ANALYSIS, MANIFOLDS AND PHYSICS

Part II

Revised and Enlarged Edition

NORTH-HOLLAND

ANALYSIS, MANIFOLDS AND PHYSICS
Part II

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by

YVONNE CHOQUET-BRUHAT
Membre de l' Académie des Sciences,
Université de Paris VI, Département de Mécanique
Paris, France

CÉCILE DeWITT-MORETTE
University of Texas, Department of Physics
and
Center for Relativity, Austin, Texas, USA

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PREFACE TO THE SECOND EDITION

Twelve problems have been added to the first edition; four of them are supplements to problems in the first edition. The others deal with issues that have become important, since the first edition of Volume II, in recent developments of various areas of physics. All the problems have their foundations in Volume I of the 2-Volume set *Analysis, Manifolds, and Physics.*

It would have been prohibitively expensive to insert the new problems at their respective places. They are grouped together at the end of this volume, their logical place is indicated by a number in parenthesis following the title.

The new problems are:

- "The isomorphism $\mathbb{H} \oplus \mathbb{H} \simeq M_4(\mathbb{R})$. A supplement to Problem I.4 and I.3 (I.17)." Its logical place is the seventeenth problem of Chapter I.
- The problem "Lie derivative of spinor fields (III.15)" belongs to Chapter III.
- "Poisson–Lie groups, Lie bialgebras, and the generalized classical Yang–Baxter equation (IV.14)" has been contributed by Carlos Moreno and Luis Valero. It belongs to Chapter IV.

  Additions to Chapter V on Riemannian and Kählerian manifolds include:
- "Volume of the sphere S^n. A supplement to Problem V.4 (V.15)"
- "Teichmuller spaces (V.16)"
- "Yamabe property on compact manifolds (V.17)"

  To Chapter V bis on Connections are added:
- "The Euler class. A supplement to Problem V bis 6 (V bis 13)"
- "Formula of Laplacians at a point of the frame bundle (V bis 14)"
- "The Berry and Aharanov–Anandan phases (V bis 15)" based on notes by Ali Mostafazadeh.

  To Chapter VI on Distributions:
- "A density theorem. A supplement to Problem VI.6 on 'spaces $H_{s,\delta}(\mathbb{R}^m)$' (VI.17)"
- Tensor distributions on submanifolds, multiple layers, and shocks (VI.18)"
- "Discrete Boltzman equation (VI.19)"

A fair number of misprints have been corrected. An updated list of errata for Volume I is included.

Naturally more problems are on our drawing boards. We would like to think of them as contributions to a third edition.

Most of the new problems were completed during a visit of Y. Choquet-Bruhat to the Center for Relativity of the University of Texas, made possible by the Jane and Roland Blumberg Centennial Professorship in Physics held by C. DeWitt-Morette. Help and comments from M. Berg, M. Blau, M. Godina, S. Gutt, M. Smith, R. Stora, X. Wu-Morrow and A. Wurm are gratefully acknowledged.

PREFACE

This book is a companion volume to our first book, *Analysis, Manifolds and Physics* (Revised Edition 1982). In the context of applications of current interest in physics, we develop concepts and theorems, and present topics closely related to those of the first book. The first book is not necessary to the reader interested in Chapters I–V bis and already familiar with differential geometry nor to the reader interested in Chapter VI and already familiar with distribution theory. The first book emphasizes basics; the second, recent applications.

Applications are the lifeblood of concepts and theorems. They answer questions and raise questions. We have used them to provide motivation for concepts and to present new subjects that are still in the developmental stage. We have presented the applications in the forms of the problems with solutions in order to stress the questions we wish to answer and the fundamental ideas underlying applications. The reader may also wish to read only the questions and work out for himself the answers, one of the best ways to learn how to use a new tool. Occasionally we had to give a longer-than-usual introduction before presenting the questions. The organization of questions and answers does not follow a rigid scheme but is adapted to each problem.

This book is coordinated with the first one as follows:

1. The chapter headings are the same – but in this book, there is no Chapter VII devoted to infinite dimensional manifolds per se. Instead, the infinite dimensional applications are treated together with the corresponding finite dimensional ones and can be found throughout the book.
2. The subheadings of the first book have not been reproduced in the second one because applications often use properties from several sections of a chapter. They may even, occasionally, use properties from subsequent chapters and have been placed according to their dominant contribution.
3. **Page numbers in parentheses refer to the first book.** References to other problems in the present book are indicated [Problem Chapter Number First Word of Title].

The choice of problems was guided by recent applications of differential geometry to fundamental problems of physics, as well as by our personal interests. It is, in part, arbitrary and limited by time, space, and our desire to bring this project to a close.

The references are not to be construed as an exhaustive bibliography; they are mainly those that we used while we were preparing a problem or that we came across shortly after its completion.

The book has been enriched by contributions of Charles Doering, Harold Grosse, B. Kent Harrison, N.H Ibragimov, and Carlos Moreno, and collaborations with Ioannis Bakas, Steven Carlip, Gary Hamrick, Humberto La Roche and Gary Sammelmann. Discussions with S. Blau, M. Dubois-Violette, S.G. Low, L.C. Shepley, R. Stora, A. H. Taub, J. Tits and Jahja Trisnadi are gratefully acknowledged.

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CONTENTS

CONVENTIONS

(1) $\{f_n\}_{\mathbb{N}} := \{f_n : n \in \mathbb{N}\}$.

(2) Commutative diagram

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "h"'] & y \arrow[d, "g"] \arrow[dl, phantom, "\bullet"] \\ & z \end{tikzcd} \Leftrightarrow \begin{cases} f : x \to y,\; g : y \to z \\ h = g \circ f \end{cases}.$$

(3) Integer part: if $d/2 = 3.5$, then $[d/2] = 3$.

(4) $A\backslash B$ and A/B sometimes mean left and right coset, respectively; but usage varies and is determined in each context.

(5) Exterior product, exterior derivative, interior product

$$(\alpha \wedge \beta)(v_1, \dots, v_{p+q}) = \frac{1}{p!q!} \sum_{\Pi} (\operatorname{sign} \Pi) \Pi [\alpha(v_1, \dots, v_p) \\ \times \beta(v_{p+1}, \dots, v_{p+q})],$$

$$(\alpha \mathbin{\bar{\wedge}} \beta)(v_1, \dots, v_{p+q}) = \frac{1}{(p+q)!} \sum_{\Pi} (\operatorname{sign} \Pi) \Pi [\alpha(v_1, \dots, v_p)\beta \\ \times (v_{p+1}, \dots, v_{p+q})].$$

When operating on a p-form $\bar{d} = d/(p+1)$ and $\bar{\imath}_v = p i_v$. Note that Kobayashi and Nomizu (Vol. I, p. 35) use what we call $\bar{\wedge}$.

(6) Riemann tensor, Ricci tensor

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) v^\lambda = R_{\alpha\beta}{}^\lambda{}_\mu v^\mu,$$

i.e.

$$R_{\alpha\beta}{}^\lambda{}_\mu = \partial_\alpha \Gamma_\beta{}^\lambda{}_\mu - \partial_\beta \Gamma_\alpha{}^\lambda{}_\mu + \Gamma_\alpha{}^\rho{}_\mu \Gamma_\beta{}^\lambda{}_\rho - \Gamma_\beta{}^\rho{}_\mu \Gamma_\alpha{}^\lambda{}_\rho.$$

$$R_{\beta\mu} := R_{\alpha\beta}{}^\alpha{}_\mu.$$

These conventions agree with Misner, Thorne, and Wheeler and differ from those of our first book *Analysis, Manifolds and Physics*.

(7) The Dirac representation of the gamma matrices

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} \qquad \eta_{\mu\nu} = \mathrm{diag}(+,+,+,-)$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix}.$$

Majorana representation of the gamma matrices

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} \qquad \eta_{\mu\nu} = \mathrm{diag}(+,+,+,-)$$

$$\gamma_1' = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\gamma_3' = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma_4' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that in Vol. I, p. 176, we give the Dirac representation of the gamma matrices for $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$.

I. REVIEW OF FUNDAMENTAL NOTIONS OF ANALYSIS

1. GRADED ALGEBRAS

For applications and references see, for instance, Problems II 1, Supersmooth mappings and III 14, Graded bundles.

A $\mathbb{Z}_2$ **graded algebra** A is a vector space over the field of real or complex numbers which is the direct sum of two subspaces A_+ (called even) and A_- (called odd) graded algebra

$$A = A_+ \oplus A_-$$

endowed with an associative and distributive operation, called product, such that

$$A_r A_s = A_{r+s} (\text{mod. } 2), \quad r, s = 0, 1, \qquad A_0 = A_+, \qquad A_1 = A_- .$$

A $\mathbb{Z}_2$ graded algebra is called **graded commutative** if any two odd elements anticommute and if even elements commute with all others: graded commutative

$$ab = (-1)^{d(a)d(b)} ba, \quad a, b \in A$$

where $d(a) = r$ if $a \in A_r$ is the **parity** of a. parity

We shall consider in this section only graded commutative algebras, so we shall omit the word "commutative".

The algebras we shall use will be endowed with a locally convex Hausdorff topology for which sum and product are continuous operations.

For example, the exterior (Grassmann) algebra over a finite dimensional vector space X (p. 196) is a graded algebra.

A generalization used in physics, which we shall call a (Bryce) **DeWitt algebra** is the algebra B of formal series with a unit e and an infinite number of generators z^I, $I \in \mathbb{N}$, with the usual sum and product laws and the anticommutation property (Bryce) DeWitt algebra B

$$z^I z^J = -z^J z^I .$$

An element $a \in B$ is written (notion of convergence is irrelevant)

$$a = \sum_{p \in \mathbb{N}} a(p), \qquad a(p) = \frac{1}{p!} a_{I_1 \ldots I_p} z^{I_1} \ldots z^{I_p} .$$

body
soul
degree

$a(0) = a_0 e$ is called the **body** of a, $a_S = \Sigma_{p\geq 1} a(p)$ its **soul**. The numbers a_0, $a_{I_1 \dots I_p}$ are real or complex, $a_{I_1 \dots I_p}$ is totally antisymmetric in $I_1 \dots I_p$; the **degree** of $a(p)$ is p.
B_+ consists of the formal series which contain only terms of even degree, B_- consists of those with only terms of odd degree. B_+ is a subalgebra of B, while B_- is not.

Show that if $ab = 0$ for all $b \in B_+$ [resp. $b \in B_-$] then $a = 0$.
Are these properties true in a finitely generated Grassmann algebra?

Answer: If $ab = 0$ for all b belonging to B_+, or to the even part of a finitely generated Grassmann algebra we see that $a = 0$ by taking $b = e$. Suppose now $ab = 0$ for all $b \in B_-$. In particular $az^I = 0$ for each z^I, $I \in \mathbb{N}$. Suppose a coefficient $a_{I_1 \dots I_p} \neq 0$. Choose $z^J \not\in (z^{I_1}, \dots, z^{I_p})$. We have

$$a_{I_1 \dots I_p} z^{I_1} \dots z^{I_p} z^J \neq 0, \quad \text{hence} \quad a \neq 0.$$

If there is a finite number N of generators the hypothesis $ab = 0$ for all odd b implies only

$$a = cz^1 \dots z^N, \quad c \text{ arbitrary numbers.} \qquad \blacksquare$$

B is endowed with a locally convex, metrizable, Hausdorff topology by the countable family of seminorms (cf. for instance, p. 424)

$$\|a\|_{I_1 \dots I_p} = |a_{I_1 \dots I_p}|.$$

The sum of formal series (in particular the decomposition $B = B_+ \oplus B_-$) and their product have the required continuity.

Show that: The partial sums

$$a_m = \sum_{p=0}^{m} a(p)$$

converge to a, in the B-topology, when m tends to infinity.

Answer: If $\| \;\|_{I_1 \dots I_p}$ is a seminorm on B we have exactly

$$\|a - a_m\|_{I_1 \dots I_p} = 0 \quad \text{if} \quad m > p.$$

Let $f(x) = \Sigma_{n=0}^{\infty} c_n x^n$ be a numerical series with radius of convergence ρ. Show that $f(a) = \Sigma\, c_n a^n$ is a well-defined formal series in B, depending continuously on a, if $|a_0| < \rho$.

Answer: We have $a = a_0 e + a_S$, so

$$a^n = \sum_{p=0}^{n} C_n^p a_0^{n-p} a_S^p.$$

Since $f(x)$ is convergent for $|x| < \rho$, the numerical series $\Sigma_{n \geq p} c_n C_n^p a_0^{n-p}$ are convergent for $|a_0| < \rho$. We denote their sum by α_p and we write

$$\sum c_n a^n = \sum \alpha_p a_S^p = \sum b(q).$$

Each term on the right-hand side is well defined: $b(q)$ is obtained by finite sums and products since a term of order q arises from a_S^p only when $p \leq q$.
In a similar spirit one proves that the inverse in B of an element a with $a_0 \neq 0$ is the formal series:

$$a^{-1} = a_0^{-1}\left(1 + \sum_n (-1)^n (a_S/a_0)^n\right).$$

2. BEREZINIAN

A **graded matrix** on a graded algebra A is a rectangular array of elements of A, together with a parity attached to each row and column. A square graded matrix with p even and q odd rows and columns is said to be of **order (p, q)**.
A graded matrix $X = (x_j^i)$ is called **even** [resp. **odd**] if for all i, j:

graded matrix

order (p, q)

even, odd

$$\mathrm{d}(x_j^i) + \mathrm{d}(i\text{th column}) + \mathrm{d}(j\text{th row}) = 0 \text{ [resp. 1](mod. 2)};$$

one then says that $\mathrm{d}(X) = 0$[resp. $\mathrm{d}(X) = 1$].

1) *We shall always suppose that in a graded matrix X of order (p, q)*

$$X = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$$

the p even rows and columns are written first.
Give the conditions on the parities of the elements of R, S, T, U for X to be even [resp. odd].

Answer: The parities of the columns of R, T are even, those of S, U odd, while the rows of R, S are even and of T, U odd. Thus $\mathrm{d}(X) = 0$ if and only if the elements of R, U are even and the elements of T, S odd. The opposite condition holds for $\mathrm{d}(X) = 1$.

2) *Show that the space $Mat_{p,q}(A)$ of graded matrices of order (p, q) forms a $\mathbb{Z}_2$ graded algebra.*

Answer: The space $\mathrm{Mat}_{p,q}(A)$ obviously forms a vector space over $\mathbb{R}$ or $\mathbb{C}$ (like A), and each element can be written as the sum (usual sum of matrices) of an even and an odd one.
The elements in the product are defined by the usual law

$$XX' = \begin{pmatrix} RR' + ST' & RS' + SU' \\ TR' + UT' & TS' + UU' \end{pmatrix}.$$

It is easy to check that if X and X' have a parity, then

$$\mathrm{d}(XX') = \mathrm{d}(X) + \mathrm{d}(X')(\mathrm{mod.}\, 2).$$

3) *Let B be a DeWitt algebra. Denote by $GL_{p,q}(B)$ the multiplicative group of even invertible graded matrices of order (p, q).*

a) *Let* $X = \begin{pmatrix} R & S \\ T & U \end{pmatrix} \in Mat_{p,q}(B)$, $\mathrm{d}(X) = 0$.
Show that X is invertible if and only if R and U are invertible.

Berezinian

b) The determinant of a square matrix with even elements in B is well defined by the usual polynomial. The **Berezinian** of a matrix $X \in GL_{p,q}(B)$ is the mapping $GL_{p,q}(B) \to B$ given by

$$\mathrm{Ber}\, X = \det(R - SU^{-1}T)(\det U)^{-1}.$$

Show that Ber X is even valued and invertible.

c) *Show that*

$$\mathrm{Ber}(XY) = \mathrm{Ber}\, X\, \mathrm{Ber}\, Y.$$

Answer a: Under the hypothesis the body of X is

$$X_0 = \begin{pmatrix} R_0 & 0 \\ 0 & U_0 \end{pmatrix}$$

which is invertible if R_0 and U_0 are invertible.

Answer b: Ber X is even because R and U have even elements, S and T odd elements. It is invertible because

$$(\mathrm{Ber}\, X)_0 = (\det R_0)(\det U_0)^{-1} \neq 0.$$

Answer c: The proof is straightforward, in a number of steps (cf. for instance, Leites, p. 16) using in particular the decomposition

$$\begin{pmatrix} R & S \\ T & U \end{pmatrix} = \begin{pmatrix} \mathbb{1}_p & SU^{-1} \\ 0 & \mathbb{1}_q \end{pmatrix} \begin{pmatrix} R - SU^{-1}T & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} \mathbb{1}_p & 0 \\ U^{-1}T & \mathbb{1}_q \end{pmatrix}$$

and the fact that any matrix of the form

$$\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$$

is a product of matrices of the same type, but with a matrix A having only one nonzero element.

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B.S. DeWitt, *Supermanifolds* (Cambridge University Press, London, 1984) and Appendix of "The spacetime approach to quantum field theory", in: *Relativité, Groupes et Topologie II*, eds. B.S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).

D.A. Leites, "Introduction to the theory of supermanifolds", Russian Mathematical Surveys **35** (1980) 1.

3. TENSOR PRODUCT OF ALGEBRAS

A real algebra A is a vector space over $\mathbb{R}$ endowed with an associative product, $A \times A \to A$, bilinear with respect to the vector space structure (cf. a more general definition of algebra, p. 9).

1) *Suppose A and B are finite dimensional (as vector spaces) real algebras. Find a natural structure for $A \otimes B$.*

Answer: Let (e_i) and (e_α) be basis for A and B respectively. Then $e_1 \otimes e_\alpha$ is a basis for $A \otimes B$. We define products of such elements by

$$(e_i \otimes e_\alpha)(e_j \otimes e_\beta) = e_i e_j \otimes e_\alpha e_\beta ,$$

where juxtaposition denotes product in the relevant algebra.
The product of arbitrary elements $c = c^{i\alpha} e_i \otimes e_\alpha$, $d = d^{j\beta} e_j \otimes e_\beta$ is given by

$$cd = c^{i\alpha} d^{j\beta} (e_i e_j \otimes e_\alpha e_\beta).$$

It is easy to show that this product has the required properties and is independent of the choice of basis in A and B.

2) *Show that if A is a real algebra, then the complexified algebra $A \otimes \mathbb{C}$ is generated by the complexified vector space $A^{\mathbb{C}}$, that is, the vector space spanned by $a_i e_i$, e_i basis of A, $a^i \in \mathbb{C}$.*

Answer: A basis of $\mathbb{C}$ as a real vector space is $(1, \mathrm{i})$, and the algebra structure is determined by $\mathrm{i}^2 = -1$; a basis of $A \otimes \mathbb{C}$ is $(e_j \otimes 1, e_j \otimes \mathrm{i})$, which we can denote $(e_j, \mathrm{i}e_j)$ without breaking the product law.

3) *Example*: *Tensor products of matrices (see Problem I 4, Clifford algebras). Let A be the space of $n \times n$ matrices and B be the space of $m \times m$ matrices. Construct $a \otimes b$ for $a \in A$, $b \in B$.*

Answer: Let $a = (a^i_j)$, $b = (b^\alpha_\beta)$ be respectively an $n \times n$ and an $m \times m$ matrix. Then $a \otimes b = ((a \otimes b)^I_J)$, where the indices I and J stand for a pair of indices (i, α) or (j, β) and $(a \otimes b)^I_J = a^i_j b^\alpha_\beta$. Usually one orders pairs of indices as follows: $(1, 1), (1, 2), \ldots, (2, 1), (2, 2), \ldots$.

Note: In Problem IV 2, Obstruction, following Atiyah, Bott and Shapiro, we shall use the graded tensor product of two graded algebras defined as follows. Let $A = \sum_{i=0,1} A^i$ and $B = \sum_{i=0,1} B^i$ be two graded algebras.

graded tensor product

The **graded tensor product** $A \hat{\otimes} B$ is, by definition, the algebra whose underlying vector space is $\sum_{i,j=0,1} A^i \otimes B^j$ with multiplication defined by

$$(u \otimes x_i)(y_j \otimes v) = (-1)^{ij} u y_j \otimes x_i v,$$

where x_i, [resp. y_j] is an element of B^i [resp. A^j], u [resp. v] is an arbitrary element of A [resp. B].

The graded tensor product is again a graded algebra

$$(A \hat{\otimes} B)^k = \sum A^i \otimes B^j, \quad i + j = k \bmod 2.$$

For example, consider the odd element $e_1 \otimes 1 + 1 \otimes e_2$, its square $e_1^2 \otimes 1 + 1 \otimes e_2^2$ is even.

4. CLIFFORD ALGEBRAS

A supplement to this problem entitled "The isomorphism $H \otimes H \simeq M_4(\mathbb{R})$" can be found near the end of the book.

1. INTRODUCTION

Let V be a real $d = n + m$ dimensional vector space with a pseudo-euclidean scalar product g, invariant under the group $\mathrm{O}(n, m)$, given by $g = (g_{AB})$, $g_{AB} = 0$ if $A \neq B$, $g_{AA} = 1$, $A = 1, \ldots, n$, $g_{AA} = -1$ if $A = n + 1, \ldots, n + m$. The Clifford algebra (p. 65) $\mathscr{C}(n, m)$ is the real vector space endowed with an associative product, distributive with respect to addition, generated by d symbols[1] γ_A, a unit $\mathbb{1}$, and their products which satisfy

$$\gamma_A \gamma_B + \gamma_B \gamma_A = -2g_{AB}\mathbb{1}. \tag{1}$$

As a vector space, the Clifford algebra $\mathscr{C}(n, m)$ has dimension 2^d; a basis of $\mathscr{C}(n, m)$ is

$$(\mathbb{1}, \gamma_A, \gamma_{A_1}\gamma_{A_2}, \ldots, \gamma_{A_1}\gamma_{A_2}\cdots\gamma_{A_p}, \gamma_1\cdots\gamma_d) \quad \text{with} \quad A_i < A_j \quad \text{if} \quad i<j.$$

We set

$$\gamma_{A_1\ldots A_p} = \frac{1}{p!}\,\epsilon^{B_1\ldots B_p}_{A_1\ldots A_p}\gamma_{B_1}\cdots\gamma_{B_p};$$

thus $\gamma_{A_1\ldots A_p} = \gamma_{A_1}\gamma_{A_2}\cdots\gamma_{A_p}$ if $A_i < A_j$ for $i<j$. We set also

$$\gamma_{d+1} = \gamma_1\gamma_2\cdots\gamma_d.$$

It has been proved (p. 66) that the center of $\mathscr{C}(n, m)$ (the set of elements which commute with all elements) is trivial, namely multiples of the unit elements, $\mathbb{R}\,\mathbb{1}$, when d is even; this center is $\mathbb{R}\,\mathbb{1} \oplus \mathbb{R}\gamma_{d+1}$ when d is odd. The complexified Clifford algebra $\mathscr{C}^c(n, m)$ is the vector space $\mathscr{C}(n, m)\otimes\mathbb{C}$: it is the vector space over the complex numbers generated by the previous elements $\mathbb{1}, \gamma_A, \ldots, \gamma_1\cdots\gamma_d$. Note that $\mathscr{C}(n, m)$ and $\mathscr{C}(n', m')$ have the same complex extension if $n + m = n' + m'$.

Case $d = n + m = 2p$ even

General theorems of algebra (cf. Bourbaki II, ch. 9, § 4, n° 2) enable one to show that $\mathscr{C}^c(n, m)$ is isomorphic to the algebra $M_{2^p}(\mathbb{C})$ over the complex numbers of $2^p \times 2^p$ complex matrices, that is, of linear transformations of the complex vector space $\mathbb{C}^{2^p}$. A set of such $2^p \times 2^p$ matrices Γ_A representing the fundamental elements γ_A are called **gamma matrices of O(n, m) or gamma matrices for $\mathscr{C}(n, m)$.** They satisfy the identities

gamma matrices of O(n, m), for $\mathscr{C}(n, m)$

$$\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = -2g_{AB}\mathbb{1}_{2^p}.$$

It can be proved that if (Γ_A) and (Γ'_A) are two sets of gamma matrices of O(n, m) there exists an invertible $2^p \times 2^p$ matrix M such that

$$\Gamma'_A = M\Gamma_A M^{-1}, \qquad A = 1, \ldots, n+m.$$

The algebra over the real numbers generated by the (eventually complex) matrices Γ_A is a faithful representation of $\mathscr{C}(n, m)$, $n + m = 2p$.

[1]We prefer to use the notation γ_A instead of e_A as on p. 64 in order to distinguish the element of the Clifford algebra from the element e_A of V.

Case $d = 2p+1$ *odd*
One also finds a representation of $\mathscr{C}^c(n,m)$ on a complex vector space of dimension 2^p, but this representation is unfaithful. The algebra $\mathscr{C}^c(n,m)$, $n+m=2p+1$, is isomorphic to $M_{2^p}(\mathbb{C}) \oplus M_{2^p}(\mathbb{C})$. The corresponding representation of $\mathscr{C}(n,m)$ may or may not be unfaithful (cf. 2c) in the following subsection).

In all cases it is interesting to know whether there exists a representation on a real vector space of the real Clifford algebra; the problem is to find, if possible, real matrices Γ_A. In this problem we shall show how the gamma matrices can be constructed recursively, and we shall study their structure, which will give a direct proof of properties quoted before, as well as other details such as the periodicity, in $(m-n)$ of period 8, of $\mathscr{C}(m,n)$.

2. GAMMA MATRICES IN LOW DIMENSIONS

a) *Show that* $\mathscr{C}(1,0)$ *is isomorphic to* $\mathbb{C}$, *and give an algebra isomorphic to* $\mathscr{C}(0,1)$.

Pauli matrices

b) *We recall the* **Pauli matrices** ($-\sigma_2$ *is also used*)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Give a faithful representation of $\mathscr{C}(2,0)$ *and a real, faithful representation of* $\mathscr{C}(0,2)$ *and* $\mathscr{C}(1,1)$.

c) *Is it possible to deduce from the Pauli matrices a faithful representation of* $\mathscr{C}(0,3)$? *Of* $\mathscr{C}(3,0)$?

Answer 2a: $\mathscr{C}(0,1)$ is generated by a unit $\mathbb{1}$ and an element γ such that $\gamma^2 = 1$. It is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.
$\mathscr{C}(1,0)$ is generated by a unit $\mathbb{1}$ and an element γ such that $\gamma^2 = -1$. It is isomorphic to $\mathbb{C}$.

Answer 2b: The Pauli matrices anticommute and satisfy

$$\begin{aligned} &\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{1}_2, \\ &\sigma_1\sigma_2 = -\mathrm{i}\sigma_3, \quad \sigma_2\sigma_3 = -\mathrm{i}\sigma_1, \quad \sigma_3\sigma_1 = -\mathrm{i}\sigma_2, \\ &\sigma_1\sigma_2\sigma_3 = -\mathrm{i}\mathbb{1}_2. \end{aligned} \tag{2}$$

The algebra $\mathscr{C}(2,0)$ is generated by $\mathbb{1}_2$ and two of the matrices $\mathrm{i}\sigma_1$, $\mathrm{i}\sigma_2$, $\mathrm{i}\sigma_3$; a basis of this 4-dimensional real vector space is

$$\mathbb{1}_2, \mathrm{i}\sigma_1, \mathrm{i}\sigma_2, \mathrm{i}\sigma_3.$$

The product laws (2) show that $\mathscr{C}(2,0)$ is isomorphic to the algebra $\mathbb{H}$ of quaternions (see also Problem V 10, Invariant geometries).
The algebra $\mathscr{C}(0,2)$ can be generated by $\mathbb{1}_2$ and the two real matrices σ_1, σ_3; a basis of this vector space is $\mathbb{1}_2, \sigma_1, \sigma_3, \mathrm{i}\sigma_2$. It is identical to the vector space $M_2(\mathbb{R})$ of 2×2 real matrices; $\mathscr{C}(0,2)$ is isomorphic to the algebra $M_2(\mathbb{R})$.
The algebra $\mathscr{C}(1,1)$ can be generated by $\mathbb{1}_2$, one of the two real matrices σ_1 or σ_3, and $\mathrm{i}\sigma_2$; it is also isomorphic to $M_2(\mathbb{R})$.

Answer 2c: The Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are gamma matrices for $\mathscr{C}(0,3)$; together with $\mathbb{1}_2$ they generate an algebra which is, by formula (2), an 8-dimensional vector space on the reals, isomorphic to $\mathscr{C}(0,3)$. The matrices $\mathrm{i}\sigma_1, \mathrm{i}\sigma_2, \mathrm{i}\sigma_3$ are gamma matrices for $\mathscr{C}(3,0)$, but the quaternionic algebra $\mathbb{H}$ they generate, together with $\mathbb{1}_2$, is only a 4-dimensional real vector space; the representation is unfaithful. A faithful representation is obtained by taking two copies of $\mathbb{H}$.

3. GAMMA MATRICES IN ARBITRARY DIMENSIONS

a) *Set, if* $n+m=d=2p$

$$\gamma_{d+1}=\gamma_1\ldots\gamma_d;$$

show that

$$(\gamma_{d+1})^2=(-1)^{n+p}\mathbb{1}.$$

Thus if Γ_A are gamma matrices for $\mathscr{C}(n,m)$, and

$$\Gamma_{d+1}=\Gamma_1\ldots\Gamma_d,$$

then

$$(\Gamma_{d+1})^2=(-1)^{n+p}\mathbb{1}_{2^p}.$$

Answer 3a: Straightforward computation using the fundamental relation (1).

b). *Show that if* Γ_a, $a=1,\ldots,d$, *is a set of gamma matrices of* $\mathrm{O}(n,m)$, $n+m=2p$ *then* $(\Gamma_a, k\Gamma_{d+1})$ *is a set of gamma matrices of* $\mathrm{O}(n,m+1)$ *if* $k^2=(-1)^{n+p}$ *[resp. of* $\mathrm{O}(n+1,m)$ *if* $k^2=(-1)^{n+p+1}$*].*

Answer 3b: Γ_{d+1} anticommutes with all Γ_a if d is even, and $k^2(\Gamma_{d+1})^2=\mathbb{1}_{2^p}$ if $k^2=(-1)^{n+p}$ [resp. $k^2(\Gamma_{d+1})^2=-\mathbb{1}_{2^p}$ if $k^2=(-1)^{n+p+1}$].

c) *Show that if* $n+m=d=2p$ *is even then we have (the sign* $\simeq$ *means isomorphic to)*

$$\mathscr{C}(n, m) \otimes \mathscr{C}(n', m') \simeq \mathscr{C}(n + n', m + m') \quad \text{if} \quad k^2 = (-1)^{n+p} = 1,$$
$$\mathscr{C}(n, m) \otimes \mathscr{C}(n', m') \simeq \mathscr{C}(n + m', m + n') \quad \text{if} \quad k^2 = (-1)^{n+p} = -1.$$

Construct a representation by real matrices of $\mathscr{C}(1, 3)$.

Answer 3c: The $d + d'$ elements

$$(\gamma_a \otimes \mathbb{1}', \gamma_{d+1} \otimes \gamma_{a'}), \qquad a = 1, \ldots, d, a' = 1, \ldots, d' \tag{3}$$

satisfy the relation (cf. Problem I 3, Tensor product)

$$(\gamma_a \otimes 1')^2 = (\gamma_a)^2 \otimes 1' = (1 \otimes 1') g_{aa},$$
$$(\gamma_{d+1} \otimes \gamma_{a'})^2 = (-1)^{n+p} (1 \otimes 1') g_{a'a'}$$

and anticommute, because γ_a anticommute with γ_{d+1}. The algebra generated by these elements, with the product deduced from the products in $\mathscr{C}(n, m)$ and $\mathscr{C}(n', m')$, is therefore a representation of $\mathscr{C}(n + n', m + m')$ [resp. $\mathscr{C}(n + m', n' + m)$], depending on the sign of $k^2 = (-1)^{n+p}$]. It is an isomorphism because $\mathscr{C}(n, m) \otimes \mathscr{C}(n', m')$ has dimension $2^d \times 2^{d'} = 2^{d+d'}$, the same dimension as $\mathscr{C}(n + n', m + m')$.
We have in particular (here $n = 1, p = 1$)

$$\mathscr{C}(1, 3) \simeq \mathscr{C}(1, 1) \otimes \mathscr{C}(0, 2).$$

We obtain real gamma matrices for $\mathscr{C}(1, 3)$ by using formulas (3) with gamma matrices of $\mathscr{C}(1, 1)$ and $\mathscr{C}(0, 2)$; these give the Dirac matrices in the so-called **Majorana representation** see [Problem I 8, Weyl]. (Note that the tensor product of two matrices (A^i_j), $(B^{i'}_{j'})$ has elements $A^i_j B^{i'}_{j'} = (A \otimes B)^{ii'}_{jj'}$.)

Majorana representation

$$\gamma_0 = -\mathrm{i}\sigma_2 \otimes 1_2 = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix}$$

$$\gamma_1 = \sigma_1 \otimes 1_2 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}$$

$$\gamma_2 = \sigma_3 \otimes \sigma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}$$

$$\gamma_3 = \sigma_3 \otimes \sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}.$$

d) The previous construction shows that gamma matrices for an arbitrary group $O(n, m)$ *can be obtained from the Pauli matrices* σ_j *or* $\mathrm{i}\sigma_j$. *Show that*:
i) *All the constructed gamma matrices, as well as the algebra they generate* ($\mathbb{1}$ *not included*) *have a zero trace.*
Prove this property directly.

ii) *The matrices* $\mathbb{1}_{2^p}, \Gamma_A, \ldots, \Gamma_{1,\ldots,2^p}$ *are linearly independent on* $\mathscr{C}$.
iii) *The matrices* Γ_A *of* $\mathrm{O}(n, m)$ *are antihermitian for* $A = 1, \ldots, n$ *and hermitian for* $A = n + 1, \ldots, d$, *in the choice of Clifford algebra defined by* (1).

Answer 3d i: The Pauli matrices have a zero trace, and the trace of a tensor product is the tensor product of the traces.
To prove directly that the trace of $\Gamma_{A_1 \ldots A_p}$ is zero we proceed as follows.
If p is even

$$\Gamma_{A_1} \ldots \Gamma_{A_p} = -\Gamma_{A_2} \ldots \Gamma_{A_p} \Gamma_{A_1} \quad \text{for all } A_i \text{ different,}$$

but

$$\operatorname{Tr} \Gamma_{A_1} \ldots \Gamma_{A_p} = \operatorname{Tr} \Gamma_{A_2} \ldots \Gamma_{A_p} \Gamma_{A_1} = -\operatorname{Tr} \Gamma_{A_1} \ldots \Gamma_{A_p} = 0.$$

If p is odd

$$\Gamma_{A_1} \ldots \Gamma_{A_p} = \epsilon \Gamma_{A_1} \ldots \Gamma_{A_p} \Gamma_{A_{p+1}} \Gamma_{A_{p+1}} = -\epsilon \Gamma_{A_{p+1}} \Gamma_{A_1} \ldots \Gamma_{A_p} \Gamma_{A_{p+1}},$$
$$\epsilon = \pm 1$$

and the same argument as in the case p even gives

$$\operatorname{Tr} \Gamma_{A_1} \ldots \Gamma_{A_p} = 0.$$

Answer 3d ii: Suppose there exist complex numbers not all zero such that

$$\lambda \mathbb{1}_{2^p} + \lambda^A \Gamma_A + \ldots + \lambda^{1 \ldots 2p} \Gamma_1 \ldots \Gamma_{2p} = 0.$$

By taking the trace one obtains $\lambda = 0$; by taking the trace with product of the gamma matrix Γ_A one obtains $\lambda^A = 0$, and so on.
Answer 3d iii: The Pauli matrices, gamma matrices of $\mathscr{C}(0, 2)$, are all hermitian, while the $i\sigma_j$ are antihermitian. A tensor product of matrices which are either hermitian or antihermitian is either hermitian or antihermitian. The conclusion follows by inspection.

4. PERIODICITY MODULO 8

a) *Express* $\mathscr{C}(0, n + 8)$, *for arbitrary* n, *in terms of tensor products of* $\mathscr{C}(2, 0), \mathscr{C}(0, 2)$ *and* $\mathscr{C}(n, 0)$.
Show that

$$\mathscr{C}(0, n + 8) \simeq M_{16}(\mathbb{R}) \otimes \mathscr{C}(0, n), \tag{4}$$

where $M_{16}(\mathbb{R})$ *is the algebra of real* 16×16 *matrices (using the relation* $\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R})$, *where* $\mathbb{H}$ *is the algebra of quaternions).*

b) *Show that, if* $m > n$

$$\mathscr{C}(n, m) \simeq M_{2^n}(\mathbb{R}) \otimes \mathscr{C}(0, m - n). \tag{5}$$

These two relations show that all the Clifford algebras are determined through the properly euclidean ones, and their classification depends upon $n - m$ modulo 8. Write the classification table.

Answer 4a: Using the result of paragraph 4 with $k^2 = -1$ we obtain:

$$\begin{aligned}\mathscr{C}(0, n+8) &\simeq \mathscr{C}(0,2) \otimes \mathscr{C}(n+6, 0)\\ &\simeq \mathscr{C}(0,2) \otimes \mathscr{C}(2,0) \otimes \mathscr{C}(0, n+4) \simeq \cdots\\ &\simeq \mathscr{C}(0,2) \otimes \mathscr{C}(2,0) \otimes \mathscr{C}(0,2) \otimes \mathscr{C}(2,0) \otimes \mathscr{C}(0,n)\\ &\simeq M_2(\mathbb{R}) \otimes \mathbb{H} \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \otimes \mathscr{C}(0,n).\end{aligned}$$

The tensor product of vector spaces is commutative, up to an isomorphism. It is easy to see that if $M_n(\mathbb{R})$ denotes the algebra of $n \times n$ real matrices

$$M_n(\mathbb{R}) \otimes M_{n'}(\mathbb{R}) \simeq M_{nn'}(\mathbb{R}),$$

and it is known that (the proof is given in a problem at the end of the book)

$$\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R});$$

formula (4) follows.

Answer 4b: The proof of (5) is analogous:

$$\mathscr{C}(n,m) \simeq (\otimes \mathscr{C}(1,1))^n \otimes \mathscr{C}(0, m-n) \simeq M_{2^n}(\mathbb{R}) \otimes \mathscr{C}(0, m-n).$$

The table reads, with $d = n + m$, abbreviating $M_p(\mathbb{R})$ to $\mathbb{R}(p)$, and $\mathbb{H}(p)$ denoting the algebra of $p \times p$ quaternionic matrices:

$(m-n) \bmod 8$	0	1	2	3
$\mathscr{C}(n,m)$	$\mathbb{R}(2^{d/2})$	$\mathbb{R}(2^{(d-1)/2}) \oplus \mathbb{R}(2^{(d-1)/2})$	$\mathbb{R}(2^{d/2})$	$\mathbb{C}(2^{(d-1)/2})$
$(m-n) \bmod 8$	4	5	6	7
$\mathscr{C}(n,m)$	$\mathbb{H}(2^{(d/2)-1})$	$\mathbb{H}(2^{(d-1)/2-1}) \oplus \mathbb{H}(2^{(d-1)/2-1})$	$\mathbb{H}(2^{(d/2)-1})$	$\mathbb{C}(2^{(d-1)/2})$

Note that for d even the Clifford algebra is isomorphic to either the algebra of real matrices or the algebra of quaternionic matrices: for $n = 1$, $\mathscr{C}(1,m)$ admits a real representation for $m = 1, 2, 3 \bmod. 8$, thus $d = 2, 3, 4, 10, 11, 12$, etc.

Note that $\mathscr{C}(n,m)$ and $\mathscr{C}(m,n)$ are *not* isomorphic, unless $|m-n| = 0 \bmod 4$.

The explicit construction of the table is done in [M. Berg et al., see p. 39] where the Clifford algebra is not (1) but $\gamma_A \gamma_B + \gamma_B \gamma_A = 2 g_{AB} \mathbb{1}$.

5. GRADING OF A CLIFFORD ALGEBRA

a) *Give a $\mathbb{Z}_2$-grading to a Clifford algebra.*
b) *Show that the even parts of $\mathscr{C}(n,m)$ and $\mathscr{C}(m,n)$ are isomorphic.*
c) *Show that*

$$\mathscr{C}_+(p,q) = \mathscr{C}(p, q-1).$$

Answer 5a: A Clifford algebra $\mathscr{C}$ is a direct sum

$$\mathscr{C} = \mathscr{C}_+ \oplus \mathscr{C}_-, \tag{6}$$

where $\mathscr{C}_+$, called the even part of $\mathscr{C}$, is generated by the even products of elements of a basis:

$$\mathbb{1}, \gamma_{A_1}\gamma_{A_2}, \ldots, \gamma_{A_1} \cdots \gamma_{A_p}, \ldots \quad A_i < A_j,$$

and $\mathscr{C}_-$, called the odd part, is generated by the odd products:

$$\gamma_A, \gamma_{A_1}\gamma_{A_2}\gamma_{A_3}, \ldots \quad A_i < A_j.$$

Note $\gamma_1 \ldots \gamma_d \in \mathscr{C}_+$ if d is even; $\gamma_1 \ldots \gamma_d \in \mathscr{C}_-$ if d is odd. We have obviously

$$\mathscr{C}_+\mathscr{C}_+ \subset \mathscr{C}_+, \quad \mathscr{C}_-\mathscr{C}_- \subset \mathscr{C}_+, \tag{7}$$

$$\mathscr{C}_+\mathscr{C}_- \subset \mathscr{C}_-, \quad \mathscr{C}_-\mathscr{C}_+ \subset \mathscr{C}_-. \tag{8}$$

The laws (7), (8) show that the decomposition (6) gives a $\mathbb{Z}_2$ grading to $\mathscr{C}$. Note that this $\mathbb{Z}_2$ grading is not $\mathbb{Z}_2$ commutative (Problem I 1).

Answer 5b: Suppose that $\mathscr{C}(n, m)$ is the Clifford algebra generated by $\mathbb{1}$ and γ_A, $A = 1, \ldots, n+m$, with a product such that

$$\gamma_A\gamma_B + \gamma_B\gamma_A = -2g_{AB}\mathbb{1}.$$

Consider a vector space V over the reals generated by $\mathbb{1}$, elements denoted $\mathrm{i}\gamma_A$, $A = 1, \ldots, n+m$, and their products defined through the product in the algebra $\mathscr{C}(n, m)$ and the products of complex numbers, namely,

$$(\mathrm{i}\gamma_A)(\mathrm{i}\gamma_B) \equiv -\gamma_A\gamma_B. \tag{9}$$

We endow the vector space V with the structure of an algebra through the product so defined. We deduce from (9) that

$$(\mathrm{i}\gamma_A)(\mathrm{i}\gamma_B) + (\mathrm{i}\gamma_B)(\mathrm{i}\gamma_A) = 2g_{AB}\mathbb{1},$$

hence V, together with the product we have defined, is the algebra $\mathscr{C}(m, n)$. It results also from (9) that the algebras $\mathscr{C}_+(m, n)$ and $\mathscr{C}_+(n, m)$ are identical:

$$\mathscr{C}_+(n, m) = \mathscr{C}_+(m, n). \tag{10}$$

Answer 5c: Let $\mathbb{1}$ and $(\gamma_A) = (\gamma_0, \gamma_a)$, $a = 1, \ldots, p+q-1$ be generators of $\mathscr{C}(p, q)$. Then

$$\gamma_0^2 = -\mathbb{1}$$

and the γ_a's are, together with $\mathbb{1}$, generators of $\mathscr{C}(p, q-1)$. On the other hand, $\mathscr{C}_+(p,q)$ is generated by $\mathbb{1}$ and the even products $\gamma_0\gamma_a$, $\gamma_a\gamma_b$. We deduce the isomorphism

$$\mathscr{C}_+(p,q) = \mathscr{C}(p, q-1) \tag{11}$$

from the relation

$$(\gamma_0\gamma_a)(\gamma_0\gamma_b) = -\gamma_0^2\gamma_a\gamma_b = \gamma_a\gamma_b.$$

The equalities (11) and (10) imply

$$\mathscr{C}(p, q-1) = \mathscr{C}(q, p-1),$$

which can also be read from the table of §4.

5. CLIFFORD ALGEBRA AS A COSET OF THE TENSOR ALGEBRA

Let V be a real vector space with a symmetric bilinear form $V \times V \to \mathbb{R}$ given by $(v,w) \mapsto b(v,w)$. Let us denote by $\mathfrak{C} = \sum_{k\geq 0}(\otimes V)^k$ the tensor algebra of V. We say that the subset $\mathfrak{T}$ of the algebra $\mathfrak{C}$ is generated by the subset $\{\tau_a,\ a \in A \text{ family of indices}\}$ of $\mathfrak{C}$ if the elements of $\mathfrak{T}$ are the finite sums

$$i = \sum t^a \otimes \tau_a, \quad t^a \in \mathfrak{C},$$

1) *Show that the subset $\mathfrak{T}$ of $\mathfrak{C}$ is a left ideal of $\mathfrak{C}$.*
2) *Let $\mathfrak{T}$ be generated by elements of the form*

$$u \otimes u + b(u,u).$$

Show that the coset space $\mathfrak{C} \setminus \mathfrak{T}$ is isomorphic to
a) *The exterior algebra of V if $b = 0$.*
b) *The Clifford algebra $\mathscr{C}(V,g)$ if b is a nondegenerate pseudo-euclidean scalar product g.*

left ideal

Answer 1: It is obvious from the definition that if $i \in \mathfrak{T}$ then $t \otimes i \in \mathfrak{T}$ for every $t \in \mathfrak{C}$. Thus $\mathfrak{J}$ is a **left ideal** of $\mathfrak{C}$ (p. 8).
Answer 2: There is a natural map from V into $\mathfrak{C} \setminus \mathfrak{T}$, namely $v \mapsto [v]$, where $[v]$ is the equivalence class of $v \in V \subset \mathfrak{C}$ in $\mathfrak{C} \setminus \mathfrak{T}$. It is easy to see that the equivalence class $[v \otimes w]$ depends only on $[v]$ and $[w]$ and thus defines a product for these elements of $\mathfrak{C} \setminus \mathfrak{T}$:

$$[v][w] \overset{\text{def}}{=} [v \otimes w].$$

To prove the isomorphisms stated in a) and b) we compute

$$\begin{aligned}[v][w]+[w][v] &= [v \otimes w]+[w \otimes v] = [v \otimes w + w \otimes v] \\ &= [(v+w)\otimes(v+w)] - [v \otimes v] - [w \otimes w].\end{aligned}$$

On the other hand, by the definition of $\mathfrak{T}$

$$[v \otimes v] + [b(v, v)] = 0; \tag{1}$$

thus, since b is bilinear and symmetric, we have

$$[v][w]+[w][v] = -2b(v, w). \tag{2}$$

a) If $b = 0$ relation (2) shows that there is an isomorphism between $\mathfrak{S}\backslash\mathfrak{T}$ and the exterior algebra of V.
b) If b is an inner product, there is an isomorphism between V, the subspace of $\mathfrak{S}\backslash\mathfrak{T}$ defined by $\{[v], v \in V\}$ and the subspace $\{\gamma_v, v \in V\}$ of the Clifford algebra $\mathscr{C}(V, g)$.

6. FIERZ IDENTITY

Here we use the generic term spinors for both spinors and pinors. For the definition of spinors and cospinors see pp. 415, 418 and [Problem IV 2, Obstruction].

The following identity is useful in many computations, particularly in supergravity. *Let φ, ψ be arbitrary spinors on a $d = 2p$ dimensional vector space with pseudo-euclidean product g, and $\bar\lambda$, $\bar\mu$ arbitrary cospinors. Prove the equality of the two scalars* (juxtaposition denotes the duality product of a spinor and a cospinor)

$$(\bar\lambda\varphi)(\bar\mu\psi) = \frac{1}{2p}\sum_I (\bar\lambda\Gamma_I\psi)(\bar\mu\Gamma^I\varphi), \tag{1}$$

where Γ_I, $I = 1, \ldots, 2^d$ denotes a basis $\mathbb{1}_{2^p}$, Γ_A, $\Gamma_{AB}, \ldots, \Gamma_1 \ldots \Gamma_d$ of a representation of the Clifford algebra $\mathscr{C}$ by $2^p \times 2^p$ complex (or real) matrices and $\Gamma^I = (\Gamma_I)^2\Gamma_I$.

Answer: Let S be the (complex) vector space of spinors, S^* its dual. We know (Problem I 4) that the complexified Clifford algebra $\mathscr{C}^c$ is isomorphic to the algebra $S^* \otimes S$ of linear mappings $S \to S$.
The left-hand side of (1) defines a quadrilinear map $(S^* \times S) \times (S^* \times S) \to \mathbb{C}$ by $(\bar\lambda, \varphi; \bar\mu, \psi) \mapsto (\bar\lambda\varphi)(\bar\mu\phi)$. This is also a quadrilinear map if considered as $(\bar\lambda, \psi; \bar\mu, \varphi) \mapsto (\bar\lambda\varphi)(\bar\mu\psi)$. It can thus be written as

$$(\bar\lambda\varphi)(\bar\mu\psi) = \sum_{I,J} \alpha^{IJ}(\bar\lambda\Gamma_I\psi)(\bar\mu\Gamma_J\varphi),$$

where the α^{IJ} are numbers which we shall determine. The above equality can be written, if a, b denote indices for components in S and S^* respectively, as

$$\sum_{I,J} \alpha^{IJ} \Gamma^a_{I\ b} \Gamma^c_{J\ d} = \delta^a_d \delta^c_b.$$

We determine the α^{IJ} by taking traces of products with various Γ_K, as follows:

$$\sum_{I,J} \alpha^{IJ} \Gamma_{K\ a}^{\ b} \Gamma^a_{I\ b} \Gamma^c_{J\ d} = \Gamma_{K\ a}^{\ b} \delta^a_d \delta^c_b. \tag{2}$$

We know that

$$\operatorname{tr} \Gamma_K \Gamma_I = 0 \qquad \text{if } K \neq I$$

so (2) reduces to

$$\sum_J \alpha^{KJ} \operatorname{tr}(\Gamma_K)^2 \Gamma_J = \Gamma_K$$

from which we deduce

$$\begin{aligned} &\alpha^{KJ} = 0 \qquad \text{if } K \neq J, \\ &\alpha^{KK} = \{\operatorname{tr}(\Gamma_K)^2\}^{-1}. \end{aligned}$$

Each matrix Γ_K is a product of gamma matrices, and hence has square $\pm \mathbb{1}_{2^p}$. If

$$(\Gamma_K)^2 = \varepsilon_K \mathbb{1}_{2^p},$$

then

$$\operatorname{tr}(\Gamma_K)^2 = 2^p \varepsilon_K.$$

The conclusion follows.

Example $d = 4$: we take as matrices Γ_I

$$\mathbb{1}_4,\ \Gamma_\alpha,\ \Gamma_{\alpha\beta},\ \xi = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3,\ \xi\Gamma_\alpha.$$

Then

$$\begin{gathered} (\mathbb{1}_4)^2 = \mathbb{1}_4, \quad (\Gamma_\alpha)^2 = -g_{\alpha\alpha} \mathbb{1}_4, \quad (\Gamma_{\alpha\beta})^2 = -g_{\alpha\alpha} g_{\beta\beta} \mathbb{1}_4, \\ \xi^2 = -\mathbb{1}_4, \quad (\xi^2 \Gamma_\alpha)^2 = -g_{\alpha\alpha} \mathbb{1}_4. \end{gathered}$$

The formula can then be written:

$$\begin{aligned} (\bar{\lambda}\varphi)(\bar{\mu}\psi) = \tfrac{1}{4}\{&(\bar{\lambda}\psi)(\bar{\mu}\varphi) - (\bar{\lambda}\xi\psi)(\bar{\mu}\xi\varphi) - (\bar{\lambda}\Gamma_\alpha\psi)(\bar{\mu}\Gamma^\alpha\varphi) \\ &- (\bar{\lambda}\xi\Gamma_\alpha\psi)(\bar{\mu}\xi\Gamma^\alpha\varphi) - (\bar{\lambda}\Gamma_{\alpha\beta}\psi)(\bar{\mu}\Gamma^{\alpha\beta}\varphi)\}. \end{aligned}$$

7. PIN AND SPIN GROUPS

To be in agreement with modern physics notation we shall modify somewhat the terminology used (p. 67). We shall give the fundamental properties of $\mathrm{Spin}(n, m)$ and $\mathrm{Pin}(n, m)$, for arbitrary dimensions. We treat in the first paragraph the case $n + m$ even, which is easier, using a matrix representation. In paragraph two, we treat the general case by using the grading of the Clifford algebra, as done by Atiyah, Bott and Shapiro in the euclidean case.

Recall the following definitions. Let $\mathrm{GL}(V)$ denote the group of isomorphisms (linear, invertible maps) from a real vector space V onto itself; V has dimension $d = n + m$ and g is a pseudo-euclidean scalar product on V of signature (n, m) (i.e., with n plus signs and m minus signs):

Orthogonal group $\mathrm{O}(n, m) = \{L \in \mathrm{GL}(V), g(Lu, Lv) = g(u, v)\}$.

Special orthogonal group $\mathrm{SO}(n, m) = \{L \in \mathrm{O}(n, m), \det L = 1\}$.

Identity (connected) component of $\mathrm{O}(n, m)$: $\mathrm{SO}_0(n, m)$.

(special) orthogonal group identity component

In the euclidean case (n or $m = 0$), $\mathrm{SO}(n, m) = \mathrm{SO}_0(n, m)$ but in the general case they are not equal. In the Lorentz case, $\mathrm{SO}(m, 1)$ and $\mathrm{SO}(1, m) = L(m + 1)$ have two connected components. The Lorentz group $\mathrm{O}(1, m)$ has four connected components defined as follows:

Let $\mathrm{SO}_0(1, m) \equiv L_0(m + 1)$:

$L_0(m+1)$: $L^0_0 > 0$ $\det L = +1$ **orthochronous proper Lorentz group**

orthochronous proper Lorentz group

$tL_0(m+1)$: $L^0_0 < 0$ $\det L = -1$ (t a time reflection)

$sL_0(m+1)$: $L^0_0 > 0$ $\det L = -1$ (s a space reflection)

$stL_0(m+1)$: $L^0_0 < 0$ $\det L = +1$

$L_0(m + 1)$ and $stL_0(m + 1)$ are both orientation preserving.

1. CASE $d = n + m$ EVEN.

Consider the group of elements Λ of the Clifford algebra $\mathscr{C}(n, m)$ which are invertible and satisfy the relation

$$\Lambda \gamma_v \Lambda^{-1} \in \mathscr{C}_V, \qquad \forall v \in V \tag{1.1}$$

where $\gamma_v = v^A \gamma_A$ is the element of $\mathscr{C}$ corresponding to $v \in V$ and $\mathscr{C}_V$ the linear subspace of $\mathscr{C}$ isomorphic to V spanned by the γ_v's.

$\{e_A; A = 1, \ldots, d\}$ is a basis for V and $\{\gamma_A; A = 1, \ldots, d\}$ a basis for the isomorphic space $\mathscr{C}_V$, whose products satisfy

$$\gamma_A \gamma_B + \gamma_B \gamma_A = -2 g_{AB} \mathbb{1}.$$

Clifford group

1a) *Show that there exists a non-injective homomorphism $\mathcal{H}$ from the group defined by* (1.1) *into the* $O(n, m)$ *group of* (V, g). *Show that this homomorphism is not surjective if $n + m = d$ is odd. Show it is surjective if d is even, the group is then called the* **Clifford group** $\Gamma(n, m)$.

Answer 1a: Let $\{e_A,\ A = 1, \ldots, d\}$ be a basis of V; consider the $d \times d$ matrix L with elements $L^B{}_A$ defined* by

$$\Lambda \gamma_A \Lambda^{-1} = L_A^B \gamma_B. \tag{1.2}$$

L_A^B are real numbers, since $\Lambda \gamma_A \Lambda^{-1} \in \mathscr{C}_V$ by hypothesis and $\{\gamma_B;\ B = 1, \ldots, d\}$ is a basis of $\mathscr{C}_V$.

The mapping $\Lambda \to L$ is a mapping into $O(n, m)$ because if Λ satisfies (1.2) for $A = 1, \ldots, d$, we have also

$$\Lambda \gamma_A \gamma_C \Lambda^{-1} = \left(L_A^B \gamma_B\right)\left(L_C^D \gamma_D\right),$$

and, using $\gamma_A \gamma_D + \gamma_D \gamma_A = -2 g_{AD} \mathbb{1}$, we find

$$g_{AC} = L_A^B L_C^D g_{BD} \quad \text{i.e., } L \in O(n, m).$$

The mapping $\Lambda \to L$ is a homomorphism because

$$\Lambda' \Lambda \gamma_A \Lambda^{-1} \Lambda'^{-1} = \Lambda' L_A^B \gamma_B \Lambda'^{-1} = L'^C_B L_A^B \gamma_C;$$

thus $\Lambda' \Lambda \mapsto L' L$. This homomorphism cannot be injective, since for any $k \in \mathbb{R}$, Λ and $k\Lambda$ have the same image in $O(n, m)$.

parity operator

It is not surjective if d is odd: the (**space time**) **parity operator** $P = (P_A^A = -1,\ P_A^B = 0$ if $B \neq A)$ belongs to $O(n, m)$, but there exists no $\Lambda \in \mathscr{C}(n, m)$ which satisfies

$$\Lambda \gamma_A \Lambda^{-1} = -\gamma_A, \quad A = 1, \ldots, d$$

since this would imply

$$\Lambda \gamma_{d+1} + \gamma_{d+1} \Lambda = 0, \quad \gamma_{d+1} = \gamma_1 \cdots \gamma_d;$$

but for d odd we know that γ_{d+1} commutes with $\Lambda \in \mathscr{C}(n, m)$, hence $\Lambda \gamma_{d+1} = 0$ which is impossible since Λ and γ_{d+1} are invertible.

It can be proved the homomorphism is surjective if d is even as follows: let Γ_A be a set of gamma matrices, and let $L \in O(n, m)$. Then it is clear that

$$\Gamma'_B = L_B^A \Gamma_A$$

*Note that (p. 176) we use the convention $L^B{}_A \gamma_B$.

is also a set of gamma matrices. Therefore, if $n + m$ is even there exists a matrix M such that:

$$\Gamma'_B = M\Gamma_B M^{-1}.$$

If d is even, the matrices Γ_A can be chosen all real or all with quaternion elements. The matrices Γ'_A are of the same nature as the Γ_A, i.e., all real or all with quaternion elements, and so is M, which represents then also an element of the Clifford group.

1b) *Show that if γ_v is invertible, then γ_v satisfies* (1.1). *What is the corresponding* $\mathrm{O}(n, m)$ *transformation? Show that this transformation changes the orientation if d is even.*

Answer 1b: For all γ_w and invertible γ_v (i.e., v such that $g(v, v) \neq 0$), we have

$$\begin{aligned}\gamma_v \gamma_w \gamma_v^{-1} &= -2g(v, w)\gamma_v^{-1} - \gamma_w \\ &= \frac{2g(v, w)}{g(v, v)}\gamma_v - \gamma_w;\end{aligned}$$

the mapping $L: V \to V$ corresponding to γ_v is given by

$$w \mapsto Lw = \frac{2g(v, w)}{g(v, v)} v - w.$$

Let for instance $v = (1, 0, \ldots, 0)$, then whatever the signature of g:

$$(w^1, w^2, \ldots, w^d) \mapsto (w^1, -w^2, \ldots, -w^d).$$

Take for instance a Lorentzian signature $g_{00} = 1$, $g_{AA} = -1$ for $A \neq 0$, and $\gamma_v = \gamma_0$, then

$$L_A^B = 2g_{0A}\delta_0^B - \delta_A^B$$

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = P \times t, \qquad t \text{ being time reflection.}$$

In all cases, the mapping $V \to V$ corresponding to γ_v is the negative of the symmetry with respect to the hyperplane orthogonal to v:

$$\mathcal{H}(\gamma_v) = Ps_v, \qquad P \text{ parity operator.}$$

The determinant of the transformation is 1 if d is odd, and -1 if d is even.

1c) *Show that the kernel of $\mathcal{H}: \Gamma(n, m) \to \mathrm{O}(n, m)$, $n + m = d$ even, is $\mathbb{R}\mathbb{1}$, $\mathbb{1}$ the unit of $\mathscr{C}(n, m)$.*

Answer 1c: $\Lambda \in \Gamma(n, m)$ is in the kernel of $\mathscr{H}$ if $\mathscr{H}(\Lambda) = \mathbb{1}$, the unit of $\mathrm{O}(n, m)$ that is

$$\Lambda \gamma_A = \gamma_A \Lambda, \qquad A = 1, \ldots, d;$$

hence Λ commutes with $\mathscr{C}(n, m)$. The result follows [cf. Problem I 4, Clifford algebras].

We deduce from this result that if two elements $\Lambda_1, \Lambda_2 \in \Gamma(n, m)$ have the same image L under $\mathscr{H}$, then

$$\Lambda_1 = c\Lambda_2, \qquad c \in \mathbb{R}.$$

pinor group

1d) If $d = n + m$ is even, $\mathscr{C}(n, m)$ and hence $\Gamma(n, m)$ admit a faithful representation by $2^p \times 2^p$ real or complex matrices. The **pinor group Pin (n, m)** is by definition the subgroup of $\Gamma(n, m)$ such that, Λ being in such a matrix representation,

$$|\det \Lambda| = 1.$$

Show that Pin(n, m) is a double covering of $\mathrm{O}(n, m)$ *under the homomorphism* $\mathscr{H}$: $\Lambda \mapsto L$.

Answer 1d: It is obvious that if $\Lambda \in \mathrm{Pin}(n, m)$, then also $-\Lambda \in \mathrm{Pin}(n, m)$ because

$$\det \Lambda = \det(-\Lambda),$$

since Λ is a $2^p \times 2^p$ matrix.

We show that the kernel of the homomorphism $\mathscr{H}\colon \mathrm{Pin}(n, m) \to \mathrm{O}(n, m)$ is indeed $\pm \mathbb{1}_{2^p}$, as follows:

$$\mathscr{H}(\Lambda) = \mathbb{1},$$

implies [cf. 1c]

$$\Lambda = c\mathbb{1}_{2^p}, \qquad c \in \mathbb{R};$$

and

$$|\det \Lambda| = 1$$

implies then

$$c^{2^p} = 1, \qquad \text{hence } c = \pm 1.$$

On the other hand, the mapping $\mathscr{H}\colon \mathrm{Pin}(n, m) \to L$ is surjective because the mapping $\mathscr{H}\colon \Gamma(n, m) \to L$ is surjective [cf. 1a] and because, given $M \in \Gamma(n, m)$, there exists $k \in \mathbb{R}$ such that

$$|\det(kM)| = 1, \qquad \text{namely } k = \pm |\det M|^{-1/2^p}.$$

Therefore $\mathscr{H}$: $\mathrm{Pin}(n, m) \rightarrow \mathrm{O}(n, m)$ is a double covering.

The **spinor group Spin(n, m)** is the subgroup of elements of $\mathrm{Pin}(n, m)$ whose image under $\mathscr{H}$ is an orientation preserving element of $\mathrm{O}(n, m)$. It is a double covering of $\mathrm{SO}(n, m)$. spinor group

1e) An $\mathrm{O}(n, m)$ transformation can be shown to be the product of a certain number of symmetries with respect to nonisotropic hyperplanes [Riesz pp. 74–75, Artin pp. 129–130]

$$L = S_{v_1} S_{v_2} \ldots S_{v_n} .$$

Show that if $\Lambda \in \Gamma(n, m)$, $n + m$ even, Λ is a product of elements γ_v.
Denote by $\boldsymbol{\Gamma}_+(n, m)$ [resp. $\boldsymbol{\Gamma}_-(n, m)$] the elements of the Clifford group which are in the even part $\mathscr{C}_+(n, m)$ of the Clifford algebra [resp. the odd part $\mathscr{C}_-(n, m)$]. $\Gamma_\pm(n, m)$
Show that

$$\Gamma(n, m) = \Gamma_+(n, m) \cup \Gamma_-(n, m).$$

That is, an element of the Clifford group is either even or odd. Show the same is true for the pinor group and that the spinor group is the even part.

Answer 1e: Let $L \in \mathrm{O}(n, m)$, $n + m$ even:

$$L = S_{v_1} \ldots S_{v_N} = (PS_{v_1} \ldots PS_{v_N}) P^N ,$$

since the parity operator $P = -\mathbb{1} = P^{-1}$ commutes with each S_v.
We know that if d is even we have

$$\mathscr{H}(\gamma_{d+1}) = P, \qquad \gamma_{d+1} = \gamma_1 \ldots \gamma_d ;$$

hence, due to the results of 1b), the element $\Lambda \in \Gamma(n, m)$, given by

$$\Lambda = \gamma_{v_1} \ldots \gamma_{v_N} \gamma_{d+1}^N ,$$

satisfies

$$\mathscr{H}(\Lambda) = L .$$

The general solution of the above equation is

$$\Lambda = c\gamma_{v_1} \ldots \gamma_{v_N} \gamma_{d+1}^N = \gamma_{cv_1} \ldots \gamma_{v_N} \gamma_{d+1}^N .$$

We see on this formula that an element of $\Gamma(n, m)$ is a product of elements in $\mathscr{C}_V$. In particular, we see that Λ is either even (when N is even) or odd (if N is odd), since γ_{d+1} is even.
The element L is in $\mathrm{SO}(n, m)$ if it is orientation preserving, hence if N is even. If L changes the orientation of V, then N is odd.

The pinor group is a subgroup of the Clifford group, hence satisfies also

$$\mathrm{Pin}(n, m) = \mathrm{Pin}_+(n, m) \cup \mathrm{Pin}_-(n, m),$$

where

$$\mathrm{Pin}_+(n, m) \equiv \mathrm{Spin}(n, m)$$

is constituted of even elements of the Clifford algebra. We have

$$\mathscr{H}(\mathrm{Spin}(n, m)) = \mathrm{SO}(n, m).$$

$\mathrm{Spin}^+(1, m)$

In the case of Lorentzian signature, one defines $\mathbf{Spin}^+(1, m)$ as the subgroup of $\mathrm{Spin}(1, m)$, whose image by $\mathscr{H}$ preserves also the time orientation.

2. CASE OF $m + n = d$ OF ARBITRARY PARITY.

Inspired by the results of Section 1, we introduce the automorphism α of $\mathscr{C}(n, m)$ defined by

$$\alpha|_{\mathscr{C}_V} = -\,\mathrm{Id}.$$

Using the grading of $\mathscr{C}(n, m)$ [cf. I 4, Clifford algebras], we have

$$\begin{aligned} \alpha(\Lambda_+) &= \Lambda_+ \quad &&\text{if } \Lambda_+ \in \mathscr{C}_+(n, m) \\ \alpha(\Lambda_-) &= -\Lambda_- \quad &&\text{if } \Lambda_- \in \mathscr{C}_-(n, m). \end{aligned}$$

2a) *Show that the elements $\Lambda \in \mathscr{C}(n, m)$ that are invertible and satisfy the relation*

$$\alpha(\Lambda)\gamma_v \Lambda^{-1} \equiv \mathrm{ad}\, \Lambda\gamma_v \in \mathscr{C}_V \tag{2.1}$$

Clifford group
twisted adjoint representative

form a group called the **Clifford group** $\Gamma(n, m)$. *Its representation* ad *on* $\mathscr{C}_V$ *is called its* **twisted adjoint representation.**
Show that the Clifford group so defined contains, in the case of d even, the Clifford group defined in 1. *They will be shown to be identical* in §2*b*.

Answer 2a: α is an automorphism of $\mathscr{C}(n, m)$, namely

$$\alpha(\Lambda_1 \Lambda_2) = \alpha(\Lambda_1)\alpha(\Lambda_2),$$

and we have

$$\mathrm{ad}(\Lambda_1 \Lambda_2) = \mathrm{ad}\, \Lambda_1 \circ \mathrm{ad}\, \Lambda_2.$$

Therefore $\Gamma(n, m)$ defined by (2.1) is a group and ad a representation of this group on $\mathscr{C}_V$.

We have defined in 1, in the case of d even, $\Gamma(n, m)$ as the group of invertible elements of $\mathscr{C}(n, m)$ that satisfy

$$\Lambda\gamma_v \Lambda^{-1} \in \mathscr{C}_V, \tag{2.2}$$

and we have shown that Λ is then either even or odd. Hence (2.2) implies (2.1). We shall show in §2b that elements of $\mathscr{C}(n, m)$ satisfying (2.1) are also either even or odd, whatever is the parity of $n+m$, hence the two definitions of the Clifford group in the case $n+m$ even are identical.

2b) i) *Show that the mapping* $\tilde{\mathscr{H}}$: $\Lambda \mapsto L$ *defined by*

$$\alpha(\Lambda)\gamma_A\Lambda^{-1} = L_A^B\gamma_B \tag{2.3}$$

is a surjective homomorphism

$$\tilde{\mathscr{H}}: \Gamma(n, m) \to \mathrm{O}(n, m).$$

ii) *Show that the kernel of* $\tilde{\mathscr{H}}$ *is* $\mathbb{R}^\times \mathbb{1}$, *with* $\mathbb{R}^\times$ *multiplicative group of nonzero real numbers.*
Show that if $\Lambda \in \Gamma(n, m)$, *it is either even or odd.*
iii) *Show that* $\tilde{\mathscr{H}}(\gamma_v)$ *changes the orientation of* V *whether* $m+n$ *is even or odd.*

Answer 2b: i) A proof analogous to the one given in 1 shows that (2.3) implies that

$$L = (L_A^B) \in \mathrm{O}(n, m),$$

and that $\tilde{\mathscr{H}}$: $\Lambda \mapsto L$ is a homomorphism.

To show that $\tilde{\mathscr{H}}$ is surjective, we remark that

$$\alpha(\gamma_1)\gamma_A(\gamma_1)^{-1} = \begin{cases} -\gamma_A & \text{if } A = 1, \\ \gamma_A & \text{if } A \neq 1. \end{cases} \tag{2.4}$$

Thus $\gamma_1 \in \Gamma(n, m)$. The transformation of V corresponding to γ_1 is the reflection in the hyperplane perpendicular to e_1, and not its negative as in question 1. Applying the same argument to any element of the basis of $\mathscr{C}_V$ we obtain all the orthogonal reflections in hyperplanes of V and thus generate $\mathrm{O}(n, m)$.
ii) Let $\Lambda_0 \in \operatorname{Ker} \tilde{\mathscr{H}}$. Eq. (2.3) implies

$$\alpha(\Lambda_0)\gamma_A = \gamma_A\Lambda_0. \tag{2.5}$$

We decompose Λ_0 into its even and odd parts

$$\Lambda_0 = \Lambda_{0+} + \Lambda_{0-};$$

then (2.5) reads

$$(\Lambda_{0+} - \Lambda_{0-})\gamma_A = \gamma_A(\Lambda_{0+} + \Lambda_{0-}).$$

The odd part of this equation yields $\Lambda_{0-}\gamma_A = -\gamma_A\Lambda_{0-}$. However no odd element can anticommute with all the γ_A's; therefore $\Lambda_{0-} = 0$.

Therefore

$$\Lambda_{0+}\Lambda = \Lambda\Lambda_{0+}, \qquad \text{for all } \Lambda \in \mathscr{C}(n,m).$$

We know that the center of $\mathscr{C}(n,m)$ is $\mathbb{R}\mathbb{1}$ if $n+m$ is even and $\mathbb{R}\mathbb{1}+\mathbb{R}\gamma_{d+1}$ if $n+m=d$ is odd. But if d is odd, γ_{d+1} is also odd; hence the only elements in $\mathscr{C}_+(n,m)$ that commute with $\mathscr{C}(n,m)$ are in $\mathbb{R}\mathbb{1}$. Those elements which are in $\Gamma_+(n,m)$ are of the form $c\mathbb{1}$, $c \in \mathbb{R}^\times$. We deduce from the above results, by a method analogous to the one used in 1, that every $\Lambda \in \Gamma(n,m)$ is a product of N elements in $\mathscr{C}_V \cap \Gamma(n,m)$ hence is either even or odd.
iii) $\tilde{\mathscr{H}}(\gamma_v)$ changes the orientation, since it is a symmetry.

norm on Γ

2c) *Define a "***norm***" ν on $\Gamma(n,m)$ by the condition*

$$\nu(\Lambda) = \left|g(v_1,v_1)\dots g(v_N,v_N)\right|$$

for

$$\Lambda = \gamma_{v_1}\dots\gamma_{v_N}$$

and

$$\nu(c\mathbb{1}) = c^2.$$

Show that the subset of elements of $\Gamma(n,m)$ such that

$$\nu(\Lambda) = 1$$

pinor group

spinor group

is a subgroup of $\Gamma(n,m)$. It is called the **pinor group** Pin(n,m). *Show that the homomorphism $\tilde{\mathscr{H}}$ restricted to* Pin(n,m) *is a* 2*-to-*1 *surjective homomorphism onto* O(n,m). *The* **spinor group** Spin(n,m) *is the subgroup of elements whose image by $\tilde{\mathscr{H}}$ is in* SO(n,m), *that is, of elements that are in $\mathscr{C}_+(n,m)$.*
Show that if $n+m$ is even the groups just defined are the ones defined in 1.

Answer 2c: ν is a homomorphism from $\Gamma(n,m)$ into $\mathbb{R}^\times$:

$$\nu(\Lambda_1\Lambda_2) = \nu(\Lambda_1)\nu(\Lambda_2),$$

hence the subset of elements of $\Gamma(n,m)$ such that $\nu(\Lambda) = 1$ is a subgroup of $\Gamma(n,m)$; we call if Pin(n,m).
The restriction of $\tilde{\mathscr{H}}$ to Pin(n,m) is surjective (cf. §2b) because, given $L \in \mathrm{O}(n,m)$, there exists $\Lambda \in \Gamma(n,m)$ such that

$$\tilde{\mathscr{H}}(\Lambda) = L$$

and, given Λ, there exists $k \in \mathbb{R}^\times$ such that

$$\nu(k\Lambda) = 1, \quad \text{namely, } k = \pm\big(\nu(\Lambda)\big)^{-1/2};$$

such $k\Lambda$ project also on L by $\tilde{\mathscr{H}}$.
The previous reasoning indicates that the homomorphism $\tilde{\mathscr{H}}$ is 2 to 1.

To avoid discussions about the uniqueness of $\nu(\Lambda)$, we proceed directly:

$$\tilde{\mathscr{H}}(\Lambda_1) = \tilde{\mathscr{H}}(\Lambda_2) \text{ if and only if } \tilde{\mathscr{H}}\left(\Lambda_1 \Lambda_2^{-1}\right) = \mathbb{1}.$$

We know that the center of $\tilde{\mathscr{H}}$ in $\Gamma(n, m)$ is $\mathbb{R}\mathbb{1}$; hence

$$\Lambda_1 \Lambda_2^{-1} = c\mathbb{1}, \quad c \in \mathbb{R}.$$

$$\nu(c\mathbb{1}) = 1 \text{ is equivalent to } c^2 = 1, \text{ i.e., } c = \pm 1.$$

In the case $n + m$ even, the pinor and spinor groups just defined are the same as those defined in 1 because

$$\gamma_v^2 = -g(v, v)\mathbb{1};$$

therefore, if we consider the matrix representation of $\mathscr{C}(n, m)$, we have

$$\begin{aligned} \Gamma_v^2 &= -g(v, v)\mathbb{1}_{2^p} \\ |\det \Gamma_v|^2 &= (g(v, v))^{2^p}. \end{aligned}$$

If

$$\Lambda = \Gamma_{v_1} \ldots \Gamma_{v_N},$$

then

$$|\det \Lambda| = \left|g(v_1, v_1) \ldots g(v_N, v_N)\right|^{2^{p-1}}.$$

Then the conditions $|\det \Lambda| = 1$ and $\nu(\Lambda) = 1$ are identical.

3. EXAMPLES

3a) *Construct* $\Gamma(0, 1)$ *and* $\Gamma(1, 0)$ *and their* Pin *and* Spin *subgroups. Show that* $\mathrm{Pin}(0, 1) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $\mathrm{Pin}(1, 0) \simeq \mathbb{Z}_4$. *Show that* $\mathrm{Spin}(0, 1) \simeq \mathrm{Spin}(1, 0) \simeq \mathbb{Z}_2$.

3b) *Construct* $\mathrm{Pin}(0, 2)$, $\mathrm{Pin}(2, 0)$, $\mathrm{Pin}(1, 1)$ *and their* Spin *subgroups.* $\mathrm{Pin}(0, n)$ *and* $\mathrm{Pin}(n, 0)$ *are often labelled* $\mathbf{Pin}^+(n)$ *and* $\mathbf{Pin}^-(n)$. Pin$^\pm$

3c) *Show that* $\mathrm{Spin}(n, m)$ *is isomorphic to* $\mathrm{Spin}(m, n)$. $\mathrm{Pin}(n, m)$ *is not necessarily isomorphic to* $\mathrm{Pin}(m, n)$.

Answer 3a: In Problem I 4, Clifford, we have computed $\mathscr{C}(0, 1)$ *and* $\mathscr{C}(1, 0)$. The generators of $\mathrm{Pin}^\pm(1)$ are $\mathbb{1}$ and γ such that $\gamma^2 = \pm 1$. Their multiplication tables are

	$\mathbb{1}$	γ	$-\mathbb{1}$	$-\gamma$
$\mathbb{1}$	$\mathbb{1}$	γ	$-\mathbb{1}$	$-\gamma$
γ	γ	$\mathbb{1}$	$-\gamma$	$-\mathbb{1}$
$-\mathbb{1}$	$-\mathbb{1}$	$-\gamma$	$\mathbb{1}$	γ
$-\gamma$	$-\gamma$	$-\mathbb{1}$	γ	$\mathbb{1}$

$\mathrm{Pin}(0, 1) \equiv \mathrm{Pin}^+(1)$

	$\mathbb{1}$	γ	$-\mathbb{1}$	$-\gamma$
$\mathbb{1}$	$\mathbb{1}$	γ	$-\mathbb{1}$	$-\gamma$
γ	γ	$-\mathbb{1}$	$-\gamma$	$\mathbb{1}$
$-\mathbb{1}$	$-\mathbb{1}$	$-\gamma$	$\mathbb{1}$	γ
$-\gamma$	$-\gamma$	$\mathbb{1}$	γ	$-\mathbb{1}$

$\mathrm{Pin}(1, 0) \equiv \mathrm{Pin}^-(1)$

which are identical to the multiplication tables of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$, respectively.

	0,0	0,1	1,0	1,1
0,0	0,0	0,1	1,0	1,1
0,1	0,1	0,0	1,1	1,0
1,0	1,0	1,1	0,0	0,1
1,1	1,1	1,0	0,1	0,0

$\mathbb{Z}_2 \times \mathbb{Z}_2$

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\mathbb{Z}_4$

The following table gives $\tilde{\mathscr{H}}(\Lambda) = L$ for $\Lambda \in \mathrm{Pin}^{\pm}(1)$ and makes it possible to select the elements of $\mathrm{Pin}^{\pm}(1)$ which are in $\mathrm{Spin}^{\pm}(1)$. Since $\pm\gamma\gamma\gamma^{-1} = \pm\gamma$ whether $\gamma \in \mathrm{Pin}^{+}(1)$ or $\gamma \in \mathrm{Pin}^{-}(1)$,

Λ	$\tilde{\mathscr{H}}(\Lambda)$	$\det L$
$\pm\mathbb{1}$	1	1
$\pm\gamma$	-1	$-1.$

The elements in $\mathrm{Pin}^{\pm}(1)$ such that $\tilde{\mathscr{H}}(\Lambda)$ is orientation preserving are $\pm\mathbb{1}$. Hence

$$\mathrm{Spin}^{\pm}(1) \text{ is isomorphic to } \mathbb{Z}_2.$$

Answer 3b: We first show that all invertible elements of a $\mathscr{C}$, a Clifford algebra $\mathscr{C}(n, m)$ with $n + m = 2$, that are either even or odd are in its Clifford group. Let γ_1 and γ_2 be the generators of $\mathscr{C}(n, m)$ with

$$\gamma_1^2 = \epsilon_1 \mathbb{1}, \qquad \gamma_2^2 = \epsilon_2 \mathbb{1}, \qquad \epsilon_1, \epsilon_2 = \pm 1$$
$$\Lambda_+ \in \mathscr{C}_+ \leftrightarrow \Lambda_+ = a\mathbb{1} + b\gamma_1\gamma_2$$
$$\Lambda_- \in \mathscr{C}_- \leftrightarrow \Lambda_- = c\gamma_1 + d\gamma_2.$$

Λ_- is invertible, with $\Lambda_-^{-1} = \gamma_{ce_1+de_2}^{-1}$ if $ce_1 + de_2$ is not isotropic; i.e., $c^2\epsilon_1 + d^2\epsilon_2 \neq 0$. It is easy to check that Λ_+ is invertible if $a^2 + b^2\epsilon_1\epsilon_2 \neq 0$. (Note that both conditions always hold for the algebras $\mathscr{C}(0, 2)$ and $\mathscr{C}(2, 0)$). It is straightforward to check that the same conditions imply the existence of 2×2 matrices $L_\pm$ such that $\tilde{\mathscr{H}}(\Lambda_\pm) = L_\pm$, while

$$\Lambda_+\gamma_A = \gamma_A\Lambda_+ \quad \text{or} \quad \Lambda_-\gamma_A = -\gamma_A\Lambda_-, \qquad A = 1, 2;$$

hence that $\Lambda_+ \in \Gamma_+(n, m)$, $\Lambda_- \in \Gamma_-(n, m)$.

In Problem I 4, Clifford we have given basis for $\mathscr{C}(0, 2)$, $\mathscr{C}(2, 0)$, and $\mathscr{C}(1, 1)$ in terms of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$. Let $\gamma_1, \gamma_2 \in \mathscr{C}_V(n, m)$, $n + m = 2$. An element of $\mathrm{Pin}(n, m)$, $n + m = 2$ is of the form either $\Lambda = a\mathbb{1} + b\gamma_1\gamma_2$ or $\Lambda = c\gamma_1 + d\gamma_2$, $a, b, c, d \in \mathbb{R}$ and such that $\det \Lambda = \pm 1$.

	γ_1	γ_2	$\gamma_1\gamma_2$	$\tilde{\mathscr{H}}(\gamma_1) = \mathscr{H}(\gamma_2)$	$\tilde{\mathscr{H}}(\gamma_2) = \mathscr{H}(\gamma_1)$
$\mathrm{Pin}(0, 2) \equiv \mathrm{Pin}^{+}(2)$	σ_1	σ_3	$i\sigma_2$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathrm{Pin}(2, 0) \equiv \mathrm{Pin}^{-}(2)$	$i\sigma_1$	$i\sigma_2$	$i\sigma_3$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathrm{Pin}(1, 1)$	σ_1	$i\sigma_2$	σ_3	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

For $n+m$ even, both $\tilde{\mathscr{H}}$ and $\mathscr{H}$ are surjective and they coincide when restricted to $\mathrm{Spin}(n, m)$. The elements of $\mathrm{Spin}(0, 2)$ and $\mathrm{Spin}(2, 0)$ are

$$\Lambda = a\mathbb{1} + b\gamma_1\gamma_2, \quad a, b \in \mathbb{R}, \quad a^2 + b^2 = 1.$$

The elements of $\mathrm{Spin}(1, 1)$ are

$$\Lambda = a\mathbb{1} + b\gamma_1\gamma_2, \quad a, b \in \mathbb{R}, \quad a^2 - b^2 = \pm 1.$$

Answer 3c: We have shown that $\mathrm{Spin}(n, m)$ is a subgroup of $\Gamma_+(n, m)$. The isomorphism $\mathrm{Spin}(n, m) \simeq \mathrm{Spin}(m, n)$ results from $\mathscr{C}_+(n, m) = \mathscr{C}_+(m, n)$ [cf. I 4, Clifford]. We deduce from the periodicity modulo 8 of the Clifford algebras that

$$\mathscr{C}(m, n) = \mathscr{C}(n, m) \quad \text{if } m - n = 4k, \qquad k \in \mathbb{N}.$$

If $m - n \neq 0 \bmod 4$, the group $\mathrm{Pin}(m, n)$ and $\mathrm{Pin}(n, m)$ may not be isomorphic; we have seen examples in the previous section. In particular, $\mathrm{Pin}^+(n)$ and $\mathrm{Pin}^-(n)$ are not isomorphic if $n \neq 4k$, $k \in \mathbb{N}$.

References for Problems I 7 to I 11 may be found at the end of Problem I 11.

8. WEYL SPINORS, HELICITY OPERATOR; MAJORANA PINORS, CHARGE CONJUGATION

1. WEYL SPINORS. HELICITY OPERATOR

Suppose $d = n + m = 2p$, $\gamma_A\gamma_B + \gamma_B\gamma_A = -2g_{AB}\mathbb{1}$.

1a) *Show that the element $\gamma_1 \dots \gamma_d$ of the Clifford algebra $\mathscr{C}(n, m)$ defines a linear operator $\xi \colon S \to S$ where S is a space of spinors; i.e., the space of a linear representation of* $\mathrm{Spin}(n, m)$ *by $2^p \times 2^p$ matrices, $p = [d/2]$.*

1b) *Show that ξ is an hermitian or antihermitian operator with eigenvalues ± 1 or $\pm \mathrm{i}$. Decompose S as a direct sum of eigenspaces*

$$S = S_+ \oplus S_-.$$

Show that $\mathrm{Spin}(n, m)$ *preserves S_+[resp. S_-]. The operator ξ is called the* **helicity operator**, sometimes **parity** operator. helicity parity operator

1c) A **Weyl spinor** of **positive helicity** [resp. **negative**] is defined as an equivalence class (ψ, Λ) with $\psi \in S_+$[resp. S_-] and $\Lambda \in \mathrm{Spin}$ [cf. Problem IV 2, Obstruction]. Weyl spinor helicity

If ψ is a differentiable field of Weyl spinors of some given helicity on an *orientable, pseudo-riemannian manifold* (M, g), *show that*

$$\Gamma^A \nabla_A \psi$$

is a Weyl spinor field of opposite helicity.

1d) *Give a representation of* $\mathscr{C}_+(n, m)$ *and* $\mathrm{Spin}(n, m)$ *by* $2^{p-1} \times 2^{p-1}$ *matrices.*

Answer 1a: We choose a representation of $\mathscr{C}(n, m)$ on $\mathbb{C}^{2^p}$ by gamma matrices and set

$$\xi = \Gamma_1 \ldots \Gamma_d.$$

ξ is a representative of a spinor–cospinor (p. 418), invariant under a change of spin frames because it anticommutes with the gamma matrices for d even and thus commutes with even products.

$$\Lambda^{-1} \xi \Lambda = \xi, \qquad \Lambda \in \mathrm{Spin}(n, m).$$

Answer 1b: We have

$$\tilde{\xi} = \tilde{\Gamma}_d \ldots \tilde{\Gamma}_1, \quad \sim \text{ is transposition and complex conjugation,}$$

so

$$\tilde{\xi} = \pm \xi.$$

We know [cf. Problem I 4, Clifford §3] that

$$\xi^2 = (-1)^{p+n} \mathbb{1}_{2^p};$$

thus the image of $k\xi$ with $k^2 = (-1)^{n+p}$ by the homomorphism $\mathscr{H}$ is the parity operator $P = -\mathbb{1}$; it preserves the orientation since $n + m$ is even, hence $P \in \mathrm{SO}(n, m)$ and $\xi \in \mathrm{Spin}(n, m)$.

Let $\psi \in S$. Then

$$\xi \psi = \lambda \psi$$

implies that

$$\xi^2 \psi = (-1)^{p+n} \psi = \lambda \xi \psi = \lambda^2 \psi;$$

thus

$$\begin{aligned} \lambda^2 &= (-1)^{p+n} \\ \lambda &= \pm 1 \quad \text{if } n + p \text{ is even.} \\ \lambda &= \pm \mathrm{i} \quad \text{if } n + p \text{ is odd.} \end{aligned}$$

One denotes by S_+ [resp. S_-] the eigenspace with eigenvalue $+1$ or $+\mathrm{i}$ [resp. -1 or $-\mathrm{i}$]. Since ξ is hermitian or antihermitian, $S = S_+ \oplus S_-$.

Elements $\Gamma_- \in \mathscr{C}_-$[resp. $\Gamma_+ \in \mathscr{C}_+$] anticommute [resp. commute] with ξ. Thus, if $\xi\psi = \lambda\psi$,

$$\xi\Gamma_-\psi = -\Gamma_-\xi\psi = -\lambda\Gamma_-\psi \qquad [\text{resp. } \xi\Gamma_+\psi = \lambda\Gamma_+\psi].$$

Hence,

$$\mathscr{C}_+S_+ = S_+, \qquad \mathscr{C}_+S_- = S_-, \qquad \mathscr{C}_-S_+ = S_-, \qquad \mathscr{C}_-S_- = S_+.$$

Answer 1c: If ψ takes its values in S_+, the same is true of $\nabla_A\psi$. Therefore, $\Gamma^A\nabla_A\psi$ takes its values in S_-. The massless Dirac equation

$$\Gamma^A\nabla_A\psi = 0$$

is meaningful for Weyl spinors of given helicity while the Dirac equation with a mass term

$$\Gamma^A\nabla_A\psi + m\psi = 0, \qquad m \text{ scalar}$$

is not.

Answer 1d (Kirillov p. 278): We choose a basis of $S = \mathbb{C}^{2^p}$ adapted to the splitting $S = S_+ \oplus S_-$. In such a basis ξ takes the form

$$\xi = k\begin{pmatrix} \mathbb{1}_{2^{p-1}} & 0 \\ 0 & -\mathbb{1}_{2^{p-1}} \end{pmatrix}, \qquad k^2 = (-1)^{n+p},$$

and the gamma matrices, anticommuting with ξ, are of the form

$$\Gamma_A = \begin{pmatrix} 0 & C_A \\ D_A & 0 \end{pmatrix},$$

where C_A and D_A are $2^{p-1} \times 2^{p-1}$ matrices. Since [cf. Problem I 4, Clifford §5] $\Gamma_A = \epsilon_A \tilde{\Gamma}_A$, $\epsilon_A = \pm 1$, we have

$$D_A = \epsilon_A \tilde{C}_A.$$

$\mathscr{C}_+(n, m)$ is generated by even products of the gamma matrices; hence every element $\lambda \in \mathscr{C}_+(n, m)$ can be represented by a $2^p \times 2^p$ matrix Λ with $2^{p-1} \times 2^{p-1}$ blocks Λ^+, Λ^-:

$$\Lambda = \begin{pmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{pmatrix}.$$

$\lambda \mapsto \Lambda^+$ and $\lambda \mapsto \Lambda^-$ are two representations of $\mathscr{C}_+(n, m)$, sometimes called **half-spinor representation.** half-spinor representation

The matrices Λ that are in Spin(n, m) are of the form [cf. Problem I 7, Pin and Spin groups]

$$\Lambda = \Gamma_{v_1} \ldots \Gamma_{v_{2q}}$$

with

$$\Gamma_{v_i} = v^{A_i} \Gamma_{A_i}$$

and satisfy

$$|\det \Lambda| = 1.$$

Therefore,

$$\Lambda = v^{A_1} \dots v^{A_{2q}} \begin{pmatrix} C_{A_1} \dots \tilde{C}_{A_{2q}} & 0 \\ 0 & D_{A_1} \dots \tilde{D}_{A_{2q}} \end{pmatrix}$$

and

$$\left| v^{A_1} \dots v^{A_{2q}} \right|^{2^p} |\det C_{A_1} \dots C_{A_{2q}}| \, |\det D_{A_1} \dots D_{A_{2q}}| = 1.$$

Due to the relation between C_A and D_A this equation is equivalent to

$$\left| v^{A_1} \dots v^{A_{2q}} \right|^{2^{p-1}} |\det C_{A_1} \dots C_{A_{2q}}| = 1;$$

that is,

$$|\det \Lambda^+| = 1 \qquad (\text{also } |\det \Lambda^-| = 1).$$

An example (the case of Spin$(0,2)$) can be found in [Problem I 9, Representations].

Odd-dimensional spaces do not admit Weyl spinors; the representation of Spin(n, m) on $\mathbb{C}^{2^p}$, $p = [(n+m)/2]$ is irreducible if $n+m$ is odd.

2. CHARGE CONJUGATION. MAJORANA PINORS

The group Pin$(1,3)$, which acts on ordinary space-time pinors, admits, like the Clifford algebra $\mathscr{C}(1,3)$, a real representation. If such a representation is chosen, it is meaningful to speak of real pinors: if a representative $\psi_{(i)}$ is real, the same is true of another representative $\Lambda\psi_{(i)}$ of the same pinor.

Majorana pinors

Real pinors are called **Majorana pinors**. It is also meaningful to speak of the complex conjugate ψ^* of a pinor ψ, represented in each pin frame by the complex conjugate of the representative of ψ.

Moreover, each pinor ψ can be written as

$$\psi = \psi_1 + \mathrm{i}\psi_2$$

with ψ_1 and ψ_1 Majorana pinors.

A Majorana particle has no connection with Majorana pinors. See reference to M. Berg et al. p. 39.

2a) *Give a definition, independent of the representation of* Pin$(1,3)$, *of the operator C defined in a real representation by C: $\psi \to \psi^*$. This opera-*

*tor is called "***charge conjugation***". Give its definition when the gamma matrices are the Dirac matrices (see conventions).* charge conjugation

2b) *Is it possible to extend the definition of charge conjugation to an arbitrary* $O(1, d-1)$ *group?*

Answer 2a: The operator defined in a real representation of $\mathscr{C}(1,3)$ by $\psi \to \psi^*$ in each pin frame is an antilinear (i.e., $(\lambda\psi + \mu\varphi)^* = \lambda^*\psi^* + \mu^*\varphi^*$) map $S \to S$, which is also involutive (i.e., its square is the identity). If we look in an arbitrary representation of $\mathscr{C}(1,3)$ for an antilinear operator $\mathbb{C}^4 \to \mathbb{C}^4$ by $\psi \mapsto \psi^c = C\psi$ such that ψ^c is a representative of a pinor when ψ is, we must have

$$C\Lambda\psi = \Lambda\psi^c = \Lambda C\psi; \tag{1}$$

that is C must commute with pin transformations. An antilinear operator on ψ is a linear operator D on ψ^*

$$C\psi = D\psi^*, \qquad D \in \mathscr{L}(S, S). \tag{2}$$

Equation (1) is equivalent to

$$D\Gamma_A^* = \Gamma_A D, \qquad A = 0, \ldots, 3. \tag{3}$$

Since the Γ_A^* are a set of gamma matrices of $O(1,3)$ if the Γ_A are, we know there exists such an operator D, determined up to product by a complex number μ.

If we want C to be involutive, we must have

$$(\psi^c)^c \equiv D(\psi^c)^* = DD^*\psi = \psi, \quad \text{i.e.,} \quad DD^* = \mathbb{1}. \tag{4}$$

Once D is known for a choice (Γ_A) of gamma matrices it exists and is determined for another choice $\hat{\Gamma}_A$: if we have

$$\hat{\Gamma}_A = M\Gamma_A M^{-1}, \quad \text{hence} \quad \hat{\Gamma}_A^* = M^*\Gamma_A^* M^{*-1},$$

the operator

$$\hat{D} = M^{-1}DM^*$$

satisfies (3), for the matrices $\hat{\Gamma}_A$, and (4).

In the Majorana representation of the gamma matrices we have $D = 1$. In the Dirac representation

$$D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The operator of charge conjugation C: $\psi \mapsto D\psi^*$ has, in any representation, eigenvalues ± 1. The eigenspace with eigenvalue 1 is the space of Majorana pinors. The space S is the direct sum of the two eigenspaces $S = S_1 \oplus S_2$.

In the Dirac representation, ψ is a Majorana pinor if $\psi^1 = (\psi^3)^*$, $\psi^2 = (\psi^4)^*$.

Remark 1: In our $O(1,3)$ case the only pinor that is both Weyl and Majorana is the zero pinor: a real pinor cannot be in an eigenspace of the helicity operator, ξ, since for $O(1,3)$ the eigenvalues of ξ are $\pm \mathrm{i}$.

Answer 2b: We can take as the definition of a charge conjugation operator C, in the general $O(n, m)$ case, $n + m$ even, the conditions (1) (equivalently (3)), (2), and (4). If ψ is a pinor, $\psi^c = C\psi$ will then also be a pinor and charge conjugation will be an involutive operator. For arbitrary n and m, it is not always possible to find such a D. We know that D is defined up to the product by a complex number μ, so if $DD^* > 0$ we can rescale D to obtain $DD^* = \mathbb{1}$; but if $DD^* < 0$ we can only obtain $DD^* = -\mathbb{1}$, since $\mu\mu^* > 0$.

If the Γ_A are real, it follows from (3) that a possible choice of D is $D = \mathbb{1}$. There exists a charge conjugation operator with representative C: $\psi \to \psi^*$ in this real representation. The eigenvalues of C are ± 1, as in the $O(1,3)$ case. The eigenspace with eigenvalue 1 is the space of Majorana pinors. The space S is the direct sum of the two eigenspaces $S = S_1 \oplus S_2$.

With the given definitions, $\mathrm{Pin}(n, m)$ and $\mathrm{Pin}(m, n)$ do not always simultaneously have charge conjugation operators. We shall now restrict the charge conjugation to spinors. We can then treat also the case $n + m$ odd, as well as enlarge the group, which admits a charge conjugation operator, namely the group $\mathrm{Spin}(m, n)$, if $\mathrm{Spin}(n, m)$ admits one.

The operator C: $\psi \mapsto C\psi = \psi^c$ must satisfy (1) for $\Lambda \in \mathrm{Spin}(m, n)$; we still set

$$C\psi = D\psi^*, \qquad D \text{ a linear operator.}$$

(1) is now satisfied, since Λ is even, if we have

$$D\Gamma_A^*\Gamma_B^* = \Gamma_A\Gamma_B D;$$

that is

$$D\Gamma_A^* = \epsilon\Gamma_A D \tag{5}$$

with $\epsilon = +1$ or $\epsilon = -1$.

The involutivity of C still gives

$$DD^* = \mathbb{1}. \tag{6}$$

The existence of D determined up to product by $\mu \in \mathbb{C}$ satisfying (5) is again a consequence of the fact that the $\epsilon\Gamma_A^*$'s are a set of gamma matrices if the Γ_A's are; but, again, it is not always possible to satisfy condition (6). However, if D exists for Spin(n, m), it also exists for Spin(m, n) – as expected, since these groups are isomorphic – because if Γ_A are gamma matrices of O(n, m), $\hat{\Gamma}_A = \mathrm{i}\Gamma_A$ are gamma matrices of O(m, n) and if the Γ_A satisfy (5) the $\hat{\Gamma}_A$ satisfy

$$D\hat{\Gamma}_A^* = -\epsilon\hat{\Gamma}_A D.$$

Examples: The Clifford algebra $\mathscr{C}(3, 1)$ admits a representation by pure imaginary matrices, generated by $\hat{\Gamma}_A = \mathrm{i}\Gamma_A$, Γ_A the gamma matrices of the Majorana representation of $\mathscr{C}(1, 3)$. The corresponding operator D is $\mathbb{1}$. A pinor is Majorana if it is real.

We give a table, taken from Coquereaux, listing the representation spaces for Dirac (i.e., general), Weyl or Majorana spinors in hyperbolic signature and dimension $d = 4$ to 11. Coquereaux' table includes representation spaces for Majorana spinors in dimensions $d = 8$ and $d = 9$ because he combines the results obtained for $\mathscr{C}(n, m)$ and $\mathscr{C}(m, n)$.

d	4	5	6	7	8	9	10	11
Dirac	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^8$	$\mathbb{C}^8$	$\mathbb{C}^{16}$	$\mathbb{C}^{16}$	$\mathbb{C}^{32}$	$\mathbb{C}^{32}$
Weyl	$\mathbb{C}^2$	/	$\mathbb{C}^4$	/	$\mathbb{C}^8$	/	$\mathbb{C}^{16}$	/
Majorana	$\mathbb{R}^4$	/	/	/	/	/	$\mathbb{R}^{32}$	$\mathbb{R}^{32}$
Weyl–Majorana	/	/	/	/	/	/	$\mathbb{R}^{16}$	/

References for Problems I 7 to I 11 may be found at the end of Problem I 11.

9. REPRESENTATIONS OF SPIN(n, m), $n+m$ ODD

1. *Show that the Clifford algebra $\mathscr{C}(n, m)$ can be identified with the subalgebra of $\mathscr{C}(n, m+1)$ defined by the relation*

$$\begin{aligned}(\Lambda_+ + \Lambda_-)\gamma_{n+m+1} &= \gamma_{n+m+1}(\Lambda_+ - \Lambda_-),\\ \Lambda = \Lambda_+ + \Lambda_- \in \mathscr{C}(n, m+1) &= \mathscr{C}_+(n, m+1) \oplus \mathscr{C}_-(n, m+1).\end{aligned} \tag{1}$$

Answer 1: Let $\mathbb{1}$, γ_A, $A = 1, \ldots, n+m$, and γ_{n+m+1} be generators of the Clifford algebra $\mathscr{C}(n, m+1)$. Then $\mathbb{1}$ and γ_A are generators of $\mathscr{C}(n, m)$. Relation (1) expresses that $\Lambda \in \mathscr{C}(n, m+1)$ is in the subalgebra generated by $\mathbb{1}$ and γ_A since

$$\Lambda = c\mathbb{1} + \sum_{q=1}^{n+m} (c^{A_{i_1} \dots A_{i_q}} \gamma_{A_{i_1}} \dots \gamma_{A_{i_q}}) + d^{A_{i_q} \dots A_{i_q}} \gamma_{A_{i_1}} \dots \gamma_{A_{i_q}} \gamma_{n+m+1}$$

satisfies (1) if and only if

$$d^{A_{i_1} \dots A_{i_q}} = 0.$$

2. *Define* Pin(n, m), $n + m$ *odd, as the subgroup of* Pin$(n, m + 1)$, *whose image by the twisted homomorphism* $\tilde{\mathscr{H}}$ *is the subgroup of* O$(n, m + 1)$, *which leaves invariant the axis* e_{n+m+1}.
Give a representation of Pin(n, m), $n + m = 2p - 1$, by $2^p \times 2^p$ *matrices.*

Answer 2: An element $\Lambda \in \text{Pin}(n, m + 1)$, whose image by $\tilde{\mathscr{H}}$ leaves invariant the axis e_{n+m+1}, satisfies

$$\alpha(\Lambda)\gamma_A \Lambda^{-1} = L_A^B \gamma_B, \qquad A, B = n, \dots, m + n, \tag{2}$$
$$\alpha(\Lambda)\gamma_{n+m+1}\Lambda^{-1} = \gamma_{n+m+1}. \tag{3}$$

(3) says that $\Lambda \in \mathscr{C}(n, m)$, as identified in (1), while (2) expresses that $\Lambda \in \text{Pin}(n, m)$.

From a representation of Pin$(n, m + 1)$, $n + m + 1 = 2p$, by $2^p \times 2^p$ matrices we deduce a representation of Pin(n, m) by $2^p \times 2^p$ matrices.

3. *Deduce from a representation of* Spin$(n, m + 1)$, $n + m + 1 = 2p$, *by* $2^{p-1} \times 2^{p-1}$ *matrices a representation of* Spin(n, m) *by such matrices.*

Answer 3: In a basis adapted to the splitting of the representation space $\mathbb{C}^{2p} = S_+ \oplus S_-$, we have seen [Problem I 8, Weyl spinors] that the representative of an element $\Lambda \in \text{Spin}(n, m + 1) \subset \mathscr{C}_+(n, m + 1)$ reads

$$\Lambda = \begin{pmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{pmatrix}$$

where Λ^+ and Λ^- are $2^{p-1} \times 2^{p-1}$ matrices, while $\gamma_{n+m+1} = \gamma_{2p}$ is represented by

$$\Gamma_{2p} = \begin{pmatrix} 0 & C_{2p} \\ \tilde{C}_{2p} & 0 \end{pmatrix}.$$

An element of Spin(n, m) is represented by a $2^{p-1} \times 2^{p-1}$ matrix, for instance Λ^+ such that, together with its canonically associated Λ^- [cf. Problem I 8, Clifford], it satisfies

$$\Lambda^+ C_{2p} = C_{2p}\Lambda^-, \qquad \Lambda^- \tilde{C}_{2p} = \tilde{C}_{2p}\Lambda^+.$$

In the case of euclidean signature, we have $\Lambda = \tilde{\Lambda}$ [cf. Problem I 4, Clifford]. The second relation is the hermitian transpose of the first.

Example: Consider the representation of $\mathscr{C}(0,2)$ with generators $\Gamma_1 = \sigma_1$, $\Gamma_2 = \sigma_2$. Then

$$\xi = \Gamma_1 \Gamma_2 = -\mathrm{i}\sigma_3 = -\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

ξ is written in a basis adapted to the splitting of helicities. Spin(0, 2) is represented by 2×2 matrices $\Lambda = a\mathbb{1} + b\xi$ such that $|\det \Lambda| = 1$; i.e., $a^2 + b^2 = 1$. The representatives Λ_+ [resp. Λ_-] are the complex numbers $a - ib$ [resp. $a + ib$], with $a^2 + b^2 = 1$. It was shown (p. 68) that the element of SO(2) corresponding to $a = \cos\varphi$, $b = \sin\varphi$ is the rotation of angle 2φ, in accordance with the fact that Spin(0, 2) is a double covering of SO(2).

Spin(0, 1) is represented by elements such that the corresponding SO(2) transformation leaves e_2 invariant, hence is the identity (it is also easy to check that $\Lambda\Gamma_2 = \Gamma_2\Lambda$ implies in this case $b = 0$). Spin(0, 1) is isomorphic to $\mathbb{Z}_2$.

4. *Show that when $n + m$ is odd the group* $\mathrm{O}(n, m)$ *is the direct product*

$$\mathrm{O}(n, m) = \mathrm{SO}(n, m) \times \mathbb{Z}_2.$$

Define $P(n, m)$ as a double covering of $\mathrm{O}(n, m)$.

Answer 4: When $n + m$ is odd, we can use the parity operator $P = -\mathbb{1}$ of $\mathrm{O}(n, m)$ to write $L \in \mathrm{O}(n, m)$ as a pair

$$L = (L_+, \epsilon)$$

with $L_+ \in \mathrm{SO}(n, m)$, $\epsilon = +1$ if $L = L_+$, and $\epsilon = -1$ if $L \notin \mathrm{SO}(n, m)$. If $L_1 = (L_{1+}, \epsilon_1)$ and $L_2 = (L_{2+}, \epsilon_2)$ we have

$$L_1 L_2 = (L_{1+} L_{2+}, \epsilon_1 \epsilon_2)$$

because P commutes with $\mathrm{SO}(n, m)$. The same argument cannot be used for $n + m$ even because P is then orientation preserving; P cannot be replaced by a reflection because these transformations do not commute with $\mathrm{SO}(n, m)$. When $n + m$ is even, $\mathrm{O}(n, m)$ is a semidirect product $\mathrm{SO}(n, m) \ltimes \mathbb{Z}_2$ (see reference M. Berg et al., p. 39).

Since $\mathrm{O}(n, m) = \mathrm{SO}(n, m) \times \mathbb{Z}_2$, $n + m$ odd, we define a double covering by setting

$$\mathrm{Pin}(n, m) = \mathrm{Spin}(n, m) \times \mathbb{Z}_2;$$

the corresponding homomorphism

$$\tilde{\mathscr{H}}\colon \mathrm{Pin}(n, m) \to \mathrm{O}(n, m)$$

is given by

$$(s_+, \epsilon) \mapsto (L_+, \epsilon), \qquad \epsilon = \pm 1,\ s_+ \in \mathrm{Spin}(n, m).$$

10. DIRAC ADJOINT

We have seen that a choice among the isomorphic spaces of pinors attached to a pseudo-euclidean vector space V is defined by a pair (ρ_0, e), where ρ_0 is an orthonormal frame in V and e is the unit of the group $\mathrm{Pin}(n, m)$, homomorphic to the group $\mathrm{O}(n, m)$ which leaves invariant the scalar product on V.

We consider the hyperbolic case $n = 1$, $m = d - 1$ and label the gamma matrices $\Gamma_0, \ldots, \Gamma_{d-1}$.

1) We say that ψ is a spinor if it is an equivalence class of triples $(\psi_{(i)}, \rho_{(i)}, \Lambda_{(i)})$ with $\mathscr{H}(\Lambda_{(i)}) = L_{(i)}$ when $\rho_{(i)} = L_{(i)}\rho_{(0)}$ and $\Lambda_{(i)} \in \mathrm{Spin}^+(1, d-1)$, that is, $L_{(i)} \in L^+(1, d-1)$, so $\rho_{(i)}$ has the same orientation and time orientation as $\rho_{(0)}$.

Show that

$$\bar{\psi}_{(i)} = \tilde{\psi}_{(i)}\Gamma_0, \qquad \tilde{\psi}_{(i)} \in \mathbb{C}^{2^p}, \qquad p = [d/2],$$

Dirac adjoint

where $\sim$ denotes transposition and complex conjugation, are the components of a covariant spinor (a cospinor), called the **Dirac adjoint** *of ψ and denoted $\bar{\psi}$.*

2) *Extend the definition of the Dirac adjoint when ψ is a pinor.*

3) *Define the Dirac adjoint of a pinor field on a manifold, when possible.*

Answer 1: Using the definition and the hermiticity properties of the gamma matrices (see Problem I 4, Clifford §3 d iii) we find:

$$\Gamma_0\Gamma_A\Gamma_0^{-1} = -\tilde{\Gamma}_A, \qquad A = 0, \ldots, d-1.$$

We deduce from the definition of an element Λ of $\mathrm{Pin}(1, d-1)$ that

$$\tilde{\Lambda}^{-1}\tilde{\Gamma}_A\tilde{\Lambda} = L_A{}^B\tilde{\Gamma}_B.$$

Thus

$$\tilde{\Lambda}^{-1}\Gamma_0\Gamma_A\Gamma_0^{-1}\tilde{\Lambda} = L_A{}^B\Gamma_0\Gamma_B\Gamma_0^{-1}.$$

A little algebraic manipulation shows that Γ_A commutes with $\Lambda^{-1}\Gamma_0^{-1}\tilde{\Lambda}^{-1}\Gamma_0$, so

$$\Lambda^{-1}\Gamma_0^{-1}\tilde{\Lambda}^{-1}\Gamma_0 = a_\Lambda \mathbb{1}_{2^p}, \qquad a_\Lambda \in \mathbb{C}.$$

Taking the determinant of this equation, we find that

$$(a_\Lambda)^{2^p} = 1.$$

Since a_Λ takes discrete values, it is constant for Λ in a connected component of the pinor group, and it is equal to 1 in the connected component of unity. For such Λ's

$$\tilde{\Lambda}\Gamma_0 = \Gamma_0 \Lambda^{-1}.$$

This relation shows that $\bar{\psi}$, with representatives $\tilde{\psi}_{(i)}\Gamma_0$, is a covariant spinor if ψ, with representatives $\psi_{(i)}$, is a spinor.

Answer 2: In order to obtain the relation

$$\bar{\psi}_{(i)} = \bar{\psi}_{(j)}\Lambda^{-1} \quad \text{when} \quad \psi_{(i)} = \Lambda\psi_{(j)}$$

we set, for any $\Lambda \in \mathrm{Pin}(1, d-1)$,

$$\bar{\psi}_{(i)} = a_\Lambda \tilde{\psi}_{(i)}\Gamma_0.$$

Note that $a_\Lambda = 1$ for $\Lambda \in \mathrm{Pin}^+(1, d-1)$ (the set of elements projecting on orthochronous Lorentz transformations) because $a_\Lambda = 1$ for $\Lambda = e$ and $\Lambda = \Gamma_0$, which projects on the space reversal transformation.
$a_\Lambda = -1$ otherwise.
a_Λ is a representation of $\mathrm{Pin}(1, d-1)$ in $\mathbb{Z}_2 = \{1, -1\}$.

Answer 3: It will be possible to define the Dirac adjoint of a pinor field on a hyperbolic manifold when it admits a time orientation. A pin frame (Pb IV §2) is future [resp. past] oriented if it projects onto a future [resp. past] oriented Lorentz frame; to each pin frame $z_{(i)}$ is associated the number $a_{z(i)} = 1$ if it is future oriented, $a_{z(i)} = -1$ otherwise. The Dirac adjoint of the pinor field ψ (with representative $\psi_{(i)}$ in the pin frame $z_{(i)}$) is then the copinor field

$$\bar{\psi}_{(i)} = a_{z(i)}\tilde{\psi}_{(i)}\Gamma_0.$$

References for Problems I 7–11 can be found at the end of Problem I 11.

11. LIE ALEBRA OF PIN(n, m) AND SPIN(n, m)

We perform in arbitrary dimensions the exercise done (p. 176) in four dimensions, since it is not obvious that one obtains the same numerical factors in all dimensions.
$\mathrm{Pin}(n, m)$ and $\mathrm{Spin}(n, m)$ are $d(d-1)/2$ dimensional Lie groups $d = n + m$, 2-fold coverings of $\mathrm{O}(n, m)$ and $\mathrm{SO}(n, m)$ by a homomorphism $\mathscr{H}$. They have the same Lie algebras, homomorphic to the Lie algebra

$\mathscr{O}(n,m)$ and in fact isomorphic to it, since the derivated mapping $\mathscr{H}'$ is an isomorphism between the tangent spaces at the unit elements of the groups. We have [Problem I 7, Pin]

$$\mathscr{H}\colon \operatorname{Spin}(n,m) \to \operatorname{SO}(n,m) \quad \text{by } \Lambda \mapsto L$$

with

$$\alpha(\Lambda)\Gamma_A\Lambda^{-1} = \Lambda\Gamma_A\Lambda^{-1} = L_A{}^B\Gamma_B, \qquad \Lambda \in \mathscr{C}_+, \qquad |\det \Lambda| = 1.$$

Thus $\mathscr{H}'(\mathbb{1})$ is determined by the mapping between tangent spaces:

$$\mathscr{H}'(\mathbb{1})\colon\ T_{\mathbb{1}_{\operatorname{Pin}(n,m)}} \to T_{\mathbb{1}_{\operatorname{O}(n,m)}} \quad \text{by } \Lambda \mapsto l$$

with

$$\lambda\Gamma_A - \Gamma_A\lambda = l_A{}^B\Gamma_B, \qquad l^{AB} = -l^{BA}, \tag{1}$$
$$\operatorname{tr}\lambda = 0. \tag{2}$$

With $\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = -2g_{AB}\mathbb{1}$ the general solution[1] of (1) is

$$\lambda = \tfrac{1}{4}\, l^{AB}\Gamma_A\Gamma_B + \mu,$$

where μ is a solution of the homogeneous equation

$$\mu\Gamma_A - \Gamma_A\mu = 0.$$

Thus μ is such that

$$\mu = a\mathbb{1}, \qquad a \in \mathbb{C}.$$

Since (2) implies $\operatorname{tr}\mu = 0$, we have $a = 0$, and $\mu = 0$.

The isomorphism between the Lie algebras $\mathscr{S}pin(n,m)$ and $\mathscr{O}(n,m)$ is therefore

$$\lambda = \tfrac{1}{4}\, l^{AB}\Gamma_A\Gamma_B$$

where $l = (l^{AB}) \in \mathscr{O}(n,m)$ and $\lambda \in \mathscr{S}pin(n,m)$.

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[1]Note that the sign of the solution depends on the choice of sign for the Clifford algebra, and the choice between $L_A{}^B\Gamma_B$ and $L^B{}_A\Gamma_B$.

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12. COMPACT SPACES

The proof[1] of the Tychonoff theorem (p. 21) is based on the following criteria which use closed subsets to establish fundamental properties of compact spaces often stated in terms of open subsets.
Let $\mathscr{G}$ be an open cover of X and let $\mathscr{F}$ be the family of complementary closed sets of X. Then

$$\cup\mathscr{G} = X \Leftrightarrow \cap\mathscr{F} = \emptyset.$$

1) *Show that a topological space X is compact if and only if every family of closed subsets of X with empty intersection has a finite subfamily $\mathscr{F}_n$ with empty intersection.*

2) *Show that X is compact if and only if every family $\mathscr{F}$ of closed subsets of X has a nonempty intersection provided every finite subfamily $\mathscr{F}_n \subset \mathscr{F}$ of closed subsets of X has a nonempty intersection.*

Answer 1: By definition, X is compact (p. 15) if and only if for every family $\mathscr{G}$ of open sets which covers X there is a finite subfamily $\mathscr{G}_n$ which covers X. Let $\mathscr{F}$ be a family of closed sets with empty intersection: the

[1] See, for instance, Casper Goffman and George Pedrick, *First course in functional analysis* (Prentice-Hall, Englewood Cliffs, N.J., 1965).

family $\mathscr{G}$ of complementary open sets covers X, and thus admits a finite subfamily $\mathscr{G}_n$ which covers X. The family $\mathscr{F}_n \subset \mathscr{F}$ of complementary closed sets then has empty intersection. Suppose X is compact. Then

$$\cap \mathscr{F} = \emptyset \quad \Rightarrow \quad \cup \mathscr{G} = X \quad \Rightarrow \quad \exists n, \ \cup \mathscr{G}_n = X \quad \Rightarrow \quad \cap \mathscr{F}_n = \emptyset. \tag{1}$$

Similar arguments show the converse implication.

Answer 2: a) The implication

$$X \text{ compact} \quad \Rightarrow \quad \{\cap \mathscr{F}_n = \emptyset, \forall n \quad \Rightarrow \quad \cap \mathscr{F} = \emptyset\}$$

follows from the first criterion. Indeed, if there is a family $\mathscr{F}$ of closed subsets of X with empty intersection such that every finite subfamily $\mathscr{F}_n$ has nonempty intersection the space X cannot be compact by (1).
b) Conversely, assume X noncompact; then there exists $\mathscr{F}$ such that $\cap \mathscr{F} = \emptyset$, and all $\mathscr{F}_n$ are such that $\cap \mathscr{F}_n \neq \emptyset$, which is contrary to the hypothesis.

For an example of compactness see the next problem I 13.

13. COMPACTNESS IN WEAK STAR TOPOLOGY

Let X be a Banach space, and $Y = X^*$ its dual. A basis of the weak star topology on Y is given by the finite intersections of open sets of the type

$$U_{x,I} = \{y \in Y; \langle x, y \rangle \in I\},$$

where x is an arbitrary element of X, $\langle \ , \ \rangle$ is the duality between X and Y and I is an open set in $\mathbb{R}$.
Denote by $Z = \Pi(\mathbb{R}_x; x \in X)$ the topological product (cf. p. 20) of copies $\mathbb{R}_x$ of $\mathbb{R}$ indexed by $x \in X$.
1) *Show that Y can be identified with a subset of Z and that its weak star topology is the one induced by the topology of Z.*

2) *Show that the subspace K of Z defined by $K = \{z; z_x \in R_x, |z_x| \leqq \|x\|\}$ is compact.*

3) *Show that the ball of Y, strongly closed and bounded in the norm topology*

$$\|y\| \leqq c$$

is compact in the weak star topology.

Answer 1: An element y of Y is a continuous linear form $X \to \mathbb{R}$ given by

$$y\colon x \to y_x \equiv \langle y, x \rangle \in \mathbb{R}$$

and can be identified with the element

$$z = \Pi(y_x; x \in X) \in Z.$$

The weak star topology of Y is the trace of the topology of Z, by the definition of their respective bases.

Answer 2: Each of the subsets $|z_x| \leqq \|x\|$ is compact in $\mathbb{R}_x$, since it is a closed and bounded interval of $\mathbb{R}$. The product K is therefore compact (Tychonoff theorem).

Answer 3: If the result is true for $c = 1$, it is true for all $c > 0$ and all strongly bounded closed subsets of Y.
Let B be the unit ball

$$B\colon \|y\| \leqq 1.$$

The norm on the dual Y of X is (p. 58) by definition

$$\|y\| = \operatorname*{Sup}_{\|x\|=1} |\langle y, x \rangle|.$$

Hence we have

$$B = K \cap Y \subset Z.$$

Since K is a compact subset of Z the closure $\bar{B}$ of B on Z in the Z-topology is a compact subset of K. To show that $\bar{B} \subset Y$ (and therefore $\bar{B} = B$) we now show that if $z \in \bar{B}$ it can be identified with a continuous linear form on X. Since $z \in \bar{B}$, the closure of B in the Z-topology, there exists for every $\varepsilon > 0$ and pair of elements of X an element $y \in Y$ such

$$|\langle y, x_1 \rangle - z_{x_1}| < \varepsilon, \ |\langle y, x_2 \rangle - z_{x_2}| < \varepsilon,$$
$$|\langle y, x_1 + x_2 \rangle - z_{x_1+x_2}| < \varepsilon.$$

Thus

$$|z_{x_1+x_2} - z_{x_1} - z_{x_2}| < 3\varepsilon.$$

Analogously, for any pair $x \in X$, $a \in \mathbb{R}$

$$|z_{ax} - az_x| < 2\varepsilon.$$

Since ε is arbitrarily small, $x \mapsto z_x$ is a linear map, continuous since $|z_x| \leqq \|x\|$.

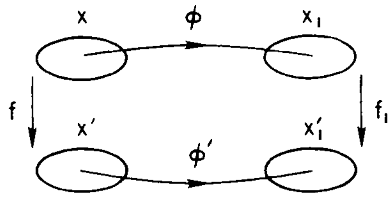
14. HOMOTOPY GROUPS, GENERAL PROPERTIES

homotopic

Two continuous maps f and f_1 between topological spaces X and X' are **homotopic** if they can be deformed into each other, that is, more precisely, if there is a continuous map $F: X \times I \to X'$, $I = [0, 1]$, such that $F(., 0) = f$, $F(., 1) = f_1$ (cf. the case $X = [0, 1]$ p. 20, homotopic paths). It is straightforward to check that this relation is reflexive, symmetric and transitive, and hence is an equivalence relation, which we denote $f \simeq f_1$.

1) *Show that if X is homeomorphic to X_1 and X' to X'_1, then there is a bijective correspondence between the homotopy classes of maps $X \to X'$ and $X_1 \to X'_1$.*

2) *Let φ be an homeomorphism of X homotopic to the identity mapping. Show that the mappings $f: X \to X'$ and $f \circ \varphi$ are in the same homotopy class.*



Answer 1: Let φ and φ' be respectively homeomorphisms of X onto X_1 and X' onto X'_1. A bijective correspondence between continuous maps $X \to X'$ and $X_1 \to X'_1$ is given by

$$f_1 = \varphi' \circ f \circ \varphi^{-1}.$$

This correspondence defines a bijective correspondence between homotopy classes because if $f \simeq h$ and F is the homotopy $X \times I \to X'$, then $f_1 \simeq h_1$ with homotopy map $\phi_1(., t) = \varphi' \circ F(., t) \circ \varphi^{-1}$.

Answer 2: Denote by ϕ the continuous mapping $X \times I \to X$ such that $\phi(., 0) = \varphi$ and $\phi(., 1) = \mathrm{Id}$. We define a continuous mapping $F: X \times I \to X'$ by

$$F = f \circ \phi, \qquad F(x, t) = f(\phi(t, x));$$

this mapping is such that

$$F(., 0) = f \circ \varphi, \qquad F(., 1) = f.$$

It thus defines an homotopy between f and $f \circ \varphi$.

The elements of the **kth homotopy group** $\pi_k(X, x_0)$ of a topological space X with base point $x_0 \in X$ are the homotopy classes of continuous maps

kth homotopy group π_k

$$I^k \to X, \qquad \partial I^k \to x_0, \quad x_0 \in X$$

where I^k is the cube $0 \le x^i \le 1$, $i = 1, \ldots, k$ and ∂I^k is its boundary.

Equivalently, the elements of $\pi_k(X, x_0)$ can be viewed as the homotopy classes of maps from the k-sphere S^k to X which map the north pole to x_0.

3) *Let f_1 and f_2 be two continuous mappings $I^k \to X$, $\partial I^k \to x_0$. Define a mapping $f_1 + f_2$: $I^k \to X$ by $x \mapsto (f_1 + f_2)(x)$*

$$\begin{aligned} (f_1 + f_2)(x) &= f_1(2x^1, x^2, \ldots, x^k), & 0 \le x^1 \le 1/2, \\ (f_1 + f_2)(x) &= f_2(2x^1 - 1, x^2, \ldots, x^k), & 1/2 < x^1 \le 1. \end{aligned} \tag{1}$$

Show that $f_1 + f_2$ is continuous and maps ∂I^k into x_0. Use the above definition to endow $\pi_k(X, x_0)$ with a group structure.

4) *Is this group abelian?*

5) *Show that $\pi_k(X, x_0)$ is independent of x_0, up to an isomorphism, if X is pathwise connected.*

Answer 3: $f_1 + f_2$ is continuous because f_1 and f_2 both take the value x_0 for $x^1 = 1/2$. It is obvious that $(f_1 + f_2)(\partial I^k) = x_0$.
If $f_1 \simeq f_1'$ and $f_2 \simeq f_2'$ then $f_1 + f_2 \simeq f_1' + f_2'$: to check this consider the homotopy maps $f_1(., t)$ and $f_2(., t)$ and their sum defined as above. Thus the sum defined by (1) acts on homotopy classes. We show it is a group operation (p. 7).

a) It is associative: define

$$\begin{aligned} h(x) \equiv ((f_1 + f_2) + f_3)(x) &= f_1(4x^1, x^2, \ldots), & 0 \le x^1 \le 1/4 \\ &= f_2(4x^1 - 1, x^2, \ldots), & 1/4 < x^1 \le 1/2 \\ &= f_3(2x^1 - 1, x^2, \ldots), & 1/2 < x^1 \le 1 \end{aligned}$$

and

$$\begin{aligned} k(x) \equiv (f_1 + (f_2 + f_3))(x) &= f_1(2x^1, x^2, \ldots), & 0 \le x^1 \le 1/2 \\ &= f_2(4x^1 - 2, x^2, \ldots), & 1/2 < x^1 \le 3/4 \\ &= f_3(4x^1 - 3, x^2, \ldots), & 3/4 < x^1 \le 1. \end{aligned}$$

These two mappings are homotopic because they are related to each other by a homeomorphism ϕ of the cube, homotopic to the identity: $h = k \circ \phi$, with $\phi: I^k \to I^k$ by $\phi(x) = (\xi^1, \ldots, \xi^k)$

$$\begin{aligned} &\xi^1 = 2x^1 && 0 \le x^1 \le 1/4 \\ &\xi^1 = x^1 + 1/4, && 1/4 < x^1 \le 1/2 \\ &\xi^1 = x^1/2 + 1/2, && 1/2 < x^1 \le 1 \\ &\xi^2 = x^2, \ldots, \xi^k = x^k. \end{aligned}$$

The homotopy of ϕ and the identity is shown by the existence of the continuous mapping $t\phi + (1-t)\mathrm{Id}$.

b) The operation has a neutral element, the homotopy class of the constant map $I^k \to x_0$. Maps in this class, i.e., maps which can be deformed to a point, are called **contractible maps.**

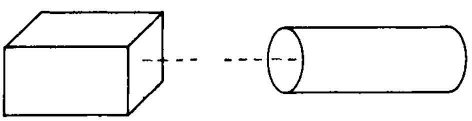
contractible maps

c) Each element has an inverse: the inverse of the homotopy class of the map f is the homotopy class of the maps $-f$ given by

$$(-f)(x) = f(1 - x^1, x^2, \ldots, x^k).$$

The proofs of b) and c) are in the same spirit as the proof of a). For details see Steenrod, p. 74. For $k = 1$ the inverse is just the homotopy class of paths with the opposite orientation.

Answer 4: It can be proved that the group π_k is abelian for $k \ge 2$ by considering a homeomorphism of the cube which interchanges the parts $x^1 \le 1/2$ and $x^1 \ge 1/2$: the cube is homeomorphic to a cylinder and the homeomorphism which interchanges the two halves of the cylinder is obtained by rotation.



The group may be nonabelian for $k = 1$: consider a space $\mathbb{R}^2$ with two holes. The path obtained by going first around one hole and then around the other cannot be continuously deformed into the path circling first the second hole and then the first one.
The operation of the group π_1 is often denoted multiplicatively. A closed path is also called a **loop**.

loop

Answer 5: Suppose there is a path c in X joining y_0 and x_0, that is, a continuous map c: $[0, 1] \to X$ such that $c(0) = y_0$, $c(1) = x_0$. Then to each loop f at x_0 (i.e., closed path passing through x_0) one can associate a loop h at y_0, by going first from y_0 to x_0 by the path c, then making the path f, then going back to y_0 by the inverse path c^{-1}. More precisely, we set

$$h = c^{-1}fc,$$

where product of paths is defined as product of loops, that is,

$$(fc)(t) = c(2t), \qquad 0 \le t < 1/2$$
$$(fc)(t) = f(2t-1), \qquad 1/2 < t \le 1$$

fc is a continuous map because $f(0) = c(1)$ and

$$c^{-1}(t) = c(1-t).$$

If $f \simeq f_1$ it is not difficult to check that $h \simeq h_1$.
Thus all groups $\pi_1(X, x_0)$, $x \in X$ are isomorphic if X is pathwise connected. The isomorphism is not necessarily canonical, but depends only on the homotopy class of the chosen path c, with fixed end points x_0, y_0. The same result holds for higher homotopy groups $\pi_k(X, x_0)$ (for details of proof see Steenrod, p. 84).

A topological space is said to be **k-simple** if it is arcwise connected and if for any two points x_0, y_0 the isomorphism from $\pi_k(X, x_0)$ onto $\pi_k(X, y_0)$ is independent of the path c. It can be proved that an arcwise connected space is 1-simple if and only if $\pi_1(X)$ is abelian; if $\pi_1(X) = 0$ then X is k simple for every k. k-simple

A few examples of homotopy groups

$\pi_k(\mathbb{R}^n) = 0,$	$\forall k, n$	
$\pi_k(S^n) = 0,$	$1 \le k < n,$	$\pi_n(S^n) = \mathbb{Z}$
$\pi_k(O_m \backslash O_n) = 0,$	$1 \le k < n$	
$\pi_k(S^1) = 0,$	$k > 1$	
$\pi_3(S^2) = \mathbb{Z};$	$\pi_{n+1}(S^n) = \mathbb{Z}_2,$	$n \ge 3$
$\pi_1(SO^3) = \mathbb{Z}_2,$	$\pi_3(SO^3) = \mathbb{Z},$	$\pi_4(SO^3) = \mathbb{Z}_2$
$\pi_1(SO_4) = \mathbb{Z}_2,$	$\pi_3(SO_4) = \mathbb{Z} \oplus \mathbb{Z},$	$\pi_4(SO_4) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

For any Lie group $\pi_2(G) = 0$.
For any semi-simple Lie group $\pi_3(G) = \mathbb{Z}$.
A more complete list of homotopy groups of spheres and Lie groups can be found in reference 2, appendix A 6.

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15. HOMOTOPY OF TOPOLOGICAL GROUPS

For two mappings $f: X \to G$ and $h: X \to G$ into a group G there is a natural product, denoted $f \cdot h$, different from the product defined in the previous problem, namely

$$(f \cdot h)(x) = f(x) \cdot h(x),$$

where the dot denotes product in G.

1. *Show that if f, f_1, h, h_1 are mappings $X \to G$, such that f is homotopic to f_1 and h to h_1, that is $f \simeq f_1$, $h \simeq h_1$, then*

$$f \cdot h \simeq f_1 \cdot h_1.$$

2. *Show that if the sum is the operation defined in the previous problem, with f and h continuous mappings, $I^k \to G$, $\partial I^k \to x_0 \in G$ then*

$$f + h \simeq f \cdot h.$$

3. *Let G_e be the arcwise connected component of e in G and $g \in G_e$. Show that either the left or the right translations of G by g induces an isomorphism $\pi_k(G, g) \to \pi_k(G, e)$.*

It can be proved that $\pi_1(G_e)$ is abelian, and that G_e is k-simple for every k.

Answer 1: If F and H are the homotopy maps $X \times I \to G$ which relate f and f_1, and h and h_1 respectively, then $(x, t) \mapsto F(x, t) \cdot H(x, t)$ is the homotopy map we are looking for.

Answer 2: If f_0 denotes the constant map $I^k \to x_0$, we have

$$f \simeq f + f_0 \quad \text{and} \quad h \simeq f_0 + h.$$

It is straightforward to check from the definition that

$$f + h = (f + f_0) \cdot (f_0 + h);$$

the conclusion follows.

Answer 3: It is clear that the left (or right) translation by g induces a mapping of $\pi_k(G, e)$ into $\pi_k(G, g)$ because it preserves homotopy classes: if ϕ is the homotopy map between f and h, $\phi(., 0) = f$, $\phi(., 1) = h$, then $g \cdot \phi$ is a homotopy map between $g \cdot f$ and $g \cdot h$. It is straightforward to show that it is an isomorphism.

References: cf. Problem I 14.

16. SPECTRUM OF CLOSED AND SELF-ADJOINT LINEAR OPERATORS

domain
range

An operator A acting on a Banach space X (p. 57) is a mapping from a subset $D(A) \subset X$, called the **domain** of A, onto a subset $R(A) \subset X$, called the **range** of A. If A is a linear operator, $D(A)$ and $R(A)$ are linear subspaces of X. In this problem we consider linear operators.

closed operator

A **closed operator** is an operator A such that if $\{x_n\} \subset D(A)$ converges to $x \in X$ and $\{Ax_n\}$ converges to $y \in X$, then $x \in D(A)$ and $Ax = y$ (p. 63).

transpose

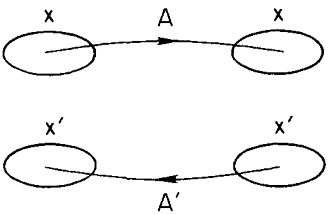
1) *Show that if A is a closed operator and if $D(A)$ is dense in X one can define the* **transpose** *of A, denoted by A', as a mapping on X', the dual of X, by the property*

$$\langle A'x', x\rangle = \langle x', Ax\rangle, \qquad x \in D(A).$$

Answer 1: If $D(A)$ is dense in X there is at most one $y' \in X'$ such that

$$\langle y', x\rangle = \langle x', Ax\rangle, \qquad \forall x \in D(A), \qquad x' \in X' \text{ given.}$$

If this element $y' \in X'$ exists we say that $x' \in D(A')$, and set $A'x' = y'$.



If X is a Hilbert space it can be canonically identified with its dual through the scalar product:

$$X \to X' \quad \text{by} \quad x \mapsto x' = (x|.).$$

adjoint

The **adjoint** A^* of a densely defined operator on a Hilbert space X is defined by

$$(A^*y|x) = (y|Ax).$$

2) *Show that A^* is a closed operator.*

Answer 2: If $y_n \in D(A^*)$

$$(A^*y_n|x) = (y_n|Ax).$$

If $\{y_n\}$ converges to $y \in X$ and A^*y_n converges to z we have, by the continuity of the scalar product,

$$(z|x) = (y|Ax), \qquad \text{i.e. } z = A^*y \text{ by definition.} \qquad \blacksquare$$

self-adjoint

An operator A is called **self-adjoint** if $A = A^*$. It is necessarily closed and densely defined.

spectrum

3) The **spectrum** of a closed operator A on a Banach space X is the set $\sigma(A)$ of numbers λ such that $A - \lambda\,\mathrm{Id}$ is either noninjective or nonsurjective on X (i.e., $R(A - \lambda\,\mathrm{Id}) \neq X$). This definition extends the definition (p. 60) of the spectrum for continuous operators.
If $A - \lambda\,\mathrm{Id}$ is noninjective, then the equation

$$Ax - \lambda x = 0 \qquad \text{has a solution } x \neq 0;$$

eigenvalue
eigenspace
point spectrum

λ is called an **eigenvalue** of A, and the space of the corresponding x's, which is of at least dimension 1, is called an **eigenspace.**
The set of eigenvalues is the **point spectrum** of A.
It can happen, if X is infinite dimensional, that $A - \lambda\,\mathrm{Id}$ is injective but not surjective. In this case, if $R(A - \lambda\,\mathrm{Id})$ is dense in X [resp. not dense] one says that λ is in the **continuous spectrum** of A [resp. in the **residual spectrum].**

continuous spectrum
residual spectrum

a) *Show that the spectrum $\sigma(A)$ of a self-adjoint real operator A on a Hilbert space X is real.*
b) *Show that the eigenspaces corresponding to different eigenvalues are orthogonal.*

Answer 3a: The proof is simple if $A - \lambda\,\mathrm{Id}$ is noninjective. Indeed

$$(x|Ax - \lambda x) = (x|Ax) - \bar{\lambda}\|x\|^2.$$

Since A is self-adjoint, $(x|Ax) = (Ax|x)$ is real, Thus if $\mathrm{Im}\,\lambda$ denotes the imaginary part of λ

$$\mathrm{Im}\,\lambda\|x\|^2 = \mathrm{Im}(x|Ax - \lambda x).$$

If $Ax - \lambda x = 0$ for $x \neq 0$ then $\mathrm{Im}\,\lambda = 0$. If one uses the physicist's convention $(x\,|\,\lambda x) = \lambda\|x\|^2$ the result is obviously the same.
If $A - \lambda\,\mathrm{Id}$ is injective but not surjective we use the following fundamental theorem. For its proof see, for instance [Reed and Simon, and Brezis, p. 29] who treat the more general case of Banach spaces.

Theorem: Let T be a closed, densely defined operator on a Hilbert space X. The following properties are equivalent:

(i) T is surjective, i.e., $R(T) = X$;

(ii) There exists a number $c > 0$ such that

$$\|x\| \le c\|T^*x\|, \qquad \forall x \in D(T^*):$$

(iii) $\ker(T^*) = \{0\}$ and $R(T^*)$ is closed.

Returning to the case $T = A - \lambda\,\mathrm{Id}$, $T^* = A - \bar{\lambda}\,\mathrm{Id}$, we have

$$\mathrm{Im}\,\bar{\lambda}\,\|x\|^2 = \mathrm{Im}(x|Ax - \bar{\lambda}x). \tag{1}$$

First we note that if $\lambda \in \sigma(A)$, then $\bar{\lambda} \in \sigma(A)$. Hence we shall assume $\mathrm{Im}\,\bar{\lambda} > 0$. It follows from (1) that

$$\mathrm{Im}\,\bar{\lambda}\,\|x\|^2 \le |(x|Ax - \bar{\lambda}x)|. \tag{2}$$

Hence we have an inequality of the form

$$\|x\| \le c\|Ax - \bar{\lambda}x\|, \qquad c > 0$$

which implies $A - \lambda\,\mathrm{Id}$ surjective, contrary to the hypothesis.
This concludes the proof that the spectrum of a self-adjoint real operator is real.
Answer 3b: To prove that eigenspaces corresponding to different eigenvalues are orthogonal we write

$$\lambda_1(x_1|x_2) = (Ax_1|x_2) = (x_1|Ax_2) = \lambda_2(x_1|x_2)$$

which implies if $\lambda \neq \lambda_2$ that $(x_1|x_2) = 0$. ■

The number λ is said to belong to the **essential spectrum** $\sigma_e(A)$ if $\lambda \in \sigma(A + K)$ for every compact operator[1] K on X. essential spectrum

An operator A on a Banach space X is called **Fredholm** if Fredholm

a) A is closed, $D(A)$ is dense in X and $R(A)$ is closed.
b) The kernel and **cokernel** (kernel of the transpose, p. 60) are finite dimensional. cokernel

The **index** of a Fredholm operator is index

$$i(A) = \dim\ker(A) - \dim\mathrm{coker}(A)$$

A classical theorem on Fredholm operators is:
If K is a compact operator and A is a Fredholm operator, the operator $A + K$ is Fredholm, with index $(A + K) =$ index A. In particular, if $A = \mathrm{Id}$, index $(\mathrm{Id} + K) = 0$.
It can be proved (see, for instance [Schechter, Reed and Simon]) that if A

[1] A compact operator is an operator which is continuous on X and such that the image of a bounded set is a compact set (p. 61).

is a closed operator then $\lambda \not\in \sigma_e(A)$ if and only if $A - \lambda\,\mathrm{Id}$ is a Fredholm operator of index zero.
4) *Show that if A is a real self-adjoint operator on a Hilbert space X then $\lambda \not\in \sigma_e(A)$ if and only if the dimension of the kernel of $A - \lambda\,\mathrm{Id}$ is finite and $R(A - \lambda\,\mathrm{Id})$ is closed in X.*

Answer 4: The implication $\lambda \not\in \sigma_e(A) \Rightarrow \dim\ker(A - \lambda\,\mathrm{Id})$ finite and $R(A - \lambda\,\mathrm{Id})$ closed in X follows from the previous statement. For the converse implication we can use the definitions and the fact that since $\sigma(A)$ is real

$$A - \lambda\,\mathrm{Id} = (A - \lambda\,\mathrm{Id})^* \quad \text{if} \quad \lambda \in \sigma(A).$$

Remark: We quote the following important results.
Let A be a compact operator on a Banach space X of infinite dimension. Then

(i) $0 \in \sigma(A)$, all other elements of $\sigma(A)$ are eigenvalues of A.
(ii) $\sigma(A) - \{0\}$ is either finite (possibly empty), or a sequence which converges to 0.

Let A be a compact self-adjoint operator on the separable Hilbert space X. Then X admits an hilbertian basis of eigenvectors of A.

We recall that for a continuous, hence bounded, self-adjoint operator, we have

$$\sigma(A) \subset [m, M], \quad m = \operatorname*{Inf}_{\|x\|=1} (Ax, x), \quad M = \operatorname*{Sup}_{\|x\|=1} (Ax, x)$$

with m and M both in $\sigma(A)$.
It can be proved (cf. for instance, Schechter, p. 38) that if A is a self-adjoint operator on a Hilbert space X and $\sigma(A) \subset [\nu, \infty]$ then

$$(Ax, x) \geq \nu \|x\|^2, \qquad x \in D(A).$$

We shall see applications in [Problem VI 15, Bounds].

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II. DIFFERENTIAL CALCULUS ON BANACH SPACES

1. SUPERSMOOTH MAPPINGS

We have given (p. 71) the definition of the (Fréchet) derivability of a mapping f between locally convex vector spaces, and (p. 79) the definition of C^k mappings between Banach spaces. If X and Y are Banach spaces the space $\mathscr{L}(X, Y)$ of continuous linear maps $X \to Y$ is a Banach space, but if X or Y is only a locally convex topological vector space the space $\mathscr{L}(X, Y)$ does not have a good topology: it is known for instance that the dual of a Fréchet space (p. 26) is never a Fréchet space. It is therefore more convenient to use a weaker notion of differentiability, the Gateaux differential, and to give new definitions for C^1 and C^k mappings (cf. for instance, Milnor, p. 1024).

Definition: Let X and Y be locally convex spaces. Let $f: U \to Y$ be a continuous map from an open set $U \in X$ into Y. Let $x \in U$ and $v \in X$. The **(Gateaux) derivative** of f in the direction v is the vector

Gateaux derivative

$$f'(x; v) \equiv f'_x(v) = \lim_{t=0} \frac{f(x+tv) - f(x)}{t}$$

if the limit exists. The mapping f is C^1 in U if this limit exists for each $x \in U$, each $v \in X$ and is continuous as a function: $U \times X \to Y$.
It can be proved that if f is C^1 in this sense, $f'_x: v \mapsto f'_x(v)$ is a linear map and the derivative of a composition of C^1 maps obeys the usual chain rule.
Moreover $f'_x = 0$, $x \in U$, U connected, implies f is constant in U. On the other hand the inverse function theorem (p. 90) and the existence and uniqueness theorem for ordinary differential equations (p. 95) are in general *not true* if X and Y are not Banach spaces.

Definition: A map $f: U \to Y$ is C^2 if it is C^1 and if for each $v_1 \in X$ the mapping

$$U \to Y \quad \text{by} \quad x \mapsto f'(x, v_1) \quad \text{is } C^1,$$

that is, if this mapping admits a (Gateaux) derivative in each direction $v_2 \in X$:

$$\lim_{t=0} \frac{f'(x + tv_2, v_1) - f'(x, v_1)}{t},$$

second derivatives

where the limit, denoted $f''_x(v_1, v_2)$ and called the **second derivative in the directions** $(\boldsymbol{v}_1, \boldsymbol{v}_2)$, depends continuously on $x \in U$, $v_1 \in X$, $v_2 \in X$.

Higher derivatives are defined similarly. Fundamental properties of derivatives are:
If f is of class C^r in U, then for each $x \in U$ the mapping

$$X \times X \times \cdots \times X \to X \quad \text{by} \quad (v_1, \ldots, v_r) \mapsto f_x^{(r)}(v_1, \ldots, v_r)$$

is a symmetric r-linear map.
Any composition of C^r maps is a C^r map.
If f is C^{n+1} in U, and if the straightline segment joining x and $x + h$ lies entirely in U then the **Taylor formula** holds:

Taylor formula

$$f(x + h) = f(x) + f'_x(h) + \tfrac{1}{2} f''_x(h, h) + \cdots + \frac{1}{n!} f_x^{(n)}(h, \ldots, h) + R_n$$

with remainder R_n given by

$$R_n = \frac{1}{n!} \int_0^1 (1 - t)^n f_{x+th}^{(n+1)}(h, \ldots, h)\, \mathrm{d}t.$$

We now consider the special case of $\mathbb{Z}_2$ graded commutative algebras, [Problem I 1, Graded algebras], with locally convex Hausdorff topologies for which sum and product are continuous operations.

1. HYPERDIFFERENTIABLE MAPPINGS $f\colon A \to A$

Let $A = A_+ + A_-$ be a $\mathbb{Z}_2$ graded algebra.

left right hyper-differentiable

A mapping $f\colon A \to A$ is called **left** [resp. **right**] **hyperdifferentiable** at $a \in A$ if it is differentiable (in the sense just defined) and if there exists an element $u \in A$ such that the linear mapping $f'_a\colon A \to A$, by $h \mapsto f'_a \cdot h$ is the left [resp. right] product by u:

$$f'_a \cdot h = uh \qquad [\text{resp. } hu].$$

Show that the C^2 and left [resp. right] hyperdifferentiable maps in an open set $\mathcal{U} \subset A$ are locally affine mappings in $\mathcal{U}$ if A satisfies the following hypothesis
(i) *A admits a unit e*
(ii) *$cb = 0$ for all $c \in A_-$ implies $b = 0$.*

Answer 1: Suppose that for $a \in \mathcal{U}$

$$f' \cdot h = u(a)h.$$

If f is C^2, the second derivative f''_a is a symmetric quadratic from

$$(k, h) \mapsto f''_a(k, h) = (u'(a) \cdot k)h.$$

By the hypothesis it is A-linear in h, so, by symmetry, also A-linear in k.
Therefore

$$f''_a(k, h) = v(a)kh.$$

Since A is a graded algebra we can have $v(a)$ symmetric in h and k only if $v(a)hk = 0$ for $h, k \in A_-$. It follows from the hypothesis that $v(a)h = 0$ and $v(a) = 0$, i.e., $f''_a = 0$, $a \in \mathcal{U}$.

2. SUPERDIFFERENTIABLE MAPPINGS

A mapping $f : A \to A$ is called **superdifferentiable** [resp. G^1] at $a \in A$ if it is differentiable [resp. C^1] and there exist elements $u, v \in A$ such that super-differentiable

$$f'_a(h) = h_+u + h_-v, \qquad \forall h = h_+ + h_- \in A_+ \oplus A_-. \tag{1}$$

a) *Show that u is uniquely defined if A admits a unit e, and that v is uniquely defined if $h_-b = 0$ for all $h_- \in A_-$ implies $b = 0$.*

b) *Show that when B is a DeWitt algebra, superdifferentiability is equivalent to the requirement that f'_a, is B_+ linear.*

Answer 2a: Take $h_+ = e$, the hypothesis on A shows that $f'_a \cdot (e + h_-) = u + h_-v$ define u and v uniquely.

Answer 2b: If f'_a is given by (1), it is obviously B_+ linear. Conversely, if f'_a, is B_+ linear, then $f'_a(h_+) = f'_a(e)h_+$.
We show now that $f'_a(h_-) = h_-v(a)$. For $x, y, z \in B_-$, we have $f'_a(xyz) = -f'_a(yxz)$ since $yx = -xy$. But, since xy and $yz \in B_+$, we have also $f'_a(xyz) = f'_a(x)yz$. Thus $f'_a(x)yz = -f'_a(y)xz$. This being true for all $z \in B_-$, we have for all $x, y \in B_-$

$$f'_a(x)y = -f'_a(y)x. \tag{2}$$

In particular

$$f'_a(x)x = 0, \qquad x \in B_-.$$

Thus taking $x = z^I$, one of the generators, we find that there exists an element $q_{(I)} \in B$ such that

$$f'_a(z^I) = q_{(I)}z^I \qquad \text{(no summation).}$$

The element $q_{(I)}$ is defined modulo the ideal generated by z^I. We show that we can construct a q such that

$$f_a'(z^I) = qz^I, \qquad \forall I.$$

Let q_1 be the solution of

$$f_a'(z^1) = q_1 z^1$$

which does not contain z^1.
We have by (2)

$$(f_a'(z^2) - q_1 z^2)z^i = 0, \qquad i = 1, 2.$$

Thus there exists $v_1 \in B$ containing neither z^1 nor z^2 such that

$$f_a'(z^2) - q_1 z^2 = v_1 z^1 z^2.$$

We choose

$$q_2 = q_1 + v_1 z^1.$$

We can construct v_n and q_n by induction on n:

$$(f_a'(z^{n+1}) - q_n z^{n+1})z^i = 0, \qquad i = 1, 2, \ldots, n+1$$

implies the existence of v_n such that

$$f_a'(z^{n+1}) - q_n z^{n+1} = v_n z^1 \ldots z^{n+1}$$

and we set

$$q_{n+1} = q_n + v_n z^1 \ldots z^n.$$

We have

$$f_a'(z^i) = q_{n+1} z^i, \qquad i = 1, \ldots, n+1,$$

if this same equation held for $i = 1, \ldots, n$ with q_{n+1} replaced by q_n. We have

$$q_{n+1} = q_1 + v_1 z^1 + v_2 z^2 + \cdots + v_n z^1 \ldots z^n.$$

The limit q, when n tends to infinity, exists in the topology of formal series [cf. Problem I 1 (Graded algebras)] since q_n and q_m, $m \geq n$, differ only by terms of order $\geq n$.

3. G^p-MAPPING

A C^p mapping is called G^p if $f_a^{(p)}(h_1, \ldots, h_p)$ is a polynomial in $h_{1+}, \ldots, h_{p+}, h_{1-}, \ldots, h_{p-}$ with coefficients in A.

a) *Show that a* C^p *and* G' *mapping is also* G^p *if* A *admits a unit and* $f^{(p)}(a)$ *vanishes when restricted to* $(A_-)^p$.
Show that if A *is a DeWitt algebra* B, *then a* C^p *and* G^1 *mapping is always* G^p. A G^p mapping, with p some convenient number is called **supersmooth.** supersmooth
A mapping which is G^p for any p is called G^∞.

b) *Prove that a polynomial on* A

$$P\colon a \to \sum_{p=0}^{n} c_p a^p, \qquad c_p \in A$$

is a G^∞ *mapping.*

c) *Let* B *be a DeWitt algebra. Prove that a mapping* $f\colon U \to B$ *defined by a formal series*

$$f(a) = \sum c_n a^n \qquad \textit{with } U = \{a \in B;\ |a_0| < K\}$$

is G^∞ *on* U *if* K *is such that the numerical series* $\sum |c_n| K^n$ *converges.*

d) *For* U *open in* B, *a mapping* $f\colon U \to B$ *given by the formal series*

$$f(a) = \sum c_n a^n, \quad a \in B, \quad c_n \in \mathbb{C}\ (\textit{or } \mathbb{R})$$

can obviously be written

$$f(a) = \sum c_n \left(a_+^n + n a_+^{n-1} a_-\right);$$

thus when f *is restricted to* B_- *it is an affine map*

$$f(a_-) = c_0 + c_1 a_-, \quad c_0, c_1 \in \mathbb{C} \textit{ or } \mathbb{R}. \tag{3}$$

Prove that conversely if a C^2 *mapping from an open set* $U \subset B$ *into* B *is* G^1 *then the restriction of* f *to* B_- *is of the form* (3), *but with arbitrary* $c_0, c_1 \in B$. *Does this last result hold for an arbitrary graded algebra?*

Answer 3a: Analogous to proofs of 1). For details see references.

Answer 3b:

$$a^n = a_+^n + n a_+^{n-1} a_-,$$

$$(a^n)'(h) = \sum_{q=0}^{n-1} a^{n-1-q} h a^q = h_+ n a^{n-1} + h_- n a_+^{n-1}.$$

Answer 3c: The mapping f is well defined (cf. above). For a, $a + h \in U$, a straightforward computation using the fact that $a_- h_- + h_- a_- = 0$ and $f(a) = \sum c_n (a_+^n + n a_+^{n-1} a_-)$ shows that

$$\begin{aligned}\lim_{t=0}\frac{f(a+th)-f(a)}{t} &= \frac{1}{t}\sum c_n((a+th)^n - a^n)\\ &= h_+\sum c_n n a^{n-1} + h_-\sum c_n n a_+^{n-1}\\ &= h_+(\partial_+ f)(a) + h_-(\partial_- f)(a).\end{aligned}$$

$\partial_+\partial_-$ The mapping f is thus G^1 with the formal series $\partial_+ f$ and $\partial_- f$ well defined in U.

$$(\partial_+ f)(a) = \sum c_n n a^{n-1}, \qquad (\partial_- f) = \sum c_n n a_+^{n-1}.$$

We deduce from these formulas that f''_a vanishes on $(A_-)^2$, that is

$$(\partial_-\partial_- f)(a) = 0;$$

thus f is G^2. An inductive proof shows that f is G^∞, with partial derivatives

$$\begin{aligned}(\partial_+^p f)(a) &= \sum c_n n(n-1)\ldots(n-p+l)a^{n-p},\\ (\partial_+^{p-1}\partial_- f)(a) &= \sum c_n n(n-1)\ldots(n-p+1)a_+^{n-p}.\end{aligned}$$

All partial derivatives which involve two or more ∂_- are zero.

Answer 3d: If $a, b \in A$ and f is G^2 in A we have, by the Taylor formula

$$f(a) = f(b) + f'_b(a-b) + \frac{1}{2!}\int_0^1 (1-t)^2 f''_{b+t(a-b)}(a-b)^2\,\mathrm{d}t,$$

where the action of f'' on $(a-b)^2$ is multiplication in the algebra. Taking $b = a_+$ gives $(a-b)^2 = (a_-)^2 = 0$ and

$$f(a) = f(a_+) + f'_{a_+}a_-.$$

If A is a DeWitt algebra every G^1 and C^2 mapping is also G^2.

4. SUPERDIFFERENTIABLE MAPPING $f\colon A^n \to A^p$

Extend naturally the definition of superdifferentiability to mappings $A^n \to A^p$, or more generally $A_+^m \times A_-^n \to A_+^p \times A_-^n$.

Answer: A C^1 mapping $f\colon \Omega \to A_+^p \times A_-^q$, with Ω an open set of $A_+^m \times A_-^n$, is called G^1 if $f'_a \cdot h$ is obtained from h by the action of an $(m+n)$ $\times(p+q)$ matrix with elements in A: when $h \in A_+^m \times A_-^n$,

$$f'_a(h) = \sum_{i=1}^{m} h_+^i(\partial_{+i} f)(a) + \sum_{j=1}^{n} h_-^j(\partial_{-j} f)(a).$$

The elements $(\partial_{+i}f)(a)$ and $(\partial_{-j}f)(a)$ of A are called **partial derivatives** of f. partial derivatives

5. *Show that a G^{n+1} mapping $f\colon A_+^m \times A_-^n \to A$ is of the form*

$$f(a) = f_0(a_+) + \sum_{i=1}^{m} f_i(a_+)a_-^i + \sum_{i\neq j} f_{ij}a_-^i a_-^j + \cdots + f_{1\ldots n}(a_+)a_-^1 \ldots a_-^n$$

where $f_0, f_i, \ldots, f_{1\ldots n}$ are G^{n+1} functions on A_+^m.

Answer: Use the Taylor formula.

We thank R. Schmidt for calling our attention to Gateaux derivatives.

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2. BEREZIN INTEGRATION; GAUSSIAN INTEGRALS

I. DEFINITIONS

a) Let B be a DeWitt algebra $B = B_+ \oplus B_-$. Let $y \mapsto f(y)$ be a supersmooth mapping from B_- into B. Then (cf. Problem II 1, Supersmooth mappings)

$$f(y) = f_0 + f_1 y \qquad \text{with} \qquad f_0 \text{ and } f_1 \text{ in } B.$$

By definition the **Berezin integral** of f on B_- is Berezin integral on B.

$$\int_{B_-} f(y)\,\mathrm{d}y = cf_1,$$

where c is a constant independent of f, chosen equal to 1 by Berezin, to $(2\pi\mathrm{i})^{1/2}$ by DeWitt, and $(2\pi)^{1/2}$ in section II of this problem.
Now let f be a supersmooth mapping $B_-^q \to B$ by $y \mapsto f(y)$, $y = (y^1, \ldots, y^q)$. Then

$$f(y) = f_0 + f_\alpha y^\alpha + \cdots + f_{1\ldots q}y^1 \ldots y^q, \qquad f_0, f_\alpha, \ldots, f_{1\ldots q} \in B$$

and the Berezin integral of f on B^q_- is by definition

$$\int_{B^q_-} f(y)\,\mathrm{d}y = c^q f_{1\ldots q}.$$

Let Y be an invertible $q \times q$ matrix with elements in B_+. It determines an isomorphism $B^q_- \to B^q_-$ by $\hat{y} \mapsto y = Y\hat{y}$.
Show the change of variables formula

$$\int_{B^q_-} f(y)\,\mathrm{d}y = \int_{B^q_-} f(Y\hat{y})(\det Y)^{-1}\,\mathrm{d}\hat{y}.$$

More generally it can be proved for instance by induction on q (cf. Vladimirov and Volovich, p. 756) that if $\hat{y} \mapsto y = F(\hat{y})$ is a supersmooth mapping $B^q_- \to B^q_-$ such that $F'(0)$ is invertible (since F is supersmooth F' is a $q \times q$ matrix with elements in B, in fact B_+ since F' is a linear map from B^q_- into B^q_-), then

$$\int_{B^q_-} f(y)\,\mathrm{d}y = \int_{B^q_-} (f \circ F)(\hat{y})(\det F'(\hat{y}))^{-1}\,\mathrm{d}\hat{y}.$$

integrable

integral on B_+

b) Let f be a mapping from B_+ into B. We call it **integrable** if its restriction to $\mathbb{R}$, the body of B, is a B-valued integrable function for the Lebesgue measure of $\mathbb{R}$. Its **integral** is then the element of B:

$$\int_{B_+} f(u)\,\mathrm{d}u = \int_{\mathbb{R}} f(u_B)\,\mathrm{d}u_B, \qquad u_B \in \mathbb{R}.$$

Analogously the integral of a mapping $f\colon B^p_+ \to B$ is, if the right-hand side is defined:

$$\int_{B^p_+} f(u)\,\mathrm{d}u = \int_{\mathbb{R}^p} f(u_B)\,\mathrm{d}u_B, \qquad u = (u^1, \ldots, u^p), \qquad \mathrm{d}u_B = \mathrm{d}u^1_B \ldots \mathrm{d}u^p_B.$$

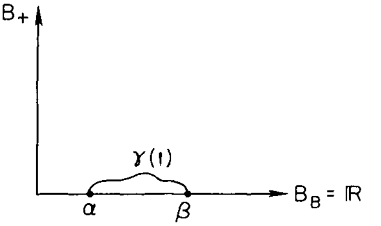
Give a formula for the change of variables in B^p_+, $u = \varphi(\hat{u})$, with φ a supersmooth bijective mapping $B^p_+ \to B^p_+$ such that $u_B = \varphi(\hat{u}_B)$.

c) *Let f be a supersmooth mapping $B^p_+ \times B^q_- \to B$. Define its integrability and its integral. Give a formula for the change of variables*

$$u = \varphi(\hat{u}), \qquad y = a(\hat{u}) + Y(\hat{u})\hat{y}, \qquad u \in B^p_+, \quad y \in B^q_-$$

where φ is a supersmooth mapping $B^p_+ \to B^p_+$ such that $\varphi(\hat{u}_B) = u_B$ and preserves the orientation, a is a supersmooth mapping $B^p_+ \to B^q_-$ and $Y(\hat{u})$ is a $q \times q$ matrix with elements in B_+, each a supersmooth function on B^p_+.

d) Let f be a supersmooth mapping from $U \subset B_+$ into B and γ a C^1 path in U, mapping $t \mapsto \gamma(t)$ from an interval $[\alpha, \beta]$ of $\mathbb{R}$ into U, and such that $\gamma(\alpha) = \gamma_B(\alpha) = \alpha$, $\gamma(\beta) = \gamma_B(\beta) = \beta$. integral over γ



The **integral** $I_\gamma(f)$ of f over γ is naturally defined by

$$I_\gamma(f) = \int_\alpha^\beta f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t$$

with $\gamma'(t)$ an element of B_+ depending continuously on t.

Give sufficient conditions on γ and U to have the equality:

$$I_\gamma(f) = I_{\gamma_0}(f),$$

where γ_0 is the soulless path $[\alpha, \beta] \mapsto \mathbb{R}$ by the identity map, i.e.

$$I_{\gamma_0}(f) = \int_\alpha^\beta f(\tau)\,\mathrm{d}\tau.$$

Answer 1a: We deduce from the expression of f, using the antisymmetry $y^\alpha y^\beta = -y^\beta y^\alpha$ that the term of order q in $f(Y\hat{y})$ is

$$f(Y\hat{y})_{1\ldots q} = f_{1\ldots q}(\det Y).$$

Thus, using the definition of the Berezin integral, we find

$$\int_{B_-^q} f(Y\hat{y})(\det Y)^{-1}\,\mathrm{d}\hat{y} = c^q f_{1\ldots q} = \int_{B_-^q} f(y)\,\mathrm{d}y.$$

Note that if $(y^1, \ldots, y^q)$ is just a permutation π of $(\hat{y}^1 \ldots \hat{y}^q)$ then the previous formula may also be written

$$\int_{B_-^q} f(y)\,\mathrm{d}y = (-1)^{\operatorname{sign}\pi} \int_{B_-^q} g(\hat{y})\,\mathrm{d}\hat{y}, \quad g = f \circ \pi.$$

Answer 1b: By definition

$$\int_{B_+^p} f(u)\,\mathrm{d}u = \int_{\mathbb{R}^p} f(u_B)\,\mathrm{d}u_B.$$

If $u_B = \varphi(\hat{u}_B)$ is a diffeomorphism of $\mathbb{R}^p$ the classical formula gives

$$\int_{\mathbb{R}^p} (f \circ \varphi)(\hat{u}_B)|(J\varphi)(\hat{u}_B)|\,\mathrm{d}\hat{u}_B = \int_{\mathbb{R}^p} f(u_B)\,\mathrm{d}u_B.$$

If φ is orientation preserving, i.e. the restriction of $J\varphi$ to $\mathbb{R}^p$ is positive, we can suppress the absolute value, and since the body of B is a subalgebra of B we have

$$\int_{B_+^p} (f \circ \varphi)(\hat{u})(J\varphi)(\hat{u})\, d\hat{u} = \int_{B_+^p} f(u)\, du.$$

Note that this formula can be generalized to mappings f whose restriction to $\mathbb{R}^p$ has support in an open set U of $\mathbb{R}^p$ and φ is an orientation preserving diffeomorphism $U \to V$.

Answer 1c: If f is a supersmooth mapping $B_+^p \times B_-^q \to B$ by $x = (u, y) \mapsto f(u, y)$ then

$$f(u, y) = f_0(u) + f_\alpha(u) y^\alpha + \cdots + f_{1 \ldots q}(u) y^1 \ldots y^q.$$

If the usual Lebesgue integral of the right-hand side exists we can define

$$\int_{B_+^p \times B_-^q} f\, du\, dy = c^q \int_{\mathbb{R}^p} f_{1 \ldots q}(u_B)\, du_B.$$

Under a change of variables of the indicated type we have, denoting $x = (u, y) = X(\hat{x})$, $\hat{x} = (\hat{u}, \hat{y})$

$$(f \circ X)(\hat{u}, \hat{y}) = (f_0 \circ \varphi)(\hat{u}) + \cdots + (f_{1 \ldots q} \circ \varphi)(\hat{u}) \det Y(\hat{u}) \hat{y}^1 \ldots \hat{y}^q,$$

thus, using previous results,

$$\int_{B_+^p \times B_-^q} (f \circ X) \operatorname{Ber} X(\hat{u}, \hat{y})\, d\hat{u}\, d\hat{y} = \int_{B_+^p \times B_-^q} f(u, y)\, du\, dy$$

where (cf. Problem I 2, Berezinian)

$$\operatorname{Ber} X = (J\varphi)(\det Y)^{-1}.$$

Remark: A more general theory of integration on "foliated submanifolds" of $B_+^p \times B_-^q$ can be developed, using the idea of question 3, i.e., defining integrals over immersed submanifolds of B_+^p, considered as mappings $G \to B_+^p$ where G is an open subset of $\mathbb{R}^p$; we considered the particular case $G = \mathbb{R}^p$ and the identity mapping. More general formulas for the change of variables can then be obtained [DeWitt, p. 41]. See the references.

Answer 1d: We have

$$\gamma(t) = \gamma_B(t) + \gamma_S(t).$$

We suppose that the real function $t \mapsto \tau = \gamma_B(t)$ is a diffeomorphism $[\alpha, \beta] \to [\alpha, \beta]$. By making the change of variables in the classical integral we find

$$I_\gamma(f) = \int_\alpha^\beta f(\tau + \gamma_S(\tau))(1 + \gamma_S'(\tau))\, d\tau.$$

We expand the integrand in a formal series of functions of τ, using the Taylor formula, valid if $\tau + \lambda\gamma_S(\tau) \in U$ for $\lambda \in [0, 1]$ (hence γ can be continuously deformed to γ_0 in U)

$$f(\tau + \gamma_S(\tau))(1 + \gamma_S'(\tau)) = \Big(\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(\tau)\gamma_S^n(\tau)\Big)(1 + \gamma_S'(\tau))$$

and we find, since $\gamma_S \in B_+$:

$$f(\tau + \gamma_S(\tau))(1 + \gamma_S'(\tau)) = f(\tau) + \left[\sum_{n=0}^{\infty} \frac{1}{(n+1)!} f^{(n)}(\tau)\gamma_S^{n+1}(\tau)\right]';$$

from which the result follows since $\gamma_S(\alpha) = \gamma_S(\beta) = 0$.
Note that the result obtained here is analogous to a result in path integration in the complex plane.
For more general formulations using superdifferential forms and Stokes formula, see Vladimirov and Volovich, Rogers, or Bruzzo.

II. COMPUTATION OF GAUSSIAN INTEGRALS

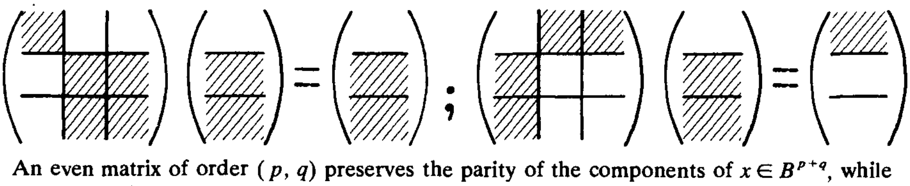
a) A quadratic form Q on $B_+^p \times B_-^q$

$$Q(x) = x^i M_{ij} x^j$$

is defined by a matrix with elements in B, M_{ij}, $i, j = 1, \ldots, p + q$. The matrix M is supposed to be even, i.e., of the type

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

elements of A and D [resp. B and C] even [resp. odd].
The action of M on $B_+^p \times B_-^q$ then preserves the parities.



An even matrix of order (p, q) preserves the parity of the components of $x \in B^{p+q}$, while an odd matrix of order (p, q) inverses their parity. The shaded areas cover even elements. The matrix on the left is even, the one on the right is odd.

An even graded matrix M of order (p, q) is said to be supersymmetric if

$$x^i M_{ij} x^j = x^j M_{ij} x^i \quad \text{for every } x^i, x^j, \quad \text{no summation.} \tag{1}$$

Show that

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is supersymmetric if and only if

$$A = \tilde{A}, \quad \text{i.e., } A_{ij} = A_{ji}; \ D = -\tilde{D}; \ C = -B. \tag{2}$$

Show that if M is supersymmetric and invertible, then q is even.

Answer 2a: (2) follows from (1) by using the fact that the elements of A and D are even, the elements of B and C are odd, $(x^1, \ldots, x^p)$ are even, $(x^{p+1}, \ldots, x^{p+q})$ are odd and the fact that odd elements anticommute. M is invertible if A and D are invertible; the antisymmetry of D implies then that q is even.

b) *Let M be an even invertible supersymmetric matrix of order (p, q). Show, by appropriate change of variables, that*

$$I \equiv \int_{B_+^p \times B_-^q} \exp\left(-\tfrac{1}{2} x^i M_{ij} x^j\right) \mathrm{d}^{p,q} x = (2\pi)^{(p+q)/2} (\operatorname{Ber} M)^{-1/2}.$$

($\mathrm{d}^{p,q}x$ *stands for the* $\mathrm{d}u\,\mathrm{d}y$ *of the previous section, with the notation* $u = (x^1, \ldots, x^p) \in B_+^p$ *and* $y = (x^{p+1}, \ldots, x^{p+q}) \in B_-^q$.

Answer 2b: The linear change of variables $x = X\hat{x}$ by

$$X = \begin{pmatrix} \mathbb{1}_p & 0 \\ D^{-1}\tilde{B} & \mathbb{1}_q \end{pmatrix} \quad \text{that is} \quad \begin{array}{c} u = \hat{u} \\ y = D^{-1}\tilde{B}\hat{u} + \hat{y} \end{array}$$

block diagonalizes M. Indeed if $x = (u, y) \in B_+^p \times B_-^q$ we have

$$Q(x) \equiv Q(u, y) = {}^{\mathrm{T}}u(Au + By) + {}^{\mathrm{T}}y(-\tilde{B}u + Dy).$$

Thus, under the indicated change, using the symmetries of A and D

$$(Q \circ X)(\hat{x}) \equiv (Q \circ X)(\hat{u}, \hat{y}) = {}^{\mathrm{T}}\hat{u}(A + BD^{-1}\tilde{B})\hat{u} + {}^{\mathrm{T}}yDy = \hat{x}^i \hat{M}_{ij} \hat{x}^j$$

with

$$\hat{M} = \begin{pmatrix} \hat{A} & 0 \\ \mathbf{0} & D \end{pmatrix} \qquad \text{with } \hat{A} = A + BD^{-1}\tilde{B}.$$

The Berezinian of the change of variables is unity, therefore

$$\int_{B_+^p \times B_-^q} \exp\left(-\tfrac{1}{2} x^i M_{ij} x^j\right) \mathrm{d}^{p,q} x = \int_{B_+^p \times B_-^q} \exp\left(-\tfrac{1}{2} \hat{x}^i \hat{M}_{ij} \hat{x}^j\right) \mathrm{d}^{p,q} \hat{x}.$$

We shall further reduce $\hat{A}$, a symmetric matrix, and D, an antisymmetric one, to canonical forms. It is known that there exist orthogonal transformations which reduce real matrices of such types respectively to diagonal or block diagonal ones. On the other hand we can always reduce symmetric or antisymmetric invertible matrices to their bodies because we have the identity, for any such matrix N

$$N \equiv \widetilde{\left(\mathbb{1} + N_B^{-1} N_S\right)}^{1/2} N_B^{1/2} N_B^{1/2} \left(\mathbb{1} + N_B^{-1} N_S\right)^{1/2}$$

We perform all appropriate transformations by considering the linear mapping $B_+^p \times B_-^q \to B_+^q \times B_-^q$ given by $\bar{x} = L\hat{x}$ with

$$L = \begin{pmatrix} O_1(\mathbb{1}_p + \hat{A}_B^{-1} \hat{A}_S)^{1/2} & 0 \\ 0 & O_2(\mathbb{1}_q + D_B^{-1} D_S)^{1/2} \end{pmatrix},$$

where O_1 and O_2 are such that $\det O_1 = 1$ and $\det O_2 = 1$, and $O_1 \hat{A}_B \tilde{O}_1 = \operatorname{diag}(a_1 \dots a_p)$, $a_i \in \mathbb{R}$

$$O_2 D_B \tilde{O}_2 = \operatorname{diag}\left(\begin{pmatrix} 0 & d_1 \\ -d_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & d_{q/2} \\ -d_{q/2} & 0 \end{pmatrix}\right).$$

We have then

$$(Q \circ X \circ L)^{-1}(\bar{u}, \bar{y}) = \sum_{i=1}^{p} a_i (\bar{u}^i)^2 + 2d_1 \bar{y}^1 \bar{y}^2 + \dots + 2d_{q/2} \bar{y}^{q-1} \bar{y}^q$$

while

$$\operatorname{Ber} L^{-1} = \det(\mathbb{1}_p + \hat{A}_B^{-1} \hat{A}_S)^{-1/2} \det(\mathbb{1}_q + D_B^{-1} D_S)^{1/2}.$$

Our integral therefore is

$$I = \int_{B_+^p \times B_-^q} \exp\left(-\tfrac{1}{2} \sum a_i (\bar{u}^i)^2 - d_1 \bar{y}^1 \bar{y}^2 - \dots - d_{q/2} \bar{y}^{q-1} \bar{y}^q\right) \times \operatorname{Ber} L^{-1} \, \mathrm{d}\bar{u} \, \mathrm{d}\bar{y}.$$

We expand the exponential to find the term in $\bar{y}_1 \dots \bar{y}_q$, then we integrate over $\mathbb{R}^p$, after replacing $\bar{u}^i$ by a real variable: the integral exists in the usual sense if all $a_i > 0$, and (taking $c = \sqrt{2\pi}$ in the Berezin integral) it is

$$I = (-1)^{q/2} (2\pi)^{(p+q)/2} (\operatorname{Ber} L^{-1}) (a_1 \dots a_p)^{-1/2} d_1 \dots d_{q/2}.$$

We note that

$$a_1 \dots a_p = \det \hat{A}_B, \quad d_1 \dots d_{q/2} = (\det D_B)^{1/2};$$

thus, using the expression of $\operatorname{Ber} L^{-1}$, and the evenness of $\hat{A}$ and D:

$$I = (2\pi)^{(p+q)/2} (\det \hat{A})^{-1/2} (\det D)^{1/2},$$

i.e.,

$$I = (2\pi)^{(p+q)/2} (\operatorname{Ber} M)^{-1/2}.$$

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3. NOETHER'S THEOREMS I

In this problem we consider systems with a finite number of degrees of freedom (questions 1–5) and systems with an infinite number of degrees of freedom (question 6). In the first case a dynamical variable is a path q: $[a, b] \to M$, in the second case a mapping u: $V \to M$; V and M are finite dimensional smooth manifolds. We prove the two fundamental Noether theorems. In questions 3 and 4 the lagrangian is taken to be invariant under a Lie group G of diffeomorphisms of M; this implies the existence of conserved quantities (identified with elements of the dual of the Lie algebra of G). In question 5 the action admits an infinite dimensional invariance group; this implies the existence of identities.

Let L: $TM \times [a, b] \to \mathbb{R}$ be the lagrangian (p. 169) of a dynamical system with configuration space the smooth manifold M. The action S is the mapping $C^1([a, b], M) \to \mathbb{R}$ by $q \mapsto S(q)$ with

$$S(q) = \int_a^b L(q(t), \dot{q}(t), t)\, \mathrm{d}t.$$

1) *Show that a diffeomorphism* $\varphi: M \to M$ *induces a diffeomorphism* $T\varphi: TM \to TM$.

2) *Show that if* $q_0: t \mapsto q_0(t)$ *is a critical point of* S *and* φ *a diffeomorphism of* M *then* $r_0 = \varphi(q_0)$ *is a critical point of* $S \circ \varphi^{-1}$.

3) Recall that a necessary condition for a smooth (C^2) mapping q to be a critical point of S is that it satisfy the **Euler equations** Euler equations

$$\mathscr{E}_L \equiv -\frac{\mathrm{d}}{\mathrm{d}t} L'_{\dot{q}}(q(t), \dot{q}(t), t) + L'_q(q(t), \dot{q}(t), t) = 0,$$

where $L'_{\dot{q}}$ denotes the partial derivative along the fibre T_qM and L'_q the partial derivative on M. One calls a solution of the Euler equations on a connected interval of $\mathbb{R}$ a **trajectory** of the dynamical system. trajectory
Suppose that L *is invariant under the* 1*-parameter group* $T\varphi_s$ *of diffeomorphisms of* TM *induced by the* 1*-parameter group of diffeomorphisms* φ_s *of* M, *that is*

$$L(\varphi_s \circ q, \varphi'_s \cdot v_q, .) = L(q, v_q, .), \quad \forall q \in M, v_q \quad \text{tangent to } M \text{ at } q. \; (1)$$

Let the duality between TM *and* T^*M *be denoted by a dot. Show that the numerical function defined by*

$$t \mapsto L'_{\dot{q}}(q(t), \dot{q}(t), t) \cdot \frac{\partial}{\partial s} \varphi_s(q(t))$$

is constant along each trajectory of the dynamical system.
Show that the conclusion is the same if (1) *holds but the trajectories are those of the dynamical system with lagrangian*

$$L_1 = L + \frac{\mathrm{d}f(q)}{\mathrm{d}t},$$

where f *is an arbitrary smooth function on* M.

4) *Suppose that the lagrangian* L *is invariant under a Lie group* G *of diffeomorphisms of* M. *Define the* **momentum mapping** *as a map from the space of trajectories of the dynamical system into the dual* $\mathscr{G}^*$ *of the Lie algebra of* G. momentum mapping

5) *Let* $q^{-1}TM$ *be the bundle with base* $[a, b]$ *and fibre* $T_{q(t)}M$ *at* $t \in [a, b]$. *We recall that the derivative of the mapping* $q \mapsto S(q)$ *acting on a smooth section* h *of this bundle with compact support in* (a, b) *is*

$$S'(q) \cdot h = \int_a^b \mathscr{E}_L(q) \cdot h \, \mathrm{d}t.$$

Suppose that $S'(q) \cdot h \equiv 0$ *for all* h *of the form*

$$h = \mathrm{D}u$$

with D *some linear differential operator mapping smooth sections* u *of a vector bundle* E *over* (a, b) *into smooth sections of* $q^{-1}TM$. *Find an identity satisfied by* $\mathscr{E}_L(q)$.

6) *Extend the previous results to lagrangians defined for smooth mappings* $u: V \to M$, *with* V *and* M *finite dimensional manifolds.*

Answer 1: If φ is a C^1 map, its derivative $\varphi'(q)$ at q is a linear map $T_qM \to T_{\varphi(q)}M$. If φ is a diffeomorphism (of class C^p), $\varphi'(q)$ is an isomorphism and $T\varphi: (q, v_q) \mapsto (\varphi(q), \varphi'(q) \cdot v_q)$ is a (fibered) diffeomorphism of TM of class C^{p-1}.

Answer 2: The mappings $S \circ \varphi^{-1}$ is defined by

$$(S \circ \varphi^{-1})(r) = \int_a^b (L \circ T\varphi^{-1})(r, \dot{r}, t)\, \mathrm{d}t.$$

We have (letting primes denote derivatives of maps)

$$(S \circ \varphi^{-1})'(r) = S'(\varphi^{-1}(r)) \cdot (\varphi^{-1})'(r);$$

thus $(S \circ \varphi^{-1})'(r_0) = 0$ if $S'(q_0) = 0$.

Answer 3: $(\partial/\partial s)\varphi_s(q(t))$ is a tangent vector to M at $\varphi_s(q(t))$. We set

$$A \equiv L'_{\dot{q}}(\varphi_s(q(t)), \frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(q(t)), t) \cdot \frac{\partial}{\partial s}\varphi_s(q(t));$$

we have

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(L'_{\dot{q}}(\varphi_s(q(t)), \frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(q(t)), t)\right) \cdot \frac{\partial}{\partial s}\varphi_s(q(t))$$
$$+ L'_{\dot{q}}(\varphi_s(q(t)), \frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(q(t)), t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial s}\varphi_s(q(t)).$$

On the other hand

$$\frac{\partial}{\partial s} L(\varphi_s(q(t)), \frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(q(t)), t) = L'_q(\varphi_s(q(t)), \frac{\mathrm{d}\varphi_s}{\mathrm{d}t}(q(t)), t) \cdot \frac{\partial \varphi_s}{\partial s}(q(t))$$
$$+ L'_{\dot{q}}(\varphi_s(q(t)), \frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(q(t)), t) \cdot \frac{\partial}{\partial s}\frac{\mathrm{d}\varphi_s}{\mathrm{d}t}.$$

Finally, since $\mathrm{d}/\mathrm{d}t$ and $\partial/\partial s$ commute

$$\frac{\mathrm{d}A}{\mathrm{d}t} \equiv \mathscr{E}_L \cdot \frac{\partial}{\partial s}\varphi_s(q(t)) + \frac{\partial}{\partial s} L\left(\varphi_s(q(t)), \frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(q(t)), t\right).$$

Therefore $\mathrm{d}A/\mathrm{d}t = 0$ if L is invariant under φ_s, that is if

$$L(\varphi_s \circ q, T\varphi_s \circ \dot{q}, .) = L(q, \dot{q}, .),$$

and if $\mathscr{E}_L = 0$; thus A is constant on a trajectory.
If $L_1 = L + (\mathrm{d}/\mathrm{d}t)f(q)$ and L is invariant, the conclusion still holds for the trajectories of L_1, since L_1 and L have the same Euler equations.

