ASTR 610: Solutions to Problem Set 2

Problem 1: The Sound-Speed of the Photon-Baryon fluid
Consider a Universe that consists solely of baryons and photons (no dark matter, no dark energy, no neutrinos). Show that, during the radiation era, the sound speed of the photon-baryon fluid can be written as

$$c_{\rm s} = \frac{c}{\sqrt{3}} \left[\frac{3}{4} \frac{\bar{\rho}_{\rm b}(z)}{\bar{\rho}_\gamma(z)} + 1 \right]^{-1/2}$$

where $\bar{\rho}_{\rm b}(z)$ and $\bar{\rho}_\gamma(z)$ are the mean energy densities of baryons and photons at redshift z, and c is the speed of light.

ANSWER: The sound speed is defined as $c_{\rm s}^2 = (\partial P/\partial \rho)_S$. Using that $\rho = \rho_\gamma + \rho_{\rm b}$ while the pressure is dominated by that of the radiation, $P = P_\gamma = \frac{1}{3}\rho_\gamma c^2$, we have that

$$c_{\rm s} = \left(\frac{{\rm d}P_\gamma}{{\rm d}\rho_\gamma} \frac{\partial \rho_\gamma}{\partial \rho} \right)^{1/2} = \frac{c}{\sqrt{3}} \left[1 + \frac{\partial \rho_{\rm b}}{\partial \rho_\gamma} \right]^{-1/2}$$

Using that $\rho_{\rm b} = \bar{\rho}_{\rm b}(z) = \bar{\rho}_{\rm b,0} a^{-3}$ and $\rho_\gamma = \bar{\rho}_\gamma(z) = \bar{\rho}_{\gamma,0} a^{-4}$, we have that

$$\frac{\partial \rho_{\rm b}}{\partial \rho_\gamma} = \frac{\partial \rho_{\rm b}}{\partial a} \frac{\partial a}{\partial \rho_\gamma} = \frac{-3\bar{\rho}_{\rm b,0} a^{-4}}{-4\bar{\rho}_{\gamma,0} a^{-5}} = \frac{3}{4} \frac{\bar{\rho}_{\rm b,0} a^{-3}}{\bar{\rho}_{\gamma,0} a^{-4}} = \frac{3}{4} \frac{\bar{\rho}_{\rm b}(z)}{\bar{\rho}_\gamma(z)}$$

Combing the above we find that, indeed,

$$c_{\rm s} = \frac{c}{\sqrt{3}} \left[\frac{3}{4} \frac{\bar{\rho}_{\rm b}(z)}{\bar{\rho}_\gamma(z)} + 1 \right]^{-1/2}$$

Problem 2: Free Streaming
Consider a flat ΛCDM cosmology with $\Omega_{\rm m,0} = 0.3$ and $h = 0.7$, and with a CMB temperature (at present) of 2.7K. Assume that the dark matter particles decouple at $z_{\rm dec} = 10^{10}$ and have a mass of 2 Gev.

a) At what redshift do the dark matter particles become non-relativistic?

ANSWER: The dark matter particles become non-relativisty when $3k_{\rm B}T = mc^2$. Using that $T = T_{\rm CMB} = 2.7{\rm K}(1+z)$ we have that

$$(1 + z_{\rm NR}) = \frac{mc^2}{3k_{\rm B}2.7} = \frac{2 \times 10^9 {\rm eV} \times 1.6 \times 10^{-12} {\rm erg\, eV}^{-1}}{3 \times 2.7{\rm K} \times 1.381 \times 10^{-16} {\rm ergK}^{-1}} = 2.9 \times 10^{12}$$

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b) Show that the comoving free-streaming length at matter-radiation equality can be written as

$$\lambda_{\rm fs}(t_{\rm eq}) = \frac{2ct_{\rm NR}}{a_{\rm NR}} \left[\left(\frac{a_{\rm dec}}{a_{\rm NR}}\right)^{1/2} \left\{ 2 + \ln\left(\frac{a_{\rm eq}}{a_{\rm dec}}\right) \right\} - 1 \right]$$

Hint: use that, during the radiation dominated era $a = a_{\rm NR}(t/t_{\rm NR})^{1/2|}$

ANSWER: The comoving free streaming length is given by

$$\lambda_{\rm fs} = \int_0^{t_{\rm eq}} \frac{v(t)}{a(t)} {\rm d}t = \int_0^{t_{\rm NR}} \frac{v(t)}{a(t)} {\rm d}t + \int_{t_{\rm NR}}^{t_{\rm dec}} \frac{v(t)}{a(t)} {\rm d}t + \int_{t_{\rm dec}}^{t_{\rm eq}} \frac{v(t)}{a(t)} {\rm d}t \equiv I_1 + I_2 + I_3$$

