Graph theory package for Giac/Xcas

Reference manual

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June 2021

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Introduction

This document[1] contains an overview of graph theory commands built in Giac computation kernel and supported within the Xcas GUI. For each command, the calling syntax is presented along with a detailed description of its functionality, followed by examples.

Angular brackets "〈" and "〉" in the calling syntax indicate that the enclosed portion may be omitted. The vertical bar "|" stands for *or*. Data types are enclosed by "<" and ">", such as `<real>` or `<string>`.

Although the development focus was on simplicity, the implemented algorithms are reasonably fast. Some difficult tasks, such as traveling salesman problem, optimal graph colorings, minimum vertex covers, and graph isomorphism, rely on optional third party libraries, precisely the GNU Linear Programming Kit (GLPK) and nauty.

1. This manual was written in GNU TeXmacs, a scientific document editing platform. All examples were entered as interactive Giac sessions.

Chapter 1
Constructing graphs

1.1 General graphs

The commands `graph` and `digraph` are used for constructing general graphs.

1.1.1 Undirected graphs

`graph(n|V,⟨opts⟩)`
`graph(V,E,⟨opts⟩)`
`graph(E,⟨opts⟩)`
`graph(V,T,⟨opts⟩)`
`graph(T,⟨opts⟩)`
`graph(V,T1,T2,T3,..,Tk,⟨opts⟩)`
`graph(T1,T2,T3,..,Tk,⟨opts⟩)`
`graph(A,⟨opts⟩)`
`graph(V,E,A,⟨opts⟩)`
`graph(V,P,⟨opts⟩)`
`graph(str)`

Undirected graphs are created by using the command `graph`. It takes from one to three mandatory arguments and returns an undirected graph $G(V, E)$. Throughout this manual, an edge $e \in E$ with endpoints $u, v \in V$ is denoted by $e = uv$ or $e = \{u, v\}$. The order of the endpoints does not matter when G is undirected; hence $uv = vu$.

The following arguments may be passed to the `graph` command.

- number n or list of vertices V (a vertex may be any atomic object, such as an integer, a symbol or a string); it must be the first argument if used
- set of edges E (each edge is represented by the list of its endpoints), a permutation, a trail of edges or a sequence of trails; it can be either the first or the second argument if used
- trail $T =$ `trail` $(u, v, ...)$ or a sequence of trails $T_1, T_2, ..., T_k$
- permutation P of vertices
- adjacency or weight matrix A
- a string `str` representing a special graph

The following optional arguments may be appended as `opts`.

- `directed=true|false` — sets the graph to be (un)directed
- `weighted=true|false` — sets the graph to be (un)weighted
- `color=<int>|<list>` — sets the color(s) of the vertices. If a single integer is given, then it is used as the color for all vertices. If a list is given, then it should represent colors for all vertices in the order as returned by `vertices`
- `coordinates=<list>` — sets 2D or 3D coordinates of the vertices

Special graphs supported in Giac are listed in the table below.

special graph	GIAC name
Balaban 10-cage	`balaban10`
Balaban 11-cage	`balaban11`
Barnette-Bosák-Lederberg graph	`barnette-bosak-lederberg`
Bidiakis cube	`bidiakis`
Biggs-Smith graph	`biggs-smith`
2nd Blanuša snark	`blanusa`
Brinkmann graph	`brinkmann`
Brouwer-Haemers graph	`brouwer-haemers`
Bull graph	`bull`
Butterfly graph	`butterfly`
Clebsch graph	`clebsch`
Chvátal graph	`chvatal`
Coxeter graph	`coxeter`
Desargues graph	`desargues`
Diamond graph	`diamond`
Dodecahedral graph	`dodecahedron`
Double star snark	`double-star`
Doyle graph	`doyle`
Dürer graph	`duerer`
Dyck graph	`dyck`
Errera graph	`errera`
F26A graph	`f26a`
Folkman graph	`folkman`
Foster graph	`foster`
Franklin graph	`franklin`
Frucht graph	`frucht`
Gewirtz graph	`gewirtz`
Goldner-Harary graph	`goldner-harary`
Golomb graph	`golomb`
Gosset graph	`gosset`
Gray graph	`gray`
Grinberg graph	`grinberg`
Grötzsch graph	`groetzsch`
Harborth graph	`harborth`
Harries graph	`harries`
Harries–Wong graph	`harries-wong`
Heawood graph	`heawood`
Herschel graph	`herschel`
Higman-Sims graph	`higman-sims`
Hoffman graph	`hoffman`
Hoffman-Singleton graph	`hoffman-singleton`
Icosahedral graph	`icosahedron`
Kittell graph	`kittell`
Krackhardt kite graph	`krackhardt`
Levi graph (Tutte 8-cage)	`levi`
Ljubljana graph	`ljubljana`
Markström graph	`markstroem`

McGee graph	`mcgee`
Meredith graph	`meredith`
Meringer graph	`meringer`
Moser spindle	`moser`
Möbius–Kantor graph	`moebius-kantor`
Nauru graph	`nauru`
Octahedral graph	`octahedron`
Pappus graph	`pappus`
Petersen graph	`petersen`
Perkel graph	`perkel`
Poussin graph	`poussin`
Robertson graph	`robertson`
Robertson-Wegner graph	`robertson-wegner`
Shrikhande graph	`shrikhande`
Schläfli graph	`schlaefli`
Sousselier graph	`sousselier`
Sylvester graph	`sylvester`
Szerekes snark	`szerekes`
Tetrahedral graph	`tehtrahedron`
Tietze graph	`tietze`
Truncated icosahedral graph	`soccerball`
Tutte graph	`tutte`
Tutte 12-cage	`tutte12`
Wagner graph	`wagner`
Walther graph	`walther`
Watkins snark	`watkins`
Wells graph	`wells`
Wiener-Araya graph	`wiener-araya`
Wong graph	`wong`

1.1.2 Directed graphs

The `digraph` command is used for creating directed graphs, although the same is possible by using the `graph` command with the option `directed=true`. The calling syntax for `digraph` is the same as for `graph` with the above option appended to the sequence of arguments. However, creating special graphs is not supported by `digraph` since these are all undirected.

Edges in directed graphs are called **arcs**. Note that in a directed graph uv and vu are treated as distinct arcs, also denoted as (u, v) and (v, u).

1.1.3 Examples

Creating vertices An empty graph (without edges) can be created simply by entering the number of vertices or the list of vertex labels.

```
> graph(5)
```

an undirected unweighted graph with 5 vertices and 0 edges

```
> graph([a,b,c])
```

an undirected unweighted graph with 3 vertices and 0 edges

Graph constructors often have to generate vertex labels. In such cases, ordinal integers are used, which are 0-based in e.g. xcas mode and 1-based in e.g. maple mode. Note that the examples appearing throughout this manual are entered by using the default, xcas mode.

Creating edges and arcs Edges/arcs must be specified as a set which is thus distinguished from a (adjacency or weight) matrix. If a set of edges/arcs is entered as the single argument, then the required vertices are created automatically, in the order of appearance.

```
> graph(%{[a,b],[b,c],[a,c]%})
```

an undirected unweighted graph with 3 vertices and 3 edges

Edge weights may also be specified.

```
> graph(%{[[a,b],2],[[b,c],2.3],[[c,a],3/2]%})
```

an undirected weighted graph with 3 vertices and 3 edges

If the graph contains isolated vertices (not connected to any other vertex) or a particular order of vertices is desired, the list of vertices has to be specified as the first argument.

```
> graph([d,b,c,a],%{[a,b],[b,c],[a,c]%})
```

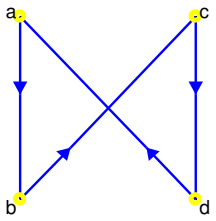
an undirected unweighted graph with 4 vertices and 3 edges

Creating paths and trails A directed graph can also be created from a list of n vertices and a permutation of order n. The resulting graph consists of a single directed cycle with the vertices ordered according to the permutation.

```
> G:=graph([a,b,c,d],[1,2,3,0])
```

a directed unweighted graph with 4 vertices and 4 arcs

```
> draw_graph(G)
```



Alternatively, one may specify edges as a `trail`.

```
> digraph([a,b,c,d],trail(b,c,d,a))
```

a directed unweighted graph with 4 vertices and 3 arcs

Using trails is also possible when creating undirected graphs. Also, some vertices in a trail may be repeated, which is not allowed in a path.

```
> G:=graph([a,b,c,d],trail(b,c,d,a,c))
```

an undirected unweighted graph with 4 vertices and 4 edges

```
> edges(G)
```

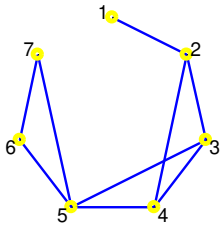
$$[[a,c],[a,d],[b,c],[c,d]]$$

It is possible to specify several trails in a sequence, which is useful when designing more complex graphs.

```
> G:=graph(trail(1,2,3,4,2),trail(3,5,6,7,5,4))
```

an undirected unweighted graph with 7 vertices and 9 edges

```
> draw_graph(G)
```



Specifying adjacency or weight matrix A graph can be created from a single square matrix $A = [a_{ij}]_n$ of order n. If it contains only ones and zeros and has zeros on its diagonal, then A is interpreted as the adjacency matrix of the desired graph. Otherwise, if an element outside the set $\{0, 1\}$ is encountered, then A is interpreted as a weight matrix, causing the edges in the resulting graph to be weighted accordingly. In each case, exactly n vertices will be created and i-th and j-th vertex will be connected if and only if $a_{ij} \neq 0$. If A is symmetric, then the resulting graph is undirected, otherwise it is directed.

```
> G:=graph([[0,1,1,0],[1,0,0,1],[1,0,0,0],[0,1,0,0]])
```

an undirected unweighted graph with 4 vertices and 3 edges

```
> edges(G)
```

$$[[0, 1], [0, 2], [1, 3]]$$

```
> G:=graph([[0,1.0,2.3,0],[4,0,0,3.1],[0,0,0,0],[0,0,0,0]])
```

a directed weighted graph with 4 vertices and 4 arcs

```
> edges(G,weights)
```

$$[[[0, 1], 1.0], [[0, 2], 2.3], [[1, 0], 4], [[1, 3], 3.1]]$$

A list of vertex labels can be specified alongside a matrix.

```
> graph([a,b,c,d],[[0,1,1,0],[1,0,0,1],[1,0,0,0],[0,1,0,0]])
```

an undirected unweighted graph with 4 vertices and 3 edges

When creating a weighted graph, one can specify the list of n vertices and the set of edges, followed by a square matrix A of order n. Then for every edge from i-th to j-th vertex, the element a_{ij} of A is assigned as its weight. The remaining elements of A are ignored.

```
> G:=digraph([a,b,c],%{[a,b],[b,c],[a,c]%},[[0,1,2],[3,0,4],[5,6,0]])
```

a directed weighted graph with 3 vertices and 3 arcs

```
> edges(G,weights)
```

$$[[[a, b], 1], [[a, c], 2], [[b, c], 4]]$$

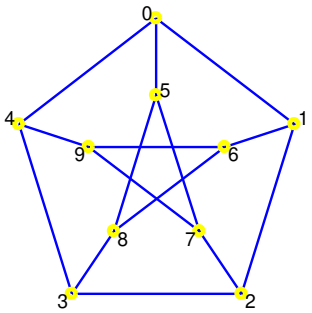
Creating special graphs If the `graph` command is supplied with a name of a special graph, it constructs and returns the latter.

```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

Some special graphs are constructed together with the corresponding layouts. These are displayed when the graphs are drawn using the default settings.

```
> draw_graph(P)
```



1.2 Cycle and path graphs

1.2.1 Cycle graphs

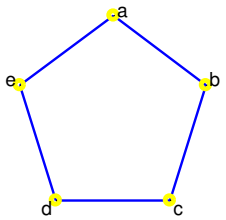
`cycle_graph(n|V)`

The command `cycle_graph` is used for constructing cycle graphs [34, p. 4]. It takes a positive integer n or a list of distinct vertices V as its only argument and returns the graph consisting of a single cycle $C_{|V|}$ on the specified vertices in the given order. If n is specified, then C_n is returned.

```
> G:=cycle_graph(["a","b","c","d","e"])
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> draw_graph(G)
```



1.2.2 Path graphs

`path_graph(n|V)`

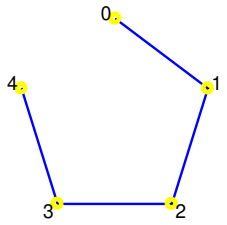
The command `path_graph` is used for constructing path graphs [34, pp. 4]. It takes a positive integer n or a list of distinct vertices V as its only argument and returns a graph consisting of a single path on the specified vertices in the given order. If n is specified, then a path graph on n vertices is returned.

Note that a path, by definition, is a walk with no repeated vertices. Walks with no repeated edges but possibly repeated vertices are called **trails** (see the command `trail`).

```
> P:=path_graph(5)
```

an undirected unweighted graph with 5 vertices and 4 edges

```
> draw_graph(P,circle)
```



1.2.3 Trails of edges

`trail(v1,v2,..,vn)`
`trail2edges(T)`

If the dummy command `trail` is called with a sequence of vertices $v_1, v_2, ..., v_n$ as arguments, it returns the symbolic expression representing the trail which visits the specified vertices in the given order. The resulting symbolic object is recognizable by some commands, for example `graph` and `digraph`.

Note that a trail, by definition, is a walk with no repeated edges. Hence some vertices in the sequence $v_1, v_2, ..., v_k$ may be repeated, but the sets $\{v_i, v_{i+1}\}$ in undirected graphs resp. the pairs (v_i, v_{i+1}) in digraphs must be mutually distinct for $i = 1, 2, ..., n-1$, since they represent edges resp. arcs.

Any trail T is easily converted to the corresponding list of edges by calling the `trail2edges` command, which takes the trail as its only argument.

```
> T:=trail(1,2,3,4,2):; graph(T)
```

"Done", an undirected unweighted graph with 4 vertices and 4 edges

```
> trail2edges(T)
```

$[[1, 2], [2, 3], [3, 4], [4, 2]]$

1.3 Complete graphs

1.3.1 Complete (multipartite) graphs

`complete_graph(n|V)`
`complete_graph(n1,n2,..,nk)`

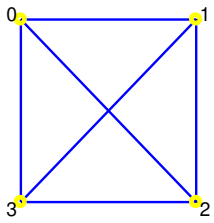
The command `complete_graph` is used for constructing complete (multipartite) graphs. It can be called with a single argument, a positive integer n or a list of distinct vertices V, in which case it returns the complete graph $K_{|V|}$ [34, pp. 2] on the specified vertices. If integer n is specified, then K_n is returned.

If `complete_graph` is given a sequence of positive integers $n_1, n_2, ..., n_k$ as its argument, it returns a complete multipartite graph $K_{n_1 n_2 \cdots n_k}$ with partitions of size $n_1, n_2, ..., n_k$.

```
> K4:=complete_graph(4)
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> draw_graph(K4)
```




```
> K3:=complete_graph([a,b,c])
```

an undirected unweighted graph with 3 vertices and 3 edges

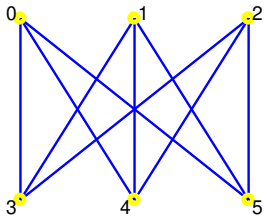
```
> edges(K3)
```

$$[[a,b],[a,c],[b,c]]$$

```
> K33:=complete_graph(3,3)
```

an undirected unweighted graph with 6 vertices and 9 edges

```
> draw_graph(K33)
```



1.3.2 Complete trees

`complete_binary_tree(n)`
`complete_kary_tree(k,n)`

The commands `complete_binary_tree` and `complete_kary_tree` are used for construction of complete binary trees and complete k-ary trees, respectively.

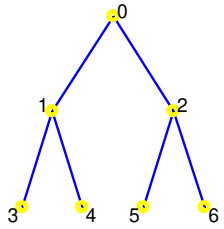
`complete_binary_tree` takes a positive integer n as its only argument and returns a complete binary tree with depth n on $2^{n+1}-1$ vertices. The lowest-indexed vertex is the root of the tree.

`complete_kary_tree` takes positive integers k and n as its arguments and returns the complete k-ary tree on $\frac{k^{n+1}-1}{k-1}$ vertices with depth n. The vertex with smallest number is the root of the tree. The command `complete_kary_tree(2,n)` is equivalent to `complete_binary_tree(n)`.

```
> T1:=complete_binary_tree(2)
```

an undirected unweighted graph with 7 vertices and 6 edges

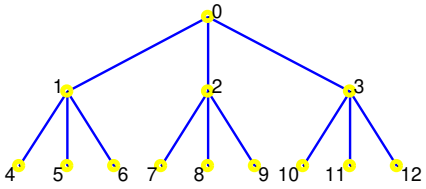
```
> draw_graph(T1)
```




```
> T2:=complete_kary_tree(3,2)
```

an undirected unweighted graph with 13 vertices and 12 edges

```
> draw_graph(T2)
```



1.4 Sequence graphs

1.4.1 Creating graphs from degree sequences

`sequence_graph(L)`

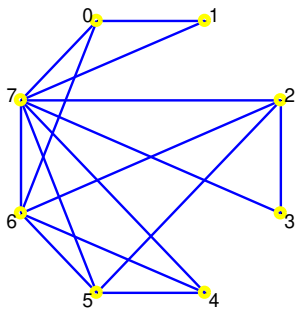
The command `sequence_graph` is used for constructing graphs from degree sequences. It takes a list L of positive integers as its only argument and, if L represents a graphic sequence, the corresponding undirected graph G with $|L|$ vertices is returned. If the argument is not a graphic sequence, then an error is returned.

Sequence graphs are constructed in $O(|L|^2 \log |L|)$ time by applying the algorithm of Havel and Hakimi [38].

```
> G:=sequence_graph([3,2,4,2,3,4,5,7])
```

an undirected unweighted graph with 8 vertices and 15 edges

```
> draw_graph(G,circle)
```




```
> degree_sequence(G)
```

$[3, 2, 4, 2, 3, 4, 5, 7]$

1.4.2 Validating graphic sequences

`is_graphic_sequence(L)`

The command `is_graphic_sequence` is used for determining whether a list of integers represents the degree sequence of some graph. It takes a list L of positive integers as its only argument and returns `true` if there exists a graph $G(V, E)$ with degree sequence $\{\deg v : v \in V\}$ equal to L and `false` otherwise. The algorithm, which has the complexity $O(|L|^2)$, is based on the theorem of ERDŐS and GALLAI.

```
> is_graphic_sequence([3,2,4,2,3,4,5,7])
```

true

1.5 Intersection graphs

1.5.1 Interval graphs

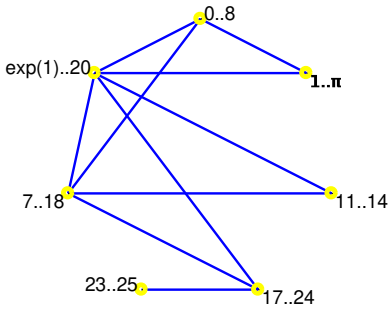
`interval_graph(L)`

The command `interval_graph` is used for construction of interval graphs. It takes a sequence or list L of real-line intervals as its argument and returns an undirected unweighted graph with these intervals as vertices (the string representations of the intervals are used as labels), each two of them being connected with an edge if and only if the corresponding intervals intersect.

```
> G:=interval_graph(0..8,1..pi,exp(1)..20,7..18,11..14,17..24,23..25)
```

an undirected unweighted graph with 7 vertices and 10 edges

```
> draw_graph(G)
```



1.5.2 Kneser graphs

`kneser_graph(n,k)`
`odd_graph(d)`

Commands `kneser_graph` and `odd_graph` are used for generating Kneser graphs and odd graphs, respectively.

`kneser_graph` takes two positive integers $2 \leqslant n \leqslant 20$ and $k < n$ as its arguments and returns the Kneser graph $K(n, k)$ with $\binom{n}{k}$ k-subsets of the set $\{1, 2, ..., n\}$ as vertices. Two vertices are connected if and only if their associated subsets are disjoint. Each vertex is labeled by the list of elements in the corresponding subset.

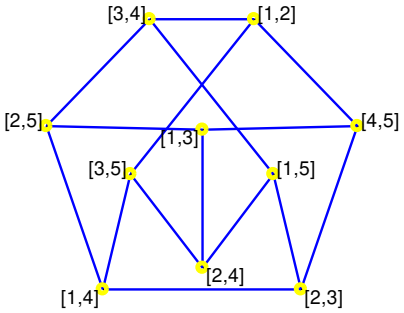
`odd_graph` takes an integer $2 \leqslant d \leqslant 10$ and returns d-th odd graph $O_d = K(2d-1, d-1)$.

Kneser graphs quickly get exceedingly large, hence n is restricted as above.

```
> K:=kneser_graph(5,2)
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> draw_graph(K,spring)
```




```
> G:=kneser_graph(12,5)
```

an undirected unweighted graph with 792 vertices and 8316 edges

```
> odd_graph(8)
```

an undirected unweighted graph with 6435 vertices and 25740 edges

1.204 sec

1.5.3 Johnson graphs

`johnson_graph(n,k)`

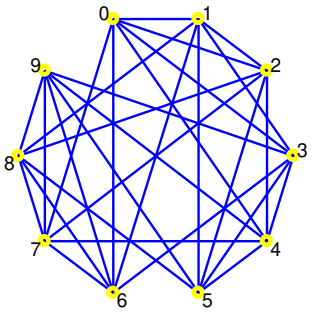
The command `johnson_graph` is used for generating Johnson graphs. It takes two positive integers $2 \leqslant n \leqslant 20$ and $k < n$ as its arguments and returns the Johnson graph $J(n,k)$ with $\binom{n}{k}$ k-subsets of a set of n elements as vertices. Two vertices are connected if and only if their associated subsets have exactly $k-1$ elements in common.

Johnson graphs quickly get exceedingly large, hence n is restricted as above.

```
> J52:=johnson_graph(5,2)
```

an undirected unweighted graph with 10 vertices and 30 edges

```
> draw_graph(J52)
```




```
> chromatic_number(J52)
```

5

$J(n,2)$ is isomorphic to the Kneser graph $K(n,2)$.

```
> is_isomorphic(J52,graph_complement(kneser_graph(5,2)))
```

true

$J(n,k)$ is isomorphic to $J(n,n-k)$.

```
> is_isomorphic(J52,johnson_graph(5,3))
```

true

The clique number of $J(n,k)$ is equal to $1-\frac{\lambda_{\max}}{\lambda_{\min}}$, where $\lambda_{\min}$ and $\lambda_{\max}$ are its least and greatest eigenvalue. The list of eigenvalues can be obtained by the command `graph_spectrum`.

```
> sp:=graph_spectrum(J52)
```

$$\begin{pmatrix} -2 & 5 \\ 1 & 4 \\ 6 & 1 \end{pmatrix}$$

```
> ev:=tran(sp)[0]
```

$[-2,1,6]$

```
> clique_number(J52) == 1-max(ev)/min(ev)
```

true

1.6 Special graphs

1.6.1 Hypercube graphs

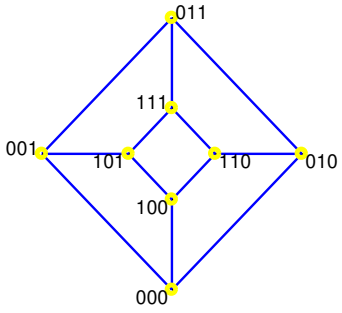
`hypercube_graph(n)`

The command `hypercube_graph` is used for constructing hypercube graphs. It takes a positive integer n as its only argument and returns the hypercube graph of dimension n on 2^n vertices. The vertex labels are strings of binary digits of length n. Two vertices are joined by an edge if and only if their labels differ in exactly one character.

```
> H:=hypercube_graph(3)
```

an undirected unweighted graph with 8 vertices and 12 edges

```
> draw_graph(H,planar)
```



1.6.2 Star graphs

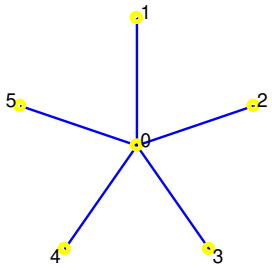
`star_graph(n)`

The command `star_graph` is used for constructing star graphs. It takes a positive integer n as its only argument and returns the star graph with $n+1$ vertices, which is equal to the complete bipartite graph `complete_graph(1,n)` i.e. a n-ary tree with one level.

```
> G:=star_graph(5)
```

an undirected unweighted graph with 6 vertices and 5 edges

```
> draw_graph(G)
```



1.6.3 Wheel graphs

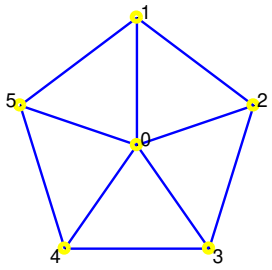
`wheel_graph(n)`

The command `wheel_graph` is used for constructing wheel graphs. It takes a positive integer n as its only argument and returns the wheel graph with $n+1$ vertices.

```
> G:=wheel_graph(5)
```

an undirected unweighted graph with 6 vertices and 10 edges

```
> draw_graph(G)
```



1.6.4 Web graphs

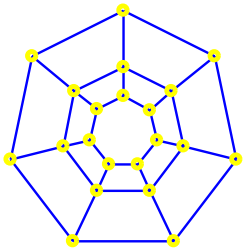
`web_graph(a,b)`

The command `web_graph` is used for constructing web graphs. It takes two positive integers a and b as its arguments and returns the Cartesian product of `cycle_graph(a)` and `path_graph(b)`.

```
> G:=web_graph(7,3)
```

an undirected unweighted graph with 21 vertices and 35 edges

```
> draw_graph(G,labels=false)
```



1.6.5 Prism graphs

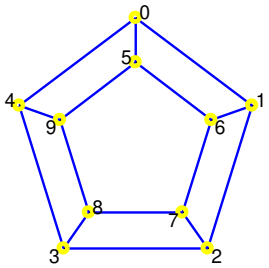
`prism_graph(n)`

The command `prism_graph` is used for constructing prism graphs. It takes a positive integer n as its only argument and returns the prism graph with parameter n, namely `petersen_graph(n,1)`.

```
> G:=prism_graph(5)
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> draw_graph(G)
```



1.6.6 Antiprism graphs

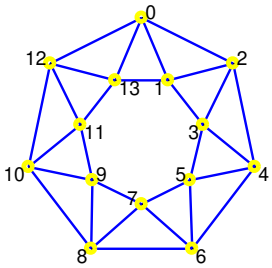
`antiprism_graph(n)`

The command `antiprism_graph` is used for constructing antiprism graphs. It takes a positive integer n as its only argument and returns the antiprism graph with parameter n, which is constructed from two concentric cycles of n vertices by joining each vertex of the inner to two adjacent nodes of the outer cycle.

