ASTR 610: Solutions to Problem Set 3

Problem 1: Spherical Collapse
According to the SC model, the parametric solution to the evolution of a mass shell is

$$r = A\,(1 - \cos\theta)$$

$$t = B\,(\theta - \sin\theta)$$

where $A^3 = G\,M\,B^2$, which implies that

$$1 + \delta = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3}$$

Show that at early times (when $\theta \ll 1$) one has that

$$\delta_{\rm i} = \frac{3}{20}\,(6\pi)^{2/3}\,\left(\frac{t_{\rm i}}{t_{\rm max}}\right)^{2/3}$$

Hint: use Taylor series expansions of $\sin\theta$ and $\cos\theta$ and the fact that $t_{\rm max} = \pi B$.

ANSWER: We have that

$$\begin{aligned}\sin\theta &\simeq \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...\\ \cos\theta &\simeq 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...\end{aligned}$$

where we can ignore the higher-order terms, since at early times $\theta \ll 1$. Hence,

$$
\begin{aligned}
(\theta - \sin\theta)^2 &= \left(\frac{\theta^3}{6} - \frac{\theta^5}{120}\right)^2 = \frac{\theta^6}{36}\left[1 - \frac{\theta^2}{10} + \frac{\theta^4}{400}\right] \simeq \frac{\theta^6}{36}\left[1 - \frac{\theta^2}{10}\right] \\
(1 - \cos\theta)^3 &= \left(\frac{\theta^2}{2} - \frac{\theta^4}{24}\right)^3 = \frac{\theta^6}{8}\left[1 - \frac{\theta^2}{6} + \frac{\theta^4}{144} - \frac{\theta^2}{12} + \frac{\theta^4}{72} - \frac{\theta^6}{1728}\right] \simeq \frac{\theta^6}{8}\left[1 - \frac{\theta^2}{4}\right]
\end{aligned}
$$

Combining, we find that

$$
\begin{aligned}
1 + \delta_{\rm i} &= \frac{9}{2}\frac{\frac{\theta^6}{36}\left[1 - \frac{\theta^2}{10}\right]}{\frac{\theta^6}{8}\left[1 - \frac{\theta^2}{4}\right]} \\
&\simeq \left[1 - \frac{\theta^2}{10}\right] \times \left[1 + \frac{\theta^2}{4}\right] \\
&\simeq 1 + \frac{3\,\theta^2}{20}
\end{aligned}
$$

from which we see that, to good approximation, $\delta_{\rm i} = 3\theta^2/20$. If we now use that $t = B\,(\theta - \sin\theta) \simeq B\,\theta^3/6$, we see that

$$
\theta_{\rm i} \simeq \left(\frac{6\,t_{\rm i}}{B}\right)^{1/3} = \left(\frac{6\,\pi\,t_{\rm i}}{t_{\rm max}}\right)^{1/3}
$$

where we have used that $t_{\rm max} = \pi\,B$. Substituting the above expression for $\theta_{\rm i}$ into the expression for $\delta_{\rm i}$, one finally obtains that

$$
\delta_{\rm i} = \frac{3}{20}\,(6\pi)^{2/3}\,\left(\frac{t_{\rm i}}{t_{\rm max}}\right)^{2/3}
$$

Problem 2: The Zel'dovich Approximation

In this problem we seek to characterize the displacement $\psi(t)$ defined by

$$
\vec{x}(t) = \vec{x}_{\rm i} + \psi(t)
$$

where $\vec{x}(t)$ is the comoving coordinate of a particle. Obviously we have that

$$\psi(t) = \int_{t_i}^{t} \frac{v(t)}{a(t)}\, \mathrm{d}t$$

where $v(t)$ is the particle's peculiar velocity. Under the Zel'dovich approximation, the gradient of the potential (which defines the direction in which the particle moves), can be written as $\nabla\Phi(t) = f(t)\nabla\Phi_{\mathrm{i}}$, where $f(t)$ is some function (to be determined) of time.

a) Use the linearized Euler equation for a pressureless fluid to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a\vec{v}\right) = -\nabla\Phi$$

ANSWER: The linearized Euler equations for a pressureless fluid is given by

$$\frac{\partial\vec{v}}{\partial t} + \frac{\dot{a}}{a}\,\vec{v} = -\frac{\nabla\Phi}{a}$$

