VARIABLE SETS ETENDU
AND
VARIABLE STRUCTURE IN TOPOI

by

F. William Lawvere

Notes by Steven Landsburg
of Lectures and Conversations
Spring 1975, The University of Chicago

Department of Mathematics The University of Chicago

I. Introduction and Historical Background

The year 1963 saw five major developments in the foundations of mathematics whose synthesis by Lawvere and Tierney seven years later was to lead to a new concept of the notion of "set". These were:

(1) The formulation of non-standard analysis by Robinson,

(2) The independence proofs in set theory given by Cohen,

(3) The appearance of Kripke's Semantics for the Intuitionistic Predicate Calculus,

(4) The presentation of the Elementary Theory of the Category of Sets by Lawvere,

and (5) The development of the general theory of topoi by Giraud.

Although each of these five events occurred independently of the others, they were related insofar as each concerned the contrast between what might be called "variable" and "constant" sets.

In particular, consider the work of Robinson and Cohen. In each of these, we begin with a system S of presumably constant sets. Then a construction is made, producing a new system S' satisfying the same axioms of constancy, and yet definitely different. In both cases, the construction passes through a system of variable sets, and then "freezes the variation" at some point.

The difference between the work of Robinson and that of Cohen is in their ways of interpreting variation within constancy (this will be made clear in the next few pages).

Consider models of the notion of variable quantity. There are different types of quantity (vector, scalar, etc.) and different types of variation (discrete, continuous, etc.). Let R be a type of continuous scalar quantity. Then any variable quantity of type R may be analyzed as having a domain of variation $\mathcal{D}$, which can be viewed in one of two ways. It can be viewed as consisting of points (or mainly of points) in which case the variation is a point-to-point mapping $\mathcal{D} \to R$, and notions such as continuity, measurability, and other properties of the variation depend on placing additional structure on $\mathcal{D}$ and R; or it can be viewed as consisting mainly of parts, such as a lattice of open sets or of measurable sets, in which case the variation is a lattice homomorphism from a lattice of parts of R into a lattice $\underline{\mathcal{D}}$ of parts of $\mathcal{D}$, and properties such as continuity and measurability are intrinsic. To see that the second view does arise in practice, let (X, μ) be a measure space, and consider $L^2(X, \mu)$, which is a system of variable quantities and is treated as such all the time, but where the notion of "value at a point" makes no sense, so that the first view cannot possibly apply. Note that the first view may be considered as a special case of the second by setting a new domain of variation $\underline{\mathcal{D}} = 2^{\mathcal{D}}$.

In what follows, $\underline{2}$ represents the 2-point lattice.

Suppose that $\mathcal{D}$ consists of points. Then to every point of $\mathcal{D}$ there corresponds an infinite-union-preserving lattice homomorphism $\underline{\mathcal{D}} \to \underline{2}$ in the obvious way, and the correspondence is bijective. We may now define an "ideal point" of $\mathcal{D}$ to be <u>any</u> lattice homomorphism $\underline{\mathcal{D}} \to \underline{2}$, regardless of whether it preserves infinite unions. (If $\mathcal{D}$ is a discrete topological space, these are just ultrafilters; for an arbitrary space, they are the points of its Wallman Compactification.)

Returning to the work of Robinson and Cohen, let $\mathcal{S}$ be a system of presumably constant sets, and let I be a set in $\mathcal{S}$. Let $\mathcal{S}/\mathrm{I}$ be the comma category (i.e. a typical object in $\mathcal{S}/\mathrm{I}$ is a morphism $\mathrm{E} \to \mathrm{I}$ in $\mathcal{S}$ and an arrow is a commutative diagram

$$\begin{tikzcd}
\mathrm{E} \arrow[dd] \arrow[dr] & \\
 & \mathrm{I} \\
\mathrm{E}' \arrow[ur] &
\end{tikzcd}$$

).

An object of $\mathcal{S}/\mathrm{I}$ can be viewed as a set varying over I (by looking at the fiber over each point of I). Looking at this set over an ideal point of I freezes the variation. For example, let I be the set of integers, $f : \mathrm{I} \to \mathrm{I}$ the identity map. Evaluating f at some ideal point freezes the variation; this is precisely Robinson's method. The Cohen method is similar, taking $\underline{\mathcal{D}}$ to be the collection of regular open sets in some 2^I. Thus the essential difference between Robinson's and Cohen's approaches was simply that Robinson utilized the first concept of variability and Cohen the second.

One is led, then, by these considerations and others, to attempt to unify the concept of "variable set". So far, only the language of categories has led to success in this attempt.

II The Notion of an Abstract Set

Preliminary to establishing a formal theory of sets, one must decide what is to be regarded as primary: sets and membership, or mappings and composition. The former view is the traditional one, and it has led to the distinction between an element x and the singleton set containing x, untold agonizing over whether or not the rationals are "actually" a subset of the reals, debates over whether the number 5 is in fact equal to the set $\{0,1,2,3,4\}$, the "discovery" that ordered pairs have elements which in turn have elements, and the consideration of countless other equally bizarre matters. The mathematician confronting these burning issues and marvelous breakthroughs is often led to the understandable conclusion that formalism is pedantry, and the result is that people become isolated from the study of foundations.

The view that membership is primary also leads one to believe that membership is global and absolute, whereas in fact it is local and relative. Given any sets X and Y, this view always makes it possible to ask whether X is a member of Y. However, this question does not generally arise in practice. In practice, what almost always happens is that we start with a set X and then look "inside" of X to pick out a subset A and an element x; and then we ask whether or not x is an element of A. This is the sort of question which constantly <u>does</u> arise in mathematics, and this is the sense in which membership is local. Given another set B which is "far away" (i.e. not contained in X) there is no reason why it should even be meaningful to ask the question of whether it is true that $x \in B$.

Membership, then, is local rather than global; it is also relative rather than absolute. To illustrate what we mean by this, consider the question of whether a given element of $\mathbb{Q}$ is also a member of $\mathbb{R}$. This question can only be answered relative to the inclusion mapping $\mathbb{Q} \hookrightarrow \mathbb{R}$ which is constructed simultaneously with $\mathbb{R}$ itself. As another example, $\mathbb{Z}_3$ consists of sets of integers, but we may still speak of 0, 1, and 2 as members of $\mathbb{Z}_3$ by speaking relative to known mappings $\mathbb{Z}_3 \rightleftarrows \{0,1,2\}$.

We now describe the concept of a <u>constant abstract set</u>: a set X has elements, which in turn have <u>no</u> internal structure (note that this makes impossible the formulation of many traditional set-theoretic difficulties, in particular Russell's Paradox). X itself has no internal structure other than equality or inequality between its members, and no external structure except cardinality.

Our next concept is that of a <u>mapping</u> $f: X \to Y$ between abstract sets. It is assumed that f associates to each $x \in X$ a unique value $f(x) \in Y$. Two mappings are <u>equal</u> if they have the same domain and codomain, and the same value at each element of the domain.

A mapping $f: X \to Y$ lends to both S and Y additional internal structure, beyond that with which they were originally endowed. On X we get a binary relation R_f given by $x_1 \sim x_2$ iff $fx_1 = fx_2$. On Y we get a predicate d_f given by $y \in d_f$ iff for some x, $fx = y$. We can also associate with each $y \in Y$ the cardinality of the pre-image of y. One way to do this is to form the category $\mathcal{Y}_f$, whose objects are the elements of Y and whose morphisms $y \to y'$ are taken to be in one-one correspondence with mappings from the pre-image of y to the pre-image of y'.

The basic admissible global structure is composition of mappings between abstract sets, and this forms a category.

