed by McMullen, the answer is no. An example is the region enclosed by a pentagram. McMullen has recently proved that the sets with the given property are just those compact star-shaped sets such that every point not in the set is visible from some point of the kernel.

**31.** Suppose  $K$  is a compact convex set in  $\mathbb{R}^d$ . Consider the family of all nonempty intersections of the form  $K \cap \cap \{K + t | t \in T\}$  ( $T \subset \mathbb{R}^d$  arbitrary). Is this family closed in the Hausdorff metric?

If we allow positive homotheties  $\mu K + t$  instead of just translates, for some choices of  $K$  the resulting family of intersections will not be closed.

R. Jamison

**32.** Relative Helly Numbers. Suppose  $F$  is a finite subset of  $\mathbb{R}^2$ . A set of the form  $C \cap F$  where  $C$  is convex can be regarded as a relative convex subset of  $F$ . The relative Helly number  $h(F)$  of  $F$  is the smallest integer for which Helly's theorem works for the relative convex sets in  $F$ . It can happen that  $h(F) = \text{card}(F) + 1$  – if  $F$  lies on a circle, say. This is because points of intersection that should be there are missing. Can one enlarge  $F$  to a finite set  $E$  so that  $h(E)$  is small, say  $h(E) \leq c$  where  $c$  is a constant independent of  $F$ ?

*Remarks.* (a) One cannot take  $c=3$  since any finite set  $E$  with  $h(E)=3$  can be easily shown to consist of a collinear set with possibly one point on each side of the line. I do not know if one can take  $c=4$ .

(b) If  $E$  is allowed to be countable, a simple inductive construction yields a positive answer with  $c=3$ . But this is avoiding the combinatorial nature of the problem.

(c) One can also ask similar questions in all dimensions and for Radon's and Tverberg's theorem as well.

R. Jamison

**33.** Some fractional Helly type problems. Let  $\mathfrak{A}$  be a family of  $n$ -sets. The family  $\mathfrak{A}$  is  $S(\beta)$ , for  $0 \leq \beta \leq 1$ , if there is a subfamily  $\mathfrak{B}$  of  $\mathfrak{A}$  such that  $\cap \mathfrak{B} \neq \emptyset$  and  $|\mathfrak{B}| \geq \beta \cdot n$ . The family  $\mathfrak{A}$  is  $E(c,r)$  for  $0 < c < 1$  and integer  $r$ , if  $n \geq r$  and the number of  $r$ -membered subfamilies  $\mathfrak{B}$  of  $\mathfrak{A}$  for which  $\cap \mathfrak{B} \neq \emptyset$  is at least  $\beta \cdot \binom{n}{r}$ .

It has been shown for each  $c > \frac{1}{2}$  there exists a  $\beta = \beta(c) > 0$  such that for any family  $\mathfrak{A}$  of rectangles in the plane with sides parallel to the axes: If  $\mathfrak{A}$  is  $E(c,2)$  then  $\mathfrak{A}$  is  $S(\beta)$ . Also,  $\beta \rightarrow 1$  as  $c \rightarrow 1$ . (For  $c = \frac{1}{2}$  there is no such  $\beta > 0$ , this is shown by taking half the members of  $\mathfrak{A}$  to be lines parallel to the  $x$ -axis and the other half to be lines parallel to the  $y$ -axis.) What happens in  $\mathbb{R}^3$ ?

*Conjecture 1.* If  $\frac{2}{3} < c < 1$  then there is a  $\gamma = \gamma(c) > 0$  such that for any family  $\mathfrak{A}$  of boxes with sides parallel to the axes in  $\mathbb{R}^3$ : If  $\mathfrak{A}$  is  $E(c,2)$  then  $\mathfrak{A}$  is  $S(\gamma)$ . ( $c = \frac{2}{3}$  will not give a positive  $\gamma$ ). The above mentioned result is based on a Turán-Erdős type result:

*Lemma:* For each  $\varepsilon > 0$  there is a  $\delta = \delta(\varepsilon) > 0$  such that any graph with  $n$  vertices and  $\frac{n^2}{2} (\frac{1}{2} + \varepsilon)$  edges contains at least  $\delta n^3$  triangles.

Prove or disprove the related

*Conjecture 2.* For each  $\varepsilon > 0$  there is an  $\alpha = \alpha(\varepsilon) > 0$  such that any graph with at least  $\frac{n^2}{2} (\frac{2}{3} + \varepsilon)$  edges contains at least  $\alpha \cdot n^4$  tetrahedrons. A proof of conjecture 2 will result in a proof of conjecture 1. Regarding convex sets and their complements there is the following

*Problem.* Find the largest  $q > 0$  such that any family  $\mathfrak{A}$  of  $n$  sets in  $\mathbb{R}^2$  each of which is either convex and compact or the complement of such a set, if  $\mathfrak{A}$  is  $E(1,3)$  then  $\mathfrak{A}$  is  $S(q)$ . A common sense argument (Danzer) and Helly's theorem show that for  $q = \frac{1}{2}$  one has a positive result.

M. Katchalski

*Comment.* Part of these conjectures has been solved in the affirmative by Katchalski.

**34.** Within the set of extreme points and extreme edges (the extreme 1-skeleton) of an  $n$ -dimensional convex body find a realisation of the complete  $n+1$  graph (the 1-skeleton of the  $n$ -simplex) by arcs. (For polytopes this was established by B. Grünbaum.)

D.G. Larman

**35.** Let  $\beta_d$  be defined as the least number such that any finite set  $X$  in  $E^d$  with  $|X| \geq \beta_d$  can be partitioned into two subsets  $A, B$  such that

$$\text{conv}(A \setminus x) \cap \text{conv}(B \setminus x) \neq \emptyset$$

for all  $x \in X$ . Show that  $B_d = 2d + 3$ .

This is known for  $d = 1, 2, 3$ ; in general there exists a constant  $c > 0$  such that

$$d + c\sqrt{d} \leq \beta_d \leq 2d + 3.$$

D.G. Larman

**36.** Let  $S$  be a centrally symmetric convex body in  $E^n$  with centre 0 and let  $1 < k < n$ . Suppose also

- (a) The extreme points  $\text{ext } S$  of  $S$  cannot be contained in a finite number of proper subspaces of  $E^n$ .  
 (b)  $\text{ext } S$  is the union of infinitely many  $k$ -dimensional ellipsoids, each with centre 0.  
 (c) There exists an infinite collection  $\{C_\beta\}_{\beta \in B}$  of  $k$ -elliptic cylinders, each with centre 0, which circumscribe  $S$  and which cannot be divided into a finite set of subcollections each with a common generator.  
 (d) each  $C_\alpha$  lies on the boundary of each  $C$ .

Does  $S$  exist? (If  $S$  does not exist then this would be helpful in the study of finite dimensional Banach space  $Y$  whose ball of operators  $(Y \rightarrow Y)$  is such that the extreme points form a semigroup.)

D.G. Larman

- 37.** Let  $K$  be a convex body in  $E^d$  and let  $F$  be a  $k$ -dimensional subspace. Let  $K(F)$  be the shadow boundary of  $K$  in direction  $F$  and let  $\hat{K}(F)$  be the orthogonal projection of  $K(F)$  in direction  $F$ . We say that  $K(F)$  is well behaved if

$$K(F) = N \cup \bigcup_{i=1}^{\infty} M_i$$

where  $N$  is a set of zero  $d - k - 1$  dimensional Hausdorff measure and on each  $M_i$ , there is a two sided Lipschitz condition between  $M_i$  and its projection in  $\hat{K}(F)$ .

*Conjecture.* Let  $K$  be a convex body in  $E^d$ . Then, for almost all  $k$ -dimensional subspaces  $F$ ,  $K(F)$  is well behaved.

D.G. Larman

- 38.** Say that a graph  $G$  with  $n$  vertices and no loops is 1-embeddable in  $E^d$  if there exists an embedding of the vertices of  $G$  into  $E^d$  such that two vertices are joined by an edge if and only if they are at distance greater than 1 apart. Let  $\dim G$  denote the smallest  $d$  such that  $G$  is 1-embeddable in  $E^d$ . Find

$$\max \{\dim G : v(G) = n\}.$$

D.G. Larman

- 39.** Let  $E^2$  be tiled by bounded convex polygons. Can this tiling be realized by the boundary complex of a convex body in  $E^3$  minus one vertex?

D.G. Larman

- 40.** If  $n + 2$  distinct points in  $E^n$  realize only two distances find all possible ratios between the two distances. I believe that T.T. Seidel has published some work on this.

D.G. Larman

- 41.** Prove the Borsuk conjecture in  $E^n$  for sets which only realise two distances.

D.G. Larman

- 42.** Let  $L$  be the set of points of the lattice  $\mathbb{Z}^d$  contained in a convex body  $K$ . Obviously for the volume  $V$  we have  $V(K) \geq V(\text{conv } L)$ . Find an upper bound for

$V(K)$ . One might conjecture that  $V(K) \leq (\cup W_i)$  where  $\{W_i\}$  is the set of all lattice cubes of sidelength 1 which contain at least one point of  $L$ . A necessary assumption for this conjecture to hold is that  $L$  is not contained in two parallel lattice planes.

J. Linhart

*Comment.* J.M. Wills remarks that the inequalities

$$V(K) < V(\text{conv}(K + C) \cap \mathbb{Z}^d), \quad V(K) < G(K + \frac{1}{2}C)$$

( $C := \{x : |x_i| \leq 1\}$ ,  $G :=$  number of lattice points) are easily obtained.

**43.** Suppose  $K$  is a convex body in  $\mathbb{R}^n$  and we let  $l_k(n)$  denote the average of all chords of  $K$  in the direction of the unit vector  $u$ . Hence we have  $l_k(u)\sigma_k(u) = V(K)$ , where  $\sigma_k(u)$  denotes the  $(n-1)$ -dimensional measure of the projection of  $K$  into the hyperplane orthogonal to  $u$ . Prove the following dual of the Bieberbach inequality:

$$\frac{\omega_{n-1}}{n\omega_n} \int_{\Omega} l_k(u) ds(u) \leq \omega_n^{\frac{n-1}{n}} V^{\frac{1}{n}}(K)$$

with equality if and only if  $K$  is an  $n$ -ball. Here  $\Omega$  denotes the boundary of the unit ball in  $\mathbb{R}^n$ ,  $\omega_n$  denotes the volume of the unit ball, and  $ds(u)$  denotes the  $(n-1)$ -dimensional volume element on  $\Omega$  at  $u$ .

E. Lutwak

**44.** Let  $n \geq 4$  be a positive integer, denote by  $A_n$  the set of combinatorial types of simplicial 3-polytopes with  $n$  vertices and by  $B_n \subset A_n$  the set of those types which arise from the simplex by repeated stellar subdivision. With  $\alpha_n := \text{card } A_n$ ,  $\beta_n := \text{card } B_n$ , I conjecture that  $\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = 0$ .

P. Mani

**45.** Let  $C \subset E^n$  be a convex body,  $\text{int}(C) \neq \emptyset$ . Is there a regular  $n$ -dimensional crosspolytope all of whose vertices lie in the boundary of  $C$ ?

P. Mani

*Comment.* Solutions in two particular cases have been obtained by R. Forneau and Ch. Leytem [Comment. Math. Univ. Carolinae 19 (1978), 151–164]. Leytem has written his thèse de license on this topic. See also Hadwiger, Larman, and Mani [J. Combinatorial Th. B 24 (1978), 290–293].

**46.** Let  $P$  be an  $n$ -dimensional convex polytope, and denote by  $\Gamma$  the group of all combinatorial automorphisms of  $P$ . Is  $\Gamma$  isomorphic to a subgroup of the rotation group  $O_n$ ?

P. Mani

**47.** Let  $C \subset E^n$  be a convex body and assume  $p \in \text{int}(C)$ . Suppose that for all hyperplanes  $H$  of  $E^n$  with  $p \in H$  we have  $p = (\frac{1}{2})(s(H^+ \cap C) + s(H^- \cap C))$ . Is  $C$

centrally symmetric, and  $p$  its centre? ( $H^+$  and  $H^-$  are the two halfspaces bounded by  $H$ , and  $s(X)$  is the Steinerpoint of  $X$ ).

### P. Mani

**48.** Hadwiger characterizes (non-negative) linear combinations of Quermaßintegrals of convex bodies in  $E^d$  as those functions  $f$  which are 1. additive:  $f(P \cup Q) + f(P \cap Q) = f(P) + f(Q)$ ,  $P, Q$ ,  $P \cup Q$  convex, 2. continuous w.r.t. the Hausdorff metric (monotone, respectively) 3. rigid motion invariant. Suppose we relax the last condition to 3'. translation invariant. Since every linear combination of mixed volumes satisfies the conditions, it is probably unreasonable to ask for a complete classification of such functions. Nevertheless, do they behave like mixed volumes, in the sense that they satisfy polynomial relations  $f(\lambda_1 P_1 + \dots + \lambda_r P_r) = \sum_{x_1, \dots, x_r} \lambda_1^{x_1} \dots \lambda_r^{x_r} f_{x_1, \dots, x_r}(P_1, \dots, P_r)$  where the functions  $f_{x_1, \dots, x_r}$  are independent of  $\lambda_1, \dots, \lambda_r$ ?

### P. McMullen

*Comment.* The theory developed by McMullen [Proc. London Math. Soc. **35** (1977), 113–135] yields this and even much more general results. Solutions by different methods have been obtained by W. Spiegel [Geom. Dedicata **7** (1978), 9–19] and Ch. Meier [Arch. Math. **29** (1977), 210–217]. Meier's paper contains a more general result on the decomposition of a Minkowski sum of polytopes.

**49.** Let  $K$  be (for simplicity) a proper pointed polyhedral cone in  $E^d$  with apex 0, and for faces  $F \leq G$  of  $K$  let  $\beta(F, G)$  and  $\gamma(F, G)$  be the internal and external angles of  $G$  at  $F$ . The functions

$$\varphi_r(K) = \sum_{r - \text{faces } F} \beta(0, F) \gamma(F, K)$$

play the rôle of the spherical Quermaßintegrals of  $K$ . The monotone linear combinations of the  $\varphi_r$ 's are the non-negative combinations of the functions

$$\psi_r = \varphi_r + \varphi_{r+2} + \dots,$$

which measure those  $(d - r + 1)$ -dimensional linear subspaces which meet  $\text{int } K$ . Is it true that any function defined on convex cones which satisfies the obvious analogues of 1), 2) and 3) of problem 48 is a linear combination of the  $\varphi_r$ 's (non-negative linear combination of the  $\psi_r$ 's, respectively)?

### P. McMullen

*Comment.* McMullen remarks that he can prove this, provided problem 74 of Schneider has an affirmative answer.

**50.** Let  $K$  be a convex body in  $E^3$ , and for each unit vector  $u$ , let  $p(u)$  be the perimeter of the orthogonal projection of  $K$  in direction  $u$ , and  $s(u)$  the length of the shadow boundary of the projection on  $K$ . (Is  $s(u)$  defined for all but a set of  $u$  of measure 0?) Let  $P(K) = \int p(u) du$ ,  $S(K) = \int s(u) du$ . Clearly  $S(K) \geq P(K)$ , and it can easily be shown that equality characterizes balls. For polyhedra, it can be proved that  $S(K) = \frac{4}{\pi} P(K)$ . Is it always true that  $1 \leq S(K)/P(K) \leq \frac{4}{\pi}$ ? One possible approach would be to show that  $S(K)$  is lower semi-continuous. If true, what

bodies attain the upper bound? (Other bodies than polyhedra can attain the bound).

### P. McMullen

**51.** Can one characterize continuous translation invariant valuations on the compact convex sets in  $E^d$ , as, for example, generalized (in some suitable sense) linear combinations of mixed volumes? This question may be easier to answer for monotone valuations. Such valuations  $\varphi$  have polynomial expansions

$$\varphi(\lambda P) = \sum_{r=0}^d \lambda^r \varphi_r(P) \quad \text{for } \lambda \geq 0$$

and  $\varphi_r$  is a homogeneous valuation of degree  $r$ . If  $\varphi$  is continuous, so is  $\varphi_r$ . If  $\varphi$  is monotone, is  $\varphi_r$  also? (For  $r=0,1,d$ , this is easy.)

### P. McMullen

*Comment* (by P. McMullen). Let  $\varphi$  be a continuous translation invariant valuation on  $\mathcal{K}^d$ , which is homogeneous of degree  $r$  [we use Proc. London Math. Soc. 35 (1977), 113–135]. If  $r=d-1$ , there are sequences  $(L_i), (M_i) \subseteq \mathcal{K}^d$ , with

$$\varphi(K) = \lim \{V(K, \dots, K, L_i) - V(K, \dots, K, M_i)\};$$

this generalizes a result of H. Hadwiger [Publ. Math. Debrecen 2 (1951), 81–94].

If  $r=1$ , and  $\varphi$  is uniformly continuous, there are  $L, M \in \mathcal{K}^d$ , with

$$\varphi(K) = V(K, L, \dots, L) - V(K, M, \dots, M);$$

moreover, such  $\varphi$  are always uniformly continuous. Monotone valuations of degree 1 are uniformly continuous, so we have a variant of the result of W.J. Firey [Israel J. Math. 24 (1976), 274–281]. The case of non-uniformly continuous valuations raises an interesting and apparently very deep problem: Let  $C$  be the closed cone of support functions (restricted to  $S^{d-1}$ ). Then  $cl(C-C) = \mathcal{C}(S^{d-1})$ , with the usual sup norm. The Riesz representation theorem describes the continuous linear functionals on  $\mathcal{C}(S^{d-1})$ ; these correspond to the uniformly continuous valuations of degree 1. Is there a “Riesz representation theorem” for those linear functionals which are merely continuous on  $C$ ? There may even be useful generalizations of this problem to other familiar Banach spaces.

**52.** The lattice point enumerator  $G$  ( $G(P) := \text{card}(P \cap \mathbb{Z}^d)$ ) has a polynomial expansion

$$G(nP) = \sum_{r=0}^d n^r G_r(P)$$

for lattice polytopes  $P$  and non-negative integers  $n$ . It is not hard to see that  $G_r$  cannot be monotone, even though  $G$  is. Is  $G_r$  non-negative?

### P. McMullen

*Comment.* A negative answer and further results on the functions  $G_r$  are given by J. Wills (Comment. Math. Helv. 53 (1978), 508–524). This paper also contains a

counterexample to the conjecture  $G_r \leq V_r$  for lattice polytopes, which was proposed by McMullen [Math. Proc. Cambridge Philos. Soc. **78** (1975), 247–261].

**53.** If  $P$  is a  $d$ -polytope, let  $L_k(P)$  be the sum of the  $k$ -volumes of the  $k$ -faces of  $P$ , and  $V_k(P)$  the intrinsic  $k$ -volume of  $P$  (normalized  $(d-k)$ -th quermassintegral).  $V_k/L_k$  is thus an “average” external angle at the  $k$ -faces of  $P$ . The example of the cartesian product of a “small  $(d-k)$ -simplex and a “large”  $k$ -polytope shows that  $V_k/L_k$  can be as near to  $(d-k+1)^{-1}$  as we wish. Is this the sharp upper bound?

P. McMullen

*Comment.* An affirmative answer was given by Wm.J. Firey and R. Schneider (Geom. Dedicata **8** (1979), 99–103). The inequality was also obtained by J. Linhart, who gives an application in Arch. Math. **29** (1977), 558–560. The inequality  $L_k > (d-k+1)V_k$  holds for general convex bodies, if  $L_k$  denotes the  $k$ -dimensional Hausdorff measure of the  $k$ -skeleton. This was proved by R. Schneider [Bull. London Math. Soc. **10** (1978), 84–85].

**54.** Is there a natural and useful (isometry) invariant measure on the space  $\mathcal{K}^d$  of compact convex sets in  $E^d$  (with the Hausdorff metric)? One would want to say (for example) that almost all bodies are non-smooth. Perhaps more easily treated is the compact metric space of affine equivalence classes of compact convex sets. A suitable measure on this space would be almost as useful.

P. McMullen

*Comment.* One might think of Hausdorff measures, but for these the following can be shown. Let  $h$  be a Hausdorff function, and denote by  $\mu^h$  the corresponding Hausdorff measure on  $\mathcal{K}^d$ . Then either  $\mu^h(\mathcal{K}^d) = 0$ , or for some convex body  $A$  the set of convex bodies contained in  $A$  is of non- $\sigma$ -finite measure. There arises the new problem to determine the generalized Besicovitch dimension [in the sense of Larman, Proc. London Math. Soc. (3) **17** (1967), 178–192] of  $\mathcal{K}^d$  and of interesting subsets.

Results on “almost all” convex bodies have been obtained, not in the sense of measure, but in the sense of category. Every convex body in  $\mathcal{K}^d$ , with the exception of those in a certain  $F_\sigma$  set of first category in  $\mathcal{K}^d$ , is smooth and strictly convex. This has been proved by V. Klee [Math. Ann. **139** (1959), 51–63] and P. Gruber [Math. Ann. **229** (1977), 259–266]. In the latter paper it is also proved that the set of all convex bodies with a boundary of class  $C^2$  is of first category. Further category results have been obtained (independently) by Zamfirescu, Kenderov, and Schneider [Math. Ann. **240** (1979), 177–181].

We should like to enclose the following remark of P. McMullen: But the original questions remain unanswered. There are actually Borel regular metric measures on the space  $\mathcal{K}_0^d$  of compact convex sets in  $E^d$  with mean width 2, and Steiner point at the origin; these would have no atoms, assign positive measure to non-empty open sets, and measure 1 to  $\mathcal{K}_0^d$  itself. (For this, see Rogers, Hausdorff measures, p. 68.) Rogers leads me to understand that the measures could even be rotation invariant. But it is still not clear that such measures can be “nice”; for example, would they assign measure 0 to the family  $\mathcal{P}_0^d$  of polytopes in  $\mathcal{K}_0^d$ ?

**55.** A convex body  $K$  which tiles  $E^d$  by translates has the following properties.  
 a)  $K$  is a polytope with centrally symmetric facets (and so is centrally symmetric – we exclude  $d=2$  as trivial). Hence the facets of  $K$  occur in “zones”, with facet normals in a 2-plane, and with adjacent facets sharing a  $(d-2)$ -face. (This is due to H. Groemer.)

b) Each zone contains 4 or 6 facets.

If  $K$  satisfies (a) and (b), it may be conjectured that  $K$  tiles  $E^d$ . This is true if  $K$  is a zonotope (vector sum of line segments).

P. McMullen

*Comment.* Compare problem 10 of Danzer.

**56.** The graph of a  $d$ -polytope  $P$  contains a refinement of the complete graph on  $d+1$  vertices; moreover, any vertex of  $P$  can be preassigned as a principal vertex, corresponding to a vertex of the complete graph. Can any two vertices of  $P$  be preassigned as principal vertices? For  $d=3$ , three vertices can be preassigned, but not four; for  $d \geq 4$  even three cannot be preassigned (S. Gallivan and E.R. Lockeberg, unpublished.)

P. McMullen

**57.** Let  $P$  be a  $d$ -polytope, and  $B(P)$  its boundary complex. A strong  $k$ -chain in  $B(P)$  is a sequence  $C = F_0 \cup F_1 \cup \dots \cup F_n$  of  $k$ -faces of  $P$ , such that  $F_{i-1} \cap F_i$  is a  $(k-1)$ -face ( $i=1, \dots, n$ ). E.R. Lockeberg has conjectured: Let  $d_1, \dots, d_r$  be positive integers with sum  $d$ . Then if  $P$  is any  $d$ -polytope and  $x, y \in \text{vert } P$ , then there are  $d_i$ -chains  $C_i$  in  $B(P)$ , with  $C_i \cap C_j = \{x, y\}$  for  $i \neq j$ . For all  $d_i = 1$  this is Balinski's theorem; for  $d_1 = 1$ ,  $d_2 = d - 1$ , this is (I believe) an old conjecture.

P. McMullen

**58.** Let  $K_0^d$  be the space of compact convex sets in  $E^d$ , with Steiner point at 0, and mean width  $\alpha_d$ , the mean width of the line segment of length 2. Is the diameter of  $K_0^d$  (in the Hausdorff metric) equal to 1? Equivalently, if  $K \in K_0^d$ , does  $K$  lie in the unit ball?

P. McMullen

**59.** Let  $K$  be a convex body in  $E^d$ , and let  $a, b$  be two points such that parallel chords of  $K$  through  $a, b$  have the same length. Is  $K$  centrally symmetric, with centre  $\frac{1}{2}(a+b)$ ? (It is enough to consider  $d=2$ .)

P. McMullen

**60.** Let  $K$  be a convex body of  $E^d$  which admits a dissection into several bodies congruent to one another and similar to  $K$ . Is it true that, if  $d \geq 3$ , then  $K$  is combinatorially equivalent to a product of simplices?

P. McMullen and G. Valette

**61.** Let

$$h(A, \varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi)$$

be the support function of a plane convex body A. One can establish the estimates

$$|a_k|, |b_k| < \frac{c_1 \ln k + c_2}{1+k^2} \max_{\varphi} \{|h(A, \varphi)|\},$$

where  $c_1, c_2$  are constants independent of A. These allow us to embed the cone of support functions with the Blaschke-Hausdorff-metric via a linear homeomorphism into the Hilbert space

$$H_\alpha = \left\{ f(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi), \right. \\ \left. \sum_{k=0}^{\infty} (1+k^2)^{1+\alpha} (a_k^2 + b_k^2) < \infty \right\} (0 < \alpha < \frac{1}{2}).$$

*Question.* Does there also exist an isomorphic and homeomorphic embedding of the cone of support functions into a Hilbert space for dimensions  $d > 2$ ?

Chr. Meier

**62.** A 3-pseudomanifold ( $3 - pm$ ) is a finite connected simplicial 3-complex K such that every 2-cell in K belongs to precisely two 3-cells of K, the link of every 1-cell is a circuit, and the link of every vertex in K is a closed 2-manifold. Does there exist a  $3 - pm$  K such that the link of no vertex in K is a sphere?

U. Pachner

*Comment.* Altshuler (Trans. Amer. Math. Soc. **241** (1978), 213–237) proves that for every finite set  $\Sigma$  of closed 2-manifolds, there exists a  $3 - pm$  K such that the set of all the (topological) links of vertices of K is precisely  $\Sigma$ .

**63.** Let K, L be two centrally symmetric convex polytopes. Does every combinatorial equivalence  $\varphi: F(K) \rightarrow F(L)$  necessarily take opposite vertices of K to opposite vertices of L?

M.A. Perles

**64.** Let  $K \subset \mathbb{R}^d$  be a convex body. A subset  $S \subset \text{int } K$  is called a lattice with respect to the Hilbert geometry on  $\text{int } K$  if S has an empty derived set and the line through any two points of S intersects S in a one-dimensional lattice, i.e. a set isometric to a lattice in  $\mathbb{R}$ . Which convex bodies  $K \subset \mathbb{R}^d$  possess a d-dimensional lattice on  $\text{int } K$  with respect to the Hilbert geometry on  $\text{int } K$ ?

C.M. Petty

**65.** *Conjecture.* Each set of 15 points in general position in  $\mathbb{R}^4$  can be partitioned into 3 subsets, each with 5 points, so that the simplices thus formed have a common interior point.

J.R. Reay

**66.** When are Tverberg partitions unique? When does every Tverberg partition give a unique single point as the intersection of the convex hulls of the partition?

J.R. Reay

**67.** For  $M \subset \mathbb{R}^d$  let  $c(M) := \inf \{\lambda \geq 0 \mid M + \lambda \text{ conv } M \text{ is convex}\}$ . It is known that  $0 \leq c(M) \leq d$  for arbitrary  $M \subset \mathbb{R}^d$  and  $d^{-1} \leq c(\text{bd } K) \leq 1$  for every convex body  $K \subset \mathbb{R}^d$ . Let  $k \in \{0, 1, \dots, d-2\}$  and let  $\mathcal{P}^d$  be the set of all convex polytopes in  $\mathbb{R}^d$ . Find sharp bounds for  $c(\text{skel}_k P)$ , where  $P \in \mathcal{P}^d$  and  $\text{skel}_k P$  denotes the union of all the  $k$ -dimensional faces of  $P$ .

R. Schneider

**68.** Let  $C^d$  denote the set of all compact subsets of  $E^d$ . By a “measure of convexity” on  $C^d$  we understand a real-valued function on  $C^d$  which attains one extreme value exactly on the set of convex bodies and which is invariant under nonsingular affine transformations of  $E^d$ . Find “geometrically reasonable” measures of convexity and determine those sets (if they exist) which, in the sense of this measure of convexity, are most remote from being convex. Are there measures of convexity which are continuous with respect to Hausdorff metric? For a given measure  $f$  of convexity, determine sharp bounds for  $f(\text{bd } K)$ , where  $K$  runs through the convex bodies of  $\mathbb{R}^d$ .

R. Schneider

**69.** Let  $(C^d, \rho)$  denote the metric space consisting of the set  $C^d$  of nonempty compact subsets of Euclidean space  $E^d$  together with the Hausdorff metric  $\rho$  defined by

$$\rho(K, L) = \max \left\{ \sup_{x \in K} \inf_{y \in L} \|x - y\|, \sup_{x \in L} \inf_{y \in K} \|x - y\| \right\}.$$

*Conjecture.* Every isometric map of  $(C^d, \rho)$  into itself is induced by an isometry of  $E^d$ .

R. Schneider

*Comment.* Under the additional assumption of surjectivity the conjecture has been proved by P. Gruber (Geom. Dedicata, to appear). An analogous result for  $\mathcal{K}^d$  but with the symmetric difference metric instead of  $\rho$  was given by P. Gruber (Mathematika 25 (1978), 270–278).

**70.** The following is true: If  $K, \bar{K}$  are  $d$ -dimensional convex bodies in  $E^d$  satisfying

$$V(K, \dots, K, L) = V(\bar{K}, \dots, \bar{K}, L)$$

for all two-dimensional convex bodies  $L$  in  $E^d$ , then  $K$  is a translate of  $\bar{K}$ . (In fact, it suffices to assume the above equality merely for those  $L$  which are congruent to a fixed triangle  $T$ , provided that at least one angle of  $T$  is an irrational multiple of  $\pi$ .)

I have an analytical proof (using spherical harmonics). Is there a simple geometric proof?

R. Schneider

**71.** Let  $K$  be a smooth convex body in  $E^d$ . *Conjecture.*  $K$  is an ellipsoid if and only if each nonempty intersection of  $K$  with a translate of  $K$  is a summand of  $K$ .

R. Schneider

**72.** A zonoid is a convex body in  $\mathbb{R}^d$  which can be approximated by finite sums of segments. Let

$$\lambda(d) = \sup_{\mathcal{Z}} \inf \{\lambda > 0 \mid E \subset Z \subset \lambda E \text{ for some ellipsoid } E \text{ with centre 0}\},$$

where  $Z$  runs through all those zonoids with centre 0 whose polar reciprocal is also a zonoid. It is known that  $\lambda(d) > 0$  for  $d \geq 2$  [Proc. Amer. Math. Soc. **50** (1975), 365–368]. Is it true that  $\lambda(d) \rightarrow 0$  for  $d \rightarrow \infty$ ?

R. Schneider

**73.** The  $r$ -skeleton, denoted by  $\text{skel}_r K$ , of a convex body  $K$  in  $E^d$  is the set of all points in  $K$  which are not the centre of a  $(k+1)$ -dimensional ball contained in  $K$ . For  $1 \leq r \leq s \leq d$ , let

$$c(d,r,s) = \inf_K \mathcal{H}^r(\text{skel}_r K)^{\frac{1}{r}} \mathcal{H}^s(\text{skel}_s K)^{-\frac{1}{s}}$$

where  $\mathcal{H}^r$  denotes  $r$ -dimensional Hausdorff measure in  $E^d$ , and where  $K$  runs through the convex bodies  $K$  with  $\mathcal{H}^s(\text{skel}_s K) < \infty$ . Is  $c(d,r,s) > 0$ ? This is true for  $c(d,d-1,d)$  (by the classical isoperimetric inequality) and for  $c(d,r,d-1)$  (W.J. Firey and R. Schneider).

Define  $c^*(d,r,s)$  analogously, but with  $K$  running only through the convex polytopes. Is  $c(d,r,s) = c^*(d,r,s)$ ? Even the inequality  $c^*(d,r,s) > 0$  is only known for special values of  $r$  and  $s$  [Eggleston, Grünbaum, Klee: Comment. Math. Helvet. **39** (1964), 165–188].

R. Schneider

**74.** Let  $\mathfrak{G}^d$  be the set of spherically convex polytopes in the unit sphere of  $E^d$ . Let  $\varphi$  be a real-valued function on  $\mathfrak{G}^d$  which is (1) rotation-invariant, (2) simply additive (i.e.,  $\varphi(P \cup Q) = \varphi(P) + \varphi(Q)$ , if  $P, Q, P \cup Q \in \mathfrak{G}^d$  and  $\text{int}(P \cap Q) = \emptyset$ ), (3) continuous (with respect to the Hausdorff metric based on spherical distance). Is  $\varphi$  a constant multiple of the spherical volume?

R. Schneider

**75.** Determine all convex polytopes  $P$  with the property that every polytope combinatorially equivalent to  $P$  admits a circumscribed ellipsoid.

R. Schneider

**76.** Eggleston, Grünbaum, and Klee [Comment. Math. Helvet. **39** (1964), 165–188] defined

$$\zeta_s(K) := \liminf_{\substack{P \rightarrow K \\ P \in \mathcal{P}^d}} \mathcal{H}^s(\text{skel}_s P)$$

for  $s = 0, \dots, d-1$ , where  $K \subset E^d$  is a convex body,  $\mathcal{P}^d$  is the set of convex polytopes in  $E^d$ ,  $\mathcal{H}^s$  denotes  $s$ -dimensional Hausdorff measure, and  $\text{skel}_s K$  is the set of all points in  $K$  which are not centres of  $(k+1)$ -dimensional balls in  $K$ . Is

$$\zeta_s(K) = \mathcal{H}^s(\text{skel}_s K) \quad \text{for } 0 < s < d-1?$$

R. Schneider

**77.** Let  $K \subset E^3$  be a convex body with the property that almost every 2-plane (in the sense of the invariant measure on the space of 2-planes) intersects  $K$  in a polygon. Describe  $K$  geometrically. In particular, is the set of extreme points of  $K$  countable? If not, is it of one-dimensional Hausdorff measure zero? The problem generalizes to  $d$ -dimensional space and  $k$ -dimensional planes.

R. Schneider

**78.** Supplement to problem 50. Determine the convex bodies in  $E^3$  for which  $s$  is constant.

R. Schneider

**79.** In my talk at the conference I explained the concept of a “net” of a 3-polytope  $P$  (3-dimensional convex polytope). Each net corresponds to a maximal tree of cut-edges in the edge-graph of  $P$ . On the other hand, not every maximal tree corresponds to a net—for if we “cut”  $P$  along such a set of edges and “open out” the resulting surface, it may not be possible to embed it in the plane. Intuitively, one part of the resulting surface may overlap another part. We refer to this situation as an overlapping or improper net. The problem is to prove that every 3-polytope possesses a net that is not improper.

It is not difficult to show that there exist 3-polytopes which possess proper nets but do not possess proper Hamiltonian nets.

In posing this problem I claim no originality; in fact I believe that it has been discussed for several years, but no solution has been found.

G.C. Shephard

**80.** Is there a possibility of characterizing quermassintegrals of convex polytopes without using continuity or monotony assumptions?

W. Spiegel

**81.**  $K_0$  and  $K_1$  are arbitrary convex bodies in  $E^d$ . Of all affine images of  $K_0$  which are contained in  $K_1$ ,  $K_0$  has the largest volume. I can show that, for each direction  $\mu$ , the width of  $K_0$  in direction  $u$  is at least  $\frac{1}{d}$  times the width of  $K_1$  in direction  $u$ . One can symmetrize to obtain an estimate for the ratio of the volumes of  $K_0$  and  $K_1$ , but this is an inefficient procedure. I conjecture that

$$\frac{\text{vol}(K_0)}{\text{vol}(K_1)} \geq \frac{1}{d^d}.$$

Further conjecture:  $-\frac{1}{d}K_1 \subseteq K_0$  (translated).

R.H.K. Thomas

**82.** The floating body problem (Ulam: “A collection of Mathematical Problems”, p. 38). I have found that very few people have heard of this old problem.  $K$  is a convex body of density  $< 1$  such that, when it is floating in water, it will float in any orientation, i.e. “neutral equilibrium”. Is  $K$  a ball?

R.H.K. Thomas

**83.** Determine or estimate the function  $f(p,q;d)$  such that one has: If  $P \subset Q$  are

convex sets in  $\mathbb{R}^d$ , with volumes  $p,q$ , then  $Q \setminus P$  contains a convex body of volume  $f(p,q;d)$ .

H. Tverberg

**84.** Is it true that whenever  $P$  is a convex  $(r-1)(d+1)$ -polytope with non-empty interior, and  $f$  is a continuous map:  $\text{bd } P \rightarrow \mathbb{R}^d$ , then there are  $r$  disjoint faces  $F_1, \dots, F_r$ , such that  $\cap f(F_i) \neq \emptyset$ ? The case  $r=2$ , which generalizes Radon's theorem, has been proved by Bajmoczy and Bárány (Budapest, still not published).

H. Tverberg

**85.** Let  $K$  be a convex body of  $E^d$  ( $d \geq 2$ ) which admits a dissection into several bodies affinely equivalent to  $K$ . Is it true that  $K$  is either a polytope or affinely equivalent with the product of a convex body and a polytope with more than one point?

G. Valette

**86.** The nerve of a finite packing of circles in the Euclidean plane is a graph, since three smooth convex bodies in  $E^2$ , whose interiors are pairwise disjoint, cannot have a point in common. Furthermore it is a planar graph: You obtain a plane straight-line realization by taking the centers of the circles as vertices and connecting segments, if the corresponding circles touch, as edges. Conversely I have the

*Conjecture.* Each finite planar graph is the nerve of some packing of circles in the plane.

G. Wegner

*Comment.* An affirmative answer to this question has been announced by W.P. Thurston at the ICM 1978 in Helsinki.

**87.** Let  $\mathfrak{K}$  denote the set of all convex bodies in  $E^n$  which are centrally symmetric with respect to the origin. Let  $P: \mathfrak{K} \rightarrow \mathfrak{K}$  denote the projection body map, i.e., for  $K \in \mathfrak{K}$  let  $PK$  be the convex body with support function  $H(u)$  at  $u \in E^n$ ,  $\|u\|=1$ , equal to the brightness of  $K$  in direction  $u$ . Now iterate this process:

$$P^k = \underbrace{P \circ \dots \circ P}_k, k = 1, 2, \dots$$

*Questions:*

- (a) How does  $P^\infty \mathfrak{K} := \cap_{k=1}^\infty P^k \mathfrak{K}$  look like?
  - (b) What can be said about the accumulation points (with respect to the Hausdorff metric) of the sequence  $K, PK, P^2 K, \dots$ , (after a suitable normalization)?
  - (c) Which  $K \in \mathfrak{K}$  satisfy  $PK = \lambda K$  with  $\lambda \geq 0$ ?
  - (d) Which relations exist between the Hausdorff distances  $d(K, L)$  for  $K, L \in \mathfrak{K}$  and  $d(PK, PL)$ ?
- For the set of polytopes in  $\mathfrak{K}$  the questions corresponding to (a) and (c) have been answered.

W. Weil

**88.** Differences of support functions. Let  $H$  be the cone of support functions of convex bodies in  $E^d$  restricted to the unit sphere  $\Omega$ , and let  $L$  be the vector space generated by  $H$ .

(a) How can one characterize the subspace  $L$  of  $C(\Omega)$ ? (More general: Which continuous functions on an open convex subset of  $E^d$  are differences of convex functions?)

(b) Find a natural topology on  $L$  such that:

- the embeddings  $C^2(\Omega) \rightarrow L \rightarrow C(\Omega)$  are continuous,
- $L$  is complete,
- on  $H$  the topology is the sup-norm-topology,
- the mixed volume (multilinearly extended to  $L$ ) is continuous in the product topology.

W. Weil

**89.** Zonal characterization of zonoids. Let  $H_K$  be the support function of a convex body  $K$  with the following property: For each  $u \in \Omega$  (unit sphere) there is a  $0 < \alpha < 1$  and a zonoid  $Z(u)$  with support function  $H_{Z(u)}$  such that  $H_K(v) = H_{Z(u)}(v)$  for all  $v \in \{w \in \Omega \mid \langle w, u \rangle < \alpha\}$ . Is  $K$  a zonoid?

W. Weil

**90.** Borel sets in  $E_q^d$ . Let  $E_q^d$  be the space of all  $q$ -planes in  $E^d$  with the usual topology,  $q \in \{1, \dots, d-1\}$ . Let  $K$  be a convex body in  $E^d$ ,  $\beta$  a Borel set in the boundary of  $K$ ,  $\omega$  a Borel set in the unit sphere. Let  $B(\beta)$  be the set of all  $q$ -planes  $E \in E_q^d$  which touch  $K$  such that  $E \cap \beta \neq \emptyset$ , and let  $A(\omega)$  be the set of all  $q$ -planes  $E$  which touch  $K$  such that  $K \cap E$  has an outer unit normal in  $\omega$ . Are  $A(\omega)$  and  $B(\beta)$  Borel sets in  $E_q^d$ ?

W. Weil

*Comment.* Counterexamples have been constructed by G.R. Burton (to appear).

**91.** For each lattice polytope  $P \subset E^d$  with  $\dim P = d$  let  $G(P) := \text{card}(P \cap \mathbb{Z}^d)$ ,  $V(P)$  the volume and  $F(P)$  the surface area of  $P$ .

*Conjecture:*

$$V(P) - \left( \frac{1}{2} - \frac{1}{d} \right) F(P) < G(P).$$

This is a refinement of a result of H. Hadwiger [Montash. Math. 76 (1972), 410–418, Math. Z. 127 (1972), 363–364] for the case of lattice polytopes.

J.M. Wills

**92.** Let  $W_i$  denote Minkowski's quermassintegrals. Consider  $f_{ij} := W_i/W_j$  for  $i < j$ . In contrast to the  $W_i$ 's the  $f_{ij}$ 's are neither additive nor monotone. What are the properties of the  $f_{ij}$ 's which correspond to the additivity and monotony of the  $W_i$ 's? *Conjecture:* Perhaps the  $f_{ij}$  satisfy a sort of "pseudomonotony". A result of this sort is the inequality

$$f_{01}(K) < df_{01}(K') \text{ for convex bodies } K, K' \text{ in } E^d \text{ with } K \subset K'$$

[cf. Arch. Math. **21** (1970), 557–560].

J.M. Wills

**93.** For a convex set  $K \subset E^d$  let  $G(K) := \text{card}(K \cap \mathbb{Z}^d)$  be its lattice point number. Minkowski proved for centrally symmetric convex bodies the following two theorems (in a slightly modified formulation):

- (1)  $G(\text{int } K) = 1$  and  $K$  strictly convex implies  $G(K) \leq 2^{d+1} - 1$ ,
- (2)  $G(\text{int conv}(K \cap \mathbb{Z}^d)) = 1$  implies  $G(K) \leq 3^d$ .

What are the analogues 1'), 2') for general convex bodies?

*Conjecture.* 2')  $G(\text{int conv}(K \cap \mathbb{Z}^d)) = 1$  implies  $G(K) \leq \binom{2d+1}{d}$ . For  $d = 2$  this was proved by Scott.

J.M. Wills

**94.** Does there exist a finite family of  $d$ -polytopes  $\{P_1, P_2, \dots, P_n\}$ , possibly neighborly, in  $E^d$ , such that  $P_1, \dots, P_n$  have mutually disjoint interiors, yet for every  $i$ ,  $1 \leq i \leq n$ , and every facet  $F$  of  $P_i$ , there exist a  $j$  so that  $P_i \cap P_j$  is a  $(d-1)$ -dimensional subset of  $F$ ? Equivalently, can the family be such that no point of  $E^d \setminus \bigcup_{i=1}^n P_i$  sees some facet of some polytope via  $E^d \setminus \bigcup_{i=1}^n P_i$ ?

J. Zaks

**95.** Find, for  $d \geq 4$ , a neighborly family of  $2^d$   $d$ -simplices in  $E^d$ . This will prove  $f(d, d+1) \geq 2^d$ , where equality has been repeatedly conjectured (Bagmihl, Baston, Klee).

J. Zaks

**96.** Is it true that every simple 3-polytope with even faces only contains a circuit which meets every face? Is the shortness exponent of the family of these polytopes less than one?

J. Zaks

**97.** Is every simple 3-polytope Hamiltonian provided it has only two types of faces,  $a$ -gons and  $b$ -gons, where  $\{a; b\} \neq \{5; 8\}$ ?

J. Zaks

**98.** Is it true that if  $P$  is a simple 3-polytope, then the polytope  $P'$ , obtained from  $P$  by “cutting all edges” (i.e., by replacing each edge by a hexagon), is Hamiltonian?

J. Zaks

**99.** Let  $Q$  be the union of finitely many rational polytopes (with vertices at rational points) in  $E^d$ , and let  $C$  be a cell complex which subdivides  $Q$  (i.e.  $|C| = Q$ ), such that all the members of  $C$  are polytopes having each an equivalent rational polytope. Does there exist a cell complex  $C'$ , combinatorially equivalent to  $C$ , such that  $C'$  subdivides  $Q$  and each member of  $C'$  is a rational polytope? The case where  $C$  is a triangulation has been done by Beynon (Canad. J. Math., recently), and the case where  $C$  is a cell complex containing simplicial polytopes only follows easily from it.

J. Zaks

**100.** Let  $f: E^n \rightarrow E^m$ ,  $2 \leq n < m$ , be a function (not necessarily continuous) satisfying the following “distance 1 preserving” property:  $\forall x, y \in E^n$ ,  $d(x, y) = 1$  implies  $d(f(x), f(y)) = 1$ . Does it follow that  $f$  is an isometry (onto its image)? The corresponding case  $n = m = 1$  is false, and the case  $n = m \geq 2$  has been shown by Bedly (Ph.D. Thesis, Mich. St. Univ., 1977, supervised by L.M. Kelly). Even the first case of  $m = 2$ ,  $m = 3$  is open.

J. Zaks

*Comment.* For  $n = m$  see also Kuz'minyh (Soviet Math. Dokl. **17** (1976), 43–45) and the references given there.

**101.** Let  $M$  be a space and  $\{T_i\}_{i \in I}$  a family of subsets of  $M$ , each of which possesses a topology  $\tau_i$ . Study (for various  $M$  and  $\{T_i\}_{i \in I}$ ) the sets  $S$  of  $M$  which are in the following sense “convex”: For each  $i \in I$ ,  $T_i \cap S$  is either connected (with respect to  $\tau_i$ ), or empty. In particular, characterize for instance the “convex” sets in  $M = \mathbb{R}^n$  if  $\{T_i\}_{i \in I}$  is the family of all  $d$ -dimensional flats in  $\mathbb{R}^n$  ( $1 < d < n$ ). (For  $d = 1$ , “convex” means convex!)

T. Zamfirescu

**102.** Find a connected graph  $G$  such that for each 3 vertices of  $G$  there exists a longest path avoiding these 3 vertices. (Examples with 12, resp. 93, vertices are known in case 3 is replaced by 1, resp. 2.)

T. Zamfirescu

**103.** Find a 4-connected graph  $G$  such that for each vertex of  $G$  there exists a longest path (circuit) avoiding this vertex. (Examples with 12, 32, resp. 36, vertices are known in the case that 4-connectivity is replaced by 1-, 2-, resp. 3-connectivity.)

T. Zamfirescu

**104.** Let  $\mathfrak{F}$  be a class of convex bodies in  $\mathbb{R}^n$ . We say that  $M \subset \mathbb{R}^n$  is  $\mathfrak{F}$ -convex if for each two points  $x, y \in M$ , there exists  $F \in \mathfrak{F}$  such that  $x, y \in F$  and  $F \subset M$ . Let  $\mathfrak{K}_{\mathfrak{F}}$  be the family of all  $\mathfrak{F}$ -convex sets in  $\mathbb{R}^n$ . Of course,  $\mathfrak{F} \subset \mathfrak{K}_{\mathfrak{F}}$ . Study the  $\mathfrak{F}$ -convexity for remarkable classes  $\mathfrak{F}$ . Characterize the (or find large families of) classes  $\mathfrak{F}$  with  $\mathfrak{F} = \mathfrak{K}_{\mathfrak{F}}$ .

T. Zamfirescu

*Comment.* The case where  $\mathfrak{F}$  is the class of rectangles in  $\mathbb{R}^2$  has been investigated by R. Blind, G. Valette and T. Zamfirescu (to appear).

**105.** Let  $M \subset \mathbb{R}^2$ ,  $0 < \alpha < 1$ , and  $f: M \rightarrow M$  be such that for each  $x, y, z \in M$ ,

$$A(f(x), f(y), f(z)) \leq \alpha A(x, y, z),$$

where  $A(a, b, c)$  means the area of the triangle abc. Let  $L(u)$  be the set of limit points of  $\{f^n(u)\}_{n=1}^\infty$  and  $\mathcal{L} = \bigcup_{u \in M} L(u)$ . Prove that  $\mathcal{L}$  is included in a line. (The assertion is known to be true for bounded  $M$ .)

T. Zamfirescu

*Notes added in proof (July 1979)*

Recently the following contributions to some of the problems have come to our knowledge.

**Problem 4.** For a related result, see

G. Bielig and Chr. Schulz, Hamiltonian circuits on simple 3-polytopes and shortness exponents. (to appear)

**Problem 21.** The conjecture has been proved by G. Lettl (Dissertation, TU Wien); a simplified proof was presented by P.M. Gruber at the 2<sup>nd</sup> Siegen Geometry Symposium, July 1979.

**Problem 30.** The result mentioned in the comment appears in  
P. McMullen, Sets homothetic to intersections of their translates. *Mathematika* 25 (1978), 264–269.

**Problem 55.** The conjecture was proved by

P. McMullen, Convex polytopes which tile space by translation (to appear). This result implies a negative answer to problem 10 and, therefore, a positive solution to problem 11.

**Problem 56.** The result mentioned in the problem appear in  
S. Gallivan, E.R. Lockeberg, and P. McMullen, Complete subgraphs of the graphs of convex polytopes (to appear).

**Problem 59.** An affirmative answer was obtained by

C.A. Rogers, An equi-chordal problem. *Geom. Dedicata* (to appear), and a generalization to higher dimensions by

D.G. Larman and N.K. Tamvakis, A characterization of centrally symmetric convex bodies in  $E^n$ . *Geom. Dedicata* (to appear).

**Problem 69.** The conjecture has been proved by P.M. Gruber and G. Lettl.

**Problem 76.** In

G.R. Burton, The measure of the s-skeleton of a convex body (to appear), it is proved that  $\zeta_s(K) \geq \mathcal{H}^s(\text{skel}_s K)$ . This paper contains further results on  $\mathcal{H}^s(\text{skel}_s K)$ .

**Problem 77.** An example in

Leoni Dalla and D.G. Larman, Convex bodies with almost all k-dimensional sections polytopes (to appear),

shows that the set of extreme points of  $K$  can be uncountable and even of Hausdorff dimension one.

**Problem 90.** The counterexamples mentioned in the comment appear in  
G.R. Burton, Subspaces which touch a Borel subset of a convex surface (to appear).

This paper also contains some related results.

## II. Differential geometry and kinematics

# Tom J. Willmore

## Mean-value theorems in riemannian geometry

### 1. Introduction

This paper describes joint work of Professor Alfred Gray (University of Maryland) and myself. It is well known that harmonic functions on  $\mathbb{R}^n$  have the mean-value property, namely, if  $S^{n-1}(r)$  denotes a sphere of radius  $r$  in  $\mathbb{R}^n$  and if  $V(S^{n-1}(r))$  denotes its volume, then

$$(1.1) \quad \frac{1}{V(S^{n-1}(r))} \int_{S^{n-1}(r)} f(m+u) du = f(m).$$

One way of generalizing the result to give a mean-value theorem for arbitrary functions is Pizzetti's formula [CH], p. 287, [PI.1], [PI.2], [OL]. This mean-value theorem for arbitrary functions on  $\mathbb{R}^n$  of class  $C^\omega$  asserts

$$(1.2) \quad M_m(r, f) = \Gamma(\frac{1}{2}n) \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k! \Gamma(\frac{1}{2}n+k)} (\Delta^k f)_m.$$

A second way of generalizing the theorem is to replace  $\mathbb{R}^n$  by an arbitrary riemannian manifold  $M$ . In that case, let  $m \in M$  and denote by  $S_m(r)$  the  $(n-l)$ -dimensional volume of the geodesic sphere of radius  $r$  centred at  $m$ . Then the mean-value of  $f$  over  $S_m(r)$  is given by

$$M_m(r, f) = \frac{1}{S_m(r)} \int_{\exp_m S^{n-1}(r)} f \omega$$

when  $\omega$  is the volume element of  $\exp_m(S^{n-1}(r))$ . Here the exponential map sends a small neighbourhood of the origin of  $M_m$  onto a neighbourhood of  $m$  in  $M$ . Geometrically  $M_m(r, f)$  is the mean-value of  $f$  over a geodesic sphere of radius  $r$  in  $M$ . The riemannian manifolds for which

$$(1.3) \quad M_m(r, f) = f(m)$$

have been completely characterized. They are the harmonic spaces invented by Ruse [CR], and described by the books [BE], [RWW]. Examples of such spaces are euclidean spaces and symmetric spaces of rank 1.

In the present paper we combine these two mean-value theorems. We obtain the Taylor expansion for  $M_m(r, f)$  in powers of  $r$ , where  $f$  is an arbitrary  $C^\omega$ -function on a  $C^\omega$ -manifold  $M$ .

Our technique will be to use the exponential map  $\exp_m$  to transfer formulas between  $M$  and the euclidean space  $M_m$ . In this way we obtain a formula similar to (1.2), but unfortunately the right hand side is not expressed in terms of the Laplacian  $\Delta$  of the riemannian manifold, but in terms of another operator  $\tilde{\Delta}_m$ , which we call the *Euclidean Laplacian*. If  $(x_1, \dots, x_n)$  is any system of normal coordinates at  $m$ , then  $\tilde{\Delta}_m$  is given by

$$(1.4) \quad \tilde{\Delta}_m = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

On the other hand the ordinary Laplacian  $\Delta$  of the riemannian manifold  $M$  is given by

$$(1.5) \quad \Delta = \frac{1}{\theta} \sum_{i=1}^n \frac{\partial}{\partial y_i} \left( g^{ij} \theta \frac{\partial}{\partial y_j} \right),$$

where  $(y_1, \dots, y_n)$  is any coordinate system,  $g_{ij}$  are the components of the metric tensor relative to  $(y_1, \dots, y_n)$ ,  $(g^{ij})$  is the inverse matrix of  $(g_{ij})$ , and  $\theta = \sqrt{\det(g_{ij})}$ . Although  $(\Delta f)_m = (\tilde{\Delta}_m f)_m$ , it is in general false that  $(\Delta^k f)_m = (\tilde{\Delta}_m^k f)_m$  for  $k > 1$ .

The relationship between  $\Delta$  and  $\tilde{\Delta}_m$  is investigated in section 2. We give formulas for  $\tilde{\Delta}_m^k$  in terms of  $\Delta^k$  and certain other terms involving curvature. In principle such formulas are possible for any  $k$ , but the computations become very complicated very quickly, and so we treat only the case  $k = 2$ .

In section 3 we derive our mean-value formula, and then in section 4 we give several applications.

## 2. The Laplacian and the Euclidean Laplacian

Let  $M$  be a riemannian manifold of class  $C^\omega$ ; all functions and tensor fields on  $M$  will also be class  $C^\omega$ , unless stated otherwise. We now give a formula for the Laplacian  $\Delta$  (defined by (1.5)) and the Euclidean Laplacian  $\tilde{\Delta}_m$  (defined by (1.4)). We recall that (1.5) is valid for any coordinate system. In contrast to this the formula (1.4) for  $\tilde{\Delta}_m$  holds only for a normal coordinate system centered at  $m$ , and it is for this reason we write the subscript  $m$ . Nevertheless (1.4) is independent of the choice of normal coordinates at  $m$ .

In order to compare  $\tilde{\Delta}_m$  and  $\Delta$  it will be necessary to use the power series expansions of  $g^{ij}$  and  $\theta$  with respect to a normal coordinate system  $(x_1, \dots, x_n)$ . The expansion for  $\theta$  to terms of order 6 is given in [GV]. Also the expansion for  $g_{ij}$  to terms of order 5 is given in [GR]; from this the expansion  $g^{ij}$  can be computed. Specifically, we shall need the following formulas:

$$(2.1) \quad \begin{aligned} \theta &= 1 - \frac{1}{6} \sum_{i,j=1}^n \rho_{ij}(m) x_i x_j - \frac{1}{12} \sum_{i,j,k=1}^n (\nabla_i \rho_{jk})(m) x_i x_j x_k \\ &\quad + \frac{1}{24} \sum_{i,j,k,l=1}^n \left\{ -\frac{3}{5} \nabla_{ij}^2 \rho_{kl} + \frac{1}{3} \rho_{ij} \rho_{kl} - \frac{2}{15} \sum_{a,b=1}^n R_{iajb} R_{kalb} \right\} (m) \\ &\quad \times x_i x_j x_k x_l + \frac{1}{120} \sum_{i,j,k,l,h=1}^n \left\{ -\frac{2}{3} \nabla_{ijk}^3 \rho_{lh} + \frac{5}{3} \nabla_i \rho_{jk} \rho_{lh} \right. \\ &\quad \left. - \frac{2}{3} \sum_{a,b=1}^n \nabla_i R_{jakb} R_{lahb} \right\} (m) x_i x_j x_k x_l x_h + \dots \end{aligned}$$

$$(2.2) \quad \begin{aligned} g^{pq} &= \delta_{pq} + \frac{1}{3} \sum_{i,j=1}^n R_{ipjq}(m) x_i x_j + \frac{1}{6} \sum_{i,j,k=1}^n \nabla_i R_{jpqk}(m) x_i x_j x_k \\ &\quad + \frac{1}{120} \sum_{i,j,k,l=1}^n \left\{ 6 \nabla_{ij}^2 R_{kpql} + 8 \sum_{s=1}^n R_{ipjs} R_{kqls} \right\} (m) x_i x_j x_k x_l \\ &\quad + \frac{1}{90} \sum_{i,j,k,l,h=1}^n \left\{ \nabla_{ijk}^3 R_{lphq} + 6 \sum_{s=1}^n \nabla_i R_{jpks} R_{lqhs} \right\} (m) \\ &\quad \times x_i x_j x_k x_l x_h + \dots \end{aligned}$$

Here  $\rho_{ij}$  and  $R_{ijkl}$  are the components of the Ricci tensor and curvature tensor of  $M$ . We choose the signs so that  $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ .

**Lemma 2.1.** *If  $f$  is a  $C^2$  real-valued function on a neighbourhood of  $m \in M$ , then*

$$(2.3) \quad (\Delta f)_m = (\tilde{\Delta}_m f)_m.$$

*Proof:* From the expansions (2.1) and (2.2) we have  $\theta(m) = 1$ ,  $g^{ij}(m) = \delta_{ij}$ , and  $\frac{\partial \theta}{\partial x_i}(m) = \frac{\partial g^{ij}}{\partial x_k}(m) = 0$ . Then (2.3) follows easily from these facts upon expanding (1.5).

The relation between  $(\Delta^2 f)_m$  and  $(\tilde{\Delta}_m^2 f)_m$  is more complicated, because certain curvature terms arise. To explain these, let  $\tau$  denote the scalar curvature of  $M$ . Then if  $f$  is a  $C^2$  real-valued function on  $M$  we can consider the (pointwise) inner product  $\langle df, d\tau \rangle$ . Also  $\langle \nabla^2 f, \rho \rangle$  will denote the (pointwise) inner product of the Hessian of  $f$  and the Ricci curvature  $\rho$  of  $M$ . (Both are symmetric bilinear forms.)

**Lemma 2.2.** *If  $f$  is a  $C^4$  real-valued function on a neighbourhood of  $m \in M$ , then*

$$(2.4) \quad (\tilde{\Delta}_m^2 f)_m = (\Delta^2 f)_m + \frac{1}{3} \langle d\tau, df \rangle(m) + \frac{2}{3} \langle \nabla^2 f, \rho \rangle(m).$$

*Proof:* Using lemma (2.1) we observe that

$$(2.5) \quad (\Delta^2 f)_m = (\tilde{\Delta}_m(\Delta f))_m \\ = \sum_{i,k,l=1}^n \frac{\partial^2}{\partial x_i^2} \left( \frac{1}{\theta} \frac{\partial}{\partial x_k} \left( \theta g^{kl} \frac{\partial f}{\partial x_l} \right) \right)_m.$$

We expand the right hand side of (2.5) using the first three terms in the expansions (2.1) and (2.2) together with the expansion for  $f$  in normal coordinates. After some calculation we obtain (2.4). In particular we have

$$\begin{aligned} (\Delta^2 f)_m &= \sum \frac{\partial^2}{\partial x_i^2} \left( \frac{1}{\theta} \frac{\partial}{\partial x_k} \left( \theta g^{kl} \frac{\partial f}{\partial x_l} \right) \right)_m \\ &= \sum \frac{\partial^2}{\partial x_i^2} \left\{ \frac{\partial(\log \theta)}{\partial x_k} g^{kl} \frac{\partial f}{\partial x_l} + \frac{\partial g^{kl}}{\partial x_k} \frac{\partial f}{\partial x_l} + g^{kl} \frac{\partial^2 f}{\partial x_k \partial x_l} \right\}_m \\ &= \sum \left\{ \frac{\partial^3(\log \theta)}{\partial x_i^2 \partial x_k} g^{kl} \frac{\partial f}{\partial x_l} + 2 \frac{\partial^2(\log \theta)}{\partial x_i \partial x_k} g^{kl} \frac{\partial^2 f}{\partial x_i \partial x_l} + \frac{\partial^3 g^{kl}}{\partial x_i^2 \partial x_k} \frac{\partial f}{\partial x_l} \right. \\ &\quad \left. + 2 \frac{\partial^2 g^{kl}}{\partial x_i \partial x_k} \frac{\partial^2 f}{\partial x_i \partial x_l} + \frac{\partial^2 g^{kl}}{\partial x_i^2} \frac{\partial^2 f}{\partial x_k \partial x_l} + g^{kl} \frac{\partial^4 f}{\partial x_i^2 \partial x_k \partial x_l} \right\}_m \\ &= \sum \left\{ \frac{\partial^3(\log \theta)}{\partial x_i^2 \partial x_k} \frac{\partial f}{\partial x_k} + 2 \frac{\partial^2(\log \theta)}{\partial x_i \partial x_k} \frac{\partial^2 f}{\partial x_i \partial x_k} \right\}_m + (\tilde{\Delta}_m^2 f)(m) \\ &\quad + \sum \left\{ \frac{\partial^3 g^{kl}}{\partial x_i^2 \partial x_k} \frac{\partial f}{\partial x_l} + \frac{\partial^2 g^{kl}}{\partial x_i^2} \frac{\partial^2 f}{\partial x_k \partial x_l} + 2 \frac{\partial^2 g^{kl}}{\partial x_i \partial x_k} \frac{\partial^2 f}{\partial x_i \partial x_l} \right\}_m \end{aligned}$$

$$\begin{aligned}
&= \sum \left\{ -\frac{1}{12} (4\nabla_i \rho_{ik} + 2\nabla_k \rho_{ii}) \frac{\partial f}{\partial x_k} - \frac{2}{3} \rho_{ik} \frac{\partial^2 f}{\partial x_i \partial x_k} \right\}_m + (\tilde{\Delta}_m^2 f)(m) \\
&\quad + \sum \left\{ \frac{1}{6} (2\nabla_i R_{ikk} + 2\nabla_k R_{iki}) \frac{\partial f}{\partial x_l} + \frac{2}{3} R_{ikk} \frac{\partial^2 f}{\partial x_i \partial x_l} \right. \\
&\quad \left. + \frac{2}{3} R_{iki} \frac{\partial^2 f}{\partial x_k \partial x_l} \right\}_m \\
&= \sum \left\{ -\frac{1}{3} \nabla_k \rho_{ii} \frac{\partial f}{\partial x_k} - \frac{2}{3} \rho_{ik} \frac{\partial^2 f}{\partial x_i \partial x_k} \right\}_m + (\tilde{\Delta}_m^2 f)(m) \\
&\quad + \sum \left\{ -\frac{1}{3} \nabla_i \rho_{il} \frac{\partial f}{\partial x_l} \right\}_m + \frac{1}{3} \sum \left\{ \nabla_k \rho_{kl} \frac{\partial f}{\partial x_l} \right\}_m - \frac{2}{3} \sum \left\{ \rho_{il} \frac{\partial^2 f}{\partial x_i \partial x_l} \right\}_m \\
&\quad + \frac{2}{3} \sum \left\{ \rho_{kl} \frac{\partial^2 f}{\partial x_k \partial x_l} \right\}_m \\
&= (\tilde{\Delta}_m^2 f)_m - \frac{1}{3} \langle d\tau, df \rangle(m) - \frac{2}{3} \langle \nabla^2 f, \rho \rangle(m).
\end{aligned}$$

### 3. The power series expansion for the mean-value $M_m(r, f)$

If  $M$  is a Riemannian manifold and  $m \in M$ , let  $S_m(r)$  denote the  $(n-1)$ -dimensional volume of a geodesic sphere of radius  $r$  in  $M$  centered at  $m$ . Then  $M_m(r, f)$  is defined by

$$(3.1) \quad M_m(r, f) = \frac{1}{S_m(r)} \int_{\exp_m(S^{n-1}(r))} f \omega,$$

where  $\omega$  is the volume element of  $\exp_m(S^{n-1}(r))$ .

We first note

**Lemma 3.1.** *Let  $f$  be a real-valued function of class  $C^2$  defined on a neighbourhood of  $m \in M$ . Let  $\bar{\Delta}$  be the Laplacian of the Euclidean space  $M_m$ . Then  $(\bar{\Delta}_m f) \circ \exp_m = \bar{\Delta}(f \circ \exp_m)$ .*

*Proof:* This follows from the fact that the vector fields  $\partial/\partial x_i$  are just the images under  $\exp_m$  of the natural coordinate vector fields on  $M_m$ .

Now we determine the complete expansion for  $M_m(r, f)$  in terms of  $\tilde{\Delta}_m$ . Let  $J_\alpha$  be the Bessel function of order  $\alpha$ . Write  $j_p(z) = 2^p \Gamma(p+1) J_p(z)/z^p$ .

**Lemma 3.2.** *We have*

$$\begin{aligned}
(3.2) \quad \int_{\exp_m(S^{n-1}(r))} f \omega &= 2\pi^{n/2} \Gamma\left(\frac{n}{2}\right)^{-1} r^{n-1} j_{(n/2)-1}(r \sqrt{-\tilde{\Delta}_m}) [f \theta](m) \\
&= 2\pi^{n/2} r^{n-1} \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k! \Gamma(\frac{1}{2}n+k)} \tilde{\Delta}_m^k [f \theta](m).
\end{aligned}$$

*Proof:* To compare the left hand side of (3.2) we must first change variables so that we can carry out the integration over an ordinary sphere in  $M_m$ . This change of variables is carried out for the function 1 in [GV]; exactly the same technique yields

$$(3.3) \quad \int_{\exp_m(S^{n-1}(r))} f \omega = \int_{S^{n-1}(r)} (f\theta)(\exp_m(ru)) du.$$

(This change of variables is not quite so obvious as it appears; note that  $\omega$  is the volume element of  $\exp_m(S^{n-1}(r))$  and not  $M$ . However everything works out because of the Gauss lemma.)

The right hand side of (3.3), being an integral over an ordinary sphere, can be computed using the mean-value theorem of [ZA]:

$$(3.4) \quad \frac{1}{V(S^{n-1}(r))} \int_{S^{n-1}(r)} h du = \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k! \Gamma(\frac{1}{2}n+k)} \bar{\Delta}^k[h](0)$$

$$= j_{(n/2)-1}(r \sqrt{-\bar{\Delta}})[h](0).$$

Using the fact that  $V(S^{n-1}(r)) = 2\pi^{n/2} \Gamma\left(\frac{n}{2}\right)^{-1} r^{n-1}$  we can rewrite (3.4) as

$$(3.5) \quad \int_{S^{n-1}(r)} h du = 2\pi^{n/2} r^{n-1} \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k! \Gamma(\frac{1}{2}n+k)} \bar{\Delta}^k[h](0)$$

$$= 2\pi^{n/2} \Gamma\left(\frac{n}{2}\right)^{-1} r^{n-1} j_{(n/2)-1}(r \sqrt{-\bar{\Delta}})[h](0)$$

Then (3.2) follows from (3.3) and (3.5).

**Corollary. 3.3.** We have the following formula for the volume  $S_m(r)$  of a geodesic sphere of radius  $r$ :

$$S_m(r) = 2\pi^{n/2} \Gamma\left(\frac{n}{2}\right)^{-1} r^{n-1} j_{(n/2)-1}(r \sqrt{-\bar{\Delta}})[\theta](m)$$

$$= 2\pi^{n/2} r^{n-1} \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k! \Gamma(\frac{1}{2}n+k)} \tilde{\Delta}_m^k[\theta](m).$$

**Theorem 3.4.** We have

$$(3.6) \quad M_m(r, f) = \frac{j_{(n/2)-1}(r \sqrt{-\bar{\Delta}_m})[f\theta](m)}{j_{(n/2)-1}(r \sqrt{-\bar{\Delta}_m})[\theta](m)}$$

$$= \frac{\sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k! \Gamma(\frac{1}{2}n+k)} \tilde{\Delta}_m^k[f\theta](m)}{\sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k! \Gamma(\frac{1}{2}n+k)} \tilde{\Delta}_m^k[\theta](m)}$$

*Proof:* This is immediate from lemma 3.2 and corollary 3.3.

In principle we know from section 2 that  $\tilde{\Delta}_m$  can be expressed in terms of  $\Delta$ . By dividing the two power series on the right hand side of (3.6) we now compute the first few terms in the power series expansion for  $M_m(r,f)$ .

**Theorem 3.5.** *We have*

$$(3.7) \quad M_m(r,f) = \left[ f + \left( \frac{1}{2n} \Delta f \right) r^2 + \frac{1}{24n(n+2)} (3\Delta^2 f - 3 \langle df, d\tau \rangle \right. \\ \left. + \frac{4}{n} (\Delta f) \tau - 2 \langle \nabla^2 f, \rho \rangle) r^4 + O(r^6) \right] (m).$$

The first two terms in an expansion similar to (3.7) have been computed by J. Lelong-Ferrand [LF].

#### 4. Applications

We have already remarked that a harmonic space is characterized [WI] by the property that for any harmonic function  $f$  we have

$$M_m(r,f) = f.$$

We are now in a position to sharpen this result as follows.

**Theorem 4.1.** *Suppose that for each  $m \in M$*

$$(4.1) \quad M_m(f,r) = f(m)$$

*for sufficiently small  $r > 0$ . Then  $f$  is a harmonic function. If (4.1) holds for all harmonic functions defined near  $m$  then  $M$  is a harmonic space.*

The proof follows immediately from theorem 3.5 and [WI].

If we write

$$(4.2) \quad M_m(r,f) = \left\{ f + \sum_{n=1}^{\infty} a_n r^{2n} \right\}_m$$

where  $a_1$  and  $a_2$  are given explicitly by theorem 3.5, it follows that a necessary and sufficient condition for the space to be harmonic is  $a_n = 0$ ,  $n = 1, 2, \dots$ . The condition  $a_1 = 0$  gives  $\Delta f = 0$ , and  $a_2 = 0$  together with  $a_1 = 0$  gives

$$(4.3) \quad 3 \langle df, d\tau \rangle + 2 \langle \nabla^2 f, \rho \rangle = 0.$$

It is easily checked that a sufficient condition that (4.3) is automatically satisfied when  $\Delta f = 0$  is that  $M$  is an Einstein space, i.e.  $\rho = \lambda g$ . Conversely this condition can also be shown to be necessary, by constructing suitable harmonic functions via the Cauchy-Kowalevsky theorem. Thus we have

**Theorem 4.2.** *Suppose that for each  $m$*

$$(4.4) \quad M_m(r,f) = f(m) + O(r^6) \text{ as } r \rightarrow 0.$$

*Then  $f$  is a harmonic function. If (4.4) holds for all harmonic functions on  $M$  defined near  $m$ , then  $M$  is an Einstein space.*

Conversely

**Theorem 4.3.** Let  $f$  be a harmonic function and  $M$  be an Einstein space. Then we have

$$M_m(r,f) = f(m) + O(r^6) \text{ as } r \rightarrow 0.$$

Furthermore

**Theorem 4.4.** Let  $M_m(r,f) = f(m) \left[ 1 + \frac{\lambda r^2}{2n} + O(r^4) \right]$  as  $r \rightarrow 0$  for each  $m$ , where  $\lambda$  is a constant function. Then  $f$  is an eigenfunction of the Laplacian, i.e.  $\Delta f = \lambda f$ .

This follows immediately by comparing the given form of  $M_m(r,f)$  with the general expression in theorem 3.5. This theorem generalizes a result previously obtained by A.G. Walker [WA] who used an entirely different method to obtain the result in the particular case where  $M$  is a 3-dimensional manifold of constant curvature.

Since an irreducible symmetric space is an Einstein space, it follows from theorem 4.3 that for such a space

$$(4.5) \quad M_m(r,f) = f(m) + O(r^6) \text{ as } r \rightarrow 0$$

when  $f$  is a harmonic function. For symmetric spaces of rank one we have

$$M_m(r,f) = f(m).$$

It would be interesting to obtain a more precise form of (4.5) for irreducible symmetric spaces of rank  $> 1$ .

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# Peter Buser und Hermann Karcher<sup>1</sup>

## Diskrete Gruppen und kleine Krümmung nach Gromov

### 1. Einleitung

M. Gromov hat das folgende erstaunliche Resultat bewiesen [3]:

**1.1. Satz.** *Falls eine kompakte Riemannsche Mannigfaltigkeit  $M$   $\varepsilon$ -flach (1.2) ist, mit genügend kleinem, nur von der Dimension  $n$  abhängigen  $\varepsilon > 0$ , so ist eine geeignete endliche Überlagerung von  $M$  eine Nilmannigfaltigkeit, d.h. Quotient einer nilpotenten Liegruppe.*

Hierin heißt eine kompakte Riemannsche Mannigfaltigkeit  $\varepsilon$ -flach, wenn die Riemannschen Schnittkrümmungen  $K$  mit dem Durchmesser  $d(M)$  von  $M$  abgeschätzt werden können:

$$1.2 \quad |K| \leq \varepsilon \cdot d(M)^{-2}.$$

Wie alle «pinching»-Voraussetzungen ist auch diese unempfindlich gegenüber Multiplikation der Metrik mit einer Konstanten. Offenbar ist jede kompakte flache ( $K=0$ ) Mannigfaltigkeit  $\varepsilon$ -flach für jedes  $\varepsilon > 0$ . Dies sind jedoch bei weitem nicht alle Beispiele: Falls  $M$  kompakter Quotient  $\Gamma \backslash G$  einer nilpotenten Liegruppe  $G$  ist, so gibt es zu jedem  $\varepsilon > 0$  eine linksinvariante Metrik auf  $G$  derart, daß  $\Gamma \backslash G$   $\varepsilon$ -flach ist (2). Daß diese Beispiele tatsächlich wesentlich allgemeiner als die flachen sind, sieht man etwa so: Flache Mannigfaltigkeiten haben Bieberbachgruppen als Fundamentalgruppen, Gruppen also, die u.a. freie abelsche Untergruppen vom Rang  $\dim M$  und endlichem Index besitzen; diskrete uniforme Untergruppen  $\Gamma$  nilpotenter Liegruppen  $G$  besitzen diese Eigenschaft im allgemeinen nicht.

Der Gedankengang von Gromovs Beweis läßt sich so zusammenfassen: Ausgehend von einer als hinreichend  $\varepsilon$ -flach vorausgesetzten Mannigfaltigkeit  $M$  konstruiert er zunächst eine nilpotente Untergruppe  $\Gamma$  von endlichem Index in der Fundamentalgruppe  $\pi_1(M)$ . Diese Konstruktion kann so eingerichtet werden, daß sich  $\Gamma$  in eine nilpotente Liegruppe  $G$  derselben Dimension wie  $M$  diskret und uniform einbetten läßt. Dann wird eine an das bisherige Vorgehen möglichst gut angepaßte linksinvariante Metrik für  $G$  definiert, mit deren Hilfe ein Diffeomorphismus zwischen  $\Gamma \backslash G$  und einer endlichen Überlagerung von  $M$  angegeben wird. Eine ausführliche Darstellung der z.T. langwierigen Beweise ist in Vorbereitung. In diesem Vortrag gehe ich nur auf einige Teilresultate ein, bei denen ich glaube, daß die Beweisargumente auch für die zuhörenden Nicht-Differentialgeometer attraktiv sind.

### 2. Linksinvariante Metriken für nilpotente Liegruppen

Es genügt uns, an einem Beispiel zu erläutern, wieso die Nilpotenz die Existenz  $\varepsilon$ -flacher Metriken für jedes  $\varepsilon > 0$  ermöglicht. Betrachte die folgende nilpotente

<sup>1</sup> Diese Arbeit entstand im Rahmen des Sonderforschungsbereichs Theoretische Mathematik (SFB 40) in Bonn. Sie wurde wesentlich gefördert durch Diskussionen mit M. Gromov während der Arbeitstagung 1977. Auf der Siegener Geometrietagung 1978 wurde über den hier dargestellten Teil des Beweises berichtet.

Gruppe von oberen Dreiecksmatrizen ( $n > 2$ ):

$$G = \left\{ \begin{pmatrix} 1 & \dots & a_{ij} \\ 0 & \dots & 1 \end{pmatrix}; a_{ij} \in \mathbb{R}, 1 \leq i < j < n \right\}$$

und darin die ganzzahlige Untergruppe  $\Gamma = \left\{ \begin{pmatrix} 1 & \dots & n_{ij} \\ 0 & \dots & 1 \end{pmatrix}; n_{ij} \in \mathbb{Z} \right\}$ .  $\Gamma \backslash G$  ist offenbar kompakt, und  $\Gamma$  ist keine Bieberbachgruppe, weil der Rang freier abelscher Untergruppen zu klein ist. Wir definieren eine Familie von Skalarprodukten für die Liealgebra  $\mathfrak{G} = \left\{ A = \begin{pmatrix} 0 & \dots & a_{ij} \\ 0 & \dots & 0 \end{pmatrix}; a_{ij} \in \mathbb{R} \right\}$  von  $G$ :

$$2.1 \quad \|A\|_q^2 = \left\| \begin{pmatrix} 0 & \dots & a_{ij} \\ 0 & \dots & 0 \end{pmatrix} \right\|_q^2 = \sum_{i < j} a_{ij}^2 q^{2(j-i)}.$$

Die Skalarprodukte (2.1) werden durch Linkstranslation zu Riemannschen Metriken für  $G$  fortgesetzt. Jeder kompakte Quotient von  $G$ , z.B.  $\Gamma \backslash G$ , hat nun beliebig kleinen Durchmesser, wenn nur  $q$  genügend klein gewählt wird. Außerdem zeigen wir, daß die Krümmungen von  $G$  unabhängig von  $q$  beschränkt sind, so daß daher  $\Gamma \backslash G$   $\varepsilon$ -flach ist für beliebig kleine  $\varepsilon > 0$ . Zunächst gilt für alle  $A, B \in \mathfrak{G}$ :  $\|[A, B]\|_q \leq 2 \cdot (n-2) \cdot \|A\|_q \cdot \|B\|_q$ , weil für die Standardbasis  $\{E_{ij}\}_{1 \leq i < j \leq n}$  von  $\mathfrak{G}$  ( $E_{ij}$  hat nur an der Stelle  $(i,j)$  eine 1, sonst Nullen) gilt:  $\|[E_{ij}, E_{kl}]\|_q = (\delta_{jk} + \delta_{il}) \cdot \|E_{ij}\|_q \cdot \|E_{kl}\|_q$ . Weiter liefert jede Abschätzung für die Norm der Lieklammer (hier:  $\leq 2 \cdot (n-2)$ ) auch Schranken für den Krümmungstensor der zugehörigen linksinvarianten Metrik: Der linksinvariante Zusammenhang  $D^L$  für  $G$  ist ein Riemannscher Zusammenhang für jede linksinvariante Metrik auf  $G$ , sein Krümmungstensor ist 0 und sein Torsionstensor  $T$  berechnet sich für linksinvariante Vektorfelder aus der Lieklammer:  $T(X, Y) = -[X, Y]$ . Schreibt man den Levi-Civita-Zusammenhang der linksinvarianten Metrik als  $D_x Y = D_x^L Y + U(X, Y)$ , so gilt für den Differenztensor  $U$ :

$$\begin{aligned} U(X, Y) - U(Y, X) &= -T(X, Y) \text{ und } \langle U(X, Y), Z \rangle + \langle Y, U(X, Z) \rangle = 0, \text{ also} \\ \langle U(X, Y), Z \rangle &= \frac{1}{2} \{ -\langle T(X, Y), Z \rangle + \langle Y, T(X, Z) \rangle + \langle X, T(Y, Z) \rangle \}, \\ \|U\| &\leq \frac{3}{2} \|T\|. \end{aligned}$$

Für linksinvariante Vektorfelder gilt  $D_y Z = U(Y, Z)$ , daher hat man für den Krümmungstensor von  $D$ :

$$R(X, Y)Z = U(X, U(Y, Z)) - U(Y, U(X, Z)) - U(T(X, Y), Z), \text{ also ist}$$

$$2.2 \quad \|R\| \leq 2 \cdot \|U\|^2 + \|U\| \cdot \|T\| \leq 6 \cdot \|T\|^2$$

die gewünschte Abschätzung des Krümmungstensors durch die Norm der Lieklammer, in unserem Beispiel:  $\|R_q\|_q \leq 24 \cdot (n-2)^2$ .

### 3. Produkte für kurze geodätische Schleifen

Zunächst erinnern wir an die bekannte Tatsache, daß in einer vollständigen Riemannschen Mannigfaltigkeit  $M$  jede geschlossene Kurve mit Basispunkt  $p$

homotop ist zu einer geodätischen Schleife in  $p$ . Mit dem Satz von Arzéla-Ascoli und mit Hilfe geodätischer Sehnen kann man die Homotopie sogar so wählen, daß keine Kurve der Homotopie länger ist als die betrachtete geschlossene Kurve. Daher wird die Fundamentalgruppe  $\pi_1(M, p)$  bereits von den (geodätischen) Schleifen in  $p$  erzeugt, deren Länge  $\leq 2d(M) + \eta$  ist. (Denn jede geschlossene Kurve ist homotop zu einem Produkt aus Schleifen der Länge  $\leq 2d(M) + \eta$ : wähle aufeinanderfolgende Teilpunkte im Abstand  $\leq \eta$  auf der Kurve und verbinde die Teilpunkte durch hin- und zurück durchlaufene Geodätische der Länge  $\leq d(M)$  mit  $p$ .) Falls konjugierte Punkte in  $M$  immer einen Abstand  $> 2d(M) + \eta$  haben, also nur endlich viele geodätische Schleifen in  $p$  mit Längen zwischen  $2d(M)$  und  $2d(M) + \eta$  existieren, so kann man sogar  $\eta$  so klein wählen, daß gar keine Schleifen in  $p$  mit Längen  $> 2d(M)$  und  $\leq 2d(M) + \eta$  vorkommen. Schließlich, falls konjugierte Punkte mindestens den Abstand  $5 \cdot d(M)$  haben, so läßt sich zeigen, daß alle Relationen in der Fundamentalgruppe bereits von den («kurzen») Relationen erzeugt werden, die von Homotopien herrühren, deren Kurven sämtlich kürzer als  $5 \cdot d(M)$  sind. Im Falle von hinreichend  $\varepsilon$ -flachen («fast flachen») Mannigfaltigkeiten werden wir ohne Voraussetzungen über  $\pi_1(M)$  zeigen, daß es viele «kurze» geodätische Schleifen gibt, mit denen man so ähnlich arbeiten kann wie mit den aus Homotopieklassen gewonnenen Schleifen.

Dazu erinnern wir als nächstes an die Exponentialabbildung der Riemannschen Geometrie

$$\exp_p \text{ (kürzer: } \exp): T_p M \rightarrow M,$$

die die Geraden  $t \rightarrow t \cdot v$  durch 0 in  $T_p M$  längentreu auf die Geodätischen durch  $p$  in  $M$  abbildet, d.h.  $c(t) = \exp tv$  ist Geodätische in  $M$  mit  $c(0) = p$ ,  $\dot{c}(0) = v$ . Da diese Abbildung durch eine Differentialgleichung zweiter Ordnung definiert wird, wird ihre Ableitung durch eine *lineare* Differentialgleichung zweiter Ordnung (benannt nach Jacobi) beschrieben: Es sei  $t \cdot w$  Tangentialvektor an  $T_p M$  mit Fußpunkt  $t \cdot v$ , dann ist das Vektorfeld  $\mathcal{F}(t) = (d \exp)_{tv} \cdot tw$  längst  $c(t)$  Lösung der Differentialgleichung

$$\frac{D}{dt} \frac{D}{dt} \mathcal{F} + R(\mathcal{F}, \dot{\mathcal{F}}) \dot{\mathcal{F}} = 0.$$

Für derartige Differentialgleichungen gilt der

**3.1. Satz von Rauch.** *Hat man für die Schnittkrümmungen  $K$  von  $M$  Schranken  $-\lambda^2 \leq K \leq \Lambda^2$ , so gilt für die Lösungen mit  $\mathcal{F}(0) = 0$  und  $|\dot{\mathcal{F}}(0)| = 1$ :*

$$|\mathcal{F}'(0)| \cdot \frac{\sin \Lambda t}{\Lambda} \leq |\mathcal{F}(t)| \leq |\mathcal{F}'(0)| \cdot \frac{\sinh \lambda t}{\lambda}.$$

Insbesondere hat die Exponentialabbildung Höchstrang in Kugeln vom Radius  $< \frac{\pi}{\Lambda}$  ( $< \pi \varepsilon^{-\frac{1}{2}} \cdot d(M)$  im  $\varepsilon$ -flachen Fall). In Kugeln vom Radius  $t$  liegt die Längenverzerrung der Exponentialabbildung zwischen  $\frac{\sin \Lambda t}{\Lambda t}$  und  $\frac{\sinh \lambda t}{\lambda t}$  (also

nahe 1 bei kleiner Krümmung), und es folgt z.B.  $d(\exp v, \exp w) \leq |v - w| \cdot \frac{\sinh \lambda r}{\lambda r}$   
 $(r = \max(|v|, |w|))$ .

Es sei nun  $M$  eine  $\varepsilon$ -flache Mannigfaltigkeit. Dann kann die Metrik von  $M$  mit Hilfe von  $\exp$  auf die Kugel vom Radius  $\pi \cdot \varepsilon^{-\frac{1}{2}} \cdot d(M)$  in  $T_p M$  geliftet werden. Man erhält einen nicht vollständigen Riemannschen Raum  $M^*$ , der «viel größer» ist als  $M$ , weil ja  $\exp$  schon auf den «kleinen» Kugeln vom Radius  $d(M)$  surjektiv ist; außerdem sind die geodätischen Radien kürzeste Verbindungen zum Mittelpunkt. Die bei Krümmung  $\leq 0$  wohlbekannte Konstruktion von Fundamentalbereichen in der universellen Überlagerung kann man in  $M^*$  (außer in der Nähe des Randes) nachahmen: Sei  $w \in M^*$  mit  $q = \exp w$  gegeben, verbinde alle Punkte von  $M$  durch kürzeste Geodätische mit  $q$  und lifte diese mit  $\exp_p$  zu von  $w$  ausgehenden Geodätischen zu einem «Fundamentalbereich» für  $M$ . Es gibt daher im Abstand  $\leq d(M)$  von  $w$  ein  $v \in M^*$  mit  $\exp_p v = p$ , d.h. es gibt auch ohne Voraussetzungen über  $\pi_1(M, p)$  viele geodätische Schleifen in  $p$ . Obwohl Gromov nie explizit damit argumentiert, ist es zum Verständnis seines Beweises doch nützlich, sich  $M^*$  in ähnlicher Weise mit Fundamentalbereichen gepflastert zu denken, wie das bei Krümmung  $\leq 0$  in der universellen Überlagerung der Fall ist.

Gromov definiert nun für «kurze» geodätische Schleifen ein Produkt. Dazu benötigt er das

**3.2. Lange-Homotopie-Lemma** (Klingenbergs [2]). *Sind zwei Geodätische mit gleichem Anfangspunkt  $p$  und Endpunkt  $q$  homotop, so ist die Länge mindestens einer Kurve der Homotopie so groß wie der Höchstrangradius der Exponentialabbildung.*

Nennen wir also eine Homotopie *kurz*, wenn alle ihre Kurven eine Länge  $< \pi \cdot \varepsilon^{-\frac{1}{2}} \cdot d(M)$  haben, so folgt aus dem Lemma:

**3.3.** In jeder kurzen Homotopieklass von in  $p$  geschlossenen Kurven gibt es genau eine geodätische Schleife.

Diese Feststellung ist der Ausgangspunkt von Gromovs Beweis.

**3.4. Definition.**  $\alpha$  und  $\beta$  seien geodätische Schleifen in  $p$ , die Summe ihrer Längen sei  $< \pi \varepsilon^{-\frac{1}{2}} \cdot d(M)$ ,  $\beta \cdot \alpha$  sei das in der Topologie übliche Produkt, nämlich die Kurve «erst  $\alpha$  dann  $\beta$ ». Dann ist *Gromovs Produkt*  $\beta * \alpha$  die eindeutig bestimmte geodätische Schleife in  $p$ , die *kurz homotop* zu  $\beta \cdot \alpha$  ist.

Offensichtlich gilt: Ist  $\alpha^{-1}$  die Schleife  $\alpha$  mit umgekehrter Parametrisierung, so ist  $\alpha^{-1} * \alpha$  die Nullschleife in  $p$ . Ist die Summe der Längen von  $\alpha, \beta, \gamma$  noch  $< \pi \varepsilon^{-\frac{1}{2}} \cdot d(M)$ , so ist  $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$ , weil beide Schleifen kurz homotop zu  $\alpha \cdot \beta \cdot \gamma$  sind. Die kurzen Schleifen bilden keine Gruppe, weil Produkte von zu vielen Faktoren nicht unbedingt definiert sind. Aber man kann die von den kurzen Schleifen und den kurzen Relationen erzeugte Gruppe betrachten; diese ist nach dem oben gesagten dann isomorph zu  $\pi_1(M)$ , wenn  $\pi \varepsilon^{-\frac{1}{2}} \geq 5$  ist. Natürlich ist damit nicht impliziert, daß verschiedene kurze Schleifen dann auch in der erzeugten Gruppe verschieden sind.

#### 4. Fast homomorphe Abbildung der kurzen Schleifen in die Bewegungsgruppe von $T_p M$

Jeder geschlossenen Kurve  $c: [0,1] \rightarrow M$  mit  $c(0) = c(1) = p$  ordnen wir eine Bewegung  $m(c): T_p M \rightarrow T_p M$  zu mit Hilfe der linearen Differentialgleichung  $\frac{D}{dt} X = \dot{c}(t)$  für Vektorfelder  $X$  längs  $c$ , nämlich  $m(c): X(0) \rightarrow X(1)$ . Es handelt sich um eine Bewegung, da für die Differenz  $Y = X_1 - X_2$  von zwei Lösungen gilt:  $\frac{D}{dt} Y = 0$ ,  $Y$  also Levi-Civita-parallel ist. Mit anderen Worten: Der Rotationsanteil der Bewegung ist gerade die übliche Riemannsche Parallelverschiebung, der Translationsanteil ist das Bild des Nullvektors, z.B. bei geodätischen Schleifen der Tangentialvektor im Endpunkt.  $m(c)$  heißt daher *affine Parallelverschiebung längs*  $c$ . Für geodätische Schleifen  $\alpha, \beta$  gilt offenbar  $m(\beta) \circ m(\alpha) = m(\beta \cdot \alpha)$ , und dies ist beinahe  $m(\beta * \alpha)$  wegen der geringen Wegabhängigkeit der Parallelverschiebung bei kleiner Krümmung, genauer wegen:

##### *Wegabhängigkeit der Levi-Civita-Parallelverschiebung*

Es seien  $c_1$  und  $c_2$  Kurven mit gleichem Anfangspunkt  $p$  und gleichem Endpunkt  $q$ , und es sei eine Homotopie vom Flächeninhalt  $\leq F$  zwischen  $c_1$  und  $c_2$  gegeben.  $\|R\| (\leq \frac{4}{3} \max |K|)$  bezeichne das Maximum der Norm des Krümmungstensors längs der Homotopie.  $X_i$  bezeichne die Levi-Civita-Parallelverschiebung eines Vektors  $X_0 \in T_p M$  längs  $c_i$  ( $i = 1, 2$ ). Dann gilt:

$$4.1 \quad |\nabla(X_i, X_2)| \leq \|R\| \cdot F.$$

##### *Wegabhängigkeit des Translationsanteils der affinen Parallelverschiebung*

Es seien  $c_1, c_2, F, \|R\|$  wie eben, außerdem sei  $L$  eine Schranke für die längste Kurve der Homotopie zwischen  $c_1$  und  $c_2$ . Dann gilt:

$$4.2 \quad |\text{trans}_1 - \text{trans}_2| \leq L \cdot \|R\| \cdot F.$$

Wir geben die kurzen *Beweise* dieser für Gromovs Argumentation grundlegenden Ungleichungen:

$c: [0,1] \times [0,1] \rightarrow M$  sei die gegebene Homotopie,  $c_1(t) = c(0,t)$ ,  $c_2(t) = c(1,t)$ ,  $c(s,0) = p$ ,  $c(s,1) = q$ . Betrachte ein Einheitsvektorfeld  $X$  längs  $c$  mit den Eigenschaften:

$X(s,0) = X_0 \in T_p M$ ,  $\frac{D}{dt} X(s,t) = 0$  (d.h. längs jedes Weges der Homotopie ist  $X$  parallel). Dann ist  $s \rightarrow X(s,1)$  eine Verbindungskurve auf der Einheitskugel in  $T_q M$  zwischen  $X(0,1) = X_1$  und  $X(1,1) = X_2$ , und eine Schranke für die Länge dieser Kurve ist eine Abschätzung des Winkels der Vektoren. Daher folgt die Behauptung 4.1 aus:

$$\int_0^1 \left| \frac{D}{ds} X(s,1) \right| ds \leq \int_0^1 \int_0^1 \left| \frac{D}{dt} \frac{D}{ds} X(s,t) \right| dt ds \left( \text{mit } \frac{D}{ds} X(s,0) = 0 \right)$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \left| R \left( \frac{\partial c}{\partial t}, \frac{\partial c}{\partial s} \right) X \right| dt ds \left( \text{wegen } \frac{D}{dt} X = 0 \right) \\
&\leq \|R\| \cdot |X| \cdot \int_0^1 \int_0^1 \left| \frac{\partial c}{\partial t}, \frac{\partial c}{\partial s} \right| dt ds = \|R\| \cdot F.
\end{aligned}$$

Im affinen Fall definiere ein Vektorfeld  $Y$  längs  $c$  durch  $Y(s,0)=0$ ,  $\frac{D}{dt} Y(s,t)=\frac{\partial c}{\partial t}$  ( $s,t$ ) ( $Y$  ist affin parallel längs der Wege der Homotopie). Wie eben genügt es, die Länge der Kurve  $s \rightarrow Y(s,1)$  in  $T_q M$  abzuschätzen; dabei berücksichtigen wir  $\frac{\partial c}{\partial s}(s,1)=0$  und  $|Y(s,t)| \leq \int_0^t \left| \frac{\partial c}{\partial t}(s,t) \right| dt \leq L$ , also beweist

$$\begin{aligned}
\int_0^1 \left| \frac{D}{ds} Y(s,1) \right| ds &\leq \int_0^1 \int_0^1 \left| \frac{D}{dt} \frac{D}{ds} Y(s,t) - \frac{D}{dt} \frac{\partial}{\partial s} c(s,t) \right| dt ds \\
&= \int_0^1 \int_0^1 \left| R \left( \frac{\partial c}{\partial t}, \frac{\partial c}{\partial s} \right) Y \right| dt ds \quad (\text{wegen } \frac{D}{dt} Y - \frac{\partial}{\partial t} c = 0) \\
&\leq \|R\| \cdot L \cdot F
\end{aligned}$$

die Behauptung 4.2.

Um nun Gromovs Produkt kurzer geodätischer Schleifen und die Komposition der entsprechenden Bewegungen möglichst übersichtlich vergleichen zu können, verwenden wir die folgenden linksinvarianten Metriken für die orthogonale Gruppe und die Bewegungsgruppe:

**4.3. Definition.** Für  $A, B \in O(n)$  setze

$$d(A, B) = \max_{(Def)} \{ | \not{x} (AX, BX) |; X \in \mathbb{R}^n, |X|=1 \} = d(B^{-1} \circ A, id).$$

Für Bewegungen  $\tilde{A}, \tilde{B}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  ( $\tilde{A}(X) = A \cdot X + a$ , entsprechend  $\tilde{B}$ ) setze  $\tilde{d}(\tilde{A}, \tilde{B}) = \max_{(Def)} (d(A, B), |a - b|) = \tilde{d}(\tilde{B}^{-1} \circ \tilde{A}, id)$ .  
Abkürzungen:  $d(A, id) = \|A\|$ ,  $\tilde{d}(\tilde{A}, id) = \|\tilde{A}\|$ .

**4.4. Bemerkung.** Für die so definierte Metrik auf  $O(n)$  sind auch die Rechtstranslationen Isometrien:  $d(A, B) = d(A \cdot C, B \cdot C)$ . Die Metrik ist eine Finslermetrik, und die zugehörige Norm auf dem Tangentialraum  $T_{id} SO(n)$ , dem Vektorraum der schiefsymmetrischen Matrizen, ist gegeben durch

$$|S| = \max_{(Def)} \{ |S \cdot X|; X \in \mathbb{R}^n, |X|=1 \};$$

$$\text{z.B. ist f\"ur } S = \begin{pmatrix} 0 & a_1 & & \\ -a_1 & 0 & & \\ & & 0 & \\ & & & 0 & a_2 \\ & & & -a_2 & 0 \end{pmatrix} \quad |S| = \max_{i=1,2} |a_i|.$$

Die 1-Parameteruntergruppen von  $SO(n)$  sind bis zur Länge  $\pi$  kürzeste Verbindungen ihrer Endpunkte, und die Exponentialabbildung  $S \rightarrow \exp S = e^S = \text{id} + \sum_{k=1}^{\infty} S^k/k!$  vergrößert Abstände nicht:  $|S - T| \geq d(\exp S, \exp T)$ .

Mit  $\text{Ad } S: T \rightarrow [S, T] = ST - TS$  ist  $e^{\text{Ad } S}$  Normisometrie von  $T_{\text{id}} SO(n)$ , und  $d(e^S e^T e^{-S} e^{-T}, \text{id}) \leq |[S, T]| \leq 2|S| \cdot |T|$ , weil der Tangentialvektor der Kurve  $t \rightarrow S(t) = e^{t \cdot \text{Ad } T} \cdot S$  die Länge  $|[T, S]|$  hat und  $\exp S(t)$  eine Verbindungskurve von  $e^S$  nach  $e^T e^S e^{-T}$  ist.

Wir fassen die Wegabhängigkeit der Parallelverschiebungen zusammen in dem

**4.5. Satz.** Die folgenden Ungleichungen gelten bei Krümmung  $\leq 0$  für beliebige geodätische Schleifen  $\alpha, \beta$ , auf  $\epsilon$ -flachen Manningfaltigkeiten wenigstens für solche der Länge  $< \pi \epsilon^{-\frac{1}{2}} \cdot d(M)$ . Mit  $r(\alpha), t(\alpha)$  bezeichnen wir Rotations- und Translationsanteil der Bewegung  $m(\alpha)$ . Für die Schnittkrümmungen sei vorausgesetzt  $|K| \leq \Lambda^2$  (also  $\|R\| \leq \Lambda^2$  bei  $K \leq 0$  bzw.  $\|R\| \leq \frac{4}{3} \cdot \Lambda^2$  im allgemeinen).

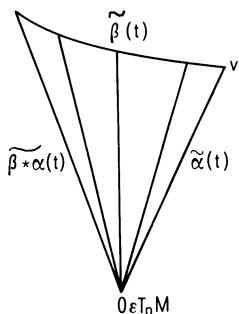
$$d(r(\beta) \cdot r(\alpha), r(\beta * \alpha)) \leq \Lambda^2 \cdot |t(\alpha)| \cdot |t(\beta)|$$

$$|r(\beta) \cdot t(\alpha) + t(\beta) - t(\beta * \alpha)| \leq (|t(\alpha)| + |t(\beta)|) \cdot \Lambda^2 \cdot |t(\alpha)| \cdot |t(\beta)|.$$

Die Abbildung  $\alpha \rightarrow m(\alpha)$  ist also bis auf quadratische Fehler ( $\sim \|R\| \cdot |t(\alpha)| \cdot |t(\beta)|$ ) ein Homomorphismus der kurzen Schleifen mit Gromov-Produkt in die Bewegungsgruppe von  $T_p M$ .

*Beweis.* Wegen  $m(\beta) \circ m(\alpha) = m(\beta \cdot \alpha)$  muß nur eine geeignet abschätzbare Homotopie zwischen der Schleife  $\beta * \alpha$  und der geschlossenen Kurve  $\beta \cdot \alpha$  angegeben werden.

Es sei  $v$  Anfangstantentialvektor der Schleife  $\alpha$ , also  $\exp v = \exp 0 = p$ . Liftinge nun die Schleife  $\beta$  mit  $\exp$  nach  $M^*$  beginnend mit  $\tilde{\beta}(0) = v$ . Man erhält in  $M^*$  ein geodätisches Dreieck, dessen dritte Seite durch  $\exp$  auf die Schleife  $\beta * \alpha$  projiziert wird, weil  $\exp$  das Dreieck zusammen mit den radialen geodätischen Sehnen auf eine verkürzende Homotopie (die «natürliche») zwischen der Kurve  $\beta \cdot \alpha$  und dem Bild der dritten Seite projiziert. Die Schranke  $L$  in 4.2 ist also  $|t(\alpha)| + |t(\beta)|$ . Es bleibt, die Fläche  $F$  dieser Homotopie abzuschätzen:



Wir betrachten das eingespannte Flächenstück mit der induzierten Metrik. Da das Flächenstück eine Schar von Geodätischen trägt, ist seine Krümmung höchstens so groß wie die entsprechenden Schnittkrümmungen der Metrik von  $M$ , also  $\leq 0$  bzw.  $\leq \Lambda^2$ . Da im zweiten Fall vorausgesetzt wird, daß die Summe

der Längen von  $\alpha$  und  $\beta$  noch  $< \frac{\pi}{\Lambda}$  ist, gelten die Toponogow-Alexandrowschen Winkelvergleichssätze für das Dreieck und daher auch der

**4.6. Flächenvergleichssatz** (Alexandrow). *Die Fläche des eingespannten Dreiecks ist höchstens so groß wie die Fläche eines Dreiecks mit denselben Seitenlängen in einem Raum der konstanten Krümmung max K. Bei  $K \leq 0$  hat ein*

Dreieck mit zwei Seitenlängen  $a, b$  also einen Flächeninhalt  $\leq \frac{1}{2} \cdot a \cdot b$ , bei  $K \leq \Lambda^2$

hat ein Dreieck mit den Seitenlängen  $a, b \leq \frac{\pi}{3\Lambda}$  einen Flächeninhalt  $\leq 0.7 \cdot a \cdot b$ .

Einsetzen dieser Abschätzungen für F in 4.1 und 4.2 gibt die Behauptung von Satz 4.5.

## 5. Genauere Untersuchung von Kommutatoren

In der orthogonalen und in der Bewegungsgruppe hat man die folgenden quadratischen Abschätzungen für Kommutatoren, die zur Folge haben, daß iterierte Kommutatoren von Elementen mit kleinem Rotationsanteil gegen die Identität konvergieren — im diskreten Fall nach *endlich* vielen Schritten:

$$5.1 \quad \| [A, B] \| \leq 2 \cdot \| A \| \cdot \| B \|$$

$$\| [\tilde{A}, \tilde{B}] \| \leq \max (\| [A, B] \|, \| A \| \cdot |b| + \| B \| \cdot |a|) \leq \| A \| \cdot \| \tilde{B} \| + \| \tilde{A} \| \cdot \| B \|.$$

Dies läßt erwarten, daß unter geeigneten Voraussetzungen auch iterierte Kommutatoren aus kurzen Schleifen (mit Gromovs Produkt) konvergieren, also nach *endlich* vielen Iterationen trivial sind, weil es ja kürzeste Schleifen gibt. Für dieses Ziel sind die Abschätzungen des vorherigen Abschnitts nicht gut genug, weil man nur dann eine Homotopie mit brauchbar abschätzbarer Fläche zwischen  $[\beta, \alpha] = \beta * \alpha * \beta^{-1} * \alpha^{-1}$  und  $\beta \cdot \alpha \cdot \beta^{-1} \cdot \alpha^{-1}$  angeben kann, wenn man die Länge der Schleife  $[\beta, \alpha]$  schon vorher abschätzen kann. Dazu verwenden wir die folgenden genauen Informationen über die Exponentialabbildung: Aus dem Satz von Rauch folgt für orthogonale  $A: T_p M \rightarrow T_p M$  und  $v \in T_p M$  (bei  $-\Lambda^2 \leq K$ ):

$$5.2 \quad d(\exp v, \exp A \cdot v) \leq \| A \| \cdot \frac{\sin h \Lambda \cdot |v|}{\Lambda \cdot |v|}.$$

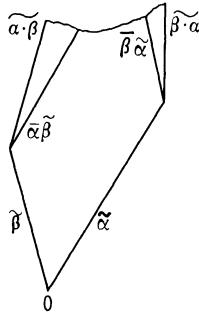
Verfeinerte Beweise [6] liefern bei  $|K| \leq \Lambda^2$  den folgenden

**5.3. Vergleichssatz für Parallelverschiebungen in M und  $T_p M$ .** Es sei  $w(t)$  ein paralleles Vektorfeld längs der Geodätschen  $t \rightarrow \exp t \cdot v$ , dann gilt:

$$d(\exp(v + w(0)), \exp w(1)) \leq \frac{1}{3} |v| \cdot |w| \cdot \sin |w| \cdot \Lambda \cdot \sinh \Lambda \cdot (|v| + |w|).$$

(Es wird also die euklidische Parallelverschiebung  $w(0) \rightarrow v + w(0)$  in  $T_p M$  und die Riemannsche Parallelverschiebung  $w(0) \rightarrow w(1)$  längs  $t \rightarrow \exp t \cdot v$ , jeweils gefolgt von  $\exp$ , verglichen.)

Um nun  $|t([\beta, \alpha])|$  abzuschätzen, genügt es, die Kurven  $\beta \cdot \alpha$  und  $\alpha \cdot \beta$  mit  $\exp$  zu von 0 ausgehenden Kurven  $\tilde{\beta} \cdot \alpha$  und  $\alpha \cdot \tilde{\beta}$  nach  $M^*$  zu liften und die Länge einer geeigneten Verbindungskurve der Endpunkte (in  $M^*$ ) abzuschätzen. Dazu



betrachten wir außerdem noch die Kurven  $\bar{\beta} \cdot \tilde{\alpha}$  bzw.  $\tilde{\alpha} \cdot \bar{\beta}$ , die dadurch entstehen, daß der Anfangsvektor  $b$  von  $\bar{\beta}$  längs  $\tilde{\alpha}$  (bzw. der Anfangsvektor  $a$  von  $\tilde{\alpha}$  längs  $\bar{\beta}$ ) parallel verschoben wird und dann die Geodätsche der Länge  $|b|$  (bzw.  $|a|$ ) in der erhaltenen Richtung angefügt wird. Der Abstand der Endpunkte von  $\bar{\beta} \cdot \tilde{\alpha}$  und  $\tilde{\alpha} \cdot \bar{\beta}$  (bzw. von  $\tilde{\alpha} \cdot \bar{\beta}$  und  $\bar{\beta} \cdot \tilde{\alpha}$ ) wird mit 5.2, der Abstand der Endpunkte von  $\bar{\beta} \cdot \tilde{\alpha}$  und  $\tilde{\alpha} \cdot \bar{\beta}$  wird mit Satz 5.3 abgeschätzt, die Summe dieser Schranken ist  $\geq |t([\beta, \alpha])|$ . Schließlich hat man wieder (sogar mehrere brauchbare) verkürzende Homotopien mit Hilfe geodätscher Sehnen von  $[\beta, \alpha]$  nach  $\beta \cdot \alpha \cdot \beta^{-1} \cdot \alpha^{-1}$ , die man sich in  $M$  aus drei Dreiecken mit zwei Seitenlängen  $|t(\alpha)|, |t(\beta)|$  bzw.  $|t(\alpha)|, \leq |t(\beta)| + |t([\beta, \alpha])|$  bzw.  $|t(\beta)|, |t([\beta, \alpha])|$  zusammengesetzt vorstellen kann. Die längste Kurve dieser Homotopie hat die Länge  $2(|t(\alpha)| + |t(\beta)|) = L$ , und für den Flächeninhalt gilt:  $F \leq 0.7 \cdot (2|t(\alpha)| \cdot |t(\beta)| + (|t(\alpha)| + |t(\beta)|) \cdot |t([\beta, \alpha])|)$ . Damit kann man 4.1 und 4.2 benutzen, um 5.1 auf Schleifen zu verallgemeinern:

#### 5.4. Satz. Für Gromov-Kommutatoren kurzer geodätscher Schleifen gilt:

$$\begin{aligned} \Lambda \cdot |t([\beta, \alpha])| &\leq \frac{2}{3} \Lambda \cdot |t(\alpha)| \cdot \Lambda \cdot |t(\beta)| \cdot \sinh \Lambda(|t(\alpha)| + |t(\beta)|) \\ &+ ||r(\alpha)|| \cdot \sinh \Lambda |t(\beta)| + ||r(\beta)|| \cdot \sinh \Lambda |t(\alpha)|, \\ ||r([\beta, \alpha])|| &\leq 2 \cdot ||r(\alpha)|| \cdot ||r(\beta)|| + 2\Lambda^2 \cdot |t(\alpha)| \cdot |t(\beta)| \\ &+ 2 \Lambda (|t(\alpha)| + |t(\beta)|) \cdot \Lambda |t([\beta, \alpha])|. \end{aligned}$$

Diese Ungleichungen werden übersichtlicher, wenn wir erstens die Norm einer Bewegung so an die Krümmungsschranke  $\Lambda$  anpassen, daß die Definition unempfindlich wird gegen Multiplikation der Riemannschen Metrik von  $M$  mit einer Konstanten:

#### 5.5. Neue Definition. $\|m(\alpha)\| = \max_{(Def)} (\|r(\alpha)\|, 3\Lambda \cdot |t(\alpha)|)$ ,

und wenn wir uns zweitens auf «kurze» Schleifen, nämlich  $|t(\alpha)|, |t(\beta)| \leq \frac{1}{8\Lambda}$  beschränken ( $\frac{2}{3} \sinh \frac{1}{4} \leq 0.17$ ,  $8 \cdot \sinh \frac{1}{8} \leq 1.003$ ).

#### 5.6. Korollar.

$$\Lambda \cdot |t([\beta, \alpha])| \leq 0.17 \cdot \Lambda |t(\alpha)| \cdot \Lambda |t(\beta)| + 1.003 \cdot (\|r(\alpha)\| \cdot \Lambda |t(\beta)| + \|r(\beta)\| \cdot \Lambda |t(\alpha)|),$$

$$\begin{aligned}\|r([\beta, \alpha])\| &\leq 2 \cdot \|r(\alpha)\| \cdot \|r\beta\| + 2\Lambda|t(\alpha)| \cdot \Lambda|t(\beta)| + \frac{1}{4}\Lambda|t([\beta, \alpha])|, \\ \|m([\beta, \alpha])\| &\leq 2,4 \|m(\alpha)\| \cdot \|m(\beta)\|.\end{aligned}$$

Aus  $\|m(\alpha)\|, \|m(\beta)\| \leq \frac{1}{3}$  folgt daher

$$\Lambda|t(\alpha)|, \Lambda|t(\beta)| \leq \frac{1}{9\Lambda} \text{ und } \|m([\beta, \alpha])\| \leq 0,8 \cdot \min(\|m(\alpha)\|, \|m(\beta)\|),$$

d.h. iterierte Kommutatoren derartiger Schleifen konvergieren gegen 0, sind also nach endlich vielen Schritten trivial.

## 6. Existenz von kurzen Schleifen mit kleinem Rotationsanteil

Um aus der Konvergenz der Kommutatoren geometrische Konsequenzen ziehen zu können, muß man sich kurze Schleifen mit einem Rotationsanteil  $\leq \frac{1}{3}$  verschaffen. Dies gelingt durch Iteration.

**6.1. Lemma.** Es sei  $A \in S0(n)$ ,  $0 < \varphi < \pi$  und  $m > \left(\frac{2\pi}{\varphi}\right)^{\frac{n}{2}}$ , dann gibt es unter den  $m$  Iterierten  $A, A^2, \dots, A^m$  mindestens eines,  $A^k$ , mit  $\|A^k\| \leq \varphi$ .

*Beweis.* Es genügt, festzustellen, wie viele Iterierte von  $A$  höchstens existieren mit paarweisem Abstand  $> \varphi$ . Alle Iterierten  $A^j$  liegen auf einem Torus der Dimension  $\leq \frac{n}{2}$  in  $S0(n)$ , und die Exponentialabbildung von  $S0(n)$ , eingeschränkt auf den Tangentialraum dieses Torus, ist längentreu und schon auf der Finsler-Kugel  $B_{\pi} = \{S; |S| \leq \pi\}$  surjektiv. Wenn andererseits die Kugeln vom Radius  $\frac{\varphi}{2}$  um die Iterierten von  $A$  disjunkt sind, so kann es offenbar höchstens  $\text{vol}(B_{\pi})/\text{vol}(B_{\frac{\varphi}{2}}) = \left(\frac{2\pi}{\varphi}\right)^{\dim \text{Torus}} \leq \left(\frac{2\pi}{\varphi}\right)^{\frac{n}{2}}$  derartige Iterierte von  $A$  geben. Um mit den Schleifenprodukten dasselbe zu erreichen, muß man nur noch den Homotopiefehler berücksichtigen.

**6.2. Satz.** Falls zu gegebenem  $0 < \eta \leq \frac{1}{3}$  eine Schleife  $\alpha$  der Länge  $\leq \frac{1}{\Lambda} \cdot \left(\frac{\eta}{6,5}\right)^{\frac{n}{2}+1}$  existiert (im  $\varepsilon$ -flachen Fall  $\frac{1}{\Lambda} = \pi \cdot \varepsilon^{-\frac{1}{2}} d(M)$ ), so setze  $\varphi = \eta \cdot (1 - \frac{1}{18}\eta)$  und wähle  $m > 2\left(\frac{2\pi}{\varphi}\right)^{\frac{n}{2}}$ . Dann gibt es unter den  $m$  Iterierten  $\alpha, \alpha * \alpha, \dots, \alpha^{*m}$  mindestens eine  $\alpha^{*k}$ , mit  $\|m(\alpha^{*k})\| \leq \eta \leq \frac{1}{3}$ .

*Beweis.* Der Flächeninhalt der natürlichen Homotopie zwischen der Schleife  $\alpha^{*k}$  und der geschlossenen Kurve  $\alpha \cdot \dots \cdot a$  ( $k$ -mal) ist  $\leq 0,7|t(\alpha)|^2 (1 + 2 + \dots + k - 1)$ .

Daher folgt aus Lemma 6.1 und Satz 4.5  $\|r(\alpha^{*k})\| \leq \varphi + \Lambda^2 \cdot |t(\alpha)|^2 \cdot \frac{m^2}{2}$ , also

$$\|m(\alpha^{*k})\| \leq \max(\varphi + \Lambda^2 \cdot |t(\alpha)|^2 \cdot \frac{m^2}{2}, 3\Lambda|t(\alpha)| \cdot m).$$

Falls nun  $3\Lambda \cdot |t(\alpha)| \cdot m \leq \eta$  ist, so ist auch  $\varphi + \Lambda^2 \cdot |t(\alpha)|^2 \frac{m^2}{2} \leq \varphi + \frac{1}{18}\eta^2 = \eta$ . Dafür ist die Bedingung

$$\Lambda \cdot |t(\alpha)| \leq \frac{\eta}{6} \left( \frac{\varphi}{2\pi} \right)^{\frac{n}{2}} = \frac{\eta}{6} \left( \frac{\eta}{2\pi} \left( 1 - \frac{1}{18}\eta \right) \right)^{\frac{n}{2}}$$

hinreichend, erst recht also die Bedingung

$$\Lambda |t(\alpha)| \leq \left( \frac{\eta}{6,5} \right)^{\frac{n}{2} + 1}$$

in Satz 6.2.

