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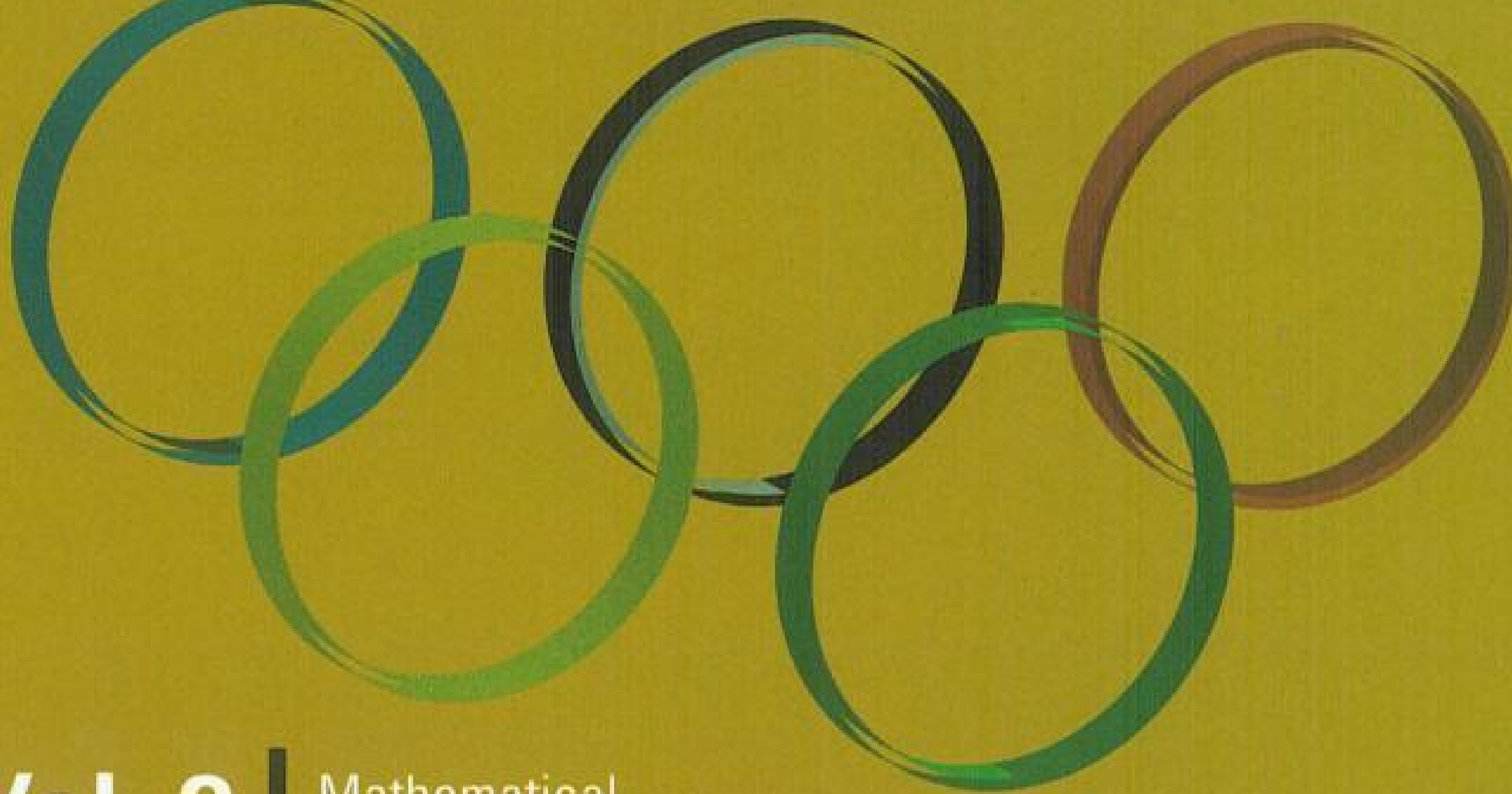

Vol. 3 | Mathematical Olympiad Series

Graph Theory

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World Scientific
ECNUP East China Normal University Press

Published by

East China Normal University Press
3663 North Zhongshan Road
Shanghai 200062
China

and

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

GRAPH THEORY
Mathematical Olympiad Series — Vol. 3

ISBN-13 978-981-4271-12-7 (pbk)
ISBN-10 981-4271-12-8 (pbk)

Printed in Singapore by B & Jo Enterprise Pte Ltd

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Chapter 1 Definition of Graph

Graph theory is a branch of mathematics on the study of graphs. The graph we consider here consists of a set of points together with lines joining certain pairs of these points. The graph represents a set that has binary relationship.

In recent years, graph theory has experienced an explosive growth and has generated extensive applications in many fields.

We often encounter the following phenomena or problems:

In a group of people, some of them know each other, but others do not.

There are some cities. Some pairs of them are connected by airlines and others are not.

There is a set of points in the plane. The distance between some of them is one and others are not one.

All the above phenomena or problems contain two aspects: one is object, such as people, football teams, cities, points and so on; and the other is a certain relationship between these objects, such as "knowing each other", "having a contest", "the distance between" and so on. In order to represent these objects and the relationships, we could use a point as an object, which is called a *vertex*. If any two objects have a relationship, then there is a line joining them, which is called an *edge*. Then we have constructed a graph.

We call the figure a *graph*[a]. For instance, the three graphs G_1,

---

a) The general definition of graphs: a graph is a triplet (V, E, ψ), where V and E are two disjoint sets, V is nonempty and ψ is a mapping from $V \times V$ to E. The sets V, E, ψ are vertex set, edge set and incidence function, respectively.

G_2, G_3 in Fig. 1.1 are isomorphic, which contain some vertices and edges joining them, representing some objects and the relationships between them.

Fig. 1.1 shows three graphs G_1, G_2, G_3, where vertices are represented by small circles.

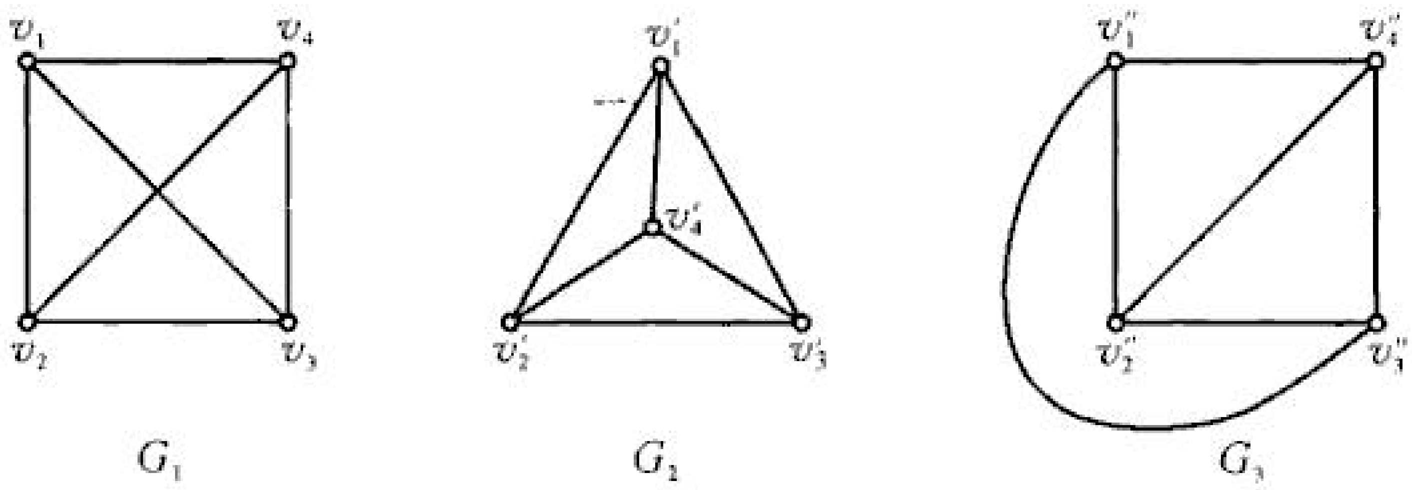

Fig. 1.1

We can see that in the definition of graphs there are no requirement on the location of the vertices, the length and the curvature of the edges, and the fact whether the vertices and the edges are in the same plane or not. However, we do not allow an edge passing through the third vertex and also not let an edge intersect itself. In graph theory, if there is a bijection from the vertices of G to the vertices of G' such that the number of edges joining v_i and v_j equals the number of edges joining $v_i'$and v_j', then two graphs G and G' are *isomorphic* and considered as the same graph.

A graph $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$, that is, all the vertices of G' are the vertices of G and the edges of G' are the edges of G.

For instance, the graphs G_1, G_2 in Fig. 1.2 are the subgraphs of G.

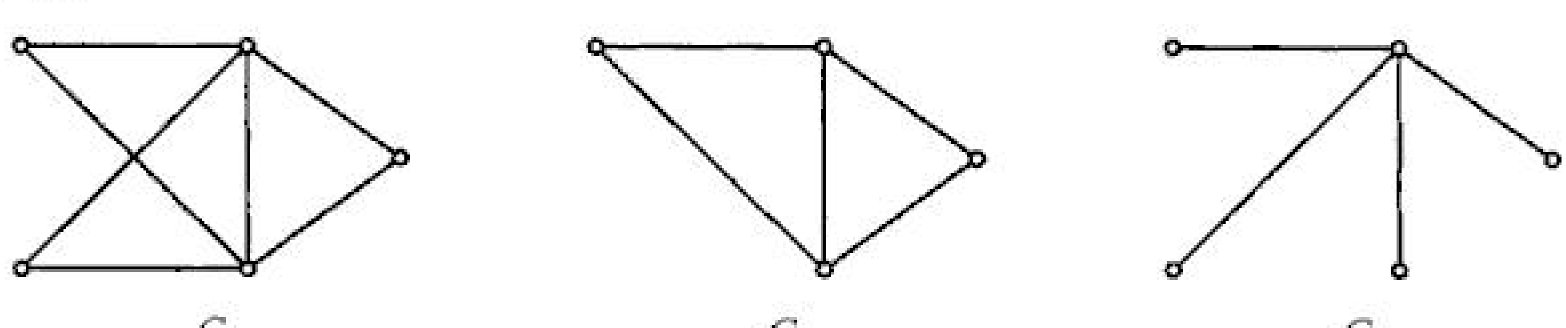

Fig. 1.2

If there is an edge joining v_i and v_j in graph G, then v_i and v_j are *adjacent*. Otherwise, they are nonadjacent. If the vertex v is an end of the edge e, then v is *incident* to e. In Fig. 1.3, v_1 and v_2 are adjacent, but v_2 and v_4 are not. The vertex v_3 is incident to the edge e_4.

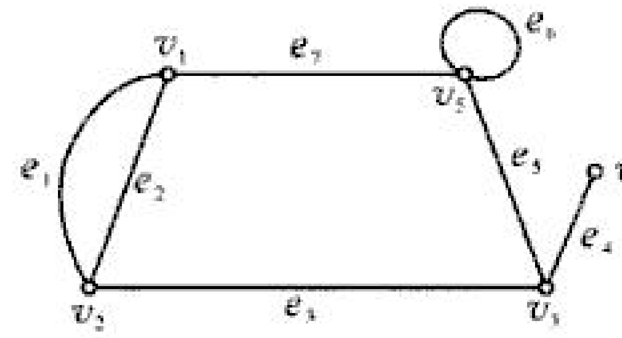

Fig. 1.3

We called the edge a *loop* if there is an edge joining the vertex and itself. For instance, the edge e_6 in Fig. 1.3 is a loop.

Two or more edges with the same pair of ends are called *parallel edges*. For instance, the edges e_1, e_2 in Fig. 1.3 are parallel edges.

A graph is *simple* if it has no loops or parallel edges. The graphs G_1, G_2, G_3 in Fig. 1.1 are simple, whereas the graph in Fig. 1.3 is not. In a simple graph, the edge joining v_i and v_j is denoted by (v_i, v_j). Certainly, (v_i, v_j) and (v_j, v_i) are considered as the same edge.

A *complete* graph is a simple graph in which any two vertices are adjacent. We denote the complete graph with n vertices by K_n. The graphs K_3, K_4, K_5 in Fig. 1.4 are all complete graphs. The number of edges of the complete graph K_n is $\binom{n}{2} = \frac{1}{2}n(n-1)$.