A <u>one-element set</u> 1 is a set such that for any given set X there is a unique mapping $X \to 1$.

<u>Definition.</u> X_1 and X_2 have the <u>same cardinality</u> if there are mappings $f_1: X_1 \to X_2$ and $f_2: X_2 \to X_1$ such that $f_1 f_2 = 1_{X_2}$ and $f_2 f_1 = 1_{X_1}$, (i.e. if they are isomorphic in the category of abstract sets).

<u>Lemma.</u> Any two one-point sets have the same cardinality.

We can now ignore the abstract "elements" of a set and redefine:

<u>Definition.</u> x is an <u>element</u> of X if x is a mapping $x: 1 \to X$.

It is a general theorem of category theory that if X_1 and X_2 are isomorphic objects of a category $\mathcal{C}$, then there is a functor $F: \mathcal{C} \to \mathcal{C}$ with $F(X_1) = X_2$, $F(X_2) = X_1$, $F(A) = A$ for any $A \neq X_1, X_2$ such that F is an isomorphism of categories. Thus if Q is any statement in categorical language, then $Q(X_1)$ is true if and only if $Q(X_2)$ is true. This makes precise our earlier contention that cardinality is the only external structure on a set.

The notion of "abstract set" is considerably less abstract than, for example, the usual notion of "quantity", since quantities have no elements and no mappings defined between them. This in turn is less abstract than the notion of "true vs. false" since quantities can at least be compared to each other with respect to magnitude.

As an illustration, consider a basket which may or may not contain any apples. The grossest possible statement which can be made concerning the existence of apples in the basket is "yes, there are apples" (or "no, there are no apples"). As we pass to gradually more sophisticated theories of the apples in the basket, we can first say precisely how many apples there are (using the notion of quantity) and then, at a higher level, form an abstract set A whose elements correspond to the apples in the basket. This last theory really is more sophisticated than the purely quantitative one in that it allows us to distinguish a particular apple and say that it has a certain property (say that of being rotten) via a map $1 \to A$.

The process of abstraction starts with real things, passes to finite structures, then to finite abstract sets, on to finite discrete quantities (i. e. the natural numbers) and finally to the distinction of true vs. false (the last step is accomplished by associating 0 with "false" and all other natural numbers with "true").

III The Uses of Abstract Sets

Abstract sets are used primarily to parameterize things and ideas. We generally believe we have "enough" abstract sets when we are able to parameterize mathematical things and ideas. Following are examples of parameterization by a set X:

P1: Elements of a set Y, by a map $X \to Y$. A parameterized family of elements of Y can be seen as a generalized element of Y defined over X.

P2: Abstract sets, by a map $E \to X$ (so that the fibers correspond to the abstract sets being parameterized).

P3: Pairs of elements of two sets Y_1, Y_2, by pairs of maps

$$\begin{tikzcd} & Y_1 \\ X \arrow[ur] \arrow[dr] & \\ & Y_2 \end{tikzcd}$$

P4: Maps $A \to B$, by a map $X \times A \to B$.

P5: Operations on mappings $A \to B$ by a map $X \times B^A \to Y$.

Given a concept, we can ask whether there is an X which perfectly (i.e. one-one and onto) parameterizes it. In case P1 above, the answer is yes; take $X = Y$ and the identity map from Y to itself. In P2 the answer is no, unless one is of the opinion that all sets are countable. (This is because the set E is required to be the disjoint union of the sets parameterized. Such disjoint unions do not generally exist, but they do exist in a universe where everything is countable.) The answers to P3 and P4 are affirmative since we take as axioms the existence of products and exponentiation.

If a concept can be perfectly parameterized by an abstract set, then that concept can become the domain of a mapping. In particular, the affirmative answer in the case of P3 makes possible the reflection of algebra within set theory. P4 is the general case of algebra; it makes an affirmative answer in the case of P5 possible, and this in turn makes it possible to reflect analysis within set theory (since we can parameterize operations on mappings such as sup, $\int$, $\frac{d}{dx}$, etc.). We can also reflect such notions as composition $B^A \times C^B \to C^A$.

Besides parameterization, the other use for abstract sets arises from the fact that various general concepts of quantity become concentrated into particular sets, so that some variable quantities can also be expressed as mappings.

We take as axioms that this is true of the following concepts of quantity:

Q1: true vs. false,

Q2: finite structures,

Q3: the continuum (i.e. continuous scalar quantity).

Quantity can always be viewed as a particular reflection of general structure. For example, a distinguished subset of a set is a general structure, and every element participates in this structure; here the quantitative reflection is in terms of true vs. false. Thus Q1 makes possible an affirmative answer to our question concerning perfect parameterization in:

P6: Parameterizing parts of Y, where a <u>part</u> of Y is a monomorphism $\nu : A \hookrightarrow Y$.

If ν is such a part and $y: 1 \to Y$ is an element of Y, then we write $y \in \nu$ if there is a $y': 1 \to A$ such that

$$\begin{tikzcd}
1 \arrow[r, "y'"] \arrow[dr, "y"'] & A \arrow[d, "\nu"] \\
& Y
\end{tikzcd}$$

commutes.

We can now map $\varphi_\nu : Y \to \binom{\text{true}}{\text{false}}$ by $\varphi_\nu(y) = \text{true}$ if and only if $y \in \nu$.

IV Finiteness and Natural Numbers

For completeness, we begin with the following standard definitions:

A topos is a category with finite limits which has the following: (1) a distinguished element Ω, called the subobject classifier or truth-value object, and a map $1 \xrightarrow{\text{true}} \Omega$ such that for any monic $\nu : A \hookrightarrow X$ there is a unique map φ_ν (called the characteristic function of ν) making

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\nu"'] & 1 \arrow[d, "\text{true}"] \\ X \arrow[r, "\varphi_\nu"'] & \Omega \end{tikzcd}$$

a pullback. (2) For every object B an object Ω^B and an arrow $\varepsilon_B : \Omega^B \times B \to \Omega$ such that for any $\varphi : A \times B \to \Omega$ there is a unique $g : A \to \Omega^B$ making

$$\begin{tikzcd} \Omega^B \times B \arrow[r, "\varepsilon_B"] & \Omega \\ A \times B \arrow[u, dashed, "g \times 1"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commute.

It follows from these axioms that there is, for any objects A and B, a natural one-one correspondence $\hom(B \times A, \Omega) \cong \hom(A, \Omega^B)$, and we refer to the image of an arrow under this correspondence as its exponential transpose.

Let B be any object in a topos and $\Delta: B \to B \times B$ the diagonal. The Kronecker delta δ_B is the characteristic function of Δ, i.e.

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[d, "\text{true}"] \\
B \times B \arrow[r, "\delta_B"'] & \Omega
\end{tikzcd}$$

is a pullback.

The singleton map $\{\cdot\}_B : B \to \Omega^B$ is the unique map such that

$$\begin{tikzcd}
\Omega^B \times B \arrow[r, "\varepsilon_B"] & \Omega \\
B \times B \arrow[u, "\{\cdot\} \times 1"] \arrow[ur, "\delta_B"'] &
\end{tikzcd}$$

commutes, and this map is always monic.

Let $f: A \to C$ be any arrow in any category. An image of f is a monic $m: M \to C$ with $f = mk$ for some $k: A \to M$ and such that if $f = nh$ for any monic n, then m factors through n, i.e. we have

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "k"] \arrow[ddr, "h"'] & & C \\
& M \arrow[ur, "m"] \arrow[d, dashed] & \\
& N \arrow[uur, "h"'] &
\end{tikzcd} .$$

Images, when they exist, are unique, and in a topos every arrow has an image.

