SOME ALGEBRAIC PROBLEMS IN THE CONTEXT OF FUNCTORIAL SEMANTICS OF ALGEBRAIC THEORIES

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The categorical approach to universal algebra, initiated in [FSAT], has been extended from finitary to infinitary operations in [SAEC], from sets to arbitrary base categories through the use of triples (monads) in [AFT] and [AMT] and from one-sorted theories over 1-dimensional categories to Γ-sorted theories over 2-dimensional categories in [BT]. But despite this generality, there is still enough information in the machinery of algebraic categories, algebraic functors, adjoints to algebraic functors, the semantics and structure superfunctors, etc. to allow consideration of specific problems analogous to those arising in group theory, ring theory, and other parts of classical algebra. The approach also suggests new problems. As examples of the latter we may mention Linton's considerations of general "commutative" theories [AEC], Barr's discussion of general "distributive" laws [CG], and Freyd's construction of Kronecker products of arbitrary theories and tensor products of arbitrary algebras [AFG-TPP]. It is our purpose here to indicate some of the "specific" aspects of the approach, and also to mention some of the representative problems which seem to be open. We restrict ourselves to the case of finitary single-sorted theories over sets.

An elegant exposition of part of the basic machinery appears in [AGA] - we content ourselves here with a brief summary. An <u>algebraic</u>

theory is a category $\mathbb{A}$ having as objects

$$1,\ A,\ A^2,\ A^3,\ \ldots\ldots\ldots$$

and, for each n = 0, 1, 2, 3, ..., n morphisms

$$A^n \xrightarrow{\pi_i^{(n)}} A, \qquad i = 0, 1, \ldots, n-1,$$

such that for any n morphisms

$$A^m \xrightarrow{\theta_i} A, \qquad i = 0, 1, \ldots, n-1,$$

in $\mathbb{A}$ there is exactly one morphism

$$A^m \xrightarrow{\langle \theta_0,\ \theta_1, \ldots,\ \theta_{n-1} \rangle} A^n$$

in $\mathbb{A}$ so that

$$\langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle \pi_i^{(n)} = \theta_i, \qquad i = 0, 1, \ldots, n-1.$$

The arbitrary morphisms $A^n \xrightarrow{\varphi} A$ are called the <u>n-ary operations</u> of $\mathbb{A}$. The <u>algebraic category</u> associated with $\mathbb{A}$ is the <u>full</u> subcategory

$$\mathbb{A}^b \subset \mathcal{S}^{\mathbb{A}}$$

consisting of those covariant set-valued functors which are product-preserving; its objects are called $\mathbb{A}$-algebras and its morphisms $\mathbb{A}$-homomorphisms. Clearly there is a full embedding $\mathbb{A}^{op} \xrightarrow{\subset} \mathbb{A}^b$ which preserves coproducts; its values are the <u>finitely-generated free</u> $\mathbb{A}$-algebras, where "free" refers to the left adjoint of the functor "underlying"

$$\mathbb{A}^b \xrightarrow{U_{\mathbb{A}}} \mathcal{S}$$

whose value at the algebra X is the value of X at A:

$$X\, U_{\mathbb{A}} = A\, X.$$

The underlying functor is a particular <u>algebraic functor</u>, where the latter means a functor

$$\mathbb{A}^{b} \xrightarrow{f^{b}} \mathbb{B}^{b}$$

induced by composition of functors from a <u>theory morphism</u> $\mathbb{B} \xrightarrow{f} \mathbb{A}$, where a theory morphism is just a functor f such that

$$(\pi_i^{(n)})f = \pi_i^{(n)} \quad \text{for all } i \in n \in \omega.$$

Clearly all the theory morphisms determine a category $\mathcal{T}$, and every algebraic functor preserves the underlying functors. Hence $f \rightsquigarrow f^{b}$ determines a <u>semantics</u> functor

$$\mathcal{T}^{op} \longrightarrow (\mathrm{Cat}, \mathcal{S})$$

where the category on the right has as morphisms all commutative triangles

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\Phi"] \arrow[dr, "U"'] & & \mathcal{X}' \arrow[dl, "U'"] \\ & \mathcal{S} & \end{tikzcd}$$

