Outline for three lectures
Como, University of Insubria
Sala Casartelli (Sala dei Nobel)
10 January 2008

COHESIVE TOPOSES:
combinatorial and infinitesimal cases

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11:30

I. Generalized Spaces

Etendues

Adequately separable (QD)

Empty generating objects

Additional requirement: algebra of functions

Source of the algebra of functions in classifying topos

Generalized space derived by reflecting from topos of spaces

Galois contradiction between non-cohesion and figure shapes

15:00

II. Topos of Spaces as Category of Cohesion

Adjoint strings preserving products on left and sums on right

Quality

Canonical extensive quality, the Hurewicz category

Canonical intensive quality, substance and micro form

The cooling map, rough graphs

Infinitesimal generation of figure shapes and even of topos

16:30

III. Bornological and simplicial cohesion

Partial equality of discrete and codiscrete

Pre-toposes satisfying the axiom of choice

Fundamental lemma of functional analysis

Aufhebung calculation

Round Table Discussion

17:30

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I. Generalized Spaces

It is often said that the basic idea of a topos is that it is a generalized space. In fact, both historically, and in terms of intrinsic content, that is an oversimplification that has distracted attention from the development of a useful class of toposes which can be more accurately described as consisting of *objects* that are generalized spaces. In order to display more sharply this contrast, let me first survey briefly some of the kinds of generalized space that are related to individual toposes.

Of course, the original connection between spaces and toposes was via the **Leray-Lazard** notion of sheaf, generalizing well-established experience in the field of complex analysis to the supposedly default notion of "space" based on a partially ordered set, satisfying conditions appropriate to a notion of "open region". Although in analysis even today linear sheaves are the most frequent, it was recognized already in the 1950's that those are effectively considered as linear objects in the non-linear category of set-like sheaves. The dual description in terms of local homeomorphisms versus special "presheaves", led in exploring its needed generalizations, to the dominance of the presheaf point of view, necessarily accompanied by the central role of the special "sheaf-condition", which became identified with a compatibility with coverings in a way which indeed held true not only for pure topology, but in smooth and analytic geometry as well. Perhaps the first and obvious generalization of the category of sheaves on the poset was its opposite, the actions of a group (or as it turned out more generally) the actions of a monoid with cancellation; this was no idle generalization at all, because of the intimate relation between topological spaces, covering spaces, fundamental groups, Eilenberg-Mac Lane spaces, etcetera. Within the general setting of topos (we will recall in a moment some of the other paths that led to that setting) Grothendieck defined the notion of **Etendue** as a topos that is locally of the kind defined by a poset of open regions. It turned out that an étendue is always obtained if we start with a site that, as a category, consists only of monomorphisms, and that conversely any étendue has a (possibly not sub-canonical) site of monomorphisms; of course, when the site is subcanonical, its morphisms remain monomorphic even within the resulting topos as a whole.

The topos of sheaves on an analytic space, or a smooth space does NOT fully express the space, any more than does the lattice of open regions itself, because a crucial additional datum is the particular sheaf of holomorphic or smooth functions (which of course is incorporated in the linearized theory, provided we can construe it as being a monoidal abelian category). This is even more striking in the case of algebraic schemes, where the product of the underlying toposes is quite different from the product of schemes, leading to the fact that an algebraic group is not a topological group. I believe that much of this mis-match can be alleviated if one considers that the base topos for algebraic geometry over a field which is not algebraically closed does not consist of the Cantorian abstract sets, but rather of Galois sets that are sheaves on the opposite of the category of finite extension fields. That is only a fragment of a task which category theorists have been avoiding for too long, namely the re-foundation of algebraic geometry, taking into account the most up-to-date insights from category theory, based of course on Grothendieck's pioneering work which has not been much further developed but only applied, by the geometers of the last 50 years.

Already in 1950 the first topos to be considered explicitly as a category, (simplicial sets) challenged - even as it served - the traditional concept of space. Even more strongly, Grothendieck's 1960 techniques of construction for analytic spaces heralded the realization that categories of space need objects for which any description as a "topological space plus something else" was ackward if not imprecise. In the ensuing half century, simplicial sets have been extensively applied by algebraic topologists, led by Kan, but category theorists have developed only to a rudimentary degree the precise distinguishing features that they have qua topos.