Answer 4: We have just seen that to the generator $\xi = (\partial/\partial s)\varphi_s(q(t))|_{s=0}$ of a 1-parameter group of diffeomorphisms of M which leaves L invariant, and to each trajectory q of the dynamical system, there corresponds a number

$$L'_{\dot{q}}(q(t), \dot{q}(t), t) \cdot \xi$$

which does not depend on t.
If G is a Lie group of diffeomorphisms which acts effectively on M, the generators ξ of the 1-parameter subgroups are the elements of a vector space isomorphic to the Lie algebra $\mathscr{G}$ of G. Given a trajectory q, the linear mapping

$$\xi \mapsto L'_{\dot{q}}(q(t), \dot{q}(t), t) \cdot \xi$$

defines an element a of $\mathscr{G}^*$. The mapping $q \mapsto a$ is called the **momentum mapping**. momentum mapping

Answer 5: If $S'(q) \cdot h = \int_a^b \mathscr{E}_L(q) \cdot \mathrm{D}u \, \mathrm{d}t = 0$ for all u, then for $u \colon t \mapsto u(t) \in E$ with compact support in (a, b) we have

$$S'(q) \cdot h = \int_a^b D^* \mathscr{E}_L(q) \cdot u \, \mathrm{d}t = 0$$

with D^* the adjoint operator of D, mapping sections of $q^{-1}T^*M$ into sections of E^*. As a consequence

$$D^* \mathscr{E}_L(q) = 0 \qquad \forall q.$$

Answer 6: Let $u \colon V \to M$ be a smooth mapping between smooth finite dimensional manifolds V and M. The differential $\mathrm{D}u$, or u', is a section of the vector bundle $T^*V \otimes u^{-1}TM$ (cf. Problem V 11, Harmonic maps), with base V and typical fibre at $x \in V$ the vector space $T^*_x V \otimes T_{u(x)}M$. A lagrangian function of derivatives up to first order is obtained by composition of a smooth map $L: M \times (T^*V \otimes TM) \times V \to \mathbb{R}$ with $(u, \mathrm{D}u)$, and the action $S(u)$ is obtained by integrating the lagrangian over V, assumed to be oriented, and[1] relatively compact with boundary ∂V:

$$S(u) = \int_V L(u(x), (\mathrm{D}u)(x), x)\tau(x)$$

where τ is a volume form on V.

The derivative at $u \in C^2(V, M)$ of the mapping $S: C^1(V, M) \to \mathbb{R}$ given by $u \mapsto S(u)$ is the linear map on sections h of the vector bundle $u^{-1}TM$ with base V and fibre $T_{u(x)}M$ at x given by (we omit x in the right-hand side for brevity)

$$S'(u) \cdot h = \int_V \mathscr{E}_L(u) \cdot h\tau + \int_{\partial V} \omega, \tag{2}$$

where $\mathscr{E}_L(u)$, the Euler operator on u, is given by

$$\mathscr{E}_L(u) \equiv L'_u(u, \mathrm{D}u, .) - \operatorname{div}(L'_{\mathrm{D}u}(u, \mathrm{D}u, .)).$$

For each given u, $\mathscr{E}_L(u)$ is a section $x \mapsto (\mathscr{E}_L(u))(x)$ of the vector bundle $u^{-1}T^*M$, the dual of $u^{-1}TM$; the dot on the right-hand side of eq. (2) denotes the duality between these finite dimensional vector bundles. The divergence is relative to the volume form τ; the explicit expression in local coordinates x^i of V and y^α of M where $u = (u^\alpha(x^i))$, L has expression $\bar{L}(u^\alpha, \partial_i u^\alpha, x^i)$, and $\tau = \rho(x_i)\,\mathrm{d}x^1 \ldots \mathrm{d}x^d$, is

$$\bar{\mathscr{E}}_L(u)_\alpha = \frac{\partial \bar{L}}{\partial u^\alpha} - \frac{1}{\rho}\frac{\partial}{\partial x^i}\left(\rho \frac{\partial \bar{L}}{\partial(\partial u^\alpha/\partial x^i)}\right).$$

The $d-1$ form ω has for its differential

$$\mathrm{d}\omega = \operatorname{div}(L'_{\mathrm{D}u} \cdot h)\tau$$

with $L'_{\mathrm{D}u}$ a section of $TV \otimes u^{-1}T^*M \to V$. This can be written

$$\omega = i_X\tau, \qquad X = h \cdot L'_{\mathrm{D}u}, \qquad \text{a vector field on } V,$$

and in coordinates (cf. p. 207)

$$\omega = \frac{1}{(p-1)!}\frac{\partial L}{\partial(\partial_i u^\alpha)} h^\alpha \tau_{ij_2 \ldots j_d}\,\mathrm{d}x^{j_2}\,\mathrm{d}x^{j_d}.$$

Remark: The addition to $L\tau$ of an exact differential does not modify the Euler operators: L and

$$L_1 = L + \operatorname{div} f(u)$$

[1] If $V \cup \partial V$ is compact – i.e., if V is relatively compact – the integral exists without hypotheses other than smoothness on L and u; in other cases further hypotheses have to hold for the integral to exist.

have the same $\mathscr{E}_L(u)$. The derivatives $S'(u)$ and $S_1'(u)$ are equal on h's with compact support in V (i.e., vanishing on ∂V).

Let φ_s be a 1-parameter group of diffeomorphisms of M which leaves L invariant, that is,

$$L(\varphi_s \circ u, \varphi_s{}' \cdot v_u, .) = L(u, v_u, .) \qquad \forall u \in M, v_u \in T_u M.$$

We replace the scalar A of answer 3 by the vector field on V

$$J_s(x) = L'_{\mathrm{D}u}(\varphi_s(u(x)), \mathrm{D}\varphi_s(u(x)), x) \cdot \frac{\partial \varphi_s}{\partial s}(x);$$

in coordinates the components of J_s are

$$J_s^i(x^h) = \frac{\partial \bar{L}}{\partial(\partial u^\alpha / \partial x^i)} (\varphi_s^\beta(u^\gamma(x^k)), \partial_i \varphi_s^\beta(u^\gamma(x^k)), x^i) \frac{\partial \varphi_s^\alpha(x^k)}{\partial s}.$$

We have

$$\operatorname{div} J_s = \operatorname{div}(L'_{\mathrm{D}u}) \cdot \frac{\partial \varphi_s}{\partial s} + L'_{\mathrm{D}u} \cdot \mathrm{D} \frac{\partial \varphi_s}{\partial s}.$$

A computation analogous to that in the one-dimensional case leads to

$$\operatorname{div} J_s = \mathscr{E}_L \cdot \frac{\partial \varphi_s}{\partial s} + \frac{\partial L(\varphi_s \circ u, \mathrm{D}(\varphi_s \circ u), .)}{\partial s} = 0.$$

Thus

$$\operatorname{div} J_s = 0$$

if $\mathscr{E}_L = 0$ and L is invariant under φ_s. One writes $J = J_s|_{s=0}$, and calls J a **conserved current** associated to the 1-parameter group φ_s. More generally if G is a Lie group of effective transformations of M and ξ is an infinitesimal generator of G, i.e., an element of $\mathscr{G}$, the linear mapping

conserved current

$$\xi \mapsto L'_{\mathrm{D}u} \cdot \xi$$

defines a mapping between the space of solutions of Euler equations and $\mathscr{G}^*$ valued conserved currents on M. Such a mapping is called a **momentum map.**

momentum map

The property of current conservation still holds for lagrangians L_1 whose sum with a divergence $L = L_1 + \operatorname{div} f(u)$ is invariant, since L and L_1 have the same Euler equations. Of course J must be constructed with L.

Remark: Another proof in the present context of Noether's theorem on current conservation is to consider the derivative $S'(u) \cdot h$. If L is invariant under the 1-parameter group φ_s of diffeomorphisms of M the same is true of $S(u)$ for any open submanifold $U \subset V$, and the corresponding derivative in the direction $h = \partial \varphi_s / \partial s|_{s=0}$ is zero. Thus

$$\int_U \mathscr{E}_L(u) \cdot h\tau + \int_{\partial V} \omega = 0, \qquad h = \left.\frac{\partial \varphi_s}{\partial s}\right|_{s=0};$$

if u satisfies $\mathscr{E}_L(u) = 0$ we therefore have

$$\int_{\partial V} \omega = 0 \qquad \text{for all boundaries } \partial V.$$

Thus

$$\mathrm{d}\omega = 0,$$

that is,

$$\operatorname{div} X = 0, \qquad \text{where } X = h \cdot L'_{\mathrm{D}u}, \text{ and } h = \left.\frac{\partial \varphi_s}{\partial s}\right|_{s=0}, \qquad \text{i.e., } X = J.$$

A section h such that

$$S'(q) \cdot h = 0$$

infinitesimal invariance

is called an **infinitesimal invariance**. Infinitesimal invariances of the type $\mathrm{D}v$, with v an arbitrary section of a bundle over V, come in general from infinite dimensional invariance groups of the lagrangian L, or of L plus a divergence in V relative to its volume element (the two corresponding actions have the same derivative when restricted to h's with compact support in V).

When M admits an infinitesimal invariance $h = \mathrm{D}v$ the proof given in the case of dimension $V = 1$ of the identities

$$\mathrm{D}^* \mathscr{E}_L(v) \equiv 0, \qquad \forall v$$

applies without change. D^* and D are adjoint for the L^2 scalar product on V with volume element τ.

Example: $v = A$ is a Yang–Mills potential (p. 403), assumed to be globally defined on V. The Yang–Mills action is

$$S(A) = \int_V F^{\lambda\mu} \cdot F_{\lambda\mu} \tau, \qquad F_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda + [A_\lambda, A_\mu].$$

If h is a $\mathscr{G}$-valued 1-form with compact support in V,

$$S'(A) \cdot h = \int_V \mathscr{E}_{\mathrm{Y.M.}} \cdot h\tau = -\int_V (\hat{\nabla} F^{\lambda\mu} \cdot h_\mu)\tau,$$

$$\hat{\nabla}_\lambda F^{\lambda\mu} = \nabla_\lambda F^{\lambda\mu} + [A_\lambda, F^{\lambda\mu}].$$

$S(A)$ is invariant under a change of gauge:

$$A \to U^{-1}AU + U^{-1}\,\mathrm{d}U, \qquad U: V \to G,$$

so

$$S'(A)\cdot h = 0 \quad \text{if} \quad h = \hat{\nabla} u \equiv \mathrm{d}u + [A, u], \qquad u: V \to \mathscr{G}$$

$$\int_V \hat{\nabla}_\lambda F^{\lambda\mu} \cdot \hat{\nabla}_\mu u\tau = -\int_V \hat{\nabla}_\mu \hat{\nabla}_\lambda F^{\lambda\mu} \cdot u\tau,$$

and we find the identities

$$\hat{\nabla}_\mu \hat{\nabla}_\lambda F^{\lambda\mu} \equiv 0.$$

These can also be derived as a consequence of the Bianchi identities.

References can be found at the end of Problem II 5, Invariance.

4. NOETHER'S THEOREMS II

We consider as in the previous problem (question 6) lagrangians for smooth mappings $u: V \to M$ between finite dimensional manifolds. These lagrangians are defined by mappings

$$L: M \times (T^*V \otimes TM) \times V \to \mathbb{R}$$

and a volume form τ on V. The corresponding action, for a C^1 mapping $u: V \to M$, with differential $\mathrm{D}u$ is the integral of the d-form $L\tau$:

$$S = \int_V L(u(x), (\mathrm{D}u)(x), x)\tau(x).$$

We shall consider in this problem invariance under diffeomorphisms of V (questions 1, 2) or $V \times M$ (question 3).

1) *Let $f: x \mapsto y = f(x)$ be a diffeomorphism of V.*
The lagrangian $L\tau$ is said to be **invariant under** *f if we have the equality of forms* invariant

$$L(u, f'\,\mathrm{D}u, f^{-1})f^{-1*}\tau = L(u, \mathrm{D}u, .)\tau.$$

a) *Justify this definition by constructing the image of the function $x \mapsto L(u(x)(\mathrm{D}u)(x), x)$ by the diffeomorphism f, extended to a diffeomorphism*

$$M \times (T^*V \otimes M) \times V \to \mathbb{R}.$$

b) *Suppose that the lagrangian $L\tau$ is invariant under a* 1*-parameter group of diffeomorphisms f_s of V. Obtain the corresponding infinitesimal invariance law.*

c) *Show that this infinitesimal invariance implies the conservation law*

$$\operatorname{div} J = 0, \qquad J = -L'_{\mathrm{D}u} X(\mathrm{D}u) + LX,$$

where X is the generator of the 1*-parameter group f_s and u is a C^2 solution of the Euler equations of the lagrangian $L\tau$. Give an example.*

d) *Let V be a product $S \times \mathbb{R}$ and let $(x, t) \in V$. Let f_s be the* 1*-parameter group of translations of V:*

$$f_s \colon x \mapsto \hat{x} = x, \qquad t \to \hat{t} = t + s.$$

Give the necessary and sufficient condition for a lagrangian $L\tau$ to be invariant under f_s. Give the corresponding conserved current J.

energy *The* **energy** *of the field u is the quantity (when the integral exists)*

$$E_u(t) = \int_{S_t} J^0 \sigma_t, \qquad S_t = S \times \{t\},$$

where σ_t is the $(d-1)$-form induced on S_t by τ and J^0 the component of J in the "time" direction $\mathbb{R}$. Show that when S is compact (without boundary) the energy $E_u(t)$ is independent of t, when u satisfies the Euler equations. Discuss the case of a non-compact S.

e) *Consider the lagrangian on scalar functions on a manifold $V = S \times \mathbb{R}$ with pseudo-riemannian metric g of hyperbolic signature, $S \times \{t\}$ being space-like and $\{x\} \times \mathbb{R}$ time-like, with associated contravariant tensor $g^{\#}$:*

$$L\tau = (g^{\#}(\mathrm{D}u, \mathrm{D}u) + m^2 u)\tau, \qquad \tau \text{ volume element of } g.$$

Suppose g is invariant under the 1*-parameter group of diffeomorphisms*
stationary *of V defined by $(x, t) \mapsto (x, t + s)$; such a metric is called* **stationary**. *Construct the corresponding conserved current.*

Give another example of a lagrangian $L\tau$ on scalar functions on a manifold V with a metric g, where g and $L\tau$ are invariant by a 1*-parameter group of diffeomorphisms of V. Construct the corresponding current.*

Answer 1a: The lagrangian $L\tau$ is a d-form on V, L is a function on V obtained by the composition of maps:

$$V \to M \times (T^*V \otimes TM) \times V \to \mathbb{R}$$

$$x \mapsto (u(x), (\mathrm{D}u)(x), x) \mapsto L(u(x), (\mathrm{D}u)(x), x).$$

If $f: W \to V$ is a diffeomorphism, its derivative defines a diffeomorphism f^* of T^*V on T^*W.
We consider then the following sequence of maps:

$$W \to V \to M \times (T^*V \otimes TM) \times V \to M \times (T^*W \otimes TM) \times W \to \mathbb{R}$$

by (recall that $W = V$)

$$\begin{aligned} y \mapsto x = f(y) &\mapsto (u(x), \mathrm{D}u(x), x) \mapsto (u(x), f^*(x)\mathrm{D}u(x), f^{-1}(x)) \\ &\mapsto L(u(x), f^*(x)\mathrm{D}u(x), f^{-1}(x)) \end{aligned}$$

which justifies the definition.

A computation in coordinates also gives the result: let

$$\varphi(x^i) = \bar{L}(u^\alpha(x^i), \partial_i u^\alpha(x^i), x^i)$$

with $\bar{L}$ the representation of L in coordinates x^i in V and y^α of M, and let $x^i = f^i(\hat{x}^j)$ represent locally the diffeomorphism of V. We then set

$$\hat{\varphi}(\hat{x}^j) = \bar{L}(u^\alpha(f^i(\hat{x}^j)), \partial_l(u^\alpha(f^i(\hat{x}^j)))\frac{\partial \hat{x}^l}{\partial x^i}, f^i(\hat{x}^k));$$

we have

$$\hat{\varphi}(\hat{x}) = \varphi(x) \quad \text{if} \quad x = f(\hat{x}).$$

We denote by $v^\alpha(\hat{x}^j)$ the function $u^\alpha(f^i(\hat{x}^j))$ and we have

$$\hat{\varphi}(\hat{x}) = \bar{L}(u^\alpha(\hat{x}^j), \partial_l u^\alpha(\hat{x}^j)\frac{\partial \hat{x}^l}{\partial x^i}, f^i(\hat{x}^k)).$$

The action $S(u)$ on a domain of a chart of V with image a subset U of $\mathbb{R}^d$ was

$$S(u) = \int_U \varphi(x^i)\rho(x^i)\,\mathrm{d}x^1 \ldots \mathrm{d}x^d \in \mathbb{R};$$

by the theorem on the change of variables the value of this integral is equal to

$$S(u) = \hat{S}(v) = \int_{f^{-1}(U)} \hat{\varphi}(\hat{x}^j)\rho(f^i(\hat{x}^j))\left(\frac{D(x^1, \ldots, x^d)}{D(\hat{x}^1, \ldots, \hat{x}^d)}\right)\mathrm{d}\hat{x}^1 \ldots \mathrm{d}\hat{x}^d.$$

The lagrangian $\hat{L}(v, \mathrm{D}v, .)\hat{\tau}$ under the integral sign is the transform of the original one. They are the same lagrangian if and only if

$$\hat{L}(v, \mathrm{D}v, .)\hat{\tau} = L(v, \mathrm{D}v, .)\tau$$

which gives the indicated invariance law.

Example: Consider the action on a ball B of $\mathbb{R}^2$:

$$S(u)=\int_B \left(m^2u^2+\left(\frac{\partial u}{\partial x^1}\right)^2+\left(\frac{\partial u}{\partial x^2}\right)^2+\lambda x^1\frac{\partial u}{\partial x^1}+\mu x^2\frac{\partial u}{\partial x^2}\right)\mathrm{d}x^1\,\mathrm{d}x^2$$

and a rotation $B\to B$; $x=f_s(y)$:

$$x^1=y^1\cos s+y^2\sin s,$$
$$x^2=-y^1\sin s+y^2\cos s.$$

The lagrangian is invariant by the rotation if and only if $\lambda=\mu$.

Answer 1b: If L is invariant under a 1-parameter group of diffeomorphisms f_s of V with generator $X=\partial f_s/\partial s|_{s=0}$, then

$$L(u,f_s^*\,\mathrm{D}u,f_s^{-1})(f_s^{-1*}\tau) \tag{1}$$

is independent of s. The corresponding infinitesimal invariance law is obtained by taking the derivative of (1) with respect to s and putting $s=0$. Since $f_s|_{s=0}=$ Identity, $f_s'|_{s=0}=$ Identity, then

$$\frac{\partial}{\partial s}f_s|_{s=0}=X,\quad\text{and}\quad\frac{\partial}{\partial s}f_s^*|_{s=0}=DX.$$

The result is, up to multiplication by τ

$$X_\tau(L)\equiv\frac{\partial L}{\partial(\mathrm{D}u)}\mathrm{D}u\cdot\mathrm{D}X-X\cdot\partial L-L\operatorname{div}X=0,$$

where ∂L is the partial derivative of L with respect to its explicit dependence on x and $-\operatorname{div}X\tau$ is the Lie derivative of τ with respect to X; that is in local coordinates

$$\operatorname{div}X\equiv\nabla_iX^i\equiv\frac{1}{\rho}\frac{\partial}{\partial x^i}(\rho X^i),$$

$$X_\tau(L)\equiv\frac{\partial L}{\partial(\partial_iu^\alpha)}\partial_iX^j\partial_ju^\alpha-X^i\partial_iL-L\nabla_iX^i=0.$$

Answer 1c: The components of the vector J, in a natural frame on V are

$$J^i=-\frac{\partial L}{\partial(\partial_iu^\alpha)}X^j\partial_ju^\alpha+LX^i.$$

We have, therefore

$$\nabla_iJ^i=-X^j\partial_ju^\alpha\partial_i\left(\frac{\partial L}{\partial\partial_iu^\alpha}\right)-\frac{\partial L}{\partial(\partial_iu^\alpha)}(\partial_iX^j\partial_ju^\alpha+X^j\partial_i\partial_ju^\alpha)$$
$$+L\nabla_iX^i+X^i\,\mathrm{D}_iL, \tag{2}$$

where $\mathrm{D}_i L$ is the derivative:

$$\mathrm{D}_i L = \partial_i L + \frac{\partial L}{\partial(u_j^\alpha)}\,\partial_i u_j^\alpha + \frac{\partial L}{\partial u^\alpha}\,\partial_i u^\alpha, \qquad u_j^\alpha = \partial_j u^\alpha.$$

(2) can therefore be written

$$\nabla_i J^i \equiv -X_\tau(L) - X^j \partial_j u^\alpha \mathscr{E}_{L,\alpha}(u), \qquad \mathscr{E}_{L,\alpha}(u) \equiv (\mathscr{E}_L(u))_\alpha$$

which implies $\nabla_i J^i = 0$ when $\mathscr{E}_L^\alpha(u) = 0$ and $X_\tau(L) = 0$.

Answer 1d: If the diffeomorphism $f_s\colon S \times \mathbb{R} \to S \times \mathbb{R}$ is

$$\hat{x} = x \qquad \text{and} \qquad \hat{t} = t + s$$

the mapping f_s^* is the identity mapping. Therefore, a lagrangian $L(u, \mathrm{D}u, .)\tau$ is invariant by f_s if and only if in coordinates $x^\alpha = (x^i, x^0)$ adapted to the product structure of $S \times \mathbb{R}$, the lagrangian $\bar{L}(u^a, \mathrm{D}_\alpha u^a, x^i, x^0)\rho(x^i, x^0)$ does not depend explicitly on x^0. The corresponding conserved current is

$$J^\alpha = -\frac{\partial L}{\partial(\partial_\alpha \cdot u^a)}\,\partial_0 u^a + \delta_0^a L, \qquad \alpha = 0, 1, \ldots, d-1.$$

The conservation law is

$$\mathrm{D}_\alpha J^\alpha \equiv \frac{1}{\rho}\,\partial_\alpha(J^\alpha \rho) = 0$$

which implies for any domain Ω of V with boundary $\partial\Omega$:

$$0 = \int_\Omega \mathrm{D}_\alpha J^\alpha \tau = \int_{\partial\Omega} J^\alpha \sigma_\alpha, \tag{3}$$

where σ_α is the $(d-1)$-form

$$\sigma_\alpha = \rho\, \mathrm{d}x^0 \wedge \cdots \wedge \widehat{\mathrm{d}x^\alpha} \wedge \cdots \wedge \mathrm{d}x^{d-1},$$

where $\mathrm{d}x^\alpha$ is suppressed. If we take $\Omega = S \times [t_0, t_1]$ and S is compact we have $\partial\Omega = S_{t_0} \cup S_{t_1}$ and (3) reads

$$\int_{S_{t_1}} J^0 \rho\, \mathrm{d}x^1 \ldots \mathrm{d}x^{d-1} = \int_{S_{t_0}} J^0 \rho\, \mathrm{d}x^1 \ldots \mathrm{d}x^{d-1}.$$

If S is not compact, but J^α has compact support we have the same formula. If J^α is a limit of functions with compact support we may have the same formula. Otherwise supplementary terms can appear.

Answer 1e: In local coordinates, $x^0 \in \mathbb{R}$, x^i, with $i = 1, 2, \ldots, d-1$, coordinates on S,

$$L = g^{\alpha\beta}\, \partial_\alpha u\, \partial_\beta u + m^2 u^2.$$

L does not depend explicitly on x^0 (since $\partial_0 g^{\alpha\beta} = 0$), and the conserved current is

$$\begin{aligned} J^0 &= L - 2g^{\alpha 0}\, \partial_\alpha u\, \partial_0 u = -g^{00}(\partial_0 u)^2 + g^{ij}\, \partial_i u\, \partial_j u + m^2 u^2, \\ J^i &= -2g^{\alpha i}\, \partial_\alpha u\, \partial_0 u. \end{aligned}$$

The energy density J^0 on S_t is positive.

Note: Compare the current obtained here with the relations on pp. 513–514.

Example: The lagrangian for scalar functions

$$L\tau \equiv (\eta^{\alpha\beta}\, \partial_\alpha u\, \partial_\beta u + m^2 u)\tau, \qquad \tau = \mathrm{d}x^0 \ldots \mathrm{d}x^{d-1}$$

with the Minkowski metric $\eta^{\alpha\beta} = \mathrm{diag}(-1, + \cdots + 1)$ is invariant under Lorentz transformations, and so is L. The conserved current corresponding, for instance, to the infinitesimal boost

$$X = x^0 \partial_1 - x^1 \partial_0$$

is

$$\begin{aligned} J^1 &= 2\eta^{a1}\, \partial_a u\, Xu + x^0 L, \\ J^0 &= 2\eta^{a0}\, \partial_a u\, Xu - x^1 L, \\ J^i &= 2\eta^{ai}\, \partial_a u\, Xu, \qquad i = 0, 1. \end{aligned}$$

2a) *Consider the action defined by the lagrangian* $L\tau$:

$$S(u) = \int_V L(u(x), (\mathrm{D}u)(x), x)\tau(x).$$

Suppose that L *admits the infinitesimal invariance* $X_\tau(L) = 0$ (*cf.* 1*b*), *with* X *a tangent vector field to* V, *tangent also to* ∂V. *Show that* S *admits the infinitesimal invariance* $h = X(u) \equiv X^j\, \partial_j u$.

2b) *Give an identity satisfied by the Euler operators when* S *is invariant by all diffeomorphisms of* V, *hence, when* $S'(u) \cdot h = 0$ *for all* h *of the form* $X(u)$.

Answer 2a: We have

$$S'(u) \cdot h = \int_V (L'_u h + L'_{\mathrm{D}u}\, \mathrm{D}h)\tau\,.$$

In local coordinates, and for $h = X(u)$:

$$S'(u) \cdot h = \int_V \left(\frac{\partial \bar{L}}{\partial u^\alpha} X^j \partial_j u^\alpha + \frac{\partial \bar{L}}{\partial(\partial_i u^\alpha)} \partial_i(X^j \partial_j u^\alpha) \right) \tau$$

which can be written, using the expression of $X_\tau(L)$ and an easy computation

$$S'(u) \cdot h = \int_V (X_\tau(L) + \mathrm{D}_i(LX^i))\tau.$$

We have

$$\int_V \mathrm{D}_i(LX^i)\tau = \int_{\partial V} LX^i n_i \sigma = 0$$

since X is tangent to ∂V. Therefore,

$$S'(u) \cdot h = \int_V X_\tau(L)\tau.$$

Note that $h = X(u)$ is the Lie derivative in the direction X of the mapping u.

Answer 2b: We restrict X to be of compact support in V, h is then also with compact support, and we have

$$S'(u) \cdot h = \int_V \mathscr{E}_L(u) \cdot h\tau = \int_V \mathscr{E}_{L,\alpha}(u) X^j \partial_j u^\alpha \tau, \qquad \mathscr{E}_{L,\alpha}(u) \equiv (\mathscr{E}_L(u))_\alpha .$$

If $S'(u) \cdot h = 0$ for all X we have therefore $\mathscr{E}_{L,\alpha}(u)\, \partial_j u^\alpha = 0$. The Euler operators satisfy the identity (vanishing of a section of T^*V)

$$\mathscr{E}_{L,\alpha}(u) \mathrm{D} u^\alpha \equiv 0.$$

Example: Consider on a 2-dimensional manifold V the lagrangian $L\tau$ with two unknown functions u^1 and u^2 (i.e., $M = \mathbb{R}^2$)

$$L = \int_V f(u^1, u^2)\, \mathrm{d}u^1 \wedge \mathrm{d}u^2 = \int_V f(u^1, u^2)(\partial_1 u^1 \partial_2 u^2 - \partial_2 u^1 \partial_1 u^2)\, \mathrm{d}x^1\, \mathrm{d}x^2.$$

A straightforward computation checks that

$$\mathscr{E}_{L,1} \partial_1 u^1 + \mathscr{E}_{L,2} \partial_1 u^2 \equiv 0, \qquad \mathscr{E}_{L,1} \partial_2 u^1 + \mathscr{E}_{L,2} \partial_2 u^2 \equiv 0.$$

Remark: When the unknown is not a mapping $V \to M$ but a section of some more general fibre bundle over V the invariance of the lagrangian by all diffeomorphisms of V leads to differential identities on $\mathscr{E}_L(u)$ (cf.

an example in Problem II 7, Stress energy tensor), if the Lie derivative of u with respect to X is a differential operator on X.

3) *Let* $f_s\colon V \times M \to V \times M$ *be a* 1*-parameter group of diffeomorphisms of* $V \times M$ *with generator* $Y = \mathrm{d}f_s/\mathrm{d}s|_{s=0}$. *Denote by* $X \oplus \xi$ *the direct sum decomposition of* Y *corresponding to the product structure of the image; in local coordinates* (x^i) *on* V, (y^α) *on* M, *the representative* $\bar{f}_s$ *of* f_s *is*

$$\bar{f}_s\colon (x^i, y^\alpha) \mapsto (\bar{f}^j_s(x^i, y^\alpha), \bar{f}^\beta_s(x^i, y^\alpha)),$$

the vector Y *has components* X^i, ξ^α *with*

$$X^i = \frac{\mathrm{d}\bar{f}^i_s}{\mathrm{d}s}\bigg|_{s=0}, \qquad \xi^\beta = \frac{\mathrm{d}\bar{f}^\beta_s}{\mathrm{d}s}\bigg|_{s=0}.$$

The lagrangian for mappings $u\colon V \to M$,

$$L = \int_V L(u(x), (\mathrm{D}u)(x), x)\tau$$

admits the infinitesimal invariance Y *if*

$$Y_\tau(L) \equiv X_\tau(L) + \xi(L) = 0,$$

where $X_\tau(L)$ *is given in* 1*b and (cf. Problem II* 3, *Noether I)*

$$\xi(L) = L'_u \cdot \xi + L'_{\mathrm{D}u} \cdot \mathrm{D}\xi.$$

Give the corresponding conservation law for a solution of Euler's equations.

Answer 3: A straightforward computation analogous to previous ones shows that

$$\operatorname{div} J \equiv (\xi - X(u)) \cdot \mathscr{E}_L(u) + Y_\tau(L)$$

with

$$J = L'_{\mathrm{D}u} \cdot (\xi - X(u)) + LX,$$

i.e.,

$$J^i = (\xi^\alpha - X^j \partial_j u^\alpha) \frac{\partial L}{\partial(\partial_i u^\alpha)} + LX^i.$$

Thus

$$\operatorname{div} J = 0 \qquad \text{if} \qquad \mathscr{E}_L(u) = 0.$$

Remark: We also have $\operatorname{div} J = 0$ when $\mathscr{E}_L(u) = 0$ if we suppose only that $Y_\tau(L) = 0$ when $\mathscr{E}_L(u) = 0$.

References can be found at the end of Problem II 5, Invariance.

5. INVARIANCE OF THE EQUATIONS OF MOTION

1) *Show that an arbitrary infinitesimal invariance of the action is also an infinitesimal invariance of the equations of motion.*

Answer 1: Let $S(u) = \int_V L(u(x), \mathrm{D}u(x), x)\tau$ be the action defined by a lagrangian L [Problem II 4, Noether II]. The equations of motion are by definition the Euler equations $\mathscr{E}_L(u(x)) = 0$. The Euler operator $\mathscr{E}_L(u(x))$ is such that the derivative of the action with respect to u at an arbitrary u is, for all h with compact support on V,

$$S'(u) \cdot h \equiv \int_V \mathscr{E}_L(u(x)) \cdot h(x)\tau. \tag{1}$$

S admits the infinitesimal invariance k if

$$S'(u) \cdot k = 0 \text{ for all } u. \tag{2}$$

We deduce from (1) that the second derivative

$$S''(q) \cdot (h, k) \equiv \int_V (\mathscr{E}'_L(q(x)) \cdot k(x))) \cdot h(x)\tau. \tag{3}$$

where $\mathscr{E}'_L(u(x))$ is the Jacobi operator (denoted J_q, p. 87). But by (2)

$$S''(u) \cdot (h, k) \equiv S''(u) \cdot (k, h) = 0.$$

Thus

$$\int_V (\mathscr{E}'_L(q(x)) \cdot k(x)) \cdot h(x)\tau = 0 \qquad \forall h \text{ with compact support.}$$

Therefore

$$\mathscr{E}'_L(q(x)) \cdot k(x) = 0. \tag{4}$$

Remark: An infinitesimal invariance – for instance, a Killing vector field of a Lie group action on Y – does not in general have compact support. Thus, in general

$$S'(q) \cdot k = \int_V (\mathscr{E}_L(u(x)) \cdot k(x) + \operatorname{div} K(x))\tau \tag{5}$$

with

$$K = k \cdot \partial L / \partial(\mathrm{D}q). \tag{6}$$

2) *Conversely, let k be an infinitesimal invariance of the Euler equations.*

on, off shell

a) *Show that if k has a compact support, or the boundary term in* (5) *vanishes, the action S admits this infinitesimal invariance, when v is a solution of $S'(v) = 0$. Such an invariance valid only when v is a solution of $S'(v) = 0$ is said to be* **on shell.**[1]

b) *Example: Let L be the lagrangian for a free Dirac particle:*

$$L = \tfrac{1}{2}\left\{ \bar{\psi}\left(\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi\right) - \left(\frac{\partial \bar{\psi}}{\partial x^k}\gamma^k - m\bar{\psi}\right)\psi \right\}. \tag{7}$$

$\bar{\psi}$ is the Dirac adjoint of ψ [Problem I 10, Dirac].
Let G be the group of transformations f_s; $\psi \mapsto \psi_s$ defined by

$$\psi_s = \psi + s\varphi \qquad \text{with } \varphi \text{ a given spinor such that } \mathscr{E}_L(\varphi) = 0. \tag{8}$$

Show that $\mathscr{E}_L(\psi)$ is invariant under G. *Show that the action $S(\psi, \bar{\psi}) = \int L \, \mathrm{d}x$ is not invariant under* G; *show that, on shell, S is invariant under* G. *Compute the generator X_y of the group of transformations defined by* (8) *and the corresponding conservation law.*

Answer 2a: The derivative of $S'(u)\cdot k$ is $S''(u)\cdot(k, h)$, which can be written as (3) for all h with compact support. Thus, if (4) holds and k has compact support or is such that boundary terms vanish

$$S''(u)\cdot(k,k) = 0 \qquad \forall u.$$

The property $S(f_s v) = S(v)$ is then a consequence of the Taylor formula (expansion in s) and the property $S'(v) = 0$, satisfied when v is a critical point of S.

Answer 2b: The Euler–Lagrange equations for a Dirac particle are

$$0 = \mathscr{E}_L(\psi) = \gamma^k \frac{\partial \psi}{\partial x^k} + m\psi \tag{9a}$$

and its conjugate

$$0 = \mathscr{E}_L(\bar{\psi}) = \frac{\partial \bar{\psi}}{\partial x^k}\gamma^k - m\bar{\psi}. \tag{9b}$$

One checks readily that if φ and $\bar{\varphi}$ solve (9a) and (9b), respectively,

$$\mathscr{E}_L(\psi_s) = \mathscr{E}_L(\psi) \qquad \text{and} \qquad \mathscr{E}_L(\bar{\psi}_s) = \mathscr{E}_L(\bar{\psi}).$$

[1]The expression on (the mass) shell [resp. off (the mass) shell] was used first for properties of real [resp. virtual] particles which hold only when $(\text{mass})^2 = (\text{energy})^2 - (\text{momentum})^2$ [resp. even when the mass–energy–momentum relation is not satisfied].

Alternatively, one can check that the following condition (derived in paragraph 1) is satisfied:

$$\mathscr{E}_L'(\psi(x)) \cdot k(x) = 0,$$

where $\mathscr{E}_L'(\psi(x))$ is the Jacobi operator along a trajectory of L. The action is invariant only on shell; indeed,

$$S(\psi_s, \bar{\psi}_s) = S(\psi, \bar{\psi}) + \frac{s}{2} \int \left(\bar{\varphi}\left(\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi\right) - \left(\frac{\partial \bar{\psi}}{\partial x^k} \gamma^k - m\bar{\psi}\right)\varphi \right)\tau.$$

The generator of the group of transformations defined by (8) is

$$\xi = \varphi^\alpha(x) \frac{\partial}{\partial \psi^\alpha} + \bar{\varphi}^\alpha(x) \frac{\partial}{\partial \bar{\psi}^\alpha}$$

and the quantity,

$$\begin{aligned} J^i_\varphi &= \varphi^\alpha(x) \frac{\partial L}{\partial \psi^\alpha_i} + \bar{\varphi}^\alpha(x) \frac{\partial L}{\partial \bar{\psi}^\alpha_i} \\ &= \tfrac{1}{2}(\bar{\psi}\gamma^i)\varphi - \tfrac{1}{2}\bar{\varphi}(\gamma^i \psi), \end{aligned}$$

is conserved for ψ and $\bar{\psi}$ solutions of the Euler equation (9):

$$\begin{aligned} 2 \frac{\partial}{\partial x^i} J^i_\varphi &= \left(\frac{\partial \bar{\psi}}{\partial x^i} \gamma^i\right)\varphi + (\bar{\psi}\gamma^i) \frac{\partial \varphi}{\partial x^i} - \frac{\partial \bar{\varphi}}{\partial x^i} (\gamma^i \psi) - \bar{\varphi}\left(\gamma^i \frac{\partial \psi}{\partial x^i}\right) \\ &= \left(\frac{\partial \bar{\psi}}{\partial x^i} \gamma^i - m\bar{\psi}\right)\varphi - \bar{\varphi}\left(\gamma^i \frac{\partial \psi}{\partial x^i} + m\psi\right) \end{aligned}$$

since φ, $\bar{\varphi}$ are solutions of the Dirac equation. Hence div $J_\varphi = 0$ when ψ and $\bar{\psi}$ are solutions of the Dirac equation.

Note: In this question the manifold V is $\mathbb{R}^4$, the fields and their derivatives are square integrable on $\mathbb{R}^4$, the boundary terms vanish for this reason.

Many other developments related to Noether theorems are of interest in physics: lagrangians depending on higher order derivatives, equivalence between invariance and conservation laws for non-degenerate lagrangians, etc. They can be found in the references or their bibliography.

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6. STRING ACTION

string

A **string** is a 1-dimensional object which sweeps out an oriented 2-dimensional submanifold W ("world sheet") of the spacetime in which it moves.

Let (M, g) be this spacetime, a d-dimensional C^∞ manifold with riemannian or pseudo-riemannian metric g.

Let h be a metric on W, $\mathrm{d}\mu(h)$ the volume element defined by h, f an embedding $W \to M$, and let

$$S(h, f, g) = \int_W (h \otimes g) \cdot (\nabla f \otimes \nabla f)\, \mathrm{d}\mu(h)$$

$$= \int_W h^{ab} g_{\mu\nu} \partial_a f^\mu \partial_b f^\nu \, \mathrm{d}\mu(h)$$

be the energy of this mapping.

1) *Show that S is invariant under conformal rescaling of h.*

2) *Give a necessary condition for h to be a critical point of S, when f and g are given.*

Show that under this condition the action is equal to

$$S = 2 \int_W \mathrm{d}\mu(h).$$

Answer 1: $k = \lambda h$ implies $k^{ab} = \lambda^{-1} h^{ab}$, $\mathrm{d}\mu(k) = \lambda\, \mathrm{d}\mu(h)$.

Answer 2: For h to be a critical point of S it is necessary that $S'_h \equiv T = 0$, with

$$T_{ab} = g_{\mu\nu} \partial_a f^\mu \partial_b f^\nu - \tfrac{1}{2} h_{ab} (h^{cd} \partial_c f^\mu \partial_d f^\nu g_{\mu\nu}).$$

The condition $T = 0$ is satisfied by

$$h = \lambda f^* g, \qquad \lambda \text{ arbitrary function.}$$

If W is endowed with the metric f^*g we have

$$h^{ab} g_{\mu\nu} \partial_a f^\mu \partial_b f^\nu = 2$$

and

$$S = 2\int_W \mathrm{d}\mu(h).$$

7. STRESS–ENERGY TENSOR, ENERGY WITH RESPECT TO A TIMELIKE VECTOR FIELD

I. LAGRANGIANS DEPENDING ON A METRIC

Let S: $g \mapsto S(g) = \int_V L(J^k(g), x)\,\mathrm{d}\mu(g)$ be a C^1 mapping from the space $\mathcal{M}$ of C^k riemannian metrics on a given compact smooth manifold V into $\mathbb{R}$; we denote by $J^k(g)$ the ***k*-jet** of g, i.e., the set of partial derivatives of g of order $\leq k$. $J^k(g)$ k-jet

The derivative $S'(g)$ of this mapping at a point $g \in \mathcal{M}$ is a linear map: $T_g\mathcal{M} \to \mathbb{R}$.

The tangent space $T_g\mathcal{M}$ is the space of C^k symmetric 2-tensor fields over V, since the space $\mathcal{M}$ of C^k metrics over V is an open set of their vector (Banach) space.

Suppose that $S'(g)\cdot h$ can be written

$$S'(g)\cdot h = \int_V \mathscr{E}_L(g)\cdot h(x)\,\mathrm{d}\mu(g), \tag{1}$$

where the dot on the right-hand side denotes duality between $(\otimes T_x V)^2$ and $(\otimes T_x^* V)^2$. The quantity $\mathscr{E}_L(g)$ (Euler operator for g, cf. Problem II 3, Noether) will now be denoted by T and called the **stress–energy tensor**. stress–energy tensor

We can also write

$$\partial S/\partial g = T \tag{2}$$

with the convention that $\partial S/\partial g$ is a linear mapping on h given by the L^2 scalar product with volume element $\mathrm{d}\mu(g)$:

$$S'(g)\cdot h = (T, h) = \int_V T(x)\cdot h(x)\,\mathrm{d}\mu(g(x)).$$

1) *Show that if L is invariant under isometries of (V, g), that is, diffeomorphisms f: $V \to V$ and replacements of g by f^*g, then T satisfies the identities (where ∇ is the riemannian covariant derivative)*

$$\nabla\cdot T \equiv 0, \qquad \text{i.e., } \nabla_\alpha T^{\alpha\beta} \equiv 0.$$

2) *Show that if L is invariant under conformal rescaling of g, i.e., under* $g \mapsto \lambda g$, *then*

$$\operatorname{tr}_g T \equiv g^{\alpha\beta} T_{\alpha\beta} = 0.$$

3) *Comment on the extensions of these properties to noncompact V and pseudo-riemannian metrics.*

Answer 1: If $S(g)$ is invariant under isometries of g then it admits the infinitesimal invariance (where $\mathscr{L}$ denotes the Lie derivative and X is an arbitrary vector field)

$$h = \mathscr{L}_X g.$$

Thus

$$S'(g) \cdot \mathscr{L}_X g \equiv \int_V T(x) \cdot \mathscr{L}_X g \, \mathrm{d}\mu(g) \equiv \int_V T_{\alpha\beta}(\nabla^\alpha X^\beta + \nabla^\beta X^\alpha) \, \mathrm{d}\mu(g) = 0$$

which implies, since V is compact and T symmetric,

$$\int_V X_\alpha \nabla_\beta T^{\alpha\beta} \, \mathrm{d}\mu(g) = 0 \qquad \forall X_\alpha$$

and therefore

$$\nabla_\beta T^{\alpha\beta} = 0.$$

Answer 2: The infinitesimal invariance corresponding to conformal rescaling is

$$h = \rho g, \qquad \rho \text{ an arbitrary function}$$

Thus

$$\int_V T(x) \cdot \rho(x) g(x) \, \mathrm{d}\mu(x) = 0 \qquad \forall \rho,$$

and therefore

$$T \cdot g = \operatorname{tr}_g T = T_{\alpha\beta} g^{\alpha\beta} = 0.$$

Answer 3: If V is noncompact, some hypothesis must be made to insure the convergence of $S(g)$. However, the derivative $S'(g)$ is still given by the L^2 scalar product (1) on V if we demand that h have compact support in V. We then have the same definition and properties for T.
The signature of the metric is irrelevant.

II. LAGRANGIAN DEPENDING ON THE METRIC AND OTHER FIELDS

1) *Suppose now that the lagrangian L depends not only on the metric g on V, but also on some other field u, a section of a vector bundle E over V. Suppose that the action does not depend explicitly on* $x \in V$

$$S(g, u) \equiv \int_V L(J^k g, J^l u)\, \mathrm{d}\mu(g),$$

and that the derivative of S at the point (g, u) *can be written*

$$S'(g, u)\cdot(h, v) = \int_V (\mathscr{E}_L(g)\cdot h + \mathscr{E}_L(u)\cdot v)\, \mathrm{d}\mu(g),$$

where $\mathscr{E}_L(u)$ *is a section of the dual bundle to E over V and* $\mathscr{E}_L(g)$, *denoted by T as in I, is a contravariant 2-tensor field on V.*
Suppose that S is invariant under diffeomorphisms of V, that is

$$S(g, u) = S(f^* g, f^* u),$$

where $f^* u$ denotes the field on V induced from u by the diffeomorphism f.
Show that if u satisfies $\mathscr{E}_L(u) = 0$, *the stress–energy tensor T satisfies the equation*

$$\nabla \cdot T = 0 \qquad \text{i.e.,}\ \nabla_\alpha T^{\alpha\beta} = 0.$$

2) *Examples: compute T and* $\nabla \cdot T$ *in the following cases:*
a) *the nonlinear scalar field, with lagrangian*

$$L = -g^{\alpha\beta}\partial_\alpha u \partial_\beta u + u^{p+1},$$

b) *the Yang–Mills field, with lagrangian*

$$L = F^{\alpha\beta}\cdot F_{\alpha\beta},$$

where the dot denotes the scalar product in the Lie algebra (p. 403) in which the field $F = \mathrm{D}A$ takes its values, and $F = \mathrm{D}A$ means $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta]$.

Answer 1: Since we have assumed that diffeomorphisms induce transformations of the field u, we have a definition of the Lie derivative of u with respect to a vector field X on V. The invariance of S under diffeomorphisms implies the infinitesimal invariance under generators of 1-parameter groups of diffeomorphism, that is

$$S'(g, u)\cdot(h, v) = 0 \qquad \text{if} \qquad h = \mathscr{L}_X g,\ v = \mathscr{L}_X u. \tag{3}$$

Eq. (3) reads

$$S'(g,u) = \int_V (T \cdot \mathscr{L}_X g + \mathscr{E}_L(u) \cdot \mathscr{L}_X u)\, \mathrm{d}\mu(g) = 0.$$

It reduces to eq. (1) when $\mathscr{E}_L(u) = 0$, hence

$$\nabla \cdot T = 0 \quad \text{if} \quad \mathscr{E}_L(u) = 0.$$

More precisely we can compute $\nabla \cdot T$ in terms of $\mathscr{E}_L(u)$ as follows.
We already know that $(\mathscr{L}_X g)_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha$. We write $\mathscr{L}_X u = \mathrm{D}_u X$, where D_u is a linear partial differential operator on X, depending on u; since $\mathscr{L}_X u$, like u, is a section of E, the operator D_u maps vector fields, i.e., sections of TV, into sections of E. We denote by D_u^*, the L^2-adjoint operator, mapping sections of E into sections of TV. Eq. (3) gives

$$\int_V (-2\nabla \cdot T + \mathrm{D}_u^* \mathscr{E}_L(u)) \cdot X\, \mathrm{d}\mu(g) = 0 \quad \forall X.$$

We therefore have the identity

$$-\nabla \cdot T + \tfrac{1}{2}\mathrm{D}_u^* \mathscr{E}_L(u) \equiv 0. \tag{4}$$

If $\mathscr{E}_L(u) = 0$, i.e., $\mathscr{E}_L(u) = 0$ is the zero section of E, then $\mathrm{D}_u^* \mathscr{E}_L(u) = 0$, and $\nabla \cdot T = 0$

2) *Examples*:
a) For a scalar field u we have $f^*u = u \circ f^{-1}$ and $\mathscr{L}_X u = X \cdot \partial u = X^\alpha \partial_\alpha u$. Thus D_u: $X \mapsto X^\alpha \partial_\alpha u$ contains no differentiation of X and D_u^*: $v \mapsto \mathrm{D}_u^* v$ is the product by ∂u, since it is defined by

$$\int \mathrm{D}_u X v\, \mathrm{d}\mu = \int X^\alpha \partial_\alpha u v\, \mathrm{d}\mu = \int (\mathrm{D}_u^* v)_\alpha X^\alpha\, \mathrm{d}\mu, \qquad v \text{ a scalar field.}$$

An easy computation gives (recall that $\mathrm{d}\mu(g)$ also contains g and that $\delta g^{\alpha\beta} = -g^{\alpha\lambda} g^{\beta\mu} \delta g_{\lambda\mu}$)

$$T_{\alpha\beta} = (\mathscr{E}_L(g))_{\alpha\beta} = \partial_\alpha u \partial_\beta u - \tfrac{1}{2} g_{\alpha\beta}(\partial^\lambda u \partial_\lambda u - u^{p+1})$$
$$\mathscr{E}_l(u) \equiv 2\nabla^\alpha \partial_\alpha u + (p+1)u^p.$$

Formula (4), or a direct calculation, gives

$$\nabla_\alpha T^{\alpha\beta} - \partial^\beta u(\nabla^\alpha \partial_\alpha u + \tfrac{1}{2}(p+1)u^p) = 0;$$

thus $\nabla_\alpha T^{\alpha\beta} = 0$ if u satisfies the equation of motion of the scalar field,

$$2\nabla^\alpha \partial_\alpha u + (p+1)u^p = 0.$$

b) Computation gives (cf. Yang–Mills field, p. 403)

$$T_{\alpha\beta} = \tfrac{1}{2} g_{\alpha\beta}(F^{\lambda\mu} \cdot F_{\lambda\mu}) - 2F_{\alpha\lambda} \cdot F_{\beta}{}^{\lambda}.$$

$$\mathscr{E}_L(A) \equiv -2\mathrm{D} \cdot F, \qquad \mathrm{D} \cdot F \equiv \mathrm{D}_\alpha F^{\alpha\beta} \equiv \nabla_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}]$$

and

$$\nabla_\alpha \mathrm{T}^{\alpha\beta} = \mathrm{D}^\beta F^{\lambda\mu} \cdot F_{\lambda\mu} - 2\mathrm{D}_\alpha F^{\alpha\lambda} \cdot F_\beta{}^\lambda - F^\alpha{}_\lambda \cdot \mathrm{D}_\alpha F_\beta{}^\lambda.$$

By changing names of indices and using the antisymmetry of F we obtain:

$$\nabla_\alpha T^{\alpha\beta} = (\mathrm{D}^\beta F^{\lambda\mu} + \mathrm{D}^\mu F^{\beta\lambda} + \mathrm{D}^\lambda F^{\beta\mu}) \cdot F_{\lambda\mu} - 2\mathrm{D}_\alpha F^{\alpha\lambda} \cdot F_\beta{}^\lambda.$$

Thus $\nabla_\alpha T^{\alpha\beta} = 0$ modulo the Bianchi identity $\mathrm{D}F = 0$ (which reduces to the closure of F in the abelian, in particular Maxwell, case) and the Yang–Mills equation $\mathrm{D} \cdot F = 0$.

III. HYPERBOLIC METRIC G ON A MANIFOLD V OF SIGNATURE $(+, -, \ldots, -)$.

We suppose that V is a product $V = M \times \mathbb{R}$, with the submanifolds $M_t = M \times \{t\}$ spacelike.
The energy–momentum vector associated to a stress–energy tensor T and a timelike or null vector field X is defined to be

$$P^\beta = X_\alpha T^{\alpha\beta}.$$

1) The energy on M_t with respect to the vector field X is defined by the integral (when it exists)

$$E(t) = \int_{M_t} P^\alpha n_\alpha \,\mathrm{d}\mu_t,$$

where n is the unit normal to M_t and $\mathrm{d}\mu_t$ is the volume element of the metric induced on M_t by g.
Suppose that T satisfies the equation $\nabla_\alpha T^{\alpha\beta} = 0$. Study the dependence of $E(t)$ on t.

2) *Determine $E(t)$ for the scalar field of II-a and for the Yang–Mills field of II-b.*

Answer 1: If $\nabla_\alpha T^{\alpha\beta} = 0$, for a symmetric tensor $T^{\alpha\beta}$, we have

$$\nabla_\alpha P^\alpha = \tfrac{1}{2} T^{\alpha\beta}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha). \tag{5}$$

Thus for any compact domain $K \subset V$ with smooth boundary ∂K, and smooth T and X, we find

$$\int_{\partial K} P^{\alpha} n_{\alpha} \, \mathrm{d}\sigma = \tfrac{1}{2} \int_{K} T^{\alpha\beta}(\nabla_{\alpha} X_{\beta} + \nabla_{\beta} X_{\alpha}) \, \mathrm{d}\mu, \tag{6}$$

where $\mathrm{d}\mu$ is the volume element of g, $\mathrm{d}\sigma$ is the volume element of the induced metric on ∂K, and n is the unit normal of ∂K.

We consider a domain $D = M \times [t_0, t_1]$ of V, and we suppose that $T^{\alpha\beta}$ is smooth and that its support is contained in a fixed compact set Σ for each $t \in [t_0, t_1]$.
We apply formula (6) and obtain, denoting by $\mathrm{d}\mu_t$ the volume element induced on M_t by g,

$$\int_{M_{t_1}} P^{\alpha} n_{\alpha} \, \mathrm{d}\mu_{t_1} - \int_{M_{t_0}} P^{\alpha} n_{\alpha} \, \mathrm{d}\mu_{t_0} = \tfrac{1}{2} \int_{t_0}^{t_1} \int_{M_t} T^{\alpha\beta}(\nabla_{\alpha} X_{\beta} + \nabla_{\beta} X_{\alpha}) \, \mathrm{d}\mu.$$

The right-hand side is zero if X is a Killing vector field of the metric g. In that case $E(t_1) = E(t_0)$.
The above formula is valid if T is smooth and has support in a set $\Sigma \times [t_0, t_1]$, Σ compact in M. It will remain valid for any tensor T which can be approximated by a sequence of tensors T_n of the previous type in such a way that the integrals written for the T_n's converge to the integrals written for T.

2) *Examples*:
a) If u is the non-linear-scalar field of II-a, we have

$$P^{\alpha} n_{\alpha} = -\tfrac{1}{2} X^{\alpha} n_{\alpha} (\partial^{\lambda} u \partial_{\lambda} u) + X^{\alpha} \partial_{\alpha} u n^{\beta} \partial_{\beta} u + \tfrac{1}{2} X^{\alpha} n_{\alpha} u^{p+1}.$$

We define a quadratic form γ by

$$\gamma^{\lambda\mu} = -g^{\lambda\mu} X^{\alpha} n_{\alpha} + (X^{\lambda} n^{\mu} + x^{\mu} n^{\lambda})$$

and we see that

$$P^{\alpha} n_{\alpha} = \tfrac{1}{2} \gamma^{\lambda\mu} \partial_{\lambda} u \partial_{\mu} u + \tfrac{1}{2} X^{\alpha} n_{\alpha} u^{p+1}.$$

The quadratic form γ is positive definite: this can most easily be seen by choosing coordinates such that $X^0 = 1$, $X^i = 0$, $n_i = 0$ (since M_t is a submanifold $x^0 = t$, constant). Then $n_0 = (g^{00})^{-1/2}$, $n^i = g^{i0}(g^{00})^{-1/2}$, and

$$\gamma^{00} = g^{00} n_0, \qquad \gamma^{i0} = 0, \qquad \gamma^{ij} = -g^{ij} n_0.$$

$P^{\alpha} n_{\alpha}$ is therefore positive for all $u \geq 0$, and positive for all u if $p+1$ is an even positive integer; the energy density $P^{\alpha} n_{\alpha}$ of the scalar field u is then positive. The energy $E(t)$ is defined if $P^{\alpha} n_{\alpha}$ is integrable on M_t; this will be the case if the restriction of ∂u to M_t is square integrable in the appropriate norm and volume and the restriction of u^{p+1} to M_t is integrable.

b) The energy density of the Yang–Mills field is

$$P^\alpha n_\alpha = \tfrac{1}{2} F^{\lambda\mu} \cdot F_{\lambda\mu} X^\alpha n_\alpha - 2X^\alpha n^\beta F_{\alpha\lambda} \cdot F_\beta{}^\lambda.$$

To simplify the writing we take time lines orthogonal to the space-like sections, so $g^{0i} = n^i = 0$, and $P^\alpha n_\alpha$ reads

$$P^\alpha n_\alpha = (\tfrac{1}{2} F^{ij} \cdot F_{ij} - F_{0i} \cdot F^{0i}) n_0$$

$$= (E^2 + H^2) n_0,$$

$$H^2 = \tfrac{1}{2} F^{ij} \cdot F_{ij}, \qquad E^2 = -g^{ij} g^{00} F_{0i} \cdot F_{0j}.$$

E^2 and H^2 are positive if the scalar product in $\mathcal{G}$ corresponds to a positive quadratic form on $\mathcal{G}$.

III. DIFFERENTIABLE MANIFOLDS

1. SHEAVES

Let X be a topological space and F a surjective mapping from the family of open sets in X onto a family $\mathscr{F}$ of sets. Suppose that for each pair (U, V) of open sets in X with $U \subset V$ there is a mapping, called the restriction mapping, $r_U^V: F(V) \to F(U)$, where $F(V)$ and $F(U)$ are sets corresponding to U and V. The family $\mathscr{F}$ together with the mappings r_U^V is called a **sheaf** over X if: sheaf

1) $r_U^V \circ r_V^W = r_U^W$ for open subsets $U \subset V \subset W$;

2) If $\{U_i\}$ is a collection of open sets in X with $U = \cup_i U_i$ then

   a) if $f, g \in F(U)$ and $r_{U_i}^U f = r_{U_i}^U g$, $\forall i$ then $f = g$,

   b) if $f_i \in F(U_i)$ and $r_{U_i \cap U_j}^{U_i} f_i = r_{U_i \cap U_j}^{U_j} f_j$, $\forall i, j$, then there exists $f \in F(U)$ such that $r_{U_i}^U f = f_i$, $\forall i$.

If the elements of $\mathscr{F}$ are groups, modules, graded algebras, etc., and the r_U^V are homomorphisms of these structures, the sheaf $\mathscr{F}$ over X is called a **sheaf of groups, modules, graded algebras,** etc. sheaf of groups modules graded algebras

Show that the family of smooth functions on the open sets of a manifold X is a sheaf of rings over X.

Answer: The set $F(U)$ associated to the open set $U \subset X$ is the set of smooth functions $U \to \mathbb{R}$. Let V and U be two open sets in X such that $U \subset V$. The mapping r_U^V associates to $f: V \to \mathbb{R}$ in $F(V)$ its restriction to U. It is straightforward to check that it satisfies the listed properties.

2. DIFFERENTIABLE SUBMANIFOLDS

Definition: $N^n \subset M^m$ is said to be a **smooth submanifold** of M^m if for every $x \in N^n$ there exists an open neighborhood U of x in M^m and a diffeomorphism smooth submanifold

$$f: U \to V \text{ open in } \mathbb{R}^m$$

such that for $\bar{\mathbb{R}}^n$ defined as the set $\mathbb{R}^n \times \{0\}^{m-n}$

$$f^{-1}(V \cap \bar{\mathbb{R}}^n) = U \cap N^n, (x^1, \ldots, x^n, 0, \ldots, 0) \in \bar{\mathbb{R}}^n.$$

Show that if $\{(U_i, \phi_i)\}$ is an atlas of M^m, and f_i satisfies the above property on each U_i, then the set $\{(U_i \cap N^n, f_i)\}$ is an atlas for N^n.

inherited structure

This differentiable structure is said to be **inherited** from the differentiable structure of M^m.

Answer: One checks that $\{U_i \cap N^n\}$ covers N^n; the differentiability of

$$f_j(U_i \cap U_j \cap N^n) \to f_i(U_i \cap U_j \cap N^n)$$

is a consequence of the differentiability of the diffeomorphisms f_i.

Show that the submanifold $N \subset \mathbb{R}^2$ defined by $x^2 = |x^1|$ cannot inherit the differentiable structure of $\mathbb{R}^2$.

Answer: If N inherits a differentiable structure from $\mathbb{R}^2$ there exists a diffeomorphism f from a neighborhood U of zero in $\mathbb{R}^2$ onto V in $\mathbb{R}^2$ such that

$$f^{-1}: V \to U \quad \text{by} \quad (y^1, y^2) \mapsto (x^1, x^2) \text{ is a diffeomorphism}$$

and $U \cap N$ is the curve of $\mathbb{R}^2$

$$(x^1, x^2) = f^{-1}(y^1, 0), \qquad y^1 \in V \cap \mathbb{R}.$$

This curve is differentiable at each of its points, while the submanifold

$$|x^1| = x^2$$

is a curve which is not differentiable at the origin.

3. SUBGROUPS OF LIE GROUPS. WHEN ARE THEY LIE SUBGROUPS?

We have defined (p. 242) a Lie subgroup H of a Lie group G as a subgroup of G which is also a submanifold of G.
We have defined a submanifold as a regular embedding (p. 239).
Show that a Lie subgroup H of a Lie group G is closed in G. Give an example of a subgroup of a Lie group which is not a Lie subgroup.
It can be proved that, conversely, any closed subgroup H of a Lie group G is a Lie subgroup.

Answer: Let $\mathscr{G}$ and $\mathscr{H}$ be the Lie algebras of G and H. We shall show first that there exists an open neighborhood V_O of O in $\mathscr{G}$ such that the exponential mapping (p. 160) is a diffeomorphism from V_O onto a neighborhood V_e of e in G, and

$$\exp(V_O \cap \mathscr{H}) = V_e \cap H .$$

Select a neighborhood N_O of O in $\mathscr{H}$ such that exp is a diffeomorphism from N_O onto N_e, a neighborhood of e in H. Since H is a submanifold of G there exists a neighborhood U_e of e in G such that $N_e = U_e \cap H$. We restrict U_e to an eventually smaller neighborhood V_e such that the exponential mapping is a diffeomorphism from V_O onto V_e. If $X \in V_O \cap \mathscr{H} \subset N_O$ we have $\exp X \in \exp V_O$ and $\exp X \in H$. This gives the required property for V_O.

Let $(h_n) \subset H$ be a sequence which converges in G to an element $g \in G$. Then $h_n g^{-1}$ converges to e: for every neighborhood W_e of e there exists an integer N such that for $n \geq N$, $h_n g^{-1} \subset W_e$.
We choose W_e small enough to have

$$(x, y) \in W_e \times W_e \Rightarrow xy^{-1} \in V_e .$$

Then for $n > N$,

$$h_n h_N^{-1} \in V_e \cap H, \qquad \text{so} \qquad h_n h_N^{-1} = \exp X_n, \quad X_n \in V_O \cap \mathscr{H}.$$

We may always choose V_O bounded. Then there is a subsequence of the sequence X_n which converges to an element $X \in \mathscr{H}$; $h_n h_N^{-1}$ converges to $\exp X \in H$, and h_n converges to $h_N \exp X \in H$. Thus H is closed in G.

Example of a subgroup of a Lie group which is not a Lie subgroup. Let G be the 2-torus T^2

$$G = S^1 \times S^1 = U(1) \times U(1).$$

Let $H_\lambda = \{(\exp(2i\pi t), \exp(2i\pi\lambda t)\}$, $t \in \mathbb{R}$.
Then H is a subset and a subgroup of G. It is not a submanifold if λ is irrational; the closure of H in G is in this case the whole of G.

4. CARTAN–KILLING FORM ON THE LIE ALGEBRA $\mathscr{G}$ OF A LIE GROUP G

The mapping $\mathscr{A}d$: $\mathscr{G} \to \mathscr{L}(\mathscr{G}, \mathscr{G})$ (p. 167) is defined through the derivative at $e \in G$ of the mapping Ad: $G \to \mathscr{L}(\mathscr{G}, \mathscr{G})$. Thus, if $X \in \mathscr{G}$, $\mathscr{A}d\, X$ is a linear mapping $\mathscr{G} \to \mathscr{G}$. We denote by tr the trace of such an endomorphism and define the Cartan–Killing symmetric bilinear form on $\mathscr{G}$ by

$$B(X, Y) = \operatorname{tr}(\mathscr{A}d\, X\, \mathscr{A}d\, Y), \qquad X, Y \in \mathscr{G}.$$

An automorphism σ of $\mathscr{G}$ is a bijective linear map $\sigma \in GL(\mathscr{G})$ which preserves the Lie bracket (i.e., σ is an isomorphism $\mathscr{G} \to \mathscr{G}$)

$$\sigma[X, Y] = [\sigma X, \sigma Y].$$

1) *Show that the Cartan–Killing form B is invariant under automorphisms of $\mathscr{G}$, that is*

$$B(X, Y) = B(\sigma X, \sigma Y).$$

Show that B is invariant under the adjoint maps $\mathrm{Ad}(g)$, $g \in G$ (p. 167).

2) *Prove the formulae*

$$B(X, [Y, Z]) = B(Y, [Z, X]) = B(Z, [X, Y]).$$

3) *Give the expression of the Cartan–Killing form in the basis e_i of $\mathscr{G}$.*

Answer 1: If $\sigma \in GL(\mathscr{G})$ we have

$$\mathscr{A}d(\sigma X) = \sigma \circ \mathscr{A}d\, X \circ \sigma^{-1}.$$

Indeed, (cf. p. 167) the action of $\mathscr{A}d\, X$ on $\mathscr{G}$ is

$$\mathscr{A}d\, X\colon \mathscr{G} \to \mathscr{G} \quad \text{by} \quad Y \mapsto [X, Y], \text{ by hypothesis,}$$

so $\mathscr{A}d(\sigma X)\colon \mathscr{G} \to \mathscr{G}$ by $Y \to [\sigma X, Y] = \sigma[X, \sigma^{-1}Y]$, since σ preserves the bracket.
We therefore have

$$\operatorname{tr} \mathscr{A}d(\sigma X)\mathscr{A}d(\sigma Y) = \operatorname{tr}(\sigma\, \mathscr{A}d\, X\, \mathscr{A}d\, Y\, \sigma^{-1}) = \operatorname{tr} \mathscr{A}d\, X\, \mathscr{A}d\, Y. \quad \blacksquare$$

The invariance of B under the adjoint maps is a consequence of the fact that $L_g R_g^{-1}\colon h \mapsto ghg^{-1}$ is an isomorphism of G and the following lemma:
Lemma:
Let $f\colon G \to \tilde{G}$ be an isomorphism of Lie groups, that is, a diffeomorphism such that:

$$f(xy) = f(x)f(y) \qquad \forall x, y \in G. \tag{1}$$

Then f determines an isomorphism $\mathscr{G} \to \tilde{\mathscr{G}}$ of their Lie algebras.

Proof: A left invariant vector field X on G has an image under f which is a left invariant vector field, since relation (1) implies that f commutes with the left translations:

$$f \circ L_x = \tilde{L}_{f(x)} \circ f$$

and, by derivation

$$f' \circ L'_x = \tilde{L}'_{f(x)} \circ f',$$

$$L'_x X = X \quad \text{implies} \quad \tilde{X} = f'X = f' \circ L'_x X = \tilde{L}'_{f(x)} \circ f' X = \tilde{L}'_{f(x)} \tilde{X}.$$

The proof is completed by using $f'[X, Y] = [f'X, f'Y]$ (p. 135).

Answer 2: The mapping $\mathcal{A}d$: $\mathcal{G} \to GL(\mathcal{G})$ is a homomorphism of Lie algebras with the usual bracket in the Lie algebra of matrices, that is

$$\mathcal{A}d[X, Y] = [\mathcal{A}d\, X, \mathcal{A}d\, Y] = \mathcal{A}d\, X\, \mathcal{A}d\, Y - \mathcal{A}d\, Y\, \mathcal{A}d\, X.$$

Thus, using the fact that $\operatorname{tr} AB = \operatorname{tr} BA$ and $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$

$$\begin{aligned} \operatorname{tr}(\mathcal{A}d[X, Y]\, \mathcal{A}d\, Z) &= \operatorname{tr}(\mathcal{A}d\, X\, \mathcal{A}d\, Y\, \mathcal{A}d\, Z - \mathcal{A}d\, Y\, \mathcal{A}d\, X\, \mathcal{A}d\, Z) \\ &= \operatorname{tr}(\mathcal{A}d\, Z\, \mathcal{A}d\, X\, \mathcal{A}d\, Y - \mathcal{A}d\, X\, \mathcal{A}d\, Z\, \mathcal{A}d\, Y) = \operatorname{tr}(\mathcal{A}d[Z, X]\, \mathcal{A}d\, Y). \end{aligned}$$

Answer 3: We have (p. 168)

$$(\mathcal{A}d\, e_i)^l_k = c_i{}^l{}_k,$$

thus

$$B(e_i, e_j) = c_i{}^l{}_k c_j{}^k{}_l.$$

If $c_i{}^k{}_l = -c_i{}^l{}_k$ and $c_i{}^l{}_k c_j{}^k{}_l = 0$ when $i \neq j$, then B is negative.
It can be proved that the Cartan–Killing form is nondegenerate if and only if G is **semi-simple**, i.e., admits no invariant abelian subgroup. The Cartan–Killing form is negative definite if G is semi-simple and compact. semi-simple

5. DIRECT AND SEMIDIRECT PRODUCTS OF LIE GROUPS AND THEIR LIE ALGEBRA

1) *Direct products: Let G be a Lie group which is the direct product of two Lie groups A and B. Show that the Lie algebra $\mathcal{G}$ is the direct sum $\mathcal{G} = \mathcal{A} \oplus \mathcal{B}$ with $[\mathcal{A}, \mathcal{B}] = 0$. Give an $\mathcal{A}d\, G$ invariant scalar product on $\mathcal{G}$.*

Answer: $G = A \times B$ means that G is the direct product of the manifolds A and B, with the product law

$$(a_1, b_2)(a_2, b_2) = (a_1 a_2, b_1 b_2),$$
$$a_1, a_2 \in A; \qquad b_1, b_2 \in B.$$

A left invariant vector field X on G is a pair of tangent vector fields to A and B, $X = (X_{\mathscr{A}}, X_{\mathscr{B}})$ such that

$$L'_g X = X;$$

but

$$L'_g X = (L'_a X_{\mathscr{A}}, L'_b X_{\mathscr{B}}).$$

Thus

$$\mathscr{G} = \mathscr{A} \times \mathscr{B} \cong \mathscr{A} \oplus \mathscr{B}$$

by identifying the subspace of $\mathscr{G}$ of pairs $(X_{\mathscr{A}}, O_{\mathscr{B}})$ [resp.$(O_{\mathscr{A}}, X_{\mathscr{B}})$] with $\mathscr{A}$ [resp. $\mathscr{B}$].
The Lie bracket in $\mathscr{G}$ is

$$[X, Y] = ([X_{\mathscr{A}}, Y_{\mathscr{A}}], [X_{\mathscr{B}}, Y_{\mathscr{B}}]),$$

as is true in general for the bracket of vector fields on a product manifold; this can be checked using the definition on p. 134.
The scalar product of a direct sum

$$(X_{\mathscr{A}} + X_{\mathscr{B}})(Y_{\mathscr{A}} + Y_{\mathscr{B}}) \overset{\text{def}}{=} X_{\mathscr{A}} Y_{\mathscr{A}} + X_{\mathscr{B}} Y_{\mathscr{B}}$$

is Ad G invariant if the scalar products in A and B are respectively Ad A and Ad B invariant. ■

2) *Semidirect products*: Let A and B be groups, and t: $(B \times A) \to A$ be a group of automorphisms of A indexed by B:

$$t_b: A \to A \quad \text{by} \quad a \mapsto t_b(a) = t(b, a).$$

semidirect product

The **semidirect product** $G = A \circledS B$ is the space of ordered pairs (a, b) with the product law

$$(a_1, b_1)(a_2, b_2) = (a_1 t_{b_1}(a_2), b_1 b_2). \tag{1}$$

If A is a Lie group and t a Lie group of diffeomorphisms of A defined by B, then $G = A \circledS B$ is a Lie group.
Show that the Poincaré group is the semi-direct product of $\mathbb{R}^4$ with the Lorentz group. Show that the Euclidean group $\mathbb{E}_n = \mathbb{R}^n \circledS \mathrm{SO}(n)$.

3) *Write the Galileo group as a semi-direct product. Obtain it by a limiting procedure from the Poincaré group.*

4) *Let* $A' = A \times \{\mathbb{1}_B\}$ *and* $B' = \{\mathbb{1}_A\} \times B$ *and let* $a' = (a, \mathbb{1}_B)$ *and* $b' = (\mathbb{1}_A, b)$. *Show that with the semidirect product law* (1)

$$b'a'b'^{-1} = (t_b(a))'. \tag{2}$$

Similarly for the Lie algebras, let

$$\mathscr{A}' = \mathscr{A} \oplus O_{\mathscr{B}} \quad \text{and} \quad \mathscr{B}' = O_{\mathscr{A}} \oplus \mathscr{B};$$

$$X'_{\mathscr{A}} = (X_{\mathscr{A}}, O_{\mathscr{B}}) \quad \text{and} \quad X'_{\mathscr{B}} = (O_{\mathscr{A}}, X_{\mathscr{B}}).$$

Let $\tau_b \colon \mathscr{A} \to \mathscr{A}$ *be such that*

$$t_b(\exp X_{\mathscr{A}}) = \exp(\tau_b(X_{\mathscr{A}})),$$

Show that

$$(\tau_b(X_{\mathscr{A}}))' = \mathrm{Ad}(b')(X'_{\mathscr{A}}).$$

5) *Write down a group multiplication of* $G = \mathbb{R} \circledS \mathbb{R}_+$ *where* $\mathbb{R}_+$ *is the abelian multiplicative group of positive numbers. Compute the Lie bracket of its Lie algebra.* The calculation given in the answer is used in the construction of the quantum observables of a system whose configuration space is $\mathbb{R}_+$ (see Isham, and Klauder).

6) Show that the Lie algebra $\mathscr{G}$ of $G = A \circledS B$ *is* $\mathscr{G} = \mathscr{A} \circledS \mathscr{B}$, *where the semi-direct product Lie algebra is defined by*

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2] + \mathscr{A}d(Y_1)X_2 - \mathscr{A}d(Y_2)X_1, [Y_1, Y_2]),$$
$$X \in \mathscr{A},\ Y \in \mathscr{B}. \tag{3}$$

where $\mathscr{A}d(Y)X = [Y, X]$.

Answer 2: A **Poincaré transformation** (a, Λ) of the four-dimensional Minkowski space X is defined by (margin: Poincaré transformation)

$$(a, \Lambda) \colon X \to X \quad \text{by} \quad y^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \qquad \Lambda \in \mathrm{SO}(3,1).$$

Performing another transformation gives the product law

$$(a_2, \Lambda_2)(a_1, \Lambda_1) = (a_2 + \Lambda_2 a_1, \Lambda_2\Lambda_1).$$

Therefore the Poincaré group is the semidirect product of the four dimensional translation group and the homogeneous Lorentz group SO(3, 1).

A **euclidean transformation** of $\mathbb{R}^n$ is defined similarly, leading to the same product law for $\Lambda \in \mathrm{SO}(3)$. (margin: euclidean transformation)

Answer 3: A Galileo transformation $g = (b, a, \boldsymbol{v}, R)$ of $\mathbb{R}^3 \times \mathbb{R}$ is defined by $g: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}$ with $(x, t) \mapsto (y, s)$, where

$$\begin{cases} y^i = R^i{}_j x^j - v^i t + a^i \\ s = t - b. \end{cases}$$

It follows that the product $g_1 g_2 = g$ is

$$g = (b_1 + b_2, a_1 + b_2 \boldsymbol{v}_1 + R_1 a_2, \boldsymbol{v}_1 + R_1 \boldsymbol{v}_2, R_1 R_2).$$

This is the group law of $(\mathbb{R}^1 \times \mathbb{R}^3) \circledS \mathbb{E}_3$ with the group of automorphisms of $\mathbb{R}_1 \times \mathbb{R}_2$ indexed by $\mathbb{E}_3$ defined by

$$E_E(b, a) = (b, b\boldsymbol{v} + Ra), \qquad E = (\boldsymbol{v}, R) \in \mathbb{E}_3, (b, a) \in \mathbb{R}^1 \times \mathbb{R}^3.$$

Note that the inverse element of g is

$$g^{-1} = (-b, -R^{-1}(a - b\boldsymbol{v}); -R^{-1}\boldsymbol{v}, R^{-1}).$$

Set $x^0 \equiv ct$ in the Poincaré transformation and let c tend to infinity. Let Λ^i_0 tend to zero in such a way that $\Lambda^i_0 c$ tends to v^i, and a^0 tend to infinity in such a way that a^0/c tends to b. The corresponding limit of the Poincaré transformation is a Galileo transformation.

Answer 4: The inverse of a pair (a, b) is the pair $(a, b)^{-1}$ such that

$$(a, b)(a, b)^{-1} = (\mathbb{1}_A, \mathbb{1}_B).$$

Hence it is given by

$$(a, b)^{-1} = (t_b^{-1}(a^{-1}), b^{-1}).$$

Let $a' = (a, \mathbb{1}_B)$ and $b' = (\mathbb{1}_A, b)$. Then

$$b'a'b'^{-1} = (t_b(a), b)(t_b^{-1}(\mathbb{1}_a), b^{-1}) = (t_b(a)\mathbb{1}_A, \mathbb{1}_B) = (t_b(a))'. \tag{4}$$

Let

$$\exp(\tau_b X_{\mathscr{A}}) = t_b(\exp X_{\mathscr{A}}).$$

It follows from (4) that

$$(\exp(\tau_b X_{\mathscr{A}}))' = b'(\exp X_{\mathscr{A}})' b'^{-1}.$$

Hence

$$(\tau_b(X_{\mathscr{A}}))' = \mathrm{Ad}(b')X'_{\mathscr{A}}.$$

Answer 5: Note that the multiplicative group $\mathbb{R}_+$ is isomorphic to the additive group $\mathbb{R}$, with the isomorphism given by $\exp: \mathbb{R} \to \mathbb{R}_+$. A group law of $\mathbb{R} \circledS \mathbb{R}_+$ is defined by an automorphism of $\mathbb{R}$ indexed by $\mathbb{R}_+$. Let $g \equiv (v, \lambda) \in \mathbb{R} \circledS \mathbb{R}_+$, consider the following left actions of $\mathbb{R} \circledS \mathbb{R}_+$:

$$L_g(x, y) = (x + \log \lambda, y - \lambda\, \mathrm{e}^x v) \text{ defined on } \mathbb{R} \times \mathbb{R},$$

and

$$L_g(x, y) = (\lambda x, y\lambda^{-1} - v) \text{ defined on } \mathbb{R}_+ \times \mathbb{R}.$$

These two actions define the same group law

$$(v_2, \lambda_2)(v_1, \lambda_1) = (v_2 + v_1\lambda_2^{-1}, \lambda_2\lambda_1). \tag{5}$$

Let $b \in \mathscr{L}(\mathbb{R})$, the Lie algebra of $\mathbb{R}$, and $r \in \mathscr{L}(\mathbb{R}_+)$, the Lie algebra of $\mathbb{R}_+$. It is easier to compute $[(b_2, r_2), (b_1, r_1)]$ by decomposing

$$(b, r) = (b, 0) + (0, r)$$

and computing separately the four Lie brackets $[(b_2, 0), (b_1, 0)]$, $[(b_2, 0), (0, r_1)]$, $[(0, r_2), (b_1, 0)]$ and $[(0, r_2), (0, r_1)]$.
We shall do the calculation by two different methods:
i) Algebraic method using the Campbell Hausdorff formula (see appendix) together with the group law (5):
ii) Computing the brackets of the left invariant vector fields on $G = \mathbb{R} \circledS \mathbb{R}_+$ (representation theory).
i) We spell out the computation of only two of the four Lie brackets which make up $[(b_2, r_2), (b_1, r_1)]$:

$$\begin{aligned} &\exp(tb_2, 0)\exp(sb_1, 0)\exp(-tb_2, 0)\exp(-sb_1, 0) \\ &\qquad = \exp(\tfrac{1}{2}ts[(b_2, 0), (b_1, 0)]). \end{aligned} \tag{6}$$

By virtue of the group law (5) the left-hand side is equal to $(0, 1) = \exp(0, 0)$.
Identifying the left- and right-hand side of (6) yields $[(b_2, 0), (b_1, 0)] = (0, 0)$. Similarly,

$$\begin{aligned} \exp(tb_2, 0)\exp(0, sr_1)\exp(-tb_2, 0)\exp(0, -sr_1) &= (tb_2 - e^{-sr_1}tb_2, 1) \\ &= \exp(tb_2(1 - e^{-sr_1}), 0) \\ &= \exp(stb_2r_1 + \text{higher order terms},\ 0), \end{aligned}$$

which yields $[(b_2, 0), (0, r_1)] = (b_2 r_1, 0)$.
A similar calculation gives

$$[(0, r_2), (b_1, 0)] = (-r_2 b_1, 0)$$

and

$$[(0, r_2), (0, r_1)] = (0, 0).$$

Finally

$$[(b_2, r_2), (b_1, r_1)] = (b_2 r_1 - r_2 b_1, 0).$$

ii) The set of left invariant vector fields L_X on G can be considered either

as the Lie algebra $\mathscr{L}(G)$ of G or as a representation of the Lie algebra obtained in the previous paragraph from the exponential map from $\mathscr{L}(G)$ into G.
The left invariant vector field L_X on G is the generator of the one parameter group of transformations $\{R_{g(t)};\, \mathrm{d}g(t)/\mathrm{d}t|_{t=0} = X\}$:

$$L_X(h) = \frac{\mathrm{d}}{\mathrm{d}t}(R_{g(t)}h)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(L_h g(t))|_{t=0} = L'_h X.$$

We shall compute $hg(t)$ with $h = (v, \lambda)$ for $g(t)$ in the two following cases

$$g(t) = \exp tX = \exp(tb, 0) = (tb, 1) \tag{7}$$

and

$$g(t) = \exp tY = \exp(0, tr) = (0, \mathrm{e}^{tr}). \tag{8}$$

In the first case, using the group multiplication (5)

$$hg(t) = (v, \lambda)(tb, 1) = (v + \lambda^{-1}tb, \lambda),$$

$$L_{(b,0)}(v, \lambda) = \frac{\mathrm{d}}{\mathrm{d}t}(hg(t))|_{t=0} = (\lambda^{-1}b, 0) = \lambda^{-1}b\frac{\partial}{\partial v} + 0\frac{\partial}{\partial \lambda},$$

$$L_X(v, \lambda) = \lambda^{-1}b\, \partial/\partial v. \tag{9}$$

In the second case

$$hg(t) = (v, \lambda)(0, \mathrm{e}^{tr}) = (v, \lambda\, \mathrm{e}^{tr}),$$

$$L_{(0,r)}(v, \lambda) = \frac{\mathrm{d}}{\mathrm{d}t}(hg(t))|_{t=0} = (0, \lambda r) = 0\frac{\partial}{\partial v} + \lambda r\frac{\partial}{\partial \lambda},$$

i.e.,

$$L_Y(v, \lambda) = \lambda r\, \partial/\partial\lambda. \tag{10}$$

We can consider $L_X = \lambda^{-1}b\, \partial/\partial v$ as the representative of $X = (b, 0)$ and $L_Y = \lambda r\, \partial/\partial\lambda$ as the representative of $Y = (0, r)$.
The brackets $[L_X, L_X]$, $[L_X, L_Y]$, $[L_Y, L_X]$ and $[L_Y, L_Y]$ are the representatives of $[X, X]$, $[X, Y]$, $[Y, X]$ and $[Y, Y]$ respectively. It follows from (9) and (10) that

$$\begin{array}{ll}
[L_X, L_X] = 0 & \text{hence } [X, X] = (0, 0) \\
[L_X, L_Y] = b_2 r_1 \lambda^{-1}\partial/\partial v & \text{hence } [X, Y] = (b_2 r_1, 0) \\
[L_Y, L_X] = -r_2 b_1 \lambda^{-1}\partial/\partial v & \text{hence } [Y, X] = (-r_2 b_1, 0) \\
[L_Y, L_Y] = 0 & \text{hence } [Y, Y] = (0, 0)
\end{array}$$

Finally,

$$[(b_2, r_2), (b_1, r_1)] = (b_2 r_1 - r_2 b_1, 0).$$

Answer 6: As in answer 5, we decompose $(X, Y) = (0, Y) + (X, 0)$. We shall show that

$$[(X_1, 0), (X_2, 0)] = ([X_1, X_2], 0)$$
$$[(0, Y_1), (0, Y_2)] = (0, [Y_1, Y_2]),$$
$$[(X_1, 0), (0, Y_2)] = (-\mathscr{A}d(Y_2)X_1, 0)$$
$$[(0, Y_1), (Y_2, 0)] = (\mathscr{A}d(Y_1)X_2, 0),$$

which combine to give (3). Let

$$a_i(t) = \exp(tX_i), \qquad b_i(t) = \exp(tY_i), \qquad i = 1, 2.$$

Then

$$\begin{aligned}[(X_1, 0), (X_2, 0)] &= \mathscr{A}d((X_1, 0))(X_2, 0)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}(a_1(t))(X_2, 0)]|_{t=0} \qquad\qquad \text{(p. 167)}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}(a_1(t), \mathbb{1}_B)(a_2(s), \mathbb{1}_B)(a_1^{-1}(t), \mathbb{1}_B)|_{t=s=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}(a_1(t)a_2(s)a_1^{-1}(t), \mathbb{1}_B)|_{t=s=0}\\ &= (\mathscr{A}d(X_1)X_2, 0) = ([X_1, X_2], 0).\end{aligned}$$

A similar calculation gives the other brackets. For instance the calculation of $[(X_1, 0), (0, Y_2)]$ requires the computation of

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}((a_1(t), \mathbb{1}_B)(\mathbb{1}_A, b_2(s))(a_1^{-1}(t), \mathbb{1}_B))|_{t=s=0}\\ &\quad= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}a_1(t)t_{b_2(s)}(a_1^{-1}(t), b_2(s))|_{t=s=0}\\ &\quad= (-\mathscr{A}d(Y_2)x_1, 0).\end{aligned}$$

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See also references at the end of [Problem III 7, Homogeneous].

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6. HOMOMORPHISMS AND ANTIHOMOMORPHISMS OF A LIE ALGEBRA INTO SPACES OF VECTOR FIELDS

Given a faithful left action of a Lie group G on a manifold X by $\sigma_g: X \to X$, let K^γ be the Killing vector field (p. 165) on X generated by $\gamma \in \mathscr{L}(G)$

$$K_x^\gamma = \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_{g(t)}(x))|_{t=0} \text{ with } g(t) = \exp t\gamma. \tag{1}$$

Show that the mapping $F: \gamma \to K^\gamma$ is an anti-isomorphism, i.e.,

$$[K^{\gamma_1}, K^{\gamma_2}] = K^{[\gamma_2, \gamma_1]}.$$

Let v_γ^L and v_γ^R be respectively the vector fields on G generated by the left action $L_{g(t)}$ and the right action $R_{g(t)}$, $g(t) = \exp t\gamma$. Show that the mapping $F: \gamma \mapsto v_\gamma^L$ is an anti-isomorphism and $F: \gamma \mapsto v_\gamma^R$ is an isomorphism.

Answer: Let c^i_{jk} be the structure constants of G, i.e.,

$$[\gamma_j, \gamma_k] = c^i_{jk}\gamma_i.$$

It has been proved (p. 164) that

$$[K^{\gamma_j}, K^{\gamma_k}] = -c^i_{jk}K^{\gamma_i}.$$

In other words

$$[F(\gamma_j), F(\gamma_k)] = -c^i_{jk}F(\gamma_i).$$

The mapping F is linear, since

$$\begin{aligned} K_x^{\gamma_1+\gamma_2} &= \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_{g(t)}(x))|_{t=0} \qquad \text{with } g(t) = \exp t(\gamma_1 + \gamma_2) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_x(g(t)))|_{t=0} \qquad \text{with } \sigma_x(g(t)) \equiv \sigma_{g(t)}(x) \equiv \sigma(g(t), x) \\ &= \sigma_x'(e)(\gamma_1 + \gamma_2) = K_x^{\gamma_1} + K_x^{\gamma_2}. \end{aligned}$$

F is a linear mapping between finite dimensional spaces and satisfies

$$[F(\gamma_j), F(\gamma_k)] = F(-c^i_{jk}\gamma^i) = F(-[\gamma_j, \gamma_k]).$$

Hence F is an anti-isomorphism.
It has also been proved (p. 164) that

$$[v^L_{\gamma_j}, v^L_{\gamma_k}] = -c^i_{jk}v^L_{\gamma_i} \qquad \text{(right invariant vector fields)}$$

and

$$[v^R_{\gamma_j}, v^R_{\gamma_k}] = c^i_{jk}v^R_{\gamma_i} \qquad \text{(left invariant vector fields).}$$

It follows by a similar argument that

$$F\colon \gamma \to v_\gamma^L \quad \text{is anti-isomorphism}$$
$$F\colon \gamma \to v_\gamma^R \quad \text{is an isomorphism.}$$

Various conventions are used to ease the fact that $F\colon \gamma \to K^\gamma$ is an *anti-*isomorphism. For instance, Isham defines $g(t) = \exp(-t\gamma)$. Note that he also denotes without comma the Lie bracket $[\gamma_j \gamma_k]$ of $\mathscr{L}(G)$ and with comma the commutator $[K^{\gamma_j}, K^{\gamma_k}]$ of vector fields. With his conventions

$$\left[K^{\gamma_1}, K^{\gamma_2}\right] = K^{[\gamma_1\gamma_2]}.$$

Note also that labels L and R are sometimes used to denote left invariant and right invariant vector fields rather than generators of left and right translations.

7. HOMOGENEOUS SPACES; SYMMETRIC SPACES

Let G be a connected Lie group and H a closed subgroup of G. The quotient space G/H of left cosets gH is called a **homogeneous space**. It can be proved (Chevalley I p. 114) that G/H can be given the structure of a differentiable manifold.

homogeneous space

1) *Show that G/H is not necessarily a Lie group. Define an action of G on G/H.*

2) *Let $\mathscr{G}$ and $\mathscr{H}$ be the Lie algebras of G and H. Show that $\mathscr{H}$ is a Lie subalgebra of $\mathscr{G}$.*

3) *Show that if G is compact there exists a subspace $\mathscr{M} \subset \mathscr{G}$ such that*

$$\mathscr{G} = \mathscr{H} \oplus \mathscr{M}, \qquad \mathscr{H} \cap \mathscr{M} = 0, \qquad \mathrm{Ad}(H)\mathscr{M} \subset \mathscr{M}. \tag{1}$$

A Lie algebra $\mathscr{G}$ and a Lie subalgebra $\mathscr{H}$ satisfying these properties is called a **reductive pair**. The corresponding homogeneous space is said to be reductive. For examples see Problem V 9, Invariant geometries.

reductive pair

4) *Assume that H is a compact, invariant, Lie subgroup of the compact Lie group G, i.e., $ghg^{-1} \in H$ for every $h \in H$ and $g \in G$. Show that there exists a Lie subalgebra $\mathcal{K}$ of $\mathscr{G}$ such that*

$$\mathscr{G} = \mathscr{H} \oplus \mathscr{K}, \qquad \mathscr{H} \cap \mathscr{K} = 0, \qquad \mathrm{Ad}(H)\mathscr{K} \subset \mathcal{K}, \qquad [\mathscr{H}, \mathscr{K}] = 0. \tag{2}$$

symmetric space
involutive automorphism

5) A **symmetric space** is an homogeneous space G/H with an **involutive automorphism** σ on G (i.e., a map $\sigma\colon G \to G$ satisfying $\sigma(gk) =$

$\sigma(g)\sigma(k)$ and $\sigma^2 = Id$), such that

$$G_0^\sigma \subset H \subset G^\sigma, \tag{3}$$

where G^σ is the closed subgroup of G consisting of all the elements left fixed by σ and G_0^σ is the connected component of G^σ which contains the identity.
Show that $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ *is a symmetric space.*

6) Let t be a transitive left action of a connected Lie group G on a differentiable manifold M,

$$t\colon G \times M \to M \qquad \text{by} \qquad t(g, x) \equiv t_g(x) \equiv gx, \qquad t_{g_1 g_2} = t_{g_1} t_{g_2} \tag{4}$$

and such that

$$t(e, x) = x.$$

G-space homogeneous

The manifold M is called a **transitive G-space** or a **homogeneous space**. *Show that M can be identified with a homogeneous space in the sense defined above.*

7) *Show that the cotangent bundle T^*S^1 is a $\mathbb{E}_2$-space, where $\mathbb{E}_2 = \mathbb{R}^2 \circledS \mathrm{SO}(2)$ is the euclidean group whose product is defined by $(m_2, n_2, \phi_2)(m_1, n_1, \phi_1) = (m, n, (\phi_2 + \phi_1) \bmod 2\pi)$, with*

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} m_2 \\ n_2 \end{pmatrix} + \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 \end{pmatrix} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}.$$

symmetric

8) A riemannian manifold M is said to be **symmetric** if:
a) there is an isometry group G which acts transitively on M,
b) there is a subgroup $H_p \subset G$ which leaves $p \in M$ invariant,
c) there is an isotrometry $s_p \in H_p$ of M onto itself which reverses the geodesics through p.

riemannian space

Show that such **riemannian symmetric space** *is a particular case of a symmetric space in the sense defined above in question* 5.

Answer 1: The product in G does not induce a product in G/H unless H is an invariant (normal) subgroup, i.e., $gHg^{-1} = H$ for all $g \in G$. Indeed if $g_1, g_2 \in G$ and $h_1, h_2 \in H$ we have

$$g_1 h_1 g_2 h_2 \not\in g_1 g_2 H$$

unless $h_1 g_2 = g_2 h_1'$ for some $h_1' \in H$. If H is a normal subgroup, the space of left cosets G/H, equal to the space of right cosets $G\backslash H$, is a Lie group. One can define a left action L_k of $k \in G$ on G/H by

$$L_k gH = (kg)H.$$

One cannot define a right action unless H is a normal subgroup.

Answer 2: $\mathcal{H}$ is a linear subspace of $\mathcal{G}$ and is given the structure of a Lie algebra by

$$[X, Y]_{\mathcal{H}} \equiv [X, Y]_{\mathcal{G}} \in \mathcal{H} \qquad \text{for } X, Y \in \mathcal{H}$$

since H is a subgroup of G.

Answer 3: Since G is compact, it admits at least one bi-invariant (left invariant and right invariant) positive definite metric. Hence $\mathcal{G}$ admits at least one scalar product invariant under the transformations $\{\mathrm{Ad}(g);\, g \in G\}$, i.e.,

$$(X|Y) = (\mathrm{Ad}(g)X|\mathrm{Ad}(g)Y).$$

Let $\mathcal{M}$ be the orthogonal complement of $\mathcal{H}$ in this scalar product. Then

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}, \qquad \mathcal{H} \cap \mathcal{M} = 0.$$

On the other hand

$$\mathrm{Ad}(h)\mathcal{H} = \mathcal{H} \text{ since } H \text{ is a Lie group}$$

and $\mathrm{Ad}(h)\mathcal{M}$ is orthogonal to $\mathrm{Ad}(h)\mathcal{H}$ because the scalar product in $\mathcal{G}$ is $\mathrm{Ad}(g)$ invariant. Hence

$$\mathrm{Ad}(h)\mathcal{M} \subset \mathcal{M} \text{ for every } h \in H.$$

Answer 4: Conditions (2) are a special case of conditions (1). Since H is invariant in G,

$$[X, Y] \in \mathcal{H} \text{ for every } X \in \mathcal{H} \text{ and } Y \in \mathcal{G}, \qquad \text{i.e., } [\mathcal{H}, \mathcal{G}] \subset \mathcal{H}.$$

Let

$$Y = Y_1 + Y_2, \qquad Y_1 \in \mathcal{H}, \qquad Y_2 \in \mathcal{K}.$$

$[X, Y_1] \in \mathcal{H}$, and $[X, Y_2] \in \mathcal{H}$; in fact, $[X, Y_2] = 0$, since $Y_2 \in \mathcal{K}$ implies $\mathrm{Ad}(h)Y_2 \in \mathcal{K}$. Hence $[\mathcal{H}, \mathcal{K}] = 0$.
It follows from the Jacobi identity and the previous property that

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K},$$

i.e., $\mathcal{K}$ is a Lie subalgebra of $\mathcal{G}$. It is isomorphic to the Lie algebra of the group G/H.

Answer 5: Let $G = \mathrm{SO}(n+1)$ and $H = \mathrm{SO}(n)$. Let $\sigma\colon G \to G$ by $g \to sgs^{-1}$ where

$$s = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1}_n \end{pmatrix}, \quad \mathbb{1}_n \text{ the unit element of } \mathrm{SO}(n). \tag{5}$$

It is easy to show that the map σ is an involutive automorphism of G: $\sigma(gk) = \sigma(g)\sigma(k)$ and $\sigma(\sigma(g)) = g$. The subgroup $G^\sigma \subset G$ consisting of all the elements left fixed by σ is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}, \quad h \in \mathrm{SO}(n). \tag{6}$$

The group $\mathrm{SO}(n)$ is connected, hence G^σ is connected, and $G_0^\sigma = G^\sigma$; we can identify G^σ with $H = \mathrm{SO}(n)$. The homogeneous space $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ is symmetric. See also [Problem III 8, Examples].

isotropy, stability, stationary, little group

Answer 6: Let $H_x \subset G$ be the **isotropy group of x** (aliases: **stability subgroup, stationary subgroup, little group**), i.e., the subgroup of G which leaves x invariant. If the action of G on M is transitive, the isotropy group of different points are isomorphic. Indeed, let $y = gx$. The isotropy group H_y at y is then conjugate to the isotropy group H_x, at x:

$$H_y = gH_xg^{-1}, \qquad y = gx.$$

The isotropy group is a closed subgroup of the connected Lie group G, so G/H_x is a homogeneous space.
There is a natural identification of M with G/H_x. Let $y = gx$ be the point in M obtained (4) by the action t_g of $g \in G$ on x. Then g determines y, but y does not determine g since

$$y = (gh)x \quad \text{for any} \quad h \in H_x.$$

There is a diffeomorphism between M and G/H_x, and one writes

$$M = G/H, \quad \text{where } H \text{ is isomorphic to } H_x.$$

Answer 7: Let (l, ψ) be the coordinates of $x \in T^*S^1 = \mathbb{R} \times S^1$. Let (m, n, ϕ) be the coordinates of $g \in \mathbb{E}_2$, and set

$$t_g(l, \psi) := (l + m\sin(\phi + \psi) - n\cos(\phi + \psi), (\phi + \psi) \bmod 2\pi).$$

One checks that t_g is a left action of $\mathbb{E}_2$ which acts transitively on T^*S^1. The isotropy group $H_x \subset \mathbb{E}_2$ which leaves $(l, \psi) \in T^*S^1$ invariant is such that

$$t_h(l, \psi) = (l, \psi) \quad \text{for} \quad h \in H_x,$$

hence it consists of the elements of $\mathbb{E}_2$ such that

$$\phi = 0, \ n/m = \tan\psi$$

and is isomorphic to $\mathbb{R}$.

Hence

$$T^*S^1 = \mathbb{E}^2/\mathbb{R}.$$

Answer 8: Let M be a symmetric riemannian manifold.
It follows from answer 6 that M is a homogeneous space. Moreover, let

$$\sigma(g) = s_p \circ g \circ s_p, \qquad g \in G$$

we shall check that σ is an involutive automorphism of G such that

$$G_0^\sigma \subset H_p \subset G^\sigma,$$

where $g \in G^\sigma \subset G$ implies $\sigma(g) = g$ and G_0^σ is the identity component of G^σ.
First, the relations $\sigma(gk) = \sigma(g)\sigma(k)$ and $\sigma(\sigma(g)) = g$ follow from the fact that $s_p \circ s_p = 1$. Next we show that if $h \in H_p$ then $h \in G^\sigma$; that is, we show that the isometries $\sigma(h)$ and h of M into M are identical, $\sigma(h) = h$. Indeed $\sigma(h)$ and h coincide at p

$$\sigma(h)(p) = p \quad \text{and} \quad h(p) = p, \qquad \text{i.e., } (\sigma(h) \circ h^{-1})(p) = \mathbb{1}$$

and their differentials coincide on T_pM

$$(\sigma(h))'(p) = -\mathbb{1} \circ h'(p) \circ (-\mathbb{1}) = h'(p); \qquad \text{i.e., } (\sigma(h))' \circ h'^{-1}|_{T_pM} = \mathbb{1}.$$

It follows that $\sigma(h) \circ h^{-1} = \mathbb{1}$ in a **normal neighborhood** of p (neighborhood admitting a system of normal coordinates at p). Since M is connected, it can be covered by overlapping neighborhoods and $\sigma(h) \circ h^{-1} = \mathbb{1}$. normal neighborhood
Hence if $h \in H_p$, then $h \in G^\sigma$, i.e., $H_p \subset G^\sigma$.
Finally, G_0^σ and H_p have the same Lie algebra. Since, by definition, G_0^σ is connected, it is included in H_p, and

$$G_0^\sigma \subset H_p \subset G^\sigma.$$

Remark: In cosmology one sometimes says that a space is homogeneous if it admits isometries which are only locally transitive, that is, transitive between points in the same neighborhood.

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8. EXAMPLES OF HOMOGENEOUS SPACES, STIEFEL AND GRASSMANN MANIFOLDS

1. *Show that* $\mathrm{O}(n+1)/\mathrm{O}(n)$ *can be identified with the sphere* S^n.

Answer 1: $\mathrm{O}(n+1)$ acts transitively on the unit sphere S^n. The isotropy group of some given point x_0 is isomorphic to $O(n)$, so S^n can be identified with $\mathrm{O}(n+1)/\mathrm{O}(n)$. (See Problems III 7, Homogeneous spaces answer 6, and V 10, Invariant geometries.)

2. *Stiefel manifolds.* a) *A k-frame in the vector space* $\mathbb{R}^n$ *is a set of k linearly independent vectors* $\{\boldsymbol{v}_k\}$. *Show that the set of k-frames in* $\mathbb{R}^n$ *can be identified with the Stiefel manifold* $V'_{n,k}$

$$V'_{n,k} = GL(n, \mathbb{R})/GL(n, k, \mathbb{R}),$$

where $GL(n, k, \mathbb{R})$ is the subgroup of $GL(n, \mathbb{R})$ leaving fixed k vectors.

b) *Show that the set of k-frames orthonormal in a given euclidean scalar product on* $\mathbb{R}^n$ *can be identified with the Stiefel manifold* $V_{n,k}$:

$$V_{n,k} = \mathrm{O}(n)/\mathrm{O}(n-k).$$

c) *Show that* $V_{n,k}$ *can be identified with the set of orthonormal* $k-1$ *frames tangent to* S^{n-1}.

d) *Show that* $V_{n,k}$ *is a fibre bundle with base* $V_{n,k-1}$, *typical fibre* S^{n-k} *and structure group* $\mathrm{O}(n+1-k)$.

Answer 2: a) $GL(n, \mathbb{R})$ acts transitively on the set of k-frames, and the subgroup (isotropy group) leaving fixed each vector of a given k-frame is isomorphic to $GL(n, k, \mathbb{R})$. Thus

$$V'_{n,k} = GL(n, \mathbb{R})/GL(n, k, \mathbb{R}).$$

b) The same type of reasoning applies to orthonormal k-frames, but the subgroup of $\mathrm{O}(n)$ which leaves fixed k orthonormal vectors is identical with the group of rotations of the space spanned by the $(n-k)$ orthogonal complement, thus isomorphic to $\mathrm{O}(n-k)$.

c) Translate each k-frame along its first vector to the end point of this vector on S^{n-1}: this gives a $k-1$ frame of vectors tangent to S^{n-1}. This is a reversible process.

d) To define the bundle structure we consider a natural inclusion $i\colon H' \to H$, with $H' = \mathrm{O}(n-k)$, $H = \mathrm{O}(n+1-k)$. We define the projection $p\colon V_{n,k} \to V_{n,k-1}$ by the mapping of left cosets

$$p(gH') = gi(H'), \qquad g \in \mathrm{O}(n).$$

The fibre at a point $gH \in V_{n,k-1}$ is the set in $V_{n,k}$ with projection gH, that is the cosets of the form $g'H'$ with $g' = gH$. It is therefore isomorphic to $H/H' \simeq S^{n-k}$. One defines a right action of $\mathrm{O}(n+1-k)$ on $V_{n,k}$, which preserves fibres, as the natural action of H on the cosets which constitute the fibre of $V_{n,k}$.

3) *Grassmann manifolds. Let* $\mathrm{Grass}(n, k)$ *denote the set of k-dimensional linear subspaces (k-planes through the origin) of a space* $\mathbb{R}^n$.

a) *Show that one can identify* $\mathrm{Grass}(n, k)$ *with the homogeneous space*

$$\mathrm{Grass}(n, k) = \mathrm{O}(n)/\big(\mathrm{O}(k) \times \mathrm{O}(n-k)\big).$$

The projective n-space is $\mathrm{Grass}(n+1, 1)$.

b) *Define the manifold* $\mathrm{Grass}^+(n, k)$ *of oriented k-planes.*

c) *Show that $V_{n,k}$ is a bundle over* $\mathrm{Grass}(n, k)$ *with group and fibre* $\mathrm{O}(k)$.

Answer 2: a) We endow $\mathbb{R}^n$ with a euclidean scalar product so that we have a group $\mathrm{O}(n)$ operating on $\mathbb{R}^n$. This group operates transitively on k-planes. Let P_k be a given k-plane, and P_{k-n} its orthogonal complement. The subgroup of $\mathrm{O}(n)$ leaving P_k fixed is the direct product of the subgroup of $\mathrm{O}(n)$, isomorphic to $\mathrm{O}(n-k)$, leaving each of its elements fixed, and another subgroup, isomorphic to $\mathrm{O}(k)$, leaving fixed the elements of P_{n-k}. $\mathrm{Grass}(n+1, 1)$ is the space of lines through the origin in $\mathbb{R}^{n+1}$, or equivalently the space of pairs of antipodal points on S^n.

b) The group SO(n) acts transitively on oriented k-planes. Clearly

$$\text{Grass}^+(n, k) = \text{SO}(n)/(\text{SO}(k) \times \text{SO}(n-k)).$$

c) Identify $V_{n,k}$ with $\text{O}(n)/\text{O}(n-k)$.

9. ABELIAN REPRESENTATIONS OF NONABELIAN GROUPS

commutator subgroup commutant

Let G be a nonabelian group and $[G, G]$ its **commutator subgroup,** [**commutant**] *i.e., the group generated by elements of the form $(g_1 g_2)(g_2 g_1)^{-1}$, $g_1, g_2 \in G$.*

1. *Show that $G/[G, G]$ is an abelian group.*
2. *Examples: Let $G = \text{GL}(n, \mathbb{R})$. Compute $[G, G]$ and $G/[G, G]$. Do the same for $G = \text{GL}(n, \mathbb{C})$.*
3. *Construct the one-dimensional unitary representation of* GL($n, \mathbb{R}$) *and* GL($n, \mathbb{C}$).

Answer 1: a) $H \equiv [G, G]$ is a subgroup of G by definition. It is appropriately called the commutator subgroup of G because a generating element reduces to unity if g_1 and g_2 commute. Its Lie algebra $\mathscr{L}(H)$ is spanned by the elements $[X_i, X_j]$ where $X_i, X_j \in \mathscr{L}(G)$.

b) Next we show that H is a normal subgroup of G. Indeed set $h(g_1, g_2) = g_1 g_2 (g_2 g_1)^{-1}$, then $\forall g \in G$,

$$\begin{aligned} gh(g_1, g_2)g^{-1} &= gg_1g^{-1}gg_2g^{-1}gg_1^{-1}g^{-1}gg_2^{-1}g \\ &= (gg_1g^{-1})(gg_2g^{-1})(gg_1g^{-1})^{-1}(gg_2g^{-1})^{-1} \in h. \end{aligned}$$

Since H is a normal subgroup of G, the quotient space G/H is a group.

c) Finally we show that the group G/H is abelian.

An element $[k] \in G/H$ is the set kH. The group multiplication in G/H is defined by

$$(k_1 H)(k_2 H) = (k_1 k_2)H.$$

The two sets $k_1 k_2 H$ and $k_2 k_1 H$ are identical since for every $h_1 \in H$ there exists $h_2 \in H$ such that

$$k_2 k_1 h_2 = k_1 k_2 h_1,$$

namely

$$h_2 = (k_2 k_1)^{-1}(k_1 k_2)h_1.$$

Answer 2: Examples:
For $G = \mathrm{GL}(n, \mathbb{R})$, $[G, G] = \mathrm{SL}(n, \mathbb{R})$, $G/[G, G] \simeq \mathbb{R}^\times \times \{1, -1\}$ where $\mathbb{R}^\times$ is the space of strictly positive numbers treated as a multiplicative group, called $\mathbb{R}_+$ in [Problem III 6, Direct].
For $G = \mathrm{GL}(n, \mathbb{C})$, $[G, G] = \mathrm{SL}(n, \mathbb{C})$, $G/[G, G] \simeq \mathbb{C}^\times \times S^1$ where $\mathbb{C}^\times$ is the space of complex numbers without the origin, treated as a multiplicative group.

Answer 3: A one-dimensional unitary representation (i.e., a **character**) of G is a homomorphism $\chi: G \to \mathbb{C}^\times$ with $|\chi(g)| = 1$. For $G = \mathbb{R}^\times$ the characters are $\{U_t\}$, $t \in \mathbb{R}$, where characters

$$U_t: \mathbb{R}^\times \to \mathbb{C}^\times \quad \text{by} \quad r \mapsto U_t(r) = r^{it}.$$

The characters of $\mathbb{Z}_2 = \{1, -1\}$ are $\{U_n\}$ where

$$U_n: \mathbb{Z}_2 \to \mathbb{C}^\times \quad \text{by} \quad U_n(z) = z^n, \qquad n \in \mathbb{Z}.$$

The characters of S^1 represented by $\theta \in \mathbb{R} \bmod 2\pi$ are $\{U_n\}$ where

$$U_n: S^1 \to \mathbb{C}^\times \quad \text{by} \quad U_n(\exp(\mathrm{i}\theta)) = \exp(\mathrm{i}n\theta), \qquad n \in \mathbb{Z}$$

Characters of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ are $\{U_{(t,n)}\}$ where

$$U_{(t,n)}: g \mapsto |\det g|^{it}\left(\frac{\det g}{|\det g|}\right)^n, \qquad t \in \mathbb{R}, n \in \mathbb{Z}.$$

It can be shown that $\{U_{(t,n)}\}$ is the complete set of characters of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$.

Remark: A group which is equal to its commutator subgroup is called **perfect**. perfect
The **abelianization** of a group G is $G/[G, G]$. abelianization

10. IRREDUCIBILITY AND REDUCIBILITY

Definition: A representation $T: x \to T_x$ of a topological group G in a Hilbert space H is said to be **irreducible** if it has no proper invariant closed subspace [i.e., there is no closed (linear) subspace $H_1 \subset H$, $H_1 \neq \{\emptyset\}$ and $H_1 \neq H$, such that $T_x u \in H_1$ for every $x \in G$ if $u \in H_1$]. irreducible

1) *Show that the restriction of T to an invariant subspace H_1 is a representation of G.* It is called the **subrepresentation** of G on H_1 and is denoted ${}^{H_1}T$. sub-representation

2) *Show that if T is a unitary representation the orthogonal complement H_2 of a subspace H_1 is invariant if and only if H_1 is invariant.*

3) *Denote by P_1 the projection operator $H \to H_1$. Show that when T is unitary, H_1 is invariant if and only if $P_1 T_x = T_x P_1$ for every $x \in G$. Give a counterexample with $G = \mathbb{R}$ and $H = \mathbb{R}^2$ with the euclidean scalar product to show that if T is not unitary the orthogonal complement of an invariant subspace is not in general invariant.*

4) A representation T of G in H is said to be the direct sum $T = \Sigma_i \oplus T_i$, of representations T_i of G in H_i if H_i are invariant subspaces of H such that $H = \Sigma_i \oplus H_i$ (topological sum) and if each T_i is a subrepresentation of T. One says that T is fully reducible if it is the direct sum of irreducible representations. A reducible but not fully reducible finite dimensional
indecomposable
representation is called an **indecomposable** representation.
Show that every finite dimensional unitary representation is fully reducible.

5) *Let T be a finite dimensional irreducible representation of G. Show that the only operators which commute with all T_x are scalar multiples of the identity.*

Answer 1: $\forall x \in G$, T_x restricts to a continuous linear operator on H_1, ${}^{H_1}T_x \in \mathscr{L}(H_1, H_1)$, which satisfies the properties required of a representation.

unitary

Answer 2: T_x **unitary** is equivalent to $(T_x)^* = (T_x)^{-1}$; i.e., since T is a representation of G

$$(T_x)^* = T_{x^{-1}};$$

therefore denoting (,) the scalar product in H

$$(T_x v, u) = (v, (T_x)^* u) = (v, T_{x^{-1}} u), \qquad \forall u, v \in H, x \in G.$$

If H_1 is invariant, $u \in H_1$ implies $T_{x^{-1}} u \in H_1$. Thus if $v \in H_2$ we have $(v, T_{x^{-1}} u) = 0$, so $(T_x v, u) = 0$ $\forall u \in H_1$; thus $T_x v \in H_2$. The converse statement follows immediately since H_1 is the orthogonal complement of H_2.

Answer 3: Let H_1 be invariant. Then $T_x P_1 u \in H_1$ for every $x \in G$, $u \in H$, that is

$$P_1^* T_x P_1^* = T_x P_1, \qquad \forall x \in G.$$

By taking the adjoint we obtain

$$P_1 T_x^* P_1 = P_1^* T_x^*;$$

thus, since $P_1 = P_1^*$ and $T_x^* = T_{x^{-1}}$

$$P_1 T_{x^{-1}} P_1 = P_1 T_{x^{-1}}, \qquad \forall x^{-1} \in G.$$

Comparing (1) and (2) gives the result.
Conversely if $T_x P_1 = P_1 T_x$ $\forall x \in G$ then, since $u = P_1 u$ for $u \in H_1$, we have $T_x u = P_1 T_x u \in H_1$.
Counterexample: Represent $\mathbb{R}$ by 2×2 nonunitary matrices acting on $\mathbb{R}^2$

$$T: x \mapsto T_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R}.$$

The subspace $\binom{u_1}{0}$, $u_1 \in \mathbb{R}$, of $\mathbb{R}^2$ is invariant, but its orthogonal complement $\binom{0}{u_2}$, $u_2 \in \mathbb{R}$, is not.

Answer 4: If H_1 is a proper invariant subspace of H then the same is true of the orthogonal complement H_2, while $H = H_1 + H_2$. If H_1 [resp. H_2] contains a proper invariant subspace, then we decompose it again into two invariant subspaces. We obtain a decomposition of H in irreducible invariant subspaces by a finite number of steps if H is finite dimensional – otherwise we have problems of convergence.

Answer 5: Denote by K the kernel of the mapping $S: H \to H$. If S commutes with T_x, we have

$$ST_x K = T_x SK = 0.$$

Thus $T_x K \subset K$ is an invariant subspace of H. Since T is irreducible, this implies that $K = 0$ or $K = H$, that is, S is an isomorphism of H or vanishes identically. If S commutes with all T_x, the same is true of $S - \lambda I$, where λ is an eigenvalue of S. But $S - \lambda I$ is not an isomorphism of H, and thus $S - \lambda I = 0$.
The result just proved extends in part to the infinite dimensional case in a form known as Schur's lemma:

Schur's lemma: Let T and T' be two irreducible representations of G in H and H' respectively, and S; $H \to H'$ a bounded linear map such that Schur's lemma

$$ST_x = T'_x S, \qquad \forall x \in G.$$

Then either S is an isomorphism of H onto H', or $S = 0$.
The proof (cf. for instance, Reed and Simon) makes use of the spectral decomposition of the hermitian operator S^*S, after observing that, since T is unitary,

$$T_x S^* = T'_x S^*, \qquad \forall x \in G.$$

References: cf. Problem III 7.

11. CHARACTERS

1. *Show that every irreducible unitary representation of an abelian group G on a Hilbert space H is one dimensional.*

character

2. A one dimensional unitary representation $\hat{x}$ of G is called a **character** of the group.
Show that the set of characters of G forms a group under the usual multiplicative law of function

$$(\hat{x}_1\hat{x}_2)(x) = \hat{x}_1(x)\hat{x}_2(x), \qquad \forall x \in G.$$

Answer 1: Given a fixed $y \in G$, T_y commutes with every T_x, $x \in G$, since

$$T_xT_y = T_{xy} = T_{yx} = T_yT_x, \qquad \forall x, y \in G.$$

Thus (cf. Problem III 10, Irreducibility) $T_y = \lambda(y)I$, every one dimensional subspace H_1 of H is invariant. But since T is irreducible, H_1 coincides with H.

Answer 2: If $x \mapsto \hat{x}_1(x)$ and $x \to \hat{x}_2(x)$ are one dimensional representations of G, i.e., $x \mapsto \hat{x}_1(x)$ [resp. $\hat{x}_2(x)$] is a continuous mapping $G \mapsto \mathscr{L}(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}$ which satisfies

$$\hat{x}_1(xy) = \hat{x}_1(x)\hat{x}_1(y), \qquad \forall x, y \in G$$

then $\hat{x}_1\hat{x}_2$: $x \mapsto \hat{x}_1(x)\hat{x}_2(x)$ satisfies the same relation.

References: cf. Problem III 7.

12. SOLVABLE LIE GROUPS

commutator

commutant

1) The element $q = xyx^{-1}y^{-1}$ is called the **commutator** of the elements x, y of the group G. The set Q of elements of G which are finite products of commutators is called the **commutant** of G. *Show that Q is an invariant subgroup of G.* See [Problem III 9, Abelian] where $Q = H \equiv [G, G]$.

2) Consider the chain of commutants

$$G = Q_0 \supset Q_1 \supset \cdots \supset Q_i \supset Q_{i+1} \supset \cdots,$$

solvable

where each Q_i is the commutant of Q_{i-1}. The group G is said to be **solvable** if the chain terminates, i.e., if for some n we have that Q_n is abelian, and hence $Q_{n+1} = \mathbb{1}$.

Show that the Poincaré group in one space dimension is solvable.
It can be shown that every solvable connected Lie group can be represented as a product of one dimensional subgroups

$$G = T_1 \times \cdots \times T_n,$$

where each $G_k = T_{k+1} \times \cdots \times T_n$ is an invariant subgroup of G.

Answer 1: If $q \in Q$, i.e., $q = q_1 \ldots q_n$ with $q_i = x_i y_i x_i^{-1} y_i^{-1}$, we have, $\forall g \in G$

$$g^{-1} q g = g^{-1} x_1 y_1 x_1^{-1} y_1^{-1} \ldots x_n y_n x_n^{-1} y_n^{-1} g = q_1' \ldots q_n'$$

with q_i' the commutator of $x_i' = g^{-1} x_i g$ and $y_i' = g^{-1} y_i' g$.
Note that Q is not closed in general. See [Problem III 3, Subgroups].

Answer 2: The Poincaré group in one space dimension is the space of pairs (a, L) with $a \in \mathbb{R}^2$, $L \in \mathrm{O}(1, 1)$ with the multiplication (cf. Problem III 5, Direct)

$$(a, L)(a', L') = (a + La', LL').$$

The inverse of (a, L) is $(-L^{-1}a, L^{-1})$. The Lorentz group in one space dimension is commutative. The commutator of $x = (a, L)$ and $y = (a', L')$ is computed to be

$$q = (a - L'a + La' - a', I)$$

Thus Q_1 is the commutative group $\mathbb{R}^2$.

REFERENCE

cf. Problem III 7.

13. LIE ALGEBRAS OF LINEAR GROUPS

1) *Show that the space $\mathscr{L}(E, E)$ of linear endomorphisms of a finite dimensional vector space E together with the mapping*

$$\mathscr{L}(E, E) \times \mathscr{L}(E, E) \to \mathscr{L}(E, E) \quad \text{by} \quad (A, B) \mapsto [A, B] \equiv AB - BA$$

is a Lie algebra.

Answer 1: The space $\mathscr{L}(E, E)$ is a vector space. The operation $[\,,]$ is obviously anticommutative. It is easy to check that it satisfies the Jacobi identity.

2) *Let G be the group $GL(E)$ of linear automorphisms of E. Show that its Lie algebra is isomorphic to the space $\mathscr{L}(E, E)$, with the Lie bracket defined above.*

Answer 2: The left action L_X of $X \in GL(E)$ on $GL(E)$ is given by the product of automorphisms, i.e.,

$$L_X: Y \mapsto L_X Y = XY.$$

The space $GL(E)$ is an open set of the vector space $\mathscr{L}(E, E)$. The tangent space at $Y \in GL(E)$ is therefore isomorphic to $\mathscr{L}(E, E)$. A vector field V on $GL(E)$ is left invariant (p. 155) if

$$V(X) = L'_X V(e), \qquad e \text{ the unit of } GL(E),$$

that is,

$$V(X) = XV(e).$$

Consider two such vector fields V_A and V_B

$$V_A(X) = XA, \qquad V_B(X) = XB,$$

where A and B are given elements of $\mathscr{L}(E, E)$.
Let $f: GL(E) \to \mathbb{R}$, by $X \to f(X)$ be a C^∞ function on $\mathbb{R}$. Choose a basis (E^α_β) in $\mathscr{L}(E, E)$. Then

$$V_A f = X^\alpha_\beta A^\beta_\gamma \frac{\partial f}{\partial X^\alpha_\gamma},$$

and a straightforward computation gives

$$(V_A V_B - V_B V_A) f = X^\alpha_\mu (A^\nu_\gamma B^\mu_\nu - B^\nu_\gamma A^\mu_\omega) \frac{\partial f}{\partial X^\alpha_\gamma}.$$

Therefore

$$[V_A, V_B](X) = X(AB - BA) = X[A, B],$$

where $[A, B]$ denotes the bracket in the Lie algebra $\mathscr{L}(E, E)$.

Weyl basis

3) a) Take as a basis, called **Weyl basis,** in $\mathscr{G\ell}(n, \mathbb{R})$, the Lie algebra of $GL(n, \mathbb{R}) \equiv GL(\mathbb{R}^n)$ the $n \times n$ matrices E^i_j whose elements are

$$(E^i_j)^l_k = \delta^i_k \delta^l_j.$$

Compute the structure constants of $\mathscr{G\ell}(n, \mathbb{R})$.

b) *Determine the Lie algebra of $SO(n)$. Prove that the structure constants are totally antisymmetric.*

Answer 3) a: By the previous result, $\mathscr{G}\ell(n, \mathbb{R})$ is the space of $n \times n$ matrices, and the Lie bracket of elements of the Weyl basis are

$$[E^i_j, E^l_k] = C_{ij}{}^{hm}{}_{lk} E^h_m$$

with

$$C_{ij}{}^{hm}{}_{lk} = \delta^h_i \delta^m_k \delta_{jl} - \delta^h_l \delta^m_j \delta_{ki}.$$

Answer 3) b: The group SO(n) is the subgroup of $GL(n, \mathbb{R}) \equiv GL(\mathbb{R}^n)$ of $n \times n$ invertible real matrices X such that

$${}^{\mathrm{T}}XX = I,$$

where I is the $n \times n$ unit matrix and ${}^{\mathrm{T}}X$ the transpose of X.
The tangent space $T_I\mathrm{SO}(n)$ is a subspace of $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, the space of $n \times n$ matrices, spanned by tangents at $t = 0$ to curves $t \mapsto X(t)$, $X(0) = I$, in SO(n), that is such that

$${}^{\mathrm{T}}X(t)X(t) = I.$$

Thus, if we set $A = \mathrm{d}X(t)/\mathrm{d}t|_{t=0}$

$${}^{\mathrm{T}}A + A = 0.$$

The Lie algebra $\mathscr{S}\mathrm{O}(n)$ is therefore the vector space of antisymmetric $n \times n$ real matrices, a Lie subalgebra of $\mathscr{G}\ell(n, \mathbb{R})$. A basis for $\mathscr{S}\mathrm{O}(n)$ is constituted by the antisymmetric matrices

$$\tilde{E}_{ij} = E^i_j - E^j_i.$$

By straightforward calculation using the results of a),

$$[\tilde{E}_{ij}, \tilde{E}_{lk}] = \tilde{C}_{ij}{}^{hm}{}_{lk} \tilde{E}_{hm},$$

where $\tilde{C}_{ij}{}^{hm}{}_{lk}$ is the sum of eight terms

$$\tilde{C}_{ij}{}^{hm}{}_{lk} = \tfrac{1}{2}\{\delta^h_i \delta^m_k \delta_{jl} - ((i, j) \leftrightarrow (h, m)) - ((i, j) \leftrightarrow (l, k)) \\ + (l, k) \leftrightarrow (h, m)\} - \tfrac{1}{2}\{m \leftrightarrow h\}.$$

Therefore the structure constants $C_\beta{}^\alpha{}_\gamma$ of $\mathscr{S}\mathrm{O}(n)$ are a totallyantisymmetric 3-tensor (where a Greek index, an ordinary vector index, is substituted by a pair of latin indices).

4) It can be proved that every compact Lie group G admits at least one faithful linear representation either by orthogonal matrices or by unitary matrices (Chevalley, Theory of Lie groups, pp. 176, 211).
Show that the structure constants of a compact Lie group have a vanishing trace:

$$C_\alpha{}^\beta{}_\beta = 0.$$

Answer 4: If G is a subgroup of an orthogonal group $O(n)$ its Lie algebra $\mathcal{G}$ is a Lie subalgebra of the Lie algebra $\mathcal{O}(n)$. Therefore the structure constants are totally antisymmetric, and in particular have a vanishing trace. If G is a subgroup of $U(n)$, its Lie algebra is represented by antihermitian matrices.
A complex $n \times n$ matrix can be written as a $2n \times 2n$ real one. Indeed, $U = A + iB$ acting on $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ is written as

$$\mathcal{U} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

If U is antihermitian then $A = -{}^{\mathrm{T}}A$, $B = {}^{\mathrm{T}}B$, and $\mathcal{U}$ is thus antisymmetric. The same conclusion follows on the structure constants.

14. GRADED BUNDLES

In this problem we study vector bundles over an ordinary C^∞, d-dimensional manifold M where the typical fibre is a graded (commutative) algebra A, as defined in Problem I 1. In supersymmetric theories, before quantization, the fermionic fields are odd sections and the bosonic fields even sections of such bundles. In superstring theories, the fibre is not a vector space.

1. KOSTANT GRADED BUNDLES

The first requirement is the definition of graded C^∞ functions on M as an extension of the algebra of ordinary C^∞ functions. The extension can be defined by considering C^∞ mappings from M into A. But it is possible to give a more general definition. Kostant has used sheaf theory (Problem I 1) to construct graded C^∞ functions on open sets of M, whose local algebra is an extension of the local algebra of ordinary C^∞ functions on these open sets. It is also possible to define a local algebra of graded functions as local sections of a vector bundle over M with typical fibre A if the product can be defined independently of the choice of trivializations. We give the following definition:

Kostant graded bundle

A **Kostant graded bundle** K over M is a topological space with
1) A continuous projection $p: K \to M$
2) A covering of M by open sets U_i together with homeomorphisms

$$\varphi_i: p^{-1}(U_i) \to U_i \times A, \qquad A \text{ a graded algebra with locally convex topology}$$

which commute with projection, that is,

$$\varphi_i(z) = (x, \varphi_{i,x}(z)), \qquad \varphi_{i,x}(z) \in A, \qquad x = p(z)$$

and which have the properties that if $x \in U_i \cap U_j$ then the mapping $\varphi_{i,x} \circ \varphi_{j,x}^{-1}\colon A \to A$

(i) is a continuous linear mapping

(ii) commutes with the product in A, that is,

$$\varphi_{i,x}(z)\varphi_{i,x}(z') = (\varphi_{i,x} \circ \varphi_{j,x}^{-1})(\varphi_{j,x}(z)\varphi_{j,x}(z')).$$

The bundle is of class C^k if the mappings

$$U_i \cap U_j \to L(A, A) \text{ by } x \mapsto \varphi_{i,x} \circ \varphi_{j,x}^{-1} \text{ are of class } C^k.$$

a) *Show that the C^p sections $(p \leq k)$ of a C^k Kostant bundle form an algebra.*

b) *Give a possible form of transition mappings $B \to B$ in the case that B is a Grassmann or DeWitt algebra.*

Answer 1a: i) A Kostant bundle is a vector bundle: the fibre $K_x = p^{-1}(x)$ is endowed with a vector space structure by the homeomorphisms φ_i. Indeed, we can define $\alpha y + \beta z \in K_x$ by

$$\varphi_{i,x}(\alpha y + \beta z) = \alpha\varphi_{i,x}(y) + \beta\varphi_{i,x}(z), \qquad y, z \in K_x, \qquad \alpha, \beta \in \mathbb{C} \text{ (or } \mathbb{R}).$$

This definition does not depend on the trivialization: since $\varphi_{j,x} \circ \varphi_{i,x}^{-1}$ is a linear mapping we have

$$(\varphi_{j,x} \circ \varphi_{i,x}^{-1})(\alpha\varphi_{i,x}(y) + \beta\varphi_{i,x}(z)) = \alpha(\varphi_{j,x} \circ \varphi_{i,x}^{-1})\varphi_{i,x}(y) + \beta(\varphi_{j,x} \circ \varphi_{i,x}^{-1})\varphi_{i,x}(z)$$
$$= \alpha\varphi_{j,x}(y) + \beta\varphi_{j,x}(z).$$

The space of sections over an open set of M is therefore also a vector space.

ii) The product of two sections f and h is the section

$$x \mapsto (fh)(x)$$

defined for $x \in U_i$ by

$$(fh)(x) = \varphi_{i,x}^{-1}(\varphi_{i,x}(f(x))\varphi_{i,x}(h(x))).$$

The definition is independent of the index i due to hypothesis (ii).
A section f is called even [resp. odd] if $\varphi_{i,x}(f(x)) \in A_+$ [resp. A_-] for all $x \in M$. The space of even and odd sections form a graded algebra, but addition must be restricted to sections of the same type.

Answer 1b: We suppose that B is a Grassmann or DeWitt algebra and look for linear, invertible maps $f: B \to B$ which commute with the product, i.e., maps such that

(i) $f(\alpha a + \beta b) = \alpha f(a) + \beta f(b), \qquad \forall a, b \in B, \quad \alpha, \beta \in \mathbb{C}$ (or $\mathbb{R}$)

(ii) $f(ab) = f(a) f(b) \qquad \forall a, b \in B.$

We first take $a = e$ and write (ii)

$$f(b)(e - f(e)) = 0 \qquad \forall f(b) \in B.$$

Therefore $f(e) = e$ and $f(\alpha e) = \alpha e$, that is, f necessarily preserves the body of B.

If (i) and (ii) are satisfied we must then have, for all $a \in B$,

$$f\Big(a_0 e + \sum_{p \geq 1} a_{I_1 \ldots I_p} z^{I_1} \ldots z^{I_p}\Big) = a_0 e + \sum a_{I_1 \ldots I_p} f(z^{I_1}) \ldots f(z^{I_p}).$$

Therefore f is known on B if it is known on the generators. In particular

$$f(z^I z^J) = -f(z^J z^I) = f(z^I) f(z^J) = -f(z^J) f(z^I).$$

Hence f must map generators into odd terms in B. We also see that the restriction to $B(1)$ (terms of degree 1) of the image of $f(B(1))$ must be surjective for f to be invertible, since by (ii) and the bodylessness of $f(B(1))$, the image under f of a term in $B(p)$, $p > 1$, is never in $B(1)$. These remarks show that f must be of the form

$$f(z^I) = L^I_J z^J + \sum_{p > 1} f^I_{J_1 \ldots J_p} z^{J_1} \ldots z^{J_p}, \tag{1}$$

where the mapping $B(1) \to B(1)$ by $(z^I) \mapsto (L^I_J z^J)$ is continuous and invertible.

A possible choice of f satisfying the required conditions is

$$f(z^I) = L^I_J z^J, \qquad \text{a formal series in the generators}$$

if the corresponding mapping f, determined on B by conditions (i) and (ii) is well-defined, continuous and invertible. Let $a \in B$ be the formal series

$$a = a_0 e + \sum_{p \geq 1} \frac{1}{p!} a_{I_1 \ldots I_p} z^{I_1} \ldots z^{I_p}.$$

We must have

$$f(a) = a_0 e + \sum_{p \geq 1} \frac{1}{p!} a_{I_1 \ldots I_p} L^{I_1}_{J_1} \ldots L^{I_p}_{J_p} z^{J_1} \ldots z^{J_p}.$$

Therefore $f(a)$ is defined as a formal series, and also continuous in a, if and only if each ordinary series

$$a_{I_1 \ldots I_p} L^{I_1}_{J_1} \ldots L^{I_p}_{J_p}$$

is convergent for any numbers $a_{I_1 \ldots I_p}$. In particular, for each J the series, $a_I L^I_J$ must be convergent for any choice of numbers a_I; this will be true if and only if, for each J, $L^I_J \neq 0$ for only a finite number of indices I. The corresponding transition function $U_i \cap U_j \to L(B, B)$, $x \mapsto f_x$ is determined by

$$f_x(z^I) = L^I_J(x) z^J.$$

In the case of a finite number of generators the Kostant bundles thus obtained are isomorphic to ordinary exterior bundles.
M. Batchelor's theorem, proved in the context of Kostant's original definition, says that the category of Kostant bundles (when locally trivial, which is included in our definition) is isomorphic to the category of exterior bundles if A is a finitely generated Grassmann algebra. This theorem can perhaps be proved in our context by using the fact that if transition functions $U_i \cap U_j \to L(B, B)$ by $x \mapsto f_x$ are given by the general formula (1), the obstruction to their extension to M lies entirely in the first term.

c) A **graded manifold** is a pair (M, K), with M a C^∞, ordinary manifold, and K a Kostant bundle over M, also called the **fundamental bundle**. A **graded chart** is a triple (U_i, ϕ_i, φ_i) where (U_i, ϕ_i) is a chart of M and φ_i a local trivialization of K over U_i. graded manifold fundamental bundle graded chart
A **graded function** f of M is a section of the fundamental bundle. graded function
Give the definition of a C^p graded function f. Give the necessary and sufficient conditions for f to be C^p when A is a Grassmann or DeWitt algebra.

Answer 1c: The representative of f in a graded chart is the mapping from an open set of $\mathbb{R}^d$ into A

$$f_i \equiv \varphi_i \circ f \circ \phi_i^{-1}, \qquad f_i \colon \phi_i(U_i) \to A.$$

A graded function is C^p (with $p \leq k$ if K is C^k) if each of the f_i is C^p. If A is a Grassmann or DeWitt algebra B, a representative $f_i = \bar{f}$ is

$$\bar{f} = f_0 e + \sum \frac{1}{p!} f_{I_1 \ldots I_p} z^{I_1} \ldots z^{I_p}$$

where each of the f_0, $f_{I_1 \ldots I_p}$ are ordinary numerical functions on an open set Ω of $\mathbb{R}^d$; the function $\bar{f} \colon \Omega \to B$ is C^p if and only if each of the functions $f_0, \ldots, f_{I_1 \ldots I_p}$ is a C^p function.

2. GRADED VECTOR OR AFFINE BUNDLES

Let $P \to M$ be an ordinary, C^k, principal fibre bundle over M with projection π, Lie group G, and transition functions σ_{ij}:

$$\sigma_{ij}: U_i \cap U_j \to G$$

(recall that $\sigma_{ij}(x) \in G$ defines an automorphism of G by its left action on G).
The vector bundles associated to P can be extended as follows to graded bundles.
a) *Let r be a linear representation of G on $\mathbb{C}^n$ or $\mathbb{R}^n$. Define an action of $r(g)$ on A^n, when A is an arbitrary locally convex vector space over $\mathbb{C}$ (or $\mathbb{R}$).*

b) *Let $E \to M$ be a vector bundle with typical fibre $\mathbb{C}^n$ (or $\mathbb{R}^n$) associated to P through the representation r. Construct a graded bundle $\mathscr{E} \to M$ with typical fibre A^n, associated to P through the same representation.*

c) *Let $K \to M$ be a Kostant bundle with typical fibre a Grassmann or DeWitt algebra B and transition functions which preserve the unit of B. Define $K^n \to M$ as the vector bundle with base M and fibre at x the n^{th} direct product of fibres, K_x^n, and construct a K-graded extension $\mathscr{E}_K$ of the vector bundle $E \to M$. Define the K-graded tangent bundle to M.*

A-valued sections

d) *When K is the trivial bundle $K = M \times A$, the sections of the corresponding graded extension of a vector bundle E will be called* **A-valued sections** *of $\mathscr{E}_K$. Show that A-valued tangent vectors define a linear endomorphism of the algebra of C^∞ functions. Study its properties. Consider the case where B is a DeWitt algebra.*

e) *Define A-valued covariant vectors, metrics, spinors, connections.*

Answer 2a: $r(g) \in GL(n, \mathbb{C})$ is an $n \times n$ matrix with complex elements $r^i{}_j$. It acts on $a = (a^i)$, $i = 1, \ldots, n \in A^n$ by the ordinary law

$$r(g)a = (r^i_j a^j).$$

Answer 2b: Since G acts on the left on A^n through the representation r, it is possible to follow the general procedure given on p. 383 to define the graded bundle $\mathscr{E} \to M$ as the quotient of $P \times A^n$ under the action τ_g of G:

$$\tau_g: P \times A^n \to P \times A^n \quad \text{by} \quad (p, a) \to (\tilde{R}_g p, r^{-1}(g)(a)).$$

Answer 2c: Since K_x is a vector space over $\mathbb{C}$ (or $\mathbb{R}$), the product of

$y \in K_x$ by $\alpha \in \mathbb{C}$ (or $\mathbb{R}$) is well defined, and the action of r on $y \in K_x^n$ also: if $y = (y^1, \ldots, y^n) \in K_x^n$

$$r(g)y = (r_i^j y^i) \in K_x^n.$$

We give below the construction of the graded extension $\mathscr{E}_K$, following essentially the same ideas as on p. 383, but with the fibre F replaced by K_x, which itself depends on x.

The ***K*-graded extension** $\mathscr{E}_K$ of E is $\mathscr{E}_K \equiv \cup_{x \in M} \mathscr{E}_x$ with $\mathscr{E}_x$ the set of equivalence classes $\{(\rho, \tau)\}$, $\rho \in G$, $\tau \in K_x$ *K*-graded extension

$$(\rho, \tau) \simeq (\rho', \tau') \quad \text{if} \quad \rho' = g\rho, \qquad \tau' = r(g)\tau \text{ for some } g \in G.$$

$\mathscr{E}_K$ projects onto M by

$$\pi: \mathscr{E}_x \to x.$$

Each $\mathscr{E}_x$ is a vector space.

To endow $\mathscr{E}_K$ with a topology and a vector bundle structure, we choose a local section of P with representative ρ_i over each open set U_i domain of a chart of a graded atlas of M over which P is trivializable. The following mapping φ_i is bijective:

$$\varphi_i: \pi^{-1}(U_i) \to U_i \times A^n$$

defined for $y \in \mathscr{E}_x$ by

$$y \mapsto (x, t_i),$$

where $(\rho_i, (\varphi_{i,x}^{-1})^n t_i)$ is the representative with first element ρ_i of the equivalence class defining y. (Note that $(\varphi_{i,x}^{-1})^n$ acts on each element in A by $\varphi_{i,x}^{-1}$ when $t_i \in A^n$.) The topology of $\mathscr{E}_K$ is defined to be such that each of these maps is a homeomorphism: the consistency of the definition rests on the fact that $U_i \cap U_j$ is open in U_i.

The vector bundle structure of $\mathscr{E}_K$ is defined by the family of local trivializations φ_i, with the resulting transition functions

$$s_{ij}: (\varphi_j^{-1} \circ \varphi_i)^n r(\sigma_{ji}).$$

(Note that since φ_i is a linear map, and $\varphi_i^n = \varphi_i I$, I the unit matrix, φ_i commutes with $r(\sigma_{ji})$.)

A change of choice of the local sections ρ_i of P gives an equivalent vector bundle structure to $\mathscr{E}_K$.

Remark: It is possible to define even-valued or odd-valued extensions by replacing the vector space A^n by A_+^n or A_-^n.

If the bundles K and E are C^k the same is true of the graded extension $\mathscr{E}_K$ and we can speak of its C^p sections ($p \leq k$).

In agreement with the preceding definitions we have:

graded tangent
cotangent

A **graded tangent** [respectively **cotangent**] vector at $x \in M$ to the graded manifold (M, K) is an equivalence class (ρ, V) where ρ is a linear frame in the ordinary tangent space at x and $V \in K_x^n$ and the equivalence relation is

$$V' = g^{-1}V \quad \text{if} \quad \rho' = \rho g, \quad g \in GL(d, \mathbb{R})$$
$$[\text{respectively} \quad V' = gV \quad \text{if} \quad \rho' = g\rho, \quad g \in GL(d, \mathbb{R})]$$

even
odd

The vector is called **even** if $V \in (K_x^+)^n$, **odd** if $V \in (K_x^-)^n$.

A-valued
tangent vectors
components

Answer 2d: When K is the trivial bundle $M \times A$, the **A-valued tangent vectors** have a representative $V \in A^d$ in each linear frame on M.
The v^α, $\alpha = 1, \ldots, d$, $(v^\alpha) = V \in A^d$ are called the **components** of v in the frame ρ.
A C^∞ A-valued tangent vector v to M (i.e., a C^∞ section of the graded extension with $K = M \times A$ of the vector bundle TM, assumed to be C^∞) defines an endomorphism of the algebra $C^\infty(M, A)$ of A-valued C^∞ functions on M: in a chart (U, ϕ) of M,

$$v(f) = v^\alpha \partial_\alpha (f \circ \phi), \qquad \partial_\alpha = \partial / \partial x^\alpha;$$

the result is chart independent due to the definitions.
The endomorphisms ∂_α are a basis of the A-valued tangent space T_xM.
This endomorphism is additive and obeys the graded Leibniz rule, when v and f have a parity (i.e., are either odd or even)

$$v(fh) = v(f)h + (-1)^{d(f)d(v)} fv(h)$$

and is zero on constant functions:

$$v(a) = 0 \qquad \text{if } a \text{ is the constant map } x \mapsto a \in A.$$

Conversely, if A is a DeWitt algebra B, every endomorphism of $C^\infty(M, B)$ which enjoys these properties is a B-valued C^∞ vector field on M (see for instance the proof in [3]).

B-supertangent
vector

A **B-supertangent vector** is an endomorphism of $C^\infty(M, B)$ which satisfies the above properties except $v(a) = 0$; this is replaced by

$$v(\lambda e) = 0 \qquad \text{for every constant map } x \mapsto \lambda e, \ \lambda \in \mathbb{R}.$$

The endomorphisms $\partial / \partial z^J$ defined by

$$\frac{\partial}{\partial z^J} f = i_{z^J} f = \sum \frac{1}{(p-1)!} f_{JI_1 \ldots I_{p-1}} \, dz^{I_1} \ldots dz^{I_{p-1}}$$

are B-supertangent vectors.

Answer 2e: The differential, also called gradient, of an A-valued C^1 function f on M is a covariant A-valued vector, with components in a chart $\partial_\alpha(f \circ \phi^{-1})$. When A is a DeWitt algebra B it is easy to see that a B-valued covariant vector is an exact differential if and only if each of its projections on a subspace of B generated by a given subset of its generators is an exact differential.

Analogous definitions apply to tensor field and scalar densities of various weights.

In particular, an A-valued metric on M is a covariant symmetric A-valued 2-tensor field g which is nondegenerate, that is, such that at each $x \in M$, the mapping $T_xM \to T_x^*M$ by $v \mapsto u = g_x(v, .)$ is an endomorphism. This will be the case if and only if in a local chart at x (and hence in all charts) the A-valued components $g_{\alpha\beta}$ determine an isomorphism of vector spaces by the mapping

$$(v^\beta) \mapsto (u_\alpha = g_{\alpha\beta} v^\beta), \qquad A^d \to A^d.$$

The inverse linear map is a matrix with A-valued elements $g^{\alpha\beta}$, which is a representative in the chart of a contravariant tensor $g^\#$.

A B-valued metric g, with B a DeWitt algebra, is nondegenerate if and only if its body is nondegenerate. Indeed, the mapping is degenerate if the body of g is degenerate, since the body of g maps the body of v into the body of u:

$$g_{\alpha\beta}(0) v^\beta(0) = u_\alpha(0).$$

The body $g(0) = g_0 e$, with g_0 a numerical metric, is nondegenerate if g_0 is nondegenerate, and the same is then true of g: the inverse $\bar{g}^\#$ of a representative $\bar{g}$ is given by

$$(g^{\alpha\beta}) = \bar{g}^\# = \bar{g}_0^{-1}\left(e + \sum (-1)^n (\bar{g}_S \bar{g}_0^{-1})^n\right).$$

Each $g^{\alpha\beta}$ has a meaning as a formal series in the generators z^I.

Remark: In a DeWitt algebra a matrix X is simultaneously left and right invertible and the inverses are equal:

$$XX^{-1} = X^{-1}X = Ie.$$

Graded spinor fields

Graded spinor fields on an ordinary pseudo-riemannian manifold admitting a spin structure are sections of the graded extension of the usual spin bundle.

Graded connections

Graded connections can be defined as graded Lie algebra valued 1-forms

on the principal bundle P with the appropriate equivariant property, or as a collection of local sections of a graded affine bundle.
In particular, an A-valued connection on M associated with the principal bundle $P \to M$ with group G is a family of 1-forms ω_i taking their values in $\mathcal{G} \otimes A$, with $\mathcal{G}$ the Lie algebra of G; ω_i is defined on U_i, and satisfies in $U_i \cap U_j$

$$\omega_j(x) = \mathrm{Ad}(\sigma_{ji}^{-1}(x))\omega_i(x) + (\sigma^*_{ji}\theta_{\mathrm{MC}})(x)e,$$

where $\mathrm{Ad}(\sigma_{ji}^{-1}(x))$ is the linear map $\mathcal{G} \otimes A \to \mathcal{G} \otimes A$ canonically deduced from the linear map on $\mathcal{G}$, and $\sigma^*_{ji}\theta_{\mathrm{MC}}$ is the pullback on M of the Maurer–Cartan 1-form on G; the unit e of A is assumed to exist.

Remark: The affine bundle of connections on $U \subset M$ is not associated in the usual way to P; σ^*_{ji} involves the derivative of the mapping σ_{ji}: $U_i \cap U_j \to G$.
Such an association can be described in terms of the first jet of P.

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IV. INTEGRATION ON MANIFOLDS

1. COHOMOLOGY. DEFINITIONS AND EXERCISES

For applications and generalizations, see for instance (IV 2, Obstruction; IV 4, Cohomology of groups, IV 6, Short exact sequences, V bis 6, Euler–Poincaré characteristic).

The building blocks of p-chains can be either simplices or rectangles in $\mathbb{R}^p$. In the book *Analysis, Manifolds and Physics* we used rectangles (p. 217); in this problem we use simplices and we recall first the basic definitions. See for instance [Patterson], [Singer and Thorpe] or [Nash and Sen] for introductions to simplicial complices.

1. HOMOLOGY

A **geometric open m-simplex** $\sigma^m \equiv (p_0, \ldots, p_m)$ $(m \in \mathbb{N})$ is a set of points x in some space $\mathbb{R}^n$, $n \geq m$, defined in terms of $m+1$ linearly independent points $p_0, \ldots, p_m$ by

geometric simplex $\sigma^m \equiv (p_0, \ldots, p_m)$

$$x = \sum_{\alpha=0}^{m} t^\alpha p_\alpha, \qquad \text{i.e.} \quad x^i = \sum_{\alpha=0}^{m} t^\alpha p_\alpha^i, \qquad i = 1, \ldots, n,$$

where

$$t^\alpha \in \mathbb{R}, \qquad \sum_\alpha t^\alpha = 1 \quad \text{and} \quad 0 < t^\alpha < 1 .$$

A 1-simplex is a line segment, a 2-simplex is a triangle, etc. Thus σ is a convex open subset of an affine subspace of $\mathbb{R}^n$ of dimension m generated by the vectors $\{p_\alpha\}$. The number m is called the **dimension of the simplex** σ^m.

dimension of the simplices

A **0-simplex** is defined to be a point. Note that it is not open in $\mathbb{R}^n$ if $n \neq 0$.

0-simplex

An **open k-face** of a geometric m-simplex σ^m is an open k-simplex defined in terms of $k+1$ of the points defining σ^m. The 0-faces, called **vertices**, are the points $p_0, \ldots, p_m$. The 1-faces are called **edges**.

face

vertex

edge

An **orientation** can be assigned to a simplex by ordering its vertices. If σ^m is an oriented simplex, $-\sigma^m$ is the same set of points with opposite orientation.

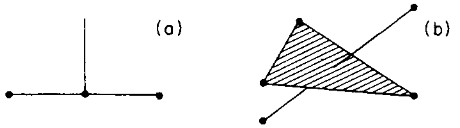
orientation

complex K

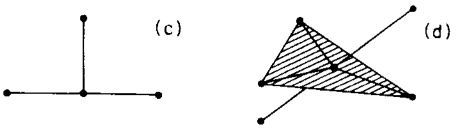
dimension of complices

A **(geometrical simplicial) complex** K is a finite set of disjoint 0 simplices and open m-simplices ($m = 1, \ldots, p$) in $\mathbb{R}^n$ such that, if $\sigma^m \in K$, then all faces of σ^m are in K. The number p is called the **dimension** of K.

1a i) *Show that the following sets are not complices*



but the following ones are complices.



Answer 1a i): In set (a), one of the 0-faces in the closure is not included. In set (b) the 2-simplex has a point in common with one of the 1-simplices; they are not disjoint. On the other hand sets (c) and (d) are complices: all the faces are included; all the simplices are disjoint because an open set and its boundary are disjoint. ■

polyhedron

Given a complex K, the **associated polyhedron** $|K|$ is uniquely defined as the set of points in the collection of simplices in K. Given a polyhedron $|K|$, there are infinitely many complexes K covering it. The process of constructing a covering complex of a given polyhedron is called its **triangulation**. A smoothly **triangulated manifold** is a triple (M, K, h) where M is a smooth manifold, K a simplicial complex and $h: |K| \to M$ is a homeomorphism.

triangulation

triangulated manifold

See examples in [Problem V bis 6, Euler–Poincaré].

chain

Let K be a simplicial complex; a k-**chain** in K with coefficients in an abelian group G is a formal sum

$$C_k = \sum g_i \sigma_i^k\,, \quad g_i \in G\,, \quad \sigma_i^k \text{ a } k-\text{simplex in } K\,.$$

$C_k(K, G)$

The set $\boldsymbol{C_k(K, G)}$ of k-chains in K with coefficients in G whose operation is here denoted by $+$, together with the addition

$$C_k + C_k' = \sum (g_i + g_i')\sigma_i^k$$

form an abelian group, called the group of k-chains with coefficients in G, integral k-chains if $G = \mathbb{Z}$, real k-chains if $G = \mathbb{R}$.

boundary ∂

Let $\sigma^{k+1} \equiv (p_0, \ldots, p_{k+1})$ be an oriented $k+1$ simplex, the **boundary** $\boldsymbol{\partial\sigma^{k+1}}$ is the k-chain defined by

$$\partial\sigma^{k+1} = \sum_{\alpha=0}^{k+1} (-1)^\alpha (p_0, \ldots, \hat{p}_\alpha, \ldots, p_{k+1})$$

where ^ over a symbol means that the symbol should be deleted.

The boundary of the chain $\sum g_i \sigma_i^k$ is $\sum g_i \partial \sigma_i^k$.

1a ii) *Show that the maps*

$$C_{k+1}(K, G) \xrightarrow{\partial} C_k(K, G) \xrightarrow{\partial} C_{k-1}(K, G)$$

satisfy $\partial^2 \equiv \partial \circ \partial = 0$.

Answer 1a ii: By computation. ■

The space of k-cycles Z_k and the space of k-boundaries B_k are subsets of C_k defined like the corresponding subsets of the space of chains expressed as formal sums of rectangles (p. 233):

cycles
boundary
homology
group

$$\begin{aligned} Z_k(K, G) &= \{c \in C_k(K, G);\ \partial c = 0\} \\ B_k(K, G) &= \left\{\partial c;\ c \in C_{k+1}(K, G)\right\} \\ H_k(K, G) &= Z_k(K, G)/B_k(K, G). \end{aligned}$$

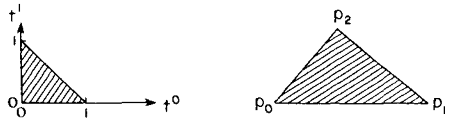
There are infinitely many triangulations of a triangulable manifold M. However inserting segments in a triangulation creates only bounding cycles or cycles homologous to existing cycles (e.g., see examples in Problem V bis 6, Euler–Poincaré characteristic 1b ii); thus the homology groups $H_k(K, G)$ are independent of such insertions. More generally it can be proved that the homology groups $H_k(K, G)$ are independent of the triangulation K of M and we shall label $H_k(M, G)$ the homology groups for any triangulation of M.

b) The **standard k-simplex** is the convex set $\Delta^k \in \mathbb{R}^{k+1}$ consisting of all $(k+1)$-tuples $(t^0, \ldots, t^k)$ of real numbers such that $\sum_\alpha t^\alpha = 1$ and $0 < t^\alpha < 1$.

standard
k-simplex

1b) *Show that a geometric simplex σ^k can be defined as the image of the standard k-simptex Δ^k by an homeomorphism.*

Answer 1b: For simplicity take $k = 2$. The standard 2-simplex is the set of 2-tuples t^0, t^1, t^2 such that $0 < t^0 < 1, 0 < t^1 < 1, 0 < t^0 + t^1 < 1$, with t^2 determined by $t^0 + t^1$.



A geometric 2-simplex defined by (p_0, p_1, p_2) can be defined as an homeomorphism from a standard 2-simplex into the triangle $(p_0, p_1, p_2) \subset \mathbb{R}^2$. ■

singular k-simplex

c) Any continuous map σ from Δ^k into a topological space X defines a **singular k-simplex** $[\sigma]$ in X

$$[\sigma] := (\sigma, \Delta^k).$$

1c i) *Compare singular k-simplices and geometric k-simplices with elementary k-chains and elementary domains of integration* (p. 217).

Answer 1c i): A geometric simplex is a particular case of a singular simplex (σ, Δ^k) when σ is a homeomorphism.
An elementary domain of integration on X is a particular case of an elementary chain (f, rectangle) when f is a diffeomorphism. ■

1c ii) *Give a definition of the j-face of a singular simplex so that it is equivalent to the previous definition of j-faces when the singular simplex is a geometric simplex.*

Answer 1c ii): The j-face of a singular k-simplex $\sigma\colon \Delta^k \to X$ is the singular $(k-1)$-simplex

$$\sigma \circ \phi_j\colon \Delta^{k-1} \to X,$$

where

$\phi_j\colon \Delta^{k-1} \to \Delta^k$ is the embedding defined
$\phi_j(t^0, \ldots, \hat{t}^j, \ldots, t^k) = (t^0, \ldots, t^j = 0, \ldots, t^k).$ ■

1c iii) *Define singular chains and singular homology groups with coefficients in $\mathbb{R}$ or $\mathbb{Z}$.*

Answer 1c iii: A singular k-chain $C \in C_k(X, G)$ is a formal sum of singular k-simplices $[\sigma_i^k] \equiv (\sigma_i, \Delta^k)$ with coefficients in G ($\mathbb{R}$ or $\mathbb{Z}$)

$$C = \sum g_i [\sigma_i^k].$$

The boundary of C

$$\partial C = \sum g_i \partial [\sigma_i^k],$$

where the boundary of k-simplex is the alternate sum of its j-faces

$$\partial[\sigma] = [\sigma \circ \phi_0] - [\sigma \circ \phi_1] + \cdots \pm [\sigma \circ \phi_k].$$

Once the boundary of a (singular) chain $C \in C_k(X, G)$ is defined, the spaces $Z_k(X, G)$ of (singular) cycles, the spaces $B_k(X, G)$ of (singular) boundaries and the (singular) homology groups $H_k(X, G)$ are defined as before. ■

2. COHOMOLOGY

Cosimplices and **cochains** are defined by passing to dual spaces $C^k(X, G)$ of cochains on X with values in $G(\mathbb{R}$ or $\mathbb{Z})$. $C^k(X, G)$ is defined as the set of maps $f\colon C_k(X, G) \to G$ such that for any finite formal sum one has cosimplex cochain

$$f\left(\sum g_i\sigma_i\right) = \sum g_i f(\sigma_i), \quad g_i \in \mathbb{R} \text{ [resp. } \mathbb{Z}], \quad f(\sigma_i) \in \mathbb{R} \text{ [resp. } \mathbb{Z}].$$

a) Give a necessary condition that must be satisfied by the abelian groups G and G_1 in order that this definition of cochains be extendible to mappings $f\colon C_k(X, G_1) \to G$.
b) Consider the case in which G is the abelian multiplicative group $\mathbb{Z}_2 = (-1, 1)$. Define the space of $\mathbb{Z}_2$ cochains on X as homomorphic mappings $f\colon C_k(X, \mathbb{Z}_2) \to \mathbb{Z}_2$.

Answer 2a: Let $\sum g_i\sigma_i \in C_k(X, G_1)$; in order to define a map f such that

$$f\left(\sum g_i\sigma_i\right) = \sum g_i f(\sigma_i) \in G, \qquad g_i \in G_1, \qquad f(\sigma_i) \in G,$$

the group G_1, must have a left action on G.

Answer 2b: If G is the abelian group $(-1, 1)$, its operation is denoted multiplicatively. The sum of the chains $c_k = \sum g_i\sigma_i^k$ and $c'_k = \sum g'_i\sigma_i^k$ is then

$$c_k + c'_k = \sum g_i g'_i \sigma_i^k.$$

A cochain in $C^k(X, \mathbb{Z}_2)$ is a mapping $f\colon C_k(X, \mathbb{Z}_2) \to \mathbb{Z}_2$ such that the image of a finite formal sum is

$$f\left(\sum g_i\sigma_i\right) = \prod f(\sigma_i)^{g_i}, \qquad g_i = \pm 1.$$

With this definition the space of k-cochains $C^k(X, \mathbb{Z}_2)$ is an abelian group like $C^k(X, G)$, $G = \mathbb{R}$ or $\mathbb{Z}$, but its operation will be denoted multiplicatively.

The **coboundary operator** d maps the space of k cochains into the space of $k + 1$ cochains: coboundary operator

$$C^k(X, G) \xrightarrow{d} C^{k+1}(X, G).$$

Let $f \in C^k(X, G)$; the coboundary $df \in C^{k+1}(X, G)$ is defined by the equation

$$(df)(\sigma^{k+1}) = f(\partial\sigma^{k+1}).$$

For another sign convention see [Milnor and Stasheff, Remark, p. 258].

Exercise: Verify that $d \circ d = 0$.

cocycle

The space of **cocycles** $\mathbb{Z}^k(X, G)$ is the space of k cochains with vanishing coboundaries. It inherits from $C^k(X, G)$ an abelian group structure.

coboundary

The space of **coboundaries** $B^k(X, G)$ is the space of k cochains which are boundaries of $k - 1$ cochains.

cohomology

The **cohomology** group is $H^k(X, G) = Z^k(X, G)/B^k(X, G)$.
The de Rham cohomology group $H^k(M, \mathbb{R})$ (pp. 222–223) (quotient spaces of vector spaces of closed forms by vector spaces of exact forms on a compact differential manifold M) is isomorphic to the dual of $H_k(M, \mathbb{R})$. See in [Milnor and Stasheff, p. 259] the necessary conditions for a cohomology group to be the dual of the homology group.

The direct sum

$$H^*(X, G) := \oplus H^k(X, G), \qquad k = 0, \ldots, \dim X,$$

can be given the structure of an associative graded algebra by the cup product defined as follows.

cup product

front k-face

Given singular cochains $f_k \in C^k(X, G)$ and $f_l \in C^l(X, G)$ with $G = \mathbb{R}$ or $\mathbb{Z}$ the **cup product** $f_k f_l \equiv f_k \cup f_l \in C^{k+l}(X, G)$ is defined as follows. Let $\sigma\colon \Delta^{k+l} \to X$ be a singular simplex. The **front k-face** of $[\sigma]$ is the singular simplex $[\sigma \circ \alpha_k] = (\sigma\alpha_k, \Delta^k)$ where α_k is the injection map

$$\alpha_k(t^0, \ldots, t^k) = (t^0, \ldots, t^k, 0, \ldots, 0).$$

back l-face

Similarly the **back l-face** of $[\sigma]$ is the singular simplex $[\sigma \circ \beta_l] = (\sigma \circ \beta_l, \Delta^l)$ where β_l is the injection map

$$\beta_l(t^k, \ldots, t^{k+l}) = (0, \ldots, 0, t^k, \ldots, t^{k+l}).$$

The cup prduct $f_k \cup f_l$ is defined on a $(k + l)$-simplex $[\sigma]$ by

$$\langle f_k \cup f_l, [\sigma]\rangle = \langle f_k, [\sigma \circ \alpha_k]\rangle\langle f_l, [\sigma \circ \beta_l]\rangle.$$

It is defined on $C_k(X, G)$ by linearity. Let $g_1, g_2 \in G$,

$$\langle f_k \cup f_l, g_1[\sigma_1] + g_2[\sigma_2]\rangle := g_1\langle f_k \cup f_l, [\sigma_1]\rangle + g_2\langle f_k \cup f_l, [\sigma_1]\rangle.$$

The cup product is bi-additive:

$$(f_k + f_k') \cup f_l = f_k \cup f_l + f_k' \cup f_l.$$

It is associative but not commutative.
The constant cocyle $1 \in C^0(X, G)$ serves as the **identity element**, for cup products: identity element

$$\langle 1, [\sigma] \rangle = 1 \quad \text{for all } [\sigma].$$

Example. See Problem IV 2, Obstruction, paragraph 4.

b) *Show that the cup product of cochains induces a cup product of cohomology classes.*

Answer 2b: Let f be a k-cocycle and f' an l-cocycle. Let $[f]$ and $[f']$ be their cohomology classes.
We can define

$$[f] \cup [f'] := [f \cup f']$$

provided that $[(f + dg) \cup (f' + dg')] = [f \cup f']$. Now

$$(f + dg) \cup f' = f \cup f' + dg \cup f';$$

and

$$[(f + dg) \cup f'] = [f \cup f']$$

provided that $dg \cup f'$ is a coboundary. We shall prove that $dg \cup f' = d(g \cup f')$ for f' a cocycle. Indeed it follows from the definitions that d obeys the anti-Leibniz rule

$$d(f \cup f') = df \cup f' + (-1)^k (f \cup df').$$

Hence the definition $[f] \cup [f'] := [f \cup f']$ makes sense. ∎

c) *Show that it is possible to interpret the exterior product $\wedge$ of exterior forms on a differentiable manifold M as a cup product.*

Answer 2c: A $(k + 1)$-form ω can be defined (p. 226) by duality on the space of differentiable chains with coefficients in $\mathbb{R}$:

$$\langle \omega, c \rangle := \int_c \omega.$$

Let c be an elementary $(k + l)$-chain (p. 217), i.e.,

$$c_{k+l} := (f, P^{k+l})$$

where P^{k+l} is the naturally oriented subset of $\mathbb{R}^{k+l}$ defined by

$$a^i \leq x^i \leq b^i, \qquad i = 1, \ldots, k + l,$$

and $f\colon\ P^{k+l} \to M$ is a differentiable mapping

$$\int_{c_{k+l}} \omega = \int_{P^{k+l}} f^* \omega.$$

It is natural to define the **front k-face** and **back l-face** of the elementary $(k+1)$-chain as follows. Let

P^k be the subset of P^{k+l} such that $x^{k+1} = x^{k+2} = \cdots = x^{k+l} = 0$,
P^l be the subset of P^{k+l} such that $x^1 = \cdots = x^k = 0$;

let $\alpha_k \colon P \to P^k$ and $\beta_l \colon P \to P^l$, with $P \equiv P^{k+l}$.
We call the front k-face of c_{k+l} the elementary k-chain $c_k := (f \circ \alpha_k, P)$.
We call the back l-face of c_{k+l} the elementary l-chain $c_l := (f \circ \beta_l, P)$.
Given a k-form η on c_k and an l-form ξ on c_l, the product $\eta \cup \xi$ on c_{k+l} defined by

$$\begin{aligned} \langle \eta \cup \xi, c_{k+l} \rangle &:= \langle \eta, c_k \rangle \langle \xi, c_l \rangle \\ &= \int_{c_k} \eta \int_{c_l} \xi \\ &= \int_P (f \circ \alpha_k)^* \eta \int_P (f \circ \beta_l)^* \xi \\ &= \int_{P^k} f^* \eta \int_{P^l} f^* \xi \\ &= \int_P f^*(\eta) \wedge f^*(\xi) = \int_P f^*(\eta \wedge \xi). \qquad \blacksquare \end{aligned}$$

Acknowledgements and references can be found at the end of Problem IV 3.

2. OBSTRUCTION TO THE CONSTRUCTION OF SPIN AND PIN BUNDLES; STIEFEL–WHITNEY CLASSES

1. INTRODUCTION: SPINOR FIELDS ON A MANIFOLD; SPIN AND PIN STRUCTURES

Let V be a d-dimensional vector space with a pseudo-euclidean scalar product g of signature (n, m), n plus signs, m minus signs, invariant under the group $\mathrm{O}(n, m)$, $n + m = d$. We have constructed a double cover of $\mathrm{SO}(n, m)$, namely $\mathrm{Spin}(n, m)$, by a representation on $\mathbb{C}^{2^p}$, $p = [d/2]$, linked with a choice of an oriented orthonormal frame in (V, g). In the same way that a vector u in V can be considered as an equivalence class of pairs $(u_{(i)}, \rho_{(i)})$ with $u_{(i)} \in \mathbb{R}^d$ and $\rho_{(i)}$ a linear frame in V with the equivalence relation

$$(u_{(i)}, \rho_{(i)}) \cong (u_{(j)}, \rho_{(j)}).$$

if and only if

$$u_{(i)} = L u_{(j)} , \qquad \rho_{(j)} = L \rho_{(i)} , \qquad L \in Gl(d) ,$$

a **spinor** ψ relative to (V, g) can be defined as an equivalence class of triples (p. 415) $(\psi_{(i)}, \rho_{(i)}, \Lambda_{(i)})$ with $\psi_{(i)} \in \mathbb{C}^{2^p}$, $\rho_{(i)}$ an orthonormal frame, and $\Lambda_{(i)} \in \mathrm{Spin}(n, m)$, with the equivalence relation spinor

$$\psi_{(i)} = \Lambda \psi_{(j)} , \qquad \rho_{(j)} = L \rho_{(i)} , \qquad \Lambda = \Lambda_{(i)} \Lambda_{(j)}^{-1} , \qquad \mathcal{H}(\Lambda) = L .$$

The pair $(\rho_{(i)}, \Lambda_{(i)})$ can be called a **spin frame**. The 2^p complex numbers defining $\psi_{(i)}$ are the components of ψ in this frame. Note that the two elements of a spin frame are not independent: if $(\rho_{(0)}, e)$ is a spin frame, called fiducial, then $(\rho(i), \Lambda(i))$ is another spin frame if and only if $\mathcal{H}(\Lambda_{(i)}) = L$ when $\rho(i) = L\rho_{(0)}$: to each L correspond two possible Λ's. If a particular pair (ρ, Λ) is chosen as spin frame, there is always one corresponding fiducial frame $(\rho_{(0)}, e)$, with $\rho(0) = \mathcal{H}(\Lambda^{-1})\rho$. Different choices of fiducial frames correspond to different, isomorphic spaces of spinors attached to V. spin frame

A **cospinor** $(\varphi_{(i)}, \rho_{(i)}, \Lambda_{(i)})$ is defined analogously, by the equivalence relation: cospinor

$$\varphi_{(i)} = \Lambda^{-1} \varphi_{(j)} , \qquad \rho_{(j)} = L \rho_{(i)} , \qquad \Lambda = \Lambda_{(i)} \Lambda_{(j)}^{-1} , \qquad \mathcal{H}(\Lambda) = L .$$

a) *Show that, if d is even, the d gamma matrices Γ_A define a spinor–cospinor 1-form on V.*

Answer 1a: The d gamma matrices are given numerical matrices such that for any $\Lambda \in \mathrm{Spin}(n, m)$, they satisfy [see Problem I 7, Pin §2]

$$\Lambda \Gamma_A \Lambda^{-1} = L_A^B \Gamma_B , \qquad \mathcal{H}(\Lambda) = L .$$

The triple $(\Gamma_A, \rho, \Lambda)$, $\rho = L\rho_{(0)}$ satisfies the relevant relation under change of spin and orthonormal frame. One defines with the Γ_A's a **spinor–cospinor 1-form** on V in a spin frame and an arbitrary linear frame by setting, as a definition, spinor–cospinor 1-form

$$\Gamma_\alpha = e_\alpha^A \Gamma_A ,$$

where (e_α^A) is the matrix transforming an orthonormal frame into an arbitrary linear one. Note, however, that it is meaningless to speak of components of a spinor in an arbitrary linear frame. ■

Let M be a differentiable manifold with a pseudo-riemannian metric g, of signature (n, m). The tangent space at each point x is a vector space with a pseudo-euclidean scalar product, and it is possible to define a space of spinors at x as one of the isomorphic spaces previously considered. It will be possible to define spinor fields on M if there exist fields of spin frames over M. Such fields exist over each open set U of M over which the bundle $\mathcal{O}$ of orthonormal frames is trivial: choosing a section $x \mapsto \rho_0(x)$ of this bundle, we can define a spin frame at each point $x \in U$ by the pair $(\rho_{(0)}(x), e)$; other spin frames at x are defined by the equivalence relation given before.

If the bundle SO of oriented orthonormal frames is not trivial, the problem will be to choose spin frames $(\rho_{(i)}, \Lambda_{(i)})$ above each open set $U_i \subset M$ over which SO is trivial in such a way that the chosen space of spinors at each $x \in M$ is uniquely defined, that is

$$\mathcal{H}(\Lambda_{(j)}(x)\Lambda_{(i)}^{-1}(x)) = L_{ji}(x) \quad \text{if} \quad \rho_{(j)}(x) = L_{ji}(x)\rho_{(i)}(x) \qquad x \in U_i \cap U_j\,. \tag{1}$$

b) *Show that the problem of defining spinor fields on M is the problem of constructing a principal fibre bundle S over M, with typical fibre* Spin(n, m), *such that the image under $\mathcal{H}$ of its transition functions are the transition functions of* SO, *the two bundles being locally trivialized over the same atlas of M.*

$\mathcal{H}$-extension prolongation

spin structure

Such a bundle projects on SO and is said to be an **$\mathcal{H}$-extension (prolongation)** of SO. A bundle $\mathcal{S}$ of spin frames, together with its projection on SO, is called a **spin structure** over the pseudo riemannian manifold.

Answer 1b): Suppose there exists a principal fibre bundle $\mathcal{S}$ over M with typical fibre Spin(n, m) and transition functions, relative to a covering $\{U_i\}$ of M

$$\gamma_{ij}\colon U_i \cap U_j \mapsto \mathrm{Spin}(n, m)$$

such that, L_{ij} being the transition functions of SO,

$$\mathcal{H}(\gamma_{ij}(x)) = L_{ij}(x)\,. \tag{2}$$

Trivializations of SO over U_i and U_j are defined respectively by local cross sections $\rho_{(i)}$ and $\rho_{(j)}$, and the transition functions are mappings

$$U_i \cap U_j \to \mathrm{SO}(n, m) \quad \text{by} \quad x \mapsto L_{ij}(x)$$

with

$$\rho_{(i)}(x) = L_{ij}(x)\rho_{(j)}(x) \quad \text{in} \quad U_i \cap U_j\,.$$

Let us denote by $z_{(i)} = (x, \Lambda_{(i)}(x))$ the image of a point $z \in p^{-1}(x) \subset \mathcal{S}$ under the trivialization of $\mathcal{S}$ over U_i. The transition function γ_{ij} is

$$\gamma_{ij}\colon x \mapsto \gamma_{ij}(x) = \Lambda_{(i)}(x)\Lambda_{(j)}^{-1}(x)\,.$$

The hypothesis (2) implies that (1) is satisfied, and conversely. The projection Π of $z \in p^{-1}(x) \subset \mathcal{S}$ onto $\rho \in \mathrm{SO}$ is defined in local trivializations by

$$(\Pi(z))_{(i)} = \mathcal{H}(\Lambda_{(i)}(x));$$

it is independent of the trivialization. ∎

Examples: See for instance [Geroch, Isham, DeWitt].

If (M, g) is a non-orientable riemannian manifold, it does not admit an $\mathrm{SO}(n, m)$ bundle linked to g. Indeed (p. 386), the existence of a metric g implies that the principal bundle of linear frames on M is reducible to an $\mathrm{O}(n, m)$ bundle linked to g. If M is not orientable, the $\mathrm{O}(n, m)$ bundle linked to g is not reducible to an $\mathrm{SO}(n, m)$ bundle.

In the following, we use the short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Pin}(n, m) \to \mathrm{O}(n, m) \to \mathbb{1} ;$$

hence, we use the surjective homomorphism from $\mathrm{Pin}(n, m)$ onto $\mathrm{O}(n, m)$, labelled $\hat{\mathcal{H}}$ in [Problem I 7, Clifford], now labelled $\mathcal{H}$ – reserving $\hat{\mathcal{H}}$ for the bundle homomorphism.

We shall now consider an $\mathrm{O}(n, m)$ bundle ξ of orthonormal frames over (M, g) and investigate under which conditions it admits an $\mathcal{H}$-extension $\tilde{\xi}$. The definitions of **pin frame, pin bundle, pin structure** are similar to the definitions of spin frame, spin bundle, spin structure, $\mathrm{O}(n, m)$ replacing $\mathrm{SO}(n, m)$. **Pinors,** like spinors, are defined via a linear representation of the group, now the Pin group. Another notation (p. 416) is Spin[resp. Spin_0] for what is called here Pin[resp. Spin]. Whereas $\mathrm{Spin}(n, m)$ and $\mathrm{Spin}(m, n)$ are isomorphic, we have established in [Problem I 7, Clifford] that $\mathrm{Pin}(n, m)$ is not always isomorphic to $\mathrm{Pin}(m, n)$. We set

pin frame
pin bundle
pin structure

pinors

$$\mathrm{Pin}^+(n) = \mathrm{Pin}(0, n) \quad \text{and} \quad \mathrm{Pin}^-(n) = \mathrm{Pin}(n, 0) .$$

We use the convention

$$\gamma_A \gamma_B + \gamma_B \gamma_A = -2 g_{AB} \mathbb{1} .$$

If a property or a discussion is valid for both $\mathrm{Pin}^+(n)$ and $\mathrm{Pin}^-(n)$, we write $\mathrm{Pin}(n)$.

Let ξ be an $\mathrm{O}(n)$ bundle over (M, g); let $\{g_{ij}\}$ be the set of its transition functions relative to a covering $\mathcal{U} = \{U_i\}$ of M. The set $\{\gamma_{ij}\}$ of $\mathrm{Pin}(n)$ valued functions on $U_i \cap U_j$ is a set of transition functions of a $\mathrm{Pin}(n)$-bundle over M if, for any triple (i, j, k) such that $U_{ijk} := U_i \cap U_j \cap U_k \neq \emptyset$,

$$\gamma_{ij}(x)\gamma_{jk}(x) = \gamma_{ik}(x) , \qquad x \in U_{ijk} .$$

The concepts and theorems of cohomology developed in the previous problems can be adapted to the problem of determining whether or not one can construct a set of transition functions defined on a cover $\mathscr{U}$ of a manifold M.

2. DEFINITIONS

abstract simplicial complex K

abstract p-simplex

jth face

An **abstract simplicial complex** K is a collection of finite subsets of an ordered, countable set $\mathscr{I}$ subject to the following condition: if $\sigma \in K$, then every subset of σ is also in K. An element $\sigma^p \equiv \{i_0, \ldots, i_p\}$, $i_0, \ldots, i_p \in \mathscr{I}$, of K is called an **(abstract) p-simplex**. A simplex is oriented, or ordered, by the ordering of its defining set. The **jth face** of σ^p is given by

$$\partial_j \sigma \equiv \{i_0, \ldots, \hat{i}_j, \ldots, i_p\} .$$

vertex

A zero-simplex is called a **vertex**.

Exercise: Show that if $j \leq k$, then $\partial_k \partial_j = \partial_j \partial_{k+1}$.

boundary

The **boundary** $\partial\sigma$ of σ is the alternate formal sum of its faces

$$\partial\sigma = \sum_{j=0}^{p} (-1)^j \partial_j \sigma .$$

Exercise: Show that $\partial^2 = 0$.

nerve

Let $\mathscr{U} = \{U_i; i \in \mathscr{I}\}$ be an open cover of a manifold M. The **nerve** $\mathscr{N}$ of $\mathscr{U}$ is the abstract simplicial complex whose vertices belong to $\mathscr{I}$ and such that a set $\{i_0, \ldots, i_p\}$ of distinct elements of $\mathscr{I}$ is a p-simplex of $\mathscr{N}$ if and only if

$$U_{i_0, i_1, \ldots, i_p} := U_{i_0} \cap \cdots \cap U_{i_p} \neq \emptyset .$$

We shall occasionally abbreviate the phrase "nerve of a cover $\mathscr{N}$ of M" to "nerve of M".

locally finite

locally finite

If the covering is **locally finite** (i.e., if each U_i meets only a finite number of the U_j) the corresponding nerve is said to be a **locally finite** simplicial complex.

simple

An open cover is said to be **simple** if all the patch intersections are contractible.

For instance if $M = S^1$, a simple cover consists of at least 3 patches.

nerve

Čech homology

The **nerve** $\mathscr{N}$ of a cover $\mathscr{U}$ is the building block of the **Čech homology** $\check{H}_*(\mathscr{N}, \mathbb{Z})$.

Abstract cochains, abstract coboundaries, Čech cohomology are defined according to the basic cohomology pattern [see, for instance, MacLane].

3. EXISTENCE OF PIN BUNDLES OVER AN O(N) BUNDLE

There exists a principal Pin(n) bundle over M, with transition functions $\gamma_{ij}: U_{ij} \to \mathrm{Pin}(n)$, $U_{ij} = U_i \cap U_j$, $\{U_i\}$ cover of M, if for any 2-simplex (i, j, k) of $\mathcal{N}$

$$\gamma_{ij}(x)\gamma_{jk}(x) = \gamma_{ik}(x)$$

or equivalently

$$\gamma_{jk}(x)(\gamma_{ik}(x))^{-1}\gamma_{ij}(x) = e\,, \qquad \text{the unit element of } \mathrm{Pin}(n)\,.$$

a) *Show that if* $\{\gamma_{ij}\}$ *is a set of lifts* $U_{ij} \to \mathrm{Pin}(n)$ *of a set of transition functions* g_{ij} *of an* $\mathrm{O}(n)$ *bundle over* M *then*

$$p_{ijk}(x) = \gamma_{jk}(x)(\gamma_{ik}(x))^{-1}\gamma_{ij}(x)$$

defines a 2*-cochain on the nerve* $\mathcal{N}$ *of a simple cover* $\mathcal{U}$ *of* M *with values in* $\mathbb{Z}_2$.

Answer 3a:

$$\mathcal{H}(p_{ijk}(x)) = g_{jk}(x)[g_{ik}(x)]^{-1}g_{ij}(x) = e\,, \qquad \text{the unit element of } \mathrm{O}(n)$$

hence $p_{ijk}(x) = \pm e$, e the unit element of Pin(n).

Since U_{ijk} is connected, $p_{ijk}(x)$ is constant on U_{ijk} and we can define the cochain p on $\mathcal{N}$ with values in $\mathbb{Z}_2$ by

$$p: (i, j, k) \to p(i, j, k) \in \mathbb{Z}_2$$

with

$$p(i, j, k)e = p_{ijk}(x)\,, \qquad x \in U_{ijk}\,.$$

Here the group $\mathbb{Z}_2$ is the abelian multiplicative group $\{1, -1\}$. ■

b) *Show that the cochain* p *is a cocycle.*

Answer 3b: The cochain p is a cocycle if

$$(\mathrm{d}p)(i, j, k, l) = 1\,, \qquad \text{the unit element of } \mathbb{Z}_2 = \{1, -1\}\,.$$

For a multiplicative group, the equation $\mathrm{d}p(\sigma) = p(\partial\sigma)$ reads (cf. Problem IV 1)

$$(\mathrm{d}p)(I, j, k, l) = p(j, k, l)[p(i, k, l)]^{-1}p(i, j, l)[p(i, j, k)]^{-1}\,.$$

Replacing p by its expression in terms of the γ_{ij} and using the fact that $\mathbb{Z}_2$ is abelian yields

$$(\mathrm{d}p)(i, j, k, l) = 1\,.$$ ■

c) *Show that the cohomology class* $[p] \in H^2(\mathcal{N}, \mathbb{Z}_2)$ *represented by the cocycle p is independent of the choice of transition functions* g_{ij} *of the* O(n) *bundle over M and of the liftings* γ_{ij}, *provided* $\mathcal{H} \circ \gamma_{ij} = g_{ij}$.

Answer 3c: Let $\{g_{ij}^1\}$ and $\{g_{ij}^2\}$ be two sets of transition functions of the O(n) bundle over M, and let $\{\gamma_{ij}^1\}$, $\{\gamma_{ij}^2\}$, $p_{ijk}^1(x)$, $p_{ijk}^2(x)$, $p^1(i, j, k)$, $p^2(i, j, k)$ be defined as before for each set of transition functions. We shall prove that the cocycles p^1 and p^2 are in the same cohomology class, i.e., there exists a 1-cochain f such that

$$p^1(p^2)^{-1} = \mathrm{d}f$$

by constructing f explicitly.