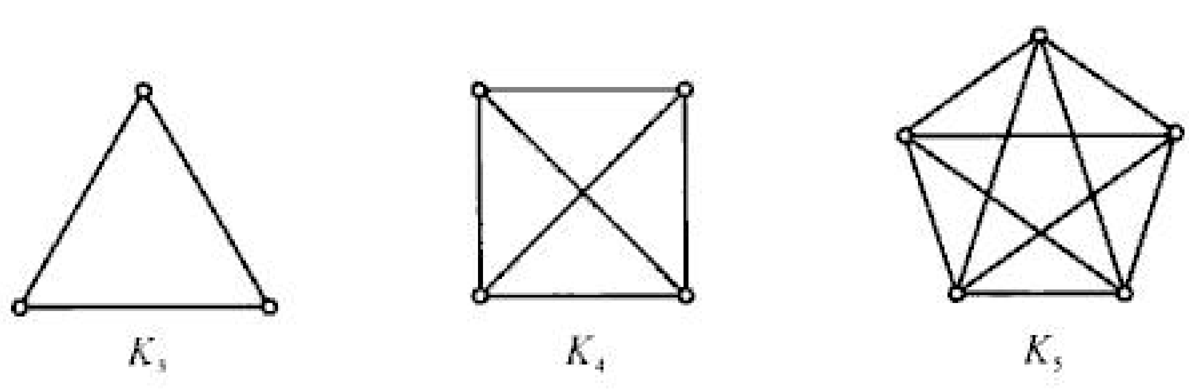

Fig. 1.4

A graph is *finite* if both the number of the vertices $|V|$ ($|V|$ is also said to be the order of G) and the number of edges $|E|$ are finite. A graph is *infinite* if $|V|$ or $|E|$ is infinite.

In this chapter, unless specified, all graphs under discussion should be taken to be finite simple graphs.

These fundamental concepts mentioned above help us to consider and solve some questions.

Example 1 There are 605 people in a party. Suppose that each of them shakes hands with at least one person. Prove that there must be someone who shakes hands with at least two persons.

Proof We denote the 605 people by 605 vertices v_1, v_2, ..., v_{605}. If any two of them shake hands, then there is an edge joining the corresponding vertices.

In this example we are going to prove that there must be someone who shakes hands with at least two persons. Otherwise, each of them shakes hands with at most one person. Moreover, according to the hypothesis each of them shakes hands with at least one person. Thus we have each of them just shakes hands with one person. It implies that the graph G consists of several figures that every two vertices are joined by only one edge.

Suppose that G have r edges. So G has $2r$ (even) vertices. It contradicts the fact that the number of vertices of G is 605 (odd).

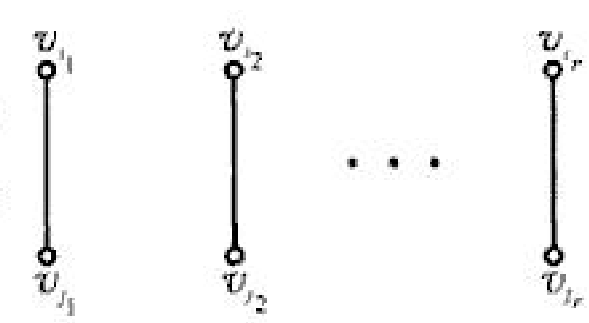

Fig. 1.5

We complete the proof.

Example 2 Is it possible to change the state in Fig. 1.6 to the state in Fig. 1.7 by moving the knights several times? (In the figures, W stands for white knight, and B stands for black knight. knight should be moved by following the international chess regulation)

Solution As Fig. 1.8 shows, the nine squares are numbered and each of them is represented by a vertex in the plane. If the knight can be moved from one square to anther square, then there is an edge joining the two corresponding vertices, as Fig. 1.9 shows.

W		W
B		B

Fig. 1.6

W		B
B		W

Fig. 1.7

1	4	7
2	5	8
3	6	9

Fig. 1.8

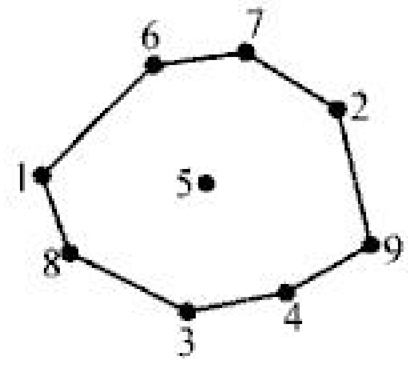

Fig. 1.9

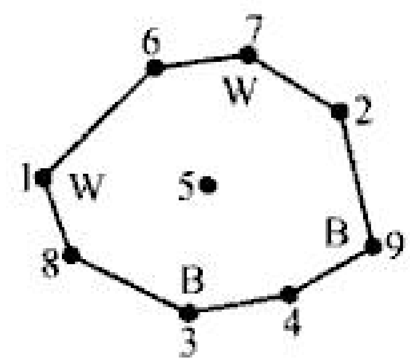

Fig. 1.10

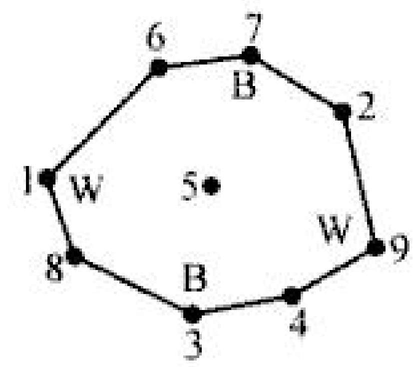

Fig. 1.11

Thus the beginning state in Fig. 1.6 and the state in Fig. 1.7 are represented by the two graphs as in Fig. 1.10, Fig. 1.11, respectively.

Obviously, the order of the knight on the circle cannot be changed from the state that two white knight are followed by two white knight into the state that white knight and black knight are interlaced. So it is impossible to change the states as required.

Example 3 There are n people $A_1, A_2, \ldots, A_n$ taking part in a mathematics contest, where some people know each other and any two people who do not know each other would have common acquaintance. Suppose that A_1 and A_2 know each other, but do not have common acquaintance. Prove that the acquaintances of A_1 are as many as those of A_2.

Proof Denote the n people $A_1, A_2, \ldots, A_n$ by n vertices $v_1, v_2, \ldots, v_n$. If two people know each other, then there is an edge joining the two corresponding vertices. Then we get a simple graph G. The vertices of G satisfy that any two nonadjacent vertices have a common neighbor. We shall prove two adjacent vertices v_1 and v_2 have the same number of neighbors.

The set of neighbors of the vertex v_1 is denoted by $N(v_1)$ and the set of neighbors of the vertex v_2 is denoted by $N(v_2)$. If there is a vertex v_i in $N(v_1)$ and $v_i \neq v_2$, then v_i is not in $N(v_2)$. Otherwise A_1 and A_2 have the common acquaintance A_i. Thus v_2 and v_i have a common neighbor v_j and $v_j \neq v_1$. So $N(v_2)$ contains v_j, as Fig. 1.12 shows. For v_i, v_k in $N(v_1)$, which are

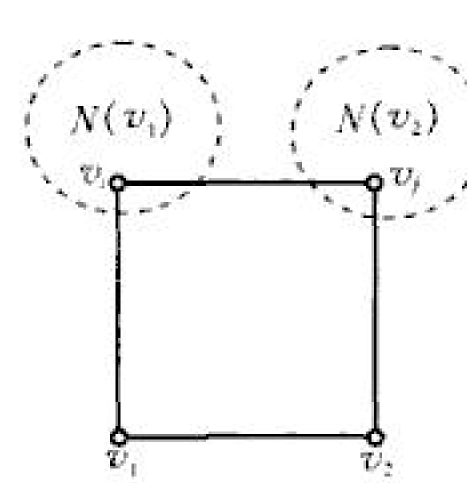

Fig. 1.12

distinct from v_2, both of them cannot be adjacent to a vertex v_j in $N(v_2)$, which is distinct from v_1. Otherwise, two nonadjacent vertices v_1, v_j have three common neighbors v_2, v_i, v_k. Therefore v_k in $N(v_1)$, which is distinct from v_k, must have a neighbor v_l in $N(v_2)$, which is distinct from v_j. So the number of vertices in $N(v_1)$ is not greater than that of $N(v_2)$. Similarly the number of vertices in $N(v_2)$ is not greater than that of $N(v_1)$. Thus the edges incident to v_1 are as many as those incident with v_2.

Example 4 Nine mathematicians meet at an international mathematics conference. For any three persons, at least two of them can have a talk in the same language. If each mathematician can speak at most three languages, prove that at least three mathematicians can have a talk in the same language. (USAMO 1978)

Proof Denote the 9 mathematicians by 9 vertices v_1, v_2, $\ldots$, v_9. If two of them can have a talk in the ith language, then there is an edge joining the corresponding vertices and color them with the ith color. Then we get a simple graph with 9 vertices and edges colored. Every three vertices have at least one edge joining them and the edges incident to a vertex are colored in at most three different colors. Prove that there are three vertices in graph G, any two of which are adjacent to the three edges colored with the same color. (This triangle is called *monochromatic* triangle.)

If the edges (v_i, v_j), (v_i, v_k) have the ith color, then the vertices v_j, v_k are adjacent and edge (v_j, v_k) has the ith color. Thus for vertex v_1, there are two cases:

(1) The vertex v_1 is adjacent to v_2, $\ldots$, v_9. By the pigeonhole principle, at least two edges, without loss of generality, denoted by (v_1, v_2), (v_1, v_3), have the same color. Thus triangle $\triangle v_1v_2v_3$ is a monochromatic triangle.

(2) The vertex v_1 is nonadjacent to at least one of v_2, $\ldots$, v_9. Without loss of generality, we suppose that v_1 is nonadjacent to v_2. For every three vertices there is at least one edge joining them, so there are at least seven edges from vertices v_3, v_4, $\ldots$, v_9 to the

vertex v_1 or v_2. From that we know at least four vertices of v_3, v_4, ..., v_9 are adjacent with vertex v_1 or v_2. Without loss of generality, we suppose that v_3, v_4, v_5, v_6 are adjacent to v_1, as it is shown in Fig. 1.13. Thus there must be two edges of (v_1, v_3), (v_1, v_4), (v_1, v_5), (v_1, v_6) which have the same color. Suppose (v_1, v_3), (v_1, v_4) have the same color, then $\triangle v_1v_3v_4$ is a monochromatic triangle.

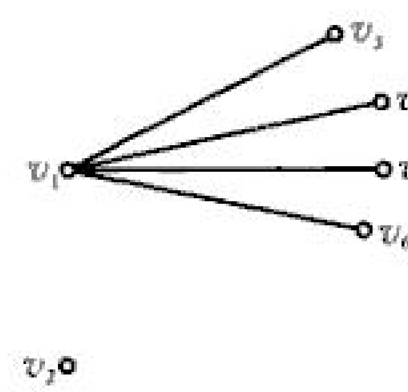

Fig. 1.13

Remark If the number 9 in the question is replaced by 8, then the proposition is not true. Fig. 1.14 gives a counterexample. Denote the 8 vertices by v_1, v_2, ..., v_8 and 12 colors by 1, 2, ..., 12, and there is no monochromatic triangle in the graph.

Fig. 1.14

The following example is the third question of national senior middle school mathematics contest in 2000.

Example 5 There are n people, any two of whom have a talk by telephone at most once. Any $n-2$ of them have a talk by telephone 3^m times, where m is a natural number. Determine the value of n. (China Mathematical Competition)

Solution Obviously $n \geqslant 5$. Denote the n persons by the vertices A_1, A_2, ..., A_n. If A_i, A_j have a talk by telephone, then there is an edge (A_i, A_j). Thus there is an edge joining two of the n vertices. Without loss of generality, we suppose that it is (A_1, A_2).

Suppose there is no edge joining A_1 and A_3. Consider $n-2$ vertices A_1, A_4, A_5, ..., A_n; A_2, A_4, A_5, ..., A_n and A_3, A_4,

$A_5, \ldots, A_n$. We know the number of edges joining any of A_1, A_2, A_3 to all of $A_4, A_5, \ldots, A_n$ is equal and we denote it by k.

