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FUNCTION COMPLEXES IN HOMOTOPICAL ALGEBRA

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§1. INTRODUCTION

1.1 Summary

IN [1] QUILLEN introduced the notion of a *model category* (a category together with three classes of maps: weak equivalences, fibrations and cofibrations, satisfying certain axioms (1.4 (iv))) as a general framework for "doing homotopy theory". To each model category **M** there is associated a *homotopy category*. If $\mathbf{W} \subset \mathbf{M}$ denotes the subcategory of the weak equivalences, then this homotopy category is just the *localization* $\mathbf{M}[\mathbf{W}^{-1}]$, i.e. the category obtained from **M** by formally inverting the maps of **W**, and it thus depends only on the weak equivalences and not on the fibrations and the cofibrations. Moreover, if two model categories are connected by a pair of adjoint functors satisfying certain conditions, then their homotopy categories are equivalent.

The homotopy category of a model category **M** does *not* capture the "higher order information" implicit in **M**. In the pointed case, however, Quillen was able to recover some of this information by adding some further structure (a loop functor, a suspension functor and fibration and cofibration sequences) to the homotopy category. His fundamental comparison theorem then stated that, if two pointed model categories are connected by a pair of adjoint functors satisfying certain conditions, then their homotopy categories are equivalent in a manner which respects this additional structure.

The aim of the present paper is to go back to an arbitrary model category **M** and construct a *simplicial homotopy category* which *does* capture the "higher order information" implicit in **M**. This simplicial homotopy category is defined as the *hammock localization* $L^H(\mathbf{M}, \mathbf{W})$ (for short $L^H\mathbf{M}$) of [2]. It is a *simplicial category* (1.4) with the following basic properties:

(*i*) *The simplicial homotopy category* $L^H\mathbf{M}$ *depends* (by definition) *only on the weak equivalences and not on the fibrations and cofibrations.*

(*ii*) *If two model categories are connected by a pair of adjoint functors satisfying Quillen's conditions, then their simplicial homotopy categories are weakly equivalent* (1.4).

(*iii*) *The "category of components" of the simplicial homotopy category of* **M** *is just the homotopy category of* **M**.

(*iv*) *If* $\mathbf{M}_*$ *is a closed simplicial model category* [1], *then*, as one would expect, *the full simplicial subcategory* $\mathbf{M}_*^{cf} \subset \mathbf{M}_*$ *generated by the objects which are both cofibrant and fibrant is weakly equivalent* (1.4) *to* $L^H\mathbf{M}$.

(*v*) "$L^H\mathbf{M}$ *provides* **M** *with function complexes*", *i.e. for every two objects* $X, Y \in \mathbf{M}$, *the simplicial set* $L^H\mathbf{M}(X, Y)$ *has the correct homotopy type for a function complex, in the sense that, for every cosimplicial resolution* X^* *of* X *and every simplicial resolution* Y_* *of* Y (4.3), *it has the same homotopy type as* $\operatorname{diag} \mathbf{M}(X^*, Y_*)$.

1.2 Application

The hammock localization enables one to construct simplicial monoids which are analogs of "the space of self homotopy equivalences" of an object $X \in \mathbf{M}$, something that seems difficult to do using resolutions. In fact there are two obvious candidates:

(i) the *homotopy automorphism complex* $haut_{L^H\mathbf{M}}X$, which is the simplicial submonoid of $L^H\mathbf{M}(X, X)$ consisting of the components which are invertible in $\pi_0 L^H\mathbf{M}(X, X)$, and

(ii) the simplicial monoid $L^H\mathbf{W}(X, X)$.

Fortunately (4.6) *their classifying complexes have the same homotopy type whenever* **M** *is a closed model category.*

Actually it was our interest in such analogs of the space of self homotopy equivalences for arbitrary model categories that led to the present paper, as well as to [2, 4].

1.3 Organization of the paper

After fixing some notation and terminology (in 1.4), we explain (in §2) what exactly we mean by *simplicial categories* and by *weak equivalences* between them and recall (in §3) the definition of the *hammock localization* of [2]. Our main results are then formulated in §4 and §5, those dealing with *function complexes* in §4, while §5 concentrates on *simplicial homotopy categories.* Then comes §6 which is, in some sense, the key section of the paper. It contains various properties of *cosimplicial and simplicial resolutions* and, most importantly, a proof of their *existence.* The remaining sections (§7 and §8) are devoted to the completion of the proofs of the results of §4 and §5.

1.4 Notation, terminology, etc.

We will freely use the notation, terminology and results of [1, 2] (especially in the proofs), except for the slight changes in the terminology indicated in (i), (ii) and (iii) below. In (iv) we recall the definition of a model category and fix some related notation and terminology.

(*i*) *Simplicial sets.* These will not, as usual, be necessarily small, but only homotopically small in a sense that will be made precise in 2.2.

(*ii*) *Simplicial categories.* These will be required (see 2.1) to have *the same objects in each dimension*, but *their "simplicial hom-sets" need only be homotopically small.*

(*iii*) *Weak equivalences between simplicial categories.* These will be (see 2.4) functors which induce an *equivalence between the "categories of components"* and *weak homotopy equivalences on the "simplicial hom-sets"*. This is a generalization of the notion of weak equivalence of [2].

(*iv*) *Model categories.* A *model category* consists of a category **M**, together with three subcategories **W**, $\underset{\sim}{\mathrm{Fib}}$ and $\underset{\sim}{\mathrm{Cof}} \subset \mathbf{M}$ which *contain all isomorphisms* of **M**, satisfying the following axioms [1, I, §1]:

M0 **M** *is closed under finite direct and inverse limits.*

M1 *Given a solid arrow diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with $i \in \underset{\sim}{\mathrm{Cof}}$, $p \in \underset{\sim}{\mathrm{Fib}}$ *and either* i *or* $p \in \mathbf{W}$. *Then the dotted arrow exists.*

M2 *Any map* $f \in \mathbf{M}$ *can be factored* $f = pi$ *and* $f = qj$ *with* $i, j \in \underset{\sim}{\mathrm{Cof}}$, $p, q \in \underset{\sim}{\mathrm{Fib}}$ *and* $i, q \in \mathbf{W}$.

M3 *Any pullback of a map in* $\underset{\sim}{\mathrm{Fib}}$ *is again in* $\underset{\sim}{\mathrm{Fib}}$ *and any pushout of a map in* $\underset{\sim}{\mathrm{Cof}}$ *is again in* $\underset{\sim}{\mathrm{Cof}}$.

M4 *Any pullback of a map in* $\mathbf{W}\cap\underset{\sim}{\mathrm{Fib}}$ *is again in* $\mathbf{W}\cap\underset{\sim}{\mathrm{Fib}}$ *and any pushout of a map in* $\mathbf{W}\cap\underset{\sim}{\mathrm{Cof}}$ *is again in* $\mathbf{W}\cap\underset{\sim}{\mathrm{Cof}}$.