Since $\{g_{ij}^1\}$ and $\{g_{ij}^2\}$ are transition functions there exists some $\lambda_i: U_i \to \mathrm{O}(n)$ such that (Problem V bis 7, Equivalent)

$$g_{ij}^2(x) = (\lambda_i(x))^{-1} g_{ij}^1(x) \lambda_j(x) .$$

Define $\tilde{\lambda}_i: U_i \to \mathrm{Pin}(n)$ such that $\mathcal{H} \circ \tilde{\lambda}_i = \lambda_i$, and

$$f_{ij}: U_{ij} \to \mathrm{Pin}(n) \quad \text{by} \quad f_{ij}(x) = \gamma_{ij}^1(x)[\tilde{\lambda}_i(x)\gamma_{ij}^2(x)\tilde{\lambda}_j(x)^{-1}]^{-1} .$$

Then

$$\mathcal{H}(f_{ij}(x)) = e , \qquad \text{the unit element of } \mathrm{O}(n) .$$

Hence for the reasons given in paragraph 3a), $f_{ij}(x)$ defines a $\mathbb{Z}_2$-2-cochain f on $\mathcal{N}$. The coboundary df of f satisfies

$$\mathrm{d}f(i, j, k) = f(j, k)[f(i, k)]^{-1} f(i, j) .$$

The desired equality, namely $p^1(p^2)^{-1} = \mathrm{d}f$ follows from

$$p_{ijk}^1(x)[p_{ijk}^2(x)]^{-1} = f_{jk}(x)[f_{ik}(x)]^{-1} f_{ij}(x)$$

which itself follows from the definitions and the fact that $\mathcal{H}(f_{ij}(x))$ and $\mathcal{H}(p_{ijk}(x))$ are in the center of O(n). ∎

d) *Show that an* O(n) *bundle over M admits a Pin-bundle extension if and only if* $[p]$ *is trivial.*

Answer 3d: Assume that the Pin-bundle extension exists. Then there are liftings γ_{ij} of g_{ij} such that $p(i, j, k) = 1$, the unit element in $\mathbb{Z}_2$; hence, the cohomology class of p is trivial. Conversely if $[p]$ is trivial, then either $p(i, j, k) = 1$ for all 2-simplices, or there is a 1-cochain f with values in $\mathbb{Z}_2$ such that $p(i, j, k)(\mathrm{d}f(i, j, k))^{-1} = 1$ for all simplices, i.e.,

$$f(j, k)[f(i, k)]^{-1} f(i, j) = p(i, j, k) \qquad \text{for all simplices} .$$

Define

$$h_{ij}\colon U_{ij} \to \mathrm{Pin}(n) \quad \text{by} \quad h_{ij}(x) = \gamma_{ij}(x) f^{-1}(i, j).$$

It is easy to check that

$$h_{ik}(x) = h_{ij}(x) h_{jk}(x)$$

and also that

$$\mathscr{H}(h_{ij}) = \mathscr{H}(\pm\gamma_{ij}) = g_{ij};$$

that is, $\{h_{ij}\}$ is a set of transition functions of a Pin-bundle extension of the O(n) bundle. ■

Remark: One can generalize (cf. Greub and Petry) the results derived in paragraphs a) through d) when O(n) is replaced by a topological group G and Pin(n) by a topological group Γ such that there is a continuous homomorphism $\mathscr{H}$: $\Gamma \to G$ with discrete kernel K contained in the center of Γ. A continuous homomorphism satisfying these properties is said to be **central**. central homomorphism

Remark: We shall assume in the following that M admits a simple cover, and we write $H^2(M, \mathbb{Z}_2)$ instead of $H^2(\mathcal{N}, \mathbb{Z}_2)$.

4. STIEFEL-WHITNEY CLASSES

We shall indicate why $[p] = w_2$ for Pin$^+(n)$ bundle and $[p] = w_2 + w_1 \cup w_1$ for Pin$^-(n)$ bundle where w_1 and w_2 are the first and second Stiefel–Whitney classes of the O(n) bundle.

The Stiefel–Whitney classes of an O(n) bundle [Hirzebruch] are, by definition, the Stiefel–Whitney classes of the associated vector bundle under the natural action of O(n) on $\mathbb{R}^n$. The Stiefel–Whitney classes of such vector bundles can be defined by the following set of axioms [Milnor and Stasheff, p. 37].

Axiom 1: To each vector bundle ξ with base $B(\xi)$ there corresponds a sequence of cohomology classes

$$w_i(\xi) \in H^i\big(B(\xi); \mathbb{Z}_2\big), \qquad i = 0, 1, 2, \ldots,$$

called the **Stiefel–Whitney** classes of ξ. The class $w_0(\xi)$ is the cohomology class of the constant cocycle 1 (which is also the identity element for cup products – See IV 1, p. 133) Stiefel–Whitney

$$1 \in H^0(B(\xi); \mathbb{Z}_2),$$

and $w_i(\xi)$ equals zero for i greater than n if ξ is an n-plane bundle. Here $\mathbb{Z}_2$ is a ring, hence $\mathbb{Z}_2 = \{0, 1\}$.

Axiom 2: *Naturality*. If f: $B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then

$$w_i(\xi) = f^* w_i(\eta) .$$

Axiom 3: *The Whitney product theorem*. If ξ and η are vector bundles over the same base space, then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta) ,$$

Whitney sum

where the cup product has been defined in [Problem IV 1, Cohomology] and where the **Whitney sum** of two bundles ξ and η over the same base B space is defined as follows. Let d: $B \to B \times B$ denote the diagonal embedding. Then $\xi \oplus \eta = d^*(\xi \times \eta)$. Note that each fibre of $\xi \oplus \eta$ is canonically isomorphic to the direct sum of the corresponding fibres of ξ and η.

The Whitney product theorem says:

$$w_1(\xi \oplus \eta) = w_1(\xi) + w_1(\eta) ,$$
$$w_2(\xi \oplus \eta) = w_2(\xi) + w_1(\xi) \cup w_1(\eta) + w_2(\eta) , \text{ etc.}$$

Axiom 4. For the twisted line bundle over the circle, the first Stiefel–Whitney class is nontrivial.

a) First Stiefel–Whitney class of $O(n)$ bundles.
i) *Show that the determinants of the transition functions* $g_{ij}(x) \in O(n)$, $x \in U_{ij} \subset M$ *of an* $O(n)$ *bundle over* M *define a cocycle* g *on the nerve of* M *with values in* $\mathbb{Z}_2$.
ii) *Show that the cohomology class* $[g] = w_1(O(n)$ *bundle*).

Answer 4a-i: Since U_{ij} is connected, det $g_{ij}(x)$ does not depend on $x \in U_{ij}$ and we can define

$$g: (i, j) \mapsto \det g_{ij}(x) \in \mathbb{Z}_2 , \qquad x \in U_{ij} .$$

It is easy to check that

$$dg(i, j, k) = g(j, k)(g(i, k))^{-1} g(i, j) = 1 ;$$

that is, g is a cocycle with values in $\mathbb{Z}_2$.

Answer 4a-ii: We shall check that $[g]$ satisfies the four axioms of a Stiefel–Whitney class.

Axiom 1: The cohomology class $[g]$, represented by the cocycle g, is independent of the choice of transition functions of the $O(n)$ bundle.

Indeed, let $\{g_{ij}^1\}$ and $\{g_{ij}^2\}$ be two sets of transition functions of an $O(n)$ bundle; it is straightforward to check that there is a zero-cochain f such that

$$g^1(g^2)^{-1} = \mathrm{d}f .$$

Hence, $[g] \in H^1(M, \mathbb{Z}_2)$.

Axiom 2: Naturality of $[g]$. Let $\tilde{f}$ be a bundle morphism from an $O(n)$ bundle ξ^1, over M_1 with transition functions g_{ij}^1: $U_{ij}^1 \to O(n)$ to an $O(n)$ bundle ξ^2 over M_2 with transition functions g_{ij}^2: $U_{ij}^2 \to O(n)$. Denote by f the induced mapping $M_1 \to M_2$. We shall show that $f^*[g^2] = [g^1]$.
The bundle map $\tilde{f}$: $\xi^1 \to \xi^2$ which covers f: $M_1 \to M_2$ is fibre preserving:

$$\Pi_2 \circ \tilde{f} = f \circ \Pi_1 .$$

It is easy to prove [see, e.g., Steenrod p. 12] that given a system of local cross sections σ_i^2: $U_i^2 \to \xi^2$ there exists one and only one system of local cross sections σ_i^1: $U_i^1 \to \xi^1$ such that

$$\tilde{f} \circ \sigma^1 = \sigma^2 \circ f , \qquad \text{i.e., } \sigma_i^1(x) = \tilde{f}^{-1}(\sigma_{\tau(i)}^2(f(x))) \tag{3}$$

where $\tau(i)$ is defined as follows:

$$f(U_i^1) \subset U_j^2 \qquad \text{for some } j \text{ which is set equal to } \tau(i) .$$

Systems of local cross sections define trivializations by

$$\overset{\Delta}{\varphi}_{i,x}(\sigma_i(x)) = e , \qquad \text{the identity in the structure group } G, \tag{4}$$

which, in turn, define transition functions

$$g_{ij}(x) = \overset{\Delta}{\varphi}_{i,x} \circ \overset{\Delta}{\varphi}{}_{j,x}^{-1} , \qquad x \in U_{ij} .$$

We shall now establish the relationship between $g_{ij}^1(x)$ and $g_{\tau(i)\tau(j)}^2(f(x))$. It follows from (4) that

$$\overset{\Delta}{\varphi}{}_{i,x}^1(\sigma_i^1(x)) = \overset{\Delta}{\varphi}{}_{\tau(i),f(x)}^2(\sigma_{\tau(i)}^2(f(x))) .$$

It follows from (3) that

$$\overset{\Delta}{\varphi}{}_{i,x}^1(\tilde{f}^{-1}(\sigma_{\tau(i)}^2(f(x)))) = \overset{\Delta}{\varphi}{}_{\tau(i),f(x)}^2(\sigma_{\tau(i)}^2(f(x))) ,$$

i.e.,

$$\overset{\Delta}{\varphi}{}_{i,x}^1 \circ \tilde{f}^{-1} = \overset{\Delta}{\phi}{}_{\tau(i),f(x)}^2 \quad \text{and} \quad \tilde{f} \circ (\overset{\Delta}{\varphi}{}_{i,x}^1)^{-1} = (\overset{\Delta}{\varphi}{}_{\tau(i),f(x)}^2)^{-1} ;$$

hence,

$$g_{ij}^1(x) = g_{\tau(i)\tau(j)}^2(f(x)) \tag{5}$$

and

$$f^*[g^2] = [g^1]\,,$$

where $f^*: H^2(\mathcal{N}_2, \mathbb{Z}_2) \to H^2(\mathcal{N}_1, \mathbb{Z}_2)$ is the mapping induced by $f: M_1 \to M_2$.

Axiom 3: The Whitney product theorem. Let

$$\begin{array}{lll} \xi^1 \text{ be a principal} & \mathrm{O}(m) \text{ bundle with transition functions} & g^1_{ij} \\ \xi^2 & \mathrm{O}(n) \text{ bundle} & g^2_{ij} \\ \xi & \mathrm{O}(n) \oplus \mathrm{O}(m) \text{ bundle} & g_{ij} \\ \xi^\lambda & \mathrm{O}(n+m) \text{ bundle} & g^\lambda_{ij} \end{array} \tag{6}$$

over the same base, with

$$g = \begin{pmatrix} g^1 & 0 \\ 0 & g^2 \end{pmatrix}$$

and

$$g^\lambda_{ij} = \lambda \circ g_{ij}\,.$$

The identity inclusion λ

$$\lambda\colon G \equiv \mathrm{O}(n) \oplus \mathrm{O}(m) \to G_\lambda \equiv \mathrm{O}(n+m)$$

maps a block diagonal matrix consisting of an $n \times n$ matrix and an $m \times m$ matrix into the same matrix now thought of as an $(n+m) \times (n+m)$ matrix.

It is straightforward to check that

$$[g^\lambda] = [g^1] \cup [g^2]\,.$$

Axiom 4: $[g]$ (Möbius band) $\neq 0$.

Hence, $[g] = w_1$. This construction of w_1 shows that w_1 is the obstruction class for orientability. $w_1 = 0 \iff M$ is orientable; $w_1 = 1 \iff M$ is not orientable.

b) In section 3, we established that $[p] \in H^2(M, \mathbb{Z}_2)$. We shall prove that $[p]$ satisfies the second, third and fourth Stiefel–Whitney axioms. This is *not*, however, sufficient to prove that $[p] = w_2$ because $[p]$ *fails to vanish on all line bundles whereas, according to the first axiom,* $w_2 = 0$ *on line bundles.*

i) *Naturality of* $[p]$. With the notation of section 3 and paragraph 4a-ii, let γ^2_{ij} be liftings of the g^2_{ij} and set

$$\gamma^1_{ij}(x) = \gamma^2_{\tau(i)\tau(j)}(f(x))\,.$$

It is easy to check that the γ^1_{ij} are liftings of the g^1_{ij}. Indeed

$$\mathcal{H}(\gamma^1_{ij}(x)) = \mathcal{H}(\gamma^2_{\tau(i)\tau(j)}(f(x))) = g^2_{\tau(i)\tau(j)}(f(x)) = g^1_{ij}(x)\,.$$

Thus the obstruction cocycle p^1 is given by

$$\begin{aligned} p^1(i,j,k) &= \gamma^1_{jk}(x)(\gamma^1_{ik}(x))^{-1}\gamma^1_{ij}(x) \quad \text{for} \quad x \in U^1_i \cap U^1_j \cap U^1_k \\ &= \gamma^2_{\tau(j)\tau(k)}(f(x))(\gamma^2_{\tau(i)\tau(k)}(f(x)))^{-1}\gamma^2_{\tau(i)\tau(j)}(f(x)) \\ &= p^2(\tau(i),\tau(j),\tau(k)) \quad \text{for} \quad x \in f(U^1_i) \cap f(U^1_j) \cap f(U^1_k)\,. \end{aligned}$$

Hence

$$[p^1] = f^*[p^2]\,,$$

where f^* is the cohomology mapping induced by $f\colon M_1 \to M_2$.

ii) *The Whitney sum product.* With ξ^L, ξ and $\xi^\wedge$ defined by (6) and

$\tilde{\xi}^1$	a principal	$\mathrm{Pin}(n)$ bundle over ξ^1 with transition functions	γ^1_{ij}
$\tilde{\xi}^2$		$\mathrm{Pin}(m)$ bundle over ξ^2	γ^2_{ij}
$\tilde{\xi}$		$\mathrm{Pin}(n) \oplus \mathrm{Pin}(m)$ bundle over ξ	γ_{ij}
$\tilde{\xi}^\wedge$		$\mathrm{Pin}(n+m)$ bundle over $\xi^\wedge$	$\gamma^\wedge_{ij}$

where all γ's are lifts of the corresponding g's: $\mathcal{H}(\gamma) = g$;

$$\gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix},$$

$$\gamma^\wedge_{ij} = \tilde{\lambda} \circ \gamma_{ij}\,;$$

$\tilde{\lambda}$, in contrast with λ, is not necessarily the identity inclusion, not even necessarily an homomorphism, but

$$\tilde{\lambda}\colon \Gamma \equiv \mathrm{Pin}(n) \oplus \mathrm{Pin}(m) \to \Gamma^\wedge \equiv \mathrm{Pin}(n+m)$$

is such that the following diagram is commutative:

$$\begin{tikzcd} \Gamma \arrow[r, "\tilde{\lambda}"] \arrow[d, "\mathcal{H}"'] & \Gamma^\wedge \arrow[d, "\mathcal{H}^\wedge"] \\ G \arrow[r, "\lambda"] & G^\wedge \end{tikzcd} \tag{7a}$$

and where

$$\tilde{\lambda}(\Lambda_k \Lambda) = \tilde{\lambda}(\Lambda_k)\tilde{\lambda}(\Lambda) \tag{7b}$$

whenever Λ_k is in the kernel of Γ.

One can use the graded tensor product defined in [Problem I 3, Tensor] to construct a mapping $\tilde{\lambda}$, satisfying (7), as follows:

Let V_1 and V_2 be two real vector spaces, Q_1 and Q_2 two quadratic forms on V_1 and V_2, respectively; let i_Q be the canonical map from V into the Clifford algebra $C(V, Q)$. The mapping

$$f: V_1 \oplus V_2 \to C(V_1, Q_1) \hat{\oplus} C(V_2, Q_2)$$

onto the graded tensor product of the Clifford algebras $C(V_i, Q_i)$ by

$$f(x_1, x_2) = i_{Q_1}(x_1) \otimes \mathbb{1} + \mathbb{1} \otimes i_{Q_2}(x_2)$$

induces an isomorphism of the $\mathbb{Z}_2$-graded algrebras

$$C(V_1 \oplus V_2, Q_1 \oplus Q_2) \approx C(V_1, Q_1) \hat{\otimes} C(V_2, Q_2) .$$

deg Λ Let **deg** $\Lambda = 0, 1$ according to the parity of Λ [Problem I 3, Tensor]; and let the dot in $\deg \Lambda^1 \cdot \deg \Lambda^2$ denote numerical multiplication.

The mapping

$$\tilde{\lambda}: (\Lambda^1, \Lambda^2) \mapsto (-1)^{\deg \Lambda^1 \cdot \det \Lambda^2} \Lambda^1 \hat{\otimes} \Lambda^2 \tag{8}$$

satisfies conditions (7).

4b ii α) *Let* $\Lambda \equiv (\Lambda^1, \Lambda^2) \in \mathrm{Pin}(n) \oplus \mathrm{Pin}(m)$. *Show that*

$$\tilde{\lambda}(\Lambda\Lambda') = (-1)^{\deg \Lambda^1 \cdot \Lambda'^2} \tilde{\lambda}(\Lambda)\tilde{\lambda}(\Lambda') , \qquad \Lambda \equiv (\Lambda^1, \Lambda^2), \Lambda' \equiv (\Lambda'^1, \Lambda'^2) . \tag{9}$$

Show in general that if $\tilde{\lambda}$ *satisfies* (7), *it is a homomorphism from the kernel of* Γ *into the kernel of* Γ^λ.

Answer 4b ii α:

$$\begin{aligned}\tilde{\lambda}((\Lambda^1, \Lambda^2)(\Lambda'^1, \Lambda'^2)) &= \tilde{\lambda}(\Lambda^1\Lambda'^1, \Lambda^2\Lambda'^2) \\ &= (-1)^{(\deg \Lambda^1 + \deg \Lambda'^1)\cdot(\deg \Lambda^2 + \deg \Lambda'^2)} \Lambda^1\Lambda'^1 \hat{\otimes} \Lambda^2\Lambda'^2 \qquad (8)\end{aligned}$$

whereas

$$\tilde{\lambda}(\Lambda^1, \Lambda^2)\tilde{\lambda}(\Lambda'^1, \Lambda'^2) = (-1)^{\deg \Lambda^1 \cdot \deg \Lambda^2 + \deg \Lambda'^1 \cdot \deg \Lambda'^2} (\Lambda^1 \hat{\otimes} \Lambda^2)(\Lambda'^1 \tilde{\otimes} \Lambda'^2)$$

and by the definition of the graded tensor product of graded algebras

$$(\Lambda^1 \hat{\otimes} \Lambda^2)(\Lambda'^1 \hat{\otimes} \Lambda'^2) = (-1)^{\deg \Lambda^2 \deg \Lambda'^1} \Lambda^1\Lambda'^1 \hat{\otimes} \Lambda^2\Lambda'^2 .$$

Equation (9) follows readily from the last three equations. ∎

Let $\Lambda_k = (k^1, k^2)$ be in the kernel K of $\mathrm{Pin}(n) \oplus \mathrm{Pin}(m)$ with $k^1 = \pm e^1$ in the kernel of $\mathrm{Pin}(n)$ and $k^2 = \pm e^2$ in the kernel of $\mathrm{Pin}(m)$. It follows from (7a) that

$$\begin{aligned}(\mathscr{H}^\lambda \circ \tilde{\lambda})(k^1, k^2) &= (\lambda \circ \mathscr{H})(k^1, k^2)\\ &= \lambda(e^1 e^2) \text{ for } e^1, e^2 \text{ the unit elements of } \mathrm{O}(n),\ \mathrm{O}(m), \\ &\quad \text{respectively} \\ &= e \text{ the unit element of } \mathrm{O}(n+m)\,.\end{aligned}$$

Hence,

$$\tilde{\lambda}(k^1, k^2) = \pm e \text{ the unit element of } \mathrm{Pin}(n+m)$$

and

$$\tilde{\lambda}\colon \text{kernel of } \mathrm{Pin}(n) \oplus \mathrm{Pin}(m) \to \text{kernel of } \mathrm{Pin}(n+m)\,.$$

It follows from (7b) that $\tilde{\lambda}$ is a homomorphism from the kernel of Γ onto the kernel of Γ^λ. ■

4b ii β) *Let the non-homomorphism of $\tilde{\lambda}$ be diagnosed by the properties of*

$$\theta_{ijk}(x) \overset{\text{def}}{=} \tilde{\lambda}(\gamma_{ij}(x))\tilde{\lambda}(\gamma_{jk}(x))\tilde{\lambda}((\gamma_{ij}(x)\gamma_{jk}(x))^{-1})\,. \tag{10}$$

Show that $\{\theta_{ijk}(x)\}$, (i, j, k) in the nerve $\mathscr{N}$ of the cover $\mathscr{U}$ of M, defines a 2-cocycle θ on $\mathscr{N}$ with values in $\mathbb{Z}_2$.
Show that if $\tilde{\lambda}$ is given by (8) *then*

$$\theta(i, j, k) = (-1)^{\deg \gamma^1_{ij}(x) \cdot \deg \gamma^2_{jk}(x)}\,. \tag{11}$$

Let $p = (p^1, p^2)$ and p^λ be the obstruction cocycles of $\tilde{\xi}$ and $\tilde{\xi}^\lambda$ constructed in section 3; *show that*

$$p^\lambda_{ijk}(x) = \theta_{ijk}(x)\tilde{\lambda}(p_{ijk}(x))\,. \tag{12}$$

Answer 4b ii β: Following the argument developed in paragraphs 3a), 3b), and 4a-i), we check that

$$\begin{aligned}\mathscr{H}^\lambda(\theta_{ijk}(x)) &= e \in G^\lambda \\ \theta_{ijk}(x) &= \pm e \in \Gamma^\lambda.\end{aligned}$$

Hence, $\theta_{ijk}(x)$ defines a 2-cochain on $\mathscr{N}$ with values in $\mathbb{Z}_2$,

$$\theta(i, j, k)e \overset{\text{def}}{=} \theta_{ijk}(x)\,.$$

The 2-cochain θ is a 2-cocycle since

$$\begin{aligned}(\mathrm{d}\theta)(i, j, k, l) &= \theta(j, k, l)(\theta(i, k, l))^{-1}\theta(i, j, l)(\theta(i, j, k))^{-1} \\ &\qquad\qquad\qquad\qquad \text{by the definition of } d \\ &= 1 \qquad\qquad\qquad\quad \text{by the definition of } \theta.\end{aligned}$$

Inserting (8) into (10) yields (11).

Applying $\tilde{\lambda}$ to

$$\gamma_{ik} = p_{ijk}^{-1}\gamma_{ij}\gamma_{jk}$$

and using (7b) gives

$$\gamma_{ik}^{\lambda} = \tilde{\lambda}(p_{ijk})^{-1}\tilde{\lambda}(\gamma_{ij}\gamma_{ik}) . \tag{13}$$

On the other hand

$$\gamma_{ik}^{\lambda} = p_{ijk}^{\lambda-1}\tilde{\lambda}(\gamma_{ij})\tilde{\lambda}(\gamma_{jk}) . \tag{14}$$

Equation (12) follows from (13), (14) and the definition (10) of θ_{ijk}.
It follows from (12)

$$p^{\lambda}(i, j, k) = \theta(i, j, k)\tilde{\lambda}(p(i, j, k)) ,$$

where we use the symbol $\tilde{\lambda}$ also for maps of cocycles defined by elements $p_{ijk}(x) \in \Gamma$. ■

Inserting θ given by (11) and using the definition of paragraph 4b), we get

$$p^{\lambda}(i, j, k) = (-1)^{\deg \gamma_{ij}^{1}(x)\cdot \deg \gamma_{jk}^{2}(x)}\tilde{\lambda}(p^{1}(i, j, k)p^{2}(i, j, k)) . \tag{15}$$

Now, because the restriction of $\tilde{\lambda}$ to the kernel $\{e, -e\}$ of Γ is a homomorphism

$$\tilde{\lambda}(p^{1}(i, j, k)p^{2}(i, j, k)) = p^{1}(i, j, k)p^{2}(i, j, k) . \tag{16}$$

Inserting (16) into (15) and applying the isomorphism between the multiplicative group S^0 and the additive group $\mathbb{Z}_2$

$$\phi: \{1, -1\} \to \{0, 1\} ,$$

we get

$$\begin{aligned}
&\phi(p^{\lambda}(i, j, k)) \\
&\quad = \deg \gamma_{ij}^{1}(x) \deg \gamma_{jk}^{2}(x) + \phi(p^{1}(i, j, k)) + \phi(p^{2}(i, j, k)) \\
&\quad = \phi(\det g_{ij}^{1}(x))\phi(\det g_{jk}^{2}(x)) + \phi(p^{1}(i, j, k)) + \phi(p^{2}(i, j, k)) .
\end{aligned} \tag{17}$$

Hence,

$$[p^{\lambda}] = [g^{1}] \cup [g^{2}] + [p^{1}] + [p^{2}] . \tag{18}$$

and $[p]$ satisfies the Whitney product theorem. ■

iii) *The nontriviality of $[p]$ for the twisted line bundle over the circle can be checked by explicit construction.* The fact that $[p] \in H^2(M, \mathbb{Z}_2)$ satisfies the naturality condition $[p^1] = f^*[p^2]$ for $f: M_1 \to M_2$, the Whitney product theorem, and the non-triviality condition imply that

$$[p] = aw_2 + bw_1 \cup w_1 \qquad \text{where } a, b \in \mathbb{Z}_2 ,$$

and depend only on the group. We shall compute a and b for the Pin groups [See Problem I 7, Clifford], $\mathrm{Pin}^+(n)$ and $\mathrm{Pin}^-(n)$. It will be sufficient to compute $[p](\mathrm{Pin}^{\pm}(1))$ and $[p](\mathrm{Pin}^{\pm}(2))$. Indeed, if $n \geq 2$, there are natural maps: $(\mathrm{Pin}^{\pm}(n) \to \mathrm{Pin}^{\pm}(n+1)$ which cover the natural inclusion $\mathrm{O}(n) \to \mathrm{O}(n+1)$; the obstruction classes of the $(\mathrm{Pin}^{\pm}(n+1))$ bundles are the same as the obstruction classes of the $(\mathrm{Pin}^{\pm}(n))$ bundles. Now, according to Axiom 1, $w_2 = 0$ for a line bundle, hence

$$[p](\mathrm{Pin}^{\pm}(1)) = bw_1 \cup w_1$$

and we determine b for $[p](\mathrm{Pin}^+(1))$ and for $[p](\mathrm{Pin}^-(1))$ as follows:

$$\mathrm{Pin}^+(1) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

and

$$\mathrm{Pin}^-(1) \simeq \mathbb{Z}_4 .$$

The short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Pin}^{\pm}(1) \to 0(1) \to 1$$

induces a long exact sequence

$$0 \to H^1(M, \mathbb{Z}_2) \to H^1(M, \mathrm{Pin}^{\pm}(1)) \to H^1(M, \mathbb{Z}_2) \to H^2(M, \mathbb{Z}_2) \to \cdots .$$

The long exact sequence splits for $\mathrm{Pin}^+(1) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, namely

$$0 \to H^1(M, \mathbb{Z}_2) \to H^1(M, \mathbb{Z}_2 \times \mathbb{Z}_2) \to H^1(M, \mathbb{Z}_2) \to 0 \to H^2(M, \mathbb{Z}_2) \to \cdots .$$

Thus, $[p](\mathrm{Pin}^+(1)) = 0$.

The long exact sequence does not split for $\mathrm{Pin}^-(1)$ and the Bockstein map B: $H^1(M, \mathbb{Z}_2) \to H^2(M, \mathbb{Z}_2)$ maps

$$w_1 \mapsto w_1 \cup w_1 .$$

Thus $[p](\mathrm{Pin}^-(1)) = w_1 \cup w_1$.

It can be proved that one can construct* a $\mathrm{Pin}^-(2)$ bundle but not a $\mathrm{Pin}^+(2)$ bundle over the real projective plane $\mathbb{R}P(2)$ for which $w_1 \cup w_1 = 1$ and $w_2 = 1$ [see, for instance, Milnor and Stasheff]; thus it follows that

$$[p](\mathrm{Pin}^+(2)) = [p](\mathrm{Pin}^+(n)) = w_2 ,$$

$$[p](\mathrm{Pin}^-(2)) = [p](\mathrm{Pin}^-(n)) = w_2 + w_1 \cup w_1 .$$

We refer the reader to [Karoubi] for the construction of the obstruction class of $\mathrm{Pin}(n, m)$ when both n and m are different from zero. For a

*Use the well-known triangulation of $\mathbb{R}P(2)$ – see for instance [Patterson p. 100] – and construct $[p]$ explicitly.

physical consequence of the fact that the obstruction classes of the $\mathrm{Pin}^+(n)$ bundles and the $\mathrm{Pin}^-(n)$ bundles are different, we refer the reader to [Carlip, DeWitt-Morette].

Acknowledgements and References may be found at the end of Problem IV 3. Sections 3 and 4 use results and arguments from [Greub and Petry]. However, Greub and Petry assume $\mathrm{Pin}(1) = \mathrm{O}(1) \times \mathrm{O}(1)$; hence, they investigate only the obstruction for $\mathrm{Pin}^+(n)$. We are indebted to S. Carlip for this remark.

3. INEQUIVALENT SPIN STRUCTURES

INTRODUCTION

We have given in Problem IV 2 a necessary and sufficient condition – the vanishing of the second Stiefel–Whitney class – for the existence of a $\mathrm{Spin}(n)$ bundle $\tilde{\xi}$, extension of a given $\mathrm{SO}(n)$ bundle ξ over a manifold M. This extension – not necessarily unique – has been obtained by lifting the transition functions $g_{ij}: U_i \cap U_j \to \mathrm{SO}(n)$ of ξ to transition functions $\gamma_{ij}: U_i \cap U_j \to \mathrm{Spin}(n)$.
The elements $\tilde{p}$ of $\tilde{\xi}$ are then the equivalence classes

$$\tilde{p} = [i, x, \Lambda_i]\,, \qquad x \in U_i\,, \quad \Lambda_i \in \mathrm{Spin}(n)$$

with the equivalence relation

$$(i, x, \Lambda_i) \simeq (j, x, \Lambda_j) \qquad \text{if } x \in U_i \cap U_j\,, \quad \Lambda_j = \gamma_{ji}(x)\Lambda_i\,,$$

while the elements of ξ were

$$\begin{array}{lll} p = [i, x, L_i]\,, & & L_i \in \mathrm{SO}(n) \\ (i, x, L_i) \simeq (j, x, L_j) & \text{if } x \in U_i \cap U_j\,, & L_j = g_{ji}(x)L_i\,. \end{array}$$

There is a mapping $\tilde{\mathscr{H}}: \tilde{\xi} \to \xi$ defined by

$$\tilde{\mathscr{H}}(\tilde{p}) = p\,, \qquad L_i = \mathscr{H}(\Lambda_i)\,;$$

the definition does not depend on the chosen representative of $\tilde{p}$ if the γ_{ij} are lifts of the g_{ij}, i.e., such that $\mathscr{H}(\gamma_{ij}(x)) = g_{ij}(x)$, $x \in U_i \cap U_j$. The mapping $\tilde{\mathscr{H}}$ is a bundle morphism, i.e., commutes with the right actions of the relevant groups:

$$\tilde{\mathscr{H}}(\tilde{p}\Lambda) = \tilde{\mathscr{H}}(\tilde{p})\mathscr{H}(\Lambda)\,.$$

More generally a pair $(\tilde{\xi}, \tilde{\mathscr{H}})$ with $\tilde{\xi}$ a Spin(n) bundle over a manifold M and $\tilde{\mathscr{H}}$ a bundle morphism from $\tilde{\xi}$ onto an SO(n) bundle ξ over M is called a **spin structure** over ξ. See for instance [Milnor] for alternative definitions.

spin structure

Two spin structures $(\tilde{\xi}_\alpha, \tilde{\mathscr{H}}_\alpha)$ and $(\tilde{\xi}_\beta, \tilde{\mathscr{H}}_\beta)$ on ξ are said to be **equivalent** if there exists an isomorphism $f\colon \tilde{\xi}_\beta \to \tilde{\xi}_\alpha$ such that $\tilde{\mathscr{H}}_\alpha \circ f = \tilde{\mathscr{H}}_\beta$.

equivalent spin structures

1) *An example of inequivalent spin structures on a trivial* SO(n) *bundle.*
a) *Show that any* Spin(n) *bundle on a solid torus T diffeomorphic to* $\mathbb{R}^N \times S^1$ *is trivial for* $n > 2$.

Answer 1a: To show that a Spin(n) bundle over T is trivial is equivalent to showing that it is trivial over S^1. A principal bundle is trivial if and only if it admits a global (continuous) cross section (p. 133). It is possible to construct a global cross section on a Spin($n > 2$) bundle over S^1 because

$$\begin{cases} S^1 & \text{is a 1-complex and} \\ \mathrm{Spin}(n>2) & \text{is a path connected group.} \end{cases}$$

Indeed, the 1-complex S^1 consists of a 0-skeleton (the 0-simplices) and a 1-skeleton (the 1-simplices). The spin-bundle restricted to the 0-skeleton admits cross sections. Let us choose one of them. It is possible to extend this cross section to a cross section over the 1-skeleton if and only if there is a path between any 2 points of Spin(n). ■

More generally one can show [Problem V bis 8, Universal Bundle, paragraph 3] that a G-bundle over X is trivial if X is an n-complex and G is $(n-1)$ **connected**, that is if $\Pi_k(G)$ is trivial for $k \leq n-1$.

n-connected group

Remark: A trivialization of a Spin(n) bundle $\tilde{\xi}$ determines a trivialization of the SO(n) bundle ξ. Indeed

$$\xi \simeq (\tilde{\xi} \times \mathrm{SO}(n))/(\text{equivalence defined by } (u\Lambda, a) \sim (u, \mathscr{H}(\Lambda)a)\,. \tag{1}$$

$u \in$ Spin(n) bundle, $\Lambda \in \mathrm{Spin}(n)$, $a \in \mathrm{SO}(n)$, $\mathscr{H}\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$.
Since $\tilde{\xi} \simeq S^1 \times \mathrm{Spin}(n)$, the isomorphism (1) provides a trivialization of ξ.

b) *Show that it is possible to construct two inequivalent* Spin(n), $n > 2$, *structures on a solid torus T diffeomorphic to* $\mathbb{R}^N \times S^1$.

Answer 1b: We give here a rough, heuristic construction; the correct one requires an atlas over S^1 and is done in paragraph c).

Let $s = [0, 1]$,

$$\tilde{F}_\alpha : S_1 \times \text{Spin}(n) \to S_1 \times \text{SO}(n)$$
$$\text{by } (\exp 2\pi i s, \Lambda) \mapsto (\exp 2\pi i s, \mathcal{H}(h_\alpha(\Lambda)\Lambda))$$

and a similar definition of $\tilde{F}_\beta$, with

$$\begin{aligned} h_\alpha &: [0, 1] \to \text{Spin}(n) \quad \text{such that} \quad h_\alpha(0) = e,\ h_\alpha(1) = e \\ h_\beta &: [0, 1] \to \text{Spin}(n) \quad \text{such that} \quad h_\beta(0) = e,\ h_\beta(1) = -e\,. \end{aligned}$$

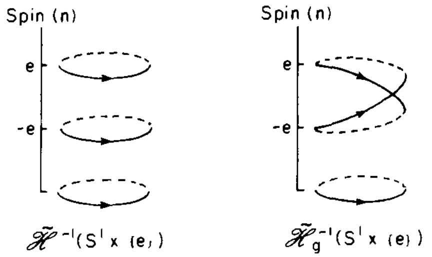
We shall show that there is no isomorphism f such that $\tilde{F}_\beta \circ f = \tilde{F}_\alpha$. Indeed the preimages by $\tilde{F}_\alpha$ and $\tilde{F}_\beta$ of the set $S^1 \times \{e\} \subset S^1 \times \text{SO}(n)$ are, respectively

$$\tilde{F}_\alpha^{-1}(S^1 \times \{e\}) = (S^1 \times \{e\}) \cup (S^1 \times \{-e\}) \subset S^1 \times \text{Spin}(n)$$

and

$$\begin{aligned} \tilde{F}_\beta^{-1}(S^1 \times \{e\}) = \{&\exp(2\pi i s), (h_\beta(s))^{-1}\} \\ \cup \{&\exp(2\pi i s), -(h_\beta(s))^{-1}\}\,, \qquad s \in [0, 1]\,. \end{aligned}$$

The first preimage is disconnected whereas the second one is connected, hence there is no isomorphism f such that $\tilde{F}_\beta \circ f = \tilde{F}_\alpha$. A correct construction using an atlas to cover S^1 is not expected to change this result and we expect to be able to construct at least two inequivalent spin structures on a solid torus.



Two different spin structures correspond to two different prescriptions for patching the pieces of the double covering.

Remark: We establish for use later on that

$$\tilde{F}(\tilde{p}\Lambda) = \tilde{F}(\tilde{p})\mathcal{H}(\Lambda)\,, \qquad \tilde{F} = \tilde{F}_\alpha \text{ or } \tilde{F}_\beta\,. \tag{2}$$

Indeed let $\tilde{p} = (\exp 2\pi i s, \Lambda)1)$, then

$$\begin{aligned} \tilde{F}(\tilde{p})\mathcal{H}(\Lambda) &= (\exp 2\pi i s, \mathcal{H}(h(s)\Lambda_1))\mathcal{H}(\Lambda) \\ &= \tilde{F}(\tilde{p}\Lambda)\,. \end{aligned}$$

c) *Construct inequivalent spin structures* $(\tilde{\xi}, \mathscr{H}_\alpha)$ *on a solid torus* T.

Answer 1c: Let $\mathscr{U} = \{U_1, U_2, U_2\}$ be a simple cover of S^1. Let $(U_i, \tilde{\phi}_i)$ [resp. (U_i, ϕ_i)] define local trivializations of a Spin(n) bundle $\tilde{\xi}$[resp. SO(n) bundle] over S^1 such that

$$\overset{\Delta}{\tilde{\phi}}_i \circ \overset{\Delta}{\tilde{\phi}}{}_j^{-1}|_x = \gamma_{ij}(x) , \qquad \overset{\Delta}{\phi}_i \circ \overset{\Delta}{\phi}{}_j^{-1}|_x = g_{ij}(x) \tag{3}$$

$$\mathscr{H}(\gamma_{ij}(x)) = g_{ij}(x) . \tag{4}$$

Denote by h a family $\{h_i\}$ such that

$$h_i \colon U_i \to \text{Spin}(n) ,$$
$$h_j(x)\Lambda_j = \pm\gamma_{ji}(x)h_i(x)\Lambda_i , \qquad x \in U_{ij} \equiv U_i \cap U_j . \tag{5}$$

Define

$$\tilde{\mathscr{H}}_\alpha \colon \tilde{\xi} \to \xi$$

by

$$(\tilde{\mathscr{H}}_\alpha \circ \tilde{\phi}_i^{-1})(x, \Lambda_i) = \phi_i^{-1}(x, \mathscr{H}(h_i^\alpha(x)\Lambda_i)) , \qquad x \in U_i, \tag{6}$$

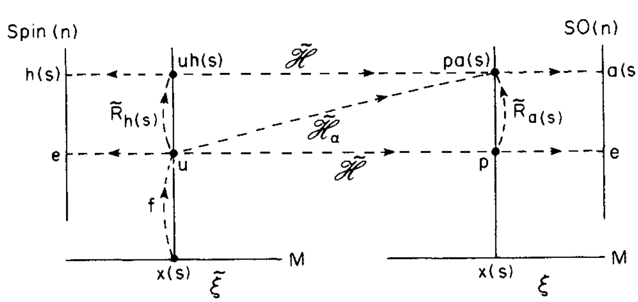
where $\{h_i^\alpha\}$ satisfies (5) with definite signs. Eq. (6) makes sense on the bundle $\tilde{\xi}$ whose elements are the equivalence classes $\tilde{p} = [i, x, \Lambda_i]$, $x \in U_i$, because

$$(\tilde{\mathscr{H}}_\alpha \circ \tilde{\phi}_i^{-1})(x, \Lambda_i) = (\tilde{\mathscr{H}}_\alpha \circ \tilde{\phi}_j^{-1})(x, \Lambda_j) , \qquad \Lambda_j = \gamma_{ji}(x)\Lambda_i .$$

Indeed

$$\begin{aligned}\phi_j^{-1}(x, \mathscr{H}(h_j^\alpha(x)\Lambda_j)) &= \phi_i^{-1} \circ \phi_i \circ \phi_j^{-1}(x, \mathscr{H}(\pm\gamma_{ji}(x)h_i^\alpha(x)\Lambda_i)) \\ &\qquad\qquad \text{by (5)} \\ &= \phi_i^{-1}(x, \mathscr{H}(\pm h_i^\alpha(x)\Lambda_i)) \qquad \text{by (3) and (4).}\end{aligned}$$

■



The pair $(\tilde{\xi}, \tilde{\mathscr{H}}_\alpha)$ is a spin structure.

2) *Show that the number of inequivalent spin structures on an* SO(n) *bundle* ξ *is equal to the number of elements in the first* $\mathbb{Z}_2$*-cohomology group* $H^1(\xi, \mathbb{Z}_2)$.

Answer 2: Let

$$\begin{aligned}\delta_{ij}^{\alpha\beta} &= \gamma_{ij}(x) h_j^\alpha(x) \Lambda_j (h_i^\alpha(x)\Lambda_i)^{-1} (\gamma_{ij}(x) h_j^\beta(x) \Lambda_j (h_i^\beta(x)\Lambda_i)^{-1})^{-1} \\ &\equiv \delta^{\alpha\beta}(i, j) e\,. \end{aligned} \tag{7}$$

The 1-cochain $\delta^{\alpha\beta}$ on the nerve $\mathscr{N}$ of S^1 defined by $\mathscr{U}$ with values in $\mathbb{Z}_2 = \{1, -1\}$ is a cocycle. Indeed

$$\begin{aligned} d\delta^{\alpha\beta}(i, j, k) &= \delta^{\alpha\beta}(j, k)(\delta^{\alpha\beta}(i, k))^{-1}\delta^{\alpha\beta}(i, j) \\ & \qquad\qquad \text{by the definition of } d \\ &= 1 \qquad\qquad \text{by (7) and property of } \{\gamma_{ij}\}. \end{aligned}$$

The 1-cocycle $\delta^{\alpha\beta}$ defines a cohomology class $\delta(\alpha, \beta) \in H^1(\mathscr{N}, \mathbb{Z}_2)$. We shall show that, given a spin structure $(\tilde{\xi}, \tilde{\mathscr{H}}_\alpha)$ and $w \in H^1(\mathscr{N}, \mathbb{Z}_2)$, there is one and only one spin structure $(\tilde{\xi}, \tilde{\mathscr{H}}_\beta)$ such that $\delta(\alpha, \beta) = w$.
To prove uniqueness we shall prove in paragraph i) $\delta(\alpha, \gamma) = \delta(\alpha, \beta) \;\Leftrightarrow\; \tilde{\mathscr{H}}_\beta \simeq \tilde{\mathscr{H}}_\gamma$ or equivalently

$$\delta(\gamma, \beta) = 0 \quad \Rightarrow \tilde{\mathscr{H}}_\gamma \simeq \tilde{\mathscr{H}}_\beta \tag{8}$$

$$\tilde{\mathscr{H}}_\gamma \simeq \tilde{\mathscr{H}}_\beta \quad \Rightarrow \delta(\gamma, \beta) = 0\,. \tag{9}$$

To prove existence we shall prove in paragraph ii) that given $\tilde{\mathscr{H}}_\alpha$ and $w \in H^1(\mathscr{N}, \mathbb{Z}_2)$, there exists $\tilde{\mathscr{H}}_\beta$ such that $\delta(\alpha, \beta) = w$.
i) Uniqueness. To prove (8) we note that if $\delta(\alpha, \beta) = 0$, then either

$$\delta^{\alpha\beta}(i, j) \equiv 1\,, \qquad \text{for all pairs } (i, j); \tag{10}$$

or there is a 0-cochain λ on $\mathscr{N}$ with values in $\mathbb{Z}_2$ such that

$$\left.\begin{aligned} \delta^{\alpha\beta}(i, j)(d\lambda(i, j))^{-1} = 1 \qquad & \text{for all pairs } (i, j), \text{ i.e.,} \\ \delta^{\alpha\beta}(i, j) = \lambda(j)(\lambda(i))^{-1} \qquad & \text{for all 1-simplices } (i, j). \end{aligned}\right. \tag{11}$$

We note that (10) is a particular case of (11) when $\lambda(i) = 1$ for all i. We shall show that if $\delta^{\alpha\beta}(i, j) = \lambda(j)(\lambda(i))^{-1}$ for all 1-simplices then $\tilde{\mathscr{H}}_\alpha \simeq \tilde{\mathscr{H}}_\beta$. Indeed, inserting (11) into (7) we get

$$\gamma_{ij}(x)\lambda(j) h_j^\alpha(x) \Lambda_j (\lambda(i) h_j^\alpha(x) \Lambda_i)^{-1} = \gamma_{ij}(x) h_j^\beta(x) \Lambda_j (h_i^\beta(x) \Lambda_i)^{-1}\,. \tag{12}$$

Hence

$$h_i^\beta(x) = \lambda(i) h_i^\alpha(x) \qquad \text{for all } i \tag{13}$$

and, by (6)

$$(\tilde{\mathscr{H}}_\beta \circ \tilde{\phi}_i^{-1})(x, \Lambda_i) = \phi_i^{-1}(x, \mathscr{H}(\lambda(i) h_i^\alpha(x)\Lambda_i)) \tag{14}$$

$$\equiv \phi_i^{-1}(x, (\mathscr{H} \circ \tilde{\lambda})(h^\alpha(x)\Lambda_i)) \tag{15}$$

$$\equiv (\tilde{\mathscr{H}}_\alpha \circ \tilde{\lambda} \circ \phi_i^{-1})(x, \Lambda_i) \tag{16}$$

with obvious definitions for $\mathscr{H} \circ \tilde{\lambda}$ and $\tilde{\mathscr{H}}_\alpha \circ \tilde{\lambda}$.
Thus $\tilde{\mathscr{H}}_\beta \simeq \tilde{\mathscr{H}}_\alpha$. ■

Now assume $\tilde{\mathscr{H}}_\beta = \tilde{\mathscr{H}}_\alpha \circ \tilde{\lambda}$, reverse the sequence of eqs. (13)–(16); inserting (13) into (7) gives $\delta^{\alpha\beta}(i, j) = \lambda(j)(\lambda(i))^{-1}$.
ii) Existence. Let a be a 1-cocycle in $\mathcal{N}$ with values in $\mathbb{Z}_2$ such that

$$\gamma_{ij}(x) h_j^\beta(x) \Lambda_j (h_i^\beta(x)\Lambda_i)^{-1} = a(i, j) \gamma_{ij}(x) h_j^\alpha(x) \Lambda_j (h_i^\alpha(x)\Lambda_i)^{-1} .$$

It is straightforward to check that $\{h_i^\beta(x)\}$ defines a spin structure $(\tilde{\xi}, \tilde{\mathscr{H}}_\beta)$, namely one checks that $\tilde{\mathscr{H}}_\beta$ satisfies (6) with $\tilde{\phi}$ and ϕ satisfying (3) and (4).

3) *Show that there are only two inequivalent spin structures on an* SO(n) *bundle over a solid torus T.*

Answer 3: We shall count the elements in $H^1(S^1, \mathbb{Z}_2)$. Consider the simple cover of S^1 consisting of U_1, U_2, U_3 such that the only nonempty intersections are $U_{12} \equiv 12$, $U_{23} \equiv 23$, $U_{31} \equiv 31$. The representatives of the elements of $H^1(S^1, \mathbb{Z}_2)$ are the maps

$$f_1: \begin{cases} 12 \\ 23 \\ 31 \end{cases} \to \begin{cases} 1 \\ 1, \\ 1 \end{cases} \quad f_2: \begin{cases} 12 \\ 23 \\ 31 \end{cases} \to \begin{cases} -1 \\ 1, \\ 1 \end{cases} \quad f_3: \begin{cases} 12 \\ 23 \\ 31 \end{cases} \to \begin{cases} 1 \\ -1, \\ 1 \end{cases}$$

$$f_4: \begin{cases} 12 \\ 23 \\ 31 \end{cases} \to \begin{cases} 1 \\ 1, \\ -1 \end{cases} \quad f_5: \begin{cases} 12 \\ 23 \\ 31 \end{cases} \to \begin{cases} -1 \\ -1, \\ 1 \end{cases} \quad f_6: \begin{cases} 12 \\ 23 \\ 31 \end{cases} \to \begin{cases} -1 \\ 1, \\ -1 \end{cases}$$

$$f_7: \begin{cases} 12 \\ 23 \\ 31 \end{cases} \to \begin{cases} 1 \\ -1, \\ -1 \end{cases} \quad f_8: \begin{cases} 12 \\ 23 \\ 31 \end{cases} \to \begin{cases} -1 \\ -1 \\ -1. \end{cases}$$

The maps f_1, f_5, f_6, f_7 belong to the same cohomology class, namely 0 and the maps f_2, f_3, f_4 and f_8 belong to the same cohomology class, namely 1. Let us check for instance that $[f_1] = [f_7]$ but $[f_1] \neq [f_8]$.

$$\text{Set } \lambda\colon \begin{cases} 1 \\ 2 \to \\ 3 \end{cases} \begin{cases} 1 \\ 1 \\ -1 \end{cases}, \qquad \text{then } f_7(d\lambda)^{-1}\colon \begin{cases} 12 \\ 23 \to \\ 21 \end{cases} \begin{cases} 1 \\ 1 \\ 1 \end{cases}.$$

On the other hand there is no λ such that

$$d\lambda\colon \begin{cases} 12 \\ 23 \to \\ 31 \end{cases} \begin{cases} -1 \\ -1 \\ -1 \end{cases}. \qquad \blacksquare$$

Remark: One can check that if $f(i, j) = \lambda(i)(\lambda(j))^{-1}$, then $[f] = 0$.

Since there are 2 elements in $H^1(S^1, \mathbb{Z}_2)$ there are 2 inequivalent spin structures on an $SO(n)$ bundle over T.

4) *Construct inequivalent spin structures on nontrivial bundles.*

Answer 4: Having rephrased in paragraph 2a) iii) the construction of inequivalent spin structures given in paragraph 2a) ii), we can generalize readily the construction of inequivalent spin structures when the Spin(n) bundle is trivial to the case when it is not.

5) *Construct inequivalent connections on the* Spin(n) *bundle* $\tilde{\xi}$ *over* T *corresponding to a given connection* ω *on* ξ.

Answer 5: Different spin structures determine different connections on the Spin bundle (p. 419). Indeed let $(\tilde{\xi}, \tilde{\mathcal{H}}_\alpha)$ be a spin structure over ξ and let $\boldsymbol{\omega}$: $T\xi \to \mathscr{L}(\mathrm{SO}(n))$ be a connection 1-form on ξ. The corresponding connection on $\tilde{\xi}$ is (p. 419)

$$\boldsymbol{\sigma}_\alpha = \mathcal{H}'^{-1}(e) \circ \tilde{\mathcal{H}}^*_\alpha \boldsymbol{\omega}\,,$$

where

$$\tilde{\mathcal{H}}^*_\alpha\colon T^*\xi \to T^*\tilde{\xi}\,.$$

$\tilde{\mathcal{H}}_\alpha \boldsymbol{\omega}$ is the 1-form on $\tilde{\xi}$ with values in $\mathscr{L}(\mathrm{SO}(n))$ pull back of $\boldsymbol{\omega}$. Under the map $\mathcal{H}'^{-1}(e)$ it becomes a 1-form σ_α on $\tilde{\xi}$ with values in $\mathscr{L}(\mathrm{Spin}(n))$.

For the computation of the pull back $\bar{\sigma}_\alpha$ of $\underset{\sim}{\sigma_\alpha}$ by local sections f_i on U_i canonically defined by the local trivialization ϕ_i we refer the reader to p. 419.

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helped grasp *la substantifique moelle* of several *theorèmes fins* and make it understandable to non-specialists.

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C.W. complexes

Often used in these references, C.W. complexes have not been used here. **C.W. complexes** are built in stages, each stage being obtained from the preceding one by adjoining cells of a given dimension. Considered as a C.W. complex a polyhedron frequently requires fewer cells than a simplicial triangulation [e.g., Spanier, p. 400].

4. COHOMOLOGY OF GROUPS

1. DEFINITIONS AND EXERCISES

cochain

Suppose that a group G acts on a group M by a left action $\sigma: G \times M \to M$ denoted by $\sigma(g, m) \equiv \sigma_g(m) \equiv gm$. An ***n*-dimensional cochain** is a mapping

$$c: G \times G \times \cdots \times G(n+1)\text{factors} \to M$$

such that

$$c(gg_0, gg_1, \ldots, gg_n) = gc(g_0, g_1, \ldots, g_n) \quad \text{for all } g, g_i \in G\,. \quad (1)$$

a) *Show that the space $C(G, M)$ of cochains $c: C^{n+1} \to M$ forms a group if σ_g commutes with the product in M. Show in particular that σ_g is not a free action.*

Answer 1a: Let $c_1, c_2 \in C(G, M)$ and let c_1c_2 be the product defined pointwise. The product c_1c_2 is also in $C(G, M)$ if it satisfies (1). We have

$$c_1(gg_0, \ldots, gg_n)c_2(gg_0, \ldots, gg_n) = \sigma_g(c_1(g_0, \ldots, g_n)) \times \sigma_g(c_2(g_0, \ldots, g_n))\,.$$

Thus if

$$\sigma_g(m_1)\sigma_g(m_2) = \sigma_g(m_1m_2)\,, \quad (2)$$

then

$$(c_1c_2)(gg_0, \ldots, gg_n) = \sigma_g(c_1c_2(g_0, \ldots, g_n)).$$

It follows from (2) that for all $g \in G$

$$\sigma_g(m)\sigma_g(1) = \sigma_g(m)\,,$$

i.e.,

$$\sigma_g(1) = 1 \quad \text{for all } g \in G\,.$$

σ_g leaves the unit element of M invariant; it is not a free action.

b) The **coboundary operator** d

coboundary operator

$$d\colon C^n(G, M) \to C^{n+1}(G, M)$$

is defined by

$$\begin{aligned} dc(g_0, \ldots, g_{n+1}) = c(g_1, \ldots, g_{n+1}) \\ &\times (c(g_0, g_2, \ldots, g_{n+1}))^{-1} \ldots \\ &\times (c(g_0, \ldots, \hat{g}_k, \ldots, g_{n+1}))^{(-1)^k} \ldots \\ &\times (c(g_0, \ldots, g_n))^{(-1)^{n+1}} \end{aligned} \tag{3}$$

Check that $ddc \equiv 1$.

A cochain c is called a **coboundary** of the cochain b if $c = db$. A cochain c is called a **cocycle** if $dc = 1$.

coboundary cocycle

Show that if M *is an abelian group the n-dimensional cocycles form a group* $Z^n(G, M)$; *show that the coboundaries of* $(n-1)$*-dimensional cochains form a normal subgroup* $B^n(G, M)$ *of* $Z^n(G, M)$.

Answer 1b: Let c_1 and c_2 be two cocycles; then

$$\begin{aligned} d(c_1c_2)(g_0, \ldots, g_{n+1}) &= c_1(g_1, \ldots, g_{n+1})c_2(g_1, \ldots, g_{n+1}) \\ &\times (c_1(g_0, g_2, \ldots, g_{n+1})c_2(g_0, g_2, \ldots, g_{n+1}))^{-1} \ldots\,. \end{aligned}$$

The given properties follow from the fact that if M is abelian this equality can be written

$$d(c_1c_2) = (dc_1)(dc_2)\,. \qquad \blacksquare$$

The ***n*-cohomology group of G with values in the abelian group M** is the factor group

cohomology

$$H^n(G, M) = Z^n(G, M)/B^n(G, M)\,.$$

c) *Let c be an n-cochain, and define $\tilde{c}$: $G \times \cdots \times G(n$ times$) \rightarrow M$ by*

$$\tilde{c}(h_1, \ldots, h_n) = c(e, h_1, h_1h_2, \ldots, h_1h_2 \ldots h_n). \tag{4}$$

Conversely show that given $\tilde{c}$ relation (4) *and the cochain condition* (1) *determine a cochain c. Show that when M is abelian a 2-cochain $\tilde{c}$ is a cocycle if and only if $\tilde{c}$ satisfies*

$$\tilde{c}(h_1h_2) = \tilde{c}(h_1)(h_1\tilde{c}(h_2)). \tag{5}$$

Answer 1c: From (4) and (1) we deduce

$$c(g_0, \ldots, g_n) = g_0\tilde{c}(g_0^{-1}g_1, g_1^{-1}g_2, \ldots, g_{n-1}^{-1}g_n). \tag{6}$$

Let c be a 1-cochain. Then the cocycle condition $dc = 1$, written in terms of $\tilde{c}$ gives

$$\begin{aligned} 1 = dc(e, h_1, h_1h_2) &= c(h_1, h_1h_2)(c(e, h_1h_2))^{-1}c(e, h_1) \\ &= h_1\tilde{c}(h_2)(\tilde{c}(h_1h_2))^{-1}\tilde{c}(h_1) \equiv \widetilde{dc}(h_1, h_2). \end{aligned}$$

The last equality defines $\widetilde{dc}$ and the cocycle condition $\widetilde{dc} = 1$ gives equation (5) when M is abelian. ∎

homogeneous

The cochains c are **homogeneous** and the $\tilde{c}$, also called cochains, are not. We give in the next section an example of cochains easier to state in terms of $\tilde{c}$ than in terms of c.

2) Cocycle condition for mappings f: $G \times X \rightarrow K$ where the group G acts on a space X by left actions and f is a mapping on $G \times X$ with values in an abelian group K. For an example see the following problem (IV 5 Lifting).

a) *Use the definition of cochain* (2) *and coboundary operators* (5) *to define a cocycle condition for mappings f: $G \times X \rightarrow K$.*

Answer 2a: Let M be the space of mappings from X to K. The mappings

$$f\colon G \times X \rightarrow K$$

can be considered as mappings

$$F\colon G \rightarrow M,$$

by setting

$$(F(g))(x) \equiv F(g)(x) = f(g, x) \in K. \tag{7}$$

The group operation in K induces a group structure on M by

$$(u_1u_2)(x) = u_1(x)u_2(x), \qquad u_1, u_2 \in M, x \in X$$

The action of G on X, denoted $x \mapsto gx$, defines an action l_g of G on M by

$$(l_g(u))(x) := u(gx), \qquad u \in M; u(x), u(gx) \in K. \tag{8}$$

By analogy with the construction in paragraph 1c), the mapping $F: G \to M$ together with the action l_g will be called a cochain and it will be called a cocycle, by analogy with (5), if

$$F(g_1 g_2) = F(g_1)(l_{g_1} F(g_2)),$$

i.e.,

$$f(g_1 g_2, x) = f(g_1, x) f(g_2, g_1 x). \tag{9}$$

A mapping f satisfying eq. (9) is said to satisfy the **cocycle condition.** cocycle condition

References for Problems IV 4 and 5 are listed at the end of Problem IV 5.

5. LIFTING A GROUP ACTION

Given a K-principal bundle P with base space X and an action of G on X, one may wish to "lift" the G action on the base to a fibre preserving G action on the bundle. The questions arise as to the existence and uniqueness of such lifts. For example, let the base X of P be a Lorentz bundle over a manifold M. Let the typical fibre K of P be $\mathbb{Z}_2$. We shall set up the problem so that the existence of a spin bundle over M is stated in terms of the existence of appropriate lifts $\tilde{R}_k$ of the right action $\tilde{R}_g$ of the Lorentz group $L(n)$ on the Lorentz bundle X.
A right action

$$\tilde{R}_k: P \to P,$$

i.e. such that

$$\tilde{R}_{k_1} \circ \tilde{R}_{k_2} = \tilde{R}_{k_2 k_1}, \tag{1}$$

is said to be a lift of the right action

$$\tilde{R}_g: X \to X$$

if

$$\pi \circ \tilde{R}_k = \tilde{R}_g \circ \pi. \tag{2}$$

Equivalently $\tilde{R}_k$ is a lift of $\tilde{R}_g$ if the following diagram is commutative:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{R}_k} & P\,. \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{\tilde{R}_g} & X\,. \end{array}$$

In the case where X is itself a principal fibre bundle with base manifold M, and $\tilde{R}_g$ is a fibre preserving right action on X, for instance the right action of its structure group G, we say that the lifting to P is appropriate if the projection on X of $\tilde{R}_k(p)$ and p lie in the same fibre of X.
Consider a local trivialization of P over a fibered open set of X with $k = \mathbb{Z}_2 = \{1, -1\}$:

$$\Phi: U \to V \times \mathbb{Z}_2, \qquad V = \pi_M^{-1}(v) \subset X, \qquad v \subset M.$$

Show that the most general appropriate lift $\tilde{R}_k$ of $\tilde{R}_g$ is such that

$$\Phi(\tilde{R}_k(p)) = (\tilde{R}_g(x), f(g, x)\overset{\Delta}{\Phi}(p)), \qquad p \in P, \pi(p) = x, \qquad (3)$$

where

$$f: G \times X \to \mathbb{Z}_2$$

satisfies the cocycle condition (Problem IV 4, Cohomology eq. (9) for the cocyle condition of a left action)

$$f(g_1 g_2, x) = f(g_2, g_1 x) f(g_1, x)\,. \qquad (4)$$

Answer: In a local trivialization of P we have

$$\Phi(\tilde{R}_k(p)) = (y, q), \qquad y = \pi(\tilde{R}_k(p) \in X, \qquad q \in \mathbb{Z}_2$$

and moreover if the lifting is appropriate

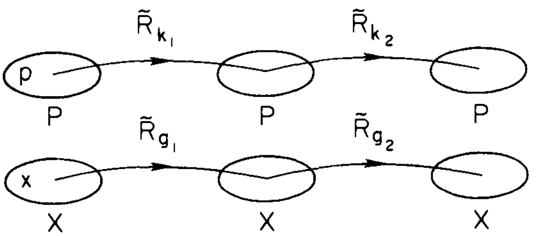
$$\pi_M(y) = \pi_M(x), \qquad \text{with } x = \pi(p)\,.$$

Hence there exists $g \in G$ such that $y = R_g(x)$.
We define $f(x, g) \in \mathbb{Z}_2$ by

$$f(x, g) = q(\Phi(p))^{-1} = \overset{\Delta}{\Phi}(\tilde{R}_k(p))\Phi^{-1}(p)\,,$$

and obtain the formula (3).
We shall now look at the conditions to impose to f in order to satisfy (1).

Since $k \in \mathbb{Z}_2$, it is not necessary to distinguish right and left action. We shall nevertheless write the factors in the "right" order.
Let $\tilde{R}_{g_2} \circ \tilde{R}_{g_1} = \tilde{R}_{g_3}$, i.e., $g_3 = g_1 g_2$.



$$f(g_3, x) = f(g_1 g_2, x) = \overset{\Delta}{\Phi}(\tilde{R}_{k_2}(p))\overset{\Delta}{\Phi}{}^{-1}(\tilde{R}_{k_1}(p))\overset{\Delta}{\Phi}(\tilde{R}_{k_1}(p))\overset{\Delta}{\Phi}{}^{-1}(p)$$
$$= f(g_2, \tilde{R}_{g_1}x)f(g_1, x) .$$

This is the cocycle condition for the right action $\tilde{R}_k$ to be a lift of a right action $\tilde{R}_g$. Note that in [Problem IV 4, Cohomology, eq. (9)] we established the cocycle condition for the lift of a left action.

It is then straightforward to check that the lift $\tilde{R}_k$ of $\tilde{R}_g$ on X can be identified with the right action $\tilde{R}_\Lambda$ of a spin bundle over the manifold M, base of the bundle X.

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6. SHORT EXACT SEQUENCE; WEYL HEISENBERG GROUP*

1) A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of vector spaces A, B, C with linear mappings f, g, is said to be **exact** if the kernel of g is equal to the image of f. exact sequence
Show that the short sequence

*Written in collaboration with Humberto J. LaRoche.

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{1}$$

is exact only if f is injective and g is surjective.

Answer 1: Follows from the definitions.

2) *If A, B, C are groups, the element 0 in the definition of kernels will be replaced by the neutral group element* 1 *and linear maps by homomorphisms. Let H be a normal subgroup of G. Show that the following sequence of groups is exact:*

$$1 \to H \xrightarrow{i} G \xrightarrow{\pi} G/H \to 1\,,$$

where i is the inclusion mapping and π the natural projection.

Answer 2: Straightforward.

extension

3) Let $1 \to G_0 \to G \to G_1 \to 1$ be an exact sequence of groups. The group G is often called an **extension** of G_1 by G_0. In general the group G cannot be reconstructed merely from G_0 and G_1. Additional information is needed. It can also happen that an extension of G_1 by G_0 is not possible. In the following problem we shall construct an extension G of the group of translations τ^2 on $\mathbb{R}^2$ by $\mathbb{R}$ known as the (Weyl) Heisenberg group.

We choose a notation appropriate to the role of the Heisenberg group in quantum mechanics; we denote points in $\mathbb{R}^2$ by

$$(q, p) \equiv s \in \mathbb{R}^2\,.$$

The group τ^2 of translations of $\mathbb{R}^2$ acts as follows,

$$l_{(a,b)}(q, p) = (q + a, p - b)\,, \qquad (a, b) \equiv A \in \mathbb{R}^2\,.$$

a) *Let γ^A be the generator of the translations by (at, bt). Let $\sigma = \mathrm{d}q \wedge \mathrm{d}p$ be the symplectic form on $\mathbb{R}^2$. Show that $\mathrm{i}_{\gamma^A}\sigma$ is an exact differential; i.e., show that $\boldsymbol{\gamma}^A$ is a Hamiltonian vector field* (p. 269) and [Problem IV 11, Poisson].

Answer 3a: The Lie derivative of σ with respect to γ^A vanishes, since σ is invariant by translation:

$$\mathrm{d}(q + at) \wedge \mathrm{d}(p - bt) = \mathrm{d}q \wedge \mathrm{d}p\,.$$

Hence (p. 269 and Problem IV 11, Poisson) the vector field $\boldsymbol{\gamma}^A$ is locally Hamiltonian.

A locally Hamiltonian vector field on $\mathbb{R}^2$ is globally Hamiltonian; hence

$$\mathrm{i}_{\gamma^A}\sigma = \mathrm{d}P_A \tag{2}$$

for some function P_A on $\mathbb{R}^2$ determined by γ^A up to a constant function.

b) *Show that the set $\mathscr{P}$ of possible functions P_A together with the Poisson bracket $\{P_A, P_B\}$ defines a Lie algebra which is not isomorphic to the Lie algebra $\mathscr{L}(\tau^2)$ of the translation group τ^2.* See also [Problem IV 11, Poisson].

Answer 3b: In natural coordinates

$$\gamma^{(a,b)} = a\frac{\partial}{\partial q} - b\frac{\partial}{\partial p} \tag{3}$$

and

$$\mathrm{d}P_A = \mathrm{i}_{\gamma^A}\sigma = a\,\mathrm{d}p + b\,\mathrm{d}q. \tag{4}$$

Hence the function P_A is determined up to an additive constant k. Set

$$P_{A,k} = ap + bq + k, \qquad k \in \mathbb{R}, \qquad A \in \mathbb{R}^2 \tag{5}$$

and, setting $P_{A_1,k_1} \equiv P_1$,

$$\{P_1, P_2\} \equiv \frac{\partial P_1}{\partial q}\frac{\partial P_2}{\partial p} - \frac{\partial P_1}{\partial p}\frac{\partial P_2}{\partial q} = b_1 a_2 - a_1 b_2. \tag{6}$$

$\{P_1, P_2\}$ is of the form (5) with $a = 0$, $b = 0$, $k = b_1 a_2 - a_1 b_2$.
Because Poisson brackets satisfy the Jacobi identity, the set $\mathscr{P}$ together with the Poisson bracket $\{\,,\,\}$ is indeed a Lie algebra. As a vector space $\mathscr{P}$ is $\mathbb{R}^2 \times \mathbb{R}$. It cannot be isomorphic to $\mathscr{L}(\tau^2)$, which has $\mathbb{R}^2$ as underlying vector space. Moreover, the subspace $\mathbb{R}^2 \times \{0\}$ of $\mathscr{P}$ (i.e., $k = 0$) is not a Lie subalgebra of $\mathscr{P}$; it cannot be isomorphic to $\mathscr{L}(\tau^2)$.

Note that $\mathscr{L}(\tau^2)$ is the trivial Lie algebra:

$$\left[\gamma^{A_1}, \gamma^{A_2}\right] = 0, \qquad \forall A^1, A^2 \in \mathbb{R}^2.$$

c) *Show that the Lie algebra $\mathscr{L}(G)$ of the* **Heisenberg group** *G defined by the group law* Heisenberg group

$$(a_1, b_1, k_1)(a_2, b_2, k_2) = (a_1 + a_2, b_1 + b_2, k_1 + k_2 + \tfrac{1}{2}(b_1 a_2 - a_1 b_2)) \tag{7}$$

is isomorphic to the Lie algebra $\mathscr{P}$ equipped with Poisson brackets.

Answer 3c: The Lie algebra $\mathscr{L}(G)$ of the Heisenberg group G on $\mathbb{R}^2 \times \mathbb{R}$

can be obtained from the group law. Let $(\alpha, \beta, \kappa) \in \mathscr{L}(G)$ and

$$\exp(\varepsilon(\alpha, \beta, \kappa)) = (a, b, k).$$

Using

$$\exp(\varepsilon\xi)\exp(\varepsilon\zeta)\exp(-\varepsilon\xi)\exp(-\varepsilon\zeta) = \exp(\varepsilon^2[\xi, \zeta] + \text{higher order in } \varepsilon)$$

we obtain

$$[(\alpha_1, \beta_1, \kappa_1), (\alpha_2, \beta_2, \kappa_2)] = (0, 0, \beta_1\alpha_2 - \alpha_1\beta_2). \tag{8}$$

The map: $P_{a,b,k} \to (\alpha, \beta, \kappa)$ is a Lie algebra isomorphism.

central

d) An extension G of a group $G_1 = G/H$ by a group H is said to be **central** if f embeds H into the center of G in the exact sequence

$$1 \to H \xrightarrow{f} G \xrightarrow{\pi} G/H \to 1.$$

The sequence

$$1 \to \mathbb{R} \xrightarrow{i} G \xrightarrow{\pi} \tau^2 \to 1$$

where $i(k) \to (0, 0, k)$ and

$$\pi(a, b, k) = (a, b)$$

is exact; $\tau^2 = G/\mathbb{R}$ but G is not the direct product $\tau^2 \times \mathbb{R}$, since the Lie algebra $\mathscr{L}(G)$, given by (8) is not trivial.
Show that the Heisenberg group G defined in c) *is a central* $\mathbb{R}$*-extension of* τ^2.

Answer 3d: The Heisenberg group is a central extension of the translation group since $(0, 0, k) \in \mathbb{R}$ commutes with all its elements. ∎

In summary the extension of the group of translations provides an isomorphism between $\mathscr{L}(G)$ and $\mathscr{P} = \{P_{A,k},\ A \in \mathbb{R}^2,\ k \in \mathbb{R}\}$.
If the basic classical observables of a system are the functions of q and p in the set $\mathscr{P}$, then a unitary irreducible representation of G on square integrable functions yields natural quantum observables. If all the unitary irreducible representations of G are unitarily equivalent then the quantum observables are uniquely defined by the classical observables.

e) *Construct a unitary representation of the Heisenberg group and show its relationship with the quantum mechanical representation of the operator* $\hat{q}$ *and* $\hat{p}$ *with* $\hat{q} =$ *multiplication by* q, $\hat{p} = -i\hbar\, d/dq$.

Answer 3e: Quantization means finding irreducible weakly continuous unitary representations $\mathscr{U}$ of a group G, on the space $L^2(\mathbb{R})$ if the configuration space of the system is $\mathbb{R}$.

Set $\mathscr{U}(a,0,0) = U(a)$, $\mathscr{U}(0,b,0) = V(b)$ and $\mathscr{U}(0,0,k) = \exp(\mathrm{i}\mu k)$. Then, it follows from the group law that

$$\begin{aligned} U(a_1)U(a_2) &= U(a_1 + a_2) \\ V(b_1)V(b_2) &= V(b_1 + b_2) \\ U(a)V(b) &= V(b)U(a)\exp(-\mathrm{i}\mu ab). \end{aligned}$$

This representation is called the **Weyl representation** of the canonical commutation relations. Weyl representation

If we set $U(a) = \exp(+\mathrm{i}a\hat{p})$, $V(b) = \exp(-\mathrm{i}b\hat{q})$, then

$$[\hat{q},\hat{q}] = 0, \quad [\hat{p},\hat{p}] = 0, \quad [\hat{q},\hat{p}] = \mathrm{i}\mu. \qquad \blacksquare$$

4) The standard procedure for extending a group G_1 by $\mathbb{R}$ is to obtain a new Lie algebra $\mathscr{L}(G)$ defined on the set $\mathscr{L}(G_1) \oplus \mathbb{R}$ by the Lie bracket

$$[(A_1,k_1),(A_2,k_2)] = ([A_1,A_2], z(A_1,A_2)), \quad A_1, A_2 \in \mathscr{L}(G_1),\ k \in \mathbb{R}, \tag{9}$$

where $z(A_1, A_2)$ is an antisymmetric bilinear map and [,] the Lie bracket in $\mathscr{L}(G_1)$. Show that (9) defines indeed a Lie algebra structure on $\mathscr{L}(G_1) \oplus \mathbb{R}$ if and only if

$$z(A,[B,C]) + z(B,[C,A]) + z(C,[A,B]) = 0. \tag{10}$$

Answer 4: Straightforward, (10) expresses that (9) satisfies the Jacobi identity.

We shall show in the following problem (IV 7, Cohomology, paragraph 4) that z is a cocycle on the Lie algebra with values in $\mathbb{R}$.

References for Problems IV 6 and IV 7 can be found at the end of Problem IV 7.

7. COHOMOLOGY OF LIE ALGEBRAS*

See [Problems V bis 11, Cocycles, V bis 12, Virasoro] for applications.

1) Let $\mathscr{G}$ be a Lie algebra over a field $\mathscr{F}$. A **$\mathscr{G}$-module** is a vector space M over $\mathscr{F}$ together with a homomorphism $\mathscr{G}$-module

$$\mathscr{G} \to L(M,M) \qquad \text{(linear transformations of } M\text{);}$$

*We acknowledge fruitful discussions with M. Dubois-Violette and I. Bakas.

we shall denote the linear transformation corresponding to $\gamma \in \mathscr{G}$ by

$$m \mapsto \gamma \cdot m\,, \qquad m \in M\,.$$

cochains on $\mathscr{G}$

The n-dimensional **cochains on $\mathscr{G}$** with values in M are the n-linear alternating functions on $\mathscr{G}^n$ with values in M:

$$f\colon \mathscr{G} \times \cdots \times \mathscr{G}\ (n\text{-factors}) \to M \text{ linearly in all factors} \tag{1}$$

and $f(\gamma_1, \ldots, \gamma_n)$ totally antisymmetric in $\gamma_1, \ldots, \gamma_n$.

Show that the set $C^n(\mathscr{G}, M)$ of n-cochains makes up a vector space over $\mathscr{F}$.

Answer 1: Straightforward from the definitions.

0-cochains

2) The space of **0-cochains** $C^0(\mathscr{G}, M)$ is identified with M.
Let f be an n-dimensional cochain on $\mathscr{G}$ with values in M and $\gamma \in \mathscr{G}$; define $\gamma \cdot f$ as the n-linear function on $\mathscr{G}$ with values in M:

$$(\gamma \cdot f)(\sigma_1, \ldots, \sigma_n) = \gamma \cdot (f(\sigma_1, \ldots, \sigma_n)). \tag{2}$$

a) *Show that $\gamma \cdot f$ is an n-cochain.*
b) *Let f be an $n+1$ cochain, and let f_σ be the n-cochain defined by*

$$f_\sigma(\sigma_1, \ldots, \sigma_n) = f(\sigma, \sigma_1, \ldots, \sigma_n)\,.$$

Show that

$$\gamma \cdot f_\sigma = (\gamma \cdot f)_\sigma$$

Answer 2:
a) $\gamma \cdot f$ is a multilinear alternating function.
b) Straightforward.

coboundary operator

3) The **coboundary operator** d: $C^n(\mathscr{G}, M) \to C^{n+1}(\mathscr{G}, M)$ is defined by

$$\begin{aligned}(\mathrm{d}f)(\gamma_0, \ldots, \gamma_n) = \sum_{i=0}^{n} (-1)^i \gamma_i \cdot f(\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_n) \\ + \sum_{p<q} (-1)^{p+q} f([\gamma_p, \gamma_q], \gamma_0, \ldots, \\ \hat{\gamma}_p, \ldots, \hat{\gamma}_q, \ldots, \gamma_n).\end{aligned} \tag{3}$$

Show that

$$d^2 f = 0\,. \tag{4}$$

Answer 3: From the definition and by calculation.

Once cochains and coboundary operators satisfying properties (4) are defined, cohomology on algebra is defined in the usual fashion.

4) When the homomorphism $\mathscr{G} \to L(M, M)$ is the trivial one, i.e., $\gamma \to 0$ for all $\gamma \in \mathscr{G}$, the space M is sometimes called a **$\mathscr{G}$-module with zero operators**. $\mathscr{G}$-module with zero operators

Show that a bilinear function f: $\mathscr{G} \times \mathscr{G} \to \mathbb{R}$ such that

$$f(\gamma, \sigma) = -f(\sigma, \gamma) \tag{5}$$

$$f(\gamma, [\sigma, p]) + f(\sigma, [p, \gamma]) + f(p, [\gamma, \sigma]) = 0 \tag{6}$$

is a 2-cocycle.

Answer 4: The previous definitions apply with $M = \mathbb{R}$. If f satisfies (5) it is an alternating function on $\mathscr{G} \times \mathscr{G}$ and hence a 2-cochain. If f satisfies (6) it is a 2-cocycle on $\mathbb{R}$ with zero operator:

$$(\mathrm{d}f)(\gamma, \sigma, p) = -f([\gamma, \sigma], p) + f([\gamma, p], \sigma) - f([\sigma, p], \gamma) = 0\,. \tag{7}$$

5) Let $\mathscr{G}$ be an ideal of the Lie algebra $\mathscr{L}$ (i.e., $\mathscr{G} \subset \mathscr{L}$, $[\mathscr{G}, \mathscr{L}] \subset \mathscr{G}$)
a) *Show that the mapping $\gamma \mapsto [\gamma, \lambda]$, $\gamma \in \mathscr{L}$, $\lambda \in \mathscr{G}$ defines an homomorphism $\mathscr{L} \to L[\mathscr{G}, \mathscr{G}]$. Give the corresponding coboundary operator d on cochains on $\mathscr{L}$ with values in $\mathscr{G}$.*

b) *Consider the set of cochains on $\mathscr{G}$ with values in $C^1(\mathscr{L}, \mathscr{G})$. Show that the mapping*

$$r\colon \mathscr{G} \to L(M, M)\,, \qquad M = C^1(\mathscr{L}, \mathscr{G})$$

defined by

$$(r(\gamma) \cdot m)(\beta) = [\gamma, m(\beta)] - m([\gamma, \beta])\,, \qquad \gamma \in \mathscr{G}, \beta \in \mathscr{L} \tag{8}$$

is an homomorphism.

Write the corresponding coboundary operator d, for instance for a 1-cochain f.

This operator is analogous, in a finite dimensional case, to the (classical) **BRST operator** considered in physics with $\mathscr{L}$ the Lie algebra of G-invariant vector fields on a principal fibre bundle with group G and $\mathscr{G}$ the Lie algebra of such vertical vector fields (Lie algebra of the gauge group, cf. [Problem V bis 11, Cocycles] and its references). BRST operator

Answer 5a: The linear mapping $\gamma \mapsto [\gamma, .]$ is a mapping from $\mathscr{L}$ into $L(\mathscr{G}, \mathscr{G})$ since $[\gamma, \lambda] \in \mathscr{G}$, when $\gamma \in \mathscr{L}$, $\lambda \in \mathscr{G}$. It is a reduction to $\mathscr{G}$ of the adjoint representation of $\mathscr{L}$ on $\mathscr{L}$ (p. 167), hence an homomorphism, as can be checked directly from the Jacobi identity:

$$[[\gamma_1, \gamma_2], \lambda] = [\gamma_1, [\gamma_2, \lambda]] - [\gamma_2, [\gamma_1, \lambda]] .$$

The definition (3) gives the coboundary operator d: $C^n(\mathscr{L}, \mathscr{G}) \to C^{n+1}(\mathscr{L}, \mathscr{G})$

$$\begin{aligned}(\mathrm{d}f)(\gamma_0, \ldots, \gamma_n) = &\sum_{i=0}^{n} (-1)^i [\gamma_i, f(\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_n)] \\ &+ \sum_{p<q} (-1)^{p+q} f([\gamma_p, \gamma_q], \gamma_0, \ldots, \hat{\gamma}_p, \ldots, \hat{\gamma}_q, \ldots, \gamma_n).\end{aligned}$$

Answer 5b: We have by the definition of r

$$\begin{aligned}(r(\gamma_1)) \cdot (r(\gamma_2) \cdot m)(\beta) &= [\gamma_1, (r(\gamma_2) \cdot m)(\beta)] - (r(\gamma_2) \cdot m)([\gamma_1, \beta]) \\ &= [\gamma_1, [\gamma_2, m(\beta)] - [\gamma_1, m([\gamma_2, \beta])] \\ &\quad - [\gamma_2, m([\gamma_1, \beta])] + m(\gamma_2, [\gamma_1, \beta]);\end{aligned}$$

the homomorphism property follows from annullations and use of the Jacobi identity.

Let $f \in C^1(\mathscr{G}, C^1(\mathscr{L}, \mathscr{G}))$ be a 1-cochain with values in $C^1(\mathscr{L}, \mathscr{G})$

$$f(\gamma): \mathscr{L} \to \mathscr{G} \qquad \text{by} \qquad \beta \mapsto f(\gamma; \beta) \in \mathscr{G}$$

then $\mathrm{d}f \in C^2(\mathscr{G}, C^1(\mathscr{L}, \mathscr{G}))$ is a 2-cochain with values in $C^1(\mathscr{L}, \mathscr{G})$ given by (3) with $\gamma_i \cdot f$ replaced by $r(\gamma_i) \cdot f$ given by (8), thus

$$(\mathrm{d}f)(\gamma_0, \gamma_1; \beta) = r(\gamma_0) \cdot f(\gamma_1; \beta) - r(\gamma_1) \cdot f(\gamma_0; \beta) + f([\gamma_0, \gamma_1]; \beta) ,$$

i.e.,

$$\begin{aligned}(\mathrm{d}f)(\gamma_0, \gamma_1; \beta) = &[\gamma_0, f(\gamma_1; \beta)] - [\gamma_1; f(\gamma_0; \beta)] \\ &- f(\gamma_1; [\gamma_0, \beta]) + f(\gamma_0; [\gamma_1, \beta]) + f([\gamma_0, \gamma_1]; \beta) .\end{aligned}$$

REFERENCES. SEE ALSO [PROBLEM V BIS II, COCYCLES]

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8. QUASI-LINEAR FIRST-ORDER PARTIAL DIFFERENTIAL EQUATION

Consider the first-order partial differential equation

$$\frac{\partial u}{\partial t} + f(u)\frac{\partial u}{\partial x} = 0\,,$$
$$u\colon (t,x) \mapsto u(t,x) \in \mathbb{R}\,, \qquad (t,x) \in \mathbb{R}^2\,. \tag{1}$$

1) *Write the exterior differential system equivalent to* (1), *determine its characteristic system and the bicharacteristics intersecting a curve γ given by $t = 0$, $u = \varphi(x)$.*

2) *Determine an integral manifold of the exterior system passing through γ. Determine a solution u of* (1) *taking the initial value $u(0,x) = \varphi(x)$. Does this solution exist for all t?*

Answer 1: (p. 250) The system is composed of the 1-form and the 0-form on $\mathbb{R}^5$

$$p + f(u)q = 0 \tag{1}$$

$$\mathrm{d}u - p\,\mathrm{d}t - q\,\mathrm{d}x = 0; \tag{2}$$

its closure is obtained by addition of the 1 and 2 forms

$$\mathrm{d}p + f'(u)q\,\mathrm{d}u + f(u)\,\mathrm{d}q = 0 \tag{3}$$

$$\mathrm{d}p \wedge \mathrm{d}t + \mathrm{d}q \wedge \mathrm{d}x = 0; \tag{4}$$

the characteristic system is eq. (1) and the associated Pfaff system of (3), (4), namely

$$\frac{\mathrm{d}x}{f(u)} = \mathrm{d}t = \frac{-\mathrm{d}p}{pf'(u)} = \frac{-\mathrm{d}q}{qf'(u)} = \frac{\mathrm{d}u}{p + f(u)q}$$

$$p + f(u)q = 0\,.$$

In particular, along a bicharacteristic (p. 252)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(u)\,, \qquad \mathrm{d}u = 0\,, \qquad \text{thus } u = \text{constant;}$$

hence the bicharacteristics project along straight lines in the plane (x, t). The bicharacteristic passing through the point $t = 0$, $x = y$, $u = \varphi(y)$ is the curve $c_y\colon \mathbb{R} \to \mathbb{R}^3$ given by $\tau \mapsto (t(\tau, y), x(\tau, y), u(\tau, y))$ such that

$$\begin{aligned} t(\tau, y) &= \tau \\ u(\tau, y) &= u(0, y) = \varphi(y) \\ x(\tau, y) &= y + \tau f(\varphi(y)) \end{aligned} \tag{5}$$

Answer 2: The subset of $\mathbb{R}^3$ spanned by the bicharacteristics c_y, $y \in \gamma$, $-\epsilon_1(y) < \tau < \epsilon_2(y)$ generate an integral manifold of (1), if the mapping $U \to \mathbb{R}^3$ determined by (5) is an immersion, where U is the open set of $\mathbb{R}^2$

$$y \in \gamma\,, \qquad -\epsilon_1(y) < \tau < \epsilon_2(y)\,.$$

This is the case when the mapping has rank 2 at each point of U. It is always true when f and φ are C^1 and $\epsilon_1(y)$, $\epsilon_2(y)$ are small enough positive numbers, since the jacobian matrix of the mapping is the continuous function of τ

$$\begin{pmatrix} 1 & 0 \\ 0 & \varphi'(y) \\ f(\varphi(y) & 1 + \tau f' \cdot \varphi'(y) \end{pmatrix}$$

which has rank 2 for $\tau = 0$.
However, for large enough $\epsilon_1(y)$ (or $\epsilon_2(y)$) the mapping may cease to be an immersion (case where $1 + \epsilon_i f' \circ \varphi'(y) = 0$).
A solution $u(t, x)$ of (1) passing through γ is obtained by elimination of y between the relations

$$u(t, y) = \varphi(y), \tag{6a}$$

$$x(t, y) = y + t(f \circ \varphi)(y) \tag{6b}$$

by the implicit function theorem we can deduce from (6b) a C^1 function

$$y = \Psi(t, x)$$

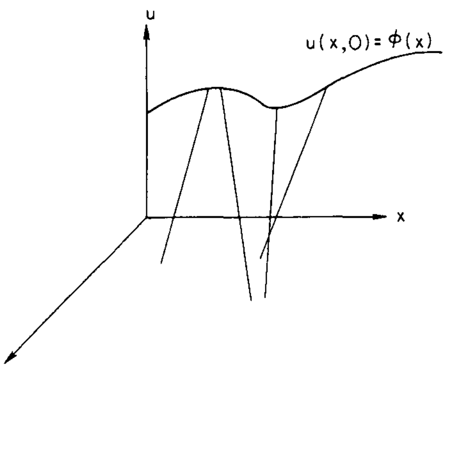
if t is small enough; the solution is then

$$u(t, x) = \varphi(\psi(t, x))\,.$$

We note that

$$\frac{\partial u}{\partial x} = \varphi' \frac{\partial \psi}{\partial x} = \frac{\varphi'}{1 + tf'\varphi'}$$

thus $\partial u/\partial x$ becomes infinite at the points where the mapping ceases to be an immersion.



9. EXTERIOR DIFFERENTIAL SYSTEMS*

Exterior differential systems may be used in a number of applications to differential equations. We illustrate some of these applications in the following examples.

1. DIFFERENTIAL EQUATIONS AS EXTERIOR DIFFERENTIAL SYSTEMS

Differential equations may be written in terms of exterior differential systems by a simple procedure. Given a (set of) differential equation(s), one first defines new variables, in terms of the derivatives of the original (dependent) variables, until one has a set of equations involving first derivatives only. (This choice of variables is not unique.) We denote the manifold of all independent and dependent variables as M. We now construct, by inspection, a set of differential forms on the cotangent bundle of M such that restricting them to the integral submanifold (parameterized by the independent variables) and annulling them (p. 229) gives the set of differential equations. How this is to be done is best illustrated by an example. We consider the **Korteweg–de Vries equation**, $u_t + u_{xxx} + 12uu_x = 0$, where u is a function of x and t and where subscripts indicate derivatives. Defining $z = u_x$ and $p = z_x$ enables us to write the original equation as $u_t + p_x + 12uz = 0$. We now write three 2-forms in the cotangent bundle with base space variables t, x, u, z, and p:

Korteweg–de Vries equation

$$\alpha^1 = \mathrm{d}u \wedge \mathrm{d}t - z\,\mathrm{d}x \wedge \mathrm{d}t\,,$$
$$\alpha^2 = \mathrm{d}z \wedge \mathrm{d}t - p\,\mathrm{d}x \wedge \mathrm{d}t\,,$$
$$\alpha^3 = -\mathrm{d}u \wedge \mathrm{d}x + \mathrm{d}p \wedge \mathrm{d}t + 12zu\,\mathrm{d}x \wedge \mathrm{d}t\,.$$

[See for example Estabrook and Wahlquist 1978.] Restricting these to the submanifold on which u, z, and p are functions of x and t, and thus substituting $\mathrm{d}u = u_x\,\mathrm{d}x + u_t\,\mathrm{d}t$, etc., yields a set of 2-forms in the submanifold, which, when set equal to zero (annulled) yield the differential equations above.

The set of forms representing the differential equations forms an ideal (p. 232), usually denoted by I.

a) *Give two 2-forms corresponding to the one-dimensional heat equation* $u_{xx} = u_t$.

*Contributed by B. Kent Harrison.

Answer 1a: Put $w = u_x$. Then $w_x = u_t$. Annulling the 2-forms

$$\alpha^1 = \mathrm{d}u \wedge \mathrm{d}t - w\,\mathrm{d}x \wedge \mathrm{d}t\,,$$
$$\alpha^2 = \mathrm{d}w \wedge \mathrm{d}t + \mathrm{d}u \wedge \mathrm{d}x\,, \tag{1}$$

after one requires u and w to be functions of x and t, yields the original differential equations (Harrison and Estabrook 1971).

Occasionally one will be able to recast the forms into another set of forms by using a basis of 1-forms and their exterior derivatives.

sine-Gordon equation

b) The **sine-Gordon equation** $\phi_{uv} = \sin\phi$ may be written as a pair of 2-forms by defining $r = \phi_u$:

$$\alpha = \mathrm{d}\phi \wedge \mathrm{d}v - r\,\mathrm{d}u \wedge \mathrm{d}v\,,$$
$$\beta = \mathrm{d}r \wedge \mathrm{d}u - \sin\phi\,\mathrm{d}v \wedge \mathrm{d}u\,.$$

We define four 1*-forms*: $\xi_1 = \mathrm{d}u$, $\xi_2 = A\,\mathrm{d}u$, $\xi_3 = \sin\phi\,\mathrm{d}v$, *and* $\xi_4 = B\,\mathrm{d}v$, *where A and B are functions of r and ϕ. We note the algebraic relations*

$$\xi_1 \wedge \xi_2 = \xi_3 \wedge \xi_4 = 0 \tag{2}$$

and require that, when α and β are annulled, the $\mathrm{d}\xi_a$ *be expressed in terms of hook products of the ξ_b with constant coefficients. Find a solution for A and B.*

Answer 1b: We note that $\mathrm{d}\xi_2 = \mathrm{d}A \wedge \mathrm{d}u$, so that putting $A = r$ will yield $\beta = \mathrm{d}\xi_2 - \xi_3 \wedge \xi_1$. We now see that, if we write $B = \cos\phi$, then $\mathrm{d}\xi_3 = \alpha\cos\phi + \xi_2 \wedge \xi_4$. Finally we have $\mathrm{d}\xi_1 = 0$ and $\mathrm{d}\xi_4 = -\alpha\sin\phi - \xi_2 \wedge \xi_3$. Annulling α and β is now equivalent to annulling the forms:

$$\begin{aligned} &\mathrm{d}\xi_1(=0)\,,\\ &\mathrm{d}\xi_2 - \xi_3 \wedge \xi_1(=0)\,,\\ &\mathrm{d}\xi_3 - \xi_2 \wedge \xi_4(=0)\,,\\ &\mathrm{d}\xi_4 + \xi_2 \wedge \xi_3(=0)\,. \end{aligned} \tag{3}$$

Eqs. (2) and (3) can be used to reconstruct the sine-Gordon equation. (We put $\xi_1 = \mathrm{d}u$; then (2) gives $\xi_2 = f\,\mathrm{d}u$, etc.) (Harrison 1984).

2. INVARIANCE GROUPS OF SETS OF FORMS AND DIFFERENTIAL EQUATIONS

It is well known that sets of differential equations may exhibit invariance under certain transformations of the variables, such as translation, scale, or rotation invariance. This may be discussed succinctly by use of the corresponding ideal of forms I. One simply treats the variable transformations as infinitesimal with generators along some direction $\boldsymbol{v}$ in $T(M)$ and considers transformation of the forms by means of a Lie derivative along $\boldsymbol{v}$ (pp. 147–152, 206). To find the invariance group, or isogroup, one leaves $\boldsymbol{v}$ unspecified, but requires that (Harrison and Estabrook 1971)

$$\mathscr{L}_v I \subset I \tag{4}$$

in order that, when I is annulled, the change of I along $\boldsymbol{v}$ also is annulled. This condition gives a (usually overdetermined) set of linear equations for the components of $\boldsymbol{v}$ and their derivatives, which may be solved. (This is a more general statement than is given in the text, pp. 261, 263.) The independent infinitesimal operators found in this way form a Lie algebra (Harrison and Estabrook 1971, Estabrook 1980, Schutz 1980, p. 87). They may be exponentiated (p. 160) to find the isogroup.

Find the generators of the isogroup for the heat equation by using the 2-forms given above, and identify the obvious symmetries.

Answer 2: We write eq. (4) as

$$\mathscr{L}_v \alpha^1 = a\alpha^1 + b\alpha^2, \tag{5a}$$

$$\mathscr{L}_v \alpha^2 = f\alpha^1 + g\alpha^2, \tag{5b}$$

where a, b, f, and g are undetermined functions, and write

$$\boldsymbol{v} = A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x} + C\frac{\partial}{\partial u} + D\frac{\partial}{\partial w}.$$

We expand eq. (5a) (pp. 148–150), noting that $\mathscr{L}_v \, \mathrm{d}x^C = \mathrm{d}v^C$, where $\mathrm{d}v^C$ is to be expanded in the $\mathrm{d}x^D$, and get

$$\begin{aligned}
&\mathrm{d}C \wedge \mathrm{d}t + \mathrm{d}u \wedge \mathrm{d}A - D\,\mathrm{d}x \wedge \mathrm{d}t - w\,\mathrm{d}B \wedge \mathrm{d}t - w\,\mathrm{d}x \wedge \mathrm{d}A \\
&\quad = a(\mathrm{d}u \wedge \mathrm{d}t - w\,\mathrm{d}x \wedge \mathrm{d}t) + b(\mathrm{d}w \wedge \mathrm{d}t) + \mathrm{d}u \wedge \mathrm{d}x.
\end{aligned}$$

We expand the $\mathrm{d}A$, etc., in terms of derivatives, identify coefficients of identical 2-forms, eliminate a and b, and simplify, obtaining

$$\begin{aligned}
A_w &= 0, \\
C_w - wB_w &= A_x + wA_u, \\
C_x - D - wB_x &= -w(C_u - wB_u).
\end{aligned} \tag{6a}$$

A similar treatment of eq. (5b) yields the equations

$$\begin{aligned}
&A_x + C_w = 0, \\
&-A_u + B_w = 0, \\
&C_u + B_x = D_w + A_t, \\
&D_x - C_t = -w(D_u + B_t).
\end{aligned} \tag{6b}$$

The first of eqs. (6a) and then the first and second of (6b) can immediately be integrated once; the second of (6a) then shows that $A = A(t)$, $B = B(t, x, u)$, and $C = C(t, x, u)$. The last of (6a) gives D in terms of the other functions. Continuing in this manner gives these results:

$$
\begin{aligned}
A &= 2k_6t^2 + 2k_4t + k_1 \,, \\
B &= 2k_6xt + k_4x - 2k_5t + k_2 \,, \\
C &= g(x, t) + u(-\tfrac{1}{2}k_6x^2 + k_5x - k_6t + k_3) \,, \\
D &= g_x(x, t) + u(-k_6x + k_5) + w(-\tfrac{1}{2}k_6x^2 + k_5x - 3k_6t + k_3 - k_4) \,,
\end{aligned}
$$

where the k_i are arbitrary constants and $g(x, t)$ is any solution of the original heat equation. Inspection shows that setting g and all $k_i = 0$ except k_1 yields simply the time translation symmetry. Similarly, the generator corresponding to k_2 represents space translation, that for k_3 is the scale change of the dependent variable u, that for k_4 is the scale change $x \rightarrow ax$ and $t \rightarrow a^2t$, and the $g(x, t)$ generator simply represents addition of any solution (which obviously leaves the equation invariant). The generators corresponding to k_5 and k_6 do not represent obvious symmetries (Harrison and Estabrook 1971).

3. PSEUDOPOTENTIALS

Pseudopotentials are additional variables, q^i, introduced in a fibre space attached to each point of the manifold, forming a local fibre bundle (Estabrook and Wahlquist 1978, Estabrook 1980). The generators α^μ of the ideal I can be lifted into the bundle in the obvious way, with all components unchanged, so that we have an ideal I' on the bundle, with forms of the form $\psi_\mu \wedge \alpha^\mu + \tau_\mu \wedge \mathrm{d}\alpha^\mu$, where the coefficients ψ_μ and τ_μ may involve the q^i and $\mathrm{d}q^i$. We also require, in I', the existence of new forms ω^i with the property that $\mathrm{d}\omega^i \subset I'$. We speak of *prolonging* the system by including the new variables q^i. The new forms may be, for example, of the type (Wahlquist and Estabrook 1976), $\omega^i = -\mathrm{d}q^i + F^i_A\, \mathrm{d}x^A$, where the x^A are the independent variables of the system and the F^i_A are functions of the original variables on M and of the q^i. Annulling the ω^i gives equations which may be solved for the q^i (if a solution of the differential equations is known).

If the F^i_A are not functions of the q^i, then $\mathrm{d}\omega^i \subset I$ and the q^i are simply the well-known potentials. If they are functions of the q^i, then the requirement $\mathrm{d}\omega^i \subset I'$ yields equations involving commutators of the F^i_A, which eventually lead to a prolongation structure as described below.

Many such treatments have sought only a single pseudopotential (Wahlquist and Estabrook 1973, Harrison 1978). But it is frequently useful to consider a set of N pseudopotentials, with N unspecified, and to take the forms ω^i to be linear in the q^i (Wahlquist and Estabrook 1976). Such a case occurs when the forms generating I may be written in terms of 1-forms ξ_a, with a set of algebraic forms as in eq. (2) above and with expressions for the $d\xi_a$ in terms of hook products of the ξ_b, with constant coefficients, as in eq. (3) above. It is then very advantageous to write $\omega = -dq + (B^a \xi_a)q$, where ω and q are vectors with components ω^i and q^i and where the B^a are $N \times N$ constant matrices. We then have

$$\begin{aligned} d\omega &= B^a \, d\xi_a q - B^a \xi_a \wedge dq \\ &= B^a \, d\xi_a q - B^a \xi_a \wedge (-\omega + B^b \xi_b q) \\ &= B^a \, d\xi_a q - B^a B^b \xi_a \wedge \xi_b q + B^a \xi_a \wedge \omega \\ &= (B^a \, d\xi_a - \tfrac{1}{2}[B^a, B^b] \xi_a \wedge \xi_b) q + B^a \xi_a \wedge \omega \,, \end{aligned}$$

whereupon the condition $d\omega^i \subset I'$ now simply becomes

$$B^a \, d\xi_a - \tfrac{1}{2}[B^a, B^b] \xi_a \wedge \xi_b = 0 \qquad (\mathrm{mod}\, I) \tag{7}$$

in which one simply substitutes the equations from I and sets the resulting expressions to zero. One usually obtains in this way an incomplete Lie algebra for the matrices B^a, although use of the Jacobi identity sometimes provides more relations. More frequently, however, the set of equations for B^a, called a **prolongation structure**, represents an infinite algebra, such as a Kac–Moody algebra (Estabrook and Wahlquist 1976, Pirani, Robinson, and Shadwick 1979, and Estabrook 1983). **prolongation structure**

a) *Apply this approach to the forms* (2) *and* (3) *for the sine-Gordon equation. Find the equations for the* B^i, *guess a two-dimensional representation for the* B^i, *and write out the equations for* ω^1 *and* ω^2. *Annul these equations, put* $q^2 = Qq^1$, *and show that a Riccati equation for* dQ, *written in terms of the* ξ_a, *results.*

Answer 3a: Eq. (7), with eqs. (2) and (3), becomes simply

$$\begin{aligned} &B^2 \xi_3 \wedge \xi_1 + B^3 \xi_2 \wedge \xi_4 + B^4 \xi_3 \wedge \xi_2 - [B^1, B^3] \xi_1 \wedge \xi_3 \\ &\quad -[B^1, B^4] \xi_1 \wedge \xi_4 - [B^2, B^3] \xi_2 \wedge \xi_3 - [B^2, B^4] \xi_2 \wedge \xi_4 = 0 \,, \end{aligned}$$

which gives the prolongation structure

$$\begin{aligned} [B^2, B^3] &= -B^4 \qquad [B^1, B^3] = -B^2 \\ [B^2, B^4] &= B^3 \qquad\;\; [B^1, B^4] = 0 \,. \end{aligned} \tag{8}$$

If we write $B^1 = k\tau^1$, $B^2 = \tau^2$, $B^3 = k^{-1}\tau^3$, and $B^4 = k^{-1}\tau^1$, where k is a nonzero parameter and the τ^a are 2×2 matrices, then

$$[\tau^1, \tau^3] = -\tau^2\,, \qquad [\tau^2, \tau^3] = -\tau^1\,, \qquad \text{and} \qquad [\tau^1, \tau^2] = -\tau^3\,,$$

and a representation is

$$\tau^1 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tau^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau^3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This gives

$$\omega^1 = -\mathrm{d}q^1 + \tfrac{1}{2}(-kq^1\xi_1 + q^2\xi_2 + k^{-1}q^2\xi_3 - k^{-1}q^1\xi_4)\,,$$
$$\omega^2 = -\mathrm{d}q^2 + \tfrac{1}{2}(kq^2\xi_1 - q^1\xi_2 + k^{-1}q^1\xi_3 + k^{-1}q^2\xi_4)\,.$$

The Riccati equation for $\mathrm{d}Q$ is

$$\mathrm{d}Q = \tfrac{1}{2}(-\xi_2 + k^{-1}\xi_3) + (k\xi_1 + k^{-1}\xi_4)Q - \tfrac{1}{2}(\xi_2 + k^{-1}\xi_3)Q^2\,.$$

Note: the change of variable $Q = \tan(\tfrac{1}{2}y)$ gives an equation which may be written as the annulling of the form

$$\begin{aligned}\Omega &= -\mathrm{d}y + (k \sin y)\xi_1 - \xi_2 + (k^{-1}\sin y)\xi_4 + (k^{-1}\cos y)\xi_3 \\ &= -\mathrm{d}y + (k\sin y - r)\,\mathrm{d}u + k^{-1}\sin(y+\phi)\,\mathrm{d}v\,. \qquad (9)\end{aligned}$$

(Estabrook and Wahlquist 1973; cited in Harrison 1983, 1984, 1986).

4. BÄCKLUND TRANSFORMATIONS

In the above, we can think of the I' as an augmentation of the ideal I. We may also think of I as a subideal into which I' is projected. In some situations, it may be possible to project an ideal I' into more than one subideal – I_1 and I_2, say. This then sets up a correspondence between integral manifolds in I_1 and those in I_2. Such a correspondence is called a **Bäcklund transformation** (BT) (Estabrook and Wahlquist 1978, Pirani, Robinson, and Shadwick 1979, Dodd and Morris 1980). It may be used as a method for generating solutions to a set of forms (or differential equations); if one knows a solution in I_1, one can use the transformation to find, or to generate, a solution in I_2. In many cases, I_1 and I_2 may be isomorphic, and then we speak of an auto-Bäcklund transformation. It should be noted that Bäcklund transformations between equations exist rather rarely.

Bäcklund transformation

In practice, to search for a BT, one searches first for a prolongation structure and for the associated pseudopotential(s) q^i. One then assumes that the new variables (in I_2 – usually just the dependent variables) are functions of the old variables (in I_1) and of the q^i and requires them to satisfy the appropriate equations, making use of the equations for q^i. (For example see Wahlquist and Estabrook 1976.) When a system can be expressed in terms of a set of 1-forms ξ_a as above, then one may try simply writing the new ξ_a' as a linear combination of the old ones, with coefficients as functions of the q^i, and requiring them to satisfy the appropriate equations (Harrison 1983, 1984). This will give a set of differential equations for the coefficients; solution will give the BT. (It has been found on at least one occasion that a BT can be found by this method only when one assumes the coefficients to be functions of another variable as well, perhaps a combination of independent variables invariant under the isogroup of the equations (Harrison 1978, 1983).)

We find the (auto-)Bäcklund transformation for the sine-Gordon equation. Assume a new set of 1*-forms* ξ_a', *which are to satisfy eqs.* (2) *and* (3), *and which are given as linear combinations of the* ξ_b. *Let the coefficients be functions of the single pseudopotential* y *defined in eq.* (9) ($\Omega = 0$). *By substitution into eqs.* (2) *and* (3), *and by using the same equations for the old* ξ_a, *find and solve the differential equations for the coefficients. By substitution back into the explicit expressions for the* ξ_a, *find expressions for the first derivatives of the new field* ϕ' *in terms of* ϕ, ϕ', *and the derivatives of* ϕ. *Assume that the independent variables remain the same. Hints: one solution, of course, will be the identity, up to scale factors; ignore it. Also note from eq.* (2), *that one may immediately see that* ξ_1' *and* ξ_2' *will be linear combinations of only* ξ_1 *and* ξ_2, *while* ξ_3' *and* ξ_4' *will be combinations of* ξ_3 *and* ξ_4. *One should find equations showing that* y *is a particular function of* ϕ *and* ϕ', *which may be used to eliminate* y *in the final equations.*

Answer 4: (Harrison 1984) We write $\xi_1' = a\xi_1 + b\xi_2$, $\xi_2' = e\xi_1 + f\xi_2$, $\xi_3' = l\xi_3 + m\xi_4$, and $\xi_4' = p\xi_3 + q\xi_4$. Substitution of ξ_1' into the first of eqs. (3) (for the *new* forms) and use of (9) yields the equation (where the dot means d/dy):

$$0 = (k \sin y\xi_1 - \xi_2 + k^{-1} \sin y\xi_4 + k^{-1} \cos y\xi_3) \wedge (\dot{a}\xi_1 + \dot{b}\xi_2) + b\xi_3 \wedge \xi_1.$$

Some terms vanish by eq. (2); we set the coefficients of the remaining 2-forms equal to zero, obtaining easily $b = 0$ and $\dot{a} = 0$. Thus a is constant (assumed nonzero in order that $\xi_1' \not\equiv 0$). The second equation of eqs. (3), treated in a similar fashion, shows that f is constant and gives the equations

$$\begin{aligned} m &= (ka)^{-1}\dot{e} \sin y, \\ l &= (ka)^{-1}(\dot{e} \cos y + kf). \end{aligned}$$

The last two of eq. (3) yield the following equations:

$$\begin{aligned} &0 = k\dot{l} \sin y - ep, \quad &&0 = k\dot{p} \sin y + el, \\ &0 = k\dot{m} \sin y - eq, \quad &&0 = k\dot{q} \sin y + em, \end{aligned}$$

$$0 = -\dot{l} - fp - m\,, \qquad 0 = -\dot{p} + fl - q\,,$$
$$0 = -\dot{m} - fq + l\,, \qquad 0 = -\dot{q} + fm + p\,.$$

The nontrivial solution of these equations is

$$-p = m = a^{-1} \sin 2y\,,$$
$$q = l = a^{-1} \cos 2y\,,$$

and we have also $f = -1$ and $e = 2k \sin y$, where we have chosen the sign of f to be negative. Requiring u and v to be unchanged gives $a = 1$. By using the explicit expressions for ξ_3', ξ_4' and ξ_3, ξ_4 in terms of the fields ϕ and ϕ', we find that $y = \frac{1}{2}(\phi' - \phi)$, where we have dropped an arbitrary phase factor. The equation for ξ_2', on substitution for y, r and r', gives

$$\phi_u' = -\phi_u + 2k \sin \tfrac{1}{2}(\phi' - \phi)\,,$$

the first equation of the BT. The second,

$$\phi_v' = \phi_v + 2k^{-1} \sin \tfrac{1}{2}(\phi' + \phi)\,,$$

is obtained merely by evaluating $y_v = k^{-1} \sin(y + \phi)$ from eq. (9). It now may be seen quickly and directly that, if ϕ is a solution of the sine-Gordon equation, ϕ' as defined by the BT equations is also a solution.

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10. BÄCKLUND TRANSFORMATIONS FOR EVOLUTION EQUATIONS*

Consider the evolution

$$u_t = F(x, u, u_1, \ldots, u_n)\,, \qquad (1)$$
$$\bar{u}_t = H(x, \bar{u}, \bar{u}_1, \ldots, \bar{u}_n) \qquad (2)$$

in one space variable x, where $u_1, u_2, \ldots$ and $\bar{u}_1$, $\bar{u}_2$ are successive derivatives of u and $\bar{u}$ with respect to x. Eqs. (1) and (2) are said to be **related by a Bäcklund transformation**, if there exists an ordinary differential equation

Bäcklund transformation

$$\Phi(x, u, \ldots, u_p, \bar{u}, \ldots, \bar{u}_q) = 0\,, \qquad \frac{\partial \Phi}{\partial u_p} \cdot \frac{\partial \Phi}{\partial \bar{u}_q} \neq 0\,, \qquad (3)$$

which is invariant along the trajectories of eqs. (1), (2), that is

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \equiv \Phi_*(F) + \Phi_{\bar{*}}(H) = 0 \quad \text{on } [\Phi]\,, \qquad (4)$$

where Φ_* and $\Phi_{\bar{*}}$ are the differential operators of orders p and q:

$$\begin{aligned} \Phi_* &= \frac{\partial \Phi}{\partial u} + \frac{\partial \Phi}{\partial u_1}\,\mathrm{D} + \cdots + \frac{\partial \Phi}{\partial u_p}\,\mathrm{D}^p\,, \\ \Phi_{\bar{*}} &= \frac{\partial \Phi}{\partial \bar{u}} + \frac{\partial \Phi}{\partial \bar{u}_1}\,\mathrm{D} + \cdots + \frac{\partial \Phi}{\partial \bar{u}_q}\,\mathrm{D}^q\,, \end{aligned} \qquad (5)$$

*Contributed by N.H. Ibragimov.

with

$$\mathrm{D} = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \bar{u}_1 \frac{\partial}{\partial \bar{u}} + u_2 \frac{\partial}{\partial u_1} + \bar{u}_2 \frac{\partial}{\partial \bar{u}_1} + \cdots$$

invariance Bäcklund transformation

and where $[\Phi]$ denotes the differential manifold given by the equations $\Phi = 0$, $\mathrm{D}\Phi = 0$, $\mathrm{D}^2\Phi = 0, \ldots$. If $H \equiv F$, one says that (3) defines an **invariance Bäcklund transformation** of eq. (1). To apply this definition to systems of evolution equations, when $u = (u^1, \ldots, u^m)$, $F = (F^1, \ldots, F^m)$ and $\bar{u} = (\bar{u}^1, \ldots, \bar{u}^m)$, $H = (H^1, \ldots, H^m)$ in (1) and (2), we take $\Phi = (\Phi^1, \ldots, \Phi^m)$ in (3) and replace (4) by

$$\frac{\mathrm{d}\Phi^\alpha}{\mathrm{d}t} \equiv \Phi^\alpha_* \cdot F + \Phi^\alpha_{\bar{*}} \cdot H = 0 \qquad (\alpha = 1, \ldots, m) \quad \text{on } [\Phi]\,, \tag{4'}$$

where

$$\begin{aligned} \Phi^\alpha_* \cdot F &= \sum_{\beta=1}^m (\Phi^\alpha_*)_\beta F^\beta\,, \qquad (\Phi^\alpha_*)_\beta = \frac{\partial \Phi^\alpha}{\partial u^\beta} + \frac{\partial \Phi^\alpha}{\partial u^\beta_1}\mathrm{D} + \frac{\partial \Phi^\alpha}{\partial u^\beta_2}\mathrm{D}^2 + \cdots\,; \\ \Phi^\alpha_{\bar{*}} \cdot H &= \sum_{\beta=1}^m (\Phi^\alpha_{\bar{*}})_\beta H^\beta\,, \qquad (\Phi^\alpha_{\bar{*}})_\beta = \frac{\partial \Phi^\alpha}{\partial \bar{u}^\beta} + \frac{\partial \Phi^\alpha}{\partial \bar{u}^\beta_1}\mathrm{D} + \frac{\partial \Phi^\alpha}{\partial \bar{u}^\beta_2}\mathrm{D}^2 + \cdots\,. \end{aligned} \tag{5'}$$

1) *Show that a substitution*

$$\bar{u} = \varphi(x, u) \tag{6}$$

converts eq. (1) *into eq.* (2) *if and only if it is a Bäcklund transformation relating eqs.* (1) *and* (2).

Answer 1: To say that (6) converts eq. (1) into (2) is to say that

$$\frac{\partial \varphi(x, u)}{\partial u} F(x, u, u_1, \ldots, u_n) = H(x, \varphi(x, u), \mathrm{D}\varphi(\mathrm{x}, \mathrm{u}), \ldots, \mathrm{D}^n\varphi(x, u)) \tag{7}$$

identically in $x, u, u_1, \ldots, u_n$. To prove that (6) is a Bäcklund transformation, we check that (7) is equivalent to eq. (4) with $\Phi = \varphi(x, u) - \bar{u}$ (here $\Phi_* = \partial\varphi/\partial u$, $\Phi_{\bar{*}} = -1$).

2) *Check that the first-order ordinary differential equation*

$$u_1 + \bar{u}_1 + A(u - \bar{u}) = 0\,, \qquad \text{where } A = \sqrt{a - 2(u + \bar{u})},\ a = \text{const.}\,,$$

defines an invariance Bäcklund transformation of the Korteweg–de Vries equation $u_t = u_3 + 6uu_1$.

Answer 2: Here $F = u_3 + 6uu_1$, $H = \bar{u}_3 + 6\bar{u}\bar{u}_1$, and $\Phi = u_1 + \bar{u}_1 + A(u - \bar{u})$. We have by (5) $\Phi_* = D + A + (\bar{u} - u)/A$, $\Phi_* = D - A + (\bar{u} - u)/A$, and therefore

$$\Phi_*(F) + \Phi_{\bar{*}}(H) = u_4 + u_4 + A(u_3 - u_3) + \frac{\bar{u} - u}{A}\,(u_3 + \bar{u}_3) + 6uu_2$$
$$+ 6\bar{u}\bar{u}_2 + 6u_1^2 + 6\bar{u}_1^2 + 6A(uu_1 + \bar{u}\bar{u}_1)$$
$$+ 6\,\frac{\bar{u} - u}{A}\,(uu_1 + \bar{u}\bar{u}_1)\,.$$

To simplify the verification of eq. (4) we note that $\mathrm{D}\Phi = \Phi_1 + \alpha\Phi$, $\mathrm{D}^2\Phi = \Phi_2 + \alpha\Phi + \beta\mathrm{D}\Phi$, $\mathrm{D}^3\Phi = \Phi_3 + \alpha\Phi + \beta\mathrm{D}\Phi + \gamma\mathrm{D}^2\Phi$ with regular coefficients α, β, γ, where

$$\Phi_1 = u_2 + \bar{u}_2 + A(u_1\bar{u}_1) + (u - \bar{u})^2\,,$$
$$\Phi_2 = u_3 + \bar{u}_3 + A(u_2 - \bar{u}_2) + 3(u - \bar{u})(u_1 - \bar{u}_1)\,,$$
$$\Phi_3 = u_4 + \bar{u}_4 + A(u_3 - \bar{u}_3) + 4(u - \bar{u})(u_2 - \bar{u}_2) + 3(u_1 - \bar{u}_1)^2\,,$$

and rewrite the system $\Phi = 0$, $\mathrm{D}\Phi = 0$, $\mathrm{D}^2\Phi = 0$, $\mathrm{D}^2\Phi = 0$ as $\Phi = 0$, $\Phi_1 = 0$, $\Phi_2 = 0$, $\Phi_3 = 0$. Then we use the identity

$$\Phi_*(F) + \Phi_{\bar{*}}(H) = \Phi_3 + \frac{\bar{u} - u}{A}\,\Phi_2 + 3(u + \bar{u})\Phi_1 + 3\Phi^2$$
$$+ 3(\bar{u} - u)\left(A + \frac{u + \bar{u}}{A}\right)\Phi\,.$$

3) The **Bonnet equation** (alias the **sine Gordon equation**) $2u_{\xi\eta} = \sin(2u)$ is invariant under the Bianchi–Lie transformation $u \mapsto \bar{u}$ given by the first-order equations $u_\xi + \bar{u}_\xi = \sin(u - \bar{u})$, $u_\eta - \bar{u}_\eta = \sin(u + \bar{u})$. *Show that this is an invariance Bäcklund transformation.*

Bonnet sine-Gordon equation

Answer 3: We rewrite the Bonnet equation in coordinates $t = \xi + \eta$, $x = \xi - \eta$ as an evolutionary system $u_t = v$, $v_t = u_{xx} + \sin u \cos u$ for the two-component vector (u, v). Bianchi–Lie transformation takes the form

$$\bar{u}_x + v - \sin u \cos \bar{u} = 0\,, \quad u_x + \bar{v} + \sin \bar{u} \cos u = 0\,. \tag{8}$$

Thus, we have to check eq. (4′) for two-component vectors F, H, Φ with components $F^1 = v$, $F^2 = u_2 + \sin u \cos u$; $H^1 = \bar{v}$, $H^2 = \bar{u}_2 + \sin \bar{u} \cos \bar{u}$; and $\Phi^1 = \bar{u}_1 + v - \sin u \cos \bar{u}$, $\Phi^2 = u_1 + \bar{v} + \sin \bar{u} \cos u$. We have by (5′)

$$(\Phi^1_*)_1 = -\cos u \cos \bar{u}\,; \qquad (\Phi^1_*)_2 = 1\,;$$
$$(\Phi^1_{\bar{*}})_1 = D + \sin u \sin \bar{u}\,, \quad (\Phi^1_{\bar{*}})_2 = 0\,;$$
$$(\Phi^2_*)_1 = D - \sin u \sin \bar{u}\,, \quad (\Phi^2_*)_2 = 0\,;$$
$$(\Phi^2_{\bar{*}})_1 = \cos u \cos \bar{u}\,, \qquad (\Phi^2_{\bar{*}})_2 = 1\,.$$

Now one can readily verify that eq. (4′) is valid. For example, $\mathrm{d}\Phi^1/\mathrm{d}t$ is given by $\Phi^1_* \cdot F + \Phi^1_{\bar{*}} \cdot H = -v \cos u \cos \bar{u} + u_2 + \sin u \cos u + \bar{v}_1 + \bar{v} \sin u \sin \bar{u}$ and vanishes after substituting the expression for u_2 obtained from (8) through differentiation and elimination.

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11. POISSON MANIFOLDS I*

0. DEFINITIONS

Poisson structures are interesting in connection with the hamiltonian formulation of classical physical systems and their quantization. Let M be a C^∞ manifold of dimension m. Let $C^\infty(M, \mathbb{R})$ be the space of C^∞ real valued functions on M. A **Poisson structure** on M is a mapping called Poisson bracket $\{\ ,\ \}$: $C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ with the following properties:

Poisson structure

a) Antisymmetry and bilinearity

$$\{f, g\} = -\{g, f\}\,,$$
$$\{\lambda f_1 + \mu f_2, g\} = \lambda\{f_1, g\} + \mu\{f_2, g\}\,, \qquad \forall \lambda, \mu \in \mathbb{R}\,.$$

b) Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0\,.$$

c) It satisfies the Leibniz rule:

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\} f_2\,.$$

Poisson manifold

A C^∞ manifold with a Poisson structure is called a **Poisson manifold.** The properties a) and b) say that the bracket $\{\ ,\ \}$ endows the associative algebra $C^\infty(M, \mathbb{R})$ with a Lie algebra structure.

Property c), together with linearity, implies that the mapping $f \mapsto \{f, g\}$ is a derivation of $C^\infty(M, \mathbb{R})$, that is a vector field (p. 117) on M. A **hamiltonian function** H is an element of $C^\infty(M, \mathbb{R})$, and we call

hamiltonian function

$$v_H\colon f \mapsto -\{f, H\} = \{H, f\}$$

hamiltonian vector field

the **hamiltonian vector field** associated with H.

*Based on notes by I. Bakas.

1. HAMILTONIAN VECTOR FIELDS

Show that the hamiltonian vector fields form a Lie algebra $\mathscr{H}$, homomorphic to the Lie algebra defined on $C^\infty(M, \mathbb{R})$ by the Poisson bracket.

Answer 1: We denote by [,] the Lie bracket of vector fields (p. 134). Let v_{H_1} and v_{H_2} be two hamiltonian vector fields on the Poisson manifold M. We have, for all $f \in C^\infty(M, \mathbb{R})$

$$\begin{aligned}[v_{H_1}, v_{H_2}]f &= (v_{H_1}v_{H_2} - v_{H_2}v_{H_1})f \\ &= v_{H_1}\{H_2, f\} - v_{H_2}\{H_1, f\} \\ &= \{H_1, \{H_2, f\}\} - \{H_2, \{H_1, f\}\},\end{aligned}$$

thus, using the Jacobi identity

$$[v_{H_1}, v_{H_2}]f = \{\{H_1, H_2\}, f\},$$

i.e.,

$$[v_{H_1}, v_{H_2}] = v_{\{H_1, H_2\}}.$$

This identity shows that hamiltonian vector fields form a Lie subalgebra $\mathscr{H}$ of the algebra of vector fields, homomorphic to the Lie algebra defined on $C^\infty(M, \mathbb{R})$ by the bracket { , }.
The homomorphism from $C^\infty(M, \mathbb{R})$ onto $\mathscr{H}$ is surjective by its definition. Its kernel is, if M is connected, the set of functions H on M such that $v_H = 0$, i.e., the set of functions on M, such that $\{H, f\} = 0$ for all f: such functions H are called **Casimir functions**. Casimir functions

2. SYMPLECTIC MANIFOLDS

Endow a symplectic manifold with a Poisson structure, whose hamiltonian vector fields coincide with the hamiltonian vector fields of the symplectic structure.

Answer 2: A **symplectic manifold** is a C^∞ $2n$-manifold M with a closed 2-form Ω of rank $2n$ (p. 268). symplectic manifold
This 2-form defines a canonical isomorphism

$$TM \to T^*M \quad \text{by} \quad v \mapsto i_v\Omega.$$

The **Poisson bracket** is defined by Poisson bracket

$$\{f, g\} = \Omega(^{\#}\mathrm{d}f, {}^{\#}\mathrm{d}g), \tag{2.1}$$

where $^{\#}$ denotes the canonical isomorphism $T^*M \to TM$ (denoted $v_{\mathrm{d}f}$ on p. 269) inverse of the isomorphism $TM \to T^*M$ defined by Ω.

Let $(\Omega_{\alpha\beta})$ be the matrix of components of Ω in local coordinates and let $\Omega^{\alpha\beta}$ be the elements of its inverse. Then

$$(i_v\Omega)_\alpha = v^\alpha \Omega_{\alpha\beta}\,, \qquad \Omega = \tfrac{1}{2}\Omega_{\alpha\beta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta\,.$$

Thus if θ is a 1-form

$$({}^\#\theta)^\alpha = \Omega^{\alpha\beta}\theta_\beta$$

and an easy calculation gives

$$\{f, g\} = \Omega^{\alpha\beta}\partial_\alpha f \partial_\beta g\,.$$

By the Darboux theorem (p. 282) there always exists a local coordinate system (x^i, y_j) in which the symplectic form Ω reads

$$\Omega = \sum_{i=1}^{n} \mathrm{d}y_i \wedge \mathrm{d}x^i\,;$$

the corresponding Poisson bracket is then

$$\{f, g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x^i}\right). \tag{2.2}$$

It is obvious that this Poisson bracket is bilinear and antisymmetric. It is clear, for instance, in expression (2.2), that it satisfies a), b) and c).

hamiltonian vector fields

If (M, Ω) is a symplectic manifold the **hamiltonian vector fields** are defined as vector fields v such that (p. 269) $i_v\Omega$ is an exact differential: the hamiltonian vector field v_H associated to the hamiltonian function H is such that

$$i_{v_H}\Omega = -\mathrm{d}H\,, \tag{2.3}$$

also written

$$v_H = -{}^\#\mathrm{d}H\,. \tag{2.4}$$

Then, for all $f \in C^\infty(M, \mathbb{R})$ we have

$$v_H f = \{H, f\}$$

as can easily be checked, for instance, in local coordinates since

$$v_H f = -\Omega^{\alpha\beta}\partial_\beta H \partial_\alpha f = \{H, f\}\,.$$

$\mathscr{H}(M, \Omega)$

Thus the two definitions of the space $\boldsymbol{\mathscr{H}(M, \Omega)}$ of hamiltonian vector fields coincide.

Casimir functions

The kernel of the homomorphism of Lie algebras $C^\infty(M, \mathbb{R}) \to \mathscr{H}(M, \Omega)$, called a set of **Casimir functions**, is in this case the set of functions with a vanishing gradient, i.e., the constant functions if M is connected: we can express this fact by saying that

$$0 \to \mathbb{R} \to C^\infty(M, \mathbb{R}) \to \mathcal{H}(M, \Omega) \to 0$$

is a short exact sequence (cf. Problem IV 6, Short).

3. DUAL OF A LIE ALGEBRA

Let $\mathcal{G}$ be the Lie algebra of a Lie group with Lie bracket $[\ ,\ ]$. Let $\mathcal{G}^*$ be the dual of $\mathcal{G}$. Let f and g belong to $C^\infty(\mathcal{G}^*, \mathbb{R})$.
a) *Show that one can define a Poisson structure $\{\ ,\ \}$ on $\mathcal{G}^*$, by denoting $\langle\ ,\ \rangle$ the duality between $\mathcal{G}^*$ and $\mathcal{G}$ and setting*

$$\{f, g\}(x) = \langle x, [\mathrm{d}f(x), \mathrm{d}g(x)]\rangle\,, \qquad x \in \mathcal{G}^*\,. \tag{3.1}$$

Give the expression of the Poisson bracket in coordinates. Show that there exists an homomorphism from the Lie algebra $\mathcal{G}$ into the Lie algebra of hamiltonian vector fields.
b) A bilinear antisymmetric mapping Θ: $\mathcal{G} \times \mathcal{G} \to \mathbb{R}$ is called a **cocycle** [cf. Problem IV 7, Cohomology eq. (6)] if cocycle

$$\Theta(X, [X, Z]) + \Theta(Y, [Z, X]) + \Theta(Z, [X, Y]) = 0\,, \qquad \forall X, Y, Z \in \mathcal{G}\,.$$

Let Θ be a cocycle on $\mathcal{G}$. Show that one can define a Poisson structure $\{\ ,\ \}_\Theta$ on $\mathcal{G}^*$ by setting

$$\{f, g\}_\Theta(x) = \langle x, [\mathrm{d}f(x), \mathrm{d}g(x)]\rangle - \Theta(\mathrm{d}f(x), \mathrm{d}g(x))\,. \tag{3.2}$$

Answer 3a: If $f \in C^\infty(\mathcal{G}^*, \mathbb{R})$, then $\mathrm{d}f(x)$ is a 1-form on $T_x\mathcal{G}^*$, identified with $\mathcal{G}^*$ since $\mathcal{G}^*$ is a vector space. Thus $\mathrm{d}f(x)$ can be identified with an element of $\mathcal{G}$, as well as $\mathrm{d}g(x)$ and therefore also $[\mathrm{d}f(x), \mathrm{d}g(x)]$. Formula (3.1) is meaningful.
The mapping so defined is obviously bilinear and antisymmetric, and obeys the Leibniz rule. It can be checked to satisfy the Jacobi identity by using this identity for the Lie bracket as follows: let $c^i{}_{jk}$ be the structure constants of $\mathcal{G}$, and x_i be the components of x in a basis of $\mathcal{G}^*$:

$$[\mathrm{d}f(x), \mathrm{d}g(x)] = c^i{}_{jk} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}\,,$$

thus

$$\{f, g\} = x_i c^i{}_{jk} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}\,.$$

Let X be an element of $\mathcal{G}$. We associate to X the C^∞ linear function on $\mathcal{G}^*$ defined by

$$H(x) = \langle x, X\rangle \quad \text{thus} \quad \mathrm{d}H = X\,,$$

that is, in dual basis of $\mathscr{G}$ and $\mathscr{G}^*$,

$$X^i = \frac{\partial H}{\partial x_i}.$$

The hamiltonian vector field corresponding to H is

$$v_H = \{H, .\} = x_i c^i{}_{jk} \frac{\partial H}{\partial x_j} \frac{\partial}{\partial x_k}.$$

A straightforward computation gives the Lie bracket of the vector fields v_{H_1}, v_{H_2}, with the linear hamiltonians $\langle x, X_1\rangle$ and $\langle x, X_2\rangle$

$$[v_{H_1}, v_{H_2}] = x_i c^i{}_{jk} c^k{}_{mh} (X_1^j X_2^m - X_2^j X_1^m),$$

thus, using the Jacobi identity for the structure constants (p. 156)

$$[v_{H_1}, v_{H_2}] = x^i c^i{}_{kh} c^k{}_{jm} X_1^j X_2^m \frac{\partial}{\partial x_h},$$

that is

$$[v_{H_1}, v_{H_2}] = x_i c^i{}_{kh} [X_1, X_2]^k \frac{\partial}{\partial x_h},$$

where the [,] on the right-hand side is the Lie bracket in $\mathscr{G}$. We can also write this identity

$$[v_{H_1}, v_{H_2}] = v_{\{H_1, H_2\}}$$

because we have here

$$\{H_1, H_2\} = \langle x, [X_1, X_2]\rangle,$$

since $\mathrm{d}H_1 = X_1$ and $\mathrm{d}H_2 = X_2$.

Answer 3b: Straightforward computation.

4. POISSON STRUCTURE DEFINED BY A CONTRAVARIANT TENSOR

Poisson structure defined by a contravariant tensor
a) *Show that on a Poisson manifold* $(M, \{\ ,\ \})$ *there exists one* C^∞ *2-contravariant antisymmetric tensor field* Λ *such that*

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g), \qquad \forall f, g \in C^\infty(M, \mathbb{R}).$$

b) *Conversely let* Λ *be a* C^∞ *field of* 2*-contravariant tensor on* M. *Give a necessary and sufficient condition for*

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g)$$

to be a Poisson bracket on M.

Answer 4*a*: We must show that $\{f, g\}(x)$, in any given point x of M, depends only on the values of the differentials $\mathrm{d}f(x)$, $\mathrm{d}g(x)$ at the point x. The mapping

$$C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}) \quad \text{by} \quad g \mapsto \{f, g\}$$

is a derivation, since it is additive and statisfies the Leibniz rule; therefore (p. 132) there exists a vector field v_f on M such that

$$\{f, g\} = v_f g \,.$$

But, we have by definition of $v_f g$

$$(v_f g)(x) = \langle v_f(x), \mathrm{d}g(x) \rangle \,.$$

Thus $\{f, g\}(x)$ depends only on g through $\mathrm{d}g(x)$ and, by the antisymmetry, on f only through $\mathrm{d}f(x)$. It is defined therefore by an antisymmetric bilinear mapping $T_x^*M \times T_x^*M \to \mathbb{R}$, i.e., a contravariant antisymmetric 2-tensor $\Lambda(x)$. We have

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) \,.$$

Λ is called the **Poisson tensor**. Poisson tensor

Answer 4b: Conversely a mapping $C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ defined by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g)$$

with Λ a contravariant 2-tensor field on M is obviously bilinear, antisymmetric, and it is easy to check that it satisfies the Leibniz rule. It can be proved by calculation that it satisfies the Jacobi identity if and only if

$$[\Lambda, \Lambda]_S = 0$$

where $[\ ,\ ]_S$ is the **Schouten–Nijenhuis bracket**: if A and B are antisymmetric contravariant tensor fields of order p and q respectively their Schouten–Nijenhuis bracket is the $p + q - 1$ antisymmetric tensor defined in local coordinates by Schouten–Nijenhuis bracket

$$[A, B]^{k_2 \cdots k_{p+q}} = \frac{1}{(p-1)!q!} \varepsilon^{k_2 \cdots k_{p+q}}_{i_1 \cdots i_p j_1 \cdots j_q} A^{\mu i_2 \cdots i_p} \partial_\mu B^{j_1 \cdots j_q}$$
$$+ \frac{(-1)^p}{p!(q-1)!} \varepsilon^{k_2 \cdots k_{p+q}}_{i_1 \cdots i_p j_2 \cdots j_q} B^{\mu j_2 \cdots j_q} \partial_\mu A^{i_1 \cdots i_p} \,.$$

Indeed we have (note that $[\Lambda, \Lambda]_S$ is a 3-tensor):

$$\sum_{\substack{\text{cyclic} \\ \text{permut}}} \{\{f_1, f_2\}, f_3\} = \tfrac{1}{2}[\Lambda, \Lambda]_S(\mathrm{d}f_1, \mathrm{d}f_2, \mathrm{d}f_3) \,.$$

5. POISSON MANIFOLDS AND SYMPLECTIC MANIFOLDS

rank of a Poisson manifold

Let (M, Λ) be a Poisson manifold. The **rank** of (M, Λ) at a point $x \in M$ is the rank at this point of the antisymmetric 2-tensor Λ, that is an even integer.
Show that if this rank is $2n$, the dimension of M, for all $x \in M$ then (M, Λ) can be identified with a symplectic manifold.

Answer 5: If Λ is of rank $2n$ it defines an isomorphism $\#$: $T^*M \to TM$ by $\omega \mapsto {}^{\#}\omega = \langle \Lambda, \omega \rangle$, i.e., in local coordinates ${}^{\#}\omega^i = \Lambda^{ij}\omega_j$. We define an exterior 2-form Ω on M by

$$\Omega(u, v) = \Lambda({}^{b}u, {}^{b}v)\,,$$

where b: $u \mapsto {}^{b}u$ is the inverse isomorphism of $\#$.
Ω is of rank $2n$ since its components in a frame are the elements of the inverse matrix of the components of Λ in the dual frame.
The Poisson bracket

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) = \Omega({}^{\#}\mathrm{d}f, {}^{\#}\mathrm{d}g)$$

satisfies the Jacobi identity by the hypothesis on Λ. A consequence is that Ω is closed: it can also be checked by direct calculation that

$$\mathrm{d}\Omega({}^{\#}\mathrm{d}f, {}^{\#}\mathrm{d}g, {}^{\#}\mathrm{d}h) = 2\{\{f, g\}, h\} + 2\{\{g, h\}, f\} + 2\{\{h, f\}, g\}\,,$$

thus $\mathrm{d}\Omega = 0$ and Ω is a symplectic form.

6. HAMILTONIAN VECTOR FIELDS AND AUTOMORPHISMS OF A POISSON MANIFOLD

Show that a hamiltonian vector field on a Poisson manifold is an infinitesimal automorphism of the Poisson structure, i.e.,

$$\mathcal{L}_{v_H}\Lambda \equiv 0\,.$$

locally hamiltonian

A vector field v on a Poisson manifold (M, Λ) associated to a 1-form α by $v = {}^{\#}\alpha$ is called **locally hamiltonian** if the 1-form is such that

$$\mathrm{d}\alpha({}^{\#}\mathrm{d}f, {}^{\#}\mathrm{d}g) = 0 \qquad \forall f, g \in C^\infty(M, \mathbb{R})\,.$$

Locally hamiltonian vector fields are also infinitesimal automorphisms of the Poisson structure: See the proof in [1, 5, 6], as well as the proof that the Lie bracket $[u, v]$ of two locally hamiltonian vector fields is hamiltonian, namely:

$$[u, v] = {}^{\#}\mathrm{d}i_u\beta = -{}^{\#}\mathrm{d}i_v\alpha$$

if $u = {}^{\#}\alpha$ and $v = {}^{\#}\beta$.

Answer 6: We have defined Λ by

$$\Lambda(\mathrm{d}f, \mathrm{d}g) = \{f, g\}$$

and an hamiltonian vector field with hamiltonian H by

$$v_H = {}^{\#}\mathrm{d}H = \Lambda(\mathrm{d}H, .\,) = -\{\,.\,, H\}\,,$$

that is

$$v_H f = -\{f, H\} = \{H, f\}\,.$$

We have therefore

$$v_H\{f, g\} = \{H, \{f, g\}\}\,;$$

thus, by the Jacobi identity

$$v_H\{f, g\} = -\{g, \{H, f\}\} - \{f, \{g, H\}\}\,,$$

that is

$$v_H\{f, g\} = \{f, v_H g\} + \{v_H f, g\}\,.$$

In other words

$$\mathscr{L}_{v_H}(\Lambda(\mathrm{d}f, \mathrm{d}g)) = \Lambda(\mathrm{d}f, \mathrm{d}\mathscr{L}H_{v_H} g) + \Lambda(\mathrm{d}\mathscr{L}_{v_H} f, \mathrm{d}g)\,.$$

On the other hand a direct calculation gives

$$\mathscr{L}_{v_H}(\Lambda(\mathrm{d}f, \mathrm{d}g)) = (\mathscr{L}_{v_H}\Lambda)(\mathrm{d}f, \mathrm{d}g) + \Lambda(\mathscr{L}_{v_H}\mathrm{d}f, \mathrm{d}g) + \Lambda(\mathrm{d}f, \mathscr{L}_{v_H}\mathrm{d}g)\,.$$

Comparison of these two formulas, and the commutativity of d and $\mathscr{L}_v$ shows that

$$(\mathscr{L}_{v_H}\Lambda)(\mathrm{d}f, \mathrm{d}g) = 0\,, \qquad \forall f, g \in C^\infty(M, \mathbb{R})\,,$$

thus

$$\mathscr{L}_{v_H}\Lambda = 0\,.$$

7. FOLIATION OF POISSON MANIFOLDS BY SYMPLECTIC LEAVES

Let M be a C^∞, m-dimensional manifold. A **field of hyperplanes** on M is a subbundle P of the tangent bundle TM, such that the fibre P_x is a vector subspace of T_xM. The dimension r_x of P_x is called the **rank** of P at x. The field P is said to be C^∞ if there exists a set of C^∞ vector fields on M whose restriction to x generates P_x, for each x.

field of hyperplanes

rank

C^∞ field of hyperplanes

integral manifold

An **integral manifold of a field of hyperplanes** P on M is a pair (N, h) where N is a C^∞ manifold and h an immersion $N \to M$, such that, for all $y \in N$:

$$h' T_y N \subset P_x \,, \qquad x = h(y) \,.$$

maximal

An integral manifold is said to be **maximal** if any integral manifold of which it is a submanifold coincides with it.

maximum dimension

An integral manifold of P passing through a point x (that is such that there exists $y \in N$ with $h(y) = x$) is said to be of **maximum dimension** at x if

$$h' T_y N = P_x \,, \qquad x = h(y)$$

completely integrable

The field P of hyperplanes is said to be **completely integrable** if for each point $x \in M$ there exists an integral manifold of maximum dimension passing through x. A necessary and sufficient condition of complete integrability of a C^∞ field of hyperplanes generated by r independent vector fields in a neighborhood of a generic point, i.e., where P_x has dimension r is given by the Frobenius theorem (p. 248).

foliation

It can be proved (cf. for instance [6] that, if P is a C^∞ completely integrable field of hyperplanes on M, then for each $x \in M$ there exists a maximal integral manifold of P through x of maximum dimension and unique. Therefore these maximal integral manifolds determine a partition of M, called a (generalized) **foliation** of M. The name foliation is sometimes reserved for the case where all the leaves (i.e., the integral submanifolds) have the same dimension.

Example: $M = \mathbb{R}^2$, P generated by the vector fields $u = \partial/\partial x^1$, $v = x^2 \partial/\partial x^2$. P is of rank two at each point $x \in \mathbb{R}^2$, except on the line $x^2 = 0$ where it is of rank 1. It is completely integrable. The foliation has two leaves of dimension 2, the submanifolds $x^2 > 0$ and $x^2 < 0$ of $\mathbb{R}^2$ (immersed in $\mathbb{R}^2$ by the canonical embedding) and one leaf of dimension 1, the submanifold $x^2 = 0$.

Consider now a Poisson manifold (M, Λ).

C_x characteristic space

The restriction at $x \in M$ of the space $\mathrm{d}H$ of differentials of C^∞ functions on M spans the cotangent space T^*M. However the vectors $({}^\#\mathrm{d}H)_x = v_{H,x}$ do not span TM if rank $\Lambda = 2p < M$, dimension of M, but they span a vector space C_x of dimension $2p$, called **characteristic space.**

The field C of hyperplanes on M defined by the C_x is invariant under an hamiltonian flow (or locally hamiltonian), since it is so of the Poisson structure. In particular the rank of Λ is constant along any trajectory of an hamiltonian vector field.

Show that C is completely integrable.

Enunciate a foliation theorem for M into symplectic leaves.

Answer 7: The idea of the proof is the following (for more details see [2, 6 or 7]: consider a point $x \in M$ where rank $\Lambda = 2p$, and hamiltonian vector fields v_{H_i}, $i = 1, \ldots, 2p$ which restrict to $2p$ independent vectors at x, thus span C_x. Since the Lie bracket $[v_{H_i}, v_{H_j}] = v_{\{H_i, H_j\}}$ is also an hamiltonian vector field there exist numbers λ_k, $k = 1, \ldots, 2$ such that

$$[v_{H_i}, v_{H_j}]_x = \sum_{k=1}^{2p} \lambda_k (v_{H_k})_x$$

and, since the Lie algebra of hamiltonian vector fields is a vector space

$$[v_{H_i}, v_{H_j}] = \sum_{k=1}^{2p} \lambda_k v_{H_k} .$$

By the Frobenius theorem the system of $2p$ vector field v_{H_i} is therefore completely integrable in a neighborhood of x, which is a generic point for this system. The integral manifold N passing through x is of the maximum dimension $2p$ at x, and is an integral manifold of the original system C, i.e., its tangent space at y is included in C_y, because C is invariant under hamiltonian flows.

The restriction to N of the Poisson tensor Λ defines a Poisson structure on N, which is symplectic since it is of rank $2p$. Thus we have:

Theorem [Weinstein [2]. A Poisson manifold (M, Λ) admits a foliation by symplectic leaves. It can be proved that a point $x \in M$ where the rank of Λ is $2p$ has an open neighborhood U which can be identified, by a Poisson diffeomorphism, with the product of a symplectic manifold V, of dimension $2p$, and a Poisson manifold W, whose rank at x is zero. In well-chosen local coordinates on $U \simeq V \times W$, x^i and y^i, $i = 1, \ldots, p$ on V and z^α, $\alpha = 1, \ldots, m - 2p$ on W the Poisson structure is

$$\Lambda = \sum_{i=1}^{p} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{\alpha,\beta=1}^{m-2p} a^{\alpha\beta} \frac{\partial}{\partial z^\alpha} \wedge \frac{\partial}{\partial z^\beta} ,$$

where the $a^{\alpha\beta}$ are functions only on the coordinates z^λ and vanish at the point x.

Remark: if the rank of Λ is constant, equal to $2p$, in U then $a^{\alpha\beta} = 0$ in U, and Λ reduces to the canonical expression corresponding to a symplectic structure.

8. KIRILLOV LOCAL LIE ALGEBRAS

Let Λ and E be a 2-contravariant antisymmetric tensor field and a vector field on M, such that the Schouten–Nijenhuis brackets

$$[\Lambda, \Lambda]_S = 2E \wedge \Lambda\,, \qquad (8.1)$$

$$[E, \Lambda]_S = \mathscr{L}_E \Lambda = 0\,. \qquad (8.2)$$

Jacobi structure

The pair (Λ, E) is called a **Jacobi structure** (cf. [5])
Show that the mapping $\{\ \}$: $C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ given by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + f i_E\, \mathrm{d}g - g i_E\, \mathrm{d}f \qquad (8.3)$$

satisfies the properties a) and b). Show that when $E \neq 0$ *it does not satisfy the property c) which implied* $\{f, g\}(x) = \Lambda(x)\,(\mathrm{d}f(x), \mathrm{d}g(x))$, *but only the weaker requirement*

c′) $\qquad \operatorname{Supp}\{f, g\} \subset \operatorname{Supp} f \cap \operatorname{Supp} g\,.$

local Lie algebras

The Lie algebras on $C^\infty(M, \mathbb{R})$ satisfying properties a), b), c′) are called **local Lie algebras** by Kirillov. See in [5] a proof that these Lie algebras are given by Jacobi structures on M.

Answer 8: A straightforward calculation (cf. [5]) shows that (8.1) and (8.2) are the necessary and sufficient condition for the bilinear and antisymmetric mapping (8.3), to satisfy the Jacobi identity. It is clear on (8.3) that the $\{\,,\}$ satisfies c′).

9. KIRILLOV ORBITS

a) *Let G be a connected and simply connected Lie group with Lie algebra* $\mathscr{G}$. *Consider on* $\mathscr{G}^*$ *the Poisson structure* (2.1). *Show that the symplectic leaves are the orbits of the coadjoint action of G on* $\mathscr{G}^*$, *defined by*

$$G \times \mathscr{G}^* \to \mathscr{G}^* \quad \text{by} \quad x \mapsto \mathrm{Ad}^*_g x\,.$$

b) *The Heisenberg Weyl group [Problems IV* 6, *Short]* G_W *that arises in the quantum mechanics of particles moving on the real line can be realized as the group of* 3×3 *real matrices of the form*

$$g = \begin{pmatrix} 1 & g^1 & g^3 \\ 0 & 1 & g^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Construct its Lie algebra $\mathscr{G}_W$, *its dual* $\mathscr{G}^*_W$ *and the coadjoint action of* G_W *on* $\mathscr{G}^*_W$. *Determine the Poisson Kirillov orbits of the coadjoint action.*

Answer 9a: The adjoint action of G on $\mathscr{G}$ is the mapping $G \times \mathscr{G} \to \mathscr{G}$ by (p. 166)

$$(g, X) \mapsto \mathrm{Ad}_g X$$

with $\mathrm{Ad}_g X$ the element of $\mathscr{G}$ defined by $L'_g(g^{-1})R_g^{-1'}(e)X_e$. The notation is often abbreviated to

$$\mathrm{Ad}_g X = gXg^{-1}$$

which is literally correct if G is a group of matrices.
The coadjoint action of G on $\mathscr{G}^*$ is defined* through the duality by

$$\langle X, \mathrm{Ad}_g^* x\rangle = \langle \mathrm{Ad} g\, X, x\rangle\ , \qquad x \in \mathscr{G}^*\ .$$

We have seen, on the other hand, that the symplectic leaves are trajectories of hamiltonian vector fields.
In the case that we consider these vector fields are

$$v_{H,x}h = \langle x, [\mathrm{d}H(x), \mathrm{d}h(x)]\rangle\ , \qquad h \in C^\infty(M, \mathbb{R})$$

and can be proved to be invariant under the coadjoint action of G on $\mathscr{G}^*$ (for details see [6 or 7]).

Answer 9b: The tangent space T_eG_W to G_W at the unit $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is spanned by the matrices $\frac{\mathrm{d}g(t)}{\mathrm{d}t}\Big|_{t=0}$, where g^1, g^2, g^3 are arbitrary functions of t, i.e., by the 3×3 matrices of the form:

$$X = \begin{pmatrix} 0 & X^1 & X^3 \\ 0 & 0 & X^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie algebra $\mathscr{G}$ can be represented by these matrices; a basis of $\mathscr{G}$ is

$$E_1 = P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie brackets for this basis are

$$[P, Q] = PQ - QP = R\ , \qquad [P, R] = [Q, R] = 0\ .$$

The dual $\mathscr{G}^*$ of $\mathscr{G}$ can be represented as the space of transposed matrices:

$$x = \begin{pmatrix} 0 & 0 & 0 \\ x_1 & 0 & 0 \\ x_2 & x_2 & 0 \end{pmatrix}$$

with the duality between $\mathscr{G}$ and $\mathscr{G}^*$ given by

$$\langle x, X\rangle = X^1x_1 + X^2x_2 + X^3x_3\ .$$

*There is also another definition which can be found in [Problem V bis 12, Virasoro].

The adjoint action of the linear group G_W on $\mathscr{G}$ is

$$\operatorname{Ad}_g X = gXg^{-1} = \begin{pmatrix} 1 & g^1 & g^3 \\ 0 & 1 & g^2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & X^1 & X^3 \\ 0 & 0 & X^2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & g^1 & g^3 \\ 0 & 1 & g^2 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & X^1 & -g^2X^1 + X^3 + g^1X^2 \\ 0 & 0 & X^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The coadjoint action on $\mathscr{G}_W^*$ is such that

$$\langle x, \operatorname{Ad}_g X\rangle = \langle \operatorname{Ad}_g^* x, X\rangle ,$$

therefore

$$\operatorname{Ad}_g^* x = \begin{pmatrix} 0 & 0 & 0 \\ x_1 - g^2x_3 & 0 & 0 \\ x_3 & x_2 + g^1x_3 & 0 \end{pmatrix}.$$

The Poisson structure on $\mathscr{G}_W^*$ is

$$\{f, h\}(x) = \langle x, [\mathrm{d}f(x), \mathrm{d}h(x)]\rangle ,$$

$\mathrm{d}f(x) = (\partial f/\partial x_i)\,\mathrm{d}x_i$ and $\mathrm{d}h(x) = (\partial h/\partial x_i)\,\mathrm{d}x_i$ are identified with elements X and Y of $\mathscr{G}$, and

$$[X, Y]^k = C^k_{i\,j}X^iY^j$$

with $C^i_{j\,k}$ the structure constants of $\mathscr{G}_W$; i.e.,

$$[X, Y]^1 = [X, Y]^2 = 0 , \qquad [X, Y]^3 = X^1Y^2 - X^2Y^1 ;$$

thus,

$$\{f, h\}(x) = x_3\left(\frac{\partial f}{\partial x_1}\frac{\partial h}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial h}{\partial x_1}\right).$$

The Poisson structure on $\mathscr{G}_W^*$ has rank 2, except on the plane $x_3 = 0$. Each other plane $x_3 = \text{constant}$ is a symplectic leaf, with symplectic form $x_3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$. These leaves are the Kirillov orbits under the coadjoint action of G on $\mathscr{G}^*$, as can be seen on the expression of Ad_g^*.

10. LIE ALGEBRA OF OBSERVABLES

Let (M, Ω) be a symplectic manifold, and G be a connected Lie group which acts on M by symplectomorphisms, i.e., diffeomorphisms leaving invariant the symplectic form Ω.

a) *Show that the Killing vector fields of this group action are locally hamiltonian vector fields* (i.e., such that $i_v\Omega = 0$, cf. p. 269).

b) *Suppose that the group G of symplectomorphisms acts effectively on M. Show that there is an isomorphism between the Lie algebra of Killing vector fields and the Lie algebra $\mathcal{G}$ of G.*
c) Suppose that the Killing vector fields of the action of G are (globally) hamiltonian: it is always the case if all closed 1-forms on M are exact, i.e., the cohomology vector space (Problem IV 1, cohomology) $H^1(M, \mathbb{R}) = 0$. To $A \in \mathcal{G}$ associate the hamiltonian vector field v_A, image of A by the isomorphism $\mathcal{G} \to$ {Killing vector fields}. Since v_A is (globally) hamiltonian there exists a function $H_A \in C^\infty(M, \mathbb{R})$ such that

$$i_{v_A}\Omega = -\mathrm{d}H_A .$$

H_A is the hamiltonian of $v_A = v_{H_A}$ and we have

$$[v_A, v_B] = v_{[A,B]} \qquad \text{by the isomorphisms } \mathcal{G} \to \mathcal{K}$$

and, by previous results

$$v_{H_{[A,B]}} = [v_A, v_B] = v_{\{H_A, H_B\}} .$$

But these equalities imply only

$$\{H_A, H_B\} = H_{[A,B]} + z(A, B)$$

with $z(A, B)$ some constant. See, for instance, [Problem IV 6, Short §3]
Show that z: $\mathcal{G} \times \mathcal{G} \to \mathbb{R}$ defined by $(A, B) \to z(A, B)$ is a 2-cocycle on $\mathcal{G}$, that is an antisymmetric bilinear map such that

$$z(A, [B, C]) + z(B, [C, A]) + z(C, [A, B]) = 0 .$$

Show that if H'_A are other hamiltonian functions corresponding to the Killing vectors v_A and

$$z'(A, B) = \{H'_A, H'_B\} - H'_{[A,B]}$$

then z and z' differ by a 2-coboundary, i.e., a mapping $\mathcal{G} \times \mathcal{G} \to \mathbb{R}$ of the form

$$z(A, B) - z'(A, B) = \langle d, [A, B] \rangle$$

with $d \in \mathcal{G}^*$ and $\langle\ ,\ \rangle$ the duality.
Show that if the second cohomology class defined by z is zero – in particular if $H^2(\mathcal{G}, \mathbb{R}) = 0$ – then there exists an isomorphism P: $A \to P_A$ from $\mathcal{G}$ onto the Lie algebra $C^\infty(M, \mathbb{R})$ with the Poisson bracket $\{\ ,\ \}$.

Answer 10a: A locally hamiltonian vector field v (called hamiltonian on p. 269) is such that

$$di_v\Omega = 0 ,$$

equivalent to, since Ω is closed

$$\mathscr{L}_v \Omega = 0 .$$

Therefore the flow of a locally hamiltonian vector field is a 1-parameter group of symplectomorphisms of (M, Ω).
Conversely a Killing vector field v of a group G of symplectomorphism of M is such that $\mathscr{L}_v \Omega = 0$, thus $d i_v \Omega = 0$, and v is locally hamiltonian.

Answer 10b: This result is true in all cases of the Killing vector fields of an action of a Lie group G on a manifold M (p. 163).

Answer 10c: $z(A, B)$ is linear in A and B, since $A \to H_A$ is a linear map. It is antisymmetric and satisfies the cocycle identity because of the Jacobi identity satisfied by $[A, B]$ and $\{H_A, H_B\}$.
If H_A and H'_A are hamiltonian functions for v_A then

$$d(H_A - H'_A) = 0 ,$$

thus

$$H_A - H'_A = C_A \in \mathbb{R} .$$

The application $\mathscr{G} \to \mathbb{R}$ by $A \to C_A$ is linear. Therefore there exists $C \in \mathscr{G}^*$ such that

$$C_A = \langle C, A \rangle .$$

Since the C_A are constants on M

$$\{H_A, H_B\} = \{H'_A, H'_B\}$$

while

$$H'_{[A,B]} = H_{[A,B]} + \langle d, [A, B] \rangle ,$$

therefore

$$z(A, B) - z'(A, B) = \langle C, [A, B] \rangle .$$

Suppose the cohomology class, the 2-cocycle z, obtained for some choice of the H_A is zero: it means that z is a coboundary, i.e., that there exist $d \in \mathscr{G}^*$ such that

$$z(A, B) = \langle d, [A, B] \rangle$$

if we replace H_A by the new hamiltionian family $H'_A = H_A + \langle d, A \rangle$ the corresponding 2-cocycle z' is zero; thus the mapping $A \to P_A = H'_A$ is an isomorphism of Lie algebras. This mapping is sometimes called a **momen-**

momentum map

tum map. The functions P_A are taken as a set of observables in the quantization on (M, Ω) (see [Isham]). They are determined up to addition of a linear form $\langle d, A\rangle$ with $\langle d, [A, B]\rangle = 0\ \forall A, B \in \mathscr{G}$. These elements are 1-coboundaries, the arbitrariness of the observables P_A is therefore classified by the elements of $H^1(\mathscr{G}, \mathbb{R})$.

Remark: If G is a semi-simple Lie group then both $H^2(\mathscr{G}, \mathbb{R})$ and $H^1(\mathscr{G}, \mathbb{R})$ vanish (cf. Jacobson, Lie algebras, Interscience, pp. 93–96).

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12. POISSON MANIFOLDS II*

INTRODUCTION

This Problem gives further properties of Poisson manifolds. A first set of questions concerns the local structure of a Poisson manifold, the transverse Poisson structure to its symplectic leaves and the characterization of the latter in the linear case, $\mathcal{G}^*$.

The second set of questions involves the notions of Realization of a Poission structure, Function groups on a symplectic manifold and Dual Pairs. The first two notions are reciprocal in some sense, and the third is a very important by-product of them. Hereafter only finite dimensional Poisson manifolds will be considered, although similar notions can be defined in an infinite dimensional context. See papers [1, 2], which bring out the importance of this notions in applications.

This problem closely follows Weinstein [3].

Theorems related to those involved in this problem have been stated by Lichnerowicz in the general framework of Jacobi manifolds [4, 5].

1. POISSON MAPS AND POISSON SUBMANIFOLDS

Definition. Let M_1, M_2 be two Poisson manifolds, and let U be an open set in M_1. A differentiable map $\varphi: U \to M_2$ satisfying

$$\{F \circ \varphi; G \circ \varphi\}_{M_1}(m) = \{F; G\}_{M_2}(\varphi(m)),$$

local Poisson map

$\forall F, G \in C^\infty(M_2)$, $\forall m \in U$, is called a **local Poisson map.**

1) *Let H be a $C^\infty(M_1)$ function, let v_H be the corresponding hamiltonian field, and let φ_t be the local 1-parameter group of local diffeomorphisms generated by v_H. Prove by straightforward calculations that φ_t is a local Poisson map (local Poisson diffeomorphism).*

Answer: The following equality holds locally

$$(v_H\{F; G\})(\varphi_t(m)) = \{v_H F; G\}(\varphi_t(m)) + \{F; v_H G\}(\varphi_t(m)).$$

Using

$$(v_H F)(\varphi_t(m)) = \frac{\mathrm{d}}{\mathrm{d}s} F(\varphi_s(\varphi_t(m)))|_{s=0},$$

we easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\{F; G\}(\varphi_t(m)) - \{F \circ \varphi_t; G \circ \varphi_t\}(m)) = 0, \quad \forall t. \qquad \blacksquare$$

*Contributed by C. Moreno.

2) *If* $J: M_1 \to M_2$ *is a surjective Poisson map, and if* $H \in C^\infty(M_2)$, *prove that the trajectories of the vector field* v_H *are the images by* J *of the trajectories of the vector field* $v_{H \circ J}$.

Answer: Letting $C: I \to M_1$ be an integral curve of $v_{H\circ J}$, we consider the map $J \circ C: I \to M_2$. If $F \in C^\infty(M_2)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(F \circ (J \circ C))(t) = \frac{\mathrm{d}}{\mathrm{d}t}(F \circ J)(C(t)) = -\{F \circ J; H \circ J\}_{M_1}(C(t))$$
$$= -\{F; H\}_{M_2}((J \circ C)(t)).$$

$J \circ C: I \to M_2$ is therefore an integral curve for v_H and any integral curve of v_H is of this form, since J is surjective. ■

Definition. Let $(M_i; \Lambda_i)$, $i = 1, 2$ be Poisson manifolds, and suppose M_1 is a submanifold of M_2. We will say that $(M_1; \Lambda_1)$ is a **Poisson submanifold** of $(M_2; \Lambda_2)$ if the inclusion $i: M_1 \to M_2$ is a Poisson map.

Poisson submanifold

3) *Let* $(M_1; \Lambda_1)$ *be a Poisson sub manifold of* $(M_2; \Lambda_2)$. *Prove that* Λ_1 *is unique and* $rg\Lambda_{1(m)} = rg\Lambda_{2i(m)}$, $\forall m \in M_1$.

Answer: $\forall f, g \in C^\infty(M_2)$ we have

$$\Lambda_{2i(m)}((\mathrm{d}f)_{i(m)}; (\mathrm{d}g)_{i(m)}) = \Lambda_{1(m)}(\mathrm{d}(f \circ i)_m; \mathrm{d}(g \circ i)_m).$$

$T_m^* M_1$ is generated by $\mathrm{d}(f \circ i)_m$, $f \in C^\infty(M_2)$. Λ_1 is therefore unique.

Let $\tilde{\Lambda}_{2i(m)}: T^*_{i(m)}M_2 \to T_{i(m)}M_2$; $\tilde{\Lambda}_{1m}: T_m^*M_1 \to T_m M_1$ be defined as in Problem IV 11, question 4, and let $T_m i: T_m M_1 \to T_{i(m)}M_2$ be the tangent map to i at m. We thereby obtain

$$\operatorname{Im} \tilde{\Lambda}_{2i(m)} = T_m i \circ \operatorname{Im} \tilde{\Lambda}_1(m).$$

As $T_m i$ is injective, $rg\Lambda_{1m} = rg\Lambda_{2i(m)}$. ■

4) *Let* M_1 *be a submanifold of the Poisson manifold* $(M_2; \Lambda_2)$. *Prove that Poisson structures* Λ_1 *such that* $(M_1; \Lambda_1)$ *is not a Poisson submanifold of* $(M_2; \Lambda_2)$ *can exist on* M_1.

Answer: For example, let $(M_2; \omega_2)$ be a symplectic manifold, and let M_1 be a submanifold of M_2, $i: M_1 \to M_2$, such that $i_* \omega_2 = \omega_1$ is not degenerate. Thus, $(M_1; \omega_1)$ is a symplectic manifold. It is a Poisson submanifold of $(M_2; \omega_2)$ if and only if $\dim M_1 = \dim M_2$, but this is not always the case for such a M_1. ■

5) *Prove that* $(M_1; \Lambda_1)$ *is a Poisson submanifold of* $(M_2; \Lambda_2)$ *if and only if* $\operatorname{Im} \tilde{\Lambda}_{2i(m)} \subset T_m i \cdot T_m M_1$; $i\colon M_1 \to M_2$ *inclusion.*

Answer: Let $(x^1, \ldots, x^{n_1}; y^1, \ldots, y^{n_2})$ be local coordinates in M_2 such that the local equation of M_1 is $y^i = 0$; $i = 1, \ldots, n_2$. $(x^1, \ldots, x^{n_1})$ are therefore local coordinates in M_1. Let $\bar{F}, \bar{G}$ be functions of the local coordinates (x^k) and let F, G be functions of the local coordinates $(x^k; y^j)$ such that $\bar{F} = F \circ i$; $\bar{G} = G \circ i$.

If $(M_1; \Lambda_1)$ is a Poisson submanifold of $(M_2; \Lambda_2)$, we get

$$\{\bar{F}; \bar{G}\}_{M_1} = i_*\{F; G\}_{M_2} .$$

The left-hand side depends only on x^k. Therefore, the right-hand side does not contain terms of the form

$$\left(\frac{\partial F}{\partial y_k}\right)_{(x;0)}; \left(\frac{\partial G}{\partial y_k}\right)_{(x,0)} .$$

Thus

$$\{F; G\}_{M_2}(x, 0) = \sum \Lambda^{ab}_{2(x;0)} \left(\frac{\partial F}{\partial x_a}\right)_{(x;0)} \left(\frac{\partial G}{\partial x_b}\right)_{(x;0)} ;$$

$$(v_F)_{(x;0)} = \sum \Lambda^{ab}_{2(x;0)} \left(\frac{\partial F}{\partial x_a}\right)_{(x;0)} \left(\frac{\partial F}{\partial x_b}\right)_{(x;0)} .$$

v_F is therefore tangent to M_1. Conversely, if every hamiltonian vector field on M_2 is tangent to M_1, we have

$$(v_F)_{(x;0)} = \sum a^i(x; 0) \left(\frac{\partial}{\partial x^i}\right)_{(x;\,0)} ;$$

and then

$$\{x^k; y^j\}_{M_2} = \{y^i, y^j\}_{M_2} = 0 .$$

Only the components $\Lambda_2^{ab}(x; 0)$, $1 \le a, b \le n_1$ are therefore nonzero, and thus $\Lambda_2(x; 0) = \Lambda_{1x}$.

A look at the local expression of the Schouten–Nijenjhuis bracket (Problem IV 11, question 4b) will enable us to write

$$[\Lambda_2, \Lambda_2]_{(x;\,y)} = 0 \Rightarrow [\Lambda_2; \Lambda_2]_{(x;0)} = 0 \Leftrightarrow [\Lambda_1; \Lambda_1]_x = 0 .$$

$(M_1; \Lambda_1)$ is therefore a Poisson submanifold of (M_2, Λ_2). ∎

2. INDUCING A POISSON STRUCTURE ON A SUBMANIFOLD M_1 OF A POISSON MANIFOLD $(M_2; \Lambda_2)$

If M_1 and $(M_2; \Lambda_2)$ satisfy certain relative conditions, it is possible to define a Poisson structure induced by $(M_2; \Lambda_2)$ on M_1. With this induced

Poisson structure G, if $\dim M_1 \neq \dim M_2$, Poisson manifold $(M_1; G)$ will not be a Poisson submanifold of $(M_2; \Lambda_2)$.

1) *Let* $(T_xM_1)^0 \subset T_x^*M_2$ *be the orthogonal space of* T_xM_1 *in* $T_x^*M_2$. *Prove that if*

$$(T_xM_1)^0 \cap \ker \tilde{\Lambda}_x = \{0\} . \tag{1}$$

then

$$T_xM_1 + \operatorname{Im} \tilde{\Lambda}_x = T_xM_2 .$$

Consequently, seeing that $\operatorname{Im} \tilde{\Lambda}_x = T_xS$ (Problem IV 11, question 7) *if* S *is a symplectic leaf of* M_2, *we have*

$$T_xM_1 + T_xS = T_xM_2 .$$

M_1 *and* S *are therefore* **transverse submanifolds** *at every point of the set* $M_1 \cap S$, *which is then a submanifold of* M_2. transverse submanifolds

Answer: Obviously, $(\ker \tilde{\Lambda}_x)^0 = \operatorname{Im} \tilde{\Lambda}_x$. The hypothesis is then equivalent to $(T_xM_1)^0 \cap (\operatorname{Im} \tilde{\Lambda}_x)^0 = \{0\}$. From linear algebra, we have

$$(T_xM_1)^0 \cap (\operatorname{Im} \tilde{\Lambda}_x)^0 = (T_xM_1 + \operatorname{Im} \tilde{\Lambda}_x)^0 ,$$

and $(\{0\})^0 = T_xM_2$. We thereby obtain (1). ■

Remark: What has been proved is actually the following equivalence

$$(T_xM_1)^0 \cap \ker \tilde{\Lambda}_x = \{0\} \Leftrightarrow T_xM_1 + \operatorname{Im} \tilde{\Lambda}_x = T_xM_2 .$$

2) *Space* $T_xM_1 \cap \tilde{\Lambda}_x T_x^*M_2$ *is then the tangent space at* x *to the submanifold* $M_1 \cap S$. *Let* Ω *be the 2-form induced on by the symplectic form of* $M_1 \cap S$. *Prove the following equality*

$$\ker \Omega_x = \tilde{\Lambda}_x((T_xM_1)^0) \cap T_xM_1 . \tag{2}$$

Answer: Let X_x be a vector in $\tilde{\Lambda}_x((T_xM_1)^0) \cap T_xM_1$. Then $X_x \in T_xM_1$, and there is some $\omega_x \in (T_xM_1)^0 \subset T_x^*M_2$ such that $X_x = \tilde{\Lambda}_x \omega_x$. We thus have

$$\Omega_x(X_x; T_xM_1 \cap \tilde{\Lambda}_x T_x^*M_2) = \omega_x[\tilde{\Lambda}_x(T_x^*M_2) \cap T_xM_1] = 0 .$$

The inclusion $\supseteq$ is therefore proved. Conversely, if $X_x \in T_xM_1 \cap \tilde{\Lambda}_x T_x^*M_2$ and $X_x \in \ker \Omega_x$, the latter equality is satisfied by some $\omega_x \in [\tilde{\Lambda}_x(T_x^*M_2) \cap T_xM_1]^0$. Therefore $X_x = \tilde{\Lambda}_x \omega_x \in \tilde{\Lambda}_x(T_xM_1)^0$. Consequently $X_x \in T_xM_1 \cap \tilde{\Lambda}_x(TM_1)^0$. The inclusion $\subseteq$ is thereby also proved. ■

3) *Let us now suppose that equality* (1) *is satisfied, and that in equality* (2), $\ker \Omega_x = \{0\}$, *i.e.*,

$$(T_xM_1)^0 \cap \ker \tilde{\Lambda}_x = \{0\} \qquad (1')$$

$$\forall x \in M_1 .$$

$$\tilde{\Lambda}_x(T_xM_1)^0 \cap T_xM_1 = \{0\} \qquad (2')$$

Prove that, in this case,

$$\tilde{\Lambda}_x(T_xM_1)^0 \oplus T_xM_1 = T_xM_2 . \qquad (3)$$

Answer: Let ω_x, ω'_x be two vectors in $(T_xM_1)^0$ such that $\tilde{\Lambda}_x\omega_x = \tilde{\Lambda}_x\omega'_x$. As a consequence of (1'), $\omega_x = \omega'_x$. Therefore $\dim \tilde{\Lambda}_x(T_xM_1)^0 = \dim(T_xM_1)^0$, giving us

$$\dim \tilde{\Lambda}_x(T_xM_1)^0 + \dim T_xM_1 = \dim T_xM_2 .$$

This equality and (2) imply (3). ■

We will suppose equalities (1'), (2') (and therefore equality (3)). Let $T_{M_1}M_2$ be the restricted bundle to M_1 of the vector bundle TM_2. Equation (3) yields the splitting of vector bundles

$$T_{M_1}M_2 = \tilde{\Lambda}(TM_1)^0 \oplus TM_1 ,$$

and the surjective morphism of vector bundles

$$\pi: T_{M_1}M_2 \to TM_1 .$$

Let $\pi^*: T^*M_1 \to T^*_{M_1}M_2$ be the dual morphism of π, and consider the sequence

$$T^*M_1 \xrightarrow{\pi^*} T^*_{M_1}M_2 \xrightarrow{\tilde{\Lambda}} T_{M_1}M_2 \xrightarrow{\pi} TM_1 .$$

Let us define a bundle morphism, $\tilde{G}: T^*M_1 \to TM_1$ as $\tilde{G} = \pi \circ \tilde{\Lambda} \circ \pi^*$. With $\omega_{M_1}, \omega'_{M_1} \in T^*M_1$, we have $\langle \omega'_{M_1}; \tilde{G}\omega_{M_1} \rangle = \langle \pi^*\omega'_{M_1}; \tilde{\Lambda}\pi^*\omega_{M_1} \rangle = \Lambda(\pi^*\omega_{M_1}; \pi^*\omega'_{M_1})$. An antisymmetric contravariant 2-tensor G is therefore defined by the relation

$$G(\omega_{M_1}; \omega'_{M_1}) = \Lambda(\pi^*\omega_{M_1}; \pi^*\omega'_{M_1}) \qquad (4)$$

Poisson induced structure

Obviously $[G; G] = 0$. G is therefore a **Poisson** structure on M_1, called the **induced structure** by Λ, when M_1 and Λ satisfy relations (1') and (2').

The symplectic leaves of $(M_1; G)$ are the symplectic manifolds $M_1 \cap S_x$, $x \in M_1$; where S_x is the symplectic leaf of M_2 through x; $M_1 = \bigcup_{x \in M_1} (M_1 \cap S_x)$, and question 3 merely put these symplectic structures together in a contravariant way.

3. LOCAL STRUCTURE OF A POISSON MANIFOLD

Definition. Let $(M_i; \Lambda_i)$, $i = 1, 2$ be two Poisson manifolds. On the manifold $M_1 \times M_2$, the 2-tensor $\Lambda = \Lambda_1 \oplus \Lambda_2$ defines a Poisson structure. The Poisson manifold $(M; \Lambda)$ will be called the **product Poisson manifold** of (M_i, Λ_i), $i = 1, 2$. product Poisson manifold

1) *Prove that the foregoing Poisson structure Λ is characterized by the following two properties*

a) *Projections π_i: $M_1 \times M_2 \to M_i$ are Poisson maps.*

b) *$\pi_i^* C^\infty(M_i)$, $i = 1, 2$ are commuting Lie subalgebras of $(C^\infty(M); \Lambda)$.*

Answer: Let $(x^i; y^a)$ be local coordinates on M adapted to the product $M_1 \times M_2$. We thus have

$$\{x^i; x^j\}_M = \Lambda_{1x}^{ij}\,; \qquad \{y^a, y^b\}_M = \Lambda_{2y}^{ab}\,; \qquad \{x^i; y^a\}_M = 0\,.$$

The result follows. ∎

2) The aim of this question is to prove the following theorem about the local structure of a Poisson manifold.

Let x_0 be an arbitrary point of a Poisson manifold $(M; \Lambda)$. There will be an open neighborhood U of point x_0, a product Poisson manifold $S \times N$, (where S is a symplectic manifold with the same rank as Λ_{x_0}, and N is a Poisson manifold), and a diffeomorphism

$$\phi = \phi_S \times \phi_N : U \to S \times N$$

such that the rank of N at $\phi_N(x_0)$ is zero. Moreover S and N are unique up to local isomorphisms.

Answer: Let us suppose ran $\Lambda_{x_0} \neq 0$. There is a $P \in C^\infty(U)$ such that $(v_P)_{x_0} \neq 0$. By "straightening out" v_P, we can find a neighborhood of x_0 and a function $Q \in C^\infty(U)$ such that

$$v_P(Q) = \{P; Q\} = 1\,.$$

Therefore $[v_P; v_Q] = 0$. According to Froebenius's theorem, v_P and v_Q generate an integrable sub-bundle of TM. This means, in particular, that if we write $p = P(x)$ and $q = Q(x)$, there are local functions $(y^2, \ldots, y^m)$, $m = \dim M$, such that $(p, q, y_2, \ldots, y_m)$ is a set of local coordinates x_0. In these coordinates, $v_P = \partial/\partial q$; $v_Q = -\partial/\partial p$, and the matrix of Λ is

$$\begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & G(p; q; y) \end{pmatrix}$$

where $G^{ab}(p; q; y) = \{y^b; y^a\}$

Note that

$$\frac{\partial G^{ab}(p;q;y)}{\partial q} = \{p; G^{ab}(p;q;y)\} = \{p; \{y^b; y^a\}\} = 0 . \quad \text{(Jacobi)} .$$

$G^{ab}(y)$ is thus independent of both p and q; and obviously $[G; G] = 0$. G therefore defines a Poisson structure of rank $rg\Lambda - 2$ on a manifold of dimension $m-2$. We see that, by iterating this procedure, we can define an open neighborhood U of x_0, a symplectic manifold S through x_0 which is isomorphic to a symplectic open of $\mathbb{R}^{2n}$, $\mathrm{rg}\Lambda_{x_0} = 2n$, a Poisson submanifold N through x_0, $rgN_{x_0} = 0$, which is diffeomorphic to an open set in $\mathbb{R}^{m-2n}$, and a Poisson diffeomorphism $\phi = \phi_s \times \phi_N$ satisfying the conclusions in the question. This proves existence. Unicity is proved when we observe that if $S_1 \times N_1 = S_2 \times N_2$, and $(S_i; N_i)$, $i = 1, 2$ has the properties desired in the question, then S_1 and S_2 are locally isomorphic because they are of the same dimension. Manifolds N_i and S_i intersect transversely and only at a point $\phi_i(x_0)$; a lemma from Weinstein [3], p. 531 proves that there is an automorphism of $(M; \Lambda)$ such that the image of any neighborhood of $\phi_1(x_0)$ in N_1 by this automorphism is another (Poisson isomorphic) neighborhood of $\phi_2(x_0)$ in N_2. ■

A natural representative for S is the symplectic leaf through x_0, but there is no natural representative for N. Weinstein's lemma nevertheless applies to any submanifold $\bar{N}$ which is transverse to S and such that $\bar{N} \cap S$ is a single point. The transverse Poisson structure to S is therefore properly defined as the equivalent class of the Poisson manifold $(N; G)$.

3) *Show that the Poisson structure of the submanifold N of M is the Poisson structure induced by the method developed in question* 3 *of the previous section.*

Answer: We need to prove that equalities (1′), (2′) also hold in this case.

$$(T_xN)^0 \cap \ker \tilde{\Lambda}_x = \{0\}(\Leftrightarrow T_xN + \operatorname{Im} \tilde{\Lambda}_x = T_xM) , \qquad (1')$$

$$\tilde{\Lambda}_x(T_xN)^0 \cap T_xN = \{0\} ; \qquad \forall x \in N . \qquad (2')$$

Using the notations in question 1, we have

$$T_xN = \mathrm{lin}\{\partial/\partial y^a\} ; \qquad (T_xN)^0 = \mathrm{lin}\{\mathrm{d}q^i; \mathrm{d}p_i\}$$

and

$$\tilde{\Lambda}_x(T_xN)^0 = \mathrm{lin}\{\partial/\partial q^i; \partial/\partial p_i\} = T_xS ; \qquad x \in U .$$

These relations show equalities (1′) and (2′).

The Poisson structure induced on N is thus defined by

$$T^*N \xrightarrow{\pi^*} T^*_N M \xrightarrow{\tilde{\Lambda}} T_N M \xrightarrow{\pi} TN\,,$$

$$\mathrm{d}y_a \to \mathrm{d}y_a \to \sum G^{ab}_{(y)}\,\partial/\partial y^b \to \sum G^{ab}_{(y)}\,\partial/\partial y^b\,,$$

(see question 3 of previous section). We can see that the Poisson structures on N coincides with the one obtained in question 1). ■

Consequently, every symplectic leaf of M in the neighborhood of a given symplectic leaf S is locally the Poisson product of S and a symplectic leaf of the Poisson manifold N.

4. TRANSVERSE POISSON STRUCTURE TO THE COADJOINT ORBITS IN $\mathscr{G}^*$

1) *Let G' and G be connected Lie groups, each with Lie algebra $\mathscr{G}$. Let us suppose G' is simply connected.*

Show that the coadjoint orbits of G and G' coincide.

Answer: G is a quotient of G' by a discrete subgroup in the center of G'. The coadjoint representation on this subgroup is trivial. Thus, the result holds. ■

2) *Let X be a vector in $\mathscr{G}$; the corresponding generator in the adjoint (coadjoint) representation written $X_{\mathscr{G}}(X_{\mathscr{G}^*})$, is defined by*

$$X_{\mathscr{G}}(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ad}(\exp tx)\cdot Y|_{t=0}\,; \qquad Y \in \mathscr{G}\,,$$

$$X_{\mathscr{G}^*}(\mu) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ad}^*(\exp - tX)\cdot \mu|_{t=0}\,; \qquad \mu \in \mathscr{G}^*$$

Show that:

$$X_{\mathscr{G}}(Y) = \mathrm{ad}\,X\cdot Y\,,$$

$$X_{\mathscr{G}^*}(\mu) = -\mathrm{ad}^*\,X\cdot\mu\,.$$

Answer: Straightforward. ■

3) In Problem IV 11, question 9, we proved that the coadjoint orbits of G on $\mathscr{G}^*$ are the symplectic leaves of the canonical Poisson structure of $\mathscr{G}^*$. *The purpose of the question is to identify the transverse Poisson structures to the orbits in $\mathscr{G}^*$.*

Suppose $\mu \in \mathscr{G}^*$, and let $G_\mu \subset G$ be the closed (Lie) subgroup defined by $\mathrm{Ad}^*G_\mu \cdot \mu = \mu$. Let $\mathscr{G}_\mu \subset \mathscr{G}$ be the Lie algebra of G_μ, and $\mathscr{G}^*_\mu$, the dual space of $\mathscr{G}_\mu$.

3i) As in Problem IV 11, question 3), we let $T_\mu\mathscr{G}^*$ and $\mathscr{G}^*$ be identical with $T_\mu^*\mathcal{G}^*$ and $\mathcal{G}$. Consequently, the Poisson tensor Λ on $\mathcal{G}^*$ defines a map

$$\tilde{\Lambda}_\mu : \mathcal{G} \to \mathcal{G}^*$$
$$X \to \tilde{\Lambda}_\mu X$$

by

$$\langle \tilde{\Lambda}_\mu X; Y\rangle = \langle \mu; [X; Y]\rangle = -\langle \mathrm{ad}^* X \cdot \mu; Y\rangle . \tag{1}$$

Therefore

$$\ker \tilde{\Lambda}_\mu = \{X \in \mathcal{G} \mid \mathrm{ad}^* X \cdot \mu = 0\} . \tag{2}$$

From (1) and (2), we obtain

$$\ker \tilde{\Lambda}_\mu = \mathcal{G}_\mu . \tag{3}$$

3ii) The tangent space at μ to the orbit $\mathrm{Ad}^* G \cdot \mu$ can be identified as

$$T_\mu(\mathrm{Ad}^* G \cdot \mu) = \{\mathrm{ad}^* X \cdot \mu | X \in \mathcal{G}\} ;$$

clearly

$$T_\mu(\mathrm{Ad}^* G \cdot \mu) \subseteq \mathcal{G}_\mu^0 ,$$

where $\mathcal{G}_\mu^0$ is the subspace in $\mathcal{G}^*$ which is orthogonal to $\mathcal{G}_\mu$ (in the duality). But $\dim \mathcal{G}_\mu^0 = \dim T_\mu(\mathrm{Ad}^* G \cdot \mu)$ and therefore

$$T_\mu(\mathrm{Ad}^* G \cdot \mu) = \mathcal{G}_\mu^0 .$$

3iii) Let V be a subspace in $\mathcal{G}^*$ supplementary to $\mathcal{G}_\mu^0$, i.e.,

$$\mathcal{G}^* = \mathcal{G}_\mu^0 \oplus V . \tag{4}$$

The affine submanifold $\mu + V$ therefore intersects the orbit $\mathrm{Ad}^* G \cdot \mu$ transversely, only at the point μ. As shown in section 3, question 2 the submanifold $\mu + V$ carries the transverse Poisson structure to the orbit. We now want to identify the linear approximation to this structure.