Add A_2 to the set $A_1, A_4, A_5, \ldots, A_n$, then there are $S = 3^m + k + 1$ edges joining the $n-1$ vertices. Take away any vertex from $n-1$ vertices, the number of edges joining the remaining $n-2$ vertices is always 3^m. So there are $k+1$ edges joining every vertex and the remaining $n-2$ vertices. Therefore,

$$S = \frac{1}{2}(n-1)(k+1).$$

Similarly, add A_3 to the set $A_1, A_4, A_5, \ldots, A_n$. We get $n-1$ vertices and the number of edges is $t = 3^m + k = \frac{1}{2}(n-1)k$.

For $S = t + 1$, we have

$$\frac{1}{2}(n-1)(k+1) = \frac{1}{2}(n-1)k + 1,$$

that is $n = 3$. A contradiction. Thus there is an edge joining A_1, A_3.

Similarly, there is also an edge joining A_2 and A_3. Moreover, there must be edges joining A_1, A_2 and all A_i $(i = 3, 4, \ldots, n)$.

For A_i, A_j $(i \neq j)$, there is an edge joining A_i and A_1. So there is an edge joining A_i and A_j. Thus it is a complete graph. Therefore,

$$3^m = \frac{1}{2}(n-2)(n-3).$$

Hence we have $n = 5$.

Example 6 There are n $(n > 3)$ persons. Some of them know each other and others do not. At least one of them does not know the others. What is the largest value of the number of persons who know the others?

Solution Construct the graph G: denote the n persons by n vertices and two vertices are adjacent if and only if the two corresponding persons know each other.

For at least one of them does not know the others, in graph G there are at least two vertices which are not adjacent. Suppose that there is no edge $e = (v_1, v_2)$ joining v_1, v_2. Thus G must be $K_n - e$ if it has the most edges. That is the graph taken away an edge e from the complete graph K_n. The largest number of vertices which is adjacent with the remaining vertices is $n - 2$. So the largest number of people who know the others is $n - 2$.

The following example is from the 29th International Mathematical Olympiad (1988).

Example 7 Suppose that n is a positive integer and A_1, A_2, $\ldots$, A_{2n+1} is a subset of a set B.

Suppose that

(1) each A_i has exactly $2n$ elements;

(2) each $A_i \cap A_j (1 \leqslant i < j \leqslant 2n + 1)$ has exactly one element;

(3) each element of B belongs to at least two A_i's.

For which values of n can one assign to every element of B one of the number 0 and 1 in such a way that A_i has 0 assigned to exactly n of its elements?

Solution At first, the words "at least" in (3) can be replaced by "exactly". If there is an element $a_1 \in A_1 \cap A_{2n} \cap A_{2n+1}$, then each of the remaining $2n - 2$ subsets A_2, A_3, $\ldots$, A_{2n-1} has at most one element of A_1. Thus there is at least one element in A_1 but not in $A_2 \cup A_3 \cup \cdots \cup A_{2n-1} \cup A_{2n} \cup A_{2n+1}$.

It contradicts (3).

Construct the complete graph K_{2n+1}, where every vertex v_i represents a subset A_i and every edge $(v_i, v_j) = e_{ij} (1 \leqslant i, j \leqslant 2n + 1, i \neq j)$ represents the common element of A_i, A_j. So the question can be changed into: what property does n satisfy such that by as signing the edges of K_{2n+1} to 0 or 1, exactly n edges of the $2n$ edges incident to any vertex v_i are assigned to 0?

K_{2n+1} has $n(2n + 1)$ edges. If the required method of assigning can be met, then there are $\frac{1}{2}n(2n + 1)$ edges which are assigned to 0. So n must be even.

Conversely, if $n = 2m$ is even, we assign the edges (v_i, v_{i-m}),

$(v_i, v_{i-m+1}), \ldots, (v_i, v_{i-1}), (v_i, v_{i+1}), \ldots, (v_i, v_{i+m}), i = 1, 2, \ldots, 2n+1$, to 0, otherwise to 1 in K_{2n+1}. Then the method can meet the requirement. (Note that $v_{(2n+1)+i} = v_i$).

Therefore, the condition of the question is satisfied if and only if n is even.

The following problem is from the IMO preseleced questions in 1995.

Example 8 There are $12k$ persons attending a conference. Each of them shakes hands with $3k+6$ persons, where any two of them shake hands with the same number of people. How many persons are there in the conference?

Solution Suppose that for any two persons, they shake hands with n people. For one person a, the set of all the persons shaking hands with a is denoted by A and the set, the other persons by B. We know from the problem that $|A| = 3k+6$, $|B| = 9k-7$. For $b \in A$, n persons shaking hands with a, b are all in A. Therefore, b shakes hands with n persons in A and $3k+5-n$ persons in B. For $c \in B$, n persons shaking hands with a, c are all in A. Thus the number of persons in A who have shaken hands with someone in B is

$$(3k+6)(3k+5-n) = (9k-7)n,$$

$$n = \frac{(3k+6)(3k+5)}{12k-1}.$$

So $16n = \dfrac{(12k-1+25)(12k-1+21)}{(12k-1)}$.

Obviously, $(3, 12k-1) = 1$. So $(12k-1) \mid 25 \times 7$. For $12k-1$ divided by 4 leaves 3, $12k-1 = 7, 5\times 7, 5^2\times 7$. By calculating $12k-1 = 5\times 7$ has the only integer solution $k = 3$, $n = 6$.

Next we construct a figure consists of 36 points. Each point is incident to 15 edges and for any two points there are 6 points adjacent to them.

Naturally, we can use 6 complete graphs K_6. Divide the 36 points into 6 teams and label the points in the same team. We get a 6×6 square matrix

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{matrix}$$

For any point in the square matrix, it only connects with 15 points in the same row, in the same column, or having the same label. It is obvious that for any two persons there are 6 persons who have shaken hands with them.

Exercise 1

1 Consider the graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_5\}$, and $E = \{(v_1, v_2), (v_2, v_4), (v_3, v_4), (v_4, v_5), (v_1, v_3)\}$. Draw the graph G.

2 Let G be a simple graph, where $|V| = n$, $|E| = e$. Prove that $e \leqslant \frac{n(n-1)}{2}$.

3 Show the following two graphs are isomorphic.

Fig. 1. 15

4 There are n medicine boxes. Any two medicine boxes have the same kind of medicine inside and every kind of medicine is contained in just two medicine boxes. How many kinds of medicine are there?

5 There are n professors $A_1, A_2, \ldots, A_n$ in a conference. Prove that these n professors can be divided into two teams such that for every A_i, the number d_i of the people whom he has acquaintance with in another team is not less than d_i' in his team, $i = 1, 2, \ldots, n$.

6 There are 18 teams in a match. In every round, if one team competes with another team then it does not compete with the same team in another round. Now there have been 8 rounds. Prove that there must be three teams that have never competed with each other in the former 8 rounds.

7 n representatives attend a conference. For any four representatives, there is one person who has shaked hands with the other three. Prove that for any four representatives, there must be one person who shakes hands with the rest of the $n-1$ representatives.

8 There are three middle schools, each of which has n students. Every student has acquaintance with $n+1$ students in the other two schools. Prove that we can choose one student from each school such that the three students know each other.

9 There are $2n$ red squares on the a big chess board. For any two red squares, we can go from one of them to the other by moving horizontally or vertically to the adjacent red square in one step. Prove that all the red squares can be divided into n rectangules.

10 There are 2000 people in a tour group. For any four people, there is one person having acquaintance with the other three. What is the least number of people having acquaintance with all the other people in the tour group?

11 In a carriage, for any $m\,(m \geqslant 3)$ travelers, they have only one common friend. (If A is a friend of B, then B is a friend of A. Anyone is not a friend of himself.) How many people are there in the carriage?

12 There are five points A, B, C, D, E in the plane, where any three points are not on the same line. Suppose that we join some points with segments, called edges, to form a figure. If there are no above five points in the figure of which any three points are the vertices of a triangle in the figure, then there cannot be seven or more than seven edges.

Chapter 2 Degree of a Vertex

The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident to v, where each loop is counted as two edges. Moreover, when there is no scope for ambiguity, we omit the letter G from graph-theoretic symbols and write, for example, $d(v)$ instead of $d_G(v)$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G, or δ and Δ for brevity.

In Fig. 2. 1, $d(v_1) = 1$, $d(v_2) = 3$, $d(v_3) = d(v_4) = 2$, $\delta = 1$, $\Delta = 3$.

A vertex is *odd* if its degree is odd, otherwise, it is *even*. In Fig. 2. 1, v_1 and v_2 are odd vertices, and v_3 and v_4 are even.

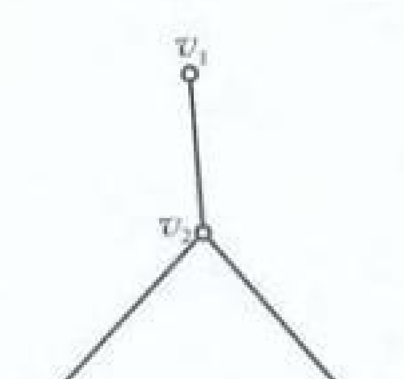

Fig. 2. 1

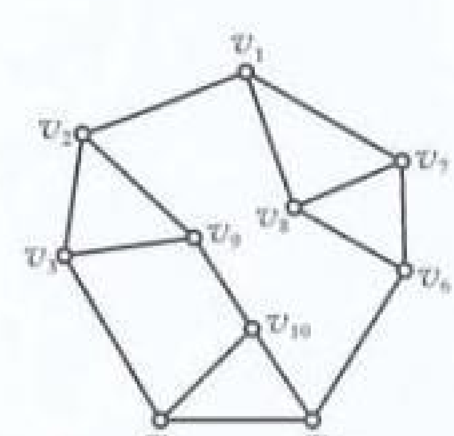

Fig. 2. 2

A graph $G = (V, E)$ is said to be *k-regular*, if $d(v) = k$ for all $v \in V$. The complete graph on n vertices is $(n-1)$-regular. Fig. 2. 2 shows a 3-regular graph.

The connection between the sum of the degrees of the vertices of a graph and the number of its edges is given as follows.

Theorem 1 For any graph G on n vertices, the sum of the degrees of all of the vertices is twice as large as the number of the edges. In symbols, if G with n edges has vertices v_1, v_2, $\ldots$, v_n, then

$$d(v_1)+d(v_2)+\cdots+d(v_n)=2e.$$

Proof The sum of the degrees of all of the vertices $d(v_1)+d(v_2)+\cdots+d(v_n)$ represents the whole number of the edges one of whose ends is $v_1, v_2, \ldots,$ or v_n. Since each edge has two ends, every edge of G is counted twice in the sum $d(v_1)+d(v_2)+\cdots+d(v_n)$. So the sum of the degrees of all of the vertices is twice as large as the number of the edges.

For instance, in Fig. 2.1, $e=4$,

$$d(v_1)+d(v_2)+d(v_3)+d(v_4)=1+3+2+2=8=2e.$$

Theorem 1 is often called the Hand-Shaking Lemma. A famous conclusion is given by Euler about two hundred years ago, that is to say, if many people shake hands when they meet, then the number of times of shaking hands is even. Then we can have the conclusion that there is an even number people who shake hands an odd number of times. The corollary is the following Theorem 2.

Theorem 2 In any graph G, the number of vertices with odd degree is even.

Proof Suppose that G has vertices $v_1, v_2, \ldots, v_n$, where $v_1, \ldots, v_t$ are odd vertices and $v_{t+1}, \ldots, v_n$ are even. According to Theorem 1,

$$d(v_1)+\cdots+d(v_t)+d(v_{t+1})+\cdots+d(v_n)=2e,$$

$$d(v_1)+\cdots+d(v_t)=2e-d(v_{t+1})-\cdots-d(v_n).$$

Since $d(v_{t+1})+\cdots+d(v_n)$ are all even, the right side of the equality is even. However $d(v_1), \ldots, d(v_t)$ are all odd, then t must be even so that $d(v_1)+\cdots+d(v_t)$ is even. That is, the number of vertices with odd degrees is even.

Example 1 Among $n\,(n>2)$ people, there are at least 2 persons, where the number of their friends are the same.

Solution We denote the n people by the vertices $v_1, v_2, \ldots, v_n$. If two persons are friends, we join the corresponding vertices. Then we get a graph. The assertion follows if we can find at least 2 vertices with the same degree in G.

