CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

F. WILLIAM LAWVERE

Toward the description in a smooth topos of the dynamically possible motions and deformations of a continuous body

Cahiers de topologie et géométrie différentielle catégoriques, tome 21, nº 4 (1980), p. 377-392.

<http://www.numdam.org/item?id=CTGDC_1980__21_4_377_0>

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE

3e COLLOQUE SUR LES CATEGORIES DEDIE A CHARLES EHRESMANN

Vol. XXI-4 (1980)

TOWARD THE DESCRIPTION IN A SMOOTH TOPOS OF THE DYNAMICALLY POSSIBLE MOTIONS AND DEFORMATIONS OF A CONTINUOUS BODY

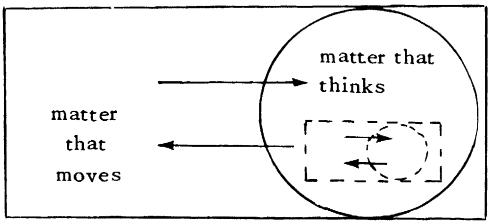
by F. William LAWVERE *

0. INTRODUCTION.

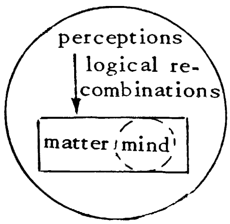
It is an honor to participate in the commemoration of the work of Charles Ehresmann because he, like other great French geometers of our times, realized clearly that in order to make possible the learning, development, and use of concrete infinite-dimensional differential geometry, it is necessary to reconstruct it as a concept, and that this reconstruction is only possible on the basis of a sharp determination of the decisive abstract general relation (DAGR) of the subject, and that in order to succeed in the latter determination, it is mandatory to develop the theory of categories. Now that category theory has indeed been advanced to a very great extent, we can show our appreciation for what we have learned of it from these geometers and others, by taking up the physics of continuous bodies and fields, which was after all the primary source of the geometry developed by their teachers such as E. Cartan. The recognition of that source, just as much as the internal axioms which we labor to perfect, is a DAGR of the subject (see Karl Marx, *Critique of Political Economy*, Section 3 «on Method», for the explanation of the role of DAGR in the reconstruction of the concrete as a concept).

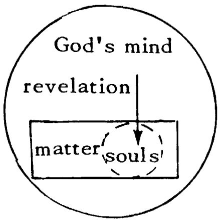
According to Lenin, the scientific «world-picture is a picture of matter-that-moves and matter-that-thinks», and moreover the special role of matter-that-thinks is to reflect the decisive relations in the world in order to provide theory as a guide to action. This materialist world-picture

*) Partially supported by the National Science Foundation and the Martin Chair of Mathematics.



is in opposition to the anti-scientific world-pictures

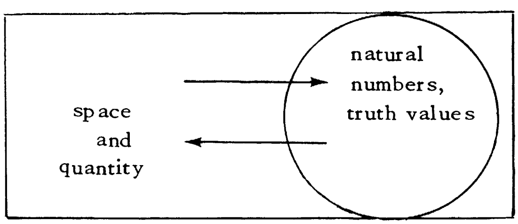




,

of subjective idealism and objective idealism respectively. Subjective idealism was concocted by Plato, Berkeley, etc. in order to prepare the way for the acceptance of objective idealism, and this destructive and anti-scientific work was carried on more recently by Mach, Russell, Brouwer, Heisenberg, etc. Note that all these idealists made special distortions of the science of mathematics as one of the bases of their attempts to get the public to accept their philosophy that the world is a figment of imagination (whether ours or «god's»).

As many have pointed out, the essential object of study in mathematics is space and quantitative relationships. Thus, as an essential part of the scientific world-picture, we have the mathematical world-picture



whose links with the remainder of the scientific world-picture should never be forgotten. Consideration of this picture shows clearly, by the way, just how wrong was the banker Kronecker and his followers who claim that the continuum is only a mental construction from N and Ω (the subjective idealizations of iteration and truth respectively), rather than primarily a concept derived directly from our historical-scientific experience with the world of matter-in-motion.