## 7. Geometrische Anwendungen

Das folgende Resultat spielt keine Rolle in Gromovs Beweis, zeigt aber, wie unmittelbar aus der Kommutatorabschätzung geometrische Ergebnisse folgen.

**7.1. Satz.** (Margulis, Heintze [5]). *M sei kompakte Riemannsche Mannigfaltigkeit und  $-1 \leq K < 0$ . Dann gibt es einen Punkt  $q \in M$ , in dem der Injektivitätsradius der Exponentialabbildung  $\geq \frac{1}{2} \cdot (\frac{1}{20})^{\frac{n}{2} + 1}$  ist. Insbesondere besitzt also das Volumen von M eine nur von der Dimension abhängige untere Schranke:*

$$\text{vol}(M) \geq \text{vol}(S^{n-1}) 2^{-n} \cdot (\frac{1}{20})^{\frac{1}{2}n^2 + n} \geq \left( \frac{\pi^n}{(n-1)!} \cdot (\frac{1}{20})^{n^2 + 2n} \right)^{\frac{1}{2}}.$$

*Beweis.* Betrachte eine kürzeste geschlossene Geodätische  $\alpha$  auf M. Falls  $\alpha$  schon länger als  $(\frac{1}{20})^{\frac{n}{2} + 1}$  ist, ist nichts mehr zu beweisen. Andernfalls wähle einen Punkt  $p$  auf  $\alpha$ , konstruiere von  $p$  aus die universelle Überlagerung  $\tilde{M}$  (z.B. durch Lifting der Metrik mit  $\exp$  von M nach  $T_p M$ ) und betrachte die maximale zyklische Untergruppe  $H$  der Deckgruppe, die die Überlagerung  $\tilde{\alpha}$  von  $\alpha$  in sich verschiebt. Weiter sei  $\tilde{\gamma}$  eine von  $p$  senkrecht zu  $\alpha$  loslaufende Geodätische und  $\tilde{\gamma}$  ihre Überlagerung ( $\tilde{\alpha}(0) = \tilde{\gamma}(0)$ ). Für jedes  $h \in H$  wächst  $d(\tilde{\gamma}(t), h(\tilde{\gamma}(t)))$  monoton mit  $t$  gegen unendlich; da außerdem nur für endlich viele  $h$  gelten kann:  $d(\tilde{\gamma}(0), h(\tilde{\gamma}(0))) < (\frac{1}{20})^{\frac{n}{2} + 1}$ , gibt es ein kleinstes  $\tau > 0$  und ein  $h_0 \in H$ , so daß für alle  $h \in H$  gilt:

$d(\tilde{\gamma}(\tau), h(\tilde{\gamma}(\tau))) \geq d(\tilde{\gamma}(\tau), h_0(\tilde{\gamma}(\tau))) = (\frac{1}{20})^{\frac{n}{2} + 1}$ . Entweder ist nun in  $\exp(\tilde{\gamma}(\tau)) = q$  der Injektivitätsradius von  $\exp$  schon  $\geq \frac{1}{2} \cdot (\frac{1}{20})^{\frac{n}{2} + 1}$ , oder es gibt eine geodätische Schleife  $\beta$  in  $q$ , die kürzer als  $(\frac{1}{20})^{\frac{n}{2} + 1}$  ist, d.h. es gibt in der Deckgruppe ein Element  $g$  mit  $d(\tilde{q}, g(\tilde{q})) < (\frac{1}{20})^{\frac{n}{2} + 1}$ , also  $g \notin H$ . Auf die geodätischen Schleifen, die  $g$  und  $h_0$  in  $q$  repräsentieren, ist Satz 6.2 und Korollar 5.6 anwendbar, d.h. geeignete Potenzen  $g^k, h_0^l$  erzeugen eine nilpotente Untergruppe der Deckgruppe von M. Daraus und da M negative Krümmung hat, schließt man, daß  $g$  und  $h$  dieselbe invariante Geodätische in M besitzen, im Widerspruch zu  $g \notin H$ . Damit

kann die Schleife  $\beta$  in  $q$  nicht kürzer als  $(\frac{1}{20})^{2^{-n+1}}$  sein, und der Satz ist bewiesen. Die nächste Anwendung ist für Gromovs Beweis wesentlich. Zunächst stellt man folgendes fest: Zu jeder Wahl von Konstanten  $\eta_1 \leq \eta_2 \leq \frac{1}{3}$  betrachte die endliche Menge  $U = \{\alpha; \alpha \text{ Schleife in } p, \|r(\alpha)\| \leq \eta_1, \|m(\alpha)\| \leq \eta_2\}$ . Wähle nun  $d$  so groß, daß für alle  $0 \neq \alpha \in U$  gilt  $(0,8)^d \cdot \eta_2 < \|m(\alpha)\|$ . Dann sind alle  $d$ -fachen Kommutatoren von Elementen aus  $U$  trivial (5.6), d.h. die von  $U$  erzeugte Gruppe  $\langle U \rangle$  ist nilpotent, jedoch mit einer vom Injektivitätsradius von  $M$  abhängigen Abschätzung des Nilpotenzgrades  $d$ . Der nächste Schritt in Gromovs Beweis kann überraschender Weise diese Abhängigkeit beseitigen!

**7.2. Satz.** *Der Nilpotenzgrad aller in der eben beschriebenen Weise erzeugten Gruppen ist  $\leq (\frac{40}{13})^{\frac{1}{2} \cdot (n^2 + n)}$ .*

*Beweis.* Wir wählen mit Gromov ein möglichst geschicktes Erzeugendensystem für  $U$ , eine sogenannte «kurze Basis»  $\{\alpha_1, \dots, \alpha_k\}$ . Betrachte die Menge  $\bar{U} = \{\alpha; \alpha \text{ Schleife in } p, \|m(\alpha)\| \leq 2 \cdot \eta_2\}$ . Wir sagen, ein Produkt aus Elementen in  $U$  sei in  $\bar{U}$  definiert, wenn es mindestens eine Klammerung des Produktes gibt, welche induktiv in  $\bar{U}$  ausgeführt werden kann. Nun wähle  $\alpha_1 \in U$  so, daß  $\|m(\alpha_1)\|$  minimal ist. Weiter seien  $\alpha_1, \dots, \alpha_j$  schon gewählt und  $U_{(j)} \subset U$  sei die Menge der in  $\bar{U}$  definierten Produkte der  $\alpha_1, \dots, \alpha_j$ , die wieder in  $U$  liegen. Dann wähle  $\alpha_{j+1} \in U \setminus U_{(j)}$  so, daß  $\|m(\alpha_{j+1})\|$  minimal ist. Nach endlich vielen Schritten ist  $U_{(k)} = U$ , also die von  $\alpha_1, \dots, \alpha_k$  erzeugte Gruppe  $= \langle U \rangle$ . Weiter gilt nach Konstruktion für die Elemente der kurzen Basis  $\|m(\alpha_1)\| \leq \|m(\alpha_2)\| \leq \dots \leq \|m(\alpha_k)\|$  und wegen 5.6 auch  $\|m([\alpha_i, \alpha_j])\| \leq 0,8 \cdot \min(\|m(\alpha_i)\|, \|m(\alpha_j)\|)$ . Daher ist  $[\alpha_i, \alpha_j] = (\alpha_i * \alpha_j) * (\alpha_i^{-1} * \alpha_j^{-1})$  in  $\bar{U}$  definiert und  $[\alpha_i, \alpha_j] \in U_{(\min(i, j) - 1)}$ . Es folgt

**7.3.** Der Nilpotenzgrad von  $\langle U \rangle$  ist  $\leq k$ , der Länge einer kurzen Basis. Schließlich gilt für je zwei Elemente  $\alpha_i, \alpha_j$  (mit  $i < j$ ) der kurzen Basis

$$(7.4) \quad \|m(\alpha_i^{-1} * \alpha_j)\| \geq \|m(\alpha_j)\| = \max(\|m(\alpha_i)\|, \|m(\alpha_j)\|).$$

*Beweis von 7.4.* Wäre  $\|m(\alpha_i^{-1} * \alpha_j)\| < \|m(\alpha_j)\|$ , so wäre  $\alpha_i^{-1} * \alpha_j \in U_{(j-1)}$  nach Definition von  $\alpha_j$ ; daraus entsteht der Widerspruch  $\alpha_j = \alpha_i * (\alpha_i^{-1} * \alpha_j) \in U_{(j-1)}$ . Aus den Ungleichungen 7.4 erhalten wir mit Satz 4.5 (d.h. wegen  $3\Lambda \cdot (\|t(\alpha_i)\| + \|t(\alpha_j)\|) \leq 1$  mit  $d(m(\alpha_i)^{-1} \circ m(\alpha_j), m(\alpha_i^{-1} * \alpha_j)) \leq \Lambda^2 \|t(\alpha_i)\| \cdot \|t(\alpha_j)\| \leq \frac{1}{9} \cdot \|m(\alpha_i)\| \cdot \|m(\alpha_j)\|$ ) die folgenden Ungleichungen für die Bewegungen  $\tilde{A}_i = m(\alpha_i)$  ( $i = 1, \dots, k$ ):

$$(7.5) \quad \|\tilde{A}_i^{-1} \circ \tilde{A}_j\| \geq \max(\|\tilde{A}_i\| - \frac{1}{27} \|\tilde{A}_j\|, \|\tilde{A}_j\| - \frac{1}{27} \|\tilde{A}_i\|).$$

Der Beweis von Satz 7.2 wird beendet durch 7.3 und das

**7.6. Lemma.** Es gibt höchstens  $(\frac{40}{13})^{\frac{1}{2} \cdot (n^2 + n)}$  Bewegungen  $A_i$ , die paarweise die Ungleichungen 7.5 erfüllen.

*Beweis von 7.6.* Es sei  $V$  der Vektorraum der  $n$ -reihigen, schiefsymmetrischen Matrizen und  $W = V \times \mathbb{R}^n$ . Auf  $W$  definieren wir eine Norm durch

$$7.7 \quad |w| = |(S, a)| = \max(|S|, 3\Lambda|a|) \text{ (vgl. 4.4).}$$

Jede Bewegung  $\tilde{A}$  mit  $\|\tilde{A}\| \leq \frac{1}{3}$  lässt sich schreiben als  $\tilde{A}(X) = \exp(S) \cdot X + a$  mit  $\|\tilde{A}\| = |(S, a)|$ .

Damit übersetzt sich  $d(e^T, e^S) \leq |T - S|$  aus 4.4 in

$$\|\tilde{A}_i^{-1} \circ \tilde{A}_j\| \leq |(S_i - S_j, a_i - a_j)|.$$

Es gibt demnach nicht mehr Bewegungen  $\tilde{A}_i$ , die paarweise 7.5 erfüllen, als es Elemente  $w_i \in W$  gibt, für die paarweise gilt:

$$7.8 \quad |w_i - w_j| \geq \max(|w_i| - \frac{1}{27}|w_j|, |w_j| - \frac{1}{27}|w_i|).$$

Schließlich gewinnen wir aus 7.8 ebenso viele Lösungen auf der Einheitssphäre von  $W$ : Betrachte jedes Paar  $w_i, w_j$  für sich; nach zentrischer Streckung sei etwa  $1 = |w_j| \leq |w_i|$ . Dann gilt:

$$\begin{aligned} \left| \frac{w_i}{|w_i|} - w_j \right| &\geq |w_i - w_j| - \left| w_i - \frac{w_i}{|w_i|} \right|_{(7.8)} \geq |w_i| - \frac{1}{27}|w_j| - \left| w_i - \frac{w_i}{|w_i|} \right| \\ &= \left| \frac{w_i}{|w_i|} \right| - \frac{1}{27}|w_j|. \end{aligned}$$

Es gibt demnach mindestens so viele *Einheitsvektoren*  $w_i$  in  $W$ , für die paarweise gilt

$$7.9 \quad |w_i - w_j| \geq \frac{26}{27}, \quad |w_i| = |w_j| = 1,$$

wie es Lösungen  $\tilde{A}_i$  von 7.5 gibt.

Da die Kugeln vom Radius  $\frac{13}{27}$  um die Lösungen  $w_i$  von 7.9 disjunkt sind und in der Kugel vom Radius  $\frac{40}{27}$  um 0 enthalten sind, besitzt 7.9 höchstens  $(\frac{40}{13})^{\dim W}$  Lösungen und 7.6 ist bewiesen.

Dieser an Argumente aus der Geometrie der Zahlen erinnernde Beweis beendet unseren Bericht über den Teil von Gromovs Arbeit, der zu unserem Thema «Diskrete Gruppen und kleine Krümmung» gehört.

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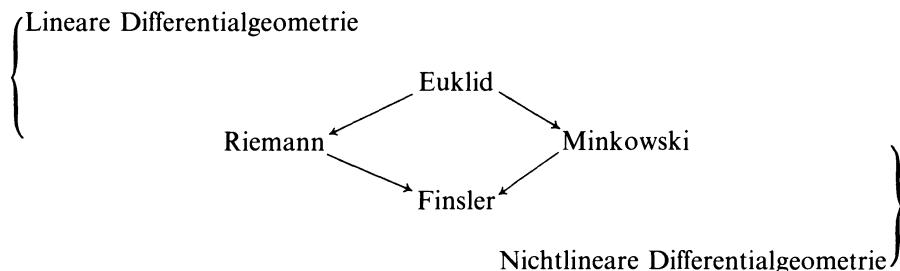
# Woldemar Barthel

## Nichtlineare Differentialgeometrie, insbesondere Minkowski-Geometrie

In diesem Übersichtsbericht zur nichtlinearen Differentialgeometrie sollen nach ihrer *Problemstellung* zunächst die wesentlichsten Phasen ihrer *historischen Entwicklung* skizziert werden. Über den Spezialfall der *Minkowskischen Differentialgeometrie* berichten wir dann ausführlicher<sup>1</sup>, während weitere Themen der Minkowski-Geometrie, die zum Teil stark in die Konvexgeometrie reichen, wieder nur skizziert werden sollen. Auf die Finsler-Geometrie können wir hier nicht näher eingehen. In einem so abgesteckten Rahmen müssen leider viele Namen und interessante Untersuchungen unerwähnt bleiben. Aber in der jeweils angegebenen Literatur finden sich häufig auch weitere ausführliche Überblicke zu den einzelnen Problemkreisen.

### §0. Zur Problemstellung der nichtlinearen Differentialgeometrie

Bei einer Verallgemeinerung der n-dimensionalen *euklidischen Geometrie* sind verschiedene Stadien möglich:



1. Wird der (flache) affine Raum durch eine Mannigfaltigkeit mit linearer Parallelverschiebung ersetzt und punktal die euklidische Metrik beibehalten, so gewinnt man die *Riemannsche Geometrie*.
2. Wird jedoch die affine Struktur des Raumes beibehalten und die euklidische Metrik durch eine allgemeinere Minkowski-Metrik ersetzt, die etwa durch einen konvexen Körper als Eichfigur gegeben ist, so erhält man die *Minkowski-Geometrie*. Dieser Übergang von der euklidischen zur Minkowskischen

<sup>1</sup> Die hier angesprochenen Probleme der Minkowskischen Differentialgeometrie, u.a. von Busemanns Arbeit [25] und der darauf aufbauenden Dissertation Bibersteins [20] angeregt, wurden zum Teil in einer Reihe von Staatsexamensarbeiten 1972–1976 und unter Berücksichtigung der direkten Schreibweise in einem viersemestrigen Vorlesungszyklus über klassische und nichtlineare Differentialgeometrie 1973–1975 sowie in Seminaren an der Universität Würzburg behandelt.

Geometrie liefert eine *Nichtlinearität der Metrik* in folgendem Sinn: Eine Minkowski-Metrik ist im allgemeinen nicht mehr durch eine Bilinearform bestimmt.

Diese Geometrien lassen eine weitere Verallgemeinerung zu:

3. Wird in der Riemannschen Geometrie punktal die euklidische Metrik durch eine Minkowski-Metrik und die lineare Parallelverschiebung durch eine nichtlineare Parallelverschiebung ersetzt, so gelangt man zur *Finsler-Geometrie*. Sie ergibt sich auch, wenn in der Minkowski-Geometrie der affine Raum durch eine Mannigfaltigkeit mit nichtlinearer Parallelverschiebung ersetzt und punktal die Minkowski-Metrik beibehalten wird.

Wir wollen diese klassischen Verallgemeinerungen der euklidischen Geometrie nun nicht nur bezüglich der *Längenmessung* und *Parallelverschiebung von Vektoren*, sondern auch bezüglich der *p-Areal-Messung* und *Parallelverschiebung zerlegbarer p-Vektoren* verstehen. Allerdings sollte man dieses Vorgehen, bei den differentialgeometrischen Strukturen

- a) Länge, p-Areal (und Volumen),
- b) Parallelverschiebung von Vektoren, zerlegbaren p-Vektoren (und n-Vektoren)

schrittweise die Linearität und – für die verschiedenen Unterdimensionen p – die gegenseitige Kopplung zu verlassen, nicht nur als Verallgemeinerung ansehen. Hierin liegt eine *Axiomatisierung dieser differentialgeometrischen Strukturen*, die nicht nur allgemeinere Geometrien liefert, sondern auch für die klassische Differentialgeometrie etwa im euklidischen und affinen Raum interessante Rückwirkungen hat und neue Gesichtspunkte zeigt.

In §2 bis §4 wollen wir speziell auf die Kurven- und Flächentheorie im 3-dimensionalen Minkowski-Raum etwas näher eingehen, also nur die Axiomatisierung der metrischen Strukturen im affinen Raum durchführen und deren Auswirkung auf Kurven und Flächen betrachten.

## §1. Zur historischen Entwicklung

Die Entwicklung der nichtlinearen Differentialgeometrie ist durch folgende Phasen gekennzeichnet:

1. Finslers Dissertation [29] führt 1918 erstmals systematisch einen allgemeinen *Längenbegriff* in die Differentialgeometrie auf einer Mannigfaltigkeit ein.
2. Berwald [19], Synge [42] und Taylor [43] erkennen 1925 unabhängig voneinander, daß zu einer solchen Differentialgeometrie auch die Struktur einer *Parallelverschiebung* gehört.
3. E. Cartan [27] gibt 1933 eine Lösung der Frage an, wie diese beiden Strukturen, Länge und Parallelverschiebung, *aneinander gekoppelt* werden können. Er verändert die Parallelverschiebung so, daß das Ricci-Lemma  $Dg_{ik} = 0$  erfüllt werden kann. Damit hat Cartan einen wichtigen Schritt getan, allerdings auch die Finsler-Geometrie in eine gewisse Einbahnstraße geführt. Seine Lösung wird nämlich durch die Konzeption erkauft, daß ein Finsler-Raum eine Linienelement-Mannigfaltigkeit sein muß.
4. Ebenfalls 1933 erkennt E. Cartan [26] einen ganz anderen Aspekt einer

metrischen Differentialgeometrie, indem er von einem allgemeinen *Arealbegriff* und einer zugehörigen *Parallelverschiebung* ausgeht.

**5.** Kawaguchi [33] behandelt 1936 erstmals eine n-dimensionale Differentialgeometrie, die auf einem allgemeinen *p-dimensionalen Areal* begründet ist. Er beschränkt sich jedoch auf den Spezialfall, daß ein sogenannter metrischer Tensor 2. Stufe existiert, eine für die Grenzdimensionen  $p=1$  und  $p=n-1$  zwar erfüllte, aber bei beliebiger Zwischendimension  $p$  geometrisch nicht motivierte Voraussetzung.

**6.** Busemann [21], [24] hat nach 1945 die Entwicklung der Finsler-Geometrie durch die Konzeption korrigiert, daß ein Finsler-Raum eine *Punkt-Mannigfaltigkeit* sein soll. Außerdem erkennt er die Bedeutung eines *Volumenbegriffs*, den er aus der Längenmessung konstruiert und der auch zu *p-Arealen* führt. Busemanns wiederholte Mahnung, daß der Weg zur Finsler-Geometrie über eine befriedigende Behandlung des Spezialfalls der *Minkowski-Geometrie* führe, wird durch seine Arbeit [25] deutlich unterstrichen.

**7.** Rund [39] gibt 1950 für eine Finslersche Punkt-Mannigfaltigkeit eine Parallelverschiebung an, die allerdings nicht längeninvariant ist. Von ihm stammt später auch die erste Monographie zur Differentialgeometrie der Finsler-Räume [40].

**8.** Eigene seit 1952 durchgeführte Untersuchungen [4], [5], [6], [7], fortgesetzt in [16] und [18], haben zu folgender Auffassung geführt:

a) Ein Finsler-Raum sollte eine *Punkt-Mannigfaltigkeit* sein, auf der zwei Arten von Strukturen vorgegeben sind, nämlich *metrische Strukturen* wie Länge, p-Areal und Volumen sowie *affine Strukturen* der Parallelverschiebung von Vektoren, zerlegbaren p-Vektoren und n-Vektoren. Dabei sollen die affinen und metrischen Strukturen durch die Invarianz der Länge, des p-Areals und des Volumens bei Parallelverschiebung *aneinander gekoppelt* sein.

b) Die Eigenschaften der differentialgeometrischen Grundstrukturen sollten nicht a priori festgelegt werden, sondern es sollte anhand konkreter differentialgeometrischer Probleme getestet werden, wieweit gewisse Eigenschaften der Grundstrukturen führen. Indem so immer wieder die Basis zur Diskussion gestellt wurde, hat sich bereits eine weitgehende Klärung der für die nichtlineare Differentialgeometrie erforderlichen Grundstrukturen ergeben.

**9.** Die *direkte und globale Beschreibung* der Finsler-Geometrie, die mittels Faserbündeln möglich ist, wird zuerst von Matsumoto [35] 1960 für die Finsler-Geometrie im Sinne von E. Cartan durchgeführt und später zusammenfassend dargestellt [36]. Von ihm stammt auch eine zweite neue Monographie zur Finsler-Geometrie [37].

**10.** Eine globale Behandlung nichtlinearer Parallelverschiebungen wird für Vektoren in einer eigenen Arbeit [13] 1962 ausgeführt und für zerlegbare p-Vektoren von Frau Haubitz [31] 1970 fortgesetzt. Für die nichtlineare Differentialgeometrie wichtige Untersuchungen des Graßmann-Kegels und des p-Areals werden in direkter Darstellung 1975 von Kern [34] und Steiner [41] durchgeführt. Auf diesen Grundlagen gewinnt Pabel [38] neue Ergebnisse zur nichtlinearen Differentialgeometrie auf Mannigfaltigkeiten, unter anderem auch zur Kurven-, p-Flächen- und Hyperflächentheorie.

**11.** Bisher war im wesentlichen von metrischer nichtlinearer Differentialgeometrie die Rede. Die Blaschkesche *affine Differentialgeometrie* ist in

[14] 1965 und in [17] 1975 auf Mannigfaltigkeiten mit linearer Parallelverschiebung verallgemeinert worden. Eine affine Hyperflächentheorie auf Mannigfaltigkeiten mit nichtlinearer Parallelverschiebung geben 1970 Frau Attallah [2] und daran anschließend 1974 Volkmer [44]. Dabei zeigen gerade die Untersuchungen von Volkmer, daß diese Verallgemeinerung auch zu neuen Einsichten in die klassische Blaschkesche Affingeometrie führt. Diese Einsichten haben einen wesentlichen Einfluß auf neuere Untersuchungen von Fräulein Arnold [1] zur Affingeometrie auf Mannigfaltigkeiten mit linearem Zusammenhang, insbesondere deren Kurven- und p-Flächentheorie.

## §2. Affiner Raum und metrische Strukturen

Ein reeller n-dimensionaler *affiner Raum*  $A^n := (A, V, \rightarrow : A \times A \rightarrow V)$  besteht bekanntlich aus einer Menge  $A$ , einem n-dimensionalen Vektorraum  $V$  über  $\mathbb{R}$  und einer Abbildung  $\rightarrow$ , die Verbindungsvektoren bildet mit den Eigenschaften der eindeutigen Abtragbarkeit von Vektoren und der geometrischen Vektoraddition.

Zunächst muß die *Tensoralgebra* über  $V$  weiterentwickelt werden. Man hat neben dem Vektorraum  $V$  der Vektoren den Dualraum  $V^*$  der Linearformen, die p-ten äußeren Potenzen  $\wedge^p V$  der p-Vektoren und  $\wedge^p V^*$  der p-Formen. Wegen  $\dim \wedge^n V = \dim \wedge^n V^* = 1$  können wir *Dichten über dem Vektorraum V* einführen: Für  $w \in \mathbb{Z}$  heißen die Elemente bei  $w \geq 0$  von  $\otimes^w \wedge^n V^*$  und bei  $w \leq 0$  von  $\otimes^{-w} \wedge^n V$  *Skalardichten* vom Gewicht  $w$  über  $V$ . Sie bilden eine Folge 1-dimensionaler Vektorräume

$$\dots \otimes^2 \wedge^n V, \wedge^n V, \mathbb{R}, \wedge^n V^*, \otimes^2 \wedge^n V^*, \dots,$$

die viele Eigenschaften mit den reellen Zahlen gemeinsam haben:

- i) Addition zweier Skalardichten vom gleichen Gewicht.
- ii) Tensormultiplikation zweier Skalardichten liefert eine Skalardichte vom Summengewicht (darin ist der Spezialfall des reellen Vervielfachens enthalten).
- iii) Tensormultiplikation mit Skalardichten ist kommutativ.
- iv) Die Inverse einer Skalardichte  $\neq 0$  ist eine Skalardichte vom negativen Gewicht.

( $\mathbb{R}$  ist unter den 1-dimensionalen Vektorräumen der Skalardichten dadurch ausgezeichnet, daß eine natürliche Basis existiert, nämlich 1.)

Die für uns wichtigsten *p-Vektor-dichten* und *p-Form-dichten* sind die Elemente von

$V$	$V^*$	$\wedge^{n-1} V$	$\wedge^{n-1} V^*$	$\wedge^n V$	$\wedge^n V^*$
und von					
$\wedge^{n-1} V^* \otimes \wedge^n V$	$\wedge^{n-1} V \otimes \wedge^n V^*$	$V^* \otimes \wedge^n V$	$V \otimes \wedge^n V^*$		

Dabei sind untereinanderstehende Vektorräume natürlich-isomorph (aber nicht identifiziert), ihre Elemente können im affinen Raum *veranschaulicht* werden durch orientierte

Strecken	Platten	$(n-1)$ -Parallelogramme	Balken	n-Parallelootope	n-Parallelootope
(also Pfeile)					

Man vergleiche dazu die abgedruckten Computer-Stereobilder<sup>2</sup>.

Im affinen Raum  $A^n$  führen wir nun ein:

Eine *Längenfunktion*  $l: V \rightarrow \mathbb{R}$ , die positiv definit, positiv homogen und differenzierbar ist.

Eine *Arealfunktion*  $F: X^{n-1}V \rightarrow \mathbb{R}$  mit den analogen Eigenschaften. Dann gibt es genau eine Arealfunktion  $f: \Lambda^{n-1}V \rightarrow \mathbb{R}$  mit  $F = f \circ \wedge$ .

Auf eine *Volumenfunktion*, d.h. die Auszeichnung einer Determinantenform  $v \in \Lambda^n V^* \setminus \{0\}$ , verzichten wir. Dies erfordert aber gerade die Benutzung von Dichten. Eine Volumenfunktion  $v$  hat eben einen ganz anderen Charakter als  $l$  und  $F$ , sie ist linear. Tatsächlich wird sie in der euklidischen Differentialgeometrie weniger als geometrische Größe, sondern mehr als Größe der linearen Algebra benutzt, um die Tensoralgebra zu vereinfachen, d.h. Dichten zu vermeiden.

Hier müßte an die Differentialrechnung für Vektorräume erinnert werden: Differenzierbarkeit und Differentiale von Abbildungen zwischen Vektorräumen. Vgl. etwa [28], Kapitel 8.

### §3. Kurventheorie im 3-dimensionalen Minkowski-Raum ( $A^3, l, F$ )

Für eine orientierte *Kurve* mit der Parameterdarstellung  $t \mapsto x(t)$  können wir die folgenden Grundgrößen definieren:

*Bogenlänge*  $t \mapsto s(t) := \int l(\dot{x}(t)) dt$  und damit Übergang zur Bogenlängenparameterdarstellung  $s \mapsto x(s)$ .

*Tangenteneinheitsvektor*  $t := \dot{x}'$  und *Krümmung*  $\kappa := f(t \wedge t')$ , die ab jetzt nullstellenfrei sein soll.

*l-Transversalvektor*  $h := \frac{\ddot{x}''}{f(\dot{x}' \wedge \dot{x}'')}$ , *Schmiegbivektor*  $\mathfrak{s} := t \wedge h$  und *Torsionsdichte*

$T := t \wedge h \wedge h' = \mathfrak{s} \wedge h'$  vom Gewicht  $(-1)$ .

Um  $t, h$  zu einer Begleitbasis der Kurve zu ergänzen, gibt es zwei natürliche Möglichkeiten, eine Vektorordnungsvektor vom Gewicht 1 (darstellbar durch einen Balken) zu konstruieren, nämlich

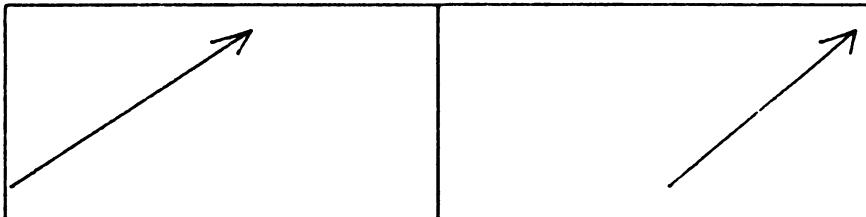
die *F-Transversalvektordichte*<sup>3</sup>  $\mathfrak{B} := t \times h = \frac{F}{2} \mathfrak{s}$ , *F-transversal* zu  $\mathfrak{s} = t \wedge h$ , und

die *l-Transversalvektordichte*  $\mathfrak{B}^* := \mathfrak{B} - dl(t)(\mathfrak{B}) \otimes t$ , *l-transversal* zu  $t$ ,  
(die allerdings im euklidischen Raum zusammenfallen und die Binormale

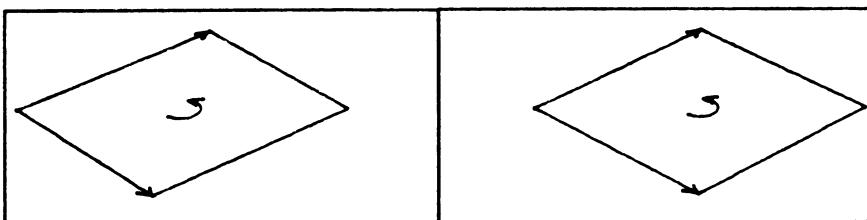
2 Diese Stereobilder wurden von Frau Haubitz nach einem von ihr zum Zeichnen allgemeiner differentialgeometrischer Figuren entwickelten Verfahren [32] am Rechenzentrum der Universität Würzburg hergestellt. Die einzelnen Bildpaare sind *zur direkten Betrachtung ohne optische Hilfsmittel* gedacht. Man sollte sie aus einem Abstand von 60 cm ansehen, indem man einen Punkt in 30 cm Entfernung fixiert. Dort erscheint dann, zunächst vielleicht unscharf, ein verkleinertes Raumbild der entsprechenden Figur.

3 Das Minkowskische *F-Vektorprodukt*  $\eta \times_3 \mathfrak{z}$  zweier Vektoren oder, was das gleiche ist, der Wert des 2-Sternoperators  $\frac{F}{2}(\eta \wedge \mathfrak{z})$  ist als Vektorordnungsvektor vom Gewicht 1 durch einen Balken darstellbar, dessen Querschnitt  $\eta \wedge \mathfrak{z}/F^2(\eta \wedge \mathfrak{z})$  das Areal  $1/F(\eta \wedge \mathfrak{z})$  hat und dessen erzeugende Gerade *F-transversal* zu  $\eta \wedge \mathfrak{z}$  ist.

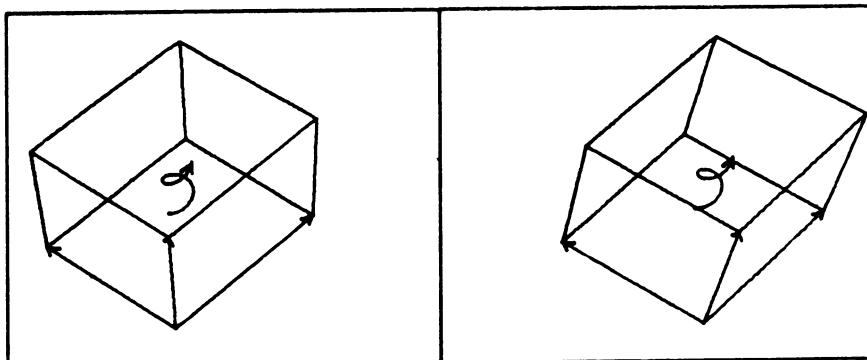
$\vee$  Vektor

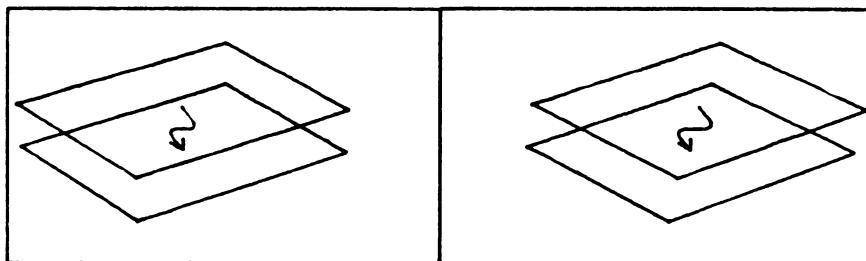
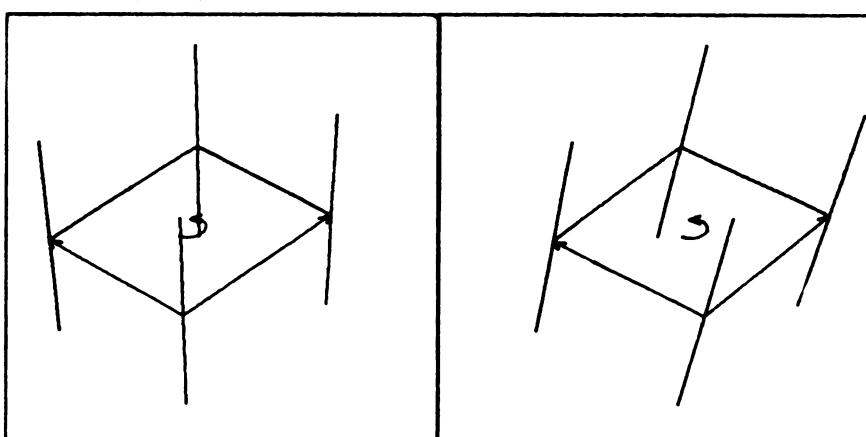
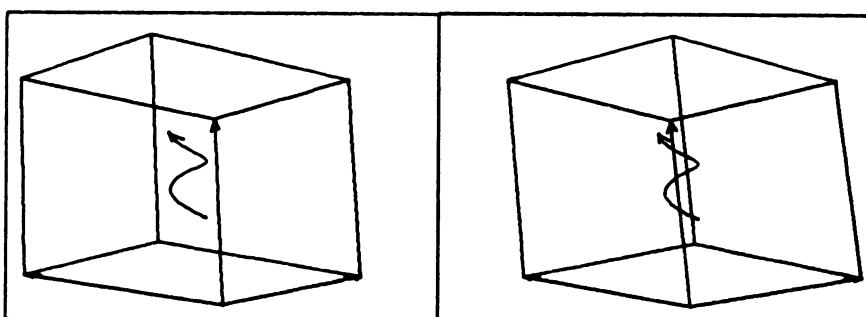


$\wedge^{n-1} \vee$   $(n-1)$ -Vektor



$\wedge^n \vee$  n-Vektor



$V^*$  Linearform $\wedge^{n-1} V^*$  ( $n-1$ )-Form $\wedge^n V^*$  n-Form

liefern). Nun bilden  $t, h\mathfrak{B}$  und  $t, h\mathfrak{B}^*$  zwei im allgemeinen verschiedene Begleitbasen der Kurve. Neben dem Schmiegbivektor  $\mathfrak{S}$  haben wir dann noch die Bivektordichten

$\mathfrak{Q} := \mathfrak{h} \wedge \mathfrak{B}$  der *Querebene* und  $\mathfrak{Q}^* := \mathfrak{h} \wedge \mathfrak{B}^* = {}_1^*t$  der *l-Transversalebene* (die im euklidischen Raum wieder zusammenfallen und die Normalebene liefern) sowie die

Bivektordichte  $\mathfrak{R} := \mathfrak{B} \wedge t = \mathfrak{B}^* \wedge t$  der *rektifizierenden Ebene*.

Wir können uns hier nicht näher mit den *Frenet-Formeln* und dem *Fundamentalsatz der Kurventheorie* beschäftigen (Bestimmtheit einer Kurve durch Bogenlänge, Krümmung, Torsionsdichte und Anfangswerte). Auf die sehr interessanten Darstellungen der Grundgrößen einer *Kurve in allgemeiner Parameterdarstellung* oder als *Schnittkurve implizit gegebener Flächen* kann ebenfalls nur hingewiesen werden, auch wenn dabei die Vorzüge der "mageren" Strukturen des Minkowski-Raumes besonders deutlich hervortreten.

Jetzt gehen wir kurz auf spezielle Kurven im  $(A^3, l, F)$  ein. Dazu bemerken wir zunächst, daß die *Grenzschnittgebilde* benachbarter

a) rektifizierender Ebenen die *Darboux-Achse* und den *Darboux-Punkt*,

b) Querebenen die *Querachse* und den *Querpunkt*,

c) l-Transversalebenen die *Krümmungsachse* und den *Schmiegmittelpunkt*

liefern (wobei die beiden letzten Fälle im euklidischen Raum wieder zusammenfallen). Damit können im Minkowski-Raum

*Böschungslinien* als Kurven mit parallelen Darboux-Achsen,

*sphärische Kurven* als Kurven mit festem Querpunkt,

*\*-sphärische Kurven* als Kurven mit festem Schmiegmittelpunkt

definiert und durch *Differentialgleichungen zwischen den Kurveninvarianten gekennzeichnet* werden, die die bekannten Kennzeichnungen im euklidischen Raum deutlich widerspiegeln. Obwohl es im  $(A^3, l, F)$  sowohl l-Kugeln als auch F-Kugeln gibt, liegen die beiden Sorten sphärischer Kurven des Minkowski-Raumes im allgemeinen nicht auf Kugeln. Hier zeigt sich recht eindrucksvoll, daß von den verschiedenen gleichwertigen Eigenschaften einer Kurve des euklidischen Raumes eine vielleicht ausgewählte Eigenschaft zur Definition dieser speziellen Kurve im Minkowski-Raum geeignet ist.

#### §4. Flächentheorie im 3-dimensionalen Minkowski-Raum $(A^3, F)$

Für eine orientierte *Fläche* mit der Parameterdarstellung  $(u^1, u^2) \mapsto x(u^1, u^2)$  können wir folgende Grundgrößen definieren:

*Tangentenvektoren*  $\mathfrak{x}_1 := \partial_1 x$ ,  $\mathfrak{x}_2 := \partial_2 x$  und

$$F\text{-Transversalvektordichte } n := \frac{\mathfrak{x}_1 \times \mathfrak{x}_2}{f(\mathfrak{x}_1 \wedge \mathfrak{x}_2)} = {}_2^* \frac{\mathfrak{x}_1 \wedge \mathfrak{x}_2}{f(\mathfrak{x}_1 \wedge \mathfrak{x}_2)} \text{ vom Gewicht 1.}$$

Dann bildet  $\mathfrak{x}_1, \mathfrak{x}_2, n$  eine Begleitbasis der Fläche. Die in den Weingartenschen Ableitungsgleichungen  $\partial_\sigma n = -\mathfrak{B}^\rho \mathfrak{x}_\rho$  auftretende  $2 \times 2$ -Matrix  $\mathfrak{B}^\rho_\sigma$  besteht aus Skalardichten vom Gewicht 1 und liefert als Invarianten

die mittlere Krümmungsdichte  $\mathfrak{H} := \frac{1}{2} \mathfrak{B}^\rho_\rho$  vom Gewicht 1 und die Gaußsche Krümmungsdichte  $\mathfrak{K} := \det \mathfrak{B}^\rho_\sigma$  vom Gewicht 2.

Auch hier können wir uns nicht mit den vollen Ableitungsgleichungen und den zwei Versionen des Fundamentalsatzes der Flächentheorie beschäftigen (Bestimmtheit einer Fläche einerseits durch zwei Grundformen und Anfangswerte sowie andererseits bis auf «relative» Bewegungen durch drei Grundformen, vgl. dazu [15]). Auf die Darstellungen der Krümmungsdichten  $\mathfrak{H}$  und  $\mathfrak{K}$  für implizit gegebene Flächen oder für Hüllflächen von Ebenenscharen kann wieder nur hingewiesen werden.

Wir stellen jetzt einige Sätze über Flächen im  $(A^3, F)$  zusammen:

Torsen sind durch  $\mathfrak{R} = 0$  gekennzeichnet.

Flächen mit lauter Nabelpunkten sind Stücke von F-Kugeln, Ebenen oder gespiegelten F-Kugeln.

Für die Parallelflächen  $\bar{x} := x + \lambda \otimes n$  zu einer Fläche mit der konstanten Distanzdicthe  $\lambda \in \wedge^3 V$  lassen sich mittlere und Gaußsche Krümmungsdichte analog wie in der euklidischen Geometrie durch  $\mathfrak{H}, \mathfrak{K}$  und  $\lambda$  ausdrücken und entsprechende Folgerungen ziehen.

Die 1. Variation des Areals  $a := \int f(x_1 \wedge x_2) du^1 du^2$  einer Fläche verschwindet bei festem Rand genau dann, wenn  $\mathfrak{H} = 0$  ist, also eine Minimalfläche vorliegt. Weiter gilt der Satz von J. Steiner, daß die Minimalflächen auch Extremalen des Areals bezüglich der Schar der Parallelflächen sind. Außerdem können Minimalflächen dadurch gekennzeichnet werden, daß ihre Koordinatenfunktionen in einem allgemeineren Sinn «harmonische» Funktionen sind.

Für die Lösungsflächen des isoperimetrischen Problems ist notwendig  $\mathfrak{H} = \text{const.}$

Da auf einer Fläche im Minkowski-Raum keine Krümmungsliniensparameter zu existieren brauchen, muß bei Beweisen meistens mit allgemeinen Parametern gerechnet werden.

## §5. Bemerkungen zur Minkowski-Geometrie

Wie man gesehen hat, sind im  $A^3$  für die Minkowskische Kurventheorie die beiden metrischen Strukturen erforderlich, nämlich Länge 1 und Areal  $F$ , während für die Minkowskische Flächentheorie nur das Areal  $F$  gebraucht wird. Interessant sind noch die Beziehungen zur affinen Differentialgeometrie. Im Minkowski-Raum können sowohl für Kurven als auch für Flächen die affinen Größen durch metrische Größen ausgedrückt werden. Allgemein zeigt sich bei allen Untersuchungen der Minkowski-Geometrie, daß im Unterschied zur euklidischen Differentialgeometrie mit den geringeren strukturellen Voraussetzungen viel sorgfältiger umgegangen werden muß. Aber dadurch ergeben sich auch tiefere Einsichten in die geometrischen Sachverhalte als im euklidischen Spezialfall.

Wir skizzieren jetzt noch einige spezielle Fragen der Minkowski-Geometrie.

Die a priori voneinander unabhängigen metrischen Strukturen eines Minkowski-Raumes können auf verschiedene Weise aneinander gekoppelt werden.

**1.** Wenn im  $A^n$  eine Volumenfunktion  $v$  gegeben ist, kann man zu jeder metrischen Struktur die *komplementäre metrische Struktur* einführen<sup>4</sup>. In dem hier betrachteten Rahmen ergeben sich dafür folgende Möglichkeiten:

- a) In einem  $(A^n, l, v)$  läßt sich das *Co-Areal*  $F$  eines orientierten  $(n-1)$ -Parallelogramms definieren als Volumen jenes positiv orientierten  $n$ -Parallelotops, das von dem  $(n-1)$ -Parallelogramm und einem Einheitsvektor aufgespannt wird, zu dem das  $(n-1)$ -Parallelogramm  $l$ -transversal ist.
- b) In einem  $(A^n, F, v)$  läßt sich die *Co-Länge*  $\bar{l}$  eines Vektors definieren als Volumen jenes positiv orientierten  $n$ -Parallelotops, das von dem Vektor und einem Einheits- $(n-1)$ -Parallelogramm aufgespannt wird, zu dem der Vektor  $F$ -transversal ist.

Es zeigt sich, daß Transversalität und Co-Transversalität einander entsprechen. In den Minkowski-Räumen  $(A^n, l, \bar{F}, v)$  und  $(A^n, \bar{l}, F, v)$  gilt dann für ein  $n$ -Parallelotop die Regel

$$\text{Volumen} = \text{Grundfläche} \times \text{transversale Höhe},$$

die also viel allgemeineren als nur euklidischen Charakter hat.

**2.** Das *Choquet-Busemannsche Areal*<sup>5</sup> gibt die Möglichkeit, in einem  $(A^n, l)$  aus der Länge ein Volumen und für alle Unterdimensionen  $0 < p < n$  ein  $p$ -Areal zu bestimmen: Das *Volumen* wird dabei durch die Forderung festgelegt, daß die  $l$ -Indikatrix in allen Minkowski-Räumen gleicher Dimension dasselbe Volumen habe (also das Volumen der euklidischen Einheitskugel). Das  *$p$ -Areal* eines  $p$ -Parallelogramms sei dann dessen  $p$ -Volumen in der von ihm aufgespannten Minkowskischen  $p$ -Ebene.

Schon der Beweis, daß die (strenge) Konvexität von  $l$  auch die (strenge) Konvexität dieses  $p$ -Areals impliziert, führt weit in die Konvexgeometrie, nämlich zum *Busemannschen Satz vom Brunn-Minkowskischen Typ*<sup>6</sup>. Diese spezielle Konstruktion eines  $p$ -Areals mit seinen nichttrivialen Eigenschaften verlor allerdings seine zentrale Stellung innerhalb der nichtlinearen Differentialgeometrie, als sich zeigte, daß eine Kopplung der  $p$ -Areale für die verschiedenen Unterdimensionen  $p$  weithin entbehrlich ist.

Während schon bei den beiden eben betrachteten Fragen Differenzierbarkeitsvoraussetzungen nicht erforderlich sind, werden die beiden folgenden Probleme der Minkowski-Geometrie explizit außerhalb der Differentialgeometrie gestellt:

**3.** Das *isodiametrische Problem* im  $A^n$  bezüglich eines symmetrischen konvexen Eichkörpers  $E$  mit positivem Lebesgueschen Maß hat die Lösung<sup>7</sup>: Unter allen nichtleeren kompakten Mengen gleichen Durchmessers hat genau die zu  $E$  homothetische maximales Lebesguesches Maß.

<sup>4</sup> Vgl. etwa [6], [5], [7] und [16], insbesondere S.31–32. Ein Komplementär-Areal (= «Dualareal») spielt schon früher in der Finsler-Geometrie im Sinne von E. Cartan eine Rolle.

<sup>5</sup> Vgl. [25], [3] und [9].

<sup>6</sup> Vgl. [22] und [11].

<sup>7</sup> Vgl. [21], S.241–246, sowie [8], insbesondere S.241 und S.243–245.

**4. Das isoperimetrische Problem im  $A^n$  bezüglich eines konvexen Eichkörpers  $E$  mit positivem Lebesgueschen Maß hat folgende Lösungen<sup>8</sup>:**

- a) Unter allen schleierlosen kompakten Mengen positiven Lebesgueschen Maßes mit gleicher äußerer Minkowskischer Relativoberfläche hat genau die zu  $E$  homothetische maximales Lebesguesches Maß.
- b) Unter allen kompakten Mengen positiven Lebesgueschen Maßes mit gleicher innerer Minkowskischer Relativoberfläche haben genau jene maximales Lebesguesches Maß, für die eine maßgleiche Teilmenge zu  $E$  homothetisch ist. Das noch nicht allgemein gelöste isoperimetrische Problem bei nichtkonvexer Eichfigur wird in [12] diskutiert.

Diese Bemerkungen zur Minkowski-Geometrie sollen darauf hinweisen, daß in einem Minkowski-Raum außer Fragen der Differentialgeometrie auch andere Probleme – von der Elementargeometrie bis zur Konvexgeometrie – von Interesse sind.

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# Karl Strubbecker

## Theorie der flächentreuen Abbildungen der Ebene

### I. Einleitung

1. Die Theorie der flächentreuen Abbildungen der Ebene ist viel weniger bekannt als die Theorie der winkeltreuen (konformen) Abbildungen der Ebene. Die konformen Abbildungen der Ebene hängen nämlich mit wichtigen Zweigen der komplexen Analysis, nämlich mit der Theorie der komplexen analytischen Funktionen einer komplexen Veränderlichen,

$$w = f(z) \quad (w = u + iv, z = x + iy)$$

zusammen, die z.B. in der Potentialtheorie eine große Rolle spielt.

Im Gegensatz dazu hängt, wie sich zeigen wird, die allgemeine flächentreue Abbildung der Ebene von einer reellen Funktion zweier reeller Variablen

$$(1) \quad z = z(x,y)$$

ab, die wir als zweimal stetig ableitbar voraussetzen. Es sollen also die partiellen Ableitungen

$$(2) \quad \begin{aligned} p &= z_x(x,y), q = z_y(x,y), \\ r &= z_{xx}, s = z_{xy} = z_{yx}, t = z_{yy} \end{aligned}$$

existieren und stetig sein.

2. Im folgenden entwickle ich in Kürze die Theorie der flächentreuen Abbildungen und zeige, daß sie in engem Zusammenhang steht mit der Differentialgeometrie der Flächen (1) in einem Raum  $R_3(x,y,z)$ , den man als den einfachsten Riemannschen Raum bezeichnen kann, weil sein Bogenelementquadrat die besonders einfache Gestalt

$$(3) \quad ds^2 = dx^2 + dy^2$$

besitzt. Für die  $z$ -Richtung ist  $dx = dy = 0$ , also  $ds^2 = 0$ ; die  $z$ -Richtung ist eine ausgezeichnete isotrope Richtung dieses Raumes  $R_3$ , der deshalb als ein isotroper Raum von drei Dimensionen bezeichnet wird.

### II. Die Formeln von G. Scheffers

3. Eine flächentreue Abbildung der Ebene ordnet einem Punkte  $E_l = (x_l, y_l)$  umkehrbar eindeutig einen Punkt  $E_r = (x_r, y_r)$  zu und kann mittels zweier Parameter  $(x, y)$  durch die Formeln

$$(4) \quad E_l \begin{cases} x_l = x_l(x,y) \\ y_l = y_l(x,y) \end{cases} \quad E_r \begin{cases} x_r = x_r(x,y) \\ y_r = y_r(x,y) \end{cases}$$

dargestellt werden. Notwendig und hinreichend für die Flächentreue der Abbildung  $E_l \rightarrow E_r$  ist dann die Gleichheit der Funktionaldeterminanten

$$(5) \quad D_l = \frac{\partial(x_l, y_l)}{\partial(x, y)} = \frac{\partial(x_r, y_r)}{\partial(x, y)} = D_r.$$

Tatsächlich wird dann jedem Flächenstück  $\Phi_l$  von Punkten  $E_l$  der «linken Bildebene»  $\pi_l(x_l, y_l)$  ein gleichsinnig flächengleiches Flächenstück  $\Phi_r$  von Punkten  $E_r$  der «rechten Bildebene»  $\pi_r(x_r, y_r)$  zugeordnet, weil für die Flächeninhalte  $F_l$  und  $F_r$

$$(6) \quad F_l = \iint dx_l dy_l = \iint \frac{\partial(x_l, y_l)}{\partial(x, y)} dx dy \\ = \iint \frac{\partial(x_r, y_r)}{\partial(x, y)} dx dy = \iint dx_r dy_r = F_r$$

gilt.

4. Als Parameter  $(x, y)$  kann man dabei nach Scheffers (1918) die Koordinaten des Mittelpunktes  $\bar{E} = (x, y)$  der beiden Punkte  $E_l = (x_l, y_l)$  und  $E_r = (x_r, y_r)$  nehmen, d.h. man kann

$$(7) \quad x = \frac{x_l + x_r}{2}, \quad y = \frac{y_l + y_r}{2}$$

setzen. Mit geeigneten Funktionen  $p(x, y)$  und  $q(x, y)$  kann man dann für die Koordinaten der Punkte  $E_l$  und  $E_r$  die Darstellung

$$(8) \quad E_l \begin{cases} x_l = x + q(x, y) \\ y_l = y - p(x, y) \end{cases} \quad E_r \begin{cases} x_r = x - q(x, y) \\ y_r = y + p(x, y) \end{cases}$$

verwenden. Für die beiden Funktionaldeterminanten  $D_l$  und  $D_r$  erhält man dann

$$(9) \quad D_l = \frac{\partial(x_l, y_l)}{\partial(x, y)} = \begin{vmatrix} 1 + q_x & q_y \\ -p_x & 1 - p_y \end{vmatrix} = 1 + (p_x q_y - p_y q_x) + (q_x - p_y) \\ D_r = \frac{\partial(x_r, y_r)}{\partial(x, y)} = \begin{vmatrix} 1 - q_x & -q_y \\ p_x & 1 + p_y \end{vmatrix} = 1 + (p_x q_y - p_y q_x) - (q_x - p_y).$$

Die Bedingung (5) für die Flächentreue  $D_l = D_r$  führt auf die Forderung

$$(10) \quad q_x - p_y = 0 \quad \text{oder} \quad p_y = q_x,$$

welche besagt, daß die Funktionen  $p(x, y)$  und  $q(x, y)$  die partiellen Ableitungen einer Funktion  $z(x, y)$  sein müssen, also

$$(11) \quad p = p(x, y) = z_x(x, y), \quad q = q(x, y) = z_y(x, y)$$

gilt. Mit dieser Bedingung stellen dann nach Scheffers die Formeln (8) stets eine gleichsinnig flächentreue Abbildung  $E_l \rightarrow E_r$  dar und umgekehrt. Dabei ist vorausgesetzt, daß der Mittenort (die Menge der Mittelpunkte  $\bar{E}$ ) flächenhaft ist, d.h.  $D_l = D_r \neq 0$  ist.

Es gibt aber auch flächentreue Abbildungen, bei denen der Mittenort bloß eine Kurve oder nur ein Punkt ist (im letzten Fall handelt es sich um die Spiegelung

an einem festen Mittelpunkt  $\bar{E}$ ). Übt man in diesen beiden Sonderfällen auf eines der Bildfelder  $\Phi_l$  oder  $\Phi_r$  noch die Spiegelung an einem beliebigen Punkt aus, so wird der neue Mittenort wieder flächenhaft.

Nach (9) und (2) hat die Funktionaldeterminante den Wert

$$(12) \quad D_l = D_r = 1 + (rt - s^2) = 1 + K.$$

Die flächentreue Abbildung ist regulär, wenn  $D_l = D_r \neq 0$ , d.h.

$$(13) \quad K = rt - s^2 \neq -1.$$

Die Flächeninhalte  $F_l$  und  $F_r$  der beiden Bildbereiche  $\Phi_l$  und  $\Phi_r$  haben dann nach (6) die gleiche Größe

$$(14) \quad F_l = F_r = \iint (1+K) dx dy \neq 0.$$

Für  $D_l = D_r = 0$ , d.h.

$$(15) \quad K = rt - s^2 = -1,$$

wird  $F_l = F_r = 0$ . Die Abbildung ist auf singuläre Weise flächentreu; die beiden Bildfelder  $\Phi_l$  und  $\Phi_r$  bestehen bloß aus zwei Kurven  $c_l, c_r$ , deren Punkte  $E_l, E_r$  einander in beliebiger Weise zugeordnet sind.

5. Man erkennt aus den Formeln (8) von Scheffers, daß jede flächentreue Abbildung der Ebene mit einer gewissen Fläche  $z = z(x,y)$  im Raum  $R_3(x,y,z)$  zusammenhängt, die bis auf Schiebungen in der isotropen  $z$ -Richtung eindeutig festliegt. Aus der Darstellung (8) der flächentreuen ebenen Abbildung erhält man nämlich umgekehrt die Formeln

$$(16) \quad \begin{cases} x = \frac{1}{2}(x_l + x_r), & p = \frac{1}{2}(y_r - y_l) = p(x,y), \\ y = \frac{1}{2}(y_l + y_r), & q = \frac{1}{2}(x_l - x_r) = q(x,y), \end{cases}$$

und daraus schließlich bis auf eine additive Konstante, die eine Verschiebung in der isotropen  $z$ -Richtung bedeutet, auch noch

$$(17) \quad z = \int p(x,y)dx + q(x,y)dy = z(x,y) + \text{const.},$$

wobei der Integrand wegen (10) ein vollständiges Differential ist. Damit ist die mit der gegebenen flächentreuen Abbildung  $\Phi_l \rightarrow \Phi_r$  der Ebene verbundene Fläche  $\Phi$  im isotropen Raum (bis auf isotrope Parallelverschiebungen eindeutig) gewonnen.

Ist umgekehrt eine Fläche (1) gegeben, so besitzen ihre Flächenelemente  $E(x,y,z,p,q)$  nach den Formeln (8) von Scheffers Bildpunkte  $E_l, E_r$ , zwischen deren Bildfeldern  $\pi_l(x_l, y_l), \pi_r(x_r, y_r)$  eine flächentreue Beziehung besteht.

Die flächentreuen Abbildungen, deren Mittenort eine Kurve oder ein fester Punkt ist, stammen dabei von den Flächenelementen  $E(x,y,z,p,q)$  her, die eine Kurve berühren oder einen festen Punkt  $P(x,y,z)$  enthalten. Zusammenfassend gilt somit der

**Satz 1.** Die zweidimensionalen Lieschen Elementvereiné (Flächenelemente einer Fläche, einer Kurve oder eines Punktes) und nur sie führen durch die Scheffersche Abbildung (8) ihrer Flächenelemente  $E(x,y,z,p,q)$  auf eine gleichsinnig flächentreue Abbildung  $(E_l, E_r)$  der Ebene  $\pi$ .