Here we have split the integral in three parts corresponding to the following periods:

$$\begin{aligned} t < t_{\rm NR} \quad & \text{for which} \quad v(t) = c \\ t_{\rm NR} < t < t_{\rm dec} \quad & \text{for which} \quad v(t) = c\left(\frac{a_{\rm NR}}{a}\right)^{1/2} \\ t_{\rm dec} < t < t_{\rm eq} \quad & \text{for which} \quad v(t) = c\left(\frac{a_{\rm NR}}{a_{\rm dec}}\right)^{1/2} \left(\frac{a_{\rm dec}}{a}\right) \end{aligned}$$

Using that for $t < t_{\rm eq}$ the scale radius evolves with time as

$$a(t) = a_{\rm NR}\left(\frac{t}{t_{\rm NR}}\right)^{1/2}$$

we have that

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{2}a_{\mathrm{NR}}\left(\frac{t}{t_{\mathrm{NR}}}\right)^{-1/2}\frac{1}{t_{\mathrm{NR}}} = \frac{1}{2}\frac{a_{\mathrm{NR}}^2}{a(t)\,t_{\mathrm{NR}}}$$

This allows us to write that

$$\frac{\mathrm{d}t}{a(t)} = \frac{2t_{\mathrm{NR}}}{a_{\mathrm{NR}}^2}\mathrm{d}a$$

Using this it is straightforward to compute the above three integrals:

$$I_1 = \int_0^{t_{\mathrm{NR}}}\frac{c}{a(t)}\,\mathrm{d}t = \frac{2ct_{\mathrm{NR}}}{a_{\mathrm{NR}}^2}\int_0^{a_{\mathrm{NR}}}\mathrm{d}a = \frac{2ct_{\mathrm{NR}}}{a_{\mathrm{NR}}}$$

$$\begin{aligned} I_2 &= c\int_{t_{\mathrm{NR}}}^{t_{\mathrm{dec}}}\left(\frac{a_{\mathrm{NR}}}{a(t)}\right)^{1/2}\frac{2t_{\mathrm{NR}}}{a_{\mathrm{NR}}^2}\,\mathrm{d}a = \frac{2ct_{\mathrm{NR}}}{a_{\mathrm{NR}}^{3/2}}\int_{a_{\mathrm{NR}}}^{a_{\mathrm{dec}}}\frac{\mathrm{d}a}{a^{1/2}} \\ &= \frac{4ct_{\mathrm{NR}}}{a_{\mathrm{NR}}^{3/2}}\left[a_{\mathrm{dec}}^{1/2} - a_{\mathrm{NR}}^{1/2}\right] = \frac{4ct_{\mathrm{NR}}}{a_{\mathrm{NR}}^{1/2}}\left[\left(\frac{a_{\mathrm{dec}}}{a_{\mathrm{NR}}}\right)^{1/2} - 1\right] \end{aligned}$$

$$\begin{aligned} I_3 &= c\left(\frac{a_{\mathrm{NR}}}{a_{\mathrm{dec}}}\right)^{1/2}\int_{t_{\mathrm{dec}}}^{t_{\mathrm{eq}}}\frac{a_{\mathrm{dec}}}{a(t)}\frac{2t_{\mathrm{NR}}}{a_{\mathrm{NR}}^2}\,\mathrm{d}a = \frac{2ct_{\mathrm{NR}}}{a_{\mathrm{NR}}^{3/2}}a_{\mathrm{dec}}^{1/2}\int_{a_{\mathrm{dec}}}^{a_{\mathrm{eq}}}\frac{\mathrm{d}a}{a} \\ &= \frac{2ct_{\mathrm{NR}}}{a_{\mathrm{NR}}}\left(\frac{a_{\mathrm{dec}}}{a_{\mathrm{NR}}}\right)^{1/2}\ln\left(\frac{a_{\mathrm{eq}}}{a_{\mathrm{dec}}}\right) \end{aligned}$$