What we have learned about mathematics should enable us to determine the DAGR in the above «vague notion of that complex whole» (Marx) which is mathematics. In the following attempt to outline part of such a determination, I have presupposed for lack of space several of those DAGR contained in the elements of category theory.

1. MOTION OF MATTER AS DESCRIBED IN A CARTESIAN-CLOSED CATEGORY OF SMOOTH OBJECTS AND SMOOTH MORPHISMS.

In order to treat mathematically the decisive abstract general relations of physics, it is nessary that the mathematical world picture involve a cartesian-closed category $\mathcal{S}$ of smooth morphisms between smooth spaces. (The same is also necessary in order to treat the calculus of variations in a self-sufficient way, as pointed out and utilized by K.T. Chen's Urbana notes on CV.)

For let $B \in \mathcal{S}$ be a space considered to represent a certain body. B could be a 0-dimensional system of particles, a 1-dimensional elastic cord, a 2-dimensional flexible shell, or a 3-dimensional solid or fluid body. Let T be a standard (1-dimensional) space used to measure time, and let E be the ordinary flat 3-dimensional space. Then a motion of B in E is often represented by a morphism $q: T \times B \to E$ which can be thought of as assigning to each $<$ time, particle in the body $>$ the corresponding place in E during the motion. (However, the diagram is valid even if $\mathcal{S}$ is generated by (duals of) atomless Boolean algebras; even though there may be no constant instants and/or particles to which apply q, the concept of a variable element of $T \times B$ would still have a rational and non-trivial significance and q could be applied to such yielding a variable point of space with the same parameter domain A.)

Describing the motion by a morphism q is necessary and useful if we want to compute by composition variable quantities (of the motion), which depend only on location. For example E is equipped with a metric $E \times E \to R$, so that if a reference point $p_0: 1 \to E$ is given, the composition

$$\begin{tikzcd} T \times B \arrow[r, "q"] \arrow[dr] & E \arrow[d, "dist(p_0, -)"] \\ & R \end{tikzcd}$$

gives the distance to p_0 at each instant of each particle of the moving and deforming body.

But, for other calculations it is necessary and useful to consider the motion as described by

$$\bar{q} = \lambda q, \quad \bar{q}: B \to E^T$$

assigning to each particle its path through E, where the space E^T of (smooth) paths exists independently of a particular body or particular motion. Since E is a flat space it has a vector space V (also an object in $\mathcal{E}$) of translations acting by a morphism $+: E \times V \to E$ (which when paired with the projection becomes an isomorphism $E \times V \xrightarrow{\sim} E \times E$ whose *inverse*, when followed by the projection to V, is denoted as subtraction of points). There is a morphism $(\dot{\ }): E^T \to V^T$ (in Newton's notation) which is again independent of a particular body and motion but which, when composed with $\bar{q}$, yields a morphism $B \to V^T$ which upon applying inverse -λ-transform, becomes $v: T \times B \to V$, the velocity morphism of the motion q.

But the same motion needs to be considered in still a third way: $\bar{\bar{q}} = \lambda(q\tau)$ for certain purposes. Here τ is merely the commutativity isomorphism $\tau: B \times T \to T \times B$ for cartesian product, so that $\bar{\bar{q}}: T \to E^B$ gives the time-dependence of the *placement* of the body in space. Again the space E^B of all possible placements is independent of any particular given motion q of B. Since B comes equipped with a mass-distribution μ and since E is convex, there is a morphism

$$\frac{1}{\mu(B)} \int_B (\ ) d\mu : E^B \to E$$

which assigns, to each (smooth) placement of B the corresponding position of the center of mass. More generally there may be a natural partition $B \to n$ (e.g., if B is the solar system considered as the union of n fluid bodies