Given $S \overset{\nu}{\hookrightarrow} A$ and $T \overset{\mu}{\hookrightarrow} A$ monic, let $S \xrightarrow{f} S+T \xleftarrow{g} T$ be the coproduct of S and T; this gives rise to a map $k: S+T \to A$ such that

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[dr, "\nu"'] & S+T \arrow[d, "k"] & T \arrow[l, "g"'] \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

commutes, and we define the *union* $S \underset{A}{\cup} T$ to be the image of k.

If $A \hookrightarrow X$ is monic, it has a characteristic function $\varphi_A : X \to \Omega$; if we identity X with $1 \times X$ this has an exponential transpose called the *name* of A and denoted $\ulcorner A \urcorner : 1 \to \Omega^X$.

In what follows, we will often denote the subobject classifier of a topos by the symbol 2.

Definition. Let $X \underset{\mu}{\hookrightarrow} Y$ and $S \underset{\nu}{\hookrightarrow} Y$ be monic. Then $\mu \leqq_Y \nu$ if for some $k: X \to S$,

$$\begin{tikzcd}
X \arrow[rr, "k"] \arrow[dr, "\mu"'] & & S \arrow[dl, "\nu"] \\
& Y &
\end{tikzcd}$$

commutes.

Thus membership is a special case of containment.

Exercise. The P's and Q's of Section III make possible the representation of infinite intersections. (Hint: Define intersection as a map $\bigcap : 2^{2^E} \to 2^E$ with the property that $x \in C$ iff $\forall S \in C[x \in S]$. This can be formulated in any topos. For any object F we define

$\text{true}_F : F \to \Omega$ as the composite $F \longrightarrow 1 \xrightarrow{\text{true}} \Omega$. Then $\ulcorner \text{true}_F \urcorner : 1 \to \Omega^F$ has a characteristic function $\forall_F : \Omega^F \to \Omega$. Now for any map $\varphi : F \to \Omega$ we have $\forall_F \ulcorner \varphi \urcorner = \text{true}$ iff $\varphi = \text{true}_F$. Use all of this to define an appropriate map $E \times 2^{2^E} \to 2$ whose exponential transpose is the desired map $\bigcap$).

As a result of this exercise, there is a smallest subset $K(X)$ of 2^X which contains $X \xrightarrow[\{\cdot\}_X]{} 2^X$, contains $\ulcorner \emptyset \urcorner$, and is closed under the binary operation $\cup$. This parameterizes perfectly the concept of "Kuratowski-finite subset of X".

<u>Definition.</u> X is <u>finite</u> if $\ulcorner X \urcorner \in K(X)$.

The image of a finite object is finite, but in a general topos a subobject of a finite object need not be finite. $\emptyset$ and 1 are always finite, and the coproduct of two finite objects is finite. In general, 2 need not be finite.

Note that if $\mathcal{S}$ is the category of abstract sets and X is any particular abstract set, then $\mathcal{S}/X$ is again a topos. An object $E \to X$ is finite in $\mathcal{S}/X$ iff each of its fibers is finite in $\mathcal{S}$. We can now list another use for abstract sets:

P_7: Parameterizing finite sets. (Since we can characterize the finite objects in $\mathcal{S}/X$, we can characterize those maps $E \to X$ which are parameterizations of finite sets.)

Q2 is interpreted to mean the existence of a set $\overline{\mathbb{N}}$ such that every finite family of sets parameterized by X corresponds to a mapping $X \xrightarrow{\varphi} \overline{\mathbb{N}}$, and this correspondence is one-one and onto. (Think of $\varphi(x)$ as the cardinality of the fiber over x.)

In addition to being able to be used for counting finite sets, the natural numbers have another aspect: one can do induction on them. One interprets this by saying that the natural numbers consist of an object $\mathbb{N}$ and maps $1 \xrightarrow{0} \mathbb{N} \xrightarrow{\sigma} \mathbb{N}$ such that for any $1 \to X \xrightarrow{t} X$ there is a unique factorization

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr] & \mathbb{N} \arrow[r, "\sigma"] \arrow[d, dashed, "f"'] & \mathbb{N} \arrow[d, dashed, "f"] \\
 & X \arrow[r, "t"'] & X
\end{tikzcd}$$

We take as an axiom (call it P8) the existence of such an object.

Consider the category $\mathcal{S}^{\circlearrowleft}$ whose objects are endomaps $X^{\circlearrowleft t}$ in $\mathcal{S}$ and whose morphisms are arrows $X^{\circlearrowleft t} \xrightarrow{f} Y^{\circlearrowleft s}$ in $\mathcal{S}$ such that $ft = sf$. P8 is sufficient to guarantee that $\mathcal{S}^{\circlearrowleft}$ is again a topos.

It is interesting to note that for any category $\mathcal{S}$ which is cartesian closed and satisfies P8, the obvious forgetful functor $\mathcal{S}^{\circlearrowleft} \to \mathcal{S}$ has a left adjoint $X \to (\mathbb{N} \times X)^{\circlearrowleft \sigma \times 1_X}$. This adjointness relation says that there is a bijective correspondence between maps $\mathbb{N} \times X \xrightarrow{f} Y^{\circlearrowleft t}$ and maps $X \xrightarrow{f_0} Y$ (think of the correspondence as being given by $f_0(x) = f(0, x)$), which is just a form of primitive recursion .

Definition. A category $\mathcal{C}$ is *locally cartesian closed* if it has pullbacks and every $\mathcal{C}/X$ is cartesian closed. (Thus every locally cartesian closed category with 1 is cartesian closed, by taking $X = 1$ in the definition.) It is a fact that every topos is locally cartesian closed.

If $\mathcal{C}$ is any category with products, we can form $\mathrm{Mon}(\mathcal{C})$, the category of monoids in $\mathcal{C}$. If $\mathcal{C}$ is also locally cartesian closed with natural number object $1 \xrightarrow{0} \mathbb{N} \xrightarrow{\sigma} \mathbb{N}$, then the obvious forgetful functor $\mathrm{Mon}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint W, called the *free monoid functor*, which has the universal property which one is led by its name to expect. We construct this functor explicitly:

Let $e : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be the evaluation map and let $h : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the exponential transpose of the composite $\mathbb{N} \times \mathbb{N}^{\mathbb{N}} \xrightarrow{e} \mathbb{N} \xrightarrow{\sigma} \mathbb{N}$. Let $g : 1 \to \mathbb{N}^{\mathbb{N}}$ be the exponential transpose of $1_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$. Now complete the diagram

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "g"'] & \mathbb{N} \arrow[r, "\sigma"] \arrow[d, dashed, "f"'] & \mathbb{N} \arrow[d, dashed, "f"] \\
 & \mathbb{N}^{\mathbb{N}} \arrow[r, "h"'] & \mathbb{N}^{\mathbb{N}}
\end{tikzcd}$$

and define $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ to be the exponential transpose of f.

Now for any $n \in \mathbb{N}$ we take the pullback of the diagram

$$\begin{tikzcd}
A_n \arrow[r, dashed] \arrow[dd, dashed] & \mathbb{N} \times \mathbb{N} \arrow[d, "+"] \\
 & \mathbb{N} \arrow[d, "\sigma"] \\
1 \arrow[r, "n"'] & \mathbb{N}
\end{tikzcd}$$

and we denote the subobject $A_n \to \mathbb{N} \times \mathbb{N} \xrightarrow{\pi_0} \mathbb{N}$ (where π_0 is the first projection) by $[n]$ or $\{ k | k < n \}$.