```
> G:=antiprism_graph(7)
```

an undirected unweighted graph with 14 vertices and 28 edges

```
> draw_graph(G)
```



1.6.7 Grid graphs

`grid_graph(m,n,⟨triangle⟩)`
`torus_grid_graph(m,n)`

The command `grid_graph` resp. `torus_grid_graph` is used for constructing rectangular/triangular resp. torus grid graphs.

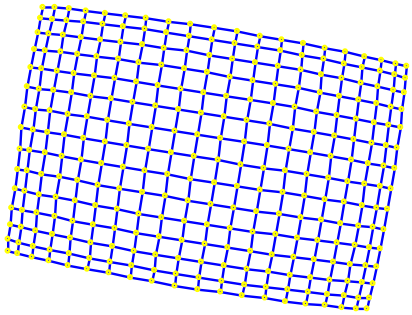
`grid_graph` takes two positive integers m and n as its arguments and returns the m by n grid on $m\,n$ vertices, namely the Cartesian product of `path_graph(m)` and `path_graph(n)`. If the option `triangle` is passed as the third argument, then the returned graph is a triangular grid on $m\,n$ vertices defined as the underlying graph of the strong product of two directed path graphs with m and n vertices, respectively [2, Definition 2, p. 189]. Strong product is defined as the union of Cartesian and tensor products.

`torus_grid_graph` takes two positive integers m and n as its arguments and returns the m by n torus grid on $m\,n$ vertices, namely the Cartesian product of `cycle_graph(m)` and `cycle_graph(n)`.

```
> G:=grid_graph(15,20)
```

an undirected unweighted graph with 300 vertices and 565 edges

```
> draw_graph(G,spring)
```

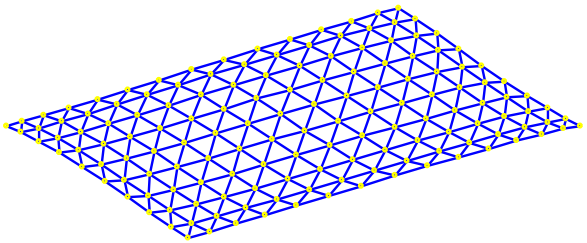


A triangular grid is created by passing the option `triangle`.

```
> G:=grid_graph(10,15,triangle)
```

an undirected unweighted graph with 150 vertices and 401 edges

```
> draw_graph(G,spring)
```

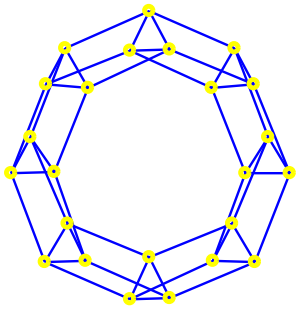


The following example demonstrates creating a torus grid graph with eight triangular levels.

```
> G:=torus_grid_graph(8,3)
```

an undirected unweighted graph with 24 vertices and 48 edges

```
> draw_graph(G,spring,labels=false)
```



1.6.8 Sierpiński graphs

`sierpinski_graph(n,k,⟨triangle⟩)`

The command `sierpinski_graph` is used for constructing Sierpiński-type graphs S_k^n and ST_k^n [42]. It takes two positive integers n and k as its arguments and optionally the option `triangle` as the third argument. It returns the Sierpiński (triangle) graph with parameters n and k.

The Sierpiński triangle graph ST_k^n is obtained by contracting all non-clique edges in S_k^n. To detect such edges the command `find_cliques` is used, which can be time consuming for $n > 6$.

```
> S:=sierpinski_graph(3,3)
```

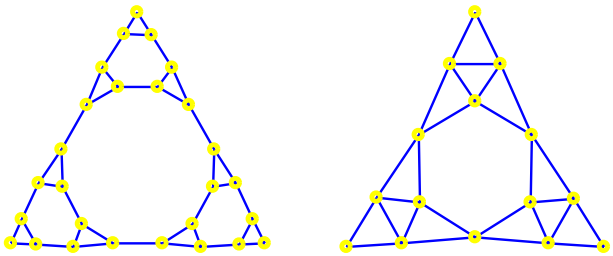
an undirected unweighted graph with 27 vertices and 39 edges

```
> ST:=sierpinski_graph(3,3,triangle)
```

an undirected unweighted graph with 15 vertices and 27 edges

```
> draw_graph(S,spring,labels=false,[0,0],size=[0,1],title="S");
  draw_graph(ST,spring,labels=false,[1.5,0],size=[0,1],title="ST");
```

S ST



1.6.9 Generalized Petersen graphs

`petersen_graph(n,⟨k⟩)`

The command `petersen_graph` is used for constructing generalized Petersen graphs. It takes two positive integers n and k as its arguments. The second argument may be omitted, in which case $k = 2$ is assumed. The return value is the graph $P(n, k)$, which is a connected cubic graph consisting of—in Schläfli notation—an inner star polygon $\{n, k\}$ and an outer regular polygon $\{n\}$ such that the n pairs of corresponding vertices in inner and outer polygons are connected with edges. For $k = 1$ the prism graph of order n is obtained.

The famous Petersen graph is equal to the generalized Petersen graph $P(5, 2)$. It can also be constructed by calling `graph("petersen")`.

To obtain the dodecahedral graph $P(10, 2)$, input:

```
> G:=petersen_graph(10)
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> is_isomorphic(G,graph("dodecahedron"))
```

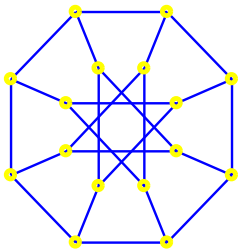
true

To obtain Möbius–Kantor graph $P(8, 3)$, input:

```
> G:=petersen_graph(8,3)
```

an undirected unweighted graph with 16 vertices and 24 edges

```
> draw_graph(G,labels=false)
```



Note that Desargues, Dürer and Nauru graphs are isomorphic to the generalized Petersen graphs $P(10,3)$, $P(6,2)$ and $P(12,5)$, respectively.

1.6.10 Snark graphs

`flower_snark(n)`
`goldberg_snark(n)`

Commands `flower_snark` and `goldberg_snark` are used for generating flower snarks and Goldberg snarks [32]. Both commands take an odd integer $n \geqslant 3$ as their only argument and return the flower snark J_n resp. the Goldberg snark on $8\,n$ vertices.

```
> J3:=flower_snark(3)
```

an undirected unweighted graph with 12 vertices and 18 edges

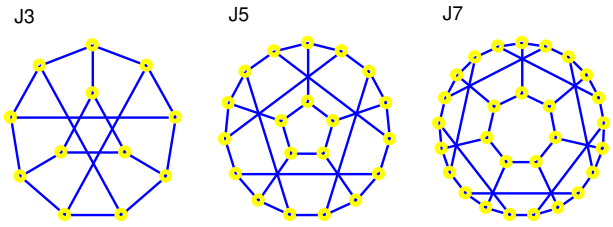
```
> J5:=flower_snark(5)
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> J7:=flower_snark(7)
```

an undirected unweighted graph with 28 vertices and 42 edges

```
> draw_graph(J3,[0,0],size=[1,0],title="J3",labels=false);
  draw_graph(J5,[1.3,0],size=[1,0],title="J5",labels=false);
  draw_graph(J7,[2.6,0],size=[1,0],title="J7",labels=false)
```



The first flower snark J_3 (which, strictly speaking, is not a snark because its girth is smaller than 5) is isomorphic to Tietze graph.

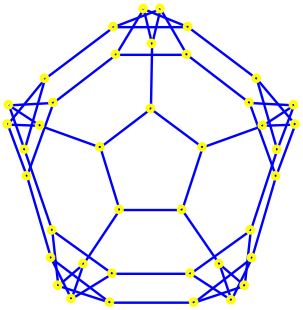
```
> is_isomorphic(J3,graph("tietze"))
```

true

```
> G:=goldberg_snark(5)
```

an undirected unweighted graph with 40 vertices and 60 edges

```
> draw_graph(G,spring,labels=false)
```



```
> chromatic_index(G) == 4
```

true

3.18 sec

```
> girth(G)
```

5

1.6.11 Paley graphs

`paley_graph(p,⟨k⟩)`

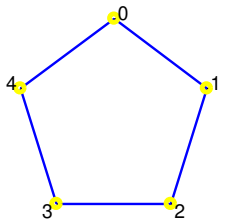
The command `paley_graph` is used for generating Paley graphs. It takes a prime number $p \geqslant 3$ as its mandatory argument and optionally a positive integer k which defaults to 1. It returns the Paley (di)graph P on p^k vertices which is constructed as follows.

- If $k = 1$ and $p \equiv 1 \pmod 4$, then P is an undirected graph with vertices $0, 1, ..., p-1$ in which the edge $\{i, j\}$, where $i < j$, is present in P if $j - i$ is a quadratic residue in $\mathbb{Z}_p$.
- If $k = 1$ and $p \equiv 3 \pmod 4$, then P is a directed graph with vertices $0, 1, ..., p-1$ in which the arc (i, j) is present in P if $j - i$ is a quadratic residue in $\mathbb{Z}_p$.
- If $k > 1$ and $q = p^k \equiv 1 \pmod 4$, then P is an undirected graph with vertices $0, 1, ..., q-1$ in which the edge $\{i, j\}$, where $i < j$, is present in P if $y - x$ is a square in the finite field $\mathrm{GF}(q)$, where x is the i-th element and y is the j-th element in $\mathrm{GF}(q)$. If g is a generator of the corresponding multiplicative group, then the elements of $\mathrm{GF}(q)$ are $0, 1, g, g^2, ..., g^{p^k-2}$.
- If $k > 1$ and $q = p^k \equiv 3 \pmod 4$, then P is a directed graph with vertices $0, 1, ..., q-1$ in which the arc (i, j) is present in P if $y - x$ is a square in the finite field $\mathrm{GF}(q)$, where x is the i-th element and y is the j-th element in $\mathrm{GF}(q)$. If g is a generator of the corresponding multiplicative group, then the elements of $\mathrm{GF}(q)$ are $0, 1, g, g^2, ..., g^{p^k-2}$.

```
> P1:=paley_graph(5)
```

an undirected unweighted graph with 5 vertices and 5 edges

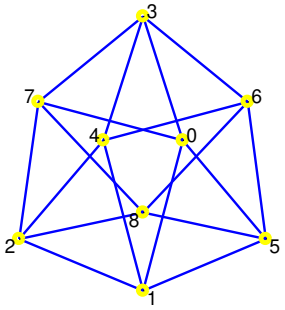
```
> draw_graph(P1)
```




```
> P2:=paley_graph(3,2)
```

an undirected unweighted graph with 9 vertices and 18 edges

```
> draw_graph(P2,spring)
```



Paley graphs are self-complementary.

```
> is_isomorphic(P2,graph_complement(P2))
```

true

```
> P3:=paley_graph(3,3)
```

a directed unweighted graph with 27 vertices and 351 arcs

Paley (di)graphs are strongly regular.

```
> is_strongly_regular(P3)
```

true

Every Paley digraph is a tournament.

```
> is_tournament(P3)
```

true

1.6.12 Haar graphs

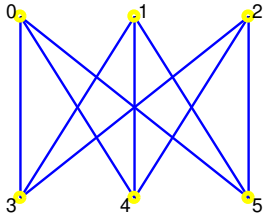
`haar_graph(n)`

The command `haar_graph` is used for generating Haar graphs [43]. It takes a positive integer n as its only argument and returns the Haar graph with index n, which is a regular bipartite graph on $2\,k$ vertices $v_0, v_1, ..., v_{2k-1}$, where $k = 1 + \lfloor \log_2 n \rfloor$, with independent sets $U = \{v_i : i = 0, 1, ..., k-1\}$ and $V = \{v_j : j = k, k+1, ..., 2\,k-1\}$. Two vertices $v_i \in U$ and $v_j \in V$ are connected if the m-th digit in the binary expansion of n is nonzero, where $m = \texttt{irem}\,(j-i, k)$.

```
> H1:=haar_graph(7)
```

an undirected unweighted graph with 6 vertices and 9 edges

```
> draw_graph(H1)
```




```
> H2:=haar_graph(69)
```

an undirected unweighted graph with 14 vertices and 21 edges

```
> is_isomorphic(H2,graph("heawood"))
```

true

```
> H3:=haar_graph(32786)
```

an undirected unweighted graph with 32 vertices and 48 edges

```
> is_isomorphic(H3,petersen_graph(16,7))
```

true

1.6.13 LCF graphs

`lcf_graph(L,⟨n⟩)`

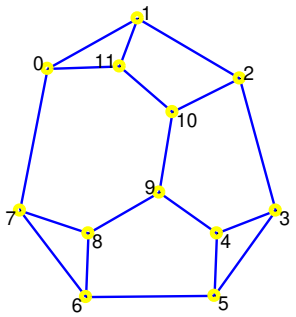
The command `lcf_graph` is used for constructing cubic Hamiltonian graphs from LCF notation. It takes one or two arguments, a list L of nonzero integers, called *jumps*, and optionally a positive integer n, called the *exponent*, which defaults to 1. The command returns the graph on $n\,|L|$ vertices obtained by iterating the sequence of jumps n times.

The following command creates the Frucht graph.

```
> F:=lcf_graph([-5,-2,-4,2,5,-2,2,5,-2,-5,4,2])
```

an undirected unweighted graph with 12 vertices and 18 edges

```
> draw_graph(F,planar)
```

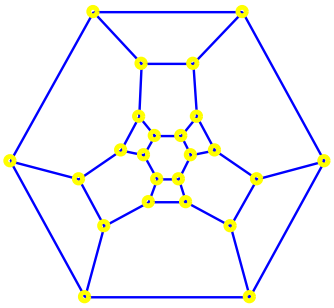


Constructing the truncated octahedral graph from its LCF notation:

```
> G:=lcf_graph([3,-7,7,-3],6)
```

an undirected unweighted graph with 24 vertices and 36 edges

```
> draw_graph(G,planar,labels=false)
```



```
> is_isomorphic(G,truncate_graph(graph("octahedron")))
```

true

1.7 Isomorphic copies of graphs

1.7.1 Creating isomorphic copies from permutations

`isomorphic_copy(G,⟨sigma⟩)`

The command `isomorphic_copy` is used for creating isomorphic copies of a graph without changing the order of its vertices. It takes one or two arguments, a graph $G(V, E)$ and optionally a permutation σ of order $|V|$. It returns a new graph where the adjacency lists are reordered according to σ or a random permutation if the second argument is omitted. The vertex labels, as well as the order of vertices, are the same as in G. This command, however, discards all vertex and edge attributes present in G.

```
> G:=path_graph([1,2,3,4,5])
```

an undirected unweighted graph with 5 vertices and 4 edges

```
> vertices(G), neighbors(G)
```

$[1, 2, 3, 4, 5], [[2], [1, 3], [2, 4], [3, 5], [4]]$

```
> H:=isomorphic_copy(G)
```

an undirected unweighted graph with 5 vertices and 4 edges

```
> vertices(H), neighbors(H)
```

$[1, 2, 3, 4, 5], [[2, 3], [1, 5], [1, 4], [3], [2]]$

```
> H:=isomorphic_copy(G,[2,4,0,1,3])
```

an undirected unweighted graph with 5 vertices and 4 edges

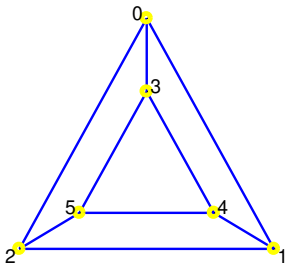
```
> vertices(H), neighbors(H)
```

$[1, 2, 3, 4, 5], [[4, 5], [5], [4], [1, 3], [1, 2]]$

```
> P:=prism_graph(3)
```

an undirected unweighted graph with 6 vertices and 9 edges

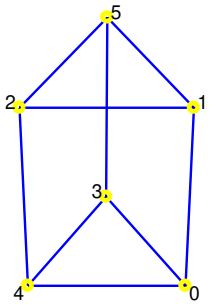
```
> draw_graph(P)
```



```
> H:=isomorphic_copy(P,[3,0,1,5,4,2])
```

an undirected unweighted graph with 6 vertices and 9 edges

```
> draw_graph(H,spring)
```



1.7.2 Permuting vertices

`permute_vertices(G,L)`
`permute_vertices(G,⟨shuffle|randperm⟩)`

The command `permute_vertices` is used for creating isomorphic copies of graphs by changing the order of vertices. It takes one or two arguments, a graph $G(V, E)$ and optionally a list L of length $|V|$ containing all vertices from V, and returns a copy of G with vertices rearranged in order they appear in L. If L is not given or if `shuffle` or `randperm` are passed as the second argument, then the vertices are shuffled randomly. All vertex and edge attributes are preserved in the copy, including any vertex position information. Hence the resulting graph will look the same as G when drawn.

```
> G:=path_graph([1,2,3,4,5])
```

an undirected unweighted graph with 5 vertices and 4 edges

```
> vertices(G), neighbors(G)
```

$$[1,2,3,4,5],[[2],[1,3],[2,4],[3,5],[4]]$$

```
> H:=permute_vertices(G,[3,5,1,2,4])
```

an undirected unweighted graph with 5 vertices and 4 edges

```
> vertices(H), neighbors(H)
```

$$[3,5,1,2,4],[[2,4],[4],[2],[1,3],[3,5]]$$

1.7.3 Relabeling vertices

`relabel_vertices(G,L)`

The command `relabel_vertices` is used for setting new labels to vertices of a graph without changing their order. It takes two arguments, a graph $G(V, E)$ and a list L of vertex labels of length $|V|$. It returns a copy of G with L as the list of vertex labels.

```
> G:=path_graph([1,2,3,4])
```

an undirected unweighted graph with 4 vertices and 3 edges

```
> edges(G)
```

$$[[1,2],[2,3],[3,4]]$$

```
> H:=relabel_vertices(G,[a,b,c,d])
```

an undirected unweighted graph with 4 vertices and 3 edges

```
> edges(H)
```

$$[[a,b],[b,c],[c,d]]$$

1.8 Subgraphs

1.8.1 Extracting subgraphs

`subgraph(G,L)`

The command **subgraph** is used for extracting subgraphs from graphs. It takes two arguments, a graph $G(V,E)$ and a list of edges L. It returns the subgraph $G'(V',L)$ of G, where $V' \subset V$ is a subset of vertices of G incident to at least one edge from L.

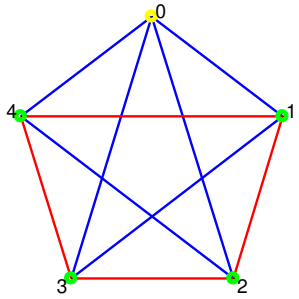
```
> K5:=complete_graph(5)
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> S:=subgraph(K5,[[1,2],[2,3],[3,4],[4,1]])
```

an undirected unweighted graph with 4 vertices and 4 edges

```
> draw_graph(highlight_subgraph(K5,S))
```



1.8.2 Induced subgraphs

`induced_subgraph(G,L)`

The command **induced_subgraph** is used for extracting induced subgraphs from graphs. It takes two arguments, a graph $G(V,E)$ and a list of vertices L. It returns the subgraph $G'(L,E')$ of G, where $E' \subset E$ contains all edges which have both endpoints in L [34, p. 3].

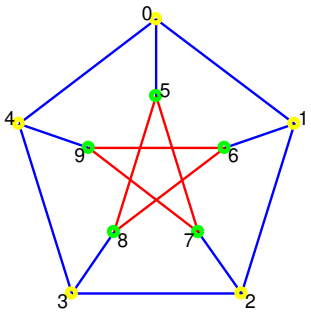
```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> S:=induced_subgraph(G,[5,6,7,8,9])
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> draw_graph(highlight_subgraph(G,S))
```



1.8.3 Underlying graphs

`underlying_graph(G)`

For every graph $G(V,E)$ there is an undirected and unweighted graph $U(V,E')$, called the **underlying graph** of G, where E' is obtained from E by dropping edge directions. To construct U, use the command `underlying_graph`. It takes a graph $G(V,E)$ as its only argument and returns an undirected unweighted copy of G in which all vertex and edge attributes, together with edge directions, are discarded.

```
> G:=digraph(%{[[1,2],6],[[2,3],4],[[3,1],5],[[3,2],7]%})
```

a directed weighted graph with 3 vertices and 4 arcs

```
> U:=underlying_graph(G)
```

an undirected unweighted graph with 3 vertices and 3 edges

```
> edges(U)
```

$[[1,2],[1,3],[2,3]]$

1.8.4 Fundamental cycles

`fundamental_cycle(G)`
`cycle_basis(G)`

The command `fundamental_cycle` is used for extracting cycles from unicyclic graphs (also called 1-trees). To find a fundamental cycle basis of an undirected graph, use the command `cycle_basis`.

`fundamental_cycle` takes one argument, an undirected connected graph $G(V,E)$ containing exactly one cycle (i.e. a unicyclic graph), and returns that cycle as a graph. If G is not unicyclic, then an error is returned.

`cycle_basis` takes an undirected graph $G(V,E)$ as its only argument and returns a basis B of the cycle space of G as a list of fundamental cycles in G, with each cycle represented as a list of vertices. Furthermore, $|B|=|E|-|V|+\kappa(G)$, where $\kappa(G)$ is the number of connected components of G. Every cycle C in G such that $C\notin B$ can be obtained from cycles in B using only symmetric differences.

The strategy is to construct a spanning tree T of G using depth-first search and look for edges in E which do not belong to the tree. For each non-tree edge e there is a unique fundamental cycle C_e consisting of e together with the path in T connecting the endpoints of e. The vertices of C_e are easily obtained from the search data. The complexity of this algorithm is $O(|V|+|E|)$.

```
> G:=graph(trail(1,2,3,4,5,2,6))
```

an undirected unweighted graph with 6 vertices and 6 edges

```
> C:=fundamental_cycle(G)
```

an undirected unweighted graph with 4 vertices and 4 edges

```
> edges(C)
```

$$[[4, 5], [2, 5], [3, 4], [2, 3]]$$

Given a tree graph G and adding an edge from the complement G^c to G one obtains a 1-tree graph.

```
> G:=random_tree(25)
```

an undirected unweighted graph with 25 vertices and 24 edges

```
> ed:=choice(edges(graph_complement(G)))
```

$$[3, 8]$$

```
> G:=add_edge(G,ed)
```

an undirected unweighted graph with 25 vertices and 25 edges

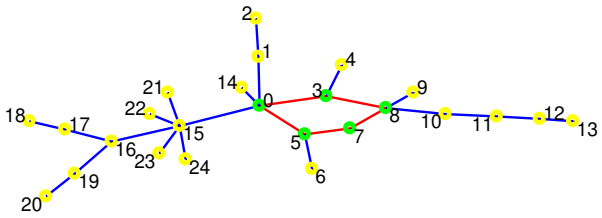
```
> C:=fundamental_cycle(G)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> edges(C)
```

$$[[5, 7], [0, 5], [7, 8], [3, 8], [0, 3]]$$

```
> draw_graph(highlight_subgraph(G,C),spring)
```

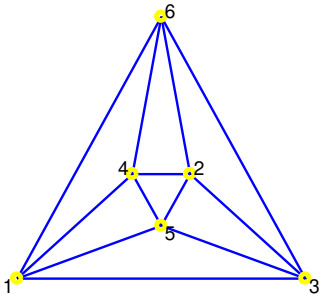


In the next example, a cycle basis of octahedral graph is computed.

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(G)
```




```
> cycle_basis(G)
```

$$[[6,3,1],[5,4,6,3,1],[4,6,3,1],[5,4,6,3],[2,5,4,6,3],[2,5,4,6],[2,5,4]]$$

Given a tree graph T, one can create a graph with cycle basis cardinality equal to k by simply adding k randomly selected edges from the complement T^c to T.

```
> tree1:=random_tree(15)
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> G1:=add_edge(tree1,rand(3,edges(graph_complement(tree1))))
```

an undirected unweighted graph with 15 vertices and 17 edges

```
> tree2:=random_tree(12)
```

an undirected unweighted graph with 12 vertices and 11 edges

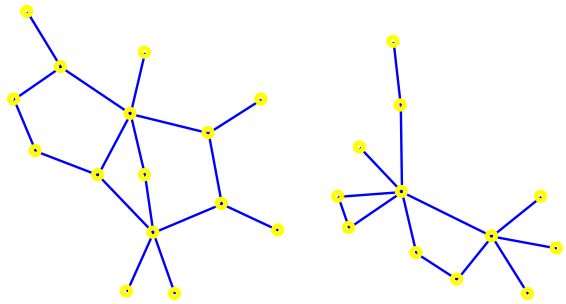
```
> G2:=add_edge(tree2,rand(2,edges(graph_complement(tree2))))
```

an undirected unweighted graph with 12 vertices and 13 edges

```
> G:=disjoint_union(G1,G2)
```

an undirected unweighted graph with 27 vertices and 30 edges

```
> draw_graph(G,spring,labels=false)
```



```
> nops(cycle_basis(G))
```

5

```
> number_of_edges(G)-number_of_vertices(G)+nops(connected_components(G))
```

5

1.8.5 Finding cycles in digraphs

`find_cycles(G)`
`find_cycles(G,length=k)`
`find_cycles(G,length=l..u)`

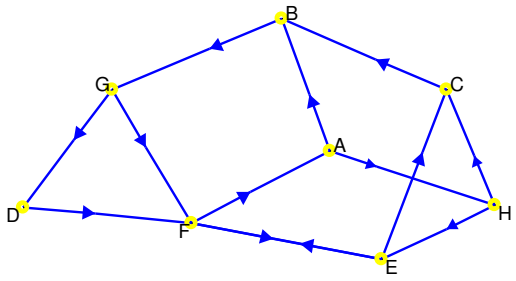
The command `find_cycles` is used for finding cycles (elementary circuits) in directed graphs. It takes a digraph $G(V,E)$ as its first argument. If it is the only input given, `find_cycles` returns the list of all cycles in G where each cycle is output as a list of its vertices. If an optional second argument `length=k` resp. `length=l..u` is given, where k, l and u are positive integers, only cycles of length k resp. of length between l and u (inclusive) are returned.