We must now return to section 2, question 2. In a neighborhood of μ we found local coordinates on M, (p_i, q_i, y_a). In the case under consideration, manifold N is an open neighborhood of μ in $\mu + V$ and S is an open neighborhood of μ in $\mathrm{Ad}^* G \cdot \mu$. If

$$i: N \subset \mu + V \to \mathcal{G}^* \equiv M$$

is the inclusion, functions $y_a \circ i$ are local coordinates on N. The sequence (1) in section 2, question 3 will be written with the following notations

$$T^*(\mu + V) \xrightarrow{\pi^*} T^*_{\mu+V}\mathcal{G}^* \xrightarrow{\tilde{\Lambda}} T_{\mu+V}\mathcal{G}^* \xrightarrow{\pi} T(\mu + V) .$$

$$\mathrm{d}(y_a \circ i) \to \mathrm{d}y_a \to \sum G^{ab}(y)\,\frac{\partial}{\partial y_b} \to \sum G^{ab}(y)\,\frac{\partial}{\partial y_b}\,.$$

Here $(\mathrm{d}y_a)_{\mu+\lambda}$, $\mu+\lambda \in N$ annuls every tangent vector to the manifold $y^a = y^a(\lambda+\mu)$, $a = 1, \ldots, \dim V$ at the point $(\lambda+\mu)$. If $\lambda = 0$, the submanifold $y^a = y^a(\mu)$; $a = 1, \ldots, \dim V$ is on the orbit $\mathrm{Ad}^*\, G \cdot \mu$, $(\mathrm{d}y_a)_\mu$ annuls every vector in $T_\mu(\mathrm{Ad}^* G \cdot \mu) = \mathscr{G}^0_\mu$. But if $\lambda \neq 0$ the tangent space to the submanifold $y^a = y^A(\lambda+\mu)$; $a = 1, \ldots, \dim V$ at $(\lambda+\mu)$ is not in general equal to $\mathscr{G}^0_\mu$. Therefore

$$(\mathrm{d}y_a)_{\lambda+\mu} \in \mathscr{G} = \mathscr{G}_\mu \oplus V^0$$

and

$$(\mathrm{d}y_a)_{\lambda+\mu} = x_0 + x_1\,; \qquad x_0 \in \mathscr{G}_\mu\,;\; x_1 \in V^0\,.$$

The transverse Poisson structure at μ is defined by (see section 2, question 3, (4)):

$$\begin{aligned} G_{\mu+\lambda}(\mathrm{d}(y_a \circ i); \mathrm{d}(y_b \circ i)) &= \Lambda_{\mu+\lambda}((\mathrm{d}y_a)_{\mu+\lambda}; (\mathrm{d}y_b)_{\mu+\lambda}) \\ &= \langle \mathrm{ad}^*(\mathrm{d}y_a)_{\mu+\lambda}(\mu+\lambda); (\mathrm{d}y_b)_{\mu+\lambda} \rangle\,. \end{aligned}$$

From the above sequence, we get

$$\mathrm{ad}^*(\mathrm{d}y_a)_{\mu+\lambda}(\mu+\lambda) \in T_{\mu+\lambda}(\mu+V) \equiv V\,.$$

If we now write $(\mathrm{d}y_b)_{\mu+\lambda} = y_0 + y_1$; $y_0 \in \mathscr{G}_\mu$; $y_1 \in V^0$ we obtain

$$\begin{aligned} G_{\mu+\lambda}&(\mathrm{d}(y_a \circ i)_{\mu+\lambda}; \mathrm{d}(y_b \circ i)_{\mu+\lambda}) = \langle \mu+\lambda; [x_0 + x_1; y_0] \rangle \\ &= \underbrace{\langle \mu; [x_0; y_0] \rangle}_{0} + \underbrace{\langle \mu; [x_1; y_0] \rangle}_{0} + \langle \lambda; [x_0; y_0] \rangle + \langle \lambda; [x_1; y_0] \rangle \\ &= \langle \lambda; [x_0; y_0] \rangle + \langle \lambda; [x_1, y_0] \rangle\,. \end{aligned}$$

We have $\mathscr{G}^* = \mathscr{G}^0_\mu \oplus V$. Therefore, $V \approx \mathscr{G}^*/\mathscr{G}^0_\mu$, and the following map

$$\begin{array}{c} \mu + \mathscr{G}^*/\mathscr{G}^0_\mu \to \mathscr{G}^*_\mu \\ \mu + \lambda \to \lambda|_{\mathscr{G}_\mu} \end{array}\;; \qquad \lambda \in \mathscr{G}^*/\mathscr{G}^0_\mu\,,$$

is a bijective map and an isomorphism, when the affine space $\mu + V \approx \mu + \mathscr{G}^*/\mathscr{G}^0_\mu$ carries the canonical vector structure with point μ as zero vector. We then obtain

$$\mu + V \approx \mathscr{G}^*_\mu \approx V\,,$$

consequently, the term $\langle \lambda; [x_0; y_0] \rangle$ in the expression $G_{\mu+\lambda}$ is precisely the canonical Poisson bracket on $\mathscr{G}^*_\mu$, because $\lambda \in V \approx \mathscr{G}^*_\mu$.

If, following Weinstein [3] we now consider the linear approximation to

Poisson structure at a point where this structure is of rank zero, we see that $\langle \lambda; [x_0; y_0] \rangle$ is precisely the linear approximation to $G_{\mu+\lambda}$ at the point μ. We thereby obtain the *Theorem*: "The linear approximation to the transverse Poisson structure at μ at the orbit $\mathrm{Ad}^* G \cdot \mu$ is isomorphic to the canonical Poisson structure of $\mathscr{G}^*_\mu$" [3, 9].

The point $\mu \in \mathscr{G}^*$ is regular if rank Λ_μ is maximal. In the neighborhood of μ, the rank of Λ is also maximal. The symplectic leaves of $\mathscr{G}^*$ in this neighborhood have maximal dimensions, and the rank of the transverse Poisson structure at all points of this neighborhood is therefore zero. Thus $G_{\mu+\lambda}$ is zero on N; which implies that the linear approximation at μ is also zero, i.e., $\langle \lambda; [x_0; y_0] \rangle = 0$; $\lambda \in N$; $x_0, y_0 \in \mathscr{G}_\mu$. It is therefore zero for every $\lambda \in \mathscr{G}^*_\mu$ (N is open) and $\mathscr{G}_\mu$ must be abelian; (regular $\mu \Rightarrow \mathscr{G}_\mu$ abelian).

If point μ is such that $[\mathscr{G}_\mu; V^0] \subset V^0$, we have $\langle \lambda_1 [x_1, y_0] \rangle = 0$; and at these points the transverse Poisson structure to the orbit $\mathrm{Ad}^* G \cdot \mu$ is therefore itself isomorphic to that of $\mathscr{G}^*_\mu$. However this cannot be the general case, because, if it were true that, for every μ, the transverse Poisson structure at μ was isomorphic to $\mathscr{G}^*_\mu$, all points μ, at which $\mathscr{G}_\mu$ is abelian, would be regular. But we know that, if $\mathscr{G}$ is an arbitrary Lie algebra, abelian $\mathscr{G}_\mu$ does not imply regular μ. However, abelian $\mathscr{G}_\mu$ does imply regular μ if $\mathscr{G}$ is semisimple ([10], section 1.11).

5. REALIZATIONS OF POISSON MANIFOLDS AND FUNCTION GROUPS

(See the Introduction)

(full) realization of a Poisson manifold

Definition. A **realization** of a Poisson manifold $(M; \Lambda)$ is a pair $(S; J)$ where S is a symplectic manifold and $J: S \to M$ is a Poisson map. The realization is **full** if J is a submersion. ■

If a realization is full, mapping J is locally surjective, and every hamiltonian flow on M has a convenient form in the coordinates of S (Clebsch variables). See [1, 2].

Realizations require consideration not only for their theoretical interest, but for their many useful applications as well.

1) *Let* $J: S \to M$ *be a realization of* (M, Λ). *Show that* $J^* C^\infty(M)$ *is a Lie subalgebra of* $C^\infty(S)$. ■

Definition. Let $U \subset \mathbb{R}^{2n}$ be a symplectic open subset of $\mathbb{R}^{2n}$. Let ϕ_i, $i = 1, \ldots, r$ be independent elements of $C^\infty(U; \mathbb{R})$, i.e., $d\phi_1 \wedge \cdots \wedge d\phi_r(m) \neq 0$. Consider the map $\phi: \phi_1 \times \cdots \times \phi_r: U \to \mathbb{R}^r$, and let $V \subset \mathbb{R}^r$ be the image of U by ϕ. ($V = \mathrm{Im}\, \phi$.) Let us define the set

$$\mathscr{F} = \{ F \circ \phi \in C^\infty(U) \mid F \in C^\infty(V) \} .$$

If $\mathscr{F}$ is a Lie subalgebra of $C^{\infty}(U)$, we will call $\mathscr{F}$ a **local function group** generated by $(\phi_1, \ldots, \phi_r)$. local function group

Function groups are useful on account of the following property, and particularly its converse, stated below.

2) *Show that a local function group $\mathscr{F}$ canonically defines a Poisson structure on V.*

Answer: Functions ϕ_i are in $\mathscr{F}$, because $\phi_i = \pi_i \circ \phi$ where $\pi_i \in C^{\infty}(V)$ is the ith projection. Thus, $\{\phi_i, \phi_j\}_U \in \mathscr{F}$. Hence, there are $W_{ij} \in C^{\infty}(V)$ such that

$$\{\phi_i; \phi_j\}_U = W_{ij} \circ \phi = W_{ij}(\phi_1 \times \cdots \times \phi_r) .$$

From Jacobi identity on U, we obtain the property that the antisymmetric contravariant 2-tensor with coordinate W_{ij} in the basis $(\partial/\partial\phi_i)_v$ of T_vV satisfies $[W; W]_V = 0$. ■

Conversely, every Poisson manifold has a full realization locally and is therefore defined by a local function group. See [3].

3) *Definition*. A **global function group** on a symplectic manifold S is a foliation Φ of S such that the set of leaves S/Φ is a manifold, and if π: $S \to S/\Phi$ is the canonical projection, the space $\pi^* C^{\infty}(S/\Phi)$ is a Lie subalgebra of $C^{\infty}(S)$. global function group

3i) The following bracket on $C^{\infty}(S/\Phi)$ obviously defines a Poisson structure on S/Φ:

$$\{F; G\}_{S/\Phi} \circ \pi = \{F \circ \pi; G \circ \pi\}_S ; \qquad F, G \in C^{\infty}(S/\Phi)$$

(cf. question 2)), and π: $S \to S/\Phi$ is a Poisson map.

3ii) Let J: $S \to M$ be a realization of M, and let us suppose that the fibres $J^{-1}(m)$, $m \in J(s)$ define a foliation Φ of S in such a way that S/Φ is a manifold. The space $\pi^* C^{\infty}(S/\Phi)$ of functions on S which are constant on the fibres of J, coincides with the space $J^* C^{\infty}(M)$. Since the latter space is a Lie subalgebra, so is $\pi^* C^{\infty}(S/\Phi)$. therefore $(S; \Phi)$ is a global function group.

3iii) *We will see in this question that each global function group on S, written $\mathscr{F}_{\Phi} = \pi^* C^{\infty}(S/\Phi)$, is in canonical association with another global function group (called its polar) and that this correspondence is involutive.*

Let $T\Phi$ be the subbundle of the tangent bundle TS whose fibre at point $x \in S$ is the subspace $T_x\Phi \subset T_xS$, of vectors tangent at x to the leaf through x, Φ_x. Let $T_x\Phi^{\perp}$ be the subspace of T_xS, whose vectors are orthogonal symplectic vectors to $T_x\Phi$, i.e.,

$$X_x \in T_x\Phi^\perp \Leftrightarrow \omega_x(X_x; T_x\Phi) = 0 .$$

Finally, let $T\Phi^\perp$ be the sub-fibre bundle of TS with fibre $T_x\Phi^\perp$ at point $x \in S$.

Prove that the Hamiltonian vector fields on S with Hamiltonian in $\mathscr{F}_\Phi$ are sections of $T\Phi^\perp$, and that $T_x\Phi^\perp$ is generated by the values of these sections of x.

Answer: If $F \in \mathscr{F}_\Phi$, and $w \in \Gamma T\Phi$ is a section of $T\Phi$, we have $\omega(v_F; w) = w(F)$. But the trajectories of w are tangent to the leaves of Φ, and F is constant on these leaves; therefore $v_F \in \Gamma T\Phi^\perp$.

Furthermore, if $\dim S = 2n$, and if r is the dimension of the leaves of Φ, the dimension of $T_x\Phi^\perp$ is $2n - r$. But Φ is locally defined by $2n - r$ functions $F_1, \ldots, F_{2n-r}$ in $\mathscr{F}_\Phi$ such that $\mathrm{d}F_1 \wedge \cdots \wedge \mathrm{d}F_{2n-r}(x) \neq 0$. Hence hamiltonian vector fields v_{F_i} are linearly independent local sections of $T\Phi^\perp$ and $T_x\Phi^\perp$ is generated by $((v_{F_i})_x;\, i = 1, \ldots, 2n - r)$. ■

Show that $\mathscr{F}_\Phi$ is a Lie subalgebra of $C^\infty(S)$ if and only if $T\Phi^\perp$ is an integrable sub-bundle of TS.

Answer: If $F, G, \{F; G\}_S \in \mathscr{F}_\Phi$, we have

$$[v_F; v_G] = -v_{\{F;G\}} ,$$

where $v_F, v_G, v_{\{F;G\}} \in \Gamma T\Phi^\perp$. $T\Phi^\perp$ is thus an involutive sub-bundle, and therefore integrable. Let $\Phi^\perp$ be the corresponding foliation, and let us suppose that the space of leaves defined by $\Phi^\perp$ is a manifold $S/\Phi^\perp$.

If $\mathscr{F}_\Phi$ is not a Lie subalgebra of $C^\infty(S)$, $T\Phi^\perp$ cannot be an integrable subbundle because, if $v_{\{F;G\}_S} \in \Gamma\Phi^\perp$ where $F, G \in \mathscr{F}_\Phi$ we must have $\{F; G\}_S \in \mathscr{F}_\Phi$.

Consequently, since $T\Phi$ is integrable, $\mathscr{F}_{\Phi^\perp}$ is a Lie subalgebra of $C^\infty(S)$, and $\pi: S \to S/\Phi^\perp$; $\pi^* C^\infty(S/\Phi^\perp) = \mathscr{F}_{\Phi^\perp}$, is a function group on S. ■

The correspondence $\Phi \to \Phi^\perp$ is obviously involutive. Functions groups on S exist by pairs, $\mathscr{F}_\Phi$ and $\mathscr{F}_{\Phi^\perp}$, and we will say that each function group in a pair is the **polar** of the other.

polar

Show that $\mathscr{F}_{\Phi^\perp}$ is the space of functions on S commuting with the functions in $\mathscr{F}_\Phi$.

Therefore, $\mathscr{F}_\Phi \cap \mathscr{F}_{\Phi^\perp}$ is the (commun) center of the Lie subalgebras $\mathscr{F}_\Phi$ and $\mathscr{F}_{\Phi^\perp}$, and hence the set of Casimir functions of the Poisson manifolds S/Φ and $S/\Phi^\perp$.

6. DUAL PAIRS

Let $S^J \to M_i$, $i = 1, 2$ be two realizations of the symplectic manifold S. Let us suppose that define global function groups on S (cf. 3ii) question 5). Let $\mathscr{F}_{\Phi_i} = J_i^* C^\infty(M_i)$ be the associated Lie subalgebras of $C^\infty(S)$. (The function groups.)

Definition. We will say that the diagram $M_2 \overset{J_2}{\longleftarrow} S \overset{J_1}{\longrightarrow} M_1$ is a **dual pair** if the function groups $\mathscr{F}_{\Phi_i}$ are polar to each other. If J_i is a surjective submersion, we will call the **dual** pair **full**. dual pair full

6i) To say that $M_2 \overset{J_2}{\longleftarrow} S \overset{J_1}{\longrightarrow} M_1$ is a dual pair is the same as saying that the (integrable) subbundles $\ker TJ_i$ are symplectic complementaries, since they coincide with $T\Phi_i$.

6ii) *If $M_2 \overset{J_2}{\longleftarrow} S \overset{J_1}{\longrightarrow} M_1$ is a full pair, show that there is a bijection between the Casimir functions on M_1 and M_2.*

Answer: Let F_1 be a $C^\infty(M_1)$ function such that $\{F_1; F\}_M = 0$, $\forall F \in C^\infty(M_1)$. We then have $\{F_1 \circ J_1; \bar{F}\}_S = 0$, $\forall F \in \mathscr{F}_{\Phi_1}$. Thus $F_1 \circ J_1 \in \mathscr{F}_{\Phi_1} \cap \mathscr{F}_{\Phi_2}$ (see end of question **5**).

Also, if $F_1' \in C^\infty(M_1)$ is a Casimir function on M_1, and if $F_1' \circ J_1 = F_1 \circ J_1 \in \mathscr{F}_{\Phi_1} \cap \mathscr{F}_{\Phi_2}$, necessarily $F_1 = F_1'$, as J_1 is surjective. Recall that $\mathscr{F}_{\Phi_1} \cap \mathscr{F}_{\Phi_2}$ is the common set of Casimir functions of the Poisson manifolds S/Φ_1 and S/Φ_2. ∎

6iii) Let $M_2 \overset{J_2}{\longleftarrow} S \overset{J_1}{\longrightarrow} M_1$ be a full dual pair, and let us suppose the fibres of J_i are connected. The connected fibre is generated by the hamiltonian vector fields with hamiltonians which are constant on the fibres of J_2. These hamiltonian vector fields can be projected on hamiltonian vector fields on M_2, whereby they become tangent to the symplectic leaves they generate. Therefore, $J_2(J_1^{-1}(m_1))$ is a connected symplectic leaf of M_2, (two points on a Poisson manifold are on the same connected symplectic leaf if and only if there is a piecewise smooth curve, each segment of which is a trajectory of a Hamiltonian vector field). If m_1' is on the same connected symplectic leaf as m_1, we have $J_2(J_1^{-1}(m_1)) = J_2(J_1^{-1}(m_1'))$. A bijective map has therefore been defined between the symplectic leaves of M_1 and M_2.

6iv) The transverse Poisson structures to Poisson manifolds M_1 and M_2 in a dual pair are anti-isomorphic at the points $J_1(m)$, $J_2(m)$, $\forall m \in S$. This is a theorem proved by Weinstein [3].

7. TWO IMPORTANT EXAMPLES OF DUAL PAIRS

Definitions. a) Let G be a connected Lie group acting on the left by symplectic diffeomorphisms on the connected symplectic manifold $(S;\omega)$.

infinitesimal generator

The **infinitesimal generator** of the action of G on M, for any $X \in \mathscr{G}$, is the (locally hamiltonian) vector field defined by

$$(X_S)_m = \frac{\mathrm{d}}{\mathrm{d}t}(\exp tX \cdot m)|_{t=0}\,.$$

momentum mapping

b) We will say that the action of G has a **momentum mapping** $J\colon S \to \mathscr{G}^*$ if each function $\hat{J}(X) \in C^\infty(S)$; defined by $\hat{J}(X),\ m = \langle J(m); X\rangle$ is a global hamiltonian for X_m, i.e.,

$$\mathrm{d}\hat{J}(X) = \mathrm{i}(X_S)\omega\,.$$

equivariant

c) A momentum mapping J is **equivariant** if

$$J(g\cdot m) = \mathrm{Ad}^*\, g^{-1}\cdot J(m)\,; \qquad \forall m \in S, \forall g \in G\,.$$

1) *Show that a momentum mapping J is equivariant if and only if map $\hat{J}\colon \mathscr{G} \to C^\infty(S)$ is a homomorphism of Lie algebras.*

Answer: Straightforward calculations. ■

2) In this question, let us suppose the action of G on S is free, i.e., if $g\cdot x = x$ for some $x \in S$ and some $g \in G$, then $g = e$, e being the identity of G. We also suppose that the action of G has an equivariant momentum mapping J and defines a foliation Φ (the common dimension of the leaves is dim G, because the action of G is free) in such a way that the set of leaves is a manifold and $\Pi\colon S \to S/G$ submersion.

Show that $\Pi^* C^\infty(S/G)$ is a Lie subalgebra of $C^\infty(S)$; and hence that S/G has a canonical Poisson structure such that $\Pi\colon S \to S/G$ is a Poisson map.

Answer: If $F, G \in \Pi^* C^\infty(S/G)$, there are unique $\bar{F}, \bar{G} \in C^\infty(S/G)$ such that $F = \bar{F}\cdot\Pi$; $G = \bar{G}\cdot\Pi$. For $\xi \in \mathscr{G}$ let $\xi_S = X_{\hat{J}(\xi)}$ be the hamiltonian vector field with hamiltonian $\hat{J}(\xi)$. We then have $X_{\hat{J}(\xi)}\{\bar{F}\circ\pi; \bar{G}\circ\pi\} = 0$; since the trajectories of $X_{\hat{J}(\xi)}$ are on leaves of Φ and $\bar{F}\circ\Pi$, $\bar{G}\circ\Pi$ are constant on each leaf. G is connected and acts freely $\{\bar{F}\circ\Pi; \bar{G}\circ\Pi\}_S$ is therefore constant on the leaves of Φ, consequently there is a unique function $\{\bar{F}; \bar{G}\}_{S/G} \in C^\infty(S/G)$ such that

$$\{\bar{F}\circ\Pi; \bar{G}\circ\Pi\}_S = \{\bar{F}; \bar{G}\}_{S/G}\circ\Pi\,.$$

$\{\bar{F}; \bar{G}\}_{S/G}$ is obviously a Poisson bracket on S/G. ■

3) We have now defined a function group $\Pi^* C^\infty(S/G)$ on the symplectic manifold S; *the purpose of this question is to find the corresponding polar function group.*

Let $T_s\Phi^\perp$ be the subspace of T_sS which is orthogonal — symplectic to the subspace $T_s(G \cdot s)$, itself tangent at s to the leaf through s of the function group $\Pi^* C^\infty(S/G)$. The subbundle $T\Phi^\perp$ is integrable because $\Pi^* C^\infty(S/G)$ is a Lie subalgebra of $C^\infty(S)$ (see **5**, question 3_3)). Let $\Phi^\perp$ be the corresponding foliation.

3i) *If* $v_s \in T_s\Phi^\perp$, *show that* $T_sJ \cdot v_s = 0$ *and conversely.*

Answer:

$$\forall \xi \in \mathcal{G}\,; \qquad \omega_S((\xi_s)_s;\, v_s) = \mathrm{d}\hat{J}(\xi)_s \cdot v_s\,.$$

But

$$\mathrm{d}\hat{J}(\xi)_s \cdot v_s = (T_sJ \cdot v_s) \cdot \xi\,.$$

Hence

$$T_sJ \cdot v_s = 0 \Leftrightarrow v_s \in T_s\Phi^\perp\,. \qquad \blacksquare$$

3ii) *Show that the momentum mapping J takes the same value on the points of a connected leaf of* $\Phi^\perp$.

Answer: Let $\Phi_S^\perp$ be a leaf through S, and let $C\colon I \to \Phi_S^\perp$ be an arc on that leaf. Then

$$t \in I\,; \qquad C'(t) = T_tC \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_t \in T_{C(t)}\Phi^\perp S\,.$$

Therefore (3i)

$$0 = T_{C(t)}J \cdot C'(t) = T_t(J \circ C)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_t\,. \qquad \blacksquare$$

(3ii) *Show that the mapping* $J\colon S \to \mathcal{G}^*$ *is a Poisson submersion.* $\blacksquare$

From this result, we see that $J^{-1}(\mu)$, $\mu \in J(S)$ is a submanifold of S and a union of connected leaves of $\Phi^\perp$ (see 3ii). We thus obtain the dual pair

$$\mathcal{G}^* \xleftarrow{J} S \xrightarrow{\Pi} S/G\,.$$

The equivariance of J implies

$$J^{-1}(\mathrm{Ad}^*\, G \cdot \mu) = G \cdot J^{-1}(\mu)\,.$$

Hence

$$\Pi J^{-1}(\mathrm{Ad}^* G\cdot\mu) = \Pi G\cdot J^{-1}(\mu)\,,$$

And $\Pi J^{-1}(\mu)$ is, by (6iii) a union of connected symplectic leaves of S/G.

3iv) *If* $S, S'\in J^{-1}(\mu)$ *and* $S' = gS$ *for some* $g\in G$, *show that*, $g\in G_\mu$ *where* $G_\mu\subset G$ *is the subgroup defined by* $\mathrm{Ad}^*\, G_\mu\cdot\mu=\mu$, *and conversely*, G_μ *acts on* $J^{-1}(\mu)$.

Answer: Straightforward ■

From this result, we see that $\Pi J^{-1}(\mu)\approx J^{-1}(\mu)/G\mu$. And therefore, $J^{-1}(\mu)/G_\mu$ carries the symplectic structure induced by this diffeomorphism.

If $F, H\in C^\infty(S/G)$, we have

$$\Lambda_{S/G}(\mathrm{d}F;\mathrm{d}H) = \omega_{S/G}(v_F;v_H)\,,$$

where $\omega_{S/G}$ is the symplectic form on the leaves of S/G. Therefore

$$\Lambda_{S/G}(\mathrm{d}F;\mathrm{d}H) = \Lambda_S(\mathrm{d}(F\circ\Pi);\mathrm{d}(H\circ\Pi)) = \omega(v_{F\circ\Pi};v_{H\circ\Pi})\,,$$

where ω is the symplectic form on S. Hence

$$\omega_{S/G}(v_F;v_H) = \omega(v_{F\circ\Pi};v_{H\circ\Pi})$$

$v_{F\circ\Pi}$, $v_{H\circ\Pi}$ are tangent to $J^{-1}(\mu)$ and v_F, v_H are their projected vector fields on $J^{-1}(\mu)/G_\mu$. We thus see that the symplectic manifold $J^{-1}(\mu)/G_\mu$ coincides with the reduced phase space of the Marsden–Weinstein theorem.

A particular case of the preceding one provides another example of a dual pair. It has been considered in [2] in an infinite dimensional context as well.

Let G be a connected Lie Group and T^*G its cotangent fibre bundle endowed with canonical symplectic structure. G acts freely and symplectically on the left and on the right on T^*G. More precisely

$$L\colon G\times T^*G\to T^*G$$
$$(g;\alpha_h)\to(T_{gh}L_{g^{-1}})^*\alpha_h$$

and

$$R\colon T^*G\times G\to T^*G$$
$$(\alpha_h;g)\to(T_{hg}R_{g^{-1}})^*\alpha_h\,.$$

Here L_g and R_g are respectively left and right translations on G.

Each of the foregoing actions has a momentum mapping. More precisely

$$J_L: T^*G \to \mathscr{G}^* ,$$
$$\alpha_g \to (T_e R_g)^* \alpha_g ,$$

and

$$J_R: T^*G \to \mathscr{G}^* ,$$
$$\alpha_g \to (T_e L_g)^* \alpha_g ,$$

i.e., a right translation to the fibre at $e \in G$, $T_e^* G \equiv \mathscr{G}^*$ for the left action and a left translation to the fibre for the right action.

4) *Show the following equalities by straightforward calculations*

$$J_L(L(g; \alpha_h)) = \mathrm{Ad}^*\, g^{-1} \cdot J_L(\alpha_h) ,$$
$$J_R(R(\alpha_h; g)) = \mathrm{Ad}^*\, g^{-1} \cdot J_r(\alpha_h) ,$$

i.e., momentum mappings J_L and J_R are equivariant.

Answer: Straightforward.

5) *Let $\mathscr{G}^*_+$ denote the canonical Poisson structure on $\mathscr{G}^*$ and $\mathscr{G}^*_-$, its opposite. Show that the momentum mappings $T^*G \xrightarrow{J_L} \mathscr{G}^*_+$ and $T^*G \xrightarrow{J_R} \mathscr{G}^*_-$ are surjective Poisson submersions.* ■

6) *Show the following equivalences by straightforward calculations*

$$J_L(\alpha_h) = J_L(\alpha_{h'}) \Leftrightarrow \alpha_{h'} = R(\alpha_h; h^{-1}h') ,$$
$$J_R(\alpha_h) = J_R(\alpha_{h'}) \Leftrightarrow \alpha_{h'} = L(h'h^{-1}; \alpha_h) . \qquad ■$$

Therefore, two points of T^*G have the same image under $J_L(J_R)$, if one of them is obtained by right translation (left translation) by an element of G.

Sets $R\backslash T^*G$ and T^*G/L are Poisson manifolds (see **5**, question 2). Hence the diagram

$$R\backslash T^*G \xleftarrow{\Pi_R} T^*G \xrightarrow{\Pi_L} T^*G/L ,$$

is a full dual pair. Likewise, the diagram

$$\mathscr{G}^*_+ \xleftarrow{J_L} T^*G \xrightarrow{J_R} \mathscr{G}^*_-$$

is also a full dual pair. We therefore have the following commutative diagram

$$\begin{tikzcd}
R\backslash T^*G \arrow[dr, "\bar{J}_L"'] & & T^*G \arrow[ll, "\Pi_R"'] \arrow[rr, "\Pi_L"] \arrow[dl, "J_L"] \arrow[dr, "J_R"'] & & T^*G/L \arrow[dl, "\bar{J}_R"] \\
& \mathcal{G}^*_+ & & \mathcal{G}^*_- &
\end{tikzcd}$$

where Poisson isomorphisms $\bar{J}_L$ and $\bar{J}_R$ are the quotient mappings of J_L and J_R respectively.

The foregoing general facts about dual pairs can be stated for this example as follows:

If $\mathcal{O}^+_\mu$ is an orbit in $\mathcal{G}^*_+$, $\Pi_L J_L^{-1}(\mathcal{O}^+_\mu)$ is a symplectic leaf in T^*G/L which is isomorphic to a reduced phase space $J_L^{-1}(\mu)/G_\mu$. Also,

$$\Pi_L J_L^{-1}(\mathcal{O}^+_\mu) = \bar{J}_R^{-1}(\mathcal{O}^-_\mu)\,,$$

where $\mathcal{O}^-_\mu$ is the orbit of μ in $\mathcal{G}^*_-$. Likewise,

$$\Pi_R \cdot J_R^{-1}(\mathcal{O}^-_\mu) = \bar{J}_L^{-1}(\mathcal{O}^+_\mu)$$

is the reduced phase space $G_\mu \backslash J_R^{-1}(\mu)$ which is isomorphic to a symplectic leaf of $R\backslash T^*G$.

8. FOR FURTHER STUDY

Much attention is currently being paid to both the theory and the applications of the mathematics of Poisson manifolds.

On the theoretical side Lichnerowicz has introduced and enlarged upon the notion of Jacobi manifolds, a generalization of both Poisson and Pfaffian manifolds. Specifically a Jacobi manifold is stratified (has generalized foliation). Each leaf of odd dimension is a pfaffian manifold, while leaves of even dimension are conform globally or conform locally symplectic manifolds. A theorem of local structure of Jacobi manifolds has been proved [4]. Their algebra of infinitesimal automorphisms and Chevalley cohomology have been determined in [4, 5].

The Poisson structure on $\mathcal{G}^*$ induces a Jacobi structure on a sphere of $\mathcal{G}^*$. The corresponding leaves are the orbits of G under the quotient coadjoint representation [7]. A theorem by Lichnerowicz [7] states that every proper contact homogeneous space is a covering of an orbit of odd dimension, and every locally conform hamiltonian symplectic homogeneous space is a covering of an orbit of even dimension.

These theorems must be considered in the same light as the Kirillov–Kostant–Souriau theorem for homogeneous symplectic spaces and coadjoint orbits.

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13. COMPLETELY INTEGRABLE SYSTEMS*

We will now proceed to define classical completely integrable hamiltonian systems. The reason for the choice of this name will become apparent from the underlying symmetries group and the classical method for integrating such systems. We will prove Arnold's theorem on the topology of the submanifolds generated by the trajectories of the system for the simplest case; then some recently stated theorems on functions in involution on coadjoint and adjoint orbits of connected reductive Lie groups, and the important Adler–Kostant–Symes theorem which generalizes the geometrical framework of the non-periodic Toda system. Finally we will indicate some recent developments in research on this subject.

1. DEFINITIONS AND ARNOLD'S THEOREM

We will speak of the **flow** of a vector field X on a manifold M when referring to 1-parameter local pseudogroup $M \to M$ generated by X.

Prove that, if X_1, X_2 are two vector fields on M such that $[X_1; X_2] = 0$, their flows commute, i.e., $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$ for each $\varphi_1 \in$ flow(X_1); $\varphi_2 \in$ flow(X_2). The image of a trajectory of X_2 (resp. X_1) by an element of the flow of X_1 (resp. X_2) will thus also be a trajectory of X_2 (resp. X_1).

*Contributed by C. Moreno.

Anwer: If $[X_1; X_2] = 0$, then X_1 is invariant under the flow of X_2 (p. 150); therefore, the flow of X_1 commutes with the flow of X_2 (p. 147). ∎

Let $(M; \Lambda)$ be a Poisson manifold. Let $F, H \in C^\infty(M)$ be the corresponding hamiltonian vector fields (p. 269 and Problem IV 11). We say that X_F generates a **1-parameter group of symmetries** of the hamiltonian system defined by H (the differential system defined by X_H) if $[X_F; X_H] = 0$. (See [1] for more general definitions of symmetries of differential systems).

1-parameter group of symmetries

F is called a **first integral** of X_H if $\{E; H\} = 0$, i.e. $(\mathrm{d}/\mathrm{d}t)F(C(t)) = 0$, for each trajectory $t \to C(t)$ of X_H.

first integral

We know that $\{F; H\} = 0$ implies $[X_F; X_H] = 0$ (Problem IV 11).

1) *Show that, conversely, if X_F generates a 1-parameter group of symmetries of the differential system defined by X_H, there is a function $\bar{F} \in C^\infty(M; \mathbb{R})$ such that* $(\mathrm{d}/\mathrm{d}t)\bar{F}(C(t)) = 0$. (We will call $\bar{F}$ a **generalized first integral** of the system X_H.)

generalized first integral

Answer: The function $\bar{F}(x; t) = F(x) - \{H; F\}(x)t$ satisfies the stipulated conditions. ∎

Remark. If we define, for each $t \in \mathbb{R}$, the following hamiltonian vector field on M (Problem IV 11):

$$X_{\bar{F}}(t) = \tilde{\Lambda}(\mathrm{d}\bar{F}(\cdot\,; t))\,, \qquad \text{we get } X_{\bar{F}}(t) = X_F\,,$$

whereby this field is easily seen to be independent of t.

2) *Let F be a first integral of X_H. Show that there exists a hamiltonian system (called a reduced system) on a Poisson manifold of* $\dim M - 2$ *such that every trajectory of X_H can be obtained from one of its trajectories by quadrature.*

Answer: Choose local coordinates on M as in Problem IV 12, 3, i.e., $p = F(x)$; $q = Q(x)$; $y = (y^1, \ldots, y^{m-2})$; $m = \dim M$. In these coordinates, the components of the tensor field Λ are

$$\Lambda(p; y) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & a(p; y) \\ 0 & -a^{\mathrm{T}}(p; y) & \Lambda_0(p; y) \end{pmatrix}$$

since $X_F = \partial/\partial q$. For each p, $\Lambda_0(p; y)$ defines a Poisson structure on a manifold of dimension $m - 2$, for which $(y^1, \ldots, y^{m-2})$ are local coordinates. Hamilton's equations are then

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = 0\,; \qquad \frac{\mathrm{d}q(t)}{\mathrm{d}t}$$

$$= -\frac{\partial H}{\partial p}(p, y) + \sum_{j=1}^{m-2} a^j(p; y)\frac{\partial H}{\partial y^j}(p; y)(\equiv A(p; y))\,,$$

$$\frac{\mathrm{d}y^i(t)}{\mathrm{d}t} = \sum_{j=1}^{m-2} \mathring{\Lambda}^{ij}(p; y)\frac{\partial H}{\partial y^j}(p; y)\,; \qquad i = 1, \ldots, m-2\,.$$

The latter, a hamiltonian system of dimension $m-2$, is a reduced system. If $y^i = y^i(t)$ is any solution to this system, the trajectories of the original system defined by X_H are then

$$p(t) = p_0\,; \qquad q(t) = \int A(p_0; y; t)\,\mathrm{d}t\,; \qquad y^i = y^i(t)\,;$$

which are clearly obtained by a quadrature from the trajectory $y^i = y^i(t)$. (See [1–3] for further information.) ■

Let $(M; \Lambda)$ be a symplectic manifold, and G a connected Lie group defining a symplectic action on M with equivariant momentum mapping J. If $H \in C^\infty(M)$ is G-invariant, the vector field X_H is projected on the hamiltonian vector field $X_{\bar{H}}$; $H = \bar{H} \circ \Pi$; Π: $M \to M/G$, of the Poisson manifold M/G (if this exists). The vector field $X_{\bar{H}}$ is tangent to the symplectic leaves of M/G (just as in Problem IV 12, 7) $\mathscr{G}^* \xleftarrow{J} M \xrightarrow{\Pi} M/G$ can be shown to be a dual pair, so that each connected symplectic leaf of M/G is isomorphic to some connected component of the reduced phase space $J^{-1}(\mu)/G_\mu$. Consequently X_H induces a hamiltonian system on this space.

We know from Lie that, if the isotropic group G_μ is solvable, the trajectories of X_H can be obtained by using quadratures from the trajectories of the induced system $X_{\bar{H}}$ on $J^{-1}(\mu)/G_\mu$. See [1].

3) *Let $(M; \Lambda)$ be a symplectic manifold of dimension $2m$. Let $F_i \in C^\infty(M)$, $i = 1, \ldots, n$ be functions in involution, i.e., $\{F_i; F_j\} = 0$ and independent, i.e.,* $\mathrm{d}F_1 \wedge \cdots \wedge \mathrm{d}F_n(m) \neq 0$. We suppose the flow *$\varphi_i$ of X_{F_i} to be complete. For each $(t_1, \ldots, t_k) \in \mathbb{R}^k$, let us define the following map*

$$M \to M\,,$$

$$(t_1, \ldots, t_k)(m) \to \varphi_1(t_1) \cdots \varphi_k(t_k)(m)\,. \tag{1}$$

Show that these maps define a symplectic action on M of the abelian group $\mathbb{R}^k$ with the equivariant momentum mapping

$$M \xrightarrow{J} \mathbb{R}^k$$

$$m \to (F_1(m), \ldots, F_k(m))\,.$$

Also prove that $J^{-1}(\mu)/G_\mu$; $\mu \in J(M)$ are the reduced phase spaces of this action.

Answer: Map (1) defines an $\mathbb{R}^k$ action on M, since flows φ_i; $i = 1, \ldots, n$ commute (See question 1). The Lie algebra of $\mathbb{R}^k$ is identical with $\mathbb{R}^k$, and the coadjoint action of the group $\mathbb{R}^k$ is trivial, since this group is abelian. F_i is a first integral for X_{F_j}, consequently J is an equivariant momentum mapping. Since $\mathrm{d}F_1 \wedge \cdots \wedge \mathrm{d}F_k(m) \neq 0$ every point $\mu \in J(M)$ is regular (i.e., $T_m J$ is surjective, $J(m) = \mu$). $\mathbb{R}^k$ acts on $J^{-1}(\mu)$, so that $J^{-1}(\mu)/\mathbb{R}^k$ is a reduced symplectic phase space for each $\mu \in J(M)$. ■

completely integrable by quadratures

In view of the above, the trajectories of X_{F_i} are obtained by quadratures from trajectories of the induced field on $J^{-1}(\mu)/\mathbb{R}^k$. If $k = n$, we know that $\dim J^{-1}(\mu)/\mathbb{R}^n = 0$, so that the system $X_{F_{i_0}}$, $i_0 \in [1, \ldots, k]$ is **completely integrable by quadratures.**

4) *Let $(M; \omega)$ be a symplectic manifold*, $\dim M = 2n$; *let $F_i \in C^\infty(M)$; $i = 1, \ldots, n$ be in involution in such a way that* $\mathrm{d}F_1 \wedge \cdots \wedge \mathrm{d}F_n(m) \neq 0$ $\forall m \in U \subset M$; *and let $(q^i; p_i)$ be canonical coordinates on M, i.e., $\omega = \Sigma\, \mathrm{d}p_i \wedge \mathrm{d}q^i$. We define $\theta = \Sigma\, p_i\, \mathrm{d}q^i$ and $N_c = \{m \in U \mid F_i(m) = C_i\}$*

a) *Prove that $(q^i; F_i)$ are local coordinates on M, that the induced l-form $i_c^*\theta$, i_c: $N_c \to M$* is closed, and that consequently $i_c^*\omega = 0$.

b) *Let $S(q^i; F_i)$ be some local function–such that $p_i = \partial S(q^i; F_i)/\partial q^i$. We also define $Q^i = \partial S(q^i; F_i)/\partial F_i$. Prove that $(Q^i; F_i)$ are canonical local coordinates on M.*

c) *Obtain the trajectories of $X_{F_{i_0}}$; $i_0 \in [1, \ldots, n]$*

Answer a: According to the stated hypothesis hamiltonians F_i, $i = 1, \ldots, n$ are independent. The transformation $F^i = F^i(q^k; p_k)$, $q^i = q^i$ is thus a change of local coordinates on M. We now write $p_i = \theta_i(q^k; F_k)$ and $i_c^*\theta = \theta_i(q^k; F_k)\,\mathrm{d}q^i$. We then have $\{p_i; p_j\} = \{\theta_i(q^k; F_k); \theta_j(q^k; F_k)\} = 0$ which is equivalent to

$$\frac{\partial \theta_i(q^k; F_k)}{\partial q^j} = \frac{\partial \theta_j(q^k; F_k)}{\partial F_i},$$

i.e., $i_c^*\theta$ is closed.

Answer b: Since $i_c^*\theta$ is closed, there is a locally defined function $S(q^k; F_k)$

such that $p_i = \partial S(q^k; F_k)/\partial q^i$. A straight-forward calculation gives us

$$\sum \mathrm{d}q^i \wedge \mathrm{d}p_i = \sum \mathrm{d}Q^i \wedge \mathrm{d}F_i\,.$$

Answer c: Consider the hamiltonian system defined by $F_{i_0} i_0 \in [1, \ldots, n]$. In coordinates $(Q^k; F_k)$ Hamilton's equations are

$$\frac{\mathrm{d}Q^i}{\mathrm{d}t} = \frac{\partial F_{i_0}(Q^k; F_k)}{\partial F_i}\,; \qquad \frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{\partial F_{i_0}(Q^k; F_k)}{\partial Q^i}\,.$$

Obviously we obtain,

$$\frac{\mathrm{d}Q^i(t)}{\mathrm{d}t} = \delta_{i_0 i}\,; \qquad \frac{\mathrm{d}P_i(t)}{\mathrm{d}t} = 0$$

and consequently

$$Q^i(t) = \delta_{i_0 i} + Q^i(t_0)\,; \qquad P_i(t) = P_i(t_0)\,.$$

The system has thus been "integrated by quadratures". ■

On a symplectic manifold $(M; \omega)$, dim $M = 2n$ there is generally no set of n independent functions $F_1, \ldots, F_n \in C^\infty(U)$ such that $\{F_i; F_j\} = 0$. Arnold's theorem states some of the conditions required for the existence of such a set.

5) We start from the same hypothesis as in question 3), where $K = n$.
a) *Let I_μ be a connected component of $J^{-1}(\mu)$. Prove that the $\mathbb{R}^n$-action on M defined in question* 3) *is a transitive action on I_μ. $I_\mu \approx \mathbb{R}^n/H$ will thus be a homogeneous manifold where H is some discrete subgroup of $\mathbb{R}^n$.*

We accept the following result, proved in [4] p. 274.:

Every discrete subgroup H of $\mathbb{R}^n$ determines a unique positive integer h and $(n-h)$ linearly independent vectors, $(a_{h+1}, \ldots, a_n)$, of $\mathbb{R}^n$ such that

$$H = \left\{ x \in \mathbb{R}^n \,|\, x = \sum_{i=h+1}^{n} m_i a_i;\ m_i \in \mathbb{Z} \right\}.$$

b) *Prove that $J^{-1}(\mu) \approx \mathbb{R}^h \times T^{n-h}$, where $T^{n-h} = \mathbb{R}^{n-h}/\mathbb{Z}^{n-h}$ is the $(n-h)$ torus. If I_μ is compact it will therefore be a torus.*

Answer a: From the definitions in question 3, it is clear that $\mathbb{R}^n$ acts on I_μ. The $\mathbb{R}^n$ orbit of a point $m \in I_\mu$ is then closed in I_μ. The map

$$\mathbb{R}^n \to I_\mu$$
$$(t_1, \ldots, t_n) \to \varphi_1(t_2) \cdots \varphi_n(t_n)(m)$$

is a local diffeomorphism $\forall m \in I_\mu$, since its tangent mapping at every point $(t_1, \ldots, t_n) \in \mathbb{R}^n$ is surjective by reason of the independence of the vector fields X_{F_i}; $i = 1, \ldots, n$. Therefore the image of $\mathbb{R}^n$ is open (and closed) in I_μ. The $\mathbb{R}^n$ action is then transitive. There is thus a closed sub-group $H \subset \mathbb{R}^n$, such that I_μ is diffeomorphic to $\mathbb{R}^n/H$. H is necessarily discrete because $n = \dim \mathbb{R}^n = \dim I_\mu$.

Answer b: This point can easily be proved using the above-mentioned result on H. ■

See Arnold [4] for further information on his theorem.

2. CONSTRUCTION OF FUNCTIONS IN INVOLUTION AND THE LAX EQUATION

In the preceding section, we saw that, as a general rule a hamiltonian system on a symplectic manifold is not completely integrable. Whittaker [15] analyzes most of the classical completely integrable systems.

The aim of this section is to show how to obtain systems of functions in involution on orbits of the coadjoint representation of a Lie group. These methods have recently been described in connection with the classical and quantum integrability of the systems considered by Calogero, Sutherland, Moser, Toda, etc. [11, 12]. They have been tested in the development of the scattering inverse method, or Lax method, which proves the classical and quantum integrability of infinite-dimensional hamiltonian systems, (Korteweg–de Vries (KdV), Kadomtsev–Petviashvili, Zakharov–Shabat, Dubrovin, etc., equations). In particular, we will see that the equations of hamiltonian dynamical systems on adjoint orbits of a reductive Lie group have Lax form.

1) *Let G be a connected Lie group acting symplectically on the symplectic manifold $(M; \omega)$ with equivariant momentum mapping J. We assume that we can define the dual pair* (*cf. IV* 12, **6**)

$$\mathscr{G}^* \xleftarrow{J} M \xrightarrow{\Pi} M/G$$

and the reduced phase spaces $J^{-1}(\mu)/G_\mu$, $\mu \in J(M)$.

Let $f, g \in C^\infty(M)$ be G-invariant functions, and f_μ, g_μ, the functions induced by f and g, respectively, on $J^{-1}(\mu)/G_\mu$. Prove that, if f and g are in involution on M, f_μ and g_μ are in involution on $J^{-1}(\mu)/G_\mu$.

Answer: Let $\bar{f}, \bar{g} \in C^\infty(M/G)$ be functions such that $f = \bar{f} \circ \pi$. We have

$$\{f; g\}_M = \{\bar{f}; \bar{g}\}_{M/G} \circ \pi .$$

If S_0 is a symplectic leaf of M/G and if $i: S_0 \to M/G$ is the inclusion, we have

$$\{\bar{f} \circ i; \bar{g} \circ i\}_{S_0} = \{\bar{f}; \bar{g}\}_{M/G} \circ i\,.$$

Since π is surjective, these two equalities prove that

$$\{f; g\}_M = 0 \Rightarrow \{\bar{f} \circ i; \bar{g} \circ i\}_{S_0} = 0\,.$$

We now observe that S_0 is isomorphic to a connected component of $J^{-1}(\mu)/G_\mu$, and $\bar{f} \circ i$, $\bar{g} \circ i$ correspond respectively to f_μ and g_μ in this isomorphism. ■

The following result was obtained and used by Mischchenko and Fomenko [5] in their study of integrable hamiltonian systems on Lie groups, and was generalized by Ratiu [6], to prove the involution of first integrals of the motion of the N-dimensional Lagrange top.

2) *Let f, h be two* Ad* *G-invariant functions on* $\mathcal{G}^*$, *i.e.*, $f(\mathrm{Ad}^*\, g \cdot \mu) = f(\mu)$; $\mu \in \mathcal{G}^*$, $g \in G$. *Let s, r be two real numbers and let λ be fixed in* $\mathcal{G}^*$. *Let us define the following two functions on* $\mathcal{G}^*$

$$f_s(\mu) = f(\mu + s\lambda)\,; \qquad h_r(\mu) = h(\mu + r\lambda)\,.$$

Prove that f_s and h_r are in involution on the Poisson manifold $\mathcal{G}^*$, *and consequently on any symplectic leaf, i.e., on the orbits of* Ad* G.

Answer: We have $f(\mathrm{Ad}^*\, tX \cdot \mu) = f(\mu)$; $\forall X \in \mathcal{G}$, $t \in \mathbb{R}$.

If we now derive at $t = 0$, we get $(\mathrm{d}f)_\mu \cdot \mathrm{ad}^*\, X\mu = 0$. Therefore $\mathrm{ad}^*(\mathrm{d}f)_\mu \cdot \mu = 0$ $((\mathrm{d}f)_\mu \in \mathcal{G}^{**} \equiv \mathcal{G})$.

If $r \neq s$, we can write

$$\mu = \frac{r}{r-s}(\mu + s\lambda) + \frac{s}{s-r}(\mu + r\lambda)\,,$$

and calculate

$$\begin{aligned}
\{f_s; h_r\}_{\mathcal{G}^*}(\mu) &= \langle \mu; [(\mathrm{d}f_s)_\mu; (\mathrm{d}h_r)_\mu]\rangle \\
&= \frac{r}{r-s}\langle(\mu + s\lambda); [(\mathrm{d}f)_{\mu+s\lambda}; (\mathrm{d}h_r)_\mu]\rangle \\
&\quad + \frac{s}{s-r}\langle(\mu + r\lambda); [(\mathrm{d}f_s)_\mu; (\mathrm{d}h)_{\mu+\lambda r}]\rangle \\
&= \frac{r}{r-s}\langle \mathrm{ad}^*(\mathrm{d}f)_{\mu+s\lambda} \cdot (\mu + s\lambda); (\mathrm{d}h_r)_\mu\rangle + \cdots = 0\,.
\end{aligned}$$

If $r = s$, the result holds by continuity. ■

3) In the following question we set out the main idea involved in the Lax method. We will see below that the hamiltonian systems on the adjoint orbits of reductive Lie groups provide one part of its geometrical foundations. This method has been very useful for obtaining first integrals in involution for hamiltonian systems of arbitrary (and even infinite) dimension.

Let $(t; X) \to B(t; X)$ *be a continuous map from* $\mathbb{R} \times M_k(\mathbb{C})$ *to the space* $M_k(\mathbb{C})$ *of complex matrices of order* k*, such that for every* $(t; X)$, $B(t; X) = -B^*(t; X)$, *where* $*$ *signifies "adjoint for matrices". Let* $t \to L(t) \in M_k(\mathbb{C})$ *be some solution of the non-linear equation*

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} = [B(t; L(t)); L(t)], \tag{1}$$

where $[\ ;\ ]$ *stands for "anticommutator for matrices" (cf. Problem III 14). Let* $t \to U(t) \in M_k(\mathbb{C})$ *be the solution to the Cauchy problem*

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = B(t; L(t)) \cdot U(t)\,; \qquad U(0) = I\,.$$

Prove that $U(t)^* \cdot U(t) = I$ *and*

$$L(t) = U(t)L(0)U^*(t)\,. \tag{2}$$

Lax pair
Lax equation

Quantities $\operatorname{Tr} L^n(t)$ *will then be time-independent.* $(L; B)$ is *called a* **Lax pair**, and (1), a **Lax equation**.

Conversely, let $t \to U(t) \in M_k(\mathbb{C})$ *be a given map such that* $U(t)^* \cdot U(t) = I$. *Let us define* $L(t)$ *by relation* (2), *where* $L(0)$ is arbitrary. *Prove that* $L(t)$ *satisfies* (1) *when* $B(t) = (\mathrm{d}U(t)/\mathrm{d}t) \cdot U^*(t)$.

Answer: The proof is straightforwardly obtained deriving (2). (See [7] for the original application of the Lax method to the KdV equation.) ■

As an example, consider the system [8] (p. 444)

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = -g(t; a(t); b(t))\,,$$

$$\frac{\mathrm{d}b(t)}{\mathrm{d}t} = g(t; a(t); b(t))\,.$$

A Lax pair for this system is

$$L(t) = \begin{pmatrix} b(t) & a(t) \\ a(t) & -b(t) \end{pmatrix}; \qquad B(t) = \begin{pmatrix} 0 & \frac{1}{2} g(\cdots) \\ -\frac{1}{2} g(\cdots) & 0 \end{pmatrix}.$$

Quantity $\sqrt{a^2(t) + b^2(t)}$ is time-independent whatever the continuous function g may be. If $g = 1$, the system is that of the 1-dimensional harmonic oscillator.

In the following two questions, we will see that hamiltonian equations on adjoint orbits of a reductive group are of the Lax type.

4) *Let G be a connected Lie group and let $\mathcal{G}$ be the corresponding Lie algebra. $\mathcal{G}^*$ will be the dual space of $\mathcal{G}$, $\mathcal{O}_{\bar{\mu}} = \mathrm{Ad}^*\, G \cdot \bar{\mu}$ and $\mu \in \mathcal{O}_{\bar{\mu}}$.*

a) *Prove that the tangent space to $\mathcal{O}_{\bar{\mu}}$ at the point μ is the subspace of $\mathcal{G}^*$ given by*

$$T_\mu \mathcal{O}_{\bar{\mu}} = \{\mathrm{ad}^*\, \xi \cdot \mu \mid \xi \in \mathcal{G}\} .$$

b) *Prove that the following* Ad^* *G-invariant differential 2-form on*

$$\begin{aligned}&\omega_{\bar{\mu}}(\mathrm{Ad}^*\, g^{-1} \cdot \bar{\mu})(\mathrm{ad}^*\, \xi \cdot \mathrm{Ad}^*_{g^{-1}}\, \bar{\mu}; \mathrm{ad}^*\, \eta \cdot \mathrm{Ad}^*_{g^{-1}} \bar{\mu})\\ &\quad = -\mathrm{Ad}^*_{g^{-1}}\, \bar{\mu} \cdot [\xi; \eta] ,\end{aligned}$$

defines a symplectic structure on $\mathcal{O}_{\bar{\mu}}$.

c) *Let us suppose $\mathcal{G}$ to be* **reductive**, *i.e., that there is a bilinear, nondegenerate form K on $\mathcal{G}$ such that* reductive

$$K(\mathrm{Ad}\, g \cdot \xi; \mathrm{Ad}\, g \cdot \eta) = K(\xi; \eta) ; \quad \xi, \eta \in \mathcal{G} ; \quad g \in G$$

(invariant by $\mathrm{Ad}\, G$*). Let $\phi: \mathcal{G} \to \mathcal{G}^*$ be the isomorphism defined by $\phi(\xi) \cdot \eta = K(\xi; \eta)$. Prove that $(\phi^* \omega_{\bar{\mu}}) = \hat{\omega}_{\bar{\xi}}$; $\bar{\xi} = \phi^{-1}(\bar{\mu})$ is a symplectic 2-form on the adjoint orbit* $\mathrm{Ad}\, G \cdot \bar{\xi}$*, invariant by* $\mathrm{Ad}\, G$*, and is given by*

$$\begin{aligned}&\hat{\omega}_{\bar{\xi}}(\mathrm{Ad}\, g \cdot \bar{\xi})(\mathrm{ad}\, \eta \cdot \mathrm{Ad}\, g\bar{\xi}; \mathrm{ad}\, \zeta \cdot \mathrm{Ad}\, g\bar{\xi})\\ &\quad = -K(\mathrm{Ad}\, g \cdot \bar{\xi}; [\eta; \zeta]) ,\end{aligned}$$

where

$$T_\xi \mathcal{O}_{\bar{\xi}} = \{\mathrm{ad}\, \eta \cdot \xi \mid \eta \in \mathcal{G}\}$$

is the tangent space to the orbit $\hat{\mathcal{O}}_{\bar{\xi}}$ at $\xi \in \hat{\mathcal{O}}_{\bar{\xi}}$.

Answer (Indications)

a) The arc $\mathrm{Ad}^* \exp t\xi \cdot \mu$; $t \in \mathbb{R}$; $\xi \in \mathcal{G}$ defines $\mathrm{ad}^*\, \xi \cdot \mu$ as a tangent vector to the orbit $\mathcal{O}_{\bar{\mu}}$ at μ. We can also see that $\mathrm{ad}^*\, \xi \cdot \mu = 0 \Leftrightarrow \xi \in \mathcal{G}_\mu$, where $\mathcal{G}_\mu$ is the Lie algebra of the Lie subgroup G_μ defined by $\mathrm{Ad}^*\, G_\mu \cdot \mu = \mu$.

b) This point can be proved by a straightforward calculation.

c) The invariance of K implies (by derivation):

$$K(\mathrm{ad}\, \zeta \cdot \xi; \eta) + K(\xi; \mathrm{ad}\, \zeta \cdot \eta) = 0 .$$

And

$$\phi(\mathrm{Ad}\, g \cdot \xi) = \mathrm{Ad}^*\, g^{-1} \cdot \phi(\xi)\,.$$

Since ϕ is linear, the latter relation implies

$$\phi(\mathrm{ad}\, \eta \cdot \xi) = -\mathrm{ad}^*\, \eta \cdot \phi(\xi)\,.$$

The expression for $\hat{\omega}_{\bar{\xi}}$ is obtained by straightforward calculations from these equalities and from its own definition, which is $\hat{\omega}_{\bar{\xi}} = (T_{\bar{\xi}}\phi)^* \omega_{\phi(\bar{\xi})}$. ∎

We now introduce some of the notations from [9]. It is very convenient to keep in mind the isomorphism ε: $\mathscr{G} \to \mathscr{G}^{**}$ defined by $\varepsilon(\xi)\beta = \beta(\xi)$, $\beta \in \mathscr{G}^*$. If $H \in C^\infty(\mathscr{G}^*)$; $(\mathrm{d}H)_\mu \in \mathscr{G}^{**}$; $\mu \in \mathscr{G}^*$, then $\varepsilon^{-1}(dH)_\mu \in \mathscr{G}$. If ϕ: $\mathscr{G} \to \mathscr{G}^*$ is the isomorphism defined by K (see the preceding question), we have $E = H \circ \phi \in C^\infty(\mathscr{G})$

$$(\mathrm{d}\phi)_\xi \xi' = \phi(\xi')\,; \qquad \forall \xi, \xi' \in \mathscr{G}$$

and

$$(\mathrm{d}H)_{\phi(\xi)} = \varepsilon(\phi^{-1}(\mathrm{d}E)_\xi)\,.$$

gradient of E

We will speak of the **gradient of E** when referring to the vector field on $\mathscr{G}$ defined as follows

$$(\mathrm{grad}\, E)_\xi = \phi^{-1}\, (\mathrm{d}E)_\xi = \varepsilon^{-1}(\mathrm{d}H)_{\phi(\xi)}\,.$$

5) *Let H be a C^∞ function on $\mathscr{G}$; let $\bar{H}$ be the restriction of H to any orbit $\mathcal{O}_{\bar{\mu}}$ in $\mathscr{G}^*$; and let $\bar{E}$ be the restriction of $E = H \cdot \phi$ to any orbit $\mathcal{O}_{\bar{\xi}}$; $\bar{\xi} = \phi^{-1}(\bar{\mu})$ in $\mathscr{G}$. Obtain the expressions of the hamiltonian vector fields $X_{\bar{H}}$ and $X_{\bar{E}}$. Conclude that the flow of $X_{\bar{E}}$ satisfies a Lax-type equation.*

Answer a: $X_{\bar{H}}$ is defined by $(\mathrm{i}(X_{\bar{H}})\omega)_\mu = (\mathrm{d}\bar{H})_\mu$; $\mu \in \mathcal{O}_{\bar{\mu}}$. For any $\eta \in \mathscr{G}$ we must then have

$$\begin{aligned}\omega_\mu((X_{\bar{H}})_\mu; \mathrm{ad}^*\, \eta \cdot \mu) &= (\mathrm{d}\bar{H})_\mu .\, \mathrm{ad}^*\, \eta \cdot \mu \\ &= (\mathrm{d}H)_\mu \cdot \mathrm{ad}^*\, \eta \cdot \mu = -\mu[\varepsilon^{-1}(\mathrm{d}H)_\mu; \eta]\,.\end{aligned}$$

We thus get $(X_{\bar{H}})_\mu = \mathrm{ad}^*\, \varepsilon^{-1}(\mathrm{d}H)_\mu \cdot \mu = \mathrm{ad}^*(\mathrm{grad}\, E)_{\phi^{-1}(\mu)} \cdot \mu$.

Answer b: $X_{\bar{E}}$ is defined by $(\mathrm{i}(X_{\bar{E}})\hat{\omega})_\xi = (\mathrm{d}\bar{E})_\xi$; $\xi \in \hat{\mathcal{O}}_{\bar{\xi}}$; $\xi = \phi^{-1}(\mu)$. For any η in $\mathscr{G}$ we then have

$$\begin{aligned}\hat{\omega}_\xi(X_{\bar{E}}; \mathrm{ad}\, \eta \cdot \xi) &= (\mathrm{d}\bar{E})_\xi \cdot \mathrm{ad}\, \eta \cdot \xi = (\mathrm{d}E)_\xi \cdot \mathrm{ad}\, \eta \cdot \xi \\ &= (\mathrm{d}H)_{\phi(\xi)} \cdot \phi(\mathrm{ad}\, \eta \cdot \xi) = k(\xi; [\varepsilon^{-1}(\mathrm{d}H)_{\phi(\xi)}; \eta])\,.\end{aligned}$$

We thus obtain

$$(X_{\tilde{E}})_\xi = -\mathrm{ad}(\mathrm{grad}\, E)_\xi \cdot \xi\,.$$

Answer c: If $\varphi(t;\ \xi_0) \in \mathscr{G}$ is an integral curve of $X_{\tilde{E}}$, it satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi(t;\ \xi_0) &= -\mathrm{ad}(\mathrm{grad}\, E)_{\varphi(t;\ \xi_0)} \cdot \varphi(t;\ \xi_0)\\ &= -[(\mathrm{grad}\, E)_{\varphi(t;\ \xi_0)};\ \varphi(t;\ \xi_0)]\,.\end{aligned}$$

If we transform $(\mathrm{grad}\, E)_{\varphi(t;\ \xi_0)}$ and $\varphi(t;\ \xi_0)$ into antisymmetric matrices (into anti-self-adjoint operators) by means of a finite (infinite) dimensional unitary representation of G, the latter equation will be of Lax type exactly as described in question 3). ■

The flow of the non-periodic Toda system satisfies the foregoing equation on some adjoint orbit of the group of lower-triangular matrices with positive diagonal entries. By analyzing the geometrical framework of this system, Adler–Kostant–Symes obtained, in particular, the theorem explained below.

Our exposition closely follows papers [9; 10].

Let $\mathcal{N}$ be a Lie subalgebra of $\mathscr{G}$, and let $\mathscr{K}$ be a complementary subspace of $\mathcal{N}$ in $\mathscr{G}$. We write $i_{\mathcal{N}}\colon \mathcal{N} \to \mathscr{G}$ and $i_{\mathscr{K}}\colon \mathscr{K} \to \mathscr{G}$ for the inclusions of $\mathcal{N}$ and $\mathscr{K}$, respectively, in $\mathscr{G}$. $\Pi_{\mathcal{N}}\colon \mathscr{G} \to \mathcal{N}$ and $\Pi_{\mathscr{K}}\colon \mathscr{G} \to \mathscr{K}$ will be the projections defined by the splitting $\mathscr{G} = \mathscr{K} \oplus \mathcal{N}$. We thus have $\mathscr{G}^* = \mathscr{K}^* \oplus \mathcal{N}^*$. If $\mu \in \mathscr{G}^*$; $\mu = \mu|_{\mathscr{K}} + \mu|_{\mathcal{N}}$, where $(|)$ represents restriction, then $\mu|_{\mathscr{K}} \in \mathscr{K}^*$ and $\mu|_{\mathcal{N}} \in \mathcal{N}^*$. $\rho \in \mathscr{K}^*(\nu \in \mathcal{N}^*)$ defines an element of $\mathscr{G}^*$, also written $\rho(\nu)$, if we impose $\rho(\mathcal{N}) = 0$; $\nu(\mathscr{K}) = 0$ and linearity. Let $i_{\mathscr{K}^*}\colon \mathscr{K}^* \to \mathscr{G}^*$, $i_{\mathcal{N}^*}\colon \mathcal{N}^* \to \mathscr{G}^*$ be the inclusions of $\mathscr{K}^*$ and $\mathcal{N}^*$, respectively, in $\mathscr{G}^*$ and let $\Pi_{\mathscr{K}^*}\colon \mathscr{G}^* \to \mathscr{K}^*$; $\Pi_{\mathcal{N}^*}\colon \mathscr{G}^* \to \mathcal{N}^*$ be the corresponding projections defined by $\Pi_{\mathscr{K}^*}\mu = \mu|_{\mathscr{K}}$ and $\Pi_{\mathcal{N}^*}\mu = \mu|_{\mathcal{N}}$. The dual mappings $i^*_{\mathcal{N}}\colon \mathscr{G}^* \to \mathcal{N}^*$; $i^*_{\mathscr{K}}\colon \mathscr{G}^* \to \mathscr{K}^*$ then satisfy $i^*_{\mathcal{N}} = \Pi_{\mathcal{N}^*}$ and $i^*_{\mathscr{K}} = \Pi_{\mathscr{K}^*}$. If $\mathrm{ad}^{\mathcal{N}}$ and ad signifies the adjoint representations of $\mathcal{N}$ and $\mathscr{G}$ respectively, we obviously have $\mathrm{ad}^{\mathcal{N}}\,\xi = \mathrm{ad}\,\xi \cdot i_{\mathcal{N}}$ and $\mathrm{ad}^{\mathcal{N}^*}\xi = \Pi_{\mathcal{N}^*}\,\mathrm{ad}^*\,\xi$. Consequently, if G is a connected Lie group with Lie algebra $\mathscr{G}$ and N is the connected Lie subgroup of G with Lie algebra $\mathcal{N}$, we have

$$\mathrm{Ad}^N g = \mathrm{Ad}\ g \cdot i_{\mathcal{N}}\,; \qquad \mathrm{Ad}^{N*} g = \Pi_{\mathcal{N}^*} \cdot \mathrm{Ad}^*\ g\,; \qquad g \in N\,.$$

Let $\varepsilon_{\mathscr{K}}\colon \mathscr{K} \to \mathscr{K}^{**}$; $\varepsilon_{\mathcal{N}}\colon \mathcal{N} \to \mathcal{N}^{**}$ be the canonical isomorphisms. If $f \in C^\infty(\mathscr{G}^*)$ then $(\mathrm{d}f)_\xi \in \mathscr{G}^{**}$; $\xi \in \mathscr{G}$. We also have $\mathscr{G}^{**} = \mathscr{K}^{**} \oplus \mathcal{N}^{**} \approx \mathscr{K} \oplus \mathcal{N}$, for the isomorphism $\varepsilon_{\mathscr{K}} \oplus \varepsilon_{\mathcal{N}}$. The partial differentials of f are then

$$(\mathrm{d}^{\mathscr{K}^*} f)_\xi = (\mathrm{d}f)_\xi|_{\mathscr{K}^*} \in \mathscr{K}^{**}\,; \qquad (\mathrm{d}^{\mathcal{N}^*} f)_\xi = (\mathrm{d}f)_\xi|_{\mathcal{N}^*} \in \mathcal{N}^{**}\,.$$

If $\nu \in \mathcal{K}^*$ we have

$$\begin{aligned}\langle \varepsilon_{\mathcal{K}} \circ \Pi_{\mathcal{K}} \circ \varepsilon^{-1}(\mathrm{d}f)_\xi ; \nu \rangle &= \langle \nu ; \Pi_{\mathcal{K}} \circ \varepsilon^{-1}(\mathrm{d}f)_\xi \rangle \\ &= \langle \nu ; \varepsilon^{-1}(\mathrm{d}f)_\xi \rangle = (\mathrm{d}f)_\xi \cdot \nu .\end{aligned}$$

We thus get

$$(\mathrm{d}^{\mathcal{K}^*} f)_\xi = \varepsilon_k \circ \Pi_{\mathcal{K}} \circ \varepsilon^{-1}(\mathrm{d}f)_\xi$$

and similarly

$$(\mathrm{d}^{\mathcal{N}^*} f)_\xi = \varepsilon_{\mathcal{N}} \circ \Pi_{\mathcal{N}} \circ \varepsilon^{-1}(\mathrm{d}f)_\xi .$$

6) *Let $\mathcal{O}_{\bar\nu}^N$ be the* Ad^{N^*} *N-orbit of a point $\bar\nu \in \mathcal{N}^*$. Let f, g be functions in $C^\infty(\mathcal{G}^*)$ and $\overline{f \circ i_{\mathcal{N}^*}}; \overline{h \circ i_{\mathcal{N}^*}}$, their restrictions to $\mathcal{O}_{\bar\nu}^N$. Obtain the expressions for the hamiltonian vector fields $X_{\overline{f\circ i_{\mathcal{N}^*}}}$; $X_{\overline{h\circ i_{\mathcal{N}^*}}}$ and for the Poisson bracket*

$$\{\overline{f \circ i_{\mathcal{N}^*}}; \overline{h \circ i_{\mathcal{N}^*}}\}_\nu^{\mathcal{N}^*} , \qquad \nu \in \mathcal{O}_{\bar\nu}^N .$$

Answer a: From question 5) and the above notations, we obtain

$$\begin{aligned}(X_{\overline{f\circ i_{\mathcal{N}^*}}})_\nu &= \mathrm{ad}^{\mathcal{N}^*} \varepsilon_{\mathcal{N}}^{-1} \mathrm{d}(\overline{f \circ i_{\mathcal{N}^*}})_\nu \cdot \nu \\ &= \mathrm{ad}^{\mathcal{N}^*} \varepsilon_{\mathcal{N}}^{-1}((\mathrm{d}f)_{i_{\mathcal{N}^*}\nu} \circ (\mathrm{d}i_{\mathcal{N}^*})_\nu)\nu = \mathrm{ad}^{\mathcal{N}^*} \varepsilon_{\Pi}^{-1}(\mathrm{d}^{\mathcal{N}^*} f)_\nu \cdot \nu \\ &= \mathrm{ad}^{\mathcal{N}^*} \Pi_{\mathcal{N}} \varepsilon^{-1}(\mathrm{d}f)_\nu \cdot \nu = \Pi_{\mathcal{N}^*}\, \mathrm{ad}^*\, \Pi_{\mathcal{N}} \varepsilon^{-1}(\mathrm{d}f)_\nu \cdot \nu .\end{aligned}$$

Answer b: From question 4, b) we get

$$\begin{aligned}\{\overline{f \circ i_{\mathcal{N}^*}}; \overline{h \circ i_{\mathcal{N}^*}}\}_\nu^{\mathcal{N}^*} &= \omega_\nu^{\mathcal{N}^*}(X_{\overline{f\circ i_{\mathcal{N}^*}}}; X_{\overline{h\circ i_{\mathcal{N}^*}}}) \\ &= -\nu[\Pi_{\mathcal{N}} \circ \varepsilon^{-1}(\mathrm{d}f)_\nu ; \Pi_{\mathcal{N}} \circ \varepsilon^{-1}(\mathrm{d}h)_\nu] .\end{aligned}$$ ∎

7) (*Adler–Kostant–Symes theorem*). *Let $\mathcal{G} = \mathcal{K} \oplus \mathcal{N}$ be as above; let $\mathcal{K}$ and $\mathcal{N}$ be two Lie subalgebras of $\mathcal{G}$. We now consider the set*

$$A = \{f \in C^\infty(\mathcal{G}^*) | f(\mathrm{Ad}^*\, G \cdot \mu) = f(\mu);\ \mu \in \mathcal{G}^*\} ,$$

of Ad^* *G-invariant functions, on $\mathcal{G}^*$. a) Prove that for each $\bar\nu \in \mathcal{N}^*$ the set*

$$\bar A_{\bar\nu} = \{\overline{f \circ i_{\mathcal{N}^*}} \in C^\infty(\mathcal{O}_{\bar\nu}^{N^*}) | f \in A\} ,$$

is in involution on $\mathcal{O}_{\bar\nu}^{\mathcal{N}^*}$. Consequently the set

$$B = \{f \circ i_{\mathcal{N}^*} \in C^\infty(\mathcal{N}^*) | f \in A\}$$

is in involution on the Poisson manifold $\mathcal{N}^*$. b) Prove that the vector field on $\mathcal{O}_{\bar\nu}^N$, with hamiltonian $\overline{f \circ i_{\mathcal{N}^*}}$ is

$$(X_{\overline{f\circ i_{\mathcal{N}^*}}})_\nu = -\mathrm{ad}^*(\Pi_{\mathcal{K}} \varepsilon^{-1}(\mathrm{d}f)_\nu) \cdot \nu$$

Answer b: We saw that

$$f \in A \Leftrightarrow \mathrm{ad}^* \, \varepsilon^{-1}(\mathrm{d}f)_\nu \cdot \nu = 0; \; \nu \in \mathscr{G},$$

thus

$$\mathrm{ad}^* \, \Pi_{\mathcal{N}} \varepsilon^{-1}(\mathrm{d}f)_\nu \cdot \nu + \mathrm{ad}^* \, \Pi_{\mathscr{K}} \varepsilon^{-1}(\mathrm{d}f)_\nu \cdot \nu = 0 .$$

Therefore the required expression for $(X_{\overline{f \circ i_{\mathcal{N}^*}}})_\nu$ may be obtained from question 6.

Answer a: We have

$$\begin{aligned}
\{\overline{f \circ i_{\mathcal{N}^*}}; \overline{h \circ i_{\mathcal{N}^*}}\}_\nu^{\mathcal{N}^*} &= -(\mathrm{ad}^* \, \Pi_{\mathcal{N}} \varepsilon^{-1}(\mathrm{d}f)_\nu) \cdot \nu \cdot \Pi_{\mathcal{N}} \varepsilon^{-1}(\mathrm{d}h)_\nu \\
&= -\mathrm{ad}^* \, \Pi_{\mathcal{N}} \varepsilon^{-1}(\mathrm{d}h)_\nu \cdot \nu \cdot \Pi_{\mathscr{K}} \varepsilon^{-1}(\mathrm{d}f)_\nu \\
&= \mathrm{ad}^* \Pi_{\mathscr{K}} \varepsilon^{-1}(\mathrm{d}h)_\nu \cdot \nu \cdot \Pi_{\mathscr{K}} \varepsilon^{-1}(\mathrm{d}f)\nu \\
&= -\nu[\Pi_{\mathscr{K}} \varepsilon^{-1}(\mathrm{d}f)_\nu; \Pi_{\mathscr{K}} \varepsilon^{-1}(\mathrm{d}h)_\nu] = 0 .
\end{aligned}$$

The latter equality holds because $\nu \in \mathcal{N}^* \subset \mathscr{G}^*$ and the bracket is in $\mathscr{K}$. Therefore $\{f \circ i_{\mathcal{N}^*}; h \circ i_{\mathcal{N}}{}^*\}^{\mathcal{N}^*} = 0$ holds on the Poisson manifold $\mathcal{N}^*$. ■

Let us now suppose that we can define a bilinear form on K as in question 4c, and let ϕ: $\mathscr{G} \to \mathscr{G}^*$ be the isomorphism defined by K. Obviously $\phi^{-1}\mathscr{K}^* = \mathcal{N}^\perp$ and $\phi^{-1}\mathcal{N}^* = \mathscr{K}^\perp$, where $\mathcal{N}^\perp$ and $\mathscr{K}^\perp$ are the K-orthogonal subspaces to $\mathscr{K}$ and $\mathcal{N}$, respectively. We have $\mathscr{G} = \mathscr{K}^\perp \oplus \mathcal{N}^\perp$. It is easy to see that $\phi^{-1} \circ \Pi_{\mathcal{N}^*} = \Pi_{\mathscr{K}^\perp} \circ \phi^{-1}$ and that the Ad^{N^*} action of N on $\mathcal{N}^*$ translated to $\mathscr{K}^\perp$, by means of ϕ is

$$A(n) = \phi^{-1} \circ \mathrm{Ad}^{N^*} n \circ \phi = \Pi_{k^\perp} \circ \mathrm{Ad}\, n^{-1} ; \qquad n \in N$$

$$A(\eta) = -\Pi_{k^\perp} \, \mathrm{ad}\, \eta ; \qquad \eta \in \mathcal{N} .$$

By means of these relations and ϕ, we can read question 7 on $\mathscr{K}^\perp$. The Adler–Kostant–Symes theorem often appears in the following form in specific problems of mathematical physics.

8) Hypothesis and notations are as in question 7).

Let us consider the set

$$C = \{F \in C^\infty(\mathscr{G}) | F(\mathrm{Ad}\, g \cdot \eta) = F(\eta); \eta \in \mathscr{G}\}$$

of Ad *G-invariant functions on $\mathscr{G}$. Prove that for each $\bar{\xi} \in \mathscr{K}^\perp$ the set of functions*

$$\bar{C}_{\bar{\xi}} = \{\overline{F \circ i_{\mathscr{K}^\perp}} \in C^\infty(A(N)\bar{\xi}) | F \in C\}$$

is in involution on $A(N)\bar{\xi}$. *Consequently the set*

$$D = \{F \circ i_{\mathcal{K}^\perp} \in C^\infty(\mathcal{K}^\perp) | F \in C\}$$

is in involution on the Poisson manifold $\mathcal{K}^\perp$.

Prove that the vector field on $A(N)\bar{\xi}$ *with hamiltonian* $\overline{F \circ i_{\mathcal{K}^\perp}}$ *is*

$$(X_{\overline{F \circ i_{\mathcal{K}^\perp}}})_\xi = [\Pi_k(\text{grad } E)_\xi ; \xi] ; \qquad \xi \in \mathcal{K}^\perp . \qquad \blacksquare$$

3. FOR FURTHER STUDY

a) The observation that the systems Calogero, Sutherland, Toda, etc. are hamiltonian systems defined on symplectic manifolds which are reduced spaces of some other hamiltonian systems having obvious first integrals in involution and such that they induce a complete and independent set of first integrals in involution on the reduced system, led to a fuller understanding of the proof of their complete integrability and of the manner of effecting the integrations themselves. This insight is systematically developed in the review [12] (where the intimate connection between integrability and the structure of simple Lie algebras [11, 13] is also proved) and can be verified in all the involutions theorems given in [9].

Another approach developed by Adler and van Moerbeck involves the systematic use of Kac–Moody algebras. This method has proved to be more suitable for dealing with the integrability of a richer class of systems [9, 10, 13]. This is currently an actively developing field of research.

b) Many other integrable systems have recently been obtained, in particular by the method of "dressing transforms" [14] in connection with the Kadomtsev–Petviashvili equation. Understanding these systems requires some change in the geometrical framework. Their symmetries groups have a Poisson structure (closely related to the classical Yang-Baxter equation). The geometrical framework within which the above systems can be understood invariably includes a description of the Poisson structure of their symmetries groups. This implies, in particular, that the dual structure of the Poisson structure of the system must be taken as fundamental. Many other new aspects of these systems are revealed when they are quantized. This is also a fast-growing field of research.

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V. RIEMANNIAN MANIFOLDS. KÄHLERIAN MANIFOLDS

1. NECESSARY AND SUFFICIENT CONDITIONS FOR LORENTZIAN SIGNATURE

1) *Show that the following statements are equivalent*:
a) g *is hyperbolic, i.e., of signature* $(1, d-1) = (+, -, \dots, -)$ *and* $\partial/\partial x^0$ *is time-like.*
b) g_{00} *is positive and* $g^{ij}v_i v_j$ *is negative definite.*

Answer: a) $\Rightarrow$ b); indeed

$$g_{00} = g(\partial/\partial x^0, \partial/\partial x^0) > 0 \quad \text{since } \partial/\partial x_0 \text{ is timelike}\,.$$

Since $g_{00} > 0$, we can write

$$\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \left(\sqrt{g_{00}}\,\mathrm{d}x^0 + \frac{g_{0i}}{\sqrt{g_{00}}}\,\mathrm{d}x^i\right)^2 + \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{1}$$

with $\gamma_{ij} \equiv -g_{0i}g_{0j}/g_{00} + g_{ij}$.
We note that (γ_{ij}) is the inverse matrix of (g^{ij}): we have $g^{ij}\gamma_{jk} = \delta^i_k$. Indeed $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\beta$ implies

$$\begin{cases} g^{ij}g_{jk} + g^{i0}g_{0k} = \delta^j_k \\ g^{ij}g_{j0} + g^{i0}g_{00} = 0, \qquad \text{i.e., } g^{i0} = -\dfrac{1}{g_{00}}\,g^{ij}g_{j0}\,. \end{cases}$$

Inserting this value of g^{i0} in the previous equation gives

$$\delta^i_k = g^{ij}\left(g_{jk} - \frac{1}{g_{00}}\,g_{j0}g_{k0}\right) = g^{ij}\gamma_{jk}\,. \tag{2}$$

Next we note that the quadratic form defined by γ_{ij} is negative definite. Indeed, we deduce from the identity (1) that $-\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ admits a decomposition as a sum of squares of $d-1$ linearly independent one forms:

$$\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = -\sum_{i=1}^{d-1}(a^i_j\,\mathrm{d}x^j)^2\,.$$

Therefore γ_{ij} is negative definite, namely $\gamma_{ij}u^iu^j < 0$ for all $u^i \neq 0$. The same is true of its inverse $g^{ij}v_iv_j$ since they coincide in an orthonormal basis.

Answer: b)$\Rightarrow$a); indeed

$$0 < g_{00} = g(\partial/\partial x^0, \partial/\partial x^0) \qquad \text{implies } \partial/\partial x^0 \text{ timelike}.$$

Since $g_{00} > 0$, the quadratic form ds^2 can be decomposed as in eq. (1). If $g^{ij}v_iv_j$ is a negative definite quadratic form, so is the inverse quadratic form $\gamma_{ij}\, dx^i\, dx^j$. It follows from (1) that g is hyperbolic of signature $(1, d-1)$.

2) *Show that the following statements are equivalent*:
a) *g is hyperbolic of signature $(1, d-1)$ and $\{\partial/\partial x^i\}$ is spacelike.*
b) *g^{00} is positive and $g_{ij}v^iv^j$ is negative definite.*

Answer: We note that $g^{00} = g(\boldsymbol{n}, \boldsymbol{n})$ where $\boldsymbol{n}$ is the gradient of the "time function", x^0. Indeed for such an n

$$n_i = \partial x^0/\partial x^i = 0\,, \qquad \text{for all } i\,,$$

and $n_0 = 1$, so

$$g(\boldsymbol{n}, \boldsymbol{n}) = g^{\alpha\beta}n_\alpha n_\beta = g^{00}\,. \tag{3}$$

To prove a)$\Rightarrow$b), we prove first that g hyperbolic together with $\partial/\partial x^i$ spacelike imply $g^{00} > 0$. Let the quadratic form $g_{\alpha\beta}u^\alpha u^\beta$ be decomposed as follows

$$g_{\alpha\beta}u^\alpha u^\beta \equiv U(u^0)^2 - \sum_i (a_{ih}u^h + \lambda_i u^0)^2\,, \tag{4}$$

with

$$\sum_i a_{ih}a_{ik} = -g_{hk}\,, \tag{5}$$

$$\sum_i a_{ih}\lambda_i = -g_{h0}\,, \tag{6}$$

$$U = g_{00} + \sum_i (\lambda_i)^2\,. \tag{7}$$

Since all the $\partial/\partial x^i$ are spacelike, the quadratic form defined by $-g_{hk}$ is positive definite. Equation (5) is the decomposition of a positive definite symmetric matrix $G = \tilde{A}A$. For such a decomposition, A is real, i.e., the a_{ih} are real. Since g_{hk} is definite, $\det g_{hk} \neq 0$ and $\det a_{ih} \neq 0$; the matrix (a_{ih}) is real and invertible and λ_i is real by (6).

Inserting in (7) the value of λ_i given by (6) yields

$$\begin{aligned} U &= g_{00} + \sum_{h,i,k} (a^{-1})_{hi}(a^{-1})_{ki} g_{h0} g_{k0} \\ &= (g^{00})^{-1}\Big[g^{00}g_{00} - \sum_{h,k} (g_{hk})^{-1} g_{h0} g_{k0} g^{00}\Big] \\ &= (g^{00})^{-1}\Big[1 - \sum_{h,k} g_{h0}(g^{h0} + (g_{hk})^{-1} g_{k0} g^{00})\Big] \\ &= (g^{00})^{-1}. \end{aligned} \tag{8}$$

We have $U > 0$ since g is of signature $(1, d-1)$, hence $g^{00} > 0$, i.e., n timelike.

b)$\Rightarrow$a). Indeed $g_{ij}v^i v^j$ negative definite implies that the hyperplane generated by the $\partial/\partial x^i$ is spacelike. Since $g^{00} > 0$ the quadratic form ds^2 can be decomposed as in eq. (9), and g is hyperbolic of signature $(1, d-1)$.

3) *Show that the following statements are equivalent.*
a) *g is hyperbolic of signature $(1, d-1)$, g_{00} and g^{00} are positive.*
b) *g is hyperbolic of signature $(1, d-1)$, $\partial/\partial x^0$ is timelike, $\partial/\partial x^i$ spacelike.*
c) *g is hyperbolic of signature $(1, d-1)$, $g_{ij}v^i v^j$ and $g^{ij}v_i v_j$ are negative definite.*

Answer: a)$\Leftrightarrow$b). Indeed $g_{00} = g(\partial/\partial x^0, \partial/\partial x^0) > 0 \Leftrightarrow \partial/\partial x^0$ timelike. $g^{00} = g(\boldsymbol{n}, \boldsymbol{n})$ where $\boldsymbol{n}$ is normal to the hyperplane generated by the vectors $\partial/\partial x^i$ (see answer 2).

$$g^{00} > 0 \Leftrightarrow n \text{ timelike} \Leftrightarrow \text{hyperplane generated by } \partial/\partial x^i \text{ spacelike}.$$

b)$\Rightarrow$c).

$$\begin{aligned} &g_{00} > 0 \Rightarrow g^{ij}v_i v_j \quad &&\text{negative definite (see answer 1)} \\ &g^{00} > 0 \Rightarrow g_{ij}v^i v^j \quad &&\text{negative definite (see answer 2)}. \end{aligned}$$

c)$\Rightarrow$b) by decomposing

$$g_{\alpha\beta}v^\alpha v^\beta \text{ and } g^{\alpha\beta}v_\alpha v_\beta \text{ as in (1) and (2)}.$$

Remark: $\partial/\partial x^0$ timelike and $\partial/\partial x^i$ spacelike, i.e., $g_{00} > 0$ and $g_{ii} < 0$ do not imply that the metric is hyperbolic. For example

$$\begin{aligned} ds^2 &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - 4dx^2\,dx^3 \\ &= (dx^0)^2 - (dx^1)^2 + 3(dx^2)^2 - (dx^3 + 2\,dx^2)^2 \end{aligned}$$

has signature $(2,2)$; the metric is not hyperbolic.

Remark: $g_{00} > 0$ and $g^{ij}v_iv_j$ negative definite imply that the metric is hyperbolic but do not imply that $\partial/\partial x^i$ is spacelike. For example

$$\begin{aligned} ds^2 &= (dx^0)^2 + 4\,dx^0\,dx^1 + 3(dx^1)^2 \\ &= (dx^0 + 2\,dx^1)^2 - (dx^1)^2 . \end{aligned}$$

Here $g_{00} > 0$, $g_{\alpha\beta} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $\quad g^{\alpha\beta} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$

$$g^{ij}v_iv_j = g^{11}(v_1)^2 < 0 \text{ but } g_{11} = 3 > 0 \text{ and } \partial/\partial x^i \text{ is not spacelike} .$$

2. FIRST FUNDAMENTAL FORM (INDUCED METRIC)

timelike

null, spacelike

Let $f: X^d \to Y^{d+1}$ be an embedding. Let $\boldsymbol{n}$ be a normal vector to the hypersurface $f(X) \subset Y$. Let $\boldsymbol{g}$ be a Lorentzian metric on Y. The hypersurface $f(X)$ is said to be **timelike** [resp. **null, spacelike**] if its normal is a spacelike vector [resp. null, timelike], i.e., $\boldsymbol{g}(g^{\alpha\gamma}n_\gamma, g^{\beta\delta}n_\delta) < 0$ [resp. $= 0, > 0$].

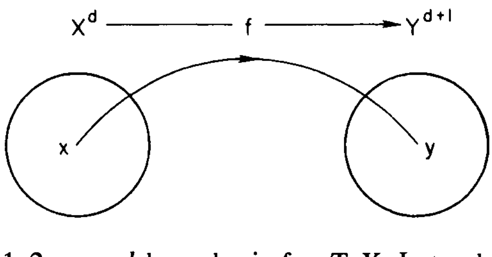
Show that

a) *if $f(X)$ is timelike, then the first fundamental form $f^*\boldsymbol{g}$ is hyperbolic,*

b) *if $f(X)$ is spacelike, then $f^*\boldsymbol{g}$ is negative definite,*

c) *if $f(X)$ is null, then $f^*\boldsymbol{g}$ is degenerate.*

Answer:



Let $\{\boldsymbol{e}_i\}$, $i = 1, 2, \ldots, d$ be a basis for T_xX. Let $\boldsymbol{n}$ be the normal to $f(X)$ at $y = f(x)$

$$\langle \boldsymbol{n}, f'\boldsymbol{v} \rangle = 0 \qquad \text{for all } \boldsymbol{v} \in T_xX . \tag{1}$$

Let $e_0 \in T_y Y$ be the contravariant vector corresponding to $\boldsymbol{n}$:

$$e_0^\alpha = g^{\alpha\beta} n_\beta \,.$$

Assume first that $e_0 \not\in T_y f(X)$. Then $\{e_0, f' e_i\}$ is a basis for $T_y Y$, and the components of g in this basis are

$$g_{\alpha\beta} = \left| \frac{g(e_0, e_0)}{0} \middle| \frac{0}{(f^* g)(e_i, e_j)} \right|$$

since $g(e_0, f' e_i) = 0$ by (1).

First of all if $e_0 \not\in T_y f(X)$ then $f(X)$ cannot be null, i.e., $g(e_0, e_0)$ cannot vanish. Indeed $g(e_0, e_0) = 0$ together with $g(e_0, f' e_i) = 0$ implies $g(e_0, v) = 0$, $\forall v \in T_y Y$, which contradicts the fact that a metric is not degenerate.

a) If $f(X)$ is timelike, $g(e_0, e_0) < 0$. Since g has signature $(1, d)$, the first fundamental form $f^* g$ must have signature $(1, d-1)$; it is hyperbolic.

b) If $f(X)$ is spacelike, $g(e_0, e_0) > 0$ and $f^* g$ must have signature $(0, d)$; it is negative definite.

c) Consider now the case where $e_0 \in T_y f(X)$ and let $w \in T_x X$ be the vector such that

$$f' w = e_0 \in T_y f(X) \,.$$

Then $f(X)$ is a null hypersurface since by (1) $g(e_0, f' v) = 0$, $\forall v \in T_x X$. In particular

$$0 = g(f' w, f' w) = g(e_0, e_0) \,.$$

On the other hand, since

$$0 = g(f' w, f' v) = (f^* g)(w, v) \,,$$

there is a nonzero vector w such that $f^* g(w, v) = 0$ for all $v \in T_x X$, and the first fundamental form is degenerate.

3. KILLING VECTOR FIELDS

Let g be a riemannian metric of arbitrary signature, and ξ be a Killing vector field for this metric.
Prove the following formula, useful in particular in the study of Killing vector fields with given asymptotic properties:

$$\nabla_\beta \nabla_\alpha \xi_\lambda = R_{\lambda\alpha\beta}{}^{\mu}\, \xi_\mu \,. \tag{1}$$

Answer: A Killing vector field ξ for the isometry group of g satisfies (p. 152)

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0. \tag{2}$$

On the other hand, by the definition of the Riemann tensor

$$\nabla_\alpha \nabla_\beta \xi_\lambda - \nabla_\beta \nabla_\alpha \xi_\lambda = R_{\alpha\beta\lambda}{}^{\mu} \xi_\mu. \tag{3}$$

We deduce from (2) that

$$\nabla_\alpha \nabla_\beta \xi_\lambda + \nabla_\beta \nabla_\alpha \xi_\lambda = \nabla_\alpha \nabla_\beta \xi_\lambda - \nabla_\beta \nabla_\lambda \xi_\alpha \tag{4}$$

Thus, using (3) and (4)

$$\begin{aligned}
\nabla_\alpha \nabla_\beta \xi_\lambda + \nabla_\beta \nabla_\alpha \xi_\lambda &= \nabla_\alpha \nabla_\beta \xi_\lambda - R_{\beta\lambda\alpha}{}^{\mu} \xi_\mu - \nabla_\lambda \nabla_\beta \xi_\alpha \\
&= \nabla_\alpha \nabla_\beta \xi_\lambda - R_{\beta\lambda\alpha}{}^{\mu} \xi_\mu + \nabla_\lambda \nabla_\alpha \xi_\beta \\
&= \nabla_\alpha \nabla_\beta \xi_\lambda - R_{\beta\lambda\alpha}{}^{\mu} \xi_\mu + R_{\lambda\alpha\beta}{}^{\mu} \xi_\mu + \nabla_\alpha \nabla_\lambda \xi_\beta \\
&= \left(R_{\lambda\beta\alpha}{}^{\mu} + R_{\lambda\alpha\beta}{}^{\mu}\right) \xi_\mu.
\end{aligned} \tag{5}$$

Adding (3) and (5) and using the algebraic identity for the Riemann tensor (p. 309) gives (1).

4. SPHERE S^n

A supplement to this problem entitled "Volume of the sphere S^n" can be found near the end of the book.

The sphere S^n is the submanifold of $\mathbb{R}^{n+1}$ given by the equation $\sum_{i=1}^{n+1} (x^i)^2 = 1$.

1) *Polar coordinates: Show that the mapping given by*

$$\begin{aligned}
x^{n+1} &= \cos\theta^n \\
x^n &= \sin\theta^n \cos\theta^{n-1} \\
x^{n-1} &= \sin\theta^n \sin\theta^{n-1} \cos\theta^{n-2} \\
&\ \vdots \\
x^2 &= \sin\theta^n \ldots \sin\theta^2 \cos\theta^1 \\
x^1 &= \sin\theta^n \sin\theta^{n-1} \sin\theta^{n-2} \ldots \sin\theta^1
\end{aligned} \tag{1}$$

is an analytic diffeomorphism from the open set $\Omega \subset \mathbb{R}^n$:

$$\Omega\colon\ 0 < \theta^1 < 2\pi, \qquad 0 < \theta^i < \pi, \qquad i = 2, \ldots, n$$

onto the open set $U \subset S^n$ obtained by deleting from it the closed subset $\{x^1=0\} \cap \{x^2 \geqq 0\}$.
Determine the metric on S^n induced by the canonical metric of $\mathbb{R}^{n+1}$ and its volume element. Show that the metric is conformally flat.

2) *Stereographic coordinates: Let $N=(0,0,\dots,1)$ be the north pole of S^n considered as the submanifold of $\mathbb{R}^{n+1}$ defined by $\Sigma_{i=1}^{n+1}(x^i)^2=1$. Consider the mapping $S^n-\{N\}\to\mathbb{R}^n$, $x\mapsto X$, defined by the intersection of the straight line (Nx) with the plane tangent to S^n at the south pole, $x^{n+1}=-1$, that is*

$$\frac{1-x^{n+1}}{2}=\frac{x^1}{X^1}=\dots=\frac{x^n}{X^n}\,.$$

Show that this mapping is an analytic diffeomorphism from $S^n-\{N\}$ onto $\mathbb{R}^n$.
Compute the metric of S^n in the stereographic coordinates (X^i).
Construct an atlas on S^n.
Give a correspondence between polar and stereographic coordinates.

3) *Determine the scalar curvature of S^n.*

Answer 1: The mapping is a C^∞, and even (real) analytic, mapping from Ω into S^n. It maps Ω into U: since $\sin\theta_i>0$, $i=2,\dots,n$ on Ω, $x^1=0$ can be obtained only for $\sin\theta^1=0$, i.e. $\theta^1=\pi$, which implies $x^2<0$. The mapping is surjective onto U and invertible: if $x\in U$

$$\theta^n=\cot^{-1}\frac{x^{n+1}}{(\Sigma_{i=1}^n(x^i)^2)^{1/2}}\,,\qquad \theta^{n-1}=\cot^{-1}\frac{x^n}{\{\Sigma_{i=1}^{n-1}(x^i)^2\}^{1/2}}\,,\dots,$$

$$\theta^2=\cot^{-1}\frac{x^3}{((x^1)+(x^2)^2)^{1/2}}\,.$$

(Note that if $x^1=0$ then $x^2<0$, all denominators are $\neq 0$)

$$\theta^1=\cot^{-1}\frac{x^2}{x^1}\,,\qquad 0<\theta^1<\pi\qquad \text{if } x^1>0$$

$$\theta^1=\pi\qquad\qquad \text{if } x^1=0$$

$$\theta^1=\cot^{-1}\frac{x^2}{x^1}\,,\qquad \pi<\theta^1<2\pi\qquad \text{if } x^1<0\,.$$

Note that the whole of S^n is obtained from the formulas (1) if we complete Ω to be $0\leqq\theta^i\leqq\pi$, $i=2,\dots,n$ and $0\leqq\theta^1<2\pi$, but the mapping is no longer bijective.

The image of the hyperplane $\theta^n = 0$ [resp. $\theta^n = \pi$] is the north [resp. south] pole $x^{n+1} = 1$ [resp. $x^{n+1} = -1$], $x^i = 0$, $i = 1, \ldots, n$. The image of the hyperplane $\theta^i = 0$ is a submanifold $S^n \cap \{x^j = 0, j = 1, \ldots, i-1\} \simeq S^{n-i}$.

The metric induced on S^n is

$$\begin{aligned} ds^2 &= \sum_{i=1}^{n+1} (dx^i)^2 \\ &= (d\theta^n)^2 + (\sin^2\theta^n)(d\theta^{n-1})^2 + \sin^2\theta^n \sin^2\theta^{n-1}(d\theta^{n-2})^2 \\ &\quad + \cdots + \sin^2\theta^n \ldots \sin^2\theta^1 (d\theta^1)^2 . \end{aligned}$$

The volume element is

$$\tau = (\sin\theta^n)^{n-1} \ldots \sin\theta^1 \, d\theta^1 \ldots d\theta^n$$

Set $r = \tan(\theta^n/2)$. Then the metric of S^n reads

$$ds^2 = 4\cos^2\frac{\theta^n}{2}\,[dr^2 + r^2((d\theta^{n-1})^2 + \sin^2\theta^{n-1}(d\theta^{n-2})^2 + \ldots)] .$$

The quantity between brackets is the metric of $\mathbb{R}^n$ in polar coordinates.

Answer 2: If $x \in \mathbb{R}^{n+1}$ with $x^{n+1} \neq 1$, the given formulae define an analytic mapping $x \to X \in \mathbb{R}^n$ with

$$X^i = \frac{2x^i}{1 - x^{n+1}} , \qquad i = 1, \ldots, n .$$

If $x \in S^n \subset \mathbb{R}^{n+1}$ then $\Sigma_{i=1}^n (x^i)^2 = 1 - (x^{n+1})^2$, and a simple computation gives

$$x^{n+1} = 1 - \frac{2}{v^2} , \qquad v = 1 + \tfrac{1}{4}\sum_{i=1}^n (X^i)^2 .$$

The mapping $S^n - \{N\} \to \mathbb{R}^n$ is surjective, and invertible.
The inverse mapping is the analytic map:

$$x^i = \frac{X^i}{v} .$$

A simple computation gives the metric, conformally flat, of $S^n - \{N\}$ in stereographic coordinates

$$ds^2 = \sum_{i=1}^n (dx^i)^2 + (dx^{n+1})^2 = \sum_{i=1}^n \frac{(dX^i)^2}{v^2} .$$

To construct an analytic atlas on S^n we take as other chart the stereographic coordinates $(\bar{X}^i)$ relative to the south pole S, with domain

$S^n - \{S\}$, $x^{n+1} \neq -1$. We have

$$\bar{X}^i = \frac{4X^i}{1 + x^{n+1}} ,$$

$$x^i = \frac{\bar{X}^i}{\bar{v}} , \qquad x^{n+1} = \frac{2}{\bar{v}} - 1 , \qquad \bar{v} = 1 + \tfrac{1}{4} \sum_{i=1}^{n} (\bar{X}^i)^2 .$$

The transition functions are indeed analytic in the image of $S^n - (\{S\} \cup \{N\})$, given by

$$\bar{X}^i = \frac{4X^i}{\Sigma_{i=1}^n (X^i)^2} , \qquad \left(\sum_{i=1}^{n} (X^i)^2\right)^{1/2} \left(\sum_{i=1}^{n} (\bar{X}^i)^2\right)^{1/2} = 4 .$$

The correspondence between polar and stereographic coordinates is given by

$$x^{n+1} = \cos\theta^n = \frac{2}{\bar{v}} - 1 = \frac{1 - \frac{1}{4}\Sigma (\bar{X}^i)^2}{1 + \frac{1}{4}\Sigma (\bar{X}^i)^2} = \frac{\frac{1}{4}\Sigma (X^i)^2 - 1}{\frac{1}{4}\Sigma (X^i)^2 + 1} ,$$

$$\bar{X}^i = \frac{2\sin\theta^n}{1 + \cos\theta^n} p^i = 2\tan\frac{\theta^n}{2} p^i , \qquad \bar{v} = \left(\cos^2\frac{\theta^n}{2}\right)^{-1}$$

where p^i are the expressions in $\theta^{n-1}, \dots, \theta^1$, polar coordinates on S^{n-1}, such that $\Sigma_{i=1}^n (p^i)^2 = 1$ (i.e., $p^n = \cos\theta^{n-1}$, $p^{n-1} = \sin\theta^{n-1}\cos\theta^{n-2}, \dots$, etc.).

Answer 3: If two metrics on an n-dimensional manifold are conformal

$$g = e^{2\phi}\tilde{g} ,$$

their scalar curvatures are related by (cf. p. 351, with sign convention changed)

$$R = e^{-2\phi}(\tilde{R} - 2(n-1)\Delta_{\tilde{g}}\phi - (n-1)(n-2)\tilde{g}^{ij}\partial_i\phi\,\partial_j\phi) .$$

The metric $\tilde{g}$ here is flat, so $\tilde{R} = 0$, and

$$\phi = -\log\left(1 + \tfrac{1}{4}\sum (X^i)^2\right) .$$

Thus

$$\partial_i\phi = -\tfrac{1}{2}X^i\left(1 + \tfrac{1}{4}\sum (X^j)^2\right)^{-1}$$

$$\Delta_{\tilde{g}}\phi = -n\left(1 + \tfrac{1}{4}\sum (X^j)^2\right)^{-2}$$

and

$$R = n(n-1) .$$

5. CURVATURE OF EINSTEIN CYLINDER

Let $\Sigma^{n+1} = \mathbb{R} \times S^n$ *be the Einstein cylinder with metric*

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}\omega^2 ,$$

where $d\omega^2$ *is the metric of the sphere* S^n. *Compute its connection in terms of a connection on* S^n, *its Ricci tensor and scalar curvature.*

Answer: set $X^0 = T$, denote local coordinates on S^n by X^i and the Christoffel symbols of the metric $\mathrm{d}\omega^2$ in these coordinates [resp. of the metric $\mathrm{d}s^2$ in coordinates $(X^\alpha) = (X^0, X^i)$] by $\bar{\Gamma}^k_{i\,j}$ [resp. $\Gamma^\lambda_{\alpha\beta}$]. Then

$$\Gamma^k_{i\,j} = \bar{\Gamma}^k_{i\,j} , \qquad \Gamma_{\alpha\ \beta}^{\ \lambda} = 0 \qquad \text{if } \alpha,\ \beta \text{ or } \lambda = 0 .$$

Thus, for the Ricci tensor

$$R_{ij} = \bar{R}_{ij} , \qquad R_{0i} = R_{00} = 0$$

and scalar curvature

$$R = \bar{R} = n(n-1) .$$

6. CONFORMAL TRANSFORMATION OF YANG–MILLS, DIRAC AND HIGGS OPERATORS IN d DIMENSIONS

The causal structure of a space-time depends only on the field of light cones so only on the metric up to a conformal factor. Moreover the zero rest mass free field equations for each spin value, at least in dimension 4, are conformally invariant if interpreted suitably. These facts are the origin of many important developments in physics, an extreme point of view being I.E. Segal's chronogeometry. From a mathematical side the conformal mapping from Minkowski space-time into a bounded open set of the Einstein cylinder is a powerful tool to obtain global existence of solutions of nonlinear field equations on Minkowski space-time, and their decay properties through local existence theorems on the cylinder.

1. YANG–MILLS OPERATOR

Let $f: (V, g) \to (\hat{V}, \hat{g})$ be a conformal diffeomorphism of d-dimensional differentiable manifolds, that is a diffeomorphism $V \to V$ such that there exists a strictly positive function Ω on V and

$$f^* g = \Omega^2 \hat{g} .$$

Let $\hat{A}$ be a 1-form on $\hat{V}$, with values in a Lie algebra $\mathscr{G}$, representing a Yang–Mills connection (cf. p. 350), and let $\hat{F} = \mathrm{d}\hat{A} + \frac{1}{2}[\hat{A}, \hat{A}]$ represent the Yang–Mills curvature. Let

$$\hat{D} \cdot \hat{F} \equiv \hat{\nabla} \cdot \hat{F} + [\hat{A}, \hat{F}]$$

be the Yang–Mills operator on $(\hat{V}, \hat{g})$.
Compute $D \cdot F$ on (V, g), when $A = f^* \hat{A}$.
We use dot for contraction and semicolon for Lie bracket with contraction.

Answer: If $A = f^* \hat{A}$, then $\mathrm{d}A = f^* \mathrm{d}\hat{A}$, $f^*[\hat{A}, \hat{A}] = [A, A]$ (cf. p. 135), thus $F = f^* \hat{F}$.
The corresponding contravariant tensors $F^{\#}$ and $\hat{F}^{\#}$ are therefore linked by

$$F^{\#} = f'(\Omega^{-4} \hat{F}^{\#}) .$$

In local coordinates on V and $\hat{V}$ identified through the diffeomorphism f we have

$$F^{\lambda\mu} = \Omega^{-4} \hat{F}^{\lambda\mu} .$$

In such coordinates the connections of $\hat{g}$ and g are linked by

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \hat{\Gamma}_{\lambda}{}^{\nu}{}_{\mu} + \Omega^{-1}(\delta^{\nu}_{\lambda} \partial_{\mu} \Omega + \delta^{\nu}_{\mu} \partial_{\lambda} \Omega - g_{\lambda\mu} g^{\nu\rho} \partial_{\rho} \Omega) ;$$

thus

$$\nabla_{\lambda} F^{\lambda\mu} \equiv \hat{\nabla}_{\lambda}(\Omega^{-4} \hat{F}^{\lambda\mu}) \equiv \Omega^{-4}(\hat{\nabla}_{\lambda} \hat{F}^{\lambda\mu} + (d-4) \hat{F}^{\lambda\mu} \Omega^{-1} \partial_{\lambda} \Omega)$$

and

$$D \cdot F \equiv f'(\Omega^{-4}(\hat{D} \cdot \hat{F} + (d-4) \hat{F}^{\#} \cdot \Omega^{-1} \partial \Omega)) .$$

The Yang–Mills equation $D \cdot F = 0$ is invariant under conformal transformation if and only if $d = 4$.
Note that the relation above can be written

$$D \cdot F \equiv f'(\Omega^{-d} \hat{D} \cdot (\Omega^{d-4} \hat{F})) ,$$

but if we introduce a "weighted Yang–Mills field" by

$$\tilde{F}^{\#} = \Omega^{d-4} \hat{F}^{\#}$$

we would lose the relation $\hat{F} = \hat{D}\hat{A}$ without being able to find an analogous one by weighting A.

2. DIRAC OPERATOR

In this question and the next we consider two conformal metrics on the same manifold V, that is we take the diffeomorphism f of the previous question to be the identity map. This restriction is made to simplify the writing, and also to avoid ambiguities in the definition of the image of spinor fields under diffeomorphisms[1].

Let g and $\hat{g}$ be two conformal metrics, $g = \Omega^2 \hat{g}$ on a manifold V.

To an orthonormal frame $O_g = \{\theta^\alpha\}$ on (V, g) corresponds an orthonormal frame $O_{\hat{g}} = \{\hat{\theta}^\alpha\}$ on $(V, \hat{g})$ given by the relation

$$\theta^\alpha = \Omega \hat{\theta}^\alpha .$$

The bundles of Lorentz frames on (V, g) and $(V, \hat{g})$ are therefore isomorphic. A spin structure on (V, g) is also a spin structure over $(V, \hat{g})$, and conversely. It is therefore natural to say that a spinor field ψ on (V, g) is equal to a spinor field $\hat{\psi}$ on $(V, \hat{g})$ if the representatives of ψ and $\hat{\psi}$ are the same $\mathbb{C}^k$ vector $(k = [d/2])$ in a given (arbitrary) spin frame.

Remark: Penrose (cf. reference in Problem V 7, Conformal) chooses instead to say that $\psi = \hat{\psi}$ on V if the $\mathbb{C}^k$ vectors, representing them in spin frames are linked by the relation $\hat{\psi}^A = \Omega^{1/2} \psi^A$.

a) *Show that, with our definition, the Dirac operators on* (V, g) *and* $(V, \hat{g})$ *are linked by*

$$\nabla\!\!\!/ \psi \equiv \Omega^{-(d+1)/2} \hat{\nabla\!\!\!/} \hat{\psi} , \qquad \hat{\psi} = \Omega^{(d-1)/2} \psi .$$

b) Suppose now that ψ is a spinor multiplet, that is, a section of a vector bundle with typical fibre $\mathbb{C}^k \times \mathbb{C}^l$, $k = [d/2]$, where $\mathbb{C}^l$ is the space of a unitary representation r of a Yang–Mills group G. In a spin frame, and for a local trivialization of the principal bundle on which the Yang–Mills connection is defined, the spinor multiplet is represented by $\mathbb{C}^k \times \mathbb{C}^l$ valued functions $(\psi^{A,I})$ on $U \subset V$. By a change of spin frame and trivialization characterized by a mapping $U \to \mathrm{Spin}(d) \times G$, $x \mapsto (\Lambda(x), u(x))$, $(\psi^{A,I})$ transforms by

$$\psi^{A,J}(x) \to \overset{-1}{\Lambda}{}^A_B(x) r^J_I(u(x)) \psi^{B,I}(x) .$$

Show that the relation given in a) holds if the covariant derivatives ∇ *are replaced by the gauge covariant ones.*

[1] For this definition, and the Lie derivative of spinor fields see Bourguignon, Rend Sem. Mat. Torino Vol. 44, 3 (1986).

Answer 2a: A straightforward computation shows that the connections in orthonormal frames of two conformal metrics

$$g = \eta_{\alpha\lambda}\theta^{\alpha}\theta^{\lambda}, \qquad g = \eta_{\alpha\lambda}\hat{\theta}^{\alpha}\hat{\theta}^{\lambda}, \qquad \theta^{\alpha} = \Omega\hat{\theta}^{\alpha}$$

are linked by

$$\omega_{\lambda\beta\alpha} = \Omega^{-1}\hat{\omega}_{\hat{\lambda}\hat{\beta}\hat{\alpha}} + \Omega^{-2}(\eta_{\alpha\lambda}\partial_{\hat{\beta}}\Omega - \eta_{\beta\lambda}\partial_{\hat{\alpha}}\Omega).$$

We choose for g and $\hat{g}$ the same gamma matrices, in their respective orthonormal frames, that is

$$\Gamma^{\alpha} = \hat{\Gamma}^{\hat{\alpha}}.$$

The spin connections are then found to satisfy the relation

$$\sigma_{\lambda} = \Omega^{-1}\hat{\sigma}_{\hat{\lambda}} + \tfrac{1}{4}\Gamma^{\alpha}\Gamma^{\beta}(\eta_{\alpha\lambda}\partial_{\hat{\beta}}\Omega - \eta_{\beta\lambda}\partial_{\hat{\alpha}}\Omega).$$

Thus

$$\nabla_{\lambda}\psi = \Omega^{-1}\hat{\nabla}_{\hat{\lambda}}\psi + \tfrac{1}{4}\Omega^{-2}\partial_{\hat{\beta}}\Omega\big(\Gamma_{\lambda}\Gamma^{\beta} - \Gamma^{\beta}\Gamma_{\lambda}\big)\psi$$

and, since $\Gamma^{\lambda}\Gamma_{\lambda} = -d$ and $\Gamma^{\beta}\Gamma_{\lambda} = -2\delta^{\beta}_{\lambda} - \Gamma_{\lambda}\Gamma^{\beta}$,

$$\not{\nabla}\psi = \Gamma^{\lambda}\nabla_{\lambda}\psi = \Omega^{-1}\hat{\not{\nabla}}\psi + \tfrac{1}{2}(1-d)\Gamma^{\beta}\partial_{\hat{\beta}}\Omega;$$

thus

$$\not{\nabla}\psi = \Omega^{-(d+1)/2}\hat{\not{\nabla}}\hat{\psi}, \qquad \hat{\psi} = \Omega^{(d-1)/2}\psi.$$

Answer 2b: The gauge covariant derivative is (cf. p. 403)

$$D\psi = \nabla\psi + TA\psi$$

with T the representation of the Lie algebra $\mathcal{G}$ induced from the representation r of $\mathcal{O}$. Thus

$$\not{D}\psi = \Gamma^{\alpha}D_{\alpha}\psi = \not{\nabla}\psi + \Gamma^{\alpha}TA_{\alpha}\psi.$$

We have $A_{\alpha} = \Omega^{-1}A_{\hat{\alpha}}$, $\Gamma^{\alpha} = \Gamma^{\hat{\alpha}}$, thus again

$$\not{D}\psi = \Omega^{-(d+1)/2}\hat{\not{D}}\hat{\psi}, \qquad \hat{\psi} = \Omega^{(d-1)/1}\psi.$$

3. HIGGS OPERATOR

If φ is a scalar function on V one has the identity (p. 351)

$$\nabla^{\alpha}\nabla_{\alpha}\varphi - \frac{d-2}{4(d-1)}R\varphi \equiv \Omega^{-(d+2)/2}\left(\hat{\nabla}^{\alpha}\hat{\nabla}_{\alpha}\hat{\varphi} - \frac{d-2}{4(d-1)}\hat{R}\hat{\varphi}\right)$$

with R and $\hat{R}$ the scalar curvatures of g and $\hat{g} = \Omega^{-2}g$ and

$$\hat{\varphi} = \Omega^{(d-2)/2}\varphi.$$

Show that an analogous identity holds when φ is a scalar m-multiplet, associated to a representation ρ of G, and ∇ is replaced by the gauge covariant derivative.

Answer 3: The gauge covariant derivative is the 1-form-scalar multiplet

$$D_\alpha \varphi = \partial_\alpha + SA_\alpha \varphi$$

with S the representation of $\mathscr{G}$ on $\mathbb{C}^m$ induced by ρ.
Thus

$$D^\alpha D_\alpha \varphi = (\nabla^\alpha + SA^\alpha)(\nabla_\alpha + SA_\alpha)\varphi .$$

Using orthonormal frames (θ^α) and $(\theta^{\hat{\alpha}} = \Omega^{-1}\theta^\alpha)$ for g and $\hat{g}$, as in (2), we have for the contravariant and covariant components of the 1-form A on V:

$$A^\alpha = \Omega^{-1}A^{\hat{\alpha}} , \qquad A_\alpha = \Omega^{-1}A_{\hat{\alpha}} .$$

A straightforward computation shows that

$$\nabla_\alpha A^\alpha = \Omega^{-2}\hat{\nabla}_{\hat{\alpha}}A^{\hat{\alpha}} + (d-2)\Omega^{-3}A^{\hat{\beta}}\hat{\partial}_{\hat{\beta}}\Omega ,$$

$$A^\alpha \nabla_\alpha \varphi = \Omega^{-(d+2)/2}A^{\hat{\alpha}}\hat{\nabla}_{\hat{\alpha}}\hat{\varphi} + \frac{4-d}{2}\,\Omega^{-(4+d)/2}A^{\hat{\alpha}}\hat{\partial}_{\hat{\alpha}}\Omega ,$$

and

$$(SA^\alpha)(SA_\alpha) = \Omega^{-2}(SA^{\hat{\alpha}})(SA_{\hat{\alpha}}) .$$

Reassembling these identities gives, in any frame:

$$D^\alpha D_\alpha \varphi - \frac{d-2}{4(d-1)}\,R\varphi = \Omega^{-(d-2)/2}\Big(\hat{D}^\alpha \hat{D}_\alpha \hat{\varphi} - \frac{d-2}{4(d-1)}\,\hat{R}\hat{\varphi}\Big) ,$$

$$\hat{\varphi} = \Omega^{(d-2)/2}\varphi .$$

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See also references of problems 7 and 8.

7. CONFORMAL SYSTEM FOR EINSTEIN EQUATIONS

In contradistinction with other field equations in dimension 4 the Einstein equations are not conformally invariant even though the spin 2, zero rest mass, free field equations are conformally invariant. However it is possible to deduce from these equations a "conformally regular" system (cf. [2–4]), through the use of the Weyl tensor (questions 1, 2, 3). In question 4, which could be considered as more relevant for chapter VI, it is shown that in a convenient gauge this system is hyperbolic. It is hoped that this formulation will help to obtain information on global existence and asymptotic behaviour of the gravitational field. The system consiered here is a variant of the system deduced by Friedrich from the Newmann–Penrose formalism and used by him towards these goals.

1) *Let (M, g) be a smooth d-dimensional pseudo-riemannian manifold, and ω be a smooth nonvanishing function on M. Denote by* $\mathrm{Riem}(g) = (R^\alpha{}_{\beta\gamma\delta})$, $\mathrm{Ricc}(g) = (R_{\alpha\beta})$ *and R respectively the Riemann tensor, the Ricci tensor and the scalar curvature of g. Set (Friedrich* 1981)

$$d^\alpha{}_{\beta\lambda\mu} = \omega^{-1} W^\alpha{}_{\beta\lambda\mu} , \tag{1.1}$$

where $W = (W^\alpha{}_{\beta\lambda\mu})$ is the Weyl tensor (denoted C, p. 351) of g. Show that if

$$\mathrm{Ricc}(\omega^{-2}g) = \Lambda\omega^{-2}g , \qquad \Lambda \text{ a constant}$$

then

$$\sum_{(\alpha,\lambda,\mu)} \nabla_\alpha W^\beta{}_{\gamma\lambda\mu} = \sum_{(\alpha,\lambda,\mu)} \partial_\rho \omega(\delta^\beta_\alpha d^\rho{}_{\gamma\lambda\mu} - g_{\alpha\gamma} d^{\rho\beta}{}_{\lambda\mu}) , \tag{1.2}$$

where $\Sigma_{(\alpha,\lambda,\mu)}$ denotes the sum over circular permutation of indices,

$$\sum_{(\alpha,\lambda,\mu)} \nabla_\alpha(\omega^{-1} W^\alpha{}_{\gamma\lambda\mu}) = 0 \tag{1.3}$$

and

$$\frac{1}{d-2} (\nabla_\lambda L_{\beta\mu} - \nabla_\mu L_{\beta\lambda}) = \partial_\rho \omega d^\rho{}_{\beta\lambda\mu} , \tag{1.4}$$

with

$$L_{\beta\mu} = R_{\beta\mu} - \frac{1}{2(d-1)} g_{\beta\mu} R . \tag{1.5}$$

Answer 1: On a d-dimensional manifold the Weyl tensor is related to the Riemann tensor by the identity (p. 351)

$$W^{\alpha}{}_{\beta\lambda\mu} \equiv R^{\alpha}{}_{\beta\lambda\mu} - \frac{1}{(d-2)}\left(R^{\alpha}{}_{\lambda}g_{\beta\mu} - R^{\alpha}{}_{\mu}g_{\beta\lambda} + g^{\alpha}{}_{\lambda}R_{\beta\mu} - g_{\beta\mu}R^{\alpha}{}_{\lambda}\right)$$
$$+ \frac{1}{(d-1)d-2} R\left(g^{\alpha}{}_{\lambda}g_{\beta\mu} - g^{\alpha}{}_{\mu}g_{\beta\lambda}\right). \tag{1.6}$$

It has the symmetries and antisymmetries of the Riemann tensor

$$W^{\alpha}{}_{\beta\lambda\mu} = -W^{\alpha}{}_{\beta\mu\lambda}, \qquad W^{\alpha\beta}{}_{\lambda\mu} = -W^{\beta\alpha}{}_{\lambda\mu}, \tag{1.7a}$$
$$\sum_{(\beta,\lambda,\mu)} W^{\alpha}{}_{\beta\lambda\mu} \equiv 0 \qquad \text{(sum over circular permutation of } \beta,\ \lambda,\ \mu)$$

and is traceless:

$$W^{\alpha}{}_{\beta\alpha\mu} = 0. \tag{1.7b}$$

If $(M, \tilde{g})$ is an Einstein space, that is, if

$$\mathrm{Ricc}(\tilde{g}) = \Lambda\tilde{g}, \quad \text{i.e., } \tilde{R}_{\alpha\beta} = \Lambda\tilde{g}_{\alpha\beta}, \quad \tilde{R} = \Lambda d$$

then one deduces from the Bianchi identities (p. 307) that

$$\sum_{(\alpha,\lambda,\mu)} \tilde{\nabla}_{\alpha}\tilde{W}^{\beta}{}_{\gamma\lambda\mu} = 0. \tag{1.8}$$

If $\tilde{g}$ and g are conformal metrics, $\tilde{g} = \omega^{-2}g$, they have the same Weyl tensor (p. 351); thus

$$\sum_{(\alpha,\lambda,\mu)} \tilde{\nabla}_{\alpha}W^{\beta}{}_{\gamma\lambda\mu} = 0. \tag{1.9}$$

The Christoffel symbols of g and $\tilde{g}$ are related by

$$\tilde{\Gamma}_{\alpha}{}^{\lambda}{}_{\beta} = \Gamma_{\alpha}{}^{\lambda}{}_{\beta} - \omega^{-1}(\delta^{\lambda}_{\alpha}\partial_{\beta}\omega + \delta^{\lambda}_{\beta}\partial_{\alpha}\omega - g^{\lambda\mu}g_{\alpha\beta}\partial_{\mu}\omega); \tag{1.10}$$

hence, using also the relations (1–7), one finds that

$$\sum_{(\alpha,\lambda,\mu)} \tilde{\nabla}_{\alpha}W^{\beta}{}_{\gamma\lambda\mu} \equiv \sum_{(\alpha,\lambda,\mu)} \nabla_{\alpha}W^{\beta}{}_{\gamma\lambda\mu} - \omega^{-1}\partial_{\rho}\omega(\delta^{\beta}_{\alpha}W^{\rho}{}_{\gamma\lambda\mu}$$
$$- g_{\alpha\gamma}W^{\rho\beta}{}_{\lambda\mu}) \tag{1.11}$$

which gives (1–2) by comparison with (1–9).
Using again the antisymmetries and the tracelessness of W, one deduces from (1.2) that

$$\omega\nabla_{\alpha}(\omega^{-1}W^{\alpha}{}_{\gamma\lambda\mu}) \equiv \nabla_{\alpha}W^{\alpha}{}_{\gamma\lambda\mu} - (\partial_{\alpha}\omega)\omega^{-1}W^{\alpha}{}_{\gamma\lambda\mu} = 0. \tag{1.12}$$

If we compute $\nabla_{\alpha}W^{\alpha}_{\gamma\lambda\mu}$ using (1.6) and the contracted Bianchi identity for $\mathrm{Riem}(g)$ we find, with the notation (1.5),

$$\nabla_\alpha W^\alpha{}_{\beta\lambda\mu} = \frac{1}{(d-2)} (\nabla_\lambda L_{\beta\mu} - \nabla_\mu L_{\beta\lambda}) . \tag{1.13}$$

We obtain (1.4) by comparing (1.12) and (1.13).

2) *Henceforth we take* $d = 4$.

a) *Show that if* $(W) = (W^\beta{}_{\gamma\lambda\mu})$ *is a* 4*-tensor satisfying the symmetries and tracelessness of the Weyl tensor then* (*Penrose*, 1965)

$$\omega^{-1} \sum_{(\alpha,\lambda,\mu)} \tilde{\nabla}_\alpha W^\beta{}_{\gamma\lambda\mu} \equiv \sum_{(\alpha,\lambda,\mu)} \nabla_\alpha(\omega^{-1} W^\beta_{\gamma\lambda\mu}) , \tag{2.1}$$

where $\tilde{\nabla}$ *is the covariant derivative in the metric* $\tilde{g} = \omega^{-2} g$, *i.e.*,

$$\tilde{g}_{\alpha\beta} = \omega^{-2} g_{\alpha\beta} .$$

A tensor W having these properties is said to represent a **spin-2 field**, and the above identity expresses the conformal covariance of the equation satisfied by such a field. spin-2 field

b) *Show that if* $\mathrm{Ricc}(\omega^{-2} g) = \Lambda \omega^{-2} g$, *then the function* ω *and the traceless tensor* $S = (S_{\alpha\beta})$

$$S_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{4} g_{\alpha\beta} R \tag{2.2}$$

satisfy the equations (*Friedrich* 1981)

$$2\nabla_\alpha \partial_\beta \omega - \tfrac{1}{2} g_{\alpha\beta} \nabla^\lambda \partial_\lambda \omega + \omega S_{\alpha\beta} = 0 , \tag{2.3}$$

$$\nabla_\alpha \nabla^\lambda \partial_\lambda \omega + 2 S_{\alpha\beta} \partial^\beta \omega + \tfrac{1}{3} R \partial_\alpha \omega + \tfrac{1}{6} \omega \partial_\alpha R = 0 . \tag{2.4}$$

Show that, conversely, if $\omega \neq 0$, *eqs.* (2.3) *and* (2.4) *imply that* $\tilde{S}_{\alpha\beta} = 0$ *and* $\tilde{R} =$ *constant, where*

$$\tilde{g}_{\alpha\beta} = \omega^{-2} g_{\alpha\beta} ,$$

$$\tilde{S}_{\alpha\beta} \equiv \tilde{R}_{\alpha\beta} - \tfrac{1}{4} \tilde{g}_{\alpha\beta} \tilde{R} , \qquad \tilde{R} = \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} , \qquad (\tilde{R}_{\alpha\beta}) = \mathrm{Ricc}(\tilde{g}) .$$

Answer 2*a*: We shall check (2.1) for arbitrary ω when $d = 4$ by checking the equality

$$\sum_{(\alpha,\lambda,\mu)} W_{\beta\gamma\lambda\mu} \delta^\rho_\alpha = \sum_{(\alpha,\lambda,\mu)} (g_{\alpha\beta} W^\rho{}_{\gamma\lambda\mu} - g_{\alpha\gamma} W^\rho{}_{\beta\lambda\mu}) . \tag{2.5}$$

Both sides are zero for $\beta = \gamma$, and zero also if two of the indices α, λ or μ are equal. We show they are equal for $\beta \neq \gamma$, for instance $\beta = 0$, $\gamma = 1$, and the indices α, λ, μ are all different, for instance $1, 2, 3$, by choosing an orthonormal frame at a point p; then $g_{\alpha\beta} = 0$, $\alpha \neq \beta$, $g_{11} = -1$, etc. An easy computation shows that the equality (2.5) reduces to

$$-W^{\rho}{}_{023} = -W_{0123}\delta^{\rho}_{1} - W_{0112}\delta^{\rho}_{3} - W_{0131}\delta^{\rho}_{2}\,.$$

The equality is obvious for $\rho = 0$ or $\rho = 1$. For $\rho = 2$ or $\rho = 3$ it results from the tracelessness and the antisymmetries of the Weyl tensor. For instance, in the orthonormal frame we have

$$W_{0323} + W_{0121} = 0\,.$$

Answer 2b: A straightforward computation (cf. p. 351) shows that the left-hand side of (2.3) is $\omega\tilde{S}_{\alpha\beta}$, while

$$\tilde{R} \equiv \omega^2 R + 6\omega\nabla^{\lambda}\partial_{\lambda}\omega - 12\partial^{\lambda}\omega\partial_{\lambda}\omega\,.$$

The left-hand side of (2.4) is

$$\tfrac{1}{6}\omega^{-1}(\partial_{\alpha}\tilde{R} + 12\omega\tilde{S}_{\alpha\beta}\partial^{\beta}\omega)\,.$$

3) *Obtain an overdetermined system of partial differential equations, regular even for $\omega = 0$, whose solutions (g, ω) satisfy for $\omega \neq 0$ the equations*

$$\mathrm{Ricc}(\omega^{-2}g) = \Lambda\omega^{-2}g \qquad \text{(I)}$$

assuming the scalar curvature R of g given and introducing the tensor S and d as auxiliary unknowns.

Answer 3: We consider the system II, with unknown g, ω, s, d, and a given function:

$$\begin{aligned}
&(1)\quad R_{\alpha\beta}(g) - s_{\alpha\beta} - \tfrac{1}{4}g_{\alpha\beta}r = 0\,,\\
&(2)\quad 2\nabla_{\alpha}\partial_{\beta}\omega - \tfrac{1}{2}g_{\alpha\beta}\nabla^{\lambda}\partial_{\lambda}\omega + \omega s_{\alpha\beta} = 0\,,\\
&(3)\quad \nabla_{\alpha}\nabla^{\lambda}\partial_{\lambda}\omega + 2s_{\alpha\beta}\partial^{\beta}\omega + \tfrac{1}{3}r\partial_{\alpha}\omega + \tfrac{1}{6}\omega\partial_{\alpha}r = 0\,, \qquad \text{(II)}\\
&(4)\quad \tfrac{1}{2}(\nabla_{\lambda}s_{\beta\mu} - \nabla_{\mu}s_{\beta\lambda}) - \partial_{\alpha}\omega d^{\alpha}{}_{\beta\lambda\mu} + \tfrac{1}{24}(g_{\beta\mu}\partial_{\lambda}r - g_{\beta\lambda}\partial_{\mu}r)\,,\\
&(5)\quad \sum_{(\gamma,\lambda,\mu)} \nabla_{\gamma}d^{\beta}{}_{\lambda\mu} = 0\,,
\end{aligned}$$

together with the algebraic relations

$$\begin{aligned}
&(6)\quad g^{\alpha\beta}s_{\alpha\beta} = 0\\
&(7)\quad d^{\alpha}{}_{\beta\lambda\mu} = -d^{\alpha}{}_{\beta\mu\lambda}\,,\quad d_{\alpha\beta\lambda\mu} = -d_{\beta\alpha\lambda\mu}\,,\quad \sum_{(\beta,\lambda,\mu)} d^{\alpha}{}_{\beta\lambda\mu} = 0\,.
\end{aligned}$$

If (g, ω) satisfy I then $(g, \omega, s = S(g),\ d = \omega^{-1}W)$, $\omega \neq 0$, satisfy II. Note that by (1) and (6) the scalar curvature R of g is equal to r.

Conversely if (g, ω, s, d) satisfy II then by (1) and (6) $R = r$, $s = S(g)$. Then (2) implies $\tilde{S}_{\alpha\beta} = 0$ if $\tilde{g}_{\alpha\beta} = \omega^{-2} g_{\alpha\beta}$, $\omega \neq 0$, and (3) implies $\partial_\alpha \tilde{R} = 0$, thus $\tilde{R} = \text{constant}$.

4) *Construct a hyperbolic system satisfied by the solutions (g, ω, s, d) of the system II, in a convenient gauge for the hyperbolic metric g.*

Answer 4: It is known that $R_{\alpha\beta}$ reduces to a second-order quasidiagonal hyperbolic operator, ${}^{(h)}R_{\alpha\beta}$, with principal part a wave-like operator $g^{\lambda\mu}\partial^2_{\lambda\mu}$, if the local coordinates are harmonic, i.e.,

$$\Phi^\lambda \equiv g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} = 0 .$$

We shall replace equations (1) of II by

$${}^{(h)}R_{\alpha\beta} = s_{\alpha\beta} + \tfrac{1}{4} g_{\alpha\beta} r . \tag{1'}$$

We deduce from the other equations a system of the same type for the other unknown by a generalization of the formula which gives the de Rham laplacian (p. 318) on exterior forms out of their differentials d and **codifferentials** δ (p. 296, 317). We now use the exterior covariant derivative D of a tensor-valued p-form (p. 373). If, for instance, $\theta = (\frac{1}{2}\theta^\beta{}_{\gamma\lambda\mu}\, \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu)$ is a mixed tensor valued 2-form, $\mathrm{D}\theta$ is the mixed tensor valued 3-form given by codifferential

$$(\mathrm{D}\theta)^\beta_\gamma = \tfrac{1}{2}\nabla_\rho \theta^\beta{}_{\gamma\lambda\mu}\, \mathrm{d}x^\rho \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu = \frac{1}{6!} \sum_{(\rho,\lambda,\mu)} \nabla_\rho \theta^\beta{}_{\gamma\lambda\mu}\, \mathrm{d}x^\rho \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu .$$

The **covariant codifferential** Δ generalizes the usual codifferential δ (p. 317). For instance, for the above 3-form, $\Delta\theta$ is the mixed tensor valued 1-form: covariant codifferential

$$(\Delta\theta)^\beta_\gamma = -\nabla^\alpha \theta^\beta{}_{\gamma\alpha\mu}\, \mathrm{d}x^\mu ,$$

while

$$(\Delta \mathrm{D}\theta)^\beta_\gamma = -\frac{1}{2!} \nabla^\alpha \sum_{(\alpha,\lambda,\mu)} \nabla_\alpha \theta^\beta{}_{\gamma\lambda\mu} .$$

A computation analogous to the one done for usual exterior forms shows that the covariant laplace operator $\square \equiv \Delta\mathrm{D} + \mathrm{D}\Delta$ is always quasidiagonal, with principal part $-g^{\alpha\rho}\partial^2_{\alpha\rho}$ in local coordinates. For a scalar valued 0-form (i.e., a function)

$$\square = -\nabla^\alpha \partial_\alpha .$$

We first rewrite some of the equations II.

We set

$$\nabla^{\lambda}\partial_{\lambda}\omega = \sigma \quad \text{(scalar function)} \tag{0}$$

and write (3):

$$\mathrm{d}\sigma + 2s\cdot\nabla\omega + \tfrac{1}{3}r\,\mathrm{d}\omega + \tfrac{1}{6}\omega\,\mathrm{d}r = 0$$

which implies

$$\begin{aligned}-\Box\sigma + 2(\nabla^{\alpha}s_{\alpha\beta})\partial^{\beta}\omega + 2s_{\alpha\beta}\nabla^{\beta}\partial^{\alpha}\omega + \tfrac{1}{3}r\sigma\\ + \tfrac{1}{2}\partial^{\alpha}r\partial_{\alpha}\omega - \tfrac{1}{6}\omega\Box r = 0\,.\end{aligned} \tag{3'}$$

We remark that we could eliminate $\nabla^{\beta}\partial_{\alpha}\omega$ by using (2) and $\nabla^{\alpha}s_{\alpha\beta}$ by using (4), (6) and (7), which imply

$$\nabla_{\lambda}s^{\lambda}_{\mu} = \tfrac{1}{4}\partial_{\mu}r \tag{4'}$$

and then obtain the equation

$$-\Box\sigma + \tfrac{1}{3}r\sigma + \partial_{\alpha}r\partial^{\alpha}\omega - \tfrac{1}{6}\omega\Box r - \omega s^{\alpha\beta}s_{\alpha\beta} = 0\,. \tag{3''}$$

We consider (4) as an equation for a covariant vector valued 1-form $s_{\mu} = s_{\lambda\mu}\,\mathrm{d}x^{\lambda}$, while $\mathrm{d} = (\mathrm{d}^{\alpha}{}_{\beta}) = (\tfrac{1}{2}\,\mathrm{d}^{\alpha}{}_{\beta\lambda\mu}\,\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\mu})$ is a 2-form

$$\mathrm{D}s_{\beta} = 2\partial_{\alpha}\omega\,\mathrm{d}^{\alpha}{}_{\beta} + \tfrac{1}{12}g_{\beta}\wedge\mathrm{d}r\,, \qquad g_{\beta} = g_{\beta\mu}\,\mathrm{d}x^{\mu}\,.$$

Eq. (4′) reads (equality of covariant vectors)

$$\Delta s_{\mu} = \tfrac{1}{4}\partial_{\mu}r\,.$$

Thus

$$-\Box s_{\beta} = \Delta(2\partial_{\alpha}\omega\,\mathrm{d}^{\alpha}{}_{\beta} + \tfrac{1}{12}g_{\beta}\wedge\mathrm{d}r) + \tfrac{1}{4}\mathrm{D}\partial_{\beta}r\,. \tag{4''}$$

In the case of the tensor d we remark that due to the tracelessness of d, (5) implies

$$\nabla_{\alpha}\,\mathrm{d}^{\alpha}{}_{\beta\lambda\mu} = 0$$

and therefore, due to its symmetries:

$$\nabla^{\lambda}\,\mathrm{d}^{\beta}{}_{\gamma\lambda\mu} = 0$$

that is

$$\Delta\,\mathrm{d}^{\beta}{}_{\gamma} = 0\,. \tag{5'}$$

From (4) and (4′) it follows that

$$\Box\,\mathrm{d}^{\beta}{}_{\gamma} = 0\,. \tag{5''}$$

characteristic matrix

Recall (p. 569) that to compute the **characteristic matrix** of a system of N partial differential equations $f_I = 0$, $I = 1, \ldots, N$ with N unknown functions u_J, $J = 1, \ldots, N$ one must first choose a set of $2N$ integers m_I, n_J.

The set of $2N$ integers is admissible if the equation $f_I = 0$ contains no derivative of order greater than $m_I - n_J$ of the unknown u_J. The principal part of the unknown u_J in the equation $f_I = 0$ is then by definition of order $m_I - n_J$, it is zero if f_I contains only derivatives of u_J of order strictly less than $m_I - n_J$. If the derivatives of order $m_I - n_J$ of u_J appear linearly in f_I the system is said to be **quasilinear**. The characteristic matrix is then the $N \times N$ matrix whose element at the (I, J) place is the polynomial obtained by taking the principal part of the unknown u_J in the equation $f_I = 0$, and replacing a derivation $\partial/\partial x^\alpha$ by the component X_α of the covariant vector X. It can be shown that the **characteristic determinant**, – i.e., the determinant of the characteristic matrix – is independent of the choice of admissible indices, if it is not identically zero.

quasilinear

characteristic determinant

Example: Consider the 2 equations with 2 unknown u_1, u_2 on $\mathbb{R}^2$

$$f_1 \equiv \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} + u_2 = 0\,,$$

$$f_2 = \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_1}{\partial t^2} = 0\,.$$

a) Choose $m_1 = m_2 = 2$, $n_1 = n_2 = 0$, the principal parts of u_1 and u_2 in each equation are then of order 2. The characteristic matrix is

$$\begin{pmatrix} X_0^2 - X_1^2 & 0 \\ X_0^2 & X_0^1 - X_1^2 \end{pmatrix}$$

and the characteristic determinant is

$$(X_0^2 - X_1^2)^2\,.$$

b) Choose the admissible set $m_1 = 2$, $n_1 = 0$, $m_2 = 3$, $n_2 = 1$. The characteristic matrix is the diagonal matrix

$$\begin{pmatrix} X_0^2 - X_1^2 & \\ 0 & X_0^2 - X_1^2 \end{pmatrix}$$

with the same determinant.

The possibility to obtain diagonal characteristic matrices is important for the proof of well posedness of the Cauchy problem (cf. [5, 6]).

The integers m_I and n_J are often called **Leray weights**, or **Leray indices**. We consider now the system III consisting of 1′, 0, 3″, 5″, we give to all equations except III 1 Leray weight zero, to all unknown except g Leray weight 2; we give Leray weight 1 to eq. III-1 and Leray weight 3 to g. The matrix of principal parts is then diagonal, with elements $g^{\alpha\beta}X_\alpha X_\beta$.

Leray weights

Leray indices

The system III consisting of 1′, 0, 3″, 4″, 5″ is a quasidiagonal, second-order, hyperbolic system.

Remark: Using an analogous artifact we can keep eq. (3′), instead of (3″), by giving weight 3 to ω, and 1 to (0).

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8. CONFORMAL TRANSFORMATION OF NONLINEAR WAVE EQUATIONS

1) *Let $\Sigma^{n+1} = \mathbb{R} \times S^n$; let $(T, \bar{X}^i)$ be coordinates on $\Sigma^{n+1} - \{\mathbb{R} \times P\}$ with $(\bar{X}_i)$ stereographic coordinates on S^n minus the south pole P (cf. Problem V4, Sphere S^n). Let (t, x^i), $i = 1, \ldots, n$, be rectilinear coordinates on $\mathbb{R}^{n+1}$. Show that the following formulae determine a mapping $\phi: \mathbb{R}^{n+1} \to \Sigma^{n+1}$ which is an analytic diffeomorphism onto its image V:*

$$\bar{X}^0 = T = \tan^{-1} u + \tan^{-1} v\,, \qquad \text{with } -\pi/2 < \tan^{-1} u < \pi/2\,. \quad (1)$$

$$\bar{X}^i = \frac{\bar{A}}{r} x^i\,, \qquad \text{with } \bar{A} = r^{-1}\{|(1+u^2)(1+v^2)|^{1/2} - (1+uv)\}\,, \quad (2)$$

where

$$u = t + r\,, \qquad v = t - r\,, \qquad r = (\Sigma(x^i)^2)^{1/2}\,.$$

2) *Denote by $\tilde{g}$ the metric on $\phi(\mathbb{R}^{n+1})$, which is the pullback by ϕ^{-1} of the Minkowski metric η on $\mathbb{R}^{n+1}$,*

$$\tilde{g} = \phi^{-1*}\eta\,.$$

Show that $\tilde{g}$ is conformal to the canonical metric g on Σ^{n+1} defined in Problem V 5

$$g = \Omega^2 \tilde{g}\,.$$

3) *Since ϕ is an analytic diffeomorphism $\mathbb{R}^{n+1} \to V \subset \Sigma^{n+1}$ it defines an analytic map $T^*V \to T^*\mathbb{R}^{n+1}$. Determine this map and show that it extends to an analytic map $T^*\Sigma^{n+1} \to \mathbb{R}^{n+1}$.*

4) *Denote by $\tilde{u}$ an $\mathbb{R}^N$ valued function on V, i.e., an N-multiplet of scalar functions on V, and, by u its pullback by ϕ on $\mathbb{R}^{n+1}$. Denote by*

$$\Box u = \left(\sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2} - \frac{\partial^2}{\partial t^2}\right) u$$

the d'alembertian of u, and by $\Box_{\tilde{g}} \tilde{u}$ the d'alembertian of $\tilde{u}$ in the metric $\tilde{g}$. Show that

$$(\Box u)(x) = (\Box_{\tilde{g}} \tilde{u})(p)\,, \qquad p = \phi(x)\,.$$

What is the analogous equality for the operator $\Box u - f(u, \partial u, \partial^2 u)$, where ∂u is the gradient of u, $\partial^2 u$ the set of its second derivatives and f a mapping $\mathbb{R}^{N(n+2)(n+3)/2} \to \mathbb{R}^N$?

5) *Use the preceding results to transform the problem of the global solution on $\mathbb{R}^{n+1}$ of the equation*

$$\Box u - f(u, \partial u, \partial^2 u) = 0$$

into a local problem on $\mathbb{R}^{n+1}$ involving the operator $\Box_g$.

6) *Find conditions on f and n under which the preceding local problem extends to Σ^{n+1}.*

Answer 1: $\Sigma^{n+1} - \{\mathbb{R} \times P\}$ is represented in the coordinates $\bar{X}^i, \bar{X}^0$ by the whole of $\mathbb{R}^n \times \mathbb{R}$: the formulae (1), (2) define a mapping ϕ: $\mathbb{R}^{n+1} \to \Sigma^{n+1}$. It is obvious, except perhaps at $r = 0$, that this mapping is analytic. The mapping is analytic even at $r = 0$ because

$$\frac{\bar{A}}{r} \simeq \frac{2}{1 + t^2}$$

when $r \simeq 0$.
The image $\phi(\mathbb{R}^{n+1})$ is the open set of Σ^{n+1}

$$\phi(\mathbb{R}^{n+1}) = \left\{2 \tan^{-1} \frac{\bar{\rho}}{2} - \pi < T < \pi - 2 \tan^{-1} \frac{\bar{\rho}}{2}\right\},$$

$$\bar{\rho}^2 = \sum_{i=1}^{n} (\bar{X}^i)^2\,. \tag{3}$$

ϕ is bijective onto its image: its inverse is the analytic mapping

$$t = \frac{1}{2}\left(\tan \frac{T + \alpha}{2} + \tan \frac{T - \alpha}{2}\right), \qquad r = \frac{1}{2}\left(\tan \frac{T + \alpha}{2} - \frac{T - \alpha}{2}\right), \tag{4}$$

with

$$\alpha = 2\tan^{-1}\frac{\bar{\rho}}{2}. \tag{5}$$

(Note that (2) is equivalent to $\bar{X}^i = \bar{\rho}x^i/r$, $\bar{\rho} = \bar{A}(t,r)$, while (5) gives

$$\tan\alpha = \frac{\bar{\rho}}{1-\bar{\rho}^2/4} = \frac{2r}{1+uv} = \frac{u-v}{1+uv}, \qquad \alpha = \tan^{-1}u - \tan^{-1}v).$$

Answer 2: Let $\eta = -\sum_{i=1}^{n}(\mathrm{d}x^i)^2 + \mathrm{d}t^2 = -\mathrm{d}r^2 - r^2\,\mathrm{d}\omega^2 + \mathrm{d}t^2$ with $\mathrm{d}\omega^2$ the metric of the sphere S^{n-1} in polar coordinates. A straightforward computation using the results of 1) gives

$$\tilde{g} = \phi^{-1*}\eta = (\cos T + \cos\alpha)^{-2}(-\mathrm{d}\alpha^2 - \sin^2\alpha\,\mathrm{d}\omega^2 + \mathrm{d}T^2).$$

That is, if g is the canonical metric of Σ^{n+1}

$$\tilde{g} = \Omega^{-2}g, \qquad \Omega = \cos T + \cos\alpha.$$

Note that $\Omega > 0$ on $V = \phi(\mathbb{R}^{n+1}) \subset \Sigma^{n+1}$, and $\Omega = 0$ on ∂V.
In the coordinates $(T, \bar{X}^i)$

$$\Omega = \cos T + \bar{a}, \qquad \bar{a} = \frac{1-\bar{\rho}^2/4}{1+\bar{\rho}^2/4}. \tag{2.1}$$

If we make analogous computations in $\Sigma^{n+1} - \{\mathbb{R}\times N\}$, in the coordinates (T, X^i), with N the north pole of S^n we find analogous results with

$$\Omega = \cos T - a, \qquad \bar{a} = \frac{1-\rho^2/4}{1+\rho^2/4}. \tag{2.2}$$

The functions defined by (2.1) and (2.2) coincide in $\Sigma^{n+1} - \mathbb{R}\times\{P \cup N\}$ (recall $(\rho\bar{\rho})^2 = 4$), they define an analytic function on Σ^{n+1}.
Note that we can also write

$$\eta = \phi^*\tilde{g} = (\Omega\circ\phi^{-1})^{-2}\phi^*g,$$

where $\Omega\circ\phi^{-1}$ is the expression of Ω in coordinates (t, x^i) (valid only on $\mathbb{R}^{n+1}$):

$$(\Omega\circ\phi^{-1})^2 = \frac{4}{(1+u^2)(1+v^2)}, \qquad u = t+r, \qquad v = t-r.$$

Answer 3: The map ϕ associates to each covector u at $p \in V$ a covector at $x = \phi^{-1}(p) \in \mathbb{R}^{n+1}$, by a linear map $E(p) \equiv \phi^*(p)$. This mapping $E(p)$ is represented in the natural frame of the coordinates $\bar{X}^N = (T, \bar{X}^i)$ of Σ^{n+1} and x^α of $\mathbb{R}^{n+1}$ by the matrix with elements

$$\bar{E}^\mu_\alpha = \frac{\partial\bar{X}^\mu}{\partial x^\alpha}, \qquad \bar{X}^\mu = \phi^\mu(x^\alpha).$$

Using the expression of ϕ given above, we find:

$$\bar{E}^{\mu}_{\alpha} = \Omega \bar{I}^{\mu}_{\alpha} + \bar{K}^{\mu}\Lambda_{\alpha}$$

with

$$\bar{I}^0_0 = \cos T\,, \qquad \bar{I}^0_0 = \bar{I}^i_0 = 0\,, \qquad \bar{I}^i_j = (1 + \bar{\rho}^2/4)\delta^i_j - \tfrac{1}{2}\bar{X}^i\bar{X}^j\,,$$
$$\bar{K}^0 = \sin T\,, \qquad \bar{K}^i = -\bar{X}^i\,,$$
$$\bar{\Lambda}^0 = \sin T\,, \qquad \bar{\Lambda}_i = -\tfrac{1}{2}(1 + \bar{a})\bar{X}^i\,.$$

In the frames of the coordinate X of Σ^{n+1} we find analogous formulae. In the common domain of the two charts,

$$E^{\mu}_{\alpha} = \frac{\partial X^{\mu}}{\partial \bar{X}^{\mu}}\,\bar{E}^{\mu}_{\alpha}\,.$$

We deduce from these formulae that ϕ^* extends to an analytic mapping E: $T^*\Sigma^{n+1} \to \mathbb{R}^{n+1}$ (since $T^*\mathbb{R}^{n+1}$ is a trivial bundle we can identify all its fibres): if $\bar{u}(p)$ are the components in the natural frame of a covector $u(p)$ at $p \in \Sigma^{n+1}$, then $\xi = E(u(p))$ is the element of $\mathbb{R}^{n+1}$ with components

$$\xi_{\alpha} = \bar{E}^{\mu}_{\alpha}(p)\bar{u}_{\mu}(p)\,.$$

Note that the $E^{\mu}_{\alpha'}$, for fixed α, define a tangent vector to Σ^{n+1}; when $p = \phi(x)$ this vector is the image $\phi'\partial/\partial x^{\alpha}$ of the frame vector $\partial/\partial x^{\alpha}$ of $\mathbb{R}^{n+1}$.

Note also that we have

$$g(E_0, E_0) = -\Omega^2\,, \qquad E^i_{0'} = -\bar{X}^i \sin T\,.$$

Answer 4: Denote by $\tilde{u}$ an $\mathbb{R}^n$-valued function on V, and by u its pullback on $\mathbb{R}^{n+1}$

$$u = \phi^* u = \tilde{u} \circ \phi\,, \qquad \text{i.e., } u(x) = \tilde{u}(p)\,, \qquad p = \phi(x)\,.$$

The differentials of u and $\tilde{u}$ at $x \in \mathbb{R}^{n+1}$ and at $p = \phi(x)$ are linked by

$$(\partial u)(x) = E(p)\tilde{\nabla}\tilde{u}(p)\,,$$

where $\tilde{\nabla}$ is the covariant derivative relative to $\tilde{g}$.

The second covariant derivatives with respect to the metrics η and $g = \phi^{-1*}\eta$ are linked by

$$(\partial^2 u)(x) = (E(p) \otimes E(p))(\tilde{\nabla}^2\tilde{u}(p))\,.$$

That is, in coordinates $\tilde{X}^{\mu}$ on V

$$\partial_{\alpha}\partial_{\beta}u = \bar{E}^{\mu}_{\alpha}\bar{E}^{\nu}_{\beta}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\tilde{u}$$

which implies the equality of the scalars

$$(\Box u)(x) = (\Box_{\tilde{g}} \tilde{u})(p) .$$

Thus $(\Box u)(x) = f(u(x), \partial u(x), \partial^2 u(x))$ is equivalent to

$$(\Box_{\tilde{g}} \tilde{u})(p) = \tilde{F}(p, \tilde{u}(p), (\tilde{\nabla}\tilde{u})(p), (\tilde{\nabla}^2 \tilde{u})(p))$$

with the definition

$$\tilde{F}(p, \tilde{u}(p), \tilde{\nabla}\tilde{u}(p), \tilde{\nabla}^2\tilde{u}(p)) = f(\tilde{u}(p), E(p)\tilde{\nabla}\tilde{u}(p), E(p) \otimes E(p)\tilde{\nabla}^2\tilde{u}(p)) .$$

Answer 5: Let $\tilde{u}$ be an $\mathbb{R}^N$ valued function on V and set $\tilde{u} = \Omega^{(n-1)/2} v$. Since $g = \Omega^2 \tilde{g}$ we have (cf. p. 352, with change of sign convention for R)

$$\Box_g v - \frac{n-1}{4n} R v = \Omega^{-(3+n)/2}\left(\Box_{\tilde{g}} \tilde{u} - \frac{n-1}{4n} \tilde{R}\tilde{u}\right) .$$

But

$$\begin{aligned} &\tilde{R} = 0 && (\tilde{g} \text{ flat}) \\ &R = n(n-1) && (\text{scalar curvature of } \Sigma^{n+1}) . \end{aligned}$$

Thus the function $u\colon x \to u(x)$ with

$$u(x) = (\Omega^{(n-1)/2} v)(p) , \qquad p = \phi(x)$$

will satisfy the original equation on $\mathbb{R}^{n+1}$ if and only if v satisfies on V

$$\Box_g v - \frac{(n-1)^2}{4} v = \Omega^{-(3+n)/2}(p)\tilde{F}(p, \tilde{u}, \tilde{\nabla}\tilde{u}, \tilde{\nabla}^2\tilde{u}) \equiv F(p, v, \nabla v, \nabla^2 v) .$$

We have

$$\tilde{\Gamma}^{\rho}_{\mu\ \nu} = \Gamma^{\rho}_{\mu\ \nu} - \Omega^{-1}(\delta^{\rho}_{\mu} \partial_{\nu} \Omega + \delta^{\rho}_{\nu} \partial_{\mu} \Omega - g^{\rho\sigma} g_{\mu\nu} \partial_{\sigma} \Omega) .$$

Thus

$$\tilde{\nabla}_{\mu} \tilde{u} = \Omega^{(n-1)/2}\left(\nabla_{\mu} v + \frac{n-1}{2} \Omega^{-1} \nabla_{\mu} \Omega v\right) ,$$

$$\begin{aligned} \tilde{\nabla}_{\mu} \tilde{\nabla}_{\nu} \tilde{u} = \Omega^{(n-1)/2}\Big\{ &\nabla_{\mu} \nabla_{\nu} v \\ &+ \Omega^{-1}\left[\frac{n+1}{2} (\nabla_{\mu} \Omega \nabla_{\nu} v + \nabla_{\nu} \Omega \nabla_{\mu} v) - g_{\mu\nu} g^{\rho\lambda} \nabla_{\lambda} \Omega \nabla_{\rho} v\right] \\ &+ \frac{n-1}{2} \Omega^{-2}\Big[\frac{n+1}{2} \nabla_{\mu} \Omega \nabla_{\nu} \Omega - g_{\mu\nu} g^{\rho\lambda} \nabla_{\lambda} \Omega \nabla_{\rho} \Omega \\ &+ \Omega \nabla_{\mu} \nabla_{\nu} \Omega\Big]\Big\} \end{aligned}$$

and with the notations of section 4:

$$F(p, v(p), \nabla v(p), \nabla^2 v(p)) \equiv \Omega(p)^{-(3+n)/2} f(\Omega^{(n-1)/2} v(p),$$
$$\Omega^{(n-1)/2} E(p)\Big(\nabla v + \frac{n-1}{2}\,\Omega^{-1}\nabla\Omega v\Big), \Omega^{(n-1)/2} E(p) \otimes E(p)$$
$$\times (\nabla^2 v + \cdots))$$

where $\nabla^2 v + \cdots$ is the expression in parentheses above.

Answer 6: The operator $\square_g - ((n-1)/4)^2$ is defined on Σ^{n+1}. The equation obtained will extend to Σ^{n+1} if the same is true of the right-hand side.
Ω is an analytic function on Σ^{n+1}, but not necessarily positive. The expression above can be smooth only if Ω appears with a non-negative integer power.
We first show that $E(p)\Omega^{-1}\nabla\Omega$ and $E(p)\otimes E(p)\Omega^{-1}\nabla^2\Omega$ extend to analytic functions on Σ^{n+1}: a straightforward computation gives, in the coordinates X^μ [resp. $\bar{X}^\mu$]

$$E^\mu_\alpha \partial_\mu \Omega = \Omega Y_\alpha$$

with Y_α the set of scalars

$$\begin{aligned} Y_0 &= a \sin T & &[\text{resp. } -\bar{a}\sin T] \\ Y_{i'} &= -\tfrac{1}{2}\cos T(1+a)X^i & &[\text{resp. } -\tfrac{1}{2}\cos T(1+\bar{a})\bar{X}^i] \\ E^\mu_\alpha E^\nu_\beta \nabla_\mu \nabla_\nu \Omega &= \Omega(a\eta_{\alpha\beta} - V_\alpha V_\beta) & &[\text{resp. } -\bar{a}\Omega\eta_{\alpha\beta} - \Omega V_\alpha V_\beta] \end{aligned}$$

with V_α the set of scalars

$$\begin{aligned} V_0 &= -(1 - a\cos T) & &[\text{resp.} = -(1+\bar{a}\cos T)] \\ V_1 &= \tfrac{1}{2}\sin T(1+a)X^i & &[\text{resp. } \tfrac{1}{2}\sin T(1+\bar{a})\bar{X}^i]\,. \end{aligned}$$

We deduce from this result that F can be written

$$F(p, v, \nabla v, \nabla' v) \equiv (\Omega)^{-(3+n)/2} f(\Omega^{(n-1)/2} v, \Omega^{(n-1)/2} v', \Omega^{(n-1)/2} v'')\,,$$

where v' [resp. v''] is a linear sum in v, ∇v, [resp. $v, \nabla v, \nabla^2 v$] with coefficients which extend analytically to Σ^{n+1}. We suppose that f is a smooth function of its arguments vanishing together with its derivative at $(0,0,0)$

$$f(0,0,0) = 0 \qquad \text{and} \qquad \mathrm{d}f(0,0,0) = 0\,. \tag{1}$$

Then there exists a smooth function h such that

$$f(\Omega^{(n-1)/2} u, \Omega^{(n-1)/2} v, \Omega^{(n-1)/2} w) = \Omega^{n-1} h(\Omega^{(n-1)/2}, u, v, w)$$

and F extends to a smooth function on Σ^{n+1} if $n-1-(3+n)/2=(n-5)/2$ is a non-negative integer, i.e., if n is odd and $n \geq 5$.
Under the above conditions the global existence of a solution of the Cauchy problem on M_{n+1} for the equation

$$\Box u = f(u, \partial u, \partial^2 u)$$

results then from the local existence up to the time $T=\pi$ of the equation on Σ_{n+1}

$$\Box_g v - \left(\frac{n-1}{4}\right)^2 v = F(p, v, \nabla v, \nabla^2 v)\,.$$

This existence is insured for small enough data in appropriate Sobolev spaces: $v|_{T=0} \in H_s(S^n)$, $\partial_0 v|_{T=0} \in H_{s-1}(S^n)$, $s > n/2 + 2$. This appartenance implies fall-off properties of the Cauchy data at space infinity. The corresponding solution of M_{n+1} has decay properties at time and null infinity (for details see the reference).

REFERENCE

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9. MASSES OF "HOMOTHETIC" SPACE-TIME

1) *Show that if the metric g satisfies $R(g)=0$ and is asymptotically euclidean in the chart (x^i) the same is true of the metric $\hat{g} = \lambda^2 \varphi^* g$ where φ is a diffeomorphism given in the chart (x^i) by $\bar{x}^i = \lambda x^i$, and λ is a constant.*

mass of a metric

2) The **"mass" of a metric** g on a 3-dimensional asymptotically euclidean manifold is given by (cf. for instance the review article "Positive energy theorems" in "*Relativité, Groupes et Topologie* II" B. DeWitt and R. Stora eds, North-Holland, 1983)

$$m = \frac{1}{2} \int_S (\partial_i g_{ij} - \partial_j g_{ii}) \epsilon_{jkl} \, \mathrm{d}x^k \wedge \mathrm{d}x^l\,,$$

where S is the 2-sphere at infinity of $\mathbb{R}^3$.
Show that the mass $\hat{m}$ of $\hat{g}$ is $\hat{m} = \lambda m$.

Answer 1:

$$\hat{g}_{ij}(\bar{x}^h) = \lambda^2 \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial x^m}{\partial \bar{x}^j} g_{lm}(x^h) = g_{ij}(x^h) = g_{ij}(\lambda^{-1}\bar{x}^h)\,,$$

thus

$$g_{ij}(x^h) = \delta_{ij} + o_{ij}(r^{-1}) \quad \text{implies} \quad \hat{g}_{ij}(\bar{x}^h) = \delta_{ij} + o_{ij}(\bar{r}^{-1}) .$$

On the other hand

$$\bar{\partial}_h \hat{g}_{ij}(\bar{x}^h) = \lambda^{-1}\partial_h g_{ij} , \qquad \hat{g}^{ij}(\bar{x}^h) = g^{ij}(\lambda^{-1}\bar{x}^h) ,$$
$$\hat{\Gamma}^{\,l}_{ij}(\bar{x}^h) = \lambda^{-1}\Gamma^{\,l}_{ij}(\lambda^{-1}\bar{x}^h) ,$$
$$R(\hat{g}) = \lambda^{-2}R(g) .$$

Answer 2:

$$\hat{m} = \tfrac{1}{2}\int_S \lambda^{-1}(\partial_i g_{ij} - \partial_j g_{ii})(\bar{x}^h)\varepsilon_{jkl}\mathrm{d}\bar{x}^h \wedge \mathrm{d}\bar{x}^k , \qquad \hat{m} = \lambda m .$$

Example

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) ;$$

set $\bar{r} = \lambda r$, $\bar{t} = \lambda t$

$$\hat{\mathrm{d}}s^2 = \left(1 - \frac{2m\lambda}{\bar{r}}\right)\mathrm{d}\bar{t}^2 - \left(1 - \frac{2m\lambda}{\bar{r}}\right)^{-1}\mathrm{d}\bar{r}^2 - \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) .$$

One concludes from these results that if there exists a space-time with $m < 0$, there exist space-times with arbitrarily large negative mass.

10. INVARIANT GEOMETRIES ON THE SQUASHED SEVEN SPHERES*

The (round) seven sphere is the submanifold of the 8-dimensional euclidean space $\mathbb{E}^8$ (i.e., $\mathbb{R}^8$ together with the euclidean metric) defined by

$$\sum_{\mu=0}^{7} (x^\mu)^2 = 1 , \qquad x \in \mathbb{E}^8 .$$

1) *Show that the (round) seven sphere is the 7-dimensional reductive homogeneous space* SO(8)/SO(7) (see Problem III 7, Homogeneous spaces).

Answer 1: Let $\mathbb{S}^7_0$ be the (round) seven sphere, thus labelled in anticipation of the "squashed" seven spheres, which will be labelled $\mathbb{S}^7_1$, $\mathbb{S}^7_2$ and $\mathbb{S}^7_3$ (see paragraph 6).

*Based on the unpublished work of G.S. Sammelmann. Discussions with L.C. Shepley, P. Spindel and S.G. Low are acknowledged.

The action of the connected group SO(8) on $\mathbb{E}^8$ induces an action on $\mathbb{S}_0^7$. Indeed, if $x \in \mathbb{S}_0^7 \subset \mathbb{E}^8$, $g \in \mathrm{SO}(8)$, and $y = gx$ by matrix multiplication, then $y \in \mathbb{S}_0^7$. A straightforward calculation shows that this action is transitive: any point on the sphere can be taken to any other point by a rotation.

The subgroup of SO(8) which leaves a point of $\mathbb{S}_0^7$ fixed is isomorphic to SO(7). We shall use the same notation for the group SO(7) and for a subgroup of SO(8) isomorphic to SO(7) whenever the context makes the meaning clear. For instance, let $N \in \mathbb{S}_0^7 \subset \mathbb{E}^8$ be the point of coordinate $x^0 = 1$, $x^i = 0$. We shall use the same notation h for the matrix $h \in \mathrm{SO}(7)$ and for the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & h \end{smallmatrix}\right)$ in the subgroup of SO(8) isomorphic to SO(7) which leaves N invariant.
Let $N \in \mathbb{S}_0^7$ and $g \in \mathrm{SO}(8)$; a point $x = gN \in \mathbb{S}_0^7$ is identified with the left coset $g\mathrm{SO}(7)$, where $\mathrm{SO}(7) \subset \mathrm{SO}(8)$ leaves N fixed. Hence

$$\mathbb{S}_0^7 = \mathrm{SO}(8)/\mathrm{SO}(7).$$

The identification of $\mathbb{S}_0^7$ with SO(8)/SO(7) depends on the choice of N. The homogeneous space SO(8)/SO(7) is reductive, since SO(8) is compact.

2) *Show that $\mathbb{S}_0^7$ is parallelizable, and construct explicitly left and right siebenbeins on $\mathbb{S}_0^7$.*

parallelizable

Answer 2: An n-dimensional manifold is said to be **parallelizable** if it admits n linearly independent smooth vector fields. We shall construct left and right siebenbeins by using the following theorem. For the proof see, e.g., [Brickell et al.]
If there exists an $\mathbb{R}$-bilinear mapping ν: $\mathbb{R}^{k+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that
 i) $\nu(v, z) = 0$ implies that $v = 0$ or $z = 0$;
 ii) there exists $e \in \mathbb{R}^{k+1}$ such that $\nu(e, z) = z$ for all $z \in \mathbb{R}^{n+1}$,
then $\mathbb{S}^n$ admits k independent vector fields. One can check that such mappings ν exist for $k = n = 1, 3, 7$, namely

quaternions
octonions

$n = 1$, ν determined by the product on $\mathbb{C}$

$n = 3$, ν determined by the product on $\mathbb{Q}$ (**quaternions**)

$n = 7$, ν determined by the product on $\mathbb{O}$ (**octonions**, Cayley numbers).

The space $\mathbb{O}$ is a vector space of 8 dimensions over the real numbers. Let $\{e_0, e_i\}_{i=1...7}$ be a basis for this real vector space; the left product function ν on $\mathbb{O}$ is denoted by $\nu(v, z) = v \cdot z$. It is bilinear and such that

$$\begin{aligned} &e_0 \cdot e_0 = e_0 \,, \qquad e_0 \cdot e_i = e_i \cdot e_0 = e_i \,, \\ &e_i \cdot e_j = -\delta_{ij} e_0 + a_{ij}{}^k e_k \,, \end{aligned} \tag{1}$$

where $a_{ijk} = a_{ij}{}^k$ are real numbers totally antisymmetric in ijk and equal to 1 for

$$ijk = 123,\ 516,\ 624,\ 435,\ 471,\ 673,\ 572$$

and zero for triples which are not permutations of the above ones.
It may be checked that $\nu(v, z) = \nu \cdot z = 0$ implies $v = 0$ or $z = 0$ and that $\nu(e_0, z) = e_0 \cdot (z^0 e_0 + z^i e_i) = z$ for all z. One defines:
The **complex conjugate** of z: $z^* \equiv z^0 e_0 - z^i e_i$. The $\{e_i\}$ are called the **imaginary elements**.
The **Cayley inner product**: $(x \mid y) \equiv \frac{1}{2}(x^* \cdot y + y^* \cdot x)$.
The **Cayley norm**:

complex conjugate
imaginary elements
Cayley inner product
Cayley norm

$$|z|^2 \equiv (z \mid z) = \sum_{\mu=0}^{n} (z^\mu)^2 e_0$$

It may be verified by direct calculation that for $z = x \cdot y$

$$|z| = |x|\,|y| \,.$$

For $z = x \cdot y$ it follows from bilinearity that

$$z^\mu = B^\mu{}_\nu(x) y^\nu = (B_\rho)^\mu{}_\nu x^\rho y^\nu, \quad \text{with } (\beta_\rho)^N{}_\nu = (e_\rho \cdot e_\nu \mid e_\mu)$$

One can check that the matrices B_ρ satisfy

$$\begin{aligned} &\tilde{B}_\rho B_\rho = \mathbb{1} \\ &\tilde{B}_\rho B_\sigma + \tilde{B}_\sigma B_\rho = 0 \,, \qquad \rho \neq \sigma \\ &\tilde{B}(x) B(x) = |x|^2 \mathbb{1} \end{aligned} \tag{2}$$

where $\tilde{B}$ denotes the transpose of B.

Explicitly, it follows from the definition (1) that

$$(B_\rho)^\mu{}_\nu x^\rho = B^\mu{}_\nu(x) = \begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 & -x^4 & -x^5 & -x^6 & -x^7 \\ x^1 & x^0 & -x^3 & x^2 & -x^7 & x^6 & -x^5 & x^4 \\ x^2 & x^3 & x^0 & -x^1 & -x^6 & -x^7 & x^4 & x^5 \\ x^3 & -x^2 & x^1 & x^0 & x^5 & -x^4 & -x^7 & x^6 \\ x^4 & x^7 & x^6 & -x^5 & x^0 & x^3 & -x^2 & -x^1 \\ x^5 & -x^6 & x^7 & x^4 & -x^3 & x^0 & x^1 & -x^2 \\ x^6 & x^5 & -x^4 & x^7 & x^2 & -x^1 & x^0 & -x^3 \\ x^7 & -x^4 & -x^5 & -x^6 & x^1 & x^2 & x^3 & x^0 \end{pmatrix} \tag{3}$$

The first column of $B(x)$ consists of the components of x. If $x \neq 0$ the remaining columns consist of 7 smooth orthogonal vector fields $l_i(x)$ on $\mathbb{E}^8 - \{0\}$:

$$x = \nu(e_0, x) = e_0 \cdot x = x^0 e_0 + x^i e_i$$

and

$$l_i(x) \equiv \nu(e_i, x) = e_i \cdot x = -x^i e_0 + x^0 e_i + a_{ij}{}^k x^j e_k . \tag{4}$$

(left) siebenbein

If $|x| = 1$, the set $\{l_i\}$ is a particular **(left) siebenbein** on $\mathbb{S}_0^7$.

Alternatively we could have used the right product function

$$\nu_r(v, z) = z \cdot v$$

right siebenbein

and constructed a particular **right siebenbein** $\{r_i\}$ on $\mathbb{E}^8$, and thus on $\mathbb{S}_0^7$:

$$x = \nu_r(e_0, x) \quad \text{and} \quad r_i(x) = \nu_r(e_i, x) = x \cdot e_i = -x^i e_0 + x^0 e_i - a_{ij}{}^k x^j e_k . \tag{5}$$

We identify $e_0 \in \mathbb{O}$ with the point $N \in \mathbb{S}_0^7$ left fixed by SO(7).

Remark: Formulae (4) and (5) are particular cases with $a = e_0$ of $l_i^a(x) = (e_i \cdot a) \cdot (a^* \cdot x)$ and $r_i^a(x) = (x \cdot a^*) \cdot (a \cdot x)$ which depend on a, an arbitrary octonion of unit norm, because the algebra of octonions is not associative.

Remark: Let $J^\alpha{}_\beta$ be the 28 linearly independent vector fields defined by

$$J^\alpha{}_\beta(x) = x^\alpha e_\beta - x^\beta e_\alpha , \qquad \alpha, \beta = 0, \ldots, 7 .$$

The left and right siebenbein can be expressed in terms of the $J^\alpha{}_\beta$ as follows:

$$l_i = J^0{}_i + \tfrac{1}{2} a_{ij}{}^k J^j{}_k ,$$
$$r_i = J^0{}_i - \tfrac{1}{2} a_{ij}{}^k J^j{}_k .$$

3a) *Construct the Lie algebra $\mathscr{L}(\mathrm{SO}(8))$ in terms of the left and right siebenbein vector fields on $\mathbb{S}_0^7$.*

3b) *Show that $\mathscr{L}(\mathrm{SO}(8)) = \mathscr{L}(\mathrm{SO}(7)) \oplus \mathcal{M}$ for $\mathcal{M}$ isomorphic to the tangent space $T_N \mathbb{S}_0^7$, where $N \in \mathbb{S}_0^7$ is left invariant under the action of* SO(7) *on $\mathbb{S}_0^7$.*

Answer 3a: We shall identify an element $\gamma \in \mathscr{L}(\mathrm{SO}(8))$ with the Killing vector field v_γ on $\mathbb{E}^8$ generated by γ. Recall (p. 164, and Problem III 6, Homomorphisms of a Lie algebra) that if

$$[\gamma_\alpha, \gamma_\beta] = c^\gamma{}_{\alpha\beta}\gamma_\gamma \qquad \text{for } \{\gamma_\alpha\} \text{ a basis of } \mathscr{L}(\mathrm{SO}(8)) \tag{6a}$$

then

$$[v_\alpha, v_\beta] = -c^\gamma{}_{\alpha\beta} v_\gamma \,. \tag{6b}$$

The vector fields $\{l_i\}$ do not satisfy eq. (6b), but we note that

$$\begin{aligned} &[[l_i, l_j], l_k] = 4\delta_{ij}^{km} l_m \,, \qquad \delta_{ij}^{km} = \delta_i^k \delta_j^m - \delta_j^k \delta_i^m \,, \\ &[[l_i, l_j], [l_k, l_l]] = 4\delta_{ij}^{km}[l_m, l_l] - 4\delta_{ij}^{lm}[l_m, l_l] \,. \end{aligned} \tag{7}$$

Thus the set $\{\frac{1}{2}l_i, \frac{1}{4}[l_i, l_j]\}$ satisfy eq. (6b) and can be identified with 28 Killing vector fields v_α, $\alpha = 1, \ldots, 28$. Each vector field v_α can be identified explicitly with an element $\gamma_\alpha \in \mathscr{L}(\mathrm{SO}(8))$ as follows. The action of SO(8) on $\mathbb{E}^8$ is an isometry, i.e., the Lie derivative of the euclidean metric g with respect to the Killing vector field v_α corresponding to a generator of SO(8) vanishes

$$0 = (\mathscr{L}_{v_\alpha} g)_{\mu\nu} = \frac{\partial v_\alpha{}^\mu}{\partial x^\nu} + \frac{\partial v_\alpha{}^\nu}{\partial x^\mu} \,.$$

The vector field v_α defines an antisymmetric 8×8 matrix γ_α

$$(\gamma_\alpha)^\mu{}_\nu = \frac{\partial v_\alpha{}^\mu(x)}{\partial x^\nu} \,. \tag{8}$$

Let $v_1, v_2, \ldots, v_7$ be equal to $l_1, l_2, \ldots, l_7$, respectively; then $\gamma_1, \gamma_2, \ldots, \gamma_7$ are equal to $B_1, B_2, \ldots, B_7$, respectively, where the matrices $\{B_0, B_1, \ldots, B_7\}$ can be read off from (3).

We now check that if $\{v_\alpha\}$ satisfy (6b), then the $\{\gamma_\alpha\}$ defined by (8) satisfy (6a). Indeed, $v_\alpha (\alpha = 8, \ldots, 28)$ are identified with $[l_i, l_j]$ and

$$\partial [l_i(x), l_j(x)]^\mu / \partial x^\nu = -[B_i, B_j]^\mu{}_\nu \,. \tag{9}$$

Answer 3b: Since SO(8)/SO(7) is a reductive homogeneous space (see answer 1))

$$\mathscr{L}(\mathrm{SO}(8)) = \mathscr{L}(\mathrm{SO}(7)) \oplus \mathscr{M} \,, \tag{10}$$

where

$$\mathscr{L}(\mathrm{SO}(7)) \cap \mathscr{M} = \{0\} \qquad \text{and} \qquad \mathscr{A}d(\mathscr{L}(\mathrm{SO}(7)))\mathscr{M} \subset \mathscr{M} \,.$$

First we check that $\{\frac{1}{4}[l_i, l_j]\}$ is a basis for $\mathscr{L}(\mathrm{SO}(7))$. Indeed $\mathscr{L}(\mathrm{SO}(7)) = \mathscr{L}(\mathrm{Spin}(7))$. Now, the 8 matrices B_ρ defined by $(B_\rho)^\mu{}_\nu x^\rho = B^\mu{}_\nu(x)$, where $B(x)$ is given by (3) generate a representation of the Clifford algebra $\mathscr{C}(7, 0)$ (p. 65 and [Problem I 4, Clifford algebra]). The equation $B_0 = \mathbb{1}$ and (2) together with $\tilde{B}_i = -B_i$ yield

$$B_iB_j + B_jB_i = -2\delta_{ij}\,, \qquad i, j = 1, \ldots, 7\,. \tag{11}$$

Given an explicit representation of $\mathscr{C}(7,0)$, the construction of $\mathscr{L}$(Spin (7)) is straightforward:
Let $\{\Lambda_a(t)\}_t$ be a one-parameter subgroup of Spin (7), i.e.,

$$\Lambda_a(t)B_i\Lambda_a^{-1}(t) = a_i^j(t)B_j\,, \tag{12a}$$

$$\det \Lambda_a(t) = 1\,. \tag{12b}$$

The derivative of (12a) with respect to t at $t = 0$ is

$$\lambda_a B_i - B_i\lambda_a = \alpha_i^j B_j\,, \qquad \begin{cases} \lambda_a = \mathrm{d}\Lambda_a(t)/\mathrm{d}t|_{t=0}\,, \\ \alpha_i^j = \mathrm{d}a_i^j(t)/\mathrm{d}t|_{t=0}\,. \end{cases}$$

The solution of this equation is easily shown (p. 177 and [Problem I 11, Lie algebra]) by repeated applications of (11) and use of (12b) to be equal to

$$\lambda_a = -\tfrac{1}{4}\alpha^{ij}B_iB_j\,, \qquad \alpha^{ij} = \alpha_i^j\,.$$

The transformation $B \mapsto \Lambda B\Lambda^{-1}$ is an isometry; hence the matrix a is orthogonal and α is antisymmetric. To form a basis for $\mathscr{L}$(Spin (7)) we need 21 such linearly independent λ_a. An obvious choice is the set $\{-\frac{1}{4}[B_i, B_j]\}$, which by (9) can be identified with the Killing vectors

$$\{\tfrac{1}{4}[l_j, l_i]\} \qquad i, j = 1, \ldots, 7\,.$$

We now exhibit the isomorphism between $\mathcal{M}$ and $T_N\mathbb{S}_0^7$. A basis for $\mathcal{M}$ is the set $\{\gamma_\alpha, \alpha = 1, \ldots, 7\}$ defined by (8). Note that $\mathrm{Ad(SO(7))}\mathcal{M} \subset \mathcal{M}$ as a consequence of eq. (7). We have already shown the correspondence between $\{\gamma_\alpha, \alpha = 1, \ldots, 7\}$ and $\{\frac{1}{2}l_i\}$. But the l_i's are seven everywhere independent vector fields, so $\{\frac{1}{2}l_i(N)\}$ is a basis for $T_N\mathbb{S}_0^7$.
This result can be understood in a more general context as follows. Let π denote the projection map

$$\pi\colon \mathrm{SO}(8) \to \mathrm{SO}(8)/\mathrm{SO}(7) \simeq \mathbb{S}_0^7 \qquad \text{by } g \mapsto gN\,.$$

Recalling that $\mathscr{L}$(SO(8)) is naturally isomorphic to $T_e\mathrm{SO}(8)$, the derivative of this equation gives a map

$$\pi'_e\colon \mathscr{L}(\mathrm{SO}(8)) \to T_N\mathbb{S}_0^7\,.$$

This is a linear map between vector spaces, so

$$\mathscr{L}(\mathrm{SO}(8)) \approx \operatorname{Ker}\pi' \oplus \operatorname{Im}\pi'\,. \tag{13}$$

We will show that $\operatorname{Ker}\pi' = \mathrm{SO}(7)$ and that $\operatorname{Im}\pi'$ is spanned by $\{l_i(N)\}$. First consider $\operatorname{Ker}\pi'$. Write $v \in T_e\mathrm{SO}(8)$ as $v = (\mathrm{d}/\mathrm{d}t)g(t)|_{t=0}$ for a one-parameter subgroup $g(t)$. Then

$$\pi' v = \frac{\mathrm{d}}{\mathrm{d}t}\,(g(t)N)\big|_{t=0} = 0 \qquad \text{if and only if } g(t)N = N \text{ near } t = 0\,.$$

In that case, $g(t)$ is in the isotropy group SO(7) of N, so $v \in \mathscr{L}(\mathrm{SO}(7))$. Now consider Im π'. Recall that N is identified with the octonion e_0, and that for an infinitesimal rotation, $\delta e_\mu = t\omega_{\mu\nu} e_\nu$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$. So $(\mathrm{d}/\mathrm{d}t)(g(t)N)|_{t=0} = \omega_{0i} e_i$. But $l_i(N) = e_i \cdot e_0 = e_i$, so $\{l_i\}$ spans Im π'. The decomposition (13) then corresponds to (10).

Remark: The isomorphism of the Lie algebra $\mathscr{L}(\mathrm{SO}(7))$ and $\mathscr{L}(\mathrm{Spin}(7))$ does not imply the isomorphism of the groups. In particular, here, the group action of $\mathrm{SO}(7) \subset \mathrm{SO}(8)$ on S^7_0 has a fixed point; and the group action of $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ on S^7_0 is transitive. The homogeneous space SO(8)/Spin (7) is isomorphic to the round seven sphere with antipodal points identified.

4) *Construct* SO(8) *invariant metrics on* $\mathbb{S}^7_0$.