A vertex is at most adjacent to other $n-1$ vertices in a simple graph on n vertices, so $d(v) \leqslant n-1$, for all $v \in V$. Hence the degree of a vertex in G can take only the following values:

$$0, 1, 2, \ldots, n-1.$$

However, not all of them are feasible. Note that a vertex with degree zero could not be adjacent to any other vertex and that the vertex with degree $n-1$ must be adjacent to any other $n-1$ vertices. So in G, only the following degrees are possible:

$$0, 1, 2, \ldots, n-2,$$

or

$$1, 2, 3, \ldots, n-1.$$

According to the pigeonhole principle, there are at least 2 vertices with the same degree.

Example 2 There are 24 pairs of contestants taking part in the International Table Tennis Mixed Doubles Contest. Some athletes shake hands before the game, and the two in one pair do not shake hands with each other. After the game, one male athlete asks everybody the number of hand-shaking, and all the answers are different. How many people does the male contestant's female partner shake hands with?

Solution The 48 vertices v, v_0, v_1, $\ldots$, v_{46} represent the 48 contestants where the male contestant is represented by v, with edges joining two people who had shaken hands before, then we can get a graph G. In graph G, $d(v_i) \leqslant 46$, $i = 0, 1, 2, \ldots, 46$, and $d(v_i) \neq d(v_j)$, if $i \neq j$. So except v, the degree of the other vertices are

$$0, 1, 2, \ldots, 45, 46.$$

Without loss of generality, we suppose that $d(v_i) = i$, for $i = 0, 1, 2, \ldots, 46$. Vertex v_{46} is adjacent to every vertex except v_0, so v_0 and v_{46} are partners. Deleting v_{46}, v_0 and the edges which are incident to

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Clearly, any graph with n vertices is n-partite graph.

Suppose that there is a k-partite graph $G = (V_1, V_2, \ldots, V_k; E)$ with $|V_i| = m_i$. A graph G is said to be a complete k-partite graph if any two vertices of G satisfy $u \in V_i$, $V \in V_j$, $i \neq j$, where $i, j = 1, 2, \ldots, k$, u and v are adjacent. We denote G by $K_{m_1, m_2, \ldots, m_k}$. Fig. 3.1 shows us a complete bigraph $K_{2,3}$. There are m^2 and $m(m+1)$ edges in complete bigraphs $K_{m,m}$ and $K_{m,m+1}$, so the number of edges of the graph is $\left[\frac{n^2}{4}\right]$, where n is the number of vertices in G. (Here $[x]$ denotes the largest integer no more than x.) Complete bigraphs $K_{m,m}$ and $K_{m,m+1}$ contain no triangle. In Theorem 1, we can see that these two kinds of graphs contain the most number of edges among the graphs without triangles.

Theorem 1 If a graph G with n vertices contain no triangle, the largest number of edges of G is $\left[\frac{n^2}{4}\right]$.

Proof Assume that v_1 is the vertex with the maximum degree in G, $d(v_1) = d$ and we denote d vertices adjacent to v_1 by

$$v_n, v_{n-1}, \ldots, v_{n-d+1}.$$

Since G contains no triangle, any two of $v_n, v_{n-1}, \ldots, v_{n-d+1}$ are not adjacent. So the number of edges of G satisfies

$$\begin{aligned} e &\leqslant d(v_1) + d(v_2) + \cdots + d(v_{n-d}) \\ &\leqslant (n-d) \cdot d \leqslant \left(\frac{n-d+d}{2}\right)^2 \\ &= \frac{n^2}{4}. \end{aligned}$$

Since e must be an integer, $e \leqslant \left[\frac{n^2}{4}\right]$.

The upper bound can be met only when $G = K_{m,m}$ if $n = 2m$ and $G = K_{m,m+1}$ if $n = 2m + 1$.

We can also use induction to prove the theorem, and we leave it as an exercise.

Example 1 Suppose that there are 20 vertices and 101 edges in a

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(1) If we join the points randomly and get 13 line segments. Prove that there must exist four points so that each of them is adjacent to any other three points.

(2) If there are only 12 line segments joining these points. Draw a graph to show that the conclusion of (i) is not true.

(3) Can the conclusion of (i) be modified so that there must exist four copies of K_4? Give a counterexample or prove it.

Solution

(1) We can phrase the problem in the language of graph theory: There are 6 vertices and 13 edges in a graph G, prove that the G contains K_4.

It is easy to calculate $e_4(6) = 12 < 13$. According to Theorem 2, we know that G must contain K_4.

(2) Consider the complete 3-partite graph $K_{2,2,2}$. According to Fig. 3.4, we choose any 4 vertices from $K_{2,2,2}$ and there must be 2 vertices belonging to one part. These two vertices are not adjacent. So the 4 vertices we choose arbitrarily cannot form a K_4.

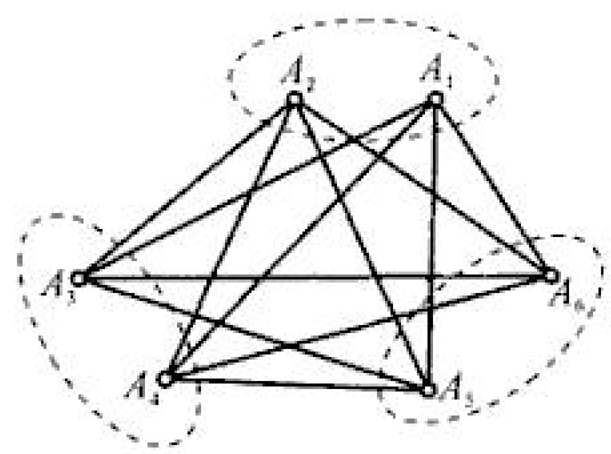

Fig. 3.4

Remark (1) Of course, we can use Theorem 2 to prove it and there are many other ways. Here we list two different methods.

(1) Since the sum of the degrees of 6 vertices is $2 \times 13 = 26$, there are at least 2 vertices whose degrees are 5 among the 6 vertices. Otherwise, the sum of degrees is $5 + 5 \times 4 = 25 < 26$. Without loss of generality, suppose that $d(A_1) = d(A_2) = 5$, there are 9 edges incident to A_1 or A_2. According to Fig. 3.5, there are $13 - 9 = 4$ edges joining A_3, A_4, A_5, A_6. Two ends of any of the four edges together with A_1, A_2 can form a K_4.

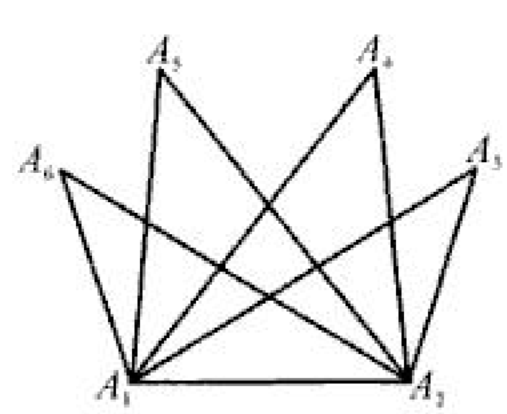

Fig. 3.5

(2) Since there are 15 edges in a complete graph with 6 vertices, we delete two edges. We discuss the problem in two different cases.

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$$\leqslant \frac{1}{2} \cdot \frac{1}{n}\left(\sum_{i=1}^{n} d(v_i)\right)^2 - e$$
$$= \frac{2}{n}e^2 - e.$$

So

$$\frac{2}{n}e^2 - e \leqslant \binom{n}{2},$$
$$e^2 - \frac{n}{2}e - \frac{1}{4}n^2(n-1) \leqslant 0.$$

We solve the above inequality and get

$$e \leqslant \frac{n}{4}(1 + \sqrt{4n-3}).$$

Remark This problem tells us an upper bound of the number of edges of graphs which contain n vertices and no quadrangle. But it is not the maximum number. For general n, we can do a further research into its maximum number. Example 4 has showed when $n = 8$, the maximum number is 11.

Example 6 There are n vertices and l edges in a graph. Then $n = q^2 + q + 1$, $l \geqslant \frac{1}{2}q(q+1)^2 + 1$, $q \geqslant 2$, $q \in \mathbf{N}$.

We know that any four points in the graph do not lie on one plane and every point must lie on at least one line. So there exists a point that lies on at least $q+2$ lines. Prove that the graph must contain a quadrangle in the space, consisting of four points A, B, C, D and four lines AB, BC, CD, DA. (China Mathematical Competition in 2003)

Solution The condition that any four points cannot lie on a plane is to ensure that there are no three points on a line. So in terms of graph theory, we only need to prove that the graph contains a quadrangle. To solve this problem, we need to use the idea of Example 5, but we cannot use it directly. Consider the removal of the $d(v_1)$ vertices which are adjacent to v_1 ($d(v_1) \geqslant q+2$). There will be $\binom{n-d(v_1)}{2}$ pairs of vertices left.

x and y by $d(x, y)$, $d(x_j, x_k)$ is more than $\frac{\sqrt{2}}{2}$ for all j, k and $\angle x_i x_j x_k \geqslant 90^\circ$, then

$$d(x_i, x_k) \geqslant \sqrt{d^2(x_i, x_j) + d^2(x_j, x_k)} > 1.$$

Since the diameter of the vertex set S is 1, among any four vertices in G there is at least one pair whose vertices are not adjacent. It means that G contains no K_4.

According to Theorem 2 the number of edges of G is no more than $e_3(n) = \left[\frac{n^3}{3}\right]$.

We can construct a vertex set $\{x_1, x_2, \ldots, x_n\}$ which contains $\left[\frac{n^3}{3}\right]$ vertex pairs so that the distance of two vertices in each pair is more than $\frac{\sqrt{2}}{2}$. The construction is as follows. Choose r so that $0 < r < \frac{1}{4}\left(1 - \frac{\sqrt{2}}{2}\right)$. Then draw three circles whose radii are all 1 and the distance of any two of their centers is all $1 - 2r$. As Fig. 3.12 shows us, we put $x_1, x_2, \ldots, x_{\left[\frac{n}{3}\right]}$ in a circle, $x_{\left[\frac{n}{3}\right]+1}, \ldots, x_{\left[\frac{2n}{3}\right]}$ in another circle and $x_{\left[\frac{2n}{3}\right]+1}, \ldots, x_n$ in the third circle so that the distance of x_1 and x_n is 1. Obviously, the diameter of this set is 1. Furthermore, $d(x_i, x_j) > \frac{\sqrt{2}}{2}$ if and only if x_i and x_j belong to different circles. So there exist exactly $\left[\frac{n^3}{3}\right]$ vertex pairs (x_i, x_j) such that $d(x_i, x_j) > \frac{\sqrt{2}}{2}$.

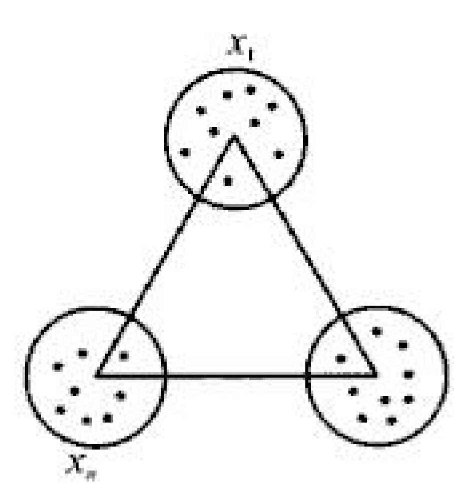

Fig. 3. 12

Exercise 3

1 Prove that if a bigraph $G = (X, Y; E)$ is δ-regular, then

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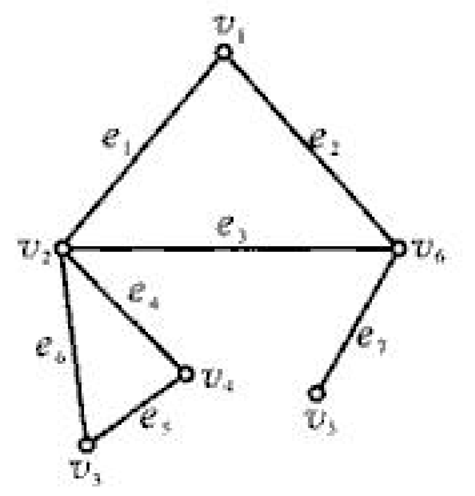

Fig. 4. 1

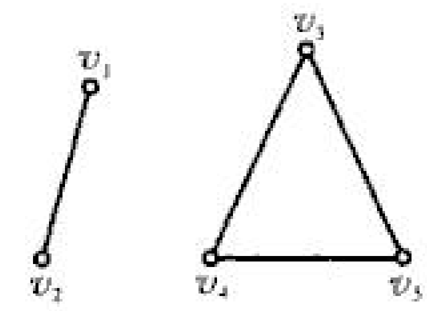

Fig. 4. 2

A connected graph which contains no cycle is called a *tree*. We usually denote a tree by T.