The strategy is to use TARJAN's algorithm for enumerating elementary circuits in a digraph [72]. The algorithm runs in $O(|V|\,|E|\,(C+1))$ time, where C is the number of cycles in G.

```
> purge(A,B,C,D,E,F,G,H):;
> DG:=digraph(%{[A,B],[A,H],[B,G],[C,B],[D,F],[E,C],[E,F],[F,A],[F,E],[G,D],[G,
  F],[H,C],[H,E]%})
```

a directed unweighted graph with 8 vertices and 13 arcs

```
> draw_graph(DG,spring)
```




```
> find_cycles(DG)
```

$[[A,H,E,F],[A,H,E,C,B,G,F],[A,H,E,C,B,G,D,F],[A,H,C,B,G,F],[A,H,C,B,G,D,F],[A,B,G,F],[A,B,G,D,F],[B,G,F,E,C],[B,G,D,F,E,C],[F,E]]$

```
> find_cycles(DG,length=4)
```

$[[A,H,E,F],[A,B,G,F]]$

```
> find_cycles(DG,length=6..7)
```

$[[A,H,E,C,B,G,F],[A,H,C,B,G,F],[A,H,C,B,G,D,F],[B,G,D,F,E,C]]$

1.9 Operations on graphs

1.9.1 Graph complement

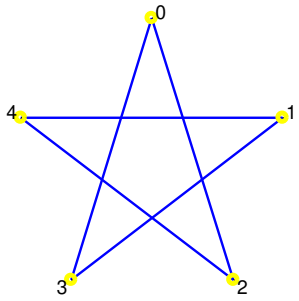
`graph_complement(G)`

The command `graph_complement` is used for constructing complement graphs. It takes a graph $G(V,E)$ as its only argument and returns the complement graph $G^c(V,E^c)$ of G, where E^c is the largest set containing only edges/arcs not present in G.

```
> G:=graph_complement(cycle_graph(5))
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> draw_graph(G)
```



1.9.2 Graph switching

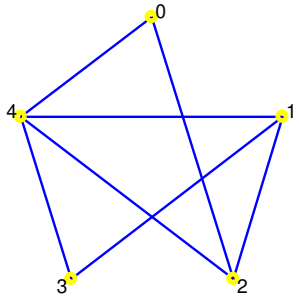
`seidel_switch(G,L)`

The command **seidel_switch** is used for Seidel switching in graphs. It takes two arguments, an undirected and unweighted graph $G(V,E)$ and a list of vertices $L \subset V$. The result is a copy of G in which, for each vertex $v \in L$, its neighbors become its non-neighbors and vice versa. The edges whose endpoints are both in the set, or both not in the set, are not changed.

```
> S:=seidel_switch(cycle_graph(5),[1,2])
```

an undirected unweighted graph with 5 vertices and 7 edges

```
> draw_graph(S)
```




```
> purge(A,B,C,D,E,F,X,Y):;
```

```
> G:=graph([A,B,C,D,E,F,X,Y],%{[A,B],[B,C],[B,X],[B,Y],[C,D],[C,Y],[D,X],[E,Y],
  [F,Y],[X,Y]%})
```

an undirected unweighted graph with 8 vertices and 10 edges

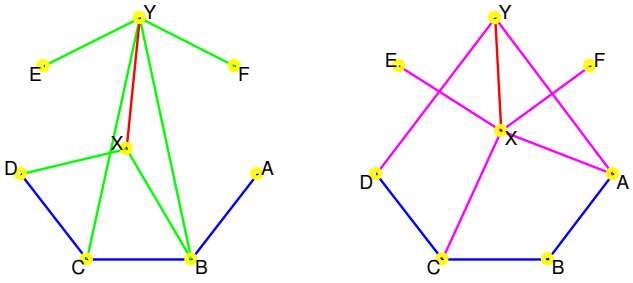
```
> G:=highlight_edges(G,[[X,B],[X,D],[Y,B],[Y,C],[Y,E],[Y,F],[X,Y]],[green$6,
  red]):;
```

```
> H:=seidel_switch(G,[X,Y])
```

an undirected unweighted graph with 8 vertices and 10 edges

```
> H:=highlight_edges(H,[[X,A],[X,C],[X,E],[X,F],[Y,A],[Y,D],[X,Y]],[magenta$6,
  red]):;
```

```
> draw_graph(G,circle=[Y,F,A,B,C,D,E],[0,0],size=[1,0]);
  draw_graph(H,circle=[Y,F,A,B,C,D,E],[1.5,0],size=[1,0]);
```



1.9.3 Transposing graphs

`reverse_graph(G)`

The command **reverse_graph** is used for reversing arc directions in digraphs. It takes a graph $G(V,E)$ as its only argument and returns the reverse graph $G^T(V,E')$ of G where $E' = \{vu : uv \in E\}$, i.e. returns the copy of G with the directions of all edges reversed.

Note that `reverse_graph` is defined for both directed and undirected graphs, but gives meaningful results only for directed graphs.

G^T is also called the **transpose graph** of G because adjacency matrices of G and G^T are transposes of each other (hence the notation).

```
> G:=digraph(6, %{[1,2],[2,3],[2,4],[4,5]%})
```

a directed unweighted graph with 6 vertices and 4 arcs

```
> GT:=reverse_graph(G)
```

a directed unweighted graph with 6 vertices and 4 arcs

```
> edges(GT)
```

$$[[2,1],[3,2],[4,2],[5,4]]$$

1.9.4 Union of graphs

`graph_union(G1,G2,..,Gn)`

The command `graph_union` is used for constructing unions of graphs. It takes a sequence of graphs $G_k(V_k, E_k)$ for $k = 1, 2, ..., n$ as its argument and returns the graph $G(V, E)$ where $V = V_1 \cup V_2 \cup \cdots \cup V_n$ and $E = E_1 \cup E_2 \cup \cdots \cup E_n$.

```
> G1:=graph([1,2,3],%{[1,2],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> G2:=graph([1,2,3],%{[3,1],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> G:=graph_union(G1,G2)
```

an undirected unweighted graph with 3 vertices and 3 edges

```
> edges(G)
```

$$[[1,2],[1,3],[2,3]]$$

1.9.5 Disjoint union of graphs

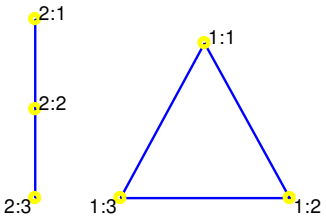
`disjoint_union(G1,G2,..,Gn)`

The command `disjoint_union` is used for constructing a disjoint union of graphs. It takes a sequence of graphs $G_k(V_k, E_k)$ for $k = 1, 2, ..., n$ as its only argument and returns the graph obtained by labeling all vertices with strings `k:v` where $v \in V_k$ and all edges with strings `k:e` where $e \in E_k$ and calling `graph_union` subsequently. As all vertices and edges are labeled differently, it follows $|V| = \sum_{k=1}^{n} |V_k|$ and $|E| = \sum_{k=1}^{n} |E_k|$.

```
> G:=disjoint_union(cycle_graph([1,2,3]),path_graph([1,2,3]))
```

an undirected unweighted graph with 6 vertices and 5 edges

```
> draw_graph(G)
```



1.9.6 Joining two graphs

`graph_join(G,H)`

The command `graph_join` is used for joining two graphs together. It takes two graphs G and H as its arguments and returns the graph $G+H$ which is obtained by connecting all the vertices of G to all vertices of H. The vertex labels in the resulting graph are strings of the form `1:u` and `2:v` where u is a vertex in G and v is a vertex in H.

```
> G:=path_graph(2)
```

an undirected unweighted graph with 2 vertices and 1 edge

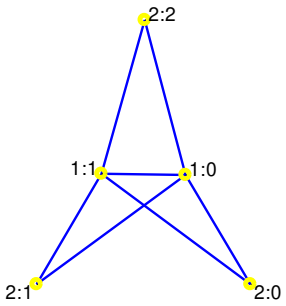
```
> H:=graph(3)
```

an undirected unweighted graph with 3 vertices and 0 edges

```
> GH:=graph_join(G,H)
```

an undirected unweighted graph with 5 vertices and 7 edges

```
> draw_graph(GH,spring)
```



1.9.7 Power graphs

`graph_power(G,k)`

The command `graph_power` is used for computing powers of graphs. It takes two arguments, a graph $G(V,E)$ and a positive integer k. It returns the k-th power G^k of G with vertices V such that $v,w \in V$ are connected with an edge if and only if there exists a path of length at most k in G.

The graph G^k is constructed from its adjacency matrix A_k which is obtained by adding powers of the adjacency matrix A of G:

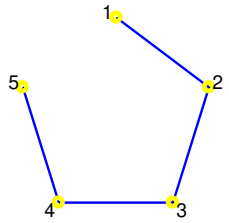
$$A_k = \sum_{i=1}^{k} A^k.$$

The above sum is obtained by assigning $A_k \leftarrow A$ and repeating the instruction $A_k \leftarrow (A_k + I)\, A$ for $k-1$ times, so exactly k matrix multiplications are required.

```
> G:=graph(trail(1,2,3,4,5))
```

an undirected unweighted graph with 5 vertices and 4 edges

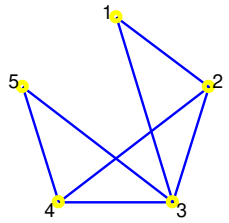
```
> draw_graph(G,circle)
```




```
> P2:=graph_power(G,2)
```

an undirected unweighted graph with 5 vertices and 7 edges

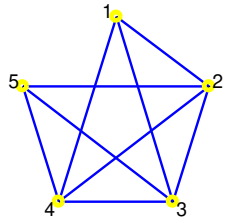
```
> draw_graph(P2,circle)
```




```
> P3:=graph_power(G,3)
```

an undirected unweighted graph with 5 vertices and 9 edges

```
> draw_graph(P3,circle)
```



1.9.8 Graph products

`cartesian_product(G1,G2,..,Gn)`
`tensor_product(G1,G2,..,Gn)`

There are two distinct operations for computing a product of two graphs: the Cartesian product and the tensor product. These operations are available in Giac as the commands `cartesian_product` and `tensor_product`, respectively.

`cartesian_product` takes a sequence of graphs $G_k(V_k, E_k)$ for $k = 1, 2, ..., n$ as its argument and returns the Cartesian product $G_1 \times G_2 \times \cdots \times G_n$ of the input graphs. The Cartesian product $G(V, E) = G_1 \times G_2$ is the graph with list of vertices $V = V_1 \times V_2$, labeled with strings `v1:v2` where $v_1 \in V_1$ and $v_2 \in V_2$, such that (`u1:v1`,`u2:v2`)$\in E$ if and only if u_1 is adjacent to u_2 and $v_1 = v_2$ **or** $u_1 = u_2$ and v_1 is adjacent to v_2.

`tensor_product` takes a sequence of graphs $G_k(V_k, E_k)$ for $k = 1, 2, ..., n$ as its argument and returns the tensor product $G_1 \times G_2 \times \cdots \times G_n$ of the input graphs. The tensor product $G(V, E) = G_1 \times G_2$ is the graph with list of vertices $V = V_1 \times V_2$, labeled with strings `v1:v2` where $v_1 \in V_1$ and $v_2 \in V_2$, such that (`u1:v1`,`u2:v2`)$\in E$ if and only if u_1 is adjacent to u_2 **and** v_1 is adjacent to v_2.

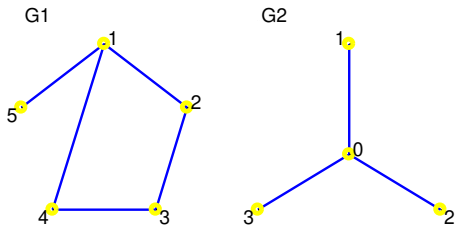
```
> G1:=graph(trail(1,2,3,4,1,5))
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> G2:=star_graph(3)
```

an undirected unweighted graph with 4 vertices and 3 edges

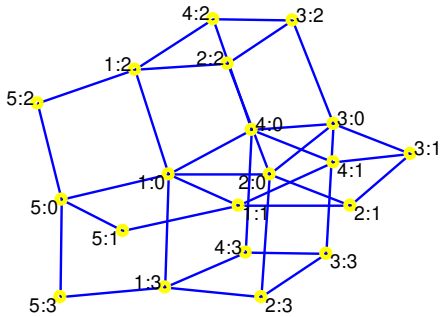
```
> draw_graph(G1,circle,[0,0],size=[0,1],title="G1");
  draw_graph(G2,[1.5,0],size=[0,1],title="G2")
```




```
> G:=cartesian_product(G1,G2)
```

an undirected unweighted graph with 20 vertices and 35 edges

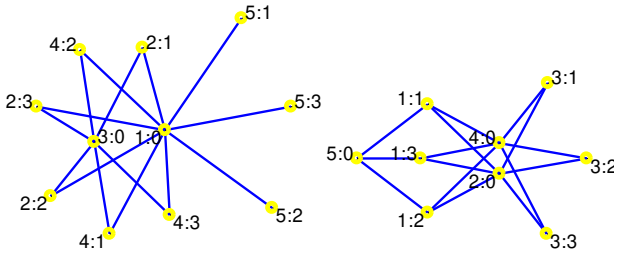
```
> draw_graph(G,spring)
```




```
> T:=tensor_product(G1,G2)
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> draw_graph(T,spring)
```



1.9.9 Transitive closure graph

`transitive_closure(G,⟨weighted⟩)`

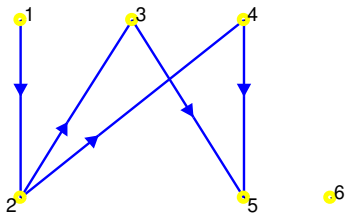
The command `transitive_closure` is used for constructing transitive closure graphs. It takes one or two arguments, a graph $G(V, E)$ and optionally the argument `weighted`. The command returns the transitive closure $T(V, E')$ of the input graph G by connecting $u \in V$ to $v \in V$ in T if and only if there is a path from u to v in G. If G is directed, then T is also directed. When `weighted` is specified, T is weighted such that the weight of edge $v\ w \in E'$ is equal to the length (or cost, if G is weighted) of the shortest path from v to w in G.

The lengths/weights of the shortest paths are obtained by the command `allpairs_distance` if G is weighted resp. the command `vertex_distance` if G is unweighted. Therefore T is constructed in at most $O(|V|^3)$ time if `weighted` is specified and in $O(|V|\,|E|)$ time otherwise.

```
> G:=digraph([1,2,3,4,5,6],%{[1,2],[2,3],[2,4],[4,5],[3,5]%})
```

a directed unweighted graph with 6 vertices and 5 arcs

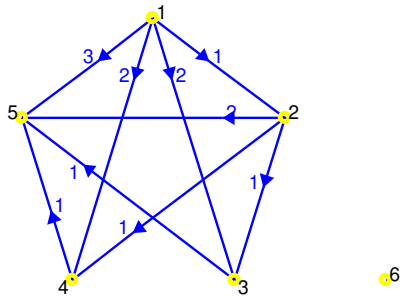
```
> draw_graph(G)
```




```
> T:=transitive_closure(G,weighted)
```

a directed weighted graph with 6 vertices and 9 arcs

```
> draw_graph(T)
```



1.9.10 Line graph

`line_graph(G)`

The command `line_graph` is used for constructing line graphs [34, p. 10]. It takes a graph $G(V, E)$ as its only argument and returns the corresponding line graph $L(G)$ with $|E|$ distinct vertices, one vertex for each edge in E. If G is undirected, then two vertices v_1 and v_2 in $L(G)$ are adjacent if and only if the corresponding edges $e_1, e_2 \in E$ have a common endpoint. If G is directed, then v_1 and v_2 are adjacent if and only if the corresponding arcs e_1 and e_2 form a directed path (e_1, e_2), i.e. if the head of e_1 coincides with the tail of e_2.

The vertices in $L(G)$ are labeled with strings in form `v-w`, where $e = vw \in E$.

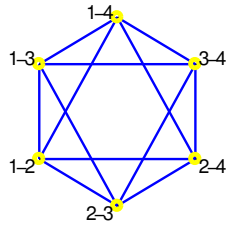
```
> K4:=complete_graph([1,2,3,4])
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> L:=line_graph(K4)
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(L,spring)
```



1.9.11 Plane dual graph

`plane_dual(G)`
`plane_dual(F)`
`is_planar(G,⟨F⟩)`

The command `plane_dual` is used for constructing the dual graph of an undirected biconnected planar graph. To determine whether a graph is planar [34, p. 12] use the command `is_planar`.

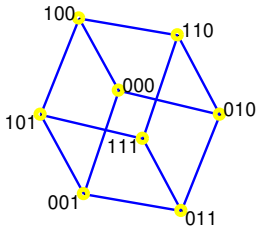
`plane_dual` takes a biconnected[1.1] planar graph $G(V, E)$ or the list F of faces of a planar embedding of G as its only argument and returns the graph H with faces of G as its vertices. Two vertices in H are adjacent if and only if the corresponding faces share an edge in G. The algorithm runs in $O(|V|^2)$ time.

`is_planar` takes one or two arguments, the input graph G and optionally an unassigned identifier `F`. It returns `true` if G is planar and `false` otherwise. If the second argument is provided and G is planar and biconnected, then the list of faces of G is assigned to `F`. Each face is represented as a list of its vertices. The strategy is to use the algorithm of DEMOUCRON et al. [33, p. 88], which runs in $O(|V|^2)$ time.

```
> H:=hypercube_graph(3)
```

an undirected unweighted graph with 8 vertices and 12 edges

```
> draw_graph(H,spring)
```

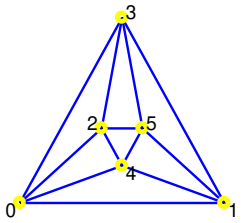


The cube has six faces, hence its plane dual graph D has six vertices. Also, every face obviously shares an edge with exactly four other faces, so the degree of each vertex in D is equal to 4.

```
> D:=plane_dual(H)
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(D,planar)
```




```
> is_planar(graph("petersen"))
```

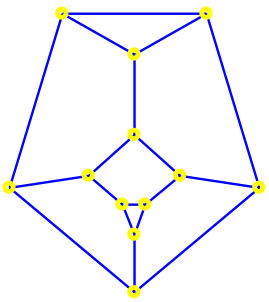
false

```
> is_planar(graph("duerer"))
```

true

1.1. The concept of dual graph is normally defined for multigraphs. Every planar multigraph has the corresponding dual multigraph; moreover, the dual of the latter is equal to the former. Since GIAC generally does not support multigraphs, a more specialized definition suitable for simple graphs is used; hence the requirement that the input graph is biconnected.

```
> draw_graph(graph("duerer"),planar,labels=false)
```



In the next example, a graph isomorphic to D is obtained when passing a list of faces of H to `plane_dual`. The order of vertices is determined by the order of faces.

```
> purge(F):; is_planar(H,F); F
```

$$\text{“Done”}, \text{true}, \begin{pmatrix} \text{“011”} & \text{“001”} & \text{“000”} & \text{“010”} \\ \text{“000”} & \text{“001”} & \text{“101”} & \text{“100”} \\ \text{“110”} & \text{“010”} & \text{“000”} & \text{“100”} \\ \text{“011”} & \text{“010”} & \text{“110”} & \text{“111”} \\ \text{“111”} & \text{“110”} & \text{“100”} & \text{“101”} \\ \text{“101”} & \text{“001”} & \text{“011”} & \text{“111”} \end{pmatrix}$$

```
> is_isomorphic(plane_dual(F),D)
```

true

1.9.12 Truncating planar graphs

`truncate_graph(G)`

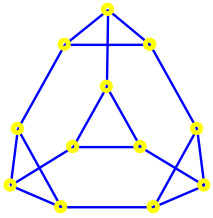
The command `truncate_graph` performs truncation of biconnected planar graphs. It takes a biconnected planar graph $G(V, E)$ as its only argument and returns the graph obtained by truncating the respective polyhedron, i.e. by “cutting off” its vertices. The resulting graph has $2|E|$ vertices and $3|E|$ edges. The procedure of truncating a graph by subdividing its edges is described in [5].

The algorithm requires computing a planar embedding of G, which is done by applying Demoucron's algorithm. Hence its complexity is $O(|V|^2)$.

```
> G:=truncate_graph(graph("tetrahedron"))
```

an undirected unweighted graph with 12 vertices and 18 edges

```
> draw_graph(G,spring,labels=false)
```



Truncating the plane dual of G represents the **leapfrog operation** on G, which can be used for constructing fullerene graphs [5]. By performing the leapfrog operation on a fullerene graph one obtains a larger fullerene. For example, the dual of the Errera graph is a fullerene (see here); hence by truncating Errera graph (i.e. the dual of its dual) one obtains a fullerene.

```
> G:=truncate_graph(graph("errera"))
```

an undirected unweighted graph with 90 vertices and 135 edges

```
> purge(F):; is_planar(G,F)
```

"Done", true

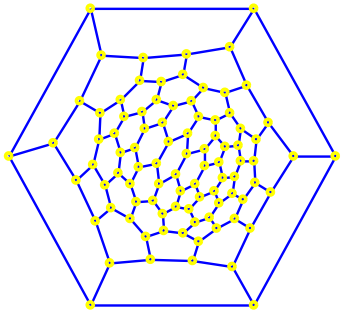
Now F contains a list of faces of the graph G. Since G is a fullerene, every face is a 5- or 6-cycle.

```
> set[op(apply(length,F))]
```

$\{5, 6\}$

When drawing fullerenes, it is recommended to use the circular method which usually produces best results. Any face of the planar embedding of a given fullerene be chosen as the outer face, as in the example below.

```
> draw_graph(G,circle=rand(F))
```



As an another example, the C_{180} fullerene is obtained by performing two leapfrog operations on the dodecahedral graph [65].

```
> G:=truncate_graph(plane_dual(graph("dodecahedron")))
```

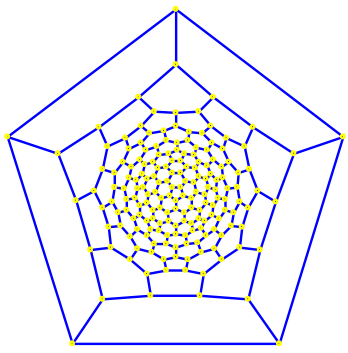
an undirected unweighted graph with 60 vertices and 90 edges

```
> C180:=truncate_graph(plane_dual(G))
```

an undirected unweighted graph with 180 vertices and 270 edges

In order to obtain a symmetric drawing of C_{180}, a 5-edge face is used as the outer face.

```
> purge(F):; is_planar(C180,F):;
  for f in F do if length(f)==5 then break; fi; od:;
  draw_graph(C180,circle=f)
```



1.10 Random graphs

1.10.1 Random general graphs

```
random_graph(n|L,p|m)
random_digraph(n|L,p|m)
random_graph(n|L,[p0,p1,...])
random_graph(n|L,f)
random_graph(n|L,d,k)
```

The commands `random_graph` and `random_digraph` are used for generating general (di)graphs at random according to various models, including the preferential attachment. Both commands take two arguments: a positive integer n or a list of labels L of length n. The second argument is a positive real number $p<1$ or a positive integer m. The return value is a (di)graph on n vertices (with elements of L as vertex labels) selected uniformly at random, i.e. a (di)graph in which each edge/arc is present with probability p or which contains exactly m edges/arcs chosen uniformly at random (Erdős–Rényi model).

Erdős–Rényi model is implemented according to Batagelj and Brandes [6, algorithms 1 and 2]. The corresponding algorithms run in linear time and are suitable for generating large graphs.

`random_graph` can also generate graphs with respect to a given probability distribution of vertex degrees if the second argument is a discrete probability density function given as a list of probabilities or weights $[p_0, p_1, ..., p_{n-1}]$ or as a weight function $f : \mathbb{N} \cup \{0\} \to [0, +\infty\rangle$ such that $f(i) = p_i$ for $i = 0, 1, ..., n-1$. Trailing zeros in the list of weights, if present, may be omitted. The numbers p_i are automatically scaled by $1/\sum_{i=1}^{n-1} p_i$ to achieve the sum of 1 and a graph with that precise distribution of vertex degrees is generated at random using the algorithm described in [58, p. 2567] with some modifications. First, a degree sequence d is generated randomly by drawing samples from the given distribution and repeating the process until a graphic sequence is obtained. Then the algorithm for constructing a feasible solution from d [38] is applied. Finally, the edges of that graph are randomized by choosing suitable pairs of non-incident edges and "rewiring" them without changing the degree sequence. Two edges uv and wz can be rewired in at most two ways, becoming either uz and wv or uw and vz (if these edges are not in the graph already). Letting m denote the number of edges, at most

$$N = \left\lceil \left(\log_2 \frac{m}{m-1} \right)^{-1} \right\rceil < m$$

such choices is made, assuring that the probability of rewiring each edge at least once is larger than $1/2$. The total complexity of this algorithm is $O(n^2 \log n)$.