M5 *Let* $f, g \in \mathbf{M}$ *be maps such that* gf *is defined. If two of* f, g *and* gf *are in* $\mathbf{W}$, *then so is the third.*

The maps of $\mathbf{W}$, $\underset{\sim}{\mathrm{Fib}}$, $\mathbf{W}\cap\underset{\sim}{\mathrm{Fib}}$, $\underset{\sim}{\mathrm{Cof}}$, $\mathbf{W}\cap\underset{\sim}{\mathrm{Cof}}$ are called weak equivalences, fibrations, trivial fibrations, cofibrations, trivial cofibrations respectively and are sometimes denoted by $\xrightarrow{\sim}$, $\twoheadrightarrow$, $\overset{\sim}{\twoheadrightarrow}$, $\rightarrowtail$, $\overset{\sim}{\rightarrowtail}$.

An object $X \in \mathbf{M}$ is called *cofibrant* if the map $\varphi \to X$ ($\varphi =$ initial object) is a cofibration and is called *fibrant* if the map $X \to *$ ($* =$ terminal object) is a fibration. The full subcategory of $\mathbf{M}$ generated by the fibrant (resp. cofibrant) objects will be denoted by $\mathbf{M}^f$ (resp. $\mathbf{M}^c$) and we abbreviate $\mathbf{M}^{cf} = \mathbf{M}^c \cup \mathbf{M}^f$, $\mathbf{W}^f = \mathbf{W} \cap \mathbf{M}^f$, $\mathbf{W}^c = \mathbf{W} \cap \mathbf{M}^c$.

§2. SIMPLICIAL CATEGORIES

We start with a brief discussion of *simplicial categories* and *weak equivalences* between them.

2.1 Simplicial categories

By a *simplicial category* we will mean something slightly different from usual. We assume, as is often done, that they have *the same objects in each dimension*. However, we do not require that the "simplicial hom-sets" be small, but only that they be *homotopically small* in the sense explained below. *A simplicial category which is "discrete" then is just an ordinary category.*

2.2 Homotopically small simplicial sets

A (not necessarily small) simplicial set X will be called *homotopically small* if $\pi_n(X; v)$ is small for every vertex $v \in X$ and every integer $n \geq 0$. This is clearly equivalent to requiring that X contain a small simplicial subset U with the property that, for every small simplicial subset $V \subset X$ containing U, there is a small simplicial subset $W \subset X$ containing V, such that the inclusion $U \to W$ is a weak homotopy equivalence. Clearly *the homotopy type of such a* U *is unique* and it thus makes sense to talk of the *homotopy type* of a homotopically small simplicial set and of *weak homotopy equivalences* between homotopically small simplicial sets.

The following proposition shows that, in a simplicial category in the sense of 2.1, one can "do homotopy theory" as usual.

2.3 Proposition. *Let* $\mathbf{C}$ *be a simplicial category and let* $\mathbf{E} \subset \mathbf{C}$ *be a small simplicial subcategory*, i.e. the objects of $\mathbf{E}$ form a small set and, for every two objects $X, Y \in \mathbf{E}$, the simplicial set $\mathbf{E}(X, Y)$ is also small. *Then there is a small simplicial subcategory* $\mathbf{D} \subset \mathbf{C}$ *containing* $\mathbf{E}$ *such that, for every two objects* $X, Y \in \mathbf{D}$, *the inclusion* $\mathbf{D}(X, Y) \to \mathbf{C}(X, Y)$ *is a weak homotopy equivalence.*

The proof is straightforward.

2.4 Weak equivalences between simplicial categories

A *weak equivalence* S: $\mathbf{C} \to \mathbf{D}$ between two simplicial categories is a functor which

(i) induces an *equivalence* $\pi_o\mathbf{C} \approx \pi_o\mathbf{D}$ between the "categories of components", and;

(ii) induces, for every two objects $X, Y \in \mathbf{C}$, a *week homotopy equivalence* $\mathbf{C}(X, Y) \sim \mathbf{D}(SX, SY)$.

Similarly two simplicial categories will be called *weakly equivalent* if they can be connected by a finite string of weak equivalences. Clearly *weakly equivalent simplicial categories are equivalent from the point of view of homotopy theory.*

We end with observing that there is also the slightly stronger notion of

2.5 Homotopy equivalences between simplicial categories

A functor S: $\mathbf{C} \to \mathbf{D}$ between simplicial categories is called a *homotopy equivalence* if there is a functor T: $\mathbf{D} \to \mathbf{C}$ (a *homotopy inverse* of S) such that the compositions TS and ST are, in the following sense, homotopic to the identity functors of $\mathbf{C}$ and $\mathbf{D}$ respectively. Two functors S_1, S_2: $\mathbf{C} \to \mathbf{C}'$ are *homotopic* if there exists a natural equivalence t: $\pi_o S_1 \approx \pi_o S_2$ such that, for every two objects $X, Y \in \mathbf{C}$, the diagram

$$\begin{tikzcd} & \mathbf{C}'(S_1X, S_1Y) \arrow[dd, "(tX^{-1}, tY)"] \\ \mathbf{C}(X, Y) \arrow[ur, "S_1"] \arrow[dr, "S_2"'] & \\ & \mathbf{C}'(S_2X, S_2Y) \end{tikzcd}$$

commutes up to homotopy, where (tX^{-1}, tY) denotes a map obtained by composition with representatives of tX^{-1} and tY.

It is not difficult to verify that indeed *every homotopy equivalence is a weak equivalence.*

§3. SIMPLICIAL LOCALIZATIONS OF CATEGORIES

The simplicial localizations of [1, 2] assigned to a small category $\mathbf{C}$ and subcategory $\mathbf{W} \subset \mathbf{C}$, weakly equivalent (2.4) small simplicial categories $L\mathbf{C}$ and $L^H\mathbf{C}$. In preparation for the formulation and proof of our results we recall the definition of $L^H\mathbf{C}$. Although in our applications the category $\mathbf{C}$ is not necessarily small, it turns out (see §4 and §5) that $L^H\mathbf{C}$ is still well defined and is a simplicial category in the sense of 2.1.

3.1 The hammock localization of a small category

Let $\mathbf{C}$ be a small category and let $\mathbf{W} \subset \mathbf{C}$ be a subcategory which contains all the objects. The *hammock localization* of $\mathbf{C}$ with respect to $\mathbf{W}$ then is the (small) simplicial category $L^H(\mathbf{C}, \mathbf{W})$ (for short $L^H\mathbf{C}$) defined as follows [2]:

(1) $L^H\mathbf{C}$ has the same objects as $\mathbf{C}$ in every dimension.