of functors. Switching functor categories a bit shows that <u>structure</u>, the left adjoint of semantics, may be calculated as follows: Given a set valued functor U, the n-ary operations of its algebraic structure are just the <u>natural transformations</u> $U^n \xrightarrow{\varphi} U$, where U^n

is the n-th cartesian power of U in the functor category $\mathcal{S}^{\mathcal{X}}$, i.e. φ is a way of assigning an operation to every value of U in such a way that all morphisms of $\mathcal{X}$ are homomorphisms with respect to it. Several applications of Yoneda's Lemma show that if in fact $U = U_{\mathbb{A}}$, $\mathcal{X} = \mathbb{A}^{b}$ for some theory $\mathbb{A}$, then the algebraic structure of U is isomorphic to $\mathbb{A}$. As a corollary every functor $\mathbb{A}^{b} \longrightarrow \mathbb{B}^{b}$ which preserves underlying sets is induced by one and only one theory morphism $\mathbb{B} \longrightarrow \mathbb{A}$. More generally, if we denote by $\amalg_n$ the <u>free theory</u> generated by one n-ary operation, then the n-ary operations of the algebraic structure of any $\mathcal{X} \xrightarrow{U} \mathcal{S}$ are in one-to-one correspondence with the <u>functors</u>

$$\mathcal{X} \xrightarrow{\quad \Phi \quad} \amalg_n^{b}$$

for which $U = \Phi \, U_{\amalg_n}$.

Algebraic functors are faithful and possess left adjoints. In fact (as pointed out by M. André and H. Volger), if $\mathbb{B} \xrightarrow{f} \mathbb{A}$ is a morphism of theories then the usual (left) Kan adjoint

$$\mathcal{S}^{\mathbb{B}} \dashrightarrow \mathcal{S}^{\mathbb{A}}$$

corresponding to f actually takes product-preserving functors into product preserving functors, and so restricts to a functor f_* with

$$f_* \dashv f^{b} .$$

Thus we have the commutative diagram of functors

$$\begin{tikzcd}
\mathbb{B}^{op} \arrow[r, "f^{op}"] \arrow[d, hook] & \mathbb{A}^{op} \arrow[d, hook] \\
\mathbb{B}^{b} \arrow[r, "f_*"] \arrow[d, hook] & \mathbb{A}^{b} \arrow[d, hook] \\
\mathcal{S}^{\mathbb{B}} \arrow[r, dashed] & \mathcal{S}^{\mathbb{A}}
\end{tikzcd}$$.

Explicitly, for any $\mathbb{B}$-algebra Y, the underlying set of the "relatively free" $\mathbb{A}$-algebra Yf_* is the colimit of

$$(f, A) \longrightarrow \mathbb{B} \xrightarrow{Y} \mathcal{S}$$

where the first factor of this composite is the obvious forgetful functor from the category whose morphisms are triples θ, φ, θ' with θ, θ' operations in $\mathbb{A}$ and φ a morphism in $\mathbb{B}$ such that

$$\begin{tikzcd}
\cdot \arrow[rr, "\varphi f"] \arrow[dr, "\theta"'] & & \cdot \arrow[dl, "\theta'"] \\
& A &
\end{tikzcd}$$

is commutative in $\mathbb{A}$. In particular, free algebras can be computed by such a direct limit by taking $\mathbb{B}$ = the initial theory, i.e., the dual of the category of finite sets and maps. For the unique f in this case we also write $f_* = F_{\mathbb{A}}$.

Given two theories $\mathbb{A}$ and $\mathbb{B}$, the category of all product preserving functors $\mathbb{B} \longrightarrow \mathbb{A}^b$ has an obvious underlying set functor, whose algebraic structure is denoted by $\mathbb{A} \otimes \mathbb{B}$, the <u>Kronecker product</u> of $\mathbb{A}$ with $\mathbb{B}$. The Kronecker product is a coherently associative functor $\mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ having the initial theory as unit object; it also satisfies $\mathbb{A} \otimes \mathbb{B} \cong \mathbb{B} \otimes \mathbb{A}$. The foregoing semantical definition of $\mathbb{A} \otimes \mathbb{B}$ is equivalent to the following wholly "theoretical" definition:

$$\mathbb{A} \otimes \mathbb{B} = \mathbb{A} * \mathbb{B}/R,$$

where $\mathbb{A} * \mathbb{B}$ denotes the coproduct in $\mathcal{T}$ and R is the congruence relation generated by the conditions that each $A^n \xrightarrow{\varphi} A$ in $\mathbb{A}$ should be a "homomorphism" with respect to each $A^m \xrightarrow{\psi} A$ in $\mathbb{B}$ [$(A^n)^m \xrightarrow{\sigma} (A^m)^n \xrightarrow{\psi^n} A^n$ being defined as the operation of ψ on A^n, σ being the transpose isomorphism.] and that, symmetrically, each $\mathbb{B}$-operation is an "$\mathbb{A}$-homomorphism". A famous example is: if $\mathbb{G}$ is the theory of groups, $\mathbb{G} \otimes \mathbb{G}$ is the theory of abelian groups.

Having briefly described some of the main tools of the functorial semantics point of view in general algebra, we now make some methodological remarks which this point of view suggests. First, many problems will take the forms: Characterize, in terms of $\mathcal{T}$, those $\mathbb{A}$ for which $\mathbb{A}^b$ has a given property stated in terms of (Cat, $\mathcal{S}$), or characterize those $f \in \mathcal{T}$ for which f^b has a given property; or for which f_* has a given property. (Properties of $\mathbb{A}$ may be viewed as properties of $U_{\mathbb{A}}$ or of $F_{\mathbb{A}}$ and as such may have natural "relativizations" to properties of f^b or f_*). Properties of diagrams in $\mathcal{T}$ may be "semantically" defined via arbitrary "mixtures" of the processes $f \rightsquigarrow f^b$, $g \rightsquigarrow g_*$, and algebraic structure from properties in (Cat, $\mathcal{S}$), and direct descriptions in $\mathcal{T}$ of such properties of diagrams may be sought. Most of the solved and unsolved problems mentioned below are of this general sort. For example, light would be shed on many situations in algebra if one could give a computation entirely in terms of $\tilde{\mathcal{J}}$ of the algebraic structure of

$$\mathbb{G}^b \xrightarrow{g^b} \mathbb{M}^b \xrightarrow{f_*} \mathbb{R}^b \xrightarrow{U_{\mathbb{R}}} \mathcal{S}$$

for any given diagram

$$\mathbb{G} \xleftarrow{g} \mathbb{M} \xrightarrow{f} \mathbb{R}$$

in $\tilde{\mathcal{J}}$. A case in point is that where $\mathbb{G}$ = theory of groups, $\mathbb{M}$ = theory of monoids, $\mathbb{R}$ = theory of rings with g and f the obvious inclusions; what is sought in the example is in this case the full algebraic structure of group rings - this is a very "rich" theory, having <u>linear</u> "p-th power" unary operations for all p and more generally an n-ary <u>multilinear</u> operation for every element of the free group on n letters (e.g. convolution corresponds to the binary operation of group multiplication). Are these multilinear operations a generating set for the theory in question? Probably this case is simpler than the example in general, since it is equivalent to the structure of

$$\mathbb{G}^b \xrightarrow{U_{\mathbb{G}}} \mathcal{S} \xrightarrow{F_{\mathbb{A}}} \mathbb{A}^b \longrightarrow \mathcal{S}$$

where $\mathbb{A}$ is the theory of abelian groups, and $\mathbb{A}^b$ has a convenient tensor product.

Sometimes the problem is in the other direction: for example, the product $\mathbb{A} \times \mathbb{B}$ of course has an easy description in terms of

$\mathcal{T}$, but a bit of computation is needed to deduce from general principles that $(\mathbb{A} \times \mathbb{B})^b$ consists of algebras which canonically split as sets into a product X x Y, where X carries the structure of an $\mathbb{A}$-algebra and Y the structure of a $\mathbb{B}$-algebra).