Using that

$$\frac{\mathrm{d}}{\mathrm{d}t}(a\vec{v}) = a\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \vec{v}\,\frac{\mathrm{d}a}{\mathrm{d}t} = a\frac{\partial\vec{v}}{\partial t} + a\vec{v}\cdot\nabla\vec{v} + \dot{a}\vec{v} = a\left(\frac{\partial\vec{v}}{\partial t} + \frac{\dot{a}}{a}\,\vec{v}\right)$$

Here we have used the relation between the Lagrangian and Eulerian derivatives, and the fact that, in the linearized Euler equation, the $\vec{v}\cdot\nabla\vec{v}$ term may be ignored. Combining this with the linearlized Euler equations, it is immediately evident that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a\vec{v}\right) = -\nabla\Phi$$

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b) Use the fact that, at early times, the Universe behaves as an EdS cosmology to show that

$$\vec{v} = -\frac{\nabla\Phi_{\mathrm{i}}}{a}\int \frac{D(a)}{a}\,\mathrm{d}t$$

Hint: use that $\Phi_{\vec{k}} \propto D(a)/a$.

ANSWER: The fact that $\Phi_{\vec{k}} \propto D(a)/a$ implies that $\Phi \propto D(a)/a$, and therefore also $\nabla\Phi \propto D(a)/a$. This allows us to write that

$$\nabla\Phi = \frac{D(a)\,a_{\rm i}}{D(a_{\rm i})\,a}\,\nabla\Phi_{\rm i}$$

Since at early times the Universe behaves as an EdS cosmology, for which $D(a) = a$, we have that $D(a_{\rm i})/a_{\rm i} = 1$, so that

$$\nabla\Phi = \frac{D(a)}{a}\,\nabla\Phi_{\rm i}$$

Using what we inferred under **a)**, we therefore have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(a\,\vec{v}) = -\frac{D(a)}{a}\,\nabla\Phi_{\rm i}$$

Integrating this equation yields

$$\int \mathrm{d}(a\,\vec{v}) = -\nabla\Phi_{\rm i}\int \frac{D(a)}{a}\,\mathrm{d}t$$

from which it is immediately evident that

$$\vec{v} = -\frac{\nabla\Phi_{\rm i}}{a}\int \frac{D(a)}{a}\,\mathrm{d}t$$

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c) Use the fact that $D(a)$ is a solution of the linearized fluid equation of a pressureless fluid to show that

$$\frac{D(a)}{a} = \frac{1}{4\pi G\bar{\rho}_{\rm i}}\,\frac{\mathrm{d}(a^2\dot{D})}{\mathrm{d}t}$$

Hint: you may use that the scale factor is normalized such that $a_{\rm i} = 1$.

ANSWER: Since $D(a)$ is a solution of the linearized fluid equation for a pressureless fluid, we have that

$$\ddot{D} + 2\,\frac{\dot{a}}{a}\,\dot{D} = 4\pi G\,\bar{\rho}(a)\,D$$

Using that $\bar{\rho}(a) = \bar{\rho}_{\rm i}(a_{\rm i}/a)^3 = \bar{\rho}_{\rm i} a^{-3}$, where we have used that $a_{\rm i} = 1$, the above equation reduces to

$$\ddot{D} + 2\,\frac{\dot{a}}{a}\,\dot{D} = 4\pi G\,\bar{\rho}_{\rm i}\,\frac{D(a)}{a^3}$$

Next we use that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\dot{D}\right) = a^2\ddot{D} + 2a\dot{a}\dot{D} = a^2\left(\ddot{D} + 2\frac{\dot{a}}{a}\dot{D}\right)$$

to write that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\dot{D}\right) = a^2\,4\pi G\,\bar{\rho}_{\rm i}\,\frac{D(a)}{a^3} = 4\pi G\bar{\rho}_{\rm i}\,\frac{D(a)}{a}$$

Rearranging shows that

$$\frac{D(a)}{a} = \frac{1}{4\pi G\,\bar{\rho}_{\rm i}}\,\frac{\mathrm{d}(a^2\dot{D})}{\mathrm{d}t}$$

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d) Use the above results to show that the displacement

$$\psi(t) = -\frac{D(a)}{4\pi G\bar{\rho}_{\rm i}}\,\nabla\Phi_{\rm i}$$

ANSWER: Under **b)** we derived that

$$\vec{v} = -\frac{\nabla\Phi_{\rm i}}{a}\int\frac{D(a)}{a}\,\mathrm{d}t$$

while under **c)** we demonstrated that

$$\frac{D(a)}{a} = \frac{1}{4\pi G\,\bar{\rho}_{\rm i}}\,\frac{\mathrm{d}(a^2\dot{D})}{\mathrm{d}t}$$