$B = \bigcup_{i \in n} B_i$ which retain their separateness throughout the motions considered). Then (assuming each B_i has substantial mass), $B \to n$ induces a morphism $E^B \to E^n$ by restricting the above center-of-mass formula to each B_i, $i \in n$. All the last was independent of particular motion. But now the just-constructed $E^B \to E^n$ can be composed with any particulat motion, when the latter is taken in the $\bar{\bar{q}}$ guise, to yield .he time-dependence of the center(s)-of-mass under the motion

$$\begin{tikzcd} T \arrow[r, "\bar{\bar{q}}"] \arrow[dr] & E^B \arrow[d] \\ & E^n \end{tikzcd}$$

This construction suggests how the state space $X \to Q$ for a certain body n may involve more than just the tangent bundle of the configuration space $Q = E^n$, since states may involve internal motions of the B_i.

Further examples of quantities whose computation requires the three versions of the motion are:

(1) The temperature experienced by each particle of B as B moves through a temperature field θ:

$$\begin{tikzcd} T \times B \arrow[r, "q"] \arrow[dr] & E \arrow[d, "\theta"] \\ & R \end{tikzcd}$$

(2) The mean position of particles of B over the last Δt duration, where the size of Δt is fixed:

$$\begin{tikzcd} B \arrow[r, "\bar{q}"] \arrow[dr] & E^T \arrow[d] \\ & E^T \end{tikzcd} \qquad x \rightsquigarrow \left(t \rightsquigarrow x(t-\Delta t) + \frac{1}{\Delta t}\int_{t-\Delta t}^{t} x\, dt \right).$$

(3) The deformation rate $T \times B \to T^* \otimes \Lambda^1(V)$ computed as the inverse λ-transform of the composite

$$B \longrightarrow \Lambda^1(V)^T \xrightarrow{(\dot{\ })} T^* \otimes \Lambda^1(V)^T ,$$

where the second makes sense for any vector space W in place of $\Lambda^1(V)$ (the purpose of tensoring with the «one-dimensional» T^* being to maintain correct physical dimensions). However the construction of the deformation $B \to \Lambda^1(V)^T$ out of $q: T \times B \to E$ will be described in a little more detail; in fact time is only a passive parameter in this particular construction, so the essential thing is to construct $grad(q): B \to \Lambda^1(V)$ from a given placement $q: B \to E$. Now applying the tangent bundle functor $(\ )^D$ and the special property of that functor for flat spaces E (Section 2), we get the composite

$$B^D \xrightarrow{q^D} E^D \xleftarrow{\approx} E \times V \xrightarrow{\pi} V$$

which by the amazing further right-adjoint property of infinitesimal objects like D (Section 3) transforms into

$$B \xrightarrow{grad(q)} \Lambda^1(V) \hookrightarrow V_D$$

where such gradients always factor through the linear portion $\Lambda^1(V)$ of the non-linear V-valued differential-form representator V_D since of course derivatives are linear on fibers.

2. REPRESENTABILITY OF TANGENT BUNDLE. SPECIAL FORMULA FOR THE TANGENT BUNDLE OF A FLAT SPACE. AND RAPID INTUITIVE AND RIGOUROUS PROOFS OF ALL BASIC RESULTS OF DIFFERENTIAL AND INTEGRAL CALCULUS.

In a topos $\mathcal{E}$ of smooth spaces and smooth morphisms, there will be a commutative-ring-object R (which may be thought of as the endomorphism ring of the abelian group of translations of the geometric line) which by abuse is often referred to as *the line* R. We don't assume R is a field, since the important mathematical properties we wish to emphasize are to be invariant under passage to a topos of variable spaces and morphisms more general than $\mathcal{E}/Y$ or $\mathcal{E}^G$ (where Y is a parameter object in $\mathcal{E}$ or where G is a group object («$\mathcal{E}$-Lie group»); however the assumption that R is a *local* ring would be stable under such passage. Often it is reasonable to assume that $\mathcal{E}$ is strongly generated by the subcategory $\mathfrak{A}$ of spaces A which can be realized as (domains of) equalizers

$$A \longrightarrow R^n \rightrightarrows R^m$$

of pairs of morphisms between finite cartesian powers of the line R ; such A could be called «algebraic varieties», «analytic varieties», «smooth varieties», or «families of smooth varieties», etc., depending on the nature of $\mathcal{E}$.