According to the definition of tree, tree is obviously a simple graph. A tree with eight vertices is shown in Fig. 4. 3. Clearly, a graph without cycles must be composed of one or several trees whose vertices are disjoint. We call such graph a *forest*.

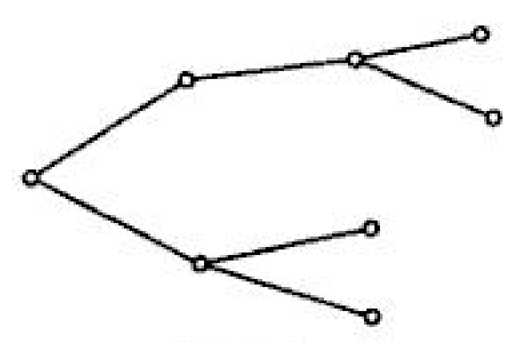

Fig. 4. 3

The graph in Fig. 4. 4 is a forest, which is composed of three trees. The vertex with degree 1 is called a *pendant* vertex (or leaf).

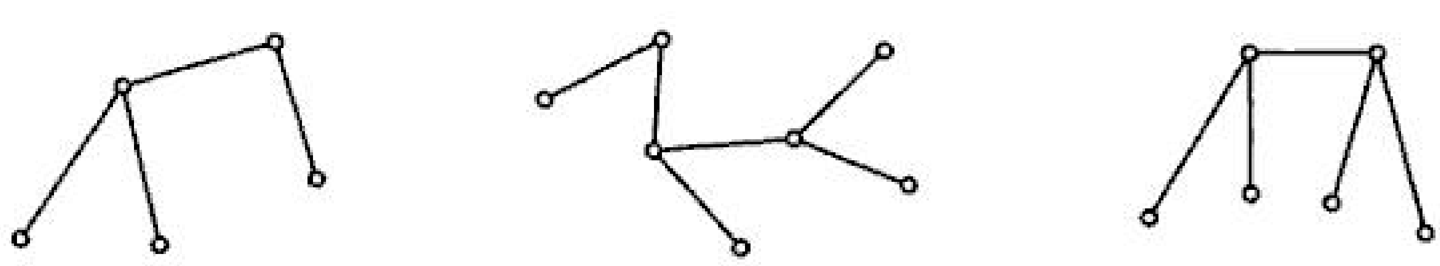

Fig. 4. 4

Theorem 1 If a tree T has no less than 2 vertices, then T contains at least two pendant vertices.

Proof 1 Suppose we start from some vertex u, along the edges of T, every edge can only be passed by once. Since a tree has no cycle, it cannot return to the vertices which have been passed. It means that each vertex can be passed at most once. If the vertex we pass is not a pendant vertex, because its degree is more than 1, we can continue. But the number of vertices of T is finite, so it is impossible to continue forever. If we cannot continue further at v, then v is a pendant

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cycle, we delete an edge on the cycle and obtain a graph G_2, and so on until the graph obtained has no cycle. The graph is certainly a tree. It has $n-1$ edges, so graph G has at least $n-1$ edges.

The above tree obtained is called the generating tree of graph G. Adding several edges to the generating tree, we can get the original graph.

Example 4 In a certain region, twenty players of a tennis club have played fourteen singles. Every person plays at least once. Prove that there are six pairs singles, in which twelve players are distinct.

Proof This question has occurred in Chapter 2. Here we prove it from the viewpoint of tree.

Denote twenty players by twenty vertices. If two persons have played a game, add an edge between them. There are fourteen edges in all, each vertex is incident to at least one edge. Our conclusion is equivalent to: it is possible to find six edges so that any two of them are not adjacent.

Suppose the graph has n connected branches, among which the ith branch has v_i vertices, e_i edges. Clearly, $e_i \geqslant v_i - 1$, so

$$\sum_{i=1}^{n} e_i \geqslant \sum_{i=1}^{n} (v_i - 1) = \sum_{i=1}^{n} v_i - n.$$

But $\sum_{i=1}^{n} e_i = 14$, $\sum_{i=1}^{n} v_i = 20$, so $14 \geqslant 20 - n$, $n \geqslant 20 - 14 = 6$. Since every vertex is incident to at least one edge, it is impossible that there exists a connected branch, which contains only one isolated vertex. Hence choose an edge from every connected branch, which promises that they are not adjacent and the number of edges is at least six. We finish the proof.

From Fig. 4.5, the number of vertices is twenty, the number of edges is fourteen. Choose arbitrarily seven edges, then there must be two edges that are in the same connected branch and adjacent. Hence six is the best possible.

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$$v_{11}^{(1)}, v_{11}^{(2)}, \ldots, v_{11}^{(179)}, v_{11}^{(180)},$$

as required.

Exercise 4

1 If the number of vertices of a connected graph G is no less than 2, then there exists at least two vertices in graph G. After removing the two vertices and their adjacent edges, the graph is still connected. (A graph without vertices is also considered as a connected graph)

2 On a coordinate plane, eleven vertical lines and eleven horizontal lines constitute a graph. The vertices of the graph are the points of intersection of the vertical and horizontal lines (lattice points), the edges are the vertical and horizontal segments between two lattice points. How many edges at least should be removed so that the degree of each vertex is less than four? How many edges at most should be deleted so that the graph keeps connected?

3 If graph G has n vertices and $n-1$ edges, then graph G is a tree. Is this proposition correct? Why?

4 A tree T has three vertices of degree 3, one vertex of degree 2 and other vertices are all pendant vertices. (1) How many pendant vertices are there in T? (2) Draw two trees which satisfy the above requirement of degrees.

5 A tree has n_i vertices whose degrees are i, $i = 1, 2, \ldots, k$. If the numbers $n_2, \ldots, n_k$ are all known, what is n_1? If n_r $(3 \leqslant r \leqslant k)$ is not known, and n_j $(j \neq r)$ is known, what is n_r?

6 Let $d_1, d_2, \ldots, d_n$ be n positive integers, $n \geqslant 2$, and $\sum_{i=1}^{n} d_i = 2n-2$. Prove that there exists a tree where the degrees of its vertices are $d_1, d_2, \ldots, d_n$.

7 There are $n (n \geqslant 3)$ segments on the plane where any three of

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$v_0 v_1 \ldots v_0$.

If μ_1 is G itself, then the proposition holds. Otherwise we can find a subgraph G_1 by deleting μ_1 in G, then every vertex in G_1 is also an odd vertex. Since G is connected, then there must exist a common vertex u in μ_1 and in G_1 and a cycle μ_2 from u to u in G_1. So μ_1 and μ_2 still constitute a cycle. Repeat the above process. Since G has only a finite number of edges in all, the cycle finally obtained is graph G itself.

Now we prove the second case. Suppose G is connected, and the number of odd vertices is 2. Let u, v be the two odd vertices, add an edge e between u and v, we get graph G'. Hence the number of odd vertices is 0 in G', so G' is a cycle. Therefore, after deleting e, G is a chain.

We call a cycle in a graph an Euler tour if it traverses every edge of the graph exactly once. A graph is Eulerian if it admits an Euler tour.

Furthermore, there is the following question: If the number of odd vertices in a connected graph G is not 0 or 2, then by lifting one's pen how many times can G be drawn? We know that the number of odd vertices is even, so we have the following conclusion.

Theorem 2 If G is connected and has $2k$ odd vertices, then graph G can be drawn by lifting one's pen k times and at least k times.

Proof Divide these $2k$ odd vertices into k pairs: v_1, v'_1; v_2, v'_2; $\ldots$; v_k, v'_k, add an edge e_i between v_i and v'_i and obtain G'. Graph G' has no odd vertex, so G' is a cycle. Delete these k added edges, then this cycle is divided into at most k parts, i.e. k chains. This indicates that G can be drawn by lifting one's pen k times.

Suppose G is divided into h chains, each chain has at most two odd vertices. Hence $2h \geqslant 2k$, i.e. $h \geqslant k$. Graph G can be drawn by lifting one's pen at least k times.

Example 1 Fig. 5.3 is a plane graph of a building and there is a living room. After entering the front door into the living room, there are four other rooms. If you enter from the front door, can you enter

Note By the conclusion of this example, the famous Brouwer fixed point theorem follows.

Example 6 A graph which consists of a convex n-polygon and $n-3$ disjoint diagonal lines in the polygon is called a subdivision graph.

Prove that there exists a subdivision graph which is a cycle drawn without lifting one's pen (i. e. start from a vertex, go through each segment only once and return the starting point) if and only if $3 \mid n$. (The 5th China Mathematical Competition)

Proof First prove by induction that the condition $3 \mid n$ is sufficient.

When $n = 3$, clearly the proposition holds.

Suppose for any convex $3k$-polygon, there exists a subdivision graph which is a cycle drawn without lifting one's pen. For a convex $3(k+1) = 3k+3$ -polygon $A_1A_2A_3 \ldots A_{3k+3}$, join A_4A_{3k+3}. Since $A_4A_5 \ldots A_{3k+3}$ is a convex $3k$-polygon, by induction, $A_4A_5 \ldots A_{3k+3}$ contains a subdivision graph which is a cycle drawn without lifting one's pen. Construct this subdivision graph and join A_2A_4, A_2A_{3k+3}, so we obtain a subdivision graph of a convex $3(k+1)$ -polygon $A_1A_2A_3 \ldots A_{3k+3}$. Since the subdivision graph of $A_4A_5 \ldots A_{3k+3}$ is a cycle, we start from A_{3k+3}, go through each edge of the subdivision graph only once and return to A_{3k+3}. Then go through $A_{3k+3}A_1$, A_1A_2, A_2A_3, A_3A_4, A_4A_2, A_2A_{3k+3}, and return to A_{3k+3}, again. This proves that for any convex $(3k+3)$ -polygon, there also exists a subdivision graph, which is a cycle drawn without lifting one's pen. So the sufficiency has been proved.

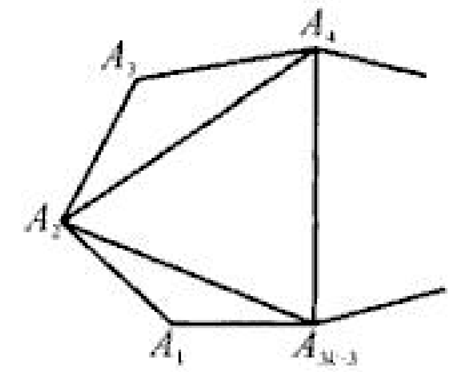

Fig. 5. 12

Next prove the necessity. Assume that a convex n-polygon has a subdivision graph, which is a cycle drawn without lifting one's pen. Then each vertex of the graph is an even vertex. Clearly a convex quadrangle and a convex pentagon do not have a subdivision graph such that each vertex is an even vertex. Thus when $3 \leqslant n < 6$, if a

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4 Choose arbitrarily n ($n > 2$) vertices, and join each vertex to all other vertices. Can you draw these segments without lifting one's pen, so that they join end to end and finally return to the starting point?

5 If at a conference, each person exchanges views with at least $\delta \geqslant 2$ persons. Prove that it is definitely possible to find k persons v_1, v_2, ..., v_k, such that v_1 changes opinion with v_2, v_2 exchanges views with v_3, ..., v_{k-1} changes opinion with v_k, and v_k exchanges views with v_1, where k is an integer greater than δ.

6 As shown in Fig. 5.16, graph G has 4 vertices, and 6 edges. They are all on a common plane. This plane is divided into 4 regions Ⅰ, Ⅱ, Ⅲ, Ⅳ, and we call them regions faces. Suppose there are two points Q_1, Q_2 on these faces. Prove that there is no line μ joining Q_1 and Q_2 which satisfies: (1) μ cuts across each edge only once; (2) μ does not go through any vertex v_j ($j = 1, 2, 3, 4$).