Additionally, to support generation of realistic networks, `random_graph` can be used with integer parameters $d > 0$ and $k \geqslant 0$ as the second and the third argument, respectively, in which case a preferential attachment rule is applied in the following way. For $n \geqslant 2$, the resulting graph $G(V, E)$ initially contains two vertices v_1, v_2 and one edge $v_1 v_2$. For each $i = 3, ..., n$, the vertex v_i is added to V along with edges $v_i v_j$ for $\min\{i-1, d\}$ mutually different values of j, which are chosen at random in the set $\{1, 2, ..., i-1\}$ with probability

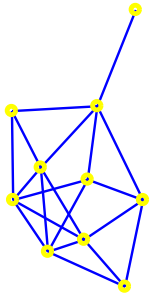
$$p_j = \frac{\deg v_j}{\sum_{r=1}^{i-1} \deg v_r}.$$

Subsequently, additional at most k random edges connecting the neighbors of v_i to each other are added to E, allowing the user to control the clustering coefficient of G. This method is due to Schank and Wagner [68, Algorithm 2, p. 271]. The time complexity of the implementation is $O(n^2 d + n k)$.

```
> G:=random_graph(10,0.5)
```

an undirected unweighted graph with 10 vertices and 20 edges

```
> draw_graph(G,spring,labels=false)
```



```
> G:=random_graph(1000,0.05)
```

an undirected unweighted graph with 1000 vertices and 24845 edges

```
> is_connected(G)
```

true

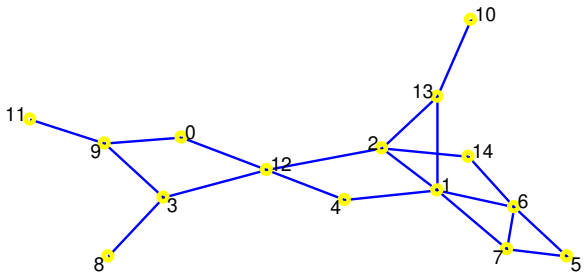
```
> minimum_degree(G),maximum_degree(G)
```

$28, 71$

```
> G:=random_graph(15,20)
```

an undirected unweighted graph with 15 vertices and 20 edges

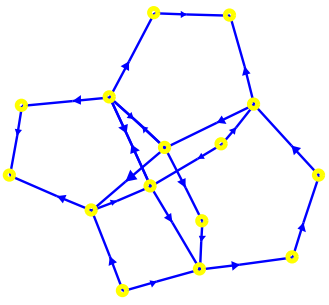
```
> draw_graph(G,spring)
```




```
> DG:=random_digraph(15,0.1)
```

a directed unweighted graph with 15 vertices and 23 arcs

```
> draw_graph(DG,labels=false,spring)
```



In the following example, a random graph is generated such that the degree of each vertex is drawn from $\{0, 1, ..., 10\}$ according to weights specified in the table below.

degree	0	1	2	3	4	5	6	7	8	9	10
weight	0	0	9	7	0	5	4	3	0	1	1

That is, the degrees are generated with probabilities $0, 0, \frac{3}{10}, \frac{7}{30}, 0, \frac{1}{6}, \frac{2}{15}, \frac{1}{10}, 0, \frac{1}{30}, \frac{1}{30}$, respectively.

```
> G:=random_graph(10000,[0,0,9,7,0,5,4,3,0,1,1])
```

an undirected unweighted graph with 10000 vertices and 21469 edges

5.413 sec

```
> frequencies(degree_sequence(G))
```

$$\begin{pmatrix} 2 & 0.3028 \\ 3 & 0.2287 \\ 5 & 0.1619 \\ 6 & 0.1324 \\ 7 & 0.1037 \\ 9 & 0.0327 \\ 10 & 0.0378 \end{pmatrix}$$

In the example below, a random graph is generated such that the vertex degrees are distributed according to the following weight function:

$$f(k) = \begin{cases} 0, & k = 0, \\ k^{-3/2}\,\mathrm{e}^{-k/3}, & k \geqslant 1. \end{cases}$$

```
> G:=random_graph(10000,k->when(k<1,0,k^-1.5*exp(-k/3)))
```

an undirected unweighted graph with 10000 vertices and 8013 edges

```
> length(connected_components(G))
```

2278

The command line below computes the average size of a connected component in G.

```
> round(mean(apply(length,connected_components(G))))
```

4

The next example demonstrates how to generate random graphs with adjustable clustering coefficient.

```
> G1:=random_graph(10000,5,10)
```

an undirected unweighted graph with 10000 vertices and 105349 edges

```
> clustering_coefficient(G1)
```

0.468517214584

```
> G2:=random_graph(10000,5,20)
```

an undirected unweighted graph with 10000 vertices and 121729 edges

```
> clustering_coefficient(G2)
```

0.615949324773

```
> G3:=random_graph(10000,10,5)
```

an undirected unweighted graph with 10000 vertices and 143378 edges

```
> clustering_coefficient(G3)
```

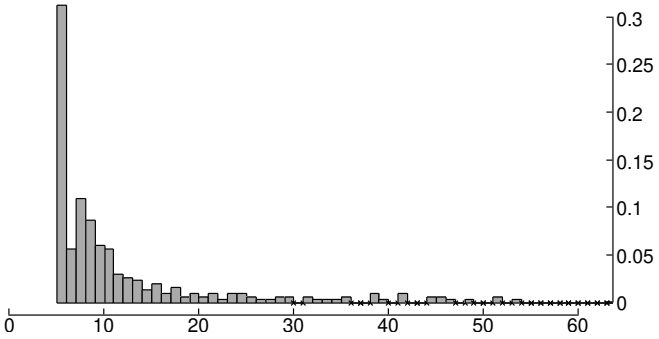
0.114122321952

The distribution of vertex degrees in a graph generated with preferential attachment rule roughly obeys the power law in its tail, as shown in the example below.

```
> G:=random_graph(300,5,2)
```

an undirected unweighted graph with 300 vertices and 1889 edges

```
> histogram(degree_sequence(G))
```



1.10.2 Random bipartite graphs

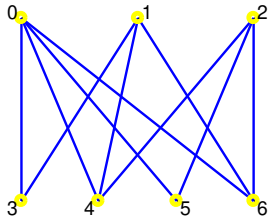
`random_bipartite_graph(n,p|m)`
`random_bipartite_graph([a,b],p|m)`

The command `random_bipartite_graph` is used for generating bipartite graphs at random. It takes two arguments. The first argument is either a positive integer n or a list of two positive integers a and b. The second argument is either a positive integer m or a positive real number $p<1$. The command returns a random bipartite graph on n vertices (or with two partitions of sizes a and b) in which each possible edge is present with probability p (or m edges are inserted at random).

```
> G:=random_bipartite_graph([3,4],0.5)
```

an undirected unweighted graph with 7 vertices and 10 edges

```
> draw_graph(G)
```




```
> G:=random_bipartite_graph(30,60)
```

an undirected unweighted graph with 30 vertices and 60 edges

1.10.3 Random trees

`random_tree(n|V)`
`random_tree(n|V,d)`
`random_tree(n|V,root⟨=v⟩)`

The command `random_tree` is used for generating tree graphs at random. It takes one or two arguments: a positive integer n or a list $V=\{v_1,v_2,...,v_n\}$ and optionally an integer $d\geqslant 2$ or the option `root⟨=v⟩`, where $v\in V$. It returns a random tree $T(V,E)$ on n vertices such that:

- if the second argument is omitted, then T is uniformly selected among all unrooted unlabeled trees on n vertices,

- if d is given as the second argument, then $\Delta(T) \leqslant d$, where $\Delta(T)$ is the maximum vertex degree in T,
- if `root[=v]` is given as the second argument, then T is uniformly selected among all rooted unlabeled trees on n vertices. If v is specified then the vertex labels in V (required) will be assigned to vertices in T such that v is the first vertex in the list returned by the command `vertices`.

Rooted unlabeled trees are generated uniformly at random using the RANRUT algorithm [59, p. 274]. The root of a tree T generated this way, if not specified as v, is always the first vertex in the list returned by `vertices`. The average time complexity of RANRUT is $O(n \log n)$ [4].

Unrooted unlabeled trees, also called **free** trees, are generated uniformly at random using Wilf's algorithm[1.2] [85], which is based on RANRUT.

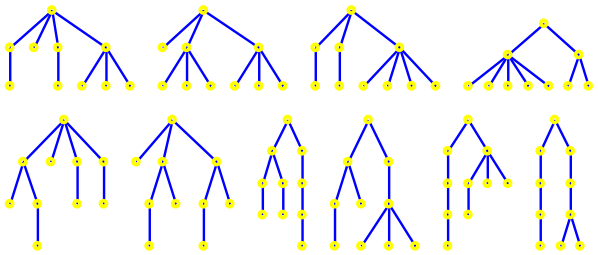
Trees with bounded maximum degree are generated using a simple algorithm which starts with an empty tree and adds edges at random one at a time. It is much faster than RANRUT but selects trees in a non-uniform manner. To force the use of this algorithm even without vertex degree limit (for example, when n is very large), one can set $d = +\infty$.

The command line below creates a forest containing 10 randomly selected free trees on 10 vertices.

```
> G:=disjoint_union(apply(random_tree,[10$10]))
```

an undirected unweighted graph with 100 vertices and 90 edges

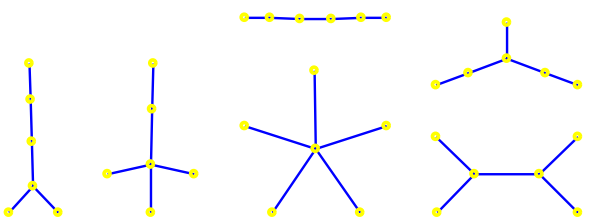
```
> draw_graph(G,tree,labels=false)
```



The following example demonstrates the uniformity of random generation of free trees. Letting $n = 6$, there are exactly 6 distinct free trees on 6 vertices, created by the next command line.

```
> trees:=[star_graph(5),path_graph(6),graph(trail(1,2,3,4),trail(5,4,6)),
  graph(%{[1,2],[2,3],[2,4],[4,5],[4,6]%}),graph(trail(1,2,3,4),trail(3,5,6)),
  graph(trail(1,2,3,4),trail(5,3,6))]:;
```

```
> draw_graph(disjoint_union(trees),spring,labels=false)
```



1.2. The original publication of Wilf's algorithm has a minor flaw in the procedure `Free` [85, p. 207]. In the formula $p = \binom{1+a_{n/2}}{2} / a_n$ in step (T1) the denominator a_n stands for the number of all rooted unlabeled trees on n vertices. However, one should divide by the number t_n of all *unrooted* unlabeled trees instead, which can be obtained from $a_1, a_2, \ldots, a_n$ by applying the formula in [61, p. 589]. This implementation includes the correction.

Now, generating a random free tree on 6 nodes always produces one of the above six graphs, which is determined by using the command `is_isomorphic`. 1200 trees are generated in total and the number of occurrences of `trees[k]` is stored in `hits[k]` for every $k = 1, 2, ..., 6$ (note that in xcas mode it is actually $k = 0, ..., 5$).

```
> hits:=[0$6]:;
```

```
> for k from 1 to 1200 do
      T:=random_tree(6);
      for j from 0 to 5 do
          if is_isomorphic(T,trees[j]) then hits[j]++; fi;
      od;
  od:;
```

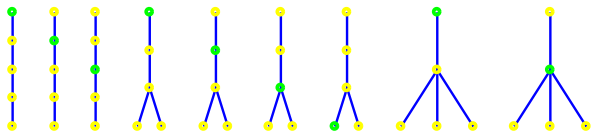
```
> hits
```

$$[186, 208, 221, 186, 189, 210]$$

To demonstrate the ability of this algorithm to select rooted trees on n vertices with equal probability, one can reproduce the example in [59, p. 281], in which $n = 5$. First, all distinct rooted trees on 5 vertices are created and stored in `trees`; there are exactly nine of them. Their root vertices are highlighted to be distinguishable. Then, 4500 rooted trees on 5 vertices are generated at random, highlighting the root vertex in each of them. As in the previous example, the variable `hits[k]` records how many of them are isomorphic to `trees[k]`.

```
> trees:=[
  highlight_vertex(graph(trail(1,2,3,4,5)),1),
  highlight_vertex(graph(trail(1,2,3,4,5)),2),
  highlight_vertex(graph(trail(1,2,3,4,5)),3),
  highlight_vertex(graph(trail(1,2,3),trail(4,3,5)),1),
  highlight_vertex(graph(trail(1,2,3),trail(4,3,5)),2),
  highlight_vertex(graph(trail(1,2,3),trail(4,3,5)),3),
  highlight_vertex(graph(trail(1,2,3),trail(4,3,5)),4),
  highlight_vertex(graph(trail(1,2,3),trail(4,2,5)),1),
  highlight_vertex(graph(trail(1,2,3),trail(4,2,5)),2)
  ]:;
```

```
> draw_graph(disjoint_union(trees),labels=false)
```



```
> hits:=[0$9]:;
```

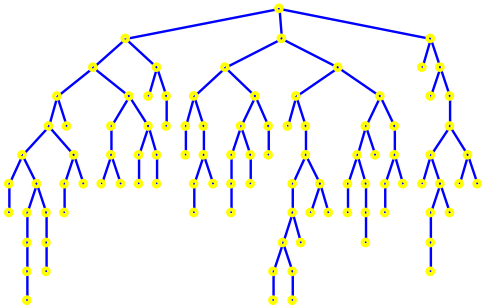
```
> for k from 1 to 4500 do
      T:=random_tree(5,root);
      HT:=highlight_vertex(T,vertices(T)[0]);
      for j from 0 to 8 do
          if is_isomorphic(HT,trees[j]) then hits[j]++; fi;
      od;
  od:;
```

```
> hits
```

$$[505, 561, 457, 496, 487, 517, 489, 500, 488]$$

In the following example, a random tree on 100 vertices with maximum degree at most 3 is drawn.

```
> draw_graph(random_tree(100,3))
```



1.10.4 Random planar graphs

`random_planar_graph(n|L,p,⟨k⟩)`

The command `random_planar_graph` is used for generating random planar graphs. It takes two or three arguments, a positive integer n or a list L of length n, a positive real number $p<1$, and optionally an integer $k\in\{0,1,2,3\}$ (by default, $k=1$). The command returns a random k-connected planar graph on n vertices (using the elements of L as vertex labels).

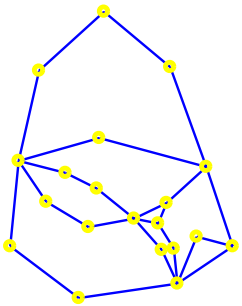
The result is obtained by first generating a random maximal planar graph and then attempting to remove each edge with probability p, maintaining the k-connectivity of the graph (if $k=0$, the graph may be disconnected). The running time is $O(n^{\lceil 1+k/2\rceil})$.

Generating a biconnected planar graph:

```
> G:=random_planar_graph(20,0.8,2)
```

an undirected unweighted graph with 20 vertices and 27 edges

```
> draw_graph(G,planar,labels=false)
```

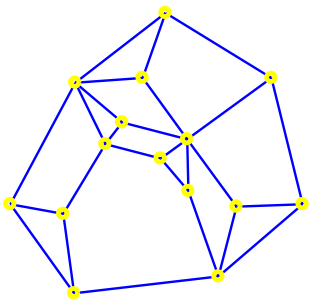


Generating a triconnected planar graph:

```
> G:=random_planar_graph(15,0.9,3)
```

an undirected unweighted graph with 15 vertices and 26 edges

```
> draw_graph(G,planar,labels=false)
```



Generating a disconnected planar graph with high probability:

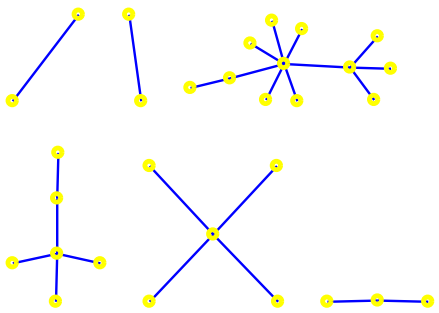
```
> G:=random_planar_graph(30,0.9,0)
```

an undirected unweighted graph with 30 vertices and 24 edges

```
> is_forest(G)
```

true

```
> draw_graph(G,spring,labels=false)
```



By default, a connected planar graph is generated, like in the following example.

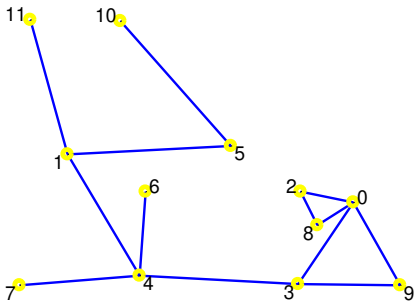
```
> G:=random_planar_graph(12,0.618)
```

an undirected unweighted graph with 12 vertices and 13 edges

```
> is_connected(G)
```

true

```
> draw_graph(G,planar)
```



1.10.5 Random graphs from a given degree sequence

`random_sequence_graph(L)`

The command `random_sequence_graph` is used for generating random undirected graphs from degree sequences. It takes a degree sequence L (a list of nonnegative integers) as its only argument and returns an asymptotically uniform random graph with the degree sequence equal to L using the algorithm developed by BAYATI et al. [8].

The algorithm slows down quickly and uses $O(|L|^2)$ of auxiliary space, so it is best used for up to several hundreds of vertices.

```
> s:=[1,3,3,2,1,2,2,2,3,3]
```

$[1, 3, 3, 2, 1, 2, 2, 2, 3, 3]$

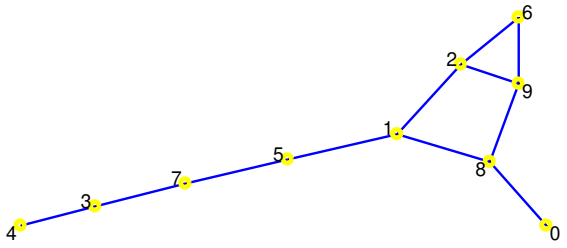
```
> is_graphic_sequence(s)
```

true

```
> G:=random_sequence_graph(s)
```

an undirected unweighted graph with 10 vertices and 11 edges

```
> draw_graph(G,spring)
```



1.10.6 Random regular graphs

`random_regular_graph(n|L,d,⟨connected⟩)`

The command `random_regular_graph` is used for generating random regular graphs. It takes two mandatory arguments, a positive integer n (or a list L of length n) and a nonnegative integer d. Optionally, the argument `connected` may be appended, forcing the generated graph to be connected. The command creates n vertices (using elements of L as vertex labels) and returns a random d-regular (connected) graph on these vertices.

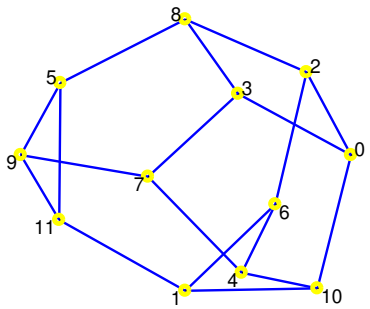
Note that a d-regular graph on n vertices exists if and only if $n > d+1$ and $n\,d$ is even. If these conditions are not met, `random_regular_graph` returns an error.

The strategy is to use the algorithm developed by Steger and Wormald [70, algorithm 2]. The runtime is negligible for $n \leqslant 100$. However, for $n > 200$ the algorithm is considerably slower. Graphs are generated with approximately uniform probability, which means that for $n \to \infty$ and d not growing so quickly with n the probability distribution converges to uniformity.

```
> G:=random_regular_graph(12,3)
```

an undirected unweighted graph with 12 vertices and 18 edges

```
> draw_graph(G,spring)
```



1.10.7 Random tournaments

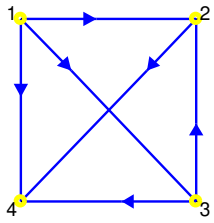
`random_tournament(n|V)`

The command `random_tournament` is used for generating random tournaments. It takes a positive integer n or a list V of length n as its only argument and returns a random tournament on n vertices. If V is provided, then its elements are used to label the vertices.

```
> G:=random_tournament([1,2,3,4])
```

a directed unweighted graph with 4 vertices and 6 arcs

```
> draw_graph(G)
```



1.10.8 Random network graphs

`random_network(a,b,⟨p⟩,⟨opts⟩)`

The command `random_network` is used for generating random networks. It takes two to four arguments: a positive integer a, a positive integer b, an optional real number p such that $0 < p \leqslant 1$ (by default $p = 0.5$) and optionally a sequence of options `opts`. The supported options are `acyclic⟨=true|false⟩` and `weights=a..b`.

The command returns a network graph with $a^2\,b$ vertices which is composed as follows (the method of generating the network skeleton is due to GOLDFARB and GRIGORIADIS [35]).

Firstly, grid graphs $F_1, F_2, ..., F_b$ (called **frames**), each of them with $a \times a$ vertices, are generated. If the option `acyclic⟨=true⟩` is used (by default is `acyclic=false`), then an acyclic orientation is computed for each frame using st-ordering (see Section 4.9.3) with two opposite corners of the frame as source and sink, otherwise all vertices in the frame are connected to their neighbors (forth and back). In addition, for each $k < b$ the vertices of F_k are connected one to one with the vertices of the next frame F_{k+1} using a random permutation of those vertices. The first vertex of the first frame is the source and the last vertex of the last frame is the sink of the network (some arcs may have to be removed to achieve that). Finally, the removal of each arc is attempted with probability $1 - p$ (unless its removal disconnects the network), making each arc present with probability p.

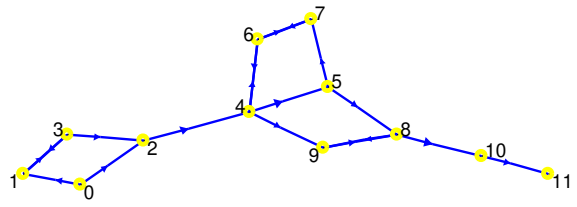
if the option `weights=a..b` is specified, arc weights in the network are randomized in the interval $[a, b] \subset \mathbb{R}$. If a, b are integers, the weights are also integers.

The command below creates a random network, consisting of 3 frames of size 2×2, in which each arc is present with the probability 0.8.

```
> N1:=random_network(2,3,0.8)
```

a directed unweighted graph with 12 vertices and 18 arcs

```
> draw_graph(N1,spring)
```




```
> is_network(N1)
```

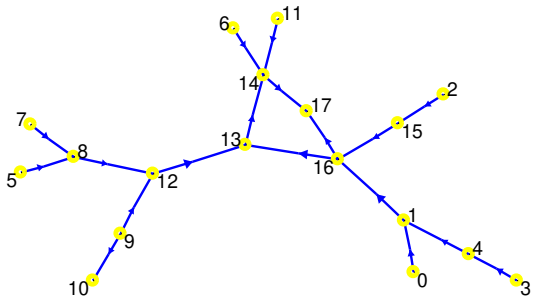
[0], [11]

In the next example, passing the option `acyclic` forces the output graph to be acyclic.

```
> N2:=random_network(3,2,0.62,acyclic)
```

a directed unweighted graph with 18 vertices and 18 arcs

```
> draw_graph(N2,spring)
```




```
> is_network(N2)
```

$[0, 2, 3, 5, 6, 7, 9, 11], [10, 17]$

```
> is_acyclic(N2)
```

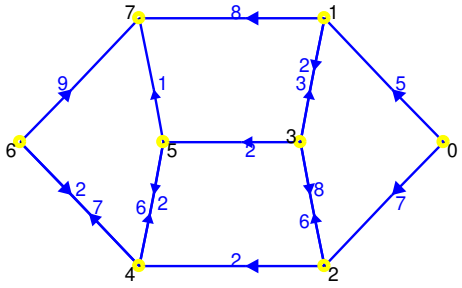
true

Arc weights can be randomized, as demonstrated in the following example.

```
> N3:=random_network(2,2,0.75,weights=1..9)
```

a directed weighted graph with 8 vertices and 15 arcs

```
> draw_graph(N3,spring)
```



1.10.9 Randomizing edge weights

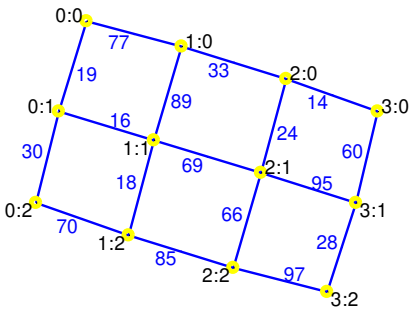
`assign_edge_weights(G,a..b)`
`assign_edge_weights(G,m,n)`

The command `assign_edge_weights` is used for assigning weights to edges of graphs at random. It takes two or three arguments: a graph $G(V, E)$ and an interval a .. b of real numbers or a sequence of two positive integers m and n. The command operates such that for, each edge $e \in E$, the weight of e is chosen uniformly from the real interval $[a, b)$ or from the set of integers lying between m and n, including both m and n. After assigning weights to all edges, a modified copy of G is returned.

```
> G:=assign_edge_weights(grid_graph(4,3),1,99)
```

an undirected weighted graph with 12 vertices and 17 edges

```
> draw_graph(G,spring)
```


0:0
1:0
2:0
3:0
0:1
1:1
2:1
3:1
0:2
1:2
2:2
3:2
77
19
33
89
14
16
24
30
60
69
18
95
66
70
28
85
97

Chapter 2
Modifying graphs

2.1 Promoting to directed and weighted graphs

2.1.1 Converting edges to arcs

`make_directed(G,⟨W⟩)`

The command `make_directed` is used for promoting undirected graphs to directed ones. It takes one or two arguments, an undirected graph $G(V,E)$ and optionally a numerical square matrix $W=[a_{ij}]$ of order $|V|$. Every edge $v_iv_j \in E$ is replaced with the pair of arcs v_iv_j and v_jv_i. If matrix W is specified, its elements w_{ij} and w_{ji} are assigned as weights of these arcs, respectively. Thus a directed (weighted) copy of G is constructed and subsequently returned.

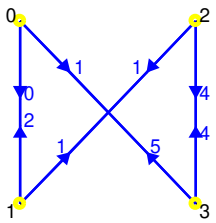
```
> G:=make_directed(cycle_graph(4))
```

a directed unweighted graph with 4 vertices and 8 arcs

```
> WG:=make_directed(cycle_graph(4),[[0,0,0,1],[2,0,1,3],[0,1,0,4],[5,0,4,0]])
```

a directed weighted graph with 4 vertices and 8 arcs

```
> draw_graph(WG)
```



2.1.2 Assigning weight matrix to unweighted graphs

`make_weighted(G,⟨W⟩)`

The command `make_weighted` is used for promoting unweighted graphs to weighted ones. It takes one or two arguments, an unweighted graph $G(V,E)$ and optionally a square matrix $W=[w_{ij}]$ of order $|V|$. If the matrix specification is omitted, a square matrix of ones is assumed. Then a copy of G is returned in which each edge/arc $v_iv_j \in E$ gets the element w_{ij} in W assigned as its weight. If G is undirected, then it is assumed that W is a symmetric matrix.

```
> G:=graph([1,2,3],%{[1,2],[2,3],[3,1]%})
```

an undirected unweighted graph with 3 vertices and 3 edges

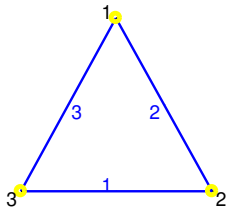
```
> W:=[[0,2,3],[2,0,1],[3,1,0]]
```

$$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

```
> H:=make_weighted(G,W)
```

an undirected weighted graph with 3 vertices and 3 edges

```
> draw_graph(H)
```



2.2 Modifying vertices of a graph

2.2.1 Adding and removing vertices

`add_vertex(G,v|L)`
`delete_vertex(G,v|L)`

The commands `add_vertex` and `delete_vertex` are used for adding new vertices to graphs and for removing existing vertices from graphs, respectively.

`add_vertex` takes two arguments, a graph $G(V, E)$ and a single label v or a list of labels L, and returns the graph $G'(V \cup \{v\}, E)$ or $G''(V \cup L, E)$ if a list L is given.