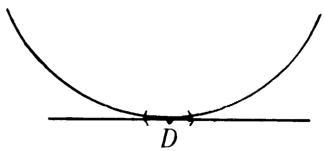
Important examples of varieties are (in addition to spheres, conics, cubic hypersurfaces, etc.) the *infinitesimal varieties* $D_k(n) \hookrightarrow R^n$ defined by the equations

$$D_k(n) = \{ < x_1, \dots, x_n > \mid \text{ any } k+1 \text{ of the } x_i \text{ have } R\text{-product} = 0 \}$$

and as an especially important case

$$D = D_1(1) \hookrightarrow R \quad \text{defined by} \quad D = \{ h \in R \mid h^2 = 0 \}$$

and which may be constructed geometrically as the intersection of a circle with one of its tangent lines:



$$\left.\begin{array}{r} x^2 + (y-1)^2 = 1 \\ y = 0 \end{array}\right] \iff x^2 = 0$$

D has the one obvious point (and in examples *only* the one *global* point) $0 \colon 1 \to D$. In fact in the useful examples, the arbitrary morphisms $R^n \to R^m$ and more generally between open subvarieties are (*not* nonstandard» points and morphisms but) precisely the *usual* points and morphisms; nonetheless in interesting examples D is not isomorphic to 1. In dealing with the geometrically- and physically-necessary categories of *variable* spaces and morphisms, one must reject the idea (fostered by the concentration in the past 100 years on *constant, abstract* sets and maps and *constant* topological spaces, etc.) that a space X somehow lives entirely on its constant points $1 \to X$. More precisely we assume (what is true in many examples) that

$$D = \{ h \in R \mid h^2 = 0 \}$$

is big enough so that for any $a_1, a_2 \in R$,

$$\forall h\ [\ h^2 = 0 \Rightarrow a_1 h = a_2 h\ ] \Rightarrow a_1 = a_2 ,$$

i.e. that although one obviously cannot divide by a single nilpotent element, one can «divide by all nilpotent elements jointly» and indeed many situations (see below) arising in practice are such that when hypotheses like $a_1 h = a_2 h$ can be proved they can actually be proved for *all* nilpotents h. This assumption of enough nilpotents can be regarded as an expression of Euler's famous principle of differential and integral calculus

$$h = 0 \quad \text{and} \quad h \neq 0.$$

Cur axiom forces the logic of the topos $\mathcal{E}$ to be *non*-Aristotelian (unless $R = \{0\}$) in the sense that

$$\{0\} \cup \{\ h \in D \mid h \neq 0\ \} \hookrightarrow D$$

cannot be an isomorphism; it was perhaps the general lack of clarity on this fact which Bishop Berkeley was able to exploit in his subjective-idealist attack on materialist science.

The assumption of enough nilpotents leads to immediate rigourous proofs of all the basic formulas of calculus, provided we define the derivative $f': R \to R$ of a morphism $f: R \to R$ as being the (unique by the axiom) one characterized by the condition

$$f(x+h) = f(x) + f'(x).h$$

for all $x \in R$, all h such that $h^2 = 0$. For example if we assume that f' is the derivative of f and g' is the derivative of g, then both the Leibniz rule

$$(fg)' = fg' + f'g$$

as well as the chain rule

$$(g \circ f)' = (g' \circ f).f'$$

are immediate algebraic calculations, wherein we apply at the last step the «cancellation of all nilpotents jointly» as assumed. Also the fundamental theorem of calculus

$$(\textstyle\int_a^{(\ )} f)' = f$$

is likewise immediate, where $\int_a^x f$ exists for all (smooth) f and is geometrically considered as the area μ_2 under the curve and where of course

$$\int_x^{x+h} f = f(x).h \quad \text{for all} \quad h \in D.$$

As a couple of further proofs (usually slandered as «nonrigourous») of some facts from geometry and electromagnetism needed in engineering, consider the following:

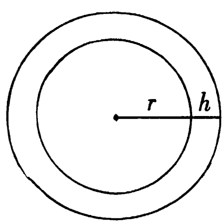
(1) Define the number $\pi \in R$ by the condition $A(r) = \pi r^2$, where $A(r)$ is the area of the disc

$$A(r) = \mu_2(\{ <x, y> \mid x^2 + y^2 \leq r^2 \}).$$

(It is clear from easy pictures that $2 < \pi < 4$.) Then we want to compute the *length* $L(r)$ of the *boundary* circle of the disc of radius r:

$$L(r) = \mu_1(\partial\{ \; \}).$$

We do this by calculating the *area* $B(r, h)$ of the band of width h in two ways:



$B(r, h) = \pi(r+h)^2 - \pi r^2 = 2\pi r h$, since $h^2 = 0$,

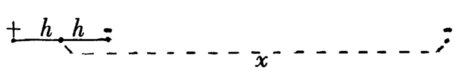
$B(r, h) = L(r).h$, since the error introduced by slitting and straighthening the band is proportional to h^2.

Hence

$$L(r)h = 2\pi r h \quad \text{for all} \quad h^2 = 0$$

and therefore $L(r) = 2\pi r$ by the principle.

(2) Consider the weak electric field E at a substantial distance x from a system of two near oppositely-charged particles $2h$ apart:



By Coulomb's law:

$$E = \left(\frac{1}{(x-h)^2} - \frac{1}{(x+h)^2}\right) = \frac{1}{x^2 - 2xh} - \frac{1}{x^2 + 2xh}$$

$$= \frac{1}{x^2}\left(\frac{1}{1-\frac{2h}{x}} - \frac{1}{1+\frac{2h}{x}}\right) = \frac{1}{x^2}\left(1 + \frac{2h}{x} - \left(1 - \frac{2h}{x}\right)\right).$$

Hence $E = \frac{4h}{x^3}$, i.e. this weak field falls off in inverse proportion to the *cube* of the distance. In fact most proofs of this sort in physics books become rigourous algebra if hypotheses like « $h << 1$ » are interpreted to mean $h^2 = 0$ and if «substantial» is interpreted to mean invertible in the appropriate ring.

Now since to describe motion of matter, the category $\mathcal{E}$ of smooth spaces needs to be cartesian closed, we can for any object X consider the space X^D of infinitesimal paths in X; we (with ample justification) will call it the *tangent bundle of* X. Since $D \hookrightarrow R$, there will be a differentiation morphism $X^R \to X^D$ which assigns to any path $R \to X$ its derivative at $t = 0$. Since clearly

$$h^2 = 0 \Rightarrow (\gamma h)^2 = 0,$$

the multiplicative monoid R acts on D, and hence by functoriality it acts on any tangent bundle. (In our smooth context, preservation of such actions by a morphism ϕ should be considered morally equivalent to *linearity* of the morphism ϕ.) The cancellation principle from which we above derived calculus, as well as the existence of a derivative in $\mathcal{E}$ for every morphism in $\mathcal{E}$, is guaranteed by the following «Kock-Lawvere axiom» which has been verified in many concrete examples of $\mathcal{E}$ (see Dubuc, these «*Cahiers*» Volume XX-3): the morphism

$$R \times R \xrightarrow{\sim} R^D \quad \text{defined by} \quad \langle a_0, a_1 \rangle \rightsquigarrow \ulcorner (h \rightsquigarrow a_0 + a_1 h) \urcorner$$

is an *isomorphism* in $\mathcal{E}$. More generally if E is any flat space with R-vector space V of translations and if G is any open subspace of E, while $k = 1, 2, 3, \dots$, then $G \times V^k \to G^{D_k(1)}$ defined by

$$\langle x, v_1, \dots, v_k \rangle \rightsquigarrow \ulcorner (h \rightsquigarrow \sum_{i=1}^{k} h^i v_i) \urcorner$$

is an isomorphism (openness of G clearly implies that G is closed under the infinitesimal translations mentioned).