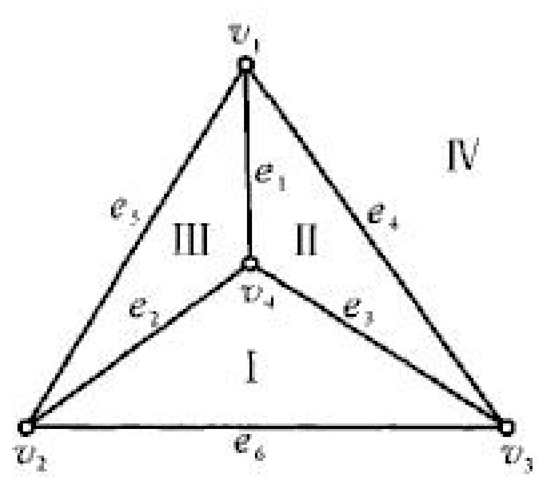

Fig. 5.16

7 Arrange n vertices v_1, v_2, ..., v_n in order on a line. Each vertex is colored in red or blue. If the ends of a segment $v_i v_{i+1}$ are colored differently, we call it a standard segment. Suppose the colors of v_1 and v_n are different. Prove that the number of the standard segments is odd.

8 Choose some points on the edges and in the interior of $\triangle ABC$. Divide $\triangle ABC$ into various small triangles. Each two small triangles has either a common vertex, or a common edge, or no common vertex at all. Use A, B or C to label those vertices in the interior of $\triangle ABC$. Use A or B to label the vertices on the edge AB of the big triangle,

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common language is labelled on each corresponding edge in Fig. 6.3.

Example 3 Determine whether the graph G in Fig. 6.4 contains a Hamiltonian chain or cycle?

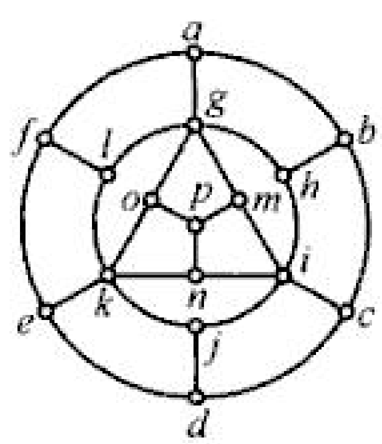

Fig. 6.4

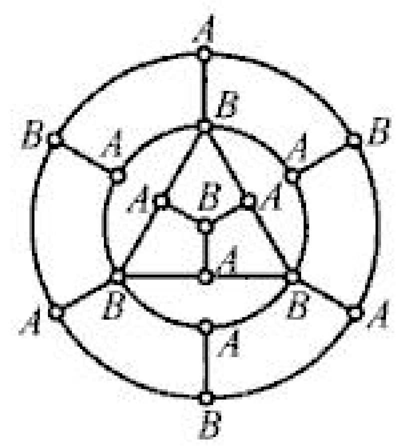

Fig. 6.5

Solution We mark one vertex in graph G as A. For example, we mark the vertex a as A and all the vertices adjacent to the vertex a as B all the vertices adjacent to B as A. Then we mark the vertices adjacent to the vertex which is marked B as A and the vertices adjacent to the vertex which is marked A as B until we mark all the vertices. As Fig. 6.5 shows us, if G contains a Hamiltonian cycle, the cycle must go through A and B in turn. So the difference between the numbers of A and B is no more than 1. But in Fig. 6.5, there are nine A vertices and seven B vertices. The difference is 2, so there are no Hamiltonian chain.

Generally, to a bigraph $G = (V_1, V_2, E)$, there is a simple method to see whether the graph contains a Hamiltonian chain or a Hamiltonian cycle.

Theorem 1 In a bigraph $G = (V_1, V_2, E)$, if $|V_1| \neq |V_2|$, G must contain no Hamiltonian cycle. If the difference between $|V_1|$ and $|V_2|$ is more than 1, G must contain no Hamiltonian chain.

We can use the same method as Example 3 to prove it.

Example 4 Fig. 6.6 shows us half of a chessboard. A knight is at the bottom right corner. Can the knight move along every square continually once only? What happens if we delete the black panes at

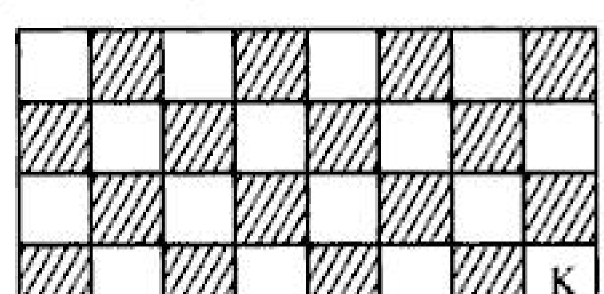

Fig. 6.6

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Without loss of generality, G is the graph which satisfies the given condition and contains the most number of edges. In other words, after adding an edge to G, G contains a Hamiltonian cycle. Otherwise, G can be added several edges until we can not add edges. After adding edges, the degrees of vertices satisfy the condition of degree. Then we get a Hamiltonian chain contains every vertex of G. We denote the chain by $v_1 v_2 \ldots v_n$, then v_1 is not adjacent to v_n. So

$$d(v_1) + d(v_n) \geqslant n.$$

Then among $v_2, v_3, \ldots, v_{n-1}$, there must be a vertex v_i so that v_1 is adjacent to v_i and v_n is adjacent to v_{i-1} as Fig. 6. 11 shows us. Otherwise, there are $d(v_1) = k$ vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ $(2 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n-1)$ adjacent to v_1, and v_n is not adjacent to $v_{i_1 - 1}$, $v_{i_2 - 1}, \ldots, v_{i_k - 1}$. So

$$d(v_n) \leqslant n - 1 - k,$$

then

$$d(v_1) + d(v_n) \leqslant k + n - 1 - k = n - 1 < n,$$

which contradicts the condition. So G contains a Hamiltonian cycle $v_1 v_2 \ldots v_{i-1} v_n v_{n-1} \ldots v_i v_1$, which also contradicts the hypothesis. We complete the proof.

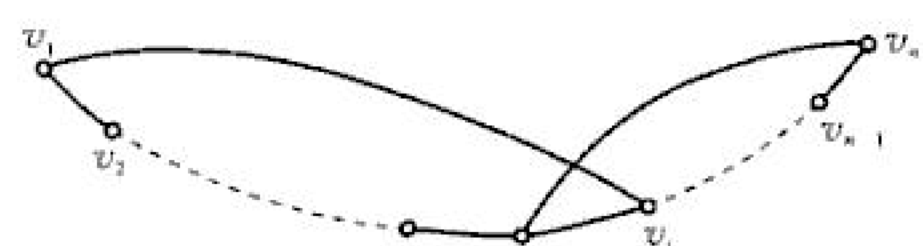

Fig. 6. 11

For the complete graph $K_n (n \geqslant 3)$, clearly there is Hamiltonian cycle.

Example 5 n persons take part in a conference. During the conference time, everyday they must sit at a round table to have dinner. Every evening, every person must sit beside different persons. How many times at most will there be such dinners?

We get two results which contradict each other. So the proof is complete.

Exercise 6

1 What is the value n so that the complete graph K_n is a Hamiltonian graph? What are the values m, n so that the complete bigraph $K_{m,n}$ is a Hamiltonian graph?

2 The graph representing the regular tetrahedron, hexahedron, octahedron or icosahedron is a Hamiltonian graph.

3 We use paper to construct an octahedron. Can we cut it into two parts so that every face is also cut into two parts and the cutting lines do not go through the vertices of the octahedron?

4 A mouse eats the cheese whose size is the same as $3 \times 3 \times 3$ cube. The way to eat it is to get through all the 27 of the $1 \times 1 \times 1$ subcube. If the mouse begins from one corner, then goes to the next subcube which has not been eaten. Can the mouse be at the center when he has eaten the cheese.

5 We divide 6 persons into 3 groups to finish 3 missions. There are 2 persons in every group. Everyone can cooperate with at least 3 persons among the other 5 persons. (1) Can the two persons of every group cooperate with each other? (2) How many distinct grouping 6 persons into 3 groups can you give?

6 A king has $2n$ ministers, among whom there are several ministers hate each other. But the number of persons every minister hate is no more than $n-1$. Can they sit in a round table so that no two adjacent ministers hate each other?

7 Among 9 children, every child knows at least four children. Can these children be arranged in a line so that every child know the child beside him?

8 A chef uses eight materials to do the cooking. He should use two materials for each dish. Every material should be used in at least

belongs to the boundaries of at most 2 faces. So $2e \geqslant 3f$, and $f \leqslant \frac{2}{3}e$.

Use Euler's Formula

$$2 = v - e + f \leqslant v - e + \frac{2}{3}e,$$

that is

$$e \leqslant 3v - 6.$$

This proves the following theorem.

Theorem 2 For a connected simple planar graph G with v $(v \geqslant 3)$ vertices and e edges, then

$$e \leqslant 3v - 6.$$

In fact, Theorem 2 also holds for disconnected simple planar graphs. Theorem 2 can be used to determine whether a graph is a planar graph or not.

Example 1 Prove that the complete graph K_5 is not a planar graph.

Proof Since $v = 5$, $e = 10$ do not satisfy $e \leqslant 3v - 6$. So K_5 is not a planar graph.

Example 2 Prove that $K_{3,3}$ is not a planar graph.

Proof Suppose that $K_{3,3}$ is a planar graph. Since we choose 3 vertices randomly in $K_{3,3}$, there must be 2 vertices which are not adjacent to each other. Therefore, each face has at least 4 edges as its boundary. By

$$4f \leqslant 2e, \ f \leqslant \frac{e}{2}.$$

Use Euler's Formula

$$2 = v - e + f \leqslant v - e + \frac{e}{2},$$

that is

$$e \leqslant 2v - 4.$$

8 If a polyhedron containing n edges exists, what is the value of n?

9 A convex polyhedron has $10n$ faces. Prove that there are n faces which have the same number of edges.

10 Prove that the graph in Fig. 7.9 has no Hamiltonian cycle.

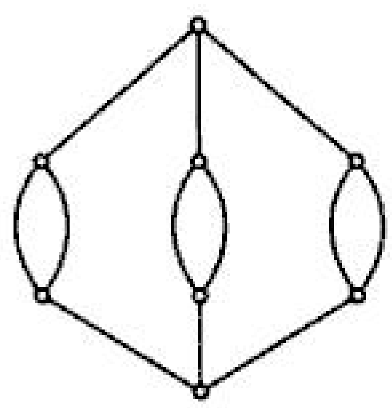

Fig. 7.9

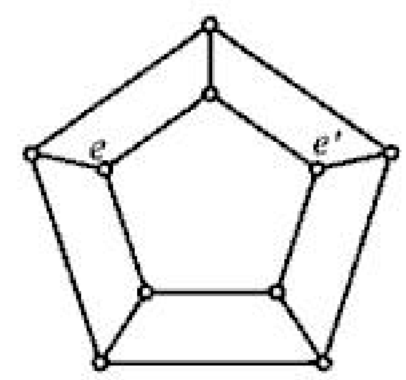

Fig. 7.10

11 The graph in Fig. 7.10 contains a Hamiltonian cycle. Prove that for any Hamiltonian cycle, if it contains edge e, then it must not contain edge e'.

12 Let $S = \{x_1, x_2, \ldots, x_n\}$ $(n \geqslant 3)$ be the set of vertices on the plane. The distance between two arbitrary vertices is no less than 1. Prove that there are at most $3n - 6$ pairs of vertices whose distances are 1.

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We construct a graph G with 9 vertices, 32 edges and in two colors. We color the edges joining the vertex v_1 and v_2, v_3, v_8, v_9 by red (a solid line) while we color the edges joining the vertex v_1 and v_4, v_5, v_6, v_7 by blue (a dotted line). We divide the vertices other than v_1 into four groups: Ⅰ: (v_2, v_3); Ⅱ: (v_4, v_5); Ⅲ: (v_6, v_7); Ⅳ: (v_8, v_9). We call Ⅰ and Ⅱ, Ⅱ and Ⅲ, Ⅲ and Ⅳ adjacent groups. Except v_1, two vertices belonging to one group are not adjacent, two vertices belonging to two different adjacent groups are joined by a solid line (red) and two vertices belonging to two different groups which are not adjacent are joined by a dotted line (blue). Fig. 8. 4 shows that there are $\binom{9}{2} - 4 = 32$ edges in the graph G which contains 16 red edges (solid lines) and 16 blue edges (dotted lines). It is not difficult to know that G contains no monochromatic triangle. So $n \geqslant 33$.