Now let us consider the category $C/\mathbb{N}$. Here the "1" object is the identity map $1_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ and the natural number object, which we denote $\mathbb{N}_{\mathbb{N}}$, is the second projection map $\pi_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Thus a typical element of $\mathbb{N}_{\mathbb{N}}$ is a commutative diagram

$$\begin{tikzcd}
\mathbb{N} \arrow[rr] \arrow[dr, "1_{\mathbb{N}}"'] & & \mathbb{N} \times \mathbb{N} \arrow[dl, "\pi_1"] \\
 & \mathbb{N} &
\end{tikzcd}$$

If A is any object of C, we identify A with the object $A \times \mathbb{N} \xrightarrow{\pi_1} \mathbb{N}$ in $C/\mathbb{N}$. Thus, letting $n \in \mathbb{N}_{\mathbb{N}}$ be the diagonal map $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$, we can form the exponential $A^{[n]}$. Now letting $\Sigma : C/\mathbb{N} \to C$ be the forgetful functor, we set $W(A) = \Sigma A^{[n]}$. This turns out to be precisely the free monoid.

To see how this works in the category of sets, $\mathcal{S}$, note that $[n]$ is just $\{(k,n) \in \mathbb{N} \times \mathbb{N} \mid k < n\} \xrightarrow{\pi_1} \mathbb{N}$. Thus a typical element of $A^{[n]}$ is a map $[n] \to A$, i.e., a commutative diagram

$$\begin{tikzcd}
\{(k,n) \mid k < n\} \arrow[rr] \arrow[dr, "1_{\mathbb{N}}"'] & & A \times \mathbb{N} \arrow[dl, "\pi_1"] \\
& \mathbb{N} &
\end{tikzcd}$$

Clearly, these diagrams are in one-one correspondence with the collection of all maps $\{(k,n) \mid k < n\} \to A$; one may view each such map as an array

$$\begin{array}{ccccc}
a_{01} & a_{02} & a_{03} & a_{04} & \cdots \\
 & a_{12} & a_{13} & a_{14} & \cdots \\
 & & a_{23} & a_{24} & \cdots \\
 & & & a_{34} & \cdots
\end{array}$$

Considering each column of this array as a word in the free monoid on A, one sees that such arrays correspond in a one-one and onto fashion to maps $\mathbb{N} \xrightarrow{f} W(A)$ such that $f(n)$ has length n. These in turn correspond to commutative diagrams

$$\begin{tikzcd}
\mathbb{N} \arrow[rr] \arrow[dr, "1_{\mathbb{N}}"'] & & W(A) \arrow[dl, "\ell"] \\
& \mathbb{N} &
\end{tikzcd}$$

where ℓ is the length map. But these are just elements in $\mathcal{S}/\mathbb{N}$ of the object $W(A) \xrightarrow{\ell} \mathbb{N}$.

We have seen that the elements of $A^{[n]}$ correspond bijectively with the elements of $W(A) \longrightarrow \mathbb{N}$; in fact, it is not hard to see that these two objects are isomorphic in $\mathcal{S}/\mathbb{N}$. Thus $W(A) = \Sigma A^{[n]}$ does indeed yield the free monoid functor in the category of sets. That it does so in any topos we leave for the reader to verify.

Since Ω^E is a monoid under the binary operation $\cup$, we can complete the following diagram:

$$\begin{tikzcd}
E \arrow[r] \arrow[dr, "\{\cdot\}_E"'] & W(E) \arrow[d, dashed, "I_E"] \\
& \Omega^E
\end{tikzcd}$$

with a unique monoid homorphism I_E .

If we denote the image of I_E by $K(E)$, it is a theorem that we get the same $K(E)$ we had earlier. Thus we can give two very different, but nevertheless equivalent, definitions of finiteness, one of which is closely related to properties of the natural number object, while the other does not concern that object at all.

<u>Examples.</u> Let $\mathcal{S}$ be the category of abstract sets and X any object in $\mathcal{S}$. Then we already noted that a finite object of $\mathcal{S}/X$ is precisely a map $E \to X$ all of whose fibers are finite.

Let G be a group in S, and let S^G be the category of abstract G-sets. Here an object is finite if and only if it is finite as an object in S.

Consider the category 2, viewed as the category of non-empty open sets in Sierpinski space. (Recall that Sierpinski space is a space of two points, one of which is open and the other is not.) We picture this category as a diagram $U \hookrightarrow 1$. Then $S^{2^{op}}$ is the category of sheaves on Sierpinski space; here an object is a map $E_1 \to E_U$ (i.e. a global section and a restriction map to a section over U) and a morphism is a commutative square. Those familiar with the work of Kripke will recognize in this the notion of "stages of knowledge about a set". The truth-value object here is a map $3 \to 2$ which can be viewed as the map

$$\begin{tikzcd}
\{\text{true} \arrow[dr] & & ? \arrow[dl] & \text{false}\} \arrow[d] \\
& \{\text{true} & & \text{false}\}
\end{tikzcd}$$

The following theorem is true: an object $E_1 \to E_U$ is finite in $S^{2^{op}}$ iff E_1 and E_U are both finite in S and $E_1 \to E_U$ is surjective.

V Extension of Theories

Let $\mathcal{E}$ be any topos. Recall that $\mathcal{E}$ is locally cartesian closed. Any arrow $f: X \to Y$ in $\mathcal{E}$ induces a functor $\Sigma_f: \mathcal{E}/X \to \mathcal{E}/Y$ in the obvious way. Moreover (and this is true if $\mathcal{E}$ is any category with pullbacks), Σ_f has a right adjoint $f^*: \mathcal{E}/Y \to \mathcal{E}/X$; if $g: F \to Y$ is an object in $\mathcal{E}/Y$ then $f^*(g)$ is the pullback of g along f.

Let $f: X \to Y$ be any arrow in any topos. Then if $T \hookrightarrow Y$ is monic we write $f^{-1}(T)$ instead of $f^*(T)$. If $S \hookrightarrow X$ is monic, we let $f(S) = \exists_f S \subset_Y Y$ be the image of $S \hookrightarrow X \xrightarrow{f} Y$. Then $f(S) \subset_Y T$ iff $S \subset_X f^{-1}(T)$. We shall call an epimorphism $f: X \to Y$ good if $f(1_X) = 1_Y$.

Definition. A regular category is a category with pullbacks and images in which every pullback of a good epimorphism is a good epimorphism. Every topos is regular.

Remark. Logicians have long thought that the essence of existential quantification is projection; however, this is merely a special case of the actual essence, which is the taking of images. This is why we have adopted the notation $\exists_f(S) = f(S)$.

To demonstrate an instance of the traditional special case, consider

$$\begin{tikzcd}
A \times X \arrow[r, "f"] & X \\
S \arrow[u, hook] & T \arrow[u, hook]
\end{tikzcd}$$

where f is projection onto X. Write $\exists_a$ instead of $\exists_f$ and define $\wedge$ as usual (see Section III). Then regularity implies that

$$\exists a[S(a, y) \wedge Q(y)] = \exists a[S(a, y)] \wedge Q(y).$$

Let $\mathcal{E}$ be any category and $\mathcal{U}$ a collection of objects in $\mathcal{E}$. An <u>element relative to</u> $\mathcal{U}$ is any morphism with its domain in $\mathcal{U}$. If U is the domain of such an element, we say that the element is defined over U and usually denote it by a symbol such as x/U.

Let S_1 and S_2 be subobjects of an object X, and let $x: U \to S_1$ be an element of S_1 defined over U. Then we write $x \in S_2$ if there is a map $y: U \to S_2$ making the diagram

$$\begin{tikzcd}
& U \arrow[dl, "x"'] \arrow[dr, "y"] & \\
S_1 \arrow[dr, hook] & & S_2 \arrow[dl, hook'] \\
& X &
\end{tikzcd}$$

commute.