This *representability* of tangent (and jet) bundle functors by objects like D leads to considerable simplifications of several concepts, constructions and calculations. For example, a first order ODE, or vector field, on X is usually defined as a *section* $\hat{\xi}$ of the projection $X^D \to X$ (induced in our context by evaluating a tangent vector v at $0: 1 \to D$). But by the λ-conversion rule $\hat{\xi}$ is equivalent to a morphism

$$\xi: X \times D \to X \quad \text{satisfying} \quad \xi(x, 0) = x,$$

i.e. to an «infinitesimal flow» of the additive group R.

The notion of a *morphism* $f: (X, \xi) \to (Y, \eta)$ between two vector fields) (on two spaces) (whose definition is for some reason usually not given but rather replaced by the misleading suggestion that there always exist induced vector fields) is simply that of an $\mathcal{E}$-morphism f which further satisfies commutativity of

$$\begin{array}{ccc} X \times D & \xrightarrow{f \times D} & Y \times D \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\ X & \xrightarrow{f} & Y \end{array}$$

If we have given actual flow $X \times R \to X$ we can restrict it to $D \hookrightarrow R$ to obtain a vector field, and indeed this process is the inverse image functor

$$\text{Flows}(\mathcal{E}) \longrightarrow \text{Vector Fields}(\mathcal{E})$$

for an essential geometric morphism of toposes, where its left and right adjoints are a useful explicit first step toward investigating the *solutions* of an arbitrary ODE.

Since $(\ )^D$ is a left exact functor, the computation of the tangent bundles of spheres (as well as of the Lie algebras of classical Lie groups, and in general of objects defined by nonlinear equations within flat spaces) leads to the expected explicit results.

We emphasize again that unlike the counter-intuitive «nonstandard analysis», the $\mathcal{E}$-morphisms between usual manifolds are typically only the standard ones; it's just that $\mathcal{E}$ has further interesting objects like D which *also* have their $\mathcal{E}$-morphisms to manifolds X and which can then be composed with morphisms $X \to Y$, etc.

But $\mathcal{E}$ also contains the (often infinite-dimensional) function space Y^X of any two spaces in $\mathcal{E}$. When $\mathcal{E}$ is generated by a category $\mathfrak{A}$, the smooth structure on Y^X is *uniquely* determined by knowing the morphisms

$$\phi : A \to Y^X, \quad A \in \mathfrak{A},$$

and how these transform under any $\alpha : A' \to A$ in $\mathfrak{A}$. But these morphisms ϕ are λ-equivalent to morphisms $A \times X \to Y$ which are in turn easily comprehensible in classical terms. The tangent bundle of a function space is trivial to compute, since in any cartesian-closed category we have

$$(Y^X)^D = (Y^D)^X,$$

i.e. a tangent vector to Y^X is just a smooth morphism from X into the tangent bundle of Y, and similarly for vector fields, etc.

3. THE AMAZING RIGHT ADJOINTS WHICH PERMIT TRANSFORMATION INTO «ORDINARY» FUNCTIONS OF INFINITESIMAL FUNCTIONALS AND THE CONSEQUENT POTENTIAL SIMPLIFICATION OF THE USUAL CALCULUS OF DIFFERENTIAL FORMS.