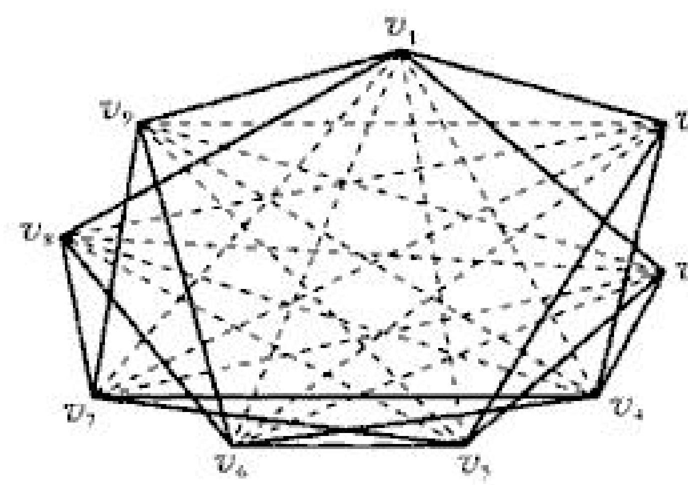

Fig. 8. 4

Next let us prove $n \leqslant 33$. Suppose 33 edges connecting have been colored. While there are 3 edges which have not been colored. Without loss of generality, we denote the 3 edges by e_1, e_2, e_3. Choose one end v_1, v_2, v_3 from e_1, e_2, e_3, respectively. Then we delete the three vertices from K_9 and the remaining 6 vertices form a graph K_6. So if we color the graph by red and blue, the graph must contain a monochromatic triangle.

So $n = 33$.

To generalize Theorem 1, we first need to increase the number of colors.

We use k colors c_1, c_2, $\ldots$, c_k to color the complete graph K_n. We call the complete graph K_n k-color complete graph K_n if every edge is colored in one color. We can imagine if n is large enough, k-color complete graph K_n must contain monochromatic triangle. We denote the least n by r_k. In Theorem 1, $r_2 = 6$. It is clear that $r_1 = 3$.

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We suppose that $A_1A_2A_3A_4A_5$ is red and $B_1B_2B_3B_4B_5$ is blue. Without loss of generality, let A_1B_1 be a blue edge. By the assumption that every triangle is not monochromatic, we can know that A_1B_5 and A_1B_2 are all red edges. So A_2B_2 and A_5B_5 are blue edges. Similarly, A_5B_1, A_5B_4, A_2B_1, A_2B_3, ... are all red edges and A_3B_3, A_4B_4 are blue edges. So A_4B_1 and A_4B_2 are blue edges and we can get a blue triangle $\triangle A_4B_1B_2$. It is a contradiction.

So the ten edges in the top and bottom faces are monochromatic.

Example 9 There are two international airlines X and Y serving 10 districts. For any two districts, there is only one company providing a direct flight (to and fro). Prove that there must be a company which can provide two tour routes so that the two routes do not pass through the same districts and each route passes through an odd number of districts.

Proof We denote the 10 districts by 10 vertices u_1, u_2, ..., u_{10}. If the flight between u_i and u_j is provided by X, then we join u_i and u_j by a red edge (a solid line); If the flight between u_i and u_j is provided by Y, then we join u_i and u_j by a blue edge (a dotted line). Then we can get a 2-color complete graph K_{10}. In order to prove the conclusion, it suffices to prove that there must be two monochromatic triangles or polygons having no common edge and an odd number of edges in K_{10}.

The 2-color complete graph K_{10} contains a monochromatic triangle. Let $\triangle u_8u_9u_{10}$ be a monochromatic triangle. By Example 1, we can know that the triangles constructed by the vertices u_1, u_2, ..., u_7 must contain a monochromatic triangle. Let $\triangle u_5u_6u_7$ be a monochromatic triangle. If the color of $\triangle u_5u_6u_7$ is the same as that of $\triangle u_8u_9u_{10}$, the conclusion holds. Then let $\triangle u_5u_6u_7$ be red and $\triangle u_8u_9u_{10}$ be blue.

The number of edges joining the vertex sets $\{u_5, u_6, u_7\}$ and $\{u_8, u_9, u_{10}\}$ is $3 \times 3 = 9$. By the Pigeonhole Principle, there must be five monochromatic edges. Let them be red edges. The five edges are induced by $\{u_8, u_9, u_{10}\}$, so there must exist a vertex which is

the vertex v_r. If P_i defeats P_j, we join v_i and v_j to get an arc (v_i, v_j). So w_r and l_r are the indegree and outdegree of v_r respectively. By Theorem 1,

$$w_1 + w_2 + \cdots + w_n = l_1 + l_2 + \cdots + l_n.$$

Note that $w_i + l_i = n - 1$ $(1 \leqslant i \leqslant n)$,

$$\begin{aligned}
& w_1^2 + w_2^2 + \cdots + w_n^2 - (l_1^2 + l_2^2 + \cdots + l_n^2) \\
= & (w_1^2 - l_1^2) + (w_2^2 - l_2^2) + \cdots + (w_n^2 - l_n^2) \\
= & (w_1 + l_1)(w_1 - l_1) + (w_2 + l_2)(w_2 - l_2) + \cdots + (w_n + l_n)(w_n - l_n) \\
= & (n-1)[(w_1 + w_2 + \cdots + w_n) - (l_1 + l_2 + \cdots + l_n)] = 0.
\end{aligned}$$

So

$$w_1^2 + w_2^2 + \cdots + w_n^2 = l_1^2 + l_2^2 + \cdots + l_n^2.$$

In a directed graph $D = (v, u)$, there exists a sequence of distinct arcs $u_1, u_2, \ldots, u_n$. If the starting point of u_i is v_i and the end point of u_i is v_{i+1} $(i = 1, 2, \ldots, n)$. We call n the length of the directed path. v_1 is the starting point of the path and v_{n+1} is the end point. If $v_1 = v_{n+1}$, we call the path a *circuit*.

Example 2 The MO space city consists of 99 space stations. Any two stations are connected by a channel. Among these channels there are 99 two-way channels and others are one-way channels. If four space stations can be arrived at from one to another, we call the set of four space stations a connected four-station group.

Design a scheme for the space city so that we get the maximum number of connected four-station groups. (Find the exact number and prove your conclusion.) (The 14th China Mathematical Olympiad)

Solution We call an unconnected four-station group a bad four-station group. A bad four-station group has three possible situations:

(1) Station A has three channels AB, AC, AD which all leave A.

(2) Station A has three channels which all arrive at A.

(3) Stations A and B, stations C and D have two-way channels but the channels AC, AD all leave A, and BC, BD all leave B.

We denote all the bad four-station groups in (1) by S and others

4 Construct a graph as follows. We denote a medicine-chest by a vertex. We denote every two medicine-chests v_i and v_j which contain one common medicine by the edges (v_i, v_j). By the hypothesis, the graph is a complete graph K_n and the number of the kinds of the medicine is equal to the number of the edges $\frac{1}{2}n(n-1)$.

5 We denote n professors by n vertices $v_1, v_2, \ldots, v_n$ and join by an edge if the two corresponding persons know each other. We divide the n persons into two groups randomly. There are finite ways to divide them. Conside the number of edges joining the vertices of the two groups. There must be a division so that S is the largest. Now $d_i \geqslant d_i'$ $(i = 1, 2, \ldots, n)$. Otherwise, if $d_1 < d_1'$, we transfer v_1 from a group to the other group. The number of S increases by $d_1' - d_1 > 0$, which contradicts the fact that S is the largest.

6 Team A had played 8 matches with 8 teams and did not play with the other 9 teams. Suppose the 9 teams had played with each other in 8 rounds. Since every team had played 8 games, the 9 teams had not played with other teams. But the 9 teams could only play 4 games, so there must be one team which had played with other teams. A contradiction. So among the 9 teams there must be two teams B and C which had not played with each other. Then A, B, C had not played with each other.

7 We denote n delegates by n vertices. If two delegates have shaken their hands, we join the corresponding vertices and get the graph G. If among any four vertices v_1, v_2, v_3, v_4 in G, every vertex has its adjacent vertex, we denote them by v_1', v_2', v_3', v_4'. By the known condition, among v_1, v_2, v_3, v_4 there is a vertex v_1 which is not adjacent to the other three vertices v_2, v_3, v_4. So $v_1' \neq v_2, v_3, v_4$. If $v_2' \neq v_1'$, among four vertices v_2, v_3, v_1', v_2', there is no vertex which is adjacent to the other three vertices. So $v_2' = v_1'$. Similarly, $v_3' = v_1'$. Among four vertices v_1, v_2, v_3, v_1', there is no vertex which is adjacent to any other vertex. So among any 4 vertices there must be one vertex which is not adjacent to the other $n-1$ vertices.

$$d(A_{k+1}) \leqslant \frac{2\left[\frac{1}{2}(k+1)(k-2)+4\right]}{k+1}$$
$$= k-2+\frac{8}{k+1}$$
$$\leqslant k-2+\frac{8}{5} < k.$$

So $d(A_{k+1}) \leqslant k-1$. Then among the remaining k vertices $A_1, A_2, A_3, \ldots, A_k$ there are at least

$$\frac{1}{2}(k+1)(k-2)+4-(k+1) = \frac{1}{2}k(k-3)+4$$

edges. By induction, the set of k vertices is stable. Also,

$$d(A_{k+1}) \geqslant \frac{1}{2}(k+1)(k-2)+4-\binom{k}{2} = 3,$$

so A_{k+1} is adjacent to at least three vertices among $A_1, A_2, \ldots, A_k$. Suppose that A_{k+1} is adjacent to vertices A_1, A_2, A_3 and that $A_{k+1}A_1 = x$, $A_{k+1}A_2 = y$, $A_{k+1}A_3 = z$. It is easy to prove A_{k+1} can be uniquely determined. If not, let A'_{k+1} be another vertex. Also $A'_{k+1}A_1 = x$, $A'_{k+1}A_2 = y$, $A'_{k+1}A_3 = z$, then A_1, A_2, A_3 are all on the perpendicular bisector of $A_{k+1}A'_{k+1}$. It contradicts the hypothesis that there are no three vertices on a common line. Then $A_{k+1}A_4, \ldots, A_{k+1}A_k$ can be determined. The set $\{A_1, A_2, \ldots, A_{k+1}\}$ is stable. The conclusion is true when $n = k+1$. In summary, the conclusion is true.