A second general methodological remark is that the structure functor often yields much more information than the usual Galois connection of Birkhoff between classes of algebras of a given type and sets of equations, precisely because in many situations it is natural to change the type. Namely, a subcategory $\mathcal{X} \subseteq \mathbb{B}^b$ of an algebraic category (even a full one) may have an algebraic structure with more operations (as well as more equations) than $\mathbb{B}$, i.e., the induced morphism $\mathbb{B} \longrightarrow \mathbb{A}_{\mathcal{X}}$ may be non-surjective, where $\mathbb{A}_{\mathcal{X}}$ denotes the algebraic structure of $\mathcal{X} \longrightarrow \mathbb{B}^b \xrightarrow{U_{\mathbb{B}}} \mathcal{S}$. An obvious example is that in which $\mathbb{B}$ is the theory of monoids and $\mathcal{X}$ is the full subcategory consisting of those monoids in which every element has a two-sided inverse. Two other examples arise from subcategories of the algebraic category of commutative rings: the algebraic structure of the full category of fields includes the theory $\mathbb{R}_\Theta$ generated by an additional unary operation Θ subject to

$$
\begin{aligned}
1^\Theta &= 1 \\
(x \cdot y)^\Theta &= x^\Theta \cdot y^\Theta \\
x^2 \cdot x^\Theta &= x \\
(x^\Theta)^\Theta &= x
\end{aligned}
$$

and similarly the algebraic structure of the category of integral domains and monomorphisms includes the theory $\mathbb{R}_e$ generated by an additional operation e subject to

$$0^e = 0$$
$$(x \cdot y)^e = x^e \cdot y^e$$
$$(x^e)^e = x^e$$
$$x^e \cdot x = x$$

The inclusion of fields in integral domains corresponds to the morphism $\mathbb{R}_e \longrightarrow \mathbb{R}_\Theta$ which, while the identity on the common subtheory $\mathbb{R}_c$ (= theory of commutative rings), takes e into the operation of $\mathbb{R}_\Theta$ defined as follows

$$x^e =_{\text{def}} x \cdot x^\Theta$$

The third general methodological remark is that, within the doctrine of universal algebra, the "natural" domain of a construction used in some classical theorem may be in fact much larger than the domain for which the theorem itself can be proved. For example, the only $\mathbb{R}_e$-algebras which can be embedded in fields are integral domains, but the usual "field of fractions" construction is just the restriction of the adjoint functor $(\mathbb{R}_e \longrightarrow \mathbb{R}_\Theta)_*$ whose domain is all of $\mathbb{R}_e^b$. To the same point, the usual construction of Clifford algebras is defined only for K-modules V equipped with a quadratic form $V \xrightarrow{q} K$; these pairs $\langle V, q \rangle$ do not form an algebraic category. But if we allow ourselves to consider quadratic forms $V \longrightarrow S$ with values in arbitrary commutative K-algebras S, we can
(i) Define the underlying set to be V x S and find that these generalized quadratic forms do constitute an algebraic category and
(ii) Extend the Clifford algebra construction to this domain and find that there it is entirely a matter of algebraic functors and

their adjoints (for this certain idempotent operations have to be introduced, as below).

Certain constructions which have the form of algebraic functors composed with adjoints to algebraic functors may also be interpretable along the line of the foregoing remark. For example the "natural" domain of the group ring construction might be said to be the larger category of all monoids, for there it becomes simply the adjoint of an algebraic functor. Similar in this respect is the construction of the exterior algebra of a module, whose usual universal property is not that of a single left adjoint, but does allow interpretation in terms of the composition of algebraic functors and the adjoint of an algebraic functor:

$$\mathbb{A}^{b} \xrightarrow{f^{b}} \mathbb{A}_{P}^{b} \overset{g_{*}}{\dashrightarrow} \mathbb{R}_{P}^{b} \xrightarrow{h^{b}} \mathbb{R}^{b}$$

where $\mathbb{A}$ is the theory of K-modules, $\mathbb{A}_P$ is the theory of modules with an idempotent K-linear operator P, $\mathbb{R}$ is the theory of K-algebras, and $\mathbb{R}_P$ is the theory of K-algebras with an idempotent K-linear unary operation P satisfying the equation

$$(x^P)^2 = 0.$$

(f, g, h) being the obvious inclusions). Thus one might claim that the natural domain of the exterior algebra functor consists really of modules with given split submodules whose elements are destined to have square zero.

The problem mentioned earlier, of computing the structure of a composition: algebraic functor followed by an adjoint of an algebraic functor, is of relevance also in the above examples, since e.g. the natural anti-automorphism of Clifford algebras is an element of the structure theory of that functor, while composing the exterior algebra functor with the forgetful functor from Lie algebras or K[x]-modules and then taking algebraic structure should yield exterior differentiation and determinant, respectively, as operations in appropriate algebraic theories.