Suppose that $S_1 \subset_X S_2$ are subobjects of X, so that we have

$$\begin{tikzcd} S_1 \arrow[r, "f"] \arrow[dr, hook] & S_2 \arrow[d, hook] \\ & X \end{tikzcd}$$

Then it is certainly the case that for any U in $\mathcal{U}$ and any x/U, $x \in S_1 \Rightarrow x \in S_2$, since we can complete the diagram

$$\begin{tikzcd} & U \arrow[dl, "x"'] \arrow[dr, dashed, "y"] & \\ S_1 \arrow[rr, "f"] \arrow[dr, hook] & & S_2 \arrow[dl, hook'] \\ & X & \end{tikzcd}$$

by setting $y = fx$. We wish to single out those collections $\mathcal{U}$ for which the converse to this is true.

<u>Definition.</u> A collection of objects $\mathcal{U}$ is said to <u>generate</u> if "extensionality" is true with respect to elements relative to $\mathcal{U}$; i.e. if whenever we are given two subobjects $S_1 \hookrightarrow X$ and $S_2 \hookrightarrow X$ for which it is the case that for all U in $\mathcal{U}$ and all x/U, $x \in S_1 \Rightarrow x \in S_2$, it follows that $S_1 \subset_X S_2$.

<u>Note:</u> Given $X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$ it is certainly the case that if $f_1 = f_2$, then for every U in $\mathcal{U}$ and every s/U, $f_1x = f_2x$. It might be tempting to define U as generating if the converse to this statement holds. (In fact, this definition is often used.) In general, this concept

is not equivalent to the notion of generating defined immediately above. However, in a regular category, the two are indeed equivalent. It follows from this that in a regular category, the good epis are precisely the coequalizers.

In searching for collections of generators, one good candidate to try is the collection of subobjects of 1. However, this does not always work.

Example. Let G be a group, $\mathcal{E} = \mathcal{S}^G$, the category of abstract G-sets. Then the subobjects of 1 do not generate. However, the singleton collection consisting of G acting on itself by translation does generate.

Example. In the category $\mathcal{S}^2$ there are three subobjects of 1, and this collection generates.

Definition. An arrow f is $\mathcal{U}$-surjective if for all U in $\mathcal{U}$ and all y/U, there exists x with $fx = y$, i.e. the following diagram can be completed.

$$\begin{tikzcd} & U \arrow[d, "y"] \arrow[dl, dashed, "x"'] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Definition. Let $\mathcal{T}$ be a collection of arrows with common codomain U. A map $f:X \to U$ is said to be $\mathcal{T}$-surjective if it factors through every t in $\mathcal{T}$ as shown:

$$\begin{tikzcd}
 & T \arrow[dl, dashed] \arrow[d, "t"] \\
X \arrow[r, "f"'] & U
\end{tikzcd}$$

$\mathcal{T}$ is said to cover U if every $\mathcal{T}$-surjective subobject $S \hookrightarrow U$ is an isomorphism.

Given $f:X \to Y$ and $y/U \in Y$, we consider diagrams of the form

$$\begin{tikzcd}
U' \arrow[r, "t"] \arrow[d, "x"'] & U \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

and ask when there are "enough" of these diagrams with U' in $\mathcal{U}$. To this end we let $\mathcal{T}_{f,y}$ be the collection of arrows t such that for some x, $fx = yt$. Then the following theorem is true:

Theorem. In a regular category with a class of generators $\mathcal{U}$, an arrow $f:X \to Y$ is good epi iff for every U in $\mathcal{U}$ and every $y:U \to Y$, $\mathcal{T}_{f,y}$ covers U. (This property is expressed by saying that f is locally $\mathcal{U}$-surjective.)

In logical terms, this says that a statement $\exists x \varphi(x)$ is true precisely when there exists a covering of the domain of free variables on each piece of which there are elements satisfying φ.

This actually occurs in ordinary logic, where, for a given formal theory with conjunction and existential quantification, we can construct a regular category $\mathcal{E}$ whose objects correspond to concepts definable in the theory (i. e. formulas of the theory) and whose morphisms correspond to the constructive functions definable in the theory (i. e. equivalence classes of terms in the theory).

As a concrete example, consider the theory of partially ordered sets. We have an object A corresponding to the idea of such a set, and an order relation $R \hookrightarrow A \times A$. (We also have objects corresponding to such concepts as $\{(x_1, x_2, x_3, x_4) \mid x_1 \geq x_4 \geq x_3\}$.)

Now suppose we add the axiom of density, $a_1 < a_2 \Rightarrow \exists a\, [a_1 < a < a_2]$. Let us interpret this axiom categorically. We write $R^{(2)} \hookrightarrow A \times A \times A$ for the subobject corresponding to $\{(a_1, a_2, a_3) \mid a_1 < a_2 < a_3\}$. Let $\pi: A \times A \times A \to A \times A$ be the projection deleting the middle term. Then we existentially quantify $R^{(2)}$ along this projection by taking the image $\pi(R^{(2)})$ of the composite $R^{(2)} \hookrightarrow A \times A \times A \xrightarrow{\pi} A \times A$. It is this image which we have called $\exists a$. Then the axiom says that we may complete the diagram

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[d, dashed] & A \times A \\
\pi(R^{(2)}) \arrow[ur, hook] &
\end{tikzcd}$$

Suppose that we also have constants $1 \underset{1}{\overset{0}{\rightrightarrows}} A$ representing greatest and least elements. We conclude that $\exists a [0 < a < 1]$ is true, and we wish to be able to demonstrate a constant a such that $0 < a < 1$. Unfortunately, there is no reason to expect the existence of such a constant. But all is not lost -- this constant <u>does</u> at least exist <u>locally,</u> in the following sense:

In keeping with the spirit of the theorem, we interpret the truth of $\exists a [0 < a < 1]$ to mean that the diagram

$$\begin{tikzcd}
 & 1 \arrow[d, "{(0,1)}"] \\
\pi(R^{(2)}) \arrow[r] & A \times A
\end{tikzcd}$$

can be lifted to a covering of 1. If we let U' be the object corresponding to $[0 < a < 1]$, then this says precisely that $U' \to 1$ is a good epimorphism.

Now there is a canonical functor $\mathcal{E} \to \mathcal{E}/U'$ taking each object A to the projection map $A \times U' \to U'$, and this functor is a logical morphism (that is, it preserves products, images, direct limits, exponentiation, the subobject classifier, etc., etc.). Moreover, the fact that $U' \to 1$ is a good epimorphism is precisely what is needed to guarantee that this functor is faithful.

Writing $A_{U'}$ for the image of A in E/U', there is a canonical map $\Delta : 1_{U'} \to U'_{U'}$. Thus the theory E/U' has an additional constant which did not exist in E; we call this constant $a: 1_{U'} \to U'_{U'}$. Faithfulness of $E \to E/U'$ simply says that the only provable statements in E/U' are those which follow from the provable statements in E and the new axiom $0 < a < 1$. In particular, the only provable statements in E/U' which do not mention the new constant a are those which were already provable in E. In logic, this is called a "conservative extension" of a theory.

A special case of this is the situation of ordinary logic where if an existential statement is true, we may adjoin to our theory a Skolem constant satisfying that statement. The lemma of logic "equivalent" to our theorem is that if the existential statement is provable in the original theory, then the extension obtained by adding the constant is conservative. (Faithfulness of topos morphisms corresponds to conservatism of extensions.)