(2) For every two objects $X, Y \in \mathbf{C}$, the simplicial set $L^H\mathbf{C}(X, Y)$ has as its k-simplices the "reduced hammocks of width k and any length" between X and Y, i.e. the commutative diagrams in $\mathbf{C}$ of the form

$$\begin{tikzcd} & C_{0,1} \arrow[r, dash] \arrow[d] & C_{0,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{0,n-1} \arrow[d] \arrow[ddr, dash] & \\ & C_{l,1} \arrow[r, dash] \arrow[d] & C_{l,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{l,n-1} \arrow[d] \arrow[dr, dash] & \\ X \arrow[uur, dash] \arrow[ur, dash] \arrow[dr, dash] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & Y \\ & C_{k,1} \arrow[r, dash] & C_{k,2} \arrow[r, dash] & \cdots \arrow[r, dash] & C_{k,n-1} \arrow[ur, dash] & \end{tikzcd}$$

in which

(i) n, the length of the hammock, is any integer ≥ 0,

(ii) all vertical maps are in $\mathbf{W}$,

(iii) in each column all maps go in the same direction; if they go to the left, then they are in $\mathbf{W}$,

(iv) the maps in adjacent columns go in different directions, and

(v) no column contains only identity maps. Faces, degeneracies and compositions are defined in the obvious manner, i.e. the i-face is obtained by omitting the i-row and the i-degeneracy by repating the i-row; if the resulting hammock is not reduced, i.e. does not satisfy (iv) and (v), then it can easily be made so by repeatedly.

(iv)′ composing two adjacent columns, whenever their maps go in the same direction, and

(v)′ omitting any column which contains only identity maps.

Note that *the hammock localization comes with an obvious functor* $p: \mathbf{C} \to L^H\mathbf{C}$.

An immediate consequence of the above definition is

3.2 PROPOSITION. *For every two objects $X, Y \in \mathbf{C}$, the components of $L^H\mathbf{C}(X, Y)$ are in* 1-1 *correspondence with the maps $X \to Y \in \mathbf{C}[\mathbf{W}^{-1}]$, i.e. $\pi_o L^H\mathbf{C} = \mathbf{C}[\mathbf{W}^{-1}]$, where $\mathbf{C}[\mathbf{W}^{-1}]$ denotes the (ordinary) localization of $\mathbf{C}$ with respect to $\mathbf{W}$, i.e. the category obtained from $\mathbf{C}$ by "formally inverting" the maps of $\mathbf{W}$.*

3.3 The simplicial localizations of not necessarily small categories

Definition 3.1 also makes sense for categories which are not necessarily small, except that in that case the simplicial sets $L^H\mathbf{C}(X, Y)$ need not be small either. This will however cause us no difficulties because, for the categories considered in this paper, these simplicial sets $L^H\mathbf{C}(X, Y)$ will turn out to be homotopically small, so that $L^H\mathbf{C}$ is a simplicial category in the sense of 2.1.

Of course *the same applies to the weakly equivalent standard simplicial localization $L\mathbf{C}$* of [3].

§4. FUNCTION COMPLEXES IN MODEL CATEGORIES

Our main result is that (i) the hammock localization (3.1) $L^H\mathbf{M}$ of a model category $\mathbf{M}$ is a simplicial category (2.1) with the same objects as $\mathbf{M}$, and (ii) the resulting homotopically small (2.2) simplicial sets $L^H\mathbf{M}(X, Y)$ have the "correct" homotopy types for being function complexes.

4.1 PROPOSITION. *Let $\mathbf{M}$ be a model category* (1.4 (iv)). *Then the hammock localization $L^H\mathbf{M} = L^H(\mathbf{M}, \mathbf{W})$* (3.1) *is a simplicial category* (2.1).

Proof. This follows immediately from propossitions 4.4 and 4.5 below.

Combining 4.1 with 3.2 one gets that the simplicial sets $L^H\mathbf{M}(X, Y)$ have indeed the "correct" set of components.

4.2 COROLLARY. *For every two objects $X, Y \in \mathbf{M}$, the components of $L^H\mathbf{M}(X, Y)$ are in* 1-1 *correspondence with the maps $X \to Y \in \mathbf{M}[\mathbf{W}^{-1}]$, i.e. $\pi_o L^H\mathbf{M} = \mathbf{M}[\mathbf{W}^{-1}]$, the classical* [1, Chap. I, 1.13] *homotopy category of* $\mathbf{M}$.

Further evidence is that (see 4.4) the simplicial sets $L^H\mathbf{M}(X, Y)$ have the same homotopy type as the less functorial "function complexes" obtained using

4.3 Simplicial and cosimplicial resolutions

Let $\mathbf{M}$ be a model category. By a *simplicial resolution* of an object $Y \in \mathbf{M}$ we then mean a simplicial object Y_* over $\mathbf{M}$ together with a weak equivalence $Y \xrightarrow{\sim} Y_o$ such that

(i) the object Y_o is fibrant

(ii) all face maps between the Y_n are trivial fibrations (and hence the objects Y_n $(n > 0)$ are also fibrant), and

(iii) for every integer $n \geq 0$ the obvious map $Y_{n+1} \to (d_*, Y_n)$ is a fibration, where (d_*, Y_n) denotes the inverse limit of the diagram which consists of

(a) for every integer i with $0 \leq i \leq n+1$, a copy (d_i, Y_n) of Y_n, and

(b) for every pair of integers (i, j) with $0 \leq i < j \leq n+1$, a copy $(d_i d_j, Y_{n-1})$ of Y_{n-1}, together with pair of maps

$$(d_j, Y_n) \xrightarrow{d_{j-1}} (d_i d_j, Y_{n-1}) \xleftarrow{d_i} (d_j, Y_n).$$

Similarly a *map* between two simplicial resolutions Y_* and $\bar{Y}_*$ of Y will be a map $Y_* \to \bar{Y}_*$ of simplicial objects which is compatible with the weak equivalences $Y \xrightarrow{\sim} Y_o$ and $Y \xrightarrow{\sim} \bar{Y}_o$.

Cosimplicial resolutions and their *maps* are, of course, defined dually.

4.4 PROPOSITION. *Let* $\mathbf{M}$ *be a model category. Then, for every two objects* $X, Y \in \mathbf{M}$ *and every cosimplical resolution* X^* *of* X *and simplicial resolution* Y_* *of* Y, *the simplicial set* $\operatorname{diag} \mathbf{M}(X^*, Y_*)$ *has the same homotopy type as* $L^H\mathbf{M}(X, Y)$. *If* X *is cofibrant, then* $\mathbf{M}(X, Y_*)$ *has also the same homotopy type and dually, if* Y *is fibrant, then so does* $\mathbf{M}(X^*, Y)$.

Proof. The second part follows from the first part and 6.3 and 6.4; a proof of the first part will be given in 7.2 and 8.1.

The usefulness of proposition 4.4 is due to proposition 4.5, which will be proved in 6.7.