It is obvious and well-known that the constructions of tensor algebras, symmetric algebras, universal enveloping algebras of Lie algebras, abelianization of groups, and of the group engendered by a monoid are all of the form f_* for a suitable morphism $f \in \mathfrak{J}$. Perhaps less well-known are the theory $\mathbb{M}_{(-)}$ of monoids equipped with a unary operation "minus" satisfying

$$-(-x) = x \,,$$

$$(-x)\cdot(-y) = x\cdot y,$$

and the functor $\qquad \mathbb{M}^b_{(-)} \xrightarrow{f_*} \mathbb{R}^b$

associated to the obvious inclusion f of $\mathbb{M}_{(-)}$ into the theory of rings; this functor has the quaternions as one of its values, the eight quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$ forming an $\mathbb{M}_{(-)}$-algebra. The quaternions also appear in another way, namely as a value of the Cayley-Dickson monad (triple) which is the composition of a certain algebraic functor with its adjoint and is defined on an appropriate algebraic category of non-associative (not even all alternative) algebras with involution.

An algebraic functor whose adjoint does not seem to have been investigated is the <u>Wronskian</u>, which assigns to each commutative algebra equipped with a derivation $x \rightsquigarrow x'$ the Lie algebra consisting of the same module with

$$[a, b] =_{def} a \cdot b' - a' \cdot b \ .$$

For example, is the adjunction always an embedding, giving an entirely different sort of "universal enveloping algebra" for a Lie algebra?

For the remainder of this paper we wish to discuss some problems exemplifying the canonical sort of the first methodological remark. Some semantically-defined subcategories of $\mathfrak{T}$ admit not only simple descriptions entirely in terms of $\mathfrak{T}$, but also can themselves be parameterized by single algebraic subcategories. Consider the full subcategory of $\mathfrak{T}$ determined by those $\mathbb{A}$ for which $U_{\mathbb{A}}$ has a <u>right</u> adjoint (as well as the usual left adjoint $F_{\mathbb{A}}$). These $\mathbb{A}$ are easily seen to be characterized by the property that for each $n = 0, 1, 2, \ldots.$ each $\mathbb{A}$-operation $A^n \longrightarrow A$ factors uniquely through one of the projections $\pi^{(n)}_i$. Such <u>unary theories</u> are in fact parameterized by the full and faithful left adjoint of the "unary core" functor

$$\mathfrak{T} \xrightarrow{\text{Un}} \mathbb{M}^b$$

where $\mathbb{M}$ is the theory of monoids with

$$\mathbb{M}(A^n, A) \cong \sum_{k=0}^{\infty} n^k$$

and where $(\mathbb{A})\text{Un} = \mathbb{A}(A, A)$ as a monoid. Thus we may also say that a theory "is a monoid" if and only if it is unary. Note that if we denote the left adjoint

$$\mathbb{M}^b \longrightarrow \mathcal{J}$$

to Un by $M \rightsquigarrow \overline{M}$, then we have

$$\overline{M_1 \times M_2} = \overline{M}_1 \otimes \overline{M}_2$$

for any two monoids M_1, M_2.

Another algebraically parameterized subcategory of $\mathcal{J}$ consists of all $\mathbb{A}$ for which $\mathbb{A}^b$ is <u>abelian</u>. We often say that such a theory "is a ring", for it must necessarily be isomorphic to a value of the full and faithful functor

$$\mathbb{R}^b \xrightarrow{\text{Mat}} \mathcal{J}$$

which assigns to each ring R the category Mat_R whose morphisms are all the finite rectangular matrices with entries from R (i.e., the algebraic theory of R-modules). Here $\mathbb{R}(A^n, A) \cong Z[x_1, \ldots, x_n] =$ the set of polynomials with integer coefficients in n non-commuting indeterminates. The functor Mat commutes with the Kronecker product operations defined in the two categories, and has a <u>left</u> adjoint given by $\mathbb{A} \rightsquigarrow Z \otimes \mathbb{A}$ where we now mean by Z the theory corresponding to the ring Z (i.e., the theory of abelian groups). Note that while a <u>quotient theory</u> of a ring is always a ring, e.g. the theory of convex sets (consisting of all stochastic matrices) is a <u>subtheory</u> of a ring which is not a ring.