### III. Die Kinematik des isotropen Raumes

6. Die von Scheffers angegebene Abbildung (8) der Flächenelemente  $E$  auf die Punktepaare  $(E_1, E_r)$  der Ebene  $\pi$  hat eine geometrische Grundlage, die man in einfacher Weise mittels der Kinematik des isotropen Raumes beschreiben kann, die nun kurz entwickelt werden soll.

Als Bewegungen des isotropen Raumes  $R_3(x,y,z)$  bezeichnen wir jene inhaltstreuen Affinitäten, welche das isotrope Bogenelement (3) ungeändert lassen. Diese isotropen Bewegungen bilden eine stetige sechsgliedrige Gruppe  $G_6$  und lauten

$$(18) \quad \begin{cases} x' = a + x \cos \phi - y \sin \phi \\ y' = b + x \sin \phi + y \cos \phi \\ z' = c + c_1 x + c_2 y + z. \end{cases}$$

Sie erscheinen im Normalriß auf die Ebene  $z=0$  (Grundriß) als gewöhnliche Bewegungen. Daher kann man auch alle Bewegungsvarianten des isotropen Raumes direkt im Grundriß als euklidische Invarianten ablesen.

Für  $\phi=0$  erhält man aus (18) die fünfgliedrige invariante Untergruppe  $G_5$  der sogenannten Grenzbewegungen des isotropen Raumes

$$(19) \quad \begin{cases} x' = a + x \\ y' = b + y \\ z' = c + c_1 x + c_2 y + z, \end{cases}$$

welche im Grundriß als gewöhnliche Parallelverschiebungen erscheinen.

Jede solche Grenzbewegung kann man (auf  $\infty^1$  Arten) zerlegen in zwei kommutative spezielle Grenzbewegungen, welche aus den beiden dreigliedrigen Gruppen

$$(20) \quad S_3^l \quad \begin{cases} x' = \alpha + x \\ y' = \beta + y \\ z' = \gamma - \beta x + \alpha y + z \end{cases} \quad S_3^r \quad \begin{cases} x' = \alpha + x \\ y' = \beta + y \\ z' = \gamma + \beta x - \alpha y + z \end{cases}$$

entnommen sind. Diese beiden Gruppen  $S_3^l$  und  $S_3^r$  sind im isotropen Raum genaue Gegenstücke der Cliffordschen Schiebungen des elliptischen Raumes. Sie sind untereinander vertauschbar, und ihre eingliedrigen stetigen Untergruppen haben geradlinige Bahnen, welche parabolische Strahlnetze mit den Gleichungen

$$(21) \quad \left\| \begin{matrix} \alpha & \beta & \gamma \\ p_{01} & p_{02} & p_{03} + p_{12} \end{matrix} \right\| = 0, \quad \left\| \begin{matrix} \alpha & \beta & \gamma \\ p_{01} & p_{02} & p_{03} - p_{12} \end{matrix} \right\| = 0$$

bilden. Wir bezeichnen diese besonderen Grenzbewegungen (20), welche sowohl die Punkte als auch die Ebenen des isotropen Raumes in einfach transitiver Weise vertauschen, als Cliffordsche Schiebungen des isotropen Raumes. Genauer heißt  $S_3^l$  die Gruppe der isotropen Linksschiebungen und  $S_3^r$  die Gruppe der isotropen Rechtsschiebungen.

#### IV. Erzeugung der Abbildung von Scheffers durch Cliffordsche Schiebungen des isotropen Raumes

7. Unterwirft man nun ein beliebiges Flächenelement  $E^*(x^*, y^*, z^*, p^*, q^*)$  des isotropen Raumes allen  $\infty^3$  Rechtsschiebungen aus  $S_3^r$ , so erhält man  $\infty^3$  Flächenelemente eines gewissen Nullsystems  $N_l$ , die wir zueinander linksparallel nennen. Ebenso erhält man aus dem Flächenelement  $E^*$ , wenn man es allen  $\infty^3$  Linksschiebungen unterwirft, die  $\infty^3$  Flächenelemente eines gewissen Nullsystems  $N_r$ , die wir zueinander rechtsparallel nennen. Die Nullsysteme  $N_l$  und  $N_r$  und die zugehörigen linearen Komplexe (Strahlgewinde), die wir als Linksgewinde  $G_l$  und Rechtsgewinde  $G_r$  bezeichnen, sind zueinander involutorisch, und ihre Gleichungen lauten, wenn das Flächenelement  $E^*$  nach (8) die beiden Bildpunkte  $E_l^* = (x_l^*, y_l^*) = (x^* + q^*, y^* - p^*)$  und  $E_r^* = (x_r^*, y_r^*) = (x^* - q, y^* + p^*)$  besitzt,

$$(22_l) \quad \left\{ \begin{array}{l} N_l: Z - z = \begin{vmatrix} X & Y & 1 \\ x & y & 1 \\ x_l^* & y_l^* & 1 \end{vmatrix} \\ G_l: dz = -(xdy - ydx) - y_l^* dx + x_l^* dy \end{array} \right.$$

und

$$(22_r) \quad \left\{ \begin{array}{l} N_r: Z - z = - \begin{vmatrix} X & Y & 1 \\ x & y & 1 \\ x_r^* & y_r^* & 1 \end{vmatrix} \\ G_r: dz = (xdy - ydx) + y_r^* dx - x_r^* dy. \end{array} \right.$$

Es folgt daraus, daß  $N_l$  und  $G_l$  alle Rechtsschiebungen gestatten, ebenso  $N_r$  und  $G_r$  alle Linksschiebungen zulassen.

8. Ist nun  $E(x, y, z, p, q)$  ein beliebiges Flächenelement im Raum (Bild 1), so gibt es in der Bildebene  $\pi$  ( $z=0$ ) genau ein Flächenelement  $E_l$ , das zu  $E$  linksparallel ist, und genau ein Flächenelement  $E_r$ , das zu  $E$  rechtsparallel ist. Die Punkte  $E_l$  und  $E_r$  sind die Nullpunkte der Ebene  $\pi$  in den  $E$  enthaltenden Nullsystemen  $N_l$  und  $N_r$ .

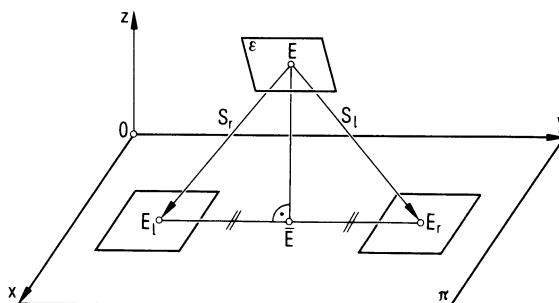


Bild 1

Konstruktion der beiden Bildpunkte  $E_l$ ,  $E_r$  des Flächenelementes  $E$  mittels Cliffordscher Schiebungen.

Diese beiden Punkte  $E_l$  und  $E_r$  stimmen nach den Formeln (22) genau mit den beiden in (8) gefundenen Schefferschen Bildpunkten des Flächenelements  $E$  überein.

Weil alle Flächenelemente  $E$  von  $N_l$  durch Rechtsschiebungen und alle Flächenelemente  $E$  von  $N_r$  durch Linksschiebungen zusammenhängen, erhält man den linken Bildpunkt  $E_l$  aus  $E$  durch die Rechtsschiebung  $S_l$  in (20) mit den Parametern

$$\alpha = q, \quad \beta = -p, \quad \gamma = px + qy - z,$$

und den rechten Bildpunkt  $E_r$  aus  $E$  durch die Linksschiebung  $S_r$  in (20) mit den Parametern

$$\alpha = -q, \quad \beta = p, \quad \gamma = px + qy - z.$$

Damit ist die angekündigte kinematische Erzeugung der beiden Bildpunkte  $(E_l, E_r)$  des Flächenelements  $E$  gefunden. Weil dabei die Kinematik des isotropen Raumes die Hauptrolle spielt, bezeichnen wir die von Scheffers nur auf analytischem Wege gefundene Abbildung (8) zweckmäßig als die kinematische Abbildung der Flächenelemente  $E$  des isotropen Raumes auf die Punktpaare  $(E_l, E_r)$  der Ebene  $\pi$ .

Zusammenfassend gilt also der

**Satz 2.** *Das in der Bildebene  $\pi$  ( $z=0$ ) liegende Flächenelement des linken kinematischen Bildpunktes  $E_l$  des Flächenelements  $E$  ist zu  $E$  linksparallel und entsteht aus  $E$  durch eine Rechtsschiebung. Ebenso ist das in  $\pi$  liegende Flächenelement des rechten kinematischen Bildpunktes  $E_r$  zu  $E$  rechtsparallel und entsteht auch  $E$  durch Linksschiebung.*

Eine unmittelbare Folge dieser kinematischen Erzeugung der beiden Bildpunkte  $(E_l, E_r)$  des Flächenelements  $E$  durch isotrope Rechts- bzw. Linksschiebungen ist der

**Satz 3.** (Hauptsatz der kinematischen Abbildung). *Bei einer Linksschiebung (20) des isotropen Raumes erfahren die linken Bildpunkte  $E_l$  der Flächenelemente  $E$  in der Bildebene  $\pi$  eine Translation, die rechten Bildpunkte  $E_r$  bleiben fest. Umgekehrt erleiden bei einer Rechtsschiebung (20) der Flächenelemente  $E$  die rechten Bildpunkte  $E_r$  eine Translation, die linken Bildpunkt  $E_l$  bleiben fest.*

Das Ergebnis von Satz 1 kann nun insgesamt auch in folgender Weise ausgesprochen werden.

**Satz 4.** *Die kinematischen Bilder  $(E_l, E_r)$  der Flächenelemente  $E$  eines zweidimensionalen räumlichen Elementvereins  $\Phi$  (der aus einer Fläche, einer Kurve oder aus einem Punkt bestehen kann) bilden zwei ebene Bildbereiche  $(\Phi_l, \Phi_r)$ , die aufeinander eigentlich flächentreu bezogen sind. Wir bezeichnen diese flächentreue Verwandtschaft  $(\Phi_l \rightarrow \Phi_r)$  kurz als das kinematische Bild des Elementvereins  $\Phi$ . Umgekehrt heißt die Fläche  $\Phi$  das kinematische Urbild der flächentreuen Verwandtschaft  $(\Phi_l \rightarrow \Phi_r)$  oder kurz die kinematische Bildfläche der flächentreuen Verwandtschaft  $(\Phi_l \rightarrow \Phi_r)$ .*

*Bemerkung 1.* Es sei bemerkt, daß die kinematische Abbildung  $E \rightarrow (E_1, E_r)$  ein Gegenstück zu der Gaußschen sphärischen Abbildung der Flächenelemente  $E$  des euklidischen Raumes auf die dazu parallelen Flächenelemente  $E'$  einer Kugel ist. Im isotropen Raum kann man die Kugel einfacher durch eine Ebene  $\pi$  ersetzen und die beiden Cliffordschen Schiebungsgruppen  $S_3^l, S_3^r$  führen durch die von ihnen erzeugten Parallelismen zu zwei verschiedenen Bildelementen  $E_1$  und  $E_r$  der Flächenelemente  $E$ .

## V. Grundsätze der isotropen Flächentheorie

9. Wegen der dargelegten Verbindung unserer Abbildung (8) mit der isotropen Kinematik ist zu erwarten, daß die Theorie der flächentreuen Abbildungen der Ebene sehr eng mit der isotropen Differentialgeometrie der Flächen  $z = z(x, y)$  zusammenhängt. Wir berichten daher zunächst in Kürze über einige Grundeigenschaften der isotropen Flächentheorie.

Die Differentialgeometrie einer Fläche

$$(23) \quad z = z(x, y)$$

im isotropen Raum wird von insgesamt vier quadratischen Grundformen beherrscht, die gegenüber isotropen Bewegungen (18) invariant sind. Diese lauten

$$(24) \quad \begin{aligned} I &= ds^2 = dx^2 + dy^2 \\ II &= dpdx + dqdy = rdx^2 + 2sdx dy + tdy^2 \\ III &= dS^2 = dp^2 + dq^2 = (rdx + sdy)^2 + (sdx + tdy)^2 \\ &\quad = (r^2 + s^2)dx^2 + 2s(r + t)dx dy + (s^2 + t^2)dy^2 \\ IV &= dx dq - dy dp = sdx^2 - (r - t)dx dy - sdy^2. \end{aligned}$$

Die erste Grundform  $I$  ist das Quadrat des isotropen Bogenelements  $ds$  der Fläche; sie bestimmt die (im Grundriß ablesbare) Flächenmetrik.

Die zweite Grundform  $II$  gibt durch ihr Verschwinden  $II = 0$  die beiden Asymptotenrichtungen der Fläche im betrachteten Punkt  $P(x, y, z(x, y))$  an und durch die Formel

$$(25) \quad \frac{1}{R} = \frac{II}{I} = \frac{rdx^2 + 2sdx dy + tdy^2}{dx^2 + dy^2}$$

die isotrope Normalkrümmung der Fläche in der Richtung  $dx:dy$  des Punktes  $P$ . Die dritte Grundform  $III = dS^2$  stellt das Quadrat des isotropen Bogenelements  $dS$  des isotropen sphärischen Bildes der Fläche dar, das man erhält, wenn man die Fläche durch parallele Tangentenebenen auf die Einheitskugel des isotropen Raumes abbildet, deren Gleichung im Hinblick auf (1)

$$(26) \quad z = \frac{1}{2}(x^2 + y^2)$$

lautet. Als isotropes sphärisches Bild des Flächenelementes  $E(x, y, z, p, q)$  der Fläche (23) erhält man nämlich auf (26) den Punkt  $E^* = (X, Y, Z)$  mit den Koordinaten

$$(27) \quad X = p, \quad Y = q, \quad Z = \frac{1}{2}(p^2 + q^2).$$

Das Bogenelementquadrat  $dS^2 = dX^2 + dY^2$  des sphärischen Bildes hat damit tatsächlich die obige Gestalt.

Schließlich gibt die vierte Grundform IV durch ihr Verschwinden ( $IV=0$ ) die beiden isotropen Hauptkrümmungsrichtungen  $dx:dy$  der Fläche im Punkt P an, welche zugleich auf der Fläche zueinander konjugiert und im isotropen Sinn normal sind. Die isotropen Krümmungslinien bilden also jenes konjugierte Netz der Fläche, das im Grundriß als Orthogonalnetz erscheint. Der Quotient

$$(28) \quad \tau_g = \frac{IV}{I}$$

stellt dabei die isotrope geodätische Torsion  $\tau_g$  der Fläche in der Richtung  $dx:dy$  des Punktes P dar.

Für die isotropen Hauptkrümmungsrichtungen hat die isotrope Normalkrümmung  $1/R$  der Fläche die beiden Extremwerte  $1/R_1$  und  $1/R_2$ , welche durch Mittelbildung auf die beiden isotropen Invarianten

$$(29) \quad K = \frac{1}{R_1 R_2} = rt - s^2$$

$$2H = \frac{1}{R_1} + \frac{1}{R_2} = r + t = \Delta z(x,y)$$

führen, die als isotrope Krümmung K und isotrope mittlere Krümmung H der Fläche bezeichnet werden. Es gilt dabei die Beziehung

$$(30) \quad KI - 2HII + III = 0.$$

Aus ihr folgt für die Windung  $\tau = dS/ds$  der Schmieglelinien ( $II=0$ ) der Fläche wegen  $KI + III = 0$ .

$$(31) \quad \tau^2 = \frac{dS^2}{ds^2} = \frac{III}{I} = -K, \quad \text{d.h.} \quad \tau = \pm \sqrt{-K}.$$

Das ist das isotrope Gegenstück der Formel von Beltrami und Enneper.

## VI. Einige besondere Flächen des isotropen Raumes

10. Man erkennt aus diesen Formeln schon die Bedeutung der isotropen Differentialgeometrie für eine Reihe von besonderen Flächen.

In der Potentialtheorie spielen, wie bekannt, die Integralflächen der ebenen Potentialgleichung

$$(32) \quad \Delta z(x,y) = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

eine große Rolle. Diese Potentialflächen  $z = z(x,y)$ , die man, wenn  $f(x+iy)$  eine beliebige analytische Funktion der komplexen Variablen  $x+iy$  ist, bekanntlich in der Gestalt

$$(33) \quad z = \Re f(x+iy) \quad \text{oder} \quad z = \Im f(x+iy)$$

darstellen kann, sind nach (29) mit den Flächen verschwindender isotroper mittlerer Krümmung  $H=0$  identisch und stimmen daher mit den Minimalflächen des isotropen Raumes überein.

*Bemerkung 2.* Man kann die ganze allgemeine Theorie der euklidischen Minimalflächen auf den isotropen Raum übertragen. Es gibt nämlich zu jeder der bekannten speziellen euklidischen Minimalflächen ein genaues Gegenstück im isotropen Raum. So erhält man z.B. als Gegenstück der berühmten Minimalfläche von Enneper im isotropen Raum die Fläche

$$(34) \quad z = \frac{3}{2} \Re(x+iy)^{2/3}$$

und als Gegenstück der Minimalfläche von Scherk im isotropen Raum die Fläche

$$(35) \quad z = \Re e^{x+iy} \quad \text{oder} \quad z = e^x \cos y.$$

11. In der affinen Differentialgeometrie spielen nach Blaschke die Flächen mit

$$(36) \quad rt - s^2 = K = \text{const.}$$

eine wichtige Rolle. Es sind dies die sogenannten uneigentlichen Affinsphären, deren sämtliche Affinnormalen untereinander (nämlich zur  $z$ -Richtung) parallel sind. Diese uneigentlichen Affinsphären sind im isotropen Raum die Flächen fester Krümmung (36). Auch sie haben in dieser interessanten Deutung mit den zu ihnen analogen euklidischen Flächen fester Gaußscher Krümmung viele Eigenschaften gemeinsam. Ihre Theorie ist aber viel einfacher als die teilweise schwierige Theorie der euklidischen Flächen fester Gaußscher Krümmung. Während man z.B. die allgemeine Fläche fester Krümmung im euklidischen Raum bisher nicht angeben kann, weil man ihre Differentialgleichung noch nicht allgemein zu lösen vermochte, kann man für die analogen Flächen (36) im isotropen Raum eine allgemeine, sehr einfache kinematische Erzeugungsweise als Cliffordsche Schiebflächen angeben, aus der dann sofort eine elegante integrallose Darstellung dieser Flächen fließt, die durch direkte Integration von (36) schon von Darboux gefunden wurde.

## VII. Beziehungen der flächentreuen Abbildungen der Ebene zur isotropen Flächentheorie

12. Wir kehren nun wieder zur Theorie der flächentreuen Abbildungen  $\Phi_l(x_l, y_l) \rightarrow \Phi_r(x_r, y_r)$  der Ebene  $\pi(x, y)$  zurück, die, wie sich sogleich erweisen wird, eng mit der isotropen Flächentheorie, nämlich mit der Differentialgeometrie ihrer kinematischen Bildfläche  $\Phi$ , zusammenhängt.

Es sei  $\Phi$  die nach (17) bestimmte Fläche  $z = z(x, y)$ , welche als kinematisches Bild die flächentreue Abbildung  $(\Phi_l(x_l, y_l) \rightarrow \Phi_r(x_r, y_r))$  mit der Darstellung (8) besitzt. Dann ist nach (8)

$$(37) \quad \begin{cases} dx_l = (1+s)dx + tdy \\ dy_l = -rdx + (1-s)dy \end{cases} \quad \begin{cases} dx_r = (1-s)dx - tdy \\ dy_r = rdx + (1+s)dy \end{cases}$$

Man erhält für die Bogenelemente  $ds_l$  und  $ds_r$  der beiden kinematischen Bilder die Formeln

$$(38) \quad \begin{aligned} ds_l^2 &= dx_l^2 + dy_l^2 = I + III + 2.IV \\ ds_r^2 &= dx_r^2 + dy_r^2 = I + III - 2.IV. \end{aligned}$$

Für die Bildkurven  $c_l, c_r$  der isotropen Krümmungslinien ( $IV=0$ ) der Fläche  $\Phi$  (und nur für sie) folgt

$$(39) \quad ds_l^2 = ds_r^2 = I + III$$

und der

**Satz 5.** Ist die flächentreue Verwandtschaft  $(\Phi_l \rightarrow \Phi_r)$  das kinematische Bild der Fläche  $\Phi$ , so sind die isometrischen Linien der Verwandtschaft ( $ds_l^2 = ds_r^2$ ) die kinematischen Bilder  $c_l, c_r$  der isotropen Krümmungslinien ( $IV=0$ ) der Fläche  $\Phi$ , und umgekehrt.

Eine Richtung  $ds$  auf der Fläche  $\Phi$  wird bei der kinematischen Abbildung in den beiden Bildfeldern  $\Phi_l$  und  $\Phi_r$  in zwei verschiedenen Richtungen  $ds_l$  und  $ds_r$  gespalten. Deren Winkel  $v$  heißt der Verzerrungswinkel der Flächenrichtung  $ds$ . Man findet für ihn die Formel

$$(40) \quad \tan v = \frac{2II}{I-III}$$

Für die Asymptotenrichtungen  $ds$  der Fläche  $\Phi$  gilt  $II=0$ ; im kinematischen Bild ist dann  $v=0$  oder  $v=\pi$ . Also gilt

**Satz 6.** Ist die flächentreue Verwandtschaft  $(\Phi_l \rightarrow \Phi_r)$  das kinematische Bild der Fläche  $\Phi$ , so entsprechen den Asymptotenlinien der Fläche  $\Phi$  die beiden durch parallele oder antiparallele Tangentenrichtungen aufeinander bezogenen Kurvenscharen der flächentreuen Verwandtschaft  $(\Phi_l \rightarrow \Phi_r)$ , und umgekehrt.

13. Ein feineres Studium der flächentreuen Verwandtschaft  $(\Phi_l \rightarrow \Phi_r)$  benötigt den Begriff ihrer Indikatrizen, der von Study stammt.

Es ist bekannt, daß jede stetig differenzierbare Punktverwandtschaft in der Ebene im Kleinen affin ist. Jede solche flächentreue Verwandtschaft  $(\Phi_l \rightarrow \Phi_r)$  kann daher in erster Annäherung in entsprechenden Punkten  $E_l, E_r$  durch eine flächentreue berührende Affinität  $(\Phi_l^* \rightarrow \Phi_r^*)$  ersetzt werden. Es gibt nun in dieser flächentreuen Affinität im Punkte  $E_l$  (bis auf gemeinsame Ähnlichkeiten) genau eine Ellipse  $i_l$ , deren affine Ellipse  $i_r$  in  $E_r$  zu ihr kongruent ist.

Diese beiden kongruenten Ellipsen  $(i_l, i_r)$  heißen nach Study die Indikatrizen der flächentreuen Verwandtschaft  $(\Phi_l \rightarrow \Phi_r)$  in dem Punktpaar  $(E_l, E_r)$ .

Diese Studyschen Indikatrizen entsprechen einander dabei in der berührenden Affinität stets so, daß der großen Achse der einen Ellipse  $i_l$  die kleine Achse der anderen Ellipse  $i_r$  zugeordnet ist und umgekehrt.

Auch die zueinander längengleichen Paare konjugierter Durchmesser der beiden Indikatrizen (die auf den Diagonalen der umschriebenen

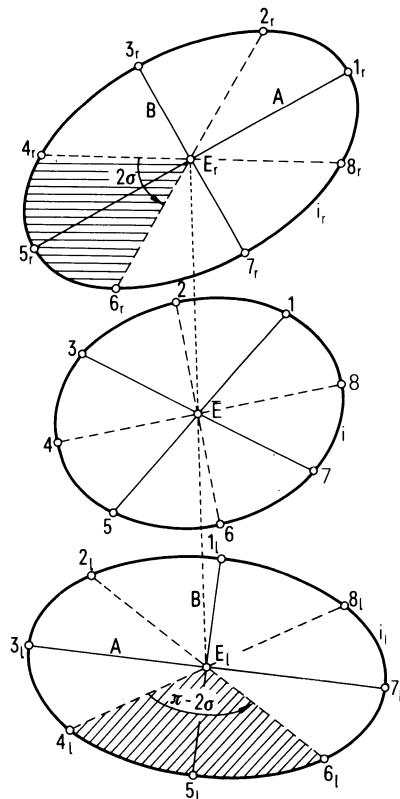


Bild 2

Die Indikatrix  $i$  und das kongruente Indikatrenpaar  $i_l, i_r$  einer flächentreuen Verwandtschaft  $E_l \rightarrow E_r$ .

Achsenrechtecke liegen) sind einander in der berührenden Affinität ( $\Phi_l^* \rightarrow \Phi_r^*$ ) zugeordnet.

Dem Indikatrenpaar  $(i_l, i_r)$  der flächentreuen Abbildungen ( $\Phi_l \rightarrow \Phi_r$ ) entspricht in der kinematischen Abbildung eine gewisse Ellipse  $i$  in der Tangentenebene  $\tau$  des Punktes  $E$  der kinematischen Bildfläche  $\Phi$ . Deutet man  $dx$  und  $dy$  in  $\tau$  als Relativkoordinaten im Punkt  $E$ , so lautet die Gleichung dieser Ellipse  $i$  einfach

$$(41) \quad I + III = 1.$$

Es sei aber betont, daß diese Ellipse i.a. nicht die Dupinsche Indikatrix der Fläche  $\Phi$  in  $E$  ist, die ja auch eine Hyperbel sein könnte.

14. Die Gestalt und die gegenseitige Lage der beiden kongruenten Indikatren  $(i_l, i_r)$  der flächentreuen Verwandtschaft  $(\Phi_l \rightarrow \Phi_r)$  hängen nun in sehr einfacher Weise mit den isotropen Krümmungsverhältnissen der zugehörigen Fläche  $\Phi$  zusammen.

Ist zunächst  $2\gamma$  der Winkel, um den die beiden Indikatren  $(i_l, i_r)$  gegeneinander gedreht sind, d.h. ist  $2\gamma$  der Winkel, den ihr großen (oder kleinen) Achsen bilden, so gilt die Formel

$$(42) \quad \tan 2\gamma = \frac{1-K}{2H}.$$

Weil nach dem Hauptsatz der kinematischen Abbildung (Satz 3) Parallelverschiebungen der beiden Bildfelder  $\Phi_l$  und  $\Phi_r$  für die Fläche  $\Phi$  nur eine isotrope Grenzbewegung bedeuten, bei der  $K$  und  $H$  ungeändert bleiben, ist durch den Winkel  $2\gamma$  die gegenseitige Lage der Indikatrizen  $i_l$  und  $i_r$  eindeutig bestimmt.

Die gemeinsame Gestalt der beiden Indikatrizen ( $i_l, i_r$ ) kann man eindeutig durch den Winkel  $2\sigma$  bzw.  $\pi - 2\sigma$  ihrer beiden längengleichen konjugierten Durchmesser kennzeichnen, d.h. durch die (in den beiden Bildfeldern supplementären) Winkel der isometrischen Bilder der beiden isotropen Hauptkrümmungsrichtungen. Man findet dafür die Formel

$$(43) \quad \sin 2\sigma = \frac{1+K}{\sqrt{4H^2 + (1-K)^2}}.$$

Durch die beiden Winkel  $2\gamma$  und  $2\sigma$  sind die isotropen Krümmungsgrößen  $H$  und  $K$  der Fläche  $\Phi$  eindeutig bestimmt (und umgekehrt), wie die Formeln

$$(44) \quad H = \frac{\cos 2\gamma}{\sin 2\sigma + \sin 2\gamma}, \quad K = \frac{\sin 2\sigma - \sin 2\gamma}{\sin 2\sigma + \sin 2\gamma}$$

zeigen. Weil durch  $H$  und  $K$  das Flächenelement 2. Ordnung der Fläche  $\Phi$  bestimmt ist, folgt daraus

**Satz 7.** *Gestalt und gegenseitige Lage der beiden Indikatrizen ( $i_l, i_r$ ) der flächentreuen Verwandtschaft ( $\Phi_l \rightarrow \Phi_r$ ) einerseits und die durch die Krümmungsgrößen  $H$  und  $K$  festgelegten Flächenelemente 2-ter Ordnung ihrer kinematischen Bildfläche  $\Phi$  andererseits bestimmen einander gegenseitig. Die beiden Indikatrizen ( $i_l, i_r$ ) der flächentreuen Verwandtschaft ( $\Phi_l \rightarrow \Phi_r$ ) kennzeichnen somit in umkehrbarer Weise die isotropen Krümmungsverhältnisse der kinematischen Bildfläche  $\Phi$ .*

*Bemerkung 3.* Die beiden Indikatrizen ( $i_l, i_r$ ) sind dann und nur dann Kreise, wenn sie zu einem isotropen Nabelpunkt  $P$  der Bildfläche  $\Phi$  gehören, d.h. alle isotropen (also z-parallel) Normalschnitte von  $\Phi$  in  $P$  dieselbe isotrope Normalkrümmung (25) haben. Für  $\Phi$  gilt dann in  $P$

$$r = t, \quad s = 0 \quad \text{oder} \quad H^2 - K = 0$$

und für die beiden Indikatrizen ( $i_l, i_r$ ) gilt

$$2\sigma = \pi - 2\tau = \frac{\pi}{2}.$$

### VIII. Einige Sonderfälle von flächentreuen Abbildungen der Ebene und ihre kinematischen Bildflächen

15. Gestützt auf diese allgemeinen Ergebnisse unserer Theorie der flächentreuen Abbildungen wenden wir uns nun einigen Sonderfällen zu.

Wir beginnen mit den singulären flächentreuen Abbildungen ( $\Phi_l \rightarrow \Phi_r$ ), deren kinematische Bildflächen  $\Phi$  im isotropen Raum Flächen der festen isotropen Krümmung

$$(45) \quad K = rt - s^2 = -1,$$

also uneigentliche Affinsphären mit  $K = -1$  sind. Nach (12) bestehen dann die kinematischen Bildbereiche  $\Phi_l$  und  $\Phi_r$  dieser Flächen (wegen  $D_l = D_r = 0$ ) bloß aus zwei Kurven  $c_l = \Phi_l$  und  $c_r = \Phi_r$ , auf denen die kinematischen Bilder  $E_l$  und  $E_r$  der Flächenelemente  $E$  von  $\Phi$  beliebig wählbar sind.

Man kann die isotrope Differentialgeometrie dieser Flächen übersichtlich und leicht aus ihren kurvenförmigen kinematischen Bildern gewinnen.

Hält man z.B. den linken Bildpunkt  $E_l$  auf  $c_l$  fest, während der rechte Bildpunkt  $E_r$  auf  $c_r$  variiert, so erhält man, weil es in  $E_l$  zu jedem Linienelement von  $c_r$  eine parallele Richtung gibt, nach Satz 6 auf der Fläche  $\Phi$  einen Asymptotenstreifen  $\sigma_l$ , der in dem durch  $E_l$  bestimmten Linksgewinde  $G_l$  enthalten ist. Das gleiche gilt, wenn man umgekehrt  $E_r$  auf  $c_r$  festhält und  $E_l$  auf  $c_l$  laufen lässt; man erhält dann auf  $\Phi$  einen in einem Rechtsgewinde  $G_r$  liegenden Asymptotenstreifen  $\sigma_r$ . Diese Streifen haben dabei wegen  $K = -1$  nach (31) die konstanten isotropen Windungen  $\tau_l = -1$  und  $\tau_r = +1$ .

Verschiebt man dann auch den bisher festgehaltenen Bildpunkt  $E_l$  stetig auf  $c_r$  (bzw.  $E_r$  stetig auf  $c_l$ ), dann erleidet der Schmiegestreifen  $\sigma_l$  (bzw.  $\sigma_r$ ) nach dem Hauptsatz der kinematischen Abbildung (Satz 3) eine stetige Linksverschiebung  $\mathfrak{L}$  (bzw. stetige Rechtsverschiebung  $\mathfrak{R}$ ), und man erkennt, daß die Flächen  $\Phi$  mit  $K = -1$  im isotropen Raum durch Cliffordsche Schiebungen ihrer Asymptotenlinien aneinander erzeugt werden können. Es gilt damit

**Satz 8.** *Die Flächen  $\Phi$  mit der festen isotropen Krümmung  $K = rt - s^2 = -1$  besitzen zwei Scharen von Schmiegestreifen  $\sigma_l$ ,  $\sigma_r$  mit den konstanten isotropen Windungen  $\tau = \pm 1$ , die in zwei zueinander involutorischen Bündeln von linearen Komplexen (22) liegen. Jede solche Fläche  $\Phi$  kann kinematisch erzeugt werden, indem man die Schmieglinien  $\sigma_l$  ( $\sigma_r$ ) einer Schar durch isotrope Cliffordsche Linksschiebungen  $\mathfrak{L}$  (Rechtsschiebungen  $\mathfrak{R}$ ) längs der anderen Schar  $\sigma_r$  ( $\sigma_l$ ) verschiebt.*

Aus dieser einfachen Erzeugung der Flächen (45) als isotrope Cliffordsche Schiebflächen folgt leicht mittels (20) ihre einfachste analytische Darstellung mittels zweier willkürlicher Funktionen  $U = U(u)$  und  $V = V(v)$ . Sie lautet

$$(46) \quad \begin{cases} x = v - u \\ y = U'(u) - V'(v) \\ z = 2(U - V) - (u + v)(U' - V') \end{cases}$$

In diesen (schon von Darboux durch Integration der partiellen Differentialgleichung (45) gefundenen) Formeln sind  $u$  und  $v$  Asymptotenparameter der Fläche.

Auch die isotropen Krümmungslinien der Fläche  $\Phi$  mit  $K = -1$  kann man im kinematischen Bild leicht gewinnen. Aus Satz 4 folgt nämlich, daß man sie erhält, wenn die Punkte  $E_l$  auf  $c_l$  und  $E_r$  auf  $c_r$  gleichsinnig oder gegensinnig längengleiche Bogen beschreiben.

Die beiden Indikatrizien ( $i_l, i_r$ ) der vorliegenden singulären flächentreuen Abbildung entstehen in zwei doppelt überdeckte, zueinander kongruente Strecken (Tangentenstrecken von  $c_l$  und  $c_r$  in den Punkten  $E_l, E_r$ ) aus. Aus deren Neigungswinkel  $2\gamma$  folgt nach (42) für die isotrope mittlere Krümmung  $H$  der Fläche  $\Phi$  die Formel

$$(47) \quad H = \cot 2\gamma.$$

16. Sehr interessant sind weiter jene flächentreuen Abbildungen ( $\Phi_l \rightarrow \Phi_r$ ), welche als kinematische Bilder zu den Flächen  $\Phi$  mit der festen isotropen Krümmung

$$(48) \quad K = rt - s^2 = +1$$

gehören. Weil diese partielle Differentialgleichung elliptisch ist, sind diese Flächen  $\Phi$  stets analytisch, also ins Komplexe fortsetzbar.

Weil nach (42) der Winkel  $2\gamma = 0$  ist, sind die beiden Indikatrizien ( $i_l, i_r$ ) der diesen Flächen  $\Phi$  kinematisch entsprechenden flächentreuen Abbildung ( $\Phi_l \rightarrow \Phi_r$ ) überall zueinander parallelgestellt (Bild 3).

Nach Study spielt diese besondere Art flächentreuer Verwandtschaften ( $\Phi_l \rightarrow \Phi_r$ ) eine wichtige Rolle bei der reellen Darstellung der  $\infty^2$  komplexen Punkte  $E$

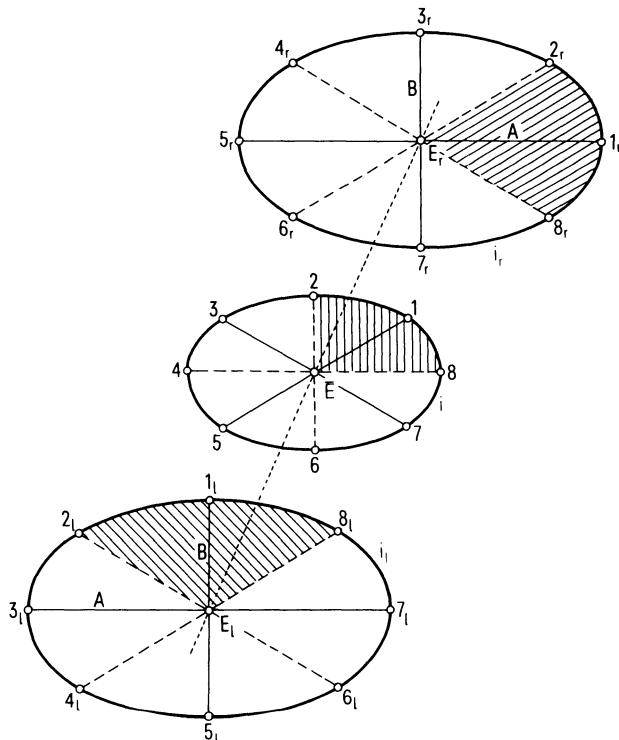


Bild 3

Die Indikatrizen der kinematischen Bilder einer Fläche der festen isotropen Krümmung  $K = +1$ .

einer komplexen analytischen Kurve  $k$  der Ebene. Nach von Staudt kann man nämlich jeden komplexen Punkt  $E$  von  $k$  durch das geordnete Paar  $(E_l \rightarrow E_r)$  der Potenzpunkte jener elliptischen Punktinvolution darstellen, deren imaginäre Doppelpunkte der Kurvenpunkt  $E$  und der dazu konjugiert komplexe Punkt  $E'$  sind. Insgesamt ist so die komplexe analytische Kurve  $k$  durch eine reelle Punktverwandtschaft  $(E_l \rightarrow E_r)$  dargestellt, die nach Study die kennzeichnende Eigenschaft hat, daß sie erstens gleichsinnig flächentreu ist und zweitens überall parallelgestellte Indikatrizen hat. Ist die ebene analytische Kurve  $k$  insbesondere reell, enthält sie also zu jedem ihrer komplexen Punkte  $E$  auch noch den konjugiert komplexen Punkt  $E'$ , so ist die sie nach von Staudt darstellende reelle flächenstreue Verwandtschaft  $(E_l \rightarrow E_r)$  involutorisch, und die reellen Punkte  $E = E'$  der Kurve  $k$  entsprechen sich selbst. Die Verwandtschaft  $(E_l \rightarrow E_r)$  ist dann die Studysche flächentreue Spiegelung an der reellen analytischen Kurve  $k$ , die ein interessantes, aber nur wenig bekanntes Gegenstück zu der von Schwarz stammenden konformen Spiegelung an  $k$  ist.

Zusammenfassend gilt also

**Satz 9.** *Mit jeder Fläche  $\Phi$  der festen isotropen Krümmung  $K = +1$  ist eine ebene analytische Kurve  $k$  verbunden und umgekehrt. Die Fläche  $\Phi$  besitzt nämlich als kinematische Abbildung  $(E_l \rightarrow E_r)$  ihrer Flächenelemente  $E$  eine eigentlich flächentreue Verwandtschaft mit gleichgestellten Indikatrizen, die mit der von Staudtschen Abbildung der komplexen Punkte  $E$  von  $k$  identisch ist. Ist die analytische Kurve  $k$  reell, so ist die Verwandtschaft  $(E_l \rightarrow E_r)$  involutorisch, nämlich die Studysche flächentreue Spiegelung an der reellen analytischen Kurve  $k$ ; die Fläche  $\Phi$  ist dann bezüglich der Bildebene  $\pi (z=0)$  symmetrisch.*

Als einfaches Beispiel sei Studys flächentreue Spiegelung an einem reellen Kreis  $k$  mit der Gleichung

$$(49) \quad x^2 + y^2 = 1$$

erwähnt, dem nach Satz 9 die bezüglich der Ebene  $z=0$  symmetrische Drehfläche mit der Gleichung (Zylinderkoordinaten  $r, \varphi, z$ )

$$(50) \quad z = \int_1^r \sqrt{r^2 - 1} \, dr = \frac{1}{2} \left( r \sqrt{r^2 - 1} - \text{Arcosh } r \right)$$

zugeordnet ist. Das ist die Evolutenfläche der Kettenfläche

$$(51) \quad 2z = \cosh 2r,$$

welche den Kreis (49) der Ebene  $z=0$  als Rückkehrkreis besitzt. Auf jeder reellen Geraden  $g$  liegt ein Paar von Spiegelpunkten  $(E_l, E_r)$  deren einfache Konstruktion Bild 4 zeigt.

17. Wir betrachten schließlich noch die flächentreuen Abbildungen  $(\Phi_l \rightarrow \Phi_r)$ , die zu isotropen Minimalflächen  $\Phi$  gehören, für welche die isotrope mittlere Krümmung

$$(52) \quad 2H = r + t = 0 \quad \text{oder} \quad \Delta z(x, y) = 0$$

ist. Diese Potentialflächen  $\Phi$  spielen in der geometrischen Funktionentheorie

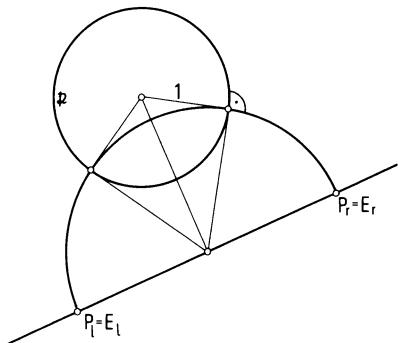


Bild 4  
Studysche flächentreue Spiegelung ( $E_l \rightarrow E_r$ ) am reellen Einheitskreis  $k$ . Jede reelle Gerade  $g$  trägt ein involutorisches Paar von Spiegelpunkten  $E_l, E_r$ .

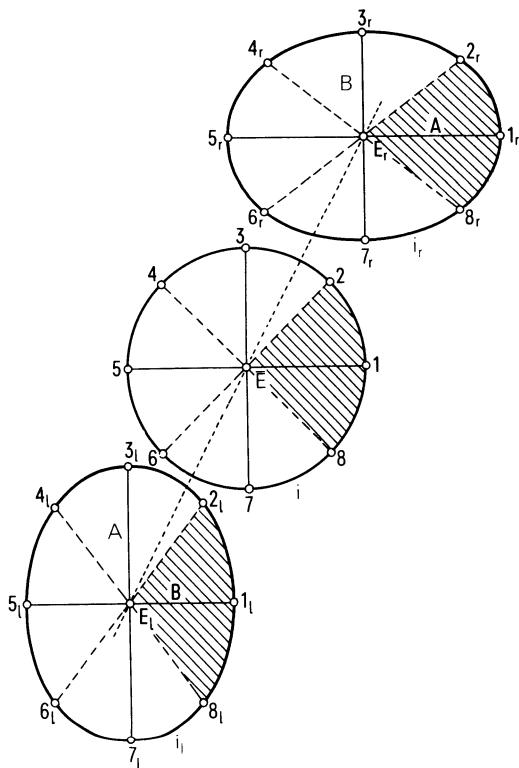


Bild 5  
Die Indikatrizen  $i_l, i_r$  des kinematischen Bildes einer isotropen Minimalfläche.

eine große Rolle. Man kann sie allgemein in der Form

$$(53) \quad z = \Re f(x + iy)$$

darstellen, wobei  $f(x + iy)$  eine beliebige komplexe analytische Funktion der komplexen Variablen  $x + iy$  ist.

Ihr kinematisches Bild ist nach (42) eine flächentreue Verwandtschaft ( $\Phi_l \rightarrow \Phi_r$ ) mit  $2\gamma = \pm \pi/2$ , deren Indikatrizen ( $i_l, i_r$ ) somit gegeneinander um einen rechten Winkel gedreht sind. Die Hauptachsen der beiden Indikatrizen sind einander folglich parallel zugeordnet und daher die kinematischen Bildrichtungen der Asymptotenlinien der isotropen Minimalflächen  $\Phi$ .

Zu jeder isotropen Minimalfläche  $\Phi$  der Form (53) gibt es die stetige Schar der assoziierten isotropen Minimalflächen  $\Phi^\alpha$  der Form

$$(54) \quad z = \Re [e^{i\alpha} f(x + iy)].$$

Darunter kommt für  $\alpha = 0$  die Minimalfläche  $\Phi^0 = \Phi$  selbst und für  $\Phi = \pi/2$  die konjugierte Minimalfläche  $\Phi^{\pi/2}$  mit der Gleichung

$$(55) \quad z = \Im f(x + iy)$$

vor. Die kinematischen Bilder  $(E_l^\alpha, E_r^\alpha)$  der Flächenelemente  $E^\alpha$  der Flächen  $\Phi^\alpha$  hängen dabei mit den Bildern  $(E_l^0, E_r^0)$  der Flächenelemente  $E^0$  der Fläche  $\Phi^0$  einfach durch deren Schwenkung um den Mittelpunkt  $\bar{E}^0$  um den Winkel  $\alpha$  zusammen.

Durch diesen einfachen «Schwenkungsprozeß» entsteht aus einer flächentreuen Abbildung  $(\Phi_l \rightarrow \Phi_r)$  mit orthogonalen Indikatrizen (und nur aus einer solchen Abbildung) immer wieder eine ebensolche flächentreue Abbildung  $(\Phi_l^\alpha \rightarrow \Phi_r^\alpha)$ .

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## Zum Satz von Holditch

Der Satz von *Holditch* [1] aus dem Jahre 1858 lautet: Werden die Endpunkte X,Y einer Strecke konstanter Länge  $a+b=d$  einmal auf einer Eilinie herumgeführt, so beschreibt ein Punkt Z dieser Strecke mit  $\overline{XZ}=b$ ,  $\overline{ZY}=a$  eine geschlossene Kurve, die keineswegs auch konvex sein muß. Für den Inhalt des so von beiden Kurven berandeten ringsförmigen Bereiches gilt

$$F_X - F_Z = ab\pi. \quad (1)$$

Hierbei werde mit  $F_X$  der Inhalt des ebenen Bereiches bezeichnet, der von der Bahnkurve des Punktes X (also unserer Eilinie) berandet wird. Entsprechendes gilt für  $F_Z$ .

Dieser Satz läßt sich leicht verallgemeinern, wenn man die Bahnen dreier kollinearer Punkte X,Y,Z der Gangebene  $E$  bei einem geschlossenen ebenen Bewegungsvorgang  $B$  mit der Drehzahl  $v$  betrachtet. Es gilt dann für die Bahnfläche des Punktes Z

$$F_Z = \frac{1}{d}(aF_X + bF_Y) - ab\pi v. \quad (2)$$

Ich hatte kürzlich [2] die polaren Trägheitsmomente von Punktbahnen bezüglich eines festen Punktes O' der Rastebene  $E'$  betrachtet, wenn diese im Steinerschen Sinne mit den augenblicklichen Drehwinkeln als Massenelementen belegt sind. Ich zeigte, daß für die polaren Trägheitsmomente  $T_{X/O'}$ ,  $T_{Y/O'}$ ,  $T_{Z/O'}$  eine ähnlich gebaute Formel gilt:

$$T_{Z/O'} = \frac{1}{d}(a T_{X/O'} + b T_{Y/O'}) - 2ab\pi v. \quad (3)$$

Für den Fall der klassischen Annahme von Holditch ergibt sich

$$T_{X/O'} - T_{Z/O'} = 2ab\pi. \quad (4)$$

Ich will nun den dualen Sachverhalt betrachten, dual im Sinne der in der euklidischen Geometrie geltenden Rudimente der projektiven Dualität.

Ich gehe von Geraden aus, die in der Gangebene  $E$  fest sind, und betrachte ihre Hüllkurven bei einem geschlossenen Bewegungsvorgang  $B$ . Mit Hilfe der Formeln von Cauchy für die Einhüllenden von Geradenscharen findet man [3] folgende Formeln:

1. Für den *Umfang* (Länge)  $L_g$  der Hüllbahn einer Geraden g gilt

$$L_g = \oint p d\varphi = 2\pi v s. \quad (5)$$

Hierin ist  $p = \overline{Pg}$  der Abstand des Momentanpols P von der Geraden g. Die Ebene  $E$  erscheint gegenüber der Ebene  $E'$  durch den Drehwinkel  $\varphi$  verdreht. v ist die Drehzahl von  $B$  mit  $\oint d\varphi = 2\pi v$ . Der Steinerpunkt S besitze von der Geraden g den Abstand  $s = \overline{Sg}$ . Dieser Punkt S wurde von J. Steiner als Schwerpunkt der Gangpolbahn P bei Belegung mit den Massenelementen  $d\varphi$  eingeführt, er existiert (im Endlichen) für  $v \neq 0$ .

2. Für die *Hüllbahnsfläche*  $F_g$  der Geraden  $g$  (berandet von der Hüllbahn von  $g$ ) gilt

$$F_g = F_{P'} - F_P + \frac{1}{2} T_{P/g}. \quad (6)$$

Hierin bedeuten  $F_P$ ,  $F_{P'}$  die Flächeninhalte der Bereiche von  $E$  bzw.  $E'$ , die von der Gangpolkurve  $P$  bzw. der Rastpolkurve  $P'$  berandet werden.  $T_{P/g} = \oint p^2 d\varphi$  ist das Trägheitsmoment von  $P$  bezüglich  $g$ .

3. Wir wollen gleich bemerken, daß eine ähnlich gebaute Formel auch für den *Inhalt* einer Punktbaahn gilt:

$$F_X = F_{P'} - F_P + \frac{1}{2} T_{P/X}. \quad (7)$$

$T_{P/X} = \oint a^2 d\varphi$  ist das polare Trägheitsmoment von  $P$  bezüglich des in  $E$  festen Punktes  $X$ . Der Abstand der Punkte  $P$  und  $X$  sei  $a = \overline{PX}$ .

## I.

Zwei Parallelkurven, im besonderen zwei parallele Gerade der Gangebene  $E$ , führen bei  $B$  wiederum zu Parallelkurven als Hüllbahnen in  $E'$ .

Für zwei parallele Gerade  $g_1$  und  $g_2$  unterscheiden sich die Abstände  $p$  und  $s$  um den festen Abstand  $\delta$  der Geraden:  $s_2 = s_1 + \delta$ ,  $p_2 = p_1 + \delta$ . Hierbei sei  $\delta$  mit Vorzeichen versehen. Mit (5) und (6) erhalten wir

$$L_{g_2} = L_{g_1} + 2\pi v \delta, \quad (8)$$

$$T_{P/g_2} = T_{P/g_1} + 2\delta L_{g_1} + 2\pi v \delta^2, \quad (9)$$

$$F_{g_2} = F_{g_1} + \delta L_{g_1} + \pi v \delta^2. \quad (10)$$

Diese Formeln sind im wesentlichen die Steinerschen Formeln für Parallelkurven von Eilinien, die auch auf höhere Momente ausgedehnt werden können [4].  $\oint d\varphi = 2\pi v$  ist hierbei als Gesamtkrümmung aufzufassen.

## II.

Wir wählen nun in der Gangebene  $E$  zwei feste Gerade  $g_1, g_2$ , die sich in einem Punkt  $X$  unter dem Winkel  $\alpha$  schneiden. Für ihre Abstände  $p_1, p_2$  vom Pol  $P$  und für die Entfernung  $a = \overline{PX}$  finden wir die elementare Beziehung

$$p_1^2 + p_2^2 - 2 p_1 p_2 \cos \alpha = a^2 \sin^2 \alpha. \quad (11)$$

Durch Integration folgt daraus

$$T_{P/g_1} + T_{P/g_2} - 2 D_{P/g_1 g_2} \cos \alpha = T_{P/X} \sin^2 \alpha, \quad (12)$$

wobei wir das Deviationsmoment  $D_{P/g_1 g_2} = \oint p_1 p_2 d\varphi$  der Gangpolbahn  $P$  bezüglich der Achsen  $g_1$  und  $g_2$  eingeführt haben. Mit den Formeln (5), (6) und (7) ergeben sich daraus zwei weitere Beziehungen zwischen den kinematischen Funktionalen:

$$F_{g_1} + F_{g_2} - D_{P/g_1 g_2} \cos \alpha = F_X \sin^2 \alpha + (F_{P'} - F_P)(1 + \cos^2 \alpha), \quad (13)$$

$$F_{g_1} + F_{g_2} - D_{P/g_1 g_2} \cos \alpha = 2F_X - \frac{1}{2} T_{P/X}(1 + \cos^2 \alpha). \quad (14)$$

## III.

Die Hinzunahme einer weiteren Geraden  $g_3$  durch den Schnittpunkt X von  $g_1$  und  $g_2$  führt bei den Winkelbezeichnungen

$$\gamma_i = \not\propto g_j g_k \ (i,j,k = 1,2,3 \text{ zyklisch}), \gamma_1 + \gamma_2 + \gamma_3 = 0$$

zur elementaren Relation

$$q_1 \sin \gamma_1 + q_2 \sin \gamma_2 + q_3 \sin \gamma_3 = 0 \quad (15)$$

zwischen den Abständen  $q_i = \overline{Qg_i}$  eines Punktes Q von den Geraden  $g_i$ . Durch Wahl von  $Q = P$  bzw.  $Q = S$ , also  $q_i = p_i = \overline{Pg_i}$  bzw.  $q_i = s_i = \overline{Sg_i}$  folgt daraus mit (5)

$$L_{g_1} \sin \gamma_1 + L_{g_2} \sin \gamma_2 + L_{g_3} \sin \gamma_3 = 0. \quad (16)$$

Bei Kenntnis der *Hüllbahnlängen* zweier Geraden können wir somit die Hüllbahnlänge einer dritten Geraden durch den gemeinsamen Punkt finden. Formel (16) kannte bereits G. Darboux [5].

Mit Hilfe der Längenformel (8) für parallele Gerade lässt sich nun auch die Hüllbahnlänge einer beliebigen Geraden h der Gangebene E angeben. Wir brauchen uns hierzu nur h parallel durch den Punkt X in die Gerade  $g_3$  verschoben zu denken (Schiebgröße  $\delta$ ). Wir finden

$$L_h = \frac{1}{\sin(\gamma_1 + \gamma_2)} (L_{g_1} \sin \gamma_1 + L_{g_2} \sin \gamma_2) + 2\pi v \delta. \quad (17)$$

Als duales Gegenstück zur Annahme und zur Formel von Holditch folgt nun: Decken sich die Hüllbahnen der Geraden  $g_1$  und  $g_2$ , d.h. wird ein Bewegungsvorgang B durch Gleiten der beiden Schenkel eines starren Winkels  $\widehat{g_1 X g_2}$  auf einer geschlossenen Kurve erzeugt, so folgt wegen  $L_{g_1} = L_{g_2}$  die Formel

$$L_h = A L_{g_1} + 2\pi v \delta \quad \text{mit } A = \frac{\sin \gamma_1 + \sin \gamma_2}{\sin(\gamma_1 + \gamma_2)} = \frac{\cos \frac{\gamma_1 - \gamma_2}{2}}{\cos \frac{\gamma_1 + \gamma_2}{2}}. \quad (18)$$

Im besonderen gilt für die Hüllbahnlänge der Geraden  $g_3$  durch X: Der Quotient der Hüllbahnlängen  $L_{g_3}/L_{g_1}$  hängt somit nur von den Winkeln  $\gamma_i$ , also von dem gleitenden starren Winkel  $\widehat{g_1 X g_2}$  und der mitgeführten Geraden  $g_3$  durch X ab. Die gemeinsame Hüllbahn von  $g_1$  und  $g_2$  bestimmt durch ihre *Umlaufzahl* die Drehzahl v von B, (Die Gesamtkrümmung der Hüllbahn ist  $2\pi v$ .) So wie beim klassischen Satz von Holditch Beschränkungen\*) über die Länge der bewegten Sehne bei Vorgabe einer beliebigen Eilinie vorgenommen werden müssen, um sie einmal herumführen zu können, so müssen wir nun auch hier

\*) R. Thüring, Verhandl. Naturforsch. Ges. Basel **63** (1952) 221–251, **67** (1956) 575–594.

verlangen, daß der starre Winkel um eine gegebene Kurve gleitend bis in seine Ausgangslage bewegt werden kann, was gewährleistet ist, wenn die Kurvenkrümmung keinen Vorzeichenwechsel erleidet.

#### IV.

Wir wollen nun entsprechende Formeln für die *Hüllbahnhalte* herleiten. Mittels (6), (7) finden wir auf Grund der elementaren Beziehung (15) für  $q_i = p_i$  zwei Paare jeweils ineinander überführbarer Formeln.

Mit den Abkürzungen

$$A_i = \frac{\sin \gamma_i}{\sin \gamma_3} \text{ für } i = 1, 2 \text{ und damit } 1 - A_1^2 - A_2^2 = 2A_1 A_2 \cos \gamma_3$$

gilt einmal

$$T_{P/g_3} = A_1^2 T_{P/g_1} + A_2^2 T_{P/g_2} + 2 A_1 A_2 D_{P/g_1 g_2}, \quad (19a)$$

$$F_{g_3} = A_1^2 F_{g_1} + A_2^2 F_{g_2} + (F_P - F_{P'}) 2 A_1 A_2 \cos \gamma_3 + A_1 A_2 D_{P/g_1 g_2}. \quad (19b)$$

Zum anderen gelangen wir (mittels (12)) zum Formelpaar

$$T_{P/g_1} \sin 2\gamma_1 + T_{P/g_2} \sin 2\gamma_2 + T_{P/g_3} \sin 2\gamma_3 + T_{P/X} 2 \sin \gamma_1 \sin \gamma_2 \sin \gamma_3 = 0, \quad (20a)$$

$$F_{g_1} \sin 2\gamma_1 + F_{g_2} \sin 2\gamma_2 + F_{g_3} \sin 2\gamma_3 + (F_X + F_{P'} - F_P) 2 \sin \gamma_1 \sin \gamma_2 \sin \gamma_3 = 0. \quad (20b)$$

Wir können nun wiederum entsprechend der Holditch-Konzeption die Übereinstimmung der Hüllbahnen von  $g_1$  und  $g_2$  fordern, was  $F_{g_1} = F_{g_2}$  und wegen (6) auch  $T_{P/g_1} = T_{P/g_2}$  zur Folge hat. Wir erhalten so gleichfalls Gegenstücke zur Formel von Holditch, in die jedoch auch Größen eingehen, die nicht nur von den erzeugten Bahnen (Punkt- bzw. Geradenhüllbahnen), sondern auch von den Polkurven von  $B$  abhängen.

#### V.

Rückblickend stellen wir fest: In die hergeleiteten Formeln weitere könnten durch Umformungen oder modifizierte Fragestellungen gefunden werden – gehen kinematische Funktionale über der Menge  $\{B\}$  der geschlossenen ebenen Bewegungsvorgänge  $B$ , sowie über der Menge der geometrischen Gebilde der beiden Ebenen  $E$  und  $E'$  linear ein.