Combining these results, we finally obtain that

$$\begin{aligned} \lambda_{\mathrm{fs}} &= \frac{2ct_{\mathrm{NR}}}{a_{\mathrm{NR}}}\left[1 + 2\left\{\left(\frac{a_{\mathrm{dec}}}{a_{\mathrm{NR}}}\right)^{1/2} - 1\right\} + \left(\frac{a_{\mathrm{dec}}}{a_{\mathrm{NR}}}\right)^{1/2}\ln\left(\frac{a_{\mathrm{eq}}}{a_{\mathrm{dec}}}\right)\right] \\ &= \frac{2ct_{\mathrm{NR}}}{a_{\mathrm{NR}}}\left[\left(\frac{a_{\mathrm{dec}}}{a_{\mathrm{NR}}}\right)^{1/2}\left\{2 + \ln\left(\frac{a_{\mathrm{eq}}}{a_{\mathrm{dec}}}\right)\right\} - 1\right] \end{aligned}$$

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c) What is the ratio between $\lambda_{\mathrm{fs}}(t_{\mathrm{eq}})$ and the comoving particle horizon, λ_{H}, at t_{NR}? Compute the actual, numerical value of $\lambda_{\mathrm{fs}}(t_{\mathrm{eq}})/\lambda_{\mathrm{H}}(t_{\mathrm{NR}})$.

ANSWER: The comoving particles horizon at $t_{\rm NR}$ is given by

$$\lambda_{\rm H} = \int_0^{t_{\rm NR}} \frac{c\,{\rm d}t}{a(t)} = \frac{2\,c\,t_{\rm NR}}{a_{\rm NR}^2} \int_0^{a_{\rm NR}} {\rm d}a = \frac{2\,c\,t_{\rm NR}}{a_{\rm NR}}$$

Hence, we have that

$$\frac{\lambda_{\rm fs}(t_{\rm eq})}{\lambda_{\rm H}(t_{\rm NR})} = \left(\frac{a_{\rm dec}}{a_{\rm NR}}\right)^{1/2} \left[2 + \ln\left(\frac{a_{\rm eq}}{a_{\rm dec}}\right)\right] - 1$$

Using that

$$\begin{aligned} a_{\rm NR} &= \frac{1}{1+z_{\rm NR}} &= \frac{1}{2.9\times 10^{12}} \\ a_{\rm dec} &= \frac{1}{1+z_{\rm dec}} &= \frac{1}{10^{10}} \\ a_{\rm eq} &= \frac{1}{1+z_{\rm eq}} &= \frac{1}{3528} \end{aligned}$$

For the latter we have used that $(1+z_{\rm eq}) = 2.4\times 10^4 \Omega_{\rm m,0} h^2 = 2.4\times 10^4 \cdot 0.3 \cdot (0.7)^2 = 3528$. Substituting these values we find that

$$\frac{\lambda_{\rm fs}(t_{\rm eq})}{\lambda_{\rm H}(t_{\rm NR})} = 286$$

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d) What is the free-streaming mass at matter-radiation equality? `Hint:` use eq. (3.80) in MBW.

ANSWER: The free streaming mass at equality is

$$M_{\rm fs} = \frac{\pi}{6}\,\bar{\rho}\,(\lambda_{\rm fs}^{\rm prop})^3 = \frac{\pi}{6}\,\bar{\rho}_0\,(\lambda_{\rm fs}^{\rm com})^3$$

Using that $\bar{\rho}_0 = \Omega_{\rm m,0}\,\rho_{\rm crit,0}$, with $\rho_{\rm crit,0} = 2.78\times 10^{11} h^{-1}\,{\rm M}_\odot/(h^{-1}\,{\rm Mpc})^3$ we find that

$$M_{\rm fs} = 4.36\times 10^{10} h^{-1}\,{\rm M}_\odot \left(\frac{\lambda_{\rm fs}^{\rm com}}{h^{-1}\,{\rm Mpc}}\right)^3$$

For the comoving free-streaming length we have that

$$\lambda_{\rm fs}^{\rm com} = 286 \frac{2\,c\,t_{\rm NR}}{a_{\rm NR}}$$

Evaluating this quantity requires that we first compute $t_{\rm NR}$. For this we use that

$$a(t) = \left(\frac{32\,\pi\,G\,\rho_{\rm r,0}}{3}\right)^{1/4} t^{1/2}$$

[see eq.(3.80) in MBW]. Using that $\Omega_{\rm r,0} = 4.2 \times 10^{-5} h^{-2}$ and that $z_{\rm NR} = 2.9 \times 10^{12}$ we find that $t_{\rm NR} = 2.83 \times 10^{-6}$s. Substitution in the equation for the free-streaming length yields that $\lambda_{\rm fs}^{\rm com} = 45.6\,{\rm pc} = 4.56 \times 10^{-5}$ Mpc. Substituting this in the expression for the free-streaming mass, and using that $h = 0.7$, we finally find that $M_{\rm fs} = 2.0 \times 10^{-3}\,{\rm M}_\odot$