Answer 4: Given an SO(7) invariant tensor on $T_N\mathbb{S}^7_0$ one can construct an SO(8) invariant tensor field on $T\mathbb{S}^7_0$ by the left action

$$\begin{aligned} &L_g\colon N \to x = gN\,, && g \in \mathrm{SO}(8) \\ &L'_g(N)\colon T_N\mathbb{S}^7_0 \to T_x\mathbb{S}^7_0 && \text{with } V_N \mapsto V_x \end{aligned}$$

and similarly for higher-order tensors.
For instance, given an SO(7) invariant scalar product g on $T_N\mathbb{S}^7_0$, the metric on $\mathbb{S}^7_0$ defined by

$$g_x(V_x, W_x) = g_N(V_N, W_N) \quad \text{where } x = gN \quad \text{and} \quad V_x = L'_g(N)V_N \tag{14}$$

is SO(8) invariant by construction.
Conversely, on any homogeneous space G/H, a G-invariant tensor is uniquely determined by its (H-invariant) value at the point identified with H.
We shall now show that modulo a scale factor, there is only one scalar product on $T_N\mathbb{S}^7_0$ invariant under the full group of rotation SO(7) on a 7-dimensional vector space, namely

$$g(V_N, W_N) = \sum \delta_{ij} V^i_N W^j_N\,, \qquad i = 1, \ldots, 7\,. \tag{15}$$

More generally, we show that up to a multiplicative constant, there is only one SO(n) invariant second rank tensor on $\mathbb{R}^n$, namely the symmetric tensor $\delta_{\mu\nu}$. Indeed, let

$$T = T_{\mu\nu}(x)\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}$$

and let $v(x)$ be the Killing vector field on $\mathbb{R}^n$ corresponding to $\gamma \in \mathrm{SO}(n)$. The tensor T is $\mathrm{SO}(n)$ invariant if its Lie derivative with respect to v vanishes

$$(\mathscr{L}_v T)_{\mu\nu} = T_{\mu\nu,\rho}v^{\rho} + T_{\rho\nu}v^{\rho}{}_{,\mu} + T_{\mu\rho}v^{\rho}{}_{,\nu} = 0\,.$$

The Killing vector field v vanishes at the center of rotation, 0. Hence

$$(\mathscr{L}_v T)_{\mu\nu}(0) = T_{\rho\nu}(0)v^{\rho}{}_{,\mu} + T_{\mu\rho}(0)v^{\rho}{}_{,\nu} = 0\,. \tag{16}$$

Let T and V be the matrices of components $T_{\rho\nu}$ and $v^{\rho}{}_{.\mu} = -v^{\mu}{}_{.\rho}$, respectively. Then (16) can be written

$$-VT + TV = 0 \qquad \text{at } 0\,.$$

Schur's lemma and the fact that the matrix T is symmetric imply that

$$T_{\mu\nu}(0) = c\delta_{\mu\nu}\,.$$

It then follows from (15) and (14) that modulo a scale factor, there is only one SO(8) invariant metric on $\mathbb{S}_0^7$.

Let $\{\lambda^i\}$ be the dual siebenbein defined by

$$\langle \lambda^i(x), l_j(x)\rangle = \delta^i_j\,. \tag{17}$$

Eq. (17) can be solved in terms of the Cayley inner product

$$\lambda^i(x) = (l_i(x)|\ \ ) = (e_i \cdot x|\ \ )\,. \tag{18}$$

The invariant metric on $\mathbb{S}_0^7$ is

$$g(x) = \delta_{ij}\lambda^i(x)\lambda^j(x)\,. \tag{19}$$

invariant connection

5) A connection on a manifold M is said to be G-**left invariant** if it defines a covariant differentiation which commutes with a left action of G on M. In general, a connection which is compatible with a metric g is uniquely determined by g together with a once contravariant, twice antisymmetric covariant tensor called the torsion tensor (p. 308: There T is assumed to be zero – the theorem remains true for an arbitrary T). If the metric and the torsion tensor are left invariant, so is the corresponding connection.

Show that left invariant torsion tensors on a symmetric space are identically zero; show that SO(8)*-invariant connections on the round seven sphere* $\mathbb{S}_0^7$ *have no torsion.*

Answer 5: A symmetric space [see Problem III 7, Homogeneous spaces] is a homogeneous space G/H with an involutive automorphism. The round seven sphere with the metric (19) is a riemannian symmetric space [Problem III 6, Homogeneous spaces question 8]. Hence there is an isometry $s_N \in H_N$, where H_N is the isotropy subgroup which leaves $N \in \mathbb{S}_0^7$ fixed, such that

$$s_N(\exp V_N) = \exp(-V_N)\,, \qquad V_N \in T_N\mathbb{S}_0^7$$

and such that the induced mappings are

$$\begin{aligned} s'_N(x)&: T_x\mathbb{S}_0^7 \to T_{s_N(x)}\mathbb{S}_0^7 &&\text{by } V_x \to -V_{s_N(x)} \\ s_N^{*-1}(x)&: T_x^*\mathbb{S}_0^7 \to T^*_{s_N(x)}\mathbb{S}_0^7 &&\text{by } \omega_x \to -\omega_{s_N(x)}\,. \end{aligned}$$

If we use the symbol s_N to designate the action on an arbitrary tensor induced by the isometry s_N (rather than explicitly writing the appropriate combination of $s'_N(x)$ and $s_N^{*-1}(x)$), then at the point N

$$s_N T(N) = (-1)^p T(N) \qquad \text{if } T \text{ is of degree } p\,.$$

A tensor is invariant under the isometry s_N if

$$s_N T(x) = T(s_N(x))\,.$$

Hence there are no invariant tensors of odd degree on $\mathbb{S}_0^7$.
The same argument applies to arbitrary symmetric spaces.

6) Let G_2 be the group of automorphisms of the algebra of octonions which leave e_0 invariant. The two following sequences of inclusions

$$\mathscr{L}(\mathrm{SO}(8)) \approx \mathscr{L}(\mathrm{Spin}(8)) \supset \mathscr{L}(\mathrm{Spin}(7)) \supset \mathscr{L}(\mathrm{Spin}(6)) \supset \mathscr{L}(\mathrm{Spin}(5)) \tag{20}$$

and

$$\mathscr{L}(\mathrm{SO}(7)) \approx \mathscr{L}(\mathrm{Spin}(7)) \supset \mathscr{L}(G_2) \supset \mathscr{L}(\mathrm{SU}(3)) \supset \mathscr{L}(\mathrm{SU}(2)) \tag{21}$$

have been used to identify the following reductive homogeneous G/H, where G and H are compact and simply connected, with manifolds $\mathbb{S}_i^7$, diffeomorphic to the round sphere $\mathbb{S}_0^7$, with increasingly smaller isometry groups

$$\begin{aligned} \mathbb{S}_1^7 &= \frac{\mathrm{Spin}(7)}{G_2} \\ \mathbb{S}_2^7 &= \frac{\mathrm{Spin}(6)}{\mathrm{SU}(3)} = \frac{\mathrm{SU}(4)}{\mathrm{SU}(3)}\,, \\ \mathbb{S}_3^7 &= \frac{\mathrm{Spin}(5)}{\mathrm{SU}(2)}\,. \end{aligned}$$

For a study of $\mathbb{S}_1^7$ see [Englert, Roman, Spindel]; for a study of $\mathbb{S}_3^7$ see [Awada, Duff and Pope; Duff, Nilson and Pope]; for $\mathbb{S}_2^7$ and a detailed study of all the squashed seven spheres $\mathbb{S}_i^7$ above see [Sammelmann].

a) *Give a basis for the Lie algebras in* (20) *in terms of the left siebenbein* $\{\ell_i\}$.

b) *Choose a basis for* $\mathscr{L}(\mathrm{SO}(7))$ *from which one can read off the inclusion* $\mathscr{L}(\mathrm{SO}(7)) \supset \mathscr{L}(G_2)$.

c) *Show that the homogeneous spaces* $\mathbb{S}_i^7$ *are* 7*-dimensional.*

Answer 6a: Using the basis for $\mathscr{L}(\mathrm{SO}(8))$ computed in 3a we have:

Basis for		dimension
$\mathscr{L}(\mathrm{SO}(8))$	$\{\frac{1}{2}l_i, \frac{1}{4}[l_i, l_j]\}, i, j = 1, \ldots, 7$	28
$\mathscr{L}(\mathrm{Spin}(7))$	$\{\frac{1}{4}[l_i, l_j]\}, i, j = 1, \ldots, 7$	21
$\mathscr{L}(\mathrm{Spin}(6))$	$\{\frac{1}{4}[l_i, l_j]\}, i, j = 1, \ldots, 6$	15
$\mathscr{L}(\mathrm{Spin}(5))$	$\{\frac{1}{4}[l_i, l_j]\}, i, j = 1, \ldots, 5$	10

Answer 6b: We note that the difference of a left siebenbein and the corresponding right siebenbein has a zero ε_0-component:

$$\tfrac{1}{2}(l_i - r_i) = a_{ij}{}^k x^j e_k\,.$$

The vector fields $\frac{1}{2}(l_i - r_i)$ correspond under the identification defined by (8) to seven (7×7) antisymmetric matrices

$$\frac{1}{2}\,\frac{\partial}{\partial x^j}\,(l_i^k - r_i^k) = (a_i)_j{}^k = (a_i)_{jk} \in \mathscr{L}(\mathrm{SO}(7))\,, \qquad i, j, k = 1, \ldots, 7\,.$$

The remaining fourteen (7×7) antisymmetric matrices necessary to make a basis for $\mathscr{L}(\mathrm{SO}(7))$ can be chosen to be the generators of the group of automorphisms G_2 of the space $\mathbb{O}$ of octonions which leave e_0 invariant. Under the identification (8) they are the matrices corresponding to

$$G_{ij} \equiv -\tfrac{1}{4}P_{ij}{}^{kl}[l_k, l_l]\,,$$

where

$$P_{ij}{}^{kl} = \tfrac{1}{2}\delta_{ij}^{kl} + \tfrac{1}{6}a_{ij}{}^m a_{km}{}^l\,.$$

The 21 matrices corresponding to G_{ij} satisfy the 7 equations

$$a_k{}^{ij}G_{ij} = 0\,;$$

14 of them are linearly independent and form a basis for $\mathscr{L}(G_2)$.

Remark:

$$G_{ij} = P_{ij}^{\ \ kl} J_{kl} ,$$

where

$$J^k{}_l(x) = x^k e_l - x^l e_k .$$

Remark: See [Günaydin and Gursey] for further study of $\mathscr{L}(G_2)$ and the sequence of inclusions:

Answer 6c: We have the following table:

$\mathscr{L}(G_2)$ is of dimension 14
$\mathscr{L}(\mathrm{SU}(3))$ is of dimension 8
$\mathscr{L}(\mathrm{SU}(2))$ is of dimension 3

which together with the table of answer 6a gives $\dim(\mathbb{S}_i^7) = 7$.

d) *Compute the number n of families of G-invariant metrics on $\mathbb{S}^7$, using the following information: with $\mathscr{M}$ defined by $\mathscr{L}(G) = \mathscr{L}(H) \oplus \mathscr{M}$, representations of H on $\mathscr{M}$ are not, in general irreducible (representations of H on $\mathscr{L}(H)$ are of course irreducible since H is simple in all our examples); the dimensions of the irreducible subspaces of $\mathscr{M}$ are as follows.*

$\mathbb{S}_i^7$	$\mathrm{Spin}(7)/G_2$	$\mathrm{Spin}(6)/\mathrm{SU}(3)$	$\mathrm{Spin}(5)/\mathrm{SU}(2)$
dimensions of irreducible subspaces of $\mathscr{M}$	7	6, 1	4, 1, 1, 1

Answer 6d: We have shown in Answer 5) that there is an isomorphism between the H-invariant scalar products on $T_N\mathbb{S}^7$ and the G-invariant metrics on $\mathbb{S}^7$. In Answer 3b), we have constructed an isomorphism between $T_N\mathbb{S}^7$ and $\mathscr{M}$.
In the case $\mathbb{S}_1^7 = \mathrm{Spin}(7)/G_2$, the representation of G_2 on $\mathscr{M}$ is irreducible, and the argument developed in Answer 5) shows that there is only one Spin (7)-invariant metric on $\mathbb{S}_1^7$, modulo a scale factor, i.e., there is a 1-parameter family of Spin (7)-invariant metrics on $\mathbb{S}_1^7$.
In the case $\mathbb{S}_2^7 = \mathrm{Spin}(6)/\mathrm{SU}(3)$, we can scale independently the SU(3) invariant scalar products defined, respectively, on the 6-dimensional irreducible subspace and on the 1-dimensional irreducible subspace of $\mathscr{M}$. There is a 2-parameter family of Spin (6)-invariant metrics on $\mathbb{S}_2^7$.
In the case $\mathbb{S}_2^7 = \mathrm{Spin}(5)/\mathrm{SU}(2)$, let $\{v_\alpha\}$, $\alpha = 1, \ldots, 7$ be a basis for $T_N\mathbb{S}_3^7$ and consider an action of SU(2) on $T_N\mathbb{S}_3^7$ which leaves invariant the three 1-dimensional subspaces generated, respectively, by v_5, v_6, v_7. The 6 scalar products $(v_i \,|\, v_j)$, i, $j = 5, 6, 7$, define 6 components of a Spin (5) invariant metric on $\mathbb{S}_3^7$ which can be scaled independently of each other

and independently of the unique (modulo scale) Spin (5)-invariant metric obtained from the scalar product $\delta_{ij}V^iV^j$, $i, j = 1, \ldots, 4$. Hence there is a 7-parameter family of Spin (5)-invariant metrics on $\mathbb{S}_3^7$.

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11. HARMONIC MAPS

Harmonic maps between riemannian or pseudo-riemannian manifolds are the direct generalization of the usual Laplace or wave operators to nonlinear fields, meaning here fields which then take values not in a vector space but in some manifold N.

Physical examples are the so-called σ-models, for instance the one which considers four mesons fields f^a constrained by the relation

$$\sum_{a=1}^{4} (f^a)^2 = \text{constant} ;$$

they constitute a field taking its values in the three sphere.
Harmonic maps appear in General Relativity, in the definition of generalized harmonic gauges, or in the construction of classes of solutions with given symmetries.
Other classical examples of harmonic maps of importance in geometry and physics are given by the extremal (with respect to volume) submanifolds of a pseudo-riemannian manifold: geodesics, minimal (or maximal) surfaces; in this case the metric of the source is not a priori given, but is the pull back of the metric of the range manifold.
Let us remark, finally, that the dynamics of a string is governed by an harmonic map from a 2-dimensional hyperbolic manifold into space-time.
There is a wealth of results – and also of open problems – about the existence of an harmonic map $f: (M, g) \to (N, h)$ in a given homotopy class of maps $M \to N$, in the case where (M, g) and (N, h) are properly riemannian manifolds (cf. [1, 2]).
In many examples, however, (σ-models, harmonic gauges) the metric g of the source manifold is of hyperbolic signature. The natural problem for hyperbolic harmonic maps is the Cauchy problem, determination of the map from its values and the values of its first derivatives on space-like submanifolds $S \subset M$ of the source (cf. [3, 4]).

1. DEFINITIONS

Let (M, g) and (N, h) be two smooth riemannian manifolds of arbitrary signature. Let

$$f: M \to N$$

be a smooth map. The **differential** of f at $x \in M$ is a linear map (p. 121) differential

$$f'(x): T_x M \to T_{f(x)} N ,$$

it is therefore an element of $T^*_x M \otimes T_{f(x)} N$. The differential itself, f', is a mapping $x \mapsto f'(x)$, that is a section of the vector bundle with base M and fibre at x the vector space $T^*_x M \otimes T_{f(x)} N$. This vector bundle – the bundle of one forms on M with values at x in $T_{f(x)} N$ – is denoted $T^* M \otimes f^{-1} TN$.
If (x^i) and (y^α) are respectively local coordinates in M and N, and f is represented in these coordinates by

$$y^\alpha = f^\alpha(x^i)$$

the derivative f' is represented by

$$(x^i) \mapsto \left(\frac{\partial f^\alpha}{\partial x^j}(x^i) \right).$$

The metrics g on M and h on N endow the fibre at x of the vector bundle $E = T^*M \otimes f^{-1}TN$ with a scalar product $G(x) \equiv g^\#(x) \otimes h(f(x))$, where $g^\#$ is the contravariant tensor canonically associated with g. In coordinates, if u and v are two sections of E,

$$G(x)(u, v) = g^{ij}(x^k) h_{\alpha\beta}(f^\lambda(x^k)) u_i^\alpha(x^k) v_j^\beta(x^k).$$

The scalar function on M

$$e(f) \equiv G(f', f')$$

energy density

is called the **energy density** of the mapping f. In local coordinates

$$e(f) = g^{ij}(x^k) h_{\alpha\beta}(f^\lambda(x^k)) \frac{\partial f^\alpha}{\partial x^i}(x^k) \frac{\partial f^\beta}{\partial x^j}(x^k).$$

This can also be written

$$e(f) = \mathrm{tr}_g\, f^*h.$$

The vector bundle $E \equiv T^*M \otimes f^{-1}TN$ is endowed with a linear connection ∇, mapping sections of E into sections of $T^*M \otimes E$, by the usual rules (cf. p. 303): if t is a section of T^*M and s of $f^{-1}TN$ we have

$$\nabla_v(t \otimes s) = {}^g\nabla_v t \otimes s + t \otimes f^* \, {}^h\nabla_v s$$

with ${}^g\nabla$ the riemannian covariant derivative in the metric g and $f^* \, {}^h\nabla_v$ defined by pull back of the riemannian connection of h. In local coordinates if $(x^i) \mapsto (u_j^\alpha(x^i))$ is a section of E, we have

$$\nabla_i u_j^\alpha(x^k) = \partial_i u_j^\alpha(x^j) + \frac{\partial f^\beta}{\partial x^i} \Gamma_\beta{}^\alpha{}_\lambda(f^\mu(x^x))$$
$$- \Gamma_{ij}^l(x^k) u_l^\alpha(x^k), \qquad (1)$$

where $\Gamma_\beta{}^\alpha{}_\lambda$ and Γ_{ij}^l denote respectively the riemannian connections of g and h. The derivation law for tensor products sections of $(\otimes TM)^p \otimes (f^{-1}TN)^q$ is deduced as usual. *Check that* $\nabla G = 0$.

Answer 1:

$$\nabla_v(g^\# \otimes h) = {}^g\nabla_v g^\# \otimes h + g^\# \otimes f^* \, {}^h\nabla_v h \equiv 0.$$

harmonic

2) The mapping f is called **harmonic** if

$$\mathrm{tr}_g \nabla f' = 0 \qquad \text{also written as} \quad \delta\, \mathrm{d}f = 0 \quad \text{or} \quad \nabla \cdot \mathrm{d}f = 0. \qquad (2)$$

a) *Write eq.* (2) *in local coordinates.*

b) *Show that f satisfies* (2) *if it is a critical point of the "energy functional"*

$$E(f)=\int_M e(f)\,\mathrm{d}\mu(g)\,.$$

$\mathrm{d}\mu(g)$ is the volume element of M which is supposed for instance compact so that the integral makes sense for all smooth maps.

Answer 2a: Applying formula (2) we obtain

$$g^{ij}\nabla_i\partial_j f^\alpha \equiv g^{ij}(\partial^2_{ij}f^\alpha - \Gamma^k_{i\,j}\partial_k f^\alpha + \partial_i f^\lambda\partial_j f^\mu\Gamma_\lambda{}^\alpha{}_\mu)\,. \tag{3}$$

Answer 2b: The energy functional is

$$f\mapsto E(f)=\int_M G(f',f')\,\mathrm{d}\mu(g)=\int_M g^{ij}(x^k)h_{\alpha\beta}(f^\lambda(x^k))\times\partial_i f^\alpha\partial_j f^\beta\,\mathrm{d}\mu(g)\,.$$

The space of smooth maps $f\colon M\to N$ is not a vector (or affine) space if N is not a vector space, but it is an infinite dimensional manifold $\mathcal{M}$. The energy is a mapping $\mathcal{M}\to\mathbb{R}$ and its derivative at $f\in\mathcal{M}$ a linear map $E'(f)\colon T_f\mathcal{M}\to\mathbb{R}$. The tangent space to $\mathcal{M}$ at f is the space of smooth sections of $f^{-1}TN$, i.e., mappings $x\mapsto u(x)\in T_{f(x)}N$. Then

$$E'(f)\cdot u\equiv\int_M g^{ij}(x^k)\Big[\frac{\partial h_{\alpha\beta}}{\partial y^\lambda}(f^\lambda(x^k))u^\lambda(x^k)\partial_i f^\alpha\partial_j f^\beta + 2h_{\alpha\beta}(f^\lambda(x^k))\partial_i u^\alpha\partial_j f^\beta\Big]\,\mathrm{d}\mu(g)$$

which gives, after integration by parts and some computation

$$E'(f)\cdot u=\int_M u_\alpha g^{ij}\nabla_i\partial_j f^\alpha\,\mathrm{d}\mu(g)\,,\qquad u_\alpha=h_{\alpha\beta}u^\beta\,;$$

thus if f is a critical point of E, that is if $E'(f)\cdot u=0$ for all u, we have

$$\operatorname{tr}_g\nabla f'\equiv g^{ij}\nabla_i\partial_j f^\alpha=0\,.$$

The section $\operatorname{tr}_g\nabla f'$ of $f^{-1}TN$ is called the **tension field** of f; a map is **harmonic** if its tension field is zero. tension field harmonic

3) *Show that if f is a C^2 harmonic mapping $(M,g)\to(N,h)$ then it satisfies the equation*

$$\operatorname{tr}_g\nabla^2 f' - \operatorname{Ricc}(g)\cdot f' + \operatorname{tr}_g(f^*\operatorname{Riem}(h)\cdot f')=0$$

that is, in local coordinates

$$g^{ij}\nabla_i\nabla_j\partial_k f^\alpha - R_i^j\partial_j f^\alpha + R_{\lambda\mu}{}^\alpha{}_\beta\partial_k f^\lambda g^{ij}\partial_i f^\mu\partial_j f^\beta = 0 .$$

Answer 3: Straightforward computation by derivation of eq. (3) and use of the Ricci identity.

4) Consider now the case where the metric g on M is not a priori given but is the pull back of the metric h of N,

$$g = f^*h .$$

a) *Show that the numerical value of $E(h)$, for any given f, is the total volume of M in the metric g (up to a constant factor).*

b) *Show that the mapping f is a critical point of the mapping $f \mapsto E(f)$, with $E(f)$ depending on f now also through g, if and only if it is a harmonic map from (M, g) into (N, h).*

Answer 4a:

$$e(f) = (g^\# \otimes h)(f', f') = \operatorname{tr}_g f^*h = \operatorname{tr}_g g = d$$

or in coordinates,

$$e(f) = g^{ij}h_{\alpha\beta}\partial_i f^\alpha\partial_j f^\beta = g^{ij}g_{ij} = d , \qquad \text{dimension of } M$$

so

$$E(f) = \mathrm{d}\,\mathrm{vol}\, M .$$

Answer 4b: The derivative of the mapping $f \mapsto E(f)$, with g depending now on f, is the linear mapping $u \mapsto (E'_f + E'_g \circ g'_f)\cdot u$ with $E'_f(f)$ the linear mapping denoted $E'(f)$ in the previous question. Denoting by $\delta g = g'_f \cdot u$ we have, since

$$g_{ij} = h_{\alpha\beta}\partial_i f^\alpha\partial_j f^\beta ,$$
$$\delta g_{ij} = h_{\alpha\beta}(\partial_i u^\alpha\partial_j f^\beta + \partial_i f^\alpha\partial_j u^\beta) ;$$

and a straightforward computation gives

$$(E'_g \circ g'_f)\cdot u = \int_M \delta g^{ij}(h_{\alpha\beta}\partial_i f^\alpha\partial_j f^\beta - \tfrac{1}{2} g_{ij}g^{kl}h_{\alpha\beta}\partial_k f^\alpha\partial_l f^\beta)\,\mathrm{d}\mu(g)$$
$$= \int_M \delta g^{ij}\left(g_{ij} - \frac{d}{2}\, g_{ij}\right)\mathrm{d}\mu(g) = \int_M \frac{d-2}{2}\, g^{ij}\delta g_{ij}\,\mathrm{d}\mu(g)$$

which gives, using a previous formula

$$(E'_g \circ g'_f) \cdot u = \frac{d-2}{2} E'_f \cdot u .$$

Thus finally

$$E'(f) \cdot u = \frac{d}{2} E'_f \cdot u .$$

Thus f is a critical point of $E(f, g(f))$ if and only if $E'_f = 0$, i.e., if f is a harmonic map.

5) a) *Suppose M is 1-dimensional, show by a direct computation that its harmonic imbedding is a geodesic of N.*

b) *Let $f: M \to N$ be an imbedding and $g = f^*h$. Show that $\nabla \mathrm{d}f$, 2-covariant tensor on M with values in TN, can be identified with the second fundamental form k(p. 312) of M as submanifold of (N, h).*

c) *Suppose $N = S \times \mathbb{R}$, M is diffeomorphic to S and the embedding $M \to N$ is given by a mapping*

$$M \to S \times \mathbb{R} \quad \text{by} \quad y \mapsto (x, t) ,$$

where the mapping $M \to S$ by $y \to x$ is a diffeomorphism.
Write the partial differential equation expressing that the imbedded submanifold M is extremal for the volume element induced by the metric of N.

Answer 5a: If M is 1-dimensional the volume element induced by the metric of N on the image of M is the length element. Thus if the embedding f is harmonic, the image of M is a geodesic (p. 321). We give now a direct computational proof: if M is one dimensional with local coordinate t, the embedding in N with local coordinates x^α is given by

$$x^\alpha = f^\alpha(t) ;$$

the induced metric on the image of M is

$$\mathrm{d}s^2 = h_{\alpha\beta} \frac{\mathrm{d}f^\alpha}{\mathrm{d}t} \cdot \frac{\mathrm{d}f^\beta}{\mathrm{d}t} \mathrm{d}t^2 = g \, \mathrm{d}t^2 .$$

The equations for the tension field read as the geodesic equations

$$\frac{1}{g}\left(\frac{\mathrm{d}^2 f^\alpha}{\mathrm{d}t^2} - \frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}t}\frac{\mathrm{d}f^\alpha}{\mathrm{d}t} + \Gamma_\lambda{}^\alpha{}_\mu \frac{\mathrm{d}f^\lambda}{\mathrm{d}t}\frac{\mathrm{d}f^\mu}{\mathrm{d}t}\right) \equiv g\left(\frac{\mathrm{d}^2 f^\alpha}{\mathrm{d}s^2} + \Gamma_\lambda{}^\alpha{}_\mu \frac{\mathrm{d}f^\lambda}{\mathrm{d}s}\frac{\mathrm{d}f^\mu}{\mathrm{d}s}\right) = 0$$

because

$$\frac{\mathrm{d}}{\mathrm{d}s} = g^{-1/2} \frac{\mathrm{d}}{\mathrm{d}t} .$$

Answer 5b: If M is an imbedded submanifold of (N, h), the second fundamental form at $x \in M$ is the covariant symmetric 2-tensor k_x on M, with values in $T^{\perp}_{f(x)}N$, the subspace of $T_{f(x)}N$ orthogonal to $f'(T_xM)$ in the metric h, given by (p. 313)

$$k_x(u_x, v_x) = (\nabla_{f'u} f'v)^{\perp}_{(x)} , \qquad u_x, v_x \in T_x ,$$

where ∇ is the covariant derivative on (N, h) and $\perp$ the projection onto $T^{\perp}_{f(x)}N$.
We have, by a straightforward generalization of pp. 312–313

$$\nabla_{f'u} f'v = f'\bar{\nabla}_u v + (\nabla_{f'u} f'v)^{\perp} ,$$

where $\bar{\nabla}$ is the covariant derivative on (M, g).
We compute in local coordinates x^{α} on M and x^a on N:

$$k_{\alpha\beta} = k(\partial_{\alpha}, \partial_{\beta}) .$$

If $u = \partial_{\alpha}$ and $v = \partial_{\beta}$ then $f'u = \partial_{\alpha} f^a \partial_a$, $f'v = \partial_{\beta} f^b \partial_b$ thus

$$\bar{\nabla}_u v = \Gamma_{\alpha}{}^{\lambda}{}_{\beta} \partial_{\lambda} , \qquad f'\bar{\nabla}_u v = \Gamma_{\alpha}{}^{\lambda}{}_{\beta} \partial_{\lambda} f^b \partial_b$$

while

$$\begin{aligned}\nabla_{f'u} f'v &= \partial_{\alpha} f^a (\partial_a \partial_{\beta} f^b + \Gamma_a{}^b{}_c \partial_{\beta} f^c) \partial_b \\ &= (\partial^2_{\alpha\beta} f^b + \Gamma_a{}^b{}_c \partial_{\beta} f^c \partial_{\alpha} f^a) \partial_b\end{aligned}$$

and therefore as announced

$$k_{\alpha\beta} = (\nabla_{\alpha} \partial_{\beta} f^b) \partial_b .$$

In particular we remark that $\nabla f'$ is orthogonal to $T_{f(x)} f(M)$, so

$$h_{ab} \nabla_{\alpha} \partial_{\beta} f^b \partial_{\gamma} f^a = 0 , \qquad \forall \alpha, \beta, \gamma ;$$

this formula can also be proved directly through integration by parts and permutation of indices.
The tension field of (M, f^*h) imbedded in (N, h) is the trace of the second fundamental form of this submanifold; if the imbedded manifold is extremal the mapping is harmonic, that is the trace of the second fundamental form vanishes.

Answer 5c: Suppose $N = S \times \mathbb{R}$ with S diffeomorphic to M and, in adapted coordinates $x^0 \in \mathbb{R}$ and coordinates (x^i) in M identified with coordinates in S by the diffeomorphism, f is represented by

$$f^0 = \varphi(x^i) , \qquad f^i = x^i ;$$

then, the vanishing of the tension field reduces to the equation

$$(\mathrm{tr}_g\ \nabla f')^0 \equiv g^{ij}(\bar{\nabla}_i \partial_j \varphi + \Gamma^0_{0\,0}\partial_i\varphi\partial_j\varphi + \Gamma^0_{i\,0}\partial_j\varphi$$
$$+ \Gamma^0_{j\,0}\partial_i\varphi + \Gamma^0_{i\,j}) = 0$$

the Christoffel symbols being those of h.

Remark: if $\varphi(x^i) = 0$ (i.e., M is the submanifold $x^0 = 0$) then we find the well-known formula:

$$\mathrm{tr}_g(\nabla f')^0 = g^{ij}\Gamma^0_{i\,j} = \frac{1}{n_0}\,\mathrm{tr}_g\,K\,.$$

Cf. references Problem V 12.

12. COMPOSITION OF MAPS

This problem uses definitions of the preceding problem (V 11 Harmonic). The composition of maps is particularly useful to transform problems about mappings between manifolds into problems for mappings which take their values in the vector space $\mathbb{R}^q$: it is a fundamental tool in the treatment of harmonic maps between properly riemannian manifolds by Eells and Sampson [1]. We indicate here an application to the case where the source manifold is of hyperbolic signature; for further study of this case see [3, 4].

1) *Show that if $f: M \to N$ and $F: N \to Q$ are arbitrary smooth maps between pseudo-riemannian manifolds (M, g), (N, h) and (Q, q) one has the identity (a dot is a scalar product in h):*

$$\nabla\, \mathrm{d}(F \circ f) \equiv \mathrm{d}F \cdot \nabla\, \mathrm{d}f + \nabla\, \mathrm{d}F \cdot (\mathrm{d}f \otimes \mathrm{d}f)\,, \tag{1}$$

that is, if (x^α), (x^a) and (x^A) are respectively local coordinates on M, N and Q

$$\nabla_{\alpha\beta}(F \circ f)^A = \partial_a F^A \nabla_\alpha \partial_\beta f^a + \nabla_a \partial_b F^A \partial_\alpha f^a \partial_\beta f^b\,.$$

Answer 1: By definition we have

$$\nabla_\alpha \partial_\beta (F \circ f)^A = \partial^2_{\alpha\beta}(F \circ f)^A - \Gamma_\alpha{}^\lambda{}_\beta \partial_\lambda (F \circ f)^A + \Gamma_B{}^A{}_C$$
$$\times\ \partial_\alpha (F \circ f)^B \partial_\beta (F \circ f)^C\,.$$

By the law of derivation of a composition map we find

$$\partial_\beta (F \circ f)^A = \partial_a F^A \partial_\beta f^a\,,$$
$$\partial^2_{\alpha\beta}(F \circ f)^A = \partial^2_{ab} F^A \partial_\alpha f^b \partial_\beta f^a + \partial_a F^A \partial^2_{\alpha\beta} f^a\,,$$

which shows that

$$\begin{aligned}\nabla_\alpha \partial_\beta (F\circ f)^A \equiv{}& \partial_a F^A(\partial^2_{\alpha\beta} f^a - \Gamma_\alpha{}^\lambda{}_\beta \partial_\lambda f^a + \Gamma_b{}^a{}_c \partial_\alpha f^b \partial_\beta f^c) \\ &+ (\partial^2_{ab} F^A - \Gamma_a{}^c{}_b \partial_c F^A + \Gamma_B{}^A{}_C \partial_a F^B \partial_b F^C)\partial_\alpha f^b \partial_\beta f^a\ ;\end{aligned}$$

the two terms containing $\Gamma_b{}^a{}_c$ add to zero and we thus have the given formula.

2) A mapping $f\colon (M, g) \to (N, h)$ is called **totally geodesic** if $\nabla\, \mathrm{d}f = 0$. *Show that the composition of two totally geodesic maps is totally geodesic; show that if f is harmonic and F totally geodesic, then $F\circ f$ is harmonic.* tot ge

Answer 2: The first statement follows readily from formula (1). We deduce from this formula the tension field of $F\circ f$

$$\tau(F\circ f) = \mathrm{d}F\cdot \tau(f) + \nabla\, \mathrm{d}F \cdot g(\mathrm{d}f, \mathrm{d}f) \tag{2}$$

and see that $\tau(F\circ f) = 0$ (i.e., $F\circ f$ is harmonic) if $\tau(f) = 0$ and $\nabla\, \mathrm{d}F = 0$. It is not sufficient for $\tau(F\circ f) = 0$ that $\tau(f) = 0$ and $\tau(F) = 0$.

3) *Suppose that $F\colon (N, h) \to (Q, q)$ is a pseudo-riemannian immersion, i.e.*

$$h = F^* q\ . \tag{3}$$

a) *Give the relation between the densities of energy $e(F\circ f)$ and $e(f)$.*
b) *Give the relation between $\tau(f)$ and $\tau(F\circ f)$.*
c) *Show that the map $f\colon M \to N$ is harmonic if and only if $\tau(F\circ f)$ is perpendicular to $F(N)$ at each point, and $F(N)$ is nonisotropic (which it will never be if h is properly riemannian).*

Answer 3a:

$$\begin{aligned} e(F\circ f) &= q_{AB}\, g^{\alpha\beta} \partial_\alpha (F\circ f)^A \partial_\beta (F\circ f)^B \\ &= q_{AB} \partial_a F^A \partial_b F^B g^{\alpha\beta} \partial_\alpha f^a \partial_\beta f^b \\ &= h_{ab}\, g^{\alpha\beta} \partial_\alpha f^a \partial_\beta f^b = e(f)\ .\end{aligned}$$

Answer 3b: Using (2) we obtain

$$\tau(F\circ f)^A = \partial_a F^A \tau(f)^a + g^{\alpha\beta} \partial_\alpha f^a \partial_\beta f^b \nabla_a \partial_b F^A\ .$$

Each tangent vector $X_{(a)}$ to Q, with components $\partial_a F^A$ is tangent to $F(N)$, and so is $\partial_a F^A \tau(f)^a$, while $Y_{(ab)} = (\nabla_a \partial_b F^A)$ is orthonormal to $F(N)$ if F is a pseudo-riemannian immersion (cf. Problem V 11, Harmonic, answer 4b). Thus $\tau(f)$ is the projection of $\tau(F \circ f)$ on the tangent plane to $F(N)$ (subspace of the tangent plane to Q).

Answer 3c: Follows from 3b).

4) An arbitrary smooth manifold N can always be embedded in a manifold $\mathbb{R}^Q$ for Q large enough. Let F be this embedding. The euclidean metric of $\mathbb{R}^Q$ induces on N the metric γ given in local coordinates by (x^a coordinates on N, x^A on $\mathbb{R}^Q$)

$$\gamma_{ab} = \partial_a F^A \partial_b F^B \delta_{AB} .$$

Set

$$t^{aA} = \gamma^{ab} \partial_b F^A ,$$

then

$$\partial_c F^B t^{aA} \delta_{AB} = \gamma^{ab} \partial_b F^A \partial_c F^B \delta_{AB} = \delta^a_c .$$

a) *Show that the quadratic form*

$$q_{AB} = \delta_{AB} - \gamma_{ab} t^{aA} t^{bB} + h_{ab} t^{aA} t^{bB} \tag{4}$$

defines a metric q in a neighborhood U of $F(N)$ which induces on N the metric h.

b) *Show that the geodesics orthogonal to $F(N)$ in the metric q coincide with the geodesics orthogonal to $F(N)$ in the flat metric of $\mathbb{R}^Q$.*

Answer 4a: By the previous relations the metric F^*q induced on N by q is

$$h_{ab} = q_{AB} \partial_a F^A \partial_b F^B .$$

Answer 4b: Let n denote the dimension of N. Consider the field over $F(N)$ of hyperplanes $T_y^\perp F(N)$ of dimension $Q - n$ orthogonal to $F(N)$, i.e., to $T_y F(N)$ at y. Consider also a smooth field of orthogonal (in the euclidean sense) Q-bein with its first n vectors at y in $T_y F(N)$ and the remaining $Q - n$ vectors e_A, $A = n + 1, \ldots, q$ in $T_y^\perp F(N)$. If U is an appropriately chosen neighborhood of $F(N)$ in $\mathbb{R}^Q$, each point X in U has one and only one euclidean projection $y = \Pi(X)$ on $F(N)$, that is it belongs to one hyperplane $T_y^\perp F(N)$. We define local coordinates Y^A for $X \in U$ by

$$Y^A = y^A\,, \qquad A = a = 1, \dots, n\,,$$

where y^A, $A = 1, \dots, n$ are local coordinates of $F^{-1}(y)$, while the coordinates Y^A, $A = n+1, \dots, q$ are given by the decomposition (note that $X - y$ is orthogonal to $F(N)$)

$$X - y = \sum_{A=n+1}^{Q} Y^A e_A\,.$$

In these coordinates the mapping $F: N \to \mathbb{R}^Q$ is given by

$$\begin{aligned} &F^A(x^1, \dots, x^n) = x^A\,, && A = 1, \dots, n \\ &F^A(x^1, \dots, x^n) = 0\,, && A = n+1, \dots, Q\,. \end{aligned}$$

Thus

$$\begin{aligned} \partial_a F^A &= \delta_a^A && a, A = 1, \dots, n \\ \partial_a F^A &= 0\,, && A = n+1, \dots, Q \\ \gamma_{ab} &= \delta_{ab} && \\ q_{AB} &= h_{AB}\,, && A, B = a, b = 1, \dots, n \\ q_{aA} &= 0\,, && a = 1, \dots, n, \quad A = n+1, \dots, Q \\ q_{AB} &= \delta_{AB}\,, && A, B = n+1, \dots, Q \end{aligned}$$

that is, in these coordinates

$$q = h_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b + \sum_{A=n+1}^{q} (\mathrm{d}x^A)^2\,,$$

recall that h_{ab} depends only on $x^1, \dots, x^n$.
The geodesics orthogonal to $F(N)$ in the euclidean metric at y are the straight lines

$$\begin{aligned} &x^A = tC^A\,, && A = n+1, \dots, q\,, && C^A \text{ some constants} \\ &x^a = y^a\,, && a = 1, \dots, n\,, && y^A \text{ constants}\,; \end{aligned}$$

they are orthogonal to $F(N)$ in the metric q since $q_{Ab} = 0$, and geodesics in the metric q because $\Gamma_A{}^C{}_B = 0$ if $C = n+1, \dots, Q$.

5) In this question and the next the definitions are the same as in question 4, but we identify N with its image $F(N)$ in $U \subset \mathbb{R}^Q$.

Show that a mapping $W: M \to U \subset \mathbb{R}^Q$, is a harmonic map

$(M, g) \to (N, h)$ if and only if it is a harmonic map $(M, g) \to (U, q)$ and it takes its values in N.

Answer: A harmonic map $W: (M, g) \to (U, q)$ satisfies the equations

$$\nabla^\alpha \nabla_\alpha W^A \equiv g^{\alpha\beta}(\partial^2_{\alpha\beta} W^A - \Gamma_\alpha{}^\lambda{}_\beta \partial_\lambda W^A + \Gamma_B{}^A{}_C \partial_\beta W^B \partial_\alpha W^C), \quad (5.1)$$

where the $\Gamma_B{}^A{}_C$ are the Christoffel symbols of the metric q. In the adapted coordinates of section 4 we have

$$\Gamma_B{}^A{}_C = 0 \qquad \text{if} \qquad A = n+1, \ldots, Q$$

$$\Gamma_B{}^A{}_C = \Gamma_b{}^a{}_c \qquad \text{if} \qquad A = a, \quad B = b, \quad C = c, \quad a, b, c = 1, \ldots, n,$$

where $\Gamma_b{}^a{}_c$ denote the Christoffel symbols of h.
W takes its values in N if and only if, in these coordinates

$$W^A = 0, \qquad A = n+1, \ldots, q. \quad (5.2)$$

Eqs. (5.1) and (5.2) imply that W is a harmonic map $(M, g) \to (N, h)$ because they imply

$$g^{\alpha\beta}(\partial^2_{\alpha\beta} W^a - \Gamma_\alpha{}^\lambda{}_\beta \partial_\lambda W^a + \Gamma_b{}^a{}_c \partial_\beta W^b \partial_\alpha W^c) = 0. \quad (5.3)$$

Conversely (5.3) and (5.2) imply (5.1).

6) *Show that if the metric g is hyperbolic on $M = S \times \mathbb{R}$, with S space-like and if $W|_S$ takes its values in N and $\partial_0 W|_S$ takes its values in $W|_S^{-1} TN$ then W, solution of* (5.1) *takes its values in N.*

Answer: To show that W takes its values in N is to show that in adapted coordinates on $U \subset \mathbb{R}^q$, $W^A = 0$, $A = n+1, \ldots, q$. Since W satisfies (5.1) and since for $A = n+1, \ldots, q$ we have $\Gamma_B{}^A{}_C = 0$ we have that W satisfies

$$g^{\alpha\beta}(\partial^2_{\alpha\beta} W^A - \Gamma_\alpha{}^\lambda{}_\beta \partial_\lambda W^A) = 0, \qquad A = n+1, \ldots, Q.$$

By the uniqueness theorem for the Cauchy problem for hyperbolic equations (p. 521) we have $W^A = 0$ if $W^A|_S = 0$ and $\partial_0 W^A|_S = 0$; the conclusion follows.

Remark: The adapted coordinates x^A, $A = 1, \ldots, q$ are not globally defined on U; they identify U with a manifold $N \times I^{q-n}$, with I some interval of $\mathbb{R}$, and N has in general no global coordinate system. These coordinates are an intermediate step in the proof that the problem of

constructing a harmonic map $(M, g) \to (N, h)$ can be reduced to a system of ordinary P.D.E.: this is done by taking global coordinates in U, for instance, the ones induced by the cartesian coordinates of $\mathbb{R}^q$.
Another procedure for operating this reduction to a usual system is by considering N as a submanifold of $(\mathbb{R}^q, e)$, with the induced metric, and using Lagrange multipliers (cf. [3, 4]).

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13. KALUZA–KLEIN THEORIES

INTRODUCTION

The rigorous nonlinear equations of a Kaluza–Klein model on a manifold V_d are equivalent to the usual Einstein–Yang-Mills equations coupled with a scalar multiplet; a cosmological constant appears in Einstein's equations in the case of a nonabelian gauge group. To enlarge the group without enlarging the dimension of the manifold V_d the theory can be extended to the case where the fibre is no longer a Lie group, but rather a homogeneous space G/H (cf. Coquereaux and Jadczyk); the class of exact solutions satisfying the original Kaluza–Klein ansatz for the group G – that is, of metrics on V_d invariant under G – is then greatly restricted.
This difficulty can be avoided for physical applications in two ways: either by considering that the full Kaluza–Klein ansatz governs only the ground state (cf, for instance, Duff, Nilsson and Pope (2)), or by relaxing this ansatz in the full nonlinear model by requiring only that the metric on V_d projects on V_4 and defining a G-connection on V_4. In this latter approach, the equations are no longer taken to be the vanishing of the Einstein tensor of the metric $\hat{g}$ on V_d, but are deduced from a "reduced" lagrangian on V_4, obtained by integration over the fibres of V_d of the

scalar curvature of $\hat{g}$ (cf, for instance, Percacci and Randjbar-Daemi or Duff, Nilsson and Pope).

In this problem we establish the equations of the original Kaluza–Klein theory. We take V_d to be a fibre bundle over V_4, with fibres the orbits of G, and with a metric $\hat{g}$ on V_d invariant under the right action of G. It is also possible to start from a manifold V_d with a metric $\hat{g}$ with a group G of isometries, but in the study of such a metric one is led to make hypothesis on the orbits; the simplest is the one we use here.

1. *Let V_d be a d-dimensional differentiable principal fibre bundle whose base is a 4-dimensional differentiable manifold V_4 and whose structure group is a Lie group G.*

a) *Suppose that V_d can be endowed with a pseudo-riemannian metric $\hat{g}$ which is invariant under the right action of G on V_d* (p. 129): *Show that $\hat{g}$ determines* i) *a G-invariant metric ξ on each fibre (as long as the orbits of the action of G are not null for $\hat{g}$)*; ii) *a connection on V_d; and* iii) *a pseudo-riemannian metric g on V_4.*
b) *Show the converse.*
c) *Write the metric $\hat{g}$ in local coordinates adapted to the fibred structure and the isometries of V_d.*

Answer 1ai): A G-invariant metric ξ on the fibre G_x at $x \in V_4$. Suppose that V_d admits a metric $\hat{g}$ which is invariant under the right action of G. Let $\hat{x}$ be a point in V_d lying over x in the base space V_4. Since the right action $\tilde{R}_y$ ($y \in G$) is transitive on the fibre at $\hat{x}$, the fibre may be viewed as the orbit G_x of G through $\hat{x}$. Such an orbit is a submanifold of V_d diffeomorphic to G. If G_x is non-null with respect to $\hat{g}$, then $\hat{g}$ induces (p. 290) a metric $\xi = i^*\hat{g}$ on G_x. It is right-invariant, since the inclusion mapping $i\colon G_x \to V_d$ commutes with $\tilde{R}_y$ and $\hat{g}$ is invariant under $\tilde{R}_y$.
ii) A connection on V_d, defined by horizontal vector spaces $H_{\hat{x}}$, $x \in V_d$. Recall that (p. 359) a connection on a principal bundle may be defined as a field of horizontal vector spaces invariant under $\tilde{R}_y$. Since the orbits G_x are not null, they admit at each point $\hat{x}$ an orthogonal hypersurface $H_{\hat{x}}$ of dimension 4. We shall use the invariance of the orbits and of the metric under $\tilde{R}_y$ to show that these hyperplanes are invariant.
The tangent space $V_{\hat{x}}$ to $G_{\hat{x}}$ at $\hat{x}$ is isomorphic to $\mathscr{G}$, the Lie algebra of G, and is spanned by the Killing vector fields of the action $\tilde{R}_y$ (p. 360). The space $H_{\hat{x}}$ is the orthogonal complement of $V_{\hat{x}}$, i.e., the space of tangent vectors u to V_d at $\hat{x}$ such that

$$\hat{g}_{\hat{x}}(u, v) = 0 \qquad \forall v \in V_{\hat{x}} .$$

Now,

$$R'_y V_{\hat{x}} = V_{\tilde{R}_y \hat{x}}$$

and since $\hat{g}$ is invariant,

$$\hat{g}_{\hat{x}} = \tilde{R}^*_y \hat{g}_{\tilde{R}_y \hat{x}} ,$$

that is,

$$\hat{g}_{\hat{x}}(u, v) = \hat{g}_{\tilde{R}_y \hat{x}}(R'_y u, R'_y v) .$$

These inequalities and the definition of $H_{\hat{x}}$ imply that

$$\tilde{R}'_y H_{\hat{x}} = H_{\tilde{R}_y \hat{x}} .$$

Further, the projection $\pi: V_d \to V_4$ given by $\hat{x} \mapsto x = \pi(\hat{x})$ defines an isomorphism $H_{\hat{x}} \to T_x V_4$, since the tangent space to V_d at $\hat{x}$ is spanned by $V_{\hat{x}}$ and $H_{\hat{x}}$, and $\pi' V_{\hat{x}} = 0$. Thus the field $H_{\hat{x}}$ of vector spaces meets the requirements (p. 359) to define a connection on V_d.

iii) The pseudo-riemannian metric g on V_4 is given by

$$g_x(u, v) = \hat{g}_{\hat{x}}(\hat{u}, \hat{v}) , \qquad u, v \in T_x V_4 ,$$

where $\hat{u}$ and $\hat{v}$ are the horizontal vectors which project onto u and v, respectively. Due to the invariance of $\hat{g}$ and $H_{\hat{x}}$, this definition is independent of the choice of $\hat{x} \in \pi^{-1}(x)$. The metric g determines the "distance" between orbits of G.

Remark: The field of 4-hyperplanes $H_{\hat{x}}$ is not in general completely integrable: (V_4, g) cannot be identified with a submanifold of $(V_d, \hat{g})$.

Answer 1b: Conversely, an invariant metric on each fibre, a field $H_{\hat{x}}$ of horizontal subspaces of a G-connection on V_d and a metric on V_4 determine an invariant metric on V_d. To define this metric, decompose each tangent vector v to V_d into its horizontal and vertical components v_{H} and v_{V}, and set

$$\begin{aligned} &\hat{g}(\hat{v}_{\mathrm{V}}, \hat{u}_{\mathrm{V}}) = \xi(\hat{v}_{\mathrm{V}}, \hat{u}_{\mathrm{V}}) , \\ &\hat{g}(\hat{v}_{\mathrm{V}}, \hat{u}_{\mathrm{H}}) = 0 , \\ &\hat{g}(\hat{u}_{\mathrm{H}}, \hat{v}_{\mathrm{H}}) = g(\pi' \hat{u}_{\mathrm{H}}, \pi' \hat{v}_{\mathrm{H}}) . \end{aligned}$$

Answer 1c: To write $\hat{g}$ in adapted coordinates we consider a local trivialization ϕ of V_d

$$\phi: \pi^{-1}(U) \to U \times G , \qquad U \text{ the domain of a chart in } V_4$$

and we choose as local coordinates, in $\pi^{-1}(U)$, $\hat{x}_M = \{$local coordinates x^α, $\alpha = 0, 1, 2, 3$ in U, and local coordinates y^m, $m = 1, \ldots, d-4$ in $G\}$. We take a moving frame $(\hat{e}_\alpha, \hat{e}_m)$ such that $\hat{e}_m$ are the vertical vector fields

$$\hat{e}_m = \overset{\Delta}{\phi}'^{-1} e_m , \qquad m = 1, \ldots, d-4 ,$$

where e_m denotes a basis of right-invariant vector fields on G, and such that $\hat{e}_\alpha$ are the horizontal vector fields (i.e., fields orthogonal to $\hat{e}_m$) which project onto a basis in V_4:

$$\pi' \hat{e}_\alpha = \pi'_{\mathrm{can}} \circ \phi' \hat{e}_\alpha = e_\alpha$$

with π_{can}: $U \times G \rightarrow U$ by $(x, y) \mapsto x$.
We can take for instance the natural basis $e_\alpha = \partial / \partial x^\alpha$, or an orthonormal basis e_a. In any case we have

$$\phi' \hat{e}_\alpha = e_\alpha - A^m_\alpha e_m .$$

Since $\phi' \hat{e}_\alpha - e_\alpha$ has a zero canonical projection on U and is invariant under the right action of G, the A^m_α depend only on the coordinates in U. $A^m_\alpha e_m$ gives the connection 1-form (cf. p. 361) on U determined by $\hat{g}$. The identification of $A^m_\alpha e_m$ with the connection 1-form differs from the usual one, which is done with left-invariant vector fields.

Remark: The vector fields $\hat{e}_m$ are invariant under the right action of G:

$$\tilde{R}'_y \hat{e}_m(\hat{x}) = \hat{e}_m(\tilde{R}_y \hat{x}) .$$

The Killing vectors $\hat{X}$ of the right action of G (i.e., the vector fields generating the transformations $\hat{x} \mapsto \tilde{R}_y \hat{x}$, (cf. p. 154)) are such that, if $t \mapsto y(t)$ is a one parameter subgroup of G,

$$\frac{\mathrm{d}(\tilde{R}_{y(t)} \hat{x})}{\mathrm{d}t} = \hat{X}(\tilde{R}_{y(t)} \hat{x}) .$$

The images of $\hat{e}_m$ and $\hat{X}$ in a trivialization are the right-invariant and the left-invariant vector fields e_m and X on G.
We have $[X, e_m] = 0$, so $[\hat{X}, \hat{e}_m] = \phi'^{-1}[X, e_m] = 0$.
The coframe dual to $(\hat{e}_\alpha, \hat{e}_m)$ (p. 136) is found to be

$$\hat{\omega}^\alpha = \pi^* \omega^\alpha = \phi^* \overset{*}{\pi}_{\mathrm{can}} \omega^\alpha , \qquad \hat{\omega}^m = \phi^*(\theta^m + A^m_\alpha \omega^\alpha) ,$$

where (ω^α) is the dual of (e_α), and (θ^m) is a basis of right-invariant 1-forms on G dual to (e_m). In this coframe the metric $\hat{g}$ reads

$$\widehat{\mathrm{d}s}^2 = g_{\alpha\beta} \omega^\alpha \omega^\beta + \xi_{mn}(\theta^m + A^m_\alpha \omega^\alpha)(\theta^m + A^n_\beta \omega^\beta) ,$$

where we have identified $\overset{*}{\pi}_{\mathrm{can}}\omega^{\alpha}$ and ω^{α}. A necessary and sufficient condition for this metric to be invariant under the right action of G is that $g_{\alpha\beta}$, ξ_{mn} and A^m_α depend only on the coordinates (x^β) of V_4, since $\mathscr{L}_X\theta = 0$ if X is a Killing field of the right action of G.

In the trivialization and the local coordinates discussed above, the space-time metric is $g_{\alpha\beta}\omega^\alpha\omega^\beta$, the G-connection is represented by the $\mathscr{G}$-valued 1-form $A = A^m_\alpha\,\omega^\alpha e_m$, and the induced metric on the fibre $x^\alpha = \text{const}$ is $\xi_{mn}\theta^m\theta^n$. In a given trivialization, the ξ_{mn} are scalar fields on U, i.e., they behave like a set of scalar functions under a coordinate change in U. For this reason, $\xi = (\xi_{mn})$ is sometimes called a **scalar multiplet**.

scalar multiplet

Let us now consider the effect of a change of local trivialization of V_d over $U_1 \cap U$. Such a change is described by a transition mapping (p. 126) $U_1 \cap U \to G$, $x \mapsto h(x) \in G$; $h(x)$ acts by left translation. Under such a change, the metric g is invariant, while A changes to

$$\mathrm{Ad}(h^{-1})A + h^*\theta_{\mathrm{MC}} = h^{-1}Ah + h^{-1}\,\mathrm{d}h\,,$$

(see pp. 365–366 for notation and proof). The scalar multiplet ξ changes to $\mathrm{Ad}(h)\otimes\mathrm{Ad}(h)\xi$, as we now show. For simplicity, take G to be the linear group $\mathrm{GL}(n, \mathbb{R})$. The index m is then a pair (i, j) $(i, j = 1, \ldots, n)$. The right-invariant 1-form $\theta^i{}_j$ is determined from its value at the unit element e of G:

$$\theta^i{}_j(e) = \mathrm{d}y^i{}_j$$

by right translation:

$$\theta^i{}_j(y) = R^*_{y^{-1}}\theta^i{}_j(e) = \mathrm{d}y^i{}_l y^{-1l}{}_j\,.$$

Under a change of trivialization determined by left multiplication by $h(x) \in G$,

$$y^i{}_l = h^i{}_j(y)y'^j{}_l\,.$$

In matrix notation ($h^i{}_j = h$, $y^i{}_j = y$, etc.), we then have

$$\begin{aligned}\theta = \mathrm{d}y\; y^{-1} &= \mathrm{d}(hy')(hy')^{-1}\\ &= \mathrm{d}h\,h^{-1} + h\,\mathrm{d}y'\;y'^{-1}h^{-1}\end{aligned}$$

or

$$\theta = h\theta' h^{-1} + \mathrm{d}h\,h^{-1}\,.$$

The transformation law for ξ then follows.

2) *Take the metric $\hat{g}$ in local coordinates to be of the form*

$$\hat{g} = g_{\alpha\beta}\, dx^\alpha\, dx^\beta + \xi_{mn}(\theta^m + A^m_\alpha\, dx^\alpha)(\theta^n + A^n_\beta\, dx^\beta)\,,$$

where the θ^m are a basis of right-invariant 1*-forms on G and $g_{\alpha\beta}$, ξ_{mn}, A^m_α depend only on the four coordinates x^λ. Compute the Ricci tensor $\hat{R}_{MN}$ of $\hat{g}$. Let $\hat{S}_{MN} = \hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R}$. Compare the equations $\hat{R}_{\alpha m} = 0$ and $\hat{S}_{\alpha\beta} = 0$, respectively, with the Yang-Mills and the Einstein equations on V_4.*

Answer 2: It is straightforward to compute the riemannian connection of $\hat{g}$ relative to the frame $\hat{\omega}^\alpha = dx^\alpha$, $\hat{\omega}^m = \theta^m + A^m_\alpha\, dx^\alpha$.
The structure coefficients $\hat{C}^M{}_{PQ}$ are defined by

$$d\hat{\omega}^M = -\tfrac{1}{2}\hat{C}^M{}_{PQ}\hat{\omega}^P \wedge \hat{\omega}^Q\,.$$

We have $d\hat{\omega}^\alpha = 0$, so

$$\hat{C}^\alpha{}_{MN} = 0\,.$$

If $C^m{}_{pq}$ are the structure constants of G, then the right-invariant 1-forms θ^m satisfy

$$d\theta^m = \tfrac{1}{2}C^m{}_{pq}\theta^p \wedge \theta^q\,.$$

Thus

$$d\hat{\omega}^m = \tfrac{1}{2}C^m{}_{pq}(\hat{\omega}^p - A^p{}_\alpha\, dx^\alpha)(\hat{\omega}^q - A^q{}_\beta\, dx^\beta) + \partial_\beta A^m_\alpha\, dx^\beta \wedge dx^\alpha\,.$$

Reading off the $\hat{C}$'s, we find

$$\hat{C}^m{}_{pq} = -C^m{}_{pq}\,,$$
$$\hat{C}^m{}_{\alpha q} = C^m{}_{pq}A^p_\alpha\,,$$

and

$$\hat{C}^m{}_{\alpha\beta} = \partial_\alpha A^m{}_\beta - \partial_\beta A^m_\alpha - C^m{}_{pq}A^p_\alpha A^q_\beta = -F^m{}_{\alpha\beta}\,,$$

where F is the curvature of the connection defined by A. The coefficients of the riemannian connection of $\hat{g}$ are given by the formula (p. 308)

$$\hat{\omega}^M{}_{PQ} = \tfrac{1}{2}g^{MN}(\partial_P g_{NQ} + \partial_Q g_{NP} - \partial_N g_{PQ})$$
$$- \tfrac{1}{2}(\hat{C}^M{}_{QP} + g^{MN}g_{LP}\hat{C}^L{}_{QN} + g^{MN}g_{LQ}\hat{C}^L{}_{PN})\,.$$

For $M, P, Q = 0, 1, 2, 3$, they are identical to the coefficients of the connection of the space-time metric $g_{\alpha\beta}$:

$$\hat{\omega}^\alpha{}_{\beta\gamma} = \omega^\alpha{}_{\beta\gamma}\,.$$

For M, P, Q all corresponding to coordinates of G,

$$\hat{\omega}^m{}_{pq} = \omega^m{}_{pq} ,$$

where $(\omega^m{}_{pq})$ is the riemannian connection of the metric induced by $\hat{g}$ on the orbit G_x. The remaining coefficients are

$$\begin{aligned}
&\hat{\omega}^\alpha{}_{\beta m} = \hat{\omega}^\alpha{}_{m\beta} = \tfrac{1}{2}F_{m\beta}{}^\alpha ,\\
&\hat{\omega}^m{}_{\alpha\beta} = -\tfrac{1}{2}F^m{}_{\alpha\beta} ,\\
&\hat{\omega}^m{}_{\beta n} = \tfrac{1}{2}\xi^{mp}D_\beta \xi_{np} + C^m{}_{rn}A^r_\beta ,\\
&\hat{\omega}^m{}_{n\beta} = \tfrac{1}{2}\xi^{mp}D_\beta \xi_{np} ,\\
&\hat{\omega}^\alpha{}_{mn} = -\tfrac{1}{2}g^{\alpha\beta}D_\beta \xi_{mn} .
\end{aligned}$$

Here D_β is the covariant derivative with respect to the metric A:

$$D_\beta \xi_{mn} = \partial_\beta \xi_{mn} - \xi_{lm}C^l{}_{pn}A^p_\beta - \xi_{ln}C^l{}_{pm}A^p_\beta .$$

The components of the curvature tensor are given by the general formula

$$\hat{R}_M{}^N{}_{QP} = \hat{\omega}^N{}_{PR}\hat{\omega}^R{}_{QM} - \hat{\omega}^N{}_{QR}\hat{\omega}^R{}_{PM} - \hat{C}^R{}_{PQ}\hat{\omega}^N{}_{RM} + \partial_P\hat{\omega}^N{}_{QM} - \partial_Q\hat{\omega}^N{}_{PM} .$$

We are concerned with the Ricci tensor with $\hat{R}_{MN} = \hat{R}_M{}^P{}_{NP}$ whose components are found to be, using the notation for symmetrization $f_{(\alpha\beta)} = \frac{1}{2}(f_{\alpha\beta} + f_{\beta\alpha})$:

$$\begin{aligned}
\hat{R}_{\alpha\beta} &= R_{\alpha\beta} + \tfrac{1}{2}F_{m\alpha}{}^\lambda F^m{}_{\beta\lambda} + \tfrac{1}{4}\xi^{mp}\xi^{nq}D_\alpha \xi_{mn}D_\beta \xi_{pq} + \tfrac{1}{2}D_{(\alpha}(\xi^{mn}D_{\beta)}\xi_{mn}) ,\\
\hat{R}_{\alpha m} &= -\tfrac{1}{2}D^\lambda F_{m\alpha\lambda} - \tfrac{1}{4}F_{m\alpha}{}^\lambda \xi^{np}D_\lambda \xi_{np} - \tfrac{1}{2}C^p{}_{mn}\xi^{nq}D_\alpha \xi_{pq}\\
&\quad - \tfrac{1}{2}C^p{}_{pn}\xi^{nq}D_\alpha \xi_{mq} ,\\
\hat{R}_{mn} &= R_{mn} - \tfrac{1}{4}F_{m\alpha\beta}F_n{}^{\alpha\beta} - \tfrac{1}{2}\xi^{pq}D_\alpha \xi_{mp}D^\alpha \xi_{nq}\\
&\quad + \tfrac{1}{4}\xi^{pq}D^\alpha \xi_{mn}D_\alpha \xi_{pq} + \tfrac{1}{2}D_\alpha D^\alpha \xi_{mn} .
\end{aligned}$$

In these formulae all fields are defined on the space-time V_4. The Ricci curvature (R_{mn}) of G_x, endowed with the induced metric $\xi_{mn}\theta^m\theta^n$, depends on x through ξ_{pq}, ξ^{pq}, but contains no space-time derivatives of ξ.

The scalar curvature is

$$\begin{aligned}
\hat{R} = R &- \tfrac{1}{4}F^m{}_{\alpha\beta}F_m{}^{\alpha\beta} + \tfrac{1}{4}\xi^{mn}\xi^{pq}D_\alpha \xi_{mp}D^\beta \xi_{nq} - \tfrac{1}{4}(\xi^{mn}D^\alpha \xi_{mn})(\xi^{pq}D_\alpha \xi_{pq})\\
&- D^\alpha(\xi^{mn}D_\alpha \xi_{mn}) - \tfrac{1}{2}D^\alpha \xi_{mn}D_\alpha \xi^{mn} + \Xi ,
\end{aligned}$$

where $\Xi \equiv R(G_x) \equiv \xi^{mn}R_{mn}$ depends on x through the ξ_{pq}, but contains no space-time derivatives of ξ.

Remark: 1. We have denoted by (ξ^{mn}) the scalar multiplet whose components in a trivialization are the inverse matrix of (ξ_{mn}):

$$\xi^{mn}\xi_{mp} = \delta^n_p .$$

Its gauge covariant derivative is

$$D_\alpha \xi^{pq} = -\xi^{mp}\xi^{nq}D_\alpha \xi_{mn} = \partial_\alpha \xi^{pq} + \xi^{nq}C^p{}_{rn}A^r_\alpha + \xi^{mq}C^q{}_{rm}A^r_\alpha .$$

Remark: 2. We denote by $\rho_{\alpha\beta}{}^m{}_n$ the curvature of the representative of the gauge connection in the vector bundle associated to the principal bundle by the adjoint representation, with connection coefficients

$$\gamma_\alpha{}^m{}_n = C^m{}_{pn}A^p_\beta .$$

Since $F_{\alpha\beta}$ is the curvature of the gauge connection, we have

$$\rho_{\alpha\beta}{}^m{}_n = C^m{}_{pn}F^p{}_{\alpha\beta}$$

(this result can also be checked by a direct calculation using the Jacobi identity for the $C^m{}_{np}$).

The scalar multiplet is a section of the corresponding tensor bundle, so

$$(D_\alpha D_\beta - D_\beta D_\alpha)\xi_{mn} = \rho_{\alpha\beta m}{}^p\xi_{pn} + \rho_{\alpha\beta n}{}^p\xi_{mp} .$$

In particular, this gives

$$D_\alpha(\xi^{mn}D_\beta \xi_{mn}) - D_\beta(\xi^{mn}D_\alpha \xi_{mn}) = 2\rho_{\alpha\beta}{}^m{}_m = 2F^p{}_{\alpha\beta}C^m{}_{pm} .$$

It is known that $C^m{}_{pm} = 0$ if the group G is compact. In this case $D_\alpha(\xi^{mn}D_\beta\xi_{mn})$ is symmetric in α and β.

The equations $\hat{S}_{MN} = 0$ on V_d are invariant under the action of G, and thus project on V_4. We have

$$\hat{S}_{\alpha\beta} \equiv \hat{R}_{\alpha\beta} - \tfrac{1}{2}\hat{g}_{\alpha\beta}\hat{R} \equiv S_{\alpha\beta} - \tau_{\alpha\beta} - s_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}\Xi = 0 ,$$

where $S_{\alpha\beta}$ is the Einstein tensor of the metric $g_{\alpha\beta}$ of V_4,

$$\tau_{\alpha\beta} = \tfrac{1}{2}(F_{m\alpha}{}^\lambda F^m{}_{\beta\lambda} - \tfrac{1}{4}g_{\alpha\beta}F^m{}_{\lambda\mu}F_m{}^{\lambda\mu})$$

is the stress-energy tensor of the Yang-Mills field F, and

$$s_{\alpha\beta} = \tfrac{1}{4}\xi^{mn}\xi^{pq}(D_\alpha\xi_{mn}D_\beta\xi_{pq} + \tfrac{1}{2}g_{\alpha\beta}D_\lambda\xi_{mp}D^\lambda\xi_{nq}) + \tfrac{1}{2}g_{\alpha\beta}D^\lambda(\xi^{mn}D_\lambda\xi_{mn})$$
$$+ \tfrac{1}{8}g_{\alpha\beta}(\xi^{mn}D_\lambda\xi_{mn})(\xi^{pq}D^\lambda\xi_{pq}) - \tfrac{1}{2}D_{(\alpha}(\xi^{mn}D_{\beta)}\xi_{mn})$$

can be interpreted as the stress-energy tensor of the scalar multiplet.

The equation

$$\hat{S}_{m\alpha} \equiv \hat{R}_{m\alpha} = 0$$

is the Yang-Mills equation with a current generated by the scalar multiplet and its interaction with F as a source.

The equation

$$\hat{R}_{mn} = 0$$

is a nonlinear wave equation for the scalar multiplet (the principal part in ξ_{mn} is $D^{\alpha}D_{\alpha}\xi_{mn}$), with the other fields as sources.
If we take for ξ_{mn} the coefficients of a fixed bi-invariant metric on G, we have $D_{\alpha}\xi_{mn}=0$. The equations $\hat{S}_{\alpha\beta}=0$ and $\hat{R}_{\alpha m}=0$ then reduce to the coupled Einstein–Yang-Mills system with a cosmological constant Ξ, but the equations $\hat{R}_{mn}=0$ must in this case be discarded, since they are not in general satisfied.

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14. KÄHLER MANIFOLDS; CALABI–YAU SPACES

almost complex

Let V_{2n} be an **almost complex** (p. 331) $2n$ dimensional manifold, that is, a C^{∞} manifold with a field J of isomorphisms of the tangent spaces, $J_x\colon T_x \to T_x$, such that $J_x^2=-1$.

1) *Show that the linear operator $J\colon T_x \to T_x$ has eigenvalues $\pm \mathrm{i}$, of multiplicity n. We shall omit the index x when the context makes the meaning clear.*

2) *Let $T_x^* \otimes \mathbb{C}$ be the complexified cotangent space to V_{2n}, i.e., the set of complex valued $\mathbb{R}$-linear forms on T_x.*

An element $\omega \in T_x^* \otimes \mathbb{C}$ is said to be of **type (1, 0)** [resp. of **type (0, 1)**] if $\forall v \in T_x$ type (1, 0) type (0, 1)

$$\langle \omega, Jv \rangle = \mathrm{i}\langle \omega, v \rangle \qquad [\text{resp. } \langle \omega, Jv \rangle = -\mathrm{i}\langle \omega, v \rangle].$$

Let θ^a, $a = 1, \ldots, 2n$ be a basis of T_x^*. Show that the elements λ^a of $T_x^* \otimes \mathbb{C}$ defined by

$$\langle \lambda^a, v \rangle = \big\langle \theta^a, (J + \mathrm{i})v \big\rangle$$

are of type $(1, 0)$. Define by a similar procedure elements $\bar{\lambda}^a$ of type $(0, 1)$. Show that exactly n of the λ^a [resp. $\bar{\lambda}^a$] are linearly independent on $\mathbb{C}$. Show that if λ^k $(k = 1, \ldots, n)$ are such a linearly independent set, then $(\lambda^k, \bar{\lambda}^k)$ $(k = 1, \ldots, n)$ are a basis of $T_x^* \otimes \mathbb{C}$.
Suppose that (V_{2n}, J) defines a **complex manifold,** *i.e., the almost complex structure J of V_{2n} is derived from a complex structure. Give a canonical basis of $T_x^* \otimes \mathbb{C}$ associated to a natural basis of T_x^*.* complex manifold

Answer 1:

$$J^2 = -\mathbb{1},$$

so $Jv = \lambda v$ implies

$$-v = \lambda J v = \lambda^2 v.$$

Thus

$$\lambda^2 = -1, \qquad \lambda = \pm \mathrm{i}.$$

Further, if $Jv = iv$, then $J(Jv) = -i(Jv)$, so the eigenvalues i and $-$i occur an equal number of times.
Note that this proves again that an almost complex manifold must be even dimensional, since complex eigenvalues of a real operator (J) appear in conjugate pairs, each of multiplicity n.

Answer 2: We have, by the definition of λ^a,

$$\begin{aligned}\langle \lambda^a, Jv \rangle &= \big\langle \theta^a, (J + \mathrm{i})Jv \big\rangle = \langle \theta^a, -v + \mathrm{i}Jv \rangle \\ &= \mathrm{i}\big\langle \theta^a, (J + \mathrm{i})v \big\rangle = \mathrm{i}\langle \lambda^a, v \rangle.\end{aligned}$$

Thus λ^a is of type $(1, 0)$. Let * denote complex conjugation. Then

$$\langle \lambda^a, v \rangle^* = \big\langle \theta^a, (J - \mathrm{i})v \big\rangle.$$

Hence if we define $\bar{\lambda}^a$ by

$$\langle \bar{\lambda}^a, v \rangle = \langle \lambda^a, v \rangle^*$$

it follows that

$$\langle \bar{\lambda}^a, Jv \rangle = -\mathrm{i}\langle \theta^a, (J-\mathrm{i})v \rangle = -\mathrm{i}\langle \bar{\lambda}^a, v \rangle$$

and $\bar{\lambda}^a$ is of type $(0,1)$.

Since $-\mathrm{i}$ is an eigenvalue of J of multiplicity n, the vector space $(J+\mathrm{i})v$, $v \in T_x$, has complex dimension n, and so has the space of 1-forms λ^a. The same is true of the space of 1-forms $\bar{\lambda}^a$.

The space $A_{1,0}$ of 1-forms of type $(1,0)$ and the space $A_{0,1}$ of 1-forms of type $(0,1)$ are both vector spaces over $\mathbb{C}$; they are subspaces of dimension n and intersection 0. Thus $T_x^* \otimes \mathbb{C}$, the vector space over $\mathbb{C}$ of dimension $2n$, is the direct sum

$$T_x^* \otimes \mathbb{C} = A_{1,0} \oplus A_{0,1} .$$

If V_{2n} admits a complex structure (p. 331) it admits local coordinates (x^i, y^i) such that J represented by

$$J = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} .$$

An element of $T_x^* \otimes \mathbb{C}$ is represented in these coordinates by $\omega = a_i\,\mathrm{d}x^i + b_j\,\mathrm{d}y^j$, where a_i, b_j are complex numbers, and an element $v \in T_x$ by $v = v^i(\partial/\partial x^i) + \omega^j(\partial/\partial x^j)$. We have

$$\langle \omega, Jv \rangle = -a_i v^i + b_j w^j .$$

Thus ω is of type $(1,0)$ if

$$-a_i v^i + b_j w^j = i(a_i w^i + b_j v^j) ,$$

i.e.,

$$b_i = \mathrm{i} a_i .$$

Hence

$$\omega = a_i\,\mathrm{d}z^i , \qquad \text{where } \mathrm{d}z^i = \mathrm{d}x^i + \mathrm{i}\,\mathrm{d}y^i .$$

Analogously we find that ω is of type $(0,1)$ if

$$\omega = a_i\,\mathrm{d}z^{*i} , \qquad \mathrm{d}z^{*i} = \mathrm{d}x^i - \mathrm{i}\,\mathrm{d}y^i .$$

A basis of 1-forms on $T_x^* \otimes \mathbb{C}$ consists of the forms $\mathrm{d}z^i$, $i = 1, \ldots, n$ of type $(1,0)$ and the forms $\mathrm{d}z^{*i}$ of type $(0,1)$.

We can link this result to the general construction: the 1-forms λ^i corresponding to $\mathrm{d}x^i$ are such that

$$\left\langle \lambda^i, \frac{\partial}{\partial x^j} \right\rangle = \left\langle \mathrm{d}x^i, (J+\mathrm{i})\,\frac{\partial}{\partial x^j} \right\rangle = \left\langle \mathrm{d}x^i, \frac{\partial}{\partial y^j} + \mathrm{i}\,\frac{\partial}{\partial x^j} \right\rangle ,$$

$$\left\langle \lambda^i, \frac{\partial}{\partial y^j} \right\rangle = \left\langle \mathrm{d}x^i, (J+\mathrm{i})\,\frac{\partial}{\partial y^j} \right\rangle = \left\langle \mathrm{d}x^i, -\frac{\partial}{\partial x^j} + \mathrm{i}\,\frac{\partial}{\partial y^j} \right\rangle .$$

Thus

$$\lambda^i = \mathrm{i}(\mathrm{d}x^i + \mathrm{i}\,\mathrm{d}y^i)\ .$$

Similarly the 1-form λ'^i corresponding to $\mathrm{d}y^i$ is found to be

$$\lambda'^i = \mathrm{d}x^i + \mathrm{i}\,\mathrm{d}y^i\ .$$

n of these forms, for instance the $\lambda'^i = \mathrm{d}z^i$, are $\mathbb{C}$-linearly independent. The $\bar{\lambda}'^i$ are $\mathrm{d}\bar{z}^i$. It follows from $\langle \bar{\lambda}'^i, v\rangle = \langle \lambda'^i, v\rangle^*$ that $\bar{\lambda}'^i = \mathrm{d}z^{i*}$, Hence $\mathrm{d}\bar{z}^i = \mathrm{d}z^{i*}$.

3) a) Let f be a complex-valued differentiable function on V_{2n}. Its differential $\mathrm{d}f$ at x is an element of $T_x^* \otimes \mathbb{C}$. Let λ^k be a basis of $A_{1,0}$ and let $\bar{\lambda}^k$ be the corresponding basis of $A_{0,1}$, i.e., the basis such that $\langle \bar{\lambda}^k, v\rangle = \langle \lambda^k, v\rangle^*$. We have

$$\mathrm{d}f = \alpha_k \lambda^k + \beta_{\bar{k}} \bar{\lambda}^k\ .$$

One defines the **Dolbeault operators** ∂ and $\bar{\partial}$ by

Dolbeault operators

$$\partial f = \alpha_k \lambda^k\ , \qquad \bar{\partial} f = \beta_{\bar{k}} \bar{\lambda}^k$$

and one sets

$$\partial_k f = \alpha_k\ , \qquad \bar{\partial}_k f = \beta_{\bar{k}}\ .$$

f is said to be holomorphic if $\bar{\partial} f = 0$.
Show that the definition coincides with the usual one if J is derived from a complex structure on V_{2n}.

b) A complex-valued exterior form ω on V_{2n} of degree m, is a section of the vector bundle $(\Lambda T^* \otimes \mathbb{C})^m$, with base V_{2n} and fibre at x the space $(\Lambda T_x^* \otimes \mathbb{C})^m$. The form ω is said to be of type (p, q) if $m = p + q$ and

$$\omega = \frac{1}{m!}\, \omega_{i_1 \ldots i_p \bar{i}_{p+1} \ldots \bar{i}_{p+q}} \lambda^{i_1} \wedge \cdots \wedge \lambda^{i_p} \wedge \bar{\lambda}^{i_p} \cdots \wedge \bar{\lambda}^{i_{p+q}}\ ,$$

where $\omega_{i_1 \ldots i_p \bar{i}_{p+1} \ldots \bar{i}_{p+q}}$ are complex-valued functions of $x \in V_{2n}$.
Show that the type of ω does not depend on the choice of the basis $(\lambda^i, \bar{\lambda}^i)$.

c) The exterior differential of ω is a section of $(\Lambda T^* \otimes \mathbb{C})^{m+1}$ defined by the usual formulas

$$\begin{aligned}\mathrm{d}\omega = \frac{1}{m!} \Big\{ & \mathrm{d}\omega_{i_1 \ldots i_p \bar{i}_{p+1} \ldots \bar{i}_{p+q}} \wedge \lambda^{i_1} \cdots \wedge \bar{\lambda}^{i_{p+q}} \\ & + \sum \omega_{i_1 \ldots i_p \ldots \bar{i}_{p+q}} (\mathrm{d}\lambda^{i_1} \wedge \lambda^{i_2} \cdots \\ & + (-1)^{m-1} \lambda^{i_1} \wedge \cdots \wedge \mathrm{d}\bar{\lambda}^{i_{p+q}} \Big\}\ .\end{aligned}$$

Show that if (V_{2n}, J) *is derived from a complex structure, then* $\mathrm{d}\omega$ *is the sum of a form of type* $(p+1, q)$ *denoted* $\partial\omega$ *and a form of type* $(p, q+1)$, *denoted* $\bar\partial\omega$:

$$\mathrm{d}\omega = \partial\omega + \bar\partial\omega .$$

Dolbeault differential

$\bar\partial$ is called the **Dolbeault differential** of ω.
Define a Dolbeault cohomology and a Dolbeault complex analogous to the De Rham cohomology and complex defined for real valued forms (pp. 223, 297).

Answer 3a: If $\lambda^k = \mathrm{d}z^k = \mathrm{d}x^k + \mathrm{i}\,\mathrm{d}y^k$ and $\bar\partial f = 0$, then

$$\mathrm{d}f = \alpha_k(\mathrm{d}x^k + \mathrm{i}\,\mathrm{d}y^k)\,, \qquad \alpha_k \in \mathbb{C}$$

which is the definition of derivability of f with respect to the complex variables z^k.

Answer 3b: The type does not depend on the choice of λ^i and $\bar\lambda^i$ because $T_x^* \otimes \mathbb{C} = A_{1,0} \oplus A_{0,1}$, the λ^i span $A_{1,0}$ and the $\bar\lambda^i$ span $A_{0,1}$. Any other choice μ^i, $\bar\mu^i$ respects the splitting.

Answer 3c: Let us take coordinates (x^i, y^j) on V_{2n} adapted to the complex structure, and choose as a basis in $T_x^* \otimes \mathbb{C}$ the 1-forms $\mathrm{d}z^i, \mathrm{d}\bar z^i$. A form of type (p, q) is

$$\omega = \frac{1}{(p+q)!}\, \omega_{i_1 \dots i_p \bar i_{p+1} \dots \bar i_{p+q}}\, \mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p} \cdots \wedge \mathrm{d}\bar z^{i_{p+q}}$$

where the $\omega_{i_1 \dots i_p \bar i_{p+1} \dots \bar i_{p+q}}$ are functions of x^i, y^i.
Using the previous definitions of ∂_k and $\bar\partial_k$ for functions, we have the decomposition of $\mathrm{d}\omega$ into a $(p+1, q)$ and a $(p, q+1)$ form

$$\mathrm{d}\omega = \frac{1}{(p+q)!}\, \{\partial_k \omega_{i_1 \dots i_p \dots \bar i_{p+q}}\, \mathrm{d}z^k \wedge \mathrm{d}z^{i_1} \cdots \wedge \mathrm{d}\bar z^{i_{p+q}}$$

$$+ \bar\partial_k \omega_{i_1 \dots i_p \dots \bar i_{p+q}}\, \mathrm{d}z^{i_1} \cdots \wedge \mathrm{d}\bar z^k \cdots \wedge \mathrm{d}\bar z^{i_{p+q}}\}$$

since $\mathrm{d}z^i = \mathrm{d}x^i + \mathrm{i}\,\mathrm{d}y^i$ and $\mathrm{d}\bar z^i = \mathrm{d}x^i - \mathrm{i}\,\mathrm{d}y^i$ have a zero differential.
The definition of $\mathrm{d}\omega$ as well as its type, is coordinate independent: the result must therefore hold for any coframe λ^i, $\bar\lambda^i$. We check this easily by verifying it for a one form; a form of type $(1,0)$ reads necessarily

$$\lambda^i = a^i_{\ j}\, \mathrm{d}z^j .$$

Thus

$$\mathrm{d}\lambda^i = \partial_k a^i_{\ j}\, \mathrm{d}z^k \wedge \mathrm{d}z^j + \bar\partial_k a^i_{\ j}\, \mathrm{d}\bar z^k \wedge \mathrm{d}z^j$$

is the sum of a form of type $(2,0)$ and a form of type $(1,1)$.

Remark: If (V_{2n}, J) is only almost complex, it is possible that $d\lambda^i$ contains a term of the form $\frac{1}{2}c^i_{jk}\, d\bar{\lambda}^j \wedge d\bar{\lambda}^k$. Thus forms of type $(p-1, q+2)$ and $(p+2, q)$, could also appear in $d\omega$.
If (V_{2n}, J) is a complex structure it is clear that $\partial^2 = 0$ and $\bar{\partial}^2 = 0$. One can define a Dolbeault cohomology analogously to the De Rham cohomology (cf. p. 223) and a Dolbeault complex (p. 297).
If, moreover, (V_{2n}, J) is endowed with a hermitian metric (p. 334 and see below) it is possible to define a metric adjoint operator to $\bar{\partial}$, a laplacian, an elliptic complex and use the Atiyah–Singer theorem (cf. for instance Eguchi, Gilkey and Hanson).
4) A **hermitian**[1] **metric** (p. 334) on an almost complex manifold (V_{2n}, J) is a riemannian metric g which is invariant under J: hermitian metric

$$g(u, v) = g(Ju, Jv)\,.$$

A **Kähler metric** (p. 334) on an almost complex manifold (V_{2n}, J) is a hermitian metric g such that the exterior 2-form ϕ defined by Kähler metric

$$\phi(u, v) = g(u, Jv)\,, \qquad u, v \in TV_{2n}\,,$$

is closed. The form ϕ is then called the **Kähler form**. Kähler form
An **almost Kähler** manifold is an almost complex manifold (V_{2n}, J) with a Kähler metric g. almost Kähler
A **Kähler manifold** is an almost Kähler manifold whose almost complex structure (V_{2n}, J) derives from a complex structure. Kähler manifold
Give conditions on $g_{i\bar{j}}$ for the following g to be a Kähler metric written in a basis λ^i, $\bar{\lambda}^i$ of $T^* \otimes \mathbb{C}$

$$g = g_{i\bar{j}}\lambda^i \otimes \bar{\lambda}^j + g_{\bar{i}j}\bar{\lambda}^i \otimes \lambda^j\,.$$

Answer 4: The tensors $\lambda^i \otimes \lambda^j$, $\lambda^i \otimes \bar{\lambda}^j$, $\bar{\lambda}^i \otimes \lambda^j$, $\bar{\lambda}^i \otimes \bar{\lambda}^j$ form a basis of $(T^*_x \otimes \mathbb{C}) \otimes (T^*_x \otimes \mathbb{C})$.
A hermitian metric on V_{2n} is a section of $(T^* \otimes \mathbb{C}) \otimes (T^* \otimes \mathbb{C})$ which is invariant under J, symmetric, and real, i.e.,

$$g(u, v) = g(v, u) = g^*(u, v)\,, \qquad \forall u, v \in T_x\,.$$

The section

$$g = \tfrac{1}{2}(g_{i\bar{j}}\lambda^i \otimes \bar{\lambda}^j + g_{\bar{i}j}\bar{\lambda}^i \otimes \lambda^j)$$

or more explicitly

$$g(u, v) = \tfrac{1}{2}(g_{i\bar{j}}\langle \lambda^i, u\rangle\langle \bar{\lambda}^j, v\rangle + g_{\bar{i}j}\langle \bar{\lambda}^i, u\rangle\langle \lambda^j, v\rangle)$$

is invariant under J because $\langle \lambda^i, Ju\rangle = i\langle \lambda^i, u\rangle$ and $\langle \bar{\lambda}^j, v\rangle = -i\langle \bar{\lambda}^j, v\rangle$.

[1] A hermitian scalar product is, in other contexts, a complex valued bilinear form such that $g(u, v) = g^*(v, u)$. Such a definition could be used here, if the invariance under J were stated in the form $g(Ju, v) = ig(u, v)$ (cf. Chern p. 9).

If g contained terms in $\lambda^i \otimes \lambda^j$ or $\bar{\lambda}^i \otimes \bar{\lambda}^j$ it could not be invariant. g satisfies $g(u, v) = g(v, u)$ and $g(u, v) = g^*(u, v)$ if and only if

$$g_{i\bar{j}} = g_{\bar{j}i} \qquad \text{and} \qquad g_{i\bar{j}} = \overset{*}{g}_{j\bar{i}} \,.$$

Remark: It is usual to abbreviate the expression of g, as in the case of riemannian metrics, to

$$g = g_{i\bar{j}} \lambda^i \bar{\lambda}^j \,.$$

The Kähler 2-form ϕ corresponding to g and J

$$\phi(u, v) = g(u, Jv) = g(Jv, u) = -g(v, Ju) = -\phi(v, u)$$

is given by

$$\phi(u, v) = \tfrac{1}{2}(g_{i\bar{j}} \langle \lambda^i, u \rangle \langle \bar{\lambda}^j, Jv \rangle + g_{\bar{j}i} \langle \bar{\lambda}^j, u \rangle \langle \lambda^i, Jv \rangle \,.$$

It is easy to show that

$$\phi = -\tfrac{1}{2} \mathrm{i} g_{i\bar{j}} (\lambda^i \otimes \bar{\lambda}^j - \bar{\lambda}^j \otimes \lambda^i) \equiv -\tfrac{1}{2} \mathrm{i} g_{i\bar{j}} \lambda^i \wedge \bar{\lambda}^j \,.$$

The metric g is Kählerian if ϕ is closed, i.e., if

$$\partial_k g_{i\bar{j}} \lambda^k \wedge \lambda^i \wedge \bar{\lambda}^j + \bar{\partial}_k g_{i\bar{j}} \bar{\lambda}^k \wedge \lambda^i \wedge \bar{\lambda}^j = 0$$

which is equivalent to

$$\partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}} = 0 \qquad \text{and} \qquad \bar{\partial}_k g_{i\bar{j}} - \bar{\partial}_j g_{i\bar{k}} = 0 \,.$$

5) *Consider a Kähler manifold. Compute the connection and curvature of its riemannian metric (which is also the Kähler metric), making use of the canonical 1-forms* $\mathrm{d}z^i, \mathrm{d}\bar{z}^i$. *Show that its Ricci tensor determines a closed 2-form* ρ.

Answer 5: We have seen in 4) that a Kähler metric, here on a complex manifold, can be written

$$g = g_{i\bar{j}} \, \mathrm{d}z^i \, \mathrm{d}\bar{z}^j = \gamma_{\alpha\beta} \theta^\alpha \theta^\beta \,,$$

where the $(\theta^\alpha) = (\mathrm{d}z^j, \mathrm{d}\bar{z}^j)$ are $2n$ independent linear complex-valued forms on V_{2n}. The usual computations determine the connection coefficients of the torsionless metric connection. Since $g_{ij} = g_{\bar{i}\bar{j}} = 0$, we find that

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{i\bar{l}} (\partial_j g_{k\bar{l}} + \partial_k g_{j\bar{l}}) \,.$$

If ϕ is closed, we can write (cf. 4))

$$\Gamma^i_{jk} = g^{i\bar{l}} \partial_j g_{k\bar{l}}$$

and

$$\Gamma^{\bar{i}}_{\bar{j}\bar{k}} = g^{\bar{i}l}\partial_{\bar{j}} g_{\bar{k}l} = (\Gamma^{i}_{jk})^* \,.$$

All other Christoffel symbols are zero.
The only nonvanishing components of the Riemann tensor are computed to be

$$R_{k\bar{m}}{}^{i}{}_{j} = -\bar{\partial}_m \Gamma^{i}_{jk}$$

and those related by symmetry and complex conjugation.
The Riemann tensor has an extra symmetry

$$R_{k\bar{m}}{}^{i}{}_{j} = R_{j\bar{m}}{}^{i}{}_{k} \,.$$

The Ricci tensor is

$$R_{\bar{m}j} = R_{i\bar{m}}{}^{i}{}_{j} = \partial_j \bar{\partial}_m (\log \det g) \,.$$

It follows from the hermiticity of g that $\det g$ is real and the Ricci tensor is hermitian, i.e.,

$$R_{j\bar{m}} = R_{\bar{m}j} \,, \qquad R_{j\bar{m}} = R^*_{m\bar{j}} \,.$$

It then follows from the expression of $R_{j\bar{k}}$ (and the properties $\partial^2 = \bar{\partial}^2 = 0$) that the 2-form

$$\rho = \mathrm{i} R_{j\bar{k}} \, \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k \,,$$

is a real-valued closed 2-form in V_{2n}, and therefore defines an element of the cohomology group $H^2(V_{2n})$.
Despite appearances, ρ need not be exact. The quantity $\det g$ is not a scalar but a density, so the expression for $R_{j\bar{k}}$ is not a globally defined differential; it will change as one moves from coordinate patch to coordinate patch.
It can be shown that the cohomology class of ρ is independent of the Kähler metric, and is equal to the first Chern class C_1 of the tangent bundle to the complex manifold.
It follows that a complex manifold cannot admit a Ricci-flat Kähler metric if $C_1 \neq 0$. Complex manifolds with a Ricci-flat Kähler metric are called **Calabi–Yau spaces**. The existence of a Ricci-flat Kähler metric on any complex manifold with $C_1 = 0$ which admits a Kähler metric has been proven by Yau. Calabi–Yau spaces

6) *Let* (V_{2n}, J, g) *be a Kähler manifold. Show that its holonomy group* (*p.* 386) (*with respect to the riemannian connection*) *is a subgroup of* $U(n)$. *Show that it is a subgroup of* $SU(n)$ *if and only if* V_{2n} *is Ricci flat.*

Answer 6: By the Ambrose–Singer theorem (p. 389) the Lie algebra of an holonomy group is spanned by elements $\Omega(u, v)$, where Ω is the curvature 2-form. It can be checked that $R^{j}_{kl\bar{m}} v^l v^{\bar{m}}$ are in the Lie algebra of $U(n)$, and in the Lie algebra of $\mathrm{SU}(n)$ if $R_{l\bar{m}} = 0$.
For a more complete proof see reference [5].

Remark: This last property has attracted the interest of physicists to six dimensional Calabi–Yau spaces. In particular superstrings can consistently propagate only on a ten-dimensional manifold. For a physically realistic model, this manifold must be of the form $V_4 \times V_6$, where V_4 is ordinary spacetime and V_6 is a six-manifold which is at least approximately Ricci flat. For four-dimensional supersymmetry to be unbroken, V_6 must also be Kähler. For more details, see reference [6].

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V BIS. CONNECTIONS ON A PRINCIPAL FIBRE BUNDLE

1. AN EXPLICIT PROOF OF THE EXISTENCE OF INFINITELY MANY CONNECTIONS ON A PRINCIPAL BUNDLE WITH PARACOMPACT BASE (p. 363)

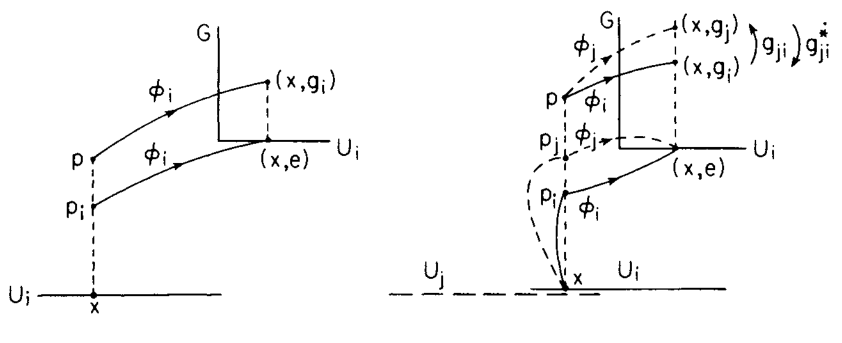
Let P be a principal bundle over X, with projection Π: $P \to X$.

Let $\{U_i\}$ be a covering of X. Construct explicitly the connection form ω_i on $\pi^{-1}(U_i)$ obtained from a given form $\bar{\omega}_i$ on U_i by a choice of a section f_i over U_i. Check explicitly that $\omega = \sum_i (\theta_i \circ \Pi)\omega_i$ is a connection on P, where $\{\theta_i\}$ is a smooth partition of unity on X. Show that the space of all connections on a principal bundle is not a vector space.

Answer: Let ϕ_i be the trivialization on $\Pi^{-1}(U_i)$ defined by a section f_i over U_i. Following the construction given on (p. 364), the pull back of ω_i on $U_i \times G$ by ϕ_i^{-1} is

$$\left(\phi_i^{-1*}\omega_i\right)_{x,g_i}(v, w) = \mathrm{Ad}\left(g_i^{-1}\right)(\bar{\omega}_i)_x(v) + (\theta_{\mathrm{MC}})_{g_i}(w), \tag{1}$$

where $(v, w) \in T_x U_i \times T_{g_i} G$, θ_{MC} is the Maurer–Cartan form on G, $(\theta_{\mathrm{MC}})_{g_i}(w) = \hat{w}$ (p. 168).



This construction can be done over each patch. If x is in the intersection of two patches, $x \in U_i \cap U_j$, the representative of $p \in \Pi^{-1}(U_i \cap U_j)$ with $\Pi(p) = x$ is (x, g_j) in the ϕ_j trivialization with

$$g_j = \overset{\Delta}{\phi}_{j,x} \circ \overset{\Delta}{\phi}{}^{-1}_{i,x} g_i = g_{ji}(x) g_i \,. \tag{2}$$

Let $\{\theta_i\}$ be a smooth partition of unity in X. Then $\{\theta_i \circ \Pi\}$ is a smooth partition of unity on the principal bundle $P = \Pi^{-1}(X)$. We define a 1-form on P with values in the Lie algebra $\mathscr{G}$ of G by setting

$$\omega = \sum_j (\theta_j \circ \Pi)\omega_j \,. \tag{3}$$

The sum is meaningful because, by definition of a partition of unity, only a finite number of θ_j's are different from zero for any given $x \in X$.
The pull back on $U_i \times G$ by ϕ_i^{-1} of the restriction of ω to $\Pi^{-1}(U_i)$ is

$$\begin{aligned}(\phi_i^{-1*}\omega)_{x,g_i} &= \sum_j (\theta_j \circ \Pi \circ \phi_i^{-1})(x, g_i)(\phi_j^{-1*}\omega_j)_{x,g_i}\,, \qquad x \in U_i\,, \qquad g_i \in G \\ &= \sum_j \theta_j(x)(\phi_i^{-1*}\omega_j)_{x,g_i}\,. \end{aligned} \tag{4}$$

Each term in the sum is well defined on $U_i \times G$. It has its support in $(U_i \cap U_j) \times G$ and is given by

$$\begin{aligned}\theta_j(x)(\phi_j^{-1*}\omega_j)_{x,g_i}(v, w) &= \theta_j(x)(\phi_j^{-1*} \circ \phi_j^* \circ \phi_j^{-1*}\omega_j)_{x,g_i}(v, w) \\ &= \theta_j(x)(g_{ji}^*\phi_j^{-1*}\omega_j)_{x,g_i}(v, w) \\ &= \theta_j(x)(\phi_j^{-1*}\omega_j)_{x,g_j}(v, g'_{ji}w), \text{ since } g_j = g_{ji}(x)g_i\,.\end{aligned}$$

Using formula (1) and the invariance of the Maurer–Cartan 1-form gives

$$\theta_j(x)(\phi_j^{-1*}\omega_j)_{x,g_i}(v, w) = \theta_j(x)(\mathrm{Ad}(g_j^{-1})(\bar{\omega}_j)_x(v) + (\theta_{\mathrm{MC}})_{g_i}(w))\,. \tag{5}$$

Reporting these values in formula (4) gives, since $\Sigma_j\, \theta_j(x) = 1$,

$$(\phi_i^{-1*}\omega)_{x,g_i}(v, w) = \sum_j \theta_j(x)(\mathrm{Ad}(g_j^{-1})(\bar{\omega}_j)_x(v) + (\theta_{\mathrm{MC}})_{g_i}(w))\,.$$

We can now check that ω is a connection (see p. 169)
1. $\phi_i^{-1*}\omega)(0, w) = \hat{w}$.
2. $\phi_i^{-1*}\omega$ depends differentially on (x, g_i).
3. $\phi_i^{-1*}\omega)_{(x,g_ih)}(v, wh) = \mathrm{Ad}(h^{-1})(\phi_i^{-1*}\omega)_{(x,g_i)}(v, w)\,.$

This calculation shows that the affine sum of two connections $\theta\omega_1 + (1-\theta)\omega_2$ is a connection, but that $\omega_1 + \omega_2$ is not a connection. Hence, the space of all connections on a principal bundle is not a vector space but an affine space.

Note, however, that this calculation is not necessary to show that the space of connections is an affine space. Indeed, let $\check{A}$ be the fundamental vector field (p. 360) defined by an element A of the Lie algebra $\mathscr{G}$, then

$$\omega_1(\check{A}) = A$$
$$\omega_2(\check{A}) = A \, .$$

$\alpha\omega_1 + \beta\omega_2$ is a connection only if $\alpha A + \beta A = A$; i.e.,

$$\beta = 1 - \alpha \, .$$

2. GAUGE TRANSFORMATIONS* (p. 364)

1) *Use the sections s_i and s_j canonically associated to the trivializations (ϕ_i, U_i) and (ϕ_j, U_j) of a principal bundle P to prove the relationship between the connection forms $\bar{\omega}_i$ and $\bar{\omega}_j$, at a point $x \in U_i \cap U_j$, in the local gauges ϕ_i and ϕ_j:*

$$\bar{\omega}_i(v) = \mathrm{Ad}(g_{ji}(x)^{-1})\bar{\omega}_j(v) + \theta_{\mathrm{MC}}(g'_{ji}(x)v) \, , \tag{1}$$

where the transition functions are

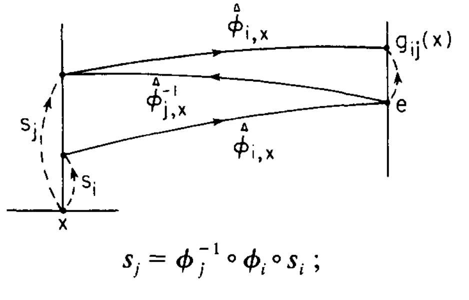
$$g_{ij}\colon U_i \cap U_j \to G \text{ by } x \mapsto g_{ij}(x) = \overset{\Delta}{\phi}_{i,x} \circ \overset{\Delta}{\phi}{}^{-1}_{j,x} \, ,$$

the vector $v \in T_x(U_i \cap U_j)$ and θ_{MC} is the Maurer–Cartan 1-form on G.

Answer 1: The definition of s_i is

$$s_i(x) = \phi_i^{-1}(x, e) \, , \qquad x \in U_i,\ e \text{ unit of } G$$

Therefore



$$s_j = \phi_j^{-1} \circ \phi_i \circ s_i \, ; \tag{2}$$

*Written in collaboration with T. Jacobson and S. Carlip.

we shall show that
$$s_j(x) = \tilde{R}_{g_{ij}(x)} s_i(x) . \tag{3}$$
Indeed in a trivialization, say ϕ_j, we have
$$\phi_j \circ s_j(x) = (x, e) , \qquad \phi_j \circ s_i(x) = (\phi_j \circ \phi_i^{-1})(x, e) = (x, g_{ji}(x)e)$$
by definition of the transition functions. They act on G by left action, however, at the identity
$$g_{ij}(x)e = eg_{ij}(x)$$
the left action is identical to a right action and we can write down eq. (3), where $\tilde{R}_{g_{ij}(x)}$ is the right action of $g_{ij}(x)$ on P. It is such that $\tilde{R}_{g_{ji}(x)} \circ \tilde{R}_{g_{kj}(x)} = \tilde{R}_{g_{kj}(x)g_{ji}(x)} = \tilde{R}_{g_{ki}(x)}$.
At x, the connection 1-forms in the local gauges corresponding to a given connection ω on the G-principal bundle are by definition
$$\bar{\omega}_i = s_i^* \omega \qquad \text{and} \qquad \bar{\omega}_j = s_j^* \omega ,$$
$$s_j(x) = \tilde{R}_{g_{ij}(x)} s_i(x) .$$
Let $x(t)$ be a curve on the base manifold X such that $x(0) = x \in U_i \cap U_j$ and $\mathrm{d}x(t)/\mathrm{d}t|_{t=0} = v$. Then, at x
$$\begin{aligned} s_j^* \omega(v) &= \omega\Big(\frac{\mathrm{d}}{\mathrm{d}t}\,(s_j(x(t)))|_{t=0}\Big) \\ &= \omega\Big(\frac{\mathrm{d}}{\mathrm{d}t}\,(\tilde{R}_{g_{ij}(x(t))} s_i(x(t)))|_{t=0}\Big) \\ &= \omega\Big(\frac{\mathrm{d}}{\mathrm{d}t}\,(\tilde{R}_{g_{ij}(x)} s_i(x(t)))|_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t}\,(\tilde{R}_{g_{ij}(x(t))} s_i(x))|_{t=0}\Big) . \end{aligned}$$
The first term on the right-hand side is equal to
$$\omega(\tilde{R}'_{g_{ij}(x)} s'_i(x)v) = \mathrm{Ad}(g_{ij}(x)^{-1})\omega(s'_i(x)v) \quad \text{(p. 361)} .$$
The argument in the second term is a vertical vector. By inserting $\tilde{R}_{g_{ij}(x)^{-1}} \tilde{R}_{g_{ij}(x)}$ in the argument, one obtains
$$\begin{aligned} \omega\Big(\frac{\mathrm{d}}{\mathrm{d}t}\,(\tilde{R}_{g_{ij}(x)^{-1} g_{ij}(x(t))} s_j(x))|_{t=0}\Big) &= \frac{\mathrm{d}}{\mathrm{d}t}\, L_{g_{ij}(x)^{-1}} g_{ij}(x(t))|_{t=0} \\ &\qquad \text{(p. 361, property 1 of definition c)} \\ &= L'_{g_{ij}(x)^{-1}} g'_{ij}(x)(v) \\ &= \theta_{\mathrm{MC}}(g'_{ij}(x)(v)) \\ &\qquad \text{by definition of the Maurer–Cartan form} \\ &= g_{ij}^*(x)\theta_{\mathrm{MC}}(v) \end{aligned}$$
and the desired relation is proved.

2) *Using the sections s_i and s_j, prove the relationship between the representatives $\bar{\Omega}_i$ and $\bar{\Omega}_j$ of the curvature in the local gauges*:

$$\bar{\Omega}_j = \mathrm{Ad}(g_{ij}(x)^{-1})\bar{\Omega}_i \,. \tag{4}$$

Answer 2: The calculation is similar to the derivation of (1), but here the term giving rise to the Maurer–Cartan form will not contribute since Ω vanishes on vertical vectors:

$$\begin{aligned} s_j^* \Omega(u, v) &= \Omega(s_j'(u), s_j'(v)) \\ &= \Omega(\tilde{R}'_{g_{ij}} s_i'(u), \tilde{R}'_{g_{ij}} s_i'(v)) \\ &= \mathrm{Ad}(g_{ij}^{-1}) s_i^* \Omega(u, v) \,. \end{aligned}$$

3. HOPF FIBERING $S^3 \to S^2$

1) a) *S^3 is the submanifold of $\mathbb{R}^4$ with equation $\Sigma_{i=1}^4 (x^i)^2 = 1$. Show that it can be represented in the space $\mathbb{C}^2$ of two complex variables by $z_1\bar{z}_1 + z_2\bar{z}_2 = 1$.*
b) *Show that S^2 can be represented as the set of equivalence classes of pairs of complex numbers $[z_1, z_2]$, not both zero, with the equivalence relation $(z_1, z_2) \simeq (\lambda z_1, \lambda z_2)$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$ (i.e., S^2 is identified with the complex projective line $\mathbb{C}P^1$).*

2) *Define a projection p: $S^3 \to S^2$ such that $p^{-1}(x)$, $x \in S^2$, is diffeomorphic to S^1. Show that it endows S^3 with a principal fibre bundle structure over S^2.*

Answer 1a: S^3 is the submanifold of $\mathbb{R}^4$ with equation $\Sigma_{i=1}^4 (x^i)^2 = 1$. It can be represented as the set of pairs (z_1, z_2) with $z_1\bar{z}_1 + z_2\bar{z}_2 = 1$ by setting $z_1 = x^1 + \mathrm{i}x^2$, $z_2 = x^3 + \mathrm{i}x^4$.

Answer 1b: The complex projective line $\mathbb{C}P^1$, set of equivalence classes $[z_1, z_2]$, is made of two sets: one in which $z_2 \neq 0$ and one with $z_2 = 0$ and thus $z_1 \neq 0$. This last set contains only 1 point since $(z_1, 0) \simeq (z_1', 0)$ for all $z_1, z_1' \in \mathbb{C}$. The first set can be identified with the complex line $\mathbb{C}$ since, if $z_2 \neq 0$, $(z_1, z_2) \simeq (z_1/z_2, 1)$. Thus $\mathbb{C}P^1$ is identified with $\mathbb{C}$ (equivalently $\mathbb{R}^2$) plus one point, that is the sphere S^2, for instance by a stereographic projection from the north pole of S^2 onto $\mathbb{R}^2$, and the additional point of $\mathbb{C}P^1$ identified with this north pole. An analytic identification of S^2 defined as the submanifold $\Sigma_{i=1}^3 (x^i)^2 = 1$ of $\mathbb{R}^3$ with the set of equivalence classes $[z_1, z_2]$ can be checked to be

$$x^1 = \frac{2\mathcal{R} z_1 \bar{z}_2}{z_1\bar{z}_1 + z_2\bar{z}_2}\,, \qquad x^2 = \frac{2\mathcal{I} z_1 \bar{z}_2}{z_1\bar{z}_1 + z_2\bar{z}_2}\,, \qquad x^3 = \frac{z_1\bar{z}_1 - z_2\bar{z}_2}{z_1\bar{z}_1 + z_2\bar{z}_2}\,. \tag{1}$$

Answer 2: We define a map $p: S^3 \to S^2$ by $(z_1, z_2) \to [z_1, z_2]$, when $z_1\bar{z}_1 + z_2\bar{z}_2 = 1$. This mapping is surjective since for any pair with z_1 and z_2 not both zero

$$(z_1, z_2) \simeq \left(\frac{z_1}{(z_1\bar{z}_1 + z_2\bar{z}_2)^{1/2}}, \frac{z_2}{(z_1\bar{z}_1 + z_2\bar{z}_2)^{1/2}}\right),$$

and the last pair satisfies the condition for it to represent a point of S^3. The reciprocal image $p^{-1}(x)$ of a point $x \in S^2$ represented by the equivalence class $[z_1, z_2]$ is the set of points of S^3 represented by a pair (z_1', z_2') with $(z_1', z_2') \simeq (z_1, z_2)$ thus

$$z_1' = \lambda \frac{z_1}{(z_1\bar{z}_1 + z_2\bar{z}_2)^{1/2}}\,, \quad z_2' = \lambda \frac{z_2}{(z_1\bar{z}_1 + z_2\bar{z}_2)^{1/2}}\,, \quad z_1'\bar{z}_1' + z_2'\bar{z}_2' = 1 \tag{2}$$

with $|\lambda| = 1$. The set $|\lambda| = 1$ in $\mathbb{C}$ is identified with S^1, or $U(1)$, and is in bijective correspondence with $p^{-1}(x)$.
Formula (2) gives the local trivialization $p^{-1}(U) = \phi(U \times S^1)$. The fibering $S^3 \to S^2$ is called "Hopf fibering" because it was discovered by H. Hopf. Other fiberings of spheres, also called Hopf fiberings, into principal bundles are

$S^7 \to S^4$,		group S^3
$S^n \to \mathbb{R}P_n$,	real projective space,	group Z_2
$S^{2n+1} \to \mathbb{C}P_n$,		group $U(1)$
$S^{4n+3} \to \mathbb{HP}_n$,	quaternion projective space,	group SU(2).

REFERENCE

N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, sixth printing 1967).

4. SUBBUNDLES AND REDUCIBLE BUNDLES*

1) *Let H be a Lie subgroup of a Lie group G. Show that the two following statements are equivalent.*

*For a physical application, see for instance Problem V bis 5, Broken symmetry.

a) *The G-principal bundle $P = (P, M, \pi, G)$ admits an H-principal subbundle $P_H = (P_H, M, \pi_H, H)$ with $P_H \subset P$, and $\pi_H = \pi|_{P_H}$.*
b) *The G-principal bundle P admits a family of local trivializations with H-valued transition functions.*

Answer: If P satisfies a), it is reducible to P_H by the definition of reducibility (p. 131). The equivalence of a) and b) results therefore from the theorem (p. 382): A principal fibre bundle (P, M, π, G) is reducible to (P_H, M, π_H, H) with H a subgroup of G if and only if it admits a family of local trivializations whose transition functions take their values in H. We recall the proof that $a \Rightarrow b$. Let (P_H, M, π_H, H) be a subbundle of (P, M, π, G). We shall show that the trivialization $(\varphi_{Hi}, U_i \subset M)$ on P_H defines an H-valued set of transition functions for P. Indeed, let $p \in P_H \subset P$ and $\pi_H(p) = x$; the fibre of P at x is

$$\pi^{-1}(x) = \{pg,\ g \in G\} \quad \text{where } pg = \widetilde{R}_g(p);$$

let (φ_i, U_i) be the trivialization of P defined by

$$\varphi_i(pg) = \varphi_i(p)g = (x, \overset{\Delta}{\varphi}_{Hi,x}(p)g), \quad p \in P_H, \quad \pi(p) = x \in U_i$$

with the standard notations (p. 125). The corresponding transition functions

$$\begin{aligned} g_{ij}(x) &\equiv \overset{\Delta}{\varphi}_{i,x} \circ \overset{\Delta}{\varphi}{}^{-1}_{j,x} = \overset{\Delta}{\varphi}_{Hi,x} g \circ g^{-1} \overset{\Delta}{\varphi}{}^{-1}_{Hj,x} \\ &= \overset{\Delta}{\varphi}_{Hi,x} \circ \overset{\Delta}{\varphi}{}^{-1}_{Hj,x} \in H \end{aligned}$$

take their values in H.

2) *Show that a G-principal bundle (P, M, π, G) is reducible to an H-principal bundle (P_H, M, π_H, H) if and only if the bundle $P \setminus H$ with typical fibre $G \setminus H$ (the space of left cosets gH) associated to P by the canonical left action of G on $G \setminus H$ admits a cross section.*

Answer 2: (p. 385)

a) Assume there exists a cross section

$$\sigma\colon M \to P \setminus H;$$

let $P_\sigma = (\mu^{-1} \circ \sigma)(M)$ where μ is the canonical map

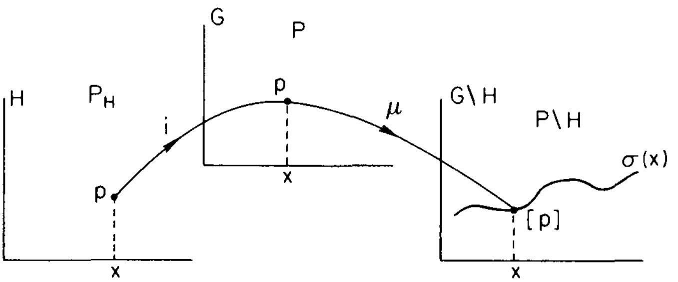
$$\mu\colon P \to P \setminus H, \quad \text{by } p \mapsto [p],\ p \simeq p' \text{ if } p' = \widetilde{R}_h p, \quad h \in H.$$

Then $P_\sigma \subset P$ and the injection identity $j\colon P_\sigma \to P$ is a bundle morphism which commutes with the right action of H on P_σ. According to the definition (p. 131) of reducible principal bundles, P is reducible to the H-principal bundle P_σ.

b) Conversely, let $P_H \subset P$ be a reduced bundle of P, then $\mu \circ i$ is constant on the fibres of P_H, and

$$\sigma = \mu \circ i \circ \pi_H^{-1}$$

defines a section $M \to P \setminus H$.



A point $[p]$ in $P \setminus H$ is the equivalence class of points $p \in P$ such that $p_1 \sim p_2$ if $p_1 = p_2 h$.

Remark: As a particular case of the previous study, we find the result that a principal bundle p is trivial (reducible to a bundle with group reduced to unity) if and only if it admits a cross section.

Remark: For a study of the existence of cross sections on the associated bundle $P \setminus H$ we refer the reader to C.J. Isham, "Space-time topology and spontaneous symmetry breaking", J. Phys. A. Math. Gen. 14 (1981) 2943–2956.

5. BROKEN SYMMETRY AND BUNDLE REDUCTION, HIGGS MECHANISM

INTRODUCTION

The action functional of a system is often invariant under a group G. We have shown (Problem II 5, Invariance) that the Euler–Lagrange equations of the system have the same symmetries as the action; hence the family of solutions of the Euler–Lagrange equations is invariant under G but a given solution is usually not invariant under G.

vacuum

A solution which minimizes the energy of the system is called **vacuum**. One expects the vacuum to be maximally symmetric and to be unique, but even the vacuum may not be invariant under G but under subgroup

H of G and possibly not unique. One says that this vacuum breaks the symmetry or that the G-symmetry of the system is **spontaneously broken** to H.

spontaneously broken symmetry

Example 1: Einstein equations are invariant under isometries but a given space-time has in general no nontrivial isometry.
The **Einstein vacuum** is Minkowski space, invariant under the maximum possible group of isometries of a 4-dimensional pseudo-riemannian manifold.

Einstein vacuum

Example 2: Let $f = (f^a)$, $a = 1, \ldots, N$ be a **scalar multiplet** on the Minkowski space time M^4, that is a mapping $M^4 \to \mathbb{R}^N$ by $x \mapsto f(x) = (f^a(x)), a = 1, \ldots, N$.

scalar multiplet

Consider the lagrangian

$$L(f) = \sum_{a=1}^{N} \left\{ \left(\frac{\partial f^a}{\partial x^0} \right)^2 - \sum_{i=1}^{3} \left(\frac{\partial f^a}{\partial x^i} \right)^2 \right\} - V \circ f, \tag{1}$$

where the potential $V\colon \mathbb{R}^N \to \mathbb{R}$ is invariant under rotations in $\mathbb{R}^N$. Take for instance V given by

$$V(v) = \left(\sum_{a=1}^{N} (v^a)^2 - k^2 \right)^2, \qquad v = (v^a) \in \mathbb{R}^N. \tag{2}$$

The **energy** of f (p. 513) and [Problem II 7, Stress energy] is the integral, which is independent of $x^0 = t$ if f is a solution of the Euler equation of L:

energy

$$\int_{x^0=t} \left\{ \frac{1}{2} \sum_{a=1}^{N} \left\{ \left(\frac{\partial f^a}{\partial x^0} \right)^2 + \sum_{i=1}^{3} \left(\frac{\partial f^a}{\partial x^i} \right)^2 \right\} + V(f(x)) \right\} \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3. \tag{3}$$

This integral attains an absolute minimum when $f(x) = v_0$, some constant vector in $\mathbb{R}^N$, and v_0 is an absolute minimum of V. If V is given by (2) the vector $v \in \mathbb{R}^N$ is an absolute minimum of V if its extremity is on the sphere S^{N-1} of radius k:

$$\sum_{a=1}^{N} (v^a)^2 - k^2 = 0. \tag{4}$$

The lagrangian (1) is invariant under the transformations

$$f(x) \to Sf(x), \quad S \in \mathrm{O}(N), \quad \text{orthogonal transformations of } \mathbb{R}^N$$

while a solution v_0 of (4), a point on $S^{N-1} \subset \mathbb{R}^N$ is invariant only under the subgroup of $\mathrm{O}(N)$, isomorphic to $\mathrm{O}(N-1)$, which leaves this point fixed.

Remark 1: The point $v = 0$ is, in this example, a maximum of V. It is invariant under $O(N)$.

Remark 2: The sphere S^{N-1} where the vacua take their values is the homogeneous space $O(N)/O(N-1)$ [Problem V 10, Invariant].

The purpose of this problem is to show how, in some cases, one exploits the H-symmetry of the vacuum.

1. EXISTENCE AND PROPERTIES OF H-INVARIANT VACUA

We consider the general case of a physical system on a (pseudo)-riemannian manifold M whose unknown quantities are a connection ω on a principal G-bundle P over M (p. 364) and a Higgs field $\tilde{\phi}$ of type (ρ, E) (p. 404), i.e., a mapping from P into a given complex – or real – vector space E, equivariant under a representation ρ of G by a group Γ of linear transformations of E:

$$\tilde{\phi}(pg) = \rho(g^{-1})\tilde{\phi}(p), \qquad p \in P, \quad g \in G, \quad \rho(g^{-1}) \in \Gamma.$$

We denote by A, F, ϕ the gauge potential, gauge field, scalar-multiplet pull backs of ω, Ω (the curvature of ω) and $\tilde{\phi}$ by a local section s: $U \to P$, $U \subset M$. We denote by $\nabla\tilde{\phi}$ the covariant derivative of $\tilde{\phi}$ in the connection ω, by $\nabla\phi$ its pull back.

If the Higgs field ϕ is coupled to the gauge field F only by the covariant derivative $\nabla\phi$ in the connection ω, the coupling is called **minimal**.

minimal coupling

We consider a potential V: $E \to \mathbb{R}$. We suppose E endowed with a hermitian – or real – scalar product, invariant under Γ. The following lagrangian defined on U by

$$L(A, \phi) = \tfrac{1}{4} g^{\mu\sigma} g^{\nu\rho} F_{\mu\nu} \cdot F_{\sigma\rho} + \tfrac{1}{2} g^{\mu\nu} \nabla_\mu\phi \cdot \nabla_\nu\phi - V(\phi) \qquad (5)$$

is a well-defined function on M because its values in U are independent of the local section s of P, since it is invariant under the transformations $F \mapsto \mathrm{Ad}(g^{-1})F$, $\phi \mapsto \rho(g^{-1})\phi$, $\nabla\phi \mapsto \rho(g^{-1})\nabla\phi$. The notations are

$\{e_\mu\}$, $\mu = 1, \ldots, d$ is a basis for T_xM,

$g_{\mu\nu}(x)$ is the metric tensor on M.

$F_{\mu\nu} = F^i_{\mu\nu} e_i$, $\{e_i\}$, $i = 1, \ldots, n$ is a basis for the Lie algebra $\mathscr{L}(G)$.

$F^i_{\mu\nu} = \partial A^i - \partial A^i + c^i A^j A^k$, c^i are the structure of constants G.

$F_{\mu\nu} \cdot F_{\alpha\beta} = g_{ij} F^i_{\mu\nu} F^j_{\alpha\beta}$, $g = (g_{ij})$ Ad-invariant scalar product on $\mathscr{L}(G)$,

$\phi = \phi^a e_a$, $\{e_a\}$, $a = 1, \ldots, N$ basis of E.

$(\nabla_\mu \phi)^a = \partial_\mu \phi^a + (\rho'_e(A^i_\mu e_i))^a{}_b \phi^b \equiv \partial_\mu \phi^a + A^i_\mu T_i{}^a{}_b \phi^b$,
$\nabla_\mu \phi \cdot \nabla_\mu \phi = h_{ab}(\nabla_\mu \phi)^a (\nabla_\mu \phi)^b$,
$h = (h_{ab})$ is a scalar product in E invariant under $\rho(g)$, $g \in G$.

The field equations are [Problem II 5, Invariance]

$$\nabla_\mu F^{i\mu\nu} + \phi^a T^i{}_{ab} \nabla^\nu \phi^b = 0 , \tag{6a}$$

$$\nabla_\mu \nabla^\mu \phi^a - \partial V / \partial \phi^a = 0 . \tag{6b}$$

If the metric on M is properly riemannian the energy of the system is (for the hyperbolic case see Problem II 7, Stress energy)

$$U(A, \phi) = \int (\tfrac{1}{4} g^{\mu\sigma} g^{\nu\rho} F_{\mu\nu} \cdot F_{\sigma\rho} + \tfrac{1}{2} g^{\mu\nu} \nabla_\mu \phi \cdot \nabla_\nu \phi + V(\phi)) \, \mathrm{d}\mu . \tag{7}$$

We suppose the scalar products in $\mathscr{L}(G)$ are positive definite and there exists a subset $\Sigma \subset E$ such that V attains its absolute minimum on Σ.

1) a) *Show that Σ is a union of orbits of Γ, representation of G by ρ.* b) *Show that each orbit can be identified with the homogeneous space $\Gamma \backslash \Gamma_H$ where Γ_H is the isotropy subgroup of a point of this orbit (Γ_H is the representative by ρ of a subgroup H of G).*

Answer 1a: $v \in \Sigma$ if and only if $V(v) = m$, the absolute minimum of V. Since V is invariant under Γ we have

$$V(\rho(g)v) = V(v) , \qquad \forall \rho(g) \in \Gamma .$$

Answer 1b: Choose a point $v_0 \in \Sigma$, denote by $\Gamma_{H_0} \subset \Gamma$ its isotropy group, i.e., the set of elements $\rho(h) \in \Gamma$ such that

$$\rho(h) v_0 = v_0 , \qquad h \subset H_0 .$$

Let $v_1 = \rho(g) v_0$ be in the orbit of v_0 under Γ. Its isotropy group Γ_{H_1} is the set of elements

$$\rho(g)\rho(h)\rho(g^{-1}) \in \Gamma$$

since

$$v_1 = \rho(g)\rho(h) v_0 = \rho(g)\rho(h)\rho(g^{-1}) v_1 .$$

If ρ is a left action $H_1 = g H_0 g^{-1}$; if ρ is a right action $H_1 = g^{-1} H_0 g$; in both cases $H_1 \simeq H_0$ and we can identify each orbit of Γ with the homogeneous space Γ / Γ_H where Γ_H is the isotropy group of an arbitrary point of the orbit.

2) *Show that if P is a trivial bundle the vacua exist and are such that*

$$F = 0 \quad i.e., \quad \Omega = 0 \text{ on } P \tag{8a}$$

$$\nabla_\mu \phi = 0 \quad i.e., \quad \nabla\tilde{\phi} = 0 \text{ on } P \tag{8b}$$

$$\phi \text{ i.e., } \tilde{\phi} \text{ takes its values in } \Sigma. \tag{8c}$$

Answer 2: If P is trivial it admits a flat connection – i.e., with zero curvature. It can be determined by choosing in a given cross section s_0 of P, the representative $A_0 = 0$. Relative to this cross section (b) reduces to $\partial_\mu \phi_0(x) = 0$; its solution is $\phi_0(x) = v_0$, constant vector in E. The condition (c) says that $v_0 \in \Sigma$.
It is clear that the energy attains its absolute minimum for the considered fields, and that they satisfy the field equations.

3) *We shall now examine under which conditions on the possibly nontrivial bundle P the system admits a vacuum of the type given in question* 2).
a) *Give necessary topological conditions for the existence of a flat connection on P.*

b) *Show that if there exists an equivariant mapping $\tilde{\phi}$: $P \to \Sigma_1$, where Σ_1 is an orbit of Γ in E, then the group G of the bundle P is reducible to the group H whose representative by ρ is the isotropy group of a point of Σ_1.*

c) *Suppose conversely that P is reducible to a principal bundle with the group H, whose image by ρ is the isotropy group of a given point $v_0 \in \Sigma_1 \subset E$.*
Denote by P_H (cf. Problem V bis 4, Subbundles) the corresponding sub-principal bundle. Given a covering of M by open sets U_i over which P_H is trivial there exist sections s_i: $U_i \to P_H \subset P$, and $s_j(x) = s_i(x)h_{ji}(x)$ with $x \in U_i \cap U_j$, $h_{ji}(x) \in H$. Show that one can define an equivariant mapping: $\tilde{\phi}$: $P \to \Sigma_1$ by setting

$$\begin{aligned} &\tilde{\phi}(s(x)) = v_0, \qquad x \in U_i, \qquad v_0 \in \Sigma_1 \\ &\tilde{\phi}(p) = \rho(g^{-1})v_0 \quad \text{if} \quad p = s_i(x)g. \end{aligned} \tag{9}$$

d) *Show that if P_H admits a flat connection the same is true of P. Show that the mapping $\tilde{\phi}$: $P \to \Sigma_1$ defined above has a vanishing covariant derivative in this connection.*

Answer 3a: If P is nontrivial it may not admit a connection with zero curvature: a necessary condition for the existence of such a connection is the

vanishing of the characteristic classes of $P(p)$. A nonzero characteristic class is also called a **topological charge**.
It can be proved [Kobayashi–Nomizu, Vol. I, p. 92] that M is simply connected P admits a flat connection if and only if P is a trivial bundle.

topological charge

Remark: if $A \neq 0$ but $F = 0$ in $U \subset M$ the equations (8b) are a completely integrable system, thus admit solutions at least locally.
We shall return to the solution of (8b) after the next question.

Answer 3b (See also answer a p. 309): Suppose there exists $\tilde{\phi}$: $P \to \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k$, such that $\tilde{\phi}(pg) = \rho(g^{-1})\tilde{\phi}(p)$. Denote by v_0 some point of Σ_1 attained by $\tilde{\phi}$, and denote by Q the subset of P such that

$$\tilde{\phi}(q) = v_0, \quad q \in Q.$$

The projection π: $P \to M$ restricted to Q still covers M: indeed let $p \in \pi^{-1}(x)$ since $\tilde{\phi}(p) \in \Sigma_1$ there exists $g \in G$ such that

$$\tilde{\phi}(pg) = \rho(g^{-1})\tilde{\phi}(p) = v_0, \quad p \in \pi^{-1}(x), \quad \text{thus } pg \in \pi_Q^{-1}(x).$$

The action of G on P induces an action of H on Q because $qh \in Q$ if $q \in Q$, $h \in H$, since

$$\tilde{\phi}(qh) = \rho(h^{-1})\tilde{\phi}(q) = v_0 \qquad \text{if } \tilde{\phi}(q) = v_0.$$

Thus Q is a principal subbundle P_H of P.

Answer 3c (See also answer b p. 310): The problem is to prove that $\tilde{\phi}(p)$ does not depend on the choice of the local sections s_i of P_H. Consider indeed the two sections s_i and s_j over $U_i \cap U_j$. We have

$$\tilde{\phi}(s_j(x)) = \rho(h_{ji}^{-1}(x))\tilde{\phi}(s_i(x)) = v_0$$

since v_0 is invariant under $\rho(h)$. On the other hand if $p = s_i(x)g$ we have $p = s_j(x)h_{ji}^{-1}(x)g$. Applying the previous definition to the section s_j gives

$$\tilde{\phi}(p) = \rho(g^{-1}h_{ji}(x))v_0 = \rho(g^{-1})v_0,$$

thus the same value $\tilde{\phi}(p)$.

Answer 3d: Suppose that P_H admits a flat connection $\bar{\omega}_H$. We define the horizontal subspace of a connection $\bar{\omega}$ on P by the general method (cf. p. 380): the horizontal subspace of $\bar{\omega}$ at $q \in P_H$ is the same as the horizontal subspace of $\bar{\omega}_H$; at $p = qg$, the horizontal subspace $\mathrm{hor}_p P$ will be (cf. p. 380)

$$\mathrm{hor}_p P = R'_g \,\mathrm{hor}_q P_H.$$

It is easy to check that the definition does not depend on the choice of $q \in P_H$. The connection $\bar{\omega}$ is flat if the connection $\bar{\omega}_H$ (cf. p. 388) is flat. Let $i: P_H \to P$ be the identity injection. The field $\tilde{\phi}_H = \tilde{\phi} \circ i$ has a vanishing covariant derivative in any H-connection. The vanishing of $\nabla\tilde{\phi}$ in the connection $\bar{\omega}$ is a particular case (with $\gamma = 0$) of results which will be proved in section 2.

2. HIGGS MECHANISM

We consider a lagrangian of type (5) and suppose that the system admits a vacuum ϕ such that (see question 2)

$$F = 0\,, \qquad \nabla\phi = 0\,, \qquad \phi \text{ takes its values in } \Sigma\,.$$

The lagrangian (5) is expressed in terms of quantities defined on the G-principal bundle P. We shall exploit the fact that P is reducible to an H-principal bundle P_H and reexpress the lagrangian in terms of quantities defined on P_H. Some of the new terms can be interpreted as massive fields and this procedure is called the (generating mass) **Higgs mechanism**.

Higgs mechanism

4) *Let $\mathscr{L}(G)$ and $\mathscr{L}(H)$ be the Lie algebras of G and H, set*

$$\mathscr{L}(G) = \mathscr{L}(H) \oplus \mathscr{K}\,.$$

See [Problem III 7, Homogeneous spaces] for properties of $\mathscr{K}$.
Let ω be a connection on P and $i^*\omega$ its pull back on P_H by the identity injection $i: P_H \to P$. Set

$$i^*\omega = \omega_H + \gamma$$

where ω_H is a 1-form with values in $\mathscr{L}(H)$ and γ is a 1-form with values in $\mathscr{K}$. Show that ω_H is a connection on P_H and γ a tensorial form (p. 373) of type (Ad, $\mathscr{K}$).

Answer: Let $h \in H \subset G$, and $\tilde{R}_h$ the right action on P_H. We have

$$\tilde{R}_h(i^*\omega) = \mathrm{Ad}(h^{-1})(i^*\omega)\,,$$

i.e.,

$$\tilde{R}_h\omega_H + \tilde{R}_h\gamma = \mathrm{Ad}(h^{-1})\omega_H + \mathrm{Ad}(h^{-1})\gamma\,.$$

It follows that ω_H is a connection on P_H. To prove that γ is horizontal, i.e., vanishes on vertical vectors $V_{H\,\mathrm{vert}} \in TP_H$ we compute

$$\begin{aligned} i^*\omega(V_{H\,\mathrm{vert}}) &= \omega(i'V_{H\,\mathrm{vert}}) = \hat{V}_{H\,\mathrm{vert}} \in \mathscr{L}(H) \\ &= \omega_H(V_{H\,\mathrm{vert}})\,. \end{aligned}$$

Hence

$$\gamma(V_{H\ \mathrm{vert}}) = 0.$$

5) *Let $\bar{Y} \in T_pP$ be the ω-horizontal lift of $Y \in T_xM$ and $\bar{Y}_H \in T_pP_H$ be its ω_H-horizontal lift. Show that*

$$\bar{Y} = i'\left(\bar{Y}_H + \frac{\mathrm{d}}{\mathrm{d}t}\, p\exp(-t\gamma(\bar{Y}_H))|_{t=0}\right). \tag{10}$$

Answer: By construction we have

$$T_pP = T_pP_H + \mathscr{K}_p,$$

where $\mathscr{K}_p$ is the subset of the vertical space at p canonically isomorphic to $\mathscr{K}$ in the decomposition

$$\mathscr{L}(G) = \mathscr{L}(H) \oplus \mathscr{K}.$$

Set

$$\bar{Y} = i'(\bar{Y}_H + Z).$$

Since $\pi'(\bar{Y}) = \pi'(\bar{Y}_H)$, it follows that Z is a vertical vector in T_pP. It follows from $i^*\omega = \omega_H + \gamma$ that

$$\begin{aligned} 0 = \omega(\bar{Y}) &= \omega(i'(\bar{Y}_H + Z)) = (\omega_H + \gamma)(\bar{Y}_H + Z) \\ &= \omega_H(\bar{Y}_H) + \gamma(\bar{Y}_H) + i^*\omega(Z) \\ &= \gamma(\bar{Y}_H) + i^*\omega(Z), \end{aligned}$$

i.e., $Z \in \mathscr{K}_p$ is the vertical vector defined by $-\gamma(\bar{Y}_H) \in \mathscr{K}$ in the canonical isomorphism $\mathscr{K} \approx \mathscr{K}_p$.

6) *Let ω_H be a flat connection on P_H; let*

$$\tilde{\phi}_H = \tilde{\phi} \circ i\colon\ P_H \to \Sigma \subset E.$$

$\tilde{\phi}_H$ is called a **vacuum solution** *if its covariant derivative with respect to ω_H vanishes,* vacuum solution

$$\nabla^H \tilde{\phi}_H(p) = 0.$$

Show that the covariant derivative $\nabla\tilde{\phi}$ defined by ω is given by

$$\nabla\tilde{\phi}(i'V) = \rho'_e(\gamma(V))\tilde{\phi}_H \quad \textit{with} \quad i^*\omega = \omega_H + \gamma, \quad V \in T_pP_H. \tag{11}$$

Answer 6: Let hor_ω label the horizontal component of a vector in the connection ω.

$$\begin{aligned}\nabla\tilde{\phi}(i'V) &= \tilde{\phi}'(\mathrm{hor}_\omega(i'V)) \quad \text{by definition,}\\ &= \tilde{\phi}'(i'\,\mathrm{hor}_{\omega_H}(V)) + \tilde{\phi}'\left(i'\frac{\mathrm{d}}{\mathrm{d}t}\, p\exp(-t\gamma(V))|_{t=0}\right)\\ &\quad \text{according to (10) and since } \gamma(V_{H\ \mathrm{vert}}) = 0,\\ &= \nabla^H\tilde{\phi}_H + \frac{\mathrm{d}}{\mathrm{d}t}\rho(\exp(t\gamma(V)))\tilde{\phi}_H(p)|_{t=0} \text{ by definition of } \tilde{\phi}\\ &= \rho'_e(\gamma(V))\tilde{\phi}_H \quad \text{since } \nabla^H\tilde{\phi}_H = 0.\end{aligned}$$

7) *Show that*

$$i^*\Omega = \Omega_H + \nabla^H\gamma + \tfrac{1}{2}[\gamma,\gamma], \tag{12}$$

where Ω and Ω_H are the curvatures defined by ω and ω_H and ∇^H the covariant derivative defined by ω_H.

With the notation of paragraph 5, let $\bar{Y}_j$ and $\bar{Y}_{Hj}$ be the horizontal lifts of $Y_j \in T_xM$ at $p \in P$ defined respectively by ω and ω_H; let

$$\bar{Y}_j = i'(\bar{Y}_{Hj} + Z_j) \in T_pP, \qquad j = 1, 2$$

where

$$Z_j(p) = \frac{\mathrm{d}}{\mathrm{d}t}\, p\exp(-t\gamma(\bar{Y}_{Hj}))|_{t=0}, \qquad \gamma = i^*\omega - \omega_H, \quad \pi(p) = x. \tag{13}$$

Let $V_j \in T_pP_H$ be such that

$$\mathrm{hor}_\omega(i'V_j) = \bar{Y}_j;$$

then

$$\begin{aligned}\Omega(i'V_1, i'V_2) &= i^*\,\mathrm{d}\omega(\bar{Y}_1, \bar{Y}_2) = \mathrm{d}i^*(\bar{Y}_1, \bar{Y}_2)\\ &= \mathrm{d}\omega_H(\bar{Y}_{H1}, \bar{Y}_{H2}) + \mathrm{d}\gamma(\bar{Y}_{H1}, \bar{Y}_{H2}) + i^*\,\mathrm{d}\omega(\bar{Y}_{H1}, Z_2)\\ &\quad + i^*\,\mathrm{d}\omega(Z_1, \bar{Y}_{H2}) + i^*\,\mathrm{d}\omega(Z_1, Z_2).\end{aligned}$$

The first two terms

$$\mathrm{d}\omega_H(\bar{Y}_{H1}, \bar{Y}_{H2}) = \Omega_H(V_1, V_2),$$
$$\mathrm{d}\gamma(\bar{Y}_{H1}, \bar{Y}_{H2}) = \nabla^H\gamma(V_1, V_2).$$

To compute the remaining three terms, we write (p. 207)

$$i^*\,\mathrm{d}\omega(\bar{Y}_{H1}, Z_2) = i^*\mathscr{L}_{\bar{Y}_{H1}}\omega(Z_2) - i^*\mathscr{L}_{Z_2}\omega(\bar{Y}_{H1}) - i^*\omega([\bar{Y}_{H1}, Z_2])$$

and two similar equations. Note first that

$$\omega(\bar{Y}_{H1}) = -\omega(Z_1) \tag{14}$$

since $\bar{Y}_1$ is ω-horizontal. Next, extend $Z_j(p)$ defined by (13) to a Killing vector field Z_j in a neighborhood of p. By definition

$$\omega(Z_j) = \hat{Z}_j = \text{constant} \in \mathscr{K} \tag{15}$$

and its Lie derivatives vanish.
We can write

$$\begin{aligned}
-i^*\omega([\bar{Y}_{H1}, Z_2]) &= i^*\omega(\mathscr{L}_{Z_2}\bar{Y}_{H1}) \\
&= i^* \lim_{t=0} \frac{1}{t}(\omega(\widetilde{R}'_{g(t)^{-1}}\bar{Y}_{H1}) - \omega(\bar{Y}_{H1}))
\end{aligned}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t)|_{t=0} = \hat{Z}_2 = -\gamma(\bar{Y}_{H2}).$$

Hence

$$\begin{aligned}
-i^*\omega([\bar{Y}_{H1}, Z_2]) &= i^* \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}(g(t))\omega(\bar{Y}_{H1}))|_{t=0} \\
&= -[\gamma(\bar{Y}_{H2}), \gamma(\bar{Y}_{H1})] \qquad \text{by (14) and (15)} \\
&= -[\gamma(V_2), \gamma(V_1)]
\end{aligned}$$

since γ is tensorial with respect to ω_H.
Finally,

$$i^*\Omega(V_1, V_2) = \Omega_H(V_1, V_2) + \nabla^H\gamma(V_1, V_2) + [\gamma(V_1), \gamma(V_2)].$$

Hence (p. 374)

$$i^*\Omega = \Omega_H + \nabla^H\gamma + \tfrac{1}{2}[\gamma, \gamma].$$

8) *The lagrangian* (5) *is defined on the base* M *of the* G*-principal bundle* P, *i.e., in terms of the pullbacks by a section* s_j: $U_j \subset M \to P$ *of* ω, Ω, *and* $\tilde{\phi}$.
Choose a section s_j *such that*

$$s_j\colon U_j \to P_H \subset P.$$

Use equations (11) *and* (12) *to reexpress the lagrangian in terms of quantities defined on* P_H *and interpret the lagrangian.*

Answer 8: The pullbacks of (11) and (12) give respectively

$$\nabla\phi(Y) = \rho'_e(\bar{\gamma}(Y))\phi, \quad s'_jY = V = i'V, \quad \bar{\gamma} = s_j^*\gamma.$$
$$F = F_H + \nabla^H\bar{\gamma} + \tfrac{1}{2}[\bar{\gamma}, \bar{\gamma}].$$

In the lagrangian evaluated at a vacuum solution ϕ and written schematically

$$L = \tfrac{1}{4}(F_H + \nabla^H \bar{\gamma} + \tfrac{1}{2}[\bar{\gamma}, \bar{\gamma}])^2 + \tfrac{1}{2}(\rho_e'(\bar{\gamma})\phi)^2$$

one notes

i) the following quadratic terms in $\bar{\gamma}$

$$\tfrac{1}{4}\nabla^H \bar{\gamma} \cdot \nabla^H \bar{\gamma} + \tfrac{1}{2}(\rho_e'(\bar{\gamma})\phi)^2$$

which can be interpreted as the lagrangian of a massive field $\bar{\gamma}$. Note that $\bar{\gamma}$ is not a gauge potential because it is not the pullback of a connection. It is sometimes called a **massive gauge potential** because it traces its origin to the gauge potential A.

massive gauge potential

ii) The field $\bar{\gamma}$ is minimally coupled to the gauge potentials $\bar{\omega}_H$ pullbacks of connections on P_H.

(Nambu) Goldstone boson

iii) The terms $\rho_e'(\bar{\gamma}^i{}_\mu(x))\phi$ are called **(Nambu) Goldstone bosons** relative to the vacuum ϕ. Since $i = 1, \ldots, \dim \mathscr{K}$, there are $(\dim G - \dim H)$ Goldstone bosons. There are as many Goldstone bosons as there are massive fields $\bar{\gamma}$.

Given the lagrangian (5) we have established necessary conditions for the existence of a vacuum, namely a flat connection, and a mapping

$$\tilde{\phi}: P \to \Sigma$$

shown to be equivalent to the existence of a principal subbundle P_H of P. In the previous problem we have shown that the existence of a principal subbundle P_H of P is equivalent to the existence of a section

$$\Psi: M \to P/H ,$$

where P/H is the associated bundle to P by the action of G on G/H. If the G-principal bundle P necessary to define the lagrangian (5) is reducible to an H-principal bundle P_H, the lagrangian (5) can be expressed in terms of massive fields $\bar{\gamma}$ and gauge potentials $\bar{\omega}_H$ pullbacks of connections on P_H.

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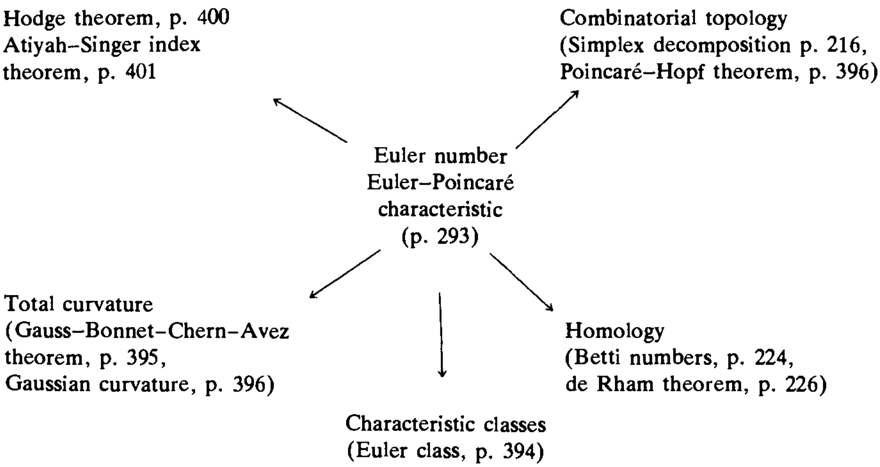
6. THE EULER–POINCARÉ CHARACTERISTIC

A supplement to this problem entitled "The Euler class" can be found near the end of the book.

INTRODUCTION

Chern (1979) called the Euler characteristic "the source and common cause of a large number of geometrical disciplines" and illustrated this relationship as follows.



This problem touches upon these various aspects of the Euler characteristic and in particular provides key elements of proofs and examples of the Poincaré–Hopf theorem and of the Gauss–Bonnet–Chern–Avez theorem. Several of the examples are given in two-dimensional spaces, thereby reproducing some classic results of curves and surfaces of euclidean spaces.

1. POINCARÉ–HOPF THEOREM

Euler number

The **Euler number** $\chi(M)$ of a compact manifold M is equal to the sum of the indices (p. 396) of the zeros of any smooth vector field v on M which has only isolated zeros. If M has a boundary, the vector field v is required to point outward at all boundary points.

a i) *Compute the indices i at* x_0 *of the vector fields drawn on figure* 1 *on a* 2-*dimensional manifold.*
a ii) *Construct a vector field with a zero of arbitrary index.*
a iii) *Show that the group of transformations generated by any of these vector fields does not act freely on* M.
a iv) *Observe that the sum of the indices* $\sum i$ *of the zeros of any vector field with isolated zeros which can be drawn on the* 2 *sphere is* 2. *Check that* $\sum i = 0$ *on the* 2 *torus.*

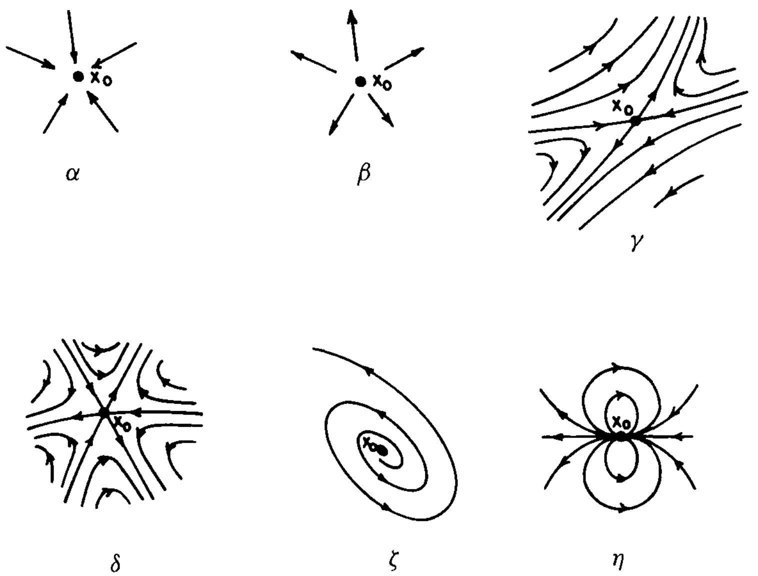


Fig. 1.

Answer 1a i: The index of a vector field v (p. 396) at an isolated zero x_0 is the degree (pp. 221, 557, 558) of the map: $x \to \hat{v}(x)$ (unit vector in the direction of v) from a small sphere $\varepsilon(x_0)$ of center x_0 into the unit sphere. The sphere $\varepsilon(x_0)$ is oriented as the boundary of the corresponding disk. For instance the index of the sink can be read off the following picture where the map: $x \to \hat{v}(x)$ maps points from the left sphere into points of the right sphere: the map is bijective and orientation preserving, thus the index is 1.

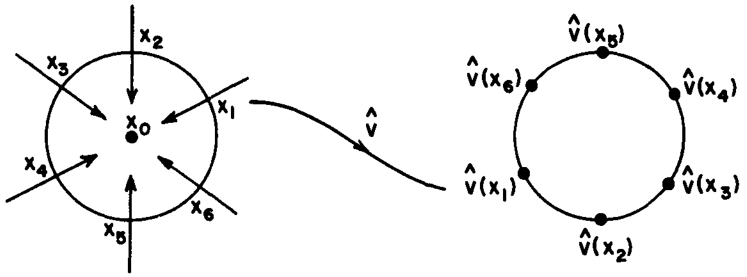


Fig. 2.

Similarly we find for the indices of the various cases drawn on fig. 1.

Vector field	a	β	γ	δ	ζ	η
Index	1	1	-1	-2	1	2

Answer 1a ii: Let $z = x + \mathrm{i}y$; the map v: $z \to z^n$, with n a positive integer, is of degree n, hence the vector field $v(x, y) = (\mathrm{Re}(x + \mathrm{i}y)^n, \mathrm{Im}(x + \mathrm{i}y)^n)$ has a zero at the origin of index n.

Answer 1a iii: The local group of transformations $\{\sigma_{g(s)}\}$ generated by any of the vector fields drawn on fig. 1 leaves x_0 fixed: $\sigma_{g(s)}(x_0) = x_0$, therefore the group does not act freely (p. 153) on M.

b) We can relate the topological properties of a 2-dimensional compact manifold M to the nature of the vector fields on M by triangulating M and constructing a vector field with the following zeros

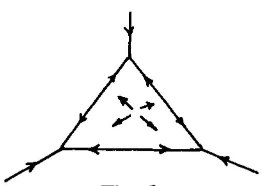


Fig. 3.

a source in each triangle

a saddle at each edge

a sink at each vertex

i) *Show that the sum of the indices of this vector field on M is*

$$\sum_M i = F - E + V ,$$

where F is the number of triangles, E the number of edges, V the number of vertices.

ii) *Show that* $F - E + V$ *is a topological invariant equal to the alternate sum of the Betti numbers of* M (p. 224):

$$F - E + V = \sum_{k=0}^{2} (-)^k b_k.$$

Show that for any n*-dimensional compact triangulable manifold* M

$$\sum_{k=0}^{n} (-1)^k \alpha_k = \sum_{k=0}^{n} (-1)^k b_k,$$

where α_k *is the dimension of the vector space* $C_k(M, \mathbb{R})$ *of real valued* k *chains on* M.
Hence

$$\sum_M i = \chi(M) \qquad \text{Poincaré–Hopf theorem (quoted p. 396).}$$

iii) *Give the Euler numbers of* S^2 *and* T^2.

Answer 1b i: The indices of sources, saddles and sinks are, respectively, $+1, -1, +1$, hence $\sum i = F - E + V$.

Inserting a segment AD in the ABC triangle does not change the alternate sum $F - E + V$.

Answer 1b ii: We shall now prove that $F - E + V = b_2 - b_1 + b_0$ or in general (see Singer and Thorpe, p. 142) for an n-dimensional compact triangulable manifold M

$$\sum_{k=0}^{n} (-1)^k \alpha_k = \sum_{k=0}^{n} (-1)^k b_k,$$

where α_k is the dimension of the vector space $C_k(M, \mathbb{R})$ of real valued k chains. For each $0 \le k \le n$, the boundary operator ∂ maps [see Problem IV 1, Cohomology] the vector space $C_k(M, \mathbb{R})$ of k-chains into $C_{k-1}(M, \mathbb{R})$:

$$\partial\colon\ C_k(M, \mathbb{R}) \to C_{k-1}(M, \mathbb{R}),$$

C_{-1} is by definition the empty space.

$$\alpha_k = \dim C_k(M, \mathbb{R}) \quad \text{by definition of } C_k(M, \mathbb{R});$$

by the rank and nullity theorem of linear algebra we have

$$\alpha_k = \dim Z_k(M, \mathbb{R}) + \dim B_{k-1}(M, \mathbb{R}),$$

where Z_k is the kernel of $\partial\colon\ C_k \to C_{k-1}$ and B_{k-1} is the image of $\partial\colon\ C_k \to C_{k-1}$.

On the other hand

$$b_k = \dim H_k(M, \mathbb{R}) = \dim Z_k(M, \mathbb{R}) - \dim B_k(M, \mathbb{R}) .$$

Hence

$$\chi(M) \equiv \sum_{k=0}^{n} (-1)^k b_k = \sum_{k=0}^{n} (-1)^k \dim Z_k + \sum_{k=0}^{n} (-1)^{k+1} \dim B_k .$$

Now $B_n(M^n) = 0$ because there is no $n+1$ chain, hence

$$\sum_{k=0}^{n} (-1)^{k+1} \dim B_k = \sum_{l=1}^{n} (-1)^l \dim B_{l-1} = \sum_{l=0}^{n} (-1)^l \dim B_{l-1} ,$$

since $B_{-1} = 0$. Finally,

$$\chi(M) = \sum_{k=0}^{n} (-1)^k (\dim Z_k + \dim B_{k-1}) = \sum_{k=0}^{n} (-1)^k \alpha_k .$$

Remark: In the case of a connected compact orientable 2-dimensional manifold M, $b_2 = 1$ because $B_2 = 0$, and H_2 has only one generator K homeomorphic to M. The Betti number $b_0 = 1$ for the following reasons: Let A and B be 2 vertices of K, then there is a sequence of vertices $A_1 = A$, $A_2, \ldots, A_{r+1} = B$ such that A_i, A_{i+1} are the vertices of a 1-simplex σ_i^1

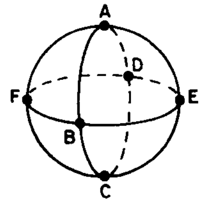
$$\partial \sum_{i=1}^{r} u\sigma_i^1 = u \sum_{i=1}^{r} (A_i - A_{i+1}) = uA - uB , \qquad u \in \mathbb{R} .$$

Therefore uA is homologous to uB. Any 0-cycle is of the form $\Sigma\, u_i\sigma_i^0$, and is homologous to $(\Sigma\, u_i)A$. Two 0-cycles uA and vB with $u \neq v$, are not homologous, because if they were $uA - vB$ would be a bounding cycle, but all bounding cycles are of the form $uA - uB$. Hence $H_0(M, \mathbb{R})$ is isomorphic to $\mathbb{R}$.
Note that the Poincaré duality theorem, i.e., $b_p = b_{n-p}$ (quoted p. 228), yields $b_0 = b_n$.

Answer 1b iii: *Examples*: The triangulations of S^2 and T^2 are as follows:

The eight curvilinear triangles ABE, AED, ADF, AFB, BCE, CED, DCF, BFC determine a triangulation of the sphere.

For the torus, the representation by means of a rectangle with opposite sides identified can be used.



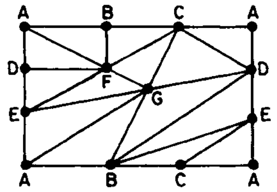


Fig. 4.

On S^2, $F - E + V = 2$; on T^2, $F - E + V = 0$.
One can also compute $\chi(S^2)$ and $\chi(T^2)$ from the Betti numbers given on (p. 224).

Remark: One can check that inserting segments in a triangulation creates only bounding cycles or cycles homologous to existing cycles; for instance, if we join FB in the torus triangulation by inserting 3 segments FH, HI, and IB, we create several bounding cycles and one 1-cycle, namely, BF + FH + HI + IB, homologous to the existing 1-cycle, AD + DE + EA.

Gauss map

c) We can prove that the sum of the indices $\sum_M i$ of a vector field v on M does not depend on the choice of v by using the properties of the Gauss map. The **Gauss map** G: $\partial M \to S^{n-1}$ assigns to each point $x \in \partial M$ the outward unit normal $n(x)$.

Let $M \subset \mathbb{R}^n$ be a compact n-manifold with smooth boundary ∂M (see paragraph d *for manifolds without boundary). Show that*

$$\deg(G, \partial M) = \sum_M i,$$

where $\sum_M i$ is the sum of the indices of any vector field v on M with isolated zeros, pointing outward at the boundary.

Answer 1c: Remove an ε-ball around each zero of the vector field v in M; let $\hat{M}$ be the new manifold. The boundary $\partial\hat{M}$ has several components: ∂M and the boundary of each ε-ball. We shall show that the sum of the degrees of the map $\hat{v}$: $x \to \hat{v}(x) \equiv v(x)/|v(x)|$ on the various components of $\partial\hat{M}$ vanishes. Indeed

$$\hat{v}|\partial\hat{M}: \partial\hat{M} \to S^{n-1}$$

is the restriction to $\partial\hat{M}$ of a smooth map

$$\hat{v}: \hat{M} \to S^{n-1}$$

which implies that the degree of $\hat{v}|\partial\hat{M}$ is zero (see Milnor 1965):

Let y be a regular value of $\hat{v}|\partial\hat{M}$. By definition (p. 557), degree $(\hat{v}, \partial\hat{M}, y) = \sum \text{sign}\, J_{\hat{v}}(x)$ for all $x \in \hat{v}^{-1}(y)$. The set $\{\hat{v}^{-1}(y)\}$ is an arc (or a finite union of arcs) whose boundary points a, b are on $\partial\hat{M}$, if $\hat{v}(a)$ points inward, $\hat{v}(b)$ points outward.

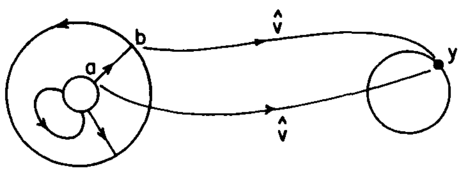


Fig. 5. $\hat{M}$ is the annulus between the 2 circles.

Thus for each pair of bounding points sign $J_{\hat{v}}(a) = -\,\text{sign}\, J_{\hat{v}}(b)$.

Now $\hat{v}||\partial M$ is homotopic to G, since, by hypothesis, v points outward at the boundary of M. Hence

$$\deg(\hat{v}, \partial M) = \deg(G, \partial M).$$

The degree of $\hat{v}$ on $\partial\varepsilon(x_0)$ with $\varepsilon(x_0)$ the ε ball centered at a zero x_0 of v is equal by definition to minus the index of v at x_0. The minus sign is introduced by the orientation of $\partial\varepsilon$ which is not the boundary of the ball ε_{x_0} but one component of the oriented (p. 218) boundary of the oriented manifold $\hat{M}$

$$\deg(\hat{v}, \partial\varepsilon(x_0)) = -i.$$

And since the sum of the degrees of $\hat{v}||\partial\hat{M}$ on all its components vanishes

$$\deg(G, \partial M) - \sum_M i = 0.$$

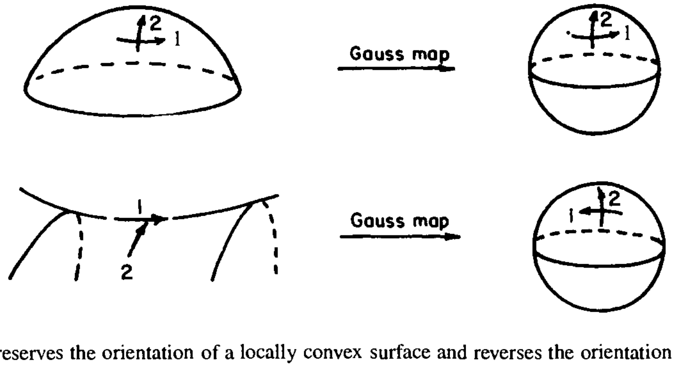


Fig. 6. G preserves the orientation of a locally convex surface and reverses the orientation of a locally saddle-like one.

d) *Let $M \subset \mathbb{R}^N$ be a compact n-manifold without boundary. Let N_ε be a smooth manifold with boundary which is a closed ε-neighborhood*[1] *of M. Show that the index sum of any vector field v on M is*

$$\sum_M i = \deg(G, \partial N_\varepsilon).$$

Hence $\sum_M i$ is independent of v.

Answer 1d: We shall only give the proof for the case where the zeros of v are nondegenerate. See [Milnor 1965] for zeros which are degenerate and more details on the proof. If the vector field v on M can be extended to a vector field w on N_ε such that w points outward on the boundary, and such that w and v vanish at the same points and have the same indices, then the result follows from the result of the previous section. Let $x \in N_\varepsilon$ and $r(x) \in M$ such that $\|x - r(x)\|$ is minimum. To construct w we set

$$w(x) = (x - r(x)) + v(r(x));$$

for ε sufficiently small $r(x)$ is well defined and smooth, $x - r(x)$ is orthogonal to $T_{r(x)}M$, w points outward along the boundary $\partial N_\varepsilon \equiv \{\|x - r(x)\| = \varepsilon\}$ and vanishes only at the zeros of v. The derivatives of the components of w at a zero x_0 of v are such that $\det(\partial w^\alpha/\partial x^\beta) = \det(\partial v^\alpha/\partial x^\beta)$. In another notation

$$\begin{array}{ll} \mathrm{d}w_{x_0}(u) = \mathrm{d}v_{x_0}(u) & \text{for all } u \in T_{x_0}M \\ \mathrm{d}w_{x_0}(u) = u & \text{for } u \text{ in the orthogonal complement of } T_{x_0}M. \end{array}$$

The determinant of $\mathrm{d}w_{x_0}$ and the determinant of $\mathrm{d}v_{x_0}$ are equal and so are the indices.

2. VAN HOVE'S SINGULARITIES[2]

The thermodynamical properties of a crystal are to a large extent determined by its frequency distribution function $g(\nu)$ where the frequency ν is defined as follows: Let $s_{(\alpha)}$ be the basic vectors defining a crystal cell. Let $\boldsymbol{h}^{(\alpha)}$ be the reciprocal lattice of the crystal defined by the duality in $\mathbb{R}^n$, usually $\mathbb{R}^3$,

$$\langle \boldsymbol{h}^{(\alpha)}, s_{(\beta)} \rangle = \delta^\alpha_\beta.$$

[1] Set of points of the euclidean space $\mathbb{R}^N$ which are at a distance $\leq \varepsilon$ from M. For the existence of such an N_ε see [Milnor 1965 problem 12, §8).

[2] See also the Reeb theorem, p. 592.

The elastic vibrations of a crystal are superpositions of normal modes, each of which is a plane wave vibration with wave vector $2\pi \boldsymbol{q}$. The frequency is a multivalued periodic function of the wave vector (i.e., the dispersion relation $\nu(\boldsymbol{q})$)

$$\nu\Big(\boldsymbol{q} + \sum n_\alpha \boldsymbol{h}^{(\alpha)}\Big) = \nu(\boldsymbol{q}), \qquad n_\alpha \in \mathbb{Z},$$

i.e., a function defined on a torus. The analytic singularities of the frequency distribution $g(\nu)$ originate from the critical points of $\nu(\boldsymbol{q})$ on each of its branches, (i.e., the zeros of its gradient, namely the zeros of the group velocity of the $\boldsymbol{q}$-wave). Van Hove used the fact that the number and the nature of the critical points of a function are constrained by the topology of its domain (Morse theorem) to constrain the occurrence and the nature of the singularities of the frequency distribution – known nowadays as the Van Hove singularities.

Let f be a differentiable function on a compact manifold M and C_λ the number of its critical points, assumed nondegenerate, of Morse index λ (p. 571); show that $\sum(-1)^\lambda C_\lambda$ is equal to the Euler–Poincaré number. Therefore the number of critical points of a function is constrained by the topology of its domain.

Answer 2: Let f: $M \to \mathbb{R}$. The critical points of f are the points $x_{(i)}$ such that

$$\frac{\partial f}{\partial x^\alpha}(x_{(j)}) = 0, \qquad \text{for every } \alpha = 1, \dots, \dim M.$$

The **index of a critical point** $x_{(j)}$ of f is the number of negative eigenvalues of the hessian $\partial^2 f/\partial x^\alpha \partial x^\beta$ at $x_{(j)}$. To assume that a critical point $x_{(j)}$ of f is nondegenerate is to assume that $\det(\partial^2 f/\partial x^\alpha_{(j)} \partial x^\beta_{(j)}) \neq 0$. The gradient of f defines a covariant vector field v of components index of a critical point

$$v_\alpha(x) = \partial f/\partial x^\alpha.$$

The zeros of v are the critical points of f. The index of v at a critical point $x_{(j)}$ is the degree of the map

$$\hat{v}\colon x \to f'(x)/|f'(x)|$$

on a small sphere $S_j \equiv \varepsilon(x_{(j)})$ centered at $x_{(j)}$:

$\deg(\hat{v}, S_j, y) = \sum \operatorname{sign} J_{\hat{v}}(x)$ for all $x \in \hat{v}^{-1}(y) \subset S_j$ by definition (p. 557).

A nondegenerate critical point of f is a zero of v where $\mathrm{d}v$ is nonsingular; hence $\hat{v}$ is bijective on S_j, and x is unique:

$$\begin{aligned}\deg(\hat{v}, S_j, y) &= \operatorname{sign} J_{\hat{v}}(x)\\ &= \operatorname{sign}\det(\partial^2 f/\partial x^\alpha \partial x^\beta)\\ &= (-1)^\lambda \qquad \text{by continuity from } x \text{ to } x_{(j)},\end{aligned}$$

where λ is the number of negative eigenvalues of the hessian of f at $x_{(j)}$. Finally, summing over all the critical points of f,

$$\sum (-1)^\lambda C_\lambda = \sum_M i = \chi(M).$$

3. GAUSS-BONNET-CHERN-AVEZ THEOREM (quoted p. 395)

Let M be an even dimensional oriented compact riemannian or pseudo-riemannian manifold with metric g; let $\chi(M)$ be its Euler number (p. 224) and γ its Euler class (p. 394), then

$$\chi(M) = \int_M \gamma.$$

A proof for some embedded manifold. Let $M \subset \mathbb{R}^n$ be a compact n-manifold with boundary ∂M. Let G be the Gauss map defined in section 1*c*)*, and let $J(x)$ be the jacobian of G at x.*
a) *Show that*

$$\int_{\partial M} J(x)\tau_{\partial M} = \deg(G) \text{ volume } S^{n-1},$$

where $\tau_{\partial M}$ is the volume element on ∂M induced by the euclidean volume element in $\mathbb{R}^n$.
b) *Assume ∂M is even dimensional. Let $a \in S^{n-1}$ be such that a and $-a$ are regular values of G. Let v be the vector field on ∂M defined by*

$$v(x) = (a|G(x))G(x) - a,$$

where $(a|G(x))$ is the euclidean scalar product.
Show that the index of v at its zeros is 1 *if G preserves the orientation and* -1 *otherwise.*
Show that the sum of the indices of v at all x where $G(x) = a$ is equal to $\deg(G)$. *Show that the sum of the indices of v at all x where $G(x) = -a$ is equal to* $\deg(G)$.
Show that if ∂M is even dimensional

$$2\deg(G) = \chi(\partial M).$$

Answer 3a: [See Guillemin and Pollack.]

$$\int_{\partial M} J(x)\tau_{\partial M} = \int_{\partial M} G^*\tau_{S^{n-1}} = \deg(G) \int_{S^{n-1}} \tau_{S_{n-1}}$$

$$= \deg(G) \operatorname{vol} S^{n-1} .$$

Answer 3b: The vector field v vanishes at the points x_0 such that

$$G(x_0) = \pm a .$$

The index of v is the degree of the map $\hat{v}: x \to \hat{v}(x)$ of a small sphere $\partial\varepsilon(x_0)$ around x_0 and

$$\deg(\hat{v}, \partial\varepsilon(x_0), y) = \sum \text{sign Jacobian of } \hat{v}(x) \text{ for all } x \in \hat{v}^{-1}(y) .$$

Since $\pm a$ are regular values of G, $\hat{v}'(x_0)$ is an isomorphism and index of v at $x_0 = \text{sign (Jacobian of } \hat{v})(x_0)$

$$v'(x) = (a|G'(x))G(x) + (a|G(x))G'(x) .$$

It follows from $(G(x)|G(x)) = 1$ that $(G(x)|G'(x)) = 0$. In particular $0 = (G(x_0)|G'(x_0)) = (\pm a|G'(x_0)$ and

$$v'(x_0) = (a| \pm a)G'(x_0) = \pm G'(x_0) .$$

Thus two cases arise:

$$(\text{Jacobian of } \hat{v})(x_0) = (\text{Jacobian of } G)(x_0)$$

or

$$\begin{aligned}(\text{Jacobian of } \hat{v})(x_0) &= (\text{Jacobian of } (-G))(x_0)\\ &= (-1)^{n-1}(\text{Jacobian of } G)(x_0)\\ &= (\text{Jacobian of } G)(x_0) \qquad \text{since } n-1 \text{ is even} .\end{aligned}$$

In both cases the index of v is 1 [respectively -1] if G preserves [reverses] the orientation.
In summary,

$(\text{index of } v)(x) = \deg v$ on $\partial\varepsilon(x)$	by definition
$= \text{sign (Jacobian of } v)(x)$	by definition and by isomorphism of $v'(x)$
$= \text{sign (Jacobian of } G)(x)$	by calculation .

It follows that

$$\sum_{\{x_0;\ G(x_0)=a\}} \text{indices of } v \text{ at } x_0 = \sum \text{sign Jacobian of } G \text{ at } x_0 = \deg G .$$

The same sum over the points x_0 such that $G(x_0) = -a$ is also equal to the degree of G. Hence by the Poincaré–Hopf theorem

$$\sum_{\partial M} i = \chi(\partial M) = \int_{\partial M} \gamma = 2\deg(G, \partial M) \quad \text{if } \partial M \text{ is even-dimensional}.$$

Because of the relation between the Euler–Poincaré characteristic and the curvature (Gauss–Bonnet theorem), the degree of the Gauss map G on ∂M is also called the **curvature integral** of ∂M.

curvature integral

Example: Let $\partial M = S^2$ and v be a vector field on ∂M with zeros of index 1 at the North and South Poles.

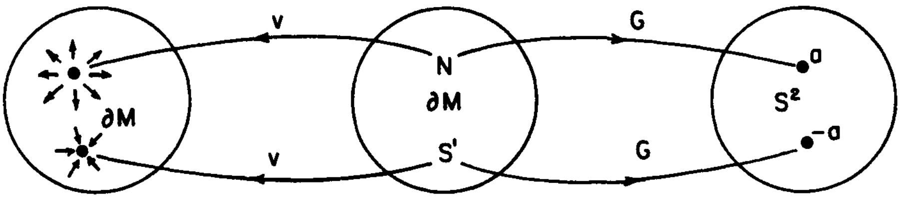


Fig. 7. A Picasso view of ∂M, its Gauss map, and the vector field v vanishing at N and S.

4. SUPERTRACE OF $\exp(-\Delta)$

a) *Show that on a compact Riemannian manifold M, the Euler number is*

$$\chi(M) = d_e - d_o , \tag{1}$$

where d_e [respectively, d_o] is the dimension of the space of harmonic forms ω of even [odd] degree, $\Delta\omega = 0$.

Answer 4a: According to Hodge's theorem (p. 400) the space of harmonic p-forms on a compact manifold M is isomorphic to $H^p(M)$, hence d_e is the sum of the even Bettí number and d_o the sum of the odd ones. The Euler–Poincaré characteristic (alternate sum of Bettí number) is equal to the analytical index of the elliptic complex $\{d_p, E_p\}$ where d_p is the exterior derivative defined on p-forms and E_p the vector bundle of p-forms on M.

The laplacian Δ on a compact manifold M is an operator with discrete spectrum $\lambda_0 = 0$, $\lambda_1, \ldots, \lambda_n, \ldots$ and $\Delta - \lambda_n$ has a finite dimensional kernel. The spectrum of the operator $\exp(-\Delta)$ is $\exp(-\lambda_0)$, $\exp(-\lambda_i), \ldots, \exp(-\lambda_n), \ldots$. Its trace in the Hilbert space of normed functions spanned by its eigenvectors is

$$\operatorname{Tr}\exp(-\Delta) = \sum_{n=0}^{\infty} \nu_n \exp(-\lambda_n)$$

$$\nu_n = \text{dim. of the space spanned by the } \lambda_n \text{ eigenvectors}.$$

b) *Show that the supertrace of* $\exp(-\Delta)$ *defined by*

$$\operatorname{Str}\exp(-\Delta) = \operatorname{Tr}\exp(-\Delta)|_{\mathscr{H}^+} - \operatorname{Tr}\exp(-\Delta)|_{\mathscr{H}^-} ,$$

where $\mathscr{H}^+$ *and* $\mathscr{H}^-$ *are the Hilbert spaces of even and odd forms on a compact manifold* M, *is a topological invariant [see McKean and Singer].*

Answer 4b: [See Getzler.] Let

$$Q = d + \delta ,$$

where δ is the metric transpose of d defined on (p. 297).
Then

$$Q: \mathscr{H}^{\pm} \to \mathscr{H}^{\mp}$$

is a self-adjoint operator such that $\Delta = Q^2$. Let $\mathscr{H}^{\pm}_{\lambda}$ be the eigenspaces of the (positive) operator Δ corresponding to the eigenvalues $\lambda \geq 0$. We have

$$Q: \mathscr{H}^{\pm}_{\lambda} \to \mathscr{H}^{\mp}_{\lambda} . \tag{2}$$

Indeed, let $f \in \mathscr{H}^{+}_{\lambda}$, then $\Delta Q f = Q \Delta f = \lambda Q f$, hence $Qf \in \mathscr{H}^{-}_{\lambda}$.
The operator Q^2 restricted to $\mathscr{H}^{\pm}_{\lambda}$ is equal to the operator multiplication by λ:

$$Q^2 | \mathscr{H}^{\pm}_{\lambda} = \lambda . \tag{3}$$

It follows from (2) and (3) that, if $\lambda \neq 0$, $\lambda^{-1/2} Q | \mathscr{H}^{\pm}_{\lambda}$ is the inverse of $\lambda^{-1/2} Q | \mathscr{H}^{\mp}_{\lambda}$ and, for $\lambda \neq 0$, $\dim \mathscr{H}^{+}_{\lambda} = \dim \mathscr{H}^{-}_{\lambda}$. Now

$$\begin{aligned} \operatorname{Str}\exp(-\Delta) &= \sum_{\lambda} \exp(-\lambda)(\dim \mathscr{H}^{+}_{\lambda} - \dim \mathscr{H}^{-}_{\lambda}) \qquad \text{(by definition)} \\ &= \dim \mathscr{H}^{+}_{0} - \dim \mathscr{H}^{-}_{0} \qquad \text{since } \dim \mathscr{H}^{+}_{\lambda} = \dim \mathscr{H}^{-}_{\lambda} \text{ for } \lambda \neq 0 \\ &= \chi(M) \qquad \text{(by eq. (1))} \end{aligned}$$

since a harmonic form is an eigenstate of Δ with 0 eigenvalue.

The Laplace–Beltrami operator is an example of "supersymmetric quantum mechanics" and displays cancellations analogous to the cancellations of supersymmetric quantum field theory (SUSY). For an explicit calculation of $\chi(M)$ by means of a path integral calculation of Str $\exp(-\Delta)$ see [Witten, DeWitt].

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7. EQUIVALENT BUNDLES

Let $\mathcal{B}$ and $\mathcal{B}'$ be two principal bundles with the same space B, base M, and group G, defined respectively by the local trivializations (U_i, ϕ_i) and (U_i, ϕ'_i), and the transition functions γ_{ij}, γ'_{ij}. The bundles are said to be **equivalent** if on each U_i there exists a continuous mapping λ_i: $U_i \to G$ such that the homomorphism $G \to G$ deduced for each $x \in U_i$ from the mapping

equivalent

$$\phi_i \circ \phi_i'^{-1}: U_i \times G \to U_i \times G$$

is the left product by $\lambda_i(x)$.

1) *Show that the transition functions of two equivalent bundles are such that*

$$\gamma'_{ij}(x) = \lambda_i^{-1}(x)\gamma_{ij}(x)\lambda_j(x) . \tag{1}$$

2) *Show that if $\mathcal{B}$ and $\mathcal{B}'$ are equivalent bundles there are homomorphisms $\mathcal{B} \to \mathcal{B}'$ and $\mathcal{B}' \to \mathcal{B}$ which induce the identity map on M.*

Remark: Principal bundles with the same group and base such that there exist such homomorphisms are also called equivalent.

Answer 1: By definition of λ_i we have

$$(\overset{\triangle}{\phi}_{i,x} \circ \overset{\triangle}{\phi}{}'^{-1}_{i,x}) = \lambda_i(x)\,, \qquad x \in U_i\,.$$

A transition function γ_{ij} is defined by

$$\overset{\triangle}{\phi}_{i,x} \circ \overset{\triangle}{\phi}{}^{-1}_{j,x} = \gamma_{ij}(x)\,, \qquad x \in U_i \cap U_j\,, \quad g \in G\,.$$

We have therefore

$$\begin{aligned}\gamma'_{ij}(x) &= \overset{\triangle}{\phi}{}'_{i,x} \circ \overset{\triangle}{\phi}{}'^{-1}_{j,x} = \overset{\triangle}{\phi}{}'_{i,x} \circ \overset{\triangle}{\phi}{}^{-1}_{i,x} \circ \overset{\triangle}{\phi}_{i,x} \circ \overset{\triangle}{\phi}{}^{-1}_{j,x} \circ \overset{\triangle}{\phi}_{j,x} \circ \overset{\triangle}{\phi}{}'^{-1}_{j,x} \\ &= \lambda_i^{-1}(x)\gamma_{ij}(x)\lambda_j(x)\,.\end{aligned}$$

Answer 2: Define a map from $p^{-1}(U_i)$ in $\mathscr{B}$ into $p^{-1}(U_i)$ in $\mathscr{B}'$ by its representative in local trivializations, given by

$$\begin{aligned} h_i\colon (x, g) \mapsto (x, \lambda_i(x)g)\,, \qquad & (x, g) \in \phi_i(p^{-1}(U_i)) \\ & (x, \lambda_i(x)g) \in \phi'_i(p^{-1}(U_i))\,. \end{aligned}$$

We check, using (1), that the maps so defined in different trivializations, coincide in $p^{-1}(U_i) \cap p^{-1}(U_j)$, and thus define a bundle map h. It obviously induces the identity on M.
It is easy to show, by expressing the right actions in local trivializations, that the mapping h is an homomorphism.

8. UNIVERSAL BUNDLES. BUNDLE CLASSIFICATION

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two bundles with fibres F_1 and F_2, groups G_1 and G_2. A mapping $\tilde{h}\colon \mathscr{B}_1 \to \mathscr{B}_2$ is said to be a **bundle homomorphism** (p. 380) if there exists a homomorphism $\mathscr{H}\colon G_1 \to G_2$ and if $\tilde{h}$ commutes with the actions of G_1 and G_2, namely: bundle homomorphism

$$\tilde{h}\tilde{R}_{g_1} = \tilde{R}_{g_2}\tilde{h} \quad \text{on} \quad \mathscr{B}_1\,, \qquad g_2 = \mathscr{H}g_1\,.$$

The map $\tilde{h}$ then induces a map h between the base spaces:

$$h = p_2 \circ \tilde{h} \circ p_1^{-1}\,,$$

and the following diagram is commutative.

$$\begin{array}{ccc} \mathscr{B}_1 & \xrightarrow{\tilde{h}} & \mathscr{B}_2 \\ p_1 \downarrow & & \downarrow p_2 \\ M_1 & \xrightarrow{h} & M_2 \end{array}$$

In question 1 we shall study the converse.

1. PULLBACK OF A BUNDLE

a) *Let $f\colon M \to M'$ be a continuous map, and let $\mathscr{B}'$ be a bundle over M' with group G and fibre F. Construct a bundle $\mathscr{B}$ over M, with the same group G and fibre F, such that there exists a homomorphism $\tilde{h}\colon \mathscr{B} \to \mathscr{B}'$ and $f = h$.*

Answer 1a: i) We can construct $\mathscr{B}$ by a pedestrian pull back of the bundle structure of $\mathscr{B}'$, namely consider a cover $\{U'_i\}$ of M', the corresponding local trivializations $p'(U'_i) \to U'_i \times F$ with the transition functions γ'_{ij}, and define $\mathscr{B}$ by the cover $\{U_i\}$, pull back by f of the open sets U'_i, the transition functions γ_{ij} by pull back of the γ'_{ij}, i.e., $\gamma_{ij} = \gamma'_{ij} \circ f$, the fibre F_x as the equivalence class of copies F_i of F defined by $(x, y_i) \simeq (x, y_j)$ if $x \in U_i \cap U_j$, $y_j = \gamma_{ij}(x) y_i$, the trivialization over U_i being $p^{-1}(U_i) = U_i \times F_i$ and the projection π, independent of the trivialization $\pi(F_x) = x$. The homomorphism $\tilde{h}\colon \mathscr{B} \to \mathscr{B}'$ is defined in the considered trivializations, by $(x, y_i) \mapsto (f(x), y_i)$; it is independent of the trivialization since $\gamma_{ij}(x) y_j = \gamma'_{ji}(f(x)) y_j$.

ii) We can define $\mathscr{B}$ as the subspace of $M \times \mathscr{B}'$ of these pairs (x, b') such that $f(x) = \pi'(b')$. It is given a bundle structure with base M by the natural projection $\pi\colon M \times \mathscr{B}' \to M$, restricted to $\mathscr{B}$. A right action of G on $F_x = \pi^{-1}(x)$, fibre at x in $\mathscr{B}$, which is identified to the fibre at $f(x)$ in $\mathscr{B}'$, is given by

$$\tilde{R}_g(x, b') = (x, \tilde{R}_g b') .$$

It completes the definition of the bundle structure of $\mathscr{B}$.
We have then obviously

$$\tilde{h}\tilde{R}_g = \tilde{R}_g \tilde{h} , \qquad g \in G$$

if the mapping $\tilde{h}\colon \mathscr{B} \to \mathscr{B}'$ is defined to be the natural projection $M \times \mathscr{B}' \to \mathscr{B}'$ by $(x, b') \mapsto b'$, restricted to $\mathscr{B}$.
The two constructions (i) and (ii) give isomorphic bundles. The bundle $\mathscr{B}$
pull back is called the **pull back** of $\mathscr{B}'$ by f, and denoted $\mathscr{B} = f^* \mathscr{B}'$.

It can be shown [Steenrod, p. 53] that if M is a paracompact manifold and if f_0 and f_1 are two homotopic maps $M \to M'$ the two pulled back bundles

$\mathscr{B}_1 = f_0^*\mathscr{B}'$ and $\mathscr{B}_2 = f_1^*\mathscr{B}'$ of the same bundle $\mathscr{B}'$ are equivalent bundles, in the sense that there exists a bundle homomorphism $\mathscr{B}_1 \to \mathscr{B}_2$ which induces the identity map $M \to M$, and conversely.
b) *Show that if M is a manifold diffeomorphic to $\mathbb{R}^n$, then any bundle over M is equivalent to a trivial bundle.*

Answer 1b: If M is diffeomorphic to $\mathbb{R}^n$ then the identity map $\mathbb{R}^n \to \mathbb{R}^n$ is homotopic to a constant map $\mathbb{R}^n \to x_0$; one says that $\mathbb{R}^n$ is **contractible to a point**. The pull back of a given bundle $\mathscr{B}'$ with base M by the identity map $M \to M$ gives the bundle $\mathscr{B}$, while the pull back of $\mathscr{B}'$ by a constant map is a trivial bundle. contractible to a point

2. UNIVERSAL BUNDLE

The statement in 1b shows that there is a way to classify the bundles with given group G, base M and fibre F if they can all be obtained as pull back of some "universal" bundle, with group G, fibre F and base some space X, by using the homotopy classes of maps $M \to X$.