No doubt the most important example of a topos as generalized space (of course still equipped with the needed linearizer) is the **étale** topos of a scheme obtained by replacing the local homeomorphism condition (which in algebraic geometry was too strong due to the lack of the implicit function theorem), by its infinitesimal analog, namely, that the derivative at each point be an invertible linear transformation. The connected objects étale over a given base form a site with the property that two morphisms agreeing on any small part of the common domain are equal, in other words, every morphism in the site is an epimorphism, although in this case the epimorphicity is not preserved by the embedding into the topos. Peter **Johnstone** showed that the toposes having an epimorphic site in this sense are just those in which every object is a quotient of a decidable or separable object, namely one in which the diagonal has a Boolean complement in the square. Thus these **adequately separable** toposes join the étendues as a class of toposes having some concrete relation to ongoing work in geometry and thus serving, when appropriately augmented with linearizers, as generalized spaces.

The categories of cohesion (that I will discuss later) embody another point of view: each object of such a category is itself a generalized space and additional quantitative structure on it, such as a function algebra, is obtained within the same category as consisting of morphisms into other objects, just as the ring of constant real numbers represents the rings of continuous of view should involve, for every object of such a category of spaces, an associated topos of "sheaves" that moreover contains an algebra object resulting from the construction and indicated by a map to a classifying topos. This is another important part of the re-foundation which needs to be worked out and made explicit because of the potential applications.

Such a doctrine of associated sheaf categories (pardon the pun!) might begin by attempting to reflect the "gros topos" E/X of an object X in E into a subcategory of special toposes which unites the étendue and adequately separable cases mentioned above. A very simple such common generalization consists of the toposes that are generated by empty objects, meaning objects that have no idempotent endomaps other than the identity. As I pointed out in my Bogota lectures (see TAC Reprints) on the basis of a few examples, this seems to be an interesting class of sites, but the characterization of the corresponding toposes is not yet clear and, moreover, the reflection from general small categories is not easy to compute although it clearly exists. That reflection preserves finite products, because the inclusion preserves exponentiation by arbitrary small categories, and thus the outline of the theory for the presheaf case seems to be clear, although the extension to general toposes is not. In any case, the special topos associated to an object X is traditionally called its "petit topos".

Progressing along the path indicated by the conviction that spaces are determined as objects in a suitable category requires renouncing another common myth: namely that of the invisibility of the inside of objects in a category. A space should involve a geometry, which is a **system of figures** in it and incidence relations between them. For any given object X in a suitable category E, a figure in X can be identified as a morphism to X from another object A considered as a figure-shape, and an incidence relation between two figures can be identified as a morphism in the category E/X (of course if E has pullbacks, more complicated relations of incidence kind have a derived meaning). As has been abundantly clear in the 60 years since Eilenberg affirmed it, general figures are **singular** in the sense of not necessarily being constrained by monomorphicity or the like. Any morphism from X to Y is "smooth" or "continuous" because as a functor that preserves shape, it also preserves the incidence relations. This simple point of view becomes Grothendieck's powerful method of Representable Functors when we replace E by a subcategory which is more tractable but adequate for E. Among figure-shapes there are often special ones called **points**. These points are usually not adequate and are usually more than just the terminal object; their exact role still needs to be clarified.

There is of course a dual analysis of objects in arbitrary categories in terms of the algebra X/E of **functions** and algebraic operations. Among function types there are typically truth values, which are typically not co-adequate. However, although they are an important derived invariant, it is important that the function algebra include functional analysis, which is in turn based on the map space structure that a geometric category should have. That map space structure is notoriously non-existent in categories that have been based on function algebras (such as Sierpinski–valued functions) as a defining structure.

In the above I have implicitly used the term **topos** to mean **U-topos** in the sense of Grothendieck, that is, a topos defined over a topos U of the very special nature of abstract sets, and so typically satisfying conditions like the axiom of choice and two-valuedness, and even the non-existence of Ulam numbers. The powerful relativization principle suggests that U itself could be a more general topos, and indeed that point of view has led to many interesting applications. Those toposes defined over U of the étendue or adequately separable kinds (and even of the more general empty-generated kind) are usefully thought of as consisting of variable "sets", that is, of variable U-objects, which in the most basic instance are just sections of the structural morphism to U. By contrast, the kind of toposes we wish to emphasize have intuitive content of cohesiveness, which provides a potentiality of lawful variation without specifying a particular variation.