Substituting the latter in the former, we find that

$$\vec{v} = -\frac{\nabla\Phi_{\rm i}}{4\pi G\bar{\rho}_{\rm i} a} \int {\rm d}(a^2\dot{D}) = -\frac{\nabla\Phi_{\rm i}}{4\pi G\bar{\rho}_{\rm i}}\, a\, \frac{{\rm d}D}{{\rm d}t}$$

Hence, for the displacement we have that

$$\begin{aligned} \psi(t) &= \int_{t_{\rm i}}^{t} \frac{v(t)}{a(t)}\,{\rm d}t = -\frac{\nabla\Phi_{\rm i}}{4\pi G\bar{\rho}_{\rm i}} \int_{D(a_{\rm i})}^{D(a)} {\rm d}D \\ &= -\frac{D(a) - D(a_{\rm i})}{4\pi G\bar{\rho}_{\rm i}}\,\nabla\Phi_{\rm i} \simeq -\frac{D(a)}{4\pi G\bar{\rho}_{\rm i}}\,\nabla\Phi_{\rm i} \end{aligned}$$

where in the last step we have used that $D(a_{\rm i}) \ll D(a)$.

Problem 3: The two-point correlation function and σ_8
Let M be the mass inside a top-hat filter. The expectation value for M, i.e., the average value obtained by putting down the top-hat filter at many different locations, is simply $\langle M\rangle = \bar{\rho}\,V$ where V is the volume of the top-hat. Similarly, one can show that

$$\langle M^2\rangle = \langle M\rangle^2 + \frac{\langle M\rangle^2}{V^2} \int_V \xi(|\vec{x}_1 - \vec{x}_2|)\,{\rm d}^3\vec{x}_1\,{\rm d}^3\vec{x}_2$$

a) Express the mass variance, $\sigma^2(M)$, in terms of the two-point correlation function $\xi(r)$.

ANSWER: The mass variance can be written in the following form

$$\sigma^2(M) = \left\langle \left(\frac{M(\vec{x},R) - \bar{M}(R)}{\bar{M}(R)}\right)^2 \right\rangle$$

(see MBW10, Eq. 6.40). Using that $\bar{M}$ is the same as $\langle M\rangle$, this is trivially rewritten as

$$\sigma^2(M) = \frac{\langle M^2\rangle}{\langle M\rangle^2} - 1$$

Hence, using the relation that is given, we immediately see that

$$\sigma^2(M) = \frac{1}{V^2} \int_V \xi(|\vec{x}_1 - \vec{x}_2|)\, \mathrm{d}^3\vec{x}_1\, \mathrm{d}^3\vec{x}_2$$

Now we write $\vec{x}_2 = \vec{x}_1 + \vec{r}$, which allows us to rewrite the above expression as

$$\sigma^2(M) = \frac{1}{V^2} \int_V \mathrm{d}^3\vec{x}_1 \int_V \xi(r) \mathrm{d}^3\vec{r}$$

Using that the universe is isotropic, and that $\int_V \mathrm{d}^3\vec{x} = V$, we finally obtain that

$$\sigma^2(M) = \frac{4\pi}{V} \int_0^R \xi(r)\, r^2\, \mathrm{d}r = \frac{3}{R^3} \int_0^R \xi(r)\, r^2\, \mathrm{d}r$$

where R is the size of the top-hat filter that (on average) encloses a mass M.

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b The first measurements of the two-point correlation function of galaxies revealed a power-law $\xi(r) = (r/r_0)^\gamma$ with $r_0 = 5h^{-1}$Mpc and $\gamma = -1.8$. Under the assumption that galaxies are unbiased tracers of the mass distribution, what does this imply for the value of σ_8?

ANSWER: Using the equation we derived above, and substituting the power-law expression for the two-point correlation function, we obtain

$$\begin{aligned} \sigma_8^2 &= \frac{3}{R^3} \int_0^R \left(\frac{r}{r_0}\right)^\gamma r^2 \mathrm{d}r && (1) \\ &= \frac{3}{R^3} r_0^{-\gamma} \int_0^R r^{2+\gamma}\, \mathrm{d}r && (2) \\ &= \frac{3}{3+\gamma} \left(\frac{R}{r_0}\right)^\gamma && (3) \end{aligned}$$

Substituting $R = 8h^{-1}$ Mpc, $r_0 = 5h^{-1}$ Mpc and $\gamma = -1.8$ yields $\sigma_8^2 \simeq 1.073$ and thus $\sigma_8 \simeq 1.04$, which is very close to unity.