Show that if a principal bundle $\mathscr{U}_G = (U, G, X)$, with group G and base X is arcwise connected and such that its first n homotopy groups vanish:

$$\pi_1(U) = \pi_2(U) = \cdots = \pi_n(U) = 0 \tag{1}$$

then every principal bundle with group G and base a paracompact manifold of dimension $\leq n$ can be obtained as the pull back of $\mathscr{U}_G$ by a mapping $f\colon M \to X$.
Such a bundle $\mathscr{U}_G$ is called ***n*-universal**. Since the pulled back bundles are equivalent for homotopic maps $M \to X$ these maps are called **classifying maps**. n-universal classifying maps

Answer 2: We shall give an idea of the construction of the homomorphism $\tilde{h}\colon \mathscr{B} \to \mathscr{U}_G$ using (1).
The mapping $\tilde{h}$, which will have to be continuous, can always be defined on fibres over isolated points of M, basis of $\mathscr{B}$, by setting

$$\tilde{h}|_{G_x} = \zeta\xi^{-1},$$

where $\xi\colon G \to G_x$ and $\zeta\colon G \to G_y$ are respectively mappings from G onto the fibre G_x over $x \in M$ and onto the fibre G_y over some point $y \in X$, which commute with the right action of G on the considered fibres.
Such an $\tilde{h}$ can be considered as a homomorphism from the restriction of $\mathscr{B}$ to fibres over the 0-cells of a complex decomposition (triangulation) of M (cf. Problem IV 1, Cohomology) into $\mathscr{U}_G$. We shall first show that $\tilde{h}$ can be extended to the restriction of $\mathscr{B}$ over the 1-cells.

Let D_1 be a 1-cell in M; since D_1 is contractible (homeomorphic to $[0, 1]$) the portion of $\mathcal{B}$ over D_1 is equivalent to the trivial bundle $D_1 \times G$, and the restriction of $\mathcal{B}$ over ∂D_1 is equivalent to $\partial D_1 \times G$. The boundary ∂D_1 consists of two points $x_0, x_1 \in M$, above which the map $\tilde{h}$ has been defined, i.e., we know $\tilde{h}|_{\partial D_1 \times G}: \partial D_1 \times G \to \mathcal{U}_G$. We define f to be the mapping $\partial D_1 \to \mathcal{U}_G$ defined by $f(x) = \tilde{h}(x, e)$, $x \in \partial D_1$. Since $\mathcal{U}_G$ is arcwise connected, f extends to a (continuous) map still denoted $f: D_1 \to \mathcal{U}_G$.
We define the map $h: D_1 \times G \to \mathcal{U}_G$ by

$$\tilde{h}(x, g) = P(f(x), g),$$

where P is the mapping

$$\mathcal{U}_G \times G \to \mathcal{U}_G \quad \text{by} \quad (y, g) \mapsto \tilde{R}_g y .$$

It is straightforward to check that $\tilde{h}$ takes the preassigned values on $\partial D_1 \times G$ and is a homeomorphism $D_1 \times G \to \mathcal{U}_G$.
Analogous reasoning will give the conclusion by induction for k cells, $k \leq n$. The boundary ∂D_k of a k-cell D_k is homeomorphic to a $k-1$ sphere S_{k-1} sphere, and maps from ∂D_k into $\mathcal{U}_G$ can be extended to D_k because $\pi_{k-1} = 0$.

3. EXISTENCE OF UNIVERSAL BUNDLES

It can be proved that every compact Lie group G is isomorphic to a subgroup of an orthogonal group O_k [cf. Chevalley, Theory of Lie groups p. 211]. It can also be proved [cf. Steenrod, p. 103] that the Stiefel manifolds $V_{n+k,k} = \mathrm{O}_{n+k} \backslash \mathrm{O}_k$ have zero homotopy groups $\pi_i(V_{n+k,k}) = 0$ for $1 \leq i < k$.
a) *Use these results to construct a n-universal bundle for a compact Lie group G.*
b) It can be proved (E. Cartan, Mostow) that every connected Lie group G is topologically $H \times \mathbb{R}^d$, where H is a compact subgroup of G.
Use this result to show the existence of universal bundles for G.

Answer 3a): Given a Lie group $G \subset \mathrm{O}_k$ we want to construct a principal bundle $\mathcal{U}_G$, with base some topological space X, such that $\pi_i(\mathcal{U}_G) = 0$, $i = 1, \ldots, N$. We shall seek $\mathcal{U}_G$ with space $\mathrm{O}_m \backslash \mathrm{O}_n$, $\mathcal{U}_G$ will then be $n-1$ universal.
The subgroup O_n of O_m leaving fixed the $m-n$ first vectors of $\mathbb{R}^m$, and

the subgroup of O_m, denoted O'_k, leaving fixed the $m-k$ last vectors, commute if $m \geq n+k$. Then $O_n \times O'_k$ is a subgroup of O_m, and $O_m \setminus O_n$ is a principal fibre bundle with group O_k and base $O_m \setminus (O_n \times O'_k)$. Since G is a subgroup of O_k, $O_m \setminus O_n$ is a principal fibre bundle with group G and base $X = O_m \setminus (O_n \times G)$.

Answer 3b: Left to the reader, or see Steenrod, p. 204.

4. CHARACTERISTIC CLASSES

The elements of the cohomology group $H^p(X_G, \mathbb{R})$ [resp. $H^p(X_G, \mathbb{Z})$ or $H^p(X_G, \mathbb{Z}_2)$] of the base space of a universal bundle $\mathscr{U}_G$ with group G, are called universal $\mathbb{R}$ [resp. $\mathbb{Z}$, $\mathbb{Z}_2$] characteristic classes. Let $\mathscr{B}$ be a principal fibre bundle with base M and group G, and f a classifying map $M \to X$. One defines here **characteristic classes** of $\mathscr{B}$ as the pull back by f of the universal characteristic classes. characteristic classes

Suppose that M and X are smooth manifolds and $f_2 = f_1 \circ \varphi$, with φ a diffeomorphism of M. Show that the bundles $\mathscr{B}_1$ and $\mathscr{B}_2$, pull back of $\mathscr{U}_G$ by f_1 and f_2, have the same $\mathbb{R}$-characteristic classes.
More generally it can be proved that if f_1 is homotopic to f_2 the bundles have the same characteristic classes.

Answer 4: By definition the characteristic classes of $\mathscr{B}_1$ and $\mathscr{B}_2$ are f_1^*c and f_2^*c, with $c \in H^p(M, \mathbb{R})$. If $f_2 = f_1 \circ \varphi$ then

$$f_2^*c = \varphi^*(f_1^*c) \in H^p(M, \mathbb{R}) \quad \text{if and only if} \quad f_2^* \in H^p(M, \mathbb{R}).$$

Remark: It is not necessarily true that two bundles with the same characteristic classes are equivalent, therefore they do not necessarily correspond to homotopic classifying maps.

For an application of universal bundles to the problem of spin structures [Problem IV 2, Obstruction] see [Avis and Isham].

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9. GENERALIZED BIANCHI IDENTITY

Let P be a principal fibre bundle with group G.
Let φ be a 1-form on P with values in the Lie algebra $\mathcal{G}$ of G.
Define a 2 form ϕ on P with values in $\mathcal{G}$ by

$$\phi = \mathrm{d}\varphi + \tfrac{1}{2}[\varphi, \varphi] . \tag{1}$$

The Bianchi identity

$$\mathcal{D}\phi \equiv \mathrm{d}\phi + [\varphi, \phi] \equiv 0 \tag{2}$$

has been proved (p. 375) when φ is a connection; prove it without this restriction.

Answer: By definition (p. 374) we have, if e_γ is a basis of $\mathcal{G}$ and $c_\alpha{}^\gamma{}_\beta$ its structure constants

$$[\varphi, \varphi]^\gamma = c_\alpha{}^\gamma{}_\beta(\varphi^\alpha \wedge \varphi^\beta)$$

therefore, if φ is a 1-form:

$$\mathrm{d}[\varphi, \varphi]^\gamma = c_\alpha{}^\gamma{}_\beta(\mathrm{d}\varphi^\alpha \wedge \varphi^\beta - \varphi^\alpha \wedge \mathrm{d}\varphi^\beta) . \tag{3}$$

By definition (1) we have

$$\phi^\gamma = \mathrm{d}\varphi^\gamma + \tfrac{1}{2}[\varphi, \varphi]^\gamma$$

which gives, after differentiation and use of (3)

$$\begin{aligned}(\mathcal{D}\phi)^\gamma \equiv{}& \tfrac{1}{2}c_\alpha{}^\gamma{}_\beta(\phi^\alpha - \tfrac{1}{2}c_\lambda{}^\alpha{}_\mu\varphi^\lambda \wedge \varphi^\mu) \wedge \varphi^\beta \\ &- \tfrac{1}{2}c_\alpha{}^\gamma{}_\beta\varphi^\alpha \wedge (\phi^\beta - \tfrac{1}{2}c_\lambda{}^\alpha{}_\mu\varphi^\lambda \wedge \varphi^\mu) + c_\alpha{}^\gamma{}_\beta\varphi^\alpha \wedge \phi^\beta .\end{aligned}$$

The vanishing of $\mathcal{D}\phi$ results from obvious cancellations (remember ϕ^α is a 2-form, thus commutes with φ^β) and from the Jacobi identity satisfied by the structure constants.

10. CHERN–SIMONS CLASSES*

We have defined (p. 390) characteristic classes on a principal fibre bundle $P(X, \pi, G)$ as projections $\bar{f}(\Omega) \equiv f(\bar{\Omega})$ on the base manifold X of closed forms $f(\Omega)$ on P, constructed with Ad G invariant polynomials on the Lie algebra $\mathcal{G}$ of G.

*For an application of Chern–Simons classes to Physics see, for instance, [Problem V bis 11, Cocycles].

We have shown that, for given P and f the cohomology class of $\bar{f}(\Omega)$ is independent of the curvature Ω, and a fortiori of the connection ω, by introducing a 1-parameter family of connections

$$\omega_t = \omega_0 + t\phi\,, \qquad \phi = \omega_1 - \omega_0 \tag{1}$$

and their curvature

$$\Omega_t = \mathrm{d}\omega_t + \tfrac{1}{2}[\omega_t, \omega_t]\,. \tag{2}$$

We have shown (p. 391) that on P for f a k-linear symmetric mapping

$$f(\Omega_1) - f(\Omega_0) = kd\int_0^1 f(\phi, \Omega_t, \ldots, \Omega_t)\,\mathrm{d}t\,, \tag{3}$$

relation which projects on X:

$$\bar{f}(\Omega_1) - \bar{f}(\Omega_0) = kd\int_0^1 \bar{f}(\phi, \Omega_t, \ldots, \Omega_t)\,\mathrm{d}t\,. \tag{4}$$

If F and A denote the representative of Ω and ω in a local trivialization of P above $U \subset X$, eq. (4) reads in U:

$$f(F_1) - f(F_0) = kd\int_0^1 f(A_1 - A_0, F_t, \ldots, F_t)\,\mathrm{d}t\,. \tag{5}$$

Relation (4) shows that if X is compact the integral of $\bar{f}(\Omega)$ on X is independent of Ω; it is a topological invariant. If X has a boundary ∂X relation (4) implies by Stokes theorem (p. 216)

$$\int_X \bar{f}(\Omega_1) - \int_X \bar{f}(\Omega_0) = \int_{\partial X} Q^{(1)}\,,$$

$$Q^{(1)} = k\int_0^1 \bar{f}(\omega_1 - \omega_0, \Omega_t, \ldots, \Omega_t)\,.$$

1) *Show that formula* (3) *holds when* ω_0 *and* ω_1 *are replaced by arbitrary* $\mathscr{G}$*-valued* 1*-forms,* $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$, *and* Ω_t *is the formal curvature*

$$\Omega_t = \mathrm{d}\omega_t + \tfrac{1}{2}[\omega_t, \omega_t]\,.$$

Does formula (4) *hold?*

Answer 1: The computation done on p. 391 applies without change to prove formula (3) on P. Note that the Bianchi identities for Ω defined by (2) remain valid. See [Problem V bis 9, Generalized].

Formula (3) projects on X to give (4) only if the forms under consideration vanish on vertical vectors: in particular this is not true of $\phi = \omega_1 - \omega_0$ if ω_0 and ω_1 are arbitrary $\mathscr{G}$ values 1-forms on P, or if ω_1 is a connection, but not ω_0.

2) a) *Show that for any connection ω with curvature 2-form Ω, $f(\Omega)$ is exact on P, namely*

$$f(\Omega) = \mathrm{d}(Tf(\omega)) \tag{6}$$

with

$$Tf(\omega) = k \int_0^1 f(\omega, \Omega_t, \ldots, \Omega_t)\,\mathrm{d}t\,,$$

$$\Omega_t = td\omega + \tfrac{1}{2}t^2[\omega, \omega]\,.$$

b) *Show that $\bar{f}(\Omega)$ is exact on each open subset $U \subset X$ over which P is a trivial bundle.*

Answer 2a: Let $\omega_0 = 0$ and $\omega_1 = \omega$ be $\mathscr{G}$-valued 1-form on P. Formula (3) gives (6).

Answer 2b: The 1-form $\omega_0 = 0$ on P is never a connection since, for a vertical vector v a connection is such that $\omega(v) = \hat{v}$. Thus the form $Tf(\omega)$ is not horizontal, the formula (6) does not project on X.
On a trivializable subset $\Pi^{-1}(U)$ there always exists a flat connection ω_0, with representative $A_0 = 0$ in the trivialization, or $A_0 = u^{-1}\,\mathrm{d}u$ in arbitrary "gauge", and curvature $\Omega_0 = 0$. The general formula (5) reads, with A_0 a flat connection:

$$f(F) = kd \int_0^1 f(A - A_0, F_t, \ldots, F_t)\,\mathrm{d}t \qquad \text{on } U\,.$$

3) *If the $2k$ form $f(\Omega)$ is identically zero one deduces from* (6) *that $Tf(\omega)$ is a closed $2k-1$ form on P, therefore defines an element of $H^{2k-1}(P, \mathbb{R})$* (*p.* 223) *and* [*Problem IV* 1, *Cohomology*], *depending on the connection ω; it is called a* **Chern–Simons class**.

Chern–Simons class

Show that if $\omega(t)$ is a 1-parameter family of connections on P, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(Tf(\omega(t))) = kf(\varphi(t), \Omega(t), \ldots, \Omega(t)) + k(k-1)\,\mathrm{d}V(t)\,, \tag{7}$$

where

$$\varphi(t) = \frac{\mathrm{d}\omega(t)}{\mathrm{d}t}\,,$$

$$V(t) = \int_0^1 t^{k-1} f(\varphi(t), \omega(t), \tilde{\Omega}(t), \ldots, \tilde{\Omega}(t))\, \mathrm{d}t,$$

with

$$\tilde{\Omega}(t) = \mathrm{d}\omega(t) + \tfrac{1}{2} t[\omega(t), \omega(t)].$$

Deduce from (7) *a condition under which the cohomology class of* $Tf(\omega(t))$ *does not depend on* t.
Show that $Tf(\omega)$ *is always closed if* $2k - 1 \geqq n$, n *dimension of* X, *and that its cohomology class is independent of* ω *if* $2k - 1 > n$.

Answer 3: It follows from (6) that if

$$f(\Omega(t), \ldots, \Omega(t)) = 0.$$

$Tf(\omega(t))$ defines a cohomology class on P. Formula (7) is deduced from (6) by a straightforward but lengthy computation (cf. Chern and Simons pp. 120–122). It says that if

$$f(\varphi(t), \Omega(t), \ldots, \Omega(t)) = 0 \tag{8}$$

then $\mathrm{d}Tf(\omega(t))/\mathrm{d}t$ reduces to an exact differential.
Hence if (6) and (8) are satisfied $[Tf(\omega(t))]$ is a cohomology class independent of t.
If $2k - 1 \geqq n$ the (horizontal (p. 373)) $2k$ form $f(\Omega)$ vanishes, therefore $Tf(\omega)$ is closed.
If $2k - 1 > n$ the $2k - 1$ form $f(\varphi(t), \Omega(t), \ldots, \Omega(t))$ is zero (it is horizontal because, unlike $\omega(t), \varphi(t)$ is horizontal); the cohomology class $[Tf(\omega(t)]$ is independent of t; it follows that $[Tf(\omega)]$ is independent of ω, indeed two arbitrary connections ω_0 and ω_1 can always be imbedded in a 1-parameter family [Problem V bis 1, Explicit proof]

$$\omega(t) = (1 - t)\omega_0 + t\omega_1,$$

and

$$[Tf(\omega(t))] = [Tf(\omega_0)] = [Tf(\omega_1)].$$

4) *We consider the case of riemannian connections on a manifold* X *of dimension* n.
a) *Show that if* f *is an* Ad O(n) *invariant polynomial of odd degree and* Ω *a riemannian curvature* 2*-form then*

$$f(\Omega) \equiv 0.$$

b) *Show that if* $\omega(t)$ *is a* 1*-parameter family of connections of conformal metrics one has*

$$f\left(\frac{\mathrm{d}\omega(t)}{\mathrm{d}t}, \Omega(t), \ldots, \Omega(t)\right) = 0$$

for any Ad $\mathcal{G}l(n, \mathbb{R})$ *invariant polynomial of even degree* $2s$. *Deduce that, if* $f(\Omega) = 0$, *the cohomology class of* $Tf(\omega)$ *is then a conformal invariant.*
c) *Show that if* f *is an* Ad $\mathcal{G}l(n, \mathbb{R})$ *invariant polynomial then* $f(\Omega)$ *itself is a conformal invariant, expressible in terms of the Weyl tensor.*

Answer 4a: The elements of the Lie algebra $\mathcal{O}(n)$ are the real, antisymmetric $n \times n$ matrices. Thus the curvature of a riemannian connexion satisfies the transposition law

$${}^{\mathrm{t}}\Omega = -\Omega\,;$$

therefore if f has degree k

$$f({}^{\mathrm{t}}\Omega) = (-1)^k f(\Omega)\,;$$

but since f is a symmetric multilinear mapping from $\mathcal{G}$ into $\mathbb{R}$ we have

$$f(V, \ldots, V) = {}^{\mathrm{t}}(f(V, \ldots, V)) = f({}^{\mathrm{t}}V, \ldots, {}^{\mathrm{t}}V)$$

thus

$$f(\Omega) = f({}^{\mathrm{t}}\Omega)$$

comparing the two relations gives $f(\Omega) = 0$ if k is odd.

Answer 4b: We consider the family of conformal metrics on X

$$g(t) = \exp(2a(t)g)\,,$$

where g is a given metric and a a scalar function on X, depending on the parameter t.
We make the computation in a local trivialization: if it is defined by a local section $s\colon X \supset U \to P$ we have, since the pull back is an homomorphism of the exterior algebra,

$$s^*f\left(\frac{\mathrm{d}\omega}{\mathrm{d}t}, \Omega(t), \ldots, \Omega(t)\right) = f\left(s^*\,\frac{\mathrm{d}\omega}{\mathrm{d}t}, s^*\Omega(t), \ldots\right).$$

Let A be the matrix valued 1-form representative of a riemannian connexion in a coordinate chart, given in terms of Christoffel symbols by

$$A^k_j = A_i{}^k{}_j\,\mathrm{d}x^i\,.$$

We have computed (p. 351) the difference of the representative $A(t)$ and A of the connections $\omega(t)$ and ω defined by $g(t)$ and g. It is given by the 3-tensor:

$$\Gamma_i{}^k{}_j(t) - \Gamma_i{}^k{}_j = \delta^k_i\,\partial_j a(t) + \delta^k_j\,\partial_i a(t) - g_{ij}g^{kl}\partial_l a(t)\,,$$

thus the derivative $\mathrm{d}A(t)/\mathrm{d}t$ is of the form

$$\bar{\varphi}(t) \equiv \frac{\mathrm{d}A(t)}{\mathrm{d}t} = I\,\mathrm{d}b + \alpha + \beta\,, \qquad b = \frac{\mathrm{d}a}{\mathrm{d}t}\,,$$

where I is the $n \times n$ unit matrix and α, β the $n \times n$ matrices of 1-forms with elements

$$\alpha_j^{\;k} = \partial_j \frac{\mathrm{d}a}{\mathrm{d}t}\, dx^k\,, \qquad \beta_j^{\;k} = -g^{kl} g_{ij} \partial_l \frac{\mathrm{d}a}{\mathrm{d}t}\, dx^i\,.$$

Using the multilinearity of f, and since $\mathrm{d}b$ is scalar valued, we have

$$\begin{aligned} f(\bar{\psi}(t), \bar{\Omega}(t), \ldots, \bar{\Omega}(t) &\equiv f(I\,\mathrm{d}b + \alpha + \beta, \bar{\Omega}(t), \ldots, \bar{\Omega}(t)) \\ &\equiv \mathrm{d}b\, f(I, \bar{\Omega}(t), \ldots, \bar{\Omega}(t)) + f(\alpha, \bar{\Omega}(t), \ldots, \bar{\Omega}(t)) \\ &\quad + f(\beta, \bar{\Omega}(t), \ldots, \bar{\Omega}(t))\,. \end{aligned}$$

The first term is zero because it is now a polynomial of odd degree in $\bar{\Omega}$. The two other terms can also be shown to be zero using the algebraic Bianchi identities (cf. Chern, p. 59).
This result together with the results of paragraph 3 imply that $[Tf(\omega(t))]$ is independent of t, i.e., is a conformal invariant.

Answer 4c: The Weyl tensor C is expressible in terms of the Riemann, Ricci and scalar curvature tensor (cf. p. 351); one can show that $f(C)$ and $f(\Omega)$ differ by an exact differential (cf. Avez).

5) *On a subset $U \subset X$ above which $\Pi^{-1}(U)$ is trivializable formula* (5) *gives, by choosing* $A_0 = 0$:

$$f(F) = kd \int_0^1 f(A, F_t, \ldots, F_t)\,\mathrm{d}t\,.$$

The $2k-1$ *form* $Q(A) \equiv \int_0^1 f(A, F_t, \ldots, F_t)\,\mathrm{d}t$ *is called a* ***local Chern–Simons form***. *It can be expressed in terms of A. Give this expression when f is a second order* Ad $Gl(n)$ *invariant polynomial.* local Chern–Simons form

Answer 5: We have $A_t = tA$,

$$F_t = tdA + \tfrac{1}{2}t^2[A, A] = tF + \tfrac{1}{2}(t^2 - t)[A, A]\,,$$

thus $f(A, F_t, \ldots, F_t)$ can be expressed as a polynomial in t with coefficients functions of A and F, or dA and $[A, A]$. If f has degree two then

$$f(A, F_t) = tf(A, dA) + \tfrac{1}{2}t^2 f(A, [A, A])$$

and the local Chern–Simons form is

$$Q(A) \equiv \int_0^1 f(A, F_t)\,\mathrm{d}t = \tfrac{1}{2} f(A, dA) + \tfrac{1}{6} f(A, [A, A])\,. \tag{9}$$

A second order Ad $GL(n)$ invariant polynomial on $\mathscr{G}l(n)$ is given by

$$f(X_1, X_2) = \mathrm{tr}(X_1, X_2)\,, \qquad X_1 \text{ and } X_2 \text{ being } n \times n \text{ matrices}\,.$$

If A is a $\mathscr{G}l(n)$ valued 1-form, (9) reads

$$Q(A) = \tfrac{1}{2}\{\mathrm{tr}(A \wedge dA) + \tfrac{1}{3}\,\mathrm{tr}(A \wedge [A, A])\}\,.$$

Chern–Simons form

6) *More generally, call* **Chern-Simons form** *a* $2k-1$ *form which is the projection on* X *of the horizontal form on* P, *given in terms of two connections* ω_1 *and* ω_0 *by*

$$Q^{(1)}(\omega_1, \omega_0) \equiv k \int_0^1 f(\omega_1 - \omega_0, \Omega_t, \ldots, \Omega_t)\,\mathrm{d}t\,.$$

Show that if $\omega_0, \omega_1, \omega_2$ *are three connections on* P *then* (*cf. Guo et al.*)

$$Q^{(1)}(\omega_1, \omega_2) + Q^{(1)}(\omega_2, \omega_0) + Q^{(1)}(\omega_0, \omega_1) = \mathrm{d}Q^{(2)}(\omega_2, \omega_1, \omega_0) \tag{10}$$

where $Q^{(2)}$ *is the* $2k-2$ *horizontal form*:

$$Q^{(2)}(\omega_2, \omega_1, \omega_0) = k(k-1) \int_{t_1+t_2 \leq 1} f(\phi_1, \phi_2, \Omega_{t_1t_2}, \ldots, \Omega_{t_1t_2})\,\mathrm{d}t_1\,\mathrm{d}t_2$$

with

$$\phi_1 = \omega_1 - \omega_0\,, \qquad \phi_2 = \omega_2 - \omega_0\,,$$

and $\Omega_{t_1t_2}$ *the curvature of the connection*

$$\omega_{t_1t_2} = \omega_0 + t_1(\omega_1 - \omega_0) + t_2(\omega_2 - \omega_0)\,.$$

For a generalization of this relation to forms $Q^{(2r)}$ of degree $2k-r$, depending on $r+1$ connections [cf. Guo et al.].

Answer 6: If

$$Q^{(2)} = k(k-1) \int_{t_1+t_2 \leq 1} f(\omega_1 - \omega_0, \omega_2 - \omega_0, \Omega_{t_1t_2}, \ldots, \Omega_{t_1t_2})\,\mathrm{d}t_1\,\mathrm{d}t_2\,,$$

we have by the properties of exterior derivation, since $\omega_1, \omega_0, \omega_2$ are 1-forms and $\Omega_{t_1t_2}$ a 2-form on which f depends in a symmetric way:

$$\mathrm{d}Q^{(2)} = k(k-1) \int_{t_1+t_2 \leqq 1} \{ f(\mathrm{d}(\omega_1 - \omega_0), \omega_2 - \omega_0, \Omega_{t_1t_2}, \ldots, \Omega_{t_1t_2})$$

$$- f(\omega_1 - \omega_0, \mathrm{d}(\omega_2 - \omega_0), \Omega_{t_1t_2}, \ldots, \Omega_{t_1t_2})$$

$$+ (k-2) f(\omega_1 - \omega_0, \omega_2 - \omega_0, \mathrm{d}\Omega_{t_1t_2}, \ldots, \Omega_{t_1t_2})\} .$$

From the expression of $\omega_{t_1t_2}$ and

$$\Omega_{t_1t_2} = \mathrm{d}\omega_{t_1t_2} + \tfrac{1}{2}[\omega_{t_1t_2}, \omega_{t_1t_2}] .$$

We deduce

$$\frac{\partial}{\partial t_i} \Omega_{t_1t_2} = \mathrm{d}(\omega_i - \omega_0) + [\omega_i - \omega_0, \omega_{t_1t_2}] , \qquad i = 1, 2 .$$

Using these expressions and the Bianchi identity

$$\mathrm{d}\Omega_{t_1t_2} = [\Omega_{t_1t_2}, \omega_{t_1t_2}] ,$$

we find that $\mathrm{d}Q^{(2)}$ can be written

$$\mathrm{d}Q^{(2)} = k \int_{t_1+t_2 \leqq 1} (k-1)$$

$$\times \left\{ f\left(\frac{\partial}{\partial t_1} \Omega_{t_1t_2} - [\omega_1 - \omega_0, \omega_{t_1t_2}], \omega_2 - \omega_0, \Omega_{t_1t_2}, \ldots \right) \right.$$

$$- f\left(\omega_1 - \omega_0, \frac{\partial}{\partial t_2} \Omega_{t_1t_2} - [\omega_2 - \omega_0, \omega_{t_1t_2}], \Omega_{t_1t_2}, \ldots \right)$$

$$\left. + (k-2) f\left(\omega_1 - \omega_0, \omega_2 - \omega_0, [\Omega_{t_1t_2}, \omega_{t_1t_2}], \Omega_{t_1t_2}, \ldots \right) \right\} \mathrm{d}t_1 \, \mathrm{d}t_2 .$$

Since f is multilinear we have

$$f\left(\frac{\partial}{\partial t_1} \Omega_{t_1t_2} - [\omega_1 - \omega_0, \omega_{t_1t_2}], \omega_2 - \omega_0, \Omega_{t_1t_2}, \ldots \right)$$

$$= f\left(\frac{\partial}{\partial t_1} \Omega_{t_1t_2}, \omega_2 - \omega_0, \Omega_{t_1t_2}, \ldots \right)$$

$$- f([\omega_1 - \omega_0, \omega_{t_1t_2}], \omega_2 - \omega_0, \Omega_{t_1t_2}, \ldots)$$

and an analogous expression for the second term.
On the other hand we know that f is Ad G invariant, thus:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} f(\mathrm{Ad}\, g(\tau) V_1, \ldots, \mathrm{Ad}\, g(\tau) V_k)|_{\tau=0} = 0$$

for any $\mathscr{G}$-valued $V_1, \ldots, V_k$ and $g(\tau)$ 1-parameter group in G and (cf. p. 167)

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \operatorname{Ad} g(\tau)V_i|_{\tau=0} = \left[\frac{\mathrm{d}}{\mathrm{d}\tau} g(\tau)|_e, V_i\right].$$

Hence, since $\omega_{t_1t_2}$ is $\mathscr{G}$-valued the sum of all terms under the integral in (11), reduces to

$$(k-1)\left\{f\left(\frac{\partial}{\partial t_1} \Omega_{t_1t_2}, \omega_2 - \omega_0, \Omega_{t_1t_2}, \ldots\right)\right.$$
$$\left. - f\left(\omega_1 - \omega_0, \frac{\partial}{\partial t_2} \Omega_{t_1t_2}, \Omega_{t_1t_2}, \ldots\right)\right\}.$$

We have therefore,

$$\mathrm{d}Q^{(2)} = k \int_{t_1+t_2\leq 1} \left\{\frac{\partial}{\partial t_1} f(\omega_2 - \omega_0, \Omega_{t_1t_2}, \ldots)\right.$$
$$\left. - \frac{\partial}{\partial t_2} f(\omega_1 - \omega_0, \Omega_{t_1t_2}, \ldots)\right\} \mathrm{d}t_1\, \mathrm{d}t_2$$
$$= k \int_0^1 \{f(\omega_2 - \omega_0, \Omega_{0t_2}, \ldots) - f(\omega_2 - \omega_0, \Omega_{(1-t_2)t_2}, \ldots)\}\, \mathrm{d}t_2$$
$$- k \int_0^1 \{f(\omega_1 - \omega_0, \Omega_{t_10}, \ldots) - f(\omega_1 - \omega_0, \Omega_{t_1(1-t_1)}, \ldots)\}\, \mathrm{d}t_1 ,$$

where we have

$$\Omega_{0t_2} = \Omega_{t_2}, \qquad \text{curvature of } \omega_0 + t_2(\omega_2 - \omega_0)$$
$$\Omega_{(1-t_2)t_2}, \qquad \text{curvature of } \omega_1 + t_2(\omega_2 - \omega_1)$$

and analogous formulas for Ω_{t_10} and $\Omega_{t_1(1-t_1)}$.
We use the linearity of f to write

$$f(\omega_2 - \omega_0, \Omega_{(1-t_2)t_2}, \ldots) = f(\omega_2 - \omega_1, \Omega_{(1-t_2)t_2}, \ldots)$$
$$+ f(\omega_1 - \omega_0, \Omega_{t_1(1-t_1)}, \ldots)$$

and then the obvious equality of the two integrals

$$\int_0^1 f(\omega_1 - \omega_0, \Omega_{t_1(1-t_1)}, \ldots)\, \mathrm{d}t_1 = \int_0^1 f(\omega_1 - \omega_0, \Omega_{(1-t_2)t_2}\, \mathrm{d}t_2$$

yields eq. (8).

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11. COCYCLES ON THE LIE ALGEBRA OF A GAUGE GROUP. ANOMALIES*

See [Problems IV 4, 5, 6 and 7] for definitions and examples.

Consider a trivial G-principal bundle over a manifold M. Let $\mathscr{A}$ be the space of gauge potentials (p. 364). An element $A \in \mathscr{A}$ is a one form on M with values in the Lie algebra $\mathscr{L}(G)$. A gauge transformation $g \in \mathscr{G}$ on $\mathscr{A}$ maps $\mathscr{A}$ into $\mathscr{A}$ by $A \mapsto A^g$ with (p. 365)

$$A^g(x) = g^{-1}(x)A(x)g(x) + g^{-1}(x)g'(x)\,, \qquad x \in M\,, \quad g\colon M \to G\,. \tag{1}$$

We shall investigate
1) the group $\mathscr{G}$ of the gauge transformations on $\mathscr{A}$, 2) its Lie algebra $\mathscr{L}(\mathscr{G})$, 3) representations of $\mathscr{G}$, 4) projective representations of $\mathscr{G}$, 5) representations of central extensions of $\mathscr{L}(\mathscr{G})$, 6) Lie algebra of $C^\infty(S^1, G)$, 7) Lie algebra of Diff S^1, 8) anomalies.

1. GROUP $\mathscr{G}$ OF GAUGE TRANSFORMATIONS

Show that the set $\mathscr{G}$ of gauge transformations is a group of right actions on $\mathscr{A}$.

The group $\mathscr{G}$ of gauge transformations is called the **gauge group**, not to be confused with the finite dimensional structure group G, also called the gauge group. gauge group

Answer 1: Applying consecutively the gauge transformations defined by g_1 and $g_2 \in \mathscr{G}$ to $A \in \mathscr{A}$,

$$(A^{g_1})^{g_2} = g_2^{-1}(g_1^{-1}Ag_1 + g_1^{-1}g_1')g_2 + g_2^{-1}g_2'$$

*Written in collaboration with I. Bakas.

is equivalent to applying the gauge transformation defined by $g_1 g_2$

$$\begin{aligned} A^{g_1 g_2} &= (g_1 g_2)^{-1} A g_1 g_2 + (g_1 g_2)^{-1} (g_1 g_2)' \\ A^{g_1 g_2} &= (A^{g_1})^{g_2} . \end{aligned} \tag{2}$$

A gauge transformation is a right action on $\mathscr{A}$. The set $\mathscr{G}$ of gauge transformations is a group. It can be identified with the group of vertical automorphisms (p. 405) of the corresponding G-principal fibre bundle which leaves the base space M invariant and commutes with the action of G on the bundle.

2. LIE ALGEBRA OF $\mathscr{G}$

Define the Lie algebra $\mathscr{L}(\mathscr{G})$; show that

$$\mathscr{L}(\mathscr{G}) = C^\infty(M, \mathscr{L}(G)) = \mathscr{L}(G) \otimes C^\infty(M, \mathbb{R}) , \tag{3}$$

where $C^\infty(M, \mathscr{L}(G))$ is the space of C^∞ mappings on M with values in $\mathscr{L}(G)$.

current algebra

The Lie algebra $\mathscr{L}(\mathscr{G})$ is often called in physics the **G-current algebra**.

Answer 2: The Lie algebra of $\mathscr{L}(\mathscr{G})$ is defined as the set of operators acting on $\mathscr{A}$ by

$$A \mapsto \left. \frac{\mathrm{d}A^{g(t)}}{\mathrm{d}t} \right|_{t=0} ,$$

where $t \mapsto g(t) \equiv g_t$ is a 1-parameter subgroup of the group $\mathscr{G}$ of gauge transformations. Since $g_0 = \mathbb{1}$, the derivative of the mapping $M \to g_0$ vanishes, and we deduce from (1)

$$\left. \frac{\mathrm{d}A^{g(t)}(x)}{\mathrm{d}t} \right|_{t=0} = [A(x), \gamma(x)] + \gamma'(x) ,$$

where $\gamma(x) = \mathrm{d}g_t(x)/\mathrm{d}t|_{t=0} \in \mathscr{L}(G)$, and γ' is a 1-form on M, with values in $\mathscr{L}(G)$, derivative of the mapping $M \to \mathscr{L}(G)$ by $x \mapsto \gamma(x)$.

We see that $\mathscr{L}(G)$ is in bijective correspondence with the mappings $M \to \mathscr{L}(G)$, $x \mapsto \gamma(x)$, i.e., elements of the space $C^\infty(M, \mathscr{L}(G))$, if the gauge group $\mathscr{G}$ is $C^\infty(M, G)$. The space $\mathscr{L}(\mathscr{G})$ is endowed with a Lie bracket by setting

$$[\gamma_1, \gamma_2](x) = [\gamma_1(x), \gamma_2(x)] , \qquad x \in M .$$

In particular, if $\{\theta_a\}$ is a chosen basis of $\mathscr{L}(G)$, f and h two arbitrary functions on M and if

$$\gamma_a = \theta_a \otimes f \qquad \text{and} \qquad \gamma_b = \theta_b \otimes h ,$$

then

$$[\gamma_a, \gamma_b](x) = f(x)h(x)[\theta_a, \theta_b],$$

i.e.,

$$[\gamma_a, \gamma_b] = f^c_{ab}\theta_c \otimes fh .$$

3. REPRESENTATION OF $\mathscr{G}$

Let Ψ be a vector space of functionals ψ on $\mathscr{A}$, that is of mappings,

$$\psi: \mathscr{A} \to \mathbb{C}, \qquad \psi \in \Psi .$$

Let U be the mapping from $\mathscr{G}$ into the space $L(\Psi, \Psi)$ of linear automorphisms of Ψ defined by

$$U(g)\psi(A) \equiv \psi^g(A) = \exp(\mathrm{i}\omega(A, g))\psi(A^g), \tag{4}$$

where A^g is given by (1) *and ω is a mapping* $\mathscr{A} \times \mathscr{G} \to S^1$, hence a cochain in the sense of [Problem IV 4, Cohomology* §2] *where X, G, K correspond respectively to $\mathscr{A}$, $\mathscr{G}$, S^1. Show that U is a homomorphism*

$$U(g_2)U(g_1) = U(g_2 g_1) \tag{5}$$

if and only if ω satisfies the cocycle condition [Problem IV 4, *Cohomology, eq.* (8)]

$\mathrm{d}\omega = 0$, *where* d *is the coboundary operator on cochains on* $\mathscr{G}$.

Answer 3: We have

$$\begin{aligned} U(g_2)U(g_1)\psi(A) &= U(g_2)\exp(\mathrm{i}\omega(A, g_1))\psi(A^{g_1}) \\ &= \exp(\mathrm{i}\omega(A^{g_2}, g_1) + \mathrm{i}\omega(A, g_2))\psi(A^{g_2 g_1}) \\ &= \exp(\mathrm{i}\omega(A^{g_2}, g_1) + \mathrm{i}\omega(A, g_2) \\ &\quad - \mathrm{i}\omega(A, g_2 g_1))U(g_2 g_1)\psi(A) . \end{aligned}$$

Eq. (5) is satisfied if

$$\omega(A^{g_2}, g_1) - \omega(A, g_2 g_1) + \omega(A, g_2) = 0 \bmod 2\pi . \tag{6a}$$

The phase ω is a mapping

$$\omega: \mathscr{A} \times \mathscr{G} \to S^1,$$

where $\mathscr{G}$ acts on $\mathscr{A}$. Eq. (6a) is the additive version of the cocycle

*In general physical situations ω takes its values in the space of local functionals on $\mathscr{A}$. [See, for instance, Dubois Violette.]

condition derived in [Problem IV 4, Cohomology, eq. (8)]. We rewrite (6a) as follows:

$$\mathrm{d}\omega = 0 . \tag{6b}$$

Remark: We refer the reader to [Jackiw] for an example in quantum mechanics of representations U satisfying (4) and (5) – namely a representation of Galilean boosts.

4. PROJECTIVE REPRESENTATIONS OF $\mathscr{G}$

projective (ray) representation

A mapping U of $\mathscr{G}$ into $L(\Psi, \Psi)$ is called a **projective (ray) representation,** *if it is associative*

$$U(g_3)(U(g_2)U(g_1)) = (U(g_3)U(g_2))U(g_1) \tag{7}$$

and if

$$U(g_2)U(g_1) = \exp(\mathrm{i}\alpha(A, g_1, g_2))U(g_2g_1) , \tag{8}$$

phase of representation

where $\alpha\colon \mathscr{A} \times \mathscr{G} \times \mathscr{G} \to S^1$ is called the **phase of the representation.**

a) *Show that, if U satisfies* (8), *it is associative if and only if α is a cocycle.*

b) *Show that if the mapping U is given by* (4) *it is always associative: U defined by* (4) *is always a projective representation, with zero phase if ω is a cocycle.*

c) *Show that if U is a projective representation and if $\alpha = d\beta$, one can find a representation $\bar{U} = \mathrm{e}^{\mathrm{i}\beta(A\cdot g)}U$ which satisfies* (5).

Answer 4a: We shall compute the action of $U(g_3)(U(g_2)U(g_1))$ on Ψ and the action of $(U(g_3)U(g_2))U(g_1)$ on Ψ and equate them:

$$\begin{aligned}
U(g_3)(U(g_2)U(g_1))\psi(A) &= U(g_3)\exp(\mathrm{i}\alpha(A, g_1, g_2))U(g_1g_2)\psi(A) \\
&= \exp(\mathrm{i}\alpha(A^{g_3}, g_1, g_2) + \mathrm{i}\alpha(A, g_1g_2, g_3)) \\
&\quad\times U(g_3g_2g_1)\psi(A) , \\
(U(g_3)U(g_2))U(g_1)\psi(A) &= \exp(\mathrm{i}\alpha(A, g_2, g_3))U(g_3g_2)U(g_1)\psi(A) \\
&= \exp(\mathrm{i}\alpha(A, g_2, g_3)) + \mathrm{i}\alpha(A, g_1, g_2g_3)) \\
&\quad\times U(g_3g_2g_1)\psi(A) .
\end{aligned}$$

The associativity holds if and only if

$$\begin{aligned}
&\alpha(A^{g_3}, g_1, g_2) - \alpha(A, g_2, g_3) + \alpha(A, g_2g_1, g_3) - \alpha(A, g_1, g_3g_2) \\
&= 0 \bmod 2\pi .
\end{aligned} \tag{9a}$$

A calculation similar to the one leading to (8) in [Problem IV 4, Cohomology] shows that (9a) can be called a 2-cocycle condition for the mapping

$$\alpha: \mathscr{A} \times \mathscr{G} \times \mathscr{G} \rightarrow S^1 \qquad \text{with } \mathscr{G} \text{ acting on } \mathscr{A}$$

and we rewrite (9a) as follows:

$$\mathrm{d}\alpha = 0 . \tag{9b}$$

Remark: See [Problem IV 6, Short exact] for an example of a projective representation (4) when α is not a trivial cocycle – namely the quantum mechanical representation of translations in phase space.

Answer 4b: We have shown in Answer 3) that if U acts on ψ by (4) then

$$U(g_2)U(g_1) = \exp(\mathrm{i}\,\mathrm{d}\omega)U(g_2 g_1) .$$

U satisfies the relation (8) with a phase $\alpha = \mathrm{d}\omega$; the cocycle condition $\mathrm{d}\alpha = 0$ is always satisfied and U is associative.

Remark: An action of $U(g)$ on $\psi(A)$ more general than (4),

$$U(g)\psi(A) = \mathscr{O}(A, g)\psi(A^g)$$

with $|\mathscr{O}(A, g)|$ not necessarily equal to 1, can define an associative projective representation (8) with a nontrivial cocycle. For properties of $\mathscr{O}$ and for applications to Physics see for instance [Faddeev, Zumino].

Answer 4c: Given a projective representation U the problem is to find $\beta(A, g)$ such that

$$\bar{U} = \exp(\mathrm{i}\beta)U \qquad \text{with } \bar{U}(g_2)\bar{U}(g_1) = \bar{U}(g_2 g_1) ,$$

i.e.,

$$\begin{aligned} &\exp(\mathrm{i}\beta(A, g_2))U(g_2)\exp(\mathrm{i}\beta(A, g_1))U(g_2)\psi(A) \\ &\quad = \exp(\mathrm{i}\beta(A, g_2 g_1)U(g_2 g_1)\psi(A) \end{aligned} \tag{10}$$

with $U(g)$ satisfying (8) and (7). Rewriting (10) and, using (8) gives

$$\beta(A, g_2) + \beta(A^{g_2}, g_1) + \alpha(A, g_1, g_2) - \beta(A, g_2 g_1) = 0 . \tag{11}$$

This equation says that the cocycle α is a coboundary, $\alpha = \mathrm{d}\beta$. Hence if $\alpha = \mathrm{d}\beta$ one can find a representation $\tilde{U}$ satisfying (5).

Remark: We can weaken the associative requirement and replace it by

$$(U(g_1)U(g_2))U(g_3) = \exp(\mathrm{i}\beta(A, g_1, g_2, g_3))U(g_1)(U(g_2)U(g_3)) ,$$

where β is obtained by requiring associativity of four-fold products. We refer the reader to Jackiw for a study and example of representations of this type.

5. CENTRAL EXTENSIONS OF $\mathscr{L}(\mathscr{G})$

Let U be a smooth mapping from $\mathscr{G} = c^\infty(M, \boldsymbol{G})$ into the space of linear maps $L(\Psi, \Psi) \equiv L$; let*

$$\phi \in T_e\mathscr{G} \simeq \mathscr{L}(\mathscr{G}) .$$

We denote by $\hat{\phi} = U'(e)\ \phi$ the element of L image of ϕ by the linear map $U'(e)$: $T_e\mathscr{G} \to L$, and by $[\hat{\phi}_1, \hat{\phi}_2]$ the usual bracket of linear maps, i.e.,

$$[\hat{\phi}_1, \hat{\phi}_2] = \hat{\phi}_1\hat{\phi}_2 - \hat{\phi}_2\hat{\phi}_1 . \tag{12}$$

a) *Show that if U satisfies* (8) *and* $U(e) = 1$ (*e unit of $\mathscr{G}$*, 1 *unit of L, i.e., the identity map*) *then*:

$$[\hat{\phi}_1, \hat{\phi}_2] = \widehat{[\phi_1, \phi_2]} + C(A, \phi_1, \phi_2)\mathbb{1} , \tag{13}$$

where $[\phi_1, \phi_2]$ is the element of $T_e\mathscr{G}$ corresponding to the Lie bracket of the elements of $\mathscr{L}(\mathscr{G})$ defined by ϕ_1 and ϕ_2, $\widehat{[\phi_1, \phi_2]}$ its image by $U'(e)$ and where C is a numerical valued 2*-cochain on $\mathscr{L}(\mathscr{G})$. (To simplify the proof, take the group G to be a linear group.)*

b) *Show that if U is associative C satisfies the cocycle condition [Problem IV* 7, *Cohomology].*

Answer 5a: We denote by $\exp t\phi$ the 1-parameter subgroup of $\mathscr{G}$ whose tangent at unity is ϕ. This subgroup exists when $\mathscr{G} = C^\infty(M, \boldsymbol{G})$ and is given by

$$(\exp t\phi)(x) = \exp t\phi(x) \in \boldsymbol{G} , \qquad x \in M , \qquad \phi(x) \in \mathscr{L}(\boldsymbol{G})$$

We recall that, for an ordinary Lie group $\boldsymbol{G}$, we have the Campbell Hausdorff expansion formula

$$\exp tu_1 \exp su_2 = \exp(tu_1 + su_2 + \tfrac{1}{2}st[u_1, u_2] + \cdots) \qquad u_1, u_2 \in \mathscr{L}(\boldsymbol{G}) . \tag{14}$$

$$(\exp t\phi_1 \exp s\phi_2)(x) = \exp(t\phi_1(x)) \exp(s\phi_2(x)) \tag{15}$$

which gives, by the Campbell-Hausdorff expansion formula for small s

*In the sense of Gateaux derivative [Problem II 1, Supersmooth], $\mathscr{G}$ being a manifold modelled on the locally convex vector space $\mathscr{L}(\mathscr{G})$.

and t (for a linear Lie group G the exponentials on the right are usual exponentials of matrices).

$$(\exp t\phi_1 \exp s\phi_2)(x) = \exp(t\phi_1(x) + s\phi_2(x) + \tfrac{1}{2}st[\phi_1(x), \phi_2(x)] + \cdots)\,;$$

in other words

$$(\exp t\phi_1 \exp s\phi_2) = \exp(t\phi_1 + s\phi_2 + \tfrac{1}{2}st[\phi_1, \phi_2] + \cdots)\,. \tag{16}$$

We now define $\exp t\hat{\phi}$ to be the image by U of the 1-parameter group $\exp t\phi$, i.e.,

$$\exp t\hat{\phi} = U(\exp t\phi)\,. \tag{17}$$

We define

$$\hat{\phi} = \frac{\mathrm{d}}{\mathrm{d}t}\, U \exp t\phi|_{t=0}\,. \tag{18}$$

We have, then, by derivation of (17) that $\hat{\phi}$ is deduced from ϕ by the linear map, $U'(e)$:

$$\hat{\phi} = U'(e)\phi\,.$$

By the hypothesis (8) on U we have

$$\exp t\hat{\phi}_1 \exp s\hat{\phi}_2 = \exp \mathrm{i}\alpha(A, \exp t\phi_1, \exp s\phi_2)\, U(\exp t\phi_1 \exp s\phi_2)\,. \tag{19}$$

Using (16), eq. (19) reads

$$\exp t\hat{\phi}_1 \exp s\hat{\phi}_2$$
$$= \exp \mathrm{i}\alpha(A, \exp t\phi_1, \exp s\phi_2)\, U(\exp(s\phi_2 + t\phi_1 + \tfrac{1}{2}st[\phi_2, \phi_2] + \cdots)) \tag{20}$$

and we have immediately, using (18)

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}s}(\exp t\hat{\phi}_1 \exp s\hat{\phi}_2)\right]_{t=s=0} = \hat{\phi}_1\hat{\phi}_2 \text{ (product of linear maps)}\,.$$

Therefore

$$[\hat{\phi}_1, \hat{\phi}_2] = \frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}s}(\exp t\hat{\phi}_1 \exp s\hat{\phi}_2 - \exp s\hat{\phi}_2 \exp t\hat{\phi}_1)|_{t=s=0}\,.$$

Computing the right-hand side of (20) we find an expression of the form

$$[\hat{\phi}_1, \hat{\phi}_2] = \widehat{[\phi_1, \phi_2]} + C(A, \phi_1, \phi_2)| \tag{21}$$

where

$$\widehat{[\phi_1, \phi_2]} = U'(e)[\phi_1, \phi_2]$$

and $C(A, \phi_1, \phi_2)$ is a bilinear and antisymmetric mapping from $\mathscr{L}(\mathscr{G})$ into L depending (20) on $\partial_{g_1}\alpha$, $\partial_{g_2}\alpha$ and also $U'(e)$, and $U''(e)$ if U is not linear.

Answer 5b: The brackets $[\phi_1, \phi_2]$ and $[\hat{\phi}_1, \hat{\phi}_2]$ both satisfy the Jacobi identity since they are brackets in the Lie algebra $\mathscr{L}(\mathscr{G})$ and in the Lie algebra of linear maps, respectively. One shows that the image $\widehat{[\phi_1, \phi_2]} = U'(e)[\phi_1, \phi_2]$ satisfies the Jacobi identity if and only if U is associative by generalizing classical methods [cf. for instance Pontryagin].

$$[\widehat{[\phi_1, \phi_2]}, \hat{\phi}_3] + [\widehat{[\phi_3, \phi_1]}, \hat{\phi}_2] + [\widehat{[\phi_2, \phi_3]}, \hat{\phi}_1] = 0 .$$

Using these results and the definition of C we obtain

$$C(A, [\phi_1, \phi_2], \phi_3) + C(A, [\phi_3, \phi_1], \phi_2) + C(A, [\phi_2, \phi_3], \phi_1) = 0$$

which expresses [Problem IV 7, Cohomology] that C is a cocycle on $\mathscr{L}(G)$:

$$\mathrm{d}C = 0 .$$

6. LIE ALGEBRA OF $C^\infty(S^1, G)$

Cocycles for the Lie algebra of the space $C^\infty(S^1, G)$ of smooth maps from the circle S^1 into a finite dimensional Lie group G, that is of C^∞ mappings from $\mathbb{R}$ into G, $\varphi \mapsto g(\varphi)$ which are periodic of period 2π. The group structure of G defines a group structure on $C^\infty(S^1, G)$ by pointwise multiplication

$$(g_1 g_2)(\varphi) := g_1(\varphi) g_2(\varphi) . \tag{22}$$

a) *Construct a basis for the Lie algebra $\mathscr{L}(C^\infty(S^1, G))$.*
b) *Identify the space of constant maps in $C^\infty(S^1, G)$.*
c) *Consider a projective representation of $\mathscr{L}(C^\infty(S^1, G))$ on a Hilbert space $\mathscr{H}$. Derive the general expression for the complex valued 2-cocycles on $\mathscr{L}(C^\infty(S^1, G))$, and compute the dimensionality of its second cohomology group $H^2(\mathscr{L}(C^\infty(S^1, \mathbb{G})); \mathbb{C})$ when G is simple.*

Answer 6a: The Lie algebra of the group $C^\infty(S^1, G)$ is the space $C^\infty(S^1, \mathscr{L}(G))$. A basis for this infinite dimensional vector space, endowed with the Fréchet topology of uniform convergence of maps and each of their derivatives is the denumerable set (we suppress the notation $\otimes$, since the meaning is clear)

$$\{\theta_a \cos n\varphi, \theta_a \sin n\varphi; \qquad n \in \mathbb{N}, \{\theta_a\} \text{ a basis of } \mathscr{L}(G)\} .$$

We shall use for simplicity in writing (expansions will then have complex conjugate coefficients)

$$G_a^n = \theta_a z^n \,, \qquad \text{with } z = e^{i\varphi},\ n \in \mathbb{Z}\,.$$

The general formula of section 1 reads now

$$[G_a^n, G_b^m] = [\theta_a, \theta_b] z^{m+n}\,.$$

Thus, if f^c_{ab} denotes the structure constants of $\boldsymbol{G}$,

$$\begin{aligned} [G_a^n, G_b^m] &= z^{n+m} f^c_{ab}\theta_c\,, \\ [G_a^n, G_b^m] &= f^c_{ab} G_c^{m+n}\,. \end{aligned} \tag{23}$$

A Lie algebra with Lie bracket (23) is called an **(untwisted) affine Kac–Moody algebra**. For applications of the Kac–Moody algebra to physics see for instance [Dolan, Goddard et al.]. Kac–Moody algebra

Answer 6b: A constant map in $\mathscr{G} = C^\infty(S^1, \boldsymbol{G})$ is a map

$$\varphi \mapsto c\,,$$

c a fixed element of $\boldsymbol{G}$. The space of constant maps in $C^\infty(S^1, \boldsymbol{G})$ is isomorphic to $\boldsymbol{G}$. The quotient space $C^\infty(S^1, \boldsymbol{G})/\boldsymbol{G}$ is called the **loop group** of $\boldsymbol{G}$, and is usually denoted loop group

$$\Omega \boldsymbol{G} \approx C^\infty(S^1, \boldsymbol{G})/\boldsymbol{G}\,.$$

Answer 6c: Let $\hat{G}_a^n$ be the representative of G_a^n in a projective representation of $\mathscr{L}(\mathscr{G})$ on a Hilbert space $\mathscr{H}$: according to Answer 5) and formula (23) we have

$$[\hat{G}_a^m, \hat{G}_b^n] = f^c_{ab}\hat{G}_c^{m+n} + C(G_a^m, G_b^n)\mathbb{1}\,, \tag{24}$$

where C is a cocycle on $\mathscr{L}(\mathscr{G})$, i.e., a bilinear antisymmetric map from $\mathscr{L}(\mathscr{G}) \times \mathscr{L}(\mathscr{G})$ into $\mathbb{R}$, which satisfies the equation

$$C([G_a^m, G_b^n], G_c^k) + C([G_b^n, G_c^k], G_a^m) + C([G_c^k, G_a^m], G_b^n) = 0\,, \tag{25}$$

i.e.,

$$f^d_{ab} C(G_d^{m+n}, G_c^k) + f^d_{bc} C(G_d^{n+k}, G_a^m) + f^d_{ca} C(G_d^{k+m}, G_b^n) = 0\,. \tag{26}$$

It is easy to show that when $\boldsymbol{G}$ is an orthogonal group* (then $f^d_{ab} = f^a_{bd} = f^b_{da}$)

$$C(G_a^l, G_b^{-l}) = \lambda l \delta_{ab} \qquad \text{for } \lambda \in \mathbb{C}\,, \quad l \in \mathbb{Z} \tag{27}$$

is a solution of (26) when $m + n + k = 0$.

*For other cases see [Goddard and Olive, paragraph 1.3].

Eq. (27) suggests that, under the same hypothesis on f^c_{ab},

$$C(G^k_a, G^m_b) = \lambda k \delta_{a,b} \delta_{m,-k} , \qquad \lambda \in \mathbb{C} , \quad k, m \in \mathbb{Z} \tag{28}$$

satisfies (26), and it is straightforward to check that, indeed, it is so.

One can check that the cocycles (28) on $\mathscr{L}(C^\infty(S^1, G))$ are nontrivial if $\lambda \neq 0$ (i.e., C is not a coboundary) by constructing another representation

$$\hat{\hat{G}}^n_a = \hat{G}^n_a + K^n_a \mathbb{1} .$$

One finds that there is no choice of K^n_a for all n which will make $[\hat{\hat{G}}^n_a, \hat{\hat{G}}^m_b] - f^c_{ab} \hat{\hat{G}}^{m+n}_c$ vanish. Hence eq. (28) provides the set of all nontrivial cocycles for $\mathscr{L}(C^\infty(S^1, G))$ and $H^2(\mathscr{L}(C^\infty(S^1, G)))$ is a space with 1 complex dimension.

Remark: All cocycles on the Lie algebra of semi-simple Lie groups are trivial [see for instance Guillemin et al., or Jacobson].

7. LIE ALGEBRA OF DIFF S^1

Let Diff S^1, *be the group of diffeomorphisms of the circle, where the group structure is defined by map composition.*

a) *Derive the Lie bracket of the Lie algebra* $\mathscr{L}(\text{Diff } S^1)$ (*not a current algebra*).

b) *Consider a projective representation* (6) *of* $\mathscr{L}(\text{Diff } S^1)$ *on a Hilbert space* $\mathscr{H}$. *Spell out the relation* (13).

c) *Derive the general expression for the complex valued* 2-*cocycles on* $\mathscr{L}(\text{Diff } S^1)$. *Show that the second cohomology group* $H^2(\mathscr{L}(\text{Diff } S^1); \mathbb{C})$ *is isomorphic to* $\mathbb{C}$.

d) *Find subalgebras of* $\mathscr{L}(\text{Diff } S^1)$ *for which the cocycles vanish.*

Remark: These two examples are instructive in many ways, but they are not generic: their cocycles take their values in $\mathbb{C}$; in general cocycles for an infinite dimensional group $\mathscr{G}$ take their values in a space of functionals. See examples for instance in [Bonora et al.] and [Dubois-Violette[1,2]].

Answer 7a: The group Diff S^1 is the set of smooth diffeomorphisms $\phi: S^1 \to S^1$ with the group law defined by map composition, $\phi_1, \phi_2 \in$ Diff S^1 implies $\phi_1 \circ \phi_2 \in$ Diff S^1. The Lie algebra $\mathscr{L}(\text{Diff } S^1)$ is the space of smooth vector fields on S^1, identified with the set of derivations of smooth functions $\varphi \mapsto f(\varphi)$ on S^1. The Lie bracket of two smooth vector fields is

$$\left[f_1(\varphi) \frac{\mathrm{d}}{\mathrm{d}\varphi} , f_2(\varphi) \frac{\mathrm{d}}{\mathrm{d}\varphi} \right] = (f_1(\varphi) f_2'(\varphi) - f_1'(\varphi) f_2(\varphi)) \frac{\mathrm{d}}{\mathrm{d}\varphi} . \tag{29}$$

Using the previous notation $z = \mathrm{e}^{\mathrm{i}\varphi}$ and $\mathrm{d}/\mathrm{d}\varphi = \mathrm{i}z\,\mathrm{d}/\mathrm{d}z$, we consider the set of vector fields on S^1

$$L_i(z) = z^{-i+1}\frac{\mathrm{d}}{\mathrm{d}z}\,, \quad i \in \mathbb{Z}\,.$$

It is a basis for $\mathscr{L}(\mathrm{Diff}\,S^1)$ and its Lie algebra is

$$[L_i, L_j] = (i-j)L_{i+j}\,. \tag{30}$$

Answer 7b: Let $\hat{L}_i$ be the representative of L_i in a projective representation of $\mathscr{L}(\mathrm{Diff}\,S^1)$ on $\mathscr{H}$: it satisfies

$$[\hat{L}_i, \hat{L}_j] = (i-j)\hat{L}_{i+j} + C(L_i, L_j)\mathbb{1}\,, \qquad \hat{L}_i \in L(\mathscr{H}, \mathscr{H})\,, \tag{31a}$$

where C is a 2-cocycle on the Lie algebra $\mathscr{L}(\mathrm{Diff}\,S^1)$, i.e., a bilinear, antisymmetric function of its argument satisfying the cocycle condition:

$$C([L_i, L_j], L_k) + C([L_j, L_k], L_i) + C([L_k, L_i], L_j) = 0\,. \tag{32}$$

Answer 7c: Complex-valued 2-cocycle on $\mathscr{L}(\mathrm{Diff}\,S^1)$ are obtained by solving eq. (32) algebraically. First set $k = -i - j$; then (32) together with (30) gives

$$(i-j)C(L_{i+j}, L_{-i-j}) + (2j+i)C(L_{-i}, L_i) - (2i+j)C(L_{-j}, L_j) = 0\,. \tag{33}$$

It is easy to show by induction that the solution of (33) is

$$C(L_k, L_{-k}) = \lambda k^3 + \mu k\,, \qquad \lambda, \mu \in \mathbb{C}\,, \quad k \in \mathbb{Z}$$

and the solution of (32) is

$$C(L_k, L_j) = (\lambda k^3 + \mu k)\delta_{j,-k}\,, \qquad \lambda, \mu \in \mathbb{C}\,. \tag{34}$$

Remark: Using (34), eq. (31a) reads

$$[\hat{L}_k, \hat{L}_j] = (k-j)\hat{L}_{k+j} + (\lambda k^3 + \mu k)\delta_{j,-k}\,. \tag{31b}$$

This algebra is a central extension [see Problem IV 6, Short exact] of $\mathscr{L}(\mathrm{Diff}\,S^1)$.

To identify the second cohomology group $H^2(\mathscr{L}(\mathrm{Diff}\,S^1); \mathbb{C})$ we need to identify which of the cocycles (34) are trivial. We know from Answer 4c) that if the cocycle in (31b) is trivial, then there exists a representation $\hat{\hat{L}}_i$ such that

$$[\hat{\hat{L}}_k, \hat{\hat{L}}_j] = (k-j)\hat{\hat{L}}_{k+j}\,. \tag{35}$$

Set

$$\hat{\hat{L}}_k := \hat{L}_k + \nu\delta_{k,0}\,, \qquad \nu \in \mathbb{C} \tag{36}$$

and determine ν so that $\{\hat{\hat{L}}_k\}$ satisfies (35). An easy calculation gives

$$[\hat{\hat{L}}_k, \hat{\hat{L}}_j] = (k-j)\hat{\hat{L}}_{k+j} + (\lambda k^3 + (\mu - 2\nu)k)\delta_{j,-k} . \tag{37}$$

Eq. (37) says that if $\lambda = 0$ one can find a representation $\{\hat{\hat{L}}_k\}$ satisfying (35) by choosing $\nu = \mu/2$.

The trivial cocycles are	$C(L_k, L_j) = \mu k \delta_{j,-k}$,	$\mu \in \mathbb{C}$.
The nontrivial cocycles are	$C(L_k, L_j) = (\lambda k^3 + \mu k)\delta_{j,-k}$,	$\lambda, \mu \in \mathbb{C}$.

The second cohomology group is

$$H^2(\mathscr{L}(\mathrm{Diff}\, S^1); \mathbb{C}) \approx \mathbb{C} .$$

Answer 7d: Consider the infinite dimensional subalgebras W_0^+ and W_0^- of $\mathscr{L}(\mathrm{Diff}\, S^1)$,

$$W_0^+ = \{L_k; k \geq 0\} ,$$
$$W_0^- = \{L_k; k \leq 0\} .$$

It follows readily from $[L_i, L_j] = (i-j)L_{i+j}$ that W_0^+ and W_0^- are Lie algebras. The cocycles

$$C(L_k, L_j) = (\lambda k^3 + \mu k)\delta_{j,-k}$$

vanish if k and j have the same sign.

The finite dimensional subalgebra generated by L_{-1}, L_0, L_1 is also a Lie algebra with vanishing cocycles:

$$C(L_0, L_{-1}) = 0 , \qquad C(L_0, L_1) = 0 ;$$
$$C(L_1, L_{-1}) = \lambda + \mu \quad \text{is a trivial cocycle} .$$

If in eq. (31b) we set $\lambda + \mu = 0$, i.e., if we work with a representation where the cocycles on the subalgebra $\{L_{-1}, L_0, L_1\}$ vanish, then (31b) reads

$$[\hat{L}_k, \hat{L}_j] = (k-j)\hat{L}_{k+j} + \lambda(k^3 - k)\delta_{j,-k} . \tag{38}$$

Virasoro algebra

This equation defines a **Virasoro algebra**. For applications of Virasoro algebras in physics see for instance [Green et al.].

8. ANOMALIES

Quantum states are equivalent classes; they are defined modulo a phase. The relevant representations are projective. If the cocycle α which characterizes a representation (8) is not a coboundary some symmetries of the classical system are lost by using such a representation. One calls

gauge, chiral, etc. **anomaly** a gauge, chiral, etc. symmetry lost by quantization. (gauge, chiral) anomaly

For example: If the lagrangian or the hamiltonian of a classical system is invariant under gauge transformations of its fields, then the fields ϕ and their conjugate momenta π satisfy a number of constraint equations

$$\boldsymbol{\phi}_a(\phi(x), \pi(x)) = 0 . \tag{39}$$

For a Yang–Mills system, for instance, the Poisson brackets of the (first class) constraints satisfy

$$\{\boldsymbol{\phi}_a(\phi(x), \pi(x)), \boldsymbol{\phi}_b(\phi(y), \pi(y))\} = \delta(x, y) f^c_{ab} \boldsymbol{\phi}_c(\phi(x), \pi(x)) , \tag{40}$$

where f^c_{ab} are the structure constants of the structure group $\mathbf{G}$. The constraints, together with the canonical Poisson brackets, generate a Lie algebra isomorphic to the Lie algebra $\mathscr{L}(\mathscr{G})$ of the gauge group. We refer the reader to [Hanson et al., Sundermeyer, Henneaux] for a study of constrained systems.

The projective representation of $\mathscr{L}(\mathscr{G})$ may introduce nontrivial cocycles. The quantum states in this representation do not satisfy the constraints. Indeed, assume that the projective representation of $\mathscr{L}(\mathscr{G})$ defined by (40) introduce a nontrivial cocycle, then we have

$$[\boldsymbol{\phi}_a(\hat{\phi}(x), \hat{\pi}(x)), \hat{\boldsymbol{\phi}}_b(\hat{\phi}(y), \hat{\pi}(y))] = \delta(x, y) f^c_{ab} \boldsymbol{\phi}_c(\hat{\phi}(x), \hat{\pi}(x)) + C_{ab}(x) . \tag{41}$$

If one requires the physical states to satisfy

$$\boldsymbol{\phi}_a|\text{physical}\rangle = 0 \qquad \text{for all } \boldsymbol{\phi}_a \tag{42}$$

then

$$[\boldsymbol{\phi}_a, \boldsymbol{\phi}_b]|\text{physical}\rangle = 0 ;$$

but this last equation cannot be satisfied if the nontrivial cocycle $C_{ab} \neq 0$.

The study of anomalies in canonical quantization is the study of the 2-cocycles C_{ab}, when the base space M is odd dimensional, say $2n - 1$. The corresponding anomalies in covariant quantization have been related to the 1-cocycle ω in eq. (4), the base M being $2n$ dimensional. For a mathematical description of the occurrence of anomalies in quantum field theory see for instance [Dubois-Violette and references therein].

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12. VIRASORO REPRESENTATION OF $\mathscr{L}(\mathrm{Diff}\, S^1)$. GHOSTS. BRST OPERATOR*

A promising approach to quantum gauge field theory is the construction of a Lagrangian which includes the so-called "ghost" and "conjugate ghost" fields in addition to the quantized classical fields. A transformation which applies to the set of classical fields, ghost and conjugate ghost fields has been introduced independently by Becchi–Rouet–Stora and Tyutin. Field theorists are interested in BRST invariant effective hamiltonian and lagrangian. In this problem we identify the ghost and conjugate ghost field in the simple case of $\mathscr{L}(\mathrm{Diff}\, S^1)$ [see Problem V bis 11, Cocycles].

1. FINITE DIMENSIONAL LIE ALGEBRA

Let $\mathscr{L}(\mathbf{G})$ be a finite dimensional Lie algebra, represented by linear operators on a vector space X. A k-cochain f, element of $C^k(\mathscr{L}(\mathbf{G}), X) = C^k$ is an antisymmetric k-linear map on $\mathscr{L}(\mathbf{G})$ with values in X. The space of cochains on $\mathscr{L}(\mathbf{G})$ with values in X is the direct sum of the spaces of k-cochains:

$$C = \bigoplus_{k=0}^{n} C^k\,, \qquad n \equiv \dim \mathbf{G}\,.$$

We introduce on C the operators $i(\theta)$ and $\epsilon(\theta^*)$ defined as follows**:

$$i(\theta)\colon C^k \to C^{k-1}\,, \qquad \theta \in \mathscr{L}(\mathbf{G})$$

by

$$(i(\theta)f)(\theta_1, \ldots, \theta_{k-1}) := f(\theta, \theta_1, \ldots, \theta_{k-1})\,, \qquad f \in C^k\,; \tag{1}$$

and

$$\epsilon(\theta^*)\colon C^k \to C^{k+1}$$

by

$$\begin{aligned}(\epsilon(\theta^*)f)&(\theta_1, \ldots, \theta_{k+1})\\ &= \sum_{l=1}^{k-1} (-1)^{l+1} \langle \theta^*, \theta_l \rangle f(\theta_1, \ldots, \hat{\theta}_l, \ldots, \theta_{k+1})\,,\\ &\theta_1, \ldots, \theta_{k+1} \epsilon \mathscr{L}(\mathbf{G})\,,\end{aligned} \tag{2}$$

where $\theta^* \epsilon \mathscr{L}^*(\mathbf{G})$, dual of $\mathscr{L}(\mathbf{G})$, and $\langle \theta^*, \theta_l \rangle \in \mathbb{R}$ is defined by the duality.

*Written in collaboration with I. Bakas.

**The signs are chosen so that (3), (4) and (5) hold. Note that the sign is not the same if the basis is labelled $\{\theta_0, \ldots, \theta_{n-1}\}$ or $\{\theta_1, \ldots, \theta_n\}$. E.g., (4) is the same as (3) in [Problem IV 7, Cohomology].

Denote by $\{\theta_a\}$, $a = 1, \ldots, n$ a basis of $\mathscr{L}(\mathrm{G})$ and by $\{\theta^{*a}\}$ the dual basis, i.e., such that

$$\langle \theta^{*a}, \theta_b \rangle = \delta^a_b \,.$$

a) *Show the following relations, where* $\circ$ *denotes map composition*

$$[i(\theta_a), i(\theta_b)]_+ \equiv i(\theta_a) \circ i(\theta_b) + i(\theta_b) \circ i(\theta_a) = 0 \,, \tag{3a}$$

$$[\epsilon(\theta^{*a}), \epsilon(\theta^{*b})]_+ = 0 \,, \tag{3b}$$

$$[\epsilon(\theta^{*a}), i(\theta_b)]_+ = \langle \theta^{*a}, \theta_b \rangle \mathbb{1} = \delta^b_a \mathbb{1} \,. \tag{3c}$$

coboundary operator

Let d be the **coboundary operator** defined on $C^k(\mathscr{L}(\mathrm{G}), X)$ by

$$\begin{aligned}(\mathrm{d}f)(\theta_1, \ldots, \theta_k) = &\sum_{l=1}^{k} (-1)^{l-1} \theta_l \cdot f(\theta_1, \ldots, \hat{\theta}_l, \ldots, \theta_k) \\ &+ \sum_{p<q} (-1)^{p+q} f([\theta_p, \theta_q], \\ &\times \theta_1, \ldots, \hat{\theta}_p, \ldots, \hat{\theta}_q, \ldots, \theta_k) \,,\end{aligned} \tag{4}$$

where a dot means the linear transformation of X defined by an element of $\mathscr{L}(\mathrm{G})$.

b) *Show that the coboundary operator* d *can be expressed in terms of* ϵ *and* i, *namely*,

$$\mathrm{d} = \sum_{a=1}^{n} \theta_a \cdot \epsilon(\theta^{*a}) - \sum_{1=a<b}^{n} i([\theta_a, \theta_b]) \circ \epsilon(\theta^{*a}) \circ \epsilon(\theta^{*b}) \,. \tag{5}$$

Answer 1a: Straightforward calculations using the definitions.

Answer 1b: Applying both sides of the equation on a $(k-1)$-cochain using (4), (2) and (1) and using the duality $\langle \theta^{*a}, \theta_b \rangle = \delta^a_b$ yields (5).

2. INFINITE DIMENSIONAL LIE ALGEBRAS

The definitions given in question (1) cannot easily be generalized to this case: in an infinite dimensional vector space V one cannot consider convergent series unless V is a topological vector space. A countable basis in a topological vector space V over $\mathbb{C}$ or $\mathbb{R}$ is a set of elements L_i, $i \in \mathbb{N}$, such that each $v \in V$ can be written in a unique way as the sum of a convergent series:

$$v = \sum_{i=1}^{\infty} v^i L_i \,, \qquad v^i \in \mathbb{C} \text{ (or } \mathbb{R}) \,.$$

General topological vector spaces do not admit countable basis. Even separable Hilbert spaces – i.e., Hilbert spaces admitting a countably dense subset – do not necessarily admit a countable basis. The space $L^2(K)$ of square integrable functions on a compact manifold (with or without boundary) admits a countable basis.

The dual V^* of a topological vector space V is the space of continuous linear maps on V; it depends on the topology chosen on V. In general, $(V^*)^* \supset V$ is not equal to V. If V is a Hilbert space $\mathscr{H}$, it can be identified with its dual. But, in general, V^* will not admit a countable basis even if V admits such a basis: for instance, the dual of the space $C^\infty(S^1)$ is the space of distributions on S^1, which does not admit a countable basis.

Cochains on $\mathscr{L}(G)$ have been defined in the finite dimensional case as antisymmetric multilinear mappings with values in a vector space X. They can be considered, equivalently, as elements of $(\mathscr{L}(G) \wedge \cdots \wedge \mathscr{L}(G))^* \otimes X$ or $\mathscr{L}^*(G) \wedge \cdots \wedge \mathscr{L}^*(G) \otimes X$. In the infinite dimensional case, caution is in order: the definition of tensor product, and hence of exterior product, can be given in inequivalent ways when the spaces are not reflexive (p. 546). Usually we shall have $\mathscr{L}^*(\mathscr{G}) \wedge \cdots \wedge \mathscr{L}^*(\mathscr{G}) \subset (\mathscr{L}(\mathscr{G}) \wedge \cdots \wedge \mathscr{L}(\mathscr{G}))^*$ and strictly smaller. To avoid lengthy justifications we shall treat explicitly the case $\mathscr{G} = \mathrm{Diff}\, S^1$, important in string theory, and even in this case content ourselves in part with formal expressions.

a) For an arbitrary Lie algebra $\mathscr{L}(\mathscr{G})$ the **adjoint representation** is the mapping $\mathscr{L}(\mathscr{G}) \to L(\mathscr{L}(\mathscr{G}), \mathscr{L}(\mathscr{G}))$ defined by

adjoint representation

$$\mathscr{A}d(L_1)L_2 = [L_1, L_2]\,, \qquad \forall L_1, L_2 \in \mathscr{L}(\mathscr{G})\,.$$

Show that there exists one (anti) representation $\mathscr{A}d^*\colon \mathscr{L}(\mathscr{G}) \to L(\mathscr{L}^*(\mathscr{G}), \mathscr{L}^*(\mathscr{G}))$ *such that*

$$\langle L_2, \mathscr{A}d^*(L_1)L^* \rangle = \langle \mathscr{A}d(L_1)L_2, L^* \rangle\,,\ L_1, L_2 \in \mathscr{L}(\mathscr{G}),\ L^* \in \mathscr{L}^*(\mathscr{G})\,. \tag{6}$$

$\mathscr{A}d^*$ *is called the* **coadjoint representation**.

coadjoint representation

Show that if $\mathscr{G}$ *is the group* $\mathrm{Diff}\, S^1$ *of* C^∞ *diffeomorphisms of* S^1, *and* $\{L_i\}$, $i \in \mathbb{Z}$, *the basis of* $\mathscr{L}(\mathscr{G})$ *considered in [Problem V bis 11, Cocycles], then*

$$\mathscr{A}d^*(L_j)L^{*k} = (2j - k)L^{*k-j}\,, \tag{7}$$

where L^{*k} *denotes the element of* $\mathscr{L}^*(\mathscr{G})$ *defined by* $\langle v, L^{*k} \rangle = v^k$, $v = \Sigma_{-\infty}^{+\infty} v^i L_i \in \mathscr{L}(\mathscr{G})$. *Show that* $\mathscr{A}d^*$ *is an antirepresentation of* $\mathscr{L}(\mathscr{G})$, *namely*

$$[\mathscr{A}d^*(L_j), \mathscr{A}d^*(L_i)] = (i - j)\mathscr{A}d^*(L_{i+j}) \quad \text{whereas}$$

$$[L_j, L_i] = (j - i)L_{i+j}\,. \tag{8}$$

Remark: $\mathscr{L}^*(\mathscr{G})$ is not a Lie algebra (the bracket of two one-forms is not a one-form but a two-form).

coadjoint representation

Remark: There is another definition for the **coadjoint representation** of $\mathscr{L}(\mathscr{G})$ namely

$$\langle L_2, \mathscr{A}d^*(L_1)L^*\rangle = -\langle \mathscr{A}d(L_1)L_2, L^*\rangle. \tag{8a}$$

With this definition the coadjoint representation is a representation – not an antirepresentation, and

$$\mathscr{A}d^*(L_j)L_k^* = (k-2j)L^{*^{k-j}}. \tag{9}$$

The group representations corresponding to these Lie algebra representations are such that

$$\langle \mathscr{A}d(g)L_2, \mathscr{A}d^*(g^{-1})L^*\rangle = \langle L_2, L^*\rangle \quad \text{corresponding to (6)},$$
$$\langle \mathscr{A}d(g)L_2, \mathscr{A}d^*(g)L^*\rangle = \langle L_2, L^*\rangle \quad \text{corresponding to (8a)}.$$

b) *Denote by Λ_n^* the vector subspace of $\mathscr{L}^*(\mathscr{G}) \wedge \cdots \wedge \mathscr{L}^*(\mathscr{G})$, n products, generated by the exterior products*

$$\{L^{*^{I_1}} \wedge \cdots \wedge L^{*^{I_n}}\}, \quad I_1, \ldots, I_n \quad \textit{ordered indices.}$$

Show that the linear mapping ρ_n^*: $\mathscr{L}(\mathscr{G}) \to L(\Lambda_n^*, \Lambda_n^*)$ determined by

$$\rho_n^*(L)(L^{*^{I_1}} \wedge \cdots \wedge L^{*^{I_n}}) = \sum_{k=1}^{n} L^{*^{I_1}} \wedge \cdots \wedge \mathscr{A}d^*(L)L^{*^{I_k}} \wedge \cdots \wedge L^{*^{I_n}}, \tag{10}$$

with the definition (8a) *for the coadjoint representation, is a homomorphism of Lie algebras.*

Answer 2a: Let $L^* \in \mathscr{L}^*(\mathscr{G})$ and $L_1 \in \mathscr{L}(\mathscr{G})$. Formula (6) gives the action of $\mathscr{A}d^*(L_1)L^*$ on any $L_2 \in \mathscr{L}(\mathscr{G})$ as a linear continuous map into $\mathbb{R}$, i.e., determines $\mathscr{A}d^*(L_1)L^*$ as an element of $\mathscr{L}^*(\mathscr{G})$, and $\mathscr{A}d^*(L_1)$ as an element of $L(\mathscr{L}^*(\mathscr{G}), \mathscr{L}^*(\mathscr{G}))$. A straightforward calculation, using the definitions and Jacobi identity shows that $\mathscr{A}d^*$ is an antirepresentation, namely

$$[\mathscr{A}d^*(L_1), \mathscr{A}d^*(L_2)] := \mathscr{A}d^*(L_1)\mathscr{A}d^*(L_2) - \mathscr{A}d^*(L_2)\mathscr{A}d^*(L_1)$$
$$= -\mathscr{A}d^*([L_1, L_2]).$$

Using the fact that, for the given basis of $\mathscr{L}(\text{Diff } S^1)$ we have

$$[L_j, L_i] = (j-i)L_{j+i}$$

we find

$$\langle v^i L_i, \mathscr{A}d^*(L_j)L^{*^k}\rangle = v^i\langle [L_j, L_i], L^{*^k}\rangle = v^i(j-i)\langle L_{j+1}, L^{*^k}\rangle$$
$$= (j-1)\delta_{j+1}^k v^i = (2j-l)\delta_l^k v^{k-l},$$

i.e.,

$$\mathscr{A}d^*(L_j)L^{*k} = (2j-k)L^{*k-l}\,.$$

Answer 2b: We need to check that

$$[\rho_n^*(L_k),\, \rho_n^*(L_l)] = (k-l)\rho_n^*(L_{k+l})\,.$$

We can check it in a few lines for $n=2$ using the defining equation (9). The calculation generalizes in a straightforward manner for arbitrary n.

3. SEMI-INFINITE FORMS

The space of **semi-infinite forms** introduced by [Feigin], applied to physics by [Frenkel] is the space Λ_∞^*, which, together with the following restrictions, is labelled $\bar{\Lambda}_\infty^*$. An element of $\bar{\Lambda}_\infty^*$ is a semi-infinite product (infinite only on one side) semi-infinite forms

$$L^{*l_1} \wedge L^{*l_2} \wedge \cdots \wedge L^{*l_{k-1}} \wedge L^{*l_k} \wedge L^{*l_{k+1}} \wedge \cdots$$

such that from an arbitrary index k on to infinity the indices decrease 1 by 1. For example,

$$L^{*3} \wedge L^{*1} \wedge L^{*(-5)} \wedge L^{*(-6)} \wedge L^{*(-7)} \wedge \cdots \wedge L^{*l_j} \ldots,$$
$$l_{j+1} = l_j - 1 \text{ for } j \geq 3$$

is an element of $\bar{\Lambda}_\infty^*$.

a) *Define $i(L_m)$ and $\epsilon(L^{*m})$ by*

$$i(L_m)(L^{*l_1} \wedge \cdots \wedge L^{*l_p} \wedge \cdots) = \sum_{p \geq 1}^{\infty} (-1)^{p-1} \langle L_m, L^{*l_p} \rangle L^{l^1} \wedge \cdots \wedge L^{*l_p} \wedge \cdots \tag{11a}$$

and

$$\epsilon(L^{*m})(L^{*l_1} \wedge \cdots \wedge L^{*l_p} \wedge \cdots) = L^{*M} \wedge L^{*l_1} \wedge \cdots \wedge L^{*l_p} \ldots\,. \tag{11b}$$

Compute their anticommutators and compare them with (3).

Answer 3a: It follows readily from the antisymmetry of the Λ-product that

$$[i(L_k),\, i(L_m)]_+ = 0\,, \tag{12a}$$

$$[\epsilon(L^{*k}),\, \epsilon(L^{*m})]_+ = 0\,. \tag{12b}$$

An explicit calculation of $[i(L_k), \epsilon(L^{*m})]_+$ operating on a generic element of $\bar{\Lambda}^*_\infty$ shows that

$$[i(L_k), \epsilon(L^{*^m})]_+ = \langle L_k, L^{*^m} \rangle = \delta^m_k. \quad \blacksquare \tag{12c}$$

b) Consider the following mapping:

$$\rho^*_\infty \colon \mathscr{L}(\mathscr{G}) \to L(\bar{\Lambda}^*_\infty, \bar{\Lambda}^*_\infty) \tag{13}$$

by

$$\rho^*_\infty(L_k)(L^{*^{I_1}} \wedge \cdots \wedge L^{*^{I_p}} \wedge \cdots)$$
$$:= \sum_{p=1}^{\infty} L^{*^{I_1}} \wedge \cdots \wedge \mathscr{A}d^*(L_k)(L^{*^{I_p}}) \wedge \cdots \qquad \text{for } k \neq 0. \tag{13a}$$

We note that for $k \neq 0$ this series terminates since for I_p smaller than a fixed number $\mathscr{A}d^*(L_k)(L^{*^{I_p}}) \sim L^{*^{I_p-k}}$ can be found in the term on which $\rho^*_\infty(L_k)$ operates. For $k = 0$

$$\rho^*_\infty(L_0)(L^{*^{I_1}} \wedge \cdots \wedge L^{*^{I_p}} \wedge \cdots) := h_0(L^{*^{I_1}} \wedge \cdots \wedge L^{*^{I_p}} \ldots), \tag{13b}$$

where h_0 is a real finite number which depends on $L^{*^{I_1}} \wedge \cdots \wedge L^{*^{I_p}} \ldots$.

Show that the mapping (13) *defines a projective representation of* $\mathscr{L}(\mathscr{G})$ *on* $\bar{\Lambda}^*_\infty$. *Compute the* 2*-cocycle* $C(L_i, L_j)$ *of this projective representation.* [*See Problem V bis* 11, *Cocycles and IV* 6, *Short exact.*]

Remark: The reason for defining $\rho^*_\infty(L_0)$ by (13b) is related to the definition, given below, of the quantum BRST operator in terms of normal ordered products.

Answer 3b: To prove that (13) defines a projective representation, one computes $[\rho^*_\infty(L_j), \rho^*_\infty(L_k)]$. It is easier to do so by considering first the case when j and k are both positive or both negative. One finds that in these cases there are no cocycles. Thus, the cocycles, if any, are proportional to $\delta_{j,-k}$. A direct straightforward calculation shows that ρ^*_∞ defines a Virasoro representation.

We have shown in the previous problem [Problem V bis 11, Cocycles, eq. (38)] that if $\hat{L}_i$ is the representative of L_i in a Virasoro representation

$$[\hat{L}_k, \hat{L}_j] = (k - j)\hat{L}_{k+j} + \lambda(k^3 - k)\delta_{j,-k}.$$

So far, λ is arbitrary because we have not specified the representation beyond eiminating the trivial cocycles. We want to compute λ for the representation ρ^*_∞ defined by (13), namely for $\hat{L}_j = \rho^*_\infty(L_j)$. Note that

$$[\hat{L}_1, \hat{L}_{-1}] = 2\hat{L}_0 \quad \text{and} \quad [\hat{L}_2, \hat{L}_{-2}] = 4\hat{L}_0 + 6\lambda;$$

hence,

$$[\hat{L}_2, \hat{L}_{-2}] - 2[\hat{L}_1, \hat{L}_{-1}] = 6\lambda .$$

Applying both sides of this equation to a generic element in $\hat{\Lambda}^*_\infty$ gives

$$\lambda = -26/12 ;$$

i.e., the representation ρ^*_∞ carries a nontrivial cocycle

$$C(L_k, L_j) = -\tfrac{26}{12}(k^3 - k)\delta_{j,-k} . \tag{14}$$

Remark: Working with the Lie algebra $\mathscr{L}(\mathscr{G} = \mathrm{Diff}\,S^1)$ is simpler than working with the diffeomorphism group of S^1 for 2 reasons:

- In a group of diffeomorphisms there are elements of the group arbitrarily close to the identity which cannot be reached by exponentiating its Lie algebra.
- $\mathrm{Diff}\,S^1$ is multiply connected, $\pi_1(\mathrm{Diff}\,S^1) = \mathbb{Z}$. See for instance [Milnor].

See [Segal] for nontrivial projective representations of $\mathrm{Diff}\,S^1$ which induce representations of $\mathscr{L}(\mathrm{Diff}\,S^1)$ with trivial cocycles.

4. NORMAL ORDERING. QUANTUM BRST OPERATORS

Normal ordering is a prescription indicated by a pair of double dots and defined as follows: normal ordering

Choose a fixed arbitrary integer k_0, then

$$:i(L_m)\epsilon(L^{*m}): \equiv \begin{cases} i(L_m)\epsilon(L^{*m}) & \text{if } m \le k_0 \\ -\epsilon(L^{*m})i(L_m) & \text{if } m > k_0 \end{cases}$$

$$:\epsilon(L^{*m})i(L_m): \equiv \begin{cases} -i(L_m)\epsilon(L^{*m}) & \text{if } m \le k_0 \\ \epsilon(L^{*m})i(L_m) & \text{if } m > k_0 \end{cases} .$$

The vector

$$w_0 = L^{*k_0} \wedge L^{*k_0 - 1} \wedge \cdots \text{ with all the slots filled in}$$

is called the **vacuum**. vacuum

One can interpret $\epsilon(L^{*m})$ as a creation operator and $i(L_m)$ as an annihilation operator. For $m \le k_0$ normal ordering means creation operator on the right, annihilation operators on the left, and the reverse situation for $m > k_0$. Normal ordering eliminates the infinity which would occur from

$$\sum_{m=-\infty}^{+\infty} [i(L_m), \epsilon(L^{*m})]_+ = \sum_m \langle L_m, L^{*m} \rangle = \sum_m \delta^m_m .$$

Indeed,

$$:\left(\sum_{m\leq k_0}+\sum_{m>k_0}\right)[i(L_m),\epsilon(L^{*^m})]:=0.$$

ghost field operator
conjugate ghost field operator

In physics,

$\eta^\alpha \equiv \epsilon(L^{*^a})$ is called a **ghost field operator**,
$\mathscr{P}_a \equiv i(L_a)$ is called a **conjugate ghost field operator**.

Let $\hat{L}_a$ be the representative of L_a be the representative of L_a in a possibly projective representation π to be chosen later.
The operator

$$\hat{Q} \equiv \hat{L}_a \cdot \eta_a - \tfrac{1}{2} f^c_{ab}{:}\mathscr{P}_c\eta^a\eta^b{:} \tag{15}$$

quantum BRST charge operator

is called the **quantum BRST charge operator** in the representation $\hat{L}_a = \pi(L_a)$. This equation has a formal resemblance with (5), which gives the coboundary operator d in terms in ϵ and i. But, in contrast with d, the quantum BRST is not necessarily nilpotent. We know [Problem V bis 11, Cocycles, Answer 7] that, if one defines the space of physical states as the kernel of the quantum BRST operator, consistency can be achieved only if the quantum BRST is nilpotent. It can be nilpotent only if the cocycle introduced by the choice of representation $L_a \to \hat{L}_a$ precisely cancels the terms introduced by $-\frac{1}{2} f^c_{ab}{:}\mathscr{P}_c\eta^a\eta^b{:}$. This is the case in particular when L_a are represented by operators on the Fock module of a bosonic string in 26 dimensions.

Interesting relations are obtained by computing the commutators of $\rho^*_\infty(L_n)$ with $i(L_m)$ and $\epsilon(L^{*^m})$ and expressing ρ^*_∞ in terms of i's and ϵ's.

We have presented the version of the BRST operator which comes naturally in an hamiltonian theory where the first class constraints together with the Poisson bracket are a representation of the Lie algebra $\mathscr{L}(\mathscr{G})$. Its relationship with the lagrangian BRST operators [see for instance Beaulieu] can be found in [Fradkin, Vilkovisky]. For the construction of the classical BRST operator, its relationship with the constraint analysis of dynamical systems, the Lie algebra cohomology and the geometric interpretation of ghosts and conjugate ghosts, see [Kostant et al., McMullan].

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VI. DISTRIBUTIONS

1. ELEMENTARY SOLUTION OF THE WAVE EQUATION IN d-DIMENSIONAL SPACETIME

C denotes the cone in $\mathbb{R}^d$ defined by

$$R \equiv \tfrac{1}{2}((x^0)^2 - r^2) = 0\,, \qquad x^0 \geq 0\,, \quad r^2 \equiv \sum_{i=1}^{d-1} (x^i)^2\,.$$

δ_R is the Dirac measure (pp. 438, 512) on C, defined by

$$\langle \delta_R, \varphi \rangle = \int_{\mathbb{R}^{d-1}} \bar{\varphi}(x^i)\, \frac{\mathrm{d}x^1 \ldots \mathrm{d}x^{d-1}}{r}\,,$$

where $\bar{\varphi}$ denotes the function induced by $\varphi \in \mathscr{D}(\mathbb{R}^d)$ on C (p. 421)

$$\bar{\varphi}(x^i) = \varphi(x^0, x^i)|_{x^0 = r}\,.$$

1) *Determine* $\partial_0 \delta_R$, $\partial_i \delta_R$ *and compare their values when* $d \geq 4$. *For which values of* d *is* $\delta_R' \equiv (1/x^0)\partial_0 \delta_R$ *defined?*
2) *Determine* $\Box \delta_R$ *when* $d = 4$.
3) *Show that when* d *is even one can define a distribution by setting*

$$E = \left(\frac{1}{x^0}\, \partial_0\right)^{(d-4)/2} \delta_R\,.$$

Show that $\Box E$ *has support* 0 *and is proportional to* δ.

Answer 1: By definition of the derivative of a distribution (p. 446)

$$\langle \partial_\alpha \delta_R, \varphi \rangle = -\langle \delta_R, \partial_\alpha \varphi \rangle\,.$$

Hence

$$\langle \partial_0 \partial_R, \varphi \rangle = -\int_{\mathbb{R}^{d-1}} \overline{\partial_0 \varphi}\, \frac{\mathrm{d}x^1 \ldots \mathrm{d}x^{d-1}}{r} \tag{1}$$

and, since

$$\overline{\partial_i \varphi} = \partial_i \bar{\varphi} - \frac{x^i}{r} \overline{\partial_0 \varphi} ,$$

$$\langle \partial_i \delta_R, \varphi \rangle = - \int_{\mathbb{R}^{d-1}} \left(-\frac{x^i}{r} \overline{\partial_0 \varphi} + \partial_i \bar{\varphi} \right) \frac{\mathrm{d}x^1 \ldots \mathrm{d}x^{d-1}}{r} . \tag{2}$$

If $d \geq 4$ and $\varphi \in \mathscr{D}$ we have the equality of convergent integrals:

$$\int_{\mathbb{R}^{d-1}} \frac{\partial_i \bar{\varphi}}{r} \mathrm{d}x^1 \ldots \mathrm{d}x^{d-1} = \int_{\mathbb{R}^{d-1}} \frac{x^i \bar{\varphi}}{r^3} \mathrm{d}x^1 \ldots \mathrm{d}x^{d-1} . \tag{3}$$

Therefore (2) can be written, if $d \geq 4$

$$\partial_i \delta_R = -\frac{x^i}{r} \partial_0 \delta_R + \frac{x^i}{r^2} \delta_R . \tag{3a}$$

We have

$$\left\langle \frac{1}{x^0} \partial_0 \delta_R, \varphi \right\rangle = - \left\langle \delta_R, \partial_0 \left(\frac{\varphi}{x^0} \right) \right\rangle = - \int_{\mathbb{R}^{d-1}} \left(\frac{\overline{\partial_0 \varphi}}{r^2} - \frac{\bar{\varphi}}{r^3} \right) \mathrm{d}x^1 \ldots \mathrm{d}x^{d-1}$$

if the integral is convergent for all $\varphi \in \mathscr{D}$, that is if $d \geq 5$.
In this case (3a) can be written

$$\partial_i \delta_R = -\frac{x^i}{x^0} \partial_0 \delta_R . \tag{3a)$'$}$$

Note: If instead of the measure δ_R we were derivating a Lorentz invariant C^1 function $f(R)$ we would have obtained

$$\partial_\alpha f(R) = \eta_{\alpha\beta} x^\beta f'(R) .$$

Formula (3a)′ is analogous if we set

$$\frac{1}{x^0} \partial_0 \delta_R = \delta_R' .$$

This definition is invariant under a Lorentz transformation

$$y^\beta = L^\beta{}_\alpha x^\alpha , \qquad \eta_{\alpha\beta} y^\alpha y^\beta \equiv \eta_{\alpha\beta} x^\alpha x^\beta$$

since by properties of change of variables on distributions (p. 456)

$$\frac{1}{y^0} \frac{\partial}{\partial y^0} \delta_R = \frac{1}{L^0{}_\alpha x^\alpha} L_0^\beta \frac{\partial}{\partial x^\beta} \delta_R ,$$

and using (3a)′

$$\frac{1}{y^0} \frac{\partial}{\partial y^0} \delta_R = \frac{1}{L^0{}_\alpha x^\alpha} \left(L_0^{\ 0} - \frac{x^i}{x^0} L_0^{\ i} \right) \frac{\partial}{\partial x^0} \delta_R .$$

$(L_\alpha{}^\beta)$ is the inverse of the Lorentz transformation $(L^\beta{}_\alpha)$, and we know that

$$L^0{}_0 = L_0{}^0 , \qquad L^0{}_i = -L^i{}_0$$

hence, as announced

$$\frac{1}{y^0}\frac{\partial}{\partial y^0}\delta_R = \frac{1}{x^0}\frac{\partial}{\partial x^0}\delta_R .$$

δ'_R is not defined if $d \le 4$, since $(1/x^0)\partial_0\delta_R$ is not defined in this case.

Answer 2: If δ'_R is defined (i.e., on $\mathbb{R}^d - \{0\}$, or on $\mathbb{R}^d$ if $d > 4$) and $\delta''_R \equiv (1/x^0)\partial_0\delta'_R$ is also defined we shall have, as in 1)

$$\partial_0\delta_R = x^0\delta'_R , \qquad \partial^2_{00}\delta_R = \delta'_R + (x^0)^2\delta''_R$$

$$\partial_i\delta_R = -x^i\delta'_R , \qquad \sum_{i=1}^{d-1}\partial^2_{ii}\delta_R = (1-d)\delta'_R + \Sigma(r^i)^2\delta''_R ,$$

thus

$$\Box\delta_R = d\delta'_R + 2R\delta''_R = (d-4)\delta'_R , \tag{4}$$

$\delta''_R = ((1/x^0)\partial_0)^2\delta_R$ is defined if $d > 6$ by

$$\langle \delta''_R, \varphi\rangle = \left\langle \delta_R, \partial_0\left(\frac{1}{x^0}\partial_0\frac{\varphi}{x^0}\right)\right\rangle$$

$$= \int_{\mathbb{R}^{d-1}} \left(\frac{\overline{\partial^2_{00}\varphi}}{r^3} - \frac{\overline{3\partial_0\varphi}}{r^4} + \frac{\overline{3\varphi}}{r^5}\right) \mathrm{d}x^1 \ldots \mathrm{d}x^{d-1} .$$

Thus (4) is valid on $\mathbb{R}^d$ if $d > 6$. It is always valid on $\mathbb{R}^d - \{0\}$.
If $d = 4$, $\Box\delta_R = 0$ on $\mathbb{R}^4 - \{0\}$.
To compute $\Box\delta_R$ on $\mathbb{R}^4$ we go back to the definition

$$\langle \Box\delta_R, \varphi\rangle = \langle \delta_R, \Box\varphi\rangle = \int_{\mathbb{R}^3} \frac{\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3}{r}\,\overline{\Box\varphi} .$$

For arbitrary d a straightforward computation gives, since $x^0 = r$ on C,

$$\overline{\Box\varphi} = -\bar{\Delta}\,\bar{\varphi} + \frac{2x^i}{r}\,\bar{\partial}_i\overline{\partial_0\varphi} + \partial_i\left(\frac{x^i}{r}\right)\overline{\partial_0\varphi} , \qquad \bar{\Delta} = \Sigma\partial^2_{ii} . \tag{4a}$$

Since $\partial_i(x^i/r) = (d-2)/r$ we have for any d

$$\frac{\overline{\Box\varphi}}{r^p} = -\frac{\bar{\Delta}\,\bar{\varphi}}{r^p} + \frac{2x^i}{r^{p+1}}\,\partial_i\psi + \frac{(d-2)\psi}{r^{p+1}} , \qquad \psi = \overline{\partial_0\varphi}$$

$$= \frac{\bar{\Delta}\,\bar{\varphi}}{r^p} + \frac{2}{r^{p+1}}\,\partial_i\left(r^{-q}\,\frac{x^i}{r}\,\psi\right) , \qquad q = \tfrac{1}{2}d - 1 ;$$

hence if $p = 1$ and $d = 4$, i.e., $q = 1$

$$\frac{\overline{\Box\varphi}}{r} = -\frac{\bar{\Delta}\bar{\varphi}}{r} + 2\partial_i(r^{-2}x^i\psi)\,.$$

Since φ has compact support, we have if $d = 4$

$$\langle \Box\delta_R, \varphi\rangle = -\int_{\mathbb{R}^3} \frac{\bar{\Delta}\varphi}{r}\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3\,;$$

we compute this integral on $\mathbb{R}^3 - B_\epsilon$ (ball of radius ϵ centered at 0) and let ϵ tend to zero and find (cf. p. 512)

$$\langle \Box\delta_R, \varphi\rangle = 4\pi\varphi(0)\,;$$

that is, when $d = 4$

$$\Box\delta_R = 4\pi\delta\,.$$

$(1/4\pi)\delta_R$ is the elementary solution with support in C of $\Box$ when $d = 4$.

Answer 3: The distribution

$$E = \left(\frac{1}{x^0}\,\partial_0\right)^{(d-4)/2}\delta_R = \delta_R^{(d-4)/2}$$

is defined for $d \geq 4$ and even, by

$$\langle E, \varphi\rangle = (-1)^{(d-4)/2}\left\langle \delta_R, \left(\partial_0\,\frac{1}{x^0}\right)^{(d-4)/2}\varphi\right\rangle$$

because the function

$$\frac{1}{r^{2+(d-4)/2}} = \frac{1}{r^{d/2}}$$

is integrable on $\mathbb{R}^{d-1}$ if $d \geq 4$.

We compute $\Box\delta_R^{(p-2)}$, $p = d/2$ on $\mathbb{R}^d - \{0\}$:

$$\Box\delta_R^{(p-2)} = \mathrm{d}\delta_R^{(p-1)} + 2R\delta_R^{(p)}$$

from the induction formula $q = 0, 1, \ldots, p$

$$\mathrm{d}\delta_R^{(p-1)} = 2R\delta_R^{(p)} = (d - 2q)\delta_R^{(p-1)} + 2(R\delta_R^{(p-q)})^{(q)}\,.$$

Since $R\delta_R \equiv 0$ we deduce

$$\Box\delta_R^{(p-2)} = 0 \quad \text{on} \quad \mathbb{R}^d - \{0\}$$

(the formula is not valid on $\mathbb{R}^d$ because $\delta_R^{(q)}$ is not defined there if $q > d/2$).

On $\mathbb{R}^d$ we have

$$\langle \Box \delta_R^{(p-2)}, \varphi \rangle = \langle \delta_R^{(p-2)}, \Box\varphi \rangle = (-1)^{p-2} \left\langle \delta_R, \overline{\left(\partial_0 \frac{1}{x^0}\right)^{(p-2)} \Box\varphi} \right\rangle .$$

Computations analogous to the ones done previously show that, if φ has compact support (divergences integrate to zero on $\mathbb{R}^{d-1}$)

$$\langle \Box \delta_R^{(p-2)}, \varphi \rangle = K\varphi(0)$$

with some constant K, thus

$$\Box \delta_R^{(p-2)} = K\delta .$$

By explicit computation K can be found to be (or see Leray p. 103 which uses another method and other notations):

$$K = 1 \cdot 3 \cdot \dots \cdot (d-3) \operatorname{vol} S^{d-2} , \qquad p = \frac{d}{2} \geq 2 .$$

The volume of the sphere S^{2n} is[1)]

$$\operatorname{vol} S^{2n} = \frac{2(2\pi)^n}{1.3 \cdot \dots \cdot (2n-1)} ,$$

therefore

$$K = 2(2\pi)^{p-1} .$$

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2. SOBOLEV EMBEDDING THEOREM

I. DEFINITIONS

We have defined (p. 486) the Sobolev space $W_p^m(U)$, m a nonnegative integer, $1 \leq p \leq \infty$, U open set of $\mathbb{R}^n$, as the space of distributions on U which, together with their derivatives of order $\leq m$ can be identified with functions in $L^p(U)$. An alternative, equivalent definition (Meyers and Serrin 1964, cf. Adams p. 52) when $p < \infty$, is that $W_p^m(U)$ is the closure in

[1)] The volume of S^{2n+1} is $2\pi^{n+1}/n!$.

the norm

$$\|f\|_{W_p^m} = \left\{ \sum_{|j| \le m} \int_U |D^j f(x)|^p \, \mathrm{d}x \right\}^{1/p}$$

of the space of C^m functions on U for which this norm is finite: in other words this last space is dense in the space $W_p^m(U)$ as previously defined, if $1 \le p < \infty$. 1) *Show that the result does not extend to $p = \infty$.*

Answer 1: The norm in $W_\infty^m(U)$ is

$$\|f\|_{W_\infty^m} = \sum_{|j| \le m} \operatorname*{Sup}_{x \in U} |D^j f(x)| \, .$$

$C_{\mathrm{B}}^m(U)$ space

Functions in $C^m(U)$ for which this norm is bounded constitute the complete (Banach) **space $C_{\mathrm{B}}^m(U)$**, not equal – even for $m = 0$ – to $W_\infty^m(U)$. $W_\infty^m(U)$ is however, like all $W_p^m(U)$, a Banach space.
$H_s(U) \equiv W_2^s(U)$ is a Hilbert space.
It is known that $C_0^\infty(\mathbb{R}^n)$ is dense in $W_p^m(\mathbb{R}^n)$, $1 \le p < \infty$. In general $C_0^\infty(U)$ is not dense in $W_p^m(U)$ (cf. p. 468).
The set of restrictions to U of functions in $C_0^\infty(\mathbb{R}^n)$, denoted $C^\infty(\bar{U})$, is dense in $W_p^m(U)$, $1 \le p < \infty$ if U has the segment property.

segment property

An open set $U \subset \mathbb{R}^n$ is said to have the **segment property** if for every $x \in \partial U$, boundary of U, there exists an open set $U_x \ni x$ and a nonzero vector v_x such that if $y \in \bar{U} \cap U_x$ then $y + tv_x \in U$ for $0 < t < 1$.
2) *Show that the open set of* $\mathbb{R}^2$

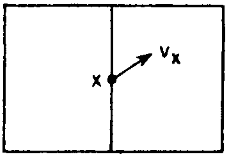
$$U = \{0 < |x^1| < 1, 0 < x^2 < 1\}$$

does not have the segment property. Give a necessary condition for an open set $U \subset \mathbb{R}^n$ with $(n-1)$ dimensional boundary to have the segment property.
Show that the function on the previous domain $U \subset \mathbb{R}^2$ given by

$$u(x^1, x^2) = \begin{cases} 1 & \text{if} \quad x^1 > 0 \\ 0 & \text{if} \quad x^1 < 0 \end{cases}$$

belongs to $W_p^1(U)$ and cannot be approximated by functions in $C^1(U)$.

Answer 2:



U is the rectangle $-1 < x^1 < 1$, $0 < x^2 < 1$, with the line $x^1 = 0$ removed. If we take x on this part of ∂U, there is no U_x, open neighborhood of x, and $v_x \ne 0$ with the required property. If $U \subset \mathbb{R}^n$ has a $(n-1)$ dimensional boundary it must lie only on one side of this boundary.

For the given function u we have $\partial_1 u = \partial_2 u = 0$ on U. Thus $u \in W_p^1(U)$ for any $1 \le p \le \infty$. But for sufficiently small $\epsilon > 0$ there is no $f \in C^1(\bar{U})$ such that $\|u - f\|_{W_p^1(U)} < \epsilon$. ■

An open set U in $\mathbb{R}^n$ is said to have the **cone property** (p. 492) if there exist $\alpha > 0$ and $h > 0$ such that at each point $x \in U$ one can draw an axially symmetric cone of vertex x, angle α, and height h. cone property

A fundamental theorem is the following.

II. SOBOLEV EMBEDDING THEOREM

Let U be an open set in $\mathbb{R}^n$ with the cone property, then the following continuous imbeddings hold for $1 \le p < \infty$.

i) If $m > n/p$, then $W_p^m(U) \hookrightarrow C_B^0(U)$
(hence $W_p^m(U) \hookrightarrow L^q(U) \quad \forall q \ge p$).

ii) If $m < n/p$, then $W_p^m(U) \hookrightarrow L^q(U), 1 \le p \le q \le np/(n - mp)$.

iii) If $m = n/p$, then $W_p^m(U) \hookrightarrow L^q(U), 1 \le p \le q < \infty$.

Moreover $W_1^n(U) \hookrightarrow C_B^0(U)$.
We shall prove parts i and ii below. See the references for the proof of part iii. The case $W_1^n(U) \hookrightarrow C_B^0(U)$ is particularly easy to prove using integration. First we give a fundamental corollary of the theorem.

3) *Deduce from this theorem the Sobolev inequalities*

$$\|f\|_{C_B^k(U)} \le C\|f\|_{W_p^{m+k}(U)}, \qquad \text{if } m > n/p$$

$$\|f\|_{W_q^k(U)} \le C\|f\|_{W_p^{m+k}(U)}, \qquad \text{if } m < n/p,\ p \le q \le np/(n - mp),$$

where the C's are constant depending only on m, p, k, U.

Answer 3: A linear mapping between normed spaces is continuous if and only if it is bounded (p. 58). In the case $k = 0$ the inequalities are just a reformulation of the theorem (in fact the proof of the theorem is by proving these inequalities). The result, for an arbitrary k, is due to the fact that the space W_p^{m+k} is the space of functions such that $Df \in W_p^{m+k-1}, \ldots, D^k f \in W_p^m$. Thus, by the theorem $D^l f \in C_B^0(U)$, $l = 1, \ldots, k$, and f can be identified with a function in $C_B^k(U)$, whose usual derivatives are identified with its derivatives in the sense of distributions (pp. 447, 451).

We shall give the proof for typical examples, general proofs can be found in Lions or Adams.

4) *Prove part i) of the embedding theorem when $p=2$ and $U=\mathbb{R}^n$ by using the Fourier transform.*

Answer 4: If $f \in W_2^m(\mathbb{R}^n) \equiv H_m(\mathbb{R}^n)$ it admits a Fourier transform, $\mathcal{F}f$, since it is already the case for functions in $L^2(\mathbb{R}^n)$.
If $\mathrm{D}^\alpha f \in L^2(\mathbb{R}^n)$, $\mathrm{D}^\alpha = (\partial/\partial x^1)^{\alpha_1} \cdots (\partial/\partial x^n)^{\alpha_n}$ we have (cf. p. 476, p. 490):
$\mathcal{F}(\mathrm{D}^\alpha f) = y^\alpha \mathcal{F}f \in L^2(\mathbb{R}^n)$, $y^\alpha = (y_1)^{\alpha_1} \ldots (y_n)^{\alpha_n}$. Thus if $f \in H_m(\mathbb{R}^n)$ we have

$$(1+|y|^2)^{m/2}\mathcal{F}f \in L^2(\mathbb{R}^n) \quad \text{thus} \quad \mathcal{F}f \in L^1(\mathbb{R}^n) \text{ if } m > n/2$$

since (Schwartz inequality)

$$\left|\int_{\mathbb{R}^n} (\mathcal{F}f)(y)\,\mathrm{d}y\right| < \left\{\int_{\mathbb{R}^n} (1+|y|^2)^m |\mathcal{F}f(y)|^2\,\mathrm{d}y\right\}^{1/2} \left\{\int_{\mathbb{R}^n} (1+|y|^2)^{-m}\,\mathrm{d}y\right\}^{1/2}$$

and

$$\int_{\mathbb{R}^n} (1+|y|^2)^{-m}\,\mathrm{d}y = K < \infty \quad \text{if} \quad m > n/2 .$$

If $\mathcal{F}f \in L^2(\mathbb{R}^n)$, f can be identified with a continuous and bounded function (p. 474)

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\mathcal{F}f)(y)\,\mathrm{e}^{\mathrm{i}x\cdot y}\,\mathrm{d}y ,$$

$$\|f\|_{C^0_{\mathrm{B}}(\mathbb{R}^n)} \leq \frac{K}{(2\pi)^n} \|(1+|y|^2)^{m/2}\mathcal{F}f\|_{L^2(\mathbb{R}^n)} .$$

Using the Plancherel theorem (p. 490) we find that,

$$\|f\|_{C^0_{\mathrm{B}}(\mathbb{R}^n)} \leq C\|f\|_{H_m(\mathbb{R}^n)} , \qquad m > n/2$$

here C is a constant depending only on n and m.

5) *Proof of part ii); to simplify the writing we first take $n=3$, $p=2$.*
a) *Suppose first that $f \in C_0^\infty(\mathbb{R}^3)$. Obtain an inequality for $|f(x)|^4$ by writing it as an integral over a coordinate line. Multiply all possible inequalities and integrate successively over x^1, x^2, x^3. Use the Schwartz inequality to prove*

$$\|f\|_{L^6} \leq C\|\nabla f\|_{L^2} , \quad \text{if} \quad f \in C_0^\infty(\mathbb{R}^3) .$$

b) *Extend the result to* $f \in H_1$. *Prove that then* $f \in L^q$, $2 \leq q \leq 6$.

c) The following question is useful when working with functions on $\mathbb{R}^n$ which are not square integrable (for instance falling off at infinity like $1/r$ in the case $n = 3$), though their derivatives are square integrable (the corresponding field has "finite energy").
Show that $\|\nabla f\|_{L^2}$ *is a norm on* $C_0^\infty(\mathbb{R}^3)$. *Give an equivalent definition of the Banach space completion of* $C_0^\infty(\mathbb{R}^3)$ *in this norm.*

Answer 5a: We have

$$f^4(x) = \int_{-\infty}^{x^1} \partial_1 f^4(\sigma, x^2, x^3)\, \mathrm{d}\sigma = \int_{-\infty}^{x^1} 4(f^3\, \partial_1 f)(\sigma, x^2, x^3)\, \mathrm{d}\sigma\,.$$

Thus, with obvious notation

$$|f^4(x)| \leq 4 \int_{-\infty}^{x^1} |f^3\, \partial^1 f|\, \mathrm{d}x^1$$

and analogous inequalities for x^2, x^3, from which we deduce

$$|f^4(x)|^3 \leq 4^3 \prod_{i=1}^{3} \omega_i^2(x)\,,$$

where the function

$$\omega_i(x) = \left\{ \int_{-\infty}^{\infty} |f^3(x)\partial_i f|\, \mathrm{d}x^i \right\}^{1/2}$$

depends only on x^j, $j \neq i$.
By integration on $\mathbb{R}^3$ we get

$$\int_{\mathbb{R}^3} f^6(x)\, \mathrm{d}x \leq 4^{3/2} \int_{\mathbb{R}^3} \prod_{i=1}^{3} \omega_i(x)\, \mathrm{d}x\,. \qquad (*)$$

Lemma: If ω_i is independent of x^i, then

$$\int_{\mathbb{R}^3} |\omega_1 \omega_2 \omega_3|\, \mathrm{d}x \leq \left\{ \left(\int_{\mathbb{R}^2} \omega_1^2\, \mathrm{d}x^2\, \mathrm{d}x^3 \right) \left(\int_{\mathbb{R}^2} \omega_2^2\, \mathrm{d}x^3\, \mathrm{d}x^1 \right) \left(\int_{R^2} \omega_3^2\, \mathrm{d}x^1\, \mathrm{d}x^2 \right) \right\}^{1/2}.$$

Proof:

$$\int_{\mathbb{R}^2} |\omega_1 \omega_2 \omega_3|\, \mathrm{d}x^1\, \mathrm{d}x^2 \leq \left\{ \int_{R^2} (\omega_1 \omega_2)^2\, \mathrm{d}x^1\, \mathrm{d}x^2 \int_{\mathbb{R}^2} \omega_3^2\, \mathrm{d}x^1\, \mathrm{d}x^2 \right\}^{1/2}$$

but, since ω_i does not depend on x^i

$$\int_{\mathbb{R}^2} (\omega_1 \omega_2)^2 \,\mathrm{d}x^1\,\mathrm{d}x^2 = \Big(\int_{\mathbb{R}} \omega_2^2\,\mathrm{d}x^1\Big)\Big(\int_{\mathbb{R}} \omega_1^2\,\mathrm{d}x^2\Big)$$

and

$$\int_{\mathbb{R}^3} |\omega_1\omega_2\omega_3|\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 \le \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \omega_1^2\,\mathrm{d}x^2\Big)^{1/2}\Big(\int_{\mathbb{R}} \omega_2^2\,\mathrm{d}x^1\Big)^{1/2}\,\mathrm{d}x^3$$
$$\times \Big(\int_{\mathbb{R}^2} \omega^2\,\mathrm{d}x^1\,\mathrm{d}x^2\Big)^{1/2}$$

which gives the result after one more application of the Schwartz inequality. ■

Using this lemma, the inequality (∗), and the definition of ω_i and once more the Schwartz inequality we obtain

$$\Big(\int_{\mathbb{R}^3} |f|^6\,\mathrm{d}x\Big)^{1/2} \le 4^3 \prod_{i=1}^{3} \|\partial_i f\|_{L^2},$$

thus

$$\|f\|_{L^6} \le \tfrac{1}{3} 4^3 \sum_{i=1}^{3} \|\partial_i f\|_{L^2}.$$

Answer 5b: Since $\|\nabla f\|_{L^2} \le \|f\|_{H_1}$ we have by the previous inequality

$$\|f\|_{L^6} \le C\|f\|_{H_1}, \qquad f \in C_0^\infty(\mathbb{R}^3).$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $H_1(\mathbb{R}^n)$ any function $f \in H_1(\mathbb{R}^3)$ is limit, in the H_1-norm, of a sequence $\{f_n\} \in C_0^\infty(\mathbb{R}^3)$. Each f_n satisfies the inequality, therefore their limit f also satisfies this inequality.

To prove that $f \in L^q$, $2 \le q \le 6$, we prove, more generally that if U is an arbitrary open set $U \subset \mathbb{R}^n$, $f \in L^p(U)$ and $f \in L^q(U)$, for given p and q, $1 \le p \le q$, then one has also $f \in L^r(u)$ for each r such that $p \le r \le q$. Indeed the Hölder inequality (p. 53) gives, writing $|f|^r = |f|^{r_1}|f|^{r_2}$, $r_1 + r_2 = r$:

$$\|f\|_{L^r}^r \le \|f\|_{L^{r_1 s}}^{r_1}\|f\|_{L^{r_2 s'}}^{r_2}, \qquad \frac{1}{s} + \frac{1}{s'} = 1.$$

We choose $r_1 = p/s$, $r_2 = q/s'$, thus

$$r = r_1 + r_2 = q + \frac{p-q}{s}, \qquad 0 < \frac{1}{s} < 1.$$

Answer 5c: The inequality of a) shows, if $f \in C_0^\infty(\mathbb{R}^3)$, that $\|\nabla f\|_{L^2} = 0$ implies $f = 0$. The semi-norm $f \mapsto \|\nabla f\|_{L^2}$ is therefore a norm on $C_0^\infty(\mathbb{R}^3)$. The completion in this norm of $C_0^\infty(\mathbb{R}^3)$ is identical with the space of measurable functions on $\mathbb{R}^3$ such that $f \in L^6(\mathbb{R}^3)$ and $\nabla f \in L^2(\mathbb{R}^3)$, with norm $\|\nabla f\|_{L^2} + \|f\|_{L^6}$.
Note that if U is a bounded open set in $\mathbb{R}^3$ the space $\{f \in L^6(U), \nabla f \in L^2(U)\}$ is identical with $H_1(U)$. It is not so if U is unbounded.

6) For functions $f \in C_0^\infty(\mathbb{R}^n)$ the following inequality can be proven along the same lines as in 2a (cf. Gagliardo-Nirenberg), if $n > p$

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \frac{q(n-1)}{2n} \|\nabla f\|_{L^p(\mathbb{R}^n)}, \qquad q = \frac{np}{n-p}.$$

Deduce from this inequality the embedding, if $m < n/p$

$$W_p^m(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \qquad p \leq q \leq \frac{np}{n-mp}.$$

Answer 6: The inequality proves, as in answer 2b, that

$$W_p^1(\mathbb{R}^n) \hookrightarrow L^{q_1}(\mathbb{R}^n), \qquad q_1 = \frac{np}{n-p}$$

and more precisely that the Banach space of distributions f on $\mathbb{R}^n$ such that

$$\nabla f \in L^p(\mathbb{R}^n),\ f \in L^s(\mathbb{R}^n) \text{ for some } 1 \leq s < +\infty$$

is continuously embedded in $L^q(\mathbb{R}^n)$, since $C_0^\infty(\mathbb{R}^n)$ is dense in the space defined above whose norm is such that

$$\|\nabla f\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^s(\mathbb{R}^n)} \geq \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

By considering a derivative of order $m-1$ of f we see that if $f \in W_p^m(\mathbb{R}^n)$ we have

$$D^{m-1} f \in L^{q_1}(\mathbb{R}^n), \qquad q_1 = \frac{np}{n-p}.$$

Since $D^{m-1}f \in L^{q_1}(\mathbb{R}^n)$ and $D^{m-2}f \in L^p(\mathbb{R}^n)$ we have $D^{m-2}f \in L^{q_2}(\mathbb{R}^n)$ with $q_2 = nq_1/(n-q_1) = np/(n-2p)$.
An easy induction achieves the proof for $q = np/(n-mp)$. Values of q between this one and p result from the proof in 2b.

7) The following theorem gives cases where the embedding given by the Sobolev theorem is not only continuous but also compact (p. 61).

Kondrakov theorem

Kondrakov theorem. Let U be a *bounded* open set of $\mathbb{R}^n$ which has the cone property. The embedding of the Sobolev theorem

$$W_p^{m+k}(U) \hookrightarrow W_q^k(U), \qquad m < \frac{n}{p},\ 1 \le p \le q < \frac{np}{n-mp}$$

is a compact embedding.

Use the fact that the composition of a continuous map and a compact map is a compact map to show that the Kondrakov theorem is a consequence of the particular case $k = 0$, $m = 1$, $q_1 = np/(n-p)$.
For the proof of this case see Adams or Aubin p. 54.

Answer 7: By the Sobolev embedding theorem we have the continuous embedding

$$W_p^m \hookrightarrow W_{p_1}^1, \qquad p_1 = \frac{np}{n-(m-1)p}.$$

We suppose that the embedding

$$W_{p_1}^1 \hookrightarrow L^{q_1}, \qquad q_1 < \frac{np_1}{n-p_1} = \frac{p}{n-mp}$$

is compact. The embedding

$$W_p^m \hookrightarrow L^{q_1}$$

is therefore compact, and that is each embedding

$$W_p^m \hookrightarrow L^q, \qquad p \le q < \frac{np}{n-mp}.$$

In fact, when U is a bounded open set, $L^q \hookrightarrow L^{q_1}$ if $q^1 \ge q$; indeed

$$\|f\|_{L^q} \le \operatorname{vol}(U)^{1/q-1/q^1} \|f\|_{L^{q_1}}.$$

The compacity $W_p^{m+k} \hookrightarrow W_q^k$ is then easy to prove going back to the definition of a compact mapping and the completude of these spaces (cf. p. 487). The Kondrakov theorem *does not hold* if U is not bounded. For instance $H_1(\mathbb{R}^n)$ is not compactly embedded in $L^2(\mathbb{R}^n)$ [for more on compact embeddings, cf. Problem VI 6, $H_{s,\delta}$ spaces].

8) The following theorem is very useful in nonlinear problems.

Interpolation theorem

Interpolation theorem (Gagliardo–Nirenberg).

Let q, r be any numbers satisfying $1 \le q \le \infty$, $1 \le r \le \infty$. Let j, m be any integers satisfying $0 \le j < m$. There exists $C > 0$ depending only on q, r, j, m, n such that any function $u \in \mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ satisfies the inequality

$$\|D^j u\|_{L^r} \le C \|D^m u\|_{L^r}^a \|u\|_{L^q}^{1-a}$$

with

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\,\frac{1}{q}$$

for all a in the interval $j/m \leq a \leq 1$ for which p is nonnegative. The theorem is not valid for $a = 1$ if $r = n/(m-j) \neq 1$. *Prove the interpolation theorem in the case $j = 1$, $m = 2$, $a = 1/2$, $p \geq 2$.*

Answer 8: (Cf. Aubin, p. 93). The inequality to prove reads

$$\|Du\|^2_{L^r} \leq C\|D^2u\|_{L^r}\|u\|_{L^q}\,, \qquad \frac{2}{p} = \frac{1}{r} + \frac{1}{q}\,.$$

We have

$$\sum_{i=1}^{n} \partial_i(u|Du|^{p-2}\partial_i u) = |Du|^p + u|Du|^{p-2}\,\Delta u + (p-2)u|Du|^{p-4} \sum_{i,j=1}^{n} \partial^2_{ij}u\partial_i u\partial_j u$$

which gives by integration if $u \in \mathscr{D}(\mathbb{R}^n)$

$$\|Du\|^p_{L^p} \leq \int_{\mathbb{R}^n} |u||Du|^{p-2}(|\Delta u| + (p-2)|D^2u|)\,\mathrm{d}x\,.$$

We have

$$|\Delta u|^2 \leq n|D^2u|^2\,,$$

hence, by using the Hölder inequality

$$\|Du\|^p_{L^p} \leq (n^{1/2} + (p-2))\|u\|_{L^q}\|Du\|^{p-2}_{L^p}\|D^2u\|_{L^r}\,, \qquad \frac{1}{q} + \frac{1}{r} = \frac{2}{p}\,,$$

which leads to the result, which $C = n^{1/2} + (p-2)$.

Note that the inequality is a fortiori valid for functions in $\mathscr{D}(U)$, U an open set of $\mathbb{R}^n$. It extends to Sobolev spaces obtained by appropriate completion if U has the relevant properties (cf. Adams pp. 66–67).

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3. MULTIPLICATION PROPERTIES OF SOBOLEV SPACES

1) *Consider a bounded open set of* $\mathbb{R}^n$ *having the cone property* [*Problem* VI 2, Sobolev, p. 379].

a) *Show that the Sobolev spaces* $W_p^m(U)$ *have the following continuous* **multiplication property**

multiplication property

$$W_p^M \times W_q^m \hookrightarrow L^p \qquad \text{by } (f, g) \mapsto fg\,.$$

$$1 < p \leq q \leq +\infty\,, \qquad m + M > \frac{n}{q}\,, \qquad m \geq 0, M \geq 0$$

b) *Extend the result to*

$$W_p^M \times W_q^m \hookrightarrow W_p^s$$

if $\quad 1 < p \leq q \leq \infty\,, \qquad m + M > \dfrac{n}{q} + s\,, \qquad m \geq s\,, \qquad M \geq s\,.$

Answer 1a: Suppose $f \in W_p^M$, $g \in W_q^m$, $q < \infty$ (the case $q = \infty$ is trivial: $W_p^M \times W_\infty^m \hookrightarrow L^p$, $\forall m \geq 0$, $M \geq 0$). Distinguish three cases:

i) $m > n/q$.
Then $g \in C_B^0$ (see definition in Problem VI 2, Sobolev); thus, since $f \in L^p$ for any $M \geq 0$, $fg \in L^p$.

ii) $m \leq n/q$ and $M \leq n/p$.
Then since $m + M > n/q$ there exists $r \geq 1$ and $s \geq 1$ such that $m > n/q - n/r$, $M > n/p - n/s$ and $1/r + 1/s = 1/p$.
For such exponents we have [Problem VI 2, Sobolev]

$$\|f\|_{L^s} \leq c\|f\|_{W_p^M}$$

$$\|g\|_{L^r} \leq c\|g\|_{W_q^m}$$

while

$$\|fg\|_{L^p} \leq \|f\|_{L^s}\|g\|_{L^r}\,.$$

The conclusion follows.

iii) $M > n/p$.
Then $f \in C_B^0$ and $fg \in L^p$ if $g \in L^p$. We know that if $g \in L^q$ on a bounded open set $U \subset \mathbb{R}^n$, then $g \in L^p$, $p \leq q$.

Answer 1b: Suppose $f \in W_p^M$ and $g \in W_q^m$, $1 < p \leq q \leq +\infty$, then $\partial_\alpha^l f \in W_p^{M-l}$, $\partial_\alpha^l g \in W_q^{m-l}$ thus

$$\partial_\alpha^k(fg) = \sum_{l=0}^{k} C_l^k \partial_\alpha^l f \partial^{k-l} g \in L^p$$

$$\text{if } m + M > \frac{n}{q} + k\,, \quad m \geq k\,, \quad M \geq k\,.$$

2) *Formulate a more restricted theorem when U is an open set of $\mathbb{R}^n$ with the cone property, not necessarily bounded. Give some examples on $\mathbb{R}^3$.*

Answer 2: The previous reasoning applies for unbounded U, except in the case iii), since if $f \in C_B^0$ and $g \in L^q$ we do not have $fg \in L^p$ for $p < q$. The restricted theorem is therefore

$$W_p^M \times W_p^m \hookrightarrow W_p^s\,, \qquad \text{by } (f, g) \mapsto fg\,,$$

$$1 < p \leq \infty\,, \qquad m + M > \frac{n}{p} + s\,, \qquad m \geq s\,, \qquad M \geq s\,.$$

Remark: When $m > n/p$ we have

$$W_p^m \times W_p^m \hookrightarrow W_p^m$$

that is W_p^m, $m > n/p$, is an algebra, called a **Sobolev algebra**. Sobolev algebra

Examples: on $\mathbb{R}^3$ we have

$$H_1 \times H_1 \hookrightarrow L^2$$

$$H_2 \times H_2 \hookrightarrow H_2\,, \qquad H_2 \text{ is an algebra}\,.$$

3) *Let F be a mapping of class C_B^m from an open set Y of $\mathbb{R}$ into $\mathbb{R}$. Let $f \in W_p^m$ on a bounded open set U of $\mathbb{R}^n$ with the cone property, $m > n/p$, and f be such that $f(U) \subset Y$. Show the* **composition theorem**: $F \circ f \in W_p^m$. composition theorem

Answer: We first take $f \in C^m(U) \cap W_p^m(U)$. Then $F \circ f \in C^m(U) \subset L^p(U)$ when U is bounded. The derivatives are given by the law of derivation of a composition of maps: we denote by ∂^k a partial derivative of order k by $\mp$ a sum with some numerical coefficient, by F', F'', . . . the derivatives of the mapping F: $Y \to \mathbb{R}$. We have

$$\begin{aligned}
\partial(F \circ f) &= F'(f)\partial f \\
\partial^2(F \circ f) &= F''(f)(\partial f)^2 \mp F'(f)\partial^2 f \\
\partial^k(F \circ f) &= F^{(k)}(f)(\partial f)^k \mp F^{(k-1)}(f)(\partial f)^{k-2}\partial^2 f \\
&\quad \mp F^{(k-2)}(f)((\partial f)^{k-3}\partial^3 f \mp (\partial f)^{k-4}(\partial^2 f)^2) \\
&\quad + \cdots \mp F'(f)\partial^k f\,.
\end{aligned}$$

We have $F^{(l)}(f)$ in $C^0_B(U)$, since $f \in C^0_B(U)$ (Sobolev theorem) $f(U) \subset Y$ and $F \in C^m_B(Y)$ (hypothesis). Now, $\partial^k(F \circ f)$ is a sum of terms. Each term is the product of a function $F^{(l)}(f)$ by a product of derivatives of f, $(\partial^{l_1} f)^{m_1} \cdots (\partial^{l_s} f)^{m_s}$ such that $l_1 m_1 + \cdots + l_m m_s = k$. By the Sobolev embedding theorem we have

$$\|\partial^l f\|_{L^{q_l}} \leq C \|\partial^l f\|_{W^{m-l}_p} \leq C \|f\|_{W^m_p}$$

if

$$p \leq q_l \leq \frac{np}{n-(m-l)p} \qquad \text{when } n > (m-l)p$$

for all q_l if $m > l + n/p$. We suppose for simplicity $n/p + 1 > m > n/p$, then all q_l are bounded by the above formula. It is straightforward to extend the proof to larger m.
We see by using the Hölder inequality that we shall have $\partial^k(F \circ f) \in L^p$ if there exist numbers $r_1 \cdots r_p$ such that

$$\frac{1}{r_1} + \cdots + \frac{1}{r_s} = 1$$

and

$$pm_i r_i \leq \frac{np}{n-(m-l_i)p}\,, \qquad i = 1, \ldots, s\,,$$

hence if

$$1 \geq \sum_{i=1}^{s} \frac{m_i(n-(m-l_i)p)}{n}\,.$$

Since we have $\Sigma\, m_i l_i = k$ and $\Sigma\, m_i \leq qk$ the inequality will be satisfied if

$$k(p - mp + n) \leq n$$

i.e., since $p - mp + n > 0$

$$k \leq \frac{n}{p - mp + n}\,.$$

The derivatives $\partial^k(F \circ f)$, $k \leq m$, will have L^p norms bounded by the W^m_p norm of f, and the C^m_B norm of F, (even for unbounded U) if

$$m \leq \frac{n}{p - mp + n}\,, \qquad \text{i.e., } m^2 p - m(n+p) + n \geq 0\,;$$

therefore if

$$m > n/p\,,$$

since n/p is the greatest root of this polynomial in m.

The proof of the theorem for general $f \in W_p^m(U)$ is obtained by approaching it with functions in $C^m(U) \cap W_p^m(U)$.

If U is unbounded the **composition theorem** still holds under the additional assumption, satisfied if $F \in C_B^1(y)$, $0 \in y$, $F(0) = 0$, that $F \circ f \in L^p(U)$ when $f \in C_B^0(U) \cap L^p(U)$. composition theorem

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4. THE BEST POSSIBLE CONSTANT FOR A SOBOLEV INEQUALITY ON $\mathbb{R}^n$, $n \geq 3$*

From rearrangement inequalities (see for example Appendix A of ref. [1]) one deduces that the infimum

$$S_n \equiv \inf_{\psi \in C_0^\infty(\mathbb{R}^n)} F_n(\psi) = \omega_{n-1}^{2/n} \inf_{\chi \in C_0^\infty(\mathbb{R}^+)} G_n(\chi) , \qquad p = 2n/(n-2) ,$$

where

$$F_n(\psi) = \|\nabla\psi\|_2^2 / \|\psi\|^2 p , \qquad \|\psi\|_p^p = \int \mathrm{d}^n x |\psi(x)|^p ,$$

$$G_n(\chi) = \int_0^\infty \mathrm{d}r \left|\frac{\mathrm{d}\chi(r)}{\mathrm{d}r}\right|^2 \Big/ \left(\int_0^\infty \mathrm{d}r |\chi(r)|^p\right)^{2/p} ,$$

is given by rotational symmetric functions. Here ω_{n-1} denotes the area of the S^{n-1} sphere; $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$.

Assume that the infimum is attained (for a proof see for instance Refs. [1, 4 or 5]) and that the minimizing function is given by the naive variational equation. Solve this nonlinear equation by transforming to new variables $z = \ln r \in (-\infty, \infty)$ and $r^{(n-2)/2}\chi(r) = \phi(z)$ [1–3] and calculate S_n. Study the degeneracy and the conformal invariance of the above functional. Do a conformal mapping of the variational equation to an n-dimensional sphere imbedded in $\mathbb{R}^{n+1}$.

Answer: Let

$$I_\psi = \|\nabla\psi\|_2^2 , \qquad J_\psi = \|\psi\|_p^p , \qquad F_n(\psi) = I_\psi / J_\psi^{2/p} ;$$

*Contributed by H. Grosse.

the naive variational equation is given by

$$-\Delta\psi(x) - \lambda|\psi(x)|^{4/(n-2)}\psi(x) = 0 \qquad \text{with } \lambda = I_\psi/J_\psi,\ x \in \mathbb{R}^n\ ,$$

since

$$\delta\left(\frac{\int |\nabla\psi|^2}{\left(\int |\psi|^p\right)^{2/p}}\right) = \delta\left(\frac{I}{J^{2/p}}\right) = \frac{\delta I}{J^{2/p}} - \frac{2}{p}\,\frac{I}{J^{(2/p)}+1}\,\delta J = 0$$

and $\delta I = -2\Delta\psi$, $\delta J = p|\psi|^{p-1}$ yield $-\Delta\psi - (I/J)\psi^{p-1} = 0$.
Take ψ to be real, positive and rotational symmetric.

By the indicated change of variables we obtain the functional

$$G_n(\phi) = \int_{-\infty}^{\infty} \mathrm{d}z\left\{\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2 + \frac{(n-2)^2}{4}\,\phi^2(z)\right\}\Big/\left\{\int_{-\infty}^{\infty} \mathrm{d}z\ \phi(z)^{2n/(n-2)}\right\}^{2/p}$$

$$= I_\phi/J_\phi^{(n-2)/n}\ ,$$

and the transformed variational equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\,\phi(z) + \frac{(n-2)^2}{4}\,\phi(z) - \lambda\phi^{(n+2)/(n-2)}(z) = 0\ , \qquad \lambda = I_\phi/J_\phi\ .$$

This equation we solve under the conditions $\phi(\pm\infty) = (\mathrm{d}/\mathrm{d}z)\phi(\pm\infty) = 0$:

$$\phi(z) = \frac{c_n}{(\operatorname{ch} z)^{(n-2)/2}}\ , \qquad \lambda c_n^{4/(n-2)} = \frac{n(n-2)}{2}\ .$$

Note that c_n drops out if one inserts $\phi(z)$ into $G_n(\phi)$. Transformation to the old variables gives

$$\chi(r) = \frac{2^{(n-2)/2}c_n}{(1+r^2)^{(n-2)/2}} \Rightarrow \psi(x) = \frac{k_n}{(1+|x|^2)^{(n-2)/2}}\ , \qquad k_n = 2^{(n-2)/2}c_n\ ,$$

and yields the best value for the infimum of F_n:

$$S_n = n(n-2)\pi\left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^{2/n}\ .$$

By translation $x \mapsto x - a$, $a \in \mathbb{R}^n$, and dilation $x \mapsto \lambda x$ we obtain an $(n+1)$-parameter family of minimizing functions:

$$\psi(x) \mapsto \mu\psi(\rho(x-a))\ , \qquad \psi(x) \mapsto \frac{1}{|x|^{n-2}}\,\psi\left(\frac{x}{|x|^2}\right).$$

The stereographic projection to $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is given by

$$\xi_0 = \frac{1-r^2}{1+r^2}\,, \qquad \xi_i = \frac{2x_i}{1+r^2}\,,$$

the Laplacian transforms into [1]

$$-\left(\frac{1+r^2}{2}\right)^2 \Delta = \left(\frac{1+r^2}{2}\right)^{-(n-2)/2}\left(\left(l^2 + \frac{n(n-2)}{4}\right)\left(\frac{1+r^2}{2}\right)^{(n-2)/2}\right),$$

where

$$l^2 = l_{\alpha\beta} l^{\alpha\beta}\,, \qquad l_{\alpha\beta} = -\mathrm{i}\xi_\alpha \frac{\partial}{\partial \xi_\beta} + \mathrm{i}\xi_\beta \frac{\partial}{\partial \xi_\alpha}\,,$$

denotes the angular momentum operator on the sphere. Define new functions

$$\sigma(\xi) = \left(\frac{1+r^2}{2}\right)^{(n-2)/2} \psi(x)\,;$$

then we get

$$\left(l^2 + \frac{n(n-2)}{4} - \lambda |\sigma(\xi)|^{4/(n-2)}\right)\sigma(\xi) = 0\,.$$

$\sigma = \text{const}$ is a solution of this equation; transforming back from (ξ, σ) to (x, ψ) yields

$$\psi(x) = K_n(1+|x|^2)^{(2-m)/2}\,, \qquad K_n = 2^{(n-2)/2} c_n\,.$$

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5. HARDY-LITTLEWOOD-SOBOLEV INEQUALITY*

The Hardy-Littlewood-Sobolev inequality [1] states that

$$\left| \int_{\mathbb{R}^n} \mathrm{d}^n x \int_{\mathbb{R}^n} \mathrm{d}^n y f(x) \frac{1}{|x-y|^\lambda} g(y) \right| \leq N_{p,\lambda,n} \|f\|_p \|g\|_q \qquad (1)$$

*Contributed by H. Grosse.

holds for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $1 < p, q < \infty$, $1/p + 1/q + \lambda/n = 2$ and $0 < \lambda < n$. An extensive discussion of the existence of optimizing functions and the determination of best possible constants was recently given in [2].

Here we illustrate the above inequality for the special case of $f = g$ and $\lambda = 1$. It can be proved by rearrangement inequalities, that the inequality will be true if it holds for spherical symmetric functions f, i.e., functions depending only on $|x|$; integration over angles leads to bounds of the following form. To simplify notations we set now $|x| = x$ and $|y| = y$ and denote $\max(x, y)$ by $(x, y)_>$. The Hardy–Littlewood–Sobolev inequality is then written

$$\left|\int_0^\infty \mathrm{d}x\, x^{n-1} \int_0^\infty \mathrm{d}y\, y^{n-1} f(x) \frac{1}{(x, y)_>} f(y)\right| \le C_n \left(\int_0^\infty \mathrm{d}x\, x^{n-1} |f(x)|^r\right)^{2/r} \tag{2}$$

with $2/r + 1/n = 2$.

Use Hölder's inequality (p. 53) to bind the left-hand side of (2). Introduce $1 = (x/y)^\sigma (y/x)^\sigma$ as a factor and use Schwarz' inequality to obtain a finite expression where $(x, y)_>$ and f appear in different integrals.

Answer: From Hölder's inequality we get

$$\left|\int_0^\infty \mathrm{d}x\, x^{n-1} \int_0^\infty \mathrm{d}y\, y^{n-1} \frac{f(x)f(y)}{(x, y)_>}\right| \le \left(\int_0^\infty \mathrm{d}x\, x^{n-1} |f(x)|^{p(1-\alpha)}\right)^{2/p} J_{q,\alpha}^{1/q},$$

$$J_{q,\alpha} = \int_0^\infty \mathrm{d}x\, x^{n-1} \int_0^\infty \mathrm{d}y\, y^{n-1} \frac{|f(x)|^{q\alpha} |f(y)|^{q\alpha}}{(x, y)_>^q}.$$

In order to separate f from $(x, y)_>$ we introduce $(x/y)^\sigma$ times its inverse and obtain from Schwarz' inequality

$$J_{q,\alpha} \le \int_0^\infty \mathrm{d}x\, x^{n-1} \int_0^\infty \mathrm{d}y\, y^{n-1} \frac{|f(x)|^{2q\alpha}}{(x, y)_>^q} \left(\frac{x}{y}\right)^{2\sigma}.$$

Homogeneity is obtained for $q = n$:

$$J_{n,\alpha} \le \int_0^\infty \mathrm{d}x\, x^{n-1} |f(x)|^{2n\alpha} \left(\frac{1}{n - 2\sigma} + \frac{1}{2\sigma}\right).$$

Optimize in σ and choose $\alpha = 1/(2n - 1)$ in order to obtain the Hardy–Littlewood–Sobolev inequality for this special case.

$$\left| \int_{\mathbb{R}^n} \mathrm{d}^n x \int_{\mathbb{R}^n} \mathrm{d}^n y f(\boldsymbol{x}) \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} f(\boldsymbol{y}) \right| \leq C_{p,n} \| f \|_p^2, \qquad \frac{2}{p} + \frac{1}{n} = 2.$$

The above steps work for various other related inequalities, like e.g. weighted ones.

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[1] M. Reed and B. Simon, "Methods of Modern Mathematical Physics II" (Academic Press, 1975).
[2] E.H. Lieb, Ann. of Math. 118 (1983) 349.

6. SPACES $H_{s,\delta}(\mathbb{R}^n)$

A supplement to this problem entitled "A density theorem" can be found near the end of the book.

1) The **space** $\boldsymbol{H_{s,\delta}}$ is the space of distributions f on $\mathbb{R}^n$ which, together with their derivatives $D^k f$, $k = 0, \ldots, s$ of order $\leqq s$ are measurable functions such that space $H_{s,\delta}$

$$\sigma^{\delta+k} D^k f \in L^2(\mathbb{R}^n), \qquad \sigma = (1 + r^2)^{1/2}, \qquad r^2 = \sum_{i=1}^n (x^i)^2.$$

Show that $H_{s,\delta}$ is a Hilbert space for the norm

$$\| f \|_{H_{s,\delta}} = \left\{ \int_{\mathbb{R}^n} \sum_{k=0}^{s} \sigma^{2\delta+2k} \left| (\mathrm{D}^k f)(x) \right|^2 \mathrm{d}x \right\}^{1/2} \tag{1}$$

($\mathrm{d}x = \mathrm{d}x^1 \ldots \mathrm{d}x^n$, $|\,|$ euclidean norm of $\mathrm{D}^k f$, that is $|\mathrm{D}^k f|^2 = \delta^{i_1 j_1} \ldots \delta^{i_k j_k} \partial^k_{i_1 \ldots i_k} f \partial^k_{j_1 \ldots j_k} f$).
Show that $H_{s,\delta} \subset H_{s',\delta'}$, if $s \geqq s'$, $\delta \geqq \delta'$.

2) *Show the embedding property*

$$H_{s,\delta}(\mathbb{R}^n) \hookrightarrow C_{m,\delta'}(\mathbb{R}^n), \quad \textit{if } s > m + n/2,\ \delta > \delta' - n/2$$

where $C_{m,\delta'}$ is the Banach space of C_m functions on $\mathbb{R}^n$ such that

$$\| f \|_{C_{m,\delta'}} = \sup_{\substack{x \in \mathbb{R}^n \\ 0 \leq k \leq m}} \sigma^{\delta'+k} \left| \mathrm{D}^k f(x) \right|$$

is finite.

3) *Show the multiplication property*

$$H_{s_1,\delta_1} \times H_{s_2,\delta_2} \to H_{s,\delta} \quad \text{by} \quad (f, h) \to fh$$

if $s_1 + s_2 > s + n/2$, $\delta_1 + \delta_2 > \delta - n/2$.

For which values of s and δ is $H_{s,\delta}$ a Banach algebra?

Answer 1: $H_{s,\delta}$ is a real vector space, whose norm (1) comes from the scalar product (p. 11) (well defined, due to the Cauchy–Schwarz inequality):

$$(f, g)_{H_{s,\delta}} = \int_{\mathbb{R}^n} \sum_{k=0}^{s} \sigma^{2\delta+2k} \mathrm{D}^k f(x) \mathrm{D}^k g(x)\, \mathrm{d}x \,.$$

It is complete, because $L^2(\mathbb{R}^n)$ is complete, and because $f \in H_{s,\delta}$ if and only if $\sigma^{\delta+k}\mathrm{D}^k f \in L^2(\mathbb{R}^n)$, $k = 0, \ldots, s$; if f_n is a Cauchy sequence in $H_{s,\delta}$ then $\sigma^\delta f_n$ converges in L^2 (and a fortiori in $\mathscr{D}'$) towards some function $g \in L^2$; since σ^δ is a C^∞ function on $\mathbb{R}^n$, f converges in $\mathscr{D}'$ towards $\sigma^{-\delta} g$ (p. 444). Analogously $\mathrm{D}^k f_n$ converges in $\mathscr{D}'$ towards a function $\sigma^{-(\delta+k)} g_k$, with $g_k \in L^2$, and by the continuity of derivation in $\mathscr{D}'$ we have $\sigma^{-(\delta+k)} g_k = \mathrm{D}^k(\sigma^{-\delta} g)$, that is

$$\sigma^{\delta+k}\mathrm{D}^k f = g_k \in L^2 \,, \qquad f \in H_{s,\delta} \,.$$

We have $\sigma^\delta \geqq \sigma^{\delta'}$ if $\delta \geqq \delta'$, both C^∞ on $\mathbb{R}^n$ thus $f \in H_{s,\delta}$ implies $f \in H_{s',\delta'}$ if $s \geqq s'$, $\delta \geqq \delta'$.

Answer 2: We show that if $f \in H_{s,\delta}$ we have also $\sigma^{\delta+k} D^k f \in H_{s-k}$ for $0 \leqq k \leqq s$. We have

$$\partial_i \sigma = x^i \sigma^{-1} \,, \qquad \text{thus } |\mathrm{D}\sigma| \leq 1 \text{ on } \mathbb{R}^n \,;$$

more generally, for each k there is a constant C such that

$$|D^k \sigma| \leq C\sigma^{-k+1} \,. \tag{1}$$

We deduce therefore from the Leibnitz formula, with $C_{m,l}$ some constants >0:

$$|\mathrm{D}^l(\sigma^\delta f)| \leqq \sigma^\delta |\mathrm{D}^l f| + C_{m,l}\sigma^{\delta-l+m}|\mathrm{D}^m f| \tag{2}$$

if $\sigma^{\delta+k}\mathrm{D}^k f \in L^2$, $0 \leqq k \leqq s$, we have a fortiori, by (2): $\mathrm{D}^l(\sigma^\delta f) \in L^2$, $0 \leqq l \leqq s$ that is $\sigma^\delta f \in H_s$, more generally $\mathrm{D}^l(\sigma^{\delta+k}\mathrm{D}^k f) \in L^2$, $0 \leqq l \leqq s-k$, thus $\sigma^{\delta+k}\mathrm{D}^k f \in H_{s-k}$.
From $\sigma^{\delta+k}\mathrm{D}^k f \in H_{s-k}$ we deduce the embedding

$$\mathrm{D}^k f \in C_{0,\delta+k} \qquad \text{if } s-k > n/2 \,,$$

thus

$$f \in C_{m,\delta} \qquad \text{if } m < s - n/2 \,.$$

We improve the embedding as follows.

We consider the diffeomorphism φ_ε of $\mathbb{R}^n$:

$$\varphi_\varepsilon : x \to y = \frac{x}{\sigma(x)^{1-\varepsilon}}, \qquad 0 < \varepsilon \leqq 1 \tag{3}$$

and denote by T_ε the automorphism of the ring of functions on $\mathbb{R}^n$ defined by

$$T_\varepsilon f = f \circ \varphi_\varepsilon^{-1} .$$

We shall prove that T_ε is an isomorphism:

i) $C^s_\delta \to C^s_{\delta/\varepsilon}$ $\quad s \in \mathbb{N}, \delta \in \mathbb{R}$
ii) $H_{s,\delta} \to H_{s,(\delta+n/2)/\varepsilon - n/2}$, $\quad s \in \mathbb{N}, \delta \in \mathbb{R}$.

We deduce from (3) that

$$\sigma^\varepsilon(x) \leqq \sigma(y) \leqq 2^{1/2}\sigma^\varepsilon(x) \tag{4}$$

and

$$\frac{\partial y^i}{\partial x^j} = \left(\delta_{ij} - (1-\varepsilon)\frac{x^i x^j}{\sigma(x)^2}\right)\sigma^{\varepsilon-1}(x)$$

and thus, for any function f on $\mathbb{R}^n$

$$\frac{\partial f}{\partial x^i}(x) = \left(\delta_{ij} - (1-\varepsilon)\frac{x^i x^j}{\sigma(x)^2}\right)\sigma^{\varepsilon-1}(x)\,\frac{\partial(T_\varepsilon f)}{\partial y^i}(y) . \tag{5}$$

The quadratic form between parenthesis is uniformly bounded if $0 \leqq \varepsilon < 1$, as well as its inverse

$$\delta_{ij} + (1-\varepsilon)\frac{x^i x^j}{1+\varepsilon|x|^2} .$$

Therefore there exist positive constants C_1 and C_2 such that

$$C_1|\sigma(x)\mathrm{D}f(x)| \leqq |\sigma(y)(\mathrm{D}T_\varepsilon f)(y)| \leqq C_2|\sigma(x)\mathrm{D}f(x)|$$

similarly, with different constants C_1, C_2,

$$C_1 \sum_{k=0}^{s} |(\sigma^{\delta+k}\mathrm{D}^k f)(x)| \leqq \sum_{k=0}^{s} |(\sigma^{\delta/\varepsilon+k}\mathrm{D}^k T_\varepsilon f)(y)|$$
$$\leqq C_2 \sum_{k=0}^{s} |(\sigma^{\delta+k}\mathrm{D}^k f)(x)| . \tag{6}$$

The isomorphism $C_{s,\delta} \to C_{s,\delta/\varepsilon}$ follows from (6). The isomorphism $H_{s,\delta} \to H_{s,(\delta+n/2)/\varepsilon-n/2}$ follows from (6) and the relation between volume elements

$$\mathrm{d}y = \det(\delta_{ij} - \sigma^{-2}x^i x^j)\sigma^{n(\varepsilon-1)}\,\mathrm{d}x .$$

We use these isomorphisms to improve the embedding ($s > m + n/2$)

$$f \in H_{s,\delta} \to T_\varepsilon f \in H_{s,(\delta+n/2)/\varepsilon - n/2}\,,$$
$$\to T_\varepsilon f \in C_{m,(\delta+n/2)/\varepsilon - n/2} \to f \in C_{m,(\delta+n/2)\varepsilon - n/2}$$

hence by letting ε tend to zero

$$f \in C_{m,\delta+n/2}\,.$$

Answer 3: The proof is analogous: if $f_1 \in H_{s_1,\delta_1}$, $f_2 \in H_{s_2,\delta_2}$ with $s_1 + s_2 > s + n/2$ we have obviously

$$f_1 f_2 \in H_{s,\delta_1+\delta_2}$$

but also, by the isomorphism

$$T_\varepsilon f_1 T_\varepsilon f_2 \in H_{s,(\delta_1+\delta_2+n)/\varepsilon - n}$$

thus

$$f_1 f_2 \in H_{s,\delta_1+\delta_2+n/2-\varepsilon n/2}$$

which gives again the result by letting ε tend to zero.
$H_{s,\delta}$ is a Banach algebra when the product of f_1, $f_2 \in H_{s,\delta}$, that is if

$$s > n/2, \quad \delta > -n/2\,.$$

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Y. Choquet-Bruhat and D. Christodoulou, Acta Matematica 146 (1981) 129–150.

7. SPACES $H_s(S^n)$ AND $H_{s,\delta}(\mathbb{R}^n)$

Denote by φ the diffeomorphism $\{S^n - \text{South pole}\} \to \mathbb{R}^n$ defined by stereographic coordinates [Problem V 4, Sphere]. Let f be a function on $\mathbb{R}^n$ with $f \in H_{s,\delta}(\mathbb{R}^n)$ (Problem VI 6, Spaces). Find, for a given s, a sufficient condition on δ for the function $h = f \circ \varphi^{-1}$ to be in $H_s(S^n)$.

Answer: $h = f \circ \varphi^{-1}$ is defined almost everywhere on S^n and measurable, as well as its derivatives of order $\leqq s$. It is in $H_s(S^n)$ if the g-norm of these derivatives (g metric of S^n) is square integrable in the g-volume element. We denote by $\tilde{g} = \varphi^{-1*}g$ the expression of the metric of S^n in stereographic coordinates. We know that if e denotes the metric of $\mathbb{R}^n$ in rectilinear coordinates [Problem V 4, Sphere]

$$\tilde{g} = \sigma^{-4} e$$

with σ given by

$$\sigma = \left\{1 + \frac{r^2}{4}\right\}^{1/2}, \qquad r^2 = \sum_{i=1}^{n} (x^i)^2 .$$

The volume elements of $(\mathbb{R}^n, e)$ and $(S^n, \tilde{g})$ are linked by

$$\mathrm{d}\mu(\tilde{g}) = \sigma^{-2n}\,\mathrm{d}\mu(e) . \tag{1}$$

Therefore $h \in L^2(S^n)$ if and only if $\sigma^{-n} f \in L^2(\mathbb{R}^n)$.
We denote by $\#$ the contravariant tensor associated to a metric, by $\partial, \tilde{\nabla}, \nabla$ the covariant derivatives in the metrics $e, \tilde{g}, g$. We have

$$g^{\#}(\nabla h, \nabla h)(p) = \tilde{g}^{\#}(\tilde{\nabla} f, \tilde{\nabla} f)(x) = \sigma^4 e^{\#}(\tilde{\nabla} f, \tilde{\nabla} f)(x) ,$$
$$x = \varphi(p), \qquad \tilde{\nabla} f = \partial f . \tag{2}$$

Thus $\nabla h \in L^2(S^n)$ if and only if $\sigma^{2-n}\partial f \in L^2(\mathbb{R}^n)$.
More generally

$$\begin{aligned} |\nabla^k h|_g^2 &= g^{i_1 j_1} \cdots g^{i_k j_k} \nabla_{i_1 \ldots i_k} h \nabla_{j_1 \ldots j_k} h \\ &= \tilde{g}^{i_1 j_1} \cdots \tilde{g}^{i_k j_k} \tilde{\nabla}_{i_1 \ldots i_k} f \tilde{\nabla}_{j_1 \ldots j_k} f \\ &= \sigma^{4k} |\tilde{\nabla}^k f|_e^2 . \end{aligned} \tag{3}$$

On the other hand

$$\tilde{\nabla}_i f = \partial_i f , \qquad \tilde{\nabla}_{ij} f = \partial^2_{ij} f - \tilde{\Gamma}^k_{ij} \partial_k f$$

with

$$\tilde{\Gamma}^k_{ij} = -2\sigma^{-1}(\delta^k_i \partial_j \sigma + \delta^k_j \partial_i \sigma - \delta^j_i \partial_k \sigma)$$

we have

$$\partial_i \sigma = \left(1 + \frac{r^2}{4}\right)^{-1/2} x^i \qquad \text{thus } |\partial_i \sigma| \leqq 4$$

$$|\tilde{\Gamma}^k_{ij}|_e \leqq C\sigma^{-1} , \qquad C = 24 .$$

Therefore

$$|\tilde{\nabla}^2 f|_e \leqq |\partial^2 f|_e + C\sigma^{-1} |\partial f|_e , \qquad C = 24n .$$

More generally if $u_{i_1 \ldots i_p}$ is a p-covariant tensor

$$\tilde{\nabla}_j u_{i_1 \ldots i_p} = \partial_j u_{i_1 \ldots i_p} - \sum_{m=1}^{p} \tilde{\Gamma}^k_{j i_m} u_{i_1 \ldots k \ldots i_p}$$

thus

$$|\tilde{\nabla} u|_e \leqq |\partial u|_e + C\sigma^{-1} |u| , \qquad C = 24 n^p .$$

By induction on k we have therefore, if $k \geqq 1$

$$|\tilde{\nabla}^k f|_e \leqq |\partial^k f|_e + \sum_{p=1}^{k-1} C_p \sigma^{-p} |\partial^{k-p} f|_e \,, \tag{4}$$

where the C_p are some positive numbers.
From (1), (3), (4) we deduce that $\nabla^k h \in L^2(S^n)$, $k \geq 1$, if

$$\sigma^{2k-p-n} \partial^{k-p} f \in L^2(\mathbb{R}^n)\,, \qquad p = 0, \ldots, k-1\,,$$

that is if

$$\partial f \in H_{k-1,k+1-n}(\mathbb{R}^n)\,.$$

We shall have $h \in H_s(S^n)$, that is $\nabla^k h \in L^2(S^n)$, $k = 0, \ldots, s$ if

$$\sigma^{-n} f \in L^2(\mathbb{R}^n)\,, \qquad \partial f \in H_{s-1,s+1-n} \tag{5a,b}$$

which will be a fortiori satisfied if

$$f \in H_{s,\delta}(\mathbb{R}^n)\,, \qquad \delta \geq s - n\,. \tag{6}$$

Even the condition (5b) is not necessary when $s > 1$. For instance let $h \in C^s(S^n)$ then $h \in H_s(S^n)$, but $\partial f = \partial(h \circ \varphi^{-1})$ will not in general be in $H_{s-1,s+1-n}$, as soon as s is large enough.

Example $\qquad h = \cos \alpha/2\,, \qquad f = \dfrac{1}{\sqrt{1+r^2}}\,,$

then

$$|\partial f|_e = \frac{r}{(1+r^2)^{3/2}}$$

and $\partial f \not\in H_{s-1,\delta}$ if $\delta \geq 2 - n/2$,
thus ∂f will not satisfy (5b) if $s \geqq n/2 + 1$.

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8. COMPLETENESS OF A BALL OF W_s^p IN W_{s-1}^p

Show that a closed bounded ball of $W_s^p(\Omega)$ is a complete metric space for the norm of $W_{s-1}^p(\Omega)$, if $p > 1$, Ω open set of $\mathbb{R}^n$.

Answer: The problem is to show that if $\{f_n\}$ is a sequence of functions in W_s^p such that $\|f_n\|_{W_s^p} \leqq C$ which is a Cauchy sequence for the W_{s-1}^p norm, that is $\forall\varepsilon, \exists N$ such that $\|f_n - f_m\|_{W^p_{s-1}} < \varepsilon$ if $n, m > N$, then the sequence converges in the W^p_{s-1} norm to a function f which is in W^p_s and such that $\|f\|_{W^p_s} \leq C$. The proof proceeds as follows:

1) By the completeness of W^p_{s-1} (cf. p. 487) the sequence f_n converges in the W^p_{s-1} norm (thus in $\mathscr{D}'$) to a function $f \in W^p_{s-1}$.

2) If $p > 1$ the space W^p_s is the dual of a Banach space $W^{p'}_{-s}$, with $1/p' = 1 - 1/p$ (p. 489). The ball

$$\|\cdot\|_{W^p_s} \leqq C$$

is compact in the weak-star topology [Problem I 13, Compactness]; therefore there exists a subsequence of $\{f_n\}$, still denoted $\{f_n\}$, such that it converges in this topology to $h \in W^p_s$. The convergence is $\forall\varepsilon, \forall g \in W^{p'}_{-s}$, $\exists N$ such that $|\langle h - f_n, g\rangle| < \varepsilon$ if $n > N$ and implies a fortiori the convergence of $\{f_n\}$ to h in $\mathscr{D}'$, thus $f = h$.

9. DISTRIBUTION WITH LAPLACIAN IN $L^2(\mathbb{R}^n)$

1) *Show that if a tempered distribution u on $\mathbb{R}^n$ (p. 476) is such that its laplacian*

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{(\partial x^i)^2}$$

is a function in $L^2(\mathbb{R}^n)$ then the same is true of each of its second partial derivatives.

2) *Show that if u is a distribution on $\mathbb{R}^n$ such that its first derivatives ∂u and its laplacian Δu are locally (i.e., on each compact set) square integrable functions, then the same is true of each of its second derivatives.*

Answer 1: If u is a tempered distribution it admits a Fourier transform $v = \mathscr{F}u$, and so does Δu, with (cf. p. 477)

$$\mathscr{F}\Delta u = -|y|^2 v\,, \qquad |y|^2 = \sum_{i=1}^{n} (y^i)^2\,.$$

If $\Delta u \in L^2(\mathbb{R}^n)$, the same is true of $\mathscr{F}\Delta u$, (p. 490):

$$-|y|^2 v \in L^2(\mathbb{R}^n)$$

from which we deduce

$$y^i y^j v \in L^2(\mathbb{R}^n)\,, \qquad \forall i, j = 1, \ldots, n$$

since $|y|^{-2} y^i y^j$ is in $L^\infty(\mathbb{R}^n)$ and $f \in L^2$, $g \in L^\infty$ implies $fg \in L^2$. Therefore, by the inverse Fourier transform

$$\frac{\partial^2 u}{\partial x^i\, \partial x^j} \in L^2(\mathbb{R}^n)\,.$$

Moreover

$$\| y^i y^j v \|^2_{L^2} \leq \| |y|^2 v \|_{L^2}\,,$$

thus

$$\left\| \frac{\partial^2 u}{\partial x^i\, \partial x^j} \right\|_{L^2} \leq \| \Delta u \|_{L^2}\,.$$

Answer 2: Let φ be a C^∞ function with compact support, equal to 1 in the compact K. Since Δu is locally square integrable we have $\varphi \Delta u \in L^2(\mathbb{R}^n)$ but

$$\varphi \Delta u \equiv \Delta(u\varphi) - u\Delta\varphi - 2 \sum_i \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^i}\,,$$

therefore, if u and ∂u are locally square integrable, $\Delta(u\varphi) \in L^2(\mathbb{R}^n)$, by answer 1),

$$\partial^2(u\varphi)\partial x^i\, \partial x^j \in L^2(\mathbb{R}^n) \quad \text{and} \quad \frac{\partial^2 u}{\partial x^i\, \partial x^j} \in L^2(K)\,,$$

since $\varphi = 1$ in K.

10. NONLINEAR WAVE EQUATION IN CURVED SPACETIME

Consider on a C^∞ manifold $V_{n+1} = V_n \times \mathbb{R}$, with V_n an n-dimensional C^∞ manifold, a hyperbolic metric which reads

$$\mathrm{d}s^2 = N^2(\mathrm{d}x^0)^2 - g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$$

in adapted coordinates (that is x^i coordinates on V_n and $x^0 \in \mathbb{R}$). Suppose $N \geq a$, $a > 0$, and g_{ij} components of a properly riemannian metric g.

1) *Show that if the hypersurfaces* $x^0 =$ *constant are harmonic, i.e.,* $\nabla^\alpha \nabla_\alpha x^0 = 0$ *then*

$$g^{\alpha\beta} \Gamma^{\,0}_{\alpha\,\beta} = 0$$

or, equivalently

$$N = (\det(g_{ij}))^{1/2} m^{-1/2}$$

with m *an arbitrary scalar density on* V_n *(independent of* x^0*).*

2) *If u is C^2 and has compact support on each $V_n \times \{t\} = S_t$, show that the equation*

$$\Box u = f\,, \qquad \Box = \nabla^\lambda \partial_\lambda$$

implies

$$\int_{S_t} \partial^2_{00} u \,\mathrm{d}\mu(m) = \int_{S_t} Nf \,\mathrm{d}\mu(g)\,,$$

where $\mathrm{d}\mu(m)$ and $\mathrm{d}\mu(g)$ *are respectively the volume elements corresponding to the scalar density m and to the metric g.*

3) *Suppose that, on V_{n+1}*

$$\Box u \geq A|u|^p\,, \qquad A>0,\ p>1\,.$$

Use the preceding result to show that u cannot exist as a C^2 function on V_{n+1} if V_n is compact and, for some $x^0 = t_0$ we have

$$\int_{S_{t_0}} \partial_0 u \,\mathrm{d}\mu(m) > 0\,.$$

Answer 1:

$$\nabla^\alpha \nabla_\alpha x^0 \equiv g^{\alpha\beta}\nabla_\beta \partial_\alpha x^0 \equiv -g^{\alpha\beta}\Gamma^{\ 0}_{\alpha\ \beta}$$

since $\partial_i x^0 = 0$ and $\partial_0 x^0 = 1$. We have here

$$g^{\alpha\beta}\Gamma^{\ 0}_{\alpha\ \beta} \equiv N^{-2}\Gamma^{0}_{0\ 0} - g^{ij}\Gamma^{0}_{i\ j} \equiv N^{-3}\partial_0 N - \tfrac{1}{2}N^{-2}g^{ij}\partial_0 g_{ij} = 0\,,$$

thus

$$N^{-1}\partial_0 N \equiv \tfrac{1}{2}(\det g)^{-1}\partial_0(\det g)$$

and therefore

$$N = (\det g)^{1/2} m^{-1/2}$$

with m independent of x^0, a scalar density on V_n since it is so of $(\det g)^{1/2}$ for each x^0 fixed, and N is a scalar.

Answer 2: We have, in the considered coordinates

$$\Box u \equiv g^{\alpha\beta}\nabla_\alpha \partial_\beta u \equiv \frac{1}{N^2}\,(\partial_0{}^2{}_0 u - g^{ij}N\partial_j N\partial_i u) - g^{ij}\bar{\nabla}_j \partial_i u$$

where $\bar{\nabla}$ denotes the covariant derivative in the metric g, because, if $g_{0i} = 0$

$$\Gamma^{k}_{i\ j} = \bar{\Gamma}^{k}_{i\ j}\,, \qquad \Gamma^{i}_{0\,0} = g^{ij}N\partial_j N\,,$$

thus

$$\Box u = \frac{1}{N^2} \partial_0{}^2{}_0 u - \frac{1}{N} g^{ij}\bar{\nabla}_j(N\partial_i u) .$$

We deduce by integration over $S_t = V_n \times \{t\}$, where u has compact support, using the Stokes formula and 1)

$$\int_{S_t} N \Box u \, \mathrm{d}\mu(g) = \int_{S_t} \frac{1}{N} \partial_0{}^2{}_0 u \, \mathrm{d}\mu(g) = \int_{S_t} \partial_0{}^2{}_0 u \, \mathrm{d}\mu(m) .$$

The result follows if $\Box u = f$.

Answer 3: We set

$$F_u(t) = \int_{S_t} u \, \mathrm{d}\mu(m) ,$$

then

$$F_u''(t) = \int_{S_t} \partial_0{}^2{}_0 u \, \mathrm{d}\mu(m) = \int_{S_t} N \Box u \, \mathrm{d}\mu(g)$$

thus, if $\Box u \geq A|u|^p$

$$F_u''(t) \geq \int_{S_t} NA|u|^p \, \mathrm{d}\mu(g) \geq C \int_{S_t} |u|^p \, \mathrm{d}\mu(m)$$

$$\text{with } C = \inf_{V_{n+1}} N^2 A = a^2 A .$$

On the other hand, if V_n is a compact manifold

$$|F_u(t)| \leq \int_{S_t} |u| \, \mathrm{d}\mu(m) \leq (\mathrm{vol}_m(V_n))^{1/p'} \Big(\int_{S_t} |u|^p \, \mathrm{d}\mu(m) \Big)^{1/p} ,$$

hence there exists $C > 0$ such that

$$F_u''(t) \geq C|F_u(t)|^p . \tag{1}$$

We first deduce from this inequality that, for all t

$$F_u''(t) \geq 0 .$$

Therefore if

$$\int_{S_{t_0}} \partial_0 u \, \mathrm{d}\mu(m) = F_u'(t_0) > 0 ,$$

we have

$$F_u'(t) \geq F_u'(t_0) > 0 , \qquad \forall t \geq t_0 ;$$

thus $F_u(t)$ is a strictly increasing function for $t \geq t_0$, and there exists $t_1 \geq t_0$ such that

$$F_u(t) > 0 , \qquad t \geq t_1 .$$

Therefore (1) can now be written

$$F_u''(t) \geq C(F_u(t))^p , \qquad t \geq t_1$$

and we have

$$\frac{\mathrm{d}}{\mathrm{d}t} (F_u')^2 = 2F_u'F_u'' \geq 2CF_u'(F_u)^p , \qquad t \geq t_1 .$$

This inequality gives by integration

$$(F_u'(t))^2 \geq \frac{2C}{p+1} \{(F_u(t))^{p+1} - (F_u(t_1))^{p+1}\} + (F_u'(t_1))^2 ,$$

that is, since the right-hand side is positive when $t \geq t_1$ an inequality of the form

$$F_u'(t) \geq \{C_1(F_u(t))^{p+1} + C_2\}^{1/2} , \qquad t \geq t_1$$

which implies

$$F_u(t) \geq y(t) , \qquad t > t_1$$

where y is the solution of the problem

$$y' = (C_1 y^{p+1} + C_2)^{1/2} , \qquad y(t_1) = F_u(t_1) ;$$

that is y such that

$$t = t_1 + \int_{y_1}^{y} (C_1 z^{p+1} + C_2)^{-1/2} \, \mathrm{d}z , \qquad y_1 = F_u(t_1) .$$

Since $C_1 = 2(p+1)^{-1}C > 0$ and $C_1 y_1^{p+1} + C_2 > 0$, the solution $y(t)$ tends to infinity, if $p > 1$, when t tends to the finite value

$$T = \int_{y_1}^{\infty} (C_1 z^{p+1} + C_2)^{-1/2} \, \mathrm{d}z .$$

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Y. Choquet-Bruhat, Comptes Rendus Ac. Sc. Paris (1987).

We give below a summary and references for results about the existence and nonexistence of global solutions of nonlinear wave equations on Minkowski spacetime M_{n+1} obtained in recent years.

1) Case $n=3$
F. John, Ann. Maths. 28 (1979) 235–261.

This article proves the blow up of solutions of

$$\square u \geq A|u|^p\,, \qquad A>0\,, \qquad 1<p<1+\sqrt{2}$$

for Cauchy data in $C_0^\infty(\mathbb{R}^3)$ which satisfy either $\int_{S_0} \partial_0 u \, d^3x > 0$ or $u|_{S_0} \leq 0$. It proves the blow up for arbitrary Cauchy data in $C_0^\infty(\mathbb{R}^3)$ for nonidentically zero solutions of the equation

$$\square u = f(u) \tag{1}$$

where f is a function on $\mathbb{R}$ such that

$$f(s) \geq A|s|^p\,, \qquad \forall s \in \mathbb{R},\, A>0$$

$$f(0)=0\,, \qquad \limsup_{s\to 0} f(s)/|s| < \infty$$

if

$$1<p<1+\sqrt{2}$$

Note that the function $f(s)=A|s|^p$, $1<p<1+\sqrt{2}$, satisfies these conditions.
It proves global existence for small Cauchy data in $C_0^\infty(\mathbb{R}^3)$ for eq. (1), if $f\in C^2(\mathbb{R})$, $f(0)=f'(0)=f''(0)$ and is Hölder continuous with exponent $>\sqrt{2}-1$ for $|s|<1$. These conditions are satisfied by $f(s)=A|s|^p$ if $p>1+\sqrt{2}$.

2) Case $n>3$
Blow up of solutions with Cauchy data of compact support has been proved when some weighted means of these data is positive and $1<p<p_0(n)$, $p_0(n)$ the positive root of the polynomial $(n-1)x^2-(n+1)x-2=0$ by T. Sideris, J. Diff. Equations 52 (1984) 378–406; For earlier related work see
W. Strauss, J. Funct. Analysis 41 (1981) 110–133;
R. Glassey, Math. 2 178 (1981) 233–261.

Global existence for small Cauchy data with fast fall off at infinity has been proved for any $n>3$ and equations

$$\square u = f(u, \partial u, \partial^2 u)$$

with f a smooth function such that

$$f(0,0,0)=0\,, \qquad f'(0,0,0)=0\,. \tag{2}$$

S. Klainerman, "Global existence for nonlinear wave equations", Comm. pure and app. Maths 33 (1980) 43–101.
D. Christodoulou, "Global solution of nonlinear hyperbolic equations for small initial data", Comm. pure and app. Maths XXXIX (1986) 267–282. This last article also contains a proof of the same theorem in the case $n=3$ when f satisfies the further condition that it vanishes when its arguments ∂u and $\partial^2 u$ are replaced respectively by a null (for the Minkowski metric) covector y and its tensor product $y\otimes y$.
3) The global existence on $\mathbb{R}^{n+1}$, $n\geq 3$, for small data in C_0^∞, under the general hypothesis

(2) on f, for the equation

$$\Box u + u = f(u, \partial u, \partial^2 u)$$

has been given in:

S. Klainerman, "Global existence of small amplitude solutions to nonlinear Klein–Gordon equations in four space time dimensions", Comm. pure and app. Maths. XXXVIII (1985) 631–641.

See also L. Hormander, Institute Mittag–Lefler reports Lund (1985) 86, 87.

11. HARMONIC COORDINATES IN GENERAL RELATIVITY

1) *Let g be a riemannian metric, of arbitrary signature on a d-dimensional manifold M.*
Show that in a coordinate system where the metric satisfies the **harmonicity conditions**

harmonicity conditions

$$F^\lambda \equiv g^{\alpha\beta}\Gamma_{\alpha\ \beta}^{\ \lambda} = 0\,, \qquad \lambda = 0, \ldots, d-1 \tag{1}$$

the Einstein equations in vacuum, $\mathrm{Ricc}(g) = 0$, *reduce to a quasi-diagonal, quasi-linear system, with principal operator* $g^{\alpha\beta}\partial^2_{\alpha\beta}$.

2) *Show that the conditions* (1) *express that the mapping* $f\colon U \to f(U)$ *by* $x \to (x^\lambda)$ *(U domain of the coordinate chart) is a harmonic map from the riemannian manifold* (U, g) *onto* $(f(U), e)$ *where e is the standard euclidean metric of* $\mathbb{R}^d$.

3) *Generalize to the case where* (1) *is replaced by an harmonicity condition with respect to an arbitrary given metric e on M.*

Answer 1: By the definition of the Riemann and Ricci tensor we have, in an arbitrary coordinate system (cf. p. 306)

$$\nabla^\mu\nabla^\beta g^{(\alpha)}_\mu - \nabla^\beta\nabla^\mu g^{(\alpha)}_\mu \equiv R^{\mu\beta}{}_\mu{}^\lambda g^{(\alpha)}_\lambda \equiv R^{\alpha\beta}\,,$$

where $g^{(\alpha)}_\mu = \delta^\alpha_\mu$ are considered as the components of a covariant vector, for fixed α (dumb index).

With this notation

$$\nabla^\mu g^{(a)}_\mu \equiv \partial^\mu g^{(\alpha)}_\mu - g^{\mu\lambda}\Gamma_{\mu\ \lambda}^{\ \alpha} = -g^{\mu\lambda}\Gamma_{\mu\ \lambda}^{\ \alpha}\,.$$

Thus

$$R^{\alpha\beta} \equiv \nabla^\mu\nabla^\beta g^{(\alpha)}_\mu + g^{\beta\lambda}\partial_\lambda F^\alpha\,;$$

since $R^{\alpha\beta}$ is symmetric we have also

$$R^{\alpha\beta} \equiv \tfrac{1}{2}(\nabla^\mu\nabla^\beta g^{(\alpha)}_\mu + \nabla^\mu\nabla^\alpha g^{(\beta)}_\mu + g^{\beta\lambda}\partial_\lambda F^\alpha + g^{\alpha\lambda}\partial_\lambda F^\beta)\,. \tag{2}$$

We remark that

$$\nabla^\beta g_\mu^{(\alpha)} + \nabla^\alpha g_\mu^{(\beta)} \equiv -g^{\beta\lambda}\Gamma_\lambda{}^\alpha{}_\rho g_\mu^p - g^{\alpha\lambda}\Gamma_\lambda{}^\beta{}_\rho g_\mu^\rho$$
$$= -(g^{\beta\lambda}g^{\alpha\nu} + g^{\alpha\lambda}g^{\beta\nu})[\nu, \lambda\mu]$$

with $[\nu, \lambda\mu] \equiv g_{\nu\alpha}\Gamma_\lambda{}^\alpha{}_\mu$ the Christoffel symbols (p. 308), hence

$$\nabla^\beta g_\mu^{(\alpha)} + \nabla^\alpha g_\mu^{(\beta)} \equiv -\tfrac{1}{2}(g^{\beta\lambda}g^{\alpha\nu} + g^{\alpha\lambda}g^{\beta\nu})\partial_\mu g_{\lambda\nu} = \partial_\mu g^{\alpha\beta}.$$

We deduce from (2), after computation

$$R^{\alpha\beta} \equiv R_{(h)}^{\alpha\beta} + \tfrac{1}{2}(g^{\beta\lambda}\nabla_\lambda F^\alpha + g^{\alpha\lambda}\nabla_\lambda F^\beta),$$

where $R_{(h)}^{\alpha\beta}$ is the second-order, quasidiagonal operator

$$R_{(h)}^{\alpha\beta} = \tfrac{1}{2}g^{\lambda\mu}\partial^2_{\lambda\mu}g^{\alpha\beta} - g^{\lambda\mu}g^{\rho\nu}\Gamma_\lambda{}^\alpha{}_\rho\Gamma_\mu{}^\beta{}_\nu.$$

The system is hyperbolic in the sense of Leray (p. 520) if g is hyperbolic, since it is quasi-diagonal with principal part $\frac{1}{2}g^{\lambda\mu}\partial^2_{\lambda\mu}g^{\alpha\beta}$. By lowering the indices we find that $R_{\alpha\beta}$ is also of the form

$$R_{\alpha\beta} \equiv R_{\alpha\beta}^{(h)} - \tfrac{1}{2}(g_{\alpha\lambda}\nabla_\beta F^\lambda + g_{\beta\lambda}\nabla_\alpha F^\lambda)$$

with $R_{\alpha\beta}^{(h)}$ a quasi-diagonal second-order hyperbolic operator with principal part $-\frac{1}{2}g^{\lambda\mu}\partial^2_{\lambda\mu}g_{\alpha\beta}$.

Answer 2: The mapping f is represented in local coordinates (x^λ) on U and $f(U)$ by the identity map

$$f^\lambda(x^\mu) = x^\lambda.$$

The equations for harmonic maps [Problem V 11, Harmonic] reduce to (1) if $f(U)$ is endowed with a flat metric for which the x^λ are canonical coordinates – where its Christoffel symbols are zero.

Answer 3: Suppose the identity map $M \to M$ is a harmonic map from (M, g) onto (M, e). Then, in local coordinates

$$\hat{F}^\lambda \equiv g^{\alpha\beta}(-\Gamma_\alpha{}^\lambda{}_\beta + \hat{\Gamma}_\alpha{}^\lambda{}_\beta) = 0,$$

where the $\hat{\Gamma}_\alpha{}^\lambda{}_\beta$ are the Christoffel symbols of e. The previous computations show that we have identities of the type

$$R^{\alpha\beta} \equiv g^{\lambda\mu}\hat{\nabla}_\lambda\hat{\nabla}_\mu g^{\alpha\beta} + \tfrac{1}{2}(g^{\beta\lambda}\hat{\nabla}_\lambda\hat{F}^\alpha + g^{\alpha\lambda}\hat{\nabla}_\lambda\hat{F}^\beta) + H^{\alpha\beta}(g, \hat{\nabla}g), \quad (3)$$

where $\hat{\nabla}$ is the covariant derivative in e. Decomposition (3) is tensorial since $\hat{F}$ is a vector and is valid on the whole manifold.

12. LERAY THEORY OF HYPERBOLIC SYSTEMS. TEMPORAL GAUGE IN GENERAL RELATIVITY

Let (M, g) be a smooth riemannian manifold of hyperbolic signature with $M = S \times \mathbb{R}$. The lines $\{x\} \times \mathbb{R}$, $x \in S$ are supposed time-like and orthogonal to the space-like submanifolds $S_t = S \times \{t\}$. In adapted coordinates, (x^i) coordinates in S and $x^0 \in \mathbb{R}$, the metric g reads

$$\mathrm{d}s^2 = -\alpha^2(\mathrm{d}x^0)^2 + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \qquad i, j = 1, \ldots, d-1\,, \tag{1}$$

$\bar{g} = (g_{ij})$ is the metric induced on S_t by g, when $x^0 = t$. α is a function called the **lapse**. lapse

Denote by $K = (K_{ij})$ the second fundamental form (p. 315) of a submanifold S_t.