Since $\mathbb{A} \longrightarrow Z \otimes \mathbb{A}$ canonically, we have the adjoint functor

$$\mathbb{A}^b \dashrightarrow (Z \otimes \mathbb{A})^b$$

from the category of $\mathbb{A}$ -algebras to the canonically associated abeliai category, and for each $\mathbb{A}$-algebra X an adjunction morphism $X \longrightarrow \overline{X}$ if we denote by $\overline{X}$ the associated $Z \otimes \mathbb{A}$ -module. The kernel of this adjunction morphism may be denoted by [X, X], suggesting notions of <u>solvability</u> for algebras over any theory $\mathbb{A}$, which do in fact agree with the usual notions for $\mathbb{A}$ = theory of groups, theory of Lie algebras, or theory of unitless associative algebras. Sometimes [X, X] may actually be the empty set; for example, if $\mathbb{A}$ is a monoid, X is a set on which the monoid acts, then $X \longrightarrow \overline{X}$ is the embedding of X into the free abelian group generated by X (equipped with the induced action of $\mathbb{A}$).

The composition

$$\mathbb{M}^b \xrightarrow{\subset} \mathcal{T} \xrightarrow{Z \otimes (\ )} \mathbb{R}^b$$

is another way of defining the monoid ring; more generally, for any theory $\mathbb{A}$ and monoid M, $\mathbb{A} \otimes M$ is the theory of $\mathbb{A}$ -algebras which are equipped with an action of M by $\mathbb{A}$ -endomorphisms. In fact, thinking of theories as generalized rings often suggests a natural extension of concepts or constructions ordinarily defined only for rings to arbitrary theories. For example consider fractions: the category whose objects are theory-morphisms $M \longrightarrow \mathbb{A}$, M any monoid, $\mathbb{A}$ any theory, admits a reflection to the subcategory in which M is a group, constructed by first ignoring $\mathbb{A}$ and forming the algebraic adjoint, and then taking a pushout in $\mathcal{T}$.

Part of the intrinsic characterization of those $\mathbb{A}$ which are rings is of course the condition that for each n, A^n is the n-fold coproduct (as well as product) of A in $\mathbb{A}$ (In fact this alone is characteristic of semi-rings). Another condition which some theories $\mathbb{A}$ satisfy is that A^n is the 2^n-fold coproduct of A; such theories turn out to be parameterized by the algebraic category of Boolean algebras.

One of the famous solved problems of our canonical type is: Which theories $\mathbb{A}$ are such that in $\mathbb{A}^b$, every reflexive subalgebra $Y \subseteq X \times X$ is actually a congruence relation? The answer is: those for which there exists at least one $\mathcal{T}$-morphism $\mathbb{B}_3 \longrightarrow \mathbb{A}$, where

$$\mathbb{B}_3 = \amalg_3/E$$

is the theory generated by one ternary operation θ satisfying the two equations E:

$$\langle x, x, z \rangle \theta = z$$
$$\langle x, z, z \rangle \theta = x$$

For example, if $\mathbb{A} = \mathbb{G}$, the theory of groups, one could define such a morphism by

$$\langle x, y, z \rangle \theta =_{def} x \cdot y^{-1} \cdot z$$

Also $\mathbb{R}$, Mat (R) for any ring R, the theory of Lie algebras, as well as certain theories of loops or lattices, share with $\mathbb{G}$ the property described.

Also by now well-known, but apparently more recently considered, is the problem: For which $\mathbb{A}$ does $\mathbb{A}^b$ have a closed (autonomous) structure with respect to the standard underlying set functor $U_{\mathbb{A}}$?

The answer is: the commutative $\mathbb{A}$, meaning those for which every operation is also a homomorphism. Since a monoid or ring is commutative as a monoid or ring if it is commutative as a theory, one is not surprised to note that in the category of commutative theories, the coproduct is the Kronecker product.