Wir wollen eine Einteilung der kinematischen Funktionale vornehmen:

1. *Rein kinematische Funktionale* über  $\{B\}$ : Es seien dies Funktionale geometrischer Gebilde in den Ebenen  $E, E'$ , die durch  $B$  allein bestimmt sind. Im wesentlichen handelt es sich um die Funktionale der auf einander rollenden Polkurven  $P, P'$ , also deren Längen  $L_P, L_{P'}$  und die Inhalte  $F_P, F_{P'}$ , der von ihnen umrandeten Bereiche usf. Auch die Steiner-Punkte  $S, S'$  in  $E$  bzw. in  $E'$ , festgelegt durch die statischen Momente, die jeweils zu den Schwerpunkten von  $P$  bzw.  $P'$  führen, sind hinzuzurechnen.

2. *Kinematische Bahnfunktionale* über  $\{B\}$  und der Menge der Elemente (Punkte, Gerade,...) von  $E$  und  $E'$ : Sie beziehen sich nur auf geometrische Gebilde, die bei  $B$  bzw. der inversen Bewegung  $B^{-1}$  von Elementen der jeweiligen Gangebene in der zugehörigen Rastebene erzeugt werden. Hierzu gehören die Längen  $L_x, L_g$  von Punkt-bzw. Geradenhüllbahnen, ebenso die umrandeten Flächen  $F_x, F_g$ , ferner die eingangs erwähnten poiaren Trägheitsmomente  $T_{x/o'}$ .

3. *Gemischte kinematische Funktionale* über  $\{B\}$  und der Menge der Elemente von  $E$  und  $E'$ : In sie gehen die  $B$  bestimmenden Polkurven  $P$  und  $P'$  aber auch Elemente von  $E$  und  $E'$  ein. Zu ihnen haben wir die Trägheits- und Deviationsmomente  $T_{P/g}, D_{P/g_1g_2}$  und das polare Trägheitsmoment  $T_{P/X}$  zu rechnen.

Die *Drehzahl*  $v$  von  $B$  nehme als topologische Größe eine Ausnahmestellung ein. Wir wollen sie bei der folgenden Formulierung generell zulassen, obwohl sie eigentlich zu Fall 1 zu zählen wäre.

Als *Verallgemeinerungen der Formel von Holditch im engeren Sinn* können wir Beziehungen (Gleichungen) zwischen kinematischen Bahnfunktionalen auffassen, bei denen mindestens drei Elemente von  $E$  beteiligt sind und keine anderen Funktionale, abgesehen von der Drehzahl  $v$ , auftreten. Beispiele sind die Formeln (1) bis (4) und (16) bis (18).

*Verallgemeinerungen im weiteren Sinn* seien Relationen zwischen kinematischen Funktionalen aller drei Typen, bei denen jedoch Funktionale aus Typ 1 und (oder) Typ 3 tatsächlich vorkommen und wiederum auf mindestens drei Elementen von  $E$  Bezug genommen wird. Beispiele sind die Formeln (12) bis (14), (19) und (20).

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## On kinematics of the n-dimensional Euclidean space

A one-parametric family of isometries of the n-dimensional Euclidean space  $E^n$  is called a *motion* of  $E^n$ . The kinematics of  $E^n$  deals with the geometry of the motions of  $E^n$ . In the cases of the dimensions  $n=2,3$  many results (excluding the recent ones) are scheduled in textbooks (for example: H.R. Müller: Kinematik, Berlin 1963; R. Beyer: Technische Raumkinematik, Berlin 1963; a.o.). The first important results in the case of arbitrary dimension  $n$  are due to H.R. Müller in [4].

The purpose of this article is to present a summary of known results (chapter I) and a theory of (instantaneously) helical motions of order  $k$  including recent results by myself (chapter II).

### I. Motions of $E^n$

A motion of  $E^n$  is described in matrix notation by

$$(1) \quad x = A\bar{x} + c, \quad AA^t = I$$

where  $A^t$  is the transposed of the orthogonal matrix  $A$  and

$$(2) \quad A:J \rightarrow O(n), \quad c:J \rightarrow \mathbb{R}^n$$

are functions of differentiability class  $C^r$  ( $r \geq 3$ ) on a real interval  $J$ . Considering a motion as a movement of the space  $\bar{E}$  against the space  $E$  the coordinate vector  $\bar{x}$  in (1) describes a point of the so-called *moving space*  $\bar{E}$  and  $x$  a point of the so-called *fixed space*  $E$ .

Let  $\bar{x}$  be fixed in  $\bar{E}$  then (1) defines by (2) a parametrized curve in  $E$  which we call the *trajectory curve* or *path* of  $\bar{x}$  under the motion. We get the (*trajectory*) *velocity vector*  $\dot{x}$  in the path-point  $x$  from (1) by differentiation (denoted by  $\cdot$ ) for  $\dot{\bar{x}}=0$  in the form:

$$(3) \quad \dot{x} = B(x - c) + \dot{c}, \quad B := \dot{A}A^{-1}.$$

Since the matrix  $A$  is orthogonal the matrix  $B$  is skew:

$$(4) \quad B + B^t = 0.$$

Therefore only in the case of even dimension is it possible that the determinant  $|B|$  may not vanish. If  $|B(t)| \neq 0$  in  $t \in J$  we get exactly one solution  $p(t)$  of the equation

$$(5) \quad B(t)(p - c(t)) + \dot{c}(t) = 0.$$

$p(t)$  is the center of the instantaneous rotation of the motion in  $t \in J$  and is called the *pole of the motion* in  $t$ . At a pole  $p$  the velocity vector vanishes by the equation (3).

If  $|B|$  does not vanish on  $J$ , by considering the regularity condition of the motion we get a differentiable curve  $p:J \rightarrow E$  of poles in the fixed space  $E$ , called the *fixed pole curve*. By (1) there is uniquely determined the *moving pole curve*  $\bar{p}:J \rightarrow \bar{E}$

from the fixed pole curve point to point on  $J$ :  $p(t) = A(t)\bar{p}(t) + c(t)$ . Müller proved in [4]: *Under the motion the fixed pole curve and the moving pole curve are rolling on each other without sliding. Merely in the case  $n=2$  the motion is determined by the pair of rolling pole curves.*

In all other cases (that means  $|B|=0$ ), especially for  $n$  odd, we obtain by the rules of Linear Algebra: For every  $t \in J$  there exist a unit vector  $e(t) \in \text{kern } B(t)$  and  $\lambda(t) \in \mathbb{R}$  so that the solutions  $y$  of the equation

$$(6) \quad B(t)(y - c(t)) + \dot{c}(t) = \lambda(t) e(t)$$

fill a uniquely determined linear subspace  $E^k(t) \subset E^n$  with the dimension  $k = n - \text{rank } B$ .  $E^k(t)$  is the axis of the instantaneous screw ( $\lambda(t) \neq 0$ ) of the motion or the axis of the instantaneous rotation ( $\lambda(t) = 0$ ) and will be called the *instantaneous axis of the motion* in  $t \in J$ .

If  $|B|=0$  on the whole interval  $J$  under the regularity conditions we obtain a generalized ruled surface of dimension  $k+1$  in the fixed space  $E$  generated by the instantaneous axes  $E^k(t)$ ,  $t \in J$ , which we call the *fixed axoid*  $\Phi$  of the motion. The fixed axoid  $\Phi$  determines the *moving axoid*  $\bar{\Phi}$  in the moving space  $\bar{E}$  generator to generator by (1).  $\Phi$  and  $\bar{\Phi}$  are mapped upon each other by the same values of parameter. In this second case Müller proved in [4]: *The axoids  $\Phi, \bar{\Phi}$  of a motion in  $E^n$  touch each other along every common pair  $E^k(t) \subset \Phi, \bar{E}^k(t) \subset \bar{\Phi}$  for all  $t \in J$  by rolling and sliding upon each other under the motion.* Such a motion is called an (*instantaneously*) *helical motion of order k* in  $E^n$  (in [4] "Schrotung k-ter Ordnung"). *In a helical motion the direction cones of  $\Phi, \bar{\Phi}$  are isometrically mapped upon each other.* A helical motion of order  $k$  is a pure rolling for  $\lambda=0$ . In the special case that the axoids of a helical motion have lines of striction these lines are mapped on each other point to point. In the case of generator dimension  $k=1$  the parameters of distribution of  $\Phi$  and  $\bar{\Phi}$  are the same function. To this result Tölke added in [5]: *The points of the instantaneous axes of a motion are characterized by a minimum of the absolute value of the trajectory velocity.* Drábek in [1] dealt with the trajectory velocities of arbitrary points. In the case of odd dimension he supplemented the matrix  $B$  in (3) by a vector product of  $n-2$  invariant vectors. In [2] he generalized investigations of Pelzer (Diss. Berlin 1959) and Frank (Diss. Karlsruhe 1968) and proved: *The points with copunctal tangents of their paths at a point  $t \in J$  of a motion in  $E^n$  lie on an algebraic curve of order n.* I was able to prove by means of the analytical treatment of the chapter II that the order of these algebraic curves is reduced to  $n-k+1$  for a helical motion of order  $k$ .

Finally Karger treated the motions of  $E^n$  with pole curves ( $|B| \neq 0$ ) respectively with ruled surfaces as axoids ( $|B|=0, k=1$ ) by means of the theory of Lie groups under the special condition that the Frenet-n-frames of the pole curves resp. the Frenet-n-frames of the unit vectors in the generating lines of the instantaneous axoids correspond to each other under the motion. For this special motions he gives the invariants (see [3]).

## II. Helical motions of order k

While the analytic treatment of a motion with pole curves is determined by the pair of the rolling pole curves (see [4], [3]) until now a corresponding treatment

of the (instantaneously) helical motions of order  $k$  has not been possible for the lack of a practicable theory of the generalized ruled surfaces (axoids). However such a theory is now available in [7] by Frank and Giering, so it is possible to give a theory of helical motions in  $E^n$  for the first time.

For this purpose we first summarize the basic properties of the axoids from the papers [7]: In any  $k$ -dimensional generator  $E^k$  of a  $(k+1)$ -dimensional generalized ruled surface (axoid, in [7] “ $(k+1)$ -Regelfläche”)  $\Phi \subset E^n$  there exists a maximal linear subspace  $K^{k-m} \subset E^k$  of dimension  $k-m$  with the property that in every point of  $K^{k-m}$  no tangent space of  $\Phi$  is determined ( $K^{k-m}$  contains all singularities of  $\Phi$  in  $E^k$ ) or there exists a maximal linear subspace  $Z^{k-m} \subset E^k$  of dimension  $k-m$  with the property that in every point of  $Z^{k-m}$  the tangent space of  $\Phi$  is orthogonal to the asymptotic bundle of the tangent spaces in the points of infinity of  $E^k$  (all points of  $Z^{k-m}$  have the same tangent space of  $\Phi$ ). We call  $K^{k-m}$  the *edge space* in  $E^k \subset \Phi$  (in [7] “Kehlraum”) and  $Z^{k-m}$  the *central space* in  $E^k \subset \Phi$  (in [7] “Zentralraum”). A point of  $Z^{k-m}$  is called a *central point*. If  $\Phi$  possesses generators all of the same type the edge spaces resp. the central spaces generate a generalized ruled surface contained in  $\Phi$  which we call the *edge ruled surface* (in [7] “Kehlregelfläche”) resp. the *central ruled surface* (in [7] “Zentralregelfläche”). For  $m=k$  the edge ruled surface degenerates in the *edge of  $\Phi$* , the central ruled surface in the *line of striction*. So the axoids with edge ruled surface generalize the tangent surfaces of  $E^3$ , the axoids with central ruled surface the ruled surfaces with line of striction of  $E^3$ .

For the analytical representation of an axoid  $\Phi$  we choose a leading curve  $y$  in the edge resp. central ruled surface  $\Omega \subset \Phi$  transversal to the generators. In [7] it is shown that there exists a distinguished moving orthonormal frame (ONF) of  $\Phi \{e_1, \dots, e_k\}$  with the properties:

- (i)  $\{e_1, \dots, e_k\}$  is an ONF of the  $E^k \subset \Phi$
- (ii)  $\{e_{m+1}, \dots, e_k\}$  is an ONF of the  $K^{k-m}$  resp.  $Z^{k-m} \subset E^k$ <sup>1</sup>
- (iii)  $\dot{e}_\sigma = \sum_{v=1}^k \alpha_\sigma^v e_v + \kappa^\sigma a_{k+\sigma}$  ( $\sigma = 1, \dots, m$ )  
 $\dot{e}_{m+\rho} = \sum_{t=1}^m \alpha_{m+\rho}^t e_t$  ( $1 \leq \rho, \gamma \leq k-m$ )  
with  $\kappa^\sigma > 0$ ,  $\alpha_\mu^v = -\alpha_v^\mu$ ,  $\alpha_{m+\rho}^{m+\gamma} = 0$
- (iv)  $\{e_1, \dots, e_k, a_{k+1}, \dots, a_{k+m}\}$  is an ONF.

A moving ONF of  $\Phi$  with the properties (i)-(iv) is called a *principal frame* of  $\Phi$ . If  $\kappa^1 > \dots > \kappa^k > 0$  the principal frame of  $\Phi$  is determined up to the signs. By a given principal frame the vectors  $a_{k+1}, \dots, a_{k+m}$  are well defined.

A leading curve  $y$  of an axoid  $\Phi$  is a leading curve of the edge resp. central ruled surface  $\Omega \subset \Phi$  too iff its tangent vector has the form

$$(7) \quad \dot{y} = \sum_{v=1}^k \zeta^v e_v + \eta^{m+1} a_{k+m+1}$$

where for  $\eta^{m+1} \neq 0$   $a_{k+m+1}$  is a unit vector well defined up to the sign with the

1 One can show that (ii) follows from (i), (iii), (iv).

property that  $\{e_1, \dots, e_{k+m}, a_{k+m+1}\}$  is an ONF of the tangent bundle of  $\Phi$ . One shows:  $\eta^{m+1}(t)=0$  in  $t \in J$  iff the generator  $E^k(t) \subset \Phi$  contains the edge space  $K^{k-m}(t)$ .

If  $\eta^{m+1} \neq 0$  we call the  $m$  magnitudes  $\delta_\sigma := \frac{\eta^{m+1}}{\kappa^\sigma}$  ( $\sigma = 1, \dots, m$ ) the *principal parameters of distribution*. These parameters are direct generalizations of the parameter of distribution of the ruled surfaces in  $E^3$  (see [7]). An axoid with central ruled surface and no principal parameter of distribution ( $m=0$ ) is a  $(k+1)$ -dimensional cylinder.

Now we find the following recent statements for the helical motions of order  $k$ :

**Theorem 1.** Under a helical motion of order  $k$  in  $E^n$  the principal ONFs of the fixed axoid  $\Phi$  and the moving axoid  $\bar{\Phi}$  correspond to each other, the edge spaces (resp. the central spaces) of  $\Phi, \bar{\Phi}$  are mapped on each other point to point. We have  $|\delta_\sigma| = |\bar{\delta}_\sigma|$  for the principal parameters of distribution  $\delta_\sigma$  of  $\Phi$  and  $\bar{\delta}_\sigma$  of  $\bar{\Phi}$  ( $\sigma = 1, \dots, m$ ).

*Proof:* Let  $\Phi$  and  $\bar{\Phi}$  be the corresponding axoids of the given helical motion of order  $k$  in  $E^n$  and  $\{\bar{e}_1, \dots, \bar{e}_k\}$  a principal ONF of the moving axoid  $\bar{\Phi}$ . Then the equations (iii) hold for  $\bar{e}_v$  with barred coefficients.  $\bar{\Phi}$  has the parameter representation on the interval  $J$  by

$$(8) \quad \bar{z}(t, x^1, \dots, x^k) = \bar{y}(t) + \sum_{v=1}^k x^v \bar{e}_v(t), \quad t \in J, \quad x^v \in \mathbb{R}$$

where  $\bar{y}$  is a leading curve of the edge resp. central ruled surface  $\bar{\Omega} \subset \bar{\Phi}$ .

The helical motion is given by (1) with (2). Set

$$(9) \quad e_v := A \bar{e}_v \quad (v = 1, \dots, k)$$

then  $\{e_1, \dots, e_k\}$  is a moving ONF of the fixed axoid by definition and spans the kernel of  $B$  by the condition (6):

$$(10) \quad Be_v = 0 \quad (v = 1, \dots, k).$$

Considering (3) and (10) we obtain from (9) by differentiation:

$$(11) \quad \dot{e}_v = A \dot{\bar{e}}_v \quad (v = 1, \dots, k).$$

From these equations and (iii) for  $\bar{e}_v$  it follows by the orthogonality of  $A$  that  $\{e_1, \dots, e_k\}$  in (9) is a principal ONF of the fixed axoid  $\Phi$  too. We find by computation

$$(12) \quad A \bar{a}_{k+\sigma} = a_{k+\sigma} \quad (\sigma = 1, \dots, m),$$

$$(13) \quad \alpha_v^\mu = \bar{\alpha}_v^\mu, \quad \kappa^\sigma = \bar{\kappa}^\sigma > 0 \quad (1 \leq \mu, v \leq k; \sigma = 1, \dots, m).$$

The leading curve  $\bar{y} \subset \bar{\Omega} \subset \bar{\Phi}$  is mapped by the motion upon the curve  $y = A \bar{y} + c \subset \Phi$ . Since  $\bar{y}$  fulfills the equation (7) with bars we find by differentiation considering the relations (6), (9), (12):

$$(14a) \quad \eta^{m+1} a_{k+m+1} = \bar{\eta}^{m+1} A \bar{a}_{k+m+1}$$

$$(14b) \quad |\eta^{m+1}| = |\bar{\eta}^{m+1}|$$

and

$$(15) \quad \dot{y} = \lambda e + A\dot{y}, \quad \zeta^v = \lambda\lambda^v + \bar{\zeta}^v, \quad e = \sum_{v=1}^k \lambda^v e_v, \quad e^2 = 1.$$

From this we get the rest of the statements in theorem 1.

We generalize a definition of W. Degen in [8] and call

$$(16) \quad \delta := \sqrt[m]{|\delta_1 \cdots \delta_m|}$$

the *parameter of distribution* of an axoid. Then we immediately read off from theorem 1:

**Theorem 2.** *The parameters of distribution agree for the fixed and the moving axoid of a helical motion in  $E^n$ .*

There are also other properties which transfer from the moving axoid to the fixed axoid. For example: Let the edge resp. central ruled surface  $\bar{\Omega}$  of the moving axoid  $\Phi$  be cylindric, that is equivalent to

$$(17) \quad \vec{\alpha}_{m+\rho} = 0 \quad (\ i = 1, \dots, m; \ \rho = 1, \dots, k - m)$$

for the coefficients in (iii). Then we obtain by the relations (13):

**Theorem 3.** *Let  $\Phi$  and  $\bar{\Phi}$  be the fixed and the moving axoid of a helical motion in  $E^n$ . The edge resp. central ruled surfaces of  $\Phi, \bar{\Phi}$  have the cylindric property exactly in common.*

Last we want to answer the question how much the pair of the fixed and the moving axoid determines a corresponding helical motion of order  $k$  in  $E^n$ . For this purpose we first complete the ONF  $\{e_1, \dots, a_{k+m}\}$  of an axoid with edge ruled surface resp. the ONF  $\{e_1, \dots, a_{k+m}, a_{k+m+1}\}$  of an axoid with central ruled surface by an arbitrary ONF  $\{a_{k+m+1}, \dots, a_n\}$  resp.  $\{a_{k+m+2}, \dots, a_n\}$  of the orthogonal complement, called a *complementary ONF*. From the orthogonality conditions we obtain by differentiation:

$$(v) \quad \begin{aligned} \dot{a}_{k+\sigma} &= -\kappa^\sigma e_\sigma + \sum_{i=1}^m \tau_i^\sigma a_{k+i} + \omega^\sigma a_{k+m+1} + \sum_{\xi=2}^{n-k-m} \gamma_\sigma^\xi a_{k+m+\xi} \\ \dot{a}_{k+m+1} &= -\sum_{i=1}^m \omega^i a_{k+i} - \sum_{\xi=2}^{n-k-m} \beta^\xi a_{k+m+\xi} \\ \dot{a}_{k+m+\xi} &= \sum_{i=1}^m \omega_i^\xi a_{k+i} + \beta^\xi a_{k+m+1} + \sum_{\varphi=2}^{n-k-m} \beta_\xi^\varphi a_{k+m+\varphi} \\ \tau_\sigma^i &= -\tau_i^\sigma, \quad \beta_\xi^\varphi = -\beta_\varphi^\xi, \quad \omega_\xi^\sigma = -\gamma_\sigma^\xi \quad (1 \leq \sigma, i \leq m; 2 \leq \varphi, \xi \leq n - k - m). \end{aligned}$$

We call a complementary ONF of an oxoid a *complementary ONF of Juza* if the coefficients in (v) satisfy the conditions

$$(18) \quad \beta^\varphi = \beta_\xi^\varphi = 0 \text{ resp. } \beta_\xi^\varphi = 0 \quad (2 \leq \varphi, \xi \leq n - k - m).$$

That is: On the *complementary axoid* of an axoid, generated by a leading curve  $y$  of the edge resp. central ruled surface and by the given complementary *ONF* of Juza, the paths of the vertices of the moving coordinate simplex

$$y, (y + a_{k+m+1}), y + a_{k+m+2}, \dots, y + a_n$$

are orthogonal trajectories of the generators (see [7]).

**Theorem 4.** Let  $\Phi, \bar{\Phi} \subset E^n$  be two  $(k+1)$ -dimensional axoids mapped upon each other by the same values of the parameters of representations (8) with principal *ONF* and a leading curve  $y$  resp.  $\bar{y}$  of the edge (resp. central) ruled surface, the edge (resp. central) ruled surfaces of  $\Phi, \bar{\Phi}$  shall correspond by the same parameter values. Then there exists a helical motion of order  $k$  in  $E^n$  with the fixed axoid  $\Phi$  and the moving axoid  $\bar{\Phi}$  and the given parameter on  $J$  iff one of the following two equivalent conditions holds:

(a) It holds (13) and (14b).

(b) The direction cones of  $\Phi, \bar{\Phi}$  are isometrically mapped upon each other. For non-cylindric axoids  $\Phi, \bar{\Phi}$  ( $m \neq 0$ ) the parameters of distribution  $\delta, \bar{\delta}$  agree, for cylinders  $\Phi, \bar{\Phi}$  ( $m = 0$ ) the orthogonal trajectories of their generators are isometrically mapped upon each other.

Then there exist exactly two different helical motions of order  $k$  if we assume the additional conditions:

(c) A given pair of leading curves  $y, \bar{y}$  of the edge (resp. central) ruled surfaces of  $\Phi, \bar{\Phi}$  corresponds point to point by the motion.

(d) A given complementary *ONF* of Juza of  $\bar{\Phi}$  is mapped on an arbitrarily given complementary *ONF* of  $\Phi$ .

*Proof:* The equivalence of the conditions (a) and (b) is easily seen from the formulas in [7]. Then first let  $\Phi, \bar{\Phi}$  be the pair of axoids of a helical motion (1) of order  $k$  in  $E^n$ . By theorem 1 and by the equations (9), (11) the assumptions of the theorem are satisfied and (a) necessarily holds. The frame vectors of the completed moving *ONFs* for principal *ONFs* of  $\Phi, \bar{\Phi}$  have the corresponding in the motion like that of (9), (12) and without loss of (geometric) generality

$$(19) \quad a_{k+m+1} = A \bar{a}_{k+m+1}, \quad a_{k+m+\xi} = A \bar{a}_{k+m+\xi} \quad (\xi = 2, \dots, n-k-m).$$

(The change of sign of  $a_{k+m+1}$  leads to the same motion by (14a).)

Let  $(b^j)$  be the skew symmetric matrix of  $B$  with respect to the moving *ONF*  $\{e_1, \dots, a_n\}$  of  $\Phi$  then we find by differentiation of the equations (9), (12), (19) and by a computation corresponding to that of the proof of theorem 1 the conditions

$$(20) \quad b_\nu^j = 0, \quad b_{k+\sigma}^{k+1} = \tau_\sigma^t - \bar{\tau}_\sigma^t, \quad b_{k+m+\xi}^{k+m+1} = \beta^\xi - \bar{\beta}^\xi, \quad b_{k+m+\xi}^{k+m+\varphi} = \beta^\varphi - \bar{\beta}^\varphi, \quad b_j^i = -b_i^j$$

$$(21) \quad b_{k+\sigma}^{k+m+1} = \omega^\sigma - \bar{\omega}^\sigma, \quad b_{k+\sigma}^{k+m+\xi} = \gamma_\sigma^\xi - \bar{\gamma}_\sigma^\xi \\ (1 \leq i, j \leq n; \nu = 1, \dots, k; 1 \leq \sigma, \iota \leq m; 2 \leq \xi, \varphi \leq n-k-m).$$

Conversely by the existence theorems of the theory of ordinary differential equations we obtain an orthonormal matrix function  $A$  on the parameter interval  $J$  such that  $\dot{A} = BA$  with  $B$  satisfying the conditions (20), (21). Then the equations (9), (12), (19) hold for  $A$  by (a), if we eventually change the sign of  $\bar{a}_{k+m+1}$ . Let  $y, \bar{y}$  be leading curves of the edge (resp. central) ruled surfaces of  $\Phi, \bar{\Phi}$

and set  $c := y - A\bar{y}$  on  $J$ . Then the mapping (1) with these  $A, c$  describes a helical motion of order  $k$  with the pair of fixed axoid  $\Phi$  and moving axoid  $\bar{\Phi}$  and with the sliding vector  $\lambda e$  in (15).

From the above definition of  $c$  one immediately sees the additional condition (c) in the theorem. The condition (d) is contained in (19) and (20).

It remains to show that there are exactly two motions under the conditions (c), (d) if there exists one motion for  $\Phi, \bar{\Phi}$ . For this we have to observe by the conditions (12), (d) that a motion merely may change the signs of the principal ONFs. Let  $A\bar{e}_v = \varepsilon_v e_v$  ( $v = 1, \dots, k$ ),  $\varepsilon_v = \pm 1$ , then we find by differentiation  $\alpha_v^\mu = \varepsilon_\mu \varepsilon_v \bar{\alpha}_v^\mu$ . Since one motion exists for which (13) holds, the comparison with (13) yields  $\varepsilon_1 = \dots = \varepsilon_k = \varepsilon = \pm 1$ . So we find as in the proof of the theorem 1 and by (19)

$$(22) \quad A\bar{e}_v = \varepsilon e_v, A\bar{a}_{k+\sigma} = \varepsilon a_{k+\sigma}, A\bar{a}_{k+m+1} = a_{k+m+1}, A\bar{a}_{k+m+\xi} = a_{k+m+\xi} \\ \varepsilon = \pm 1, v = 1, \dots, k, \sigma = 1, \dots, m, \xi = 2, \dots, n - k - m.$$

By differentiation of this equation we obtain (13), (14), (20) again and instead of (21) now

$$(23) \quad b_{k+\sigma}^{k+m+1} = \omega^\sigma - \varepsilon \bar{\omega}^\sigma, b_{k+\sigma}^{k+m+\xi} = \gamma_\sigma^\xi - \varepsilon \bar{\gamma}_\sigma^\xi, \varepsilon = \pm 1.$$

As in the above part of the proof there exists a second motion with  $\varepsilon = -1$ , (20), (22), (23). This completes the proof.

*Remarks.* (22) means a reflection at the space of dimension  $n - k - m$  through a point of the leading curve  $y$  totally orthogonal to the asymptotic bundle of the generator of the axoid. The moving axoids of the both different motions for  $\Phi, \bar{\Phi}$  correspond to each other by this reflection.

In generalizing the investigations of J. Krames in [6] we may define a *symmetric helical motion of order k* (German: *symmetrische Schrotung k-ter Ordnung*) for a given axoid. The pair of axoids of a symmetric helical motion is given by the fixed axoid and its picture by the above reflection as the moving axoid. The symmetric helical motion of order  $k$  for the axoid  $\Phi$  with the properties, that a given leading curve  $y$  of the edge resp. central ruled surface and its reflected curve  $\bar{y}$  is a corresponding pair and the complementary ONFs of Juza are corresponding, is characterized by

$$(24) \quad \lambda \lambda^v = 2\zeta^v, b_{k+\sigma}^{k+m+1} = 2\omega^\sigma, b_{k+\sigma}^{k+m+\xi} = 2\gamma_\sigma^\xi, b_j^i = 0 \text{ otherwise}, b_j^i = -b_i^j \\ (\sigma = 1, \dots, m; \gamma = 1, \dots, k; \xi = 2, \dots, n - k - m; 1 \leq i, j \leq n)$$

if we consider (15), (20), (23).

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## Bemerkungen über parabolische Kegelschnittflächen

1. Wir nennen eine Fläche  $\Phi$ , die von einer einparametrischen Schar<sup>1</sup> ( $t \rightarrow K_t$ ,  $t \in I$ ) von Kegelschnitten  $K_t$  des reellen projektiven Raumes  $\mathbb{P}^3$  erzeugt wird, eine *Kegelschnittfläche*. Die Ebene  $E_t$ , in welcher  $K_t$  liegt, soll *Stützebene* heißen. Wir setzen voraus, daß die Stützebenen weder einem Büschel noch einem Bündel angehören, so daß sie also eine Torse  $T$ , die *Stütztorse* von  $\Phi$ , einhüllen. Die zum Parameterwert  $t$  gehörige Erzeugende von  $T$  werde mit  $g_t$  bezeichnet und *Stützgerade* genannt. Die Kegelschnittfläche  $\Phi$  heißt *parabolisch*, wenn für alle  $t \in I$  die Stützgerade  $g_t$  den erzeugenden Kegelschnitt  $K_t$  berührt und dieser Berührpunkt  $x_t$  zugleich der Kehlpunkt von  $g_t$  ist. (Um diese letztere Zusatzbedingung anzudeuten, spricht man auch von parabolischen Kegelschnittflächen *im engeren Sinne*.)

In den meisten umfassenderen Arbeiten über Kegelschnittflächen (vgl. [1], [2], [3]) tritt der parabolische Fall zwar in der Klassifikation hervor, wird jedoch keiner eigenen Untersuchung unterzogen. Andererseits besitzt man aber in der projektiven Kurventheorie und in der Theorie der Flächenstreifen, wie sie in [6] dargestellt sind, ausgeprägte Beispiele für parabolische Kegelschnittflächen, die von den jeweiligen Schmiekegelschnitten dieser Gebilde erzeugt werden. Auf diesen Zusammenhang wurde erstmalig in [8] aufmerksam gemacht; wir kommen auf einige Ergebnisse dieser Arbeit zurück und vervollständigen sie.

2. Zunächst betrachten wir die *Hülltorse*  $H_t$  von  $\Phi$  längs des erzeugenden Kegelschnitts  $K_t$ —sie wird von den Tangentenebenen an  $\Phi$ , genommen in den Punkten von  $K_t$ , erzeugt. Betrachtet man  $K_t$  als Fläche 2. Klasse (duales Analogon zu einem quadratischen Kegel), so erkennt man, daß  $H_t$  als Schnitt zweier benachbarter von 4. Klasse ist, von welcher jedoch das Ebenenbüschel mit  $g_t$  als Achse abspaltet. Der Restschnitt—und nur diesen wollen wir mit  $H_t$  bezeichnen—isst von 3. Klasse. Hier zeichnet sich bereits eine Gabelung der Theorie der parabolischen Kegelschnittflächen in zwei Hauptfälle ab:

1. *Gattung*:  $H_t$  zerfällt nicht, besitzt also eine Normkurve 3. Ordnung  $C_t$  als Gratlinie.
2. *Gattung*:  $H_t$  zerfällt, und zwar nochmals in das Büschel um  $g_t$  und einen quadratischen Kegel  $K'_t$ .

Wir erwähnen, daß die Schmiekegelschnittflächen der Raumkurven zur 1. Gattung und die der Flächenstreifen zur 2. Gattung gehören.

3. In einem Punkt  $p$  von  $K_t$  ist definitionsgemäß jene Erzeugende von  $H_t$ , die durch  $p$  geht, die *konjugierte Tangente* zur Tangente an  $K_t$ . Die durch Integration der konjugierten Richtungen auf  $\Phi$  zu gewinnende Kurvenschar  $S$  bildet zusammen mit der Schar der erzeugenden Kegelschnitte ( $K_t$ ,  $t \in I$ ) ein *konjugiertes Netz*  $N$  auf  $\Phi$ . Nach einem frühen Ergebnis von Blutel [1] bilden die Kurven von  $S$  die Kegelschnitte  $K_t$  projektiv aufeinander ab (d. h. je vier Kurven

<sup>1</sup> Darunter wird eine hinreichend oft stetig differenzierbare Abbildung eines offenen, reellen Intervalls  $I$  in die jeweilige Mannigfaltigkeit der angegebenen geometrischen Objekte verstanden.

von  $S$  schneiden alle Kegelschnitte  $K_t$  unter konstantem Doppelverhältnis). Man nennt daher  $S$  auch eine *Doppelverhältnisschar* auf  $\Phi$  (vgl. [7]).

Der Kehlpunkt auf einer Erzeugenden von  $H_t$ , also einer konjugierten Tangente, erzeugt in Abhängigkeit der beiden Parameter  $s, t$  von  $\Phi$  (wobei  $s$  ein projektiver Parameter auf  $K_t$  sein möge) die *Laplace-Transformierte*  $\Psi$  von  $\Phi$  (bzgl.  $N$ ). Dieser Kehlpunkt ist aber bei Flächen der 1. Gattung der Berührpunkt mit  $C_t$  und bei denen der 2. Gattung die Spitze von  $K'_t$ . Daher gilt:

*Bei Kegelschnittflächen der 1. Gattung wird die Laplace-Transformierte  $\Psi$  (in Richtung der zu  $K_t$  konjugierten Tangenten) von einer Schar von Normkurven 3. Ordnung ( $C_t, t \in I$ ) erzeugt; diese sind durch gleiche Parameterwerte auf  $(K_t, t \in I)$  projektiv bezogen.*

*Bei Kegelschnittflächen der 2. Gattung entartet die Laplace-Transformierte  $\Psi$  in die von den Spitzen der Hüllkegel  $K'_t$  erzeugte Kurve.*

Weiter sei bemerkt, daß die Laplace-Transformierte in Richtung der Tangenten an  $K_t$  selbst in die Stütztorse entartet (im Sinne von Goursat); die konjugierten Kurven bilden auf der Stütztorse eine Doppelverhältnisschar. Man kann auch diese DV-Schar auf der Stütztorse zum Ausgangspunkt der Untersuchungen über parabolische Kegelschnittflächen machen, da sie diese eindeutig rückbestimmt.

4. Im folgenden werden wir nur noch Flächen der 1. Gattung betrachten. Die Gratlinie  $C_t$  der Hülltorse  $H_t$  nennen wir *Hauptkurve*, die von ihr erzeugte Fläche  $\Psi$  die *Begleitfläche*.

Wir geben nun ein begleitendes Bezugssystem an, welches aus  $x(t)$ —dem Kehlpunkt<sup>2</sup> der Stütztorse—, dem erzeugenden Punkt  $\eta(t)$  einer auf  $\Phi$  beliebig, jedoch transversal zu  $(K_t, t \in I)$  gewählten *Leitkurve*  $L$ , dem Pol  $t(t)$  von  $x(t) \wedge \eta(t)$  (bzgl.  $K_t$ ) und dem erzeugenden Punkt  $w(t)$  von  $K_t$  besteht. Bei geeigneter Normierung dieser Basis (und Fortlassung einiger Argumente  $t$ ) gelten dann folgende Beziehungen:

*Parameterdarstellung der Kegelschnittfläche  $\Phi$ :*

$$(1) \quad f(s,t) = 2\kappa s^2 x(t) + 2st(t) + \eta(t) \quad (s,t) \in \bar{\mathbb{R}} \times I.$$

*Ableitungsgleichungen:*

$$(2) \quad \begin{aligned} x' &= && et \\ t' &= 4\kappa ax &+ f\eta \\ \eta' &= && 4at &+ gw \\ w' &= &ux + &pt + q\eta + rw \end{aligned}$$

mit

$$(3) \quad \kappa = \text{const} \quad g = 4(f - \kappa e).$$

*Paramterdarstellung der Begleitfläche  $\Psi$ :*

$$(4) \quad l(s,t) = 2\kappa s^3 x(t) + 3s^2 t(t) + 3s\eta(t) + w(t) \quad (s,t) \in \bar{\mathbb{R}} \times I.$$

*Sterntransformation (Übergang zu anderer Leitkurve und Umnormungen):*

<sup>2</sup> Der Kürze wegen bezeichnen wir Punkte und die sie darstellenden Vektoren aus  $V^4(\mathbb{R}) \setminus \{0\}$  gleich.

$$(5) \quad \begin{aligned} x^* &= \rho x \\ t^* &= \sigma(t + 2\kappa Bx) \\ \eta^* &= \rho^{-1}\sigma^2(\eta + 2Bt + 2\kappa B^2x) \\ w^* &= \rho^{-2}\sigma^3(w + \frac{3}{2}B\eta + \frac{3}{2}B^2t + \kappa B^3x) \end{aligned}$$

mit

$$(6) \quad \frac{\rho'}{\rho} = 2\kappa eB, \quad \frac{\sigma'}{\sigma} = 2(f - \kappa e)B \quad (\sigma(t) \neq 0, \rho(t) \neq 0 \quad \text{für alle } t \in I).$$

*Transformationsformeln für die Koeffizienten in (2):*

$$(7) \quad e^* = \rho\sigma^{-1}e \quad f^* = \rho\sigma^{-1}f,$$

$$(8) \quad a^* = \rho^{-1}\sigma [a + \frac{1}{2}(f - 2\kappa e)B^2 + \frac{1}{2}B'],$$

$$(9) \quad \begin{aligned} r^* &= r + 4kB \\ j^* &= \rho^{-1}\sigma[j + \frac{1}{2}rB + kB^2] \\ p^* &= \rho^{-2}\sigma^2[p + 6jB + \frac{3}{2}rB^2 + 2kB^3] \\ v^* &= \rho^{-3}\sigma^3[v + 2pB + 6jB^2 + rB^3 + kB^4] \end{aligned}$$

mit den Abkürzungen

$$(10) \quad k = 3f - 4\kappa e, \quad j = a - \frac{1}{3}q, \quad v = -\frac{1}{\kappa}u.$$

5. Auf einige Folgerungen, die sich aus diesen Beziehungen unmittelbar ziehen lassen, sei besonders hingewiesen:

1. Mit der Fläche  $\Phi$  ist in zweiter Differentiationsordnung eine absolute Invariante  $J = \kappa e/f$  projektiv verbunden.
2. Die Differentialgleichung der (zu den erzeugenden Kegelschnitten) konjugierten Kurven lautet

$$s' + 2a + (f - 2\kappa e)s^2 = 0.$$

(Darin, daß diese vom Riccati'schen Typ ist, spiegelt sich der Satz von Blutel [1] wider, daß die konjugierten Kurven die erzeugenden Kegelschnitte projektiv aufeinander abbilden.)

3. Die Klasse der Kegelschnittflächen 1. Gattung<sup>3</sup> enthält eine Folge von fünf Unterklassen, von denen jede folgende in der vorigen enthalten ist; kennzeichnend sind dafür folgende Bedingungen:

1. Unterklasse:  $k = 0$  ( $f = \frac{4}{3}\kappa e$ )
2. Unterklasse:  $k = 0, r = 0$
3. Unterklasse:  $k = 0, r = 0, j = 0$
4. Unterklasse:  $k = 0, r = 0, j = 0, p = 0$
5. Unterklasse:  $k = 0, r = 0, j = 0, p = 0, v = 0$

(Denn diese bleiben bei Sterntransformation invariant.)

6. Im Punkt  $x(t)$  berühren sich folgende drei Kurven
1. die Gratlinie ( $x(t), t \in I$ ) der Stütztorse,

3 Ein analoges Ergebnis gilt auch für die Kegelschnittfläche der zweiten Gattung [8].

2. der erzeugende Kegelschnitt  $K_t$ ,
3. die Hauptkurve  $C_t$ ,

und alle drei haben  $E_t$  als gemeinsame Schmiegebene. Für solche Kurvenpaare reguliert die Invariante von Mehmke-Segre-Smith das Berührverhalten. Den Ableitungsgleichungen entnehmen wir:

*Die Berührinvarianten von Mehmke-Segre-Smith haben folgende Werte (wobei sich die Indizes auf obige Numerierung beziehen):*

$$M_{12} = \frac{f}{\kappa e}, M_{13} = \frac{3f}{4e}, M_{23} = \frac{3}{4}.$$

*Insbesondere berühren sich die Gratlinie und der erzeugende Kegelschnitt genau dann an jeder Stelle  $t \in I$  in zweiter Ordnung, wenn*

$$f = \kappa e$$

*gilt, die Fläche also der zweiten Gattung angehört; ferner berühren sich die Gratlinie und die Hauptkurve  $C_t$  genau dann an jeder Stelle  $t \in I$  in 2. Ordnung, wenn*

$$f = \frac{4}{3}\kappa e$$

*gilt, die Fläche also der bereits erwähnten 1. Unterkategorie angehört.*

7. Interessante Berühreigenschaften haben auch der *Hauptkegel  $\tilde{K}_t$* , der sich durch Verbinden der Hauptkurve  $C_t$  mit  $x(t)$  ergibt, und die Gratlinie. In lokalen Koordinaten lautet die Gleichung von  $\tilde{K}_t$

$$2x_2^2 - 3x_1x_3 = 0.$$

Durch kanonische Reihenentwicklung erhält man nach einiger Rechnung:

*Die Gratlinie berührt den Hauptkegel stets mindestens in 3. Ordnung; diese Ordnung erhöht sich genau dann auf mindestens die Werte 4, 5 und 6, wenn die Fläche der 1., 2. bzw. 3. Unterkategorie angehört.*

Damit sind erste geometrische Kennzeichnungen dieser Unterklassen geleistet. Darüber hinaus stellt man fest:

*Die 3. Unterkategorie besteht genau aus den Schmiegkegelschnittflächen der Raumkurven; die Fläche gehört genau dann der 4. bzw. 5. Unterkategorie an, wenn die Raumkurve eine Komplexkurve bzw. eine Kurve 3. Ordnung ist.*

Damit ist der Anschluß «nach unten», zur wohlbekannten projektiven Theorie der Raumkurven [6] hergestellt, und die Theorie der parabolischen Kegelschnittflächen erweist sich als deren natürliche Verallgemeinerung.

8. Die Schar der Hauptkegel ( $\tilde{K}_t$ ,  $t \in I$ ) hüllt eine Fläche  $\tilde{\Phi}$  ein, die der Kegelschnittfläche  $\Phi$ , von der man ausging, dual gegenüber steht. Der Stützebene  $E_t$  entspricht dabei die Spitze  $x(t)$ , und der Stützgeraden  $g_t$  die Tangente an die Gratlinie, die mit  $g_t$  identisch ist. Da  $g_t$  zugleich Erzeugende von  $\tilde{K}_t$  ist, ist  $\tilde{\Phi}$  tatsächlich dual-parabolisch (im engeren Sinne). Folglich kann die Theorie der parabolischen Kegelschnittflächen dualisiert und auf  $\tilde{\Phi}$  angewandt werden. Wir nennen  $\tilde{\Phi}$  die zu  $\Phi$  *duale Kegelfläche*.

Die duale Kegelfläche gehört—wie man leicht nachrechnet—genau dann der 2. (dualen) Gattung an, wenn

$$2f - 3\kappa e = 0 \quad (J = \frac{3}{2})$$

ist. Nehmen wir an, daß für alle  $t \in I$   $J \neq \frac{3}{2}$  ist, so gehört  $\tilde{\Phi}$  zur ersten Gattung, d.h. zwei benachbarte Hauptkegel schneiden sich in  $g_t$  und einer Normkurve 3. Ordnung, die wir mit  $\tilde{C}_t$  bezeichnen wollen; sie entspricht der Hülltorse  $H_t$  dual. Die Schnittpunkte ihrer Tangenten mit  $E_t$  erzeugen einen in  $E_t$  gelegenen Kegelschnitt  $\tilde{K}_t$ . Nach Konstruktion ist dieser Kegelschnitt das duale Gegenstück des Hauptkegels. Die so erhaltene zweimalige Dualisierung führt also wieder auf eine parabolische Kegelschnittfläche  $\tilde{\Phi} = (\tilde{K}_t, t \in I)$ , deren Stütztorse mit der von  $\Phi$  übereinstimmt. Offenbar berühren sich die beiden erzeugenden Kegelschnitte  $K_t$  und  $\tilde{K}_t$  in  $x(t)$  mit der Tangente  $g_t$ .

Die Parameterdarstellung von  $\tilde{K}_t$  lautet:

$$\tilde{l}_t = -\frac{2}{e}[3(2f - 3\kappa e)s^2 + rs + 2j]\dot{x} + 2st + \eta.$$

Hieraus liest man ab:

*Der erzeugende Kegelschnitt  $\tilde{K}_t$  der zweiten Dualfläche  $\tilde{\Phi}$  von  $\Phi$  berührt den erzeugenden Kegelschnitt  $K_t$  von  $\Phi$  genau dann im Punkte  $x(t)$  in 2. Ordnung, wenn*

$$f = \frac{4}{3}\kappa e$$

*ist, in 3. Ordnung, wenn zusätzlich*

$$r = 0$$

*ist; und beide fallen genau dann zusammen, wenn zusätzlich*

$$j = 0$$

*gilt. Ist dies an jeder Stelle  $t \in I$  erfüllt, so gehört die Fläche  $\Phi$  zur 1., 2. bzw. 3. Unterklasse.*

Im letzten Fall sind insbesondere  $\Phi$  und  $\tilde{\Phi}$  identisch, d.h. die Konfiguration aus der Kegelschnittfläche und ihrer zugehörigen dualen Kegelfläche ist in sich dual. Andererseits handelt es sich—wie wir in Abschnitt 7 gesehen haben—bei den Kegelschnittflächen der 3. Unterklasse um die Schmiegekegelschnittflächen der Raumkurven. Dies ist der tiefere Grund, weshalb der von Bol entwickelte Kalkül der projektiven Kurventheories [6] einen in bezug auf Dualität völlig symmetrischen Aufbau gestattet.

Schließlich sei bemerkt, daß im Falle der Schmiegekegelschnittflächen die beiden hier genannten Normkurven 3. Ordnung, die Hauptkurve  $C_t$  und die charakteristische Kurve  $\tilde{C}_t$  des Hauptkegels in jene beiden Kurven der Fubinischen Schar übergehen, die im Bolschen Kalkül durch

$$\beta^* - b^*\dot{x}^* \quad \text{bzw.} \quad \beta^* + b^*\dot{x}^*$$

dargestellt werden; die Harmonikal- $C_3$  ist also die vierte harmonische zwischen diesen beiden in bezug auf die Hauptkegelspitze  $x(t)$ .

9. Werfen wir noch einen Blick auf die 2. Laplace-Transformierte von  $\Phi$  in  $t$ -

Richtung, also die 2. Brennfläche der Kongruenz der konjugierten Tangenten (bzgl. der Schar  $(C_t, t \in I)$ ) auf  $\Psi$ . Für diese findet man nach einiger Rechnung die Darstellung

$$b^* = r^*w^* + u^*x^* + p^*t^* - 3j^*\eta^*.$$

(Darin sind die Größen  $B$  formal durch den Parameter  $s$  zu ersetzen.) Hieraus gewinnt man durch nochmalige Ableitung:

*Genau dann entartet die 2. Laplace-Transformierte in eine Kurve  $B$  (d.h. die Hauptkurven auf  $\Psi$  sind Kegelschattengrenzen), wenn  $\Phi$  der 1. Unterkategorie angehört. Der erzeugende Punkt  $b(t)$  von  $B$  liegt überdies genau dann für alle  $t \in I$  in  $E_t$ , wenn  $\Phi$  sogar der 2. Unterkategorie angehört.*

Der erzeugende Punkt  $b(t)$  von  $B$  hat die Darstellung:

$$b = rw + ux + pt - 3j\eta.$$

10. Es soll nun die Frage untersucht werden, bei welchen parabolischen Kegelschnittflächen 1. Gattung die Asymptotenlinien ebenfalls die Eigenschaft haben, die erzeugenden Kegelschnitte projektiv aufeinander abzubilden, also eine Doppelverhältnisschar bilden.

Zunächst berechnen wir die Differentialgleichung der Asymptotenlinien zu

$$[s' + 2a + (f - 2ke)s^2]^2 - (f - ke) [v + 2ps + 6js^2 + rs^3 + ks^4] = 0$$

Da eine Doppelverhältnisschar eine Riccati-Differentialgleichung erfordert, muß

$$v + 2ps + 6js^2 + rs^3 + ks^4 = [F(s,t)]^2$$

ein vollständiges Quadrat in  $s$  sein. Hierbei unterscheiden wir zwei Fälle

1.  $\Phi$  gehört der 1. Unterkategorie an:  $k=0$ .

Dann lauten die Bedingungen:

$$k = 0, r = 0, p^2 - 6vj = 0.$$

$\Phi$  muß also sogar der 2. Unterkategorie angehören. Nach den Ergebnissen des vorigen Abschnitts besitzt dann aber  $\Psi$  die Hauptkurven  $C_t$  als Kegelschattengrenzen, und die Lichtpunkte  $b(t)$  liegen jeweils in  $E_t$ . Nun besagt die 3. Bedingung  $p^2 - 6vj = 0$  gerade, daß  $b(t)$  auf  $K_t$  liegt. Wir haben also:

*Eine parabolische Kegelschnittfläche 1. Gattung der 1. Unterkategorie hat genau dann die Eigenschaft, daß die Asymptotenlinien beider Scharen (es genügt: die einer Schar) die erzeugenden Kegelschnitte projektiv aufeinander abbilden, wenn sie sogar der 2. Unterkategorie angehört (die Hauptkurve  $C_t$  auf der Begleitfläche  $\Psi$  also Kegelschattengrenzen sind) und der erzeugende Punkt der Lichtkurve  $b(t)$  an jeder Stelle  $t \in I$  auf dem erzeugenden Kegelschnitt  $K_t$  liegt.*

2.  $\Phi$  gehört nicht der 1. Unterkategorie an:  $k \neq 0^4$

In diesem Fall müssen wir etwas weiter ausholen. Wir betrachten zunächst—ohne weitere Voraussetzungen zu machen—die Spurpunkte  $q(s,t)$ , die von den konjugierten Tangenten (bzgl. der Schar der Hauptkurven) an  $\Psi$  in  $E_t$

<sup>4</sup> Wir beschränken uns auf ein Teilintervall von  $I$ , in dem  $k$  keine Nullstelle besitzt.

ausgeschnitten werden. Man findet nach etwas Rechnung die folgende bemerkenswerte Darstellung

$$q = Q_{ss}t - Q_s t_s + Qt_{ss}$$

mit

$$Q = v + 2ps + 6js^2 + rs^3 + ks^4.$$

In Abhängigkeit von  $s \in \bar{\mathbb{R}}$  beschreiben diese Spurpunkte eine Kurve  $S_t = (q(s,t), s \in \bar{\mathbb{R}})$ , welche rational und von höchstens vierter Ordnung ist. Offensichtlich gilt

$$q_s = Q_{sss}t,$$

so daß wir folgendes Resultat erhalten:

*Die Spurkurve  $S_t$  ist für alle parabolischen Kegelschnittflächen 1. Gattung, die nicht der 1. Unterklasse angehören eine rationale Kurve höchstens 4. Ordnung. Ihre Tangenten treffen—sofern sie definiert sind—stets den entsprechenden Punkt  $t(s,t)$  auf  $K_t$ . An der Nullstelle  $s_0$  von  $Q_{sss}$  hat  $S_t$  eine Spitze.*

Aus obigen Gleichungen entnehmen wir weiter, daß  $q \wedge q_s = 0$  genau an der Stelle  $s_0$  und den evtl. vorhandenen gemeinsamen Nullstellen von  $Q$  und  $Q_s$  gilt. Daraus erhalten wir die gewünschte Kennzeichnung:

*Die Asymptotenlinien beider Scharen bilden die erzeugenden Kegelschnitte genau dann projektiv aufeinander ab, wenn die Spurkurve  $S_t$  (für jeder  $t \in l$ ) genau drei Spitzen besitzt oder in einen Kegelschnitt degeneriert, je nach dem ob*

$$Q = k[(s - s_1)(s - s_2)]^2 \text{ mit } s_1 \neq s_2$$

oder

$$Q = k(s - s_0)^4$$

ausfällt. Im letzten Fall hat dieser die Darstellung

$$q = 12t - 4(s - s_0)t_s + t_{ss}.$$

Im ersten Fall liegt eine interessante Konfiguration vor, die aus dem erzeugenden Kegelschnitt  $K_t$  und der Spurkurve  $S_t$  besteht. Wegen

$$q = 2(FF_{ss} + F_s^2)t - 2FF_s t_s + F^2 t_{ss}$$

fallen die beiden zu  $s_1, s_2$  gehörenden Spitzen mit den entsprechenden Punkten  $t(s_i, t)$  zusammen. Man überlegt sich ferner, daß dort sogar die Tangenten von  $K_t$  und  $S_t$  übereinstimmen und daß die Schnittmultiplizität je drei beträgt. Von den acht Schnittpunkten  $K_t \cap S_t$  sind daher bereits sechs bestimmt. Die restlichen beiden fallen in  $x(t)$  zusammen, wo sich  $K_t$  und  $S_t$  also berühren. Die dritte Spitze, welche zu

$$s_0 = \frac{1}{2}(s_1 + s_2)$$

gehört, liegt nicht zugleich auf  $K_t$ , jedoch geht die Spitzentangente durch den Pol der Verbindungsgeraden der ersten beiden Spitzen.

Im übrigen erweist sich  $S_t$  als ein projektives Bild der Steinerschen dreispitzigen Hypozykloide (vgl. z.B. [11], S. 142, oder [12], S. 157).

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## Affine Trochoidenbewegungen

Mit Trochoidenbewegungen verknüpfte Fragen haben auch in neuerer Zeit wiederholt das Interesse auf sich gezogen (z.B. [3, 8, 12, 16–18, 27, 30, 31]). Wir befassen uns hier—hauptsächlich im Rahmen der äquiaffinen Kinematik (z.B. [1, 4, 5, 9, 13, 15, 19, 20, 26, 29])—mit jenen nicht parabolischen Affinbewegungen, deren Polbahnkrümmungsmittelpunkte im jeweiligen System fest sind. Im Sinne von W. Blaschke und H.R. Müller nennen wir die in Rede stehenden Zwangsläufe affine Trochoidenbewegungen. Da wir Kinematik ohne gruppenmetrische Zusatzforderungen betreiben, bedingt dies ein sich Einordnen der klassischen Trochoidenbewegungen und gibt uns zugleich die Möglichkeit nach dem Gültigkeitsbereich bekannter Resultate zu fragen.

Äquiaffine Trochoidenbewegungen sind spezielle  $S^{(m)}$ -Bewegungen [23], bei denen der Hüllpunkt der Polbahnnormalen der jeweilige Polbahnkrümmungsmittelpunkt ist. Unter den zuletzt genannten Äquiaffinbewegungen—wir sprechen von Hohenberg-Bewegungen—sind die affinen Trochoidenbewegungen durch die Existenz eines eigentlich aufeinander rollgleitenden Hüllkurvenpaars mit im jeweiligen System festen Krümmungsmittelpunkten gekennzeichnet. Das Hüllkurvenpaar ist zu vorgegebener Rollgleitzahl eindeutig bestimmt und zu den Polkurven zentrisch ähnlich. Sind die Polbahnen einer äquiaffinen Trochoidenbewegung Kegelschnitte, so handelt es sich um eine symmetrische Rollung oder um eine euklidische bzw. pseudoeuklidische Trochoidenbewegung.

Wir schließen mit einigen Bemerkungen zu den äquiaffinen Zykloidenbewegungen.

### §1. Hilfsmittel

1. Unseren Untersuchungen legen wir nichtparabolische Affinbewegungen  $A(t)$  ( $t \in I \subset \mathbb{R}$ ,  $A(t) \in C^5$ ) der reellen affinen Ebene zugrunde, bei denen (im projektiven Abschluß) an jeder Parameterstelle drei linear unabhängige Momentanpole derart existieren, daß der endliche Momentanpol  $P$  im Parameterintervall nicht fest ist und an keiner Stelle die Tangenten der von den Polen gebildeten Polbahnen  $\mathbb{P}, \mathbb{P}'$  mit einem der zum betreffenden Parameterwert gehörigen Fernpole  $P_1, P_2$  inzidieren.

Nach Einführung von Koordinaten haben wir die Darstellung

$$(1) \quad x \mapsto A(t)x = x' = C'(t)\{x - c(t)\}, \quad \text{Det } C'(t) > 0.$$

Für die zugehörige inverse Affinbewegung  $A^{-1}(t)$  haben wir

$$(1') \quad x = C(t)\{x' - c'(t)\}, \quad c' := -C'c, \quad C(t) := C'^{-1}(t).$$

Bezeichnet (Ableitungen nach dem Parameter deuten wir durch Punkte an)

$$B(t) := CC' \quad \text{bzw.} \quad B'(t) := C'C$$

die die momentanen *Fernpole* von  $A(t)$  bzw.  $A^{-1}(t)$  (als Fixrichtungen) be-

stimmende Matrix und setzen wir

$$a := \text{Spur } B, b := \text{Det } B, d := a^2 - 4b,$$

so gilt für Affinbewegungen  $A$  im Parameterintervall<sup>1</sup>

$$(2) \quad bd \neq 0, \dot{p} \neq 0, [B\dot{p}, \dot{p}] \neq 0,$$

wobei wir mit  $p$  bzw.  $p'$  den Spaltenvektor des Momentanpolen  $P$  bezeichnen. Mit  $(\text{Det } C)(\text{Det } C') = 1$  folgt ferner  $\text{Spur } B = (\text{Det } C)(\text{Det } C')$ , so daß *äquiaffine* Bewegungen  $A$  durch (2) und  $a=0$  erfaßt werden.

2. Da im anstehenden Problemkreis die Polbahnen eine ausschlaggebende Rolle spielen, wird man diesem Umstand bereits im Kalkül Rechnung tragen. Zur geometrischen Herleitung der Differentialgleichungssysteme der Polbahnen setzen wir zuerst voraus, daß der Momentanpol an der betrachteten Stelle kein Wendepunkt der Polbahnen  $P, P'$  sei. Nach [13, 20] versteht man dann unter dem *Polbahnkrümmungskegelschnitt* der Gangpolbahn  $P$  (Rastpolbahn  $P'$ ) jenen durch die Fernpole  $P_i$  gehenden Kegelschnitt, der die Polbahn  $P(P')$  im Poloskuliert. Analytisch folgt hieraus für den Krümmungskegelschnitt von  $P$  die Darstellung

$$[\ddot{p}, \dot{p}] [B(x-p), x-p] + 2[B\dot{p}, \dot{p}] [\dot{p}, x-p] = 0.$$

Der Krümmungskegelschnitt der Rastpolbahn ergibt sich durch Ersetzen aller ungestrichenen Größen durch die entsprechenden gestrichenen Größen.