For cohesive toposes, it seems to be essential that there be **degenerate** figures x in the sense that xe = x, where e is an idempotent of the figure-shape. Perhaps someone will be able to make this idea into a definition! In the case of a presheaf (such as a simplicial set for example), it is the degeneracies that are trivialized under the reflection to its "petit topos".

Also for a cohesive topos, it seems to be especially appropriate to consider how the "inert" base U over which it is defined can in fact be isolated and "defined" inside the topos itself as the part that is relatively non-cohesive. That can be viewed as a way of making precise Cantor's negation of "Mengen" to obtain the abstract sets that he called "Kardinalen". Depending on the nature of the cohesive topos of interest, the appropriate contrasting notion U of inertness may still have some traces of cohesion or variation. An appropriate way of expressing this contrast is through the equation

$$S = S^A$$

that exploits the assumption that the cohesive universe under discussion has map spaces. The equation means more precisely that the morphism from A to 1 induces as an isomorphism the inclusion of constant figures. This equation induces a Galois connection: given a family of figure shapes A, we can define U to consist of all the S that **"look discrete"** to those A, and conversely, given a suitable sub-category U, all the figure shapes A satisfying the equation for S in U can be considered as **"looking connected"**. We will consider the result of imposing conditions on these opposed classes, but note that the connected figure shapes are always closed under finite products and that the discrete objects are closed under exponentiation by arbitrary exponents; because of the latter, the left adjoint of the inclusion of U will always preserve finite products. This "connected components functor" when it exists assigns to each space a discrete space that represents its "pieces" in the sense that any map to a discrete space is constant on each piece.

In the next section we will consider both stronger conditions on such an opposed connected/discrete pair of subcategories, as well as many of the constructions of geometrical significance possible in such a context.

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II. Topos of Spaces as Category of Cohesion

We analyze a category E of cohesion by contrasting it with non-cohesion U. The contrast is implemented specifically by an **adjoint string** and some basic constructions are effected by collapsing or expanding the length of these strings. The string begins on the left with a product-preserving "pieces"-functor, left adjoint to a "discrete" inclusion. Part of the investigation of particular cases will involve the search for small adequate subcategories of figure shapes that are connected in the sense of the Galois connection with the discrete subcategory U. Except for very interesting examples involving O-minimal models of geometry and analysis, the string will extend further to the right with Cantor's bold introduction of infinite discrete spaces, namely, with a right adjoint **"points" functor**. In general, we make no special assumption about these objects of points, i.e. about the inner nature, if any, of the objects of U. Partly to insure that the points functor itself preserves colimits, we often further postulate a right adjoint for it, usually called the **inclusion of "codiscrete spaces"**. The discrete spaces and the codiscrete spaces form identical categories which are, however, opposite in the way they are placed within the category E that unites them around the points functor. There are also examples where this string of four adjoints is replaced by a string of three, because the first "pieces" functor does not exist, and instead the analysis revolves around the codiscrete inclusion; this applies in particular to the bornological topos (of great importance in functional analysis).

There is a natural map from the points functor to the pieces functor, which always arises when we have both adjoints to a full inclusion functor, and which, in this case, can be thought of as the indication to of which piece of a space any given point of the space might belong. The Nullstellensatz requires that this map be epic, namely that in a certain internal sense every non-empty piece has at least one point; this Nullstellensatz assumption is equivalent to the monomorphicity of the natural map from the discrete to the codiscrete realization of a given abstract object in U. We are led to consider the intermediate category L of those objects for which every piece contains exactly one point in the sense that the canonical map becomes invertible there. The Nullstellensatz implies that this subcategory is closed under arbitrary subobjects and thus that it is epi-reflective; it obviously ought to be co-reflective as well, because both of the aspects of p that are being identified preserve colimits. Thus we have a factorization