Problem 3: Silk Damping
Consider a Universe consisting purely of baryons and radiation (no dark matter or dark energy), and ignore any elements heavier than hydrogen. Assume that $\Omega_{\rm m,0} = 0.3$, $h = 0.7$, and that recombination in this universe happens at a redshift $z_{\rm rec} \simeq 1100$, when the ionization fraction $X_{\rm e} \equiv n_{\rm e}/(n_{\rm p} + n_{\rm H}) = 0.1$. Here $n_{\rm e}$, $n_{\rm p}$ and $n_{\rm H}$ are the number density of free electons, free protons, and hydrogen atoms. A crude estimate for the Silk damping scale at time t, based on kinetic theory, is

$$\lambda_{\rm d} = \left(\frac{ct}{3\sigma_{\rm T} n_{\rm e}}\right)^{1/2}$$

(see lecture 5).

a) Is this in physical or comoving units? Explain.

ANSWER: Silk damping describes the diffusion of photons (a random-walk process) in the presence of Thomson scattering. The mean free path of a photon is $l = 1/(n_{\rm e}\sigma_{\rm T})$, corresponding to a mean time between collisions of $\tau = l/c = (n_{\rm e}\sigma_{\rm T} c)^{-1}$. During a time t, a photon takes $N = t/\tau = n_{\rm e}\sigma_{\rm T} ct$ such steps. According to kinetic theory, the actual length over which the photon has diffused after N steps is equal to $\lambda_{\rm d} = (N/3)^{1/2} l$. Substituting we obtain that

$$\lambda_{\rm d} = \left(\frac{c\,t}{3 n_{\rm e} \sigma_{\rm T}}\right)^{1/2}$$

We thus see that the expression for the Silk damping scale is derived in a purely classical sense, without resource to an expanding space time. Consequently, $\lambda_{\rm d}$ is in physical, rather than comoving, units.

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b) Express the mean mass per particle, $\mu m_{\rm p}$, with $m_{\rm p}$ the mass of a proton, in terms of the ionization fraction $X_{\rm e}$

ANSWER: The mean mass per particle is the total mass per unit volume (of all particles), divided by the total number density of all particles. In the universe in question, the only particles available are electrons, protons and hydrogen atoms. Hence

$$\mu m_{\rm p} = \frac{n_{\rm e} m_{\rm e} + n_{\rm p} m_{\rm p} + n_{\rm H} m_{\rm H}}{n_{\rm e} + n_{\rm p} + n_{\rm H}}$$

Since the universe is globally neutral, have that $n_{\rm e} = n_{\rm p}$, while we also have that $m_{\rm e} \ll m_{\rm p} \simeq m_{\rm H}$. This allows us to rewrite the above expression as

$$\begin{aligned} \mu m_{\rm p} &\simeq \frac{(n_{\rm p} + n_{\rm H}) m_{\rm p}}{n_{\rm e} + n_{\rm p} + n_{\rm H}} \\ &= \frac{m_{\rm p}}{\frac{n_{\rm e}}{n_{\rm p}+n_{\rm H}} + 1} \end{aligned} \tag{1}$$

Using the definition for the ionization fraction, we thus see that

$$\mu = \frac{1}{X_{\rm e} + 1}$$

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c) Show that the number density of free electrons at recombination can be written as

$$n_{\rm e} \simeq 1.5 \times 10^4 \,{\rm cm}^{-3}\, X_{\rm e}\, (\Omega_{\rm m,0} h^2)$$

ANSWER: Using the definition of the ionization fraction, we can write

$$n_{\rm e} = (n_{\rm p} + n_{\rm H})\, X_{\rm e}$$

Since our universe only contains baryons, and the mass of the electron is negligble compared to that of the proton, we have that

$$\rho_{\rm m} = \rho_{\rm m,0}\,(1+z)^3 = \Omega_{\rm m,0}\,\rho_{\rm crit,0}\,(1+z)^3 = (n_{\rm p} + n_{\rm H})\, m_{\rm p}$$

Combining these two expressions we can write

$$n_{\rm e} = \frac{\Omega_{\rm m,0}\,\rho_{\rm crit,0}\,(1+z)^3}{m_{\rm p}}\, X_{\rm e}$$