1) *Express the Ricci tensor of M, at each point $(x, x^0 = t)$ in terms of the metric $\bar{g}$, second fundamental form K and lapse α of S_t, their covariant derivatives in the metric $\bar{g}$, and "time" derivatives (i.e., partial derivatives with respect to x^0).*

2) *Show that if $S_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ is the Einstein tensor then*

$$S_{00} \equiv \frac{\alpha^2}{2}\,(\bar{R} - K_i^j K_j^i + (K_i^i)^2)\,, \tag{2a}$$

$$S_{i0} \equiv \alpha(-\bar{\nabla}_j K_i^j + \bar{\nabla}_i K_l^l)\,. \tag{2b}$$

Express these identities in an intrinsic (i.e., coordinate-free) manner.

3) *Show that the quantity*

$$\partial_0 R_{ij} - \alpha^2(\bar{\nabla}_i S_{j0} + \bar{\nabla}_j S_{i0})$$

contains no third derivatives of g or α. Show that it contains second derivatives of K only through the operator

$$\square = \frac{1}{\alpha^2}\,\partial_{00}^2 - \bar{\nabla}^i \bar{\nabla}_i$$

if α is chosen such that

$$\alpha^2 \operatorname{tr} K + \alpha' = 0\,, \qquad \alpha' = \partial_0 \alpha\,.$$

Give a relation between α and $\det(\bar{g})$ *equivalent to the above equation.*

4) *Using the previous results, deduce from the Einstein equations in vacuum $R_{\alpha\beta} = 0$, in an appropriate gauge, a Leray hyperbolic system.*

5) *Examine whether a solution of the Leray system thus obtained satisfies the original Einstein equations.*

Answer 1: Let n be the unit normal to S_t, whose equation is $x^0 = t$. We have $g(n, n) = -1$, $n_i = 0$, thus

$$n_0 = (-g^{00})^{-1/2} = \alpha$$

and (cf. p. 315)

$$K_{ij} = \alpha \Gamma^{0}_{i\,j} = -\frac{1}{2\alpha} \partial_0 g_{ij} .$$

All the Christoffel symbols of g are then easily computed (overlined quantities are relative to the space metric $\bar{g}$)

$$\Gamma^{k}_{i\,j} = \bar{\Gamma}^{k}_{i\,j} ,$$

$$\Gamma^{i}_{0j} = -\alpha K^{i}_{\;j} , \qquad \Gamma^{0}_{i\,j} = -\frac{1}{\alpha} K_{ij} ,$$

$$\Gamma^{0}_{0\,i} = \frac{1}{\alpha} \partial_i \alpha , \qquad \Gamma^{i}_{0\,0} = \alpha \partial^{i} \alpha , \qquad \Gamma^{0}_{0\,0} = \frac{1}{\alpha} \partial_0 \alpha .$$

We deduce from these formulas and the definition of Ricc(g), after some simplifications:

$$R_{ij} \equiv \bar{R}_{ij} - \alpha^{-1} \partial_0 K_{ij} - 2K_{ih} K^{h}_{j} + K_{ij} K^{h}_{h} - \alpha^{-1} \bar{\nabla}_i \partial_j \alpha ,$$

$$R_{0i} \equiv \alpha(-\bar{\nabla}_h K_i^{\;h} + \partial_i K^{h}_{h}) ,$$

$$R_{00} \equiv \alpha^2 (\alpha^{-1} \bar{\nabla}^i \partial_i \alpha + \alpha^{-1} \partial_0 K^{h}_{h} - K^{j}_{i} K^{i}_{j}) ,$$

and using

$$\partial_0 g^{ij} = 2\alpha K^{ij} ,$$

the scalar curvature

$$R \equiv g^{00} R_{00} + g^{ij} R_{ij} \equiv R + (K^{h}_{h})^2 + K^{j}_{i} K^{i}_{j} - 2\alpha^{-1} \partial_0 K^{h}_{h} - 2\alpha^{-1} \bar{\nabla}_i \partial^i \alpha .$$

Answer 2: We deduce from the identities in answer 1) the following ones

$$S_{00} \equiv R_{00} - \frac{1}{2} g_{00} R \equiv \frac{\alpha^2}{2} (\bar{R} - K^{j}_{i} K^{i}_{j} + (K^{h}_{h})^2 \tag{2a'}$$

$$S_{0i} \equiv R_{0i} \equiv \alpha(-\bar{\nabla}_h K_i^{\;h} + \partial_i K^{h}_{h}) \tag{2b'}$$

which depends only on $\bar{g}$ and K on S_t, and α.

We remark that $\alpha^{-2} S_{00}$ is the expression in the chosen coordinates ($g_{0i} = 0$, $g_{00} = -\alpha^2$, thus $n^0 = \alpha^{-1}$, $n^i = 0$) of the scalar function:

$$S_{\perp\perp} \equiv S \cdot (n \otimes n) \equiv S_{\alpha\beta} n^\alpha n^\beta ,$$

while $\alpha^{-1}S_{0i}$ are the components in the chosen coordinates of the covariant vector $S_\perp$ on S_t,

$$S_\perp = i^*(S \cdot (n \otimes \pi)) ,$$

where π is the projection operator on S_t, i the inclusion map of S_t into M. In arbitrary coordinates

$$(S \cdot (n \otimes \pi))_\lambda = S_{\alpha\beta} n^\alpha n^\beta \pi^\beta_\lambda , \qquad \pi^\beta_\lambda = g^\beta_\lambda + n^\beta n_\lambda .$$

Remark: In arbitrary coordinates (x^i, x^0), with $x^0 = t$ the equation of S_t, we have

$$S_{\perp\perp} = (n_0)^2 S^{00} ,$$

$$(S_\perp)_i = -n_0 S^0_i .$$

The identities (2a′), (2b′) read, intrinsically

$$2S_{\perp\perp} \equiv \bar{R} - K \cdot K + (\mathrm{tr}\, K)^2 ,$$

$$S_\perp \equiv -\bar{\nabla} \cdot K + \bar{\nabla}\, \mathrm{tr}\, K .$$

The right-hand side depends only, for $x^0 = t$, on the two fundamental forms of S_t as a submanifold of (M, g).

Answer 3: From the definitions of the connection and the Ricci tensor one deduces the formulae [Lichnerowicz], setting $\nabla_0 \bar{g}_{ij} = \bar{g}'_{ij}$,

$$\partial_0 \bar{\Gamma}^k_{ij} = \tfrac{1}{2} \bar{g}^{kh} (\bar{\nabla}_i \bar{g}'_{jh} + \bar{\nabla}_j \bar{g}'_{ih} - \bar{\nabla}_h \bar{g}'_{ij}) , \tag{3}$$

$$\partial_0 \bar{R}_{ij} = \tfrac{1}{2} \bar{g}^{kh} \{ \bar{\nabla}_k (\bar{\nabla}_i \bar{g}'_{jh} + \bar{\nabla}_j \bar{g}'_{ih}) - \bar{\nabla}_i \bar{\nabla}_j \bar{g}'_{hk} \} . \tag{4}$$

Using these relations, the Ricci identity and the identities (2) we obtain

$$\begin{aligned}\partial_0 R_{ij} - \alpha^2 \bar{\nabla}_{(i} S_{j)0} \equiv {} & -\alpha \Box K_{ij} + \frac{\alpha'}{\alpha^2} K'_{ij} - \alpha \bar{\nabla}_i \bar{\nabla}_j K^h_h + f_{ij} - \frac{1}{\alpha} \nabla_i \nabla_j \alpha' \\ & + \frac{\alpha'}{\alpha^2} \bar{\nabla}_i \bar{\nabla}_j \alpha - \bar{\Gamma}^{k'}_{ij} \bar{\nabla}_k \alpha + K'_{ij} K^h_h + K_{ij} K'^h_h - 2K'_{im} K_j^m - 2K_{im} K'^m_j\end{aligned} \tag{5}$$

with

$$\begin{aligned} f_{ij} \equiv {} & 2(\bar{\nabla}_h \alpha)(\bar{\nabla}^h \alpha) K_{ij} + (\bar{\nabla}_h \bar{\nabla}^h \alpha) K_{ij} - (\bar{\nabla}_i \alpha) \bar{\nabla}_h K_j^h - (\bar{\nabla}_h \alpha) \bar{\nabla}_i K^h_j \\ & - (\bar{\nabla}_i \bar{\nabla}_h \alpha) K_j^h + (\bar{\nabla}_{(i} \alpha) \bar{\nabla}_{j)} K^h_h + (\bar{\nabla}_i \bar{\nabla}_j \alpha) K^h_h \\ & + \alpha (R_{ihj}{}^k K_k{}^h - R_{h(i} K_{j)}{}^h) , \end{aligned} \tag{6}$$

$\alpha' = \partial_0 \alpha$, $K'_{ij} = \partial_0 K_{ij}$; $\Box \equiv \alpha^{-2} \partial_0^2 - \bar{\nabla}_h \bar{\nabla}^h$.

We see on (5) that $\partial_0 R_{ij} - \alpha^2 \bar{\nabla}_{(i} S_{j)0}$ contains no third derivatives of the

$\bar{g}$'s. The expression (5) will contain second derivatives of the K's only through the operator $\Box$, and no third derivatives of α, if we choose α such that

$$\alpha^2 \operatorname{tr} K + \alpha' = 0 \tag{7}$$

since we have

$$\alpha \bar{\nabla}_i \bar{\nabla}_j K^h_h = \alpha^{-1} \bar{\nabla}_i \bar{\nabla}_j (\alpha^2 K^h_h) + m_{ij}$$

with

$$m_{ij} \equiv -2(\bar{\nabla}_i \alpha)\bar{\nabla}_j K^h_h - 2(\bar{\nabla}_j \alpha)\bar{\nabla}_i K^h_h - 2\alpha^{-1}(\bar{\nabla}_i \alpha)(\bar{\nabla}_j \alpha) K^h_h - 2(\bar{\nabla}_i \bar{\nabla}_j \alpha) K^h_h \,.$$

With the choice (7) expression (5) reduces to

$$\partial_0 R_{ij} - \alpha^2 \bar{\nabla}_{(i} S_{j)0} \equiv -\alpha \Box K_{ij} + f_{ij} + n_{ij} \,, \tag{8}$$

where f_{ij} is given by (6) and n_{ij} by

$$\eta_{ij} = -m_{ij} + K_{ij} K'^h_h - K^h_h \bar{\nabla}_i \bar{\nabla}_j \alpha - \bar{\Gamma}^k_{i\ j}{}' \bar{\nabla}_k \alpha$$
$$-2K'_{im} K^m_j - 2K_{im} K'^m_j \,.$$

We remark that we have

$$\frac{(\det \bar{g})'}{\det \bar{g}} = -2\alpha K^i_i$$

the relation (7) is therefore

$$\frac{(\det \bar{g})'}{\det \bar{g}} = -2\alpha' = 0$$

and its general solution for the scalar α is

$$\alpha = (\det \bar{g})^{1/2} a^{-1/2} \tag{9}$$

with a an arbitrary, positive, scalar density on S (independent of x^0).

Remark: The equation

$$\alpha' + \alpha^2 \operatorname{tr} K = 0$$

expresses that the submanifolds S_t satisfy the harmonicity condition

$$\nabla^\lambda \nabla_\lambda x^0 \equiv g^{\lambda\mu} \Gamma^{\ 0}_{\lambda\ \mu} = 0 \,.$$

Answer 4: Identity (8) shows that, with the choice (9) for α, and $g_{0i} = 0$, the equations in vacuum $R_{\alpha\beta} = 0$ imply

$$\partial_0 R_{ij} - \alpha^2 \bar{\nabla}_{(i} S_{j)0} \equiv -\alpha \Box K_{ij} + f_{ij} + n_{ij} = 0 \,, \tag{10a}$$

$$\partial_0 g^{ij} = 2\alpha K^{ij} \tag{10b}$$

These equations are equivalent if $\alpha \neq 0$ to a quasidiagonal third-order system for $\bar{g}$, with principal operator $\square \partial_0$. The operator $\square \partial_0$ is hyperbolic if $\alpha \neq 0$ and $\bar{g}$ is properly riemannian: the dual characteristic cone in the tangent space is

$$X^0\left(\frac{1}{\alpha^2}(X^0)^2 - \bar{g}_{ij}X^iX^j\right) = 0$$

it is cut in three distinct points by every straight line passing through a point where

$$\frac{1}{\alpha^2}(X^0)^2 - \bar{g}_{ij}X^iX^j > 0 .$$

The dual of the above cone, that is the light cone of the metric, determines the dependence domain of the solution, in agreement with physical expectation.

Answer 5: We shall prove:
Lemma Let $\bar{g}$ and K verify the hyperbolic system (10a,b), and α be given by (9), then the Einstein tensor $S^{\alpha\beta}$ corresponding to the metric

$$-\alpha^2(\mathrm{d}x^0)^2 + \bar{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j , \qquad \alpha = (\det \bar{g})^{1/2}a^{-1/2} \tag{11}$$

verifies a linear, homogeneous hyperbolic system.

Proof: By the Bianchi identities we have

$$\nabla_\alpha S^{\alpha\beta} \equiv 0$$

which can be written, modulo linear terms in $S^{\alpha\beta}$

$$\partial_0 S^{00} + \bar{\nabla}_i S^{i0} \simeq 0 , \tag{12a}$$

$$\partial_0 S^{j0} + \bar{\nabla}_i S^{ij} \simeq 0 . \tag{12b}$$

Eqs. (10) say that the metric (11) verifies the equations

$$\partial_0 R_{ij} - \alpha^2(\bar{\nabla}_i S_{j0} + \bar{\nabla}_j S_{i0}) = 0 .$$

We have, in the case of $g_{i0} = 0$

$$R^{00} \equiv 2S^{00} - \frac{1}{\alpha^2}\, g_{hk}R^{hk} ,$$

thus

$$S^{ij} \equiv R^{ij} - g^{ij}(g_{hk}R^{hk} - \alpha^2 S^{00})$$

and eqs. (10) imply, modulo linear terms in $S^{\alpha\beta}$

$$\partial_0 S^{ij} \simeq -\alpha^2(\bar{\nabla}^i S^{j0} + \bar{\nabla}^j S^{i0}) + 2\alpha^2 g^{ij}\bar{\nabla}_h S^{h0} + \alpha^2 g^{ij}\partial_0 S^{00}$$

from which we deduce, modulo linear terms in $S^{\alpha\beta}$, by the Bianchi identity (12b)

$$\partial_0 S^{ij} \simeq -\alpha^2(\bar{\nabla}^i S^{j0} + \bar{\nabla}^j S^{i0}) + \alpha^2 g^{ij}\bar{\nabla}_h S^{h0} . \tag{13}$$

We deduce from which eq. (13) modulo linear terms in $S^{\alpha\beta}$, and $\bar{\nabla}_i S^{\alpha\beta}$

$$\bar{\nabla}_i \partial_0 S^{ij} \simeq -\alpha^2 \bar{\nabla}_i \bar{\nabla}^i S^{j0} .$$

The Bianchi identities (12b) imply therefore, modulo linear terms in $S^{\alpha\beta}$ and $\bar{\nabla}_i S^{\alpha\beta}$

$$\Box S^{j0} \simeq 0 . \tag{14}$$

The system (12a, 13, 14) is a linear homogeneous system for the $S^{\alpha\beta}$ which can be shown to be hyperbolic by derivating equation (14) with respect to x^0,
We then obtain a third-order equation

$$\Box \partial_0 S^{j0} \simeq 0 ,$$

where the symbol $\simeq 0$ means modulo linear terms in $S^{\alpha\beta}$, their first derivatives and the second derivatives of only S^{j0} (we use (12) and (13) to eliminate second derivatives of S^{00} and S^{ij}).
By the uniqueness theorem for a solution of the Cauchy problem for hyperbolic systems we shall have $S^{\alpha\beta} = 0$ (in appropriate functional spaces of tensor fields on M) if $S^{\alpha\beta}$ is zero on S_0 together with its derivatives of order ≤ 2. This is checked to be satisfied, for solutions of the hyperbolic system (10a, 10b) if the Cauchy data $\bar{g}$ and K on S_0 satisfy the "constraints", $S^{00} = 0$, $S_i^0 = 0$, that is

$$\bar{R} - K \cdot K + (\operatorname{tr} K)^2 = 0 ,$$

$$\nabla \cdot K - \nabla \operatorname{tr} K = 0 ,$$

(one uses also the Bianchi identities).

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13. EINSTEIN EQUATIONS WITH SOURCES AS A HYPERBOLIC SYSTEM

Consider the Einstein equations with sources where the unknown are a hyperbolic metric g on a d-dimensional manifold M, and a symmetric 2-tensor field ρ:

$$R_{\alpha\beta} = \rho_{\alpha\beta}\ , \tag{1a}$$

$$\nabla_\alpha T^{\alpha\beta} = 0\ , \tag{1b}$$

with $T_{\alpha\beta} = \rho_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\rho$, $\rho = \rho_{\lambda\mu} g^{\lambda\mu}$.
Show that in the gauge used in [Problem VI 12, Leray systems] and if ρ^{ij} *is given on M then a solution* $\bar{g}_{ij}$, $\rho^{0\alpha}$ *of* (1*a*), (1*b*) *satisfies a Leray hyperbolic system.*

Answer: We choose coordinates such that $g_{0i} = 0$, $g_{00} = -\alpha^2$, $\alpha = c(\det \bar{g})^{1/2}$ and use the second fundamental form $K = -(2\alpha)^{-1}\partial_0 \bar{g}$ to write the equations deduced from the Einstein ones:

$$\partial_0 R_{ij} - \bar{\nabla}_i R_{0j} - \bar{\nabla}_j R_{0i} = \partial_0 \rho_{ij} - \bar{\nabla}_i \rho_{0j} - \bar{\nabla}_j \rho_{0i}$$

under the form

$$\Box \partial_0 g_{ij} = (2 \text{ in } g,\ 1 \text{ in } \rho_{0h})\ , \tag{1}$$

(where the notation "k in f" means that the quantity depends on f only through its derivatives of order $\leq k$).
Since $\rho = -\alpha^2\rho^{00} + g_{ij}\rho^{ij}$, the equation $\nabla_\alpha T^{\alpha 0} = 0$ gives

$$\nabla_0(\tfrac{1}{2}\rho^{00} - \tfrac{1}{2}\alpha^{-2} g_{ij}\rho^{ij} + \nabla_i \rho^{0i}) = 0\ ,$$

that is

(i) $$\tfrac{1}{2}\partial_0 \rho^{00} + \bar{\nabla}_i \rho^{0i} = (1 \text{ in } g,\ 0 \text{ in } \rho^{0\alpha})\ ;$$

and the equation $\nabla_\alpha T^{\alpha i}$ gives

$$\partial_0 \rho^{0i} + \nabla_j(\rho^{ji} - \tfrac{1}{2} g^{ij}(-\alpha^2\rho^{00} + g_{hk}\rho^{hk})) = 0\ ,$$

that is (note that an index 0 is scalar for $\bar{\nabla}$)

(ii) $$\partial_0 \rho^{0i} + \tfrac{1}{2}\alpha^2 \bar{\nabla}^i \rho^{00} = (1 \text{ in } g,\ 0 \text{ in } \rho^{0\alpha})\ .$$

We deduce from (i) and (ii)

$$\tfrac{1}{2}(\partial^2_{00}\rho^{00} - \alpha^2 \bar{\nabla}_i \bar{\nabla}^i \rho^{00}) = (2 \text{ in } g,\ 1 \text{ in } \rho^{0\alpha})\ ,$$

thus

$$\Box\rho^{00} = (2 \text{ in } g,\ 1 \text{ in } \rho^{0\alpha}) \tag{2}$$

and then

$$\Box\partial_0\rho^{0i} = (3 \text{ in } g,\ 1 \text{ in } \rho^{0\alpha})\,. \tag{3}$$

We choose as follows the Leray weights [Problem V 7, Conformal] for the unknowns:

$$m(\rho^{0\alpha}) = 2\,, \qquad m(g) = 3$$

and for the equations

$$n(1) = 0\,, \quad n(2) = 0\,, \quad n(3) = -1\,.$$

In eqs. (1) the principal part for the unknown g_{ij}, of order 3, is $\Box\partial_0 g_{ij}$, while it is zero for g_{hk} if $(h, k) \neq (i, j)$ and for $\rho^{0\alpha}$, since (1) contains no second derivatives of $\rho^{0\alpha}$ and $m(\rho^{0\alpha}) - n(1) = 2$. The only nonzero principal parts in (2) is $\Box\rho^{00}$ and in (3) it is $\Box\partial_0\rho^{0i}$. The system is therefore quasi-diagonal. The operator $\Box\partial_0$ is hyperbolic, since the cone H determined by its characteristic polynomial is, as in the vacuum case:

$$H\colon h(X) \equiv (-(-\alpha^2)(X^0)^i + \bar{g}_{ij}X^iX^j)X^0 = 0\,.$$

It is cut in 3 distinct points by any straight line through the inferior of the light cone of the hyperbolic metric g.

Remark: For a solution of the hyperbolic system (1)–(3) and $g_{0i} = 0$, $g_{00} = -c^2 \det \bar{g}$, the same arguments as in Problem VI 12 show that the quantities $S_{\alpha\beta} - T_{\alpha\beta}$ satisfy a linear, homogeneous hyperbolic system, with zero Cauchy data if the initial data for $\bar{g}$, K, ρ satisfy the constraints.

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14. DISTRIBUTIONS AND ANALYTICITY: WIGHTMAN DISTRIBUTIONS AND SCHWINGER FUNCTIONS*

The vacuum expectation values of quantum fields are known as Wightman distributions $\mathcal{W}_n$: formally

*(Extracted from lectures by K. Osterwalder). Contributed by Charles Rogers Doering.

$$\mathcal{W}_n(x_1, \ldots, x_n) = \langle \Omega, \phi(x_1) \ldots \phi(x_n)\Omega \rangle .$$

They are tempered distributions based on spacetime V^4 (for a complete discussion see Streater and Wightman (1964), Reed and Simon (1975), or Glimm and Jaffe (1981)). The Schwinger functions of euclidean field theory are the "analytic continuation" of the Wightman distributions to imaginary time values; they are tempered distributions given by analytic functions based on euclidean space R^4.

The physical basis for the ability to perform this analytic continuation is the positivity of the Hamiltonian (the generator of time translations) and the mass operator $M^2 = H^2 - \boldsymbol{P}^2$ (where P_i is the generator of translations in the ith space direction). These conditions on the translation generators imply that the Fourier transform of $\mathcal{W}_n$ has support in a restricted part of V^{4^*}, the dual of the spacetime V^4. Just as the Paley–Wiener theorem ensures that the Fourier transform of a distribution with compact support can be extended to an entire analytic function, the connection between the Schwinger functions and the Wightman distributions is established by the generalization to higher dimension of the following theorem.

Problem: *Let $W \in \mathcal{S}'(R)$ with supp $\mathcal{F}W \subset [0, \infty)$. Prove that there is a function $W(z)$, analytic in the upper-half z-plane, so that for every $f \in \mathcal{S}$*

$$W(f) = \lim_{y \downarrow 0} \int \mathrm{d}x\, W(x + \mathrm{i}y) f(x) .$$

Remark: If W is given by a function in $\mathcal{S}$, then $W(z)$ is given by the formula

$$W(z) = \int \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}kz} \int \mathrm{d}x'\, \mathrm{e}^{-\mathrm{i}kx'} W(x') ,$$

the natural analytic extension. The work in this problem is to generalize the result to $W \in \mathcal{S}'$. The distinctions among distributions, distributions given by functions and functions is critical here. Hence the notation in this problem will be the following: functions will have their arguments (x, y, z, k, p) explicitly displayed while distributions will have no argument. For example, we may say $f(x) \in \mathcal{S}$ while the distribution defined by this function is $f \in \mathcal{S}'$. This problem and its solution is a shorter version of the general case treated in Streater and Wightman (1964) (note correction to the proof in revised edition) and Reed and Simon (1975).

Solution: The solution is divided into five main sections.

1. Let $f_z(k) = (2\pi)^{-1}\chi(k)\, \mathrm{e}^{\mathrm{i}kz}$, $z \in \mathbb{C}$, $\chi(k) \in C^\infty(R)$ and

$$\chi(k) = \begin{cases} 0 , & k \leq -1 \\ 1 , & k \geq 0 . \end{cases}$$

Show that $\{f_z(k) | \operatorname{Im} z > 0\} \subset \mathscr{S}(R)$.

2. Show that for $\operatorname{Im} z > 0$, the function

$$W(z) := \mathscr{F} W(f_z)$$

is analytic.

3. For $y > 0$ let $W_y(x) = W(x + \mathrm{i}y)$ and show that there are integers n and m and a polynomial $C(y)$ so that

$$|W_y(x)| \leq C(y)\left(1 + \frac{1}{y^n}\right)(1 + x^2)^m .$$

The polynomial bound on $W_y(x)$ for each y ensures that for fixed y this function defines a distribution $W_y \in \mathscr{S}'$.

4. Show that

$$\lim_{y \downarrow 0} W_y \in \mathscr{S}'$$

and

$$\sup \mathscr{F} W_y \in [0, \infty) .$$

5. Show that

$$\lim_{y \downarrow 0} W_y = W .$$

This establishes that W is the "boundary value" of an analytic function.

To proceed:

1. For $y = \operatorname{Im} z > 0$,

$$f_z(k) = (2\pi)^{-1} \chi(k)\, \mathrm{e}^{ikx - ky}$$

is C^∞, supported in $[-1, \infty)$ and decays exponentially as $k \to +\infty$. To show the $f_z \in \mathscr{S}$, choose integers m and n. Then

$$\begin{aligned}
&\sup_k \left| (1 + k^2)^m \frac{\mathrm{d}^n}{\mathrm{d}k^n} f_z(k) \right| \\
&= (2\pi)^{-1} \sup_k \left| (1 + k^2)^m \sum_{i=0}^{n} \binom{n}{i} \chi^{(i)}(k) (\mathrm{i}z)^{n-i}\, \mathrm{e}^{\mathrm{i}kz} \right| \\
&\leq (2\pi)^{-1} \sup_{k > -1} |(1 + k^2)^m\, \mathrm{e}^{-ky}| \\
&\quad + (2\pi)^{-1} \sum_{i=1}^{n} \binom{n}{i} 2^m\, \mathrm{e}^{y} |z|^{n-i} \sup_{-1 < k < 0} |\chi^{(i)}(k)| .
\end{aligned}$$

The first term about is finite since the exponential decays faster than the polynomial diverges, and the second term is finite since χ is C^∞.

2. Since $f_z(k) \in \mathscr{S}(R)$ for each $y = \operatorname{Im} z > 0$, $W(z) := \mathscr{F}W(f_z)$ is well-defined for each $y > 0$. To establish analyticity it is sufficient to show that $W(z)$ is once (and thus infinitely) differentiable in this complex sense for $\operatorname{Im} z > 0$.

This means that we must show that

$$\lim_{|\xi| \to 0} \xi^{-1}(W(z+\xi) - W(z))$$

exists. By linearity,

$$\begin{aligned}\xi^{-1}(W(z+\xi) - W(z)) &= \xi^{-1}(\mathscr{F}W(f_{z+\xi}) - \mathscr{F}W(f_z)) \\ &= \mathscr{F}W(\xi^{-1}(f_{z+\xi} - f_z)) .\end{aligned}$$

Hence the limit exists if $\xi^{-1}(f_{z+\xi} - f_z)$ converges in $\mathscr{S}(R)$ as $|\xi| \to 0$. Since we expect that

$$\frac{\mathrm{d}f_z(k)}{\mathrm{d}z} = \mathrm{i}kf_z(k) ,$$

we will show that

$$\mathscr{S} - \lim_{|\xi| \to 0} \xi^{-1}(f_{z+\xi}(k) - f_z(k)) = \mathrm{i}kf_z(k) .$$

(Note that $\mathrm{i}kf_z(k) \in \mathscr{S}(R)$ by a proof like that in 1. above). To begin, we first remark that

$$\begin{aligned}|\mathrm{e}^{\mathrm{i}k\xi} - 1| = \left|\sum_{n=1}^{\infty} \frac{(\mathrm{i}k\xi)^n}{n!}\right| &\le |k||\xi| \sum_{n=0}^{\infty} \frac{|k\xi|^n}{(n+1)!} \\ &\le |k||\xi| \sum_{n=0}^{\infty} \frac{|k\xi|^n}{n!} \\ &= |k||\xi|\,\mathrm{e}^{|k\xi|} .\end{aligned}$$

and similarly,

$$\begin{aligned}|\mathrm{e}^{\mathrm{i}k\xi} - 1 - \mathrm{i}k\xi| &= \left|\sum_{n=2}^{\infty} \frac{(\mathrm{i}k\xi)^n}{n!}\right| \\ &\le |k|^2|\xi|^2 \sum_{n=0}^{\infty} \frac{|k\xi|^n}{(n+2)!} \le |k|^2|\xi|^2 \sum_{n=0}^{\infty} \frac{|k\xi|^n}{n!} \\ &= |k|^2|\xi|^2\,\mathrm{e}^{|k\xi|} .\end{aligned}$$

Without loss of generality we may take $|\xi| < y$. Then for any positive integers m and n,

$$
\begin{aligned}
&\sup_k \left|(1+k^2)^m \frac{\mathrm{d}^n}{\mathrm{d}k^n}\,[\xi^{-1}(f_{z+\xi}(k)-f_z(k)) - \mathrm{i}kf_z(k)]\right| \\
&\quad = \sup_k \left|(1+k^2)^m \frac{\mathrm{d}^n}{\mathrm{d}k^n}\, f_z(k)\xi^{-1}(\mathrm{e}^{\mathrm{i}k\xi} - 1 - \mathrm{i}k\xi)\right| \\
&\quad = \sup_k \left|(1+k^2)^m \sum_{i=0}^{n} \binom{n}{i} f_z^{(n-i)}(k)\xi^{-1} \frac{\mathrm{d}^i}{\mathrm{d}k^i}\,(\mathrm{e}^{\mathrm{i}k\xi} - 1 - ik\xi)\right| \\
&\quad \le \sup_k [(1+k^2)^m |f_z^{(n)}(k)||k|^2|\xi|\,\mathrm{e}^{|k\xi|}] \\
&\qquad + \sup_k [(1+k^2)^m n |f_z^{(n-1)}(k)||k|^2|\xi|\,\mathrm{e}^{|k\xi|}] \\
&\qquad + \sum_{i=2}^{n} \binom{n}{i} \sup_k [(1+k^2)^m |f_z^{(n-i)}(k)||k|^i|\xi|^{i-1}] \\
&\quad \le |\xi| \Big\{ \sup_k [(1+k^2)^{m+1}\,\mathrm{e}^{|k||\xi|}|f_z^{(n)}(k)|] \\
&\qquad + n \sup_k [(1+k^2)^{m+1}\,\mathrm{e}^{|k||\xi|}|f_z^{(n-1)}(k)|] \\
&\qquad + \sum_{i=2}^{n} |\xi|^{i-2} \binom{n}{i} \sup_k [(1+k^2)^{m+i}|f_z^{(n-i)}(k)|] \Big\}.
\end{aligned}
$$

Each $f^{(j)}(k)$ is supported on $[-1,\infty)$ and decays like e^{-ky} as $k \to \infty$. As $|\xi| < y$, we conclude that the first two terms in brackets above are finite for any m and n. The sum above is a polynomial in $|\xi|$ with finite coefficients (they are just multiples of $\mathcal{S}$-space seminorms of f_z). Thus,

$$\lim_{|\xi|\to 0} \sup_k |(1+k^2)^m \frac{\mathrm{d}^n}{\mathrm{d}k^n}\,[\xi^{-1}(f_{z+\xi}(k)-f_z(k)) - \mathrm{i}kf_z(k)] = 0\,.$$

3. Since $\mathcal{F}W$ is a tempered distribution it is continuous in some seminorm, i.e., there are integers m and n and a constant A such that for any $g \in \mathcal{S}$

$$|\mathcal{F}W(g)| \le A \sup_p \left|(1+p^2)^m \frac{\mathrm{d}^n}{\mathrm{d}p^n}\, g(p)\right|.$$

Thus,

$$
\begin{aligned}
|W_y(x)| &= |\mathcal{F}W(f_{x+\mathrm{i}y})| = |W(\mathcal{F}f_{x+\mathrm{i}y})| \\
&\le A \sup_p \left|(1+p^2)^m \frac{\mathrm{d}^n}{\mathrm{d}p^n}\,(\mathcal{F}f_{x+\mathrm{i}y})(p)\right|.
\end{aligned}
$$

The seminorm of $\mathscr{F}f_{x+\mathrm{i}y}$ is bounded as follows:

$$
\begin{aligned}
(1+p^2)^m \frac{\mathrm{d}^n}{\mathrm{d}p^n}(\mathscr{F}f_{x+\mathrm{i}y})(p) &= (1+p^2)^m \frac{\mathrm{d}^n}{\mathrm{d}p^n}\int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kp} f_{x+\mathrm{i}y}(k) \\
&= \int \mathrm{d}k\left[\left(1-\frac{\mathrm{d}^2}{\mathrm{d}k^2}\right)^m \mathrm{e}^{-\mathrm{i}kp}\right](-\mathrm{i}k)^n f_{x+\mathrm{i}y}(k) \\
&= \int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kp}\left(1-\frac{\mathrm{d}^2}{\mathrm{d}k^2}\right)^m (-\mathrm{i}k)^n f_{x+\mathrm{i}y}(k) \\
&= (-\mathrm{i})^n \sum_{i=0}^{m}\binom{m}{i}(-1)^m \int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kp}\frac{\mathrm{d}^{2i}}{\mathrm{d}k^{2i}} k^n f_{x+\mathrm{i}y}(k) \\
&= (-\mathrm{i})^n \sum_{i=0}^{m}\binom{m}{i}(-1)^m \sum_{j=0}^{2i}\binom{2i}{j} \\
&\quad \times \int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kp}\left(\frac{\mathrm{d}^j}{\mathrm{d}k^j}k^n\right)\left(\frac{\mathrm{d}^{2i-j}}{\mathrm{d}k^{2i-j}} f_{x+\mathrm{i}y}(k)\right) \\
&= (-\mathrm{i})^n \sum_{i=0}^{m}(-1)^m\binom{m}{i}\sum_{j=0}^{\min(2i,n)}\binom{2i}{j}\frac{n!}{(n-j)!}\sum_{l=0}^{2i-j}\binom{2i-j}{l} \\
&\quad \times \int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kp} k^{n-j}(2\pi)^{-1}\chi^{(l)}(k)(\mathrm{i}x-y)^{2i-j-l}\,\mathrm{e}^{\mathrm{i}kx-ky},
\end{aligned}
$$

and thus

$$
\begin{aligned}
|W_y(x)| \le (2\pi)^{-1}A\sum_{i=0}^{m}\binom{m}{i}\sum_{j=0}^{\min(2i,n)}\binom{2i}{j}\frac{n!}{(n-j)!}\sum_{l=0}^{2i-j}\binom{2i-j}{l} \\
\times \sup_k|\chi^{(l)}(k)|(1+x^2+y^2)^{2i-j-l}\int_{-1}^{\infty}\mathrm{d}k\, k^{n-j}\,\mathrm{e}^{-ky}.
\end{aligned}
$$

For $y \ge 1$ the k-integral above is bounded uniformly in $j \le n$ so there is a constant B and a polynomial $B(y)$ with

$$
\begin{aligned}
|W_y(x)| &\le B(1+y^2+x^2)^{2m} \\
&\le B(y)(1+x^2)^{2m}.
\end{aligned}
$$

For $y<1$,

$$
\begin{aligned}
\int_{-1}^{\infty}\mathrm{d}k\, k^{n-j}\,\mathrm{e}^{-ky} &= \int_{-1}^{0}\mathrm{d}k\, k^{n-j}\,\mathrm{e}^{-ky} + \int_{0}^{\infty}\mathrm{d}k\, k^{k-j}\,\mathrm{e}^{-ky} \\
&\le e + \frac{(n-j)!}{y^{n-j+1}} \le en!\left(1+\frac{1}{y^{n+1}}\right)
\end{aligned}
$$

and there is a constant C such that

$$|W_y(x)| \leq C\Big(1 + \frac{1}{y^{n+1}}\Big)(1 + x^2)^{2m} .$$

Combining the bounds for $y < 1$ and $y \geq 1$, we conclude that there is a polynomial $C(y)$ such that for all $y > 0$

$$|W_y(x)| \leq C(y)\Big(1 + \frac{1}{y^{n+1}}\Big)(1 + x^2)^{2m} .$$

Hence, for each $y > 0$, $W_y(x)$ defines a tempered distribution.

4. To show that $\lim_{y\downarrow 0} W_y \in \mathscr{S}'$ we must establish that for each $g \in \mathscr{S}$,

$$W_y(g) = \int \mathrm{d}x\, W_y(x) g(x)$$

converges as $y \downarrow 0$ and that the limit is bounded by a seminorm of g.

Since for every $y > 0$ $W_y(g)$ is given by an absolutely convergent integral, we may differentiate with respect to y under the integral. Recalling that $W_y(x) = W(z)$ is analytic so that Cauchy's theorem applies, we find

$$\begin{aligned}\frac{\mathrm{d}^l}{\mathrm{d}y^l} W_y(g) &= \int \mathrm{d}x\, \frac{\mathrm{d}^l}{\mathrm{d}y^l} W(x + \mathrm{i}y) g(x) \\ &= \int \mathrm{d}x\Big[(\mathrm{i})^l \frac{\mathrm{d}^l}{\mathrm{d}x^l} W(x + \mathrm{i}y)\Big] g(x) = (-\mathrm{i})^l \int \mathrm{d}x\, W(x + \mathrm{i}y) g^{(l)}(x) .\end{aligned}$$

As $y \downarrow 0$, the derivations of $W_y(g)$ are bounded by the *same power* of y^{-1} as $W_y(g)$:

$$\begin{aligned}\Big|\frac{\mathrm{d}^l}{\mathrm{d}y^l} W_y(g)\Big| &\leq C(y)\Big(1 + \frac{1}{y^{n+1}}\Big) \int \mathrm{d}x (1 + x^2)^{2m} |g^{(l)}(x)| \\ &\leq \pi C(y)\Big(1 + \frac{1}{y^{n+1}}\Big) \sup_x |(1 + x^2)^{2m+1} g^{(l)}(x)| .\end{aligned}$$

This means that even though the a priori bounds on $W_y(x)$ and $W_y(g)$ diverge as $y \downarrow 0$, they may be improved to convergent bounds.

Let us denote

$$I(y) = W_y(g) .$$

Then for $y \in (0, 1)$ each derivative of $I(y)$ is bounded as

$$|I^{(l)}(y)| \leq C_l \frac{1}{y^{n+1}} ,$$

where C_l is proportional to a seminorm of g. The point is that $I(y)$ cannot possibly diverge as $y\downarrow 0$ if its derivatives do not diverge in a worse manner. We can use the fundamental theorem of calculus to show this:

$$\begin{aligned} I(y) &= -\int_y^1 \mathrm{d}y_1\, I^{(1)}(y_1) + I(1) \\ &= \int_y^1 \mathrm{d}y_1 \int_{y_1}^1 \mathrm{d}y_2\, I^{(2)}(y_2) + (1-y)I^{(1)}(1) + I(1) \\ &= \cdots \\ &= (-1)^k \int_y^1 \mathrm{d}y_1 \int_{y_1}^1 \mathrm{d}y_2 \cdots \int_{y_k}^1 \mathrm{d}y_{k+1} I^{(k+1)}(y_{k+1}) + P_k(y)\,, \end{aligned}$$

where $P_k(y)$ is a polynomial of degree $k-1$ in y with coefficients made up of linear combinations of $I(1), \ldots, I^{(k)}(1)$.

Choose $k = n+1$. Then

$$\begin{aligned} |I(y)| &\le C_{n+1} \int_y^1 \mathrm{d}y_1 \int_{y_1}^1 \mathrm{d}y_2 \cdots \int_{y_{n+1}}^1 \mathrm{d}y_{n+2} \frac{1}{(y_{n+2})^{n+1}} + |P_n(y)| \\ &= C_{n+1} \int_y^1 \mathrm{d}y_1 \int_{y_1}^1 \mathrm{d}y_2 \cdots \int_{y_n}^1 \mathrm{d}y_{n+1} \frac{1}{n}\left(\frac{1}{(y_{n+1})^n} - 1\right) + |P_u(y)| \\ &= C_{n+1} \int_y^1 \mathrm{d}y_1 \int_{y_1}^1 \mathrm{d}y_2 \cdots \int_{y_{n-1}}^1 \mathrm{d}y_n \left\{ \frac{1}{n(n-1)} \left(\frac{1}{(y_n)^{n-1}} - 1\right) \right. \\ &\quad \left. - \frac{1}{n}(1-y_n) \right\} + |P_n(y)| \\ &= \cdots \\ &= (\mathrm{Const})\, C_{n+1} \left[\int_y^1 \mathrm{d}y_1 \ln y_1 + Q(y) \right] + |P_n(y)|\,, \end{aligned}$$

where $Q(y)$ is a polynomial in y. Since the integral on the last line above is bounded as $y\downarrow 0$, $I(y)$ is given by a convergent integral. Hence $W_y(g)$ converges as $y\downarrow 0$ and, since C_{n+1} and $p_n(0)$ are bounded by seminorms of g, we conclude that $\lim_{y\downarrow 0} W_y \in \mathcal{S}'$.

To see that $\mathcal{F}W_y$ (and thus also $\lim_{y\downarrow 0} \mathcal{F}W_y$) is supported on $[0,\infty)$, choose a function $g \in \mathcal{D}$ supported on $(-\infty, 0)$. These functions are dense in $\{g \in \mathscr{S} | \operatorname{supp} g \subset (-\infty, 0)\}$ so it is sufficient to show that $\mathscr{F}W_y(g)$ vanishes for this restricted set.

If $g \in \mathscr{S}$ has compact support, then its Fourier transform is an entire analytic function defined by the absolutely convergent integral

$$(\mathscr{F}g)(z) = \int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kz} g(k)\,.$$

It is left as an exercise to the reader to show that this function is complex differentiable.

If the compact support of such a function is contained in the open set $(-\infty, 0)$, then its Fourier transform vanishes exponentially in the upper half plane. To see this, note that there is an $\varepsilon > 0$ so that $k > -\varepsilon \Rightarrow k \notin \operatorname{supp} g$. Then for any integer $l \geq 0$

$$\begin{aligned}
&\sup_x |(1 + x^2)^l (\mathscr{F}g)(x + \mathrm{i}y)| \\
&= \sup_x \left| \int_{-\infty}^{-\varepsilon} \mathrm{d}k\; g(k)\, \mathrm{e}^{ky} \left(1 - \frac{\mathrm{d}^2}{\mathrm{d}k^2}\right)^l \mathrm{e}^{-\mathrm{i}kx} \right| \\
&\leq \int_{-\infty}^{-\varepsilon} \mathrm{d}k \left| \left(1 - \frac{\mathrm{d}^2}{\mathrm{d}k^2}\right)^l \mathrm{e}^{ky} g(k) \right| \\
&\leq \sum_{j=0}^{l} \binom{l}{j} \sum_{i=0}^{2j} \binom{2j}{i} \int_{-\infty}^{-\varepsilon} \mathrm{d}k\; y^i |g^{(2j-i)}(k)|\, \mathrm{e}^{ky} \\
&\leq \mathrm{e}^{-\varepsilon y} R_l(y)\,,
\end{aligned}$$

where R_l is a polynomial of degree $2l$ in y.

Then, for any $\alpha > y^{-1}$

$$\begin{aligned}
\mathscr{F}W_y(g) &= W_y(\mathscr{F}g) \\
&= \int \mathrm{d}x\, W(x + \mathrm{i}y)(\mathscr{F}g)(x) \\
&= \int \mathrm{d}x\, W(x + \mathrm{i}\alpha y)(\mathscr{F}g)(x + \mathrm{i}(\alpha - 1)y)
\end{aligned}$$

since the contour of integration may be shifted in the upper-half complex plane due to analyticity. Recalling the bounds on $W(z)$ we find

$$\begin{aligned}
|\mathscr{F}W_y(g)| &\leq C(\alpha y)\left(1 + \frac{1}{(\alpha y)^{n+1}}\right) \int \mathrm{d}x (1 + x^2)^{2m} \left| (\mathscr{F}g)(x + \mathrm{i}(\alpha - 1)y) \right| \\
&\leq C(\alpha y)\left(1 + \frac{1}{(\alpha y)^{n+1}}\right) \pi \sup_x \left| (1 + x^2)^{2m+1} (\mathscr{F}g)(x + \mathrm{i}(\alpha - 1)) \right| \\
&\leq 2\pi C(\alpha y) R_{2m+1}((\alpha - 1)y)\, \mathrm{e}^{-\varepsilon(\alpha - 1)y}\,.
\end{aligned}$$

This estimate holds for any $\alpha > y^{-1}$ so we may take $\alpha \to \infty$ and conclude that $W_y(g) = 0$.

5. We are now ready to show that W is the *boundary value* of W_y. By this we mean that for any $g \in \mathcal{S}$

$$\lim_{y \downarrow 0} W_y(g) = W(y) .$$

Since

$$W_y(g) = (\mathcal{F} W_y)(\bar{\mathcal{F}} g) \quad \text{and} \quad W(g) = (\mathcal{F} W)(\bar{\mathcal{F}} g)$$

and $\mathcal{F} W_y$ and $\mathcal{F} W$ are supported on $[0, \infty)$ we may without loss of generality consider test functions g with sup $\bar{\mathcal{F}}(g) \subset [0, \infty)$.

The regularity theorem for tempered distributions (Schwarz, 1957, 1959, Reed and Simon, 1972) ensures that $\mathcal{F} W$ has the representation

$$\mathcal{F} W(h) = \int \mathrm{d}k \sum_{n=0}^{N} a_n(x) \frac{\mathrm{d}^n}{\mathrm{d}x^n} h(x)$$

for some finite N and continuous, polynomially bounded functions $a_n(x)$. Hence,

$$\begin{aligned}
W_y(g) &= \int \mathrm{d}x \, g(x) \mathcal{F} W(f_{x+\mathrm{i}y}) \\
&= \int \mathrm{d}x \, g(x) \int \mathrm{d}k \sum_{n=0}^{N} a_n(k) \frac{\mathrm{d}^n}{\mathrm{d}k^n} f_{x+\mathrm{i}y}(k) \\
&= \sum_{n=0}^{N} \sum_{j=0}^{n} \binom{n}{j} (2\pi)^{-1} \int \mathrm{d}x \, g(x) \int \mathrm{d}k \, a_n(k) \chi^{(n-j)}(k) (\mathrm{i}x - y)^j \, \mathrm{e}^{\mathrm{i}kx - ky} \\
&= \sum_{n=0}^{N} \sum_{j=0}^{n} \binom{n}{j} \sum_{l=0}^{j} \binom{j}{l} (-y)^{j-l} \int \mathrm{d}k \, a_n(k) \chi^{(u-j)}(k) \, \mathrm{e}^{-ky} \\
&\quad \times \int \frac{\mathrm{d}x}{2\pi} (\mathrm{i}x)^l \, \mathrm{e}^{\mathrm{i}kx} g(x) ,
\end{aligned}$$

where the reversal of the order of integrations is justified by the absolute convergence of the integrals. Thus we find

$$\begin{aligned}
W_y(g) &= \sum_{n=0}^{N} \sum_{j=0}^{n} \binom{n}{j} \int \mathrm{d}k \, a_n(k) \chi^{(n-j)}(k) \frac{\mathrm{d}^j}{\mathrm{d}k_j} (\mathrm{e}^{-ky} (\bar{\mathcal{F}} g)(x)) \\
&= \int \mathrm{d}k \sum_{n=0}^{N} a_n(k) \frac{\mathrm{d}^n}{\mathrm{d}k^n} (\mathrm{e}^{-ky} (\bar{\mathcal{F}} g)(k))
\end{aligned}$$

since $\bar{\mathcal{F}} g$ and all its derivatives are supported on $[0, \infty)$ while $\chi^{(n-j)}$ is supported on $[-1, 0]$ for $j \neq n$. The conclusion is

$$W_y(g) = \mathscr{F}W(\mathrm{e}^{-(\cdot)y}(\bar{\mathscr{F}}g)(\cdot)) = W(\mathscr{F}(\mathrm{e}^{-(\cdot)y}(\bar{\mathscr{F}}g)(\cdot))) .$$

It thus only remains to be shown that

$$\mathscr{S}\text{-}\lim_{y\downarrow 0} \mathscr{F}(\mathrm{e}^{-(\cdot)y}(\bar{\mathscr{F}}g)(\cdot)) = g ,$$

or equivalently, since the inverse Fourier transform is continuous on $\mathscr{S}$, that

$$\mathscr{S}\text{-}\lim_{y\downarrow 0} \mathrm{e}^{-ky}(\bar{\mathscr{F}}g)(k) = (\bar{\mathscr{F}}g)(k) .$$

Toward this end, choose integers m and n. Then

$$\begin{aligned}
&\sup_k \left|(1+k^2)^m \frac{\mathrm{d}^n}{\mathrm{d}k^n}(1-\mathrm{e}^{-ky})(\bar{\mathscr{F}}g(k))\right| \\
&\quad \le \sum_{i=0}^{n} \binom{n}{i} \sup_k \left|(1+k^2)^m (\bar{\mathscr{F}}g)^{(n-i)}(k) \frac{\mathrm{d}^i}{\mathrm{d}k^i}(1-\mathrm{e}^{-ky})\right| \\
&\quad \le \sup_{k>0}\left(\frac{1-\mathrm{e}^{-ky}}{1+k^2}\right) \sup_k |(1+k^2)^{m+1}(\bar{\mathscr{F}}g)^{(n)}(k)| \\
&\qquad + \sum_{i=1}^{n} y^i \binom{n}{i} \sup_k |(1+k^2)^m (\bar{\mathscr{F}}g)^{(n)}(k)| .
\end{aligned}$$

The sum above is a polynomial in y with no constant coefficient – it clearly vanishes as $y\downarrow 0$. The first term may be rewritten

$$\begin{aligned}
\sup_{k>0} \frac{1-\mathrm{e}^{-ky}}{1+k^2} &= \sup_{k>0} \int_0^k \mathrm{d}k' \frac{\mathrm{d}}{\mathrm{d}k'}\left(\frac{1-\mathrm{e}^{-k'y}}{1+k'^2}\right) \\
&= \sup_{k>0}\left\{ y \int_0^k \mathrm{d}k' \frac{\mathrm{e}^{-k'y}}{(1+k'^2)} - 2\int_0^k \mathrm{d}k' \frac{k'(1-\mathrm{e}^{-k'y})}{(1+k'^2)^2}\right\} \\
&\le y \int_0^\infty \frac{\mathrm{d}k'}{1+k'^2} \xrightarrow[y\downarrow 0]{} 0 .
\end{aligned}$$

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15. BOUNDS ON THE NUMBER OF BOUND STATES OF THE SCHRÖDINGER OPERATOR*

This problem concerns the spectrum of Schrödinger-type operators $H = -\Delta + V(x)$ acting on $L^2(\mathbb{R}^n)$. We assume that

$$\lim_{|x|\to\infty} V(x) = 0$$

tends to zero at infinity. Simple modifications allow us to treat confining potentials.

1) *Start from the energy functional for $n \geq 3$, defined by (integrals are taken over $\mathbb{R}^n$)*

$$E(\psi) = T(\psi) + U(\psi)\,, \qquad T(\psi) = \int \mathrm{d}^n x\, |\nabla\psi(x)|^2\,,$$

$$U(\psi) = \int \mathrm{d}^n x\, V(x)|\psi(x)|^2\,, \qquad \int \mathrm{d}^n x\, |\psi(x)|^2 = 1\,,$$

for $\psi \in H_1(\mathbb{R}^n)$ and suitable potentials $V(x)$. Bind the kinetic energy $T(\psi)$ using Sobolev's inequality, use Hölder's inequality for the potential energy contribution $U(\psi)$. Combine both to obtain a condition excluding negative energy bound states. Observe that $-\Delta + V(x) \geq 0$ implies the absence of such states [1].

2) *Consider the reduced wave equation for a radial symmetric Schrödinger problem in $\mathbb{R}^3$ with angular momentum l*

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + V(r)\right)u_{l,E}(r) = E u_{l,E}(r)\,, \qquad u_{l,E}(0) = 0\,,$$

on $L^2(\mathbb{R}^+, \mathrm{d}r)$. Assume that there exist ν_l bound states. The zero energy solution splits the half line into ν_l disjoint intervals, according to the nodal theorem (see for instance [2]*). Take first $l = 0$; integrate the Schrödinger equation between successive nodes to obtain*

$$\int_{r_p}^{r_{p+1}} \mathrm{d}r \left\{\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 + V(r)u^2(r)\right\} = 0\,, \qquad \text{where } u(r) = u_{0,0}(r)\,.$$

Apply the reasoning of part 1) to each interval, observe that Sobolev's inequality applies to finite intervals as well. Sum up the contributions to

*Contributed by H. Grosse.

obtain a bound on the number of bound states of the form

$$\nu_0 \leq \tilde{S}_3 \int_0^\infty \mathrm{d}r\, r^2 |V|_-^{3/2}, \qquad |V|_- = \tfrac{1}{2}(|V| - V),$$

where $|V|_-$ *denotes the absolute value of the negative part of* V. *Relate* $\tilde{S}_3$ *to the Sobolev constant* [1].

3) *Generalize* 2) *to angular momentum* l: *Transform from* r *and* $u(r)$ *to* $z = \ln r \in (-\infty, \infty)$ *and* $\phi(z) = u(r)/\sqrt{r}$ *and apply a scaling argument* [1, 3].

Answer 1: From Sobolev's inequality we get $T(\psi) \geq S_n \|\psi\|_p^2$ with $p = 2n/(n-2)$. Let $V = V_+ - V_-$ be the decomposition of V into positive and negative parts; $|V|_- = V_-$. Hölder's inequality yields $V(\psi) \geq -\|V_-\|_q \|\psi\|_p^2$ for $1/q + 2/p = 1$. Combining both bounds gives a lower bound on the energy functional

$$E(\psi) \geq (S_n - \|V_-\|_{n/2}) \|\psi\|_p^2, \qquad \frac{2}{p} + \frac{2}{n} = 1,$$

from which we conclude that there is no negative energy bound state if

$$\bar{S}_n \cdot \int \mathrm{d}^n x |V|_-^{n/2} < 1, \qquad \bar{S}_n = [\pi n(n-2)]^{-n/2} \frac{\Gamma(n)}{\Gamma(n/2)}.$$

For a one-parameter family of similar conditions, see [6].

Answer 2: From the zero-energy solution of the reduced radial wave equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V(r)\right) u(r) = 0, \qquad u(0) = 0,$$

we obtain by integrating between successive zeros

$$0 = \int_{r_p}^{r_{p+1}} \mathrm{d}r \left\{ \left(\frac{\mathrm{d}u(r)}{\mathrm{d}r}\right)^2 + V(r) u^2(r) \right\}$$

$$\geq \left(\tilde{S}_3^{-2/3} - \left(\int_{r_p}^{r_{p+1}} \mathrm{d}r\, r^2 |V|_-^{3/2} \right)^{2/3} \right) \left(\int_{r_p}^{r_{p+1}} \mathrm{d}r \frac{u^6(r)}{r^4} \right)^{1/3},$$

where we used the fact that from Sobolev's lemma

$$\inf_{\psi\in C_0^\infty(\mathbb{R}^+)} \frac{\int_0^\infty \mathrm{d}r\, r^2(\mathrm{d}\psi/\mathrm{d}r)^2}{\left(\int_0^\infty \mathrm{d}r\, r^2\psi^6(r)\right)^{1/3}} = \inf_{u\in C_0^\infty(\mathbb{R}^+)} \frac{\int_0^\infty \mathrm{d}r(\mathrm{d}u(r)/\mathrm{d}r)^2}{\left(\int_0^\infty \mathrm{d}r\, u^6(r)/r^4\right)^{1/3}}$$

$$= (4\pi)^{-2/3}S_3 = \tilde{S}_3^{-1/3},$$

$$\psi(r) = u(r)/r,$$

and the same constant appears if one restricts integration to a finite interval. Adding up the inequalities

$$1 \le \tilde{S}_3 \int_{r_p}^{r_{p+1}} \mathrm{d}r\, r^2|V|_-^{3/2}, \qquad \tilde{S}_3 = \frac{16}{3\sqrt{3}\pi},$$

for each interval, leads to the stated result.

Answer 3: For angular momentum l the above functional is replaced by

$$\frac{\int_0^\infty \mathrm{d}r\{(\mathrm{d}u(r)/\mathrm{d}r)^2 + [l(l+1)r^2]u^2(r)\}}{\left(\int_0^\infty \mathrm{d}r\, u^6(r)/r^4\right)^{1/3}}$$

$$= \frac{\int_{-\infty}^\infty \mathrm{d}z\{(\mathrm{d}\phi(z)/\mathrm{d}z)^2 + (l+\tfrac{1}{2})^2\phi^2(z)\}}{\left(\int_{-\infty}^\infty \mathrm{d}z\, \phi^6(z)\right)^{1/3}} = G_l(\phi).$$

Since $G_l(\phi)$ differs from $G_0(\phi)$ only through the replacement $\phi^2/4 \to \phi^2(2l+1)^2/4$, we may get back $G_0(\phi)$ from $G_l(\phi)$ by a scale transformation: $\phi(z) = \tilde{\phi}((2l+1)z)$; this leads to

$$G_l(\phi) = (2l+1)^{4/3}G_0(\phi).$$

Since the infimum of $G_0(\phi)$ is known, we obtain finally

$$\nu_l \le \tilde{S}_3 \frac{\int_0^\infty \mathrm{d}r\, r^2|V|_-^{3/2}}{(2l+1)^2}.$$

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16. SOBOLEV SPACES ON RIEMANNIAN MANIFOLDS

1. DEFINITIONS

Let X be a paracompact C^∞ manifold of dimension n and let h be a C^∞ (properly) riemannian metric on X. We have defined (pp. 480–485) distributions, tensor distributions and their covariant derivatives on (X, h).
We have denoted by

$$|t(x)| = (t^{i_1 \dots i_r}(x^j) t_{i_1 \dots i_r}(x^j))^{1/2}$$

the norm, relative to h, of an r-tensor $t(x)$ at $x \in X$.
We denote by $\mathrm{d}\mu(h)$ the measure defined on X by the volume element of h. In local coordinates

$$\mathrm{d}\mu(h) = (\det(h_{ij}))^{1/2}\, \mathrm{d}x^1 \dots \mathrm{d}x^n .$$

A tensor field t, defined almost everywhere on X, is said to belong to $L^q(X, h)$ if the function $|t|$ is in $L^q(X, \mathrm{d}\mu(h))$. We set

$$\|t\|_{L^q(X,h)} = \left(\int_X |t(x)|^q \,\mathrm{d}\mu(h)\right)^{1/q} .$$

If E denotes the bundle of r tensors over X, the space $L^q(X, h)$ just defined in the space of L^q-sections of E.

Show that if X is compact the space $L^q(X, h)$ does not depend on the choice of h.

uniformly equivalent

Answer: Two metrics h_1 and h_2 on X are called **uniformly equivalent** if there exist numbers $A > 0$ and $B > 0$ such that, for any smooth vector field v on X

$$Ah_2(v, v) < h_1(v, v) \le Bh_2(v, v) .$$

On a compact manifold two continuous metrics are always uniformly equivalent because at each point x the positive definite quadratic forms $h_{1,x}(v, v)$ and $h_{2,x}(v, v)$ are linked by such inequalities, with A and B replaced by strictly positive and bounded $a(x)$ and $b(x)$, which attain on X their positive minimum and maximum. In each coordinate chart, there then exist other strictly positive constants A and B such that

$$A \det(h_{1,ij}) \leq \det(h_{2,ij}) \leq B \det(h_{1,ij})$$

which shows that a function in $L^q(X, h_1)$ is also in $L^q(X, h_2)$, and conversely, though the norms may be different. If X is not compact $L^q(X, h_1)$ and $L^q(X, h_2)$ may be different. Example:

$$X = \mathbb{R}^2, \qquad h_1 = \mathrm{d}x^2 + \mathrm{d}y^2, \qquad h_2 = (1 + r^2)(\mathrm{d}x^2 + \mathrm{d}y^2),$$

$r^2 = x^2 + y^2$ the function $f = [1/(1 + r^2)^2]$ is in $L^1(X, h_1)$ but not in $L^1(X, h_2)$.

We denote by $\nabla^k t$ the $k + r$ tensor, covariant derivative of the r-tensor t. The Sobolev space $W_p^m(X, h)$, m a positive integer, $p \geq 1$, of sections of the bundle of r-tensors on X is the space of r-tensor distributions on X which, together with their covariant derivatives of order $\leq m$ can be identified with tensors in $L^p(X, h)$. This space is a Banach space, with the norm

$$\|t\|_{W_p^m} = \left\{ \sum_{k \leq m} \int_X |\nabla^k t(x)|^p \, \mathrm{d}\mu(h) \right\}^{1/p}.$$

In the case of a compact manifold X the space $W_p^m(X, h)$ does not depend on h (though the norm of a particular section depends on h).

Remark: It is not necessary to endow X with a metric to define W_p^m spaces of sections of a vector bundle over X. One can instead work through the charts of an atlas. The definition will be independent of the choice of the atlas if X is compact.

In the case of spinor fields on (X, h) it may be convenient to use the covariant derivative in h, and an appropriate, noncanonical, norm in the fibre.

The **injectivity radius** of a riemannian manifold (X, h) is by definition the greatest number δ such that injectivity radius

$$\delta \leq \delta_x,$$

where $\delta_x > 0$ is the maximal radius of a geodesic sphere in X centered at x for it to be diffeomorphic by the exponential mapping to an open neighborhood of 0 in the tangent space $T_x X$.

a) *Show that on a compact manifold one has always $\delta > 0$, but only $\delta \geq 0$ if X is noncompact and h arbitrary.*

b) *Show that a riemannian manifold with nonzero injectivity radius is complete.*

euclidean at infinity

c) *A riemannian manifold (X, h) is called* **euclidean at infinity** *if it is the union of a compact set K and an open set U and if $(U, h|_U)$ is isometric to the exterior of a ball of $\mathbb{R}^n$ with its euclidean metric. Show that if (X, h) is euclidean at infinity it has a nonzero injectivity radius.*

Answer a: δ_x is a continuous function of $x \in X$ which attains its minimum at a point $x_0 \in X$ if X is compact. Therefore $\delta_x \geq \delta_{x_0} = \delta > 0$.

Answer b: Let $\{x_n\}$ be a Cauchy sequence in X: for each $\epsilon > 0$ there exists n such that

$$\mathrm{d}(x_n, x_m) < \epsilon \quad \text{if} \quad n, m \geq N .$$

We choose $\epsilon < \delta$ and we consider the geodesic ball B_δ of center x_N and radius δ. The sequence $\{x_m\}$, $m \geq N$ lies in $B_\epsilon \subset B_\delta$ and B_δ is diffeomorphic to a ball of $\mathbb{R}^n$. The image of the closure of B_ϵ is complete in $\mathbb{R}^n$, therefore the sequence $\{x_n\}$ converges in X.

Answer c: We define a compact $K_1 \supset K$ which is the set of points where distance to K is less or equal to some given number $d > 0$. A point $x \in U - K_1$ is at a distance $> d$ from K, and therefore is the center of a geodesic ball in the euclidean metric of radius at least d. On the other hand δ_x, $x \in K_1$, attains on the compact K_1 its minimum, which is therefore positive. ∎

2. DENSITY THEOREM

One denotes $C_0^\infty(X)$ the space of C^∞ tensors on X of some given order r, with compact support. It can be proved (cf. Aubin [3], Wamon [6])

(i) $C_0^\infty(X)$ is dense in $W_p^0(X, h) = L^p(X)$.

(ii) $C_0^\infty(X)$ is dense in $W_p^1(X, h)$ if (X, h) has a nonzero injectivity radius, and uniformly bounded riemann curvature.

(iii) $C_0^\infty(X)$ is dense in $W_p^m(X, h)$, $m > 1$, if (X, h) has nonzero injectivity radius and if the riemann curvature is uniformly bounded on X as well as its derivatives of order $\leq m$:

$$\sup_X |\nabla^l \operatorname{Riem}(h)| \leq M , \qquad 0 \leq l \leq m .$$

Remark: the theorem shows that $C^\infty(X)$ is always dense in $W_p^m(X)$ if X is compact.

3. EMBEDDING AND MULTIPLICATION PROPERTIES

It can be proved that the **Sobolev embedding theorem** [Problem VI 2] holds for a riemannian manifold (X, h) if it has bounded curvature and nonzero injectivity radius. It holds in particular for compact manifolds, and manifolds euclidean at infinity.

Sobolev embedding theorem

The proof (cf. Aubin pp. 44–50) rests on the theorem for $U \subset \mathbb{R}^n$ and on the construction on X of an atlas where domains of charts are geodesic balls of fixed radius $\rho, 0 < \rho < \delta$, and which is uniformly locally finite – i.e. such that there exist an integer k with the property that each point $x \in X$ has a neighborhood which has a nonempty intersection with at most k of the considered balls.

The Sobolev embedding theorem is also valid for manifolds with boundary, under appropriate hypothesis reformulating the cone condition, for instance for compact manifolds with C^1 boundary.

Kondrakov's theorem holds for compact riemannian manifolds, or compact riemannian manifolds with a boundary which satisfies some kind of cone condition, for instance is C^1.

Kondrakov's theorem

The **multiplication theorem** and the **composition theorem** are valid for riemannian manifolds for which the Sobolev embedding theorem holds. The multiplication theorem is valid under the form given in Problem VI 3 for a manifold with finite volume (for instance compact), and otherwise under the form given in Problem VI 3, 2. ·

multiplication theorem composition theorem

4) The **interpolation theorem** in its general form is valid for C^∞ compact riemannian manifolds with or without boundary, with the additional hypothesis $\int_X u \, \mathrm{d}\mu = 0$ in this last case. The particular case proved ($j = 1$, $m = 2$, $a = \frac{1}{2}$) in Problem VI 2 is valid for an arbitrary C^∞ riemannian manifold, with a quite analogous proof. Many other cases are valid for noncompact manifolds as can be seen in following the details of Aubin's proof (pp. 94–95).

interpolation theorem

Prove that if the interpolation theorem is valid for riemannian manifolds without boundary under the restriction (Aubin, p. 94) $\int_X u \, \mathrm{d}\mu = 0$, it is also true without this restriction if $j > 0$.

Answer 4: Suppose X is compact without boundary, $u \in \mathscr{D}(X)$ and

$$\int_X u \, \mathrm{d}\mu = k \neq 0 , \quad \mathrm{d}\mu \text{ volume element.}$$

Set

$$v = u - k/\operatorname{vol} X\,, \qquad \operatorname{vol} X = \int_X \mathrm{d}\mu\ ;$$

then $v \in \mathscr{D}(X)$ and

$$\int_X v\,\mathrm{d}\mu = 0\,.$$

The interpolation theorem for such functions read

$$\|\nabla^j v\|_{L^p} \leq c\|\nabla^m v\|^a_{L^r}\|v\|^{1-a}_{L^q}\,.$$

We have $\nabla^j v = \nabla^j u$ if $j > 0$, $\nabla^m v = \nabla^m u$, and

$$\|v\|_{L^q} \leq \|u\|_{L^q} + \|k/\operatorname{vol} X\|_{L^q}$$

and also

$$|k| \leq \|u\|_{L^q} \leq \|u\|_{L^q}(\operatorname{vol} X)^{1/q'}\,, \qquad \frac{1}{q'} = 1 - \frac{1}{q}\,,$$

from which we deduce

$$\|v\|_{L^q} \leq 2\|u\|_{L^q}\,,$$

hence

$$\|\nabla^j u\|_{L^p} \leq 2^{1-a}C\|\nabla^m u\|^a_{L^r}\|u\|^{1-a}_{L^q}\,, \qquad \blacksquare$$

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SUPPLEMENTS AND ADDITIONAL PROBLEMS

Note:
The numbers in parenthesis such as (I.17), (III.15) ... (IV.19) are the numbers corresponding to the places of these problems in Vol. II (e.g. (I.17) is the seventeenth problem of Chapter I.)

SUPPLEMENTS AND ADDITIONAL PROBLEMS

1. THE ISOMORPHISM $H \otimes H \simeq M_4(\mathbb{R})$

A Supplement to Problem I.4 (pp. 6–14)

In paragraphs 3 and 4 of Problem I.4 on Clifford algebras, we use the isomorphism between the real tensor product of two quaternion algebras H with the algebra of real 4×4 matrices $M_4(\mathbb{R})$. This isomorphism plays a key role in establishing the periodicity modulo 8 of Clifford algebras. A well known two dimensional representation of the quaternion basis $\{1, i, j, k\}$ consists of the matrix $\mathbf{1}_2$ together with i times the Paul matrices. Since Paul matrices cannot be all imaginary, the isomorphism $H \otimes H \simeq M_4(\mathbb{R})$ is not trivial (see also answer 2c of Problem I.4 and answer 3 of Problem I.3).

Let $\alpha = a^1 + a^2 i + a^3 j + a^4 k \in H$, *and* $\beta = b^1 + b^2 i + b^3 j + b^4 k$, *where* $i^2 = j^2 = k^2 = -1$ *and* $ij = k$, $jk = i$, $ki = j$. *Let*

$$\alpha^+ = a_1 - a_2 i - a_3 j - a_4 k.$$

The space of quaternions is a real four dimensional vector space. Let $\alpha \otimes \beta$ *act linearly on* H *by*

$$(\alpha \otimes \beta)(\gamma) = \alpha\gamma\beta^+ \quad \textit{for every } \gamma \in H. \tag{1}$$

Show that the map defined by (1), *namely*

$$\alpha \otimes \beta \mapsto \alpha_L \beta_R, \tag{2}$$

is an isomorphism $H \otimes H \to M_4(\mathbb{R})$.

Answer: Let I be the isomorphism between H and V_4

$$I(a^1 + a^2 i + a^3 j + a^4 k) = \text{ column vector } (a^1, a^2, a^3, a^4). \tag{3}$$

The isomorphism $f : H \otimes H \to M_4(\mathbb{R})$ is defined by

$$f(\alpha \otimes \beta) = M(\alpha, \beta) \tag{4}$$

where

$$M(\alpha,\beta)I(\gamma)=I(\alpha\gamma\beta^{+})\quad\text{for all } \gamma\in H. \tag{5}$$

This is an algebra isomorphism: We have (Problem I.3 tensor)

$$(\alpha\otimes\beta)(\alpha'\otimes\beta')=\alpha\alpha'\otimes\beta\beta'. \tag{6}$$

We shall prove that

$$M(\alpha,\beta)M(\alpha',\beta')=M(\alpha\alpha',\beta\beta'). \tag{7}$$

Indeed

$$\begin{aligned} M(\alpha,\beta)M(\alpha',\beta')I(\gamma) &= M(\alpha,\beta)I(\alpha'\gamma\beta'^{+}) \\ &= I(\alpha\alpha'\gamma\beta'^{+}\beta^{+})=I\big((\alpha\alpha')\gamma(\beta\beta')^{+}\big) \\ &= M(\alpha\alpha',\beta\beta')I(\gamma). \quad \blacksquare \end{aligned}$$

To show that $M(\alpha,\beta)\in M_4(\mathbb{R})$ we construct $M(\alpha,\beta)$ for all the elements in a basis of $H\otimes H$. Then we extend the result by linearity. Let the basis of $H\otimes H$ consist of

$$\begin{gathered} \mathbf{1}\otimes\mathbf{1},\ \mathbf{1}\otimes i,\ \mathbf{1}\otimes j,\ \mathbf{1}\otimes k \\ i\otimes\mathbf{1},\ i\otimes i,\ i\otimes j,\ \text{etc.} \end{gathered}$$

Let $\gamma=a+bi+cj+dk$, then

$$M(1,i)I(\gamma)=I(b-ai-dj+ck).$$

Therefore

$$M(1,i)=\begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \end{pmatrix}.$$

A similar calculation gives $M(\alpha,\beta)\in M_4(\mathbb{R})$ for all $\alpha\otimes\beta$ in the basis of $H\otimes H$. The explicit isomorphism depends on the choice of representation of the Pauli matrices, and on the convention made for tensor products of matrices (see answer 3 of Problem I.3). Let a and b be two 2×2 matrices, then $a\otimes b$ is a 4×4 matrix

$$a\otimes b=\begin{pmatrix} a^1{}_1 b & a^1{}_2 b \\ a^2{}_1 b & a^2{}_2 b \end{pmatrix}\ \text{(first choice) or } \begin{pmatrix} ab^1{}_1 & ab^1{}_2 \\ ab^2{}_1 & ab^2{}_2 \end{pmatrix}\ \text{(second choice).}$$

With the second choice and with $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, σ_1 and σ_3 as usual, (see e.g. I.4 §2b) then

$$\begin{array}{lll} M(1,i) = i\sigma_2 \otimes \sigma_3, & M(1,j) = i\mathbf{1} \otimes \sigma_2, & M(1,k) = i\sigma_2 \otimes \sigma_1, \\ M(i,1) = -i\sigma_2 \otimes \mathbf{1}, & M(i,i) = \mathbf{1} \otimes \sigma_3, & M(i,j) = \sigma_2 \otimes \sigma_2, \\ M(i,k) = \mathbf{1} \otimes \sigma_3, & M(j,1) = -i\sigma_3 \otimes \sigma_2, & M(j,i) = \sigma_1 \otimes \sigma_1, \\ M(j,j) = \sigma_3 \otimes \mathbf{1}, & M(j,k) = -\sigma_1 \otimes \sigma_3, & M(k,1) = -i\sigma_2 \otimes \sigma_2, \\ M(k,i) = -\sigma_3 \otimes \sigma_1, & M(k,j) = \sigma_1 \otimes \mathbf{1}, & M(k,k) = \sigma_3 \otimes \sigma_3. \end{array}$$

For detailed calculations leading to the periodicity modulo 8, see "The Pin groups in Physics, C.P. and T" by M. Berg, C. DeWitt-Morette, Shangjr Gwo, and E. Kramer (to be published).

2. LIE DERIVATIVE OF SPINOR FIELDS

The difficulty in defining the Lie derivative of a spinor field on a Riemannian manifold (V, g) comes from the fact that there is no natural and unique definition of the image of such a spinor field by a diffeomorphism. However there is a fairly natural definition of the image of a spinor field by an isometry near the identity. We shall take as definition for the Lie derivative of a spinor field with respect to an arbitrary vector field X the formula that we shall obtain when X is the generator of a one parameter group of isometries.

Lie derivative of spinor fields

1. a) *Let (V, g) be a pseudo Riemannian manifold with a spin structure S (Problem* IV2 *construction p.* 136). *Let f be an isometry of (V, g) near the identity. Give a definition of the reciprocal image by f of a spinor field ψ on V by using representatives in spin frames* (*p.* 135).

b) *Applications. Let ψ be a spinor field on the euclidean space $\mathbb{E}^2$, determine its reciprocal image under rotation around the origin. Same question when ψ is a spinor field on Minkovski space time $\mathbb{M}^4$ and the isometry is a rotation around a given space like axis.*

Answer 1a: Let U_i be an open set in V over which the bundle $\mathcal{O}(n, m)$ of orthonormal frames is trivial. Let $x \mapsto \rho_{x,i}$ be a given section of this bundle over U_i. Let $\psi_i(x) \in \mathbb{C}^{2^p}$, $p = (n+m)/2$, be the representative of $\psi(x)$, $x \in U_i \subset V$, in the fiducial spin frame $(\rho_{x,i}, e)$ over U_i where e is the unit of the spin group. We suppose that $f(x)$ is also in U_i and denote by $\tilde{\rho}_{f,x,i}$

the orthonormal frame at x which is the image by f^{-1} of $\rho_{f(x),i}$. Let $L_{f,x,i}$ be the element of $O(n, m)$ which sends $\rho_{x,i}$ onto $\tilde{\rho}_{f,x,i}$.
Let $\mathcal{H}$ be the two sheeted homomorphism

$$\mathcal{H}: \mathrm{Spin}(n, m) \to \mathrm{SO}(n, m) \text{ by } \Lambda \mapsto L \quad \text{such that } \Lambda \gamma_A \Lambda^{-1} = \gamma_B L^B_A.$$

We denote by $\Lambda_{f,x,i}$ the reciprocal image of $L_{f,x,i}$ by $\mathcal{H}$ which is in a neighbourhood of e. We call image $\tilde{\psi}(x)$ of $\psi(f(x))$ by f^{-1} the spinor whose representative at x in the spin frame $(\tilde{\rho}_{f,x,i}, \Lambda_{f,x,i})$ if $\psi_i(f(x))$; its representative in the fiducial spin frame $(\rho_{x,i}, e)$ is then $\Lambda^{-1}_{f,x,i}\psi_i(f(x))$.

Answer 1b: Take cartesian coordinates (x, y) on $\mathbb{E}^2$ and their natural frames as field of orthonormal frames. Take a rotation of angle θ around the origin. The corresponding isometry $f: (x, y) \mapsto (\xi, \eta)$ is given by

$$\xi = x\cos\theta + y\sin\theta, \quad \eta = -x\sin\theta + y\cos\theta.$$

The frame $\tilde{\rho}$ image of $\rho = (\partial/\partial x, \partial/\partial y)$ under f^{-1} is $\tilde{\rho} = L\rho$ where L is the element of O(2) represented by the matrix

$$L = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

An element Λ of spin(0, 2), or spin(2, 0) (cf. Problem I.4 Clifford) corresponding to L under $\mathcal{H}$ is represented by a 2×2 matrix such that, with Γ_i, $i = 1, 2$, gamma matrices

$$\Lambda \Gamma_i \Lambda^{-1} = \Gamma_j L^j{}_i,$$

equivalently

$$\Lambda\Gamma_1 = \Gamma_1 \Lambda\cos\theta + \Gamma_2\Lambda\sin\theta, \qquad \Lambda\Gamma_2 = \Gamma_1\Lambda\sin\theta - \Gamma_2\Lambda\cos\theta.$$

We look for Λ in the form $\Lambda = a\mathbb{1} + b\Gamma_1\Gamma_2$ with a and b real numbers (cf. Problem I.8, Weyl, p. 27). We find that the general solution is $a = \lambda\cos\frac{\theta}{2}$, $b = \lambda\sin\frac{\theta}{2}$, where λ will be determined by the condition $|\det\Lambda| = 1$. If we consider Spin(0, 2) we take $\Gamma_1 = \sigma_1$, $\Gamma_2 = \sigma_3$, $\Lambda = \pm(\cos\frac{\theta}{2}\mathbb{1} - \mathrm{i}\sin\frac{\theta}{2}\sigma_2)$ where the σ_i are the Pauli matrices (p. 8); choosing the + sign (then $\Lambda = \mathrm{e}$ for $\theta = 0$) we have

$$\Lambda = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{1}$$

If we consider Spin(2, 0) we can take $\Gamma_1 = \mathrm{i}\sigma_1$, $\Gamma_2 = \mathrm{i}\sigma_2$, then

$$\Lambda = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta/2} \end{pmatrix} \equiv \cos\frac{\theta}{2}\mathbb{1} - \mathrm{i}\sigma_3\sin\frac{\theta}{2}. \tag{2}$$

Remark: Equations (1) and (2) give an explicit expression for the isomorphism of Spin(0, 2) with Spin(2, 0).

The reciprocal image of ψ at (x, y) in the frame $(\partial/\partial x, \partial/\partial y)$ is

$$\tilde{\psi}(x, y) = \Lambda \psi(x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta).$$

Here ψ and $\tilde{\psi}$ are each a pair of complex valued functions on $\mathbb{R}^2$. Suppose we express these functions in polar coordinate (r, φ). The above relation reads

$$\tilde{\psi}^1(r, \varphi) = e^{-i\theta/2} \psi^1(r, \varphi - \theta), \quad \tilde{\psi}^2(r, \varphi) = e^{i\theta/2} \psi^2(r, \varphi - \theta).$$

We see that if ψ is such that $\psi(r, \varphi) = (e^{-i\varphi/2} u(r), e^{i\varphi/2} b(r))$ it is invariant by rotation.

Consider now $\mathbb{M}^4$ with cartesian coordinates (x, y, z, t) and a rotation of angle θ around Oz. The element $L \in L(4)$ linking ρ and its image $\tilde{\rho}$ is given by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The element Λ with image L by $\mathcal{H}$ and reducing to e for $\theta = 0$ is again

$$\Lambda = \cos\frac{\theta}{2} \mathbb{1} - \sin\frac{\theta}{2} \Gamma_1 \Gamma_2.$$

If we consider Spin(1, 3) and the Dirac representation (conventions p. xii, Vol. 2) of gamma matrices we find for Λ the diagonal matrix

$$\begin{pmatrix} e^{-i\theta/2} & 0 & 0 & 0 \\ 0 & e^{i\theta/2} & 0 & 0 \\ 0 & 0 & e^{-i\theta/2} & 0 \\ 0 & 0 & 0 & e^{i\theta/2} \end{pmatrix}.$$

Exercise: find other representations.

2) *Define and compute the Lie derivative of a spinor field with respect to a vector field X generator of a one parameter group of isometries of (V, g). Show that the formula defines a spinor field when X is an arbitrary vector field.*

Answer 2: Let f_t be a one parameter group of isometries of (V, g), $f_0 = \mathrm{Id}$ and let $X = \frac{\mathrm{d}f}{\mathrm{d}t}|_{t=0}$ be its generator. The Lie derivative of the spinor field ψ is naturally defined by ($\tilde{\psi}_t$ is the image by f_t)

$$(\mathcal{L}_X \psi)(x) = \lim_{t=0} \frac{1}{t} \{ \tilde{\psi}_t(x) - \psi(x) \}.$$

It is represented in the fiducial spin frame by

$$(\mathcal{L}_X \psi)_i(x) = \lim_{t=0} \frac{1}{t} \{ \Lambda^{-1}_{f_t,x,i} \psi_i(f_t(x)) - \psi_i(x) \}.$$

Since $\Lambda_{f_0,x,i} = e$, it follows that

$$(\mathcal{L}_X \psi)_i(x) = \left\{ \tfrac{\mathrm{d}}{\mathrm{d}t} (\Lambda_{f_t,x,i})^{-1} \psi_i(x) + \tfrac{\mathrm{d}}{\mathrm{d}t}\, \psi_i(f_t(x)) \right\}_{t=0}.$$

By hypothesis we have

$$\mathcal{H}(\Lambda_{f_t,x,i}) = L_{f_t,x,i},$$

therefore (cf. Problem I.11, Lie algebra, p. 38)

$$\left\{ \tfrac{\mathrm{d}}{\mathrm{d}t}\, \Lambda_{f_t,x,i} \right\}_{t=0} = \tfrac{1}{4}\, L_{AB} \Gamma^A \Gamma^B = \tfrac{1}{2} (L_{AB} - L_{BA}) \Gamma^A \Gamma^B$$

where L_{AB} is the (antisymmetric) element tangent to $\mathrm{O}(n, m)$ at its unit element defined by $\{\frac{\mathrm{d}}{\mathrm{d}t}\, L_{f_t,x,i}\}_{t=0}$.

Set $f_t^{-1} \equiv \varphi_t$. The image of a frame ρ by φ_t is the frame $\tilde{\rho}$

$$\tilde{\rho} = \nabla \varphi_t \rho.$$

In local coordinates φ_t is represented by $\varphi_t^B(x^C)$. We represent the frame ρ by a matrix e_A^B acting on the natural frame; the above equation reads then

$$\tilde{e}_A^B(x) = \frac{\partial \varphi_t^B}{\partial x^C} e_A^C(f_t(x))$$

which we write

$$\tilde{\rho}_x = L_{f_t,x} \rho_x, \quad \text{with } L_{f_t,x} = \nabla \varphi_t \rho_{f_t,x} \rho_x^{-1},$$

that is

$$L_{f_t,x} = (L_A^D) \equiv \left(\frac{\partial \varphi_t^B}{\partial x^C} e_A^C(f_t(x)) (e^{-1}(x))_B^D \right).$$

The generator of the one parameter group f_t is given by

$$X^A = \left\{ \frac{\mathrm{d} f_t^A}{\mathrm{d}t} \right\}_{t=0} = - \left\{ \frac{\mathrm{d} \varphi_t^A}{\mathrm{d}t} \right\}_{t=0}.$$

We choose local coordinates such that ρ_x coincides at the considered point x with the natural frame. Then

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}L_{f_t,x,i}\right\}_{t=0} = \left(-\frac{\partial X^D}{\partial x^A} + X^B\frac{\partial e^D_A}{\partial x^B}\right).$$

If we chose local normal (p. 326) coordinates such that at the pointy x we have, in addition to $g_{AB} = \eta_{AB}$, $\frac{\partial}{\partial x^C}g_{AB} = 0$ we can choose ρ such that we have, at the point x, $\frac{\partial e^D_A}{\partial x^C} = 0$. We deduce from these considerations that in this chosen coordinates $\{\frac{\mathrm{d}}{\mathrm{d}t}L_{f_t,x}\}_{t=0}$ is the element of the tangent space at unity of $\mathrm{O}(n, m)$ determined by the matrix $(-\frac{\partial X^B}{\partial x^C})$. We use the fact that, by its definition the Lie derivative of a spinor field is a spinor field and that the matrix $(\nabla_c X_B)$ is antisymmetric when the f_t are isometries to write as follows the formula for the Lie derivative of a spinor field

$$\mathcal{L}_X\psi = X^A\nabla_A\psi - \tfrac{1}{8}(\nabla_A X_B - \nabla_B X_A)\Gamma^A\Gamma^B\psi.$$

b) For an arbitrary vector field X the above formula defines a linear operator from the space of smooth spinor fields into itself. It is taken as definition of the **Lie derivative** with respect to X.

3) *Show that on the Euclidean space $\mathbb{E}^3$ there is no spherically symmetric spinor field, except* 0.

Answer 3: A spinor field is **spherically symmetric** if it is invariant under the rotation group, hence if its Lie derivatives with respect to 3 independent generators of this group vanish. Such generators are, in cartesian coordinates

$$Z = y\partial_x - x\partial_y, \qquad Y = x\partial_z - z\partial_z, \qquad X = z\partial_y - y\partial_z.$$

In polar coordinates on $\mathbb{E}^3$,

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta,$$

these vectors read

$$Z = -\partial_\varphi, \qquad Y = -\cos\varphi\partial_\theta + \frac{\sin\varphi\cos\theta}{\sin\theta}\partial_\varphi,$$

$$X = \sin\varphi\partial_\theta + \frac{\cos\varphi\cos\theta}{\sin\theta}.$$

We consider $\mathrm{Spin}(0, 3)$, with gamma matrices σ_i, $i = 1, 2, 3$. We must have

$$\mathcal{L}_Z\psi \equiv (x\partial_y - y\partial_x)\psi - \tfrac{1}{2}\sigma_1\sigma_2\psi \equiv \partial_\varphi\psi + \tfrac{i}{2}\sigma_3\psi = 0$$

i.e.

$$\partial_\varphi \psi^1 + \tfrac{\mathrm{i}}{2}\psi^1 = 0, \qquad \partial_\varphi \psi^2 - \tfrac{\mathrm{i}}{2}\psi^2 = 0.$$

The general solution of these equations is (result which could have been deduced from answer 1)

$$\psi^1 = \mathrm{e}^{-\mathrm{i}\varphi/2} u(r,\theta), \qquad \psi^2 = \mathrm{e}^{\mathrm{i}\varphi/2} v(r\theta).$$

We now compute

$$\mathcal{L}_Y \psi \equiv (-\cos\varphi\,\partial_\theta + \sin\varphi \cot\theta\,\partial_\varphi)\psi - \tfrac{1}{2}\mathrm{i}\sigma_2\psi = 0.$$

Using the expression found for ψ we obtain two linear equations in u and v which are also linear in $\cos\varphi$ and $\sin\varphi$. Equating to zero the terms in $\cos\varphi$ and $\sin\varphi$ gives four equations for u and v which read

$$\partial_\theta u = \tfrac{1}{2} v, \quad \cot\theta u = v, \quad \partial_\theta v = -\tfrac{1}{2} u, \quad \cot\theta v = -u.$$

The only solution of these equations is $u \equiv 0$, $v \equiv 0$.

Remark: A vector field V on $\mathbb{E}^3$ is spherically symmetric if its components in cartesian coordinates are of the form $(x\,u(r))$, $(y\,u(r))$, $(z\,u(r))$.

Remark: The quantity $[\mathcal{L}_{X_1}, \mathcal{L}_{X_2}] - \mathcal{L}_{[X_1, X_2]}$ vanishes if and only if either X_1 or X_2 is the generator of a group of conformal isometries. For the proof see [Bourguignon].

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See also Bryce DeWitt, "The spacetime approach to Quantum Field Theory", in: *Relativity, Groups, and Topology II*, eds. B.S. DeWitt and R. Stora (North-Holland, 1984) p. 552.

3. POISSON–LIE GROUPS, LIE BIALGEBRAS AND THE GENERALIZED CLASSICAL YANG–BAXTER EQUATION*

1. POISSON–LIE GROUPS

This section concerns the Drinfeld definitions of the Poisson–Lie group structure, $(\boldsymbol{G}; \Lambda)$. (Ref. [1], [2], [4], [3], [9], [7], [5], [8], [10], [11].)

1.1. Definition
A Poisson–Lie group (PLG) is a Lie group $\boldsymbol{G}$ endowed with a Poisson structure such that the mapping

$$\pi : \boldsymbol{G} \times \boldsymbol{G} \to \boldsymbol{G}$$
$$(g_1; g_2) \to g_1 \cdot g_2$$

is a Poisson map from the Poisson manifold $(\boldsymbol{G} \times \boldsymbol{G}; \{;\}_{G\times G})$ to the Poisson manifold $(\boldsymbol{G}; \{;\}_G)$, i.e., for all $\varphi_1, \varphi_2 \in C^\infty(\boldsymbol{G})$,

$$\{\varphi_1; \varphi_2\}_G \circ \pi = \{\varphi_1 \circ \pi; \varphi_2 \circ \pi\}_{G\times G}. \tag{1}$$

By the definition of the Poisson structure on a product of two Poisson manifolds expression (1) can be written:

$$\begin{aligned} &\{\varphi_1; \varphi_2\}_G(g \cdot h) \\ &= \left\{(\varphi_1 \circ \pi)_1^g; (\varphi_2 \circ \pi)_1^g\right\}_G(h) + \left\{(\varphi_1 \circ \pi)_2^h; (\varphi_2 \circ \pi)_2^h\right\}_G(g) \\ &\equiv \{\varphi_1 \circ \lambda_g; \varphi_2 \circ \lambda_g\}_G(h) + \{\varphi_1 \circ \rho_h; \varphi_2 \circ \rho_h\}_G(g), \end{aligned} \tag{2}$$

where for any given $g \in \boldsymbol{G}$ we define the mapping on $\boldsymbol{G}$: $(\varphi_i \circ \pi)_1^g(h) = (\varphi_i \circ \pi)(g; h) = \varphi_i(g \cdot h) = (\varphi_i \circ \lambda_g)(h)$ and for any given $h \in \boldsymbol{G}$, the mapping: $(\varphi_i \circ \pi)_2^h(g) = \varphi_i(g \cdot h) = (\varphi_i \circ \rho_h)(g)$.

1.2. Definition
Let, $(\boldsymbol{G}_1; \{;\}_{G_1})$, $(\boldsymbol{G}_2; \{;\}_{G_2})$ be two PLGs. Let $\Phi : \boldsymbol{G}_1 \to \boldsymbol{G}_2$ be a C^∞ mapping. We say that Φ is a Poisson–Lie morphism if it is a Lie group morphism, i.e., a Lie group homomorphism, and a Poisson morphism: for all $\varphi, \psi \in C^\infty(\boldsymbol{G}_2)$,

$$\{\varphi; \psi\}_{G_2} \circ \Phi = \{\varphi \circ \Phi; \psi \circ \Phi\}_{G_1}.$$

*Contributed by Carlos Moreno, Laboratoire Gevrey de Mathématique physique, CNRS UMR 5029, UFR Sciences et Techniques, Université de Bourgogne BP 47870, F-21078 Dijon Cedex France, Departamento de Física Teórica, Universidad Complutense, E-28040 Madrid, Spain, and Luis Valero, Instituto "Giner de los Ríos", E-28100 Madrid, Departamento de Física Teórica, Universidad Complutense, E-28040 Madrid, Spain.

I.3.

Let $(G; \{;\}_G)$, $(H; \{;\}_H)$ *be two* PLGs.

Prove that the Poisson manifold $(G \times H; \{;\}_{G\times H})$ *is itself a* PLG.

Answer. The direct product Lie group $G \times H$ is defined by

$$\Pi\big((g_1; g_2); (h_1; h_2)\big) = (g_1 \cdot g_2; h_1 \cdot h_2)$$

for all $g_1, g_2 \in G$, for all $h_1, h_2 \in H$. We need to prove that the mapping $\Pi : (G \times H) \times (G \times H) \to G \times H$ is a Poisson mapping, i.e., for all $F_1, F_2 \in C^\infty(G \times H)$

$$\begin{aligned}\{F_1; F_2\}_{G\times H}&(g_1 \cdot g_2; h_1 \cdot h_2)\\ &= \{F_1 \circ \Pi; F_2 \circ \Pi\}_{(G\times H)\times(G\times H)}\big((g_1; h_1); (g_2; h_2)\big).\end{aligned} \tag{3}$$

To do this we develop both sides of the equality (3) using the facts that $G \times H$ and $(G \times H) \times (G \times H)$ are Poisson manifolds and that G and H are PLGs. We then see that both sides are identical.

For a better understanding of the notion of a PLG we need to look more closely at its Poisson tensor Λ.

I.4.

Let $(G; \Lambda)$ *be a Poisson manifold, and* $(G \times G; \Lambda \oplus \Lambda)$ *the corresponding product of Poisson manifolds. Let* $\pi : (g; h) \in G \times G \to g \cdot h \in G$ *be the product in group* G.

Prove that π *is a Poisson mapping if and only if:*

$$\Lambda(g \cdot h) = T_h\lambda_g \cdot \Lambda(h) + T_g\rho_h \cdot \Lambda(g).$$

Answer. (a) Let us suppose that the mapping π is a Poisson morphism. The equality (1) is then verified, and in terms of the tensor Λ can be written:

$$\begin{aligned}\Lambda_{g\cdot h}&\big(d\varphi_1(g \cdot h); d\varphi_2(g \cdot h)\big)\\ &= \Lambda_h\big(d(\varphi_1 \circ \lambda_g)(h); d(\varphi_2 \circ \lambda_g)(h)\big)\\ &\quad + \Lambda_g\big(d(\varphi_1 \circ \rho_h)(g); d(\varphi_2 \circ \rho_h)(g)\big).\end{aligned} \tag{4}$$

Developing each term in the second member of this equality, we get:

$$\begin{aligned}&= \Lambda_h\big(d\varphi_1(g \cdot h) \circ T_h\lambda_g; d\varphi_2(g \cdot h) \circ T_h\lambda_g\big)\\ &\quad + \Lambda_g\big(d\varphi_1(g \cdot h) \circ T_g\rho_h; d\varphi_2(g \cdot h) \circ T_g\rho_h\big) = \cdots =\\ &= \big(T_h\lambda_g^{\otimes 2} \cdot \Lambda_h\big)\big(d\varphi_1(g \cdot h); d\varphi_2(g \cdot h)\big)\\ &\quad + \big(T_g\rho_h^{\otimes 2} \cdot \Lambda_g\big)\big(d\varphi_1(g \cdot h); d\varphi_2(g \cdot h)\big)\end{aligned}$$

for all $\varphi_1, \varphi_2 \in C^\infty(\boldsymbol{G})$.
We thus obtain the equality:

$$\Lambda(g \cdot h) = (T_h \lambda_g)^{\otimes 2} \cdot \Lambda_h + (T_g \rho_h)^{\otimes 2} \cdot \Lambda_g, \quad \text{for all } g, h \in \boldsymbol{G}. \qquad (5)$$

(b) *Conversely*. It is clear that the above reasoning can be reversed *mutatis mutandi*, i.e., if Λ verifies equality (5) it verifies also equality (4) and then the mapping π is also a Poisson mapping. ■

Remark. Note that we have not use the condition $[\Lambda; \Lambda] = 0$ in the above reasoning, i.e., we have proved that if Λ is *any* skewsymmetric contravariant tensor of degree 2 on $\boldsymbol{G}$ and if Λ and $\Lambda \oplus \Lambda$ are π-related, i.e., if they verify the equality (4), then Λ verifies equality (5), and conversely. This leads us to the following problem.

I.5.
Let Λ be a skewsymmetric contravariant 2-tensor on $\boldsymbol{G}$, i.e. $\Lambda \in \wedge^2(\boldsymbol{G})$. Let $l : \boldsymbol{G} \to \mathfrak{g} \otimes \mathfrak{g}$ and $m : \boldsymbol{G} \to \mathfrak{g} \otimes \mathfrak{g}$ be respectively the mappings:

$$l(g) = (T_g \rho_{g^{-1}} \otimes T_g \rho_{g^{-1}}) \cdot \Lambda(g); \qquad m(g) = (T_g \lambda_{g^{-1}} \otimes T_g \lambda_{g^{-1}}) \cdot \Lambda(g).$$

Prove that the following three properties of Λ are then equivalent:

(a) *For all $g, h \in \boldsymbol{G}$,*

$$\Lambda(g \cdot h) = (T_h \lambda_g)^{\otimes 2}\big(\Lambda(h)\big) + (T_g \rho_h)^{\otimes 2}\big(\Lambda(g)\big).$$

*(If Λ verifies this equality we say that Λ has the **Drinfeld property**.)*

(b) *The mapping l is a 1-cocycle on $\boldsymbol{G}$, with values in $\mathfrak{g} \otimes \mathfrak{g}$, relative to the adjoint action of $\boldsymbol{G}$ on $\mathfrak{g} \otimes \mathfrak{g}$, i.e.,*

$$l(g \cdot h) = l(g) + \mathrm{Ad}(g)^{\otimes 2} l(h)$$

for all $g, h \in \boldsymbol{G}$.

(c) *The mapping m verifies the equality*:

$$m(g \cdot h) = m(h) + \mathrm{Ad}\left(h^{-1}\right)^{\otimes 2} m(g)$$

for all $g, h \in \boldsymbol{G}$.

Answer. (a) $\Leftrightarrow$ (b) From (a) and the definition of l, we get:

$$\begin{aligned}
l(g\cdot h) &= (T_{gh}\rho_{(gh)^{-1}})^{\otimes 2}\Lambda(g\cdot h)\\
&= (T_{gh}\rho_{(gh)^{-1}})^{\otimes 2}\big((T_h\lambda_g)^{\otimes 2}\Lambda(h)+(T_g\rho_h)^{\otimes 2}\Lambda(g)\big)\\
&= \big(T_h(\lambda_g\circ\rho_{(gh)^{-1}})\big)^{\otimes 2}\Lambda(h)+\big(T_g(\rho_{(gh)^{-1}}\circ\rho_h)\big)^{\otimes 2}\Lambda(g)\\
&= \big(T_e(\lambda_g\circ\rho_{g^{-1}})\circ T_h\rho_{h^{-1}}\big)^{\otimes 2}\Lambda(h)+(T_g\rho_{g^{-1}})^{\otimes 2}\Lambda(g)\\
&= (\mathrm{Ad}_g)^{\otimes 2}\cdot l(h)+l(g).
\end{aligned}$$

We thus obtain (b) from (a). Obviously, if l verifies (b) the second equality in the above reasoning must also be verified, and (a) is therefore verified, since the mapping:

$$T_{gh}\rho_{(gh)^{-1}} : T_{gh}\boldsymbol{G} \longrightarrow T_e\boldsymbol{G} \equiv \mathfrak{g}$$

is a vector spaces isomorphism.

The equivalence (a) $\Leftrightarrow$ (c) can be proved in a similar way. ■

Remark. If $\Lambda \in \wedge^2(\boldsymbol{G})$ has one of these properties, then $\Lambda(e)=0$, $l(e)=0$, $m(e)=0$, where e is the neutral element of $\boldsymbol{G}$. In particular, this is the case when $(\boldsymbol{G};\Lambda)$ is a PLG.

From 1.4 and 1.5, we can say that $(\boldsymbol{G};\Lambda)$ is a PLG if and only if:

(a′) $[\Lambda;\Lambda]=0$, i.e., Λ is a *cocycle* in the Poisson cohomology on $\boldsymbol{G}$ defined by the Schouten bracket.

(b′) $l(g\cdot h)=l(g)+(\mathrm{Ad}_g)^{\otimes 2}l(h)$, where $l(g)=(T_g\rho_{g^{-1}})^{\otimes 2}\Lambda(g)\in\mathfrak{g}\otimes\mathfrak{g}$, for all $g,h\in\boldsymbol{G}$, i.e., the mapping $g\in\boldsymbol{G}\to(T_g\rho_{g^{-1}})^{\otimes 2}\Lambda(g)\in\mathfrak{g}\otimes\mathfrak{g}$ is a *1-cocycle* on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}\otimes\mathfrak{g}$.

The notion of a Lie algebra can be seen *as the infinitesimal* definition of the notion of a Lie group. What then is the *infinitesimal* definition of a PLG? This will be the notion of a *Lie bialgebra*, which can be reached by looking more closely at the definition (a′), (b′) of a PLG.

1.6.
Let $\Lambda\in\wedge^2(\boldsymbol{G})$ and suppose the mapping

$$l : g\in\boldsymbol{G}\longrightarrow(T_g\rho_{g^{-1}})^{\otimes 2}\Lambda(g)\in\mathfrak{g}\otimes\mathfrak{g}.$$

Now let $\epsilon\equiv T_e l:\mathfrak{g}\longrightarrow\mathfrak{g}\otimes\mathfrak{g}$ be the tangent mapping to l at the point $e\in\boldsymbol{G}$.

Prove that if Λ has the Drinfeld property, or equivalently, if l is a 1-cocycle relative to the adjoint representation of G on $\mathfrak{g}\otimes\mathfrak{g}$, then ϵ is a 1-cocycle on $\mathfrak{g}$ with values on $\mathfrak{g}\otimes\mathfrak{g}$ relative to the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}\otimes\mathfrak{g}$, i.e.,

$$\epsilon([x;y]) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x)\epsilon(y) - (\mathrm{ad}_y \otimes 1 + 1 \otimes \mathrm{ad}_y)\epsilon(x),$$

for all $x, y \in \mathfrak{g}$.

Answer. From the Definition (b) in 1.5, we easily obtain the following properties of l:
(i) $l(e) = 0$
(ii) $l(g^{-1}) = -\mathrm{Ad}_{g^{-1}} \cdot l(g)$; for all $g \in G$.
From the same definition and these two properties, we easily obtain, for all $g_1, g_2 \in G$, the equality:

$$l(g_1 \cdot g_2 \cdot g_1^{-1}) = l(g_1) + (\mathrm{Ad}_{g_1})^{\otimes 2} l(g_2) - \left(\mathrm{Ad}_{g_1 \cdot g_2 \cdot g_1^{-1}}\right)^{\otimes 2} \cdot l(g_1).$$

In this equality, we now substitute $g_2 = \exp ty$, $y \in \mathfrak{g}$, $t \in \mathbb{R}$:

$$l(g_1 \cdot \exp ty \cdot g_1^{-1}) = l(g_1) + (\mathrm{Ad}_{g_1})^{\otimes 2} l(\exp ty) - \left(\mathrm{Ad}_{g_1 \cdot \exp ty \cdot g_1^{-1}}\right)^{\otimes 2} l(g_1).$$

We then calculate the derivative of this mapping at the point $t = 0$, i.e.,

$$\begin{aligned} &T_e l \cdot \frac{\mathrm{d}}{\mathrm{d}t}(g_1 \cdot \exp ty \cdot g_1^{-1})\big|_{t=0} \\ &= (\mathrm{Ad}_{g_1})^{\otimes 2} \cdot \frac{\mathrm{d}}{\mathrm{d}t} l(\exp ty)|_{t=0} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ad}_{g_1} \cdot \mathrm{Ad}_{\exp ty} \cdot \mathrm{Ad}_{g_1^{-1}}\right)^{\otimes 2} \cdot l(g_1). \end{aligned}$$

Then:

$$\begin{aligned} T_e l \cdot (\mathrm{Ad}_{g_1}(y)) &= \mathrm{Ad}_{g_1}^{\otimes 2} \cdot T_e l(y) \\ &- \left(\mathrm{Ad}_{g_1} \cdot \mathrm{ad}_y \cdot \mathrm{Ad}_{g_1^{-1}} \otimes I + I \otimes \mathrm{Ad}_{g_1} \cdot \mathrm{ad}_y \cdot \mathrm{Ad}_{g_1^{-1}}\right) \cdot l(g_1). \end{aligned}$$

In this equality, we then substitute $g_1 = \exp sx$, $x \in \mathfrak{g}$, $s \in \mathbb{R}$, and compute the derivative of this mapping at $s = 0$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left(T_e l \cdot \mathrm{Ad}_{\exp sx}(y)\right)\big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{Ad}_{\exp sx}^{\otimes 2} \cdot T_e l(y)\right)\big|_{s=0} \\ &- \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{Ad}_{\exp sx} \cdot \mathrm{ad}_y \cdot \mathrm{Ad}_{\exp(-sx)} \otimes I \\ &\quad + I \otimes \mathrm{Ad}_{\exp sx} \cdot \mathrm{ad}_y \cdot \mathrm{Ad}_{\exp(-sx)}\big) \cdot l(\exp sx)\big|_{s=0}. \end{aligned}$$

Likewise:

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(T_e l \cdot \mathrm{Ad}_{\exp sx}(y)\big)\big|_{s=0} = T_e l \cdot \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{Ad}_{\exp sx}(y)\big)\big|_{s=0} = T_l \cdot [x; y],$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{Ad}^{\otimes 2}_{\exp sx} \cdot T_e l(y)\big)\big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{Ad}^{\otimes 2}_{\exp sx}\big)\big|_{s=0} \cdot T_e l(y)$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\Big(\big(\exp(\mathrm{ad}\, sx)\big)^{\otimes 2}\Big)\Big|_{s=0} \cdot T_e l(y) = (\mathrm{ad}\, x \otimes I + I \otimes \mathrm{ad}\, x) \cdot T_e l(y),$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{Ad}_{\exp sx} \cdot \mathrm{ad}_y \cdot \mathrm{Ad}_{\exp(-sx)} \otimes I$$
$$+ I \otimes \mathrm{Ad}_{\exp sx} \cdot \mathrm{ad}_y \cdot \mathrm{Ad}_{\exp(-sx)}\big) \cdot l(\exp sx)\big|_{s=0}$$
$$= (\mathrm{ad}_y \otimes I + I \otimes \mathrm{ad}_y) \cdot T_e l(x).$$

The required derivative at $s = 0$ is then:

$$T_e l\big([x; y]\big) = (\mathrm{ad}\, x \otimes I + I \otimes \mathrm{ad}\, x) \cdot T_e l(y) - (\mathrm{ad}\, y \otimes I + I \otimes \mathrm{ad}\, y) \cdot T_e l(x).$$

■

Given a Poisson–Lie group $(\boldsymbol{G}; \Lambda)$ the essential property of cocycle $\epsilon : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is that its transposed mapping $\epsilon^{\mathrm{t}} : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ will define a Lie algebra structure on $\mathfrak{g}^*$.

This property can be proved by solving the following problem:

I.7.
Let $\Lambda \in \wedge^2(\boldsymbol{G})$ (note that we need not assume that Λ verifies the Drinfeld property). Let us consider the mapping $l : \boldsymbol{G} \to \mathfrak{g} \otimes \mathfrak{g}$ as in 1.6 *(which need not be a* 1*-cocycle). Let us also consider the following skewsymmetric mapping on $C^\infty(\boldsymbol{G})$:*

$$\{\varphi; \psi\}_G = \Lambda(d\varphi; d\psi)$$

for all $\varphi, \psi \in C^\infty(\boldsymbol{G})$.

Prove the following relation:

$$\big\langle \mathrm{d}\{\varphi; \psi\}_G(e); x\big\rangle = \big\langle \mathrm{d}\varphi(e) \wedge \mathrm{d}\psi(e); \epsilon(x)\big\rangle \quad \Big(\equiv \big\langle \epsilon^{\mathrm{t}}\big(\mathrm{d}\varphi(e) \wedge \mathrm{d}\psi(e)\big); x\big\rangle\Big)$$

for all $x \in \mathfrak{g}$. Equivalently, prove the relation:

$$\mathrm{d}\big(\{\varphi; \psi\}_G\big)(e) = \epsilon^{\mathrm{t}}\big(\mathrm{d}\varphi(e) \wedge \mathrm{d}\psi(e)\big).$$

Answer.

$$
\begin{aligned}
&\big\langle \mathrm{d}(\{\varphi;\psi\})(e);x\big\rangle\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\big(\{\varphi\ \psi\}(\exp tx)\big)\big|_{t=0}\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\big((T_e\rho_{\exp tx})^{\otimes 2}l(\exp tx)\big(\mathrm{d}\varphi(\exp tx)\wedge \mathrm{d}\psi(\exp tx)\big)\big)\big|_{t=0}\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\big(l(\exp tx)\big(\mathrm{d}\varphi(\exp tx)\circ T_e\rho_{\exp tx}\wedge \mathrm{d}\psi(\exp tx)\circ T_e\rho_{\exp tx}\big)\big)\big|_{t=0}\\
&=\frac{\mathrm{d}}{\mathrm{d}t}l(\exp tx)|_{t=0}\big(\mathrm{d}\varphi(e)\wedge \mathrm{d}\psi(e)\big)=\epsilon(x)\big(\mathrm{d}\varphi(e)\wedge \mathrm{d}\psi(e)\big)\\
&=\big\langle \mathrm{d}\varphi(e)\wedge \mathrm{d}\psi(e);\epsilon(x)\big\rangle=\big\langle \epsilon^{\mathrm{t}}\big(\mathrm{d}\varphi(e)\wedge \mathrm{d}\psi(e)\big);x\big\rangle.
\end{aligned}
$$

The above equality is obviously verified for any PLG $(\boldsymbol{G};\Lambda)$; furthermore $\{;\}_G$ is the Lie bracket of the Poisson structure defined by Λ on the manifold $\boldsymbol{G}$.

For PLG, we can also prove:

I.8.

Let $(\boldsymbol{G};\Lambda)$ be a PLG. Let $l(g)=(T_g\rho_{g^{-1}})\Lambda(g)$ be the corresponding 1-cocycle on $\boldsymbol{G}$, with values on $\mathfrak{g}\wedge\mathfrak{g}$, relative to the adjoint representation of $\boldsymbol{G}$ on $\mathfrak{g}\wedge\mathfrak{g}$. Let $\epsilon=T_el:\mathfrak{g}\to\mathfrak{g}\wedge\mathfrak{g}$ be the corresponding 1-cocycle on $\mathfrak{g}$ with values on $\mathfrak{g}\wedge\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}\wedge\mathfrak{g}$. Let us define the following bilinear mapping on $\mathfrak{g}^*$:

$$[\xi_1;\xi_2]_{\mathfrak{g}^*}=\epsilon^t(\xi_1\wedge\xi_2)$$

for all $\xi_1,\xi_2\in\mathfrak{g}^*$.
Prove that the pair $(\mathfrak{g}^*;[;]_{\mathfrak{g}^*})$ is a Lie algebra.

Answer. It is clear that $[;]_{\mathfrak{g}^*}$ is bilinear and skewsymmetric. It remains to be proved that it also verifies the Jacobi identity. Let $\varphi_1,\varphi_2\in C^\infty(\boldsymbol{G})$ and be such that $\xi_1=\mathrm{d}\varphi_1(e)$, $\xi_2=\mathrm{d}\varphi_2(e)$, $\xi_3=\mathrm{d}\varphi_3(e)$. Then:

$$[\xi_1;\xi_2]_{\mathfrak{g}^*}=\epsilon^{\mathrm{t}}\big(\mathrm{d}\varphi_1(e)\wedge \mathrm{d}\varphi_2(e)\big)=\mathrm{d}\big(\{\varphi_1;\varphi_2\}_G\big)(e).$$

Also:

$$\big[\xi_1;[\xi_2;\xi_3]_{\mathfrak{g}^*}\big]_{\mathfrak{g}^*}=\big[\mathrm{d}\varphi_1(e);\mathrm{d}\big(\{\varphi_2;\varphi_3\}_G(e)\big)\big]_{\mathfrak{g}^*}=\mathrm{d}\big(\big\{\varphi_1;\{\varphi_2;\varphi_3\}_G\big\}_G\big)(e).$$

The result follows from the fact that $[\Lambda;\Lambda]=0$. ■

1.9.
Let $(\boldsymbol{G}; \Lambda)$ *be a* PLG *and* $\{e_i;\ i = 1, \dots, n\}$ *a basis of* $\mathfrak{g}$. *Let us define the following right invariant vectorfields on* $\boldsymbol{G}$: $x_i^\rho(g) = T_e\rho_g \cdot e_i, i = 1, \dots, n$. *Prove the following equality*:
(a) $\{\varphi; \psi\}_G(g) = \Lambda^{ij}(g)(\mathrm{L}_{x_i^\rho}\varphi)(g) \cdot (\mathrm{L}_{x_j^\rho}\psi)(g)$ *where* $\Lambda(g) = \Lambda^{ij}(g)x_i^\rho(g) \otimes x_j^\rho(g)$ *and* $\mathrm{L}_{(\cdot)}$ *is the Lie derivative.*
(b) *For the* 1*-cocycle on* $\boldsymbol{G}$, $l(g) = (T_g\rho_{g^{-1}})^{\otimes 2}\Lambda(g) \in \mathfrak{g} \otimes \mathfrak{g}$ *we have*:

$$l(g) = \Lambda^{ij}(g)e_i \otimes e_j.$$

(c) *Structure constants*: *Let* $\{e^k;\ k = 1, \dots, n\}$ *be the dual basis on* $\mathfrak{g}^*$ *of the basis* $\{e_i;\ i = 1, \dots, n\}$ *of* $\mathfrak{g}$. *Then*:

$$\left[e^j; e^k\right]_{\mathfrak{g}^*} = f_i^{jk} e^i$$

where

$$f_i^{jk} = d\Lambda^{jk}(e) \cdot e_i; \qquad \epsilon^{\mathrm{t}}(e^j \wedge e^k) = d\Lambda^{jk}(e)$$

or

$$\epsilon(x) = \left(\boldsymbol{i}_x d\Lambda^{jk}(e)\right)e_j \otimes e_k = \left\langle d\Lambda^{jk}(e); x\right\rangle e_j \otimes e_k.$$

($\boldsymbol{i}_{(\cdot)}$ *is the interior product*, $x \in \mathfrak{g}$.)

2. LIE BIALGEBRAS

We saw in the last section that any Poisson–Lie group $(\boldsymbol{G}; \Lambda)$ determines a Lie algebra structure, $[;]_{\mathfrak{g}^*}$, on the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of $\boldsymbol{G}$. This structure is defined by means of the 1-cocycle, $\epsilon : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, which is itself equivalent to the Drinfeld property of the tensor Λ, and to the cocycle property of Λ in the Poisson cohomology. Our consideration of the notion of a PLG thus leads to the following algebraic structure which can be now be discussed without referring back to the PLG structure. (Ref. [1], [2], [3].)

2.1. Definition
A Lie bialgebra structure is a triple $(\mathfrak{g}; [;]; \epsilon)$ *where* $(\mathfrak{g}; [;])$ *is a Lie algebra and* $\epsilon : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *is a* 1*-cocycle on* $\mathfrak{g}$, *with values on* $\mathfrak{g} \otimes \mathfrak{g}$, *relative to the adjoint action of* $\mathfrak{g}$ *on* $\mathfrak{g} \otimes \mathfrak{g}$, *and such that the mapping*:

$$\epsilon^{\mathrm{t}} : \mathfrak{g}^* \otimes \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$$

defines a Lie algebra structure on $\mathfrak{g}^*$:

$$[\xi; \eta]_{\mathfrak{g}^*} = \epsilon^{\mathrm{t}}(\xi \otimes \eta).$$

Exercise. Prove that $\mathrm{Im}(\epsilon) \subset \mathfrak{g} \wedge \mathfrak{g}$.

Now we want to prove that the notion of Lie bilagebra is the *infinitesimal* definition of a Poisson–Lie group. That is to say it ramains to prove that any Lie bialgebra $(\mathfrak{g}; [;]_{\mathfrak{g}}; \epsilon)$ determines a Poisson–Lie group structure on the simply-connected Lie group G with Lie algebra $\mathfrak{g}$ such that the Lie bialgebra determined by this Poisson–Lie group, 1.8, is the former one.

The next problem is to discover the necessary and sufficient condition for the respective Lie algebra structures on vector spaces $\mathfrak{g}$ and $\mathfrak{g}^*$ to determine a Lie bialgebra $(\mathfrak{g}; [;]; \epsilon)$.

2.2.
Let $(\mathfrak{g}; [;])$, $(\mathfrak{g}^*; [;]_{\mathfrak{g}^*})$ *be Lie algebras. Let us consider the following linear mappings*:

$$\beta : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}; \qquad \phi : \mathfrak{g}^* \otimes \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$$

defined by the expressions:

$$[x_1; x_2] = \beta(x_1 \otimes x_2); \qquad [\alpha_1; \alpha_2]_{\mathfrak{g}^*} = \phi(\alpha_1 \otimes \alpha_2),$$

$x_1, x_2 \in \mathfrak{g}$; $\alpha_1, \alpha_2 \in \mathfrak{g}^*$.

Prove that the following three properties are then equivalent.

(i) *The mapping* $\epsilon \equiv \phi^{\mathrm{t}} : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *is a* 1*-cocycle on* $\mathfrak{g}$, *with values on* $\mathfrak{g} \otimes \mathfrak{g}$, *relatively to the adjoint representation of* $\mathfrak{g}$ *on* $\mathfrak{g} \otimes \mathfrak{g}$, *i.e.*,

$$\epsilon([x; y]) \equiv \phi^{\mathrm{t}}([x; y]) = \mathrm{ad}_x \cdot \phi^{\mathrm{t}}(y) - \mathrm{ad}_y \cdot \phi^{\mathrm{t}}(x).$$

(ii) *The action of* $[\xi; \eta]_{\mathfrak{g}^*}$ *on* $[x; y]$ *can be written in the following three equivalent forms*:

$$\begin{aligned}
&\big\langle [\xi; \eta]_{\mathfrak{g}^*}; [x; y] \big\rangle \\
&\quad = -\big\langle [\mathrm{ad}^*{}_x\, \xi; \eta]_{\mathfrak{g}^*}; y \big\rangle - \big\langle [\xi; \mathrm{ad}^*{}_x\, \eta]_{\mathfrak{g}^*}; y \big\rangle \\
&\qquad + \big\langle [\mathrm{ad}^*{}_y\, \xi; \eta]_{\mathfrak{g}^*}; x \big\rangle + \big\langle [\xi; \mathrm{ad}^*{}_y\, \eta]_{\mathfrak{g}^*}; x \big\rangle, \\
&\big\langle [\xi; \eta]_{\mathfrak{g}^*}; [x; y] \big\rangle \\
&\quad = \big\langle \xi; [\mathrm{ad}^*{}_\eta\, x; y] \big\rangle + \big\langle \xi; [x; \mathrm{ad}^*{}_\eta\, y] \big\rangle - \big\langle \eta; [\mathrm{ad}^*{}_\xi\, x; y] \big\rangle - \big\langle \eta; [x; \mathrm{ad}^*{}_\xi\, y] \big\rangle, \\
&\big\langle [\xi; \eta]_{\mathfrak{g}^*}; [x; y] \big\rangle \\
&\quad = \big\langle \mathrm{ad}^*{}_x\, \eta; \mathrm{ad}^*{}_\xi\, y \big\rangle + \big\langle \mathrm{ad}^*{}_y\, \xi; \mathrm{ad}^*{}_\eta\, x \big\rangle \\
&\qquad - \big\langle \mathrm{ad}^*{}_x\, \xi; \mathrm{ad}^*{}_\eta\, y \big\rangle - \big\langle \mathrm{ad}^*{}_y\, \eta; \mathrm{ad}^*{}_\xi\, x \big\rangle,
\end{aligned}$$

for all $x, y \in \mathfrak{g}$; $\xi, \eta \in \mathfrak{g}^*$. *This property is known as the Drinfeld compatibility condition.*

(iii) *The mapping* $\beta^{\mathrm{t}} : \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ *is a* 1*-cocycle on* $\mathfrak{g}^*$ *with values on* $\mathfrak{g}^* \otimes \mathfrak{g}^*$ *relative to the adjoint action of* $\mathfrak{g}^*$ *on* $\mathfrak{g}^* \otimes \mathfrak{g}^*$, *i.e.*,

$$\beta^{\mathrm{t}}\big([\xi;\eta]_{\mathfrak{g}^*}\big) = \mathrm{ad}_\xi \cdot \beta^{\mathrm{t}}(\eta) - \mathrm{ad}_\eta \cdot \beta^{\mathrm{t}}(\xi).$$

Answer. (i)$\Rightarrow$(ii) The following expansion is straightforward,

$$\begin{aligned}
\big\langle [\xi;\eta]_{\mathfrak{g}^*}; [x;y]\big\rangle &= \big\langle \xi\otimes\eta; \phi^{\mathrm{t}}([x;y])\big\rangle = \big\langle \xi\otimes\eta; \mathrm{ad}_x \cdot\phi^{\mathrm{t}}(y) - \mathrm{ad}_y \cdot\phi^{\mathrm{t}}(x)\big\rangle \\
&= \big\langle (\mathrm{ad}_x)^{\mathrm{t}}\cdot(\xi)\otimes\eta + \xi\otimes(\mathrm{ad}_x)^{\mathrm{t}}\cdot\eta; \phi^{\mathrm{t}}(y)\big\rangle \\
&\quad - \big\langle (\mathrm{ad}_y)^{\mathrm{t}}\cdot(\xi)\otimes\eta + \xi\otimes(\mathrm{ad}_y)^{\mathrm{t}}\cdot\eta; \phi^{\mathrm{t}}(x)\big\rangle \\
&= \big\langle [(\mathrm{ad}_x)^{\mathrm{t}}\xi;\eta]_{\mathfrak{g}^*}; y\big\rangle + \big\langle [\xi; (\mathrm{ad}_x)^{\mathrm{t}}\eta]_{\mathfrak{g}^*}; y\big\rangle \\
&\quad - \big\langle [(\mathrm{ad}_y)^{\mathrm{t}}\xi;\eta]_{\mathfrak{g}^*}; x\big\rangle - \big\langle [\xi; (\mathrm{ad}_y)^{\mathrm{t}}\eta]_{\mathfrak{g}^*}; x\big\rangle.
\end{aligned}$$

The first relation in (ii) is thereby proved.
(ii)$\Rightarrow$(i) The above reasoning can be reversed until the second equality is reached. Since by hypothesis this equality holds for all ξ, η, x, y, we obtain the equality:

$$\phi^{\mathrm{t}}\big([x;y]\big) = \mathrm{ad}_x \cdot \phi^{\mathrm{t}}(y) - \mathrm{ad}_y \cdot \phi^{\mathrm{t}}(x).$$

Therefore ϕ^{t} is a 1-cocycle.
We now prove that the three relations in (ii) are equivalent. In so doing, we remark that:

$$-\langle \mathrm{ad}^*{}_x \cdot \xi; \mathrm{ad}^*{}_\eta \cdot y\rangle = \big\langle \mathrm{ad}_\eta \cdot (\mathrm{ad}^*{}_x \cdot \xi); y\big\rangle = -\big\langle \big[(\mathrm{ad}^*{}_x \cdot \xi);\eta\big]_{\mathfrak{g}^*}; y\big\rangle,$$

and so with all other equalities of the same kind; whereby the third relation can be obtained from the first. We then obtain the equivalence. To show the equivalence between the first and the second, we remark that:

$$\begin{aligned}
-\big\langle \eta; [x; \mathrm{ad}^*{}_\xi \cdot y]\big\rangle &= \big\langle \mathrm{ad}^*{}_x \cdot \eta; \mathrm{ad}^*{}_\xi \cdot y\big\rangle \\
&= -\big\langle \mathrm{ad}_\xi(\mathrm{ad}^*{}_x \cdot \eta); y\big\rangle = -\big\langle \big[\xi; \mathrm{ad}^*{}_x \cdot \eta\big]_{\mathfrak{g}^*}; y\big\rangle
\end{aligned}$$

and so with oll other equalities of the same kind, as above.
To prove the equivalence (iii)$\Leftrightarrow$(ii), we follow exactly the same procedure.
■

From Definition 2.1 and the equivalences proved in 2.2, we easily obtain the following result.

2.3.
Let $(\mathfrak{g}; [;])$, $(\mathfrak{g}^*; [;]_{\mathfrak{g}^*})$ *be two Lie algebras and let* β , ϕ *be the mapping defined in* 2.2. *The triple* $(\mathfrak{g}^*; [;]_{\mathfrak{g}^*}; \beta')$ *is then a Lie bialgebra if and only if the triple* $(\mathfrak{g}; [;]; \phi')$ *is a Lie bialgebra. We will say therefore that these Lie bialgebras are dual to each other.*

2.4.
Let notations as in 1.9 *be. Prove that the Drinfeld compatibility condition* (*see* 2.2 (ii)) *can be written as*:

$$c^k_{rs} f^{jk}_i = c^i_{\alpha_r} f^{j\alpha}_s - c^j_{\alpha_r} f^{i\alpha}_s - c^i_{\alpha_s} f^{j\alpha}_r - c^j_{\alpha_s} f^{i\alpha}_r .$$

2.5.
Let $(\mathfrak{g}_1; [;]_1; \epsilon_1)$, $(\mathfrak{g}_2; [;]_2; \epsilon_2)$ *be two Lie bialgebras.*
Provide the natural definition of a morphism of these Lie bialgebras.
Let $(G_1; \Lambda_1)$, $(G_2; \Lambda_2)$ *be* PLGs.
Prove that every Poisson–Lie morphism between them determines a morphism between their Lie bialgebras.

Once we realize that the notion of the duality of the Lie bialgebras $(\mathfrak{g}; [;]; \phi^{\mathrm{t}})$, $(\mathfrak{g}^*; [;]_{\mathfrak{g}^*}; \beta^{\mathrm{t}})$ involves reflexivity (see 2.2, 2.3, or the natural definition obtainable from Lie bialgebra isomorphism 2.5) we are naturaly led to look for the algebraic structure on the space $\mathfrak{p} = \mathfrak{g} \times \mathfrak{g}^*$, which is equivalent to the above notion of the duality of Lie bialgebra. We thereby come to the notion of the Manin triple.

Let us first introduce the symmetric bilinear mapping:

$$\langle ; \rangle_{\mathfrak{p}} : \mathfrak{p} \times \mathfrak{p} \longrightarrow \mathbb{R}$$

defined as:

$$\big\langle (x; \xi); (y; \eta) \big\rangle_{\mathfrak{p}} = \langle \xi ; y \rangle + \langle \eta ; x \rangle .$$

2.6.
Using the preceding notations, prove that the skewsymmetric bilinear mapping $[;]_{\mathfrak{p}} : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ *defined as*:

$$\begin{aligned} &\big[(x; \xi); (y; \eta)\big]_{\mathfrak{p}} \\ &\quad = \big([x; y] + \mathrm{ad}^*{}_{\xi} \cdot y - \mathrm{ad}^*{}_{\eta} \cdot x; [\xi; \eta]_{\mathfrak{g}^*} + \mathrm{ad}^*{}_{x} \cdot \eta - \mathrm{ad}^*{}_{y} \cdot \xi\big), \end{aligned}$$

is the only one which satisfies the following properties:
(a) *Its restrictions to subspaces* $\mathfrak{g} \times \{0\}$ *and* $\{0\} \times \mathfrak{g}^* \subset \mathfrak{p}$ *are respectively the Lie brackets on* $\mathfrak{g}$ *and* $\mathfrak{g}^*$, *i.e.*,

$$\big[(x; 0); (y; 0)\big]_{\mathfrak{p}} = \big([x; y]; 0\big); \qquad \big[(0; \xi); (0; \eta)\big]_{\mathfrak{p}} = \big(0; [\xi; \eta]_{\mathfrak{g}^*}\big).$$

(b) *It leaves invariant the symmetric bilinear form* $\langle ; \rangle_{\mathfrak{p}}$, *i.e.*,

$$\big\langle \big[(x; \xi); (y; \eta)\big]_{\mathfrak{p}}; (z; \mu)\big\rangle_{\mathfrak{p}} + \big\langle (y; \eta); \big[(x; \xi); (z; \mu)\big]_{\mathfrak{p}}\big\rangle_{\mathfrak{p}} = 0.$$

Answer. Clearly $[;]_{\mathfrak{p}}$ verifies the property (a).
We now prove that $[;]_{\mathfrak{p}}$ leaves $\langle;\rangle_{\mathfrak{p}}$ invariant. From the definitions, we get:

$$\begin{aligned}
&\big\langle\big[(x;\xi);(y;\eta)\big]_{\mathfrak{p}};(z;\mu)\big\rangle_{\mathfrak{p}}\\
&\quad=\big\langle\big([x;y]+\mathrm{ad}^*{}_{\xi}\cdot y-\mathrm{ad}^*{}_{\eta}\cdot x;[\xi;\eta]_{\mathfrak{g}^*}+\mathrm{ad}^*{}_{x}\cdot\eta-\mathrm{ad}^*{}_{y}\cdot\xi\big);(z;\mu)\big\rangle_{\mathfrak{p}}\\
&\quad=\cdots=\\
&\quad=-\big\langle\mathrm{ad}^*{}_{x}\cdot\mu;y\big\rangle+\big\langle\mu;\mathrm{ad}^*{}_{\xi}\cdot y\big\rangle-\big\langle\mu;\mathrm{ad}^*{}_{\eta}\cdot x\big\rangle\\
&\qquad-\big\langle\eta;\mathrm{ad}^*{}_{\xi}\cdot z\big\rangle+\big\langle\mathrm{ad}^*{}_{x}\cdot\eta;z\big\rangle-\big\langle\mathrm{ad}^*{}_{y}\cdot\xi;z\big\rangle.
\end{aligned}$$

In a similar way, we get:

$$\begin{aligned}
&\big\langle(y;\eta);\big[(x;\xi);(z;\mu)\big]_{\mathfrak{p}}\big\rangle_{\mathfrak{p}}=\cdots=\\
&\quad=-\big\langle\mathrm{ad}^*{}_{x}\cdot\eta;z\big\rangle+\big\langle\eta;\mathrm{ad}^*{}_{\xi}\cdot z\big\rangle+\big\langle\mathrm{ad}^*{}_{y}\cdot\xi;;z\big\rangle\\
&\qquad+\big\langle\mathrm{ad}^*{}_{x}\cdot\mu;y\big\rangle-\big\langle\mu;\mathrm{ad}^*{}_{\xi}\cdot y\big\rangle+\big\langle\mu;\mathrm{ad}^*{}_{\eta}\cdot x\big\rangle.
\end{aligned}$$

Property (b) is proved by summing these two expressions.
We now prove the property of unicity. Let $[;]_{\mathfrak{p}}$ be any skewsymmetric bilinear form on $\mathfrak{p}=\mathfrak{g}\times\mathfrak{g}^*$. In particular:

$$\begin{aligned}
\big[(x;\xi);(y;\eta)\big]_{\mathfrak{p}}&=\big[(x;0);(y;0)\big]_{\mathfrak{p}}+\big[(x;0);(0;\eta)\big]_{\mathfrak{p}}\\
&\quad+\big[(0;\xi);(y;0)\big]_{\mathfrak{p}}+\big[(0;\xi);(0;\eta)\big]_{\mathfrak{p}}.
\end{aligned}$$

$[;]_{\mathfrak{p}}$ must verify (a). Therefore,

$$\begin{aligned}
\big[(x;0);(y;0)\big]_{\mathfrak{p}}&=\big([x;y];0\big),\\
\big[(0;\xi);(0;\eta)\big]_{\mathfrak{p}}&=\big(0;[\xi;\eta]_{\mathfrak{g}^*}\big),\\
\big[(x;0);(0;\eta)\big]_{\mathfrak{p}}&=\big(A(\eta;x);B(x;\eta)\big),\\
\big[(0;\xi);(y;0)\big]_{\mathfrak{p}}&=\big(C(\xi;y);D(y;\xi)\big)
\end{aligned}$$

where: $A:\mathfrak{g}^*\times\mathfrak{g}\to\mathfrak{g}$, $B:\mathfrak{g}\times\mathfrak{g}^*\to\mathfrak{g}^*$, $C:\mathfrak{g}^*\times\mathfrak{g}\to\mathfrak{g}$ and $D:\mathfrak{g}\times\mathfrak{g}^*\to\mathfrak{g}^*$ must be bilinear mappings.
$[;]_{\mathfrak{p}}$ must be skewsymmetric; therefore: $A(\eta;x)=-C(\eta;x)$, $B(x;\eta)=-D(x;\eta)$. Consequently,

$$\begin{aligned}
\big[(x;\xi);(y;\eta)\big]_{\mathfrak{p}}&=\big([x;y]+A(\eta;x)-A(\xi;y);\\
&\qquad[\xi;\eta]_{\mathfrak{g}^*}+B(x;\eta)-B(y;\xi)\big).
\end{aligned}$$

We now determine A and B by requering $[;]_{\mathfrak{p}}$ have to remaining property (b). This requirement is clearly equivalent to imposing the following two conditions:

$$\big\langle\big[(x;0);(y;0)\big]_{\mathfrak{p}};(0;\xi)\big\rangle_{\mathfrak{p}}+\big\langle(y;0);\big[(x;0);(0;\xi)\big]_{\mathfrak{p}}\big\rangle_{\mathfrak{p}}=0,$$

and

$$\left\langle \left[(0;\xi);(0;\eta)\right]_{\mathfrak{p}};(x;0)\right\rangle_{\mathfrak{p}} + \left\langle (0;\eta);\left[(0;\xi);(x;0)\right]_{\mathfrak{p}}\right\rangle_{\mathfrak{p}} = 0.$$

From these conditions, we easily obtain:

$$B(x;\xi) = \mathrm{ad}^*{}_x \cdot \xi; \qquad A(\xi;x) = -\,\mathrm{ad}^*{}_\xi \cdot x.$$

We can then obtain no more than one skewsymmetric bilinear mapping $[;]_{\mathfrak{p}}$, which is the one required. ■

2.7.
Prove that the skewsymmetric bilinear mapping $[;]_{\mathfrak{p}}$ *in* 2.6 *verifies the Jacobi identity if and only if the brackets* $[;]$ *on* $\mathfrak{g}$ *and* $[;]_{\mathfrak{g}^*}$ *on* $\mathfrak{g}^*$ *verify the Drinfeld compatibility condition* (ii) *in* 2.2; *or equivalently, if and only if the set* $(\mathfrak{g};[;];\phi^t)$ *is a Lie bialgebra, where* ϕ *is defined by the bracket of* $\mathfrak{g}^*$, *i.e.*, $[\xi_1;\xi_2]_{\mathfrak{g}^*} = \phi(\xi_1 \otimes \xi_2)$.

Answer. From the bilinearity of $[;]_{\mathfrak{p}}$, it suffices to require that:

$$\begin{aligned}&\left[(0;\xi);\left[(x;0);(y;0)\right]_{\mathfrak{p}}\right]_{\mathfrak{p}} + \left[(x;0);\left[(y;0);(0;\xi)\right]_{\mathfrak{p}}\right]_{\mathfrak{p}}\\ &\quad + \left[(y;0);\left[(0;\xi);(x;0)\right]_{\mathfrak{p}}\right]_{\mathfrak{p}} = (0;0),\end{aligned} \tag{6}$$

and

$$\begin{aligned}&\left[(x;0);\left[(0;\xi);(0;\eta)\right]_{\mathfrak{p}}\right]_{\mathfrak{p}} + \left[(0;\xi);\left[(0;\eta);(x;0)\right]_{\mathfrak{p}}\right]_{\mathfrak{p}}\\ &\quad + \left[(0;\eta);\left[(x;0);(0;\xi)\right]_{\mathfrak{p}}\right]_{\mathfrak{p}} = (0;0).\end{aligned} \tag{7}$$

From the definition of $[;]_{\mathfrak{p}}$, we get:

$$\begin{aligned}&\left[(0;\xi);\left[(x;0);(y;0)\right]_{\mathfrak{p}}\right]_{\mathfrak{p}} = \left(\mathrm{ad}^*{}_\xi \cdot [x;y]; -\,\mathrm{ad}^*{}_{[x;y]} \cdot \xi\right),\\ &\left[(x;0);\left[(y;0);(0;\xi)\right]_{\mathfrak{p}}\right]_{\mathfrak{p}}\\ &\quad = \left([x;-\,\mathrm{ad}^*{}_\xi \cdot y] - \mathrm{ad}^*{}_{(\mathrm{ad}^*{}_y \cdot \xi)}\, x; \mathrm{ad}^*{}_x \cdot \mathrm{ad}^*{}_y \cdot \xi\right),\\ &\left[(y;0);\left[(0;\xi);(x;0)\right]_{\mathfrak{p}}\right]_{\mathfrak{p}}\\ &\quad = \left([y;\mathrm{ad}^*{}_\xi \cdot x] + \mathrm{ad}^*{}_{(\mathrm{ad}^*{}_x \cdot \xi)}\, y; -\,\mathrm{ad}^*{}_y \cdot \mathrm{ad}^*{}_x \cdot \xi\right).\end{aligned}$$

Requirement (6) is equivalent to the equalities:

$$\mathrm{ad}^*{}_\xi \cdot [x;y] - [x;\mathrm{ad}^*{}_\xi\, y] - \mathrm{ad}^*{}_{(\mathrm{ad}^*{}_y \xi)} \cdot x + [y;\mathrm{ad}^*{}_\xi \cdot x] + \mathrm{ad}^*{}_{(\mathrm{ad}^*{}_x \xi)} \cdot y = 0$$

and

$$-\,\mathrm{ad}^*{}_{[x;y]} \cdot \xi + \mathrm{ad}^*{}_x \cdot \mathrm{ad}^*{}_y \cdot \xi - \mathrm{ad}^*{}_y \cdot \mathrm{ad}^*{}_x \cdot \xi = 0.$$

The second equality is easily seen to be equivalent to the Jacobi identity for the Lie algebra $\mathfrak{g}$. From the definition of ad* and duality, the first equality is clearly equivalent to:

$$-\langle x; y; [\xi; \eta]_{\mathfrak{g}^*}\rangle - \langle \eta; [x; \mathrm{ad}^*{}_{\xi} \cdot y]\rangle$$
$$- \langle \eta; [\mathrm{ad}^*{}_{\xi} \cdot x; y]\rangle + \langle \xi; [\mathrm{ad}^*{}_{\eta} \cdot x; y]\rangle + \langle \xi; [x; \mathrm{ad}^*{}_{\eta} \cdot y]\rangle = 0,$$

for all $\eta \in \mathfrak{g}^*$, which is precisely the second condition in 2.2 (ii).
In a similar way, we can prove that the second requirement is equivalent to the first condition in 2.2 (ii). ■

The results obtained thus far can be summarized as follows:

2.8.
Let $(\mathfrak{g}; [;])$, $(\mathfrak{g}^*; [;]_{\mathfrak{g}^*})$ *be two Lie algebras. Let* $\phi : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ *be the linear mapping defined as* $[\xi; \eta]_{\mathfrak{g}^*} = \phi(\xi \otimes \eta)$, *and let* $\phi^{\mathrm{t}} : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *be the mapping transposed to* ϕ. *On the vector space* $\mathfrak{p} = \mathfrak{g} \times \mathfrak{g}^*$, *there is a Lie bracket such that its restriction to the subspaces* $\mathfrak{g} \times \{0\}$ *and* $\{0\} \times \mathfrak{g}^*$ *coincides with the Lie brackets of* $\mathfrak{g}$ *and* $\mathfrak{g}^*$ *respectively, and leaves the symmetric mapping* $\langle ; \rangle_{\mathfrak{p}}$ (2.6) *invariant if and only if* $(\mathfrak{g}; [;]; \phi^{\mathrm{t}})$ *is a Lie bialgebra. There is only one Lie bracket with the above properties, namely* $[;]_{\mathfrak{p}}$, *given in* 2.6.

Any Lie bialgebra $(\mathfrak{g}; [;]; \phi^{\mathrm{t}})$ can therefore be associated with a set

$$(\mathfrak{p} = \mathfrak{g} \times \mathfrak{g}^*; \mathfrak{g}; \mathfrak{g}^*; [;]_{\mathfrak{p}}; \langle ; \rangle_{\mathfrak{p}}),$$

where $\mathfrak{p}$ is a Lie algebra with bracket $[;]_{\mathfrak{p}}$, 2.6; $\langle ; \rangle_{\mathfrak{p}}$ is the non-degenerate, symmetric bilinear form on $\mathfrak{p}$, 2.6, which is invariant relative to the adjoint representation of $\mathfrak{p}$; and $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras of $\mathfrak{p}$ ($[;]_{\mathfrak{g}^*} = \epsilon^{\mathrm{t}} \circ \otimes$) which, as vector subspaces of $\mathfrak{p}$, are both isotropic relative to $\langle ; \rangle_{\mathfrak{p}}$. This particular structure suggests the more general one in the following definition.

2.9. Definition
A Manin triple is a set

$$(\mathfrak{p} \equiv \mathfrak{p}_1 \times \mathfrak{p}_2; \mathfrak{p}_1; \mathfrak{p}_2; [;]_{\mathfrak{p}}; \langle ; \rangle_{\mathfrak{p}})$$

where $\mathfrak{p}$ *is a Lie algebra with bracket* $[;]_{\mathfrak{p}}$; $\langle ; \rangle_{\mathfrak{p}}$ *is a non-degenerate, symmetric bilinear form on* $\mathfrak{p}$ *which is invaiant relative to the adjoint representation of* $\mathfrak{p}$; *and* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *are Lie subalgebras of* $\mathfrak{p}$, *and as vector subspaces of* $\mathfrak{p}$, *are both isotropic relative to* $\langle ; \rangle_{\mathfrak{p}}$.

In the above notation of a Manin triple, we can write:

$$\big\langle (x_1;\xi_1);(x_2;\xi_2)\big\rangle_{\mathfrak{p}} = \big\langle (x_1;0);(0;\xi_2)\big\rangle_{\mathfrak{p}} + \big\langle (0;\xi_1);(x_2;0)\big\rangle_{\mathfrak{p}}$$

$(x_1;\xi_1)\in\mathfrak{p}=\mathfrak{p}_1\times\mathfrak{p}_2$; $i=1,2$. This 2-form on $\mathfrak{p}$ is non-degenerate and consequently, the bilinear mapping $\langle (x_1;0);(0;\xi_2)\rangle_{\mathfrak{p}}$ on $\mathfrak{p}_1\times\mathfrak{p}_2$ is also non-degenerate.

We can now define the isomorphism:

$$\xi_2\in\mathfrak{p}_2 \xrightarrow{\varphi} \hat{\xi}_2\in\mathfrak{p}_1^*$$

by means of the relation:

$$\big\langle \varphi(\xi_2);x_1\big\rangle = \big\langle (0;\xi_2);(x_1;0)\big\rangle_{\mathfrak{p}},$$

and thereby the isomorphism

$$(x;\xi)\in\mathfrak{p}=\mathfrak{p}_1\times\mathfrak{p}_2 \xrightarrow{\hat{\varphi}} (x;\hat{\xi})\in\hat{\mathfrak{p}}=\mathfrak{p}_1\times\mathfrak{p}^*$$

i.e., $\hat{\varphi}$ can be defined by the properties: $\hat{\varphi}|_{\mathfrak{p}_1}=I_{\mathfrak{p}_1}$; $\hat{\varphi}|_{\mathfrak{p}_2}=\varphi$.

The image of the bracket $[;]_{\mathfrak{p}}$ by $\hat{\varphi}$ is:

$$\big[(x;\hat{\xi});(y;\hat{\eta})\big]_{\hat{\mathfrak{p}}} = \hat{\varphi}\big(\big[\hat{\varphi}^{-1}(x;\hat{\xi});\hat{\varphi}^{-1}(y;\hat{\eta})\big]_{\mathfrak{p}}\big).$$

The restriction of $[;]_{\hat{\mathfrak{p}}}$ to $\mathfrak{p}_1$ is therefore a Lie bracket and $\mathfrak{p}_1$ is a Lie algebra relative to this bracket!

Also:

$$\big[(0;\xi_1);(0;\xi_2)\big]_{\hat{\mathfrak{p}}} = \cdots = \varphi\big(\big[\varphi^{-1}(\hat{\xi}_1);\varphi^{-1}(\hat{\xi}_2)\big]_{\mathfrak{p}_2}\big).$$

We can therefore define the following bracket on $\mathfrak{p}_1^*$:

$$\big[\hat{\xi}_1;\hat{\xi}_2\big]_{\mathfrak{p}_1^*} = \varphi\big(\big[\varphi^{-1}(\hat{\xi}_1);\varphi^{-1}(\hat{\xi}_2)\big]_{\mathfrak{p}_2}\big).$$

The $\mathfrak{p}_1^*$ endowed with this bracket is thus a Lie algebra, and the bracket is the restriction to $\mathfrak{p}_1^*$ of the bracket $[;]_{\hat{\mathfrak{p}}}$.

The image of $\langle;\rangle_{\hat{\mathfrak{p}}}$ by $\hat{\varphi}$ is:

$$\big\langle (x;\hat{\xi});(y;\hat{\eta})\big\rangle_{\hat{\mathfrak{p}}} = \cdots = \big\langle \hat{\xi};y\big\rangle + \big\langle \hat{\eta};x\big\rangle.$$

This is the 2-form on $\hat{\mathfrak{p}}=\mathfrak{p}_1\times\mathfrak{p}_1^*$ of section 2.6. Clearly, $\langle;\rangle_{\hat{\mathfrak{p}}}$ is invariant by the adjoint representation of the Lie algebra $\hat{\mathfrak{p}}=\mathfrak{p}_1\times\mathfrak{p}_1^*$, this being a consequence of the invariance of $\langle;\rangle_{\mathfrak{p}}$ by the adjoint representation of the Lie algebra $\mathfrak{p}$.

The bracket $[;]_{\hat{\mathfrak{p}}=\mathfrak{p}_1\times\mathfrak{p}_1^*}$ *must* therefore be the bracket in 2.6, 2.7, 2.8 (where $\mathfrak{g}\equiv\mathfrak{p}_1$; $\mathfrak{g}^*\equiv\mathfrak{p}_1^*$).

We thereby obtain:

2.10.
Any Manin triple is isomorphic to some standard one ($\mathfrak{p} = \mathfrak{g} \times \mathfrak{g}^*$; $[;]_{\mathfrak{p}}$; $\langle ; \rangle_{\mathfrak{p}}$).

Isomorphic, in this context, means that the isomorphism $\hat{\varphi}$ exists.

3. THE SIMPLY CONNECTED POISSON–LIE GROUP CORRESPONDING TO A GIVEN LIE BIALGEBRA

Starting from a Lie bialgebra ($\mathfrak{g}$; $[;]$; ϵ), we prove that on the simply-connected Lie group G with Lie algebra $\mathfrak{g}$, this bialgebra determines one and only one Poisson–Lie group structure on G, which is such that its Lie-bialgebra is the initial one. (Ref. [1], [2], [6], [10], [11], [12].)

Let P be any contravariant tensor field, of degree p, on the Lie group G. Let X be any vector field on G. Let $\{F_t; t \in \mathbb{R}\}$ be the flow of X:

$$\frac{\mathrm{d}F_t}{\mathrm{d}t}(g) = X\big(F_t(g)\big), \tag{8}$$

$$(Xf)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\big(f \circ F_t(g)\big)\big|_{t=0}. \tag{9}$$

We must keep in mind that, by the Lie derivative theorem for tensor fields, we can write:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((TF_{-t})^{\otimes p} \circ P \circ F_t\big)(g) = \big((TF_{-t})^{\otimes p} \circ \mathrm{L}_X P \circ F_t\big)(g). \tag{10}$$

In particular:

$$(\mathrm{L}_X P)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\big((TF_{-t})^{\otimes p} \circ P \circ F_t\big)(g)\big|_{t=0}.$$

3.1.
Let G be a conected Lie group. Let $\wedge^p(G)$ be the space of skewsymmetric q-contravariant tensor fields on G.
Prove that $Q \in \wedge^2(G)$ is right-invariant if and only if $\mathrm{L}_X Q = 0$ for any left-invariant vector field X on G. Prove that a similar result holds when left and right are inverted.

Answer. This is a classical result, which can be obatined from the definitions and the flow $F_t = \rho_{\exp tx}$, $x = X(e) \in \mathfrak{g}$, $t \in \mathbb{R}$, of X. ■

With reference to the Schouten bracket, the above result says that P is right-invariant if and only if $[X; Q] = 0$, for any left-invariant vector field on G. This result can be generalized.

3.2.
Let G be a connected Lie group. Let $Q \in \wedge^q(G)$ be a right-invariant tensor field on G, and $P \in \wedge^p(G)$ a left-invariant tensor field on G.
Prove that $[P; Q] = 0$, where $[;]$ is the Schouten bracket.

Answer. Let $R \in \wedge^r(G)$. For any tensors P, Q, R, the Schouten bracket satisfies the following basic relations:

$$[P; Q \wedge R] = [P; Q] \wedge R + (-1)^{pq+r} Q \wedge [P; R];$$
$$[P; Q] = (-1)^{p\cdot q}[Q; P].$$

Let $(x_i;\ i = 1, \ldots, n)$ be any basis of the Lie algebra $\mathfrak{g}$ of G. The set of left-invariant vector fields $(x_i^\lambda;\ i = 1, \ldots, n)$ is a basis of $\wedge_\lambda^1(G)$; and the set of right-invariant vector fields $(x_i^\rho;\ i = 1, \ldots, n)$ is a basis of $\wedge_\rho^1(G)$. Therefore, the sets

$$x_{i_1}^\lambda \wedge \cdots \wedge x_{i_p}^\lambda; \quad 1 \le i_1 \le \cdots \le i_p \le n,$$
$$x_{j_1}^\rho \wedge \cdots \wedge x_{j_q}^\rho; \quad 1 \le j_1 \le \cdots \le j_q \le n$$

are respectively a basis of $\wedge_\lambda^p(G)$ and $\wedge_\rho^q(G)$.

We get the required result by induction. Let us suppose that $[\tilde{P}; \tilde{Q}] = 0$ for $\tilde{P} \in \wedge_\lambda^{p-1}(G)$, $\tilde{Q} \in \wedge_\rho^{q-1}(G)$.
From the initial relation, we get

$$\begin{aligned}
&\left[x_{j_1}^\rho \wedge \cdots \wedge x_{j_q}^\rho; x_{i_1}^\lambda \wedge \cdots \wedge x_{i_p}^\lambda\right] \\
&\quad = \left[x_{j_1}^\rho \wedge \cdots \wedge x_{j_q}^\rho; x_{i_1}^\lambda \wedge \cdots \wedge x_{i_{p-1}}^\lambda\right] \wedge x_{i_p}^\lambda \\
&\qquad + (-1)^{q(q-1)+(p-1)} x_{i_1}^\lambda \wedge \cdots \wedge x_{i_{p-1}}^\lambda \wedge \left[x_{j_1}^\rho \wedge \cdots \wedge x_{j_q}^\rho; x_p^\lambda\right].
\end{aligned}$$

Also:

$$\begin{aligned}
&\left[x_{j_1}^\rho \wedge \cdots \wedge x_{j_q}^\rho; x_{i_1}^\lambda \wedge \cdots \wedge x_{i_{p-1}}^\lambda\right] \\
&\quad = (-1)^{q(p-1)}\left[x_{i_1}^\lambda \wedge \cdots \wedge x_{i_{p-1}}^\lambda; x_{j_1}^\rho \wedge \cdots \wedge x_{j_q}^\rho\right] \\
&\quad = (-1)^{q(p-1)}\Big\{\left[x_{i_1}^\lambda \wedge \cdots \wedge x_{i_{p-1}}^\lambda; x_{j_1}^\rho \wedge \cdots \wedge x_{j_{q-1}}^\rho\right] \wedge x_{j_q}^\rho \\
&\qquad + (-1)^{pq+q} x_{j_1}^\rho \wedge \cdots \wedge x_{j_{q-1}}^\rho \wedge \left[x_{i_1}^\lambda \wedge \cdots \wedge x_{i_{p-1}}^\lambda; x_{j_q}^\rho\right]\Big\} = 0.
\end{aligned}$$

This last equality is proved from the hypothesis by induction. ∎

3.3. Definition
We say that $P \in \wedge^p(G)$ verifies the Drinfeld property if:

$$P(g \cdot h) = (T_h \lambda_g)^{\otimes p} P(h) + (T_g \rho_h)^{\otimes p} P(g), \quad \text{for all } g, h \in G. \qquad (11)$$

Exercise. As in 1.5, prove that P verifies the Drinfeld property if and only if the mapping:

$$l : g \in G \longrightarrow (T_g \rho_{g^{-1}})^{\otimes p} P(g) \in \mathfrak{g} \otimes \overset{p}{\cdots} \otimes \mathfrak{g}, \qquad (12)$$

verifies the relation:

$$l(g \cdot h) = l(g) + (\mathrm{Ad}_g)^{\otimes p} \cdot l(h); \quad \text{for all } g, h \in G, \qquad (13)$$

i.e., if and only if l is a 1-cocycle on G, with values in $\mathfrak{g} \otimes \overset{p}{\cdots} \otimes \mathfrak{g}$ relative to the adjoint representation of G on $\mathfrak{g} \otimes \overset{p}{\cdots} \otimes \mathfrak{g}$.

Let us now consider the notion (11). For any given $h \in G$, the term $(T_h \lambda_g)^{\otimes p} \cdot P(h)$ defines a left-invariant tensor field on G; and for any given $g \in G$, the term $(T_g \rho_h)^{\otimes p} \cdot P(g)$ defines a right-invariant tensor field on G.
This observation leads us to reformulate condition (11) using the result in 3.1.

3.4.
Let G be a connected Lie group, and $\mathfrak{g}$ its Lie algebra.
Prove that $P \in \wedge^p(G)$ verifies the Drinfeld property if and only if:
(i) *P is zero at $e \in G$, i.e., $P(e) = 0$.*
(ii) *$\mathrm{L}_X P$ is a left-invariant tensor field for any left-invariant vector field $X \in \wedge^1_\lambda(G)$. Equivalently,* 3.1, *$\mathrm{L}_Y \mathrm{L}_X P = 0$ for any right-invariant vector field $Y \in \wedge^1_\rho(G)$.*

Answer. P verifies the Drinfeld property if it verifies (11) or, equivalently, if it is verified (13).
Let us suppose that P verifies (11). Setting $g = h = e$ in (11), we get $P(e) = 0$. (i) is thereby proved.
We now prove (ii). Let $x \in \mathfrak{g}$ and $x^\lambda \in \wedge^1_\lambda(G)$. The flow of x^λ is $F_t = \rho_{\exp tx}$, $t \in \mathbb{R}$. Thus,

$$\begin{aligned}
(\mathrm{L}_{x^\lambda} P)(g) &= \frac{\mathrm{d}}{\mathrm{d}t} (T_{g \cdot \exp tx} \rho_{\exp(-tx)})^{\otimes p} \big(P(g \cdot \exp tx) \big) \Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \big((T_{g \exp tx} \rho_{\exp(-tx)})^{\otimes p} \circ (T_{\exp tx} \lambda_g)^{\otimes p} \cdot P(\exp tx) \big) \Big|_{t=0} \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t} \big(T_{g \cdot \exp tx} (\rho_{\exp(-tx)})^{\otimes p} \circ (T_g \rho_{\exp tx})^{\otimes p} \cdot P(g) \big) \Big|_{t=0}.
\end{aligned}$$

(In the second equality, we use (11) for $P(g \cdot \exp tx)$.) The second of the last two terms is obviously zero. The first is:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big((T_{g\exp tx}\rho_{\exp(-tx)})^{\otimes p} \circ (T_{\exp tx}\lambda_g)^{\otimes p} \cdot P(\exp tx)\big)\big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big((T_e\lambda_g)^{\otimes p} \circ (T_{\exp tx}\rho_{\exp(-tx)})^{\otimes p} \cdot P(\exp tx)\big)\big|_{t=0} \\
&= (T_e\lambda_g)^{\otimes p}\frac{\mathrm{d}}{\mathrm{d}t}\big((T_{\exp tx}\rho_{\exp(-tx)})^{\otimes p} \cdot P(\exp tx)\big)\big|_{t=0} \\
&= (T_e\lambda_g)^{\otimes p}(\mathrm{L}_{x^\lambda}P)(e).
\end{aligned}$$

We thus get the equality:

$$(\mathrm{L}_{x^\lambda}P)(g) = (T_e\lambda_g)^{\otimes p}(\mathrm{L}_{x^\lambda}P)(e),$$

(ii) is thereby proved.

Proving that, if P verifies (i) and (ii), then P verifies the Drinfeld condition will be equivalent to proving that the mapping (12) verifies (13); or equivalently, given that G is connected, the relation:

$$l(g \cdot \exp tx) = l(g) + (\mathrm{Ad}_g)^{\otimes p} l(\exp tx), \quad \text{for all } x \in \mathfrak{g}. \tag{14}$$

First, let us compute the derivative of each side of this equality (14):

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}l(g\exp tx) = \frac{\mathrm{d}}{\mathrm{d}t}\big((T_{g\cdot\exp tx}\cdot\rho_{(g\cdot\exp tx)^{-1}})^{\otimes p}\cdot P(g\cdot\exp tx)\big) = \cdots = \\
&= (T_g\rho_{g^{-1}})^{\otimes p}\frac{\mathrm{d}}{\mathrm{d}t}\big((T_{g\cdot\exp tx}\cdot\rho_{\exp(-tx)})^{\otimes p}\cdot P\big(\rho_{\exp tx}(g)\big)\big) \\
&= (T_{g\cdot\exp tx}\cdot\rho_{(g\cdot\exp tx)^{-1}})^{\otimes p}(\mathrm{L}_{x^\lambda}P)(g\cdot\exp tx) = \cdots = \\
&= \big(T_e(\rho_{(g\cdot\exp tx)^{-1}}\circ\lambda_{g\cdot\exp tx})\big)^{\otimes p}\cdot(\mathrm{L}_{x^\lambda}P)(e) \\
&= (\mathrm{Ad}_{g\cdot\exp tx})^{\otimes p}(\mathrm{L}_{x^\lambda}P)(e).
\end{aligned}$$

For the derivative of the right side of (14), we get:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(l(g) + (\mathrm{Ad}_g)^{\otimes p}\cdot l(\exp tx)\big) = \cdots = \\
&= (\mathrm{Ad}_g)^{\otimes p}\frac{\mathrm{d}}{\mathrm{d}t}\big((T_{\exp tx}\cdot\rho_{\exp(-tx)})^{\otimes p}P(\exp tx)\big) \\
&= (\mathrm{Ad}_g)^{\otimes p}(T_{\exp tx}\cdot\rho_{\exp -tx})^{\otimes p}(T_e\lambda_{\exp tx})^{\otimes p}(\mathrm{L}_{x^\lambda}P)(e) \\
&= (\mathrm{Ad}_{g\cdot\exp tx})^{\otimes p}\cdot(\mathrm{L}_{x^\lambda}P)(e).
\end{aligned}$$

We thereby prove that the derivatives of the terms on both sides of (14) are equal, i.e., for all $t \in \mathbb{R}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(l(g\cdot \exp tx) - l(g) - (\mathrm{Ad}_g)^{\otimes p}\cdot l(\exp tx)\big) = 0.$$

Thus,

$$l(g\cdot \exp tx) - l(g) - (\mathrm{Ad}_g)^{\otimes p}\cdot l(\exp tx) = l(g) - l(g) - 0 = 0,$$

where we make use of the fact $l(e) = 0$ since $P(e) = 0$. The proof is now complete. ■

3.5.
Prove that the space of contravariant skewsymmetric tensors on G *verifying the Drinfeld property has the structure of a graded Lie algebra relative to the Schouten bracket.*

Answer. We must prove that, if P and Q verify the Drinfeld property, so does $[P;\,Q]$. From 3.4 this is equivalent to proving that $[P;\,Q](e) = 0$ and $\mathrm{L}_{x^\lambda}[P;\,Q]$ is a left-invariant tensor field for any $x \in \mathfrak{g}$.
(a) From the local expresion of $[P;\,Q]$, it is clear that $[P;\,Q](e) = 0$ when $P(e) = Q(e) = 0$.
(b) For $R \in \wedge^r(G)$, the relation that defines the graduated structure of skewsymmetric tensors is:

$$(-1)^{p\cdot q}\big[[Q;\,R];\,P\big] + (-1)^{q\cdot r}\big[[R;\,P];\,Q\big] + (-1)^{p\cdot r}\big[[P;\,Q];\,R\big] = 0. \tag{15}$$

Whenever $R = X$ is a vector field, the above relation is:

$$\big[P;\,[X;\,Q]\big] + \big[[X;\,P];\,Q\big] = \big[X;\,[P;\,Q]\big],$$

that is:

$$\mathrm{L}_X[P;\,Q] = [\mathrm{L}_X P;\,Q] + [P;\,\mathrm{L}_X Q].$$

Let Y be any vector field on G. As above, we get:

$$\begin{aligned}\mathrm{L}_Y\mathrm{L}_X[P;\,Q] &= [\mathrm{L}_Y\mathrm{L}_X P;\,Q] + [\mathrm{L}_X P;\,\mathrm{L}_Y Q]\\ &\quad + [\mathrm{L}_Y P;\,\mathrm{L}_X Q] + [P;\,\mathrm{L}_Y\mathrm{L}_X Q].\end{aligned} \tag{16}$$

Let $\wedge^1_\lambda(G)$, $\wedge^1_\rho(G)$ be respectively the spaces of left and right-invariant vector fields on G. Let us suppose that $X \in \wedge^1_\lambda(G)$, $Y \in \wedge^1_\rho(G)$.
From 3.4 tensors $\mathrm{L}_X P$, $\mathrm{L}_X Q$ are left-invariant. Therefore, $\mathrm{L}_Y\mathrm{L}_X P = 0$, and $\mathrm{L}_Y\mathrm{L}_X Q = 0$. Also, $[X;\,Y] = 0$; wherefore $\mathrm{L}_X\mathrm{L}_Y P = \mathrm{L}_Y\mathrm{L}_X P = 0$ and $\mathrm{L}_X\mathrm{L}_Y Q = \mathrm{L}_Y\mathrm{L}_X Q = 0$; and $\mathrm{L}_Y P$, $\mathrm{L}_Y Q$ are right-invariant tensors. From these relations, question 3.2, and (16), we get $\mathrm{L}_Y\mathrm{L}_X[P;\,Q] = 0$.

From this relation, the equality $[P;\, Q](e) = 0$ and from 3.4, $[P;\, Q]$ verifies the Drinfeld property. The graded character of the algebra is a consequence of (15). ■

3.6.
Let $\boldsymbol{G}$ *be the simply-connected Lie group with Lie algebra* $\mathfrak{g}$. *Let* $(\mathfrak{g};\, [;\,];\, \epsilon)$ *be a Lie bialgebra* (2.1).
Prove that there is only one Poisson–Lie structure on $\boldsymbol{G}$ *with this Lie bialgebra.*

Answer. (a) From the classical Lie group and Lie algebra cohomology, the 1-cocycle $\epsilon : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$ determines a unique 1- cocycle on $\boldsymbol{G}$, $l : \boldsymbol{G} \to \mathfrak{g} \otimes \mathfrak{g}$, such that $T_e l = \epsilon$. Let us therefore define the following skewsymmetric contravariant 2-tensor:

$$\Lambda(g) = (T_e \rho_g)^{\otimes 2} \cdot l(g).$$

(b) From question 1.5, the tensor Λ verifies the Drinfeld property. From question 3.5, the 3-tensor $[\Lambda;\, \Lambda]$ also verifies this property. Therefore, from 3.4, $[\Lambda;\, \Lambda](e) = 0$, and $\mathrm{L}_{x^\lambda}[\Lambda;\, \Lambda]$, $x^\lambda \in \wedge^1_\lambda(\boldsymbol{G})$, $x \in \mathfrak{g}$ is a left-invariant tensor field, i.e.,

$$\mathrm{L}_{x^\lambda}[\Lambda;\, \Lambda](g) = (T_e \lambda_g)^{\otimes 3} \cdot \big(\mathrm{L}_{x^\lambda}[\Lambda;\, \Lambda]\big)(e).$$

To prove that $(\boldsymbol{G}\ \Lambda)$ is a Poisson–Lie group, it remains to be proved that

$$\mathrm{L}_{x^\lambda}[\Lambda;\, \Lambda](e) = 0; \qquad \text{for all } x \in \mathfrak{g},$$

in which case $\mathrm{L}_{x^\lambda}[\Lambda;\, \Lambda](g) = 0$, and therefore (3.1), the tensor $[\Lambda;\, \Lambda]$ is right-invariant. Thus:

$$[\Lambda;\, \Lambda](g) = 0,$$

as required.
(c) From the definition of the Lie derivative, we have:

$$\begin{aligned}
&\big(\mathrm{L}_{x^\lambda}[\Lambda;\, \Lambda]\big)(g) \cdot \big(\mathrm{d}\varphi_1(g);\, \mathrm{d}\varphi_2(g);\, \mathrm{d}\varphi_3(g)\big) \\
&\quad = \mathrm{L}_{x^\lambda} \cdot \Big([\Lambda;\, \Lambda](g)\big(\mathrm{d}\varphi_1(g);\, \mathrm{d}\varphi_2(g);\, \mathrm{d}\varphi_3(g)\big)\Big) \\
&\qquad - [\Lambda;\, \Lambda] \cdot \big((\mathrm{L}_{x^\lambda}\mathrm{d}\varphi_1)(g);\, \mathrm{d}\varphi_2(g);\, \mathrm{d}\varphi_3(g)\big) \\
&\qquad - [\Lambda;\, \Lambda] \cdot \big(\mathrm{d}\varphi_1(g);\, (\mathrm{L}_{x^\lambda}\mathrm{d}\varphi_2)(g);\, \mathrm{d}\varphi_3(g)\big) \\
&\qquad - [\Lambda;\, \Lambda] \cdot \big(\mathrm{d}\varphi_1(g);\, \mathrm{d}\varphi_2(g);\, (\mathrm{L}_{x^\lambda}\mathrm{d}\varphi_3)(g)\big).
\end{aligned}$$

Whence:

$$\begin{aligned}
&\big(\mathrm{L}_{x^\lambda}[\Lambda;\, \Lambda]\big)(e)\big(\mathrm{d}\varphi_1(e);\, \mathrm{d}\varphi_2(e);\, \mathrm{d}\varphi_3(e)\big) \\
&\quad = \Big(\mathrm{L}_{x^\lambda}\big([\Lambda;\, \Lambda](\mathrm{d}\varphi_1;\, \mathrm{d}\varphi_2;\, \mathrm{d}\varphi_3)\big)\Big)(e).
\end{aligned} \tag{17}$$

From question 1.7, we get:

$$\big(\mathrm{d}\{\varphi_1;\varphi_2\}\big)(e)=\epsilon^{\mathrm{t}}\big(\mathrm{d}\varphi_1(e)\otimes\mathrm{d}\varphi_2(e)\big).$$

Using for example the local expression of the Schouten bracket $[\Lambda;\,\Lambda]$, we can easily get the following equality:

$$[\Lambda;\,\Lambda](\mathrm{d}\varphi_1;\mathrm{d}\varphi_2;\mathrm{d}\varphi_3)=2\big(\big\{\{\varphi_1;\varphi_2\};\varphi_3\big\}+\text{p.c.}\big).$$

Expression (17) can then be written:

$$\begin{aligned}
&\big(\mathrm{L}_{x^\lambda}[\Lambda;\,\Lambda]\big)(e)\big(\mathrm{d}\varphi_1(e);\mathrm{d}\varphi_2(e);\mathrm{d}\varphi_3(e)\big)\\
&\quad=2\Big(\big(\mathrm{L}_{x^\lambda}\big\{\{\varphi_1;\varphi_2\};\varphi_3\big\}\big)(e)+\text{p.c.}\Big)\\
&\quad=2\Big(\Big\langle\big(\mathrm{d}\big\{\{\varphi_1;\varphi_2\};\varphi_3\big\}\big)(e);x\Big\rangle+\text{p.c.}\Big)\\
&\quad=2\Big(\Big\langle\epsilon^{\mathrm{t}}\big(\epsilon^{\mathrm{t}}(\mathrm{d}\varphi_1(e)\otimes\mathrm{d}\varphi_2(e))\otimes\mathrm{d}\varphi_3(e)\big);x\Big\rangle+\text{p.c.}\Big)\\
&\quad=2\Big(\Big\langle\big[[\mathrm{d}\varphi_1(e);\mathrm{d}\varphi_2(e)]_{\mathfrak{g}^*};\mathrm{d}\varphi_3(e)\big]_{\mathfrak{g}^*};x\Big\rangle+\text{p.c.}\Big)=0,
\end{aligned}$$

where $[;\,]_{\mathfrak{g}^*}$ is the Lie bracket on $\mathfrak{g}^*$ defined by the mapping $\epsilon^{\mathrm{t}}:\mathfrak{g}^*\otimes\mathfrak{g}^*\to\mathfrak{g}^*$. We thus obtain $\big(\mathrm{L}_{x^\lambda}[\Lambda;\,\Lambda]\big)(e)=0$, and consequently $[\Lambda;\,\Lambda]=0$. ■

4. EXACT POISSON–LIE GROUPS AND THE GENERALIZED CLASSICAL YANG–BAXTER EQUATION

This problem consider the particular case of Poisson–Lie groups where cocycles l and ϵ are exact: $l=\tilde{\partial}r$; $\epsilon=\tilde{\delta}r$, $r\in\mathfrak{g}\otimes\mathfrak{g}$. These are basic to the understanding of the mathematical structure of the integrable dynamical systems connected with the Inverse Scattering Method. The generalized classical Yang–Baxter equation and the notions of quasi-triangular exact Lie bialgebras are brought into play. (Ref. [1], [2], [3], [7].)

4.1.

We begin by introducing a few notations.

(a) Let $\boldsymbol{M}$ be a differentiable manifold. Let $\wedge^2(\boldsymbol{M})$ be the spaces of contravariant skewsymmetric 2-tensors on $\boldsymbol{M}$. Let $\Lambda\in\wedge^2(\boldsymbol{M})$. We define the vector spaces homomorphism:

$$\#:\alpha\in\wedge_1(\boldsymbol{M})\longrightarrow\#\alpha\in\wedge^1(\boldsymbol{M})$$

by the relation:

$$\langle\beta;\#\alpha\rangle=\Lambda(\beta;\alpha);\qquad\text{for all }\alpha,\beta\in\wedge^1(\boldsymbol{M}).$$

(b) Let $\boldsymbol{G}$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $r\in\mathfrak{g}\otimes\mathfrak{g}$. The canonical isomorphism:

$$r\in\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\tilde{r}\in\mathrm{Hom}(\mathfrak{g}^*;\mathfrak{g})$$

is defined by the relation:

$$\langle \eta; \tilde{r}(\xi)\rangle = \langle \eta \otimes \xi; r\rangle; \qquad \text{for all } \xi, \eta \in \mathfrak{g}^*.$$

Let $\Lambda(g) = (T_e\lambda_g)^{\otimes 2} r$, $\Lambda(e) = r$ be the left-invariant 2-tensor defined by the element $r \in \mathfrak{g} \otimes \mathfrak{g}$. Then:

$$\#_g : \xi(g) \in T_g^*\boldsymbol{G} \longrightarrow \#_g\xi(g) \in T_g\boldsymbol{G}$$

and

$$\langle \eta(g); \#_g\xi(g)\rangle = \Lambda(g)\big(\eta(g); \xi(g)\big).$$

Let us suppose that $\xi(g)$ and $\eta(g)$ are left-invariant 1-forms:

$$\xi(g) = (T_g\lambda_{g^{-1}})^{\mathrm{t}} \cdot \xi(e); \qquad \eta(g) = (T_g\lambda_{g^{-1}})^{\mathrm{t}} \cdot \eta(e),$$

where $\xi(e), \eta(e) \in T_e^*\boldsymbol{G} \equiv \mathfrak{g}$. We then get:

$$\begin{aligned}
&\big\langle (T_g\lambda_{g^{-1}})^{\mathrm{t}} \cdot \eta(e); \#_g(T_g\lambda_{g^{-1}})^{\mathrm{t}} \cdot \xi(e)\big\rangle \\
&\quad = \big((T_e\lambda_g)^{\otimes 2} r\big)\big((T_g\lambda_{g^{-1}})^{\mathrm{t}}\eta(e); (T_g\lambda_{g^{-1}})^{\mathrm{t}}\xi(e)\big) \\
&\quad = r\big(\eta(e); \xi(e)\big) = \big\langle \eta(e); \tilde{r}\big(\xi(e)\big)\big\rangle,
\end{aligned}$$

for all $g \in \boldsymbol{G}$. Therefore $\langle \eta(e); \#_e\xi(e)\rangle = \langle \eta(e); \tilde{r}(\xi(e))\rangle$, whereby we get $\#_e = \tilde{r}$.

(c) The expressions of the function $[\Lambda; \Lambda'](\alpha_1; \alpha_2; \alpha_3)$ given in the following three questions will be needed subsequently.

4.2.
Let $\boldsymbol{M}$ be a differentiable manifold and $\Lambda, \Lambda' \in \wedge^2(\boldsymbol{M})$.
Prove that the Schouten bracket can be written as:

$$[\Lambda; \Lambda'](\alpha^1; \alpha^2; \alpha^3) = -\big(\langle \mathrm{L}_{\#\alpha^1}\alpha^2; \#'\alpha^3\rangle + \langle \mathrm{L}_{\#'\alpha^1}\alpha^2; \#\alpha^3\rangle + \text{p.c.}\big), \quad (18)$$

where # and #′ are the homomorphisms defined at the begining of this section.

Answer. In the natural coordinates corresponding to a local chart on $\boldsymbol{M}$, the components of the tensor $[\Lambda\ \Lambda]$ are:

$$[\Lambda; \Lambda']^{ijk} = \frac{1}{2!}\big(\epsilon^{ijk}_{nrs} \cdot \Lambda^{tn} \cdot \partial_t(\Lambda')^{rs} + \epsilon^{ijk}_{mns} \cdot \partial_t\Lambda^{mn}(\Lambda')^{ts}\big).$$

From the definition

$$\big(\alpha^1 \wedge \alpha^2 \wedge \alpha^3\big)_{ijk} = \epsilon^{abc}_{ijk}\alpha^1_a\alpha^2_b\alpha^3_c,$$

the left-hand side of the expression we need to prove is then:

$$\begin{aligned} i_{|\Lambda;\Lambda'|}(\alpha^1\wedge\alpha^2\wedge\alpha^3) &= \frac{1}{3!}[\Lambda;\,\Lambda']^{ijk}(\alpha^1\wedge\alpha^2\wedge\alpha^3)_{ijk} \\ &= \frac{1}{3!}\frac{1}{2!}\left(\epsilon^{ijk}_{nrs}\Lambda^{tn}\cdot\partial_t(\Lambda')^{rs}+\epsilon^{ijk}_{mns}\partial_t\Lambda^{mn}\cdot(\Lambda')^{ts}\right)\cdot\epsilon^{abc}_{ijk}\cdot\alpha^1_a\cdot\alpha^2_b\cdot\alpha^3_c. \end{aligned}$$

Using the relation:

$$\epsilon^{ijk}_{mns}\epsilon^{abc}_{ijk}=3!\epsilon^{abc}_{mns},$$

we get

$$\begin{aligned} i_{|\Lambda;\Lambda'|}(\alpha^1\wedge\alpha^2\wedge\alpha^3) = \big(&\Lambda^{tn}\cdot\partial_t(\Lambda')^{rs}\cdot\alpha^1_n\cdot\alpha^2_r\cdot\alpha^3_s \\ &+\partial_t\Lambda^{mn}\cdot(\Lambda')^{ts}\cdot\alpha^1_m\cdot\alpha^2_n\cdot\alpha^3_s+\text{p.c.}\big). \end{aligned} \tag{19}$$

From the definition of the Lie derivative, we have:

$$\left\langle \mathrm{L}_{\#\alpha^1}\alpha^2;\,\#'\alpha^3\right\rangle=\mathrm{L}_{\#\alpha^1}\left\langle\alpha^2;\,\#'\alpha^3\right\rangle-\left\langle\alpha^2;\,\mathrm{L}_{\#\alpha^1}\#'\alpha^3\right\rangle \tag{20}$$

and also:

$$\begin{aligned} \left\langle \mathrm{L}_{\#\alpha^1}\alpha^2;\,\#'\alpha^3\right\rangle &= \left(\partial_t\Lambda^{ik}\right)\cdot(\Lambda')^{tl}\cdot\alpha^1_k\cdot\alpha^2_i\cdot\alpha^3_l \\ &\quad+\Lambda^{tl}\cdot(\Lambda')^{ik}\cdot\alpha^1_l\cdot\left(\partial_t\alpha^2_i\right)\cdot\alpha^3_k+\Lambda^{ik}\cdot(\Lambda')^{tl}\cdot\left(\partial_t\alpha^1_k\right)\cdot\alpha^2_i\cdot\alpha^3_l. \end{aligned}$$

The right-hand side of the expression we want to prove is then:

$$\begin{aligned} &\left\langle \mathrm{L}_{\#\alpha^1}\alpha^2;\,\#'\alpha^3\right\rangle+\left\langle \mathrm{L}_{\#'\alpha^1}\alpha^2;\,\#\alpha^3\right\rangle+\text{p.c.} \\ &\quad=\Lambda^{tl}\cdot\left(\partial_t(\Lambda')^{ik}\right)\cdot\alpha^1_k\cdot\alpha^2_i\cdot\alpha^3_l+\left(\partial_t\Lambda^{ik}\right)\cdot(\Lambda')^{tl}\cdot\alpha^1_k\cdot\alpha^2_i\cdot\alpha^3_l \\ &\qquad+\Lambda^{tl}\cdot(\Lambda')^{ik}\cdot\alpha^1_l\cdot\left(\partial_t\alpha^2_i\right)\cdot\alpha^3_k+\Lambda^{ik}\cdot(\Lambda')^{tl}\cdot\alpha^1_l\left(\partial_t\alpha^2_i\right)\cdot\alpha^3_k \\ &\qquad+\Lambda^{ik}\cdot(\Lambda')^{tl}\cdot\left(\partial_t\alpha^1_k\right)\cdot\alpha^2_i\cdot\alpha^3_l+\Lambda^{tl}\cdot(\Lambda')^{ik}\cdot\left(\partial_t\alpha^1_k\right)\cdot\alpha^2_i\cdot\alpha^3_l+\text{p.c.} \end{aligned}$$

All the terms in this expression containing a derivative $\partial_a\alpha^i_c$ cancel out, because Λ and Λ' are skewsymmetric tensors. We then get:

$$\begin{aligned} &\left\langle \mathrm{L}_{\#\alpha^1}\alpha^2;\,\#'\alpha^3\right\rangle+\left\langle \mathrm{L}_{\#'\alpha^1}\alpha^2;\,\#\alpha^3\right\rangle+\text{p.c.} \\ &\quad=\Lambda^{tl}\cdot\left(\partial_t(\Lambda')^{ik}\right)\cdot\alpha^1_k\cdot\alpha^2_i\cdot\alpha^3_l+\left(\partial_t\Lambda^{ik}\right)\cdot(\Lambda')^{tl}\cdot\alpha^1_k\cdot\alpha^2_i\cdot\alpha^3_l+\text{p.c.} \end{aligned}$$

Equality (18) follows from this relation in conjunction with (19) and (20). ■

4.3.
Let Λ, $\Lambda' \in \wedge^2_\lambda(G)$ be left-invariant tensor fields on G defined as:

$$\Lambda(g) = (T_e\lambda_g)^{\otimes 2} r; \qquad \Lambda'(g) = (T_e\lambda_g)^{\otimes 2} r',$$

where $r, r' \in \mathfrak{g} \wedge \mathfrak{g}$. Let $\mathfrak{A}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, and let P^{23} be the permutation of the second and third factors in the tensor product $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})$. Let us define the following elements of $\mathfrak{A}(\mathfrak{g})^{\otimes 3}$

$$r_{12} = r \otimes 1; \qquad r_{13} = P^{23} \cdot r_{12}; \qquad r_{23} = 1 \otimes r$$

and also the bracket:

$$[r_{ij}; r'_{mn}] = r_{ij} r'_{mn} - r'_{mn} r_{ij}; \quad i, j, m, n = 1, 2, 3,$$

using the product of $\mathfrak{A}(\mathfrak{g})$. Let us also define the element

$$\begin{aligned}[r; r'] = -\big(&[r_{12}; r'_{13}] + [r_{12}; r'_{23}] + [r_{13}; r'_{23}] \\ &+ [r'_{12}; r_{13}] + [r'_{12}; r_{23}] + [r'_{13}; r_{23}]\big).\end{aligned}$$

Prove that $[r; r']$ is an element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$, and following a similar argument as in 3.2, *prove that*

$$[\Lambda; \Lambda'](g) = (T_e\lambda_g)^{\otimes 3}[r; r'].$$

4.4.
Using the notations and the result of the preceding question, by taking into account together the result in question 4.1, *prove the following equality:*

$$[r; r'](\alpha^1; \alpha^2; \alpha^3) = -\Big(\big\langle\alpha^3; \big[\tilde{r}(\alpha^1); \tilde{r}'(\alpha^2)\big] + \big[\tilde{r}'(\alpha^1); \tilde{r}(\alpha^2)\big]\big\rangle + \text{p.c.}\Big), \tag{21}$$

where $\alpha^1, \alpha^2, \alpha^3 \in \mathfrak{g}^*$.