$$p = qs$$

where q involves an adjoint string collapsed to length two, because the left and right adjoint of q^* are isomorphic; there is also an adjoint string in s, but of lenghth only three, because there is no analog of codiscrete for it, and moreover the left adjoint part of q no longer preserves products. I call q a **quality type** because of the collapse and the functor s from p to it is called a specific quality, indeed the canonical example of an **intensive quality** because it is compatible with the right adjoint points functor. By contrast, an **extensive quality** is a map to a quality type that is compatible instead with the pieces aspect of p. Thus the left adjoint part of s is an example of extensive quality valued in the category L of spaces which we will call **infinitesimal**. Some examples deserve to be called continuous, in particular, the examples with finite Hom sets, because the number of pieces of a function space X^S is the same as (the number of pieces of X itself)^S whenever S is discrete. In that case, the **Hurewicz** category, which consists of the same objects as E, but re-enriched in S by means of [X,Y] = pieces of Y^X, is itself a quality type over U, thus enriching the idea that pieces are to be considered as coarse points; this category is not a topos, but it is still cartesian-closed and extensive, and the obvious "homotopy type" quantity from E to it strictly preserves addition, multiplication, and exponentiation. This canonical extensive quality could be thought of as **"form"**, not to be confused with "shape".

In the same spirit the canonical intensive quality s might be called **"substance"** because, as can be illustrated with examples involving generalized graphs, the "molecular structure" at each point of a space X is retained in s_* (X) even though all the "interactions" connecting different points of X have been neglected (perhaps through super-heating and rarification). (The motivation for calling this a substance comes from chemistry, where we say that a sample of water is H_2O whether it is ice or steam, expressing only that the individual molecules have a certain combinatorial configuration.)

By contrast, in the left-adjoint companion, $s_!$ (X) all the interactions have been retained, but the points have coalesced (possibly through super-cooling and compression) to form super-molecules, one for each component of X. By general principles there is again a natural "cooling map" between these two aspects of s, which retains some of the information relating the way in which the molecules are coalesced to form super-molecules. There arises the conjecture that in some cases we might be able to completely determine an object X by measuring these three aspects of substance as a functor from E into the morphism category of L; however, this functor has no obvious preservation properties so that the problem would be a difficult one. In the analogous problem at the lower level, namely considering just pieces, points, and the placement of the latter in the former as invariants, at least the functor preserves finite products. These two reconstruction problems can be illustrated by E = the category of reflexive graphs in U, where the codomains L^2 and U^2 of the invariants might be called "rough graphs" in analogy with the "rough sub-sets" sometimes considered in the poset setting.

In SDG the whole internal structure of E which we are considering, and much more, is determined by a single pointed object D deemed to represent the tangent bundle functor; the "inert" objects in U are thus those which by definition carry no non-zero vector fields. D is a connected object of the category L of infinitesimal objects, and is in fact contractible in the sense that D^D sees all the same objects of U as discrete that D itself does. The further strong condition that D is an ATOM, in the sense that $(\ )^D$ has a right adjoint, implies many further special features; its adjointness is enriched only in U (not E or even L), which was one of the motivations for the definition of U, while one of the first applications was the representability of differential forms and the accompanying possibility of Eilenberg-Mac Lane spaces for deRham cohomology (still underexploited!). (ATOM can be thought of as an acronym for "amazingly tiny objectified motion" where the amazing is a way of reading Grothendieck's upper shrieck notation, and where "objectified" is related to the idea that pure cohesion is a potential basis for motion, actual motion being a morphism in another related category.) From D can be derived, following Euler, the definition of the rig of continuous quantities as the subobject of D^D fixing zero, with automatic multiplication and with addition uniquely guaranteed by further properties. The inclusion of Aut (D) into D^D can be taken as a generic open subobject, thus by pullback defining a "topology" on every object with respect to which every map is continuous. From this Eulerian algebra based on geometric infinitesimals there will be a canonical map to the Dedekind algebra based on discrete rational numbers and truth values provided D is "real" in the sense that Aut (D) has precisely two pieces, so that the component of the identity can be considered as positive; in a rig with a sub-group called "positive", a subrig to be called "non-negative" can be defined, and thus the map to the Dedekind continuum is an instance of a Yoneda map applied in the truth-valued case. Of course, this map is not likely to be either injective or surjective.

There is a well-defined sense in which a quotient topos like L of E determines a subtopos, namely the smallest subtopos of E for which all the objects in the inverse image s^* of L are sheaves. In case this subtopos is all of E, we could say that **"E is infinitesimally generated"**. Because sheaf subcategories are always closed under exponentiation by arbitrary objects, all objects defined by equalizers inside R^n , R^R etcetera, will automatically be sheaves in the subtopos generated by L, because D itself is in L. In this sense, all of smooth, and analytic, algebraic geometry and functional analysis are taking place in infinitesimally generated toposes. However, there are many cohesive toposes, such as reflexive graphs, reversible reflexive graphs, etcetera which are infinitesimally generated in this sense, even though not infinitesimal in a traditional sense; namely, the spaces of maps between the objects consisting entirely of loops include among their retracts the standard figure types (= the representable presheaves).