4.5 PROPOSITION. *Every object of model category has a simplicial and a cosimplicial resolution.*

4.6 APPLICATION. *The simplicial localizations enable us to construct simplicial monoids which are analogs of "the space of self homotopy equivalences", something that seems difficult to do using resolutions. There are two obvious candidates*:

(*i*) *the homotopy automorphism complex* $haut_{L^H\mathbf{M}} X$ *of an object* $X \in \mathbf{M}$, *which is the simplicial submonoid of* $L^H\mathbf{M}(X, X)$ *consisting of the components of* $L^H\mathbf{M}(X, X)$ *which are invertible in* $\pi_0 L^H\mathbf{M}(X, X)$, *and*

(*ii*) *the simplicial monoid* $L^H\mathbf{W}(X, X)$. However [2, 6.4] *if* $\mathbf{W}$ *is closed in* $\mathbf{M}$ *in the sense of* [3, 3.4] (e.g. if $\mathbf{M}$ is a closed model category [1, *I*, §5]), *then the inclusion of* $L^H\mathbf{W}(X, X)$ *in* $haut_{L^H\mathbf{M}} X$ *induces a weak homotopy equivalence between their classifying complexes.* Of course [2] *the classifying complex of* $L\mathbf{W}(X, X)$ *has also the same homotopy type.* This is sometimes useful because [3] $L\mathbf{W}(X, X)$ is actually a simplicial group.

We end with observing that, if the model category $\mathbf{M}$ comes already equipped with function complexes, i.e. if $\mathbf{M}$ is the 0-dimensional part of a *closed simplicial model category* $\mathbf{M}_*$ [1, Chap. II], then proposition 4.5 implies:

4.7 COROLLARY. *If* $X \in \mathbf{M}_*$ *is cofibrant and* $Y \in \mathbf{M}_*$ *is fibrant, then* $\mathbf{M}_*(X, Y)$ *has the same homotopy type as* $L^H\mathbf{M}(X, Y)$.

In fact the following *stronger result* holds. If $\mathbf{M}_*^{cf} \subset \mathbf{M}_*$ denotes the full simplicial subcategory generated by the cofibrant fibrant objects and if $L^H\mathbf{M}_*$ and $L^H\mathbf{M}_*^{cf}$ denote the bisimplicial categories obtained by applying L^H dimensionwise with respect to the (iterated) degeneracies of the weak equivalences, then one has: (see 3.1)

4.8 PROPOSITION. *In the commutative diagram*

$$\begin{tikzcd} & \mathbf{M} = \mathbf{M}_0 \arrow[r, "\text{incl.}"] \arrow[dl] & \mathbf{M}_* \arrow[d] & \mathbf{M}_*^{cf} \arrow[l, "\text{incl.}"'] \arrow[dr] & \\ L^H\mathbf{M} \arrow[rr, "\sim"'] & & \operatorname{diag} L^H\mathbf{M}_* & & \operatorname{diag} L^H\mathbf{M}_*^{cf} \arrow[ll, "\sim"] \end{tikzcd}$$

the maps indicated by $\sim$ *are weak equivalences.*

Proof. The map on the right is a weak equivalence by [3, 6.4] and Proposition 5.3 readily implies that the left bottom map is so too. That the right bottom map is a weak equivalence will be proved in 7.4 and 8.4.

§5. SIMPLICIAL HOMOTOPY CATEGORIES

The results of the previous section, and in particular Corollary 4.2 and Proposition 4.8, suggest the following definition

5.1 Simplicial homotopy categories

A *simplicial homotopy category* of a model category $\mathbf{M}$ is an simplicial category which is weakly equivalent (2.4) to $L^H\mathbf{M} = L^H(\mathbf{M}, \mathbf{W})$.

Some examples are provided by

5.2 PROPOSITION. *Let* $\mathbf{M}$ *be a model category. Then the inclusions* (1.4(iv))

$$L^H(\mathbf{M}^c, \mathbf{W}^c \cap \underset{\sim}{\mathrm{Cof}}) \to L^H(\mathbf{M}^c, \mathbf{W}^c) = L^H\mathbf{M}^c \to L^H\mathbf{M}$$

$$L^H(\mathbf{M}^f, \mathbf{W}^f \cap \underset{\sim}{\mathrm{Fib}}) \to L^H(\mathbf{M}^f, \mathbf{W}^f) = L^H\mathbf{M}^f \to L^H\mathbf{M}$$

are all weak equivalences.

5.3 PROPOSITION. *Let* $\mathbf{M}_*$ *be a closed simplicial model category* [1, Chap. II]. *Then, for every integer* $k \geq 0$, *the map* $L^H\mathbf{M}_0 \to L^H\mathbf{M}_k$, *induced by the k-fold degeneracy, is a weak equivalence.*

Proofs. It follows from [1, 5.1] and [4, 1.2 and 1.3] that in 5.2 the maps on the left are weak equivalences.

Proposition 5.3 and the rest of Proposition 5.2 will be proved in 7.1, 7.3, 8.2 and 8.3. A further justification of Definition 5.1 is the fact that model categories which can be connected by a pair of adjoint functors having the usual properties have the same simplicial homotopy categories, i.e.

5.4 PROPOSITION. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be model categories and let* $S\colon \mathbf{M} \to \mathbf{N}$ *and* $T\colon \mathbf{N} \to \mathbf{M}$ *be a pair of adjoint functors such that* S *(the left adjoint) sends cofibrations into cofibrations and weak equivalences between cofibrant objects into weak equivalences and* T *(the right adjoint) sends fibrations into fibrations and weak equivalences between fibrant objects into weak equivalences. Then, for every cofibrant object* $X \in \mathbf{M}$ *and cosimplicial resolution* Y_* *of* Y, *the adjunction map induces an isomorphism of bisimplicial sets*

$$M(X^*, TY_*) \approx N(SX^*, Y_*).$$

If moreover, for every cofibrant object $X \in \mathbf{M}$ *and fibrant object* $Y \in \mathbf{N}$, *a map* $X \to TY \in \mathbf{M}$ *is a weak equivalence if and only if its adjoint* $SX \to Y \in \mathbf{N}$ *is so, then the induced functors*

$$L^H\mathbf{M}^c \to L^H\mathbf{N}^c \qquad \text{and} \qquad L^H\mathbf{N}^f \to L^H\mathbf{M}^f$$

are weak equivalences and hence (5.2) $\mathbf{M}$ *and* $\mathbf{N}$ *have the same simplicial homotopy categories.*

Proof. The proof of the first part is straightforward. The second part follows from the first part and [1, I, Theorem 3].

5.5 *Remark.* Of course [2, 2.2] one can everywhere replace L^H by L.

5.6 *Remark.* If $\mathbf{M}$ is a model category which *admits functorial factorizations* (1.4(iv), M2), then it is not hard to verify that *in* 5.2 *the maps on the right are homotopy equivalences* in the sense of 2.5.