11 We use the unit cube as the vertex. We join the two corresponding vertices if and only if there is a common face between the two unit cubes and we get a graph G. The number of edges of its complementary graph $\overline{G}$ is what we need. It is easy to know the number of edges of G is $3n^2(n-1)$, the number of edges of K_{n^3} is $\frac{1}{2}n^3(n^3-1)$ and the number of edges of $\overline{G}$ is

$$\frac{1}{2}n^3(n^3-1)-3n^2(n-1) = \frac{1}{2}n^6-\frac{7}{2}n^3+3n^2.$$

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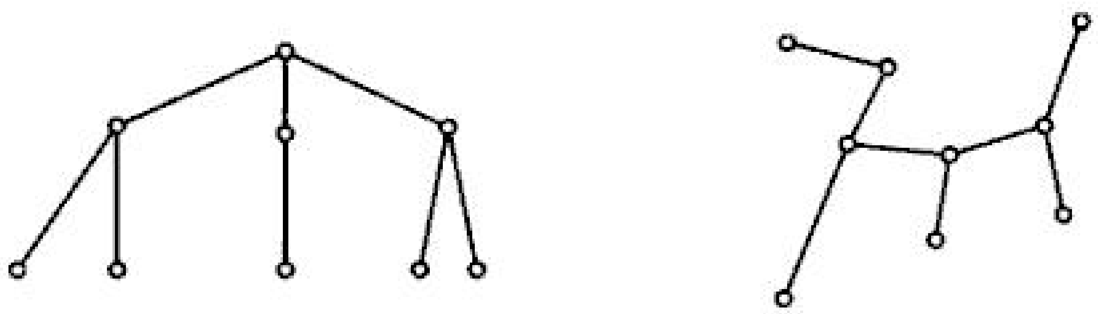

Fig. 4

5 Suppose T contains n vertices and e edges. Then $n = \sum_{i=1}^{k} n_i$, $e = n - 1$,

$$\sum_{i=1}^{n} d(v_i) = \sum_{i=1}^{k} i n_i = 2e = 2n - 2 = 2\sum_{i=1}^{k} n_i - 2.$$

So

$$n_1 = \sum_{i=2}^{k} (i-2) n_i + 2.$$

For $r \geqslant 3$, by the above equality, we can obtain

$$n_r = \frac{1}{r-2}\left[\sum_{i \neq r}^{k} (2-i) n_i - 2\right].$$

6 Among $d_1, d_2, \ldots, d_n$, there must be at least two which is equal to 1. $\left(\text{Otherwise, } \sum_{i=1}^{n} d_i \geqslant 2n - 1\right)$. We apply induction on the number of vertices n. When $n = 2$, the proposition is true. Suppose that the conclusion is true when $n = k$. When $n = k + 1$, there exists a number 1 among $d_1, d_2, \ldots, d_k, d_{k+1}$. Without loss of generality, let $d_{k+1} = 1$. It is easy to know among the $k + 1$ numbers there exists a number which is no less than 2, denoted by d_k. Consider the k numbers $d_1, d_2, \ldots, d_{k-1}, (d_k - 1)$,

$$d_1 + \cdots + d_{k-1} + (d_k - 1) = 2(k+1) - 2 - 1 - 1 = 2k - 2.$$

By induction, there exists tree T' whose vertices are $v_1, \ldots, v_k$,

$$\sum_{i=1}^{k} d(v_i) = d_1 + \cdots + d_{k-1} + (d_k - 1) = 2k - 2.$$

In T', there is an edge which is from v_k to v_{k+1}. We obtain a tree

contains a broken line μ satisfying conditions (1) and (2), then G^* is a chain (Q_1 and Q_2 are not on a face) or a cycle (Q_1 and Q_2 are on a face), i.e. the graph G^* is unicursal or it can be drawn in one strock. But if the four vertices of G^* are all odd, the graph G^* needs two strocks to draw.

7 Suppose that there are k lines and that one vertex v_i corresponds to one number a_i in the following way. If v_i is red, then $a_i = 1$; v_i is blue, then $a_i = -1$, $i = 1, 2, \ldots, n$. Then

$$-1 = a_1 a_n = (a_1 a_2)(a_2 a_3)\ldots(a_{n-1} a_n) = (-1)^k,$$

hence k is odd.

8 Use the conclusion of Exercise 7 and refer to Example 5 in Chapter 5.

9 The given graph contains 16 odd vertices B_i, C_i ($i = 1, 2, \ldots, 8$). If we want to make it a cycle, we should add at least 8 edges so that the graph becomes a cycle. Fig. 7 shows the cycle after adding 8 edges B_iC_i ($i = 1, 2, \ldots, 8$) so that the walk is the shortest.

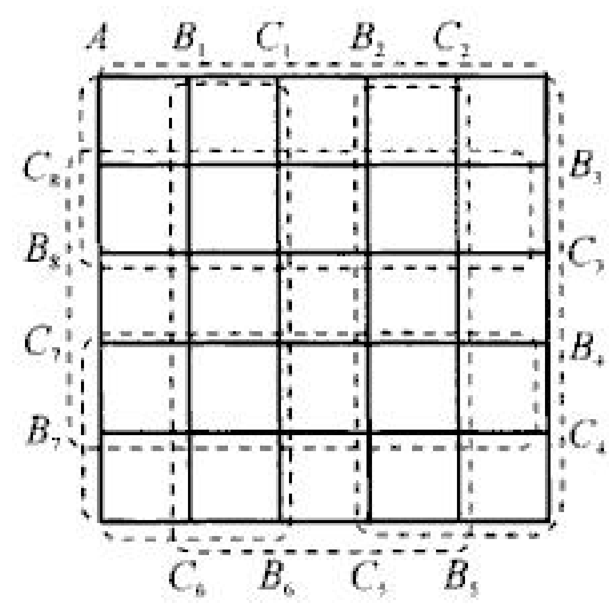

Fig. 7

Exercise 6

1 When $n \geqslant 3$, K_n is a Hamiltonian graph. When $m = n \geqslant 2$, the complete bigraph $K_{m,n}$ is a Hamiltonian graph.

2 The reader can find it on the graph.

3 A regular icosahedron consists of 20 congruent equilateral triangles. At the center of every triangle we mark a vertex. Only if two triangles have a common edge, we join the corresponding vertices and construct a regular dodecahedron which consists of 12 regular pentagons. From the study of Hamiltonian cycles, we can know that on the regular dodecahedron we can find a Hamiltonian cycle. Use scissors to cut the dodecahedron along the Hamiltonian cycle. It means

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Remove one edge, then the remaining vertex A cannot lie on cycle which contradicts the condition. So $n \geqslant 3$. It is easy to prove that $n \neq 4, 5, 6$.

If $n = 7$, remove the vertex whose degree is the largest (clearly, the degree is at least 3) to get a cycle whose length is 6. Since the vertices adjacent to this vertex may be non-adjacent on the cycle. (Otherwise, there will be a cycle whose length is 7.) The removed vertex is adjacent to at most three non-adjacent vertices on the cycle. So the degree of this vertex is at most 3. $3 \times 7 = 21$ is odd. In fact, the sum of all the degrees is even. A contradiction.

If $n = 8$, after removing the vertex whose degree is the largest, we get a cycle whose length is 7. The removed vertex is adjacent to at most three non-adjacent vertices on the cycle. So the degree of this vertex is at most 3. The degree of every vertex is 3. As Fig. 13(1) shows us, the degrees of A, C, F, O are 3. They cannot be incident to any edge. Every vertex of B, D, E, G is incident to one edge, respectively. If B is adjacent to G, D is adjacent to E (there are two edges). It is impossible. If B is adjacent to D, E is adjacent to G and the graph contains a cycle whose length is 8. A contradiction. If B is adjacent to E, D is adjacent to G and the graph contains a cycle whose length is also 8. A contradiction.

If $n = 9$, since $3 \times 9 = 27$ is not even, it is impossible that the degree of every vertex is 3. There exists one vertex whose degree is at least 4. We remove the vertex whose degree is the largest to get a cycle whose length is 8. So the removed vertex is adjacent to at most four non-adjacent vertices on the cycle. The largest degree is 4 and the smallest is 3. As Fig. 13(2) shows us, B is at least incident to one edge. Clearly we cannot join more edges between B and A, C. If B is adjacent to D, the graph contains a cycle whose length is 9. A contradiction. Similarly, B cannot be adjacent to H. If B is adjacent to F, the graph contains a cycle whose length is also 9. A contradiction. So B can only be adjacent to E or G. By symmetry, let B be adjacent to E. Similar to the above argument, we can know H is

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(2) Suppose when $n = k$, there exists a directed graph satisfying the requirement. When $n = k + 2$, first using the vertices V_1, V_2, ..., V_k, we draw a directed graph with k vertices satisfying the requirement. For another two vertices V_{k+1}, V_{k+2}, suppose that V_1, V_2, ..., V_k all point to V_{k+1} and that V_{k+2} points to V_1, V_2, ..., V_k. Suppose that V_{k+1} points to V_{k+2}. Then V_{k+1} gets to V_1, V_2, ... through V_{k+2}. (Clearly, V_{k+2} can get to V_1, V_2, ..., V_k.) V_1, V_2, ..., V_k can get to V_{k+2} through V_{k+1}. (Clearly, V_1, V_2, ..., V_k can get to V_{k+1}.) So this graph with $k + 2$ vertices still satisfy the requirement.

By (1) and (2), we know for any $4 < n \in \mathbf{N}$, there exists a scheme of changing the path among n cities satisfying the requirement.

2 Suppose G contains a circuit (v_1, v_2, ..., v_k). In v_2, v_3, ..., v_{k-1}, take the first vertex v_i so that the arc (v_{i+1}, v_1) exists. Then there exists an arc (v_1, v_i), so (v_1, v_i, v_{i+1}) is a triangular circuit.

3 We will prove that if an air route satisfies the condition f and there is no flight between two cities A and B, then we can use airline $A \to B$ or $B \to A$ so that the air route still satisfies the condition. If not, the new route does not satisfy the condition f. Then after opening the route $A \to B$, there exists a closed path $B \to C_1 \to \cdots \to C_n \to A \to B$. Similarly, after opening the route $B \to A$, there exists a closed path $A \to D_1 \to \cdots \to D_m \to B \to A$. But before opening route between A and B, there exists a route $A \to D_1 \to \cdots \to D_m \to B \to C_1 \to \cdots \to C_n \to A$. (Maybe there are some vertices C_i and D_j which are overlapped. It means that the former air route does not satisfy the condition f, because it is possible to fly from A and return to A. A contradiction.)

4 Refer to Example 4.

5 We denote n players by n vertices. If v_i defeats v_j, we can draw an arc from v_i to v_j to get a directed graph D. If there is no circuit in D, there must exist a vertex v whose indegree is 0. The vertex represents the person who wins all the games. Similarly, we can prove there is a person who loses all the game.

6 Suppose among v_1, v_2, ..., v_n, the vertex v_p has the most

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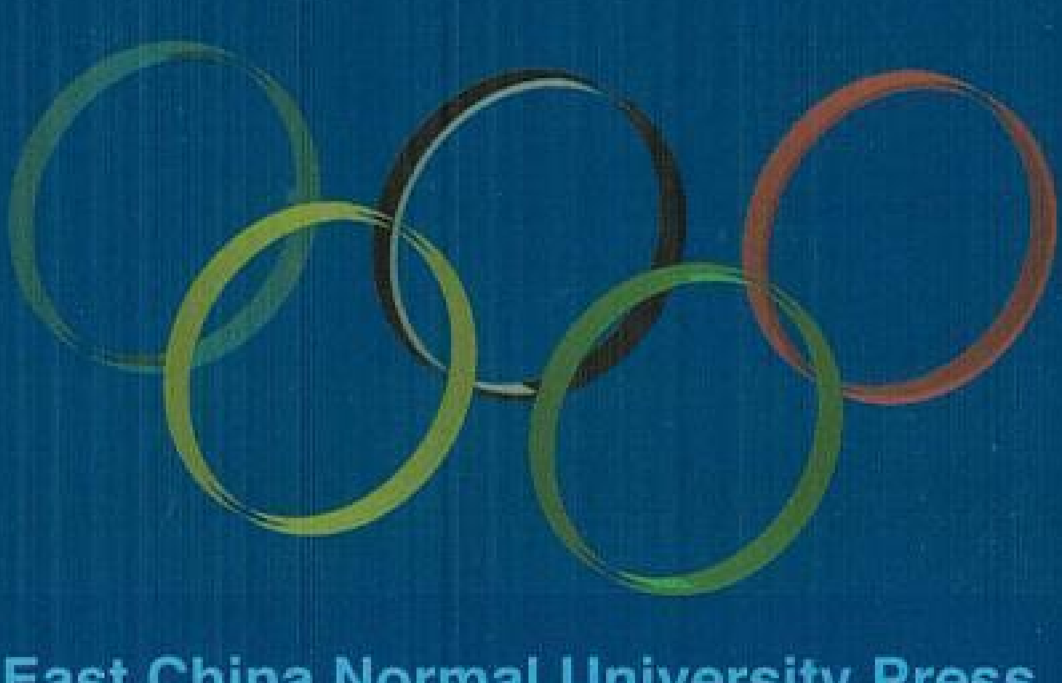
In 1736, the mathematician Euler invented graph theory while solving the Konigsberg seven-bridge problem. Over 200 years later, graph theory remains the skeleton content of discrete mathematics, which serves as a theoretical basis for computer science and network information science. This book introduces some basic knowledge and the primary methods in graph theory by many interesting problems and games.

Vol. 3 Mathematical Olympiad Series

Graph Theory



East China Normal University Press

World Scientific
www.worldscientific.com
7241 sc ISSN: 1793-8570

ISBN-13 978-981-4271-12-7(pbk)
ISBN-10 981-4271-12-8(pbk)
9 789814 271127