Less classical, but more trivial, is the question: for which $\mathbb{A}$ is the trivial algebra 1 a good generator for $\mathbb{A}^b$? The answer is: the affine $\mathbb{A}$, meaning those for which

$$A \xrightarrow{\text{diag}} A^n \xrightarrow{\varphi} A$$

is the identity morphism for every n-ary $\mathbb{A}$ -operation φ and for every n = 0, 1, 2, ... Being "equationally defined", the inclusion (of affine theories into all) clearly has a left adjoint, but more interesting seems to be the right adjoint which happens to exist; this assigns to any $\mathbb{A}$ the subtheory Aff ($\mathbb{A}$) consisting of all (tuples of) those φ which do satisfy the above condition. Noting the first four letters of the word "coreflection", we call Aff ($\mathbb{A}$) the affine core of $\mathbb{A}$. The term "affine" was suggested by the fact that

$$\mathbb{R}^b \xrightarrow{\text{Mat}} \mathcal{T} \xrightarrow{\text{Aff}} \mathcal{T}_{\text{Aff}} \xrightarrow{\subset} \mathcal{T}$$

assigns to each ring its theory of affine modules.

We now list some semantically-defined subcategories $\mathcal{G}$ of $\mathcal{T}$ for which good characterizations in terms of $\mathcal{T}$ alone do not seem to be known. They will be presented in relativized form, so that none of them are full subcategories of $\mathcal{T}$ but all of them contain all the isomorphisms of $\mathcal{T}$. With each such relativized problem $\mathcal{G}$ there is a corresponding "absolute" problem: namely to find those $\mathbb{A}$ such that the morphism f from the initial theory to $\mathbb{A}$ belongs to the class $\mathcal{C}$. We simply list the condition that arbitrary $\mathbb{B} \xrightarrow{f} \mathbb{A}$ belong to $\mathcal{C}$ in each case:

① f^b takes epimorphisms in $\mathbb{A}^b$ into epimorphisms in $\mathbb{B}^b$. The corresponding absolute question is: for which $\mathbb{A}^b$ are epimorphisms surjective? so that for example $\mathbb{G}$ has the property while $\mathbb{M}$ does not.

② f^b has a right adjoint (as well as the usual left adjoint). Note that this second category ② is included in the category ① defined above, and that the corresponding absolute question was answered with "unary theories". However the present relative question is definitely more general than just morphisms of unary theories since every morphism between rings is included in category ② as is the inclusion $\mathbb{M} \longrightarrow \mathbb{G}$ (Recall the "group of units"). Since the right adjoint of f^b would have to be represented by $f \cdot X_1$, X_1 being the free $\mathbb{A}$-algebra on one generator, the question is related to the more general one of computing, for any f, the algebraic structure of the set-valued functor $\mathbb{B} \longrightarrow \mathcal{S}$ so represented.

③ f_* is right adjoint to f^b. This very strong condition obviously implies ②. We call the f satisfying ③ Frobenius morphisms since a typical example is a morphism in $\mathcal{T}$ of the form $K \xrightarrow{f} R$ where K is a commutative ring, R is a ring, and f makes R a Frobenius K-algebra. It does not seem to be known if there are any examples in $\mathcal{T}$ of Frobenius morphisms which are not ring morphisms. In the context of triples in arbitrary categories, a characterization in terms of the existence of a "nonsingular associative quadratic form" can be given, but it is not clear what the abstrac form of this condition means when restricted back to theories (unless they are rings).

④ f^b takes finitely generated $\mathbb{A}$-algebras into finitely generated $\mathbb{B}$-algebras. A thorough understanding of this category would imply the solution of the restricted Kurosh and restricted Burnside problems as special cases. In fact the restricted Burnside problem belongs to the absolute case of the question, taking $\mathbb{A} = \mathcal{G}_r$ = theory of groups of exponent r, and the restricted Kurosh problem to the case relative to $\mathbb{B}$ = the ground field, taking $\mathbb{A}$ = theory of algebras satisfying a given polynomial identity.

⑤ The adjunction morphism $Y \longrightarrow f\cdot(Yf_*)$ is monomorphic for all $\mathbb{B}$-algebras Y. This category includes the f defined by the Lie bracket, but not that defined by the Jordan bracket, into the theory of associate algebras over a field. Since when applied to finitely generated free algebras, the adjunction reduces to f itself, it is clear that all f in category ⑤ are necessarily monomorphisms themselves. But this is not sufficient, as the morphism $Z \xrightarrow{f} Q$ from the ring of integers to the ring of rationals shows (apply f_* to an abelian group with torsion). Linton has suggested that the universally monomorphic f in $\mathcal{J}$ may coincide with category ⑤.