When Kock, Wraith, Reyes, Dubuc and others took up my 1967 program of Synthetic Differential Geometry, they arrived at the amazing previously-undreamed-of (except by Newton, Euler, etc.?) fact, and indeed from both the angle that the fact is needed axiomatically to prove such results as the stability under étale descent of the property of being a manifold, as well as from the angle that the fact is actually true in many examples of $\mathcal{E}$ constructed by the standard Cartier-Grothendieck-Gabriel-Lawvere method! (See Springer Lecture Notes 753 from the 1977 Durham Conference on Sheaves and Logic.) This fact is that all the objects $D_k(n)$ have the property of being connected and internally projective, which even more objectively means that

$$D_k(n) \in \mathfrak{D} \underset{\text{def}}{=} \{ D \in \mathcal{E} \mid (\ )^D \text{ has itself got a right adjoint} \}.$$

Until someone suggests a better notation, I am calling $(\ )_D$ these right adjoints. It is clear that $\mathfrak{D}$ is closed with respect to finite products in any topos $\mathcal{E}$, but further properties of $\mathfrak{D}$ remain to be discovered. The standard examples of $\mathcal{E}$ which are relevant actually satisfy the following ***null-stellensatz*** : every object X in $\mathcal{E}$ is a quotient $Y \twoheadrightarrow X$ of an object Y which is weakly generated by $\mathfrak{D}$ in the sense that

$$\forall A \in \mathcal{E} \quad \forall\, A \underset{y_2}{\overset{y_1}{\rightrightarrows}} Y \quad \& \quad y_1 \neq y_2$$

$$\Rightarrow \quad \exists D \in \mathfrak{D} \ \exists a : D \to A \ (y_1 a \neq y_2 a).$$

However we will not need the nullstellensatz in what follows.

Note that the fundamental transformation rule for $D \in \mathfrak{D}$ is

$$\frac{X^D \to Y}{X \to Y_D}$$

for any X, Y. We will call such a morphism a (not-necessarily linear) Y-valued *differential form of kind D on X*. Note that in its classical guise $X^D \to Y$, a differential form is an infinitesimal functional, i.e. a functional whose domain consists itself of functions with infinitesimal domain D. But in the transformed guise $X \to Y_D$, a differential form in X is just an ordinary smooth function, albeit with a highly non-classical codomain Y_D. If $Y = V$ is an object equipped with an action of the multiplicative monoid R, we can single out a subobject

$$\Lambda^1(V) \hookrightarrow V_D\,, \quad \text{where} \quad D = D_1(1),$$

by the condition that the two induced actions of R on V_D should agree there. Then an arbitrary morphism $X \to \Lambda^1(R)$ is a *differential 1-form on X* in the usual sense. Just how non-classical the objects Y_D are is indicated by the fact that there is only one point $0 : 1 \to \Lambda^1(R)$ even though the paths $R \to \Lambda^1(R)$ are as many as the functions $R \to R$, etc. These phenomena, taking place in the «gros topos of all spaces» should not be confused with the similar well-known phenomena taking place in a «petit topos» $sh(X) \hookrightarrow \mathcal{E}/X$ of (étale or other) sheaves on X, which latter sort $sh(X)$ of topos should be considered as a glorified picture of a ***single*** space X, rather than as a category of «all» spaces as $\mathcal{E}$ is.

Not only does the functoriality of the space $\Lambda^1(R)^X$ of differential forms on a variable space X (not entirely trivial to prove rigourously in the usual setting) become immediate but the process of taking the *gradient* $grad(\phi): X \to \Lambda^1(R)$ of an arbitrary function $\phi: X \to R$ on X becomes an instance of *composition* with a canonical morphism $d: R \to \Lambda^1(R)$ namely the one adjoint to the «principal part» projection

$$R^D \xrightarrow{\approx} R \times R \xrightarrow{\pi} R.$$

The formula

$$grad(\phi) = d \circ \phi \quad \text{for all } \phi$$

and others like it suggests that a drastic simplification of the usual differential form calculus may be in the offing.

There is also another subobject $Q(R) \hookrightarrow R_D$ such that morphisms $X \to Q(R)$ from X are «quadratic differential forms», i.e. (not necessarily definite) *Riemannian metrics on* X.

Question: Do physicists and engineers who work in electromagnetism and continuum mechanics somehow already unconsciously «know» that differential forms are really just glorified functions?

