ASTR 610: Solutions to Problem Set 1

Problem 1:
The Einstein-de Sitter (EdS) cosmology is defined as a flat, matter dominated cosmology without cosmological constant. In an EdS cosmology the universe is always matter dominated; it never experiences a phase of radiation domination.

a) Use the Friedmann equation to show that in an EdS cosmology

$$\left(\frac{a}{a_0}\right) = \left(\frac{3}{2} H_0 t\right)^{2/3}$$

ANSWER: The Friedmann equation can be written as

$$\left(\frac{\dot{a}}{a}\right) = H_0\, E(z) = H_0 \left[\Omega_{\Lambda,0} + (1-\Omega_0)(1+z)^2 + \Omega_{\rm m,0}(1+z)^3 + \Omega_{\rm r,0}(1+z)^4\right]^{1/2}$$

Hence, for a flat, matter dominated EdS cosmology ($\Omega_{\rm m,0} = \Omega_0 = 1.0$, and $\Omega_{\Lambda,0} = \Omega_{\rm r,0} = 0$), we have that

$$\left(\frac{\dot{a}}{a}\right) = H_0 \left(\frac{a}{a_0}\right)^{-3/2}$$

which we rewrite as

$$\frac{{\rm d}(a/a_0)}{{\rm d}t} = H_0 \left(\frac{a}{a_0}\right)^{-1/2}$$

Integration yields

$$\int \left(\frac{a}{a_0}\right)^{1/2} {\rm d}\left(\frac{a}{a_0}\right) = H_0 \int {\rm d}t = H_0 t$$

which solves to

$$\frac{2}{3}\left(\frac{a}{a_0}\right)^{3/2} = H_0 t$$

from which it is clear that

$$\frac{a}{a_0} = \left(\frac{3}{2}H_0 t\right)^{2/3}$$

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b) Show that, in an EdS cosmology, $\bar{\rho}(t) = (6\pi G t^2)^{-1}$

ANSWER: The density of an EdS cosmology evolves as

$$\rho(t) = \rho_{\mathrm{m},0}\left(\frac{a}{a_0}\right)^{-3} = \rho_{\mathrm{crit},0}\left(\frac{a}{a_0}\right)^{-3} = \frac{3H_0^2}{8\pi G}\left(\frac{3}{2}H_0 t\right)^{-2} = \frac{1}{6\pi G t^2}$$

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c) Show that, in an EdS cosmology, $H(t)\,t = 2/3$.

ANSWER: Substituting what we learned under **a)** in the Friedmann equation yields

$$H(t) = H_0\left(\frac{a}{a_0}\right)^{-3/2} = H_0\left[\left(\frac{3}{2}H_0 t\right)^{2/3}\right]^{-3/2} = \frac{2}{3t}$$

Hence, it is clear that $H(t)\,t = 2/3$.

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d) Show that, in an EdS cosmology, the proper particle horizon is $\lambda_{\mathrm{H}} = 3\,c\,t$.

ANSWER: The proper particle horizon is given by

$$\lambda_{\mathrm{H}}^{\mathrm{prop}} = a\lambda_{\mathrm{H}}^{\mathrm{com}} = a\int_0^t \frac{c\,\mathrm{d}t}{a(t)}$$

Using the expression for the scale-factor derived under **a)** yields

$$\lambda_{\mathrm{H}}^{\mathrm{prop}} = ac\left(\frac{3}{2}H_0\right)^{-2/3}\int_0^t t'^{-2/3}\,\mathrm{d}t' = 3ct$$

Problem 2: In a hypothetical universe, $\Omega_\Lambda = 0$, the CMB has a temperature of 10.0 K, and the energy density of neutrinos, which are still relativistic at the present, is 1.5 times higher than that of the photons. What is the redshift of matter-radiation equality in units of $\Omega_{\rm m,0}\, h^2$?

ANSWER: The energy density of relativistic species is the sum of the energy density of photons (radiation) and that of neutrinos. Since the later is 1.5 times that of the former, we have that

$$\rho_{\rm r}(z) = \rho_{\rm r,0}(1+z)^4 = 2.5\rho_{\gamma,0}(1+z)^4 = \frac{10\sigma_{\rm SB}}{c^3}\, T_{\gamma,0}^4\,(1+z)^4$$

The energy density of matter, on the other hand, is

$$\rho_{\rm m}(z) = \rho_{\rm m,0}(1+z)^3 = \Omega_{\rm m,0}\frac{3H_0^2}{8\pi G}(1+z)^3$$

The redshift of equality, $z_{\rm eq}$ is defined by $\rho_{\rm r}(z_{\rm eq}) = \rho_{\rm m}(z_{\rm eq})$, which implies

$$(1+z_{\rm eq}) = \frac{3H_0^2 c^3}{80\pi G\sigma_{\rm SB} T_{\gamma,0}^4}\Omega_{\rm m,0}$$