`delete_vertex` takes two arguments, a graph $G(V, E)$ and a single label v or a list of labels L, and returns the graph

$$G'(V \setminus \{v\}, \{e \in E : e \text{ is not incident to } v\})$$

or, if L is given,

$$G''(V \setminus L, \{e \in E : e \text{ is not incident to any } v \in L\}).$$

If any of the specified vertices does not belong to G, an error is returned.

```
> K5:=complete_graph([1,2,3,4,5])
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> add_vertex(K5,6)
```

an undirected unweighted graph with 6 vertices and 10 edges

```
> add_vertex(K5,[a,b,c])
```

an undirected unweighted graph with 8 vertices and 10 edges

Note that vertices already present in G will not be added. For example:

```
> add_vertex(K5,[4,5,6])
```

an undirected unweighted graph with 6 vertices and 10 edges

```
> delete_vertex(K5,2)
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> delete_vertex(K5,[2,3])
```

an undirected unweighted graph with 3 vertices and 3 edges

2.2.2 Contracting subgraphs

`contract_subgraph(G,S,⟨lab⟩)`

The command `contract_subgraph` is used for contracting subgraphs into single vertices. It takes two mandatory arguments, a graph $G(V, E)$ and a set (or list) $S \subset V$. It returns a copy of G with all the vertices in S merged into a single vertex. The neighborhood of that vertex is the union of the neighborhoods of all of merged vertices. The argument `lab`, or the list of labels of merged vertices if `lab` is omitted, becomes the label of the new vertex.

Edge attributes and directions from G are kept in the resulting graph, as well as the attributes for vertices in $V \setminus S$.

```
> C6:=cycle_graph(6)
```

an undirected unweighted graph with 6 vertices and 6 edges

```
> H:=contract_subgraph(C6,[0,1,5],a)
```

an undirected unweighted graph with 4 vertices and 4 edges

```
> vertices(H)
```

$$[a, 2, 3, 4]$$

```
> edges(H)
```

$$[[2, a], [4, a], [2, 3], [3, 4]]$$

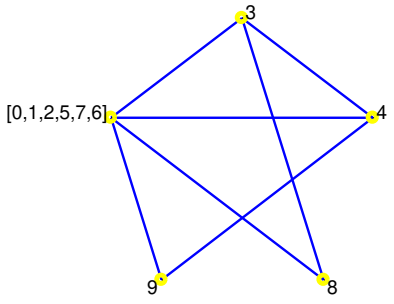
```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> H:=contract_subgraph(P,[0,1,2,5,7,6])
```

an undirected unweighted graph with 5 vertices and 7 edges

```
> draw_graph(H)
```



2.3 Modifying edges of a graph

2.3.1 Adding and removing edges

`add_edge(G,e|E|T)`
`add_arc(G,e|E|T)`
`delete_edge(G,e|E|T)`
`delete_arc(G,e|E|T)`

The commands `add_edge` or `add_arc` and `delete_edge` or `delete_arc` are used for adding new edges to graphs and for removing existing edges from graphs, respectively.

`add_edge` takes two arguments, an undirected graph G and an edge e or a list of edges E or a trail of edges T (entered as a list of vertices), and returns the copy of G with the specified edges inserted. Edge insertion implies that its endpoints will be created if they are not already present in G. The command `add_arc` works similarly to `add_edge` but applies only to directed graphs. Note that the order of endpoints in an arc is relevant. When adding edge to a weighted graph, its weight should be specified alongside its endpoints, or it will be assumed that it equals to 1.

`delete_edge` and `delete_arc` both take two arguments, the input graph G and an edge e or a list of edges E or a trail of edges T. It returns a copy of G in which the specified edges are removed. Note that this operation does not change the set of vertices of G.

Examples.

```
> C4:=cycle_graph(4)
```

an undirected unweighted graph with 4 vertices and 4 edges

```
> add_edge(C4,[1,3])
```

an undirected unweighted graph with 4 vertices and 5 edges

```
> add_edge(C4,[1,3,5,7])
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> add_arc(digraph(trail(a,b,c,d,a)),[[a,c],[b,d]])
```

a directed unweighted graph with 4 vertices and 6 arcs

```
> add_edge(graph(%{[[1,2],5],[[3,4],6]%}),[[2,3],7])
```

an undirected weighted graph with 4 vertices and 3 edges

```
> K33:=relabel_vertices(complete_graph(3,3),[A,B,C,D,E,F])
```

an undirected unweighted graph with 6 vertices and 9 edges

```
> has_edge(K33,[A,D])
```

true

```
> delete_edge(K33,[A,D])
```

an undirected unweighted graph with 6 vertices and 8 edges

Note that the original input graph is not changed.

```
> has_edge(K33,[B,D])
```

true

```
> delete_edge(K33,[[A,D],[B,D]])
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> DG:=digraph(trail(1,2,3,4,5,2,4))
```

a directed unweighted graph with 5 vertices and 6 arcs

```
> delete_arc(DG,[[2,3],[4,5],[5,2]])
```

a directed unweighted graph with 5 vertices and 3 arcs

```
> delete_arc(DG,[3,4,5,2])
```

a directed unweighted graph with 5 vertices and 3 arcs

2.3.2 Accessing and modifying edge weights

`set_edge_weight(G,e,⟨w⟩)`

The commands `get_edge_weight` and `set_edge_weight` are used to access and modify the weight of an edge in a weighted graph, respectively.

`set_edge_weight` takes three arguments: a weighted graph $G(V, E)$, edge $e \in E$ and the new weight w, which may be any number. It returns the modified copy of G.

`get_edge_weight` takes two arguments, a weighted graph $G(V, E)$ and an edge or arc $e \in E$. It returns the weight of e.

Examples.

```
> G:=set_edge_weight(graph(%{[[1,2],4],[[2,3],5]%}),[1,2],6)
```

an undirected weighted graph with 3 vertices and 2 edges

```
> get_edge_weight(G,[1,2])
```

6

2.3.3 Contracting edges

`contract_edge(G,e)`

The command `contract_edge` is used for contracting edges in undirected graphs. It takes two arguments, an undirected graph $G(V, E)$ and an edge $e = vw \in E$, and merges v and w to a single vertex, deleting the edge e. The resulting vertex inherits the label of v. The modified copy of G is returned.

Examples.

```
> K5:=complete_graph(5)
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> contract_edge(K5,[1,2])
```

an undirected unweighted graph with 4 vertices and 6 edges

To contract a set $\{e_1, e_2, ..., e_k\} \subset E$ of edges in G, none two of which are incident (i.e. when the given set is a matching in G), one can use the `foldl` command. In the following example, the complete graph K_5 is obtained from Petersen graph by edge contraction.

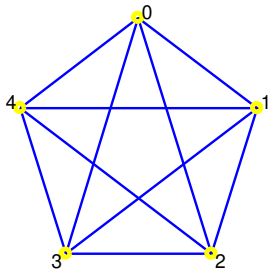
```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> G:=foldl(contract_edge,P,[0,5],[1,6],[2,7],[3,8],[4,9])
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> draw_graph(G)
```



2.3.4 Subdividing edges

`subdivide_edges(G,e|S,r)`

The command `subdivide_edges` is used for graph subdivision. It takes two or three arguments: a graph $G(V, E)$, a single edge/arc $e \in E$ or a list of edges/arcs $S \subset E$ and optionally a positive integer r (which defaults to 1). Each of the specified edges/arcs will be subdivided with exactly r new vertices, labeled with the smallest available nonnegative integers. The resulting graph, which is homeomorphic to G, is returned.

If the endpoints of the edge being subdivided have valid coordinates, the coordinates of the inserted vertices will be computed accordingly.

Examples.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> G:=subdivide_edges(G,[2,3])
```

an undirected unweighted graph with 11 vertices and 16 edges

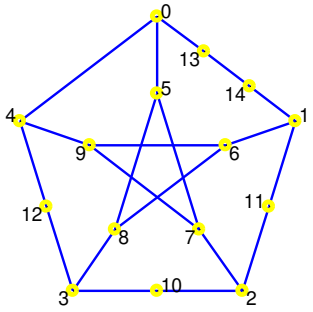
```
> G:=subdivide_edges(G,[[1,2],[3,4]])
```

an undirected unweighted graph with 13 vertices and 18 edges

```
> G:=subdivide_edges(G,[0,1],2)
```

an undirected unweighted graph with 15 vertices and 20 edges

```
> draw_graph(G)
```



2.4 Using attributes

2.4.1 Graph attributes

```
set_graph_attribute(G,tag1=value1,tag2=value2,...)
set_graph_attribute(G,[tag1=value1,tag2=value2,...])
set_graph_attribute(G,[tag1,tag2,...],[value1,value2,...])
get_graph_attribute(G,tag1,tag2,...)
get_graph_attribute(G,[tag1,tag2,...])
list_graph_attributes(G)
discard_graph_attribute(G,tag1,tag2,...)
discard_graph_attribute(G,[tag1,tag2,...])
```

The graph structure maintains a set of attributes as tag-value pairs which can be accessed and/or modified by using the commands `set_graph_attribute`, `get_graph_attribute`, `list_graph_attributes` and `discard_graph_attribute`.

The command `set_graph_attribute` is used for modifying the existing graph attributes or adding new ones. It takes two arguments, a graph G and a sequence or list of graph attributes in form `tag=value` where `tag` is a string. Alternatively, attributes may be specified as a sequence of two lists `[tag1,tag2,...]` and `[value1,value2,...]`. The command sets the specified values to the indicated attribute slots, which are meant to represent some global properties of the graph G, and returns the modified copy of G.

The graph attribute values can be fetched by using the command `get_graph_attribute` which takes two arguments: a graph G and a sequence or list of tags. The corresponding values will be returned in a sequence or list, respectively. If an attribute is not set, then `undef` is returned as its value.

To list all graph attributes of G for which the values are set, use the command `list_graph_attributes` which takes G as its only argument.

To discard a graph attribute, use the command `discard_graph_attribute`. It takes two arguments: a graph G and a sequence or list of tags to be cleared, and returns the modified copy of G.

Two tags being used by the CAS commands are directed and weighted, so it is not advisable to overwrite their values using this command; use the `make_directed`, `make_weighted` and `underlying_graph` commands instead. Another attribute used internally is name, which holds the name of the respective graph (as a string).

Examples.

```
> G:=digraph(trail(1,2,3,1))
```

a directed unweighted graph with 3 vertices and 3 arcs

```
> G:=set_graph_attribute(G,"name"="C3","message"="this is some text")
```

C3: a directed unweighted graph with 3 vertices and 3 arcs

```
> get_graph_attribute(G,"message")
```

"this is some text"

```
> list_graph_attributes(G)
```

["directed"=true,"weighted"=false,"name"="C3","message"="this is some text"]

```
> G:=discard_graph_attribute(G,"message")
```

C3: a directed unweighted graph with 3 vertices and 3 arcs

```
> list_graph_attributes(G)
```

["directed"=true,"weighted"=false,"name"="C3"]

2.4.2 Vertex attributes

```
set_vertex_attribute(G,v,tag1=value1,tag2=value2,...)
set_vertex_attribute(G,v,[tag1=value1,tag2=value2,...])
set_vertex_attribute(G,v,[tag1,tag2,...],[value1,value2,...])
get_vertex_attribute(G,v,tag1,tag2,...)
get_vertex_attribute(G,v,[tag1,tag2,...])
list_vertex_attributes(G,v)
discard_vertex_attribute(G,v,tag1,tag2,...)
discard_vertex_attribute(G,v,[tag1,tag2,...])
```

For every vertex of a graph, the list of attributes in form of tag-value pairs is maintained, which can be accessed/modified by using the commands `set_vertex_attribute`, `get_vertex_attribute`, `list_vertex_attributes` and `discard_vertex_attribute`.

The command `set_vertex_attribute` is used for modifying the existing vertex attributes or adding new ones. It takes three arguments, a graph $G(V,E)$, a vertex $v \in V$ and a sequence or list of attributes in form `tag=value` where `tag` is a string. Alternatively, attributes may be specified as a sequence of two lists `[tag1,tag2,...]` and `[value1,value2,...]`. The command sets the specified values to the indicated attributes of the vertex v and returns the modified copy of G.

The attribute values for v can be fetched by using the command `get_vertex_attribute` which takes three arguments: G, v and a sequence or list of tags. The corresponding values will be returned in a sequence or list, respectively. If an attribute is not set, then `undef` is returned as its value.

To list all attributes of v for which the values are set, use the command `list_vertex_attributes` which takes two arguments, G and v.

The command `discard_vertex_attribute` is used for discarding attribute(s) assigned to some vertex $v \in V$. It takes three arguments: G, v and a sequence or list of tags to be cleared, and returns the modified copy of G.

The attributes label, color, shape, and pos are also used internally. These hold the vertex label, color, shape, and coordinates in a drawing, respectively. If the color is not set for a vertex, the latter is drawn in yellow. The shape attribute may have one of the following values: square, triangle, diamond, star or plus. If the shape attribute is not set or has a different value, then the vertices are drawn as circles.

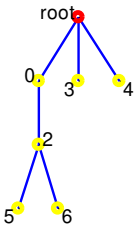
```
> T:=complete_binary_tree(2)
```

an undirected unweighted graph with 7 vertices and 6 edges

```
> T:=set_vertex_attribute(T,1,"label"="root","color"=red)
```

an undirected unweighted graph with 7 vertices and 6 edges

```
> draw_graph(T,tree="root")
```



A vertex may also hold custom attributes.

```
> T:=set_vertex_attribute(T,"root","depth"=3,"shape"="square")
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> list_vertex_attributes(T,"root")
```

["label"="root", "color"=red, "shape"="square", "depth"=3]

```
> T:=discard_vertex_attribute(T,"root","color")
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> list_vertex_attributes(T,"root")
```

["label"="root", "shape"="square", "depth"=3]

2.4.3 Edge attributes

```
set_edge_attribute(G,e,tag1=value1,tag2=value2,...)
set_edge_attribute(G,e,[tag1=value1,tag2=value2,...])
set_edge_attribute(G,e,[tag1,tag2,...],[value1,value2,...])
get_edge_attribute(G,e,tag1,tag2,...)
get_edge_attribute(G,e,[tag1,tag2,...])
list_edge_attributes(G,e)
discard_edge_attribute(G,e,tag1,tag2,...)
discard_edge_attribute(G,e,[tag1,tag2,...])
```

For every edge of a graph, the list of attributes in form of key-value pairs is maintained, which can be accessed and/or modified by using the commands `set_edge_attribute`, `get_edge_attribute`, `list_edge_attributes` and `discard_edge_attribute`.

The command `set_edge_attribute` is used for modifying the existing edge attributes or adding new ones. It takes three arguments, a graph $G(V, E)$, an edge/arc $e \in E$ and a sequence or list of attributes in form `tag=value` where `tag` is a string. Alternatively, attributes may be specified as a sequence of two lists `[tag1,tag2,...]` and `[value1,value2,...]`. The command sets the specified values to the indicated attributes of the edge/arc e and returns the modified copy of G.

The attribute values for e can be fetched by using the command `get_edge_attribute` which takes three arguments: G, e and a sequence or list of tags. The corresponding values will be returned in a sequence or list, respectively. If some attribute is not set, then `undef` is returned as its value.

To list all attributes of e for which the values are set, use the command `list_edge_attributes` which takes two arguments, G and e.

To discard attribute(s) assigned to e call the command `discard_edge_attribute`, which takes three arguments: G, e and a sequence or list of tags to be cleared, and returns the modified copy of G.

The attributes weight, color, style, width, pos, and temp are also used internally. They hold the edge weight, color, line style, line width, the coordinates of the weight label anchor (and also the coordinates of the arrow), and `true` if the edge is temporary, respectively. If the color attribute is not set for an edge, the latter is drawn in blue, unless it is a temporary edge, in which case it is drawn in light gray. The style attribute may have one of the following values: dashed, dotted or bold. If the style attribute is not set or has a different value, then the solid line style is applied when drawing the edge. The width attribute may hold an integer between 1 and 8 or one of the predefined widths: `thin`, `normal`, `thicker`, `thick`, and `very thick`.

```
> T:=cycle_graph(5)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> T:=set_edge_attribute(T,[1,2],"cost"=12.8,"message"="this is some text")
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> list_edge_attributes(T,[1,2])
```

["cost"=12.8, "message"="this is some text"]

```
> T:=discard_edge_attribute(T,[1,2],"message")
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> T:=set_edge_attribute(T,[1,2],"style"="dashed","color"=magenta,"width"=thick)
```

an undirected unweighted graph with 5 vertices and 5 edges

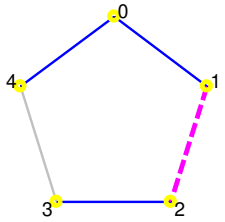
```
> list_edge_attributes(T,[1,2])
```

["color"=magenta, "style"="dashed", "width"=thick, "cost"=12.8]

```
> T:=set_edge_attribute(T,[3,4],"temp"=true)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> draw_graph(T)
```



Chapter 3
Import and export

3.1 Importing graphs

3.1.1 Loading graphs from DOT and LST files

`import_graph(filename,⟨opts⟩)`

The command `import_graph` is used for importing a graph from text file in dot format. It takes a string `filename` as its only argument and returns the graph constructed from instructions written in the file `filename`, or an error. The passed string should contain the path to a file in DOT or LST format. The format is recognized from the extension: the DOT format is associated with extensions `.dot` and `.gv`[3.1], while the LST format is associated with `.lst`.

If a relative path to the file is specified, i.e. if it does not contain a leading forward slash in Linux, the current working directory (which can be obtained by calling the `pwd` command) will be used as the reference. The working directory can be changed by using the command `cd`.

In MS Windows, paths must be entered using either the forward slash (e.g. in `C:/Users/You/path/to/file.dot`) or double backslash (e.g. in `C:\\Users\\You\\path\\to\\file.dot`) as the directory separator.

When importing graphs in DOT format, the following optional arguments may be given in a sequence `opts`.

- `style=true|false` — import or discard style-related attributes, including color, shape, style, and position (by default, `style=true`).
- `eval=labels|weights` or `eval=labels+weights` — parse labels and/or weights of the vertices and edges, entered as strings in the file, to the corresponding Giac expressions. By default, no parsing of edge weights and vertex labels is performed.

Optional arguments are ignored when importing a file in LST format.

For example, assume that the following files exist in the directory `path/to/dot/`.
`example.dot`:

```
graph "ExampleGraph" {
  a [label="Foo"];
  b [shape=square,color=red];
  a -- b [style=bold];
  b -- c [color=green];
  b -- d [style=dashed];
}
```

`octahedron.lst`:

```
1: 3 4 5 6
2: 3 4 5 6
```

3.1. Although it is recommended to use `.gv` as the extension for dot files to avoid a certain confusion between different file types, Giac uses the `.dot` extension because it coincides with the format name. This may change in the future.

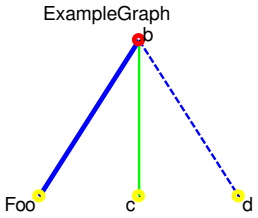
```
3: 5 6
4: 5 6
```

Now we load these files using `import_graph` and draw the imported graphs.

```
> G:=import_graph("path/to/dot/example.dot")
```

ExampleGraph: an undirected unweighted graph with 4 vertices and 3 edges

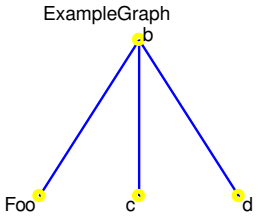
```
> draw_graph(G,tree="b")
```




```
> G:=import_graph("path/to/dot/example.dot",style=false)
```

ExampleGraph: an undirected unweighted graph with 4 vertices and 3 edges

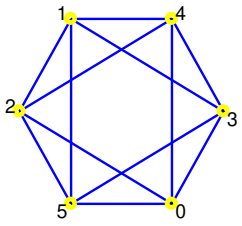
```
> draw_graph(G,tree="b")
```




```
> H:=import_graph("path/to/dot/octahedron.lst")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(H,spring)
```




```
> is_isomorphic(H,graph("octahedron"))
```

true

3.1.2 The DOT file format specification

GIAC has a basic support for the DOT language. Each dot file must correspond to exactly one graph and should consist of a single instance of the following environment:

```
strict? (graph | digraph) name? {
```

```
    ...
}
```

Keywords `strict` and `name` may be omitted, as indicated by the question marks. The former is used for differentiating between simple graphs (strict) and multigraphs (non-strict). Since Giac supports only simple graphs, `strict` is redundant.

For specifying undirected graphs the keyword `graph` is used, while the `digraph` keyword is used for undirected graphs.

The `graph`/`digraph` environment contains a series of instructions describing how the graph should be built. Each instruction must end with the semicolon (`;`) and has one of the forms given in Table 3.1.

syntax	**creates**
`vertex_name [attributes]?`	isolated vertices
`V1 <edgeop> V2 <edgeop> ... <edgeop> Vk [attributes]?`	edges and trails
`graph [attributes]`	graph attributes

Table 3.1. Syntax for creating vertices, edges, and graph attributes in the DOT language.

Here, `attributes` is a comma-separated list of tag-value pairs in form `tag=value`, and `<edgeop>` equals to `--` for undirected graphs and to `->` for directed graphs. Each of `V1`, `V2` etc. is either a vertex name or a set of vertex names in the form `{vertex_name1 vertex_name2 ...}`. If a set of vertices is specified, then each vertex in that set is connected to the neighbor operands. A vertex will be created if it does not exist yet.

Lines beginning with `#` are ignored. C-like comments are supported as well.

Using DOT syntax it is easy to specify a graph using adjacency lists. For example, the following is the contents of a file which defines the octahedral graph.

```
# octahedron
graph "octahedron" {
  1 -- {3 6 5 4};
  2 -- {3 4 5 6};
  3 -- {5 6};
  4 -- {5 6};
}
```

Also see the example in Section 3.1.1.

3.1.3 The lst file format specification

The LST file format is a practical way to save an undirected graph as the sequence of adjacency lists. Unlike the DOT format, it does not support attributes, including vertex labels and edge weights. This format is supported by e.g. The House of Graphs.

Each line of a file in LST format has the following syntax:

```
v: a1 a2 a3 ...
```

Here, v is a vertex index and a_i are its neighbors' indices. Hence this line represents the adjacency list of v (possibly empty, i.e. with no indices after the colon). It is enough to enter adjacencies with respect to the upper triangle of the adjacency matrix, i.e. one can list only the neighbors with indices $a_j > v$ on each line. Each vertex must be specified somewhere in the file, either as v or a.

Note that indices must be 1-based, while the labels in the imported graph are mode-aware (0-based in xcas mode and 1-based in maple mode) but follow the order specified in the file. For an example of lst file, see `octahedron.lst` in Section 3.1.1.

3.2 Exporting graphs

`export_graph(G,filename,⟨opts⟩)`

The command `export_graph` is used for saving graphs to disk in DOT/LST or LaTeX format. It takes two mandatory arguments, a graph G and a string `filename`, and writes G to the file specified by `filename`. The argument `filename` should be a string containing a path to the desired destination file (which is created if it does not exist). The remark on relative paths in Section 3.1.1 applies here as well.

If only two arguments are provided, the extension `.dot`/`.gv` or `.lst` must exist in `filename`. In the former case the graph is output in DOT format, while in the latter case it is output in LST format (it is thereby required that G is undirected, and all attributes will be stripped).

The optional argument `opts` is a sequence which can include the following arguments.

- `style=true|false` — if `style` is set to `true` (the default), then the style-related attributes of G are exported to the file, including color, shape, style, and position. Otherwise, these attributes are discarded
- `latex⟨=<params>⟩` — the drawing of G (obtained by calling the `draw_graph` command) is saved to the LaTeX file indicated by `filename` (the extension `.tex` may be omitted, in which case it is appended automatically). Optionally, one can specify a parameter or a list of parameters `params` which will be passed to `draw_graph`

`export_graph` returns 1 on success and 0 on failure.

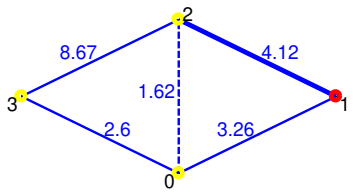
```
> G:=graph("diamond")
```

an undirected unweighted graph with 4 vertices and 5 edges

```
> G:=set_edge_attribute(G,[0,2],"style"=dashed):;
> G:=set_edge_attribute(G,[1,2],"style"=bold):;
> G:=set_vertex_attribute(G,1,"color"=red):;
> G:=assign_edge_weights(G,1..10)
```

an undirected weighted graph with 4 vertices and 5 edges

```
> draw_graph(G)
```




```
> export_graph(G,"path/to/diamond.dot")
```

1

The following is the contents of the exported file `diamond.dot`.

```
graph {
  graph [directed=false,weighted=true];
  0 [label=0,pos="1.4300725153,0.430159709003"];
  0 -- 1 [weight=3.26368919434];
  0 -- 2 [weight=1.6176089556,style=dashed];
  0 -- 3 [weight=2.60382712772];
  1 [label=1,color=red,pos="2.430159709,0.894229765084"];
  1 -- { 0 };
```

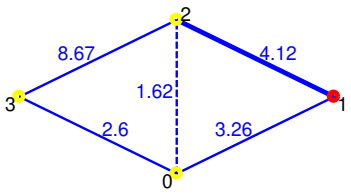
```
  1 -- 2 [weight=4.1175966193,style=bold];
  2 [label=2,pos="1.43051338931,1.3605143449"];
  2 -- { 0 1 };
  2 -- 3 [weight=8.66874152562];
  3 [label=3,pos="0.430159709003,0.894229765084"];
  3 -- { 0 2 };
}
```

The exported file can be imported back to Giac with the `import_graph` command. By drawing the imported graph, we see that all attributes are interpreted correctly, including the vertex positions generated by the `graph` constructor.

```
> H:=import_graph("path/to/diamond.dot")
```

an undirected weighted graph with 4 vertices and 5 edges

```
> draw_graph(H)
```



If the `style=false` option is used with `export_graph`, all style-related attributes are discarded.

```
> export_graph(G,"path/to/diamond.dot",style=false)
```

1

Now the contents of `diamond.dot` are as follows.

```
graph {
  graph [directed=false,weighted=true];
  0 [label=0];
  0 -- 1 [weight=3.26368919434];
  0 -- 2 [weight=1.6176089556];
  0 -- 3 [weight=2.60382712772];
  1 [label=1];
  1 -- { 0 };
  1 -- 2 [weight=4.1175966193];
  2 [label=2];
  2 -- { 0 1 };
  2 -- 3 [weight=8.66874152562];
  3 [label=3];
  3 -- { 0 2 };
}
```

One can use `export_graph` to obtain graph drawings using the Graphviz software. In this example, we construct a network, export it to DOT file, and create a layout using `dot`.