Using that the redshift of recombination is $z = 1100$, that $\rho_{\rm crit,0} = 1.879 \times 10^{-29} h^2 {\rm g\,cm^{-3}}$, and that $m_{\rm p} = 1.673 \times 10^{24}$g, we see that, at recombination,

$$n_{\rm e} \simeq 1.5 \times 10^4\,{\rm cm^{-3}}\, X_{\rm e}\,(\Omega_{\rm m,0} h^2)$$

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d) Derive the Silk damping scale, in comoving Mpc, at the epoch of recombination. Use that $X_{\rm e} \sim 0.1$ at that epoch, and that recombination occurs during the matter dominated era, so that

$$t(z) = \frac{2}{3}\,\frac{1}{H_0}\,(1+z)^{-3/2}$$

(cf Eq. (3.96) in MBW).

ANSWER: Using the expression for the Silk damping scale, which is in physical (not comoving) units (see **(a)**), we have that

$$\lambda_{\rm d}^{\rm com} = (1 + z_{\rm rec}) \left(\frac{ct}{3\sigma_{\rm T} n_{\rm e}} \right)^{1/2}$$

Using that $\sigma_{\rm T} = 6.65 \times 10^{-25} {\rm cm}^{-2}$, and that $n_{\rm e}$ is given by the expression under **(c)**, we find that $n_{\rm e} = 220.5 cm^{-3}$, where we have used that $X_{\rm e} = 0.1$, $\Omega_{\rm m,0} = 0.3$ and $h = 0.7$. Thus we have that $3\sigma_{\rm T} n_{\rm e} = 4.4 \times 10^{-22}\,{\rm cm}^{-1}$.

Using the expression for $t(z)$ given, and using that $H_0^{-1} = 9.78h^{-1}$Gyr $= 3.086 \times 10^{17} h^{-1}$s, we find that the time at recombination is equal to $t = 5.63 \times 10^{12} h^{-1}$s. Using that $c = 3 \times 10^{10}\,\mathrm{cm\,s^{-1}}$, we thus obtain that $\lambda_{\rm d}^{\rm com} = 2.57 \times 10^{25}$ cm $= 8.3$ Mpc.

More accurate derivation: In the above derivation, we computed the physical scale NOT accounting for the expansion of the Universe, and then simply converted it to comoving units at the end. This is not exactly correct. A more correct derivation is to realize that the comoving distance free-streamed by the photons is

$$\lambda_{\rm d}^{\rm com} = \int_0^{t_{\rm rec}} \frac{v_{\rm stream}(t)\mathrm{d}t}{a(t)} \tag{2}$$

where $v_{\rm stream}$ is the effective streaming velocity of the photons, which is simply

$$v_{\rm stream}(t) = \frac{\lambda_{\rm d}^{\rm phys}}{t} = \left(\frac{c}{3n_{\rm e}(t)\sigma_{\rm T} t}\right)^{1/2} \tag{3}$$

where we have made it explicit that the electron number density is a function of time. In particular, $n_{\rm e}(a) = n_{\rm rec}(a/a_{\rm rec})^{-3}$, with $n_{\rm rec}$ the electron number density at recombination. Since recombination takes place during matter domination, if we ignore the fact that the scale factor evolves differently during radiation domination, we simply have that $a(t) = a_{\rm rec}(t/t_{\rm rec})^{2/3}$. Substitution in the expression for $\lambda_{\rm d}^{\rm com}$ yields

$$\lambda_{\rm d}^{\rm com} = \left(\frac{c}{3\sigma_{\rm T}}\right)^{1/2} \int_0^{t_{\rm rec}} \frac{\mathrm{d}t}{n_{\rm rec}^{1/2}(t/t_{\rm rec})^{-1}\, t^{1/2}\, a_{\rm rec}(t/t_{\rm rec})^{2/3}} \tag{4}$$

Using that $a_{\rm rec} = (1+z_{\rm rec})^{-1}$ this reduces to

$$\begin{aligned} \lambda_{\rm d}^{\rm com} &= \left(\frac{c\, t_{\rm rec}}{3n_{\rm rec}\sigma_{\rm T}}\right)^{1/2} (1+z_{\rm rec})\, t_{\rm rec}^{-5/6} \int_0^{t_{\rm rec}} \frac{\mathrm{d}t}{t^{1/6}} && (5)\\ &= \frac{6}{5}\left(\frac{c\, t_{\rm rec}}{3n_{\rm rec}\sigma_{\rm T}}\right)^{1/2} (1+z_{\rm rec}) && (6) \end{aligned}$$

which is 6/5 times larger than the answer above (i.e., $\lambda_{\rm d}^{\rm com} = 10.0$ Mpc).

e) Compute the corresponding Silk damping mass, $M_{\rm d}$, in Solar masses.