Using that $1/H_0 = 9.78 h^{-1}$Gyr (see MBW Appendix E), one easily obtains that

$$(1+z_{\rm eq}) = 89\,\Omega_{\rm m,0} h^2$$

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Problem 3: The "tired light hypothesis" is an old hypothesis that was once used to explain the Hubble relation without having to resort to an expanding space-time. It postulates that photons 'simply' lose energy as they move through space (by some unexplained means), with the energy loss per unit distance being given by the law

$$\frac{{\rm d}E}{{\rm d}r} = -K\,E$$

where K is a constant. Show that this hypothesis gives a distance-redshift relation that is linear in the limit $z << 1$, and derive the value for K that yields a Hubble constant of $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

ANSWER: Using that $E = h\nu$ we have that $\mathrm{d}\nu/\mathrm{d}r = -K\nu$, which in intergral form reads

$$\int_{\nu_{\rm em}}^{\nu_{\rm obs}} \frac{\mathrm{d}\nu}{\nu} = -K \int_d^0 \mathrm{d}r$$

where d is the distance of the object that is emitting the light. Solving this integral equations yields that $\ln(1+z) = +K\,d$, where we have used that $\nu_{\rm em}/\nu_{\rm obs} = (1+z)$. If we now use a Taylor series expansion of $\ln(1+z)$, we see that in the limit $z << 1$ we have the linear relation $z = +K\,d$. Comparing this to the Hubble relation $z = (H_0/c)\,d$ (which follows from $z = v/c$ and $v = H_0\,d$), we infer that $K = H_0/c$. Using $H_0 = 70\mathrm{km\,s^{-1}\,Mpc^{-1}}$ this implies that $K = 2.33 \times 10^{-4}\mathrm{Mpc}^{-1}$.

Problem 4: Suppose you are a two-dimensional being living on the surface of a sphere of radius R.

a) Derive an expression for the circumference of a circle of radius r.

ANSWER: Our circle is the set of points on the sphere that are a distance r away from the 'North Pole'. This r is equivalent to what we called $R\chi$ in Lecture 2, when we defined the corresponding metric

$$\mathrm{d}l^2 = R^2(\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2)$$

In this coordinate system, we have that the circle has a circumference given by

$$C = \int \mathrm{d}l = \int_0^{2\pi} R\,\sin\chi\,\mathrm{d}\theta = 2\pi\,R\,\sin\chi$$

where we have used that the along the circle $\mathrm{d}\chi = 0$. Using that the distance r on the surface of the sphere is equal to $R\chi$, we can finally write this as $C = 2\pi R\,\sin(r/R)$.

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b) Idealize the Earth as a perfect sphere of radius $R = 6400\,\mathrm{km}$. If you could measure distances with an accuracy of ± 1 meter, how large a circle would you have to draw/construct on the Earth's surface to convincingly demonstrate that the Earth is curved, rather than flat?

ANSWER: For a flat space, you have that $C = 2\pi r$. Hence, in order to be able to tell the difference with an accuracy of 1 meter, we need to have that $2\pi r - 2\pi R \sin(r/R) = 1$meter. If we use the shorthand $x = r/R$, this implies that we seek the solution of

$$x - \sin(x) = \frac{10^{-3}}{2\pi\, 6400} = 2.49 \times 10^{-8}$$

Using the Taylor series expansion for $\sin(x)$, we can write (for $x \ll 1$) that $x - \sin(x) = x^3/6$. Solving for $r = Rx$, we find that $r = 33.9\,\mathrm{km}$. Note that $r/R = 1.75 \times 10^{-2}$, which justifies our Taylor series approximation.

Problem 5: Using that, in a homogeneous and isotropic expanding universe, the equation of motion of a shell of matter is given by

$$\ddot{R} = -\frac{GM}{R^2}$$

with $M = M(< R)$ ythe mass enclosed by that shell. Show that the Laplacian of $\Phi \equiv \phi + a\ddot{a}x^2/2$ with respect to the comoving coordinate $\vec{x}$ is equal to $4\pi G\bar{\rho}a^2\delta$, indicating that Φ is only 'sourced' by the density contrast $\bar{\rho}\delta = \rho - \bar{\rho}$. Here ϕ is the Newtonian gravitational potential, and $a = a(t)$ is the scale factor.