VI Sheaves on Heyting Algebras

We have just seen that the truth of existential statements is a local property. Another example of a local property is Kuratowski finiteness. An object X in a topos $\mathcal{E}$ is Kuratowski finite iff there is a collection $\mathcal{U}$ of objects covering 1 such that for every U in $\mathcal{U}$, the image X_U of X in $\mathcal{E}/U$ is actually finitely enumerable. That is, for each U there is an element n_U of the natural number object in $\mathcal{E}/U$ and a good epimorphism $[n_U] \to X_U$. Thus in $\mathcal{E}$, a finite set is one which can be locally enumerated by a finite number. There is, however, no guarantee that the n_U's can be patched together to give a natural number n in $\mathcal{E}$; the situation is thus entirely analogous to the more general case of existential statements, where global internal truth is equivalent to local external (or actual) truth of a particular statement about a particular object.

The importance of these local properties and others leads us to investigate categories of sheaves, which are the proper setting for the study of such properties.

<u>Definition.</u> A <u>complete Heyting Algebra</u> H is a partially ordered set in which any two elements have an infimum, any collection of elements has a supremum, and the distributive law $a\wedge(\bigwedge_n b_n) = \bigwedge_n(a\wedge b_n)$ holds.

Example. The open sets of a topological space form a complete Heyting Algebra.

Definition. Let H be a Heyting Algebra. Then a presheaf of sets on H is an assignment of a set E_U to each $U \epsilon H$ and a mapping $r_{U,U'}: E_U \to E_{U'}$ for each pair of elements U, U' with $U' < U$ (these are called the restriction mappings) satisfying the property that whenever $U'' < U' < U$ we have $r_{U',U''} \; r_{U,U'} = r_{U,U''}$.

Thus if H consists of the open sets in some topological space, this is just a presheaf in the usual sense. In extending the definition to an arbitrary Heyting Algebra, we think of every Heyting Algebra as a space, but not necessarily with enough points to make it a space in the usual sense. That is, we are looking at spaces as composed of parts instead of points. We shall elaborate on this further a little later in the paper.

Definition. A presheaf on H is a sheaf if for every subset $C \subset H$ such that $A' \subset A \epsilon C \Rightarrow A' \epsilon C$, the following condition holds: Let $M = \bigvee_{A \epsilon C} A$. Then we have a unique map $E_M \to \varprojlim_{A \epsilon C} E_A$ which commutes with all of the restriction maps; we require that this be an isomorphism. (Here the $\varprojlim$ is taken in $\mathcal{S}$.)

We can now state the following theorem:

Theorem. A topos $\mathcal{E}$ defined over $\mathcal{S}$ (that is, an object of $\mathcal{T}op_{\mathcal{S}}$ as defined more precisely in Section VII) is generated by the subobjects of 1 iff there is a complete Heyting Algebra H such that $\mathcal{E}$ is the category of sheaves on H. In fact, H can be taken to be precisely the collection of subobjects of 1.

Note that every Heyting Algebra of open sets in a topological space is complete.

Example. Consider the Sierpinski space $X = \{p, q\}$, with open sets $\emptyset \subset U = \{p\} \subset X$. A typical presheaf is any diagram $E_X \to E_U \to E_\emptyset$, and a morphism of presheaves is a commutative ladder. Thus the category of presheaves is $\mathcal{S}^3$ where 3 denotes the partially ordered set $\{\emptyset \subset U \subset X\}$.

When is a presheaf a sheaf? The requirement is that for any collection $\mathcal{V}$ with $W' \subset W \in \mathcal{V} \Rightarrow W' \in \mathcal{V}$, if $V = \bigcup_{W \in \mathcal{V}} W$, then $E_V = \varprojlim_{W \in \mathcal{V}} E_W$. If we put $\mathcal{V} = \emptyset$, this says that $E_\emptyset = \varprojlim \emptyset = 1$; moreover, it is relatively easy to see that there are no other conditions induced by this requirement. Thus the category of sheaves consists of diagrams $E_X \to E_U$; we view this as the category $\mathcal{S}^2 \hookrightarrow \mathcal{S}^3$ where $2 = \{U \subset X\}$.

VII The Category Top_S and the Notion of an Etendue

Definition. A geometric morphism $f_*: E' \to E$ is a functor between two topoi with a left adjoint f^* such that f^* preserves finite limits (being left adjoint, it necessarily preserves arbitrary direct limits). It is then a theorem that f^* also preserves all internal direct limits indexable by the natural number object $\mathbb{N}$ (although it does not necessarily preserve all countable limits), and in particular that it preserves the objects K(E).

Example. Let $Y = S^C$ where C is any small category. Map $g_*: Y \to S$ by $g_*(Y) = \varprojlim_{C \in C} Y(C)$. The left adjoint is given by $g^*(S)(C) = S$. Note that in this case g^* itself has a left adjoint $g!$ given by $g!(Y) = \varinjlim_{C \in C} Y(C)$.

Example. Let Y be the category of sheaves on a complete Heyting Algebra H. Map $g_*: Y \to S$ by $g_*(Y) = Y(1)$, where 1 is the greatest element of H. Then the left adjoint is given by letting $g^*(S)$ be the constant sheaf with value S. If in particular H is the category of open sets in a space X, then g^* has a left adjoint $g!$ iff X is locally connected. This suggests that for any topos Y over S (by which we mean a topos Y that comes equipped with a geometric morphism $g: Y \to S$), regardless of whether or not Y arises from a space, we may define Y to be locally connected iff g^* has a left adjoint.

Definition. Let $\mathcal{Y}$ be a topos and Ω its truth-value object. Then $\wedge : \Omega \times \Omega \to \Omega$ is the characteristic function of (true, true), i.e.,

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, "(\text{true},\text{true})"'] & 1 \arrow[d, "(\text{true})"] \\
\Omega \times \Omega \arrow[r, "\wedge"'] & \Omega
\end{tikzcd}$$

is a pullback.

Let $\varphi, \psi : Y \to \Omega$ be any arrows. Then the conjunction $\varphi \wedge \varphi$ is the composite arrow $Y \xrightarrow{(\varphi,\psi)} \Omega \times \Omega \xrightarrow{\wedge} \Omega$.

Definition. Let $\mathcal{Y}$ be any topos, and Ω it truth-value object. Then a Grothendieck Modal Operator (or, in the older terminology, a Grothendieck Topology) is an arrow $j : \Omega \to \Omega$ satisfying

(1) $jj = j$,

(2) $j(\varphi \wedge \psi) = j\varphi \wedge j\psi$,

(3) $j(\text{true}) = \text{true}$.

Remark. Any map φ with codomain Ω can be viewed as a proposition; then $j\varphi$ is a new proposition, which we read as "it is j-locally the case that φ". Under this interpretation, the three axioms are eminently reasonable. This is the source of the name "modal operator".

Example. Let H be a complete Heyting Algebra, $\mathcal{Y} = \mathcal{S}^{H^{op}}$.

The truth-value object Ω is given by $\Omega(B) = \{\mathfrak{A} \subset H \mid A \in \mathfrak{A} \Rightarrow A \leq B$ and $A' \leq A \in \mathfrak{A} \Rightarrow A' \in \mathfrak{A}\}$. There is a canonical Grothendieck Modal Operator $j \colon \Omega \to \Omega$ given by $j_B(\mathfrak{A}) = \{A \mid A \leq \bigvee_{A' \in \mathfrak{A}} A'\}$.

Definition. A geometric morphism $g_* \colon \mathcal{X}' \to \mathcal{X}$ is called a subtopos if it is full and faithful. It is called surjective (this usage is not standard) if g^* is faithful.

Theorem. Any geometric morphism of topoi factors uniquely as a surjective morphism followed by a subtopos.