Damit läßt sich das DGL-System der Gang- bzw. Rastpolbahn in der Form

$$(3) \quad \ddot{p} = \sigma \dot{p} - \frac{1}{1-m} B \dot{p} \quad \text{bzw.} \quad \ddot{p}' = \sigma \dot{p}' + \frac{m}{1-m} B' \dot{p}'$$

ansetzen, wobei  $\sigma = \sigma(t)$  eine «willkürliche» Parameterfunktion und  $m = m(t) \neq 1$  der Ähnlichkeitsmodul der bezüglich dem Momentanpol zentrisch ähnlichen Polbahnkrümmungskegelschnitte ist<sup>2,3</sup>.

Bewegungen  $A(t)$  mit  $m = \text{const.}$  nennen wir affine  $S^{(m)}$ -Bewegungen;  $S^{(-1)}$ -Bewegungen heißen *symmetrische Rollungen* [22, 23].

Im Falle von  $m \neq \infty$  bzw.  $m \neq 0$  ist der *Krümmungsmittelpunkt*  $M_p$  bzw.  $M_{p'}$ —als Mittelpunkt des betreffenden Polbahnkrümmungskegelschnittes—der Gang- bzw. Rastpolbahn erklärt. Unter Verwendung von (3) ergibt sich hierfür die Darstellung

$$(4) \quad m_p = p + \frac{2}{d}(1-m)\{2B\dot{p} - a\dot{p}\} \quad \text{bzw.}$$

$$m_{p'} = p + \frac{2}{d} \frac{1-m}{m} \{2B\dot{p} - a\dot{p}\}.$$

<sup>1</sup> Wie in der Kinematik üblich setzen wir auf I einheitliches Verhalten von  $A(t)$  voraus, so daß wir uns in der elliptischen bzw. hyperbolischen Affinkinematik befinden. Unter  $[a, b]$  verstehen wir die Determinante der Spaltenvektoren a,b.

<sup>2</sup> Somit ist  $m(t)$  ein *absolute Invariante*, was man natürlich sofort auch analytisch bestätigt.

<sup>3</sup> Man erkennt, daß (3) auch für die zunächst ausgeschlossenen Möglichkeiten  $m=0, \infty$  richtig ist.

3. Im Euklidischen fällt der Hüllpunkt der Polbahnnormalen (trivialerweise) mit dem Mittelpunkt des Krümmungskreises zusammen. Im Affinen gilt dies im allgemeinen nicht! Nach [20] verstehen wir unter der *Polbahnnormalen* von  $\mathbb{P}$  bzw.  $\mathbb{P}'$  die Gerade

$$n = p + \mu \{2B\dot{p} - a\dot{p}\} \quad \text{bzw.} \quad n' = p' + \mu' \{2B'\dot{p}' + a\dot{p}'\}.$$

Die Fernpunkte der Polbahnnormalen und Polbahntangente trennen also die Fernpole  $P_j$

$$p_j = -(a + (-1)^j \sqrt{d})\dot{p} + 2B\dot{p}, \quad j = 1, 2$$

harmonisch. Setzen wir mit [26]

$$(5) \quad \dot{B}\dot{p} = b_1\dot{p} + b_2B\dot{p} \quad \text{oder} \quad \dot{B}'\dot{p}' = -b_1\dot{p}' + b_2B'\dot{p}',$$

so zeigt man leicht: *Der Krümmungsmittelpunkt  $M_p(M_{p'})$  der Polbahn  $\mathbb{P}(\mathbb{P}')$  fällt genau dann beständig mit dem Hüllpunkt der Polbahnnormalen  $n$  ( $n'$ ) zusammen, wenn die Halbinvariante*

$$(6) \quad H := 2b_1 + ab_2 - a$$

*verschwindet.*

Affinbewegungen, für die  $H$  auf dem zugrundeliegenden Parameterintervall verschwindet nennen wir *Hohenberg-Bewegungen*.

Vermerken wir noch, daß die Fernpole  $P_j$  genau dann fest sind, wenn die Affinbewegung  $A$  eine Hohenberg-Bewegung ist und die Halbinvariante

$$(7) \quad I := b - 2bb_2 - ab_1$$

*verschwindet.* Wir befinden uns dann in der Ähnlichkeits- bzw. Pseudoähnlichkeitskinematik.

## §2. Problemformulierung

1. Unter (euklidischen) *Trochoidenbewegungen*  $T$  versteht man bekanntlich ebene Zwangsläufe, bei denen die Gang- und Rastpolbahn jeweils ein Kreis ist. Zugehörige Bahnkurven heißen Trochoiden. Offenbar haben wir für  $T$  folgende kennzeichnenden Eigenschaften:

- (i) Die Polbahnen sind Kegelschnitte durch die (festen) Fernpole.
- (ii) Die Polbahnen sind durch gleiche Bogendifferentiale aufeinander bezogene W-Kurven mit endlichem Krümmungsmittelpunkt.
- (iii) Der Krümmungsmittelpunkt der Gang- bzw. Rastpolbahn ist gangfest bzw. rastfest.

2. Um im Rahmen der elliptischen bzw. hyperbolischen Affinkinematik, mit nicht degenerierenden Polbahnen, ein Analogon zu den Trochoidenbewegungen  $T$  zu erklären, wird man sich an den genannten Eigenschaften orientieren.

Die Eigenschaft (i) ist unbrauchbar, da die Fernpole dann fix sind, wir uns also in der *Ähnlichkeits- bzw. Pseudoähnlichkeitskinematik* befinden.

Die Eigenschaft (ii) ist im Sinne der *natürlichen Kinematik* von Lie benutzbar [5] und wurde z.T. auch von Kowalewski verwandt [10]. Abrollvorgänge sind durch

gleiche Bogendifferentiale erklärt, also von 4. bzw. 2. Differentiationsordnung. Dies hat den Nachteil zur Folge, daß sich die klassischen Trochoidenbewegungen nicht einordnen lassen.

Da wir Affinkinematik im Sinne von Blaschke und Müller treiben, also *keine gruppenmetrischen Zusatzforderungen* stellen, wollen wir uns hier mit der Möglichkeit (iii) befassen. Dies bedingt ein sich Einordnen der klassischen Trochoidenbewegungen und gibt uns die Möglichkeit nach dem Gültigkeitsbereich klassischer Resultate zu fragen.

Eine Affinbewegung  $A$ , mit lokal wendepunktfreien Polbahnen, heiße eine *affine Trochoidenbewegung* genau dann, wenn die Gangpolbahn einen gangfesten und die Rastpolbahn einen rastfesten (endlichen) Krümmungsmittelpunkt besitzt.

3. Um die affinen Trochoidenbewegungen analytisch in den Griff zu bekommen, haben wir die Fixbedingungen

$$\dot{m}_p = 0 \quad \text{und} \quad \dot{m}_{p'} = -B(m_{p'} - p)$$

auszuwerten. Beachten wir die Beziehungen (1.3) bis (1.5) und die Relation ( $E$  = Einheitsmatrix)

$$(1) \quad B^2 = -bE + aB \quad \text{oder} \quad B'^2 = -bE - aB',$$

so liefert eine einfache Zwischenrechnung

$$(2) \quad a = 2 \frac{\dot{m}}{m}, \quad H = 0, \quad \sigma = -b_2 + \frac{\dot{d}}{d} + \frac{1+m}{2(1-m)} a.$$

Durch (2) werden die affinen Trochoidenbewegungen analytisch erfaßt und wir entnehmen diesen Beziehungen:

a) Eine äquiaffine Trochoidenbewegung ist eine Hohenberg-Bewegung mit konstantem Übertragungsverhältnis  $m$  und für die Funktion  $\sigma(t)$  in (1.3) gilt

$$(3) \quad \sigma = \frac{b}{b} - b_2.$$

b) Eine trochoidale affine  $S^{(m)}$ -Bewegung ist eine äquiaffine Hohenberg-Bewegung mit (3).

c) Unter den äquiaffinen  $S^{(m)}$ -Bewegungen, die zugleich Hohenberg-Bewegungen sind, sind die affinen Trochoidenbewegungen dadurch gekennzeichnet, daß es einen beständig auf der Polbahnnormalen gelegenen Punkt gibt, der eine Bahnkurve beschreibt.

Denn nach Voraussetzung gilt  $\dot{m} = a = b_1 = 0$ . Damit beschreibt der Punkt  $Y$  mit  $y = p + \mu(t)B\dot{p}$  genau dann eine Bahnkurve, wenn neben (3) noch  $\mu = (m-1)b^{-1}$  gilt. Der in Rede stehende Punkt ist (natürlich) gerade der Krümmungsmittelpunkt der Gangpolbahn.

Nach (1.4) haben wir für die vom Krümmungsmittelpunkt der Gangpolbahn in der Rastebene beschriebene Bahnkurve

$$m'_p = m'_{p'} + (1-m)(p' - m'_{p'})$$

und somit analog zum Euklidischen

**Satz 1.** Bei äquiaffinen Trochoidenbewegungen ist die vom Krümmungsmittelpunkt

der Gangpolbahn beschriebene Bahnkurve zur Rastpolbahn zentrisch ähnlich.

*Bemerkung.* Gemäß unserer Untersuchung liegt somit genau dann eine euklidische bzw. pseudoeuklidische Trochoidenbewegung vor, wenn neben  $m = \text{const.}$  noch

$$(4) \quad a = b_1 = 0 \quad \text{und} \quad \sigma = \frac{\dot{b}}{b} - b_2 = \frac{1}{2} \frac{\dot{b}}{b}$$

gilt.

4. Wir wollen eine Charakterisierung der Trochoidenbewegungen der (Pseudo-)Ähnlichkeitskinematik geben.

**Satz 2.** Eine affine Trochoidenbewegung ist genau dann eine Trochoidenbewegung der Ähnlichkeits- bzw. Pseudoähnlichkeitskinematik, wenn die Affinnormale der Polbahnen beständig zusammenfallen.

Zum Beweis gehen wir von der Darstellung

$$(5) \quad a_p = 3[\dot{p}, \ddot{p}]p - [\dot{p}, \dot{p}]p \quad \text{bzw.} \quad a'_{p'} = 3[\dot{p}', \ddot{p}']p' - [\dot{p}', \dot{p}']p'$$

der Richtungsvektoren der Affinnormalen von  $\mathbb{P}$  bzw.  $\mathbb{P}'$  aus. Beachten wir hierin (1), (1.3) und (1.5), so folgt

$$\begin{aligned} \frac{(1-m)}{[\dot{p}, B\dot{p}]} a_p &= \left( b_2 - \sigma + \frac{\dot{m}-a}{1-m} \right) \dot{p} + \frac{3}{1-m} B\dot{p} \quad \text{und} \\ \frac{(1-m)}{[\dot{p}, B\dot{p}]} Ca'_{p'} &= \left( m(b_2 - \sigma) + \frac{\dot{m}-m^2a}{1-m} \right) \dot{p} + \frac{3m^2}{1-m} B\dot{p}, \end{aligned}$$

so daß die Affinnormale der Polbahnen genau dann zusammenfallen, wenn

$$(6) \quad \sigma = b_2 + \frac{\dot{m}}{m} \frac{1+m}{1-m}.$$

Gilt andererseits (2), so folgt  $I=0$ . Q.e.d. Formulieren wir noch das

**Korollar 1.** Eine äquiaffine Trochoidenbewegung ist genau dann eine Trochoidenbewegung der euklidischen bzw. pseudoeuklidischen Kinematik, wenn die Affinnormale der Polbahnen beständig zusammenfallen.

Die Trochoidenbewegungen der Ähnlichkeitskinematik<sup>4</sup> lassen sich explizit bestimmen. Setzen wir in (1.1)

$$(7) \quad C'(t) = \lambda(t) \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) \end{bmatrix},$$

so folgt für den Spaltenvektor  $c'(t)$  in (1.1')

<sup>4</sup> Zur Ähnlichkeitskinematik vergleiche man [11] und die in [6] genannte Literatur.

$$\mathbf{c}'(t) = \frac{r}{\rho} (1 - \lambda\rho) \begin{bmatrix} \sin \alpha \\ \cos \alpha \end{bmatrix} \text{ mit } \alpha := \rho \int \frac{\lambda \theta dt}{1 - \rho \lambda},$$

wobei  $r(\neq 0)$  eine Integrationskonstante ist und wir für den nach (2,links) konstanten Quotienten  $m/\lambda (\neq 0)$   $\rho$  geschrieben haben. Die Polbahnen sind also Kreise. Ausführlich gilt

$$\mathbf{p} = r \begin{bmatrix} \sin \beta \\ \cos \beta \end{bmatrix}, \mathbf{p}' = \frac{r}{\rho} \begin{bmatrix} \sin \alpha \\ \cos \alpha \end{bmatrix} \text{ mit } \beta := \int \frac{\theta dt}{1 - \rho \lambda}.$$

### §3. Äquiaffine Trochoidenbewegungen

1. Die (euklidischen) Trochoidenbewegungen lassen sich dadurch kennzeichnen, daß ein eigentlich aufeinander rollgleitendes Kreispaar als Hüllkurvenpaar existiert [27, 28]<sup>5</sup>. Da der Begriff des Rollgleitens sogar projektiv ist [23], macht es Sinn nach einem affinen Analogon der gerade erwähnten Charakterisierung zu fragen.

Sei  $X \neq P$  der gemeinsame Berührpunkt zusammengehöriger *affiner Hüllkurvenpaare*  $k, k'$ , welche mit der *Rollgleitzahl*  $\lambda = \lambda(t)$  aufeinander rollen und gleiten. Dann gilt für die Kurve  $k$  bzw.  $k'$  das Differentialgleichungssystem

$$(1) \quad \dot{x} = \frac{\lambda}{1 - \lambda} B(x - p) \quad \text{bzw.} \quad \dot{x}' = -\frac{1}{1 - \lambda} B'(x' - p').$$

Bei fester Rollgleitzahl sprechen wir mit Grüssz [7] und Müller [14] von einem *eigentlichen* Rollgleiten.

Für den Krümmungskegelschnitt der Kurve  $k$  bzw.  $k'$ , im Berührpunkt  $X$ , haben wir die Darstellung

$$(2) \quad [\ddot{x}, \dot{x}] [B(y - x), y - x] + 2[B\dot{x}, \dot{x}] [\dot{x}, y - x] = 0 \quad \text{bzw.} \\ [\ddot{x}', \dot{x}'] [B'(z' - x'), z' - x'] + 2[B'\dot{x}', \dot{x}'] [\dot{x}', z' - x'] = 0.$$

Um die nachfolgenden Betrachtungen zu vereinfachen führen wir den *Ähnlichkeitsmodul*  $\tau = \tau(t)$  der vermöge  $y = x + \tau(z - x)$  zentrisch ähnlichen Krümmungskegelschnitte ein. Damit liefert (2), mit einer Funktion  $\rho(t)$ , für das Hüllkurvenpaar  $k, k'$

$$(3) \quad \ddot{x} = \rho \dot{x} - \frac{\lambda}{\lambda - \tau} B \dot{x} \quad \text{bzw.} \quad \ddot{x}' = \left( \rho + \frac{\lambda}{1 - \lambda} \right) \dot{x}' + \frac{\tau}{\lambda - \tau} B' \dot{x}',$$

wobei wir  $\tau \neq \lambda$ , d.h.  $[B\dot{x}, \dot{x}] \neq 0$  vorausgesetzt haben, was für äquiaffine Bewegungen sicher der Fall ist.

Wir legen nun Äquiaffinbewegungen zugrunde<sup>6</sup>. Damit ergibt sich vermöge (2) und (3) für die *Krümmungsmittelpunkte*  $M_k, M_{k'}$  des Hüllkurvenpaars  $k, k'$  die Darstellung

<sup>5</sup> Analoges gilt auch für die sphärische Kinematik [21].

<sup>6</sup> Genau für Äquiaffinbewegungen läßt sich die Rollgleitzahl  $\lambda$ , wie im Euklidischen [14], vermöge der Krümmungsmittelpunkte als ein Doppelverhältnis deuten, nämlich  $\lambda = DV(P; M_{k'}, M_k)$  [20], S.69.

$$(4) \quad m_k = x - \frac{\tau - \lambda}{1 - \lambda}(x - p), \quad m'_{k'} = x' - \frac{1}{\tau} \frac{\tau - \lambda}{1 - \lambda}(x' - p').$$

Wann, so wollen wir uns fragen, sind die Krümmungsmittelpunkte eigentlich aufeinander rollgleitender Hüllkurvenpaare in den jeweiligen Systemen fest? Zunächst liefern die Fixbedingungen unmittelbar die Beziehungen

$$\frac{1 - \tau}{1 - \lambda} B(x - p) - \frac{\dot{\tau}}{\lambda}(x - p) + \frac{\tau - \lambda}{\lambda} \dot{p} = 0, \quad \frac{1 - \tau}{1 - \lambda} B(x - p) - \frac{\dot{\tau}}{\tau}(x - p) + \frac{\tau - \lambda}{\lambda} \dot{p}$$

$$= 0,$$

so daß notwendig ( $\tau = 1$  ist nicht möglich)

$$(5) \quad x = p + \frac{1}{b} \frac{1 - \lambda}{1 - \tau} \frac{\tau - \lambda}{\lambda} B \dot{p} \quad \text{und} \quad \tau = \text{const.}$$

gilt. Zur Gewinnung hinreichender Bedingungen haben wir (1) zu überprüfen! Differenzieren wir (5), so folgt

$$(6) \quad \sigma = \frac{b}{b} - b_2 \quad \text{und} \quad b_1 + \frac{\tau - m\lambda}{(1 - m)(\tau - \lambda)} b = 0.$$

Die Beziehungen (1.6) und (2.2) liefern damit insbesondere

**Satz 3.** Unter den äquiaffinen Hohenberg-Bewegungen sind die affinen Trochoidenbewegungen dadurch gekennzeichnet, daß es ein (von den Polbahnen verschiedenes) eigentlich aufeinander rollgleitendes Hüllkurvenpaar mit (im jeweiligen System) festen Krümmungsmittelpunkten gibt.

Vermöge (5) und (6) hat dieses (zu gegebener konstanter Rollgleitzahl eindeutig bestimmte) Hüllkurvenpaar  $k, k'$  die (integralfreie) Darstellung

$$(7) \quad k: x = p - \frac{1}{b} \frac{1 - \lambda}{1 - m\lambda} (1 - m) B \dot{p}, \quad k': x' = p' + \frac{1}{b} \frac{1 - \lambda}{1 - m\lambda} (1 - m) B' \dot{p}'.$$

Mit (1.4) lassen sich diese Darstellungen umformen zu

$$(7') \quad k: x = m_p + \lambda \varepsilon(p - m_p), \quad k': x' = m'_{p'} + \varepsilon(p' - m'_{p'})$$

mit

$$\varepsilon := \frac{1 - m}{1 - m\lambda}$$

und somit haben wir

**Satz 4.** Bei dem gemäß Satz 2 bei einer äquiaffinen Trochoidenbewegung zur konstanten Rollgleitzahl eindeutig bestimmten Hüllkurvenpaar  $k, k'$ , geht  $k$  aus der Gangpolbahn und  $k'$  aus der Rastpolbahn jeweils durch eine zentrische Ähnlichkeit hervor.

Mit dem Korollar 1 folgt daraus noch das

**Korollar 2.** Unter den äquiaffinen Hohenberg-Bewegungen sind die euklidischen bzw. pseudoeuklidischen Trochoidenbewegungen dadurch gekennzeichnet, daß es ein (von den Polbahnen verschiedenes) eigentlich aufeinander rollgleitendes Hüllkurvenpaar  $k, k'$  mit (im jeweiligen System) festen Krümmungsmittelpunkten gibt, dessen Affinnormale zusammenfallen.

Nach Satz 4 ist dieses Hüllkurvenpaar dann ein Kreis- bzw. Pseudokreispaar.  
 3. Die Klasse der affinen Trochoidenbewegungen ist noch sehr umfangreich. Z.B. hängen die äquiaffinen Trochoidenbewegungen, wenn wir eine geometrisch unwesentliche Parametertransformation derart vornehmen, daß die Halbinvariante  $b$  konstant wird, von dem konstanten Übertragungsverhältnis  $m$  und einer willkürlichen Funktion  $b_2(t) (\neq 0)$  ab.

Wir wollen ein Beispiel geben. Offenbar ist die Affinbewegung (1.1) mit

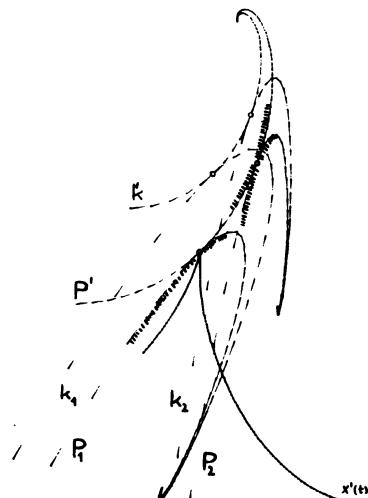
$$(8) \quad C'(t) = \begin{bmatrix} \sin 2t - \cos 2t - 1 & \sin 2t + \cos 2t \\ -\sin 2t - \cos 2t & \sin 2t - \cos 2t + 1 \end{bmatrix},$$

$$c'(t) = -\sqrt{2} e^{-t} \begin{bmatrix} \sin\left(\frac{\pi}{4} - t\right) \\ \cos\left(\frac{\pi}{4} - t\right) \end{bmatrix}$$

eine äquiaffine symmetrische Trochoidenbewegung. Genauer berechnet man sich

$$\text{Det } C' = 1, a = b - 8 = b_1 = b_2 - 2 = m + 1 = \sigma + 2 = 0.$$

Die zugehörigen Polbahnen sind logarithmische Spiralen.



In der obenstehenden Figur sind die Rastpolbahn  $\mathbb{P}'$  und zwei Lagen der Gangpolbahn  $\mathbb{P}$  gezeichnet. Ebenso wurden für das der Rollgleitzahl  $\lambda=2$

entsprechende Hüllkurvenpaar  $k, k'$  die zu den nämlichen Parameterwerten gehörigen beiden Lagen der gangfesten Kurve  $k$  gezeichnet.  $x'(t)$  ist Bahnkurve eines Punktes der Gangpolbahn.

Dieses Beispiel zeigt zugleich, daß die Polbahnen äquiaffiner Trochoidenbewegungen keine W-Kurven der äquiaffinen Gruppe sein müssen! Wir haben so die Frage nach den äquiaffinen Trochoidenbewegungen mit Kegelschnitten als Polbahnen.

#### §4. Äquiaffine symmetrische Trochoidenbewegungen

1. Im Euklidischen gibt es trivialerweise genau einen Typ symmetrischer Trochoidenbewegungen: Die symmetrischen Kreisrollungen. Insbesondere sind die zugehörigen Trochoidenbahnen dann Fußpunktcurven. Wie steht es damit im Affinen? Nach [25] gilt:

*Bei einer Affinbewegung  $A$ , mit lokal wendepunktfreien Polbahnen, ist der in Richtung der Polbahnnormalen an der Polbahnntangente gebildete Spiegelpunkt eines jeden gangfesten Punktes genau dann rastfest, wenn es sich um eine äquiaffine symmetrische Hohenberg-Bewegung handelt.*

Die äquiaffinen symmetrischen Hohenberg-Bewegungen, es gilt also  $a = b_1 = m + 1 = 0$ , stellen gewissermaßen das natürliche affine Analogon der klassischen symmetrischen Rollungen dar, denn es gilt auch

**Satz 5.** *Die Polbahnen äquiaffiner symmetrischer Hohenberg-Bewegungen haben gleiche Affinkrümmungen.*

Zum Beweis Satzes berechnen wir uns die Affinkrümmungen  $\kappa_p, \kappa_{p'}$  der Polbahnen  $P, P'$  äquiaffiner  $S^{(m)}$ -Bewegungen. Um Anschluß an die Formeln von Blaschke [2], S.14, zu erhalten, denken wir uns dabei o.B.d.A. die Determinante der Matrix  $C'(t)$  zu Eins normiert. Man findet

$$(1) \quad \kappa_p = (1-m)^{2/3} [\dot{p}, B\dot{p}]^{-2/3} \{b(1-m)^{-2} + b_1(1-m)^{-1} + \alpha\}$$

bzw.

$$(2) \quad \kappa_{p'} = \left(\frac{1-m}{m}\right)^{2/3} [\dot{p}, B\dot{p}]^{-2/3} \left\{ b\left(\frac{1-m}{m}\right)^{-2} + b_1\left(\frac{1-m}{m}\right)^{-1} + \alpha \right\},$$

wobei wir

$$(3) \quad \alpha := \frac{1}{9} \{ \sigma(\sigma + b_2) - 2b_2^2 + 3(b_2 - \dot{\sigma}) \}$$

gesetzt haben. Damit ist die Behauptung abzulesen. Q.e.d.

2. Im Sinne der Schlußbemerkung des voranstehenden Abschnitts wird man sich die Ableitungen der Affinkrümmungen der Polbahnen bestimmen. Damit finden wir die Beziehung

$$\begin{aligned} m^{2/3} \dot{\kappa}_{p'} - \dot{\kappa}_p &= (1-m)^{2/3} [\dot{p}, B\dot{p}]^{-2/3} \left\{ \frac{2}{3} (2\sigma + b_2) \left( b_1 + \frac{1+m}{1-m} b \right) \right. \\ &\quad \left. - \frac{1+m}{1-m} \dot{b} - \dot{b}_1 \right\}, \end{aligned}$$

woraus wir ablesen ((2.3) und (2.4)):

**Satz 6.** (a) Eine äquiaffine Trochoidenbewegung mit Kegelschnitten als Polbahnen ist symmetrisch, oder es handelt sich um euklidische bzw. pseudoeuklidische Trochoidenbewegungen.

(b) Eine nicht symmetrische äquiaffine Trochoidenbewegung ist genau dann euklidisch bzw. pseudoeuklidisch, wenn die Polbahnen konstante Affinkrümmungen besitzen.

Für äquiaffine Trochoidenbewegungen haben wir nach (3.7') für die Affinkrümmungen  $\kappa_k, \kappa_{k'}$ , des Hüllkurvenpaars  $k, k'$  gemäß Satz 3, die Beziehungen

$$(4) \quad \kappa_k = (\lambda \varepsilon)^{-4/3} \kappa_p, \quad \kappa_{k'} = \varepsilon^{-4/3} \kappa_{p'}.$$

Notieren wir noch das

**Korollar 3.** Für eine äquiaffine Trochoidenbewegung mit Kegelschnitten als Polbahnen existiert ein eigentlich aufeinander rollgleitendes Kegelschnittpaar  $k, k'$  (vom selben Typ wie die Polbahnen  $\mathbb{P}, \mathbb{P}'$ ) als Hüllkurvenpaar.

Wir wollen ein Beispiel geben. Die Äquiaffinbewegung (1.1) mit

$$C(t) = \begin{bmatrix} 4e^t & -3 \\ 3 & -2e^{-t} \end{bmatrix}, \quad c'(t) = - \begin{bmatrix} 2e^{-t} \\ e^{-2t} \end{bmatrix}$$

ist eine äquiaffine symmetrische Trochoidenbewegung mit Parabeln als Polbahnen. Man berechnet sich

$$\text{Det } C' = 1, \quad a = b - 8 = \sigma + 3 = m + 1 = b_1 = b_2 - 3 = 0.$$

## §5. Äquiaffine Zykloidenbewegungen

1. Im Euklidischen heißt jener Grenzfall der Trochoidenbewegungen, wo ein Kreis  $\mathbb{P}$  auf einer Geraden  $\mathbb{P}'$  abrollt, Zykloidenbewegung. Abschließend soll kurz auf ein affines Analogon eingegangen werden.

Unter einer *affinen Zykloidenbewegung* verstehen wir eine Affinbewegung  $A$ , bei der eine lokal wendepunktfreie Gangpolbahn  $\mathbb{P}$  auf einer Geraden  $\mathbb{P}'$  derart affin abrollt, daß der Krümmungsmittelpunkt  $M_p$  von  $\mathbb{P}$  dabei eine Bahnkurve beschreibt. Mit (1.4, links) folgt: Affine Zykloidenbewegungen sind Hohenberg-Bewegungen mit

$$m = 0 \quad \text{und} \quad \sigma = -b_2 + \frac{d}{d} + \frac{a}{2}.$$

Bei äquiaffinen Zykloidenbewegungen beschreibt  $M_p$  eine zu  $\mathbb{P}'$  parallele Gerade. Analog zum Satz 2 haben wir

**Satz 7.** Eine affine Zykloidenbewegung ist genau dann eine Zykloidenbewegung der Ähnlichkeits- bzw. Pseudöähnlichkeitskinematik, wenn die Gangpolbahn-normale beständig mit der Affinnormalen der Gangpolbahn zusammenfällt.

Hieraus fließt wieder—analog zum Korollar 1—eine Kennzeichnung der Zykloidenbewegungen der euklidischen bzw. pseudoeuklidischen Kinematik. Die Zykloidenbewegungen der Ähnlichkeitskinematik lassen sich explizit angeben.

2. Die Betrachtungen des 3. Abschnitts übertragen sich sinngemäß ohne weiteres. Als Beispiel formulieren wir

**Satz 3'.** Unter den äquiaffinen Hohenberg-Bewegungen sind die affinen Zykloidenbewegungen dadurch gekennzeichnet, daß es ein (von den Polbahnen verschiedenes) eigentlich aufeinander rollgleitendes Hüllkurvenpaar  $k, k'$  derart gibt, daß  $k'$  eine Gerade ist und die fest mitgeführte Kurve  $k$  einen gangfesten Krümmungsmittelpunkt besitzt.

Ferner haben wir analog zum Korollar 3 ( $\varepsilon = 1$ ) das

**Korollar 3'.** Für eine äquiaffine Zykloidenbewegung mit einem Kegelschnitt  $\mathbb{P}$  als Gangpolbahn, existiert, zu vorgegebener konstanter Rollgleitzahl, eine Gerade  $k'$ , auf der ein, mit dem Gangsystem fest verbundener, Kegelschnitt  $k$  (vom selben Typ wie  $\mathbb{P}$ ) rollgleitet.

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## Globale Geometrie der Regelflächen

Die von Kruppa [23] entwickelte natürliche Geometrie der Regelflächen hat sich in den letzten Jahren als wirksame Methode zur Untersuchung von Regelflächen des euklidischen Raumes erwiesen. Die Berichte [6], [16] vermitteln eine Übersicht über die vielfältigen *lokalen Resultate*. Ziel der vorliegenden Übersicht ist es nun, *globale Eigenschaften* von Regelflächen aufzuzeigen. Im Vordergrund werden dabei Ergebnisse aus der euklidischen Geometrie stehen, ergänzt von projektiven und nichteuklidischen Eigenschaften der geschlossenen Regelflächen.

Eine Regelfläche  $\Phi$  des dreidimensionalen euklidischen Raumes  $R^3$  mit dem Ortsvektor  $S(u) \in C^2$  ihrer *Striktionslinie* und dem normierten Erzeugendenvektor  $v_1(u) \in C^3$  besitzt die Parameterdarstellung

$$X(u, v) = S(u) + v v_1(u) \quad (1)$$

mit  $u \in [0, L]$  und  $v \in (-\infty, +\infty)$ . Die Linien  $v = \text{const.}$  beschreiben die erzeugenden Geraden (Erzeugenden) der Regelfläche  $\Phi$ .  $S$  und  $v_1$  seien regulär parametrisiert, so daß  $u$  als Bogenlänge der Striktionslinie gewählt werden kann. Ist weiter  $S$  das Bild der  $S^1$ -Sphäre in den  $R^3$ , so kann  $L$  als Länge der Striktionslinie gedeutet werden. Eine Regelfläche heißt *geschlossen*, wenn  $S(u)$  und  $v_1(u)$   $L$ -periodische Vektorfunktionen sind, also gilt

$$X(u, v) = X(u + L, v). \quad (2)$$

Weiter sei vorausgesetzt, daß  $L$  die kleinste Periode ist, d.h.  $\Phi$  *orientierbar* ist. In der natürlichen Geometrie ist eine Regelfläche  $\Phi$  durch die lokalen Invarianten  $\kappa$  (*Krümmung*, mit  $\kappa > 0$ ),  $\tau$  (*Torsion*) und  $\sigma$  (*Striktion*) bis auf Bewegungen im  $R^3$  eindeutig bestimmt. Zugrunde liegt dabei ein orthonormiertes begleitendes Dreibein  $D(v_1, v_2, v_3)$ , dessen Ursprung sich entlang der Striktionslinie bewegt und das durch

$$\dot{v}_1 = \kappa v_2$$

festgelegt ist (mit  $\cdot = \frac{d}{du}$ ). Diese Definition versagt an Stellen, an denen  $\kappa$  verschwindet (torsale Erzeugende). Torsale Erzeugende sollen in Zukunft ausgeschlossen werden.

Ist  $\sigma \neq 0$ , so heißt die Regelfläche  $\Phi$  *windschief*, ist  $\sigma \equiv 0$  ist  $\Phi$  eine Torse und  $\kappa$  bzw.  $\tau$  werden zur Krümmung  $K$  bzw. Torsion  $T$  der Gratlinie der Torse. Im folgenden sollen i.allg. nur windschiefe Regelflächen betrachtet werden, für  $\sigma \rightarrow 0$  lassen sich die gefundenen Resultate auf die geschlossenen Raumkurven des  $R^3$  übertragen, so daß Eigenschaften geschlossener Regelflächen zu einem tieferen Verständnis der *globalen Kurventheorie* führen.

### 1. Integralformeln aus der euklidischen Geometrie

Während bereits Koenigs [22] invariante Integrale für Regelflächen des dreidimensionalen euklidischen Raumes betrachtet hat, haben wohl Blaschke

[4], Haack [11] und H.R. Müller [26] die ersten *globalen* Invarianten geschlossener Regelflächen angegeben: Umschreibt man einer geschlossenen Regelfläche  $\Phi$  eine Orthogonaltrajektorie  $t$  ihrer erzeugenden Geraden, und lässt man  $t$  im Punkt  $P_1$  der Erzeugenden mit der Richtung  $v_1(u_0)$  beginnen, so trifft  $t$  nach einem Umlauf die Erzeugende  $v_1(u_0)$  in einem Punkt  $P_2$ , der i.allg. von  $P_1$  verschieden ist. Der Abstand  $l = \overline{P_1 P_2}$  ist unabhängig von der Wahl der Erzeugenden  $v_1(u_0)$  und des Punktes  $P_1$ .  $l$  ist daher eine Invariante der geschlossenen Regelfläche und wird als *Öffnungsstrecke* bezeichnet. Im Invariantensystem der natürlichen Geometrie der Regelfläche berechnet sich die Öffnungsstrecke  $l$  zu

$$l = - \oint_{\Phi} \cos \sigma \, du. \quad (3)$$

Die Öffnungsstrecke hat auch in der geometrischen Optik Bedeutung, da sie invariant gegenüber Brechung ist [11]!

Der Öffnungsstrecke dual gegenüber steht der *Öffnungswinkel*  $\lambda$  einer geschlossenen Regelfläche: Umschreibt man einer geschlossenen Regelfläche  $\Phi$  eine Berührtose deren Erzeugenden orthogonal zu den Erzeugenden von  $\Phi$  sind, und die nicht mit der Tangentenfläche einer Orthogonaltrajektorie zusammenfällt, so trifft die Tose eine Erzeugende mit der Richtung  $v_1(u_0)$  von  $\Phi$  mit dem Richtungsvektor  $p_1$  bzw. nach einem Umlauf mit dem Richtungsvektor  $p_2$ . Der Winkel  $\lambda$  zwischen  $p_1$  und  $p_2$  ist wieder unabhängig von der Wahl von  $v_1(u_0)$  und  $p_1$  und damit ebenfalls eine globale Invariante, die als Öffnungswinkel bezeichnet wird. Der Öffnungswinkel berechnet sich in der natürlichen Geometrie zu

$$\lambda = \oint_{\Phi} \tau \, du. \quad (4)$$

Die Öffnungsstrecke verschwindet für  $\sigma = \frac{\pi}{2}$ , was kennzeichnend für

Binormalenflächen ist; der Öffnungswinkel verschwindet, wenn das sphärische Erzeugendenbild  $v_1(u)$  von  $\Phi$  das sphärische Tangentenbild einer geschlossenen doppelpunktfreien sphärischen Kurve ist [17]. Der Öffnungswinkel führt zu einer geometrischen Deutung der *Gesamtkrümmung*  $\oint \kappa \, du$ , die vor allem im Bereich der Raumkurven ( $\sigma \rightarrow 0!$ ) vielfaches Interesse gefunden hat: Die Gesamtkrümmung einer geschlossenen Regelfläche ist gleich dem Öffnungswinkel der von den Zentraltangentialen  $v_3$  des begleitenden Dreibeins  $D(v_1, v_2, v_3)$  erzeugten Regelfläche (Zentraltangentialenfläche). Beim Übergang zur *Kurventheorie* geht die Zentraltangente in die Binormale der Raumkurve über, d.h. aber, die Gesamtkrümmung einer geschlossenen Raumkurve des euklidischen Raumes lässt sich als Öffnungswinkel der Binormalenfläche der Kurve deuten. Ist das Erzeugendenbild einer geschlossenen Regelfläche Tangentenbild einer geschlossenen doppelpunktfreien sphärischen Kurve, so gilt für den Öffnungswinkel  $\lambda_z$  der Zentraltangentialenfläche und damit für die Gesamtkrümmung der geschlossenen Regelfläche  $\Phi$  (in Anlehnung an den bekannten Satz von Fenchel aus der Kurventheorie)

$$\lambda_z = \oint_{\Phi} \kappa \, du \geq 2\pi. \quad (6)$$

Ähnliche Abschätzungen lassen sich auch für die gesamte ganze Krümmung  $\oint_{\Phi} \sqrt{\kappa^2 + \tau^2} du$  angeben: Ist das sphärische Zentralnormalenbild ( $v_2$ -Bild des begleitenden Dreibeins D) einer geschlossenen Regelfläche  $\Phi$  doppelpunktfrei, so ergibt sich für die gesamte ganze Krümmung [17]

$$\oint_{\Phi} \sqrt{\kappa^2 + \tau^2} du \geq 2\pi. \quad (7a)$$

Ist das Erzeugendenbild doppelpunktfrei und sphärisches Tangentenbild einer geschlossenen Raumkurve Y und erfüllt die Torsion T von Y die Bedingung  $T \geq 0$  aber T nicht identisch 0, so gilt für die gesamte ganze Krümmung der geschlossenen Regelfläche  $\Phi$  sogar

$$\oint_{\Phi} \sqrt{\kappa^2 + \tau^2} du \geq 4\pi. \quad (7b)$$

In engem Zusammenhang mit diesen Abschätzungen stehen Resultate, die als Verallgemeinerungen der Sätze von Jacobi und Scherrer aus der Kurventheorie angesehen werden können: Wird das (geschlossene) Zentralnormalenbild einer geschlossenen Regelfläche  $\Phi$  k-fach durchlaufen ( $k = 1, 2, \dots$ ), bevor das begleitende Dreibein D einmal die Striktionslinie durchlaufen hat und ist n die Nutationszahl von  $\Phi$  ( $-k < n < k$ ), d.h. die Drehzahl des Darboux-Vektors  $d = \kappa v_1 + \tau v_3$  des begleitenden Dreibeins D bei einem Umlauf längs der Striktionslinie, so umschließt das Zentralnormalenbild von  $\Phi$  ein Vielfaches der halben Oberfläche der Einheitskugel. Es gilt für den Inhalt des umschlossenen sphärischen Flächenstückes [9], [25]

$$\Omega = 2(k - n)\pi. \quad (8)$$

Die Zentralnormale  $\pm v_2$  ist übrigens der einzige feste Vektor des begleitenden Dreibeins D mit dieser Eigenschaft. Außerdem umschließen die sphärischen Bilder von fixen Vektoren orthogonal zum Darboux-Vektor Flächenstücke, deren Inhalt ein ganzzahlig Vielfaches der halben Oberfläche der Einheitskugel ist [10].

Mit Hilfe der Öffnungsstrecke und des Öffnungswinkels kann auch der bekannte Satz von Holditch für ebene (konvexe) Kurven auf Regelflächen übertragen werden: Bewegt man nämlich in der euklidischen Ebene die Endpunkte einer Strecke s konstanter Länge auf einer Eilinie E, so beschreibt ein fester Punkt P ∈ s eine i. allg. nicht konvexe Kurve K. Die Fläche zwischen E und K hängt nicht mehr von der Gestalt von K bzw. E ab, sondern nur von der Lage des Punktes P auf s. Dieses Resultat kann auf die Sphäre übertragen werden [28], wobei an die Stelle der ebenen Kurve E die sphärische Kurve V tritt, an die Stelle der Strecke s ein Großkreisbogen g der Länge γ. Um die sphärische Bewegung von g auf V analytisch beschreiben zu können, nehmen wir an, daß V die reguläre Parameterdarstellung  $v_1(u)$  besitzt und die Lage der Endpunkte von g bei dieser Bewegung durch die Parameterwerte u und f(u) beschrieben werden. Dann gilt für die entsprechenden Kurvenvektoren  $v_1(u)$  und  $v_1(f(u)) = v_1^*(u)$

$$v_1 \cdot v_1^* = \cos \gamma. \quad (9)$$

Nun wird  $v_1(u)$  als *sphärisches Erzeugendenbild* einer geschlossenen Regelfläche  $\Phi$  gedeutet, die Bedingung (9) legt dann auf  $\Phi$  zwei erzeugende Geraden mit den Richtungsvektoren  $v_1$  und  $v_1^*$  fest. Auf  $\Phi$  wählen wir nun als Leitkurve einer räumlichen Bewegung eine Kurve  $K$  mit der Parameterdarstellung

$$X(u) = S(u) + d(u) \tan \alpha(u) v_1(u) \quad (10)$$

mit  $d(u)$  als Drall von  $\Phi$  und  $\alpha(u)$  als Winkel zwischen der Kurvennormalen des sphärischen Bildes von  $\Phi$  und dem darin bewegten Großkreisbogen  $g$ . (Auf  $\Phi$  kann  $\alpha$  auch als Winkel zwischen der Tangentialebene im Punkt  $X(u_0)$  und der Tangentialebene im Striktionspunkt  $S(u_0)$  gedeutet werden.) Mit Hilfe der Punkte  $X(u)$  und  $X(f(u)) = X^*(u)$  auf  $\Phi$  wird nun folgende mit  $\Phi$  assoziierte lineare Kongruenz konstruiert:

$$\begin{aligned} Y = & \mu(X + d \tan \alpha v_1) + (1 - \mu)(X^* + d^* \tan \alpha^* v_1^*) \\ & + v(v_1 \cos \nu + v_1^* \sin \nu) \end{aligned} \quad (11)$$

mit  $\mu, v$  als Kongruenzparametern. Wird nun diese Kongruenz längs  $\Phi$  bewegt, so beschreibt jede Gerade ( $\mu$  und  $v$  const) der Kongruenz eine Regelfläche  $\tilde{\Phi}$ , deren Öffnungsstrecke  $\tilde{l}$  bzw. Öffnungswinkel  $\tilde{\lambda}$  nur von der Öffnungsstrecke  $l$  bzw. dem Öffnungswinkel  $\lambda$  der (Ausgangs-) Regelfläche  $\Phi$  und den Konstanten  $\mu, v, \nu$  abhängen. Es gilt

$$\begin{aligned} \tilde{l} &= l(\mu \cos \nu + (1 - \mu) \cos(\nu - \nu)), \\ \tilde{\lambda} &= \lambda \frac{\sin \nu + \sin(\nu - \nu)}{\sin \nu}. \end{aligned} \quad (12)$$

Dieses Resultat kann auch auf andere analog zu (11) mit  $\Phi$  assoziierte Kongruenzen, sowie auf zwei Regelflächen  $\Phi$  und  $\Phi^*$  übertragen werden, wobei die Bedingung (9) für die beiden (verschiedenen) Erzeugendenbilder  $v_1(u)$  und  $v_1^*(u)$  gelten muß (vgl. [12], [13], [19]). Für die Darstellung von  $\tilde{l}$  und  $\tilde{\lambda}$  werden dann jedoch i.allg. noch die Öffnungsstrecken von gewissen den Regelflächen  $\Phi$  und  $\Phi^*$  bewegungsinvariant zugeordneten weiteren Regelflächen benötigt.

Öffnungsstrecke und Öffnungswinkel haben große Bedeutung für die Untersuchung geschlossener Regelflächen in *Geradenkongruenzen* gefunden. So läßt sich z.B. die (halbe) Öffnungsstrecke der «äußersten» Regelfläche einer Kongruenz als Integral über die mittlere Krümmung der Kongruenz und das Flächenelement des sphärischen Bildes der Kongruenz deuten [7], [29], der Öffnungswinkel kann als Integral über das sphärische Bild der Kongruenz berechnet werden [18].

## 2. Scheitelsätze in der euklidischen Geometrie

Nun soll das Augenmerk auf Scheitelsätze und damit verwandte Fragen gelenkt werden. Als Scheitelsätze werden Aussagen über die *Mindestzahl von Extremwerten* von Invarianten oder invarianten Funktionen bezeichnet. Der wohl bekannteste Scheitelsatz ist der Vierscheitelsatz für ebene Kurven, der besagt, daß die Krümmung einer glatten ebenen (konvexen) Kurve mindestens 4 Extremwerte besitzt.

Scheitelsätze für geschlossene windschiefe Regelflächen  $\Phi$  lassen sich—analog zu den Kurven des euklidischen Raumes—nur für spezielle Flächenklassen angeben. Wir betrachten daher zunächst geschlossene windschiefe *Böschungsflächen*, die im Invariantensystem der natürlichen Geometrie der Regelflächen durch

$$\frac{\tau}{\kappa} = \tan \vartheta = \text{const.} \left( 0 < \vartheta = \frac{\pi}{2} \right) \quad (13)$$

gekennzeichnet sind. Bei geschlossenen windschiefen Böschungsflächen hat die Funktion

$$a(u) = \frac{\cos(\vartheta + \sigma)}{\sqrt{\kappa^2 + \tau^2}}$$

mindestens 4 Scheitel. Die Funktion  $a(u)$  kann als *Abstand der Momentanachse* der Dreibeinbewegung von der zugehörigen erzeugenden Geraden gedeutet werden, d.h. die Momentanachse hat bei geschlossenen windschiefen Böschungsflächen mindestens 4 Extremallagen. Ist weiter die Strikionslinie eine *konvexe* Kurve, so gibt es mindestens 3 Paare von erzeugenden Geraden, die von der zugehörigen Momentanachse gleichen Abstand haben, außerdem besitzt die Funktion

$$h(u) = \frac{\cos(\vartheta + \sigma)}{\sqrt{\kappa^2 + \tau^2}} + \frac{1}{2\pi} \frac{1}{\cos \vartheta}$$

wenigstens 4 Nulldurchgänge [14]. Die letzten Resultate führen bei geschlossenen windschiefen Böschungsflächen konstanter Breite  $B$  sogar auf einen *Sechsscheitelsatz*: Der Abstand  $a(u)$  der Momentanachse der Dreibeinbewegung von der zugehörigen erzeugenden Geraden hat mindestens 6 Extremwerte. Dabei werden als windschiefe Böschungsflächen konstanter Breite jene geschlossene Böschungsflächen mit konvexer Strikionslinie bezeichnet, die in Gegenpunkten parallele Tangentialebenen von konstantem Abstand besitzen. Gegenpunkte sind jene Punkte der Strikionslinie, in denen die Normalen der Regelfläche antiparallel sind. In der Parametrisierung nach der Bogenlänge  $u_2$  des sphärischen  $v_2$ -Bildes gilt für Gegenpunkte

$$v_2(u_2) = -v_2(u_2 + \pi).$$

Die Böschungsflächen konstanter Breite haben viele weitere bemerkenswerte Eigenschaften und sind Grundlage einer Übertragung der Zindlerkurven im dreidimensionalen euklidischen Raum [8].

Hat eine geschlossene windschiefe Regelfläche  $\Phi$  eine konvexe Strikionslinie, die gleichzeitig Krümmungslinie von  $\Phi$  ist, so besitzt die ganze Krümmung  $\omega = \sqrt{\kappa^2 + \tau^2}$  mindestens 4 Extremwerte [15]. Bei geschlossenen windschiefen Regelflächen, deren sphärisches Erzeugendensystem eine einfach durchlaufene sphärisch konvexe Kurve ist, hat die konische Krümmung

$$\kappa_2 = \frac{\tau}{\kappa}$$

mindestens 4 Extremwerte. Dabei wird als sphärisch konvexe Kurve der Schnitt eines konvexen Kegels mit der offenen Halbkugel bezeichnet [21]. Dieses Resultat lässt sich auch auf geschlossene windschiefe Regelflächen mit einfach durchlaufenem sphärisch konvexen  $v_3$ -Bild übertragen. Die Beweisführung nach der Herglotz-Methode erfolgt am einfachsten über die reziproke konische Krümmung.

### 3. Aus der projektiven und der nichteuklidischen Geometrie

Abschließend sollen nun noch einige globale Eigenschaften von geschlossenen Regelflächen aus der *projektiven* und aus der *nichteuklidischen* Geometrie angegeben werden.

Im *projektiven Raum*  $P^3$  werden zunächst geschlossene windschiefe Regelflächen betrachtet, die einer linearen Kongruenz angehören (Kongruenzregelflächen). Eine geschlossene *Kongruenzregelfläche* mit nichtreellen Leitlinien besitzt mindestens vier Stellen stationärer *Schmiegquadrik*, wird eine konvexe, geschlossene Kongruenzregelfläche von einer Geraden in 2k Punkten getroffen, so besitzt die Regelfläche mindestens 2k Stellen stationärer Schmiegquadrik. Dabei wird hier eine Regelfläche als konvex bezeichnet, wenn je zwei Erzeugende eine Treffgerade besitzen, die die Regelfläche sonst nicht mehr trifft; die Schmiegquadrik einer Regelfläche  $\Phi$  ist die Quadrik, die drei «aufeinanderfolgende» Erzeugende von  $\Phi$  enthält [2]. Für eine *geschlossene Asymptotenlinie* einer geschlossenen Regelfläche gilt weiter: Gibt es einen linearen Komplex, dem eine Schmiegertangente einer geschlossenen Asymptotenlinie aber keine Erzeugende der geschlossenen Regelfläche angehört, so besitzt die Asymptotenlinie mindestens zwei Wendepunkte. Gibt es ferner durch je zwei Erzeugende der Regelfläche einen linearen Komplex, dem eine der beiden zugehörigen Schmiegertangenten der geschlossenen Asymptotenlinie, aber keine weitere Erzeugende der Regelfläche angehört, so besitzt die Asymptotenlinie sogar mindestens 4 Wendepunkte [2]. Ähnliche Aussagen lassen sich auch für geschlossene Regelflächen, die einem linearen Komplex angehören (*Komplexregelflächen*) angeben: Gibt es durch je drei Geraden einer *Komplexregelfläche* einen linearen Komplex, der diese aber keine weiteren Geraden der Regelfläche enthält, so besitzt die Regelfläche mindestens 5 Stellen stationärer Schmiegkongruenz. Gibt es dagegen einen linearen Komplex, dem k, aber nicht alle Erzeugenden einer Kongruenzregelfläche angehören, und gibt es zu je 2 Erzeugenden einen linearen Komplex, dem außer diesen keine weiteren Erzeugenden der Regelfläche angehören, so hat die Regelfläche mindestens k Stellen (also immer mindestens 4 Stellen), in denen die Schmiegquadrik stationär ist [1].

Scheitelsätze lassen sich für *Fleknodalkurven* angeben, wobei mit Fleknodalkurven jene Ortslinien einer Regelfläche  $\Phi$  bezeichnet werden, die von den Wendepunkten der krummlinigen Asymptotenlinien von  $\Phi$  gebildet werden. Auf einer windschiefen geschlossenen Regelfläche gibt es höchstens 2 Fleknodalkurven. Fallen die beiden Fleknodalkurven zusammen und wird diese Kurve von jeder Ebene in höchstens 3 Punkten getroffen, so hat die Fleknodalkurve mindestens acht sextaktische Punkte, d.h. Punkte, in denen der Schmiegkegel die Kurve in fünfter Ordnung berührt [5].

Wird im  $P^3$  eine nichtausgeartete nullteilige Quadrik als absolute Quadrik ausgezeichnet, erhält man einen dreidimensionalen *elliptischen Raum*  $E^3$ . In dieser Geometrie lassen sich einige der bisher aufgezeigten Resultate weitgehend analog zur euklidischen Geometrie entwickeln. Über Orthogonaltrajektorien der Erzeugenden einer geschlossenen Regelfläche bzw. orthogonal umschriebene Torsen ergeben sich als Integralinvarianten geschlossener Regelflächen *Öffnungsstrecke* (*Öffnung*) und *Öffnungswinkel* (*Klaffung*), bei der zu einer geschlossenen Regelfläche  $\Phi$  polaren Regelfläche  $\Phi^*$  vertauschen Öffnung und Klaffung ihre Bedeutung, bei Zentralnormalenflächen von  $\Phi$  verschwinden Öffnung und Klaffung. In einem natürlichen Bezugssystem lassen sich diese Resultate wieder direkt auf die Geometrie der Raumkurven des  $E^3$  übertragen [3], [27], [24]. Der projektive Raum  $P^3$  wird zum einfach *isotropen Raum*  $J^3$ , wenn in  $P^3$  das in der Fernebene liegende Geradenpaar  $x^2 + y^2 = 0$  als absolute Quadrik ausgezeichnet wird. Die Öffnungsstrecke einer geschlossenen Regelfläche  $\Phi$  des  $J^3$  kann als euklidische Länge des Grundrisses der Striktionslinie von  $\Phi$  gedeutet werden, die Gesamt torsion von  $\Phi$  kann wieder als *Öffnungswinkel* der geschlossenen Regelfläche interpretiert werden. Gehört eine geschlossene windschiefe Regelfläche  $\Phi$  des  $J^3$  einer (allgemeinen) Kongruenz an, so lässt sich die Öffnungsstrecke als Integral über das Produkt von mittlerer Krümmung  $h$ , dem Abstandsquadrat  $B$  der Brennpunkte vom Mittelpunkt und dem Flächenelement des zylindrischen Bildes der Kongruenz deuten. Der Öffnungswinkel kann dabei als Integral über das Flächenelement des zylindrischen Bildes der Kongruenz aufgefasst werden.

*Scheitelsätze* für die Krümmung  $\kappa$  einer geschlossenen windschiefen Regelfläche im  $J^3$  folgen unmittelbar aus den Scheitelsätzen für geschlossene Kurven der euklidischen Ebene, da  $\kappa$  mit der (euklidischen) Kurvenkrümmung der Projektion der Striktionslinie von  $\Phi$  zusammenfällt [20].

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## Über einige geometrische Aspekte der Approximationstheorie

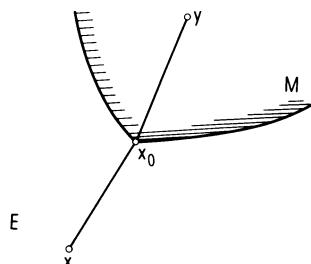
### 1. Einleitung

Die Approximationstheorie befaßt sich mit der Untersuchung von Prozessen, mit deren Hilfe Funktionen durch die Funktionen einer vorgegebenen Klasse in einem zu präzisierenden Sinne angenähert werden können. Dabei stehen naturgemäß algorithmische Gesichtspunkte im Vordergrund. Neben den Algebraisierungstendenzen, die für die moderne Analysis so charakteristisch sind, spielen in der Approximationstheorie auch der geometrischen Anschauung zugängliche Prinzipien eine wichtige Rolle. An dieser Stelle soll nur auf zwei dieser geometrischen Aspekte näher eingegangen werden: Auf die Projektionsmethode mit der verallgemeinerten Inversen stetiger linearer Abbildungen und der  $L^2$ -Approximation durch kubische Splines als Anwendungsbeispiele sowie auf die Approximation stetiger Funktionen auf den kompakten Einheitssimplizes reeller affiner Räume mit Hilfe von Bernštein-Polynomen in mehreren Variablen zur Konkretisierung des Approximationssatzes von Stone-Weierstraß.

### 2. Die Projektionsmethode

Es bezeichne  $E$  einen reellen oder komplexen Prähilbert-Raum, also einen Vektorraum über dem Körper  $\mathbb{R}$  der reellen oder  $\mathbb{C}$  der komplexen Zahlen mit der nicht ausgearteten positiven hermitischen Sesquilinearform  $(x,y) \rightsquigarrow (x|y)$  als Skalarprodukt. Durch  $x \rightsquigarrow \|x\| = \sqrt{(x|x)}$  wird auf  $E$  die zugehörende Norm definiert. Sei  $M$  eine nicht leere konvexe Teilmenge von  $E$ , welche unter der von der Norm  $\|\cdot\|$  hervorgerufenen Abstandsfunktion  $(x,y) \rightsquigarrow \|x - y\|$  ein vollständiger metrischer Raum ist. Dann existiert zu jedem Punkt  $x \in E$  genau ein Proximum in  $M$ , d.h. genau ein Punkt  $x_0 \in M$  mit der Eigenschaft

$$\|x - x_0\| = \inf_{y \in M} \|x - y\|. \quad (1)$$



Das Proximum  $x_0 \in M$  zum Punkt  $x \in E$  in  $M$  wird durch die Bedingung

$$\operatorname{Re}(x - x_0 | y - x_0) \leqq 0 \quad (y \in M) \quad (2)$$

gekennzeichnet. Ist  $M$  insbesondere ein vollständiger Untervektorraum von  $E$ , so ist  $x_0 = P_M(x)$  gerade die Orthogonalprojektion von  $x$  auf  $M$ , d.h. es gilt in diesem Falle

$$x - x_0 \in M^\perp. \quad (3)$$

### 3. Erste Anwendung: Die verallgemeinerte Inverse

Es seien  $E_1, E_2$  zwei Hilbert-Räume (also vollständige Prähilbert-Räume) über dem gleichen Körper  $\mathbb{R}$  oder  $\mathbb{C}$  und  $T: E_1 \rightarrow E_2$  eine (bezüglich der Norm-Topologien) stetige lineare Abbildung, deren Bildraum  $F = T(E_1)$  in  $E_2$  abgeschlossen ist. Zwar besitzt für jeden Punkt  $a \in E_2$  die lineare Gleichung

$$Tx = a \quad (4)$$

im Falle  $a \notin F$  keine Lösung  $x \in E_1$ , aber es existiert zumindest eine »bestmögliche« Lösung  $y \in E_1$ , welche die Bedingung

$$Ty = P_F(a) \quad (5)$$

erfüllt. Es gilt dann

$$\|Ty - a\| \leq \|Tx - a\| \quad (x \in E_1) \quad (6)$$

(»Minimaleigenschaft«) und

$$T^* \circ Ty = T^*a, \quad (7)$$

wobei wie üblich  $T^*: E_2 \rightarrow E_1$  den zu  $T$  adjungierten Operator bezeichnet. Elementare Rechnungen mit dem Satz von Pythagoras zeigen, daß die Bedingungen (5), (6), (7) für  $y \in E_1$  paarweise äquivalent sind. Die nicht leere konvexe Menge

$$M = \{y \in E_1 \mid T^* \circ Ty = T^*a\} \quad (8)$$

ist abgeschlossen im Hilbert-Raum  $E_1$ . Mithin existiert genau ein Proximum  $x_0$  zum Nullpunkt von  $E_1$  in  $M$ . Man nennt die durch  $T^+a = x_0$  definierte Abbildung  $T^+: E_2 \rightarrow E_1$  die verallgemeinerte Inverse von  $T$ . Es kann bewiesen werden, daß  $T^+$  eine stetige lineare Abbildung von  $E_2$  in  $E_1$  ist, die den gleichen Bildraum wie  $T$  besitzt und die bei invertierbarer Ausgangsabbildung  $T$  mit  $T^{-1}$  übereinstimmt. Bezeichnet  $\tilde{T}$  die Einschränkung von  $T^* \circ T$  auf den abgeschlossenen Untervektorraum  $T^*(E_2)$  von  $E_1$ , so gilt  $T^+ = \tilde{T}^{-1} \circ T^*$ .