I got the idea for the above approach to differential forms while reading some papers by K.T. Chen on the Calculus of Variations in «differential spaces». However the existence of spaces like D, Λ^1, etc., with the consequent *mathematical representability* of tangent vectors and differential forms, does *not* hold in his setting, which is essentially the cartesian-closed subcategory $\mathcal{E}_1 \hookrightarrow \mathcal{E}$ consisting of all objects weakly generated by 1, i.e. «living on their constant points». Actually the topos generated by $\mathcal{E}_1$ is smaller than $\mathcal{E}$; it contains Λ^1, etc., but *not* D, etc.

4. LAGRANGIAN DESCRIPTION OF THE DYNAMICALLY POSSIBLE MOTIONS OF A BODY.

In general the states X of a body B should be sufficient to determine their own evolution provided the general law $\mathcal{L}$ of the motion of B is known. In general the state space X may involve *histories* of motion but there will always be a given morphism $X \to Q$ to the configuration space of B, expressing the fact that each state involves a specific underlying configuration («plus whatever further information is necessary»). In most classical situations the state space is taken to be $X = Q^D$ the tangent bundle of Q, expressing that infinitesimal histories are all that is necessary. Moreover the configuration space is $Q \hookrightarrow E^B$ the (sub)space of (admissible) placements. Then

$$X = Q^D \hookrightarrow (E^B)^D = (E^D)^B = (E \times V)^B = E^B \times V^B ,$$

so that in fact $X \hookrightarrow Q \times V^B$ (and often $=$) where V^B is the space of *velocity fields* on B. In many cases the dynamically possible motions of B can be singled out from the kinetically possible Q^T by knowing the Lagrangian function $\mathfrak{L}$ which is implied by the constitutive relations for B: $\mathfrak{L} : X \to W$ gives the work $\mathfrak{L}(q, v)$ which would need to be added to the potential energy of the placement q in order to obtain the kinetic energy of the velocity field v (on the body with mass distribution μ): Thus W is a «1-dimensional» vector space with the physical dimension of work.

Since T has a canonical vector field, any motion $q : T \to Q$ yields

$$\begin{tikzcd}
T^D \arrow[r, "q^D"] \arrow[d] & Q^D \arrow[d] \\
T \arrow[r, "q"'] \arrow[u, bend left, dashed, "canon"] \arrow[ur, "{< q, \dot{q} >}"] & Q
\end{tikzcd}$$

Hence for $\mathfrak{L} : X = Q^D \to W$, one has for each $q : T \to Q$ the corresponding

$$T \xrightarrow{< q, \dot{q} >} X \xrightarrow{\mathfrak{L}} W .$$

Suppose t_1, t_2, q_1, q_2 are given and let $\bar{Q} \hookrightarrow Q^T$ be the subspace of those motions q for which $q(t_i) = q_i$, $i = 1, 2$. Then the action

$$J_{t_1}^{t_2}(q) = \int_{t_1}^{t_2} \mathfrak{L}(q, \dot{q})\, dt$$

may be considered as a morphism

$$\bar{Q} \to A \quad \text{where} \quad A = W \otimes T .$$

The gradient $grad(J) = d_A \circ J$ is then a morphism $\bar{Q} \to A_D$ (which factors through $\Lambda^1(A) \hookrightarrow A_D$). The object of $\mathfrak{L}$-possible motions is then the subobject of $\bar{Q}$ where $grad(J)$ vanishes. The Lagrangian equations of motion then follow as usual. (See Courant-Hilbert, Vol. 1, pages 184-185.) Note that when B is a continuous body, the Lagrangian

$$\mathfrak{L} : E^{B \times D} \to \mathbb{W}$$

may very well depend on the underlying placement of a state, or even on higher spatial derivatives.

Department of Mathematics
Faculty of Natural Sciences and Mathematics
State University of Buffalo
106 Diefendorf Hall
BUFFALO, N.-Y. 14214. U.S.A.