John Isbell's Adequate Subcategories

F. W. Lawvere

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For mathematicians of my age, the theory of rings of continuous functions was one of the first exciting research topics we encountered. Many results of that theory have appeared, but its ramifications for category theory are still not fully worked out. A crucial link was provided by John Isbell's contributions around 1960 on the theme of adequate subcategories. Briefly, a subcategory A of a larger category is adequate if every object X of the larger category is canonically the colimit of the category A/X of objects of A equipped with structural maps to X; John's equivalent definition was that the truncated Yoneda embedding of the whole category into the category of set-valued contravariant functors on A is actually a full embedding. The following language is suggestive: (a) The objects of A are figure-types, (b) the objects of A/X are particular figures in X, and (c) the morphisms of A/X (commutative triangles in the big category) are incidence relations between figures. Thus adequacy means that the large category in question consists of objects entirely determined by their A-figures and incidence relations, and that (d) the morphisms in the whole category are nothing but the "geometrically continuous" ones in the sense that they map figures to figures without tearing the incidence relations.

For example, if A is the category of countable compact spaces then A is adequate in many large categories constructed in attempts to capture the notion of topological space; in this case a morphism can be identified as a mapping that preserves sequential limits. That example is one of many illustrating that typical large categories of mathematics often have quite small adequate subcategories; it had been studied by Fox in 1945 at the instigation of Hurewicz, who sought a rational notion of function space for use in algebraic topology and functional analysis. In fact, for any A, each space of A-continuous morphisms has its own cohesion, described again by A-figures.

The dual notion of a co-adequate subcategory C leads to a contravariant representation of a large category that can be described in terms of (a) quantity-types, (b) functions, and (c) algebraic operations on functions. The dual of the notion of geometric continuity (that is, a name for naturality of maps of covariant functors instead of contravariant ones) is (d)"algebraic homomorphism".

These ideas of John Isbell became fused with the conceptions of Kan, Grothendieck, and Yoneda (emerging in the same period 1958-1960), to form a basic method of analyzing and constructing mathematical categories. That method was used in the early 60's by Freyd, Gabriel, Lawvere, Mitchell, and by Isbell himself, and became as natural as breathing to many algebraists and topologists during the following decades.

What does adequacy have to do with rings of continuous functions? The theory of rings of continuous functions springs from a basic philosophical hope to the effect that there should be a near-perfect duality between space and quantity. Such duality questions can be investigated for a great many different categories, but categorical considerations suggest that they need to be brought down to earth in certain respects.

John Isbell was also one of the main developers of the theory of locales. This theory revealed that the traditional notion of topological space is algebraic rather than geometric (in the sense of the above analysis) with the infinitary algebras (frames) of open sets playing the dominant role; this merely means that the Sierpinski space, together with "all" its powers, constitutes a coadequate subcategory of the category of sober spaces. John's insistent quest for smallness (as a further requirement on co-adequate subcategories C) brought this analysis qualitatively nearer to real mathematics.

If a small co-adequate subcategory is available, it can often be reduced to a single object (for example by taking the product of its objects; or more concretely, the Euclidean plane as a topological object will often serve the purpose of co-adequacy). The endomorphisms of that object then parameterize the unary operations whose preservation by C-homomorphisms serves to exclude ghosts from among detected points and figures. Even among those operations a few may be co-adequate, as the Stone-Weierstrass theorem had shown: addition, multiplication, and conjugation can replace the monoid of all continuous operations for that particular task, and there are many variations on that theme. But apart from such details of presentation, the implicit insight of Czech and Stone, of Hewitt and Nachbin, apparently includes smallness of the algebraic theory C in terms of which spaces are to be (co) analyzed.

There was an apparent barrier to the realization of that concrete insight: set theory, in its striving for larger and larger cardinals, had neglected to emphasize that all of the cardinals arising in geometry and analysis in fact satisfy a useful smallness condition: the category of countable sets is co-adequate in the category of all small sets. That is essentially John Isbell's formulation; he proved that it is equivalent to the condition that no small set has the kind of ghost elements called Ulam measures. John knew full well that his formulation for abstract sets would imply that many categories having small adequate subcategories also have small co-adequate subcategories, thus making possible the desired sort of dualities between space and quantity.

The ideas of John Isbell contributed to the enlightened understanding of mathematics by lifting some dark clouds of confusion, and they continue to be actively developed and diffused.