```
> V:=[A,B,C,D,E,F,G,H,I,J,K,L,M]:;
```

```
> purge(op(V)):;
```

```
> net:=digraph(V,%{[A,B],[B,C],[B,D],[C,H],[D,E],[D,F],[D,G],[E,H],[F,H],[G,H],
  [H,I],[H,J],[I,K],[J,L],[K,M],[L,M]%})
```

a directed unweighted graph with 13 vertices and 16 arcs

```
> net:=relabel_vertices(net,["Excavate","Lay foundation","Rough plumbing",
  "Frame","Finish exterior","Install HVAC","Rough electric","Sheet rock",
  "Install cabinets","Paint","Final plumbing","Final electric","Install
  flooring"])
```

a directed unweighted graph with 13 vertices and 16 arcs

```
> export_graph(net,"path/to/net.dot")
```

1

Now we enter the following command line in a terminal:

```
dot -Tpdf net.dot -o net_layout.pdf
```

The result is saved to `net_layout.pdf`. The drawing is shown in Figure 3.1. By default, dot spreads the graph vertically; if a horizontally oriented layout is desired, then one can set the graph attribute `rankdir` to `LR` before calling `export_graph`.

In the following example, we create a LaTeX drawing of the Sierpiński sieve graph ST_3^5.

```
> G:=sierpinski_graph(5,3,triangle)
```

an undirected unweighted graph with 123 vertices and 243 edges

```
> export_graph(G,"st53.tex",latex=[spring,size=[1,0],labels=false])
```

1

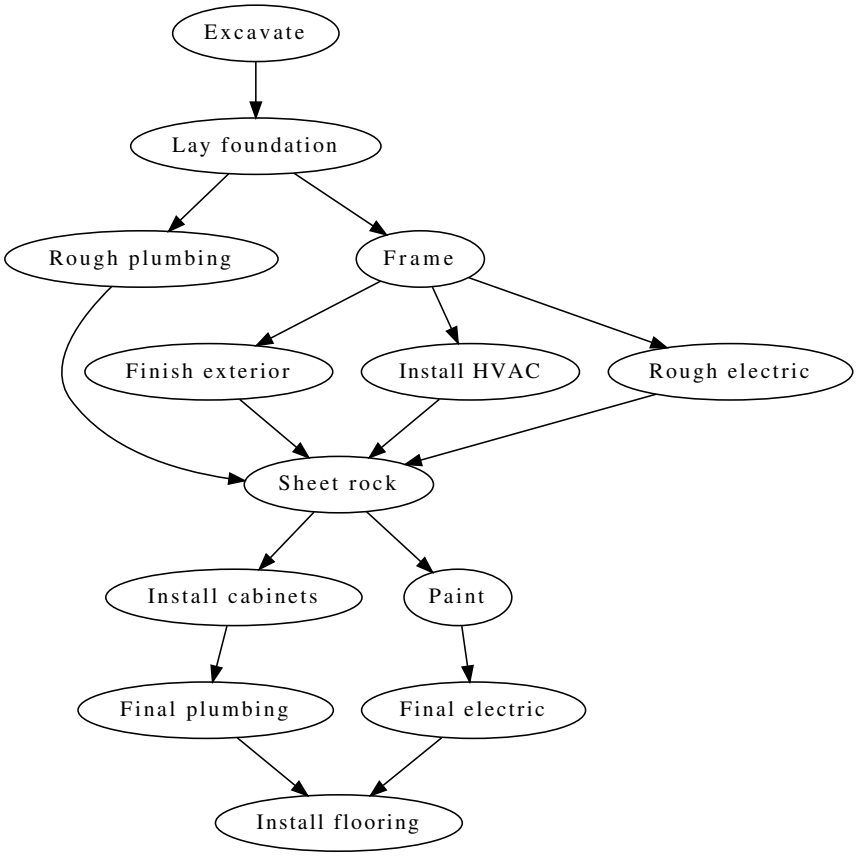


Fig. 3.1. A network graph drawn by GRAPHVIZ (dot).

Chapter 4
Graph properties

4.1 Basic properties

4.1.1 Determining the type of a graph

`is_directed(G)`
`is_weighted(G)`

The commands `is_directed` and `is_weighted` are used for determining the type of a graph: whether is it directed or not resp. weighted or not. Both commands take a graph G as their only argument. `is_directed` resp. `is_weighted` returns `true` if G is directed resp. weighted, else it returns `false`.

```
> G:=graph(trail(1,2,3,4,5,1,3))
```

an undirected unweighted graph with 5 vertices and 6 edges

```
> is_directed(G)
```

false

```
> is_directed(make_directed(G))
```

true

```
> is_weighted(G)
```

false

```
> is_weighted(make_weighted(G,randmatrix(5,5,99)))
```

true

4.1.2 Listing vertices and edges

`vertices(G)`
`graph_vertices(G)`
`edges(G,⟨weights⟩)`
`number_of_vertices(G)`
`number_of_edges(G)`

The command `vertices` or `graph_vertices` resp. `edges` is used for extracting the set of vertices resp. the set of edges from a graph. To obtain the number of vertices resp. the number of edges, use the `number_of_vertices` resp. the `number_of_edges` command.

`vertices` or `graph_vertices` takes a graph $G(V, E)$ as its only argument and returns the set of vertices V in the same order in which they were created.

`edges` takes one or two arguments, a graph $G(V, E)$ and optionally the argument `weights`. This command returns the set of edges E (in a non-meaningful order). If `weights` is specified, then each edge is paired with the corresponding weight (in this case G must be a weighted graph).

`number_of_vertices` resp. `number_of_edges` takes the input graph $G(V, E)$ as its only argument and returns $|V|$ resp. $|E|$.

```
> G:=hypercube_graph(3)
```

an undirected unweighted graph with 8 vertices and 12 edges

```
> vertices(G)
```

["000", "001", "010", "011", "100", "101", "110", "111"]

```
> number_of_vertices(G), number_of_edges(G)
```

8, 12

```
> H:=digraph([[0,2.32,0,0.25],[0,0,0,1.32],[0,0.50,0,0],[0.75,0,3.34,0]])
```

a directed weighted graph with 4 vertices and 6 arcs

```
> edges(H)
```

[[0, 1], [0, 3], [1, 3], [2, 1], [3, 0], [3, 2]]

```
> edges(H,weights)
```

[[[0, 1], 2.32], [[0, 3], 0.25], [[1, 3], 1.32], [[2, 1], 0.5], [[3, 0], 0.75], [[3, 2], 3.34]]

4.1.3 Equality of graphs

`graph_equal(G1,G2)`

Two graphs are considered **equal** if they are both (un)weighted and (un)directed and if the commands `vertices` and `edges` give the same results for both graphs. To determine whether two graphs are equal use the command `graph_equal`. It takes two arguments, graphs G_1 and G_2, and returns `true` if G_1 is equal to G_2 with respect to the above definition. Otherwise, it returns `false`.

```
> G1:=graph([1,2,3],%{[1,2],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> G2:=graph([1,3,2],%{[1,2],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> graph_equal(G1,G2)
```

false

```
> G3:=graph(trail(1,2,3))
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> graph_equal(G1,G3)
```

true

```
> G4:=digraph(trail(1,2,3))
```

a directed unweighted graph with 3 vertices and 2 arcs

```
> graph_equal(G1,G4)
```

false

4.1.4 Vertex degrees

`vertex_degree(G,v)`
`vertex_in_degree(G,v)`
`vertex_out_degree(G,v)`
`degree_sequence(G)`
`minimum_degree(G)`
`maximum_degree(G)`

The command `vertex_degree` is used for computing the degree of a vertex, i.e. counting the vertices adjacent to it. The related specialized commands are `vertex_out_degree`, `vertex_in_degree`, `degree_sequence`, `minimum_degree` and `maximum_degree`.

`vertex_degree` takes two arguments, a graph $G(V, E)$ and a vertex $v \in V$. It returns the number of edges in E which are incident to v.

When dealing with directed graphs, one can also use the specialized command `vertex_out_degree` resp. `vertex_in_degree` which takes the same arguments as `vertex_degree` but returns the number of arcs $vw \in E$ resp. the number of arcs $wv \in E$, where $w \in V$.

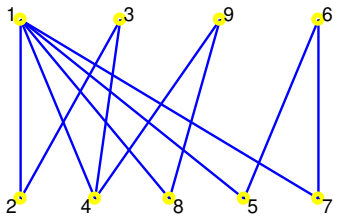
To obtain the list of degrees of all vertices $v \in V$, use the command `degree_sequence` which returns the list of degrees of vertices from V in the same order as returned by the command `vertices`. If G is a digraph, then the arc directions are ignored.

To compute the minimum vertex degree $\delta(G)$ and the maximum vertex degree $\Delta(G)$ in an undirected graph G, use the commands `minimum_degree` and `maximum_degree`, respectively.

```
> G:=graph(trail(1,2,3,4,1,5,6,7,1,8,9,4))
```

an undirected unweighted graph with 9 vertices and 11 edges

```
> draw_graph(G)
```




```
> vertex_degree(G,1)
```

5

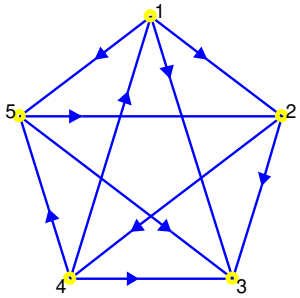
```
> degree_sequence(G)
```

$[5, 2, 2, 3, 2, 2, 2, 2, 2]$

```
> T:=random_tournament([1,2,3,4,5])
```

a directed unweighted graph with 5 vertices and 10 arcs

```
> draw_graph(T)
```




```
> vertex_out_degree(T,1)
```

3

```
> vertex_in_degree(T,5)
```

2

The command line below shows that Petersen graph is cubic (3-regular).

```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> minimum_degree(P), maximum_degree(P)
```

3, 3

```
> is_regular(P,3)
```

true

4.1.5 Regular graphs

`is_regular(G,⟨d⟩)`

The command `is_regular` is used for determining whether a graph is regular. It takes one or two arguments, a graph $G(V,E)$ and optionally a nonnegative integer or an unassigned identifier d. If G is undirected, the return value is `true` if $\delta_G = \Delta_G$, i.e. if the minimal vertex degree is equal to the maximal vertex degree in G, otherwise `false` is returned. If G is a digraph, then it is also required for each vertex $v \in V$ to have the same in- and out-degree. If the second argument is given, then G is tested for d-regularity in case d is an integer; otherwise Δ_G is assigned to d in case the latter is an unnasigned identifier and G is regular.

The complexity of the algorithm is $O(|V|)$.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> is_regular(G,d)
```

true

```
> d
```

3

```
> is_regular(G,2)
```

false

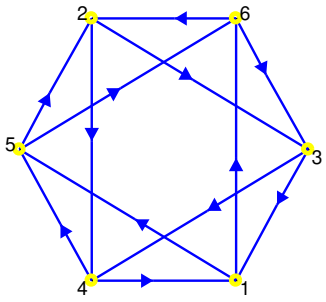
```
> is_regular(graph("groetzsch"))
```

false

```
> G:=digraph(%{[1,5],[1,6],[2,3],[2,4],[3,1],[3,4],[4,1],[4,5],[5,2],[5,6],[6,
  2],[6,3]%})
```

a directed unweighted graph with 6 vertices and 12 arcs

```
> draw_graph(G,spring)
```




```
> is_regular(G,4)
```

true

```
> H:=add_arc(delete_arc(G,[5,6]),[6,5])
```

a directed unweighted graph with 6 vertices and 12 arcs

```
> is_regular(H,4)
```

false

```
> is_regular(underlying_graph(H))
```

true

4.1.6 Strongly regular graphs

`is_strongly_regular(G,⟨srg⟩)`

The command `is_strongly_regular` is used for determining whether a graph is strongly regular. It takes one or two arguments, a graph $G(V, E)$ and optionally an unassigned identifier `srg`. It returns `true` if G is regular and there are integers λ and μ such that every two adjacent vertices resp. non-adjacent vertices in V have exactly λ resp. μ common neighbors. Otherwise, it returns `false`. If the second argument is given, then the list $[k, \lambda, \mu]$, where k is the degree of G, is assigned to `srg`.

The complexity of the algorithm is $O(k\,|V|^2)$.

```
> G:=graph("clebsch")
```

an undirected unweighted graph with 16 vertices and 40 edges

```
> is_regular(G)
```

true

```
> is_strongly_regular(G)
```

true

```
> H:=graph("shrikhande")
```

an undirected unweighted graph with 16 vertices and 48 edges

```
> purge(s):; is_strongly_regular(H,s)
```

"Done", true

```
> s
```

$[6, 2, 2]$

```
> is_strongly_regular(cycle_graph(5))
```

true

```
> is_strongly_regular(cycle_graph(6))
```

false

4.1.7 Vertex adjacency

`has_edge(G,[u,v])`
`has_arc(G,[u,v])`
`neighbors(G,⟨v⟩)`
`departures(G,⟨v⟩)`
`arrivals(G,⟨v⟩)`

The command `has_edge` is used for determining whether two vertices in an undirected graph are adjacent. For digraphs, there is an analogous command `has_arc`.

The command `neighbors` is used for obtaining the list of vertices in a graph that are adjacent to the particular vertex or the complete adjacency structure of the graph, in sparse form.

The command `departures` resp. `arrivals` is used for obtaining all neighbors of a vertex v in a digraph which are the heads resp. the tails of the corresponding arcs.

`has_edge` takes two arguments, an undirected graph $G(V, E)$ and a list `[u,v]` where $u, v \in V$. The command returns `true` if $uv \in E$ and `false` otherwise. The syntax for `has_arc` is the same, except now G is required to be directed. Note, however, that the order of vertices u and v matters in digraphs. The worst-case complexity of both algorithms is $O(\log |V|)$.

`neighbors` takes one or two arguments, a graph $G(V, E)$ and optionally a vertex $v \in V$. It returns the list of vertices adjacent to v, if given. Otherwise, it returns the list of lists of neighbors for all vertices in V, in order of `vertices(G)`. Note that edge directions are ignored in case G is a digraph.

`departures` resp. `arrivals` takes one or two arguments, a digraph $G(V, E)$ and optionally a vertex $v \in V$, and returns the list L_v containing all vertices $w \in V$ for which $vw \in E$ resp. $wv \in E$. If v is omitted, then the list of lists L_v for each $v \in V$ is returned.

```
> G:=graph(trail(1,2,3,4,5,2))
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> has_edge(G,[1,2])
```

true

```
> has_edge(G,[2,1])
```

true

```
> has_edge(G,[1,3])
```

false

```
> D:=digraph(trail(1,2,3,4,5,2,1))
```

a directed unweighted graph with 5 vertices and 6 arcs

```
> has_arc(D,[1,2])
```

true

```
> has_arc(D,[2,1])
```

true

```
> has_arc(D,%{1,2%})
```

true

```
> has_arc(D,[4,5])
```

true

```
> has_arc(D,[5,4])
```

false

```
> has_arc(D,%{4,5%})
```

false

```
> neighbors(G,3)
```

$[2, 4]$

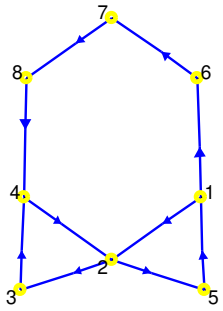
```
> neighbors(G)
```

$[[2], [1, 3, 5], [2, 4], [3, 5], [2, 4]]$

```
> G:=digraph(trail(1,2,3,4,2,5,1,6,7,8,4))
```

a directed unweighted graph with 8 vertices and 10 arcs

```
> draw_graph(G,spring)
```




```
> departures(G,2); arrivals(G,2); departures(G,1); arrivals(G,1)
```

$[3, 5], [1, 4], [2, 6], [5]$

4.1.8 Tournament graphs

`is_tournament(G)`

The command `is_tournament` is used for determining whether a graph is a tournament. It takes a graph $G(V, E)$ as its only argument and returns `true` if G is directed and for each pair of vertices $u, v \in V$ it is either $uv \in E$ or $vu \in E$, i.e. there is exactly one arc between u and v. Otherwise, it returns `false`.

```
> T1:=digraph(%{[1,2],[2,3],[3,1]%})
```

a directed unweighted graph with 3 vertices and 3 arcs

```
> is_tournament(T1)
```

true

```
> T2:=digraph(%{[1,2],[2,3],[3,1],[1,3]%})
```

a directed unweighted graph with 3 vertices and 4 arcs

```
> is_tournament(T2)
```

false

4.1.9 Bipartite graphs

`is_bipartite(G,⟨P⟩)`

The command `is_bipartite` is used for determining if a graph is bipartite. It takes one or two arguments, a graph $G(V, E)$ and optionally an unassigned identifier `P`. It returns `true` if there is a partition of V into two sets S and T such that every edge from E connects a vertex in S to one in T. Otherwise, it returns `false`. If the second argument is given and G is bipartite, the partition of V is assigned to `P` as a list of two lists of vertices, the first one containing the vertices from S and the second one containing vertices from T.

```
> K32:=complete_graph(3,2)
```

an undirected unweighted graph with 5 vertices and 6 edges

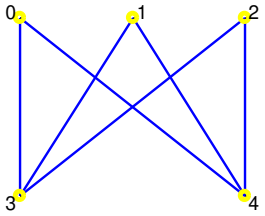
```
> is_bipartite(K32,bp)
```

true

```
> bp
```

$$[[0, 1, 2], [3, 4]]$$

```
> draw_graph(K32,bipartite)
```




```
> adjacency_matrix(K32)
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

```
> C5:=cycle_graph(5)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> is_bipartite(G)
```

false

4.1.10 Edge incidence

`indcident_edges(G,v|L)`

The command `incident_edges` is used for obtaining edges incident to a given vertex in a graph. It takes two arguments, a graph $G(V, E)$ and a vertex $v \in V$ or a list of vertices $L \subset V$, and returns the list of edges $e_1, e_2, ..., e_k \in E$ such that each of them has v as one of its endpoints.

Note that edge directions are ignored when G is a digraph. To obtain only outgoing or incoming edges, use the commands `departures` and `arrivals`, respectively.

```
> G:=cycle_graph([1,2,3,4,5])
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> incident_edges(G,1)
```

$$[[1, 2], [1, 5]]$$

```
> incident_edges(G,[2,4,5])
```

$$[[1, 2], [1, 5], [2, 3], [3, 4], [4, 5]]$$

```
> G:=random_tournament([1,2,3,4,5])
```

a directed unweighted graph with 5 vertices and 10 arcs

```
> incident_edges(G,2)
```

$$[[2,1],[2,3],[2,4],[2,5]]$$

4.2 Algebraic properties

4.2.1 Adjacency matrix

`adjacency_matrix(G)`

The command `adjacency_matrix` is used for obtaining the adjacency matrix of a graph. It takes a graph $G(V,E)$, where $V=\{v_1,v_2,...,v_n\}$, as its only argument and returns the square matrix $A=[a_{ij}]$ of order n such that, for $i,j=1,2,...,n$,

$$a_{ij}=\begin{cases}1, & \text{if the set } E \text{ contains edge/arc } v_i v_j,\\ 0, & \text{otherwise.}\end{cases}$$

Note that $\mathrm{tr}\,(A)=0$. Also, the adjacency matrix of an undirected graph is always symmetric.

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> A:=adjacency_matrix(G)
```

$$\begin{pmatrix}0&0&1&1&1&1\\0&0&1&1&1&1\\1&1&0&0&1&1\\1&1&0&0&1&1\\1&1&1&1&0&0\\1&1&1&1&0&0\end{pmatrix}$$

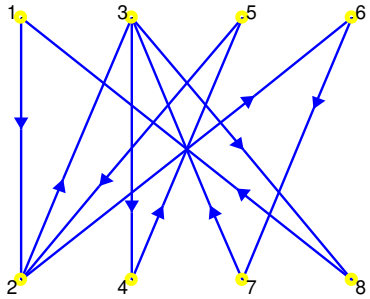
```
> transpose(A)==A
```

true

```
> D:=digraph(trail(1,2,3,4,5,2,6,7,3,8,1))
```

a directed unweighted graph with 8 vertices and 10 arcs

```
> draw_graph(D)
```




```
> A:=adjacency_matrix(D)
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
> transpose(A)==A
```

false

4.2.2 Laplacian matrix

`laplacian_matrix(G,⟨normal⟩)`

The command `laplacian_matrix` is used for computing the Laplacian matrix of a graph. It takes a graph $G(V,E)$, where $V=\{v_1,v_2,...,v_n\}$, and returns the matrix $L=D-A$, where A is the (weighted) adjacency matrix of G and

$$D=\begin{pmatrix} \deg(v_1) & 0 & \cdots & 0 \\ 0 & \deg(v_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \deg(v_n) \end{pmatrix}.$$

The number $\deg(v)$ is computed by summing weights of all edges incident to v (in unweighted graphs, all edge weights are assumed to be equal to 1). The option `normal` may be passed as the second argument. In that case, the normalized Laplacian

$$L^{\text{sym}} := I - D^{-1/2}\,A\,D^{-1/2}$$

of G is returned.

```
> G:=path_graph(4)
```

an undirected unweighted graph with 4 vertices and 3 edges

```
> A:=adjacency_matrix(G)
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
> L:=laplacian_matrix(G)
```

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

```
> diag(degree_sequence(G))-A==L
```

true

```
> laplacian_matrix(G,normal)
```

$$\begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

The smallest eigenvalue of the Laplacian matrix of an undirected graph is always zero. Moreover, its multiplicity is equal to the number of connected components in the corresponding graph [34, p. 280].

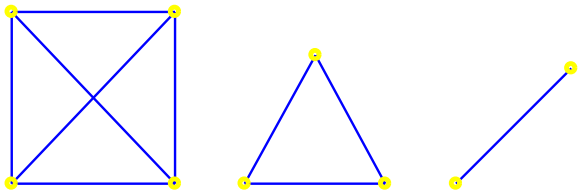
```
> sort(eigenvals(L))
```

$$0, -\sqrt{2}+2, 2, \sqrt{2}+2$$

```
> H:=disjoint_union(complete_graph(4),cycle_graph(3),path_graph(2))
```

an undirected unweighted graph with 9 vertices and 10 edges

```
> draw_graph(H,labels=false)
```



```
> eigenvals(laplacian_matrix(H))
```

$$0, 0, 0, 4, 4, 4, 3, 3, 2$$

Therefore, the multiplicity of zero eigenvalue is equal to 3. Indeed, that is also the number of connected components in H:

```
> nops(connected_components(H))
```

3

4.2.3 Incidence matrix

`incidence_matrix(G)`

The command `incidence_matrix` is used for obtaining the incidence matrix of a graph. It takes a graph $G(V, E)$, where $V = \{v_1, v_2, ..., v_n\}$ and $E = \{e_1, e_2, ..., e_m\}$, as its only argument and returns the $n \times m$ matrix $B = [b_{ij}]$ such that, for all $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$,

$$b_{ij} = \begin{cases} 1, & \text{if the vertex } v_i \text{ is incident to the edge } e_j, \\ 0, & \text{otherwise} \end{cases}$$

if G is undirected resp.

$$b_{ij} = \begin{cases} 1, & \text{if the vertex } v_i \text{ is the head of the arc } e_j, \\ -1, & \text{if the vertex } v_i \text{ is the tail of the arc } e_j, \\ 0, & \text{otherwise} \end{cases}$$

if G is directed.

```
> K4:=complete_graph([1,2,3,4])
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> edges(K4)
```

$$[[1,2],[1,3],[1,4],[2,3],[2,4],[3,4]]$$

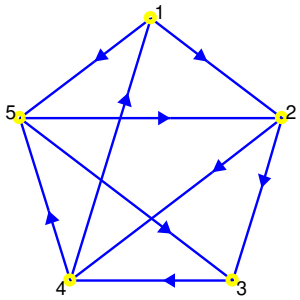
```
> incidence_matrix(K4)
```

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

```
> DG:=digraph(trail(1,2,3,4,5,3),trail(1,5,2,4,1))
```

a directed unweighted graph with 5 vertices and 9 arcs

```
> draw_graph(DG)
```




```
> edges(DG)
```

$$[[1,2],[1,5],[2,3],[2,4],[3,4],[4,1],[4,5],[5,2],[5,3]]$$

```
> incidence_matrix(DG)
```

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

4.2.4 Weight matrix

`weight_matrix(G)`

The command `weight_matrix` is used for obtaining the weight matrix of a weighted graph. It takes a graph $G(V,E)$, where $V=\{v_1,v_2,...,v_n\}$, as its only argument and returns the square matrix $M=[m_{ij}]$ of order n such that m_{ij} equals zero if v_i and v_j are not adjacent and the weight of the edge v_iv_j otherwise, for all $i,j=1,2,...,n$.

Note that $\mathrm{tr}\,(M)=0$. Also, the weight matrix of an undirected graph is always symmetric.

```
> G:=graph(%{[[1,2],1],[[2,3],2],[[4,5],3],[[5,2],4]%})
```

an undirected weighted graph with 5 vertices and 4 edges

```
> weight_matrix(G)
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 4 & 0 & 3 & 0 \end{pmatrix}$$

4.2.5 Characteristic polynomial

`graph_charpoly(G,⟨x⟩)`
`charpoly(G,⟨x⟩)`

The command `graph_charpoly` or `charpoly` is used for obtaining the characteristic polynomial of an undirected graph. It takes one or two arguments, an undirected graph $G(V, E)$ and optionally a value or symbol x. The command returns $p(x)$, where p is the characteristic polynomial of the adjacency matrix of G.

```
> G:=graph(%{[1,2],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> charpoly(G,x)
```

$$x^3 - 2\,x$$

```
> charpoly(G,3)
```

21

```
> G:=graph("shrikhande")
```

an undirected unweighted graph with 16 vertices and 48 edges

```
> charpoly(G,x)
```

$$x^{16} - 48\,x^{14} - 64\,x^{13} + 768\,x^{12} + 1536\,x^{11} - 5888\,x^{10} - 15360\,x^9 + 23040\,x^8 + 81920\,x^7 - 36864\,x^6 - 245760\,x^5 - 32768\,x^4 + 393216\,x^3 + 196608\,x^2 - 262144\,x - 196608$$

4.2.6 Graph spectrum

`graph_spectrum(G)`

The command `graph_spectrum` is used for computing graph spectra. It takes a graph G as its only argument and returns the list in which every element is an eigenvalue of the adjacency matrix of G paired with its multiplicity.