Definition. Let j be a Grothendieck Modal Operator. A monomorphism $\nu : Y' \hookrightarrow Y$ is called j-dense if the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[r, "\varphi_\nu"] \arrow[dr] & \Omega \arrow[r, "j"] & \Omega \\
& 1 \arrow[ur, "\text{true}"'] &
\end{tikzcd}$$

where φ_ν is the characteristic function of ν .

Theorem. Let $\mathcal{X} \xrightarrow{\nu} \mathcal{Y}$ be any subtopos and let Ω be the truth-value object for $\mathcal{Y}$. Then there is a unique Grothendieck Modal Operator $j \colon \Omega \to \Omega$ such that X is in the image of ν iff for every j-dense $g \colon Y' \to Y$ and every map $f \colon Y' \to X$, f factors uniquely through g as shown:

$$\begin{tikzcd}
Y' \arrow[r, "g"] \arrow[dr, "f"'] & Y \arrow[d, dashed, leftrightarrow, "\exists !"] \\
& X
\end{tikzcd}$$

Definition. Let X be a topological space and H the Heyting Algebra of open sets in X. To every point $x \in X$ we may associate a map $H \rightarrow \underline{2}$ by sending U to "true" if $x \in U$ and "false" if $x \notin U$. These mappings all preserve infinite union and finite intersection; and the space X is called sober if every such mapping arises from a point in this way. The reason for the terminology is as follows: the choice of a mapping $H \rightarrow \underline{2}$ which preserves infinite union and finite intersection can reasonably be called the process of "seeing" a point; now a space is sober if every point you can see is really there, and if you never see double.

Definition. $\mathcal{Top}_S$ is the category of topoi over $\mathcal{S}$. That is, a typical object of $\mathcal{Top}_S$ is a geometric morphism $\mathcal{Y} \rightarrow \mathcal{S}$, and a typical arrow is a commutative diagram of geometric morphisms

$$\begin{tikzcd}
\mathcal{X} \arrow[rr] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\
& \mathcal{S} &
\end{tikzcd}$$

Example. The category of sober topological spaces is embedded in the category of Heyting Algebras by associating each space to the Heyting Algebra of its open sets. The category of Heyting Algebras is in turn embedded in $\mathcal{Top}_S$ by associating to each Heyting Algebra H the category of sheaves of sets on H. Thus every sober topological space (and this includes all of the "reasonable" spaces) is an object in $\mathcal{Top}_S$.

Example. The category of groups is embedded in $\mathcal{Top}_S$ by associating to each group G the category $\mathcal{S}^G$ of abstract G-sets.

Example. Let G be a group and X a space. Let $\mathcal{Sh}(G, X)$ be the collection of sheaves on X which are acted on by G in such a way that the projection map is equivariant with respect to G. This is an object in $\mathcal{Top}_S$.

All of the above examples will turn out to belong to a very important subcategory of $\mathcal{Top}_S$, called the Etendues , which we now define.

Definition. A topos $\mathcal{X}$ is called an Etendue if there is an object C of $\mathcal{X}$ which covers 1 (i.e. $C \to 1$ is epic) such that $\mathcal{X}/C$ is the category of sheaves on some Heyting Algebra.

Example. We have seen above that every sober topological space and every group can be identified with an etendue. $\mathcal{S}^2$ is an etendue since it is the category of sheaves on Sierpinski space. We can also consider any set I as an etendue by identifying it with the category $\mathcal{S}^I = \mathcal{S}/I$.

The geometric morphisms from one topological space to another are precisely those which are induced by continuous maps. The geometric morphisms from one group to another are those functors $\mathcal{S}^G \to \mathcal{S}^H$ induced by homomorphisms $G \to H$. The geometric morphisms from one set to another are those functors $\mathcal{S}^I \to \mathcal{S}^J$ induced by ordinary functions $I \to J$.

Remark. The fact that we have included both groups and spaces in a category which is still small enough to have many very nice properties makes it much easier to formalize such notions as cohomology.

Note that $\mathcal{S}$ itself is an etendue, since it is the category of sheaves on the one-point space. This observation motivates the following definition:

Definition. Let $\mathcal{Y}$ be any object in $\mathcal{Top}_S$. Then a point in $\mathcal{Y}$ is a geometric morphism $\mathcal{S} \to \mathcal{Y}$.

Example. Let G be a group, and let G_ℓ be the G-set consisting of G acting on itself by left translation. Then G_ℓ covers 1 and $\mathcal{S}^G/G_\ell \cong \mathcal{S}$, so G is locally isomorphic to the one-point space (or, more precisely, the category of G-sets is locally isomorphic to the category of sheaves on the one-point space).

$\mathcal{S}^G$ itself has a point, namely $\mathcal{S} \xrightarrow{p} \mathcal{S}^G$, where $p(S) = S^G$ (the collection of functions $G \to S$). However, unlike the points of a topological space, this point is moved by endomorphisms (namely the actions of the elements of G). Thus we say that this point can "spin".

Remark. Our definition of etendue is not entirely standard. We have said that $\mathcal{X}$ is an etendue if there is a covering $C \to 1$ with $\mathcal{X}/C$ the category of set-valued sheaves on some Heyting Algebra H. Grothendieck adds the extra condition that H have "enough points"; that is, if we let $\overline{\mathrm{Hom}}(H, 2)$ be the set of mappings $H \to \underline{2}$ which preserve infinite unions and finite intersection, then this condition says that the map $H \to 2^{\overline{\mathrm{Hom}}(H, 2)}$ must be monic. We have dropped this condition from our definition.

VIII Applications

A. Let $\mathcal{X}$ and $\mathcal{Y}$ be objects in $\mathcal{T}op_S$. Then $\mathrm{Hom}_{\mathcal{T}op_S}(\mathcal{X}, \mathcal{Y})$ can always be viewed as the category of models in some theory T in $\mathcal{X}$ thought of as a set theory. In particular, the points of $\mathcal{Y}$ are the models in the category of sets of some theory T. (For a further explanation see Wraith's notes on Object Classifiers.)

Example. Let K be a field, $\mathcal{A}$ the category of finitely generated commutative K-algebras. Then $\mathrm{Hom}_{\mathcal{T}op_S}(\mathcal{X}, \mathcal{S}^{\mathcal{A}})$ turns out to be the category of commutative K-algebra objects in $\mathcal{X}$. If we let $\mathcal{G}$ be the subcategory of $\mathcal{S}^{\mathcal{A}}$ consisting of functors $\mathcal{A} \to \mathcal{S}$ which preserve finite products, then $\mathrm{Hom}_{\mathcal{T}op_S}(\mathcal{X}, \mathcal{G})$ gurns out to be the category of commutative K-algebra objects with no non-trivial idempotents in $\mathcal{X}$.

Example. Let $\mathcal{S}_f$ be the category of finite sets. Then $\mathrm{Hom}_{\mathcal{T}op_S}(\mathcal{X}, \mathcal{S}^{\mathcal{S}_f})$ is just $\mathcal{X}$; i.e., the collection of models for the theory $[x = x]$.

Example. Let G be a group. Then $\mathrm{Hom}_{\mathcal{T}op_S}(\mathcal{X}, \mathcal{S}^G)$ turns out to be the same as $H^1(\mathcal{X}, G)$, the category of principal homogeneous G-objects in $\mathcal{X}$. These are models of a theory which has an object U, a map $g: U \to U$ for each $g \in G$ and axioms:

$$(g_1 g_2)u = g_1(g_2 u) ,$$

$$\exists u\, [u \in U] ,$$

$$g_1 u = g_2 u \vdash \text{false} \quad \text{whenever } g_1 \neq g_2 ,$$

$$\exists x \bigvee_{g \in G} gx = y .$$

(Note that the $\bigvee$ appearing in the last axiom is not the same as existential quantification, since that would have to be over an object in the theory.)