Problem 4: Power spectrum and Mass variance
Let the matter power spectrum be a pure power-law, $P(k) \propto k^n$.

a) Using a sharp k-space filter, show that the mass variance $\sigma^2(M) \propto M^\gamma$, and give the relation between γ and n.

ANSWER: We can write the variance as

$$\sigma^2(R) = \int \Delta^2(k)\, \tilde{W}^2(kR)\, \frac{\mathrm{d}k}{k}$$

where $\Delta^2(k) \equiv (k^3/2\pi^2)P(k)$. For a sharp k-space filter we have that $\tilde{W}(kR) = 1$ for $k < 1/R$ and zero otherwise. Writing the power-spectrum as $P(k) = A(k/k_0)^n$ we thus have that

$$\sigma^2(R) = \frac{Ak_0^{-n}}{2\pi^2} \int_0^{1/R} k^{n+3}\, \frac{\mathrm{d}k}{k} = \frac{Q}{n+3} R^{-(n+3)}$$

which is valid as long as $n > -3$, and we have defined $Q = A/(2\pi^2 k_0^n)$. Using that, in general, $M = \gamma_{\rm f}\bar{\rho}R^3$, where $\gamma_{\rm f}$ is a filter-specific constant ($\gamma_{\rm f} = \gamma_{\rm SK} = 6\pi^2$ for the sharp k-space filter), we thus obtain that

$$\sigma^2(M) = \frac{Q}{n+3}\, (\gamma_{\rm SK}\bar{\rho})^{(n+3)/3}\;\; M^{-(n+3)/3}$$

from which it is clear that $\gamma = -(n+3)/3$.

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b Repeat the same exersize as under **(a)**, but this time using a Gaussian filter.

ANSWER: For the Gaussian filter we have that $\tilde{W}(kR) = \mathrm{e}^{-(kR)^2/2}$ and thus $\tilde{W}^2(kR) = \mathrm{e}^{-(kR)^2}$. Substituting in the expression for the variance we obtain

$$\begin{aligned}
\sigma^2(R) &= Q \int_0^\infty k^{n+2} \mathrm{e}^{-(kR)^2}\, \mathrm{d}k && (4)\\
&= Q\, R^{-(n+3)} \int_0^\infty x^{n+2} \mathrm{e}^{-x^2} \mathrm{d}x && (5)\\
&= \frac{Q}{2}\, R^{-(n+3)} \int_0^\infty y^{(n+1)/2} \mathrm{e}^{-y} \mathrm{d}y && (6)\\
&= \frac{Q}{2}\, \Gamma\left(\frac{n+3}{2}\right) R^{-(n+3)} && (7)\\
& && (8)
\end{aligned}$$

where $\Gamma(x)$ is the Gamma function. Hence, we have that the mass variance is given by

$$\sigma^2(M) = \frac{Q}{2}\,\Gamma\left(\frac{n+3}{2}\right)\,(\gamma_{\rm G}\bar{\rho})^{(n+3)/3}\;M^{-(n+3)/3}$$

where $\gamma_{\rm G} = (2\pi)^{3/2}$. We thus see that $\gamma = (n+3)/2$, the same as for the sharp k-space filter.

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c Give the ratio of the mass variances computed using the Gaussian filter and the sharp k-space filter for the case $n = 1$. Do NOT use mathematica (or similar), but first express your answer in terms of a special function, prior to giving the numerical value of the ratio.

ANSWER: We simply take the ratio of the expressions derived under **(a)** and **(b)**:

$$\frac{\sigma^2_{\rm SK}(M)}{\sigma^2_{\rm G}(M)} = \frac{\frac{1}{n+3}\gamma_{\rm SK}^{(n+3)/3}}{\frac{1}{2}\,\Gamma\left(\frac{n+3}{2}\right)\,\gamma_{\rm G}^{(n+3)/3}}$$

Using that $n = 1$, that $\Gamma(n+1) = n\,\Gamma(n)$, and that $\Gamma(1) = 1$, we obtain

$$\frac{\sigma^2_{\rm SK}(M)}{\sigma^2_{\rm G}(M)} = \frac{1}{2}\left(\frac{6\pi^2}{(2\pi)^{3/2}}\right)^{4/3} \simeq 2.92$$