Answer. When no confusion results, we will identify the element $\alpha_i \in \mathfrak{g}^* \equiv T_e^*(G)$, $i = 1, 2, 3$, with the invariant 1-form $(T_g\lambda_{g^{-1}})^t\alpha^i \equiv \alpha^i(g)$ as required in the following proof.
From relations (20) and (18) in 4.1, we can write:

$$\begin{aligned}[\Lambda; \Lambda'](\alpha^1; \alpha^2; \alpha^3) = &\Big(\big\langle\alpha^2; \mathrm{L}_{\#\alpha^1}\#'\alpha^3\big\rangle + \big\langle\alpha^2; \mathrm{L}_{\#'\alpha^1}\#\alpha^3\big\rangle + \text{p.c.}\Big) \\ &- \Big(\mathrm{L}_{\#\alpha^1}\big\langle\alpha^2; \#'\alpha^3\big\rangle + \mathrm{L}_{\#'\alpha^1}\big\langle\alpha^2; \#\alpha^3\big\rangle + \text{p.c.}\Big).\end{aligned} \tag{22}$$

The terms in the second line of this expression are zero because the functions $\langle\alpha^i; \#\alpha^j\rangle(g)$ are constant: every $\alpha^i(g)$ is a left-invariant 1-form, and

$\#$, $\#'$ are respectively defined through the left-invariant tensor fields $\Lambda(g)$, $\Lambda'(g)$. We thus get:

$$\begin{aligned}&[\Lambda;\Lambda'](\alpha^1;\alpha^2;\alpha^3)(e)\\&\quad=\langle\alpha^2(e);[\tilde r(\alpha^1(e));\tilde r'(\alpha^3(e))]\rangle+\langle\alpha^2(e);[\tilde r'(\alpha^1(e));\tilde r((\alpha^3)(e))]\rangle\\&\qquad+\text{p.c.}\end{aligned}$$

This expresion is precisely (21), since $[\Lambda;\Lambda'](e)=[r;r']$, 4.2, and since, at this point, we make the identification $\alpha^i\equiv\alpha^i(e)$, $i=1,2,3$. ■

(d) One of the results from question 4.3 is that the 2-tensor

$$\Lambda_\lambda(g)=(T_e\lambda_g)^{\otimes 2}r;\quad r\in\mathfrak{g}\wedge\mathfrak{g},$$

verifies the equivalence

$$[\Lambda_\lambda;\Lambda_\lambda]=0\Longleftrightarrow[r^{12};r^{13}]+[r^{12};r^{23}]+[r^{13};r^{23}]=0,$$

i.e., $(\boldsymbol{G};\Lambda_\lambda)$ is a left-invariant Poisson structure if and only if the element $r\equiv\Lambda(e)\in\mathfrak{g}\wedge\mathfrak{g}$ is a solution of the classical Yang–Baxter equation $[r;r]=0$. It is obvious that this Poisson structure can not be a Lie–Poisson structure unless $r=0$. A Lie–Poisson structure on $\boldsymbol{G}$ is defined, 1.4, by a 2-tensor Λ such that $[\Lambda;\Lambda]=0$, and Λ verifies the Drinfeld property:

$$\Lambda(g\cdot h)=T_h\lambda_g\cdot\Lambda(h)+T_g\rho_h\cdot\Lambda(g).$$

Even if the 2-tensor $\Lambda_\lambda(g)$ does not verify this property, it may be that the 2- tensor

$$\Lambda(g)=\Lambda_\lambda(g)-\Lambda_\rho(g)=(T_e\lambda_g)^{\otimes 2}r-(T_e\rho_g)^{\otimes 2}r\tag{23}$$

does. The sing $-$ in this definition is due to the fact that $\Lambda(e)=0$, from the Drinfeld condition.

From the definition of Λ, we get:

$$\Lambda(g\cdot h)=(T_e\lambda_{g\cdot h})^{\otimes 2}\cdot r-(T_e\rho_{g\cdot h})^{\otimes 2}\cdot r.$$

Also:

$$T_h\lambda_g\cdot\Lambda(h)=T_h\lambda_g\big(\Lambda_\lambda(h)-\Lambda_\rho(h)\big)=(T_e\lambda_{g\cdot h})\cdot r-T_e(\lambda_g\cdot\rho_h)\cdot r$$

and

$$T_g\rho_h\cdot\Lambda(g)=T_g\rho_h\big(\Lambda_\lambda(g)-\Lambda_\rho(g)\big)=T_e(\rho_h\cdot\lambda_g)\cdot r-T_e\rho_{g\cdot h}\cdot r.$$

From the last three expresions, we can see that the tensor Λ defined in (23) verifies the Drinfeld condition. Note that, in this computation, we have not had to suppose that Λ_λ, and consequently Λ_ρ, defines a Poisson structure on $\boldsymbol{G}$; i.e., the classical Yang–Baxter equation $[r;r]=0$ does not hold. Were we to suppose that this equation does hold, then $[\Lambda_\lambda; \Lambda_\lambda] = 0$ and $[\Lambda_\rho; \Lambda_\rho] = 0$. From question 3.2, we then get

$$[\Lambda; \Lambda] = [\Lambda_\lambda; \Lambda_\lambda] - [\Lambda_\lambda; \Lambda_\rho] - [\Lambda_\rho; \Lambda_\lambda] + [\Lambda_\rho; \Lambda_\rho] = 0.$$

Therefore, the pair $(G; \Lambda = \Lambda_\lambda - \Lambda_\rho)$, where $\Lambda_\lambda(g) = T_e\lambda_g \cdot r$, $\Lambda_\rho(g) = T_e\rho_g \cdot r$ and $[r; r] = 0$, is a Poisson–Lie group.

We no longer suppose with respect to expression (23) that $[r; r] = 0$, or equivalently Λ_λ and Λ_ρ defines a Poisson structure on G.

From the question 1.5, the mapping:

$$l : g \in G \longrightarrow (T_g\rho_{g^{-1}})^{\otimes 2}\Lambda(g)$$

is therefore a 1-cocycle on G, with values in $\mathfrak{g} \otimes \mathfrak{g}$, relative to the adjoint action of G on $\mathfrak{g} \otimes \mathfrak{g}$. We can easily compute this cocycle as determined by $r \in \mathfrak{g} \otimes \mathfrak{g}$:

$$l(g) = (T_g\rho_{g^{-1}})^{\otimes 2}\big((T_e\lambda_g)^{\otimes 2} \cdot r - (T_e\rho_g)^{\otimes 2} \cdot r\big).$$

Thus,

$$l(g) = (\mathrm{Ad}_g)^{\otimes 2} \cdot r - r.$$

In terms of the cohomology on G, with values on $\mathfrak{g} \otimes \mathfrak{g}$, relative to the adjoint representation, this last expression tells us that the 1-cocycle l is the cobondary of the zero cochain $r \in \mathfrak{g} \otimes \mathfrak{g}$:

$$l = \tilde{\partial} r.$$

The corresponding 1-cocycle on $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$ corresponding to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$ is $\epsilon = T_e l$. Therefore, for all $x \in \mathfrak{g}$,

$$\begin{aligned}\epsilon(x) &= T_e l(x) = \frac{\mathrm{d}}{\mathrm{d}t} l(\exp tx)\Big|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big((\mathrm{Ad}_{\exp tx})^{\otimes 2} \cdot r - r\big)\Big|_{t=0} = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x) \cdot r.\end{aligned}$$

Then: $\epsilon(x) = \widehat{\mathrm{ad}_x} \cdot r = \tilde{\delta} r$, where $\widehat{\mathrm{ad}}$ stands for the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$.

(e) The tensor Λ defined by the expression (23) verifies the Drinfeld property. Now we will look for a necesary and sufficient condition for $r \in \mathfrak{g} \otimes \mathfrak{g}$ whereby the tensor Λ in (23) also verifies $[\Lambda; \Lambda] = 0$, i.e., whereby the condition on r for $(G; \Lambda = \Lambda_\lambda - \Lambda_\rho)$ is a Poisson–Lie group.

From questions 1.8 and 3.6, we can see that this is equivalent to finding the condition on r such that $\epsilon^{\mathrm{t}} : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$, where $\epsilon = \tilde{\delta} r$, defines a Lie algebra structure on $\mathfrak{g}^*$.

First, let us introduce some additional notations. From the definition of isomorphism: $\sim : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathrm{Hom}(\mathfrak{g}^*; \mathfrak{g})$, 4.1 (b) we have, for all $\xi, \eta \in \mathfrak{g}^*$, $x \in \mathfrak{g}$,

$$\begin{aligned}\big\langle \xi ; \widetilde{\tilde{\delta} r(x)} \eta \big\rangle &= \big\langle \xi \otimes \eta ; \tilde{\delta} r(x) \big\rangle \\ &= -\langle \mathrm{ad}^*{}_x \, \xi \otimes \eta + \xi \otimes \mathrm{ad}^*{}_x \, \eta ; r \rangle \\ &= \big\langle \xi ; \mathrm{ad}_x \cdot \tilde{r}(\eta) - \tilde{r}(\mathrm{ad}^*{}_x \cdot \eta) \big\rangle .\end{aligned}$$

Therefore:

$$\widetilde{\tilde{\delta} r(x)} = \mathrm{ad}_x \circ \tilde{r} + \tilde{r} \circ (\mathrm{ad}_x)^{\mathrm{t}}. \tag{24}$$

We simply define $\tilde{\delta}\tilde{r} \in \mathrm{Hom}(\mathfrak{g}^*; \mathfrak{g})$ as:

$$\tilde{\delta}\tilde{r}(x) \equiv \widetilde{\tilde{\delta} r(x)} = \mathrm{ad}_x \circ \tilde{r} - \tilde{r} \circ \mathrm{ad}^*{}_x \, .$$

4.5.
Let $r \in \mathfrak{g} \otimes \mathfrak{g}$. Let

$$[;]^r_{\mathfrak{g}^*} : \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \tag{25}$$

be the bilinear mapping defined by the exact 1-cocycle $\epsilon = \tilde{r} : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$. Prove that:

$$[\xi ; \eta]^r_{\mathfrak{g}^*} = (\tilde{\delta} r)^{\mathrm{t}} (\xi \otimes \eta) = \mathrm{ad}^*{}_{\tilde{r}(\eta)} \cdot \xi + \mathrm{ad}^*{}_{\tilde{r}^{\mathrm{t}}(\xi)} \cdot \eta . \tag{26}$$

Answer. We have, for all $x \in \mathfrak{g}$,

$$\begin{aligned}\big\langle [\xi ; \eta]^r_{\mathfrak{g}^*} ; x \big\rangle &= \cdots = \big\langle \xi ; \mathrm{ad}_x \cdot \tilde{r}(\eta) - \tilde{r}(\mathrm{ad}^*{}_x \cdot \eta) \big\rangle \\ &= \cdots = \big\langle \mathrm{ad}^*{}_{\tilde{r}(\eta)} \, \xi ; x \big\rangle - \langle \mathrm{ad}_{\tilde{r}^{\mathrm{t}}(\xi)} \, x ; \eta \rangle .\end{aligned}$$

Then:

$$[\xi ; \eta]^r_{\mathfrak{g}^*} = \mathrm{ad}^*{}_{\tilde{r}(\eta)} \cdot \xi + \mathrm{ad}^*{}_{\tilde{r}^{\mathrm{t}}(\xi)} \cdot \eta .$$

Obviously, the definition of $\tilde{r}^{\mathrm{t}} : \mathfrak{g}^* \to (\mathfrak{g}^*)^* \equiv \mathfrak{g}$ is as follows:

$$\begin{aligned}\big\langle \tilde{r}^{\mathrm{t}}(\beta) ; \alpha \big\rangle &= \big\langle \beta ; \tilde{r}(\alpha) \big\rangle = \big\langle \beta \otimes \alpha ; r \big\rangle \\ &= \big\langle P^{12}(\alpha \otimes \beta) ; r \big\rangle = \big\langle \alpha \otimes \beta ; P^{12}(r) \big\rangle = \big\langle \alpha ; \widetilde{P^{12}(r)} \beta \big\rangle .\end{aligned}$$

Thus $\tilde{r}^{\mathrm{t}} = \widetilde{P^{12}(r)}$ where the components of $r \in \mathfrak{g} \otimes \mathfrak{g}$ permute under the action of P^{12}. ■

We now determine a necesary and sufficient condition on making the bilinear form, $[;]_{\mathfrak{g}^*}$, on $\mathfrak{g}^*$, defined by the exact 1-cocycle $\epsilon = \tilde{\delta} r$, skewsymmetric.

4.6.

Let $r = s + a \in \mathfrak{g} \otimes \mathfrak{g}$ where s and a are respectively the symmetric and skewsymmetric components of r.

Prove that the mapping (25) *is skewsymmetric if and only if $s \in \mathfrak{g} \otimes \mathfrak{g}$ is a zero cocycle: $\tilde{\delta} s = 0$, i.e.,*

$$\mathrm{ad}_x \cdot \tilde{s} = \tilde{s} \cdot \mathrm{ad}^*{}_x; \quad \textit{for all } x \in \mathfrak{g}, \tag{27}$$

or equivalently, if and only if

$$\mathrm{ad}^*{}_{\tilde{s}(\xi)} \cdot \eta + \mathrm{ad}^*{}_{\tilde{s}(\eta)} \cdot \xi = 0; \quad \textit{for all } \xi, \eta \in \mathfrak{g}^*. \tag{28}$$

Moreover, supposing $\tilde{\delta} s = 0$, then $\tilde{\delta} r = \tilde{\delta} a$, and

$$[\xi; \eta]^r_{\mathfrak{g}^*} = [\xi; \eta]^a_{\mathfrak{g}^*} = \mathrm{ad}^*{}_{\tilde{a}(\eta)} \, \xi - \mathrm{ad}^*{}_{\tilde{a}(\xi)} \, \eta. \tag{29}$$

Answer. Clearly, $\tilde{a}^{\mathrm{t}} = -\tilde{a}$ and $\tilde{s}^{\mathrm{t}} = \tilde{s}$.
(a) The equality (26) is therefore:

$$\begin{aligned} [\xi; \eta]^r_{\mathfrak{g}^*} &= \mathrm{ad}^*{}_{\tilde{r}(\eta)} \cdot \xi + \mathrm{ad}^*{}_{\tilde{r}^{\mathrm{t}}(\xi)} \cdot \eta \\ &= \left(\mathrm{ad}^*{}_{\tilde{s}(\eta)} \cdot \xi + \mathrm{ad}^*{}_{\tilde{s}(\xi)} \cdot \eta\right) + \left(\mathrm{ad}^*{}_{\tilde{a}(\eta)} \cdot \xi - \mathrm{ad}^*{}_{\tilde{a}(\xi)} \cdot \eta\right). \end{aligned}$$

Therefore, $[;]^r_{\mathfrak{g}^*}$ is skewsymmetric if and only if:

$$\mathrm{ad}^*{}_{\tilde{s}(\xi)} \cdot \eta + \mathrm{ad}^*{}_{\tilde{s}(\eta)} \cdot \xi = 0.$$

Expression (29) thereby is proved.
(b) On the other hand:

$$\begin{aligned} 0 &= \left\langle \mathrm{ad}^*{}_{\tilde{s}(\eta)} \cdot \xi + \mathrm{ad}^*{}_{\tilde{s}(\xi)} \cdot \eta; x \right\rangle = \cdots = \\ &= \left\langle \eta; \mathrm{ad}_x \cdot \tilde{s}(\xi)\right\rangle + \left\langle \xi; \mathrm{ad}_x \cdot \tilde{s}(\eta)\right\rangle = \left\langle \eta; \mathrm{ad}_x \cdot \tilde{s}(\xi)\right\rangle - \left\langle \eta; \tilde{s} \cdot \mathrm{ad}^*{}_x \, \xi\right\rangle. \end{aligned}$$

Therefore $\mathrm{ad}_x \circ \tilde{s} = \tilde{s} \circ \mathrm{ad}^*{}_x$, for all $x \in \mathfrak{g}$. The conditions (27), (28), (29) are then equivalent.
(c) Also, by definition, $\tilde{\delta} s(x) = \widehat{\mathrm{ad}_x} \cdot s$. By (24) this is equivalent to:

$$\tilde{\delta} \tilde{s} = \widetilde{\tilde{\delta}(x)} = \mathrm{ad}_x \circ \tilde{s} - \tilde{s} \circ \mathrm{ad}^*{}_x .$$

The relations (27), (28), (29) are then equivalent to $\tilde{\delta} s = 0$, i.e., to the invariance of s by the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$:

$$\widehat{\mathrm{ad}_x} \cdot s = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x) \cdot s = 0.$$

These results are equivalent to the following one: $[;]^r_{g^*}$ is skewsymmetric if and only if it is defined by the skewsymmetric part of r. ∎

In order to get a necessary and sufficient condition on r for the skewsymmetric mapping $[;]^r_{\mathfrak{g}*}$ to verify the Jacobi identity, we shall write an equivalent of expression (21) in 4.4 for the tensor $[a;a]$, making use of the form $[;]^a_{\mathfrak{g}*} \equiv [;]^r_{\mathfrak{g}*}$.

4.7.
Let $a \in \mathfrak{g} \wedge \mathfrak{g}$. *Let* $[a;a] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ *be the element defined in* 4.2 *and* 4.3. *Let* $[;]^a_{\mathfrak{g}*}$ *be the skewsymmetric bilinear form in* 4.5.
Prove that the expression (21) *in* 4.4 *for* $r = r' = a$ *can be written as*:

$$\frac{1}{2}\widehat{\mathrm{ad}_x}[a;a](\xi^1;\xi^2;\xi^3) = \langle [[\xi^1;\xi^2]^a_{\mathfrak{g}*};\xi^3]^a_{\mathfrak{g}*};x\rangle + \text{p.c.} \tag{30}$$

where: $\widehat{\mathrm{ad}_x} : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}^{\otimes 3}$ *is the adjoint representation of* $\mathfrak{g}$ *on* $\mathfrak{g}^{\otimes 3}$, $\widehat{\mathrm{ad}_x} = \mathrm{ad}_x \otimes 1 \otimes 1 + 1 \otimes \mathrm{ad}_x \otimes 1 + 1 \otimes 1 \otimes \mathrm{ad}_x$, *for all* $x \in \mathfrak{g}$.
In consequence, $[;]^a_{\mathfrak{g}*}$ *verifies the Jacobi identity if and only if the tensor* $[a;a]$ *is invariant by the adjoint representation of* $\widehat{\mathrm{ad}}$ *of* $\mathfrak{g}$ *on* $\mathfrak{g}^{\otimes 3}$, *equivalently, if and only if* $[a;a]$ *is a zero cocycle in the corresponding cohomology.*

Answer. We develop both sides of the expression (30); by the result in 4.4 and the Jacobi identity for $\mathfrak{g}$, we find that both sides coincide.

From the expression (29), the definition of ad, ad* and the Jacobi identity for $\mathfrak{g}$, we get:

$$\begin{aligned}\langle [[\xi^1;\xi^2]^a_{\mathfrak{g}*};\xi^3]^a_{\mathfrak{g}*};x\rangle + \text{p.c.} &= \cdots = \\ &= -\langle \mathrm{ad}^*{}_x\,\xi^3; \tilde{a}(\mathrm{ad}^*{}_{\tilde{a}(\xi^1)}\,\xi^2 - \mathrm{ad}^*{}_{\tilde{a}(\xi^2)}\,\xi^1)\rangle \\ &\quad + \langle \mathrm{ad}^*{}_x \cdot \xi^3; [\tilde{a}(\xi^1);\tilde{a}(\xi^2)]\rangle + \text{p.c.}\end{aligned} \tag{31}$$

Furthermore

$$\begin{aligned}\widehat{\mathrm{ad}_x}[a;a](\xi^1;\xi^2;\xi^3) &= -[a;a](\mathrm{ad}^*{}_x\,\xi^1;\xi^2;\xi^3) \\ &\quad - [a;a](\xi^1;\mathrm{ad}^*{}_x\,\xi^2;\xi^3) - [a;a](\xi^1;\xi^2;\mathrm{ad}^*{}_x\,\xi^3) \\ &= -[a;a](\mathrm{ad}^*{}_x\,\xi^1;\xi^2;\xi^3) + \text{p.c.}\end{aligned}$$

(in the last equality, we make use of the fact that $[a;a] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$).
We now develop the last term by referring to expression (21) in 4.3. We thus have:

$$\begin{aligned}-[a;a](\xi^1;\xi^2;\xi^3) + \text{p.c.} &= 2\langle \xi^3; [\tilde{a}(\mathrm{ad}^*{}_x\,\xi^1);\tilde{a}(\xi^2)]\rangle \\ &+ 2\langle \mathrm{ad}^*{}_x\,\xi^3; [\tilde{a}(\xi^1);\tilde{a}(\xi^2)]\rangle + 2\langle \xi^3; [\tilde{a}(\xi^1);\tilde{a}(\mathrm{ad}^*{}_x\,\xi^2)]\rangle + \text{p.c.}\end{aligned}$$

From the equality $\tilde{a} = -\tilde{a}^{\mathrm{t}}$, we also get:

$$\langle \xi^3; [\tilde{a}(\mathrm{ad}^*{}_x \xi^1); \tilde{a}(\xi^2)]\rangle = -\langle \tilde{a}(\mathrm{ad}^*{}_{\tilde{a}(\xi^2)} \xi^3); \mathrm{ad}^*{}_x \xi^1\rangle,$$

and

$$\langle \xi^3; [\tilde{a}(\xi^1); \tilde{a}(\mathrm{ad}^*{}_x \xi^2)]\rangle = \langle \tilde{a}(\mathrm{ad}^*{}_{\tilde{a}(\xi^1)} \xi^3); \mathrm{ad}^*{}_x \xi^2\rangle.$$

Finally, we obtain:

$$\begin{aligned}\widehat{\mathrm{ad}_x}[a; a](\xi^1; \xi^2; \xi^3) &= -[a; a](\xi^1; \xi^2; \xi^3) + \text{p.c.}\\ &= 2\langle \mathrm{ad}^*{}_x \xi^3; [\tilde{a}(\xi^1); \tilde{a}(\xi^2)]\rangle - 2\langle \tilde{a}(\mathrm{ad}^*{}_{\tilde{a}(\xi^1)} \xi^2); \mathrm{ad}^*{}_x \xi^3\rangle\\ &\quad + 2\langle \tilde{a}(\mathrm{ad}^*{}_{\tilde{a}(\xi^2)} \xi^1); \mathrm{ad}^*{}_x \xi^3\rangle + \text{p.c.}\end{aligned}$$

If we compare this expression with expression (31), we obtain expression (30). ■

We can summarize the results in 4.4, 4.5 and 4.6 as follows:

4.8.
Let $\mathfrak{g}$ be a Lie algebra and $r = s + a \in \mathfrak{g} \otimes \mathfrak{g}$ where s and a are respectively the symmetric and skewsymmetric parts of r. The set $(\mathfrak{g}; [\,;\,]; \epsilon = \tilde{\delta} r)$ is a (exact) Lie bialgebra if and only if $s \in \mathfrak{g} \otimes \mathfrak{g}$ and $[a; a] \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ are 0-cocycles relative to the adjoint representations of $\mathfrak{g}$ on $\mathfrak{g}^{\otimes 2}$ and $\mathfrak{g}^{\otimes 3}$ respectively.

4.9. Definitions
(a) *We will say that the skewsymmetric tensor $a \in \mathfrak{g} \wedge \mathfrak{g}$ is a solution of the generalized classical Yang–Baxter equation if the tensor $[a; a] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is a 0-cocycle in the Chevalley cohomology of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$, equivalently with values in $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$, and relative to the adjoint representation, $\widehat{\mathrm{ad}}$, of $\mathfrak{g}$ on $\mathfrak{g}^{\otimes 3}$, equivalently with values in $\mathfrak{g}^{\wedge 3}$. Also equivalently, if the tensor $[a; a]$ is $\widehat{\mathrm{ad}}$ invariant.*
Whenever $a \in \mathfrak{g} \wedge \mathfrak{g}$ is a solution of the generalized classical Yang–Baxter equation, we will say that the (exact) Lie bialgebra $(\mathfrak{g}; [\,;\,]; \epsilon = \tilde{\delta} a)$ is quasitriangular.
(b) *Whenever $[a; a] = 0$, we will say that a is a solution of the classical Yang–Baxter equation, and the Lie bialgebra $(\mathfrak{g}; [\,;\,]; \epsilon = \tilde{\delta} a)$ will be called triangular.*

4.10.
Let G be a simply-connected Lie group. Let r be any element in $\mathfrak{g} \otimes \mathfrak{g}$. Let $\epsilon = \tilde{\delta} r : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ be the exact 1-cocycle on $\mathfrak{g}$, with values on $\mathfrak{g} \otimes \mathfrak{g}$, relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$.
Prove that the 1-cocycle $l : G \to \mathfrak{g} \otimes \mathfrak{g}$ on G with values on $\mathfrak{g} \otimes \mathfrak{g}$ relative to the adjoint representation of G on $\mathfrak{g} \otimes \mathfrak{g}$ such that $T_e l = \tilde{\delta} r = \epsilon$ is the exact 1-cocycle: $l = \tilde{\partial} r$:

$$l(g) = \mathrm{Ad}^{\otimes 2} \cdot r - r; \quad \textit{for all } g \in G.$$

Answer. From the action of $\tilde{\partial}$ on the 0-cochains, it is clear that $l = \tilde{\partial} r$; consequently, l is an exact 1-cocycle. In particular,

$$l(g \cdot h) = l(g) + \mathrm{Ad}^{\otimes 2} \cdot l(h); \quad \text{for all } g, h \in G.$$

We need only prove that $T_e l = \tilde{\delta} r$, by reason of the bijective correspondence between the 1-cocycles for the Chevalley cohomology of $\mathfrak{g}$ and for the cohomology of the Lie group G. But this is the same proof as the one at the end of section 4.4. ■

4.11.
Let $r = s + a \in \mathfrak{g} \otimes \mathfrak{g}$. Let us suppose that $(\mathfrak{g}; [;]; \epsilon = \tilde{\delta} r (= \tilde{\delta} a))$ is a (quasitriangular) Lie bialgebra. (Equivalently, let us suppose that $\widehat{\mathrm{ad}}_x \cdot s = 0$ and $\widehat{\mathrm{ad}}_x [a; a] = 0$.) Let G be the simply connected Lie group whose Lie algebra is $\mathfrak{g}$. Let $(G; \Lambda)$ be the Poisson–Lie group determined by the above Lie bialgebra.
Prove that:

$$\Lambda(g) = \Lambda_r^{\lambda}(g) - \Lambda_r^{\rho}(g) = \Lambda_a^{\lambda}(g) - \Lambda_a^{\rho}(g); \quad \textit{for all } g \in G,$$

where

$$\Lambda_r^{\lambda}(g) = (T_e \lambda_g)^{\otimes 2}(r); \qquad \Lambda_r^{\rho}(g) = (T_e \rho_g)^{\otimes 2}(r).$$

Answer. By 4.10, the 1-cocycle on G, l, corresponding to the 1-cocycle $\epsilon = \tilde{\delta} r$ on $\mathfrak{g}$ is:

$$l(g) = \mathrm{Ad}_g^{\otimes 2} \cdot r - r = \mathrm{Ad}_g^{\otimes 2} \cdot a - a.$$

($\widehat{\mathrm{ad}_x} s = 0$ for all $x \in \mathfrak{g} \Rightarrow \mathrm{Ad}_g^{\otimes 2} s = s$ for all $g \in G$).
From 3.6, we know that the tensor field

$$\Lambda(g) = (T_e \rho_g)^{\otimes 2} l(g)$$

defines the Lie–Poisson structure on G corresponding to the Lie bialgebra in the question. So:

$$\begin{aligned}\Lambda(g) &= (T_e\rho_g)^{\otimes 2}(\mathrm{Ad}_g\, r - r) = \cdots = \\ &= (T_e\lambda_g)^{\otimes 2} r - (T_e\rho_g)^{\otimes 2} r = (T_e\lambda_g)^{\otimes 2} a - (T_e\rho_g)^{\otimes 2} a \\ &= \Lambda_r^\lambda(g) - \Lambda_r^\rho(g) = \Lambda_a^\lambda(g) - \Lambda_a^\rho(g).\end{aligned}$$

This tensor field is precisely the one that could be conjectured after the reasoning concerning expression (23). ■

As a consequence of this result, we obtain:

4.12.
Let $\{e_\mu;\ \mu = 1, \ldots, n\}$ be any basis on $\mathfrak{g}$, and $r = r^{\mu\nu} e_\mu \otimes e_\nu \in \mathfrak{g} \otimes \mathfrak{g}$, any element such that $(\mathfrak{g}; [\,;\,]; \epsilon = \tilde{\delta} r)$ is a quasitriangular Lie bialgebra. The corresponding Poisson bracket on the Poisson–Lie group $(G; \Lambda)$ determined by the above Lie bialgebra is:

$$\{\varphi; \psi\} = r^{\mu\nu}(\mathrm{L}_{x_\mu^\lambda}\varphi \cdot \mathrm{L}_{x_\nu^\lambda}\psi - \mathrm{L}_{x_\mu^\rho}\varphi \cdot \mathrm{L}_{x_\nu^\rho}\psi); \quad \textit{for all } \varphi, \psi \in \mathcal{C}^\infty(G)$$

where $x_\mu^\lambda(g) = T_e\lambda_g \cdot e_\mu$, $x_\nu^\rho(g) = T_e\rho_g \cdot e_\nu$.

Note, 4.6, that this bracket does not depend on the symmetric part of r since $\mathrm{Ad}_g\, s = s$, for all $g \in G$.

4.13.
Let $a, a' \in \mathfrak{g} \wedge \mathfrak{g}$ and $\Lambda_{a'}^\lambda$, Λ_a^ρ be respectively left and right-invariant 2-tensors as in 4.11. Let us suppose that a and a' are solutions of the generalized Yang–Baxter equation.
Prove that the 2-tensor $\Lambda_a^\lambda - \Lambda_{a'}^\rho$ is a Poisson tensor on the simply connected Lie group G with Lie algebra $\mathfrak{g}$, if and only if $[a; a] = [a'; a']$. In the paricular case in which a and a' are solutions of the classical Yang–Baxter equation, the tensors Λ_a^λ, $\Lambda_{a'}^\rho$, $\Lambda_a^\lambda + \Lambda_{a'}^\rho$, $\Lambda_a^\lambda - \Lambda_{a'}^\rho$ are Poisson tensors.

Answer. From the equality $\widehat{\mathrm{ad}_x} \cdot [a; a] = 0$ for all $x \in \mathfrak{g}$, we get

$$(\mathrm{Ad}_g)^{\otimes 3} \cdot [a; a] = [a; a]; \qquad \text{for all } g \in G.$$

If $[\Lambda_a^\lambda - \Lambda_{a'}^\rho; \Lambda_a^\lambda - \Lambda_{a'}^\rho] = 0$, we have: $[\Lambda_a^\lambda; \Lambda_a^\lambda] + [\Lambda_{a'}^\rho; \Lambda_{a'}^\rho] = 0$ given that, by 3.2, $[\Lambda_a^\lambda; \Lambda_{a'}^\rho] = 0$. The tensors $[\Lambda_a^\lambda; \Lambda_a^\lambda]$ and $[\Lambda_{a'}^\rho; \Lambda_{a'}^\rho]$ are respectively left-invariant and right-invariant. Moreover:

$$\left[\Lambda_a^\lambda; \Lambda_a^\lambda\right](g) = (T_e\lambda_g)^{\otimes 3}[a; a]$$

and

$$\left[\Lambda^{\rho}_{a'}; \Lambda^{\rho}_{a'}\right](g) = -(T_e\rho_g)^{\otimes 3}[a'; a'].$$

By adding these two expressions, we get:

$$(T_e\rho_g)^{\otimes 3}\big(\mathrm{Ad}_g[a; a] - [a'; a']\big) = 0; \quad \text{for all } g \in \boldsymbol{G}.$$

Thus, $\mathrm{Ad}_g^{\otimes 3}[a; a] = [a'; a']$, and ultimately, $[a; a] = [a'; a']$.
The converse now becomes obvious. ■

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4. VOLUME OF THE SPHERE S^n

A Supplement to Problem V.4 (pp. 240–243)

Use the expression for the metric of S^n in polar coordinates to compute its volume ω_n by induction on n when n is even and when n is odd. Show that in each case

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Answer. The volume ω_n of S^n is given by the integral

$$\omega_n = \int_\Omega \tau_n, \quad \text{with } \tau_n = (\sin\theta^n)^{n-1}\tau_{n-1}\,\mathrm{d}\theta^n,\ \tau_1 = \sin\theta^1\,\mathrm{d}\theta^1.$$

Therefore

$$\omega_n = I_n\omega_{n-1}, \qquad I_n \equiv \int_0^\pi (\sin\theta)^{n-1}\,\mathrm{d}\theta.$$

Integration by parts gives if $n > 1$

$$I_n = \frac{n-1}{n} I_{n-2}$$

hence, since $I_1 = \pi$ and $I_2 = 2$

$$I_{2p} = \frac{(2p-1)\times(2p-3)\cdots 3\times 1}{2p\times(2p-2)\cdots 4\times 2},$$
$$I_{2p+1} = \frac{2p\times(2p-2)\cdots 4\times 2}{(2p-1)\times(2p-3)\cdots 3\times 1}$$

and one finds using induction

$$\omega_{2p} = \frac{2(2\pi)^p}{(2p-1)(2p-3)\cdots 3\times 1}, \qquad \omega_{2p+1} = \frac{2\pi^{p+1}}{p!}.$$

Suppose that $n-1 = 2p+1$. Then by the definition of the gamma function

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Suppose that $n-1 = 2p$. We have

$$\omega_{2p} = \frac{2\pi^p}{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots\frac{3}{2}\times\frac{1}{2}}.$$

By the property $\Gamma(x+1) = x\Gamma(x)$ of the gamma function and $\Gamma(\frac{1}{2}) = \pi^{1/2}$, we have

$$\frac{1}{2} = \frac{\Gamma(3/2)}{\pi^{1/2}}, \ \ldots, \ \frac{n}{2} - 1 = \frac{\Gamma(n/2)}{\Gamma(n/2-1)},$$

from which we deduce that the given expression of ω_{n-1} holds also when n is odd.

5. TEICHMULLER SPACES

INTRODUCTION AND DEFINITIONS

Teichmuller spaces

A diffeomorphism φ of a riemannian manifold (M, g) is a confeomorphism if there exists on M a positive function f such that:

$$\varphi^* g = fg.$$

It is known that the confeomorphisms of an n-dimensional (connected) riemannian manifold form a Lie group of dimension at most $\frac{1}{2}(n+1)(n+2)$ if $n \geq 2$. It has been proved (cf. J. Ferrand 1995 and references therein) that the group of confeomorphisms of (M, g) is the isometry group of some conformal space (M, fg) except if (M, g) is a metric sphere S^n or an euclidean space E^n. Analogous properties do not necessarily hold for pseudo riemannian metrics.

The isometry group of a compact riemannian manifold is compact. The conformal group of S^n is not compact.

Two riemannian metrics g and g' on a manifold M are called **conformally equivalent** if there exists a diffeomorphism φ homotopic to the identity and a positive function f such that on M

$$\varphi^* g = fg'.$$

When $n = 2$ the space of conformally inequivalent metrics on a compact manifold M has the remarquable property to be isomorphic to a finite dimensional vector space. We will study this space.

Teichmuller space

The **Teichmuller space** $\mathcal{T}(M)$ of a compact (connected, without boundary) smooth 2-dimensional manifold M is the space of conformally inequivalent smooth riemannian metrics which can be put on M. It has come to play an important role in geometry and physics, in particular in string theory. We shall give, in the case where M is oriented, the main steps of a proof that $\mathcal{T}(M)$ can be given the structure of a finite dimensional smooth manifold diffeomorphic to $\mathbb{R}^{6G-6}$, with $G > 1$ the genus of M.

genus

The **genus** of a 2-dimensional connected compact (with or without boundary) manifold M is the maximum number G of cuts one can inflict to M along a simple curve (continuous injection of a circle into the interior of M) so that the resulting manifold remains connected (in some definitions the genus is $G+1$). It can be proved that two compact 2-dimensional manifolds without boundary are homeomorphic if and only if they have the same genus and are either both orientable or both non orientable. The sphere S^2 has genus zero, the torus T^2 has genus 1, an orientable "n holes surface" has genus n. The Betti number b_1 of an orientable connected compact

2-dimensional manifold (without boundary) of genus G is $b_1 = 2G$. Its Euler–Poincaré number (cf. Problem 6 chapter Vbis "Euler Poincaré...") is $\chi = b_0 - b_1 + b_2 = 2 - 2G$. Recall that if R is the scalar curvature of a riemannian metric on M, then $\chi = \frac{1}{2\pi}\int_M R\,\mathrm{d}v$.

1. *Denote by $\mathcal{M}$ [resp. $\mathcal{M}_s$] the space of smooth [resp. in the Sobolev space H_s] riemannian metrics on the 2-dimensional connected compact without boundary smooth manifold M. Denote by $\mathcal{P}$ [resp. $\mathcal{P}_s$] the space of smooth [resp. H_s] strictly positive functions on M.*
a) *Show that the relation $\mathcal{R}$ given in $\mathcal{M}_s \times \mathcal{M}_s$ by*

$$(g', g) \in \mathcal{R} \text{ if there exists } f \in \mathcal{P}_s \text{ such that } g' = fg$$

is an equivalence relation (p. 5) if $s > 1$. Denote by $\mathcal{M}_s/\mathcal{P}_s$ the set of these equivalent classes.
b) *Admit the fact (Kazdan and Warner) that the equation on (M, g) for the function σ*

$$\Delta_g \sigma - e^{\sigma} = R$$

has one and only one solution $\sigma \in H_s$ for each given function $R \in H_{s-2}$ if $g \in H_s$, $s \geq 2$, and $\chi < 0$. Give a bijection between $\mathcal{M}_s/\mathcal{P}_s$ and $\mathcal{M}_s^{-1}$, space of H_s metrics with scalar curvature equal to -1 on a given oriented M with $G > 1$.
c) *Show that $\mathcal{M}_s^{-1}$ is a C^∞ submanifold of $\mathcal{M}_s$. Give its tangent space at a point g.*

Answer 1. a) $\mathcal{R}$ is defined if $s > 1$ because H_s is an algebra if $s > \frac{n}{2} = 1$. It is reflexive, symmetric, and transitive.
b) If M is 2-dimensional the scalar curvatures of two conformal metrics g and fg with $f = e^{\sigma}$ are linked by (cf. p. 351)

$$R(g) \equiv \Delta_g \sigma + e^{\sigma} R(\mathrm{e}^{\sigma} g)$$

hence if g is given we shall have $R(\mathrm{e}^{\sigma} g) = -1$ if and only if σ satisfies the equation

$$\Delta_g \sigma - e^{\sigma} = R(g)$$

this equation has one and only one solution σ on the oriented manifold M if $G > 1$, since then $\chi < 0$.
If g and g' are conformal, $g' = e^{\lambda} g$, we have, by uniqueness of the solution of the above equation, $R(e^{\sigma'} g') = -1$ if and only if $\sigma = \sigma' + \lambda$, therefore to g and g' are associated the same metric $g \in \mathcal{M}_s^{-1}$. The mapping $\mathcal{M}_s/\mathcal{P}_s \to \mathcal{M}_s^{-1}$ thus defined is bijective.

c) $\mathcal{M}_s$ is an open set in the vector space of H_s symmetric 2-tensors, if $s \geq 2$. The subset $\mathcal{M}_s^{-1}$ is defined by the equation $R(g) = -1$. Multiplication properties of H_s spaces show that $g \mapsto R(g)$ is a differentiable mapping from $\mathcal{M}_s$ into H_s. Its derivative at g is the linear mapping from covariant 2-tensors into scalars given by, since $R \equiv g^{ij} R_{ij}$

$$h \mapsto DR(g).h = -h^{ij} R_{ij} + g^{ij} DR_{ij}.h, \quad h^{ij} = g^{ik} g^{jl} h_{kl}.$$

We find using the expression of the Ricci tensor for a riemannian manifold of arbitrary dimension n

$$DR_{ij}.h = \nabla_k (D\Gamma^k_{ij}.h) - \nabla_i (D\Gamma^k_{jk}.h), \quad \nabla \text{ covariant derivative in } g,$$

$$D\Gamma^k_{ij}.h = \tfrac{1}{2} g^{kl} (\nabla_j h_{il} + \nabla_i h_{lj} - \nabla_l h_{ij})$$

hence

$$DR(g).h = -h^{ij} R_{ij} - \Delta_g \operatorname{tr} h + \nabla.\nabla.h, \quad \operatorname{tr} h = g^{ij} h_{ij}, \ \nabla.\nabla.h = \nabla_i \nabla_j h^{ij}.$$

One shows that $R(g)$ is a submersion when $R(g) < 0$, hence $\mathcal{M}_s^{-1}$ a submanifold of $\mathcal{M}_s$ (cf. p. 239, 242), by proving that $DR(g)$ is then a mapping onto H_{s-2}. Indeed if $h_{ij} = \frac{1}{n} \tau g_{ij}$ then

$$DR(g).h \equiv -\frac{1}{n} \tau R - \left(1 - \frac{1}{n}\right) \Delta_g \tau = k$$

has a solution $\tau \in H_s$ for any $k \in H_{s-2}$.

Note. When M is 2-dimensional we have $R_{ij} = \frac{1}{2} g_{ij} R$, therefore for all h

$$DR(g).h \equiv -\frac{1}{2} R \operatorname{tr} h - \Delta_g \operatorname{tr} h + \nabla.\nabla.h.$$

2) *Show that the group $\mathcal{D}^{s+1}$ of H_{s+1} diffeomorphisms of M acts on $\mathcal{M}_s^{-1}$. Split the tangent space at g to $\mathcal{M}_s^{-1}$ into a tangent space to the orbit of g through the action of $\mathcal{D}^{s+1}$ and an L_2 orthogonal complement. Show that this orthogonal complement is finite-dimensional.*

Answer 2. The metrics g and $\varphi^* g$ have scalar curvatures related by

$$R(\varphi^* g) = \varphi^* R(g) \equiv \big(R(g)\big) \circ \varphi^{-1}.$$

Therefore if one belongs to $\mathcal{M}_s^{-1}$ so does the other.

The orbit $\mathcal{O}_g$ of g by $\mathcal{D}$ is the space of metrics isometric to g. The generator of a one parameter group of isometries of g is a Killing vector field.

The tangent space at g to $\mathcal{O}_g$ is therefore the set of symmetric 2-covariant tensors on M of the form $L_X g$, L_X the Lie derivative with respect to a vector field X on M. Every 2-covariant H_s tensor field admits a unique L_2 orthogonal decomposition

$$h = \underline{h} + L_X g \quad \text{where } \nabla.\underline{h} = 0$$

because the operators $X \mapsto L_X g$ and $h \mapsto \nabla.h$ are L_2 adjoint, the first one has injective principal symbol (p. 397), and the operator $X \mapsto \nabla.L_X g$ is elliptic with kernel $\{Y, L_Y g = 0\}$.
Suppose that $h \in T_g\mathcal{M}_s^{-1}$ that is $DR(g).\underline{h} = 0$. By the previous argument or direct verification, $DR(g).L_X g = 0$, hence $DR(g).h = 0$, which reduces to

$$\Delta_g \operatorname{tr}\underline{h} - \operatorname{tr}\underline{h} = 0 \quad \text{which implies } \operatorname{tr}\underline{h} = 0.$$

The L_2 orthogonal complement of the tangent space to $\mathcal{O}_g$ in $T_g\mathcal{M}_s^{-1}$ is therefore the space Σ_2^{TT} of transverse ($\nabla.h = 0$), traceless, symmetric covariant, H_s, 2 tensors on M.
On a 2-dimensional manifold the operator $\nabla.$ on traceless symmetric 2-tensors has injective symbol, i.e., is elliptic, its kernel is finite-dimensional. To determine its dimension we shall use results of analysis on complex manifolds.

3. a) *Show that an almost complex* (*cf.* p. 331) *structure on a* 2*-dimensional manifold M is always a complex structure.*
b) *Show that the linear mapping* $J : T_x M \to T_s M$ *defined by the mixed tensor* $-g^{\$}\mu_g$ *where* $g^{\$}$ *is the contravariant tensor associated to* g *and* μ_g *the volume* 2*-form* (*i.e.* $J^i_j = -g^{ik}\mu_{kj}$) *defines a complex structure on M, which is the same for two conformal metrics.*

Answer 3. a) When $n = 2$ the integrability condition for an almost complex structure to be complex is identically satisfied.
b) J is the same for g and fg because $(fg)^{\$} = f^{-1}g$ and $\mu_{fg} = f\mu_g$, if $n = 2$.
J defines an almost complex (hence here complex) structure if $J^2 = -\mathrm{Id}$. This property is frame independent. Take an orthonormal frame in $T_x M$. Then $g^{\$ij} = \delta_{ij}$ and $\mu_{ij} = \frac{1}{2}\epsilon^{12}_{ij}$, hence

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J^2 = -\mathrm{Id}.$$

4. *Show that the space* Σ_2^{TT} *of transverse traceless* 2*-covariant symmetric tensors on* (M, g) *is isomorphic to the space of* 2*-covariant tensors on* (M, J) *of type* $(0, 2)$ *and holomorphic* (cf. Chapter V, Problem 294).

Answer 4. Take coordinates where $g_{ij} = f\delta_{ij}$ (always possible locally) then J has the expression given above, $z = x^1 + \mathrm{i}x^2$ is a local coordinate for the complex structure (M, J). A 2-covariant tensor of type $(0, 2)$ reads

$$H = F(z, z)\,\mathrm{d}z \otimes \mathrm{d}z \text{ it is holomorphic if } F \text{ does not depend on } z.$$

Let $h \in \Sigma_2^{\mathrm{TT}}$. A straightforward computation, using also $\operatorname{tr} h = 0$ (cf. Fisher and Tromba), gives, with $g_{ij} = f\delta_{ij}$

$$(\nabla.h)_i = f^{-1}\frac{\partial}{\partial x^j}h_{ij}.$$

In the chosen coordinates set $h_{11} = -h_{22} = u$, $h_{12} = h_{21} = -v$, then $\nabla.h = 0$ reads as the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x^2} = -\frac{\partial v}{\partial x^1}, \qquad \frac{\partial u}{\partial x^1} = \frac{\partial v}{\partial x^2},$$

conditions for $u + \mathrm{i}v$ to be an analytic function of $z = x + \mathrm{i}y$.
We have

$$\begin{aligned} h &= u\big(\mathrm{d}x^1 \otimes \mathrm{d}x^1 - \mathrm{d}x^2 \otimes \mathrm{d}x^2\big) - v\big(\mathrm{d}x^1 \otimes \mathrm{d}x^2 + \mathrm{d}x^2 \otimes \mathrm{d}x^1\big) \\ &\equiv \mathcal{R}\mathrm{e}\big\{(u + \mathrm{i}v)\,\mathrm{d}z \otimes \mathrm{d}z\big\}. \end{aligned}$$

The $(0, 2)$ analytic tensor H defined locally by $H = (u + \mathrm{i}v)\,\mathrm{d}z \otimes \mathrm{d}z$ extends to a holomorphic tensor defined on (M, J) by

$$H = h + \mathrm{i}Jh, \quad \text{where } (Jh)_{ij} = J_i^k h_{kj},$$

Jh is a symmetric 2-tensor, as can be checked in the chosen frame.

CONCLUSION

Riemann–Roch theorem

The **Riemann–Roch theorem** says that on a 2-dimensional compact orientable manifold the vector space of $(0, 2)$ holomorphic 2-tensor is $6G - 6$ dimensional, $G > 1$ the genus of M. Therefore Σ_2^{TT} is also $6G - 6$ dimensional.

Teichmuller space

Bochner

The **Teichmuller space** $\mathcal{T}$ is the set of equivalence classes $\mathcal{M}_s^{-1}/\mathcal{D}_0^{s+1}$, $\mathcal{D}_0^{s+1}$ the subset of $\mathcal{D}^{s+1}$ of diffeomorphisms homotopic to the identity. It has been proved by Bochner that $\mathcal{D}_0$ acts freely on a riemannian manifold (M, g) with negative definite Ricci curvature (i.e. $\varphi^* g = g$ only if

φ is the identity map). This property is also a consequence of the theorem of Eells and Sampson saying that there is a harmonic mapping in each homotopy class of maps from (M, g) into (M', g') if the Riemann curvature of g' is non positive. A 2-dimensional manifold with $R(g) = -1$ has $\text{Ricci}(g) = -\frac{1}{2}g$, negative definite. Hence $\mathcal{D}_0$ acts freely on $\mathcal{M}_s^{-1}$. It is possible to endow $\mathcal{T}$ with the structure of a C^∞ manifold of dimension $6G - 6$ (dimension of the L_2 orthogonal complement of the tangent to the orbit $\mathcal{O}_g$) diffeomorphic to $\mathbb{R}^{6G-6}$. Eells and Sampson

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6. YAMABE PROPERTY ON COMPACT MANIFOLDS

INTRODUCTION

It is an old result that every 2-dimensional compact riemannian manifold admits a conformal metric with constant scalar (gaussian) curvature.

A riemannian manifold (M, g) is said to have the **Yamabe property** if there exixts on M a metric g' conformal to g with constant scalar curvature R'. This property is important in geometry for the classification of compact manifolds, and in General Relativity in relation with the solution of the constraints (cf. Y. Choquet-Bruhat and J. York, 1979). The original proof of Yamabe contained a flaw and it has required a considerable effort from the mathematicians (Trudinger, Aubin, Schoen) to obtain results valid in all cases and for every dimension. Yamabe property

In the following (M, g) denotes a smooth, compact, n-dimensional riemannian manifold with $n \geq 3$. We shall see that all such manifolds possess the Yamabe property.

1. *Show that* (M, g) *has the Yamabe property with*

$$g' \equiv \varphi^{4/(n-2)} g$$

if and only if φ *satisfies the equation, with* R *the scalar curvature of* g,

$$k_n \Delta_g \varphi - R\varphi + R' \varphi^{(n+2)/(n-2)} = 0, \quad k_n \equiv 4(n-1)/(n-2). \quad (1)$$

Answer 1. Straightforward calculation (cf. p. 351 and problem 6, chapter V, p. 247).

2. It can be proved (Palais, 1968) that a linear operator of order m on a compact smooth riemannian manifold (M, g), with smooth coefficients a_k (the hypotheses on smoothness can be weakened), $L \equiv \sum_{k=0}^{m} a_k \nabla^k$, which is elliptic (p. 397), is a Fredholm (p. 401) operator from the Sobolev space (p. 486) W_p^{s+m} into W_p^s if $s \geq 0$, $1 < p < \infty$, in particular it satisfies:

elliptic operators

Isomorphism theorem for elliptic operators. If the linear elliptic operator L with smooth coefficients is injective, then it is an isomorphism from W_p^{s+m} onto W_p^s for any (integer) $s \geq 0$, $1 < p < \infty$ (the inverse of L is the Volterra convolution (p. 471) with the Green function of L on M).

Use this theorem and the Leray–Schauder degree theory (p. 563–565 *and the application on p.* 591) *to prove that if $R < 0$ then (M, g) has the Yamabe property with $R' < 0$.*

Answer 2. Suppose $R < 0$ and let R' be some negative constant. We show the existence of a solution $\varphi > 0$ of (1) on (M, g) as follows. Consider the linear equation, with $t \in [0, 1]$ and c some number

$$\Delta_g u - u = -c + tf(v), \qquad f(v) \equiv -v + c + Rv - R' v^{(n+2)/(n-2)}. \quad (2_t)$$

Suppose $v > 0$ and $v \in W_p^s$, $s > n/p$, then (problem 3, chapter VI) $f(v) \in W_p^s$. Therefore (2_t) has one and only one solution $u \in W_p^{s+2}$, and there exists a constant C such that for all these v

$$\|u\|_{W_p^{s+2}} \leq C\|c + tf(v)\|_{W_p^s}.$$

Consider a bounded subset $\Omega \subset W_p^s$ defined by the two inequalities

$$\|v\|_{W_p^s} < K, \qquad 0 < a < v(x) < b, \ \forall x \in M.$$

Ω is an open subset of W_s^p if $C^0 \subset W_p^s$, i.e. if $s > n/p$.

Each mapping $\mathcal{F}_t : \Omega \to W_p^s$ defined by $v \mapsto u$ is a compact mapping (p. 563) because the injection $W_p^{s+2} \to W_p^s$ is compact (extension of the Kondrakov theorem quoted in Problem 2, chapter VI, p. 383). The fixed points of $\mathcal{F}_1$ are the functions u such that $1 + u$ is solution of (1); $\mathcal{F}_0$ has one fixed point in Ω, $u = c$, if $\|c\|_{W_p^s} < K$, i.e. $c(\mathrm{vol}\, M)^{1/p} < K$, and $a < c < b$. The existence of a solution of (1) (non necessarily unique) will follow from the Leray–Schauder theory if we prove that no mapping $\mathcal{F}_t : \overline{\Omega} \equiv \Omega \cup \partial\Omega$, $t \in [0, 1]$, has a fixed point on the boundary $\partial\Omega$ of Ω: we must prove that a solution of (2_t) with $v = u$ and satisfying

$$\|u\|_{W_p^s} \leq K, \quad 0 < a \leq v(x) \leq b, \ x \in M,$$

verifies the corresponding strict inequalities. A C^2 function u such that $u(x) \geq a$ on M can attain its minimum a at a point x of M only if at that point $\Delta u \geq 0$ (maximum principle, p. 500). A fixed point u of $\mathcal{F}_t$ is C^2 since $u \in W_p^{s+2} \subset C^2$ if $s > n/p$; it will verify $\Delta u < 0$ when $u = a$ if

$$P(a) \equiv Ra - R' a^{(n+2)/(n-2)} < 0.$$

If $R < 0$ on M and R' is a negative constant this inequality is satisfied as soon as $a < (|R'| \inf_M |R|)^{(n-2)/4}$. We shall have $P(b) > 0$, hence u cannot attain its upper value b, if $b > (|R'| \sup_M |R|)$. We choose $a > 0$ and $b > a$ satisfying these inequalities. Then $0 < a < u < b$ on M. We choose c between a and b.

On the other hand, if u is a fixed point of $\mathcal{F}_t$ in $\overline{\Omega}$ we have $f(u) \in W_p^0 \equiv L_p$ with an L_p norm depending only on the data and the choice of b. The inequality recalled for elliptic equations shows that there exists a constant K' such that for all fixed points of the mappings $\mathcal{F}_t$ satisfy

$$\|u\|_{W_p^2} \leq K'.$$

If we take p such that $p > n/2$ we can choose $s = 2$ in the definition of Ω and K such that $K > K'$. We have completed the proof of the existence of a solution $u \in W_p^4$, $u > 0$, of (1). One shows that this solution is smooth by remarking that, since W_p^2, $p > n/2$, is an algebra and $f(u)$ is a C^∞ function of u when $u > a > 0$ the right hand side of (1) is uniformly bounded in W_p^4 norm, hence we could have taken $s = 4$ in the definition of Ω, or by iteration any $s \geq 2$.

Remark. We can choose $s = 1$ if $p > n$.

3. A **Yamabe functional** is a mapping $J_q : \varphi \mapsto J_q(\varphi)$ given by: Yamabe functional

$$J_q(\varphi) \equiv \frac{\int_M (k_n |\nabla\varphi|^2 + R\varphi^2)\, dv}{(\int_M \varphi^{2q} dv)^{1/q}} \tag{3}$$

with $|\nabla\varphi|^2 \equiv g(\nabla\varphi, \nabla\varphi)$ and dv is the volume element of g.

a) *Show that J_q is defined and bounded below in the open set D of H_1,*

$$D \equiv \{\varphi \in H_1,\ \varphi \not\equiv 0\} \quad \text{if } 1 \leq q \leq n/(n-2).$$

Denote by μ_q the infimum

$$\mu_q = \inf_{\varphi \in D} J_q(\varphi).$$

b) *Show that* $\varphi \mapsto J_q(\varphi)$ *is* C^1 *in* D. *Show that a critical point* φ_q *of* J_q *in* D (*in particular a minimum if there is one*) *satisfies a* **Yamabe equation**:

Yamabe equation

$$k_n \Delta_g \varphi_q - R\varphi_q = \lambda_q \varphi_q^{2q-1}, \quad \lambda_q \text{ some number.}$$

Answer 3a. We have $L_p \subset H_1$ if $1 \le p \le 2n/(n-2)$ (cf. problem 2, chapter 6, p. 379). J_q is defined on D, since also $\|\varphi\|_{L_{2q}} \not\equiv 0$. It is bounded below if $q \ge 1$ because

$$J_q \ge -\left| \int_M R\varphi^2 \, dv \right| / \|\varphi\|^2_{L_{2q}} \ge -\|R\|_{L_{2q'}}, \quad \frac{1}{q'} = 1 - \frac{1}{q}.$$

Answer 3b. It is straightforward to check that J_q is C^1 on D, its first derivative at φ is the linear mapping $H_1 \to \mathbb{R}$ by $h \mapsto J'_q(\varphi).h$ given by

$$J'_q(\varphi).h = \left(\|\varphi\|_{L_{2q}}\right)^{-2q} 2 \int_M \left\{ k_n g(\nabla\varphi, \nabla h) + R\varphi h - \lambda_q (\varphi^2)^{q-1} \varphi h \right\} dv \tag{4}$$

with $\lambda_q = J_q(\varphi)(\int_M \varphi^{2q} \, dv)^{-1+1/q}$.
The condition for φ to be a critical point of J_q, i.e. $J'_q(\varphi).h = 0$ for all $h \in H_1$ is equivalent to (4), taken in the sense of distributions.

4. a) *Show that*

$$\mu_q = \inf_{\varphi \in D, \ \|\varphi\|_{L_{2q}} = 1} J_q(\varphi).$$

b) *Show that* J_q *admits a positive minimizing sequence, functions* $\varphi_{(N)}$, $N \in \mathbb{N}$, *such that* $J_q(\varphi_{(N)}) - \mu_q < N^{-1}$, $\varphi_{(N)} \in D \cap \{\|\varphi\|_{L_{2q}} = 1\}$, *and* $\varphi_{(N)} \ge 0$.

c) *Show that if* $q < n(n-2)$ *there exists a subsequence of* $\{\varphi_{(N)}\}$ *which converges to a function* φ_q *in* D *satisfying a Yamabe equation with* $\lambda_q = \mu_q$.

d) *Show that* $\varphi_q > 0$.

Answer 4a. For any q, J_q is invariant by homothetic rescaling of φ.

Answer 4b. The non existence of $\varphi_{(N)}$, given N, would lead to a contradiction, since μ_q is the infimum of J_q. We can choose $\varphi_{(N)} \ge 0$ because μ_q

is also the infimum of J_q restricted to $\varphi \geq 0$. Indeed if $\varphi \in H_1$ then also $|\varphi| \in H_1$ since $|\nabla|\varphi|| = |\nabla\varphi|$, and $J_q(\varphi) = J_q(|\varphi|)$. Therefore

$$\mu_q = \inf_{\varphi \in D} J_q(|\varphi|) \geq \inf_{\varphi \in D,\ \varphi \geq 0} J_q(\varphi) \geq \mu_q.$$

Answer 4c. Since $\|\varphi_{(N)}\|_{L_{2q}} = 1$, $q \geq 1$, we have $\|\varphi_{(N)}\|_{L_2}$ uniformly bounded when M is compact since (Holder inequality, p. 53)

$$\|\varphi\|_{L_2} \leq \|\varphi\|_{L_{2q}} \|1\|_{L_{2q'}}, \quad \frac{1}{q} + \frac{1}{q'} = 1, \ \|1\|_{L_{2q'}} = \left(\mathrm{Vol}(M, g)\right)^{1/2q'}.$$

On the other hand $J_q(\varphi_{(N)}) < \mu_q + 1$. Hence the norms $\|\varphi_{(N)}\|_{H_1}$ are uniformly bounded. It is known (**Kondrakov theorem**, p. 384) that the imbedding $H_1 \to L_p$, $p < 2n(n-2)$, is compact; hence from the bounded sequence $\varphi_{(N)}$ in H_1 one can extract a subsequence denoted also $\varphi_{(N)}$ which converges strongly in L_{2q} to a function φ_q. On the other hand it is known (p. 29) that a bounded subset of the Hilbert space H_1 is relatively compact in the weak topology, therefore there exists a subsequence $\{\varphi_{(N)}\}$ which converges weakly in H_1 to a limit which must necessarily coincide with φ_q since both are a fortiori limits in $\mathcal{D}'$ (p. 454), hence $\varphi_q \in H_1$, and $\varphi_q \in D$ since $\varphi_q \not\equiv 0$ (recall that $\|\varphi_q\|_{L_{2q}} = 1$ because φ_q is the strong limit in L_{2q} of the $\varphi_{(N)}$). We show that $\mu_q = J_q(\varphi_q)$, that is φ_q realizes the infimum of J_q and hence satisfies the Yamabe equation by using the facts

Kondrakov theorem

$$\lim_{N\to\infty} \|\varphi_{(N)}\|_{L_2} = \|\varphi_q\|_{L_2} \quad \text{by the strong convergence in } L_2.$$

$$\|\nabla\varphi_q\|^2_{L_2} = \lim_{N\to\infty} \int_M \nabla\varphi_{(N)}.\nabla\varphi_q \, dv \quad \text{by the weak convergence in } H_1,$$

therefore

$$\|\nabla\varphi_q\|^2_{L_2} \leq \lim_{N\to\infty} \sup \|\nabla\varphi_{(N)}\|_{L_2} \|\nabla\varphi_{(N)}\|_{L_2}$$

hence $J_q(\varphi_q) \leq \lim_{N\to\infty} J_q(\varphi_{(N)}) = \mu_q$ *and* $J_q(\varphi_q) = \mu_q$, *infimum of* J_q.

Answer 4d. Since $\varphi_q \in H_1$, $\varphi_q \in L_{2n/(n-2)}$ hence $\varphi_q^{2q-1} \in L_{p_0}$, with $p_0 = \frac{2n}{(n-2)(2q-1)} > 1$ if $q \leq \frac{n}{n-2}$.
The elliptic isomorphism theorem shows that $\varphi_q \in W^2_{p_0}$ since it is solution of the Yamabe equation. Now $W^2_{p_0} \subset L_{p_1}$, $p_1 = \frac{np_0}{n-2p_0} > \frac{2n}{n-2}$ if $q < \frac{n}{n-2}$. By induction we obtain a sequence of imbeddings $\varphi_q \in W^2_{p_N}$ with p_N a strictly increasing sequence. We obtain the smoothness of φ_q by taking P_N

large enough. We deduce from the Yamabe equation and the boundedness of φ_q an inequality of the form

$$\Delta_g \varphi_q - C\varphi_q \geq 0, \quad \text{with } C \geq \sup_M k_n^{-1} |R - \mu_q \varphi_q^{2q-1}| \geq 0.$$

By the maximum principle $\varphi_q \geq 0$ cannot attain its possible minimum zero without being identically zero.

Yamabe invariant

5. *Show that* $\mu \equiv \mu_{n/(n-2)}$, *is a conformal invariant.* It is called the **Yamabe invariant**.

Answer 5. When M is compact without boundary C^∞ is dense in H_1 therefore

$$\mu = \inf_{\varphi \in C^\infty,\ \varphi \not\equiv 0} J(\varphi), \quad J(\varphi) \equiv J_{n/(n-2)}(\varphi).$$

Suppose g' is conformal to g and set $g' \equiv \psi^{4/(n-2)} g$, with $\psi > 0$ a C^∞ function. If $\varphi \in C^\infty$, $\varphi \not\equiv 0$, so does $\varphi' = \varphi\psi^{-1}$ and conversely. Recall (cf. Problem 6, chapter V, conformal ..., p. 247)

$$k_n \Delta_g \varphi - R\varphi \equiv \psi^{(n+2)/(n-2)}(k_n \Delta_{g'} \varphi' - R'\varphi'), \quad \varphi \equiv \psi\varphi'. \tag{3}$$

The volume elements of g and g' are linked by

$$\mathrm{d}v \equiv \psi^{-2n/(n-2)} \mathrm{d}v'$$

hence for all functions f

$$\int_M \varphi\psi^{(n+2)/(n-2)} f \, \mathrm{d}v = \int_M \varphi' f \, \mathrm{d}v'.$$

Multiplying (3) by φ and integrating on M gives after integration by parts

$$\int_M \left(k_n |\nabla\varphi|^2 + R\varphi^2\right) \mathrm{d}v = \int_M \left(k_n |\nabla\varphi'|^2 + R'\varphi'^2\right) \mathrm{d}v'$$

hence $J(g, \varphi) \equiv J(g', \psi^{-1}\varphi)$ from which the conclusion follows.

6. The reasoning of previous paragraphs 3 does not give the proof of the existence of a conformal factor $\varphi \equiv \varphi_{n/(n-2)}$ since the injection $H_1 \to L_{2n/(n-2)}$ is not compact. Originally Yamabe attempted to prove the existence of a conformal factor by constructing φ as a limit of the φ_q when q tends to $n/(n-2)$, this can be made to work easily when $\mu < 0$.

a) *Show that by homothetic rescaling of* g *the volume of* (M, g) *can always be made equal to* 1.

Answer 6a. $\operatorname{Vol}(M, k^2 g) \equiv k^n \operatorname{Vol}(M, g)$.

b) *Show that* μ_q *is a continuous function of* $q \in (2, n/(n-2)]$. *Show that if* $1 \leq q_0 \leq q \leq n/(n-2)$ *and* $\operatorname{Vol}(M, g) = 1$ *then* $|\mu_q| \geq |\mu_{q_0}|$.

c) *Prove that if* $\mu < 0$ *there exists a sequence* $\{\varphi_{q_N}\}$, *which converges in* H_1 *to a strictly positive function* φ *when* q_N *converges to* $n/(n-2)$. *Conclude that* (M, g) *has then the Yamabe property.*
Hint. Show that the φ_q, $q \in [q_0, n/(n-2)[$, *are uniformly bounded.*

d) *Show that a sufficient condition for* $\mu < 0$ *is* $\int_M R \, dv < 0$.

Answer 6b. If $\psi \in D \cap C^\infty$ it is straightforward to prove that $J_q(\psi)$ is a continuous function of $q \in [1, n/(n-2)]$ and to deduce from this property the continuity in q of μ_q, infimum of J_q (cf. Aubin, 1982, p. 128).
On the other hand if $\operatorname{Vol}(M, g) = 1$ and $q_0 \leq q$ we have $\|\psi\|_{L_{2q_0}} \leq \|\psi\|_{L_{2q}}$ by the Holder inequality, therefore

$$|J_q(\psi)| \geq |J_{q_0}(\psi)| \quad \text{and consequently} \quad |\mu_q| \geq |\mu_{q_0}| \text{ if } q_0 \leq q.$$

It results that μ_q, $q \in (1, n/(n-2)]$, is either everywhere positive, everywhere zero or everywhere negative.

Answer 6c. Suppose $\mu < 0$. Then $\mu_q < 0$, $q \in [1, n/(n-2)]$.
We have seen that φ_q, $1 \leq q < n/(n-2)$, is continuous on M hence attains its maximum m_q at a point $x_q \in M$. At such a point we have $(\Delta\varphi_q)(x_q) \leq 0$. We deduce then from the Yamabe equation satisfied by φ_q, with $\mu_q < 0$

$$m_q \leq \frac{-R(x_q)}{|\mu_q|} \leq \frac{\sup|R|}{|\mu_{q_0}|} = K,$$
$$\text{number independent of} \quad q \in [q_0, n/(n-2)[.$$

One way of completing the proof is to deduce from the above bound, the Yamabe equation and the elliptic isomorphism theorem an inequality of the form

$$\|\varphi_q\|_{H_2} \leq C, \quad q \in [q_0, n/(n-2)[,$$

where C depends on the data and μ_{q_0} but not on q, $q_0 \leq q < n/(n-2)$.
Since H_2 is compact in H_1 we can extract from $\{\varphi_q,\ q_0 \leq q < n/(n-2)\}$ a sequence which converges, in the H_1 norm, to a function denoted $\varphi_{n/(n-2)}$

when q converges to $n/(n-2)$. This function satisfies the corresponding Yamabe equation and is in H_2, hence finally C^∞. It is proved to be strictly positive, like the other φ_q.

Answer 6d. Take $\psi \equiv 1$ on M. Then $\int_M R\,dv < 0$ implies $J(\psi) < 0$ hence $\mu < 0$.

7. When μ_q is positive we still have

$$\|\varphi_q\|_{H_1} \leq K, \quad \text{if } \|\varphi_q\|_{L_{2q}} = 1 \text{ (hence } \|\varphi_q\|_{L_2} \leq 1 \text{ if } \mathrm{Vol}(M, g) = 1).$$

Therefore there still exists a sequence extracted from $\{\varphi_q, q \in [1, n/(n-2)[\}$ which converges to a function $\varphi \in H_1$, weakly in H_1 and strongly in L_2, but the φ_q are not uniformly bounded without further hypothesis.

Show that the existence for $q \in [q_0, n/(n-2)[$ of a uniform L^r bound of φ_q, for some $r > 2n/(n-2)$, is sufficient to prove that φ is smooth, satisfies the Yamabe equation, and is positive.

Answer 7. If $\|\varphi_q\|_{L_r} = \{\|\varphi_q^{2q-1}\|L_{r/(2q-1)}\}^{1/(2q-1)} \leq C$, $r > 2q-1$, then by the Yamabe equation satisfied by φ_q and the Sobolev imbedding theorem there exist other constants C such that

$$\|\varphi_q\|_{W^2_{r/(2q-1)}} \leq C, \qquad \|\varphi_q\|_{L_{r_1}} \leq C, \qquad r_1 = \frac{nr}{n(2q-1) - 2r}.$$

Under the hypothesis made on q and r, one has $r_1 > r$. By induction one sees that φ_q is uniformly in H_2, and hence admits a subsequence which converges strongly in H_1 to φ satisfying the Yamabe equation and that φ is smooth and strictly positive (cf. the proof in answer 3d).

8) It can be proved (cf. Aubin, 1982, p. 45) that on a riemannian manifold of dimension $n \geq 3$, with bounded curvature and injectivity radius (Problem 16, Chapter 6, p. 429) $\delta > 0$, the Sobolev imbedding theorem (problem 2, Chapter 6, p. 379) $L_p \subset H_1$, $p = 2n/(n-2)$, can be written

$$\left(\|\varphi\|_{L_p}\right)^2 \leq (K_n + \epsilon)\left(\|\nabla\varphi\|_{L_2}\right)^2 + A(\epsilon)\left(\|\varphi\|_{L_2}\right)^2,$$

where K_n is the **best Sobolev constant** obtained on $\mathbb{R}^n$ (cf. Problem 4, Chapter 6, p. 389), hence independent of (M, g), ϵ is an arbitrary strictly positive number, and $A(\epsilon) > 0$ depends on (M, g) and ϵ.

8a) *Show that the best Sobolev constant* Sobolev constant

$$(K_n)^{-1} = \inf_{\varphi\not\equiv 0,\ \in C_0^\infty(\mathbb{R}^n)} \frac{\|\nabla\varphi\|^2_{L_2}}{\|\varphi\|^2_{L_{2n/(n-2)}}}$$

obtained on p. 389 *can be written*

$$K_n = 4(\omega_n)^{-2/n}\{n(n-2)\}^{-1}, \quad \omega_n \textit{ volume of the sphere } S^n.$$

Answer 8a. We have found on p. 389, with the notation $S_n = K_n^{-1}$

$$(K_n)^{-1} = n(n-2)\pi\left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^{2/n}.$$

A straightforward calculation gives (cf. an analogous one in Problem 4, Chapter V, Sphere S^n, p. 240)

$$\Gamma(n) = 2^{n-1}\pi^{-1/2}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right) \quad \text{hence}$$

$$K_n = \frac{4}{n(n-2)}\left\{\frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}}\right\}^{2/n}$$

which, due to the value recalled in problem 4, chapter V of the volume ω_n of the sphere S^n, is the indicated expression.

8b) *Show that on* S^n, $\mu = n(n-1)\omega_n^{2/n}$.

Hint: use the conformal invariance of μ and the function on $\mathbb{R}^n$ given by

$$u(x) = (1+r^2)^{(2-n)/2}, \quad r = |x|.$$

Answer 8b. The given function u is such that (cf. Problem 4, Chapter VI, The best ...)

$$\inf_{\varphi\in C_0^\infty(\mathbb{R}^n)} \frac{\|\nabla\varphi\|^2_{L^2(\mathbb{R}^n)}}{\|\varphi\|^2_{L^p(\mathbb{R}^n)}} = K_n^{-1} = \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^n)}}{\|u\|^2_{L^p(\mathbb{R}^n)}}.$$

On the other hand we know that S^n minus one point is conformal to $\mathbb{R}^n$ which has zero scalar curvature. We deduce therefore from the definition of the Yamabe invariant

$$\mu(S^n) = k_n K_n^{-1} = n(n-1)(\omega_n)^{2/n}.$$

8c) *Prove that for all manifolds (M, g) with bounded curvature and positive injectivity radius the Yamabe invariant μ satisfies the inequality*

$$\mu \le n(n-1)\omega_n^{2/n}.$$

Hint: use the family of functions $\varphi_{\alpha,\epsilon} = \eta_\epsilon u_\alpha$, where η_ϵ is a cut off function with support in a geodesic ball of radius ϵ and u_α is a function given in normal coordinates (p. 326) by $u_\alpha = \{\frac{\alpha^2+r^2}{\alpha}\}^{(2-n)/2}$.

Answer 8c (inspired from the survey article of Lee and Parker). The radial functions u_α on $\mathbb{R}^n$ have the same Sobolev quotients as u for any positive α because they read, with $r = \alpha\rho$

$$u_\alpha = \{\alpha(\rho^2+1)\}^{(2-n)/2}, \qquad \partial_r u_\alpha = (2-n)\alpha^{(2-n)/2}\rho(\rho^2+1)^{-n/2}.$$

The maximum of u_α is attained for $r = 0$ and becomes more and more sharp as α decreases: the compactly supported functions $\varphi_{\epsilon,\alpha}$, defined on a manifold via normal coordinates will have a Sobolev quotient which approximates K_n^{-1} when ϵ and α are small enough. Indeed let $\varphi_{\epsilon,\alpha} = \eta_\epsilon u_\alpha$, where η_ϵ is a C^1 function on the manifold M with support in a geodesic ball B_ϵ of radius ϵ, depending only on the radial normal coordinate r, with $\eta_\epsilon \le 1$ in B_ϵ, $\eta_\epsilon = 1$ in $B_{\epsilon/2}$ and $|\nabla\eta_\epsilon| \le C\epsilon^{-1}$ (choice possible cf. p. 529), while u_α is the function expressed in normal coordinates by the previous formula. We have then $\nabla\varphi_{\epsilon,\alpha} = (\partial_r\varphi_{\epsilon,\alpha}, 0, \dots, 0)$.
Recall that in normal coordinates there exists a constant C, depending on the curvature of (M, g), such that in B_ϵ

$$\mathrm{d}v \equiv f\,\mathrm{d}v_0, \quad \mathrm{d}v_0 \equiv r^{n-1}\,\mathrm{d}r\,\mathrm{d}\omega_{n-1}, \quad |f-1| \le C\epsilon.$$

We deduce from the definition of $\varphi_{\epsilon,\alpha}$, the properties of η_ϵ and the estimate of $\mathrm{d}v$

$$\int_M |\nabla\varphi_{\epsilon,\alpha}|^2\,\mathrm{d}v = \int_{B_\epsilon} \eta_\epsilon^2|\nabla u_\alpha|^2 + 2(\nabla\eta_\epsilon.\nabla u_\alpha)\eta_\epsilon u_\alpha + u_\alpha^2|\nabla\eta_\epsilon|^2\,\mathrm{d}v$$
$$\le (1+C\epsilon)\left\{\|\nabla u_\alpha\|^2_{L^2(\mathbb{R}^n)} + C\epsilon^{-1}\int_{\epsilon/2\le r\le\epsilon}\int_{S^n-1}\left(u_\alpha^2 + |\partial_r u_\alpha|^2\right)\mathrm{d}v_0\right\}.$$

We have $u_\alpha \le \alpha^{(n-2)/2}r^{2-n}$ and $\partial_r u_\alpha \le (n-2)\alpha^{(n-2)/2}r^{1-n}$. Hence when $r \ge \epsilon/2 u_\alpha \le C\epsilon^{2-n}\alpha^{(n-2)/2}$ and $|\partial_r u_\alpha| \le C\epsilon^{1-n}\alpha^{(n-2)/2}$. Given ϵ we can choose α small enough for the second term in the inequality above to be arbitrary small. On the other hand we know that

$$\|\nabla u_\alpha\|^2_{L^2(\mathbb{R}^n)} = K_n^{-1}\left\{\int_{\mathbb{R}^n} u_\alpha^p \,\mathrm{d}v_0\right\}^{2/p}, \quad p = \frac{2n}{n-2}.$$

Splitting the integral over $\mathbb{R}^n$ into an integral over a ball $B_{\epsilon/2}$ and over the complement, then using the properties of η_ϵ and $u_\alpha \leq \alpha^{(n-2)/2} r^{2-n}$ we obtain the estimate

$$\begin{aligned}\|\nabla u_\alpha\|^2_{L^2(\mathbb{R}^n)} &\leq K_n^{-1}\left\{\int_{B_{\epsilon/2}} u_\alpha^p \,\mathrm{d}v_0 + \omega_{n-1}\int_{r\geq\epsilon/2} \alpha^n r^{-n-1}\,\mathrm{d}r\right\}^{2/p} \\ &\leq K_n^{-1}\left\{(1+C\epsilon)\int_M \varphi_\alpha^p \,\mathrm{d}v + C\alpha^n\right\}^{2/p}.\end{aligned}$$

On the other hand if $n \geq 3$ and $\alpha \leq 1$

$$\int_M \varphi_\alpha^2 \,\mathrm{d}v \leq (1+C\epsilon)\omega_{n-1}\int_{r\leq\epsilon} \alpha^{n-2} r^{3-n}\,\mathrm{d}r \leq (1+C\epsilon)C\alpha.$$

Collecting all these results gives

$$\int_M \left\{k_n|\nabla\varphi_{\epsilon,\alpha}|^2 + R\varphi_{\epsilon,\alpha}^2\right\}\mathrm{d}v \leq k_n K_n^{-1}(1+C\epsilon)(1+C\alpha)\left\{\int_M \varphi_\alpha^p\,\mathrm{d}v\right\}^{2/p}$$

which proves the desired result.

8d) *Prove if* $\mu < n(n-1)\omega_n^{2/n}$, $\mathrm{Vol}(M, g) = 1$ *that there exists* $r > 2n/(n-2)$ *such that* $\|\varphi_q\|_{L_r}$ *is uniformly bounded for* $q \in]1, n/(n-2)[$.

Hint: use the Sobolev inequality proved by Aubin recalled above.

Answer 8d. Let d be some positive number. We deduce from the Yamabe equation for φ_q, by multiplication by φ_q^{1+2d} and integration by parts

$$\int_M \left\{k_n(1+2d)\varphi_q^{2d}|\nabla\varphi_q|^2 + R\varphi_q^{2+2d}\right\}\mathrm{d}v = \mu_q\int_M \varphi_q^{2q+2d}\,\mathrm{d}v.$$

Set $w = \varphi_1^{1+d}$. The previous equality becomes

$$\frac{1+2d}{(1+d)^2} \int_M k_n |\nabla w|^2 \, \mathrm{d}v = \int_M \left(\mu_q w^2 \varphi_q^{2q-2} - R w^2\right) \mathrm{d}v.$$

Using the Sobolev inequality with the best Sobolev constant leads to

$$\|w\|_{L_p}^2 \leq \frac{K_n}{k_n} \frac{(1+\epsilon)(1+d)^2}{1+2d} \int_M \left\{\mu_q w^2 \varphi_2^{2q-2} - (R + A(\epsilon)) w^2\right\} \mathrm{d}v,$$

$$p = \frac{2n}{n-2}.$$

By the Holder inequality

$$\int_M w^2 \varphi_q^{2q-2} \, \mathrm{d}v \leq \|w\|_{L_p}^2 \|\varphi\|_{L_{n(q-1)}} \leq \|w\|_{L_p}^2$$

because, by the hypothesis on q, $n(q-1) < 2q$ and we have $\mathrm{Vol}(M, g) = 1$. We know that μ_q depends continuously on q, hence if $\mu < \frac{k_n}{K_n}$ we can choose q_0 near enough from $n/(n-2)$, ϵ and d small enough so that the coefficient of $\|w\|_{L_p}^2$ in the right hand side is strictly smaller than one and obtain an inequality of the form

$$\|w\|_{L_p}^2 \leq C \|w\|_2^2, \quad \text{for } q \in \left[q_0, n/(n-2)\right[$$

that is $\|\varphi_q\|_{L_p(1+d)} \leq C \|\varphi_q\|_{L_{2(1+d)}} \leq C$, $d > 0$ and small enough.

8) In the case of M compact, it has been proved by Aubin when $n \geq 6$ that if (M, g) is not locally conformally flat then $\mu < n(n-1)\omega_n^{2/n}$; it has been proved by Schoen (1984) by a new method related to the positive mass theorem of General Relativity (in fact the positivity of the second term in an expansion of a Green function) that if $n = 3, 4$, or 5 or if (M, g) is locally conformally flat then $\mu < n(n-1)\omega_n^{2/n}$, except if (M, g) is conformal to S^n. **Hence the Yamabe constant satisfies the strict inequality, except if (M, g) is conformal to S^n.**
The sphere S^n has constant scalar curvature, $n(n-1)$. It is an old result that every compact riemannian manifold of dimension 2 has a conformal metric of constant scalar (gaussian) curvature. **Conclusion: every compact riemannian manifold has the Yamabe property.**

Remark. It has been proved that any smooth compact manifold M can be endowed with a smooth metric with $R < 0$, hence **the strict negativity of a Yamabe invariant does not imply any topological restriction on M.** It has first been proved by Lichnerowicz that there are manifolds which do not carry metrics with $R > 0$. It has also been proved (Gromow–Lawson, Kazdan–Warner, Schoen–Yau) that some manifolds do not carry any metric with $R \geq 0$, therefore **the positivity of a Yamabe invariant is a topological obstruction.**

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7. THE EULER CLASS

A Supplement to Problem Vbis.6 (pp. 321–334)

CHECK THE GAUSS–BONNET THEOREM FOR A 2-DIMENSIONAL MANIFOLD OF GENUS 0

Euler class
Gauss–Bonnet theorem

By definition (p. 394), the Euler class for a 2-dimensional riemannian manifold M is represented by a two form with values in the Lie algebra $O(n, m)$

$$\gamma = \frac{-1}{4\pi}\epsilon^{12}_{i_1 i_2}\bar{\boldsymbol{\Omega}}_{i_2}{}^{i_1} = \frac{-1}{2\pi}\bar{\boldsymbol{\Omega}}_2{}^1 = \frac{-1}{2\pi}\left(\bar{\boldsymbol{\Omega}}_2{}^1\right)_{kl}\mathrm{d}x^k \wedge \mathrm{d}x^l, \quad \bar{\Omega}_{i_2}{}^{i_1} = -\bar{\Omega}_{i_1}{}^{i_2}.$$

Since $\int_M \gamma$ is a topological invariant, we shall choose $M = S^2$, and compute $\bar{\boldsymbol{\Omega}}_2{}^1$ on S^2. According to Cartan structural equation (p. 306), we have in the basis $\{\boldsymbol{\theta}^k\}$ dual to a moving frame

$$\begin{aligned}\bar{\boldsymbol{\Omega}}_2{}^1 &= \frac{1}{2}R_2{}^1{}_{kl}\boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^l = \mathbf{d}\omega_2^1 + \omega_m^1 \wedge \omega_2^m, \quad k, l, m \in \{1, 2\},\\ &= \mathbf{d}\omega_2^1.\end{aligned}$$

In the riemannian connection,

$$0 = \mathbf{d}\boldsymbol{\theta}^i + \omega_j^i \wedge \boldsymbol{\theta}^j.$$

On S^2 parameterized by $0 \leq \theta < \pi$ and $\leq \varphi < 2\pi$, we can take

$$\begin{aligned}\boldsymbol{\theta}^1 &= r\,\mathrm{d}\theta, & \mathbf{d}\boldsymbol{\theta}^1 &= 0 = -\omega_2^1 \wedge \boldsymbol{\theta}^2,\\ \boldsymbol{\theta}^2 &= r\sin\theta\,\mathrm{d}\varphi, & \mathbf{d}\boldsymbol{\theta}^2 &= \frac{\cos\theta}{r\sin\theta}\boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2 = -\omega_1^2 \wedge \boldsymbol{\theta}^1.\end{aligned}$$

Therefore

$$\omega_2^1 = -\frac{\cos\theta}{r\sin\theta}\boldsymbol{\theta}^2$$

and

$$\bar{\Omega}_2{}^1 = -\frac{1}{r^2}\boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2 = -\sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi.$$

Finally

$$\int_{S^2} \gamma = \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\varphi = 2.$$

8. FORMULA FOR LAPLACIANS AT A POINT OF THE FRAME BUNDLE

INTRODUCTION

Laplacians

Laplacians are defined on forms by the formula (Vol. I, p. 318)

$$\Delta := -(\delta\mathrm{d} + \mathrm{d}\delta) \tag{1}$$

where d is the exterior derivative, and δ its metric transpose (Vol. I, p. 296). On zero form

$$\Delta = -\delta\mathrm{d} = g^{\mu\nu}\nabla_\mu\nabla_\nu = g^{\mu\nu}\partial_\mu\partial_\nu - g^{\mu\nu}\Gamma^\rho_{\mu\nu}\partial_\rho \tag{2}$$

where ∇_μ is the covariant derivative defined by the riemannian connection (Vol. I, p. 308). Lichnerowicz has generalized the coordinate expression of (1) to laplacians on arbitrary tensors.

Laplacians of elliptic complex (Vol. I, p. 398) are defined by the formula

$$\Delta_\rho := -(D_\rho^* D_\rho + D_{\rho+1} D_{\rho-1}^*) \tag{3}$$

where the D_ρ's are elliptic operators on sections of a finite sequence $\{E_\rho\}$ of vector bundles, and the D_ρ^*'s their respective metric transpose. In particular if $D = \mathrm{d}$, $D^* = \delta$, and Δ is the Laplacian of the De Rham complex.

Alternatively, we can express the Laplacian on a (possibly pseudo) riemannian manifold $(\mathbb{M}^d, g)$ by its lift on the frame bundle $O(\mathbb{M})$. It operates on any equivariant map (Vol. I, p. 404) on $O(\mathbb{M})$ with values in the typical fibre of an associated bundle. The expression for the Laplacian on the frame bundle is useful in the theory of functional integrals in riemannian spaces. [See, for instance, Cartier and DeWitt-Morette.]

PROBLEM

Let $O(\mathbb{M})$ be the frame bundle over a riemannian manifold $(\mathbb{M}^d, g)$, possibly pseudo-riemannian. Let $\rho(t)$ be the horizontal lift of a path $x(t) \in \mathbb{M}$ defined by the riemannian connection map $\sigma : O(\mathbb{M}) \to L(T\mathbb{M}, TO(\mathbb{M}))$

$$\dot{\rho}(t) = \sigma\big(\rho(t)\big)\dot{x}(t), \qquad \Pi_\rho(t) = x(t). \tag{4}$$

Choose a trivialization

$$\rho(t) = \big(x(t), u(t)\big), \qquad \rho(t_a) =: \rho_a =: (a, u_a). \tag{5}$$

If $\rho(t)$ is a solution of (4), then the frame $u(t)$ is the u_a frame parallel transported along the path $x(t)$ from a to $x(t)$. A frame is also an admissible map (Vol. I, p. 368)

$$u(t) : \mathbb{R}^d \to T_{x(t)}\mathbb{M}. \tag{6}$$

Therefore the set $\{(u(t)^{-1}\dot{x}(t))^\mu\}$ is the set of coordinates of $\dot{x}(t)$ in the frame $u(t)$. Denote by $\dot{z}(t)$ the vector with these components in the canonical basis $\{e_\mu\}$ of $\mathbb{R}^d$

$$\begin{aligned} \dot{z}(t) &:= u(t)^{-1}\dot{x}(t) = \dot{z}^\mu(t)e_\mu, \\ \dot{z}^\mu(t) &= \big(u(t)^{-1}\dot{x}(t)\big)^\mu. \end{aligned} \tag{7}$$

Since we are interested in riemannian manifolds, possibly pseudo-riemannian, the canonical basis $\{e_\mu\}$ is either euclidean, or minkovskian

$$(e_\mu | e_\nu) = h_{\mu\nu} \quad \text{with } h_{\mu\nu} \in \{\delta_{\mu\nu}, \eta_{\mu\nu}\}.$$

Rewrite equation (4) as follows

$$\dot{\rho}(t) = \sigma\big(\rho(t)\big) \circ u(t) \circ \big(u(t)\big)^{-1}\dot{x}(t).$$

and define the set of d vectors fields $\{X_{(\mu)}\}$ on $O(\mathbb{M})$ by

$$X_{(\mu)}\big(\rho(t)\big) = \sigma\big(\rho(t)\big) \circ u(t)e_\mu; \tag{8}$$

then,

$$\dot{\rho}(t) = X_{(\mu)}\big(\rho(t)\big)\dot{z}^\mu(t). \tag{9}$$

If $x(t)$ is within focal distance of a, the vectors $\{X_{(\mu)}(\rho(t))\}$ are linearly independent.

1. *Show that* $x : [t_a, t_b] \to \mathbb{M}$ *is the Cartan development map of* $z : [t_a, t_b] \to T_a\mathbb{M}$, $z(t_a) = 0$.

We recall that the Cartan development is an injective map from spaces of paths on $T_a\mathbb{M}$ into spaces of paths on $\mathbb{M}$ (or vice versa). Let $z:[t_a,t_b]\to T_a\mathbb{M}$ such that $z(t_a)=0$, then x is said to be the development of z,

$$x = \operatorname{Dev} z,$$

if $\dot{x}(t)$ parallel transported along $x(t)$ to a is equal to $\dot{z}(t)$, trivially parallel transported to the origin of $T_a\mathbb{M}$, for every $t\in[t_a,t_b]$. In particular if $z(t)=t\dot{z}(t_a)$, then $\dot{z}(t)$ is constant and $x(t)$ is the geodesic defined by

$$x(t_a)=a, \qquad \dot{x}(t_a)=\dot{z}(t_a).$$

2. *Let $\Psi:\mathbb{M}\to\mathbb{R}$ and $F:=\Psi\circ\Pi:O(\mathbb{M})\to\mathbb{R}$; Show that*

$$\Delta\Psi(a)=h^{\mu\nu}\mathcal{L}_{X_{(\mu)}}\mathcal{L}_{X_{(\nu)}}F(\rho_a), \qquad \Pi\rho=x \tag{10}$$

where the integral curves of the set $\{X_{(\mu)}\}$ are the horizontal lifts of a set of geodesics at a, tangent to a canonical basis $\{e_\mu\}$ on $T_a\mathbb{M}$. $\mathcal{L}_X$ is the Lie derivative defined by the vector field X, and $h_{\mu\nu}$ has the same signature as $g_{\mu\nu(x)}$.

Answer 1. Identify $T_a\mathbb{M}$ with $\mathbb{R}^d$. Parallel transport $\dot{x}(t)$ along $x(t)$ back to a, and compute the coordinates of this vector in the frame u_a; we have

$$u_a^{-1}\int_{t_a}^{t} u_a\big(u(s)\big)^{-1}\dot{x}(s)\,\mathrm{d}s=\int_{t_a}^{t}\big(u(s)\big)^{-1}u(s)\dot{z}(s)\,\mathrm{d}s=\int_{t_a}^{t}\dot{z}(s)\,\mathrm{d}s.$$

Remark. The development map parameterizes spaces of paths on $\mathbb{M}$ with one fixed point a by paths taking their values in $T_a\mathbb{M}$. It cannot parameterize spaces of paths on $\mathbb{M}$ with two fixed points, e.g., $x(t_a)=a$, $x(t_b)=b$. E.g., two geodesics in S^2 which intersects in *two* antipodal points are the development of two halflines with *one* common origin.

Remark. The development map preserves the quadratic form

$$\int \mathrm{d}t\, h_{\alpha\beta}\dot{z}^\alpha(t)\dot{z}^\beta(t)=\int \mathrm{d}t\, g_{\alpha\beta}\big(x(t)\big)\dot{x}^\alpha(t)\dot{x}^\beta(t),$$

and therefore the length of a path.

Answer 2. It follows from answer 1 that the integral curves of the d linearly independent vector fields $\{X_{(\mu)}\}$ are the horizontal lifts of d geodesics

$$\gamma_{(\mu)}(t_a)=a, \qquad \dot{\gamma}_{(\mu)}(t_a)=e_\mu, \qquad \{e_\mu\}\text{ the canonical basis.}$$

Indeed, it follows from (9) that, if $\dot{z}(t) = \dot{z}(t_a) = e_\mu$,

$$\dot{\rho}_{(\mu)} := X_{(\nu)}\big(\rho(t)\big)\delta^\nu_\mu = X_{(\mu)}\big(\rho(t)\big) \tag{11}$$

and $\rho_{(\mu)}(t)$ is the integral curve of $X_{(\mu)}$.

Proof of (10).

$$\mathcal{L}_{X_{(\mu)}}\mathcal{L}_{X_{(\nu)}}F = \big(F_{,\alpha\beta}X^\alpha_{(\mu)}X^\beta_{(\nu)} + F_{,\alpha}X^\alpha_{(\mu),\beta}X^\beta_{(\mu)}\big).$$

On the other hand since ρ_μ is the integral curve of X_μ, we have, using (11),

$$\frac{\mathrm{d}}{\mathrm{d}t}F\big(\rho_{(\mu)}(t,\rho_a)\big) = F_{,\alpha}X^\alpha_{(\nu)}\big(\rho(t,\rho_a)\big)e^\nu_{(\mu)}$$

and, omitting the arguments of the right hand side,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}F\big(\rho_{(\mu)}(t,\rho_a)\big) = F_{,\alpha\beta}X^\alpha_{(\rho)}e^\rho_{(\mu)}X^\beta_{(\sigma)}e^\sigma_{(\mu)} + F_{,\alpha}X^\alpha_{(\rho),\beta}e^\rho_{(\mu)}X^\beta_{(\sigma)}e^\sigma_{(\mu)}.$$

Therefore

$$h^{\mu\nu}\mathcal{L}_{X_{(\mu)}}\mathcal{L}_{X_{(\nu)}}F = \sum_\mu \frac{\mathrm{d}^2}{\mathrm{d}t^2}F\big(\rho_{(\mu)}(t,\rho_a)\big).$$

Set

$$F(\rho) = (\Psi\circ\Pi)(\rho) = \Psi(x),$$

and

$$\Pi\rho_{(\mu)}(t,\rho_a) = x_{(\mu)}(t,a);$$

then

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}F\big(\rho_{(\mu)}(t,\rho_a)\big) &= \Psi_{,\rho\sigma}\dot{x}^\rho_{(\mu)}\dot{x}^\sigma_{(\mu)} + \Psi_{,\rho}\ddot{x}^\sigma_{(\mu)}\\ &= \Psi_{,\rho\sigma}\dot{x}^\rho_{(\mu)}\dot{x}^\sigma_{(\mu)} + \Psi_{,\tau}\left(\frac{D^2}{\mathrm{d}t^2}x^\tau_{(\mu)} - \Gamma^\tau_{\rho\sigma}\dot{x}^\rho_{(\mu)}\dot{x}^\sigma_{(\mu)}\right)\\ &= \dot{x}^\rho_{(\mu)}\dot{x}^\sigma_{(\mu)}\Psi_{;\rho\sigma}\big(x_\mu(t,a)\big)\end{aligned}$$

since $x_{(\mu)}$ is a geodesic.

At $t = t_a$, i.e., at the base point of the development map,

$$\dot{x}^\rho_{(\mu)}(t_a)\dot{x}^\sigma_{(\mu)}(t_a) = e^\rho_{(\mu)}e^\sigma_{(\mu)}$$

and

$$h^{\mu\nu}\mathcal{L}_{X_{(\mu)}}\mathcal{L}_{X_{(\nu)}}F(\rho_a) = g^{\rho\sigma}(a)\Psi_{;\rho\sigma}(a).$$

Remark. Much of this problem is inspired by Elworthy's work who generalized equation (9) to a Stratonovich equation

$$d\rho(t) = X\big(\rho(t)\big)\, dz(t) \tag{12}$$

where z is brownian. He replaced (4) by (9) because brownian paths are defined on $\mathbb{R}^d$. He used (12) to construct diffusions on riemannian manifolds.

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9. THE BERRY AND AHARONOV–ANANDAN PHASES*

THE BERRY PHASE

Aharonov–Anandan phase
Berry phase

Problem 1: Let $x \mapsto H(x)$ be a smooth mapping from a manifold M into the space of self adjoint linear operators on the Hilbert space $\mathcal{H}$. For each smooth curve $C : [0, T] \to M$, let $H \circ C$ serve as a time dependent Hamiltonian for the Schrödinger equation on $\mathcal{H}$, i.e., with $\hbar$ scaled to 1

$$i\frac{d\psi}{dt} = H\big(C(t)\big)\psi(t), \quad \psi : [0, T] \to \mathcal{H}; \tag{1}$$

in physicists notations with $\hbar$ not scaled to 1:

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle.$$

a) *Show that the time evolution of ψ by* (1) *preserves its norm in $\mathcal{H}$. We denote by S the subset elements of $\mathcal{H}$ whose norm is* 1.
b) *Show that the scalar product $(\frac{d\psi}{dt}|\psi)$ is pure imaginary.*

Answer 1. a) Using the properties of the scalar product and (1) we obtain

$$\frac{d}{dt}(\psi|\psi) = \left(\frac{d\psi}{dt}\middle|\psi\right) + \left(\psi\middle|\frac{d\psi}{dt}\right) = -i(H\psi|\psi) + i(\psi|H\psi) = 0$$

since H is self adjoint.
b) We have $(\frac{d\psi}{dt}|\psi) = (\psi|\frac{d\psi}{dt})^*$ and by the previous answer $(\frac{d\psi}{dt}|\psi) = -(\psi|\frac{d\psi}{dt})$, hence $(\frac{\partial\psi}{\partial t}|\psi)$ is pure imaginary.

*Based on notes by Ali Mostafazadeh.

Problem 2: *Suppose that for each point* $x(t) = C(t) \subset M$ *the Hamiltonian*

$$H_t \equiv H\big(C(t)\big)$$

admits an eigenvalue E_t *with eigenvector* $X_t \subset S$, *both depending smoothly on* t. *Show that if*

$$\psi(t) = \mathrm{e}^{\mathrm{i}\lambda(t)} X_t, \quad \text{with } \lambda(t) \in \mathbb{R} \tag{2}$$

is a solution of the Schrödinger equation (1), *then* $\lambda(t)$ *is determined up to an additive constant.*

Remark: A solution of the form (2) is called an **adiabatic evolution** of X_0. It will be shown in paragraph 5 that solutions of (1) are not of the form (2), but only approximately so if H_t varies slowly with t. adiabatic evolution

Answer 2: Inserting (2) in (1) gives

$$\mathrm{i}\frac{\mathrm{d}X}{\mathrm{d}t} - \dot{\lambda}(t) X_t = H_t X_t = E_t X_t. \tag{3}$$

Scalar product of this equation by X_t implies

$$\dot{\lambda}(t) = \mathrm{i}\left(\frac{\mathrm{d}X}{\mathrm{d}t}\middle| X\right) - E_t. \tag{4}$$

Hence

$$\lambda(t) = \lambda(0) - \int_0^t E_\tau \,\mathrm{d}\tau + \gamma(t) \tag{5}$$

where

$$\gamma(t) = \mathrm{i}\int_0^t \left(\frac{\mathrm{d}X_\tau}{\mathrm{d}\tau}\middle| X_\tau\right) \mathrm{d}\tau \in \mathbb{R}. \tag{6}$$

Problem 3: The first integral in (5) expresses the phase change of ψ by the dynamical evolution. The second integral $\gamma(t)$ is called the **geometrical** or **Berry phase**. Suppose that the properties assumed for the Hamiltonian hold on M; namely for each $x \in M$, $H(x)$ admits an eigenvalue $E(x)$ with normalized eigenvector $X(x)$, both dependent smoothly on x. geometrical phase Berry phase

a) *Show that* $\gamma(t)$ *can be written as the integral on the curve* $\tau \mapsto C(\tau)$, $\tau \in [0, t]$, *of a* 1*-form* A *defined on* M. *Determine its transformation under a phase change in the choice of* $X(x)$.

holonomy group

b) *Interpret A as the representative on M of a connection in a $U(1)$ principal fiber bundle with base M. Interpret the Berry phase around a closed loop as an element of the holonomy group. Give its expression as a double integral independent of the choice of the phase factor in $X(x)$.*

Answer 3: a) Denote by dX the differential of the $\mathcal{H}$-valued function on M, $x \mapsto X(x)$. It is an $\mathcal{H}$-valued 1-form and the scalar product $A \equiv (\mathrm{d}X|X)$ is a $\mathbb{C}$ valued 1-form on M, in fact pure imaginary valued since $(X|X) \equiv 1$. The integrand in (5) is the 1-form induced on $\mathbb{C}$ by A.

Suppose we replace $X(x)$ by $\tilde{X}(x) = \mathrm{e}^{\mathrm{i}\varphi(x)}X(x)$ then

$$\tilde{A} \equiv (\mathrm{d}\tilde{X}|\tilde{X}) = A + \mathrm{i}\,\mathrm{d}\varphi.$$

b) Consider the trivial principal bundle B over M which admits a trivialization $\phi : B \to M \times U(1)$. The sections $s : x \mapsto (x, \mathrm{e}^{\mathrm{i}\varphi(x)})$ are in bijective correspondence with the choices $X(x)\mathrm{e}^{\mathrm{i}\varphi(x)}$ of eigenvectors of $\mathcal{H}$ corresponding to the eigenvalue $E(x)$, since $X(x)$ has been assumed given. The transformation law of A is exactly the gauge transformation of a $U(1)$ connection under the corresponding change of sections. A is the representative on M, in the trivialization ϕ, of a connection ω called the Berry–Simon connection (B.S.); in other words A is the pullback of ω by the section canonically associated to the trivialization.
Let $p(t)$ be the horizontal lift of $x(t)$, and $\rho(t) = \exp(\mathrm{i}\varphi(t))$ its map on the typical fiber in the chosen trivialization, then $p(t)$ is obtained by solving the differential equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}A\frac{\mathrm{d}x}{\mathrm{d}t}\rho, \quad \text{i.e.,} \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \mathrm{i}A\frac{\mathrm{d}x}{\mathrm{d}t}. \tag{7}$$

Proof of (7) (cf. exercise Vol. I, p. 366). A connection on a principal bundle P can be defined either by the horizontal lift $\sigma_p : T_xM \to T_xP$ or by a one form ω on the bundle, with values in the Lie algebra $L(G)$ of the structure group, which vanishes on the horizontal subspaces of the bundle. The equation giving the horizontal lift $p(t)$ of a curve $x(t)$ in the base space M is

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \sigma\big(\rho(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}, \qquad \sigma(p) : T_xM \to T_pP. \tag{8}$$

We wish to rewrite it in terms of ω, more precisely in terms of the pull back $s^*\omega$ of ω on the base space by the section s canonically associated to a trivialization ϕ (Vol. I, p. 363). For $\phi : \Pi^{-1}(U) \to U \times G$, this section

$s : U \to \Pi^{-1}(U)$ is such that

$$\phi \circ s : U \to U \times G \quad \text{by} \quad x \mapsto (x, e). \tag{9}$$

It has been established (Vol. I, p. 366) that at p_0, which is mapped into (x, e) by the trivialization ϕ:

$$\phi(p_0) = (x, e), \tag{10}$$

the connection map is, up to a sign, the pull back of ω

$$\sigma(p_0) = -s^*\omega.$$

On the other hand, if $p = \widetilde{R}_g p_0$, with $\widetilde{R}_g$ the globally defined right action on P (Vol. I, p. 129), then

$$\sigma(p) = \widetilde{R}'_g \circ \sigma(p_0) \tag{11}$$

and (8) becomes

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\widetilde{R}'_g \circ (s^*\omega) \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{8a}$$

In the trivialization corresponding to s, let $p(t) = (x(t), \rho(t))$; then, on the typical fibre

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\phi'_p \circ \widetilde{R}'_g \circ (s^*\omega) \frac{\mathrm{d}x}{\mathrm{d}t} = -\widetilde{R}'_{p(t)} \circ (s^*\omega) \frac{\mathrm{d}x}{\mathrm{d}t} = -(s^*\omega) \frac{\mathrm{d}x}{\mathrm{d}t} \rho(t). \tag{8b}$$

Set $A = s^*\omega$ and $\rho(t) = \exp(\mathrm{i}\varphi(t))$ to obtain (7). Integrating (7) we get

$$\gamma(T) = \varphi(T) - \varphi(0) = \mathrm{i} \int_0^T A \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t = \mathrm{i} \int_C A,$$

$e^{\mathrm{i}\gamma(T)} \in U(1)$ is the element of the holonomy group of the B.S. connection.

If the loop C bounds a 2-surface S, $C = \partial S$, then

$$\int_C A = \int_S \mathrm{d}A.$$

Since $U(1)$ is abelian $\mathrm{d}A$ is the representative of the curvature F of the connection represented by A. We had $A = (\mathrm{d}X|X)$ pure imaginary, hence

$$F = \mathrm{d}A = (\mathrm{d}X \hat{,} \mathrm{d}X) \quad \text{is pure imaginary},$$

where $\hat{,}$ denotes exterior product in T^*M and scalar product in $\mathcal{H}$, that is, in coordinates

$$F = (\partial_\alpha X | \partial_\beta X) \, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta.$$

A straightforward computation shows that – as predicted by the general theory for an abelian group – F is invariant by the change of trivialization corresponding to $X(x) \rightsquigarrow e^{i\varphi(x)} X(x)$.

holonomy
Berry phase

Problem 4: *Use the previous ideas to construct a possibly nontrivial $U(1)$ bundle B over M whose holonomy gives the Berry phase when the eigenvectors $X(x)$ are not defined globally on M, but coherently on the domains U_i, $i \in I$, of an open covering of M, that is $X_{(i)}$ is a mapping $U_i \to \mathcal{S} \subset \mathcal{H}$ and there exist functions $\varphi_{ij} : U_i \cap U_j \to \mathbb{R}$ such that*

$$X_{(i)}(x) = e^{i\varphi_{ij}(x)} X_{(j)}(x)$$

with

$$\exp\bigl(i\varphi_{ik}(x)\bigr) \exp\bigl(i\varphi_{kj}(x)\bigr) = \exp\bigl(i\varphi_{ij}(x)\bigr)$$

if $x \in U_i \cap U_j \cap U_k$.

Answer 4: We define the bundle by the family of trivializations $U_i \times U(1)$ with the transition functions $\exp(i\varphi_{ij}(x))$ and the connection 1-form by its representatives $A_{(i)} = (dX_{(i)}|X_{(i)})$ in each U_i. By definition the Berry phase around a closed loop is the corresponding element of the holonomy group.

Problem 5: Equation (4) is necessary for $e^{i\lambda(t)} X_t$ to be a solution of (1) when X_t is an eigenvector of H_t for each t; but it is not a sufficient condition. To study the approximate validity of (2) we suppose that H_t admits for each t a complete orthonormal set of eigenvectors $X_{n,t}$ corresponding to the eigenvalues $E_n(t)$.

Expand the solution $\varphi(t)$ of (1) as follows

$$\varphi(t) = \sum_n a_n(t) \exp\left(-i \int_0^t E_n(\tau)\, d\tau\right) X_{n,t}. \tag{12}$$

Use the Schrödinger and eigenvalue equations to obtain a differential system satisfied by the complex functions a_n.

Show that if for some n

$$\sum_m \frac{(\dot{H}_t X_m | X_m)^2}{(E_n - E_m)^2} \ll 1, \quad t \in [0, T], \tag{13}$$

adiabatic

the evolution of $X_{n,0}$ can be considered as adiabatic.

Remark: (13) expresses that H_t varies slowly with t, and no $E_m(t)$ is equal to $E_n(t)$ if $m \neq n$.

Answer 5: Inserting (12) in (1) gives, since $X_{n,t}$ is an eigenvector of H_t with eigenvalue $E_n(t)$,

$$\sum_n (\dot{a}_n X_n + a_n \dot{X}_n) = 0.$$

We expand $\dot{X}_n$:

$$\dot{X}_n \equiv \sum_m z_{nm} X_m, \quad z_{nm} = (\dot{X}_n | X_m) \in \mathbb{C}.$$

We take the derivative of the eigenvector equation $H_t X_n = E_n X_n$ with respect to t and obtain

$$\dot{H}_t X_n + H_t \dot{X}_n = \dot{E}_n X_n + E_n \dot{X}_n, \quad \text{for each } n;$$

therefore

$$(\dot{H}_t X_n | X_m) + E_m z_{nm} = \dot{E}_n \delta_n^m + E_n z_{nm}$$

hence for $n \neq m$

$$z_{nm}^{(t)} = \frac{(\dot{H}_t X_{n,t} | X_{m,t})}{E_n(t) - E_m(t)}.$$

If $\sum_m (z_{nm})^2 \ll 1$, $\dot{X}_n$ is approximately proportional to X_n, therefore the evolution of $X_{n,0}$ is approximately adiabatic.

Problem 6: *Give an expression of F for the adiabatic evolution of X_n in problem* 4 *in terms of dH, the complete set $\{X_m(x)\}$, and $E_m(x)$.*

Answer 6: In

$$F = (\mathrm{d}X_n | \mathrm{d}X_n) = (\partial_\alpha X_n | \partial_\beta X_n)\, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta,$$

we write the scalar product as a sum of products of components

$$F = \sum_m (\partial_\alpha X_n | X_m)(\partial_\beta X_n | X_m)^* \, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta;$$

the term $m = n$ vanishes identically because

$$(\partial_\beta X_n | X_m)^* = -(\partial_\beta X_n | X_n).$$

By a calculation similar to the one in the previous paragraph, we find

$$F = \sum_{m \neq n} \frac{(\mathrm{d}H X_m | X_n) \wedge (\mathrm{d}H X_n | X_m)}{(E_n - E_m)^2}.$$

THE AHARONOV–ANANDAN PHASE

Aharonov–Anandan phase

Problem 7: We shall consider now the general, nonadiabatic case. Let ψ be the evolution of some arbitrary normalized $\psi(0) \in \mathcal{S} \subset \mathcal{H}$ by the Schrödinger equation with some time dependent self adjoint Hamiltonian H_t. One says that ψ is (T, α) cyclic if there exist T and $\alpha \in \mathbb{R}$ such that

$$\psi(T) = \mathrm{e}^{\mathrm{i}\alpha}\psi(0). \tag{14}$$

A–A (Aharonov–Anandan)

The **A–A** (**Aharonov–Anandan**) phase of ψ is by definition the real number

$$\beta = \alpha + \int_0^T (H_t\psi, \psi)\,\mathrm{d}t$$

equivalently

$$\beta = \alpha + \mathrm{i}\int_0^T \left(\frac{\mathrm{d}\psi}{\mathrm{d}t}, \psi\right)\mathrm{d}t. \tag{15}$$

The formula (15) defines β for any (T, α) cyclic mapping $\psi : [0, T] \to \mathcal{S}$.

Denote by $\mathcal{P}$ the space of "rays" in $\mathcal{H}$, i.e., equivalence classes of elements of $\mathcal{H}$ by the equivalence relation

$$h \sim ch, \quad c \in \mathbb{C} - \{0\},$$

$\mathcal{H} - \{0\}$ projects onto $\mathcal{P}$ by $\Pi : h \mapsto \{h\}$, it is a $\mathbb{C}^* \equiv \mathbb{C} - \{0\}$ fiber bundle with base $\mathcal{P}$ and fiber $\mathcal{P}_p \simeq \mathbb{C}^*$ at $p = \{h\}$ the elements of $\mathcal{H} - \{0\}$ which belong to $\{h\}$.

a) *Show that $\mathcal{P}$ can be identified with the set of equivalence classes*

$$h \sim ch, \quad c \in U(1), \ h \in \mathcal{S}$$

cyclic

b) *Show that β given by* (15) *is the same for all cyclic mappings* $[0, T] \to \mathcal{S}$ *which project on the same closed curve* $[0, T] \to \mathcal{P}$.

Answer 7: a) Any element $h \in \mathcal{H} - \{0\}$ has an equivalent element $h/\|h\|$ in $\mathcal{S}$.
b) Two normalized mappings ψ_1 and ψ_2 project onto the same curve C in $\mathcal{P}$ if

$$\psi_2(t) = \mathrm{e}^{\mathrm{i}\varphi(t)}\psi_1(t).$$

If ψ_1 is cyclic the curve C is closed in $\mathcal{P}$. If ψ_1 is α_1-cyclic, ψ_2 is $\alpha_2 = \alpha_1 + \varphi(T) - \varphi(0)$ cyclic. Simple computation gives

$$\beta_2 = \alpha_1 + \varphi(T) - \varphi(0) + \mathrm{i}\int_0^T \left(\frac{\mathrm{d}\psi_1}{\mathrm{d}t}, \psi_1\right)\mathrm{d}t - \int_0^T \dot{\varphi}(t)\,\mathrm{d}t = \beta_1.$$

Problem 8: We consider the case where $\mathcal{H}$ is $\mathbb{C}^{N+1}$, with its canonical scalar product. We have defined in [Problem III 8, examples of homogeneous spaces], the real projective space. The complex projective space $\mathbb{C}P_N$, is the space of rays in $\mathbb{C}^{N+1}$ and can be defined as the space of equivalence classes of elements in $\mathbb{C}^{N+1} - \{0\}$ under the equivalence relation.

$$z \sim cz, \quad c \in \mathbb{C}^* \equiv \mathbb{C} - \{0\}.$$

a) *Give an atlas of $\mathbb{C}P_N$ which endows it with the structure of a complex holomorphic, manifold.* complex holomorphic manifold

b) *Show that $\mathbb{C}^{N+1} - \{0\} \to \mathbb{C}P_N$ is a $\mathbb{C}^*$ principal fiber bundle.* principal fiber bundle

c) *Show that the bundle $\mathbb{C}^{N+1} - \{0\} \to \mathbb{C}P_N$ reduces to a $U(1)$ principal bundle $\mathbb{C}S^N \to \mathbb{C}P_N$, where $\mathbb{C}S^N$ is the complex N-sphere, $\sum_{i=1}^{N+1} |z^i|^2 = 1$. Show that $\mathbb{C}S^N$ can be identified with the real sphere S^{2N+1}.* $\mathbb{C}P_N$

d) *Show that the bundle $\mathbb{C}S^N \to \mathbb{C}P_N$ is a $2N$ universal* (II, p. 337) $U(1)$ *bundle.* universal bundle

Answer 8: a) Denote by Π the projection $\mathbb{C}^{N+1} - \{0\} \to \mathbb{C}P_N$ by $z \mapsto p = \{z,\ c \in \mathbb{C}^*\}$. Denote by $\widehat{U}_i$ the open set $z^i \neq 0$ of $\mathbb{C}^{N+1} - \{0\}$. The $N+1$ open sets $\widehat{U}_i$ are a covering of $\mathbb{C}^{N+1} - \{0\}$, their projections $U_i = \Pi(\widehat{U}_i)$, $i = 1, \ldots, N+1$, are an open covering of $\mathbb{C}P_N$. Suppose $p \in U_i$. Let $z \in \Pi^{-1}(p)$, then

$$\left(z^{i_1}/z^i, \ldots, z^{i_N}/z^i\right) \quad \text{with } (i_1, \ldots, i, \ldots, i_N) = (1, \ldots, N+1)$$

is an element of $\mathbb{C}^N$ independent of the choice of z in $\Pi^{-1}(p)$. We choose these numbers $y^j = z^{i_j}/z^i$ as coordinates of p in U_i. The mapping $\phi_1 : p \mapsto y$ is bijective from U_i onto $\mathbb{C}^N$, as can be seen by choosing in each $\Pi^{-1}(p)$, $p \in U_i$, the element z such that $z^i = a^i$ some fixed element of C^*. The mapping $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is the map between open sets of $\mathbb{C}^N$ where $z^1 \neq 0$ and $z^2 \neq 0$ given by

$$\left(y^1_{(1)}, \ldots, y^N_{(1)}\right) \mapsto \left(y^1_{(2)}, \ldots, y^N_{(2)}\right)$$

with

$$y^1_{(2)} = \left(y^1_{(1)}\right)^{-1}, \; y^2_{(2)} = \left(y^1_{(1)}\right)^{-1} y^2_{(1)}, \; \ldots, \; y^N_{(2)} = \left(y^1_{(1)}\right)^{-1} y^N_{(1)}.$$

This is a holomorphic mapping since $y^1_{(1)} \neq 0$ in $U_1 \cap U_2$.
b) $\mathbb{C}^{N+1} - \{0\} \to \mathbb{C}P_N$ is a $\mathbb{C}^*$ principal fiber bundle, because it can be defined by the family of trivializations. $\widehat{U}_i \equiv \pi^{-1}(U_i) \to (U_i, \mathbb{C}^*)$ by $z \mapsto (p, z^i)$. The transition function $U_i \cap U_j \to \mathbb{C}$ is $p \mapsto z^i/z^j$. The commutative group $\mathbb{C}^*$ acts on the fiber $\Pi^{-1}(p)$ by $z \mapsto cz$.
c) The intersections of $\mathbb{C}S^N$ with the open sets $\widehat{U}_i$ are an open covering of $\mathbb{C}S^N$, which project onto the covering U_i, $i = 1, \ldots, N+1$, by Π since in every fiber $\Pi^{-1}(p)$ of $\mathbb{C}^{N+1} - \{0\}$ there is an element with norm 1. The trivializations are now $\widehat{U}_i|^N_{\mathbb{C}S} \to (U_i, U(1))$ by $z \mapsto (p, \frac{z^i}{|z|^i})$, since the $\frac{z^i}{|z^i|}$ take their values in $U(1)$. The same holds for the transition functions $\frac{z^i}{z^j} \frac{|z^j|}{z^i}$, which give therefore to $\mathbb{C}S^N$ the structure of a $U(1)$ bundle over $\mathbb{C}P_N$. Let $z^i = z^i + iy^i$, with x^i and y^i real numbers, $\mathbb{C}S^N$ is given by

$$\sum_{i=1}^{N+1} |z^i|^2 \equiv \sum_{i=1}^{N+1} \left(|x^i|^2 + |y^i|^2\right) = 1,$$

therefore $\mathbb{C}S^N$ can be identified with S^{2N+1}.
d) The homotopy groups π_k, $k \leq 2N$, of S^{2N+1} are zero (II, p. 43).

Problem 9: *Consider the* 1*-form defined on* $\mathbb{C}S^N$ *by the pull back through the embedding* $\mathbb{C}S^N \to \mathbb{C}^{N+1}$ *of the* 1*-form* $\omega_z = (z|\cdot)$, $z \in \mathbb{C}^{N+1}$. *Show that this* 1*-form, also denoted* ω, *is a connection* 1*-form on the* $U(1)$ *principal bundle* $\mathbb{C}S^N \to \mathbb{C}P^N$.

Answer 9: The analytic 1-form on $\mathbb{C}S^N$ given by ω_z at the point $z \in \mathbb{C}S^N$ is a connection 1-form because (cf. p. 361):
1. It takes its values in the Lie algebra $i\mathbb{R}$ of $U(1)$ because $v \in T_z\mathbb{C}S^N$ implies (z, v) imaginary.
2. It is invariant under the action of $U(1)$ (an abelian group) since

$$\omega_{cz}(cv) \equiv (cz, cv) = (z, v) \quad \text{if } |c| = 1.$$

3. If $u \in T_z\mathbb{C}S^N$ is tangent to the fiber V_z, there is a curve $t \mapsto C(t) \equiv e^{i\varphi(t)}z$ such that

$$u = \left.\frac{dC}{dt}\right|_{t=0} \equiv i\dot{\varphi}(0)z$$

therefore

$$\omega_z(u) = (z, u) = \big(z, i\dot{\varphi}(0)z\big) = i\dot{\varphi}(0)(z, z) = i\dot{\varphi}(0) \in i\mathbb{R}.$$

$i\dot{\varphi}(0)$ is the Lie algebra element $\hat{u}$ canonically associated with the vertical vector u.

The horizontal subspace of ω at $z \in \mathbb{C}S^N$ is defined by $\omega_z(v_H) = 0$, i.e., is orthogonal to z.

Problem 10: Let $\gamma : [0, T] \mapsto \mathbb{C}P_N$ by $t \mapsto p(t)$ be a closed loop in $\mathbb{C}P_N$ determined by the projection on $\mathbb{C}P_N$ of a cyclic mapping $t \mapsto \varphi(t) \in \mathbb{C}S^N$.
a) *Determine the element of the holonomy group of the connection ω on the $U(1)$ bundle $\mathbb{C}S^N \to \mathbb{C}P_N$ which is determined by this loop.*
b) *Show that it coincides with the A–A phase of the mapping ψ.*

Answer 10: a) The parallel transport of a point $q \in \mathbb{C}S^N$ along γ is defined by the solution of the differential equation

$$\frac{dq}{dt} = \sigma_q \frac{d\gamma}{dt}, \quad q(0) = \psi(0), \tag{16}$$

where σ_q is the linear mapping $T_p(\mathbb{C}P_N) \to T_q(\mathbb{C}S^N)$ determined by the connection 1-form.
In answer 3b, we have rewritten (16) in terms of the pull back of the connection 1-form, and obtained the corresponding equation in a trivialization. Suppose the loop γ lies in a coordinate patch of $\mathbb{C}P_N$, for instance U_{N+1}, that is the mapping ψ defining that loop takes its values in $\widehat{U}_{N+1} = \Pi^{-1}(U_{N+1})$, i.e., $\psi^{N+1}(t) \neq 0$, $t \in [0, T]$.
Let ϕ be the trivialization of $\mathbb{C}S^N$ over U_{N+1} given by $\phi : \widehat{U}_{N+1} \to (U_{N+1}, U(1))$ by $z \mapsto (p, \frac{z^{N+1}}{|z|^{N+1}})$. Consider the canonical section $s : p \mapsto (p, 1)$. The mapping σ_z, $z \in \mathbb{C}S^N$, is given by $T_p\mathbb{C}P_N \to T_z\mathbb{C}S^N$ by $u \mapsto (u, -(s^*\omega)_p(u))$. Equation (1) reads then (cf. 8b), with $\phi(z) = (p, e^{i\varphi})$

$$i\frac{d\rho}{dt} + (s^*\omega)_{\gamma(t)}\left(\frac{d\gamma}{dt}\right) = 0.$$

Therefore we have by integration a curve $C : [t_1, t_2] \to C(t) \in U_{N+1}$

$$\varphi(t_2) = \varphi(t_1) + \mathrm{i} \int_C s^* \omega.$$

Since s is a diffeomorphism between U_{N+1} and $s(U_{N+1}) \subset \mathbb{C}S^N$ the above formula can be written

$$\varphi(t_2) = \varphi(t_1) + \mathrm{i} \int_{s(c)} \omega. \tag{17}$$

b) If C is the closed loop γ in $\mathbb{C}P_N$ defined by the projection of the α-cyclic mapping ψ, then $s(\gamma)$ is a closed loop in $\mathbb{C}S^N$ defined by a mapping $\tilde{\psi}$ with the same projection and such that $\tilde{\psi}^{N+1}/|\tilde{\psi}|^{N+1} = 1$, hence

$$\tilde{\psi}(t) = \mathrm{e}^{\mathrm{i}\lambda(t)}\psi(t), \quad \mathrm{e}^{\mathrm{i}\lambda}(t) = \left(\psi^{N+1}(t)\right)^{-1}\left|\psi^{N+1}(t)\right|.$$

The formula (17) reads therefore

$$\varphi(T) = \varphi(0) + \mathrm{i}\int_0^T \left(\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}\middle|\tilde{\psi}(t)\right),$$

$$\varphi(T) = \varphi(0) + \mathrm{i}\int_0^T \mathrm{i}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\middle|\psi(t)\right)\mathrm{d}t - \lambda(T) + \lambda(0)$$

if ψ is α cyclic we have

$$\lambda(T) - \lambda(0) = -\alpha.$$

The element $\mathrm{e}^{\mathrm{i}\varphi(0)}\mathrm{e}^{-\mathrm{i}\varphi(t)}$ of $U(1)$ (i.e., of the holonomy group) associated to the parallel transport along γ is therefore the number β associated to ψ defined in problem 8.
If the closed loop γ is not included in one coordinate patch of $\mathbb{C}P_N$, we split it into arcs included into such patches and use the transition function to obtain the formula (17) for the element of the holonomy group associated to γ.

Problem 11: *Indicate how the results of paragraphs* 9 *and* 10 *extend to the case where $\mathcal{H}$ is a separable Hilbert space.*

Answer 11: A separable Hilbert space is isomorphic to the space l^2 of sequences of complex numbers $\{z^i\}$, $i \in N$, such that $\sum_{i=1}^{\infty} |z^i|^2$ is bounded. The scalar product of $z_1, z_2 \in \mathcal{H}$ is $(z_1|z_2) = \sum_{i=1}^{\infty} \bar{z}_1^i z_2^i$. The sphere $\mathcal{S}$ in the subspace of $\mathcal{H}$ of sequences $\{z^i\}$ such that $\sum_{i=1}^{\infty} |z^i|^2 = 1$. The previous reasonings apply, with a covering of $\mathcal{H}$ being defined by the open sets $z^i \neq 0$, $i \in \mathbb{N}$.

Problem 12: *Let M be a smooth manifold as in the beginning of this problem, but we are only given for each $x \in M$ a 1-dimensional equivalence class $p(x)$ of vectors in $\mathcal{S}$, namely we are given a smooth mapping $Y : M \to \mathcal{P}$. Denote by $B := Y^*\mathcal{S}$ the bundle obtained by pulling back $\mathcal{S}$ by Y (p. 336). (B is a $U(1)$) bundle with base M. Y is its "classifying map".) Show that the Berry–Simon connection on B is the pull back of the A.A. connection on $\mathcal{S}$.* Berry–Simon connection

Answer 12: Let $\Pi : \mathcal{S} \to \mathcal{P}$ be the projection of $\mathcal{S}$ onto its base $\mathcal{P}$. The bundle B is obtained by identification with the subset of $\mathcal{P}$ $\{\Pi^{-1}(Y(x)),\ x \in M\}$. Its base is M, its fiber at x is $\Pi^{-1}(Y(x))$, isomorphic to $U(1)$, its transition functions are obtained from the transition functions of $\mathcal{S} \to \mathcal{P}$. An equivalent definition is that the right action of $U(1)$ on the fibers is the same in $\mathcal{S} \to \mathcal{P}$ and $B \to M$.
If U_i is an open set of $\mathcal{P}$ over which $\mathcal{S}$ is trivial, then $u_i = Y^{-1}(U_i)$ is an open set of M over which B is trivial. The mapping $\hat{y} : \Pi_M^{-1}(u_i) \to \Pi^{-1}(U_i)$ is given in these trivializations by $\hat{Y} = (Y, \mathrm{Id})$, its derivatives is (Y', Id).
A section of B an open set $u_i \subset M$ is determined by a mapping $s : x \mapsto X_{(i)}(x) \in \Pi^{-1}(Y(x)) \subset \mathcal{S}$. Straightforward computations show that the pull back by s of the pull back by Y of the A.A. form on $\mathcal{S}$ is the 1-form on u_i given by

$$(A_{(i)})_x(v) = \left(s^*\hat{Y}^*\omega\right)_x(v) = \omega_{(\hat{Y}\circ s)(x)}\left(\hat{Y}'s'v\right) = \left(\mathrm{d}X_{(i)}(x)v | X_{(i)}(x)\right).$$

Therefore $A_{(i)}$ is the representative of the B.S. 1-form in u_i.

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10. A DENSITY THEOREM

A Suplement to Problem VI.6 (pp. 393–396)

Show that $\mathcal{D}(\mathbb{R}^n) \equiv C_0^\infty(\mathbb{R}^n)$ *is dense in* $H_{s,\delta}(\mathbb{R}^n)$.

$H_{s,\delta}(\mathbb{R}^n)$

Answer: We use the notations of Vol. II, problem VI 6, p. 393. We consider a truncating sequence τ_N (Vol. I, p. 434) defined by the composition of a function τ_1 of one variable and a sequence y_N of functions on $\mathbb{R}^n$; namely we set

$$\tau_N = \tau_1 \circ y_N,$$

where $\tau_1 \in C_0^\infty(\mathbb{R})$, i.e. τ_1 is a C^∞ function on $\mathbb{R}$ with compact support. We choose it such that

$$\tau_1(y) = 1 \text{ if } y < 1, \quad \tau_1(y) = 0 \text{ if } y > 2.$$

We take

$$y_N = N^{-1} \log \sigma, \qquad \sigma(x) \equiv \left(1 + |x|^2\right)^{1/2}, \quad x \in \mathbb{R}^n.$$

τ_N is then in $C_0^\infty(\mathbb{R}^n)$ with $\tau_N(x) = 1$ if $\log \sigma(x) < N$, $\tau_N(x) = 0$ if $\log \sigma(x) > 2N$. The family of functions τ_N is uniformly bounded on $\mathbb{R}^n$. We have

$$\frac{\partial \tau_N}{\partial x^i} = \frac{\partial \tau_1}{\partial y} \frac{\partial y_N}{\partial x^i} = \frac{1}{N\sigma} \frac{x^i}{\sigma} \frac{\partial \tau_1}{\partial y}.$$

Therefore the family $\sigma |\frac{\partial \tau_N}{\partial x^i}|$ is uniformly bounded on $\mathbb{R}^n$. An analogous proof shows that each $\sigma^\alpha |D^\alpha \tau_N|$ is uniformly bounded on $\mathbb{R}^n$.
Let $f \in H_{s,\delta}$. Then $\tau_N f = f_N \in H_{s,\delta}$ and has compact support. We have

$$\|f - f_N\|_{H_{s,\delta}} \equiv \left\| f(1-\tau_N) \right\|_{H_{s,\delta}} = \sum_{\alpha \le s} \int_{\mathbb{R}^n} \sigma^{2\delta + 2\alpha} \left| D^\alpha \left[f(1-\tau_N) \right] \right| \mathrm{d}x.$$

The previous results show that f_N converges to f in $H_{s,\delta}$ norm.
The proof of the density of $C_0^\infty(\mathbb{R}^n)$ is completed by regularization (Vol. I, p. 433).

Remark: The theorem would not in general hold for spaces with other types of weights on the successive derivatives.

11. TENSOR DISTRIBUTIONS ON SUBMANIFOLDS, MULTIPLE LAYERS AND SHOCKS

We have defined in volume I, p. 480–482, distributions on a paracompact C^∞ differentiable manifold V: the space $\mathcal{D}'(V)$ of distributions on V is the topological dual of $\mathcal{D}(V)$, space of C^∞ functions on V with compact support, with topology the inductive limit of the family of Frechet topologies; namely each topology is defined on a compact set K of V by the family of semi norms $p_{K_i,m}(\varphi) \equiv \sup_{x\in K_i} |D^m\varphi_i(x)|$, $K_i = u_i(K \cap U_i)$ with (U_i, u_i) the coordinate charts of a locally finite atlas of V, φ_i the representative of $\varphi \in \mathcal{D}(V)$ in this chart and D^m a partial derivative of order m. tensor distributions space of distributions

If $f \in \mathcal{D}'(V)$, then

$$f : \mathcal{D}(V) \to \mathbb{R} \quad \text{by } \varphi \mapsto \langle f, \varphi\rangle.$$

Let η be a given volume element on V (odd form, cf. Vol. I, p. 212, and more elaboration on integrals on non orientable manifolds in De Rham, 1955). We define the representative of $f \in \mathcal{D}'(V)$ in the chart (U_i, u_i) of V as the distribution f_i on the open set of $\mathbb{R}^d$ given by

$$\langle f_i, \rho_i\varphi_i\rangle_{\mathbb{R}^d} \equiv \langle f, \varphi\rangle,$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ denotes the usual duality between $\mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d)$; ρ_i is obtained from the representative η_i of η in U_i by the equation

$$\eta_i \equiv \rho_i \mathrm{d}x^0 \mathrm{d}x^1 \dots \mathrm{d}x^n, \quad d = n + 1.$$

1. a) *Show that the representatives in the intersection $U_i \cap U_j$ of two overlapping charts (U_i, u_i) and (U_j, u_j) are linked in $U_i \cap U_j$ by the classical formula, where $\psi \in \mathcal{D}(\Omega)$, $\Omega \equiv u_i(U_i \cap U_j)$*

$$\langle f_i, \psi\rangle_{\mathbb{R}^d} \equiv \left\langle f_j, \left|\frac{D(x_i)}{D(x_j)}\right| \psi \circ u_i \circ u_j^{-1}\right\rangle_{\mathbb{R}^d}.$$

b) *Show that the representatives of partial derivatives obey the usual law of change of variables.*

Answer 1: a) By the definition of a representative

$$\langle f_i, \psi\rangle_{\mathbb{R}^d} \equiv \left\langle f, \left(\rho_i^{-1}\psi\right) \circ u_i\right\rangle \equiv \left\langle f_j, \rho_j\left(\rho_i^{-1}\psi\right) \circ u_i \circ u_j^{-1}\right\rangle.$$

The given formula is then a consequence of the transformation law of a volume element.
b) This property can be proved using the transformation law given above (cf. Vol. I, p. 456).

Distribution-valued tensors

2) **Distribution-valued tensors** on a C^∞, paracompact d-dimensional manifold V have been defined (Vol. I, p. 482).
The notion of C^∞ tensor fields can be defined without introduction of a connection on V. Given a locally finite atlas on V, we can endow the space of C^∞ tensors of some given type which have compact support on V with an inductive limit of Fréchet topologies (analogous to the one previously given to $\mathcal{D}(V)$), called a **Schwartz topology**. Let $\mathcal{D}_p$ be the space of p-contravariant such tensors, with a Schwartz topology. The space of p-**covariant distribution-valued tensors**, $\mathcal{D}_p^{*\prime}$, is the dual of $\mathcal{D}_p$. We note, if $T \in \mathcal{D}_p^{*\prime}$:

Schwartz topology

covariant distribution-valued tensors

$$T : \mathcal{D}_p \to \mathbb{R} \quad \text{by } \varphi \mapsto \langle T, \varphi\rangle.$$

a) *Let $\mathcal{D}_p$ be the space of C^∞ p-contravariant tensor fields on V with compact support endowed with its Schwartz topology, show that a linear mapping*

$$T : \mathcal{D}_p \to \mathbb{R} \quad by\; \tau \mapsto \langle T, \tau\rangle,$$

is continuous if and only for each τ whitch support in the domain (U_i, u_i) of an admissible chart and each partial derivative operator D^m and each component $\tau^{\alpha_1\ldots\alpha_p}$ there exists a constant C such that

$$\left|\langle T, \tau\rangle\right| \le C \sup_{x\in U_i} \left|D^m \tau^{\alpha_1\ldots\alpha_p}\right|.$$

Answer 2a: Analogous to the one given in Vol. I, p. 436.

b) *Given on V a C^∞ volume element η and an ordinary locally integrable tensor T identify it with a distribution-valued tensor.*

Answer 2b: We set, the integrand being an integrable function on V since T is locally integrable and τ is smooth with compact support

$$\langle T, \tau\rangle \equiv \int_V T.\tau\, \eta \equiv \int_V T_{\alpha_1\ldots\alpha_p}\tau^{\alpha_1\ldots\alpha_p}\, \eta. \tag{1}$$

In a coordinate chart we have

$$\eta \equiv \rho\, \mathrm{d}x^0 \mathrm{d}x^1 \dots \mathrm{d}x^n, \quad n+1=d. \tag{2}$$

The components $T_{\alpha_1 \dots \alpha_p}$, locally integrable functions in the image of the chart in $\mathbb{R}^d$, are the representatives in $\mathbb{R}^d$ of generalized functions in the domain of the chart in V, for which we use the same notation.

3a) *Let T be a distribution valued p-covariant tensor and let $e_0, \dots, e_n$ be a basis of vector fields in an open set U of V. Define the components of T in this basis as distributions on U. If (U, u) is a chart, define the representatives of these components, given a volume element η.*

b) *Define the partial derivatives of the components of T in a chart.*

c) *Suppose that V is endowed with a C^∞ linear connection which leaves invariant the volume element η and has for representative the set of C^∞ functions $\Gamma^\lambda_{\beta\mu}$ in a coordinate chart. Define the covariant derivative of the generalized tensor T by its components in coordinate charts.*

Answer 3a): Let $\varphi \in \mathcal{D}(U)$ be a C^∞ function with compact support in U. We define $T_{\alpha_1 \dots \alpha_p} \in \mathcal{D}'(U)$ by

$$\langle T_{\alpha_1 \dots \alpha_p}, \varphi \rangle \equiv \langle T, \varphi e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \rangle, \quad \varphi \in \mathcal{D}(U).$$

It follows from the linearity and multiplication law of a distribution by a C^∞ function that the components of T transform as the components of a p-covariant tensor by change of basis. If U is the domain of a chart the representatives of the components of T are distributions on an open set of $\mathbb{R}^d$. We use the same notation for a distribution on U and its representative on $u(U)$; we have then the relation

$$\langle T_{\alpha_1 \dots \alpha_p}, \varphi \rangle = \langle T_{\alpha_1 \dots \alpha_p}, \rho \varphi \rangle_{\mathbb{R}^d}$$

where on the left φ denotes a C^∞ function on V with support in U, and on the right its representative in the chart. The definition agrees with the usual one when T is a locally integrable tensor.

b) To have a definition which coincides with the usual one when T is C^1 we define the partial derivative $\partial_\beta T_{\alpha_1 \dots \alpha_p}$ in a coordinate chart as the distribution on U such that

$$\langle \partial_\beta T_{\alpha_1 \dots \alpha_p}, \varphi \rangle \equiv -\big\langle T_{\alpha_1 \dots \alpha_p}, \rho^{-1} \partial_\beta (\rho \varphi) \big\rangle.$$

c) The covariant derivative ∇T is the $p+1$ tensor with components the distributions

$$\nabla_\beta T_{\alpha_1 \dots \alpha_p} \equiv \partial_\beta T_{\alpha_1 \dots \alpha_p} - \Gamma^\lambda_{\beta \alpha_1} T_{\lambda \alpha_2 \dots \alpha_p} - \cdots - \Gamma^\lambda_{\beta \alpha_p} T_{\alpha_1 \dots \alpha_{p-1} \lambda}.$$

It follows from the transformation laws of volume elements and linear connections that these quantities obey the transformation law of a $p+1$ covariant tensor by change of coordinates. The distribution-valued tensor ∇T, thus defined is such that, if $\tau \in \mathcal{D}_p$

$$\langle \nabla T, \tau \rangle \equiv -\langle T, \nabla.\tau \rangle, \quad \text{with } (\nabla.\tau)^{\alpha_1 \dots \alpha_p} \equiv \nabla_\beta \tau^{\beta\alpha_1 \dots \alpha_p}.$$

If the manifold V is endowed with a pseudo riemannian metric with η its volume element and Γ its riemannian connection we find the definition of Vol. I, p. 483.

4. Particularly useful in applications are some tensor distributions with support an n-dimensional submanifold Σ of V, called **multiple layers**. We suppose now that V is oriented and that Σ is defined by an equation $f=0$, with $f: V \to \mathbb{R}$ a smooth function with non vanishing gradient. We suppose there exists an open neighbourhood Ω of Σ in V, divided in disjoint open sets $\Omega_+ \equiv \{x \in \Omega, f(x) > 0\}$, $\Omega_- \equiv \{x \in \Omega, f(x) < 0\}$. We orient Σ by setting $\Sigma \equiv \partial\Omega_-$. We recall (Vol. I, p. 439) that a **Leray form** relative to Σ is an n-form ω such that

multiple layers

Leray form

$$\mathrm{d}f \wedge \omega \equiv \eta$$

Dirac measure

while the **Dirac measure** δ_Σ is the distribution of order zero (measure) with support Σ defined by

$$\langle \delta_\Sigma, \varphi \rangle \equiv \int_\Sigma \varphi\, \omega.$$

The measure δ_Σ depends through ω on the choice of f. Since f has non-vanishing gradient we can choose charts on Ω such that, in local coordinates, $f(x) \equiv x^0$ and

$$\omega \equiv \rho\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

a) *Show the following formulae*

$$\nabla Y_+ = -\nabla Y_- = \ell \delta_\Sigma, \quad \ell = \nabla f,$$

where Y_+ [resp. Y_-] is the locally integrable function equal to 1 *in Ω_+ [resp. Ω_-] and to 0 in Ω_- [resp. Ω_+].*

b) *Show there is a distribution of order* 1 *with support Σ, called δ'_Σ, such that*

$$\nabla \delta_\Sigma = \ell \delta'_\Sigma.$$

Answer 4a: Let $\tau \in \mathcal{D}_1$, then

$$\begin{aligned}\langle \nabla Y_+, \tau \rangle &= -\langle Y_+, \nabla.\tau \rangle \\ &= -\int_{\Omega_+} \nabla_\alpha \tau^\alpha \eta = -\int_{\partial\Omega_+} \tau^\alpha \ell_\alpha \omega = \int_{\partial\Omega_-} \tau^\alpha \ell_\alpha \omega.\end{aligned}$$

The result follows from the multiplication law of C^∞ functions and distributions.

Answer 4b: Let $\tau \in \mathcal{D}_1$, then

$$\langle \nabla \delta_\Sigma, \tau \rangle = -\langle \delta_\Sigma, \nabla.\tau \rangle = -\int_\Sigma \nabla_\alpha \tau^\alpha \omega.$$

Choose local coordinates such that $f(x) \equiv x^0$. Then $x^1, \dots, x^n$ are local coordinates on Σ and, if τ has compact support in the chart,

$$\int_\Sigma \nabla_\alpha \tau^\alpha \omega = \int_{x^0=0} \partial_\alpha(\rho\tau^\alpha)\, \mathrm{d}x^1 \dots \mathrm{d}x^n = \int_{x^0=0} \partial_0(\rho\tau^0)\, \mathrm{d}x^1 \dots \mathrm{d}x^n.$$

In the chosen coordinates we have therefore $\nabla_i \delta_\Sigma = 0$ and we set $\delta'_\Sigma = \nabla_0 \delta_\Sigma$. Hence in these coordinates $\nabla\delta_\Sigma = \ell\delta'_\Sigma$. This tensorial expression is valid in all coordinates, with δ'_Σ the distribution of order 1 defined above in particular coordinates.

5. A tensor T is said to be **regularly C^k discontinuous across Σ** if discontinuous tensor
1) T is C^k in Ω_+ and Ω_-.
2) $T, \nabla T, \dots, \nabla^k T$ converge uniformly to tensors denoted $T_+, (\nabla T)_+, \dots, (\nabla^k T)_+$ when $x \in \Omega_+$ tends to a point of Σ [resp. $T_-, (\nabla T)_-, \dots, (\nabla^k T)_-$ when $x \in \Omega_-$ tends to Σ].
To $T, \dots, \nabla^k T$, locally integrable tensors in $\Omega_+ \cup \Omega_-$ are associated tensor distributions in Ω. On the other hand the following ordinary tensors are defined and continuous on Σ:

$$[T] \equiv T_+ - T_-, \; \dots, \; \left[\nabla^k T\right] \equiv \left(\nabla^k T\right)_+ - \left(\nabla^k T\right)_-.$$

and such that $[T]_{\alpha^1 \dots \alpha^p} = [T_{\alpha^1 \dots \alpha^p}]$, with an analogous property for the derivatives.

a) *Define* $[T]\delta_\Sigma, \dots, [\nabla^k T]\delta_\Sigma$ *as generalized tensors of order* 0 *(measures) with support* Σ.

b) *Show there exists a tensor distribution t of order* 1 *with support Σ such that*

$$\nabla([T]\delta_\Sigma) - [\nabla T]\delta_\Sigma \equiv \ell t.$$

c) *Show there exists a tensor distribution v of order* 2 *such that*

$$\nabla(\ell \otimes t) + \ell \otimes \nabla t + \ell \otimes \ell \otimes v \equiv [\nabla\nabla T]\delta_\Sigma.$$

Answer 5a: These measures are defined by

$$\langle [\nabla^k T]\delta_\Sigma, \varphi\rangle = \int_\Sigma [\nabla^k T].\varphi\omega, \quad \varphi \in \mathcal{D}_{p+k} \text{ if } T \text{ is } p\text{-covariant.}$$

Answer 5b: By the definitions the components of $\nabla([T]\delta_\Sigma)$ are such that

$$\langle \partial_\beta([T]\delta_\Sigma)_{\alpha^1\ldots\alpha^p}, \varphi\rangle = -\int_\Sigma [T_{\alpha^1\ldots\alpha^p}]\rho^{-1}\partial_\beta(\varphi\rho)\,\omega, \quad \varphi \in \mathcal{D}(\Omega).$$

Hence in adapted coordinates, using integration by parts

$$\partial_i([T]\delta_\Sigma)_{\alpha^1\ldots\alpha^p} = \partial_i[T]_{\alpha^1\ldots\alpha^p}\delta_\Sigma,$$

from which results

$$\nabla_i([T]\delta_\Sigma)_{\alpha^1\ldots\alpha^p} = [\nabla_i T_{\alpha^1\ldots\alpha^p}]\delta_\Sigma$$

and the announced tensorial relation where t is the tensor with components in adapted coordinates such that

$$\langle t_{\alpha^1\ldots\alpha^p}, \varphi\rangle = \int_{x^0=0} \left\{[T_{\alpha^1\ldots\alpha^p}]\partial_0(\rho\varphi) - [\nabla_0 T_{\alpha^1\ldots\alpha^p}]\rho\varphi\right\} \mathrm{d}x^1 \ldots \mathrm{d}x^n.$$

Answer 5c: Analogous to the previous one.

Hadamard relation

Remark: If T is continuous across Σ and ∇T regularly discontinuous then (**Hadamard relation**)

$$[\nabla T]\delta_\Sigma = \ell \otimes t,$$

where t is a tensor valued measure with support Σ and components in adapted coordinates given by

$$t_{\alpha^1\ldots\alpha^p} = -[\nabla_0 T_{\alpha^1\ldots\alpha^p}]\delta_\Sigma.$$

shock equations

6. The **shock equations** in mechanics can be obtained by looking for dis-

tribution solutions which are regularly discontinuous tensors across a 3 dimensional submanifold Σ of the 4 dimensional space time. Consider for example the case of a perfect relativistic fluid. The dynamical equations are conservation laws

$$\nabla_\alpha T^{\alpha\beta} = 0$$

for the stress energy tensor

stress energy tensor

$$T_{\alpha\beta} \equiv (r + p)u_\alpha u_\beta + p g_{\alpha\beta}$$

r, p specific energy density and pressure, u unit 4-velocity, g space time metric. We say that the fluid undergoes a shock across Σ if T and ∇T are regularly discontinuous across Σ.

Determine the equations satisfied by the stress energy tensor of a fluid which undergoes a shock across Σ.

shock

Answer 6: If T and ∇T are regularly discontinuous across Σ the derivative of the tensor distribution T is given by:

$$\nabla_\alpha T^{\lambda\mu} \equiv \nabla_\alpha \{T^{\lambda\mu}\} + \ell_\alpha [T^{\lambda\mu}] \delta_\Sigma,$$

where $\nabla_\alpha \{T^{\alpha\beta}\}$ denotes the locally integrable function in Ω equal to $\nabla_\alpha T^{\alpha\beta}$ in Ω_+ and in Ω_-. The sum of a locally integrable tensor and a measure with support Σ is zero if and only if both are zero. The equation $\nabla_\alpha T^{\alpha\beta} = 0$ is therefore equivalent to the equations

$$\nabla_\alpha \{T^{\alpha\beta}\} \quad \text{and} \quad \ell_\alpha [T^{\alpha\beta}] = 0,$$

they are the **Rankine Hugoniot equations**.

Rankine Hugoniot equations

7. *Study* **gravitational shocks** *as a consequence of Einstein equations in vacuum (cf. Problem* 11, *chapter* 6), *supposing that the physical metric g is continuous while its partial derivatives are C^2 regularly discontinuous across Σ.*

gravitational shocks

Answer 7: In order to have covariant formulas we endow the space time V with a given C^∞ metric e (which can be the euclidean metric if permitted by the topology of V) and denote by ∂ the covariant differentiation in the metric e.
The equations satisfied in a generalized sense are Einstein equations in vacuo which read, with H depending smoothly on its arguments

$$R_{\lambda\mu} \equiv -\tfrac{1}{2} g^{\alpha\beta} \partial^2_{\alpha\beta} g_{\lambda\mu} + \partial_\lambda F_\mu + \partial_\mu F_\lambda + H_{\lambda\mu}(g, \partial g) = 0,$$

$$F^\lambda \equiv g^{\alpha\beta} \left(\Gamma^\lambda_{\alpha\beta} - E^\lambda_{\alpha\beta}\right), \quad E^\lambda_{\alpha\beta} \text{ Christoffel symbols of } e.$$

($F^{\lambda}=0$ are the generalized harmonicity conditions.)
The results of 4 show on, the one hand, that there exists a symmetric space time 2-tensor γ defined and continuous on Σ such that

$$[\partial_{\beta}g_{\lambda\mu}]=\ell_{\beta}\gamma_{\lambda\mu};$$

on the other hand they give the formulae

$$\begin{aligned}
&\partial^2_{\alpha\beta}g_{\lambda\mu}=\partial^2_{\alpha\beta}\{g_{\lambda\mu}\}+\ell_{\alpha}\ell_{\beta}\gamma_{\lambda\mu}\delta_{\Sigma},\\
&\partial_{\lambda}F_{\mu}=\partial_{\lambda}\{F_{\mu}\}+\ell_{\lambda}[F_{\mu}]\delta_{\Sigma},\\
&R_{\lambda\mu}=\{R_{\lambda\mu}\}+\big(-\tfrac{1}{2}g^{\alpha\beta}\ell_{\alpha}\ell_{\beta}\gamma_{\lambda\mu}+\ell_{\lambda}[F_{\mu}]+\ell_{\mu}[F_{\lambda}]\big)\delta_{\Sigma}.
\end{aligned}$$

The sum of a locally integrable function on V and a measure with support Σ is zero if and only if both are zero. The measure with support Σ in the above equation is zero if and only if the coefficient of δ_{Σ} is zero. We consider two cases:
1) $g^{\alpha\beta}\ell_{\alpha}\ell_{\beta}\not\equiv 0$ on Σ, then $\gamma_{\lambda\mu}=(\ell_{\lambda}[F_{\mu}]+\ell_{\mu}[F_{\lambda}])(\frac{1}{2}g^{\alpha\beta}\ell_{\alpha}\ell_{\beta})^{-1}$ such a discontinuity in ∂g across Σ is not considered as significant because it can be destroyed by a C^0, regularly C^1 discontinuous across Σ, change of coordinates.
2) $g^{\alpha\beta}\ell_{\alpha}\ell_{\beta}=0$, then ℓ is a null vector in the physical metric. In this case the Einstein equations reduce to

$$\{R_{\lambda\mu}\}=0 \quad\text{and}\quad [F_{\lambda}]=0.$$

Rankine Hugoniot equations

The second set of conditions, which are independent of the choice of the background metric e, are the gravitational **Rankine Hugoniot equations**. They read, using the expressions of F and $[\partial g]$, setting $\ell^{\alpha}=g^{\alpha\beta}\ell_{\beta}$

$$\ell^{\alpha}\theta_{\alpha\beta}=0,\quad\text{with}\quad \theta_{\alpha\beta}\equiv\gamma_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}g^{\lambda\mu}\gamma_{\lambda\mu}.$$

An extension of the given method permits a study of the propagation of gravitational shocks (see Lichnerowicz, 1973) analogous to those obtained for high frequency waves (Choquet-Bruhat, 1969).

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12. DISCRETE BOLTZMANN EQUATION

We recall that the usual **Boltzmann equation** on a pseudo riemannian manifold (V, g) reads

Boltzmann equation

$$\mathcal{L}f \equiv p^\alpha \frac{\partial f}{\partial x^\alpha} + P^\alpha \frac{\partial f}{\partial p^\alpha} = \mathcal{I}(f)$$

where x^α, p^α are local coordinates in the tangent bundle $T(V)$ to V, representing position and 4-momentum of particles; f is the distribution function of particles on $T(V)$; (p, P) is the tangent vector to the trajectories of particles in the phase space $T(V)$ between collisions: the operator $\mathcal{L}$ is the Lie derivative along such a free trajectory, $\mathcal{L}f = 0$ expresses the conservation of f in the absence of collisions. The term $\mathcal{I}(f)$ expresses the loss and gain of particles with momentum p undergoing collisions at the point x.

If no other field is present the particles follow between collisions geodesics of the metric g. In the general case we have:

$$P^\alpha \equiv -\Gamma^\alpha_{\lambda\mu} p_\lambda p_\mu + Q^\alpha$$

where Q represents the non gravitational force fields. If Q results from an electromagnetic field represented by an exterior 2-form F, then

$$Q^\alpha \equiv e F^\alpha_\beta p^\beta, \quad e \text{ the charge of the particles.}$$

In a collisionless model $\mathcal{I}(f) \equiv 0$, the equation is called the **Vlasov equation**. Suppose that the momenta of the particles can take only a finite number of values, depending on the point of V where the particle is located, take a distribution generalized function of the form

Vlasov equation

$$f(x, p) \equiv \sum_{I=1}^{N} a_I(x) \delta_{B_I(x)}(p).$$

Where a_I is a smooth function and B_I a smooth vector field on V while $\delta_{B_I(x)}(p)$ is the measure on $T(V)$ defined by (Vol. I, p. 430)

$$\langle \delta_{B_I(x)}(p), \varphi(x, p) \rangle = \int_V \varphi\big(x, B_I(x)\big)\, \eta, \quad \varphi \in \mathcal{D}\big(T(V)\big),$$

with in local coordinates $\eta \equiv \rho\, \mathrm{d}x^0 \ldots \mathrm{d}x^n$, with $\rho \equiv |\det g|^{1/2}$ when V is endowed with a pseudo-riemannian metric g. Note that the volume element of $T(V)$ is then $\rho^2\, \mathrm{d}x^0 \ldots \mathrm{d}x^n\, \mathrm{d}p^0 \ldots \mathrm{d}p^n$.

1) *Compute the partial derivatives of the generalized functions* δ_{B_I} *on* $T(V)$.

2) *Suppose that* Q *is such that* $\frac{\partial Q^\alpha}{\partial p^\alpha} = 0$ (*show that such is the case if* Q *is the Lorentz force of an electromagnetic field*). *Write the system of partial differential equations satisfied by* a_I *and* B_I *when* f *satisfies the Vlasov equation in a generalized sense: the system is called a discrete Vlasov equation.*

3) *Propose a model for a discrete Boltzmann equation.*

Answer 1: Let φ be a C^∞ function with compact support on $T(V)$. The partial derivatives of δ_{B_I} are such that (we suppress the explicit dependence of δ_{B_I} on x and p to shorten the writing, but keep it in φ to make the proof more transparent)

$$\left\langle \frac{\partial}{\partial p^\alpha}\delta_{B_I}, \varphi(x,p)\right\rangle = -\left\langle \delta_{B_I}, \rho^{-2}\frac{\partial(\rho^2\varphi(x,p))}{\partial p^\alpha}\right\rangle.$$

Hence, since ρ does not depend on p:

$$\left\langle \frac{\partial}{\partial p^\alpha}\delta_{B_I}, \varphi(x,p)\right\rangle = -\int_V \left\{\frac{\partial\varphi(x,p)}{\partial p^\alpha}\right\}_{p=B_I(x)} \eta,$$

$$\left\langle \frac{\partial}{\partial x^\alpha}\delta_{B_I}, \varphi(x,p)\right\rangle \equiv -\left\langle \delta_{B_I}, \rho^{-2}\frac{\partial}{\partial x^\alpha}\left(\rho^2\varphi(x,p)\right)\right\rangle.$$

We use the property $\rho^{-1}\frac{\partial}{\partial x^\alpha}\rho \equiv \Gamma^\lambda_{\alpha\lambda}$, together with the identity

$$\left\{\frac{\partial}{\partial x^\alpha}\varphi(x,p)\right\}_{p=B_I} = \frac{\partial}{\partial x^\alpha}\varphi\big(x, B_I(x)\big) - \frac{\partial B_I^\beta}{\partial x^\alpha}\left\{\frac{\partial\varphi(x,p)}{\partial p^\beta}\right\}_{p=B_I}$$

the definition of δ_{B_I} and the compactness of the support of φ to obtain

$$\frac{\partial}{\partial x^\alpha}\delta_{B_I} = -\Gamma^\lambda_{\alpha\lambda}\delta_{B_I} - \frac{\partial B_I^\beta}{\partial x^\alpha}\frac{\partial\delta_{B_I}}{\partial p^\beta}.$$

Answer 2: Straightforward computation leads then to the following formula

$$p^\alpha \frac{\partial}{\partial x^\alpha}\delta_{B_I} + \left(Q^\alpha - \Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu\right)\delta_{B_I}$$
$$\equiv \left(\nabla_\beta B_I^\beta + \frac{\partial Q^\alpha}{\partial p^\alpha}\right)\delta_{B_I} - \frac{\partial}{\partial p^\alpha}\left\{(p^\beta \nabla_\beta B_I^\alpha - Q^\alpha)\delta_{B_I}\right\}.$$

If Q is an electromagnetic force then $Q^\alpha = eF^\alpha_\beta p^\beta$, hence $\frac{\partial Q^\alpha}{\partial p^\alpha} = 0$. The Vlasov equation is satisfied in a generalized sense if and only if all independent measures and doublets on the left hand side vanish. This fact leads to the equations, written when $\frac{\partial Q^\alpha}{\partial p^\alpha} = 0$:

$$B_I^\beta \nabla_\beta B_I^\alpha - \left\{Q^\alpha\right\}_{p=B_I} = 0, \quad I = 1, \ldots, N,$$
$$\nabla_\beta\left(a_I B_I^\beta\right) = 0, \quad I = 1, \ldots, N.$$

3) A natural model for a discrete Boltzmann equation is obtained by replacing in the usual Boltzmann equation the collision term $\mathcal{I}(f)$ by a sum of measures:

$$\mathcal{I}(f) \equiv \sum_{I=1}^{N} \mathcal{I}_I\, \delta_{B_I}$$

where $\mathcal{I}_I$ represents the balance of loss and gain of particles with B_I momentum at the point x. For binary collisions $\mathcal{I}_I$ is of the form

$$\mathcal{I}_I \equiv \sum_{J,K,L} \left\{\sigma^{KL}_{IJ} a_K a_L - \sigma^{IJ}_{KL} a_I a_J\right\}$$

where the σ^{KL}_{IJ} are given positive functions on V.

The equations are

$$B_I^\beta \nabla_\beta B_I^\alpha - \left\{Q^\alpha\right\}_{p=B_I} = 0, \quad I = 1, \ldots, N,$$
$$\nabla_\beta\left(a_I B_I^\beta\right) = \mathcal{I}_I, \quad I = 1, \ldots, N.$$

4a) *Consider the metric and the external force Q as given. Show that the system of equations is a nonlinear, causal, Leray hyperbolic first order system, when the B_I are time like or null.* Leray hyperbolic first order

b) *Study the case where the metric is considered as a field variable satisfying Einstein equations whose source is the stress energy tensor generated by the distribution generalized function f, while Q is the Lorentz force* *corresponding to an electromagnetic field generated by f when particles have an electric charge e.*

Leray weights

Answer 4: a) Give to the equations (1), (2) and to the unknown B_I, a_I the Leray weights (problem 7, Chapter V, p. 255)

$$m(1) = 1, \ m(2) = 0 \quad \text{and} \quad n(B_I) = 2, \ n(a_I) = 1.$$

Leray criteria

The characteristic matrix is then a diagonal matrix with elements the hyperbolic polynomials (of degree 1) $B_I^\alpha X_\alpha$. If the vectors B_I are time like or null, the cones (half spaces) $B_I^\alpha X_\alpha < 0$ have a non empty intersection, which contains the interior of the future light cone. The Leray criteria for a strictly and causal hyperbolic system are satisfied.

b) The stress energy tensor and electric current generated by f are respectively the tensor and the vector on V given by

$$T_{\alpha\beta} = \sum_I a_I B_{I\alpha} B_{I\beta}, \qquad J_\alpha = \sum_I e a_I B_{I\alpha}.$$

Leray weights

causal system

Gevrey class

Consider the Einstein equations in harmonic gauge (problem 11, chapter VI, p. 405) and Maxwell equations in Lorentz gauge (p. 337). There are no Leray weights which make the characteristic matrix diagonal with elements hyperbolic polynomials, except when the space-time is 2-dimensional. The characteristic determinants is a product of terms $B_I^\alpha X_\alpha$ and $g^{\alpha\beta} X_\alpha X_\beta$, the system is causal and non strictly hyperbolic in the sense of Leray–Ohya, the Cauchy problem for such systems is well posed in a Gevrey class (cf. Leray–Ohya, 1967).

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INDEX

ERRATA TO

ANALYSIS, MANIFOLDS AND PHYSICS. PART I

Page*	Line	Errata
		Front Matter
xi	8	III. Differentiable Manifolds,
		Chapter I
6	12	Example 1: Let P be the set $\mathbb{N}$ of all
	16	$\mathbb{N}$ has no maximal element.
11	8	(p. 17)
12	−2	is open iff it is a neighborhood
13	3	x: $(N(x) - \{x\}) \cap A \neq$
	12	in A. The set A is dense
14	5	For each $x \in X$ and $U \in \mathscr{U}$ with $x \in U$ there exists $B \in \mathscr{B}$ such that $x \in B$ and $B \subset U$.
	−9	is a filter $\mathscr{F}(x)$.
15	−10	U_j such that $V_i \subset U_j$.
20	12	is simply connected when $n \geq 2$.
21	13	Exercise 2, p. 68.
23	4	vector space (p. 27) is bounded
24	caption	$d_1(P, 0) \leq a$, $d_2(P, 0) \leq a$, $d_3(P, 0) \leq a$
27	−10	continuous (p. 21)
28	3	into the neighborhood $\|\alpha x -$
31	1	p. 56
39	−16	**Haar measure** (p. 180).
46	9	a σ-field of subsets, $\mathscr{A}$
49	−2	$\|m(A)\| < \|m\|(A)$
55	1	$+\|x^2\|^p)^{1/p}$
61	17	Let X and Y be two metric spaces,
62	19	operator T on $L^2(Y)$ by
65	4	$v^2 = (v\|v)$
	22	add C_p in the margin
66	3	Let $P = C(V^3{}_{(3)})$

*Refers to the revised edition.

Page*	Line	Errata
	−8	$\forall \boldsymbol{v},\ \boldsymbol{w} \in V^n{}_{(s)}$
67	−12	So we have for k odd
69	3–10	

Answer: We shall construct a covering of the product space $\times_{\alpha\in\mathbb{N}} X_\alpha$ which has no countable subcovering, let alone a finite one. Let $A \subset \mathbb{N}$ and $V_A = \times_{\alpha\in\mathbb{N}} U_\alpha$ be an element of the topology on the product space defined as follows:

$$U_\alpha = \begin{cases} [0, 2/3) & \text{if } \alpha \in A \\ (1/3, 1] & \text{if } \alpha \notin A \end{cases}.$$

Then $\{V_A;\ A \subset \mathbb{N}\}$ is an open covering of the product space; it has no countable subcovering. Indeed let $\{V_{A_i}\}_i$ be any countable subset of $\{V_A;\ A \subset \mathbb{N}\}$. One can always find a point $x = \{x_1, x_2, \ldots\}$ in the product space such that $x \notin \{V_{A_i}\}_i$. For example, let

$$x_i = \begin{cases} 1/6 & \text{if } i \notin A_i \\ 5/6 & \text{if } i \in A_i \end{cases} \quad \text{then } x \notin \{V_{A_i}\}_i.$$

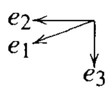
Contributed by J. Labelle.

Chapter II

72	9	mapping: $\mathrm{D}l|_{x_0} = l$, $D\,\mathrm{Id}|_{x_0} = \mathrm{Id}$, $\forall x_0$.
	−4	of f. A point where the rank is not maximal is called **critical**. If $n = p$ the determinant of $\mathrm{D}f$ is called the **Jacobian**
77	11	for the function q
85	5	A sufficient condition theorem
	6	A sufficient condition for f

Chapter III

111	−12	$x \in U \subset X$.
127	−15	admissible atlas (see p. 543)
129	8	replace "identical" by "isomorphic"
	12	bundle in particular (p. 376)
131	−5	E_1 a subspace of E
	footnote	or [Osborn II 4]
135	−6	if follows that $f^* \circ \boldsymbol{v} = \boldsymbol{v} \circ f^*$.

Page*	Line	Errata
151	−5	$\det\left[\left.\dfrac{\partial x'^i}{\partial x^j}\right\vert_{x_0^n=0}\right]$
	footnote	only identity transformations
152	17	$\mathscr{L}_{\mathbf{v}}\mathbf{g}$
	−7	The tensor $\mathscr{L}_{\mathbf{v}}\mathbf{g}$ is the strain tensor generated by the vector field $\boldsymbol{v}$
153	6	$\sigma_g \circ \sigma_h = \sigma_{gh}$ (left action of G on X) or $\sigma_g \circ \sigma_h = \sigma_{hg}$ (right action of G on X)
154	footnote	the group which defines isometries
155	20	effectively, transitively, and freely on G.
156	−1	$[\boldsymbol{v}_\alpha, \boldsymbol{v}_\beta](g)$
158	12, 13	and the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$
	−7	(p. 209)
162	−3	(or $\sigma_h \circ \sigma_g$)
164	7	$\boldsymbol{v}^k_{(\alpha)}$ as well as the generators $\boldsymbol{v}^L_{(\alpha)}$ and $\boldsymbol{v}^R_{(\alpha)}$ of
165	7	inserted in (5)
165	−3	158 and p. 353
166	2	$[\boldsymbol{v}^k_{(\alpha)}, \boldsymbol{v}^k_{(\beta)}]^j(x)$
	3	$-[D_{,\alpha}, D_{,\beta}]^i_k(e)x^k$
	5	is defined by matrix multiplication, antisymmetrized
176	−11	for every $v \in C_1$
177	9, 12	$\lambda = \frac{1}{4} V^{\alpha\beta} e_\alpha e_\beta$
186	−15	i.e., all rotations
188	−8	be nonsingular. $k_{\alpha\beta}$ defines a metric called **Cartan–Killing**.
190	−6	jacobian matrix of the embedding mapping (p. 240)
192		$\binom{a^1}{a^2} = ((z^1)^2 + (z^2)^2)^{-2} \begin{pmatrix} (z^2)^2 - (z^1)^2 & -2z^1z^2 \\ -2z^1z^2 & (z^1)^2 - (z^2)^2 \end{pmatrix} \binom{b^1}{b^2}$
		Chapter IV
196	footnote	Then if α and β are 1-forms
200	−2	By property 2
203	5	are vectors and $\boldsymbol{W}$ a pseudo-vector
	9	$\operatorname{div} f\boldsymbol{W} = \operatorname{grad} f \cdot \boldsymbol{W} + f \operatorname{div} \boldsymbol{W}$
210	11	$\phi_x\colon (X)^p$
211	16	$[\boldsymbol{\varphi}\,\hat{;}\,\boldsymbol{\psi}]$
216	−12	$\iint_{\text{surface}}$
218	?	

Page*	Line	Errata
223	18	group [See Analysis, Manifolds and Physics, Part II]
	20	Move the sentence "H^p is often called the de Rham group" and the marginal note "de Rham cohomology" back to line 12.
224	1, 2	Delete and replace by: "For the cases of 0-chains see for instance [Patterson]."
	4	Insert the definition of the Euler–Poincaré characteristic, which can be found on p. 293.
225	6	ω depends only on $x^1, \dots, x^p$.
226	−5	for $C \in H_p$, $\omega \in H^p$.
250	23, 24	delete "A differential ... system."
253	17	vector field. By the theorem on p. 248, C is completely integrable.
254	11	result in a set $\{\bar{\theta}^{(k)}\}$
271	7	equations (see example p. 263)
276	−9	$= \underline{-\mathrm{d}y_\alpha}$
		Chapter V
285	14	(p. 134)
287	−3	**manifold**. The metric is called **lorentzian**. It is
302	16	$u = C' \dfrac{\mathrm{d}u^i}{\mathrm{d}t} = \dfrac{\mathrm{d}C^i}{\mathrm{d}t}$,
306	5	In a moving frame the
313	18	$(\nabla_{\boldsymbol{v}} \boldsymbol{u})^{\parallel} = (\nabla_{\boldsymbol{u}'} \boldsymbol{v}'$
314	−5	a lorentzian metric it is a maximal
	−4	is a minimal hypersurface; in
316	−8	$\Phi^*(\mathrm{D}f) = \mathrm{D}(\Phi^* f)$
	−4	pp. 482–486.
317	6	$\mathrm{d}(^*\boldsymbol{\omega})$
331	−12	$(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n/2}, \boldsymbol{J}\boldsymbol{v}_1, \dots, \boldsymbol{J}\boldsymbol{v}_{n/2})$
	−13	$\mathscr{L}_v \eta = 2\Phi\eta.$ (1)
	−11	such that $\Phi =$ constant is the group of isometries and dilations
	−8	$= \eta_{\lambda\beta}\partial_\alpha v^\lambda + \eta_{\lambda\alpha}\partial_\beta v^\lambda = 2\Phi\eta_{\alpha\beta}$
	−7	where ϕ is obtained by contraction $2\phi = \frac{1}{2}\partial_\alpha v^\alpha$
354	12–20	Replace lines 12 to 20 by: This transformation is defined in the case of an euclidean metric if one adds to the space a point at infinity, whose image in an inversion is the origin of this inversion. In the

Page*	Line	Errata
		case of the Lorentz metric the inversion with origin point x_0, for instance $x_0 = 0$, is defined only when $x^2 = \eta_{\alpha\beta} x^\alpha x^\beta \neq 0$, that is when x is not a null (i.e. lightlike) vector. The conformal group is defined in Segal cosmos ($S^3 \times \mathbb{R}$, ref p. 356) universal cover of the compactified Minkowski spacetime (cf. problem V.8). For more on the compactification of Minkowski space see for instance R. Penrose "Conformal Treatment of Infinity" pp. 563–584 in *Relativity, Groups, and Topology* (Les Houches 1963) Eds C. DeWitt and B. DeWitt (Gordon and Breach New York 1964)
356	13	in empty space.
		Chapter V bis
359	15	subspace of $T_p(P)$. Due to
360	16	Add: We also write $\check{\boldsymbol{\gamma}} = \boldsymbol{v}_\gamma(p)$.
362	−14	a 1-form on U with values
363	7	$= \mathrm{Ad}(g^{-1})$
365	2	$= (\phi_j^{-1*}\omega)$
	3	$L_{g_{ji}}(x)$
	9,10	$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} \mapsto \begin{pmatrix} \partial x/\partial x & \partial x/\partial g \\ \partial L_{g_{ij}(x)} g/\partial x & \partial L_{g_{ij}(x)} g/\partial g \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}$
	14	$\boldsymbol{\theta}_{\mathrm{MC}}(g'_{ji}(x)\boldsymbol{v})$
369	8	delete the composition sign
371	10	then (p. 367),
	−13	corresponding to ψ and $\nabla_u \psi$
372	−13	of a covariant vector.
373	−14	Add: "$h \equiv$ hor"
375	4	$= \cdots + \frac{1}{2} \cdots$
377	4	$\mathscr{L}_{\mathrm{ver}\,\mathrm{u}}\theta$
379	14	differential 2-form $\bar{\boldsymbol{\Omega}}$ on X
380	−8, −7	Delete: "It is ... p. 359"
	−6	connection as defined on p. 359
381	−13	of the connection, as we have already seen on an
382	15	the injection f in P
	20	of G_1; thus
	−4	if $g_i = g_{ij}(x) g_j$

Page*	Line	Errata
383	13	the realization σ of G
	-4	$\tau_{g_2 g_1} =$
	-3	$P \times_G F$
384	-12	action of G_1, where (P, X, π, G) is a principal fibre bundle
385	17, 18	Delete and insert: 2) Suppose G is reducible to G, and let P_1 denote a reduced bundle with injection $f\colon P_1 \to P$. Let μ be the projection $P \to P/G_1$. The mapping $\mu \circ f$
385	$-9, -1$	Delete and insert the following

Theorem. Every fibre bundle (E, X, π, F, G) such that the base manifold X is paracompact and the fibre F is diffeomorphic to $\mathbb{R}^n$ admits infinitely many cross sections.

Proof: It is easy to show that when E is a vector bundle, i.e. when F_x is a vector space, it admits infinitely many cross sections. Indeed, let $\{\theta_i\}$ be a partition of unity on X subordinates to a locally finite covering by open sets V_i such that $V_i \subset U_j$ some open set of an atlas of X over which E is trivializable. Let σ_i be an arbitrary cross section over V_i. Then the element of F_x given by the finite sum

$$\sigma(x) = \sum_i \theta_i(x)\sigma_i(x)$$

is a cross section over X. The theorem is stronger because it does not require a canonical identification of a point of F_x with the origin of $\mathbb{R}^m$, nor the group G to be linear. For the proof[1] one uses the property that a differentiable function defined on a closed set of $\mathbb{R}^m$ can be extended to the whole of $\mathbb{R}^m$ together with Zorn's lemma, to show that every cross section defined over a closed set $\overline{Y} \subset X$ can be extended to a cross section over X.

[1]Cf. R. Godement, *Theorie des Faisceaux* (Hermann, Paris, 1958) p. 151 or Kobayashi and Nomizu, loc. cit. Vol. I, p. 58.

Page*	Line	Errata
386	21	and ρ_x^{-1} the linear
	-15	product in $\mathbb{R}^n$ and ρ_x^{-1} the linear

Page*	Line	Errata
	-9	in part A (pp. 380–381)
387	$-12, -13$	delete i.e. ... and insert: $[\mathscr{H}_0(p') = g\mathscr{H}_0(p)g^{-1}]$.
389	18	p. 388
	22	and by the subspace
	-7	horizontal field is horizontal (p. 374), and that
	-5	$= -\boldsymbol{\Omega}(\boldsymbol{u}, \boldsymbol{v})$
390	-10	$\cdots = \dfrac{1}{(2k)!} \cdots (v_{\sigma(1)}, v_{\sigma(2)}), \ldots,$
	-1	$T_xX, x = \Pi(p)$
391	1	and be unique
	9	Add: $\mathrm{d}f(\Omega)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \pi^* \, \mathrm{d}\bar{f}(\boldsymbol{\Omega})(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ $= \mathrm{d}\bar{f}(\Omega)(\pi' \operatorname{hor} \boldsymbol{v}_1, \ldots, \pi' \operatorname{hor} \boldsymbol{v}_n) =$ $\mathrm{d}f(\Omega)(\operatorname{hor} \boldsymbol{v}_1, \ldots, \operatorname{hor} \boldsymbol{v}_n) = \mathrm{D}f(\Omega)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$
	18	$+\frac{1}{2}[\;]$
	20	$+\frac{1}{2}[\;] + \frac{1}{2}[\;]$
392	4	This, together with the fact Φ being invariant
	5	on a vertical vector projects to a form $\bar{\Phi}$ on X,
	7	$= \mathrm{d}\bar{\Phi}$.
394	11	$[f(V)]^2$
395	8	curvature 2-form
	11	complexification
	-1	(p. 224)
397	-8	via a scalar product in the fibres, possibly deduced
398	5	to the scalar product in
399	11	Decomposition theorem.
	17,18	Proof: It follows formally the same lines as the proof given above for the finite dimensional space $E_{p,x}$. Its validity in the new context rests
	-7	therefore $\boldsymbol{\Delta}_p$ is an elliptic
401	4	$))\wedge\rho^*$ ($\wedge$ is exterior product)
	11,13	$\{\boldsymbol{D}_p, \boldsymbol{E}_p\}$
402	10	$\mathbf{d}\boldsymbol{x}^\nu = \bar{\boldsymbol{\Omega}}_i$,
404	figure	Replace π_i by π_1
405	5	$\mathrm{D}\tilde{\psi}_u(\boldsymbol{v}) =$
	11	$\underset{\Delta}{=} \tilde{\psi}'(\boldsymbol{v}) + \rho_e'(\omega(\boldsymbol{v}))\tilde{\psi}(u)$
406	-13	$\Phi_{i,x}\circ$

Page*	Line	Errata
407	7	$\to (\check{\phi}(s_i(x)))^{-1}$
	-12	when the connections A is flat.
408	$-10, -11$	bold face $\mathrm{d}\phi$
	-5	p: $U(1) \to L(\mathbb{C}, \mathbb{C})$
410	20	, $\gamma_1 = -(2\pi \mathrm{i})^{-1} \operatorname{tr} \bar{\boldsymbol{\Omega}}$
	-13	$C_1 = \cdots = \dfrac{1}{2\pi} \displaystyle\int_{S^1} \boldsymbol{n}\, \mathbf{d}\boldsymbol{\phi}$
	-10	$\mathrm{i}A_+ = \mathrm{i}\boldsymbol{A}_- \mathrm{i}n\, \mathbf{d}\boldsymbol{\phi}$.
	-9	define the following electromagnetic field
	-6	define the following
411	18	Do $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$
415	12	Atiyah
419	-1	$(\tilde{\mathscr{H}} \circ$
420	6	change signs before 1/4
	6,7,14,17	change signs before 1/8
421	6	$-\frac{1}{2}\gamma^\alpha \gamma^\beta (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\psi$
	7	if there is no torsion.
	$-3, -4$	$\times \mathbb{R} \times \mathbb{R}$
422	References	T. Regge, "The group manifold approach to unified gravity", in *Relativité, Groupes et Topologie II*, eds. B.S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984) pp. 933–1006.

Chapter VI

Page*	Line	Errata
435	13	exists C^∞ on U, but cannot
441	-9	in general vanish for
	-8	it vanishes if $\varphi'(a) = 0$.
	-7	Remark. $\varphi\delta'_a =$
455	-10	$T_n = \dfrac{1}{\mathrm{i}} \dfrac{\mathrm{d}}{\mathrm{d}x}$
468	16	Proof: X is solution in $\mathscr{A}$ since,
472	-2	$\mathrm{D}X = B$ *has at most one* ... *if* D^* *has an*
477	10	$= (\mathrm{i}y)^\alpha$.
478	5	p. 272
490	-6	$\cdots = \|(1 + \|x\|^2)^{m/2} \mathscr{F} f\|_2$
494	13	$\cdots \Box u = (- - -)u = 0$

Page*	Line	Errata
499	-2	$\cdots = \cdots - \int \sum \frac{\partial \varphi}{\partial x^i} \frac{\partial u}{\partial x^i} \mathrm{d}x$
512	7	$= \frac{1}{4\pi} \delta_c.$
	9	$= \frac{1}{t} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$
513	13	$\alpha^1 = \sin\theta \cos\phi, \alpha^2 = \sin\theta \sin\phi, \alpha^3 = \cos\theta$
522	-9	For a system with an infinite
523	6	$\cdots = \langle T, L^* U \rangle$ when U is C^∞ with compact support
532	8	$Y^-(x) \exp(ax)$
from 538 to 539	-2 23	Replace by:

Answer: a) The elementary kernels $E(t, s)$ are solutions of

$$(-\mathrm{d}^2/\mathrm{d}t^2 - \rho^2) E(t, s) = \delta(t, s). \tag{4}$$

The Fourier transform of this equation is

$$\mathscr{F}((-\mathrm{d}^2/\mathrm{d}t^2 - \rho^2)\delta * E)(\eta) = 1,$$

$$(\eta^2 - \rho^2)\mathscr{F} E = 1,$$

$$(\mathscr{F} E)(\eta) = \frac{1}{2\rho} \mathrm{Pv}\left(\frac{1}{\eta - \rho} - \frac{1}{\eta + \rho} \right) + K_1 \delta(\eta - \rho) + K_2 \delta(\eta + \rho). \tag{5}$$

We can choose K_1 and K_2 such that the elementary kernel is in the convolution algebra $\mathscr{D}'^{+}$ or $\mathscr{D}'^{-}$. Recall (Problem VI 7) that

$$(\mathscr{F} Y^{\pm})(\eta) = \mp \mathrm{i} \left(\mathrm{Pv} \frac{1}{\eta} \pm \mathrm{i}\pi \delta_\eta \right) = \mp \mathrm{i}(\eta \mp \mathrm{i}0)^{-1}. \tag{6}$$

Hence

$$E^+ \in \mathscr{D}'^{+} \qquad \text{if } K_1^+ = -K_2^+ = \mathrm{i}\pi/2\rho \text{ and}$$

$$\mathscr{F} E^+ = \frac{1}{2\rho} \left(\frac{1}{\eta - \rho - \mathrm{i}0} - \frac{1}{\eta + \rho - \mathrm{i}0} \right), \tag{7a}$$

$$E^- \in \mathscr{D}'^{-} \qquad \text{if } K_1^- = -K_2^- = \mathrm{i}\pi/2\rho \text{ and}$$

$$\mathscr{F} E^- = \frac{1}{2\rho} \left(\frac{1}{\eta - \rho + \mathrm{i}0} - \frac{1}{\eta + \rho - \mathrm{i}0} \right). \tag{7b}$$

Page*	Line	Errata

To compute $E^{\pm}$ from (5) using (6) one can translate (see p. 458)

$$\mathrm{Pv}\left(\frac{1}{\eta-\rho}\pm\mathrm{i}\delta(\eta-\rho)\right)$$

by ρ and the other two terms by $-\rho$:

$$E^{\pm}(t,s)=\mp Y^{\pm}(t-s)\rho^{-1}\sin(\rho t-\rho s). \tag{8}$$

The propagator $E=E^{+}-E^{-}$ is

$$E(t,s)=-\rho^{-1}\sin(\rho t-\rho s).$$

We could, of course have obtained (8) by solving (4) according to the method developed on p. 469, which says

$$E^{\pm}(t,s)=Y^{\pm}(t-s)h^{\pm}(t-s)$$

where the C^{∞} functions $h^{\pm}$ satisfy the homogeneous equation and the following boundary conditions:

$$h^{+}(0)=0,\ h^{+\prime}(0)=-1 \quad \text{and} \quad h^{-}(0)=0,\ h^{-\prime}(0)=1.$$

Equation (7) suggests the following integral representation for the ele-

541	−3	equation (9)
		Chapter VII
549	6	Hilbert space of norm
571	footnote	The Morse index is the negative of the index defined on p. 287
590	−7	$=-\mathrm{d}(\cdots+\frac{1}{2}m^2u^2)\,\mathrm{d}x)(U,V).$
593	−4	$\int_A^B \exp(-\lambda t^2/2)h(t)\,\mathrm{d}t = \frac{-1}{\lambda t}\left(h(t)-\frac{h(t)}{\lambda t^2}+\frac{h'(t)}{\lambda t}\right)\exp(-\lambda t^2/2)\Big\vert_A^B$

Page*	Line	Errata
		$$+\frac{1}{\lambda^2 t^2}\int_A^B \exp(-\lambda t^2/2)$$ $$\times\left(3\frac{h(t)}{t^2}-3\frac{h'(t)}{t}+h''(t)\right)\mathrm{d}t$$
		References
603		CHERN, S.S., *Selected Papers* (Springer-Verlag, New York, 1978).
606		ITZYKSON, C. and J.B. ZUBER, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
608		SCHUTZ, B., *Geometrical methods of mathematical physics* (Cambridge University Press, Cambridge, 1980).

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