```
> C5:=cycle_graph(5)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> gs:=graph_spectrum(C5)
```

$$\begin{pmatrix} 2 & 1 \\ \frac{\sqrt{5}-1}{2} & 2 \\ \frac{-\sqrt{5}-1}{2} & 2 \end{pmatrix}$$

```
> p:=charpoly(C5,x)
```

$$x^5 - 5\,x^3 + 5\,x - 2$$

```
> expand(roots(p))==expand(gs)
```

true

The above result indicates that `gs` and `roots(p)` are equal.

4.2.7 Seidel spectrum

`seidel_spectrum(G)`

The command `seidel_spectrum` is used for computing Seidel spectra. It takes a graph $G(V, E)$ as its only argument and returns the list in which every element is an eigenvalue of the Seidel adjacency matrix S paired with its multiplicity. The matrix S, which can be interpreted as the difference of the adjacency matrices of G and its complement G^c, is computed as $J - I - 2\,A$, where J is all-one $n \times n$ matrix, I is the identity matrix of order n, A is the adjacency matrix of G, and $n = |V|$.

```
> seidel_spectrum(graph("clebsch"))
```

$$\begin{pmatrix} -3 & 10 \\ 5 & 6 \end{pmatrix}$$

```
> seidel_spectrum(graph("levi"))
```

$$\begin{pmatrix} -5 & 9 \\ -1 & 10 \\ 3 & 9 \\ 5 & 1 \\ 23 & 1 \end{pmatrix}$$

4.2.8 Integer graphs

`is_integer_graph(G)`

The command `is_integer_graph` is used for determining whether a graph is an integral graph. It takes a graph G as its only argument and returns `true` if the spectrum of G consists only of integers. Otherwise, it returns `false`.

```
> G:=graph("levi")
```

an undirected unweighted graph with 30 vertices and 45 edges

```
> is_integer_graph(G)
```

true

```
> factor(charpoly(G,x))
```

$$x^{10}\,(x-3)\,(x-2)^9\,(x+2)^9\,(x+3)$$

```
> graph_spectrum(G)
```

$$\begin{pmatrix} -3 & 1 \\ -2 & 9 \\ 0 & 10 \\ 2 & 9 \\ 3 & 1 \end{pmatrix}$$

4.3 Graph isomorphism

4.3.1 Isomorphic graphs

`is_isomorphic(G1,G2,⟨m⟩)`

The command `is_isomorphic` is used for determining whether two graphs are isomorphic. It takes two or three arguments: a graph $G_1(V_1, E_1)$, a graph $G_2(V_2, E_2)$ and optionally an unassigned identifier `m`. The command returns `true` if G_1 and G_2 are isomorphic and `false` otherwise. If the third argument is given and G_1 and G_2 are isomorphic, then the list of pairwise matching of vertices in G_1 and G_2, representing the isomorphism between the two graphs, is assigned to `m`.

Note that the algorithm takes vertex colors into account. Namely, only vertices sharing the same color can be mapped to each other. Vertex colors can be set by calling the `highlight_vertex` command.

This command, as well as the commands `canonical_labeling` and `graph_automorphisms` described later in this section, is using the NAUTY library [55], which is one of the fastest implementations for graph isomorphism. If NAUTY is not available, then `is_isomorphic` uses ULLMANN's algorithm [80] as a fallback.

```
> is_isomorphic(graph("petersen"),kneser_graph(5,2))
```

true

In the following example, G_1 and G_3 are isomorphic while G_1 and G_2 are not isomorphic.

```
> G1:=graph(trail(1,2,3,4,5,6,1,3))
```

an undirected unweighted graph with 6 vertices and 7 edges

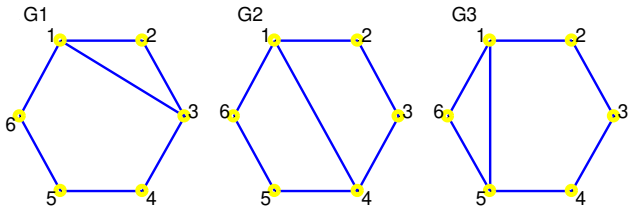
```
> G2:=graph(trail(1,2,3,4,5,6,1,4))
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> G3:=graph(trail(1,2,3,4,5,6,1,5))
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> draw_graph(G1,circle,[0,0],size=[1,0],title="G1");
  draw_graph(G2,circle,[1.3,0],size=[1,0],title="G2");
  draw_graph(G3,circle,[2.6,0],size=[1,0],title="G3");
```




```
> is_isomorphic(G1,G2)
```

false

```
> is_isomorphic(G1,G3)
```

true

```
> purge(mapping):; is_isomorphic(G1,G3,mapping):; mapping
```

"Done", "Done", $[1 = 5, 2 = 6, 3 = 1, 4 = 2, 5 = 3, 6 = 4]$

```
> H1:=highlight_vertex(G1,5):; H3:=highlight_vertex(G3,5):;
```

"Done", "Done"

```
> is_isomorphic(H1,H3)
```

false

```
> H1:=highlight_vertex(H1,1):; H3:=highlight_vertex(H3,3):;
```

"Done", "Done"

```
> is_isomorphic(H1,H3)
```

true

In the following example, D_1 and D_3 are isomorphic while D_1 and D_2 are not isomorphic.

```
> D1:=digraph(trail(1,2,3,1,4,5))
```

a directed unweighted graph with 5 vertices and 5 arcs

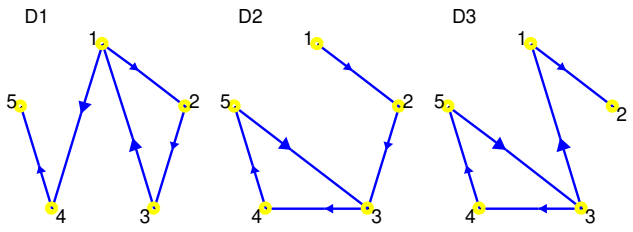
```
> D2:=digraph(trail(1,2,3,4,5,3))
```

a directed unweighted graph with 5 vertices and 5 arcs

```
> D3:=digraph([1,2,3,4,5],trail(3,4,5,3,1,2))
```

a directed unweighted graph with 5 vertices and 5 arcs

```
> draw_graph(D1,circle,[0,0],size=[1,0],title="D1");
  draw_graph(D2,circle,[1.3,0],size=[1,0],title="D2");
  draw_graph(D3,circle,[2.6,0],size=[1,0],title="D3");
```




```
> is_isomorphic(D1,D2)
```

false

```
> is_isomorphic(D1,D3)
```

true

Isomorphism testing with NAUTY is very fast and can be used for large graphs, as in the example below.

```
> G:=random_graph(10000,0.01)
```

an undirected unweighted graph with 10000 vertices and 498793 edges

```
> H:=permute_vertices(G,shuffle)
```

an undirected unweighted graph with 10000 vertices and 498793 edges

```
> vertices(G)==vertices(H)
```

false

```
> is_isomorphic(G,H)
```

true

1.567 sec

To make the edge structures of G and H slightly different, a random edge from H is "misplaced".

```
> ed:=edges(H)[rand(number_of_edges(H))]
```

$[4826, 5144]$

```
> has_edge(H,[4826,5145])
```

false

```
> H:=add_edge(delete_edge(H,ed),[4826,5145])
```

an undirected unweighted graph with 10000 vertices and 498793 edges

```
> is_isomorphic(G,H)
```

false

1.678 sec

4.3.2 Canonical labeling

`canonical_labeling(G)`

Graph isomorphism testing in NAUTY is based on computing the canonical labeling for the input graphs. The **canonical labeling** of G is a particular ordering of the vertices of G. Rearranging the vertices with respect to that ordering produces the **canonical representation** of G. Two graphs are isomorphic if and only if their canonical representations share the same edge structure.

The command `canonical_labeling` is used for computing the canonical labeling as a permutation. One can reorder the vertices by using this permutation with the `permute_vertices` command.

`canonical_labeling` takes a graph $G(V, E)$ as its only argument and returns the permutation representing the canonical labeling of G. Note that the colors of the vertices are taken into account.

In the next example it is demonstrated how to prove that G_1 and G_3 are isomorphic by comparing their canonical representations C_1 and C_3 with the `graph_equal` command. Before testing C_1 and C_3 for equality, their vertices have to be relabeled so that the command `vertices` gives the same result for both graphs.

```
> L1:=canonical_labeling(G1)
```

$[4, 3, 5, 1, 2, 0]$

```
> L3:=canonical_labeling(G3)
```

$[2, 1, 3, 5, 0, 4]$

```
> C1:=relabel_vertices(isomorphic_copy(G1,L1),[1,2,3,4,5,6])
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> C3:=relabel_vertices(isomorphic_copy(G3,L3),[1,2,3,4,5,6])
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> graph_equal(C1,C3)
```

true

4.3.3 Graph automorphisms

`graph_automorphisms(G)`

The command `graph_automorphisms` is used for finding generators of the automorphism group of a graph. It takes a graph G as its only argument and returns a list containing the generators of $\mathrm{Aut}(G)$, the automorphism group of G (see [34, p. 4] and [10, p. 115]). Each generator is given as a list of cycles, which can be turned to a permutation by calling the command `cycles2permu`.

Note that vertex colors are taken into account. Only vertices sharing the same color can be mapped to each other. The color of a vertex can be set by calling the command `highlight_vertex`.

```
> g:=graph_automorphisms(graph("petersen"))
```

$[[[3,7],[4,5],[8,9]],[[2,6],[3,8],[4,5],[7,9]],[[1,4],[2,3],[6,9],[7,8]],[[0,1],[2,4],[5,6],[7,9]]]$

```
> cycles2permu(g[2])
```

$[0,4,3,2,1,5,9,8,7,6]$

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> G:=highlight_vertex(G,4)
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> graph_automorphisms(G)
```

$[[[2,6],[3,9],[7,8]],[[1,5],[2,7],[3,9],[6,8]],[[0,3],[1,2],[5,8],[6,7]]]$

In the above result, all permutations map the vertex 4 to itself, because it is the single green-colored vertex in G (it cannot be mapped to any other vertex because colors do not match).

Frucht graph (see the page 32) is an example of a graph with automorphism group containing only the identity, so the set of its generators is empty:

```
> graph_automorphisms(graph("frucht"))
```

[]

4.3.4 Test for isomorphism against subgraphs

`is_subgraph_isomorphic(G1,G2,⟨opts⟩)`

The command `is_subgraph_isomorphic` is used for checking whether a graph is isomorphic to a subgraph of another graph. It takes two mandatory arguments, graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$, which are either both undirected or both directed. Edge weights are ignored in both G_1 and G_2. The command returns `true` if G_1 is isomorphic to a subgraph S of G_2, else it returns `false`. A sequence of optional arguments may be passed as the third argument `opts`. The following options are supported:

- `induced_subgraph` — S must be an induced subgraph of G_2

- unassigned identifier — a destination to which S is stored

The strategy is to use ULLMANN's backtracking algorithm [80] with improvements by ČIBEJ and MIHELIČ [17]. The implementation is space-efficient but may perform slower on regular graphs.

In the first example we show that the flower snark J_5 contains a 5- and 6-cycle.

```
> J5:=flower_snark(5)
```

an undirected unweighted graph with 20 vertices and 30 edges

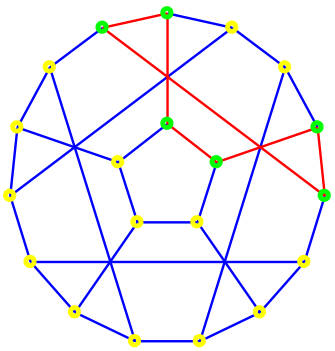
```
> is_subgraph_isomorphic(cycle_graph(5),J5)
```

true

```
> purge(S):; is_subgraph_isomorphic(cycle_graph(6),J5,S)
```

"Done", true

```
> draw_graph(highlight_subgraph(J5,S),labels=false)
```

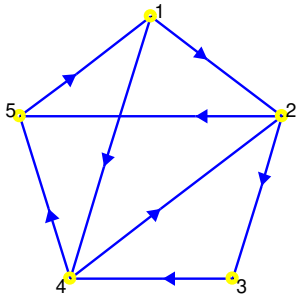


In the example below we find 3-cycle in a directed graph.

```
> G:=digraph(trail(1,2,3,4,5,1,4,2,5))
```

a directed unweighted graph with 5 vertices and 8 arcs

```
> draw_graph(G)
```



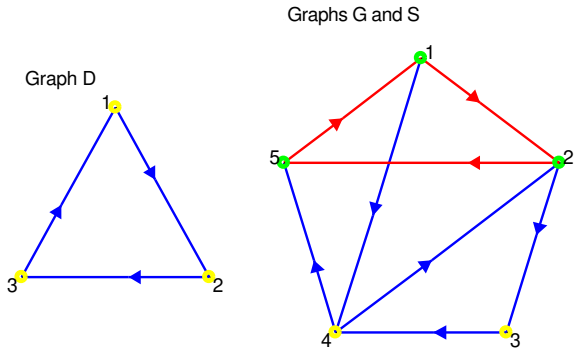

```
> D:=digraph([1,2,3],trail(1,2,3,1))
```

a directed unweighted graph with 3 vertices and 3 arcs

```
> purge(S):; is_subgraph_isomorphic(D,G,S)
```

"Done", true

```
> draw_graph(D,[0,0.2],size=[0,0.62],title="Graph D");
  draw_graph(highlight_subgraph(G,S),[1,0],size=[0,1],title="Graphs G and S");
```



In the following example, we verify that the Sylvester graph is a subgraph of the Hoffman-Singleton graph.

```
> HS:=graph("hoffman-singleton")
```

an undirected unweighted graph with 50 vertices and 175 edges

```
> G:=graph("sylvester")
```

an undirected unweighted graph with 36 vertices and 90 edges

```
> is_subgraph_isomorphic(G,HS)
```

true

The source graph may also be sought only among induced subgraphs of the destination graph, as in the example below.

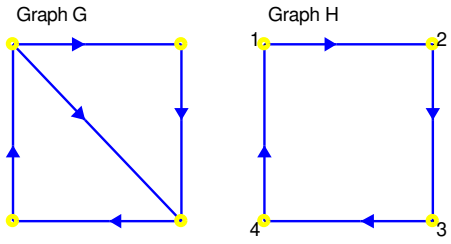
```
> G:=digraph(trail(1,2,3,4,1,3))
```

a directed unweighted graph with 4 vertices and 5 arcs

```
> H:=digraph(trail(1,2,3,4,1))
```

a directed unweighted graph with 4 vertices and 4 arcs

```
> draw_graph(G,[0,0],size=[1,0],labels=false,title="Graph G");
  draw_graph(H,[1.5,0],size=[1,0],circle=[1,2,3,4],title="Graph H")
```




```
> is_subgraph_isomorphic(H,G)
```

true

```
> is_subgraph_isomorphic(H,G,induced_subgraph)
```

false

4.3.5 Recognizing special graphs

`identify_graph(G,⟨haar_graph=<n>⟩)`

The command `identify_graph` is used for testing a graph for isomorphism against special graphs known to GIAC. It takes a graph G as its mandatory argument and returns the list of special graphs which are isomorphic to G. Each entry in the list is a list comprised of the respective constructor and a sequence of parameters.

The optional argument `haar_graph=<n>`, where n is an integer between 0 and 64 (by default $n = 25$), is used for setting an upper limit for the number of vertices in each partition of a Haar graph. Haar graphs with more than $2\,n$ vertices will not be tested for an isomorphism. If $n = 0$, then Haar graphs are not tested at all. This is also equivalent to setting `haar_graph=false`.

Note that `identify_graph` outputs only the lowest index of an isomorphic Haar graph.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> identify_graph(G)
```

[[kneser_graph, 5, 2], [odd_graph, 3], [petersen_graph, 5, 2], [graph, "petersen"]]

```
> G:=petersen_graph(16,7)
```

an undirected unweighted graph with 32 vertices and 48 edges

```
> identify_graph(G)
```

[[petersen_graph, 16, 7], [haar_graph, 32786]]

In the following example, we remove an edge together with all incident vertices from the Hoffman-Singleton graph. The remaining graph is isomorphic to the Sylvester graph, as discovered by `identify_graph`.

```
> G:=graph("hoffman-singleton")
```

an undirected unweighted graph with 50 vertices and 175 edges

```
> ed:=choice(edges(G))
```

[36, 37]

```
> H:=delete_vertex(G,concat(ed,neighbors(G,ed[0]),neighbors(G,ed[1])))
```

an undirected unweighted graph with 36 vertices and 90 edges

```
> identify_graph(H)
```

[[graph, "sylvester"]]

4.4 Graph polynomials

4.4.1 Tutte polynomial

`tutte_polynomial(G,⟨x,y⟩)`

The command `tutte_polynomial` is used for computing Tutte polynomials. It takes one or three arguments, an undirected graph $G(V, E)$ and optionally two variables or values x and y. It returns the the bivariate Tutte polynomial[4.1] T_G of G or the value $T_G(x, y)$ if the optional arguments are given. If G is weighted, then it is treated as a multigraph: the weight w of an edge e, which must be a positive integer, is interpreted as the multiplicity of e, for each $e \in E$. Note that self-loops are not supported.

The strategy is to apply the recursive definition of Tutte polynomial [36] together with the vorder heuristic proposed by Haggard et al. [37] and improved by Monagan [56]. The subgraphs appearing in the computation tree are cached and reused when possible, pruning the tree significantly. Subgraphs are cached in their canonical form, for which nauty is required.

```
> K4:=complete_graph(4)
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> tutte_polynomial(K4,x,y)
```

$$x^3 + 3\,x^2 + 4\,x\,y + 2\,x + y^3 + 3\,y^2 + 2\,y$$

```
> tutte_polynomial(K4,x,1)
```

$$x^3 + 3\,x^2 + 6\,x + 6$$

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> f:=tutte_polynomial(G)
```

$$y^6 + 9\,y^5 + 10\,y^4\,x + 35\,y^4 + 15\,y^3\,x^2 + 65\,y^3\,x + 75\,y^3 + 30\,y^2\,x^3 + 105\,y^2\,x^2 + 171\,y^2\,x + 84\,y^2 + 12\,y\,x^5 + 70\,y\,x^4 + 170\,y\,x^3 + 240\,y\,x^2 + 168\,y\,x + 36\,y + x^9 + 6\,x^8 + 21\,x^7 + 56\,x^6 + 114\,x^5 + 170\,x^4 + 180\,x^3 + 120\,x^2 + 36\,x$$

This result coincides with that in [10, p. 103], which is supposed to be correct. Alternatively, it can be verified by applying the recursive definition with an arbitrary edge $e \in E$, as below.

```
> ed:=edges(G)[0]
```

$$[0, 1]$$

```
> Gdelete:=delete_edge(G,ed)
```

an undirected unweighted graph with 10 vertices and 14 edges

```
> Gcontract:=contract_edge(G,ed)
```

an undirected unweighted graph with 9 vertices and 14 edges

```
> expand(f-tutte_polynomial(Gdelete)-tutte_polynomial(Gcontract))
```

4.1. See [36], [10, p. 97] and [11, p. 335].

0

The value $T_G(1,1)$ is equal to the number of spanning forests in G [11, p. 345]—in this case, the number of spanning trees in Petersen graph. For verification, the same number is computed by using the specialized command `number_of_spanning_trees`, which uses a much faster algorithm.

```
> tutte_polynomial(G,1,1)
```

2000

```
> number_of_spanning_trees(G)
```

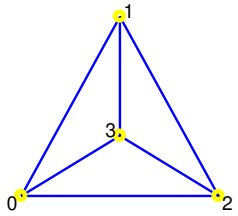
2000

For a graph G and its dual G^* the following relation holds: $T_G(x,y) = T_{G^*}(y,x)$. Therefore, if $T_G(x,y) = T_G(y,x)$ then G and G^* are isomorphic (since Tutte polynomial is a graph invariant). A simple example of such graph is tetrahedral graph. Since it is planar and biconnected, its dual can be determined by using the command `plane_dual`.

```
> G:=graph("tetrahedron")
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> draw_graph(G)
```




```
> is_biconnected(G) and is_planar(G)
```

true

```
> H:=plane_dual(G)
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> T:=tutte_polynomial(G)
```

$$x^3 + 3\,x^2 + 4\,x\,y + 2\,x + y^3 + 3\,y^2 + 2\,y$$

```
> expand(T-subs(T,[x,y],[y,x]))
```

0

```
> is_isomorphic(G,H)
```

true

Multiple edges can be specified as edge weights.

```
> M:=make_weighted(G)
```

an undirected weighted graph with 4 vertices and 6 edges

```
> M:=set_edge_weight(set_edge_weight(M,[0,1],2),[2,3],3)
```

an undirected weighted graph with 4 vertices and 6 edges

```
> edges(M,weights)
```

$$[[[0,1],2],[[0,2],1],[[0,3],1],[[1,2],1],[[1,3],1],[[2,3],3]]$$

```
> tutte_polynomial(M,x,y)
```

$$x^3+x^2\,y^2+2\,x^2\,y+3\,x^2+3\,x\,y^3+6\,x\,y^2+6\,x\,y+2\,x+y^6+3\,y^5+6\,y^4+7\,y^3+5\,y^2+2\,y$$

4.4.2 Chromatic polynomial

`chromatic_polynomial(G,⟨t⟩)`

The command `chromatic_polynomial`, is used for computing chromatic polynomials. It takes one or two arguments, an undirected unweighted graph $G(V,E)$ and optionally a variable or value t. It returns the chromatic polynomial P_G of G or the value $P_G(t)$ if the second argument is given.

P_G and the Tutte polynomial T_G satisfy the following relation (see [36] and [11, p. 346]):

$$P_G(t)=(-1)^{|V|-\kappa(G)}\,t^{\kappa(G)}\,T_G(1-t,0),$$

where $\kappa(G)$ is the number of connected components of G.

The value $P_G(k)$, where $k>0$ is an integer, is equal to the number of all distinct k-colorings of vertices in G. As shown in the example below, Petersen graph cannot be colored by using only two colors, but is 3-colorable with 120 distinct colorings (all using the same three colors).

```
> P:=chromatic_polynomial(graph("petersen"),x)
```

$$x\,(x-2)\,(x-1)\,(x^7-12\,x^6+67\,x^5-230\,x^4+529\,x^3-814\,x^2+775\,x-352)$$

```
> subs(P,x=2)
```

$$0$$

```
> subs(P,x=3)
```

$$120$$

4.4.3 Flow polynomial

`flow_polynomial(G,⟨x⟩)`

The command `flow_polynomial` is used for computing flow polynomials. It takes one or two arguments, an undirected unweighted graph $G(V,E)$ and optionally a variable or value x. It returns the flow polynomial Q_G of G or the value $Q_G(x)$ if the second argument is given.

Q_G and the Tutte polynomial T_G satisfy the following relation (see [36] and [10, p. 110]):

$$Q_G(x)=(-1)^{|E|-|V|+\kappa(G)}\,T_G(0,1-x),$$

where $\kappa(G)$ is the number of connected components of G.

The value $Q_G(k)$, where $k>0$ is an integer, is equal to the number of all nowhere-zero k-flows in G. In such flows, the total flow f_v entering and leaving vertex v is congruent modulo k, hence $f_v\in\{1,2,\ldots,k-1\}$ for all $v\in V$ [11, p. 347]. As shown in the example below, Petersen graph has zero 4-flows and 240 5-flows.

```
> Q:=flow_polynomial(graph("petersen"))
```

$$x^6-15\,x^5+95\,x^4-325\,x^3+624\,x^2-620\,x+240$$

```
> Q | x=4
```

0

```
> Q | x=5
```

240

4.4.4 Reliability polynomial

`reliability_polynomial(G,⟨p⟩)`

The command `reliability_polynomial` is used for computing reliability polynomials. It takes one or two arguments, an undirected graph $G(V,E)$ and optionally a variable or value p. It returns the all-terminal reliability polynomial R_G of G or the value $R_G(p)$ if the second argument is given. If G is weighted, then it is treated as a multigraph: the weight w of an edge e, which must be a positive integer, is interpreted as the multiplicity of e, for each $e \in E$.

R_G and the Tutte polynomial T_G satisfy the following relation [56]:

$$R_G(p) = (1-p)^{|V|-\kappa(G)}\, p^{|E|-|V|+\kappa(G)}\, T_G(1, p^{-1}),$$

where $\kappa(G)$ is the number of connected components of G.

If G is connected, then the value $R_G(p)$, where $p \in [0,1]$, is equal to the probability that G does not fail (i.e. stays connected) after removing each edge with probability p [34, pp. 354–355].

In the following example, it is clear that the graph G will stay connected with probability $(1-p)^2$ if each of its two edges is removed with probability p.

```
> G:=graph(%{[1,2],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> R:=reliability_polynomial(G,p)
```

$$p^2 - 2\,p + 1$$

```
> factor(R)
```

$$(p-1)^2$$

Adding a new edge should increase the reliability of G, since the latter is connected. Indeed, the difference $S - R$ below is positive for $0 < p < 1$.

```
> S:=reliability_polynomial(add_edge(G,[1,3]),p)
```

$$2\,p^3 - 3\,p^2 + 1$$

```
> factor(S-R)
```

$$2\,p\,(p-1)^2$$

Multiple edges can be specified as edge weights.

```
> M:=graph(%{[[1,2],2],[[2,3],1],[[3,1],1]%})
```

an undirected weighted graph with 3 vertices and 3 edges

```
> factor(reliability_polynomial(M))
```

$$(x-1)^2\,(2\,x^2+2\,x+1)$$

The following graph represents the Arpanet (early internet) in December 1970.

```
> V:=["MIT","LINCOLN","CASE","CMU","HARVARD","BBN","UCSB","UCLA","STANFORD",
  "SRI","RAND","UTAH","SDC"]:;
```

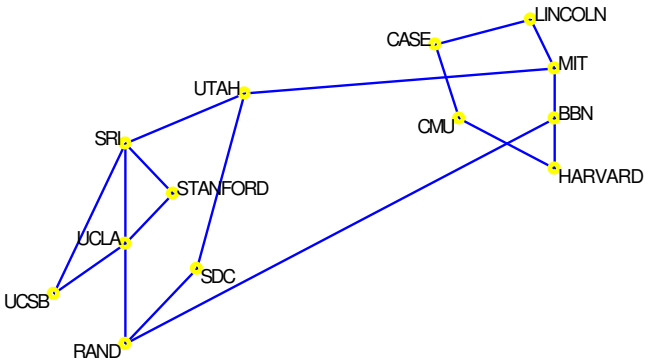
```
> A:=graph(V,trail("BBN","HARVARD","CMU","CASE","LINCOLN","MIT","UTAH","SRI",
  "STANFORD","UCLA","UCSB","SRI","UCLA","RAND","BBN","MIT"),trail("RAND","SDC",
  "UTAH"))
```

an undirected unweighted graph with 13 vertices and 17 edges

```
> Arpanet:=set_vertex_positions(A,[[1.0,1.0],[0.9,1.2],[0.5,1.1],[0.6,0.8],[1.0,
  0.6],[1.0,0.8],[-1.1,0.1],[-0.8,0.3],[-0.6,0.5],[-0.8,0.7],[-0.8,-0.1],[-0.3,
  0.9],[-0.5,0.2]])
```

an undirected unweighted graph with 13 vertices and 17 edges

```
> draw_graph(Arpanet)
```



If we were allowed to add a single edge to Arpanet in order to improve its reliability, which choice would be optimal? Below is the analysis for an edge from Stanford to CMU.