B. Consider the cagegory $\mathcal{S}^{\circlearrowleft}/\mathbb{N}^{\circlearrowleft\sigma}$. Here a typical object is a commutative diagram

$$\begin{tikzcd}
S \arrow[r, "t"] \arrow[d, "f"'] & S \arrow[d, "f"] \\
\mathbb{N} \arrow[r, "\sigma"'] & \mathbb{N}
\end{tikzcd}$$

Such an object can be identified with a functor $F\colon \omega \to \mathcal{S}$ where ω is the well-ordered set of natural numbers, by letting $F(n) = f^{-1}(n)$ and taking all arrows to be restrictions of t. That is, F is the diagram

$$S_0 \xrightarrow{t_0} S_1 \xrightarrow{t_1} S_2 \xrightarrow{t_2} S_3 \xrightarrow{t_3} \cdots$$

where $S_n = f^{-1}(n)$ and $t_n = t|_{S_n}$. This identification is an isomorphism of categories $\mathcal{S}^{\circlearrowleft}/\mathbb{N}^{\circlearrowleft\sigma} \cong \mathcal{S}^{\omega}$, and, since $\mathcal{S}^{\omega}$ is the category of sheaves on $\omega \cup \{\infty\}$ with the one-sided order topology, it shows that $\mathcal{S}^{\circlearrowleft}$ is an etendue.

Now it is a true theorem that if $\mathcal{X}$ is any topos over $\mathcal{S}$ and $C \to 1$ is epic in $\mathcal{X}$, that any geometric morphism $\mathcal{S} \to \mathcal{X}$ factors through the forgetful functor $\mathcal{X}/C \to \mathcal{X}$ as shown:

$$\begin{tikzcd}
 & \mathcal{X}/C \arrow[d] \\
\mathcal{S} \arrow[ur, dashed] \arrow[r] & \mathcal{X}
\end{tikzcd}$$

(Sketch of proof: Given

$$\begin{tikzcd}
 & \mathcal{X}/C \arrow[d] \\
\mathcal{S} \arrow[r, "x"'] & \mathcal{X}
\end{tikzcd}$$

the liftings which are topos morphisms are in 1-1 correspondence with the maps $1 \to x^*(C)$ in $\mathcal{S}$. (In fact, this statement remains true if we replace $\mathcal{S}$ by an arbitrary topos.) Thus, if $C \to 1$ is epi, then $x^*(C) \to 1$ is also epi (this is a general property of topos morphisms), so that if $\mathcal{S}$ satisfies even a weak axiom of choice, maps $1 \to x^*(C)$ exist and so by what we have noted, the desired liftings exist.)

In particular, let $f\colon \mathcal{S} \to \mathcal{S}^{\circlearrowright}$ be a geometric morphism. Then we have a commutative diagram of geometric morphisms

$$\begin{tikzcd}
 & S^{\circlearrowright}/\mathbb{N}^{\circlearrowright\sigma} \cong S^{\omega} \arrow[d, "\text{forgetful}"] \\
S \arrow[ur, "g"] \arrow[r, "f"'] & S^{\circlearrowright}
\end{tikzcd}$$

which in turn yields a commutative diagram of left adjoints

$$\begin{tikzcd}
 & S^{\circlearrowright}/\mathbb{N}^{\circlearrowright\sigma} \cong S^{\omega} \arrow[dl, "g^{*}"'] \arrow[d, "(\text{forgetful})^{*}"] \\
S \arrow[r, "f^{*}"'] & S^{\circlearrowright}
\end{tikzcd}$$

The functor $(\text{forgetful})^{*}$ takes the object $S^{\circlearrowright\, t}$ to the object

$$\begin{tikzcd}
S \times \mathbb{N} \arrow[r, "t \times \sigma"] \arrow[d, "\pi_1"'] & S \times \mathbb{N} \arrow[d, "\pi_1"] \\
\mathbb{N} \arrow[r, "\sigma"'] & \mathbb{N}
\end{tikzcd}$$

The functor g^{*} can only be either evaluation at some point n (here we are thinking of the domain of g^{*} as S^{ω}) or the functor $\varinjlim$ (which can be thought of as evaluation at infinity).

Now since (forgetful)* places the same fiber over every natural number, the composition of (forgetful)* followed by g^* is always the same functor if g is evaluation at any finite point. This shows that there are only two possibilities for f^* (and hence for f) -- (forgetful)* followed by evaluation at an arbitrary finite point, and (forgetful)* followed by evaluation at ∞. Therefore $S^{\circlearrowleft}$ has precisely two points.

Now let $\langle \mathbb{N}, + \rangle$ be the monoid of natural numbers, thought of as a one-element category. Clearly we may identify $S^{\circlearrowleft}$ as $S^{\langle \mathbb{N}, + \rangle}$ by associating the object $X \xrightarrow{+} X$ in $S^{\circlearrowleft}$ to the functor $F: \langle \mathbb{N}, + \rangle \to S$ which takes the unique object to the object X and the arrow n to the arrow t^n. With this identification, one of the possibilities for f^* is simply the forgetful functor $S^{\langle \mathbb{N}, + \rangle} \to S$. The other possibility is to take each object M in $S^{\langle \mathbb{N}, + \rangle}$, thought of as a monoid in its own right which comes equipped with a monoid homomorphism $\mathbb{N} \to M$, to the underlying set of $\mathbb{Z} \otimes_{\mathbb{N}} M$.

If we write the forgetful functor (redundantly) as $\mathbb{N} \otimes_{\mathbb{N}} -$ (this is what is called in SGA4 the "banal point"), then our two points are $\mathbb{N} \otimes_{\mathbb{N}} -$ and $\mathbb{Z} \otimes_{\mathbb{N}} -$, it is useful in some applications to abbreviate these points as $\mathbb{N}$ and $\mathbb{Z}$. The monoid structure of $\mathbb{N}$ and the group structure of $\mathbb{Z}$ then yield information about the interactions between these two points.

C. The following example of a use for etendue is due to Mumford, and makes heavy use of the fact that, while an etendue is locally a topological space, it may fail to be one globally because of spin .

In complex analysis, all curves of a given genus are parameterized by some space. If we attempt to do the same thing over an arbitrary ground field, there does not exist such a space, precisely because of the existence of Galois Groups. (If we had a space, these groups would have to act on individual points of the space, which they cannot do.) However, it is possible to parameterize by an etendue, since here we can allow the Galois Groups to "spin" the points. This example is discussed further in SGA4.

D. A set is a possible domain of variation. More interesting domains of variation are topological spaces (here we have sheaves, rings of continuous functions, etc.). The passage to etendues is another step in this generalization, and it even has geometric significance (even though the points may "spin"). In fact, every topos is a domain of variation, but etendues are easier to work with then general topoi because they have sets of points while general topoi have classes of points.

Ultimately, we may have to consider more general things. For example, we might try to imbed Top_S in a still larger category H which contained an object y such that for any object X of Top_S ,

$\mathrm{Hom}_{\mathcal{H}}(X, Y)$ would be precisely the collection of Kuratowski Finite objects of X. Thus Y would "be" the concept of finiteness. Conceivably, by taking $\mathcal{H}$ large enough, we could recognize larger classes of concepts in this way.

Such enlargements $\mathcal{Top}_S \subset \mathcal{H}$ may be analogous to the enlargement of affine spaces to more general algebraic spaces which are not determined by a single ring of global variable quantities. $X \in \mathcal{Top}_S$ is a domain of variation determined by a single topos of global variable sets; an object $\mathcal{H}$ would be determined by an interlocking system of topoi, each consisting of sets varying over a piece of a covering of the object.