Die hier wiedergegebene Definition der verallgemeinerten Inversen  $T^+$  beruht auf der »Methode der kleinsten Quadrate«. Andere äquivalente Definitionen von  $T^+$  und eine Erweiterung für stetige lineare Abbildungen  $T: E_1 \rightarrow E_2$ , deren Bildraum  $F$  in  $E_2$  nicht abgeschlossen zu sein braucht, finden sich z.B. in der Monographie von Groetsch [3].

### 4. Zweite Anwendung: $L^2$ -Approximation durch kubische Splines

Sei  $I = [a, b]$  ein kompaktes Teilintervall der reellen Zahlengeraden  $\mathbb{R}$  und  $\Delta_n = (x_j)_{0 \leq j \leq n}$  eine (nicht notwendig äquidistante) Zerlegung von  $I$  mit  $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ . Für jede natürliche Zahl  $m \geq 1$  bezeichne  $\mathfrak{S}_m(\Delta_n)$  die

Menge aller reellwertigen Funktionen  $s \in \mathcal{C}^{m-1}(I)$  mit der Eigenschaft, daß die Restriktionen  $s|_{[x_j, x_{j+1}]}$  ( $0 \leq j \leq n-1$ ) Polynomfunktionen vom Grade  $\leq m$  sind. Die Elemente  $s \in \mathfrak{S}_m(\Delta_n)$  werden Polynomsplines vom Grad  $m$  auf  $I$  zur Zerlegung  $\Delta_n$  genannt. Im Falle  $m=3$  spricht man von kubischen Splines. Zu jeder natürlichen Zahl  $k$  mit  $0 \leq k \leq n$  existiert genau ein kubischer Spline  $s_k \in \mathfrak{S}_3(\Delta_n)$  mit folgenden Eigenschaften:

$$\begin{aligned} s_k(x_j) &= \delta_{jk} \quad (0 \leq j \leq n), \\ s'_k(x_0) &= s'_k(x_n) = 0. \end{aligned} \tag{9}$$

Die sog. Kardinalsplines  $(s_k)_{0 \leq k \leq n}$  bilden zusammen mit der konstanten Funktion  $x \mapsto 1$  und der identischen Abbildung  $x \mapsto x$  eine  $\mathbb{R}$ -Basis  $(t_k)_{0 \leq k \leq n+2}$  von  $\mathfrak{S}_3(\Delta_n)$ . Demnach ist  $\mathfrak{S}_3(\Delta_n)$  ein  $(n+3)$ -dimensionaler Untervektorraum des Vektorraumes  $\mathcal{C}(I)$  aller stetigen reellwertigen Funktionen auf  $I$ .

Versieht man den reellen Vektorraum  $\mathcal{C}(I)$  mit dem Skalarprodukt  $(f,g) \mapsto f|g| = \int_I f(x)g(x)dx$  und der zugehörigen  $L^2$ -Norm  $\| \cdot \|_2$ , so ist  $\mathfrak{S}_3(\Delta_n)$  ein vollständiger Untervektorraum des reellen Prähilbert-Raumes  $\mathcal{C}(I)$ . Somit existiert zu jeder vorgegebenen Funktion  $f \in \mathcal{C}(I)$  genau ein Proximum  $S_0 \in \mathfrak{S}_3(\Delta_n)$  in  $\mathfrak{S}_3(\Delta_n)$ . Gemäß (3) wird dieses durch die Bedingungen

$$(f - S_0|t_k) = 0 \quad (0 \leq k \leq n+2) \tag{10}$$

gekennzeichnet. Stellt man  $S_0$  als Linearkombination der Funktionen  $(t_k)_{0 \leq k \leq n+2}$  dar, so erhält man aus (10) ein lineares Gleichungssystem für die reellen Koeffizienten. Durch Auflösen dieser sog. Normalgleichungen gewinnt man das Proximum  $S_0 \in \mathfrak{S}_3(\Delta_n)$  zu  $f \in \mathcal{C}(I)$  in expliziter Form.

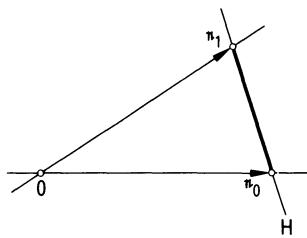
Polynomsplines eignen sich besonders gut zur Interpolation an den Knoten der Zerlegung  $\Delta_n$  von  $I$ . Interpolationssplines zeichnen sich durch Extremaleigenschaften aus, die ebenfalls mit der Projektionsmethode erfaßt werden können. Bezüglich dieser Ergebnisse sei auf den Übersichtsartikel [8] und die dort aufgeführte Literatur verwiesen.

Man wird vermuten, daß zwischen der Theorie der verallgemeinerten Inversen und der Theorie der Splinefunktionen ein Zusammenhang besteht. Dies ist tatsächlich der Fall, wie z.B. die Arbeit von Delvos [2] zeigt.

## 5. Bernstein-Polynome in mehreren Variablen

Es bezeichne  $(n_i)_{0 \leq i \leq m}$  die kanonische Basis des  $(m+1)$ -dimensionalen reellen Vektorraumes  $\mathbb{R}^{m+1}$  ( $m \geq 1$ ). Betrachtet man die Vektoren  $(n_i)_{0 \leq i \leq m}$  als Koordinatenvektoren der Grundpunkte eines homogenen Koordinatensystems des affinen Raumes  $\mathbb{R}^m$ , so wird dieser mit der Hyperebene  $H: \sum_{0 \leq i \leq m} x_i - 1 = 0$  des  $\mathbb{R}^{m+1}$  identifiziert und das kompakte Einheitssimplex, d.h. das abgeschlossene Simplex  $X_m$  des  $\mathbb{R}^m$  mit den Grundpunkten als Ecken, wird durch  $X_m = \{x \in \mathbb{R}^{m+1} \mid x_i \geq 0, \sum_{0 \leq i \leq m} x_i = 1\}$  beschrieben.

Sei  $T$  die mit der diskreten Topologie versehene Menge  $\{0, 1, \dots, m\}$ . Dann wird der Vektorraum  $\mathcal{M}(T)$  aller reellen Radon-Maße auf  $T$  von den Dirac-Maßen  $(\varepsilon_i)_{0 \leq i \leq m}$  aufgespannt. Identifiziert man die reellen Vektorräume  $\mathcal{M}(T)$  und  $\mathbb{R}^{m+1}$ , so wird  $X_m$  zum Raum  $\mathcal{M}_+^1(T)$  aller Radonschen Wahrscheinlichkeitsmaße auf  $T$  und durch die Zuordnung



$$\pi_n: T^n \ni (t_j)_{1 \leq j \leq n} \rightsquigarrow \sum_{1 \leq j \leq n} \frac{1}{n} \varepsilon_{t_j} \in X_m \quad (n \geq 1) \quad (11)$$

wird eine Folge  $(\pi_n)_{n \geq 1}$  stetiger Abbildungen definiert. Für jede Funktion  $F \in \mathcal{C}(X_m)$  und jedes Maß  $\mu = (\mu_i)_{0 \leq i \leq m} \in \mathcal{M}_+^1(T)$  gilt

$$\langle F \circ \pi_n, \mu^{\otimes n} \rangle = \sum_{\substack{k_j \geq 0 \\ k_1 + \dots + k_m \leq n}} F\left(\frac{k_1}{n}, \dots, \frac{k_m}{n}\right) b_{n,k_1, \dots, k_m}(\mu) \quad (12)$$

mit

$$b_{n,k_1, \dots, k_m}(\mu) = \binom{n}{k_1, \dots, k_m} \mu_1^{k_1} \dots \mu_m^{k_m} \left(1 - \sum_{1 \leq j \leq m} \mu_j\right)^{n - \sum_{1 \leq j \leq m} k_j}$$

und

$$\binom{n}{k_1, \dots, k_m} = \frac{n!}{\prod_{1 \leq j \leq m} k_j! (n - \sum_{1 \leq j \leq m} k_j)!}.$$

Man nennt die Abbildung  $B_n(F): X_m \ni \mu \rightsquigarrow \langle F \circ \pi_n, \mu^{\otimes n} \rangle \in \mathbb{R}$  das  $n$ -te Bernštein-Polynom von  $F$ .

## 6. Verallgemeinerte Bernštein-Polynome

Die vorstehenden, geometrisch motivierten Überlegungen führen fast zwangsläufig zu einer natürlichen Verallgemeinerung der Bernštein-Polynome.

Sei  $T$  ein beliebiger kompakter topologischer Raum. Der Dual  $\mathcal{M}(T)$  des mit der Topologie der gleichmäßigen Konvergenz versehenen Vektorraumes  $\mathcal{C}(T)$  aller stetigen reellwertigen Funktionen auf  $T$  besteht aus den reellen Radon-Maßen auf  $T$ . Versieht man  $\mathcal{M}(T)$  mit der vagen Topologie  $\sigma(\mathcal{M}(T), \mathcal{C}(T))$ , so ist die Menge  $\mathcal{M}_+^1(T)$  aller Radonschen Wahrscheinlichkeitsmaße auf  $T$  ein kompakter und konvexer topologischer Unterraum  $X$  von  $\mathcal{M}(T)$ ; vgl. Bauer [1]. Führt man mit einer stochastischen (unteren) Dreiecksmatrix  $P = (p_{nj})_{n \geq 1, j \geq 1}$ , d.h. mit einer unendlichen reellen Matrix, deren Elemente die Bedingungen

$$\begin{aligned} p_{nj} &\geq 0 \quad (n \geq 1, j \geq 1), \\ p_{nj} &= 0 \quad (j > n), \\ \sum_{j \geq 1} p_{nj} &= 1 \quad (n \geq 1) \end{aligned} \quad (13)$$

erfüllen, die Abbildungen

$$\pi_{n,P}: T^n \ni (t_j)_{1 \leq j \leq n} \rightsquigarrow \sum_{j \geq 1} p_{nj} e_{t_j} \in X \quad (n \geq 1) \quad (14)$$

ein, so wird für jedes  $F \in \mathcal{C}(X)$  durch die Zuordnung

$$B_{n,P}(F): X \ni \mu \rightsquigarrow \langle F \circ \pi_{n,P}, \mu^{\otimes n} \rangle = \int_X F d\pi_{n,P}(\mu^{\otimes n}) \in \mathbb{R} \quad (15)$$

die sog.  $n$ -te Bernštein-Funktion von  $F$  bezüglich  $P$  definiert. Man kann zeigen, daß unter der Bedingung  $\lim_{n \rightarrow \infty} \sum_{j \geq 1} p_{nj}^2 = 0$  an die Elemente der Matrix  $P$  die (gleichmäßige) Konvergenz  $\lim_{n \rightarrow \infty} \|B_{n,P}(F) - F\|_\infty = 0$  für alle Funktionen  $F \in \mathcal{C}(X)$  gesichert ist. Einzelheiten finden sich bei Grossman [4], [5], Schempp [9] und der in diesen Arbeiten zitierten Literatur. Für eine Weiterführung der Theorie sei z.B. auf Nishishiraho [6], [7] verwiesen.

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### III. Foundations of Geometry

# Hans-Joachim Arnold

## Richtungsalgebren

### 1. Affine Richtungs-Relative

Zur Kennzeichnung angeordneter affiner Geometrien eignen sich Relative (Relationenalgebren), deren Grundmenge die Menge  $\mathfrak{P}$  der Punkte ist und deren Relationenmenge  $\mathfrak{R}$  aus den als binäre Relationen auf  $\mathfrak{P}$  gedeuteten «Richtungen» bestehen. Diese Eignung kommt nicht nur darin zum Ausdruck, daß die Relative  $(\mathfrak{P}, \mathfrak{R})$  auch den nichtdesarguesschen Fall mit einbeziehen, sondern beruht vor allem darauf, daß das Rechnen mit den Relationen zwanglos auch zu einer algebraischen Charakterisierung der angeordneten projektiven Geometrien führt mit Hilfe geeigneter Multigruppen.

*Definition 1.* Wir sprechen von einem *affinen Richtungsrelativ*  $(\mathfrak{P}, \mathfrak{R})$ , wenn gegeben sind

1. eine Menge  $\mathfrak{P} = \{A, B, C, \dots\} (\neq \emptyset)$ ,  
deren Elemente *Punkte* heißen,
2. eine Menge  $\mathfrak{R} = \{a, b, c, \dots\} \subset 2^{\mathfrak{P} \times \mathfrak{P}}$ ,  
deren Elemente *Richtungen* heißen und binäre Relationen auf  $\mathfrak{P}$  sind, [wir schreiben statt  $(A, B) \in c$  auch  $AcB$  und lesen: «Von A in Richtung c nach B»], und wenn außer den Grundannahmen

$$GA1 \quad a \in \mathfrak{R} \succ a^{-1} \in \mathfrak{R},$$

$$GA2 \quad \circ \quad (= \text{Gleichheitsrelation auf } \mathfrak{P}) \in \mathfrak{R},$$

noch die folgenden Axiome gelten:

$$Ax.I \quad \begin{array}{c} \wedge \quad \wedge \quad \psi \\ A \quad B \quad c \end{array} \quad AcB \quad [\text{Def. } AB=c : \not\propto AcB],$$

$$Ax.II \quad \begin{array}{c} \wedge \quad \wedge \quad \wedge \\ A \quad B \quad C \end{array} \quad AB \subset AC \circ CB,$$

$$Ax.III \quad \begin{array}{c} \wedge \quad \wedge \\ a \quad b \end{array} \quad a \circ b = b \circ a,$$

$$Ax.IV \quad \begin{array}{c} \wedge \\ a \end{array} \quad a \circ a = a,$$

$$Ax.V \quad \begin{array}{c} \wedge \\ a \neq o \end{array} \quad a^{-1} \neq a,$$

$$Ax.VI \quad \begin{array}{c} \wedge \\ a \end{array} \quad a \circ a^{-1} = a \cup a^{-1} \cup o.$$

[Mit  $\circ$  bezeichnen wir das Relationenprodukt.]

$$\begin{aligned}\mathfrak{G} &:= \{A(a \circ a^{-1}) \mid A \in \mathfrak{P} \wedge a \in \mathfrak{R} \setminus \{o\}\}; \\ A(a \circ a^{-1}) &:= \{P \mid A(a \circ a^{-1})P\}; \\ [A(a \circ a^{-1})] \text{ IP} &\not\asymp [A(a \circ a^{-1})P]; \\ [A(a \circ a^{-1})] \parallel [B(b \circ b^{-1})] &\not\asymp a \circ a^{-1} = b \circ b^{-1}; \\ z(A, B, C) &\not\asymp AB = BC\end{aligned}$$

(I Inzidenzrelation,  $\parallel$  Parallelenrelation,  $z$  Zwischenrelation).

b) Umgekehrt gilt, daß sich jede im Hilbertschen Sinne angeordnete affine Geometrie  $(\mathfrak{P}, \mathfrak{G}, I \parallel, z)$  in der oben angegebenen Form durch ein affines Richtungsrelativ darstellen lässt. Für den Beweis dieses Satzes sei auf [2] hingewiesen.

## 2. Die zu $(\mathfrak{P}, \mathfrak{R})$ gehörige projektive Richtungsalgebra

Jedes affine Richtungsrelative  $(\mathfrak{P}, \mathfrak{R})$  induziert auf  $\mathfrak{R}$  selbst zwei nahe verwandte Strukturen, nämlich die von  $\mathfrak{R}$  und  $\circ$  (Relationenprodukt) erzeugte Halbgruppe binärer Relationen einerseits und die Multigruppe  $\mathfrak{R}(+)$  andererseits, wobei wir eine Abbildung

$$(x) \quad \begin{cases} \mathfrak{R} \times \mathfrak{R} \rightarrow 2^{\mathfrak{R}} \\ a, b \mapsto a + b \in 2^{\mathfrak{R}} \end{cases}$$

definieren, indem wir setzen  $a + b := \{c \mid c \subset a \circ b\}$ .

Ferner haben wir auf  $\mathfrak{R}$  wegen GA1 eine Involution

$$(xx) \quad \begin{cases} \mathfrak{R} \rightarrow \mathfrak{R} \\ a \mapsto \bar{a} \in \mathfrak{R} \end{cases}$$

mit  $o$  als einzigmem Fixpunkt (s. Ax.V), wenn wir setzen  $\bar{a} := a^{-1}$ .

Für das so durch  $(\mathfrak{P}, \mathfrak{R})$  definierte Quadrupel  $(\mathfrak{R}, +, \bar{\cdot}, o)$  gelten die folgenden Rechenregeln:

- (1)  $a + a = a + o = \{a\}$ ,
- (2)  $a + \bar{a} < \{a, \bar{a}, o\}$ ,
- (3)  $a + b = b + a$ ,
- (4)  $a + (b + c) = (a + b) + c$ ,
- (5)  $a \in b + c > b \in a + \bar{c}$ .

Dabei wird [wie bei Operationen des Typs  $(\mathfrak{P}^n \rightarrow 2^{\mathfrak{P}}, \text{ für } n=2)$  üblich] gesetzt

$$\begin{aligned}a + (b + c) &:= \bigcup_{x \in b + c} (a + x), \\ (a + b) + c &:= \bigcup_{y \in a + b} (y + c).\end{aligned}$$

*Definition 2.* Wir sprechen von einer «projektiven Richtungsalgebra  $(\mathfrak{R}, +, \bar{\cdot}, o)$ », wenn für eine Menge  $\mathfrak{R}$  eine zweistellige Operation des Typs (x), eine Involution

des Typs (xx) und ein Element  $\mathfrak{o} \in \mathfrak{R}$  gegeben sind mit

$$\bar{\alpha} = \alpha \not\propto \alpha = \mathfrak{o},$$

und die Rechenregeln (1)–(5) gelten.

**Satz 2.** Zu jedem affinen Richtungsrelativ  $(\mathfrak{P}, \mathfrak{R})$  gehört eine projektive Richtungsalgebra  $(\mathfrak{R}, +, -, \mathfrak{o})$  mit  $\alpha + \beta := \{c \mid c \subset \alpha \circ \beta\}$ ,  $\bar{\alpha} := \alpha^{-1}$ ,  $\mathfrak{o}$  = Gleichheitsrelation auf  $\mathfrak{P}$ . Diese projektive Richtungsalgebra bezeichnen wir mit  $\mathbb{T}(\mathfrak{P}, \mathfrak{R})$ .

### 3. Die zu einer projectiven Richtungsalgebra gehörige angeordnete projektive Geometrie

**Satz 3.** Jede projektive Richtungsalgebra  $(\mathfrak{R}, +, -, \mathfrak{o})$  definiert wie folgt eine angeordnete projektive Geometrie  $(\mathbb{P}, \mathbb{G}, \mathbb{I}, \mathbb{T})$ :

$$(6) \quad \mathbb{P} := \left\{ \mathfrak{A} \subset \mathfrak{R} \mid \bigvee_{\alpha \in \mathfrak{R} \setminus \{\mathfrak{o}\}} \mathfrak{A} = \alpha + \bar{\alpha} \right\}$$

(Menge der Punkte),

$$(7) \quad \mathbb{G} := \{ \mathfrak{A}_1 + \mathfrak{A}_2 \mid \mathfrak{A}_1 \neq \mathfrak{A}_2 \in \mathbb{P} \}$$

(Menge der Geraden).

[Dabei setzen wir  $\mathfrak{A}_1 + \mathfrak{A}_2 := \bigcup_{\alpha_i \in \mathfrak{A}_i} (\alpha_1 + \alpha_2)$ .]

$$(8) \quad \mathbb{I} : \mathfrak{A}_1 \mathbb{I} \mathfrak{A}_2 \iff \mathfrak{A}_1 \subset \mathfrak{A}_1 + \mathfrak{A}_2$$

(I bezeichnet die Inzidenzrelation);

$$(9) \quad \begin{aligned} \{ \mathfrak{A}_1, \mathfrak{A}_2 \} \mathbb{T} \{ \mathfrak{A}_3, \mathfrak{A}_4 \} : \not\propto \\ \left\{ \begin{array}{l} \alpha_3 \in \alpha_1 + \alpha_2 \wedge \alpha_4 \in \alpha_1 + \bar{\alpha}_2 \\ \text{für geeignete } \alpha_i \text{ mit } \mathfrak{A}_i = \alpha_i + \bar{\alpha}_i \in \mathbb{P} \end{array} \right. \end{aligned}$$

(T bezeichnet die Trennrelation).

Der Übergang von  $(\mathfrak{P}, \mathfrak{R})$  zu  $(\mathfrak{R}, +, -, \mathfrak{o})$ , den wir mit  $\mathbb{T}$  bezeichnet haben (s. Satz 2), induziert geometrischerseits den Projektionsfunktor, der die angeordnete affine Geometrie in ihren angeordneten projektiven Fernraum abbildet.

Dieser Funktor ist nicht surjektiv, denn die nichtdesargesschen angeordneten projektiven Ebenen können nicht als Projektionen dargestellt werden. Der Kalkül der projektiven Richtungsalgebren ist aber allgemein genug, um alle angeordneten projektiven Geometrien—einschließlich der angeordneten nichtdesargesschen projektiven Ebenen—algebraisch zu charakterisieren.

### 4. Algebraische Kennzeichnung der angeordneten projektiven Geometrien

**Lemma:** Es sei ein beliebiges affines Richtungsrelative  $(\mathfrak{P}, \mathfrak{R})$  vorgegeben und  $(\mathfrak{R},$

$(+, -, \circ) = \pi(\mathfrak{P}, \mathfrak{R})$ . Dann gibt es eine projektive Richtungsalgebra  $(\hat{\mathfrak{R}}, +, -, \circ)$  mit  $\hat{\mathfrak{R}} \supseteq \mathfrak{P} \cup \mathfrak{R}$ , so daß  $(\mathfrak{R}, +, -, \circ)$  maximale Unterlagebra von  $(\hat{\mathfrak{R}}, +, -, \circ)$  ist und gilt

1.  $\mathfrak{P} = A + \mathfrak{R}$  für ein  $A \in \hat{\mathfrak{R}}$ ,
2.  $A \subset B \nless A + c \ni B$ .

Mit Hilfe dieses Lemmas gelingt die Umkehrung des Satzes 3:

**Satz 4.** *Jede angeordnete projektive Geometrie  $(\mathbb{P}, \mathbb{G}, I, T)$  läßt sich mit Hilfe einer geeigneten projektiven Richtungsalgebra in der Form (6)–(9) darstellen.*

Hierzu eine Beweiskizze: Sei  $(\mathbb{P}, \mathbb{G}, I, T)$  eine vorgegebene angeordnete projektive Geometrie. Diese wird zunächst bezüglich einer beliebigen Hyperebene affiniert: Ist  $\mathbb{H} \subset \mathbb{P}$  die beliebig (aber fest) gewählte Hyperebene, so ist  $(\mathfrak{P}, \mathfrak{G}, I, ||, z)$  mit  $\mathfrak{P} := \mathbb{P} \setminus \mathbb{H}$  und  $\mathfrak{G}, I, ||, z$  wie üblich definiert eine angeordnete affine Geometrie, die sich gemäß Satz 1b in der Form eines affinen Richtungsrelatives  $(\mathfrak{P}, \mathfrak{R})$  darstellen läßt.

Ferner läßt sich leicht zeigen, daß der projektiven Richtungsalgebra  $(\mathfrak{R}', +, -, \circ) := \Pi(\mathfrak{P}, \mathfrak{R})$  die zu  $\mathbb{H}$  gehörige Untergeometrie bis auf Isomorphie entspricht im Sinne der in Satz 3 behandelten Zuordnung. Das Lemma liefert nun die Einbettbarkeit von  $(\mathfrak{P}, \mathfrak{R})$  und  $\Pi(\mathfrak{P}, \mathfrak{R}) = (\mathfrak{R}', +, -, \circ)$  in eine umfassende projektive Richtungsalgebra  $(\hat{\mathfrak{R}}, +, -, \circ)$ , welche bis auf Isomorphie die vorgegebene Geometrie  $(\mathbb{P}, \mathbb{G}, I, T)$  im Sinne des Verfahrens von Satz 3 induziert.

Unser Satz 4 schließt die Kennzeichnung der angeordneten projektiven Ebenen mit ein, liefert also eine *Algebraisierung auch der nicht notwendig desarguesschen angeordneten projektiven Ebenen*. Diese Anwendung von Satz 4 wird interessanter durch das folgende

**Korollar.** *Jede angeordnete projektive Ebene läßt sich (im Sinne des Verfahrens nach Satz 3) darstellen als projektive Richtungsalgebra der Dimension 3. Dabei gilt*

$$\dim \mathfrak{R} = 3 \nless \text{def}$$

Es gibt  $a_1, a_2, a_3 \in \mathfrak{R}$  mit  $\mathfrak{R} = \sum_{i=1}^3 (a_i + \bar{a}_i)$  und  $\{a_1, a_2, a_3\}$  unabhängig. Die Unabhängigkeit von  $\{a_1, a_2, a_3\}$  wird dabei in  $\mathfrak{R}$  wie folgt definiert:

$$\left. \begin{array}{l} \mathfrak{o} \in a'_1 + a'_2 + a'_3 \\ \wedge a'_i \in a_i + \bar{a}_i \end{array} \right\} \succ a'_i = \mathfrak{o} \quad \text{für } i = 1, 2, 3.$$

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# Andries Brouwer und Hanfried Lenz

## Unterräume von Blockplänen

### 1. Einleitung

Doyen und Wilson [2] haben folgendes bewiesen: Sind  $u, v$  Ordnungen von Steiner-Tripelsystemen (also  $u, v \equiv 1$  oder  $3 \bmod 6$ ) und ist  $v > 2u$ , so existieren Steiner-Tripelsysteme  $U, V$  mit  $u$  bzw.  $v$  Punkten, so daß  $U$  ein Unterraum von  $V$  ist. Die entsprechende Frage für Blockpläne mit  $k = 4$  Punkten auf jeder Geraden (und—wie immer in dieser Arbeit— $\lambda = 1$ ) scheint schwierig zu sein.  $B_u(4)$  bedeute die Menge der  $v \in \mathbb{N}$ , so daß ein  $B_u[4;v]$  existiert, d.h. ein Blockplan mit insgesamt  $v$  Punkten, einem Unterraum aus  $u$  Punkten und genau 4 Punkten auf jeder Geraden. Nach Hanani ist  $B_1(4) = B(4) = (12\mathbb{N}_0 + 1) \cup (12\mathbb{N}_0 + 4)$ , vgl. [4]. Ziel dieser Arbeit ist ein Teilresultat in Richtung auf die Vermutung

$$(\text{?}) \quad v \in B_u(4), \text{ falls } u, v \in B(4) \text{ und } v > 3u.$$

Allgemeiner bedeute  $B_u(K)$  die Menge der  $v \in \mathbb{N}$ , so daß ein  $B_u[K;v]$  existiert, d.h. ein linearer Raum (= pairwise balanced design = PBD) mit Geradenlängen aus  $K \subseteq \mathbb{N}$  und insgesamt  $v$  Punkten, mit einem Unterraum aus  $u$  Punkten. Eine *Punktklassenebene* (= group divisible design = GDD) ist eine Inzidenzstruktur, deren Punkte in Klassen (meist leider Gruppen genannt) eingeteilt sind, so daß Punkte derselben Klasse (Gruppe) unverbunden sind und je zwei Punkte verschiedener Gruppen eindeutig durch eine Gerade verbunden sind. Ein  $GD[K, L; v]$  ist ein GDD mit  $v$  Punkten, Geradenlängen aus  $K$  und Gruppengrößen aus  $L \subseteq \mathbb{N}$ . Mit  $GD(K, L)$  bezeichnen wir die Menge der  $v \in \mathbb{N}$ , für welche ein  $GD[K, L; v]$  existiert. Wenn eine Gruppe aus  $u$  Punkten vorkommt, schreiben wir  $GD_u[K, L; v]$  bzw.  $GD_u(K, L)$ . Eine *Dualnetz* (transversal design = TD)  $T[s; t]$  ist ein  $GD[s; t; st]$  mit  $s > 2$ . Jede Gerade schneidet dann jede Gruppe eindeutig. Ein  $T[s; t]$  existiert bekanntlich, wenn und nur wenn die maximale Anzahl  $N(t)$  der orthogonalen lateinischen Quadrate der Ordnung  $t$  mindestens  $s - 2$  ist.

Bisher konnten wir folgende Teilresultate der Vermutung (?) erzielen.

I. (?) ist richtig für  $u = 4$  (trivial), 13, 16, 25. Für  $u = 28$  gibt es höchstens die Ausnahmewerte  $v = 157, 169, 193$ .

II. Für  $u \geq 37$  gelten folgende groben Abschätzungen. Es sei  $(u - 1)/12$  bzw.  $(u - 4)/12 \equiv h \bmod 4$  und  $h \in \{0, 1, 2, 3\}$ , je nachdem ob  $u$  ungerade oder gerade ist. Dann gilt mit der Abkürzung

$$[X]_n^\infty := \{x \in X : n \leq x\} \text{ für } X \subseteq \mathbb{N}.$$

$$(1.1) \quad [B(4)]_{13u+36h-12}^\infty \subseteq B_u(4) \quad \text{für ungerade } u \in B(4),$$

$$(1.2) \quad [B(4)]_{3u+36h-39}^\infty \subseteq B_u(4) \quad \text{für gerade } u \in B(4).$$

III. Für gerade Differenzen  $v - u$  kann mehr gesagt werden, nämlich

$$(1.3) \quad [12\mathbb{N} + 1]_{48s+1}^\infty \subseteq B_{12s+1}(4),$$

$$(1.4) \quad [12\mathbb{N} + 4]_{36s+16}^\infty \subseteq B_{12s+4}(4).$$

Die Vermutung (?) ist also richtig unter der Zusatzvoraussetzung, daß  $u$  und  $v$  beide gerade sind.

In dieser Arbeit sollen die Teilresultate (1.3) und (1.4) für  $s > 6$  bewiesen werden. Mehr ist aus Platzgründen nicht möglich.

## 2. Das Hauptlemma mit Anwendungen

Das folgende Hauptlemma ist im wesentlichen Hanani zu verdanken. Es ist ein Sonderfall der allgemeinen rekursiven Konstruktion von Wilson in seiner Theorie der PBD [8].

*Definition 2.1.* Es sei  $K \subseteq \mathbb{N}$ ,  $m \in \mathbb{N}$ . Dann bedeute  $R_K^m$  die Menge der  $x \in \mathbb{N}$  mit

$$(2.1) \quad mx \in GD(K,m) := GD(K,\{m\}).$$

Beispiel:  $R_k^{k-1} = R_{\{k\}}^{k-1} =: R_k$  ist die Menge der  $x \in \mathbb{N}$  mit  $(k-1)x \in GD(k,k-1)$ , das heißt

$$(2.2) \quad (k-1)x + 1 \in B(k).$$

Nach Hanani ist

$$(2.3) \quad R_3 = (3\mathbb{N}_0 + 1) \cup 3\mathbb{N}, \quad R_4 = (4\mathbb{N}_0 + 1) \cup 4\mathbb{N}, \quad R_5 = (5\mathbb{N}_0 + 1) \cup 5\mathbb{N}.$$

*Hauptlemma.* Für beliebige  $K, L \subseteq \mathbb{N}$  und  $m, s \in \mathbb{N}$  gilt

$$(2.4) \quad m \text{ } GD_s(R_K^m, L) \subseteq GD_{ms}(K, mL).$$

Beweis: Wir benutzen die Methode der Hilfsmatrizen (s. etwa Jungnickel [6] in diesem Band).  $A = (a_{ik})$  sei die  $v \times b$ -Inzidenzmatrix eines  $GD_s[R_K^m, L; v]$ . Wir wählen eine Spalte mit  $x \in R_K^m$  Einsen und  $v - x$  Nullen. Nach Voraussetzung gibt es ein  $GD[K, m; mx]$  mit einer Inzidenzmatrix

$$\begin{array}{|c|c|c|c|} \hline & B_1 & B_2 & \dots \\ \hline & B_2 & & \\ \hline & \dots & & \\ \hline & B_x & & \\ \hline \end{array},$$

dabei sind die  $B_i$  ( $i = 1, \dots, x$ )  $m \times y$ -Matrizen, so daß  $B_i B_j^T$  für  $i \neq j$  aus lauter Einsen besteht und für  $i = j$  eine Diagonalmatrix ist.

Nun ersetzen wir die Einsen aus der gegebenen Spalte von  $A$  durch die Hilfsmatrizen  $B_1, \dots, B_x$  in irgendeiner Reihenfolge, und die Nullen durch  $m \times y$ -Nullmatrizen. Wenn wir das für alle Spalten von  $A$  tun, so erhalten wir die Inzidenzmatrix eines  $GD_{ms}[K, mL; mv]$ , q.e.d.

*Folgerung 2.2* Die Menge  $R_K^m$  ist abgeschlossen im Sinne von Wilson, d.h.  $B(R_K^m) = R_K^m$ .

Beweis: Für  $L = \{1\}$ ,  $s = 1$  wird aus (2.4)  $mGD(R_K^m, 1) = mB(R_K^m) \subseteq GD(K, m)$ . Nach der Definition von  $R_K^m$  folgt  $B(R_K^m) \subseteq R_K^m$ , d.h. die Behauptung. Als wohlbekannt anzusehen ist

*Hilfssatz 2.3.* Aus  $v \in B_u(R_4)$  folgt  $3v + 1 \in B_{3u+1}(4)$ .

Beweis: Wenn ein  $B[R_4; v]$  mit einem Unterraum  $U$  aus  $u$  Punkten existiert, so dürfen wir o.B.d.A. annehmen, daß  $U$  eine Gerade ist, also

$$v \in GD_u(R_4, \{1, u\}).$$

Nach dem Hauptlemma für  $m=3$  folgt  $3v \in GD_{3u}(4, \{3, 3u\})$ . Einführung eines uneigentlichen Punktes auf den Gruppen liefert

$$3v + 1 \in B_{3u+1}(\{4, 3u+1\}) \subseteq B_{3u+1}(4),$$

wie behauptet.

#### *Definition 2.4*

$$(2.5) \quad M := R_K^4 \quad \text{für } K = R_4.$$

#### *Hilfssatz 2.5*

$$(2.6) \quad 4 \cdot GD_s(M, \mathbb{N}) \subseteq GD_{4s}(R_4, 4 \cdot \mathbb{N}) \subseteq B_{4s}(R_4),$$

$$(2.7) \quad 4 \cdot GD_s(M, \mathbb{N}) + 1 \subseteq B_{4s+1}(R_4).$$

Beweis: (2.6) folgt nach dem Hauptlemma und (2.3). Einführung eines uneigentlichen Punktes auf den Gruppen liefert (2.7). Wegen Hilfssatz 2.3 gilt

#### *Folgerung 2.6*

$$(2.8) \quad 12 \cdot GD_s(M, \mathbb{N}) + 1 \subseteq B_{12s+1}(4),$$

$$(2.9) \quad 12 \cdot GD_s(M, \mathbb{N}) + 4 \subseteq B_{12s+4}(4).$$

#### *Hilfssatz 2.7*

$$(2.10) \quad M = R_{R_4}^4 = \mathbb{N} \setminus \{2, 3\}.$$

Beweis: Offenbar ist  $2, 3 \notin M$ . Der weitere Beweis erfordert mehrere Schritte.

Behauptung 1:  $R_5 = (5\mathbb{N}_0 + 1) \cup 5 \cdot \mathbb{N} \subseteq M$ .

Denn nach Definition ist  $4R_5 = GD(5, 4) \subseteq GD(R_4, 4)$ .

Behauptung 2:  $3\mathbb{N} + 1 \subseteq M$ .

Beweis: Nach Hanani, Ray-Chaudhuri und Wilson [5] existiert für jedes  $n \in \mathbb{N}$  ein auflösbarer Blockplan  $B^*[4; 12n + 4]$ , also ein  $GD[4, 4; 12n + 4]$ . Also ist

$$4(3n + 1) \in GD(4, 4) \subseteq GD(R_4, 4), \quad \text{w.z.b.w.}$$

Behauptung 3:  $R_4 \subseteq M$ .

Beweis: Es sei  $1 < t \in R_4$ . Dann ist  $N(t) > 2$  (s. etwa van Lint [7]), und es gibt ein auflösbares  $T[4; t]$ . Wir wählen die Geraden einer Parallelklasse als neue Gruppen und erhalten

$$4t \in GD(\{4, t\}, 4) \subseteq GD(R_4, 4), \quad \text{also } t \in M.$$

Behauptung 4:  $4\mathbb{N} + 2 \subseteq M$ .

Beweis: Für  $t \in 4\mathbb{N} + 1$  gibt es ein auflösbares  $T[4; t]$  mit Parallelklassen  $\mathfrak{P}_1, \dots, \mathfrak{P}_4$  von 4-Geraden. Wir führen 4 uneigentliche Punkte  $\infty_1, \infty_2, \infty_3, \infty_4$  auf  $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4$  ein und verbinden sie durch eine uneigentliche Gerade  $U$ . So entsteht ein  $B[\{4, 5, t\}; 16n + 8]$ , in dem  $U$  mit den Geraden aus  $\mathfrak{P}_5$  eine Parallelklasse bildet, deren Geraden als neue Gruppen gewählt werden können. Es folgt

$$4(t + 2) \in GD(\{4, 5, t\}, 4) \subseteq GD(R_4, 4), \quad \text{also } 4n + 2 \in M.$$

Behauptung 5:  $3\mathbb{N} + 2 \subseteq M$ .

Beweis: Es sei  $n \in \mathbb{N}$ . In einem auflösbaren  $B^*[4;12n+4]$  mit Parallelklassen  $\mathfrak{P}_1, \dots, \mathfrak{P}_{4n+1}$  führen wir uneigentliche Punkte  $\infty_1, \infty_2, \infty_3, \infty_4$  auf den Parallelklassen  $\mathfrak{P}_i$  ( $i=1,2,3,4$ ) ein und verbinden sie durch eine uneigentliche Gerade U. Die Geraden aus  $\mathfrak{P}_5$  bilden mit U die Gruppen eines  $GD[\{4,5\}; 4(3n+2)]$ . Also ist  $3n+2 \in M$ .

Behauptung 6:  $[3 \mathbb{N}]_{21}^\infty \subseteq M$ .

Beweis: Es sei  $n > 4$ . In einem  $B^*[4;12n+4]$  mit Parallelklassen  $\mathfrak{P}_1, \dots, \mathfrak{P}_{4n+1}$  führen wir 20 uneigentliche Punkte  $\infty_i$  auf  $\mathfrak{P}_i$  ein ( $i=1, \dots, 20$ ). Auf der Menge dieser uneigentlichen Punkte bilden wir ein  $T[5;4]$ , d.h. ein  $GD[5,4;20]$ . Die Gruppen dieses TD, zusammen mit den Geraden aus  $\mathfrak{P}_{21}$ , bilden die neuen Gruppen eines  $GD[\{4,5\}; 12n+24]$ , also ist  $4(3n+6) \in GD(R_4, 4)$ ,  $3n+6 \in M$ , w.z.b.w.

Nun sei, entgegen der Behauptung des Hilfssatzes 2.7,  $2,3 \neq x \notin M$ . Nach den Behauptungen 2,5 und 6 ist  $x < 21$ , und nach Behauptungen 3 und 4 ist  $x \equiv 3 \pmod{4}$ , also  $x \equiv 3 \pmod{12}$ , und daher  $x = 15$ . Nach Behauptung 1 und (2.3) ist doch  $x \in M$ , womit Hilfssatz 2.7 bewiesen ist.

Wir untersuchen nun die nach Folgerung 2.6 für unser Problem offenbar wichtigen Mengen  $GD_s(M, \mathbb{N})$ . Wir bilden zwei monotone Folgen natürlicher Zahlen  $(a_n), (s_n)$  wie folgt. Es sei  $a_1, a_2, \dots, a_n, \dots = 5, 7, 8, 9, 11, 12, 13, 16, 17, 19, 21, 23, 25, 27, 29, 31, 32, 35, 37, 40, 41, 43, 45, 47, 49, 50, 53, 54, 55, \dots$  und  $a_{n+1} = a_n + 1$  für  $a_n > 52$ . Dann ist

$$0 < a_{n+1} - a_n \leq 3 \quad \text{und} \quad N(a_n) > 3 \quad \text{für alle } n \in \mathbb{N}.$$

Nun setzen wir  $s_0 := s$ .  $s_1$  sei das kleinste  $a_n > s$ , ferner

$$s_i = a_m \Rightarrow s_{i+1} = a_{m+1} \quad \text{für alle } i \in \mathbb{N}.$$

*Hilfssatz 2.8*

$$(2.11) \quad GD_s(M, \mathbb{N}) \supseteq [\mathbb{N}]_{4s_1+s}^\infty \quad \text{für } s > 4,$$

$$(2.12) \quad GD_s(M, \mathbb{N}) \supseteq [\mathbb{N}]_{4s}^\infty \quad \text{falls } s > 5 \text{ und } N(s) > 3.$$

Beweis: I. Aus einem  $T[6;s_n]$  entfernen wir  $s_n - s$  bzw.  $s_n - w$  Punkte zweier Gruppen und erhalten ein  $GD_s[\{4,5,6\}; \{s_n, s, w\}; 4s_n + s + w]$  (im Fall  $w=0$  ist der Buchstabe w wegzulassen). Es folgt für  $w=0, \dots, s_n$ .

$$(2.13) \quad [\mathbb{N}]_{4s_n+s}^{5s_n+s} \subseteq GD_s(M, \mathbb{N}) \quad \text{für alle } n \in \mathbb{N}.$$

Im Fall  $s > 4$  ist  $s_1 \geq 7$  und  $5s_n + 1 \geq 4s_{n+1}$ , daher folgt (2.11) durch Induktion.

II. Nun sei  $s > 5$  und  $N(s) > 3$ . Dann gilt (2.13) schon für  $n=0$ , und es folgt durch Induktion

$$[\mathbb{N}]_{ss}^\infty \subseteq GD_s(M, \mathbb{N}).$$

Für  $w \in \{0, \dots, s\}$  entfernen wir  $s - w$  Punkte aus einem  $T[5;s]$  und erhalten

$$(2.14) \quad 4s + w \in GD_s(\{4,5\}, \{s, w\}) \subseteq GD_s(M, \mathbb{N}),$$

also gilt (2.12). Insbesondere gilt (2.12) für alle  $s > 52$ .

*Hilfssatz 2.9.* (2.12) gilt für alle  $s > 6$ .

Der Beweis erfordert einige Schritte.

**Behauptung 1:** Aus  $N(s) > 2$  folgt  $[N]_{4s}^{5s} \subseteq GD_s(M, \mathbb{N})$ , aus  $N(s+1) > 2$  folgt  $[N]_{4s+3}^{5s+4} \subseteq GD_s(M, \mathbb{N})$ .

Der erste Teil der Behauptung folgt aus (2.14). Im Fall  $N(s+1) > 2$  entfernen wir aus einem  $T[5; s+1]$  einen Punkt  $p$  einer Gruppe  $G$  und  $s+1-u$  Punkte einer anderen Gruppe  $H$ . Es entsteht ein GDD, das als PBD aufgefaßt werden kann. Als neue Gruppen wählen wir die 3-Geraden  $L_1, \dots, L_m$ , die Menge  $G' := G \setminus \{p\}$  und die Menge  $H'$  der  $u$  übrigbleibenden Punkte von  $H$  (falls  $H'$  nicht leer ist) sowie die einpunktigen Mengen  $\{x\}$  mit  $x \notin G' \cup H' \cup L_1 \cup \dots \cup L_m$ . Das geht wegen

$$G' \cap L_i = G' \cap H' = L_i \cap L_j = L_i \cap H' = \emptyset \quad (\text{für } i \neq j).$$

Wir erhalten ein  $GD_s[M, \mathbb{N}, 4s+u+3]$ , also für  $0 \leq u \leq s+1$

$$[N]_{4s+3}^{5s+4} \subseteq GD_s(M, \mathbb{N}),$$

wie behauptet.

**Behauptung 2:** Aus  $N(s+1) > 3$  und  $N(s) > 2$  folgt (2.12).

Beweis: Dann ist  $s_1 = s+1$  (notfalls füge man  $s+1$  in die Folge  $(a_n)$  ein). Nach Behauptung 1 ist  $[N]_{4s}^{5s+4} \subseteq GD_s(M, \mathbb{N})$  ferner  $5s+4 = 4s_1 + s$ . also gilt die Behauptung nach (2.11).

**Behauptung 3:** Aus  $s > 6$  und  $N(s-1) > 3$  folgt (2.12).

Beweis: Wenn auch  $(N(s) > 3$  ist, sind wir nach Hilfssatz 2.8 fertig. Wir dürfen daher annehmen, daß  $s \neq a_n$  für alle  $n \in \mathbb{N}$  und  $s < s_1 \leq s+2$  gilt. Durch Entfernung von Punkten zweier Gruppen aus einem  $T[6; s-1]$  erhalten wir

$$(2.15) \quad 4(s-1) + u + w \in GD_{s-1}(\{4, 5, 6\}; \{s-1, u, w\}) \quad \text{für } 2 < u, w < s,$$

$$(2.16) \quad 4(s-1) + u \in GD_{s-1}(\{4, 5\}, \{s-1, u\}) \quad \text{für } 2 < u < s.$$

Einführung eines uneigentlichen Punktes auf den Gruppen liefert mit Rücksicht auf Hilfssatz 2.7

$$(2.17) \quad [N]_{4s}^{6s-5} \subseteq B_s(M) \subseteq GD_s(M, \mathbb{N}).$$

Wegen  $s > 6$  und  $s \neq a_n$  für alle  $n \in \mathbb{N}$  ist entweder  $s = 10$  und  $s < s_1 = s+1$  oder  $s \geq 14$  und  $s < s_1 \leq s+2$ , also  $4s_1 + s \leq 6s - 4$ , und (2.12) folgt aus (2.11) und (2.17). Nun nehmen wir an, es sei  $s > 6$  und (2.12) gelte nicht für dieses  $s$ . Nach den bisherigen Behauptungen und bekannten Resultaten über  $N(x)$  (van Lint [7]) bleibt dann nur  $s = 34$  übrig.\* Denn Behauptung 2 erledigt die Werte  $s = 15, 39$  und  $52$ , und Behauptung 3 die Werte  $10, 14, 18, 20, 22, 24, 26, 28, 30, 33, 36, 38, 42, 44, 46, 48, 51$ . Die Werte  $s$  aus der Folge  $(a_n)$  waren schon durch Hilfssatz 2.8 erledigt. Für  $s = 34$  ist nach (2.11) und Teil 2 der Behauptung 1  $[N]_{139}^{\infty} \subseteq GD_{34}(M, \mathbb{N})$ , ferner  $N(33) > 2$ , also nach (2.16) für  $s = 34$

$$4 \cdot 33 + u \in GD_{33}(M, M-1) \quad \text{für } 2 < u < 34.$$

Einführung eines uneigentlichen Punktes liefert für diese  $u$

$$\begin{aligned} 4 \cdot 33 + u + 1 &\in B_{34}(M) \subseteq GD_{34}(M, \mathbb{N}), \\ [N]_{136}^{166} &\subseteq GD_{34}(M, \mathbb{N}). \end{aligned}$$

\* Zusatz bei der Korrektur: Nach Wang und Wilson ist  $N(34) \geq 3$ , daher erledigt sich auch der Fall  $s = 34$  nach Behauptung 2.

Das beweist Hilfssatz 2.9.

**Satz I.** Für  $s > 6$  gilt

$$\begin{aligned}[12\mathbb{N}+1]_{48s+1}^{\times} &\subseteq B_{12s+1}(4), \\ [12\mathbb{N}+4]_{48s+4}^{\times} &\subseteq B_{12s+4}(4).\end{aligned}$$

Das folgt unmittelbar aus Hilfssatz 2.9 und Folgerung 2.6.

### 3. Große Unterräume

Wir beginnen mit einem auflösbaren  $B^*[4;12n+4]$  mit Parallelklassen  $\mathfrak{P}_1, \dots, \mathfrak{P}_{4n+1}$ . Für die 4-Geraden aus  $\mathfrak{P}_1 \cup \dots \cup \mathfrak{P}_h$  mit  $4n+1 \geq h \in R_4 \cup \{0\}$  benutzen wir die Hilfsmatrizen

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Für die übrigen 4-Geraden benutzen wir die Hilfsmatrizen

$$B_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} \text{ ist die Inzidenzmatrix eines } T[4;3] \text{ und } A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

ist die Inzidenzmatrix eines  $GD[\{3,4\}, 3; 12]$ , das aus einem  $B[4; 16]$  durch Entfernung einer Geraden mit ihren Punkten entsteht, und das 3 Parallelklassen zu je vier 3-Geraden hat.

Die Erweiterungskonstruktion mittels Hilfsmatrizen liefert die Inzidenzmatrix eines  $GD[\{3,4\}, 3; 36n + 12]$  mit  $3h$  Parallelklassen von 3-Geraden und einer Parallelklasse von Gruppen (zu je 3 Punkten). Die Einsen in den linken oberen Ecken der Hilfsmatrizen, zusammen mit den Nullen in den linken oberen Ecken der eingesetzten Nullmatrizen, bestimmen die Inzidenzmatrix eines Unterraumes  $U$ , der zum gegebenen  $B^*[4; 12n + 4]$  isomorph ist. Nun führen wir auf den  $3h + 1$  Parallelklassen von 3-Geraden (einschl. Gruppen) uneigentliche Punkte ein und erhalten ein  $B[4; 36n + 12 + 3h + 1]$ , das  $U$  als Unterraum hat. Es folgt

$$(3.1) \quad 36n + 3h + 13 \in B_{12n+4}(4) \quad \text{für } 4n + 1 \geq h \in R_4 \cup \{0\}, \text{ also}$$

**Satz II.** Für gerade  $u \in B(4)$  gilt

$$(3.2) \quad [B(4)]_{3u+1}^{4u} \subseteq B_u(4).$$

Zusammen mit Satz I ergibt sich

**Satz III.** Für gerade  $u \in B(4)$  mit  $u \geq 88$  ist

$$(3.3) \quad [12 \mathbb{N} + 4]_{3u+4}^{\infty} \subseteq B_u(4).$$

*Schlußbemerkung.* Satz III gilt auch für  $u = 4$  (trivial), 16, 28, 40, 52, 64, 76. Satz I, ebenso wie die Hilfsformel (2.12), gilt für alle  $s \in \mathbb{N}$ . Um Platz zu sparen, verzichten wir hier auf den Beweis.

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## Die Methode der Hilfsmatrizen<sup>1</sup>

### Abstract

We give a survey on the “method of auxiliary matrices” which constructs from the incidence matrix of a given structure an incidence matrix for a new larger structure by replacing all 0’s resp. 1’s in the old matrix by suitable zero matrices resp. “auxiliary matrices”. The method is used in the construction of designs, partial designs, Hjelmslev planes, Klingenberg structures, etc.

Wir wollen in dieser Arbeit einen Überblick über eine Methode geben, die sich bei der rekursiven Konstruktion endlicher geometrischer Strukturen als sehr nützlich erwiesen hat: die «Methode der Hilfsmatrizen». Sie wird u.a. zur Konstruktion von Blockplänen, Partialblockplänen und Hjelmslev-Ebenen benutzt.

Wir benutzen die Bezeichnungen von Dembowski [2]; insbesondere bezeichnen kleine lateinische Buchstaben Punkte, große hingegen Geraden. (Die Terme «Block» und «Gerade» werden hier, entgegen [2], synonym benutzt.)

Gegeben sei eine Inzidenzstruktur  $\pi'$  mit  $v'$  Punkten und  $b'$  Geraden;  $A' = (a'_{ik})$  ( $i = 1, \dots, v'$ ;  $k = 1, \dots, b'$ ) sei eine Inzidenzmatrix für  $\pi'$ . Wir ersetzen nun die Einträge  $a'_{ik}$  durch 0 – 1-Matrizen  $A_{ik}$  geeigneter Größe (für festes  $k$  müssen z.B. alle  $A_{ik}$  die gleiche Spaltenzahl besitzen); hierbei sei  $A_{ik} \neq 0$ , wenn und nur wenn  $a'_{ik} \neq 0$  ist. Die so erhaltene Matrix  $A$  definiert dann eine neue Inzidenzstruktur  $\pi$  und die Abbildung  $\phi: \pi \rightarrow \pi'$ , die sich aus unserer Konstruktion ergibt (z.B. wird also jedem Punkt  $p$  von  $\pi$ , der zu der «Blockzeile»  $A_{i1} \cdots A_{iv'}$  gehört, der Punkt  $p'$  von  $\pi'$  zugeordnet, der zu der Zeile  $a'_{i1} \cdots a'_{iv'}$  gehört), ist ein Epimorphismus, für den jede Fahne von  $\pi'$  mindestens eine Fahne von  $\pi$  als Urbild hat. Man nennt dann auch Punkte (bzw. Geraden) von  $\pi$  mit gleichen Bild unter  $\phi$  *benachbart*. Die Nachbarklasse von  $p$  bzw.  $G$  sei mit  $p'$  bzw.  $G'$  bezeichnet. Die auf den Nachbarklassen von  $\pi$  durch

$$(1) \quad p' \perp G' : \Leftrightarrow \exists q, H : p' = q' \wedge G' = H' \wedge q \perp H$$

definierte Inzidenzstruktur ist dann isomorph zu  $\pi'$ . Umgekehrt lässt sich jedes Tripel  $(\phi, \pi, \pi')$ , wo  $\phi$  ein Epimorphismus mit der oben beschriebenen Zusatzeigenschaft ist, so erhalten, wenn man die Punkte bzw. Geraden von  $\pi$  in Nachbarklassen zusammengefaßt numeriert. Die  $A_{ik} \neq 0$  werden *Hilfsmatrizen* genannt.

Diese Methode geht (in einem Spezialfall) auf Drake und Lenz [7] zurück. Natürlich kann man nur dann interessante Ergebnisse erwarten, wenn man an  $\pi'$  und an die Hilfsmatrizen Zusatzforderungen stellt. Bisher wurden im wesentlichen 2 Klassen von Hilfsmatrizen benutzt, die beide mit Netzen zusammenhängen. In beiden Fällen ist die Bedingung

$$(2) \quad A_{ik} A_{jk}^T = cJ \quad \text{für } A_{ik}, A_{jk} \neq 0, j \neq i$$

erfüllt. Wir wiederholen zunächst die allgemeine Definition eines Netzes.

<sup>1</sup> Ausarbeitung eines Vortrages für das Geometrie-Symposion in Siegen 1978.

*Definition 1.* Eine Inzidenzstruktur mit Parallelismus  $\Sigma = (P, B, I, ||)$  heißt ein *Netz* der *Ordnung*  $s$ , des *Grades*  $r$  und vom *Typ*  $\mu$  (kurz ein  $(s, r; \mu)$ -Netz), wenn die folgenden Axiome erfüllt sind:

- (N1) Jeder Punkt liegt auf genau einer Geraden aus jeder Parallelklasse.
- (N2) Es gibt genau  $r \geq 3$  Parallelklassen, von denen mindestens eine genau  $s$  Geraden enthält.
- (N3) Je zwei Geraden aus verschiedenen Parallelklassen schneiden sich in genau  $\mu$  Punkten.

Im Falle  $\mu = 1$  liegt ein gewöhnliches  $(s, r)$ -Netz vor (vgl. [2]).

$\Sigma$  hat dann  $v := s^2\mu$  Punkte, jede Parallelklasse enthält genau  $s$  Geraden, und jede Gerade hat genau  $k := s\mu$  Punkte. Ferner gilt  $r \leq (s^2\mu - 1)/(s - 1)$ ; Gleichheit tritt genau dann auf, wenn  $\Sigma$  ein *affiner Blockplan* ist (d.h., wenn es eine Konstante  $\lambda$  gibt, so daß je zwei Punkte durch genau  $\lambda$  Geraden verbunden sind). In diesem Fall hat  $\Sigma$  die Parameter

$$(3) \quad v = s^2\mu, k = s\mu, r = (s^2\mu - 1)/(s - 1), \lambda = (s\mu - 1)/(s - 1), b = sr.$$

(man vergleiche hierzu [6], Section 5, und [2], 2.2.8).