ANSWER: The Silk damping mass is

$$\begin{aligned} M_{\rm d} &= \frac{4}{3}\pi\,\bar{\rho}_{\rm m}(z_{\rm rec})\,\left(\lambda_{\rm d}^{\rm phys}/2\right)^3 \\ &= \frac{\pi}{6}\bar{\rho}_{\rm m,0}\,(\lambda_{\rm d}^{\rm com})^3 \\ &= \frac{\pi}{6}\Omega_{\rm m,0}\rho_{\rm crit,0}\,(\lambda_{\rm d}^{\rm com})^3 \end{aligned} \tag{7}$$

Substituting $\lambda_{\rm d}^{\rm com} = 8.3{\rm Mpc}$, $\Omega_{\rm m,0} = 0.3$ and $\rho_{\rm crit,0} = 2.775{\times}10^{11}h^2{\rm M}_\odot{\rm Mpc}^{-3}$, one obtains that $M_{\rm d} \simeq 1.2\times10^{13}\,{\rm M}_\odot$.

Problem 4: The Continuity Equation `[5 points]`

The continuity equation in physical coordinates, $\vec{r}(t)$, in Lagrangian form reads

$$\frac{{\rm D}\rho}{{\rm D}t} + \rho\nabla_r\cdot\vec{u} = 0$$

Show that this equation can be written as

$$\frac{\partial\delta}{\partial t} + \frac{1}{a}\,\nabla_x\cdot[(1+\delta)\,\vec{v}] = 0$$

where δ is the overdensity, $\vec{x}$ are comoving coordinates and $\vec{v}$ are peculiar velocities. The latter are related to $\vec{r}$ and $\vec{u}$ according to $\vec{r} = a(t)\,\vec{x}$ and $\vec{u} = \dot{a}\,\vec{x} + \vec{v}$. You may use that the (physical) density ρ scales as a^{-3}.

ANSWER: The first step is to write the continuity equation in Eulerian form, which can be done by using the relation between the Lagrangian and Eulerian derivatives (MWB Eq 4.4):

$$\frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla_r\rho + \rho\nabla_r\cdot\vec{u} = 0$$

Next we transform from physical to comoving coordinates. As shown in the lecture notes and MBW (Eq. 4.7), we have that

$$\nabla_r \rightarrow \frac{1}{a}\nabla_x \equiv \frac{1}{a}\nabla$$

and

$$\left(\frac{\partial}{\partial t}\right)_r \rightarrow \left(\frac{\partial}{\partial t}\right)_x - \frac{\dot{a}}{a}\,\vec{x}\cdot\nabla$$

Substitution in our equation, and using that $\vec{u} = \dot{a}\vec{x} + \vec{v}$, we obtain that

$$\frac{\partial \rho}{\partial t} - \frac{\dot{a}}{a}\vec{x}\cdot\nabla\rho + (\dot{a}\vec{x} + \vec{v})\frac{1}{a}\nabla\rho + \rho\frac{1}{a}\nabla\cdot(\dot{a}\vec{x} + \vec{v}) = 0$$

Next, we use that $\nabla\cdot\vec{x} = 3$ to rewrite this equation in more compact form:

$$\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\nabla\cdot\rho\vec{v} = 0$$

Now we can substitute $\rho = \bar{\rho}(1+\delta)$. Note that $\bar{\rho}$ is a function of time, but not of $\vec{x}$ (homogeneous density). Hence, we obtain that

$$\frac{\partial \bar{\rho}}{\partial t}(1+\delta) + \bar{\rho}\frac{\partial \delta}{\partial t} + 3\frac{\dot{a}}{a}\bar{\rho}[1+\delta] + \bar{\rho}\frac{1}{a}\nabla\left([1+\delta]\vec{v}\right) = 0$$

Multiplying by $1/\bar{\rho}$ and using that $\partial\bar{\rho}/\partial t = -3(\dot{a}/a)\bar{\rho}$ (which follows from the fact that $\bar{\rho} \propto a^{-3}$), we now easily obtain that

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\,\nabla\cdot\left[(1+\delta)\,\vec{v}\right] = 0$$