ANSWER: The Laplacian is

$$\nabla_x^2 \Phi = \nabla_x^2(\phi + a\ddot{a}x^2/2) = a^2\, \nabla_r^2(\phi + a\ddot{a}x^2/2)$$

where $\vec{r} = a\vec{x}$ are the proper coordinates. Using the Poisson equation (which holds in proper coordinates), we can write this as

$$\nabla_x^2 \Phi = a^2 \left(4\pi G\rho + \frac{\ddot{a}}{2a}\nabla_r^2 r^2\right) = a^2 \left(4\pi G\rho + \frac{\ddot{a}}{2a}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r^2\right)$$

$$= a^2 \left(4\pi G\rho + 3\frac{\ddot{a}}{a}\right) \qquad .$$

Now we use that

$$\ddot{R} = -\frac{GM}{R^2}$$

Using that $M = \frac{4\pi}{3}\bar{\rho}R^3$ and writing that $R = a(t)R_0$, we find that

$$\ddot{a}R_0 = -\frac{4\pi G}{3}\bar{\rho}aR_0$$

which implies that

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\bar{\rho}$$

Substituting this in the equation for our Laplacian we find that

$$\nabla_x^2\Phi = a^2\left(4\pi G\rho - 4\pi G\bar{\rho}\right) = 4\pi G a^2(\rho - \bar{\rho}) = 4\pi G\bar{\rho}a^2\delta$$

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Problem 6: Einstein, in an attempt to construct a static model for the Universe, introduced the cosmological constant Λ.

a) What is the relation between Λ and the matter density ρ for a static Universe in which the energy density of relativistic matter is negligible?

ANSWER: A static equation requires that $\dot{a} = \ddot{a} = 0$. For a FRW cosmology we have that

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3\,P}{c^2}\right) + \frac{\Lambda\,c^2}{3}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K\,c^2}{a^2} + \frac{\Lambda\,c^2}{3}$$

Using the acceleration equation for non-relativistic matter (for which $P = 0$), and demanding that $\ddot{a} = 0$, we immediately infer that $\Lambda = \frac{4\pi G\rho}{c^2}$.

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b) Show that Einstein's static universe has to be positively curved (i.e., $K = +1$), and demonstrate that the present-day value for the scale-factor, a_0 (also called the radius of curvature) has to be equal to $\Lambda^{-1/2}$.

ANSWER: Substituting the solution under **a)** in the Friedmann equation we find that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K\,c^2}{a^2} + \frac{4\pi G}{3}\rho = 4\pi G\rho - \frac{K\,c^2}{a^2}$$

Using that a static Universe requires that $\dot{a} = 0$, we thus infer that for $a = a_0$

$$K = \frac{4\pi\,G\,\rho\,a_0^2}{c^2}$$

Since $\rho > 0$ (we can't have negative densities), it is immediately clear that $K > 0$. Since K only takes on -1, 0, or $+1$, we thus have that $K = +1$. Substitution in the above equation then trivially results in $a_0 = c/\sqrt{4\pi G\rho} = \Lambda^{-1/2}$.

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c) Consider once more Einstein's static universe. Suppose that some matter is converted to radiation (for example, due to stars and/or AGN). What happens to the scale factor of the universe? Explain your answer.

ANSWER: To answer this question, it is easiest to look at the acceleration equation, which reveals that ρ and P both cause a deceleration, whereas Λ is responsible for a positive acceleration. For the static universe, these are exactly balanced. But now imagine transferring some of the matter into radiation. The total energy density ρ is conserved, and simply changes from $\rho = \rho_{\rm m}$ to $\rho' = \rho'_{\rm m} + \rho'_{\rm r}$, whereby $\rho = \rho'$. However, since radiation has non-zero pressure, the pressure *increases* from zero to $P_{\rm r} = \frac{1}{3}\rho'_{\rm r}c^2 > 0$. Hence, the constribution $\rho + 3P/c^2$ increases, whereas the energy density associated with the cosmological constant remains fixed. As a consequence, we have that $\ddot{a} < 0$, and the space-time will start to collapse: Einstein's static universe is unstable.

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d) Suppose the energy density associated with the cosmological constant is equal to the present-day critical density. What is the total energy associated with Λ within 1 AU, in units of the rest-mass energy of the Sun? What does this imply for the impact of Λ on the the dynamics of the Solar system?

ANSWER: The energy associated with Λ within a radius r

$$E_\Lambda = \rho_{\rm crit}\, c^2\, \frac{4\pi}{3} r^3$$

Using that $\rho_{\rm crit} = 1.879 \times 10^{-29} h^2 \,{\rm g\,cm^{-3}}$ and $1\,{\rm AU} = 1.496 \times 10^{13}$ cm (see App E of MWB), we find that $E_\Lambda = 3.36 \times 10^{11}\, c^2\, h^2$ g. Using that the rest-mass energy of the Sun is $E_\odot = M_\odot c^2 = 1.99 \times 10^{33}\, c^2$ g, we obtain a ratio $E_\Lambda / E_\odot = 1.32 \times 10^{-22} h^2$. This is miniscule, and therefore Λ has no discernible impact on Solar system dynamics.