Similarly if, in 5.4, $\mathbf{M}$ and $\mathbf{N}$ both *admit functorial* factorizations, then *the maps* $L^H\mathbf{M}^c \to L^H\mathbf{N}^c$ *and* $L^H\mathbf{N}^f \to L^H\mathbf{M}^f$ *are both homotopy equivalences* and *the compositions*

$$L^H\mathbf{M} \to L^H\mathbf{M}^c \to L^H\mathbf{N}^c \to L^H\mathbf{N} \qquad \textit{and} \qquad L^H\mathbf{N} \to L^H\mathbf{N}^f \to L^H\mathbf{M}^f \to L^H\mathbf{M}$$

(in which the maps $L^H\mathbf{M} \to L^H\mathbf{M}^c$ and $L^H\mathbf{N} \to L^H\mathbf{N}^f$ are homotopy inverses (2.5) of the inclusions) *are homotopy inverses of each other.*

§5. SIMPLICIAL AND COSIMPLICIAL RESOLUTIONS

This section is in many respects the key section of the paper. It contains some of the basic properties of simplicial and cosimplicial resolutions as well as a proof of their existence, i.e. Proposition 4.5.

We start with showing that the "function complexes" obtained from resolutions are unique up to homotopy.

6.1 PROPOSITION. *Let $X' \overset{\sim}{\rightarrowtail} X \in \mathbf{M}$ be a trivial cofibration and let Y_* be a simplicial resolution of an object $Y \in \mathbf{M}$. Then the induced map $\mathbf{M}(X, Y_*) \to \mathbf{M}(X', Y_*)$ is a trivial fibration of simplicial sets.*

Dually one has

6.2 PROPOSITION. *Let $Y \overset{\sim}{\twoheadrightarrow} Y' \in M$ be a trivial fibration and let X^* be a cosimplicial resolution of an object $X \in \mathbf{M}$. Then the induced map $\mathbf{M}(X^*, Y) \to \mathbf{M}(X^*, Y')$ is a trivial fibration of simplicial sets.*

The proofs are straightforward, using the standard characterization of trivial fibrations of simplicial sets [1, II, 2.2].

In view of [4, 1.2 and 1.3] these propositions imply

6.3 COROLLARY. *Let $X' \xrightarrow{\sim} X \in \mathbf{M}$ be a weak equivalence between cofibrant objects and let Y_* be a simplicial resolution of an object $Y \in \mathbf{M}$. Then the induced map $\mathbf{M}(X, Y_*) \to \mathbf{M}(X', Y_*)$ is a weak homotopy equivalence.*

6.4 COROLLARY. *Let $Y \xrightarrow{\sim} Y' \in \mathbf{M}$ be a weak equivalence between fibrant objects and let X^* be a cosimplicial resolution of an object $X \in \mathbf{M}$. Then the induced map $\mathbf{M}(X^*, Y) \to \mathbf{M}(X^*, Y')$ is a weak homotopy equivalence.*

And combining these results with diagonal arguments [1, 1.4] one gets

6.5 COROLLARY. *Let X^* be a cosimplicial resolution of a cofibrant object $X \in \mathbf{M}$ and let Y_* be a simplicial resolution of a fibrant object $Y \in \mathbf{M}$. Then the simplicial sets*

$$M(X^*, Y), \qquad \mathbf{M}(X, Y_*), \qquad \textit{and} \qquad \operatorname{diag}\mathbf{M}(X^*, Y_*)$$

have all the same homotopy type.

6.6 Corollary. *Let X^* and $\bar{X}^*$ be cosimplicial resolutions of an object $X \in \mathbf{M}$ and let Y_* and $\bar{Y}_*$ be simplicial resolutions of an object $Y \in \mathbf{M}$. Then the simplicial sets*

$$\operatorname{diag} \mathbf{M}(X^*, Y_*), \quad \text{and} \quad \operatorname{diag} \mathbf{M}(\bar{X}^*, \bar{Y}_*)$$

have the same homotopy type.

We now give an outline of a

6.7 *Proof of Proposition* 4.5. (First half only).

The proof proceeds by induction on n as follows.

Given an object $Y \in \mathbf{M}$, find a fibrant object Y_o together with a weak equivalence $Y \xrightarrow{\sim} Y_o$ by choosing a factorization $Y \rightarrowtail\!\!\xrightarrow{\sim} Y_o \twoheadrightarrow e$ of the unique map $Y \to e$ (1.4(iv)).

Now let $n \geq 0$ and assume that fibrant objects Y_i have been constructed for all $0 \leq i \leq n$, together with face and degeneracy maps which satisfy the requirements for a simplicial resolution of Y, as far as they make sense. Denote by (s_*, Y_n) the direct limit of the diagram which consists of

(i) for every integer i with $0 \leq i \leq n$, a copy (s_i, Y_n) of Y_n, and

(ii) for every pair of integers (i, j) with $0 \leq i < j \leq n$, a copy $(s_i s_j, Y_{n-1})$ of Y_{n-1}, together with the pair of maps

$$(s_i, Y_n) \xleftarrow{s_{j-1}} (s_i s_j, Y_{n-1}) \xrightarrow{s_i} (s_j, Y_n).$$

Then the maps $(s_j, Y_n) = Y_n \xrightarrow{f} Y_n = (d_i, Y_n)$ with $f = s_{j-1} d_i$ for $i < j$, $f = id$ for $i = j$, $j+1$ and $f = s_j d_{i-1}$ for $i > j+1$, induce a map $(s_*, Y_n) \to (d_*, Y_n)$. Choose a factorization $(s_*, Y_n) \rightarrowtail\!\!\xrightarrow{\sim} Y_{n+1} \overset{\sim}{\twoheadrightarrow} (d_*, Y_n)$ of this map and a lengthy but essentially straightforward calculation shows that the fibrant object Y_{n+1} and the maps

$$d_i\colon Y_{n+1} \twoheadrightarrow (d_*, Y_n) \twoheadrightarrow (d_i, Y_n) = Y_n \qquad \text{and} \qquad s_i\colon Y_n = (s_i, Y_n) \rightarrowtail\!\!\xrightarrow{\sim} (s_*, Y_n) \rightarrowtail\!\!\xrightarrow{\sim} Y_{n+1}$$

have all the desired properties.

6.8 *Remark.* It should be noted that the simplicial resolution of Y constructed in 6.7 has the *additional properties*:

(i) the map $Y \to Y_o$ is a trivial cofibration,

(ii) all degeneracy maps between the Y_n are trivial cofibrations, and

(iii) the every integer $n \geq 0$, the map $(s_*, Y_n) \to Y_{n+1}$ is a trivial cofibration. Such a *simplicial resolution* will be called *special.*

Special cosimplicial resolutions are, of course, defined dually.

Special resolutions have the following useful properties.

6.9 Proposition. *Let Y_* and $\bar{Y}_*$ be simplicial resolutions of an object $Y \in \mathbf{M}$. If Y_* is a special resolution, then there exists a map of resolutions* (4.3) $Y_* \to \bar{Y}_*$.

6.10 Proposition. *Let X^* and $\bar{X}^*$ be cosimplicial resolutions of an object $X \in \mathbf{M}$. If X^* is a special resolution, then there exists a map of resolutions* (4.3) $\bar{X}^* \to X^*$.

The proofs are straightforward.