⑥ f^b reflects the existence of quasi-sections; i.e., for any $\mathbb{A}$-homomorphism h, if there is a $\mathbb{B}$-homomorphism g with $(h)f\cdot g\cdot(h)f = (h)f$, there is an $\mathbb{A}$-homomorphism $\bar{g}$ with $h\,\bar{g}\,h = h$. The absolute form of this condition applies to a ring $\mathbb{A}$ if it is semi-simple artinian. Since simplicity, chain conditions, etc. have sense in the category $\tilde{\mathcal{J}}$, it would be interesting if subcategory ⑥ could be characterized in these terms.

Finally, various completion processes on the category of theories are suggested by the adjointness of the structure functor. For example, consider the inclusion $\mathcal{S}_{\mathrm{fin}} \longrightarrow \mathcal{S}$ of finite sets into all sets. Pulling back and composing with this functor yields an adjoint pair

$$(\mathrm{Cat}, \mathcal{S}) \rightleftarrows (\mathrm{Cat}, \mathcal{S}_{\mathrm{fin}})$$

which when composed with the semantics-structure adjoint pair yields a triple (monad) on the category $\tilde{\mathcal{J}}$. This triple assigns to each theory $\mathbb{A}$ the algebraic theory $\bar{\mathbb{A}}$ consisting of all operations naturally definable on the <u>finite</u> $\mathbb{A}$ algebras. For example $\bar{\mathbb{G}}$ is the (finitary part of) the theory of profinite groups.

Burnside's general problem suggests a different "completion" for a theory $\mathbb{A}$, namely let $\tilde{\mathbb{A}}$ be the structure of (the underlying set functor of) the category of those $\mathbb{A}$-algebras which are finitely generated and in which each single element generates a finite subalgebra. We have

$$\tilde{\mathbb{A}} \cong \varprojlim_{F} \tilde{\mathbb{A}}_F$$

where F ranges over finite sets of finite cyclic $\mathbb{A}$-algebras, since structure is an adjoint. Note that this completion is not functorial unless we restrict ourselves to category ④. Since every finite $\mathbb{A}$-algebra satisfies the two finiteness conditions above, one obtains a morphism

$$\tilde{\mathbb{A}} \longrightarrow \bar{\mathbb{A}}$$

the study of which reflects one form of a generalized Burnside problem

The functorial completion can also be done relative to a given theory $\mathbb{B}_o$ by using finitely generated or finitely presented $\mathbb{B}_o$-algebras, and considering theories $\mathbb{A}$ equipped with $\mathbb{B}_o \longrightarrow \mathbb{A}$. For example, with $\mathbb{B}_o$ = a field K, the completion of $\mathbb{A}$ = K[x] is the full natural operational calculus $\overline{K[x]}$ for arbitrary operators on finite-dimensional spaces; explicitly this ring consists of all functions Θ assigning to every square matrix a over K another a^Θ of the same size, such that for every suitable rectangular matrix b and square a_1, a_2

$$a_1 b = b a_2 \Longrightarrow a_1^\Theta \cdot b = b \cdot a_2^\Theta$$

If K is the field of complex numbers, one has

$$\begin{tikzcd} & & & K[[x]] \\ K[x] \arrow[r] & \mathcal{E}(K) \arrow[r] & \overline{K[x]} \arrow[ur] \arrow[dr] & \\ & & & K^K \end{tikzcd}$$

where $\mathcal{E}$ (K) is the ring of entire functions and K[[x]] the ring of formal power series. (Formal power series also arise as algebraic structure, by restricting to the subcategory where the action of x is nilpotent). The ring $\overline{K[x]}$ would seem to have a possible role in "formal analytic geometry"; it has over formal power series the considerable advantage that substitution is always defined, so that formal endomorphisms of the formal line would be composable. This monoid is extended to $\overline{\mathbb{A}}$, (the dual of) a category of formal maps of formal spaces of all dimensions by applying the structure-semantical completion process over finite-dimensional K-vector spaces to the theory $\mathbb{A}$ of commutative K-algebras.

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