Aus einem gegebenen  $(s, r; \mu)$ -Netz  $\Sigma$  erhält man nun leicht Hilfsmatrizen mit (2) und weiteren angenehmen Eigenschaften: man numeriere nämlich die Punkte von  $\Sigma$  als  $p_1, \dots, p_v$  und die Parallelklassen von  $\Sigma$  als  $B_1, \dots, B_r$  und definiere Matrizen  $M_1, \dots, M_r$  mit  $M_h = (m_{ij}^h)$  durch

$$m_{ij}^h := \begin{cases} 1 & \text{falls } p_i \text{ und } p_j \text{ auf einer Geraden aus } B_h \text{ liegen} \\ 0 & \text{sonst} \end{cases}$$

Man erhält dann das folgende Resultat (siehe [6], 5.10 und 5.11):

**Satz 1** (Drake-Jungnickel). *Die Existenz eines  $(s, r; \mu)$ -Netzes  $\Sigma$  ist äquivalent zur Existenz von  $r$  symmetrischen 0–1-Matrizen  $M_1, \dots, M_r$  mit  $M_h = (m_{ij}^h)$  für  $i, j = 1, \dots, s^2\mu$ , die den folgenden Bedingungen genügen:*

- (4)  $M_i M_j^T = M_j^T M_i = \mu J$  für  $i \neq j$  ( $J$  ist die Matrix aus Einsen);
- (5)  $M_i M_i^T = s\mu M_i$ ;
- (6)  $m_{jj}^i = 1$  für alle  $i, j$ .

Folglich ist die Existenz eines affinen Blockplans mit den Parametern (3) gleichbedeutend mit der Existenz von  $r$  symmetrischen 0–1-Matrizen  $M_1, \dots, M_r$  mit  $M_h = (m_{ij}^h)$  für  $i, j = 1, \dots, s^2\mu$ , die den Bedingungen (4) bis (6) und

$$(7) \quad \sum_{i=1}^r M_i M_i^T = k(r - \lambda)I + k\lambda J = s^2\mu^2 I + s\mu(s\mu - 1)J/(s - 1)$$

genügen.

Mit Hilfe von Satz 1 kann man nun aus gegebenen affinen Blockplänen und geeigneten Bildstrukturen  $\pi'$  symmetrische Partialblockpläne und sogar symmetrische Blockpläne konstruieren. Das folgende Resultat ist implizit in [6], Section 4; wir erwähnen, daß in [5] und [6] mit Hilfe des Begriffs der «c-Klingenberg-Struktur» weitere Existenzsätze für symmetrische Partialblockpläne (auch auf mehr als 2 Klassen) bewiesen worden sind.

**Satz 2** (Drake-Jungnickel). *Es gebe einen symmetrischen Blockplan  $\pi'$  mit*

Parametern  $v'$ ,  $k'$ ,  $\lambda'$  und einen affinen Blockplan  $\Sigma$  mit Parametern (3), wobei noch  $k' = r$  sei. Dann gibt es auch einen symmetrischen teilbaren Partialblockplan (in der Terminologie von Raghavarao [20] ein «group divisible design», für das die duale Struktur ebenfalls ein group divisible design mit denselben Parametern ist)  $\pi$  mit Parametern

$$(8) \quad v^* = vv', k^* = kk', m = v', n = v, \lambda_1 = k\lambda \text{ und } \lambda_2 = \mu\lambda'.$$

(Die erwähnten Parameter sind in [20] erklärt.)

Wir geben folgende Beweis-Andeutung: Man konstruiere Hilfsmatrizen  $M_1, \dots, M_r$  aus  $\Sigma$  nach Satz 1 und setze sie in die Inzidenzmatrix  $A'$  von  $\pi'$  so ein, daß in jeder Zeile und Spalte von  $A'$  jedes  $M_i$  genau einmal benutzt wird; dies ist wegen  $r = k'$  nach dem Lemma von König (siehe etwa Ryser [21], 5.5.3) möglich. Die neue Matrix  $A$  beschreibt dann das gewünschte  $\pi$ .

Wählt man in Satz 2 für  $\pi'$  speziell den trivialen  $(r+1, r, r-1)$ -Blockplan, so erhält man ein Resultat von Wallis [23], das ursprünglich mit Hilfe von «Design-Graphen» bewiesen wurde:

**Satz 3** (Wallis). *Aus der Existenz eines affinen Blockplans mit den Parametern (3) folgt die Existenz eines symmetrischen Blockplans mit den Parametern*

$$(9) \quad v^* = (r+1)v, k^* = kr \text{ und } \lambda^* = k\lambda.$$

Man beachte hierbei, daß  $k\lambda = \mu(r-1)$  wegen (3) gilt. Dieser Alternativ-Beweis für Wallis' Resultat wurde ursprünglich in [19] von Lenz und Jungnickel gegeben, die auch zeigten, daß nur die trivialen Blockpläne (für  $\pi'$  gewählt) wieder zu Blockplänen führen.

Zur Anwendung der Sätze 2 und 3 benötigt man natürlich noch affine Blockpläne; wir erwähnen, daß alle bekannten derartigen Strukturen Parameter in einer der beiden folgenden Serien haben:

$$(10) \quad s \text{ Primzahlpotenz, } \mu = s^{n-1} \text{ für ein } n \in \mathbb{N};$$

$$(11) \quad s = 2, 4\mu \text{ Ordnung einer Hadamard-Matrix.}$$

Serie (10) wird etwa durch die affinen Räume (und Ebenen) und Serie (11) durch die Hadamard-3-Blockpläne realisiert. (Zu affinen Blockplänen vergleiche man den Übersichtsartikel von Shrikhande [22].) Mit Serie (10) erhält also z.B. aus Satz 3.

**Satz 4** (Wallis). *q sei Primzahlpotenz und n eine natürliche Zahl. Dann gibt es einen symmetrischen Blockplan mit den Parametern*

$$(12) \quad v = q^{n+1}(q^n + \dots + q^2 + q + 2), \quad k = q^n(q^n + \dots + q + 1) \text{ und} \\ \lambda = q^n(q^{n-1} + \dots + q + 1).$$

Wir erwähnen, daß die Methode der Hilfsmatrizen sogar die Existenz einer Differenzenmenge mit den Parametern (12) liefert (siehe Lenz und Jungnickel [19], Theorem 2).

Ein Beispiel für die Anwendung von Satz 2:  $\pi'$  sei der  $(15, 7, 3)$ -Blockplan und  $s = \mu = 2$  (nach (11) möglich). Dann erhält man einen symmetrischen Partialblockplan  $\pi$  mit  $v^* = 120$ ,  $k^* = 28$ ,  $\lambda_1 = 12$  und  $\lambda_2 = 6$ .

In [13] wurde mit ähnlichen Überlegungen (und einer Verallgemeinerung des Lemmas von König) folgende Verallgemeinerung von Satz 2 bewiesen:

**Satz 5 (Jungnickel).** Es gebe einen affinen Blockplan  $\Sigma$  mit Parametern (3) und einen Blockplan  $\pi'$  mit Parametern  $v', b', r', k', \lambda'$ , wobei noch  $r' = tr$  (für eine geeignete natürliche Zahl  $t$ ) und  $k' \leq r$  sei. Dann gibt es auch einen teilbaren Partialblockplan  $\pi$  mit Parametern

$$(13) \quad v^* = vv', b^* = vb', k^* = kk', r^* = kr', m = v', n = v, \lambda_1 = tk\lambda \text{ und } \lambda_2 = \mu\lambda'.$$

Für Anwendungen von Satz 5 verweisen wir auf [13]. Soviel zunächst über Blockpläne und Partialblockpläne. Wir wenden uns jetzt Klingenberg- und Hjelmslev-Ebenen zu. Zur Wiederholung dient

*Definition 2.*  $\pi = (P, B, I, \sim)$  sei eine Inzidenzstruktur, wobei  $\sim$  eine Äquivalenzrelation auf  $P \cup B$  ist (die «Nachbarrelation»; die Klasse von  $p$  bzw.  $G$  sei mit  $p'$  bzw.  $G'$  bezeichnet).  $\pi$  heißt eine *projektive Klingenberg-Ebene* (kurz *K-Ebene*), wenn die folgenden Axiome erfüllt sind:

$$(K1) \quad p \not\sim q \Rightarrow [p, q] = 1 \quad \text{für alle } p, q \in P;$$

$$(K2) \quad G \not\sim H \Rightarrow [G, H] = 1 \quad \text{für alle } G, H \in B.$$

(Hierbei ist  $[p, q]$  die Anzahl der Verbindungsgeraden von  $p$  und  $q$ , und dual für  $[G, H]$ .)

(K3) Definiert man  $p' \perp G'$  wie in (1), so bilden die Nachbarklassen eine gewöhnliche projektive Ebene  $\pi'$ .

$\pi$  heißt eine *projektive Hjelmslev-Ebene* (kurz *H-Ebene*), wenn außerdem noch gelten:

$$(H1) \quad p \sim q \wedge p \neq q \Rightarrow [p, q] \geq 2 \quad \text{für alle } p, q \in P;$$

$$(H2) \quad G \sim H \wedge G \neq H \Rightarrow [G, H] \geq 2 \quad \text{für alle } G, H \in B.$$

Weiterhin benötigen wir aus [14] (für *H-Ebenen*) bzw. [7] das

**Lemma 1 (Kleinfeld).**  $\pi$  sei eine endliche K-Ebene. Dann gibt es natürliche Zahlen  $t$  und  $r$  (die Parameter von  $\pi$ ), so daß gilt:

- (i)  $|p'| = |G'| = t^2$  für alle  $p \in P$  und alle  $G \in B$ ;
- (ii)  $|\{q : p \sim q \mid G\}| = |\{H : p \mid H \sim G\}| = t$  für alle  $(p, G) \in I$ ;
- (iii)  $r$  ist die Ordnung von  $\pi'$ ;
- (iv)  $t \neq 1 \Rightarrow r \leq t$  dann heißt  $\pi$  echt).

Es gilt dann

**Satz 6 (Drake-Lenz).** Eine  $(t, r)$ -K-Ebene existiert genau dann, wenn es eine projektive Ebene der Ordnung  $r$  und ein  $(t, r + 1; 1)$ -Netz gibt.

Für den Beweis verweisen wir auf [7], 3.1; hier sei nur die Konstruktionsmethode skizziert. Man konstruiere aus dem gegebenen  $(t, r + 1; 1)$ -Netz Hilfsmatrizen  $M_0, \dots, M_r$  nach Satz 1 und setze sie so in die

Inzidenzmatrix der projektiven Ebene ein, daß jedes  $M_i$  in jeder Zeile und Spalte genau einmal benutzt wird.

Wann erhält man nun so sogar eine Hjelmslev-Ebene? Offenbar ist dafür notwendig und hinreichend

$$(14) \quad \sum_{i=0}^r M_i M_i^T \geq 2J \text{ und } \sum_{i=0}^r M_i^T M_i \geq 2J.$$

(Hierbei sei  $\geq$  für Matrizen komponentenweise erklärt.)

Als unmittelbare Folgerungen erhält man erstens einen sehr einfachen Beweis für das folgende Ergebnis von [1], indem man Satz 1 für eine affine Ebene der Ordnung  $r$  anwendet:

**Satz 7 (Craig).** *Die Existenz einer  $(r,r)$ -H-Ebene ist äquivalent zur Existenz einer projektiven Ebene der Ordnung  $r$ .*

Und zweitens erhält man die erste Konstruktionsmethode für H-Ebenen, für die  $t$  keine Potenz von  $r$  ist (was lange ein offenes Problem war; die immer noch offene Frage ist, ob  $t$  ein Vielfaches von  $r$  sein muß):

**Satz 8 (Drake-Lenz).** *Es gebe eine  $(t,r)$ -H-Ebene und  $q$  sei Ordnung einer projektiven Ebene, wobei  $q := t(r+1) - 1$  ist. Dann gibt es eine  $(tq,r)$ -H-Ebene.*

Für den Beweis sei auf [7], Theorem 4.1, verwiesen. Als Beispiel erhält man etwa für  $t=r=2$ ,  $q=5$  die Existenz einer  $(10,2)$ -H-Ebene. Es lässt sich aber noch viel mehr erreichen; auf algebraische Weise erhält man (aus «desarguesschen» H-Ebenen) für jede Primzahlpotenz  $q$  eine Menge von  $q+1$   $(q^{2n} \times q^{2n})$ -Hilfsmatrizen, die (4) und (14) genügen (mit  $\mu=1$ ; diese Matrizen sind im allgemeinen nicht mehr symmetrisch). Damit kann man in Satz 8  $(tq^n,r)$ -H-Ebenen erhalten, sobald  $q$  Primzahlpotenz ist ([7], Corollary 5.1). Mit einer rekursiven Konstruktion von Hilfsmatrizen gilt sogar (siehe [7], §6):

**Satz 9 (Drake-Lenz).**  *$r,q$  seien Primzahlpotenzen mit  $2(r+1) \leq q+1 \leq r(r+1)$ . Dann gibt es eine  $(q^n r, r)$ -H-Ebene.*

Zum Beispiel folgt die Existenz von  $(3 \cdot 7^n, 3)$ - und  $(3 \cdot 8^n, 3)$ -H-Ebenen für jedes  $n$ . Die Methode der Hilfsmatrizen war in mehreren anderen Arbeiten über H-Ebenen von Bedeutung, u.a. auch beim Beweis des Drake-Törner-Theorems (vgl. [3], [4] und [8]). Sie ist auch bei der Untersuchung von Verallgemeinerungen (den «(b,c)-K-Strukturen») von großer Bedeutung. Wir verweisen hier nur auf [5], [6] und [9].

Wir wollen die Diskussion der K- und H-Ebenen mit einem letzten Beispiel für die Stärke der Methode der Hilfsmatrizen beschließen. Die von Jungnickel eingeführten «K-Matrizen» und «H-Matrizen» (siehe Definition 3 unten) haben sich als sehr hilfreich bei der Konstruktion von K- und H-Ebenen mit einer regulären Kollineationsgruppe erwiesen; sie lieferten sogar die Existenz von vielen H-Ebenen mit bisher völlig unbekannten Parametern (siehe [10] und [11]). Diese Existenzaussagen kann man aber auch (wenngleich unbequemer) mit Hilfsmatrizen erhalten (siehe Satz 10 unten). Zur Wiederholung zunächst

*Definition 3.* N sei eine abelsche Gruppe der Ordnung  $t^2$  und  $A = (a_{ik})$  ( $i = 0, \dots, r$ ;  $k = 1, \dots, t$ ) eine Matrix über N. A heißt eine  $(t,r)$ -K-Matrix, wenn

$$(15) \quad \{a_{ik} - a_{jl} : k, l = 1, \dots, t\} = N \quad \text{für alle } (i, j) \text{ mit } i \neq j,$$

und eine  $(t,r)$ -H-Matrix, wenn außerdem noch gilt:

$$(16) \quad \text{Die Multimenge } \{a_{ik} - a_{ij} : i = 0, \dots, r; j, k = 1, \dots, t; j \neq k\} \text{ enthält jedes Element von } N \setminus \{0\} \text{ mindestens zweimal.}$$

Man bemerkt, daß es in Wirklichkeit nicht auf die Ordnung der Elemente in jeder Zeile, sondern nur auf die Mengen  $A_i = \{a_{i1}, \dots, a_{ir}\}$  ( $i = 0, \dots, r$ ) ankommt. Weiterhin kann man  $0 \in A_i$  für alle  $i$  annehmen (z.B. durch Addition von  $-a_{ii}$  zu  $A_i$ ). Die Bedeutung der K-Matrizen liegt u.a. in

**Lemma 2** (Jungnickel). *r sei Primzahlpotenz und es gebe eine  $(t,r)$ -K-Matrix (bzw. H-Matrix). Dann gibt es eine  $(t,r)$ -K-Ebene (bzw. H-Ebene).*

Den Beweis findet man in [10], Section 2. Nach [18] gilt nun

**Satz 10** (Lenz). *Aus der Existenz einer  $(t,r)$ -K-Matrix (bzw. H-Matrix) folgt die Existenz von  $r+1$  symmetrischen  $(t^2 \times t^2)$ -Hilfsmatrizen  $M_0, \dots, M_r$ , die (4) (bzw. (4) und (14)) genügen (mit  $\mu = 1$ ).*

Dann folgt mit der angedeuteten Konstruktionsmethode ein Alternativbeweis für Lemma 2 oben. Als Beweisskizze geben wir hier die Definition der  $M_i$  mit Hilfe einer  $(t,r)$ -K-Matrix  $A = (A_0, \dots, A_r)$  mit  $a_{ii} = 0$  für alle  $i$ . Dazu sei etwa  $N = \{x_1, \dots, x_u\}$  mit  $u := t^2$  und  $x_1 := 0$ . Für  $i = 0, \dots, r$  sei  $\chi_i$  die charakteristische Funktion von  $A_i$ , also  $\chi_i(x) = 1$  für  $x \in A_i$  und  $\chi_i(x) = 0$  sonst. Wir definieren  $M_i = (m_{jk}^i)$  durch  $m_{jk}^i := \chi_i(x_j + x_k)$ . Die Gültigkeit von (4) bzw. (14) folgt dann leicht aus der von (15) bzw. (16).

Es sei noch bemerkt, daß K-(bzw. H-)Matrizen K-(bzw. H-)Ebenen mit einer regulären abelschen Kollineationsgruppe liefern. Die Frage, ob man auch mit der Methode der Hilfsmatrizen Aussagen über die Existenz von Kollineationsgruppen erhalten kann, ist an Beispielen bereits in [15], §4, diskutiert worden.

Die zweite Klasse von Hilfsmatrizen, die in der Konstruktion von Blockplänen, lateinischen Quadraten und PBD's häufig benutzt wird, entsteht aus «transversal designs» (ein  $(t,r)$ -TD ist die duale Struktur zu einem  $(t,r;1)$ -Netz) oder allgemeiner aus «group divisible designs». Beispiele für die Beweise wohlbekannter Resultate (etwa des Satzes von McNeish) mit Hilfe dieser Methode findet man in [15], §2. Wir wollen hier nur ein Resultat aufschreiben, dessen eine Richtung noch nicht in der Literatur steht:

**Lemma 3** Ein  $(t,r)$ -TD existiert genau dann, wenn es  $r$   $(t \times t^2)$ -Matrizen  $M_1, \dots, M_r$  mit Einträgen 0 und 1 gibt, die den Gleichungen

$$(17) \quad M_i M_j^T = J \quad \text{für } i \neq j$$

und

$$(18) \quad M_i M_i^T = t M_i$$

genügen.

Die Konstruktion der Matrizen aus dem TD findet man in [15], §2; die Umkehrung ist ebenfalls eine einfache Übungsaufgabe.

Die eben genannten Hilfsmatrizen ermöglichen aber nicht nur bequem formulierte Alternativbeweise klassischer Sätze, sondern sind auch ein wesentliches Hilfsmittel für eingehendere Strukturuntersuchungen, z.B. bei der Gewinnung unterer Schranken für die Anzahl nicht-isomorpher Blockpläne (siehe [16]) oder zur Konstruktion von Blockplänen mit Unterräumen (siehe [17]).

Es sei noch bemerkt, daß die Konstruktion mit Hilfe von Matrizen, die aus einem TD erhalten worden sind, sich theoretisch durch die in [12] eingeführten «TD-Strukturen» behandeln und durch Eigenschaften des Epimorphismus  $\phi$  charakterisieren läßt:  $\phi$  ist «abgeschlossen» (das Bild jedes Unterraums ist ein Unterraum) und «transversal» ( $a^\phi = b^\phi$  impliziert  $a = b$  oder  $[a, b] = 0$ ).

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## Verallgemeinerte Minkowski-Geometrie

Die klassischen Kreisgeometrien von Möbius, Laguerre und Minkowski kann man sich anschaulich am besten vorstellen als Geometrie der Ebenenschnitte einer Kugel, eines Zylinders und eines einschaligen Hyperboloids im dreidimensionalen projektiven Raum  $\Pi(\mathbb{R}^3)$ . Eine einheitliche algebraische Darstellung dieser Geometrien kann man mit Hilfe der 2-dimensionalen Algebren über  $\mathbb{R}$  geben. Für die Punktmenge der Möbiusgeometrie nimmt man den projektiven Abschluß  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  des Körpers der komplexen Zahlen und für die Punktemenge der Laguerre resp. der Minkowski-Geometrie den projektiven Abschluß  $\bar{\mathbb{D}}$  der Algebra  $\mathbb{D} = \mathbb{R}(j)$  mit  $j^2 = 0$  der Dualzahlen, resp. den projektiven Abschluß  $\bar{\mathbb{A}}$  der Algebra der anomal komplexen Zahlen  $\mathbb{A} = \mathbb{R}(j)$  mit  $j^2 = 1$ . Die Kreise werden dann jeweils definiert als Bildmengen von  $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$  unter einer Abbildung aus  $\mathrm{PGL}(2, A)$ , wobei  $A$  eine der Algebren  $\mathbb{C}$ ,  $\mathbb{D}$  oder  $\mathbb{A}$  ist. Hierbei lässt man  $\mathrm{PGL}(2, A)$  auf  $\bar{\mathbb{C}}$  resp.  $\bar{\mathbb{D}}$ ,  $\bar{\mathbb{A}}$  operieren.

Nun lässt die Minkowski-Geometrie noch eine andere überraschend einfache Darstellung mit Hilfe der Gruppe  $\mathrm{PGL}(2, \mathbb{R})$  zu, indem man als Punktmenge  $\mathfrak{P} = \mathbb{R} \times \mathbb{R}$  und als Menge der Kreise  $\mathfrak{K} = \{K_\gamma \subset \mathfrak{P} \mid \gamma \in \mathrm{PGL}(2, \mathbb{R})\}$  mit  $K_\gamma = \{(x, \gamma(x)) \in \mathfrak{P} \mid x \in \mathbb{R}\}$  nimmt. Die Erzeugenden, die den auf dem Hyperboloid liegenden Geraden entsprechen, sind die Mengen  $\{a\} \times \bar{\mathbb{R}}$  und  $\bar{\mathbb{R}} \times \{a\}$  mit  $a \in \bar{\mathbb{R}}$ .

Inzidenzgeometrisch wird hierbei nur benutzt, daß  $\mathrm{PGL}(2, \mathbb{R})$  scharf 3-fach transitiv auf  $\bar{\mathbb{R}}$  operiert.

Die folgenden Ausführungen sollen einen kurzen Überblick über Minkowski-Geometrien und allgemeiner über Hyperbelstrukturen unter diesem Gesichtspunkt geben.

### §1. Definitionen und Begriffsbildungen

1.1. *Definition:* Es sei  $P$  eine Menge von Punkten und  $\mathfrak{K}, \mathfrak{G}_1, \mathfrak{G}_2 \subset \mathfrak{P}(P)$  Teilmengen der Potenzmenge  $\mathfrak{P}(P)$  von  $P$ . Die Elemente aus  $\mathfrak{G}_i$  heißen Erzeugende, die aus  $\mathfrak{K}$  heißen Kreise. Zwei Punkte  $p, q$  heißen verbindbar, wenn es keine Erzeugende gibt, auf der beide liegen.

$(P, \mathfrak{K}, \mathfrak{G}_1, \mathfrak{G}_2)$  heißt eine Benz-Ebene, wenn gilt:

G1: Zu jedem  $p \in P$  und jedem  $i = 1, 2$  gibt es genau eine mit  $[p]_i$  bezeichnete Erzeugende aus  $\mathfrak{G}_i$  mit  $p \in [p]_i$ .

G2: Falls  $\mathfrak{G}_1 \neq \mathfrak{G}_2, \mathfrak{G}_i \neq \emptyset, i = 1, 2$ , dann gilt  $|G_i| \geq 2$  und  $|G_1 \cap G_2| = 1$  für alle  $G_i \in \mathfrak{G}_i$ .

B1: Jede Erzeugende schneidet jeden Kreis in genau einem Punkt.

B2: Zu je drei paarweise verbindbaren Punkten  $a, b, c \in P$  gibt es genau einen Kreis  $K \in \mathfrak{K}$  mit  $a, b, c \in K$ .

B3: (Berühraxiom) Zu jedem Kreis  $K$ , jedem Punkt  $a \in K$  und jedem mit  $a$  verbindbaren Punkt  $b \notin K \cup [a]_1 \cup [a]_2$  gibt es genau einen Kreis  $L \in \mathfrak{K}$  mit  $a, b \in L$  und  $L \cap K = \{a\}$ .

B4: Es gibt einen Kreis  $K \in \mathfrak{K}$  mit  $|K| \geq 3$ . Es ist  $P \setminus K \neq \emptyset$  und  $\emptyset \notin \mathfrak{K}$ .

$(P, \mathfrak{K}, \mathfrak{G}_1, \mathfrak{G}_2)$  heißt eine Möbius-Ebene, wenn  $\mathfrak{G}_1 = \mathfrak{G}_2 = \emptyset$ ; eine Laguerre-Ebene, wenn  $\mathfrak{G}_1 = \mathfrak{G}_2 \neq \emptyset$ ; und eine Minkowski-Ebene, wenn  $\emptyset \neq \mathfrak{G}_1 \neq \mathfrak{G}_2 \neq \emptyset$  ist.

Nimmt man aus einer solchen Ebene einen Punkt  $a \in P$  und die mit ihm inzidierenden Erzeugenden  $[a]_i$  heraus, erhält man in kanonischer Weise eine affine Ebene:

- 1.2. Es sei  $(P, \mathfrak{K}, \mathfrak{G}_1, \mathfrak{G}_2)$  eine Benz-Ebene. Dann ist für einen beliebigen festen Punkt  $a \in P$  das Paar  $(P_a, \mathfrak{K}_a \cup \mathfrak{G}_a)$  mit  $P_a := P \setminus ([a]_1 \cup [a]_2 \cup \{a\})$ ,  $\mathfrak{K}_a = \{K \setminus \{a\} | a \in K \in \mathfrak{K}\}$  und  $\mathfrak{G}_a = \{G \setminus ([a]_1 \cup [a]_2) | G \in \mathfrak{G}_1 \cup \mathfrak{G}_2 \setminus \{[a]_1, [a]_2\}\}$  eine affine Ebene.

Hierbei sichert das Berühraxiom B3 die Gültigkeit des Parallelaxioms.

- 1.3. *Definition:*  $\mathfrak{H} = (P, \mathfrak{K}, \mathfrak{G}_1, \mathfrak{G}_2)$  heißt eine Hyperbelstruktur, wenn G1, G2, B1, B2, B4 und  $\emptyset \neq \mathfrak{G}_1 \neq \mathfrak{G}_2 \neq \emptyset$  gilt.

Hyperbelstrukturen sind also gewissermaßen Minkowski-Ebenen, in denen das Berühraxiom B3 nicht notwendig gilt. Im folgenden betrachten wir nur noch Hyperbelstrukturen und lassen die Möbius und Laguerre Geometrien außer acht.

## §2. Darstellungssätze

Hyperbelstrukturen sind eng mit scharf 3-fach transitiven Permutationsmengen  $(\Gamma, M)$  verknüpft. Dieser Zusammenhang bildet den Ausgangspunkt für unsere weiteren Überlegungen.

Mit  $(\Gamma, M)$  bezeichnen wir eine Permutationsmenge (d.h.  $\Gamma$  ist eine Teilmenge der symmetrischen Gruppe  $S_M$ ), die scharf 3-fach transitiv auf  $M$  operiert, d.h. zu  $a_i, b_i \in M$  mit  $a_i \neq a_j, b_i \neq b_j$  für  $i \neq j$  mit  $i, j = 1, 2, 3$  gibt es genau ein  $\gamma \in \Gamma$  mit  $\gamma(a_i) = b_i$  für  $i = 1, 2, 3$ .

- 2.1. Es sei  $(\Gamma, M)$  eine scharf 3-fach transitive Permutationsmenge. Definiert man  $P := M \times M$  als Menge der Punkte,  $\mathfrak{K} := \{K_\gamma \subset P | \gamma \in \Gamma\}$  mit  $K_\gamma := \{(x, \gamma(x)) \in P | x \in M\}$  als Menge der Kreise und  $\mathfrak{G}_1 := \{\{x\} \times M | x \in M\}$ ,  $\mathfrak{G}_2 := \{M \times \{x\} | x \in M\}$  als die beiden Erzeugendenscharen, dann ist  $\mathfrak{H}(\Gamma) := (P, \mathfrak{K}, \mathfrak{G}_1, \mathfrak{G}_2)$  eine Hyperbelstruktur. Umgekehrt kann man zu jeder Hyperbelstruktur  $\mathfrak{H}$  eine scharf 3-fach transitive Permutationsmenge  $(\Gamma, M)$  finden, die sie in der oben angegebenen Art darstellt.

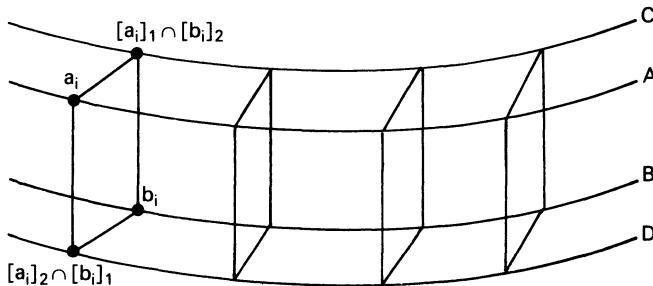
Ausgehend von einer Hyperbelstruktur kann man ein  $(\Gamma, M)$  folgendermaßen finden: Es sei  $L \in \mathfrak{K}$  ein fest gewählter Kreis. Wir setzen  $M := L$  und definieren für jedes  $K \in \mathfrak{K}$  die Permutation  $\gamma_K: x \rightarrow [[x]_1 \cap K]_2 \cap L$ . Dann operiert  $\Gamma := \{\gamma_K | K \in \mathfrak{K}\}$  scharf 3-fach transitiv auf  $M = L$ .

Diese Darstellung von Hyperbelstrukturen  $\mathfrak{H}$  ist in folgendem Sinne eindeutig: Ist  $(\Gamma', M')$  eine andere Permutationsmenge, die  $\mathfrak{H}$  darstellt, dann gibt es Bijektionen  $\alpha, \beta: M \rightarrow M'$ , so daß  $\alpha \Gamma' \beta^{-1} = \Gamma'$  ist.

Eigenschaften der Hyperbelstruktur übertragen sich natürlich auf  $(\Gamma, M)$  und

umgekehrt. Ist  $\Gamma$  eine Gruppe, so gilt in  $\mathfrak{H}$  ein besonders schöner Schließungssatz, das Rechtsecksaxiom R:

R: Für  $i=1,2,3,4$  seien  $a_i$  und  $b_i$  jeweils vier verschiedene Punkte mit  $a_i \in A$ ,  $b_i \in B$  und  $[a_i]_1 \cap [b_i]_2 \in C$  mit  $A, B, C \in \mathfrak{R}$ . Dann gibt es einen Kreis  $D \in \mathfrak{R}$  mit  $[a_i]_2 \cap [b_i]_1 \in D$  für  $i=1,2,3,4$ .



2.2.  $(\Gamma, M)$  ist genau dann eine scharf 3-fach transitive Gruppe, wenn in der zugehörigen Hyperbelstruktur  $\mathfrak{H}$  das Rechtecksaxiom R gilt.

Die Untersuchung von Hyperbelstrukturen mit R führt also auf das Studium der scharf 3-fach transitiven Gruppen. Hier gilt nun wieder ein Darstellungssatz, der es gestattet, Methoden der Algebra anzuwenden:

2.3. *Definition:*  $(F, +, \cdot)$  heißt ein Fastbereich, wenn gilt:

Fb1:  $(F, +, \cdot)$  ist ein Loop (mit 0 als neutralem Element).

Fb2:  $a + x = 0 \Rightarrow x + a = 0$ .

Fb3:  $(F^*, \cdot)$  ist eine Gruppe,  $F^* := F \setminus \{0\}$ .

Fb4:  $0 \cdot a = 0$  für jedes  $a \in F$ .

Fb5:  $a(b+c) = ab + ac$  für alle  $a, b, c \in F$ .

Fb6: Für alle  $a, b \in F$  gibt es genau ein  $d_{a,b} \in F^*$ , so daß  $a + (b + x) = (a + b) + d_{a,b} \cdot x$  für alle  $x \in F$  gilt.

Bemerkung:

Die Fastbereiche wurden von Karzel [10] eingeführt. Die Abbildungen  $\alpha: x \rightarrow a + m \cdot x$  mit  $a, m \in F$ ,  $m \neq 0$  von  $F$  in  $F$  bilden eine scharf 2-fach transitive Permutationsgruppe und umgekehrt lassen sich zu jeder auf einer Menge  $N$  scharf 2-fach transitiven Gruppe zwei lineare Verknüpfungen  $(+)$  und  $(\cdot)$  auf  $N$  finden, so daß die Gruppe dargestellt wird durch Abbildungen der Form  $\alpha$ . Ein analoger Satz gilt auch für scharf 3-fach transitiven Gruppen:

2.4. *Definition:*  $(F, +, \cdot, \sigma)$  heißt KT-Feld, wenn  $(F, +, \cdot)$  ein Fastbereich ist und  $\sigma$  ein involutorischer Automorphismus von  $(F^*, \cdot)$ , der die Funktionalgleichung

$$\sigma(1 + \sigma(x)) = 1 - \sigma(1 + x) \quad \forall x \in F \setminus \{0, -1\}$$

erfüllt.

2.5. Es sei  $(F, +, \cdot, \sigma)$  ein KT-Feld,  $\infty \notin F$ . Die Abbildungen der Form  $\alpha$  und  $\beta$  von  $\bar{F} := F \cup \{\infty\}$  auf  $\bar{F}$ :

$$\alpha: \begin{cases} x \rightarrow a + mx, & a, m \in F, m \neq 0. \\ \infty \rightarrow \infty \end{cases}$$

$$\beta: \begin{cases} x \rightarrow a + (b + mx) & a, b, m \in F, m \neq 0 \\ \infty \rightarrow a \\ -m^{-1}b \rightarrow \infty \end{cases}$$

operieren scharf 3-fach transitiv auf  $\bar{F}$ . Ist umgekehrt  $(\Gamma, M)$  eine scharf 3-fach transitive Gruppe, so läßt sich nach Auswahl eines Punktes  $\infty \in M$  auf  $M_\infty := M \setminus \{\infty\}$  eine Addition  $(+)$  und eine Multiplikation  $(\cdot)$ , sowie ein  $\sigma \in \Gamma$  finden, so daß  $(M_\infty, +, \cdot, \sigma)$  ein KT-Feld ist und  $\Gamma$  durch Abbildungen der Form  $\alpha$  und  $\beta$  dargestellt wird. Das KT-Feld ist bis auf Isomorphie durch die Gruppe  $(\Gamma, M)$  eindeutig bestimmt.

Die bekanntesten scharf 3-fach transitiven Gruppen sind die  $PGL(2, K)$ , aufgefaßt als Gruppe der gebrochen-linearen Transformationen  $x \rightarrow \frac{rx+s}{tx+u}$ ,  $ru-st \neq 0$  von  $\bar{K} := K \cup \{\infty\}$  auf  $\bar{K}$ . Das zugehörige KT-Feld ist hier selbstverständlich der kommutative Körper  $(K, +, \cdot)$  mit  $\sigma(x) = x^{-1}$ . Je nachdem, ob  $t=0$  oder  $t \neq 0$  ist, läßt sich die Abbildung  $x \rightarrow \frac{rx+s}{tx+u}$  in der Form  $\alpha: x \rightarrow a + mx$  oder  $\beta: x \rightarrow a + (b + mx)^{-1}$  schreiben.

### §3. KT-Felder und scharf 3-fach transitive Gruppen

Alle bisher bekannten KT-Felder werden in der folgenden Weise konstruiert:

3.1. Es sei  $(F, +, \cdot)$  ein kommutativer Körper und  $A$  eine Untergruppe von  $(F^*, \cdot)$ ,  $F^* := F \setminus \{0\}$  mit den Eigenschaften:

- 1)  $Q := \{a^2 \in F^* \mid a \in F^*\} \subseteq A$
- 2) Es gibt einen Monomorphismus  $\pi: F^*/A \rightarrow \text{Aut}(F, +, \cdot)$
- 3)  $\tau(x) \in xA$  für jedes  $x \in F^*$  und jedes  $\tau \in \pi(F^*/A)$

Bezeichnet man mit  $\kappa: x \rightarrow xA$  den kanonischen Homomorphismus und definiert als neue Multiplikation  $(\circ)$  auf  $F$

$$a \circ b = a \cdot a_\phi(b)$$

mit  $a_\phi = \pi\kappa(a)$  für  $a \neq 0$  und  $0_\phi = \text{id}$ , dann ist  $(F, +, \circ)$  ein (stark gekoppelter Dicksonscher) Fastkörper und  $(F, +, \circ, \sigma)$  mit  $\sigma(x) = x^{-1}$  (Inverses bezüglich  $(\cdot)$ ) ein KT-Feld.

Wegen 1) besteht  $\pi\kappa(F^*) = \{a_\phi \in \text{Aut}(F, +, \cdot) \mid a \in F^*\}$ , abgesehen von der Identität, nur aus involutorischen Automorphismen.

Neben den kommutativen Körpern umfassen die in 3.1 angegebenen Beispiele bestimmte Typen von Fastkörpern. Ob es auch KT-Felder gibt, die keine Fastkörper sind, ist bisher noch nicht bekannt. Echte Schiefkörper können nicht auftreten [6].

Die Existenz echter KT-Felder ist jedoch recht unwahrscheinlich. Wie Kerby [13] zeigte, sind alle KT-Felder  $F$  mit einer Charakteristik  $\text{char } F \equiv 1 \pmod{3}$  bereits Fastkörper. Hinzu kommt, daß auch alle Fastbereiche mit  $\text{char } F = 3$  schon Fastkörper sind [15]. Auch darüber, ob bereits alle KT-Felder dieser Charakteristiken von den in 3.1 beschriebenen Konstruktionsverfahren erfaßt werden, ist noch nichts bekannt.

Es gibt jedoch eine ganze Reihe von verschiedenartigen Zusatzbedingungen, die sichern, daß ein KT-Feld bzw. ein Fastbereich, bereits ein Fastkörper ist ([16], [17], [18]). Wir erwähnen hier die beiden folgenden:

- 3.2. Es sei  $(F, +, \cdot)$  ein Fastbereich. Wenn  $(F^*, \cdot)$  eine FC-Gruppe ist (d.h. die konjugierten Klassen von  $(F^*, \cdot)$  sind endlich), dann ist  $(F, +, \cdot)$  ein Fastkörper, Insbesondere sind endliche Fastbereiche Fastkörper.

Die endlichen Fastkörper und die endlichen KT-Felder hat Zassenhaus alle bestimmt. Alle endlichen KT-Felder, die keine Körper sind erhält man dadurch, daß man für  $A = Q = \{a^2 \in F^* \mid a \in F^*\}$  setzt. Nichttriviale Automorphismen der Ordnung 2 gibt es nur in Galoisfeldern  $GF(p^{2m})$ , so daß man im endlichen Fall nur in diesen Körpern die Multiplikation gemäß 3.1 abändern kann:

$$a \circ b = a \cdot \tau(b) \text{ für } a \in F^* \setminus Q \text{ mit } \tau(b) = b^{p^m} \text{ und } a \circ b = a \cdot b \text{ sonst.}$$

Die Gültigkeit des Berühraxioms B3 in Hyperbelstrukturen mit  $R$  impliziert, daß das zugehörige KT-Feld projektiv ist, d.h. die Gleichung  $ax + bx = c$  hat für beliebige  $a, b, c \in F$  mit  $a \neq -b$  stets eine Lösung. Hier gilt:

- 3.3. Jeder projektive Fastbereich ist ein Fastkörper.

Minkowski-Geometrien mit Rechtecksaxiom führen also auf KT-Fastkörper (KT-Felder  $(F, +, \cdot, \sigma)$ , bei denen  $(F, +, \cdot)$  ein Fastkörper ist). Beispiele von nicht projektiven KT-Fastkörpern erhält man folgendermaßen: Es sei  $F = K(t)$  der

Körper der in  $t$  rationalen Funktionen. Für  $h = \frac{f(t)}{g(t)}$ ,  $f(t), g(t) \in K[t]$ , definieren

wir:  $\text{grad } h := \text{grad } f - \text{grad } g$ . Wählt man für  $\tau \in \text{Aut } (F, +, \cdot)$  einen involutorischen Automorphismus, der den kommutativen Grundkörper  $K$  punktweise fest läßt, etwa  $\tau(t) = -t + k$  mit  $k \in K$ , und definiert  $a_\varphi = \tau^{\text{grad } a}$  für  $a \in F^*$ , dann ist  $(F, +, \circ, \sigma)$  mit  $a \circ b = a \cdot a_\varphi(b)$  und  $\sigma(a) = a^{-1}$  gemäß 3.1 (hier ist  $A = \{h \in F \mid \text{grad } h = 2n, n \in \mathbb{Z}\}$ ) ein nichtprojektiver KT-Fastkörper. Der Beweis von Kerby überträgt sich wörtlich.

Beispiele von projektiven KT-Fastkörpern, die keine Körper sind, erhält man folgendermaßen [14]: Es sei  $(K, +, \cdot)$  ein kommutativer Körper, der einen involutorischen Automorphismus  $\tau_0 \in \text{Aut } (K, +, \cdot)$  besitzt.  $\tau_0$  läßt sich auf den Körper  $K((t))$  der formalen Potenzreihen fortsetzen, indem man setzt:  $\tau(\sum_{v \geq v_0} k_v t^v) := \sum_{v \geq v_0} \tau_0(k_v) t^v$ .

$(K((t)), +, \circ, \sigma)$  mit  $a \circ b = a \cdot a_\varphi(b)$  und  $a_\varphi = \tau^{\text{grad } a}$ , wobei  $\text{grad } a = \min\{v \in \mathbb{Z} \mid a_v \neq 0 \text{ mit } a = \sum a_v t^v\}$ , ist ein projektiver KT-Fastkörper.

Untersucht man Eigenschaften der Involutionen in scharf 3-fach transitiven Gruppen, so stößt man im zugehörigen KT-Feld auf die beiden folgenden Mengen:

$$\begin{aligned} S &:= \{m \in F^* \mid \sigma(m)m = 1\} \\ R &:= \{\sigma(z)z^{-1} \mid z \in F^*\} \end{aligned}$$

Wegen  $\sigma(\sigma(z)z^{-1}) \cdot \sigma(z)z^{-1} = z\sigma(z^{-1})\sigma(z)z^{-1} = 1$  ist  $R \subseteq S$ . Die Involutionen und ihre Fixpunkte lassen sich nun angeben:

3.4. Die Involutionen von  $\Gamma$  haben die folgende Gestalt:

$$\begin{aligned} \alpha: x \rightarrow a - x &\quad a \in F \\ \beta: x \rightarrow -a + n\sigma(a+x) &\quad a \in F, n \in S. \end{aligned}$$

Die Fixpunkte von  $\alpha$  sind  $\infty$  und  $a \cdot 2^{-1}$ ; Fixpunkte von  $\beta$  existieren genau dann, wenn  $n \in R$  ist. Dann hat nämlich  $n$  die Gestalt  $n = \sigma(z)z^{-1}$  und die Fixpunkte sind  $-a \pm \sigma(z)$ .

Bei endlichen KT-Feldern  $F$  gilt entweder  $R \neq S = F^*$  ( $F$  ist dann ein Galois-Feld) oder  $R = S \neq F^*$  (dann ist  $F$  der auf Seite 398 beschriebene Fastkörper; hier gilt  $Q = R = A = S$ ).

Bei unendlichen KT-Feldern können die Fälle  $R = S$ , aber auch  $R \neq S \neq F^*$  auftreten.

#### §4. Spiegelungen und Orthogonalität

In Hyperbelstrukturen lassen sich in natürlicher Weise Kreisspiegelungen einführen. Es sei  $K \in \mathfrak{K}$ .

$$\tilde{K}: \begin{cases} P \rightarrow P \\ p \rightarrow [[p]]_1 \cap K \end{cases}$$

heißt Spiegelung am Kreis  $K$ . In der gruppentheoretischen Darstellung hat  $K$  die folgende Gestalt. Es sei  $K = K_\delta$ ,  $\delta \in \Gamma$ .

$$4.1 \quad \tilde{K}_\delta: \begin{cases} M \times M \rightarrow M \times M \\ (x, y) \rightarrow (\delta^{-1}(y), \delta(x)). \end{cases}$$

Denn ist  $p = (x, y)$  und  $K_\delta = \{(x, \delta(x)) \mid x \in M\}$ , so ist  
 $[p]_1 \cap K = (x, \delta(x)) \Rightarrow [[p]]_1 \cap K_2 = M \times \{\delta(x)\}$   
 $[p]_2 \cap K = (\delta^{-1}(y), y) \Rightarrow [[p]]_2 \cap K_1 = \{\delta^{-1}(y)\} \times M$   
 $\Rightarrow M \times \{\delta(x)\} \cap \{\delta^{-1}(y)\} \times M = (\delta^{-1}(y), \delta(x))$ .

Bei Spiegelungen gehen Kreise in Kreise über:

$$4.2 \quad \tilde{K}_\delta(K_\gamma) = K_{\delta\gamma^{-1}\delta}$$

Denn  $\tilde{K}_\delta: (x, \gamma(x)) \rightarrow (\delta^{-1}\gamma(x), \delta(x)) = (y, \delta\gamma^{-1}\delta(y))$  für  $y = \delta^{-1}\gamma(x)$ . Erzeugende werden wieder auf Erzeugende abgebildet, wobei  $G_1$  in  $G_2$  und  $G_2$  in  $G_1$  übergeht.

Der Begriff der Spiegelung gestattet es nun, ganz einfach das Senkrechtstehen einzuführen. Zwei Kreise  $K_\delta$  und  $K_\gamma$  heißen orthogonal, im Zeichen  $K_\delta \perp K_\gamma$ , wenn gilt  $\tilde{K}_\delta(K_\gamma) = K_\gamma$ . In der gruppentheoretischen Schreibweise bedeutet das:

$$4.3. K_\delta \perp K_\gamma \Leftrightarrow \gamma\delta^{-1} \text{ ist involutorisch.}$$

Aus  $K_\delta \perp K_\gamma \Rightarrow K_\gamma \perp K_\delta$ , da mit  $\gamma\delta^{-1}$  auch  $\delta\gamma^{-1}$  involutorisch ist.

4.4. Die Anzahl der Schnittpunkte  $|K_\gamma \cap K_\delta|$  zweier Kreise ist gleich der Anzahl der Fixpunkte von  $\gamma\delta^{-1}$ .

Denn  $K_\gamma \ni (x, \gamma(x)) = (x, \delta(x)) \in K_\delta \Leftrightarrow \gamma(x) = \delta(x) \Leftrightarrow \delta^{-1}\gamma(x) = x \Leftrightarrow \gamma\delta^{-1}(z) = z$  für  $z = \delta(x)$ .

Ganz allgemein können also zwei orthogonale Kreise 0, 1 oder 2 Schnittpunkte besitzen. Aus dem Schnittpunktverhalten orthogonaler Kreise kann man nun Eigenschaften des zugehörigen KT-Feldes ableiten und umgekehrt.

4.5. Es sei  $(H, \mathfrak{R}, \mathfrak{G}_1, \mathfrak{G}_2)$  eine Hyperbelstruktur mit Rechtecksaxiom und  $(F, +, \cdot, \sigma)$  das zugehörige KT-Feld. Dann gilt:

- a) Es gilt  $|K \cap L| \leq 1$  für alle  $K, L \in \mathfrak{R}$  mit  $K \perp L$  genau dann, wenn  $\text{char } F = 2$ .
- b) Es gibt zwei orthogonale Kreise  $K, L \in \mathfrak{R}$  mit  $|K \cap L| = 2$  genau dann, wenn  $\text{char } F \neq 2$ .

In diesem Fall gilt:  $K' \perp L' \Rightarrow |K' \cap L'| \in \{0, 2\}$ .

Der Fall, daß orthogonale Kreise sich immer schneiden, kann natürlich auch vorkommen. Je nach Charakteristik gibt es immer nur einen (bei  $\text{char } F = 2$ ) oder zwei Schnittpunkte (bei  $\text{char } F \neq 2$ ). Wir können diesen Fall folgendermaßen charakterisieren:

4.6. Es sei  $(H, \mathfrak{R}, \mathfrak{G}_1, \mathfrak{G}_2)$  eine Hyperbelstruktur mit Rechtecksaxiom und  $(F, +, \cdot, \sigma)$  das zugehörige KT-Feld. Dann sind die folgenden Aussagen äquivalent:

- a) Für alle  $K, L \in \mathfrak{R}$  mit  $K \perp L$  gilt  $|K \cap L| \in \{1, 2\}$ .
- b)  $R = S$  d.h. es ist  $\{\sigma(z)z^{-1} \in F^* | z \in F^*\} = \{m \in F^* | \sigma(m) \cdot m = 1\}$

Während bei endlichen Hyperbelstrukturen mit R dieser Fall nur bei den echten KT-Fastkörpern (die dann notwendig  $\text{char } F > 2$  haben) eintreten kann (vgl. [22]), lässt sich 4.6 bei unendlichen Hyperbelstrukturen mit den Körpern realisieren, bei denen jedes Element Quadrat ist. In unseren beiden in §3 angegebenen Beispielen von KT-Fastkörpern ist

$$S = \{x \in F^* | \text{grad } x \equiv 0 \pmod{2}\} \cup \{x \in F^* | \tau(x) = x\}$$

und  $R \neq S$ . Beispiele von unendlichen KT-Fastkörpern mit  $R = S$  die keine Körper sind, sind bisher noch nicht angegeben worden.

Die Beweise zu 4.5 und 4.6 lassen sich aus 4.4, 4.3 und 3.4 ablesen.

## §5. Schließungssätze

Wird eine Hyperbelstruktur durch eine scharf 3-fach transitive Gruppe, also durch ein KT-Feld beschrieben, dann gilt in ihr das Rechtecksaxiom R. Wir geben nun einige weitere Schließungssätze an, die die Beschreibung durch kommutative Körper charakterisieren:

S Symmetri axiom:

Es seien  $K, L$  Kreise. Wenn es einen Punkt  $p \in L \setminus K$  gibt mit  $\tilde{K}(p) \in L$ , dann gilt bereits  $\tilde{K}(L) = L$ ; d.h.  $L \perp K$ .

**S<sub>0</sub>** lokales Symmetrieaxiom:

Es gibt einen Kreis K mit der Eigenschaft: Ist  $p \in L \setminus K$  mit  $\tilde{K}(p) \in L$ , dann gilt  $\tilde{K}(L) = L$ .

**0 Büschelorthogonalität:**

Es gibt einen Kreis K der orthogonal ist zu allen Kreisen eines Kreisbüschels durch zwei verschiedene verbindbare Punkte.

**D Dreispiegelungssatz:**

Sind  $K_1, K_2, K_3$  Kreise durch zwei verschiedene verbindbare Punkte p und q, dann gibt es einen Kreis L durch p und q, so daß gilt  $\tilde{K}_1 \tilde{K}_2 \tilde{K}_3 = \tilde{L}$ .

Bereits in der Arbeit [3] hat Benz gezeigt, daß Hyperbelstrukturen, in denen R und S gilt, ovoidal sind, d.h. mit der Gruppe  $PGL(2, K)$ , also mit Hilfe kommutativer Körper beschrieben werden können. Später hat Artzy [2] mit geometrischen Schlüssen bewiesen, daß S das Axiom R bereits impliziert. Einen rein algebraischen Beweis in der Sprache der Permutationsmengen gab Karzel [11] an. Ebenso impliziert S die Axiome  $S_0, 0$  und D:

5.1. Gilt in einer Hyperbelstruktur das Symmetrieaxiom S, dann gelten auch die Axiome R, B3,  $S_0$ , O und D.

Über die Umkehrung ist bisher nur folgendes bekannt (vgl. [22], [23]):

5.2. Es sei eine Hyperbelstruktur mit R gegeben. Dann folgt aus der Gültigkeit von einem der Axiome  $S_0, O, D$  die Gültigkeit von S.

Ob man in 5.2 die Voraussetzung R fallen lassen kann, ist noch nicht bekannt. In diesem Zusammenhang muß man auch den Satz von Miquel mit seinen vielen verschiedenen Ausartungsfällen erwähnen. Für Möbius-Ebenen haben Chen und Schaeffer [20] die Zusammenhänge zwischen den verschiedenen Fällen des Satzes von Miquel weitgehend erhellt. Bei Laguerre und Minkowski-Ebenen resp. für Hyperbelstrukturen ist er noch ungeklärt. Kaerlein hat in [9] u.a. gezeigt, daß Minkowski-Ebenen, in denen ein bestimmter verallgemeinerter Satz von Miquel gilt, ovoidal sind.

Einen Schließungssatz, der die Hyperbelstrukturen mit Rechtecksaxiom charakterisiert, die zwar nicht mehr ovoidal sind, den ovoidalen aber noch am nächsten stehen, habe ich in [23] angegeben. Es handelt sich um eine Art «lokalen» Dreispiegelungssatz. Die zugehörigen KT-Fastkörper F sind genau die, die eine kommutative Untergruppe  $A < (F^*, \cdot)$  von Index  $[F^*:A]=2$  besitzen.

## §6. Einbettungsfragen

Hier gibt es zwei verschiedene Richtungen der Untersuchung, je nachdem, ob man eine andere geometrische Struktur in  $(H, R, \mathfrak{G}_1, \mathfrak{G}_2)$  einbetten will oder ob man umgekehrt eine gegebene Hyperbelstruktur in eine größere Struktur einzubetten versucht.

6.1. Für eine ovidale Hyperbelstruktur  $(H, R, \mathfrak{G}_1, \mathfrak{G}_2)$  ist  $(H_p, \mathfrak{G}_p)$  eine affine Ebene (vgl. 1.2). Ist  $(H, R, \mathfrak{G}_1, \mathfrak{G}_2)$  nicht-ovoidal, so ist  $(H_p, \mathfrak{G}_p)$  eine 2-Struktur (vgl. [10], [7]). Bedingungen, wann eine 2-Struktur in eine Hyperbelstruktur einbettbar ist, kann man z.B. auf solche Bedingungen zurückführen, wann eine scharf 2-fach transitive Permutationsmenge in eine scharf 3-fach transitive

Permutationsmenge eingebettet werden kann; bzw. wenn das Rechtecksaxiom R gilt, wann eine scharf 2-fach transitive Permutationsgruppe isomorph als Permutationsgruppe zur Standuntergruppe  $(\Gamma_a, M \setminus \{a\})$  einer scharf 3-fach transitiven Gruppe  $(\Gamma, M)$  ist.

6.2. Stellt man sich die Punktmenge H einer ovoidalen Hyperbelstruktur als Hyperboloid in einem dreidimensionalen projektiven Raum vor, wobei die Kreise und Paare von Erzeugenden durch Ebenenschnitte markiert werden, so kann man sich bekanntlich ausgehend von diesem Hyperboloid den umgebenden projektiven Raum wieder verschaffen. Dieses Verfahren lässt sich nun auch bei Hyperbelstrukturen anwenden. Man erhält so geometrische Räume, die nicht mehr notwendig projektiv sind. Wir denken uns die gegebene Hyperbelstruktur  $(H, \mathfrak{R}, \mathfrak{G}_1, \mathfrak{G}_2)$  mit Hilfe einer scharf 3-fach transitiven Permutationsmenge  $(\Gamma, M)$  dargestellt. Den umgebenden Raum konstruieren wir nun folgendermaßen:

Als Punktmenge  $\mathfrak{P}$  nehmen wir  $\mathfrak{P} := M \times M \cup \Gamma$ . Als Geradenmenge  $\mathfrak{G}$  sei  $\mathfrak{G} := \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3 \cup \mathfrak{G}_4 \cup \mathfrak{G}_5$  definiert:

$$\begin{aligned}\mathfrak{G}_1 &:= \{\{a\} \times M \mid a \in M\} \\ \mathfrak{G}_2 &:= \{M \times \{a\} \mid a \in M\} \\ \mathfrak{G}_3 &:= \{[(p_1, p_2); (q_1, q_2)] \mid (p_1, p_2), (q_1, q_2) \in M \times M, (p_1, p_2) \neq (q_1, q_2)\} \\ &\quad \text{mit } [(p_1, p_2); (q_1, q_2)] := \{(p_1, p_2), (q_1, q_2)\} \cup \{\gamma \in \Gamma \mid \gamma(p_1) = p_2, \\ &\quad \text{und } \gamma(q_1) = q_2\} \\ \mathfrak{G}_4 &:= \{[(p_1, p_2); \delta] \mid (p_1, p_2) \in M \times M \text{ und } \delta \in \Gamma\} \text{ mit} \\ &\quad \text{mit } [(p_1, p_2); \delta] := \{(p_1, p_2), \delta\} \cup \{\gamma \in \Gamma \mid \gamma(p_1) = p_2 \text{ und } \delta \gamma^{-1} \text{ ist} \\ &\quad \text{involutorisches}} \\ \mathfrak{G}_5 &:= \{[\delta; \varepsilon] \mid \delta, \varepsilon \in \Gamma\} \text{ mit } [\delta; \varepsilon] := \{\gamma \in \Gamma \mid \delta \gamma^{-1}, \varepsilon \gamma^{-1} \\ &\quad \text{sind involutorisch, } \varepsilon^{-1} \delta \text{ fixpunktfrei}\}\end{aligned}$$

Betrachtet man  $H = M \times M$  als «Pseudo-Quadratik» in  $\mathfrak{P}$ , dann sind die Geraden von  $\mathfrak{G}_1$  und  $\mathfrak{G}_2$  Erzeugende von H; die Geraden von  $\mathfrak{G}_3$  «Sekanten», die H in genau zwei Punkten schneiden.  $\mathfrak{G}_4$  besteht aus den «Tangenten», die H in genau einem Punkt schneiden, und  $\mathfrak{G}_5$  enthält die «Passanten», die außerhalb von H verlaufen.

Als Menge der Ebenen  $\mathfrak{E}$  definieren wir  $\mathfrak{E} := \mathfrak{E}_1 \cup \mathfrak{E}_2$  mit:

$$\begin{aligned}\mathfrak{E}_1 &:= \{\langle p_1, p_2 \rangle \mid (p_1, p_2) \in M \times M\} \\ &\quad \text{mit } \langle p_1, p_2 \rangle := \{p_1\} \times M \cup M \times \{p_2\} \cup \{\gamma \in \Gamma \mid \gamma(p_1) = p_2\} \\ \mathfrak{E}_2 &:= \{\langle \delta \rangle \mid \delta \in \Gamma\} \\ &\quad \text{mit } \langle \delta \rangle := \{\delta\} \cup \{(x, \delta(x)) \mid x \in M\} \cup \{\gamma \in \Gamma \mid \delta \gamma^{-1} \text{ involutorisch}\}.\end{aligned}$$

Im Falle, daß das Rechtecksaxiom gilt, wir es algebraisch also mit KT-Feldern resp. scharf 3-fach transitiven Gruppen zu tun haben, hat solche Räume schon Tits [21] angegeben. Er nannte sie «espace pseudocayléen centré». Nähere Untersuchungen über diese Räume scheinen aber bis jetzt noch nicht vorzuliegen.

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