6.11 Proposition. *Let Y_* be a special simplicial resolution of an object $Y \in \mathbf{M}$ and let $(Y \downarrow \mathbf{W})$ denote the category of the trivial cofibrations starting at Y. Then the functor $y\colon \mathbf{\Delta}^{op} \to (Y \downarrow \mathbf{W})$ which sends $[n]$ to $Y \xrightarrow{\sim} Y_n$ is right cofinal* [5, p. 316].

6.12 PROPOSITION. *Let X^* be a special cosimplicial resolution of an object $X \in \mathbf{M}$ and let $(\mathbf{W} \downarrow X)$ denote the category of the trivial fibrations ending at X. Then the functor x: $\Delta \to (\mathbf{W} \downarrow X)$ which sends $[n]$ to $X^n \xrightarrow{\sim} X$ is left cofinal* [5, p. 316].

Proof. (Of 6.11 only). One has to show that, for every object $Y \xrightarrow{\sim} Y' \in (Y \downarrow \mathbf{W})$, the under category $(Y \xrightarrow{\sim} Y' \downarrow y)$ is contractible. But $(Y \xrightarrow{\sim} Y' \downarrow y)$ is isomorphic to the homotopy direct limit [5, Chap. XII].

$$\underrightarrow{\mathrm{holim}}^{\Delta^{op}}[[n] \mapsto (Y \downarrow \mathbf{W})(Y \xrightarrow{\sim} Y', Y \xrightarrow{\sim} Y_n)]$$

which [1, Chap. XII, §3] has the same homotopy type as the simplicial set $(Y \downarrow \mathbf{W})$ $(Y \xrightarrow{\sim} Y', \ Y \xrightarrow{\sim} Y_*)$. But the latter is a fibre of the trivial fibration (6.1) $\mathbf{M}(Y', Y_*) \xrightarrow{\sim} \mathbf{M}(Y, Y_*)$ and hence is contractible.

§7. PROOFS (ASSUMING FUNCTORIAL FACTORIZATIONS)

It remains to prove Proposition 5.3 and parts of Propositions 4.4, 4.8 and 5.2. In this section we will prove these results *under the assumption that*, as is often the case, *the model category* $\mathbf{M}$ (resp. $\mathbf{M}_0$) *admits functorial factorizations* (1.4(iv), M2). This will simplify the proofs considerably. In §8 we will then indicate what changes have to be made if $\mathbf{M}$ (res. $\mathbf{M}_o$) does not admit functorial factorizations.

Easiest is the completion of the

7.1 *Proof of* 5.2. Combination of the functorial factoziations with [2, 3.5] readily yields that (see 5.6) the inclusions $L^H\mathbf{M}^c \to L^H\mathbf{M}$ and $L^H\mathbf{M}^f \to L^H\mathbf{M}$ are actually homotopy equivalences (2.5).

Next we finish the

7.2 *Proof of* 4.4. In view of the results of §6 it suffices to show that, for $X, Y \in \mathbf{M}^{cf}$, X^* a special cosimplical resolution of X with $X^o = X$ and Y_* a special simplicial resolution of Y with $Y_o = Y$, the simplicial set diag $\mathbf{M}(X^*, Y_*)$ has the same homotopy type as $L^H\mathbf{M}(X, Y)$. To prove this consider the sequence of maps

$$\operatorname{diag} \mathbf{M}(X^*, Y_*) \longleftarrow \mathbf{M}''(X^*, Y_*) \longrightarrow (\mathbf{W} \cap \underbrace{\mathrm{Cof}})^{-1}\mathbf{M}(\mathbf{W} \cap \underbrace{\mathrm{Fib}})^{-1}(X, Y) \xrightarrow{r} L^H\mathbf{M}(X, Y)$$

in which

(i) $M''(X^*, Y_*) = \underrightarrow{\mathrm{holim}}^{\Delta^{op}\times\Delta^{op}}\mathbf{M}(X^*, Y_*)$ [5, Chap. XII], where $\mathbf{M}(X^*, Y_*)$ is considered as a functor $\Delta^{op} \times \Delta^{op} \to \mathrm{Sets} \subset s\,\mathrm{Sets}$ and the map on the left is the weak homotopy equivalence obtained by twice applying [5, Chap. XII, 3.4],

(ii) $(\mathbf{W} \cap \underbrace{\mathrm{Cof}})^{-1}\mathbf{M}(\mathbf{W} \cap \underbrace{\mathrm{Fib}})^{-1}(X, Y)$ is, as in [2, 5.9], the nerve of the category which has as objects the sequences $X \xleftarrow{\sim} X' \to Y' \xleftarrow{\sim} Y$ in $\mathbf{M}$, and the reduction map r is a weak homotopy equivalence (by the argument of [2, §8]), and

(iii) if $\mathbf{S} = (\mathbf{W} \downarrow X)$ is the category of the trivial fibrations ending at X, $\mathbf{T} = (Y \downarrow \mathbf{W})$ is the category of the trivial cofibrations starting at Y and K: $\mathbf{S}^{op} \times \mathbf{R} \to \underbrace{\mathrm{Sets}} \subset s\,\underbrace{\mathrm{Sets}}$ is the functor given by

$$(X' \xrightarrow{\sim} X, Y \xrightarrow{\sim} Y') \mapsto \mathbf{M}(X', Y')$$

then $(\mathbf{W} \cap \underbrace{\mathrm{Cof}})^{-1}\mathbf{M}(\mathbf{W} \cap \underbrace{\mathrm{Fib}})^{-1}(X, Y) = \underrightarrow{\mathrm{holim}}^{\mathbf{S}^{op}\times\mathbf{T}}K$ and the map in the middle is the map between homotopy direct limits induced by the functor A: $\Delta^{op} \times \Delta^{op} \to \mathbf{S}^{op} \times \mathbf{T}$ given by

$$([k], [n]) \mapsto (X^k \xrightarrow{\sim} X, Y \xrightarrow{\sim} Y_n).$$

The desired result now follows from the fact that (6.11 and 6.12) A is right cofinal and that therefore, by the right cofinality theorem for homotopy direct limits (which is the obvious dual of the left cofinality theorem for homotopy inverse limits [5, Chap. XI, 9.2] and which is proved in the same manner) the middle map is a weak homotopy equivalence.