Both quotient toposes and sub-toposes determine left exact idempotents with a coherence condition, and if we consider the system of all such left-exact idempotents as ordered by the inclusion of their fixed subcategories, the above generation process appears as taking place within this unified lattice. In a paper published in 1989 in the Bulletin of the Australian Math Society, vol. 39 (no. 3, pp 421-434) Paré, Rosebrugh, and Wood proved the remarkable fact that left exact idempotents not only split, but in fact can be split in two steps, one step in which the projection is left adjoint, and the next in which the section is left adjoint. Thus, any left-exact idempotent on a topos in particular leads to a quotient topos and a sub-topos of that, in other words, to a sub-quotient. This result does not seem to have been widely utilized or developed, but surely its ramifications would be of immediate value in analyzing this question of infinitesimal generation.

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III. Bornological and simplicial cohesion

There are other modes of cohesion that seem unrelated to the actual infinitesimal examples related to SDG and which in fact do not have a simple U-valued notion of the number of pieces, so that even our definition of the category L of infinitesimal substance does not have a direct expression. On the other hand, the other natural map between discrete and codiscrete can be considered and the extent to which it is an isomorphism can be used as a measure of cohesion of the topos as a whole. For example, in the topos of presheaves on the site of all finite sets one can consider the subtopos on which the above natural map becomes an isomorphism when evaluated at the empty set. The result is to change the classifier for all Boolean algebras into the classifier for non-trivial Boolean algebras, which has the full range of homotopy types because it is the exact completion of the classical (unordered) simplicial category and contains the Cartesian-closed extensive category of all groupoids as a Poincaré reflective subcategory. This topos has the property that all subtoposes are essential and determined by finite cardinals in the obvious way. What is not obvious is the precise Aufhebung relation between these subtoposes: Given n, how large must we take m so that every n-skeletal object is not only m-skeletal, but also m-co-skeletal?

A similar construction within the category of presheaves on the site of countable sets, yields the subtopos which perceives co-discrete and discrete as isomorphic when evaluated at any finite set. In this example, unlike the previous one, there is no pieces functor. The basic figure shape is that of a **"bounded"** sequence, so the whole topos envelops in an exact way the category of bornological spaces of importance in analysis. The category of modules over the Dedekind real number object includes all the countable inductive limits of Banach spaces as a full subcategory, but at the same time is itself a Grothendieck AB5 Abelian category with the accompanying exactness properties. The site is an extensive category in which every map factors as a monomorphism following a split epimorphism and so the Yoneda embedding of the site preserves finite limits and sums and epimorphisms, but it does not preserve coequalizers. Since the category of recursive sets also satisfies the same axiom of choice it should also give rise to a superficially similar topos whose more distinctive properties were studied by Phil Mulry in his thesis.

Both the Boolean algebra classifier and the bornological topos are specific examples whose deeper and more detailed study should give more insight into what to expect from general theories of this kind of cohesion. The Aufhebung calculation is a question similar to the Ramsey theorem which has in principle sense in any combinatorial topos of presheaves on a small category with finite hom sets.

Also still not completely clarified is the "fundamental lemma of functional analysis" which compares the linearized bornological topos and the Johnstone topological topos by means of the idea of Mackey convergence and its equi-continuity adjoint, and, moreover, the clarification of the relation between completeness and duality in the same situation established by Waelbreck's 1965 theorem to the effect that distributions of compact support form the free complete bornological vector space generated by a smooth space.

The study of such examples is always related to double-dualization monads and to the failure of reasonable geometrical theorems in case so-called measurable cardinals are admitted into the category of small sets. This suggests to me that first of all "small" should not be identified with "member of some class", but should explicitly exclude the measurable cardinals. The category of all small sets is an object we frequently use and if it itself is a measurable cardinal that should not dismay us any more than that the category of all finite sets is not finite.

As the open problems mentioned above strongly indicate, there are still a great many problems in topos theory and even in pure category theory that are on the one hand of a very elementary nature, and yet whose clarification still needs to be carried out, in order to achieve that maximally-unified guide to the development of mathematical concepts toward which our subject has always been striving.