Using 7.2 we will now give a

7.3 *Proof of* 5.3. Let X, Y, X^* and Y^* be as in 7.2 and consider the commutative diagram induced by the k-fold degeneracy

$$\begin{array}{ccccccc}
\operatorname{diag}\mathbf{M}_0(X^*, Y_*) & \xleftarrow{\sim} & \mathbf{M}_0''(X^*, Y_*) & \xrightarrow{\sim} & (\mathbf{W}\cap \underset{\sim}{\mathrm{Cof}})^{-1}\mathbf{M}_0(\mathbf{W}\cap \underset{\sim}{\mathrm{Fib}})^{-1}(X, Y) & \xrightarrow{\sim} & L^H\mathbf{M}_0(X, Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{diag}\mathbf{M}_k(X^*, Y_*) & \xleftarrow{\sim} & \mathbf{M}_k''(X^*, Y_*) & \xrightarrow{\sim} & (\mathbf{W}\cap \underset{\sim}{\mathrm{Cof}})^{-1}\mathbf{M}_k(\mathbf{W}\cap \underset{\sim}{\mathrm{Fib}})^{-1}(X, Y) & \rightarrow & L^H\mathbf{M}_k(X, Y).
\end{array}$$

Clearly (see 7.2) the indicated maps are weak homotopy equivalences. Furthermore, even though $\mathbf{M}_k$ $(k>0)$ is not a model category, the fact that the maps $X\to Y\in\mathbf{M}_k$ are in natural 1-1 correspondence with the maps $X\otimes\Delta[k]\to Y\in\mathbf{M}_0$ and the maps $X\to Y^{\Delta[k]}\in\mathbf{M}_o$, together with the functoriality of the factorizations, readily implies that the conditions of [2, 8.2] are satisfied and that one therefore, as in 7.2, can apply the argument of [2, §8] to show that the right bottom map is a weak homotopy equivalence. It thus remains to show that the map on the left is a weak homotopy equivalence. But this follows easily from the results of §6 and the fact that $\mathbf{M}_k(X^*, Y_*)\approx\mathbf{M}_0(X^*\otimes\Delta[k], Y_*)$ and that $X^*\otimes\Delta[k]$ is also a cosimplicial resolution of X.

It remains to complete the

7.4 *Proof of* 4.8. The existence of functorial factorizations readily implies that $\mathbf{M}_o^{cf}$ satisfies the conditions of [2, 8.2] and the same clearly (see 7.3) applies to $\mathbf{M}_k^{cf}$ The rest of the argument of 7.3 also applies, showing that the k-fold degeneracy map $\mathbf{M}_o^{cf}\to\mathbf{M}_k^{cf}$ induces a weak equivalence $L^H\mathbf{M}_o^{cf}\to L^H\mathbf{M}_k^{cf}$. The desired result now follows immediately.

§8. PROOFS (WITHOUT FUNCTORIAL FACTORIZATIONS)

We end with indicating the changes that have to be made in the proofs of §7, if $\mathbf{M}$ (resp. $\mathbf{M}_o$) does *not* admit functorial factorizations.

We begin with

8.1 *Proof of* 4.4. If $\mathbf{M}$ does not admit functorial factorizations, then (see 7.2(vii)) part of the argument of [2, §8] does not work. What goes wrong is that, for any two words $\mathbf{m}$ and $\mathbf{n}$ in $\mathbf{M}$ and $\mathbf{W}^{-1}$, the map

$$\mathbf{m}(\mathbf{W}\cap\mathrm{Cof})^{-1}(\mathbf{W}\cap\mathrm{Fib})^{-1}\mathbf{n}(X, Y)\xrightarrow{C}\mathbf{m}\mathbf{W}^{-1}\mathbf{n}(X, Y)$$

has not any longer an obvious (homotopy) inverse. But we will prove that it is still a weak homotopy equivalence, which is all that is needed.

Note that [2, 5.1] C is the nerve of a functor $N^{-1}C$. Hence, in view of Theorem A of [6], it suffices to prove that, for every object $b\in N^{-1}\mathbf{m}\mathbf{W}^{-1}\mathbf{n}(X, Y)$, the over-category $(N^{-1}C\downarrow b)$ has a contractible nerve. Let

$$b=(X \text{ —— } \cdots \text{ —— } U\xleftarrow[w]{\sim} V \text{ —— } \cdots \text{ —— } Y).$$

Then one can apply the construction of 6.7 to the object $w\colon V\xrightarrow{\sim}U$ of the over-category $(\mathbf{W}\downarrow U)$ and obtain a simplicial object V_* together with a compatible collection of trivial fibrations $V_n \xrightarrow{\sim}\!\!\to U$ and trivial cofibrations $V \rightarrowtail\!\!\xrightarrow{\sim} V_n$ such that the compositions $V \xrightarrow{\sim} V_n \xrightarrow{\sim}\!\!\to U$ equal w. Denote by $D\colon \Delta^{op}\to(N^{-1}C\downarrow b)$ the functor which sends $[n]$ to the object

$$\begin{tikzcd} X \arrow[r, dash] & \cdots \arrow[r, dash] & U \arrow[d, "id"] & V_n \arrow[l, two heads, "\sim"'] & V \arrow[l, "\sim"'] \arrow[d, "id"] \arrow[r, dash] & \cdots \arrow[r, dash] & Y \arrow[d, "id"] \\ X \arrow[r, dash] \arrow[from=u, "id"] & \cdots \arrow[r, dash] & U & & V \arrow[ll] \arrow[r, dash] & \cdots \arrow[r, dash] & Y. \end{tikzcd}$$

As $N\Delta^{op}$ is contractible, it then remains to show that, for every object $e\in(N^{-1}C\downarrow b)$, the undercategory $(e\downarrow D)$ has a contractible nerve. The latter is naturally isomorphic to the homotopy direct limit [5, Chap. XII] $\underrightarrow{\mathrm{holim}}^{\Delta^{op}}E$ of the simplicial set E which sends $[n]$ to $(N^{-1}C\downarrow b)(e, D[n])$, when E is considered as a functor $\Delta^{op}\to$ Sets $\subset s$ Sets. Hence [5, Chap. XII, §3] $N(e\downarrow D)$ has the same homotopy type as E. But if e is the object

$$\begin{tikzcd} X \arrow[r, dash] \arrow[d, "="] & \cdots \arrow[r, dash] & U' \arrow[d] & W \arrow[l, two heads, "\sim"'] & V' \arrow[l, tail, "\sim"'] \arrow[d] \arrow[r, dash] & \cdots \arrow[r, dash] & Y \arrow[d, "="] \\ X \arrow[r, dash] & \cdots \arrow[r, dash] & U & & V \arrow[ll] \arrow[r, dash] & \cdots \arrow[r, dash] & Y \end{tikzcd}$$

then it is not hard to see that E is isomorphic to a fibre of the (6.1) trivial fibration $(\mathbf{W}\downarrow U)(W, V_*)\to(\mathbf{W}\downarrow U)(V', V_*)$ and hence is contractible.

8.2 *Proof of* 5.2. To show that the inclusion $L^H M^c\to L^H M$ is a weak equivalence (without assuming functorial factorizations), let X, Y, X^* and Y_* be as in 7.2 and note that there is a commutative diagram

$$\begin{tikzcd} \mathbf{M}(X^*, Y) \arrow[d] & \mathbf{M}'(X^*, Y) \arrow[l] \arrow[r] \arrow[d] & (\mathbf{W}^c\cap \underset{\sim}{\mathrm{Cof}})^{-1}\mathbf{M}^c(X, Y) \arrow[r] \arrow[d] & L^H M^c(X, Y) \arrow[d] \\ \operatorname{diag}\mathbf{M}(X^*, Y_*) & \mathbf{M}''(X^*, Y_*) \arrow[l] \arrow[r] & (\mathbf{W}\cap \underset{\sim}{\mathrm{Cof}})^{-1}\mathbf{M}(\mathbf{W}\cap \mathrm{Fib})^{-1}(X, Y) \arrow[r] & L^H\mathbf{M}(X, Y) \end{tikzcd}$$

in which the bottom row is as in 7.2 and the top row is similar $(\mathbf{M}'(X^*Y) = \underrightarrow{\mathrm{holim}}^{\Delta^{op}}\mathbf{M}(X^*, Y)$, where $\mathbf{M}(X^*, Y)$ is considered as a functor $\Delta^{op}\to \underset{\sim}{\mathrm{Sets}}\subset s\ \underset{\sim}{\mathrm{Sets}})$. By 7.2 and 8.1 the maps in the bottom row are all weak homotopy equivalences, and, by a slight variation on the arguments of 7.2, so are the maps in the middle and on the left in the top row. The results of §6 readily imply that the map on the left is a weak homotopy equivalence and it thus remains to show that the map on the right in the top row is one too. To do this it suffices to show [2, 6.2(ii)] that

(i) the inclusion $(\mathbf{W}^c\cap\mathrm{Cof})^{-1}\mathbf{M}^c(X, Y)\to(\mathbf{W}^c)^{-1}\mathbf{M}^c(X, Y)$ is a weak homotopy equivalence and

(ii) the pair $(\mathbf{M}^c, \mathbf{W}^c)$ admits a homotopy calculus of left fractions [2, 6.1(ii)].

To prove (i) consider the diagram

$$\begin{tikzcd} (\mathbf{W}^c\cap \underset{\sim}{\mathrm{Cof}})^{-1}\mathbf{M}^c(X, Y)^{\#} \arrow[r, "\text{incl.}"] \arrow[d] & (\mathbf{W}^c)^{-1}\mathbf{M}^c(X, Y)^{\#} \arrow[d] \arrow[dl] \\ (\mathbf{W}^c\cap \underset{\sim}{\mathrm{Cof}})^{-1}\mathbf{M}^c(X, Y) \arrow[r, "\text{incl.}"] & (\mathbf{W}^c)^{-1}\mathbf{M}^c(X, Y) \end{tikzcd}$$

in which

(iii) $(\mathbf{W}^c\cap \underset{\sim}{\mathrm{Cof}})^{-1}\mathbf{M}^c(X, Y)^{\#}$ is the nerve of the category which has as objects the pairs (u, v), where u is a diagram in $\mathbf{M}^c$ of the form $X \to A \xleftarrow{\sim} Y$ and v is a cylinder object for Y, (i.e. a factorization $YvY \to Y' \overset{\sim}{\twoheadrightarrow} Y$ of the folding map $YvY \to Y$),

(iv) $(\mathbf{W}^c)^{-1}\mathbf{M}^c(X, Y)^{\#}$ is defined similarly,

(v) the vertical maps are obtained by forgetting the cylinder objects, and

(vi) the diagonal map assigns to a diagram $X \to A \xleftarrow{\sim} Y$ in $\mathbf{M}^c$ and a cylinder object $YvY \rightarrowtail Y' \overset{\sim}{\twoheadrightarrow}$ of Y the diagram $X \to Av_YY' \overset{\sim}{\leftarrowtail} Y$.

Using the argument of 8.1 it is not hard to show that the vertical maps are weak homotopy equivalences and the desired results now follows from the (easily verifiable) fact that the diagram commutes up to homotopy.

The proof of (ii) combines the above argument with the one of [2, §8] and will be left to the reader.

8.3 *Proof of* 5.3. One combines the arguments of 7.3 and 8.1.

We end with

8.4 *Proof of* 4.8. The argument of 7.4 breaks down completely, as the $\mathbf{M}_k^{cf}$ do not any longer satisfy the conditions of [2, 8.2]. Instead we will prove directly that, for every two objects X, $Y \in \mathbf{M}_*^{cf}$, the map $\mathbf{M}_*(X, Y) \to \operatorname{diag} L^H\mathbf{M}_*(X, Y)$ is a weak homotopy equivalence (actually we only use the fact that $X \in \mathbf{M}_*^c$ and $Y \in \mathbf{M}_*^f$). To do this we factor this map

$$\begin{aligned}\mathbf{M}_*(X, Y) \to \operatorname{diag} \mathbf{M}_*'(X \otimes \Delta[*], Y) \to \operatorname{diag} \mathbf{W}^{-1}\mathbf{M}_*\mathbf{W}^{-1}(X, Y)\\ \to \operatorname{diag} L^H\mathbf{M}_*(X, Y)\end{aligned}$$

where $\mathbf{M}_*'(X \otimes \Delta[*], Y)$ is as in 8.2. The map on the right is a weak homotopy equivalence (by the arguments of 7.3 and 8.1), and so is the map on the left, because it fits into a commutative diagram

$$\begin{tikzcd} \mathbf{M}_*(X, Y) \arrow[r] \arrow[dr] & \operatorname{diag} \mathbf{M}_*'(X \otimes \Delta[*], Y) \arrow[d] \\ & \operatorname{diag} \mathbf{M}_*(X \otimes \Delta[*], Y) \end{tikzcd}$$

in which the map on the left is readily verified to be a weak homotopy equivalence, while the map on the right is the weak homotopy equivalence of [5, Chap. XII, 3.4]. It thus remains to investigate the middle map.

Choose a special cosimplicial resolution X^* of X with $X^o = X$, a map of resolutions $X \otimes \Delta[*] \to X^*$ and a special simplicial resolution Y_* of Y with $Y_o = Y$. Then the middle map fits into a homotopy commutative diagram

$$\begin{tikzcd} \operatorname{diag} \mathbf{M}_*'(X \otimes \Delta[*], Y] \arrow[rr] \arrow[d] & & \operatorname{diag} \mathbf{W}^{-1}\mathbf{M}_*\mathbf{W}^{-1}(X, Y) \arrow[d] \\ \operatorname{diag} \mathbf{M}_*'(X^*, Y) \arrow[r] & \operatorname{diag} \mathbf{M}_*''(X^*, Y_*) \arrow[r] & \operatorname{diag} (\mathbf{W} \cap \underset{\sim}{\mathrm{Cof}})^{-1}\mathbf{M}_*(\mathbf{W} \cap \underset{\sim}{\mathrm{Fib}})^{-1}(X, Y) \end{tikzcd}$$

in which $\mathbf{M}_k''(X^*, Y_*)$ is as in 7.2 and the deisred result now follows from the fact that

(i) by the arguments of 7.3, 8.1 and [2, §8], the vertical map on the right is a weak homotopy equivalence,

(ii) by the argument of 7.2 the right bottom map is so too, and

(iii) by the arguments of §6 and [5, Chap. XII, 3.4] both maps starting at $\operatorname{diag} \mathbf{M}_*'(X^*, Y)$ are weak homotopy equivalences.

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