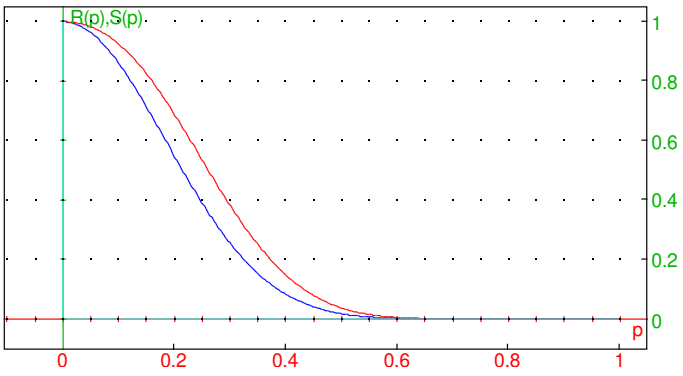
```
> R:=reliability_polynomial(Arpanet,p)
```

$$(p-1)^{12}\,(280\,p^5+310\,p^4+186\,p^3+63\,p^2+12\,p+1)$$

```
> S:=reliability_polynomial(add_edge(Arpanet,["STANFORD","CMU"]),p)
```

$$(p-1)^{12}\,(976\,p^6+1118\,p^5+703\,p^4+276\,p^3+72\,p^2+12\,p+1)$$

```
> labels=["p","R(p),S(p)"]; plot([R,S],p=0..1,color=[blue,red])
```



The improvement is defined as the area enclosed by the two curves.

```
> approx(integrate(S-R,p=0..1))
```

0.0419486688063

4.5 Connectivity

4.5.1 Connected, biconnected and triconnected graphs

```
is_connected(G)
is_biconnected(G)
is_triconnected(G)
```

The commands `is_connected`, `is_biconnected` and `is_triconnected` are used for determining if a graph is connected, biconnected or triconnected (3-connected), respectively. Each of these commands takes a graph $G(V, E)$ as its only argument and returns `true` if G possesses the required level of connectivity. Otherwise, it returns `false`.

If G is directed, the edge directions are simply ignored (the commands operate on the underlying graph of G).

The strategy for checking 1- and 2-connectivity is to use depth-first search (see [33, p. 20] and [73]). Both algorithms run in $O(|V| + |E|)$ time. The algorithm for checking 3-connectivity is, however, less efficient: it works by choosing a vertex $v \in V$ and checking if the subgraph induced by $V \setminus \{v\}$ is biconnected, moving on to the next vertex if so, and repeating the process until all vertices are visited exactly once or a non-biconnected subgraph is found for some v. In the latter case the input graph is not triconnected. The complexity of this algorithm is hence $O(|V||E|)$.

```
> G:=graph_complement(complete_graph(2,3,4))
```

an undirected unweighted graph with 9 vertices and 10 edges

```
> is_connected(G)
```

false

```
> C:=connected_components(G)
```

$[[0, 1], [2, 3, 4], [5, 6, 7, 8]]$

```
> H:=induced_subgraph(G,C[2])
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> is_connected(H)
```

true

```
> is_biconnected(path_graph(5))
```

false

```
> is_biconnected(cycle_graph(5))
```

true

```
> is_triconnected(graph("petersen"))
```

true

```
> is_triconnected(cycle_graph(5))
```

false

4.5.2 Connected and biconnected components

`connected_components(G)`
`biconnected_components(G)`

The commands `connected_components` and `biconnected_components` are used for decomposing a graph into connected and biconnected components, respectively. Both commands take a graph $G(V, E)$ as its only argument and return the minimal partition $\{V_1, V_2, ..., V_k\}$ of V such that the subgraph $G_i \subset G$ induced by V_i is connected resp. biconnected for each $i = 1, 2, ..., k$. The partition is returned as a list of lists $V_1, V_2, ..., V_k$.

If G is directed, the edge directions are simply ignored (the commands operate on the underlying graph of G).

The connected components of G are readily obtained by depth-first search in $O(|V| + |E|)$ time. To find the biconnected components of G, Tarjan's algorithm is used [73], which also runs in linear time.

```
> G:=graph_complement(complete_graph(3,5,7))
```

an undirected unweighted graph with 15 vertices and 34 edges

```
> is_connected(G)
```

false

```
> C:=connected_components(G)
```

$[[0, 1, 2], [3, 4, 5, 6, 7], [8, 9, 10, 11, 12, 13, 14]]$

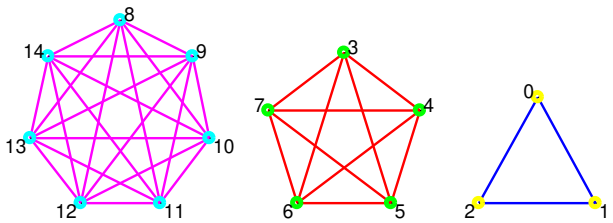
```
> G:=highlight_subgraph(G,induced_subgraph(G,C[1]))
```

an undirected unweighted graph with 15 vertices and 34 edges

```
> G:=highlight_subgraph(G,induced_subgraph(G,C[2]),magenta,cyan)
```

an undirected unweighted graph with 15 vertices and 34 edges

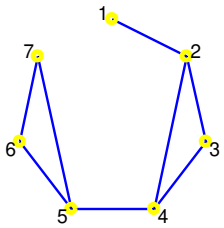
```
> draw_graph(G)
```




```
> H:=graph(trail(1,2,3,4,2),trail(4,5,6,7,5))
```

an undirected unweighted graph with 7 vertices and 8 edges

```
> draw_graph(H)
```




```
> is_biconnected(H)
```

false

```
> biconnected_components(H)
```

$[[5,6,7],[4,5],[2,3,4],[1,2]]$

4.5.3 Vertex connectivity

`vertex_connectivity(G)`

The command `vertex_connectivity` is used for computing vertex connectivity in undirected graphs. It takes an undirected connected graph $G(V,E)$ as its only argument and returns the largest integer k for which G is k-vertex-connected, meaning that G remains connected after removing fewer than k vertices from V.

The strategy is to use the algorithm by ESFAHANIAN and HAKIMI [25], which is based on the maximum-flow computing approach by EVEN [27, Section 6.2]. The algorithm makes $|V|-\delta-1+\frac{\delta(\delta-1)}{2}$ calls to `maxflow` command, where δ is the minimum vertex degree in G.

```
> vertex_connectivity(graph("petersen"))
```

3

```
> vertex_connectivity(graph("clebsch"))
```

5

```
> G:=random_planar_graph(1000,0.5,2)
```

an undirected unweighted graph with 1000 vertices and 1865 edges

```
> is_biconnected(G)
```

true

```
> vertex_connectivity(G)
```

2

2.582 sec

4.5.4 Graph rank

`graph_rank(G,⟨S⟩)`

The command `graph_rank` is used for computing graph rank. It takes one or two arguments, a graph $G(V,E)$ and optionally a set of edges $S\subset E$ (by default $S=E$), and returns $|V|-k$ where k is the number of connected components of the spanning subgraph of G with edge set S.

```
> G:=graph(%{[1,2],[3,4],[4,5]%})
```

an undirected unweighted graph with 5 vertices and 3 edges

```
> graph_rank(G)
```

3

```
> graph_rank(G,[[1,2],[3,4]])
```

2

4.5.5 Articulation points

`articulation_points(G)`

The command `articulation_points` is used for obtaining the set of articulation points (cut-vertices) of a graph. It takes a graph $G(V, E)$ as its only argument and returns the list of articulation points of G. A vertex $v \in V$ is an **articulation point** of G if the removal of v increases the number of connected components of G.

The articulation points of G are found by depth-first search in $O(|V| + |E|)$ time [33].

```
> articulation_points(path_graph([1,2,3,4]))
```

$[2, 3]$

```
> length(articulation_points(cycle_graph(1,2,3,4)))
```

0

4.5.6 Strongly connected components

`strongly_connected_components(G)`
`is_strongly_connected(G)`

The command `strongly_connected_components` is used for decomposing digraphs into strongly connected components. A digraph H is **strongly connected** if for each pair (v, w) of distinct vertices in H there is a directed path from v to w in H. The command `is_strongly_connected` can be used to determine whether a graph is strongly connected.

`strongly_connected_components` takes a digraph $G(V, E)$ as its only argument and returns the minimal partition $\{V_1, V_2, ..., V_k\}$ of V such that the subgraph $G_i \subset G$ induced by V_i is strongly connected for each $i = 1, 2, ..., k$. The result is returned as a list of lists $V_1, V_2, ..., V_k$.

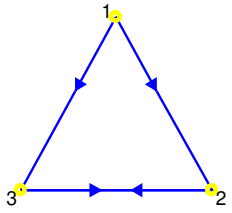
`is_strongly_connected` takes a digraph G as its only argument and returns `true` if G has exactly one strongly connected component and `false` otherwise.

The strategy is to use Tarjan's algorithm [73], which runs in $O(|V| + |E|)$ time.

```
> G:=digraph([1,2,3],%{[1,2],[1,3],[2,3],[3,2]%})
```

a directed unweighted graph with 3 vertices and 4 arcs

```
> draw_graph(G)
```




```
> is_connected(G)
```

true

```
> is_strongly_connected(G)
```

false

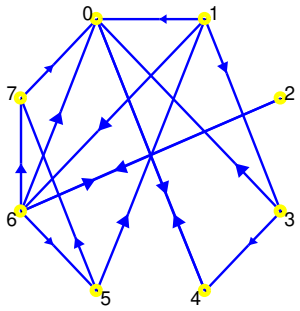
```
> strongly_connected_components(G)
```

$[[2,3],[1]]$

```
> G:=random_digraph(8,15)
```

a directed unweighted graph with 8 vertices and 15 arcs

```
> draw_graph(G)
```




```
> strongly_connected_components(G)
```

$[[0,4],[3],[7],[1,2,5,6]]$

4.5.7 Edge connectivity

`edge_connectivity(G)`

The command `edge_connectivity` is used for computing the edge connectivity of an undirected graph. It takes an undirected connected graph $G(V,E)$ as its only argument and returns the largest integer k for which G is k-edge connected, meaning that G remains connected after fewer than k edges are removed from E.

The strategy is to apply MATULA's algorithm [77, Section 13.3.1], which constructs a dominating set $D \subset V$ and calls `maxflow` command $|D|-1$ times.

```
> G:=cycle_graph([1,2,3,4,5])
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> edge_connectivity(G)
```

2

```
> K5:=complete_graph(5)
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> edge_connectivity(K5)
```

4

```
> edge_connectivity(graph("petersen"))
```

3

```
> edge_connectivity(graph("clebsch"))
```

5

4.5.8 Edge cuts

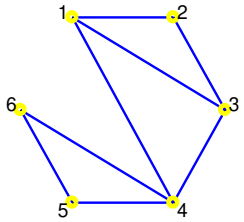
`is_cut_set(G,L)`

The command `is_cut_set` is used for determining whether a particular subset of edges of a graph is an edge cut. It takes two arguments, a graph $G(V, E)$ and a set $L \subset E$, and returns `true` if the graph $G'(V, E \setminus L)$ has more connected components than G. Otherwise, it returns `false`.

```
> G:=graph(trail(1,2,3,4,5,6,4,1,3))
```

an undirected unweighted graph with 6 vertices and 8 edges

```
> draw_graph(G)
```




```
> E:=[[1,4],[3,4]]
```

$$\begin{pmatrix} 1 & 4 \\ 3 & 4 \end{pmatrix}$$

```
> is_cut_set(G,E)
```

true

```
> is_connected(delete_edge(G,E))
```

false

4.5.9 Two-edge-connected graphs

`is_two_edge_connected(G)`
`two_edge_connected_components(G)`

The command `is_two_edge_connected` is used for determining whether an undirected graph is two-edge-connected. The command `two_edge_connected_components` is used for splitting a graph into components having this property. It takes an undirected graph $G(V, E)$ as its only argument and returns `true` if G has no bridges, i.e. edges which removal increases the number of connected components of G.

`two_edge_connected_components` takes an undirected graph $G(V, E)$ and returns the list of two-edge-connected components of G, each of them represented by the list of its vertices. To obtain a component as a graph, use the `induced_subgraph` command.

The strategy for finding bridges [74] is similar to finding articulation points. Once the bridges of G are found, it is easy to split G into two-edge-connected components by removing the bridges and returning the list of connected components of the resulting graph. Both algorithms run in $O(|V| + |E|)$ time.

```
> is_two_edge_connected(cycle_graph(4))
```

true

```
> is_two_edge_connected(path_graph(4))
```

false

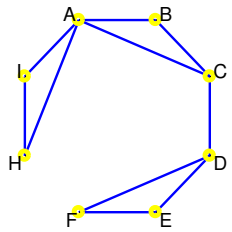
```
> purge(A,B,C,D,E,F,G,H,I):; G:=graph(%{[A,B],[B,C],[A,C],[D,E],[E,F],[D,F],[C,
  D],[A,H],[A,I],[H,I]%})
```

"Done", an undirected unweighted graph with 8 vertices and 10 edges

```
> is_two_edge_connected(G)
```

false

```
> draw_graph(G)
```

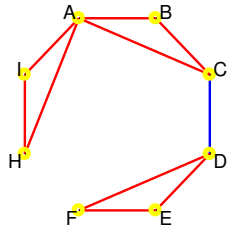



```
> comp:=two_edge_connected_components(G)
```

$$[[A, B, C, H, I], [D, E, F]]$$

To visualize the bridges of G, one can highlight the edges of each component. The non-highlighted edges are the bridges.

```
> for c in comp do G:=highlight_edges(G,edges(induced_subgraph(G,c))); od:;
> draw_graph(G)
```



4.6 Trees

4.6.1 Tree graphs

`is_tree(G)`

The command `is_tree` is used for determining whether a graph is a tree. It takes a graph $G(V, E)$ as its only argument and returns `true` if G is undirected, connected and $|V| = |E| + 1$. Otherwise, it returns `false`. The algorithm runs in $O(|V|)$ time.

```
> is_tree(complete_binary_tree(3))
```

true

```
> is_tree(cycle_graph(5))
```

false

4.6.2 Forest graphs

`is_forest(G)`

The command `is_forest` is used for determining whether a graph is a forest. It takes the a $G(V, E)$ as its only argument and returns `true` if each connected component of G is a tree. Otherwise, it returns `false`. The algorithm runs in $O(|V| + |E|)$ time.

```
> F:=disjoint_union(apply(random_tree,[k$(k=10..30)]))
```

an undirected unweighted graph with 420 vertices and 399 edges

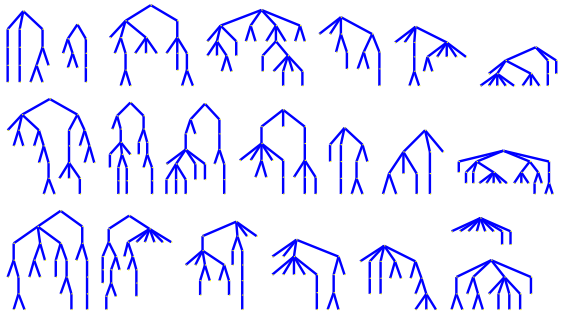
```
> is_connected(F)
```

false

```
> is_forest(F)
```

true

```
> draw_graph(F)
```



4.6.3 Height of a tree

`tree_height(G,⟨r⟩)`

The command `tree_height` is used for determining either the height of a tree with respect to the specified root node or the minimal height. It takes a tree graph $G(V, E)$ as its first argument and optionally a vertex $r \in V$, which is used as the root node. The command returns the the length of the longest path in T that has the node r as one of its endpoints. If r is not specified, the return value is a sequence comprised of the minimal tree height and an expression `root=r0`, where $r_0 \in V$ is the first vertex such that G has the minimal height with respect to r_0.

The strategy for given r is to start a depth-first search from the root node and look for the deepest node. Therefore the algorithm runs in $O(|V|)$ time. If r is not given, then r_0 is determined by pointers which move from leaf nodes inwards and get combined as they meet, until at most two of them remain, pointing to the admissible vertices. This algorithm requires $O(|V|)$ time as well.

```
> G:=random_tree(1000)
```

an undirected unweighted graph with 1000 vertices and 999 edges

```
> r:=choice(vertices(G))
```

```
> tree_height(G,r)
```

88

```
> tree_height(G)
```

46, root = 189

4.6.4 Prüfer sequences

`pruefer_code(L|T)`

The command `pruefer_code` is used for converting a tree into a Prüfer sequence and vice versa. It takes a Prüfer encoding L or a tree graph T as its only argument and returns a tree corresponding to L resp. the Prüfer encoding of T. The argument L must be a list of integers between 0 and $|L|+1$ (inclusive) in modes with 0-based indices (e.g. xcas mode) resp. between 1 and $|L|+2$ in modes with 1-based indices (e.g. maple mode). Encodings output by `pruefer_code` are also mode-aware.

The strategy is to use linear-time implementations[4.2] of the encoding/decoding algorithm [50].

The following examples were entered in xcas mode (0-based indices).

```
> T:=pruefer_code([5,6,0,4,1,2,10,3,8,11])
```

an undirected unweighted graph with 12 vertices and 11 edges

```
> is_tree(T)
```

true

```
> G:=graph(%{[1,8],[2,3],[2,7],[4,6],[5,6],[6,7],[6,8],[7,9],[8,10]%})
```

an undirected unweighted graph with 10 vertices and 9 edges

```
> is_tree(G)
```

true

```
> pruefer_code(G)
```

[1, 2, 4, 6, 6, 4, 6, 1]

```
> R:=random_tree(100)
```

an undirected unweighted graph with 100 vertices and 99 edges

```
> code:=pruefer_code(R):;
> is_isomorphic(R,pruefer_code(code))
```

true

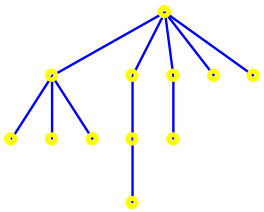
`pruefer_code` can be used for fast uniform random sampling of labeled trees. Namely, for $n>2$, each sequence of length $n-2$ in alphabet of n symbols corresponds to a unique labeled tree on n vertices. Thus the number of labeled trees on n vertices equals to n^{n-2} (also by Cayley's formula). It is hence enough to select a sequence at random and convert it to its associated tree by using `pruefer_code`. In the following example we generate a tree on 10 vertices.

4.2. See https://cp-algorithms.com/graph/pruefer_code.html.

```
> L:=randvector(10,12)
```

$$[8, 8, 2, 4, 2, 6, 3, 2, 2, 8]$$

```
> draw_graph(pruefer_code(L),labels=false)
```



In maple mode (or any other mode with 1-based indices) one should add 1 to each element of L before passing it to `pruefer_code`.

For generating unlabeled trees uniformly at random, see Section 1.10.3.

4.6.5 Lowest common ancestor of a pair of nodes

```
lowest_common_ancestor(G,r,u,v)
lowest_common_ancestor(G,r,[[u1,v1],[u2,v2],..,[uk,vk]])
```

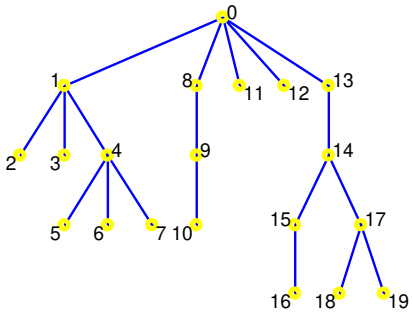
The command `lowest_common_ancestor` is used for computing the lowest common ancestor (LCA) of a pair of nodes in a tree or for each element of a list of such pairs. It takes two mandatory arguments, a tree graph $G(V, E)$ and the root node $r \in V$. There are two possibilities for specifying the nodes to operate on: either the nodes $u, v \in V$ are given as the third and the fourth argument, or a list of pairs of nodes $(u_1, v_1), (u_2, v_2), ..., (u_k, v_k)$, where $u_i, v_i \in V$ and $u_i \neq v_i$ for $i = 1, 2, ..., k$, is given as the third argument. The command returns the LCA of u and v or the list containing LCA of every pair of nodes u_i, v_i for $i = 1, 2, ..., k$. Note that this is much faster than calling `lowest_common_ancestor` k times with a single pair of vertices each time.

The strategy is to use Tarjan's offline LCA algorithm [75], which runs in nearly linear time.

```
> G:=random_tree(20)
```

an undirected unweighted graph with 20 vertices and 19 edges

```
> draw_graph(G)
```




```
> lowest_common_ancestor(G,0,16,17)
```

14

```
> lowest_common_ancestor(G,0,[[5,6],[7,8],[15,18]])
```

$$[4, 0, 14]$$

4.6.6 Arborescence graphs

`is_arborescence(G)`

The command `is_arborescence` is used for determining whether a directed unweighted graph is an arborescence (which is the digraph form of a rotted tree). It takes a digraph $G(V, E)$ as its only argument and returns `true` if there is a vertex $u \in V$ such that for any other $v \in V$ there is exactly one directed path from u to v. Otherwise, it returns `false`.

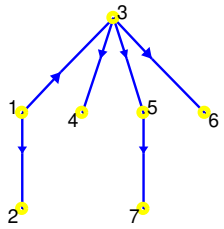
```
> T:=digraph(%{[1,2],[1,3],[3,4],[3,5],[3,6],[5,7]%})
```

a directed unweighted graph with 7 vertices and 6 arcs

```
> is_arborescence(T)
```

true

```
> draw_graph(T)
```



4.7 Networks

4.7.1 Network graphs

`is_network(G,⟨s,t⟩)`

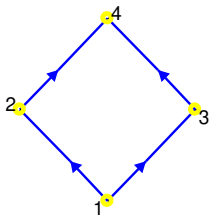
The command `is_network` is used for determining whether a graph is a flow network. In this context, a flow network is directed, connected graph with at least one vertex with in-degree 0 (the **source**) and at least one vertex with out-degree 0 (the **sink**).

`is_network` takes one or three arguments, a digraph $G(V, E)$ and optionally the source vertex s and the sink vertex t. If these vertices are given, then the command returns `true` if G is a network with respect to s and t, otherwise it returns `false`. If the graph G is given as the only argument, then the command returns a sequence of two objects, the list of all sources in G and the list of all sinks in G, respectively. If at least one of these lists is empty, then G is implicitly not a network (both lists are empty if G is not connected).

```
> N:=digraph(%{[1,2],[1,3],[2,4],[3,4]%})
```

a directed unweighted graph with 4 vertices and 4 arcs

```
> draw_graph(N,spring)
```



```
> is_network(N,1,4)
```

true

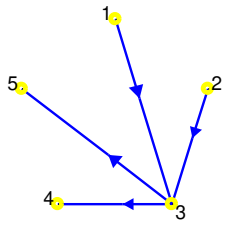
```
> is_network(N,2,3)
```

false

```
> G:=digraph(%{[1,3],[2,3],[3,4],[3,5]%})
```

a directed unweighted graph with 5 vertices and 4 arcs

```
> draw_graph(G,circle)
```




```
> is_network(G)
```

$[1, 2], [4, 5]$

4.7.2 Maximum flow

`maxflow(G,s,t,⟨M⟩)`

The command `maxflow` is used for computing the maximum flow in a network. It takes three or four arguments: a network graph $G(V, E)$, the source $s \in V$, the sink $t \in V$ and optionally an unassigned identifier `M`. It returns the optimal value for the maximum flow problem for the network (G, s, t). If the fourth argument is given, then an optimal flow is assigned to `M` in form of a matrix.

The strategy is to use the algorithm of Edmonds and Karp [24], which solves the maximum flow problem in $O(|V|\,|E|^2)$ time.

```
> A:=[[0,1,0,4,0,0],[0,0,1,0,3,0],[0,1,0$3,1],[0,0,3,0,1,0],[0$3,1,0,4],[0$6]]
```

$$\begin{pmatrix} 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

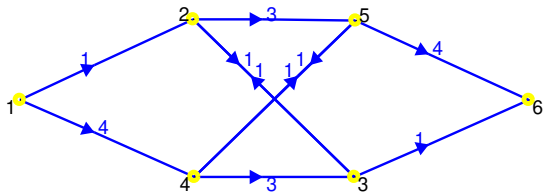
```
> N:=digraph([1,2,3,4,5,6],A)
```

a directed weighted graph with 6 vertices and 10 arcs

```
> is_network(N)
```

$[1], [6]$

```
> draw_graph(N,spring)
```




```
> purge(M):; maxflow(N,1,6,M)
```

"Done", 4

```
> M
```

$$\begin{pmatrix} 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

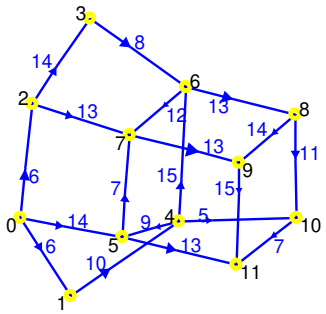
```
> N:=random_network(2,3,0.9,acyclic,weights=5..15)
```

a directed weighted graph with 12 vertices and 19 arcs

```
> is_network(N)
```

[0], [11]

```
> purge(pos):; draw_graph(N,spring,pos)
```




```
> purge(M):; maxflow(N,0,11,M)
```

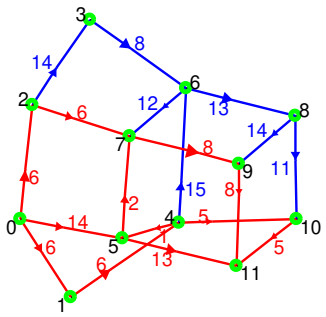
"Done", 26

To visualize the optimal flow F, one can use the `highlight_subgraph` command with the option `weights` to display the actual flow in the highlighted edges. Non-highlighted edges have zero flow. Note that we use vertex positions generated in the above drawing.

```
> flow:=set_vertex_positions(highlight_subgraph(N,digraph(vertices(N),M),
  weights),pos)
```

a directed weighted graph with 12 vertices and 19 arcs

```
> draw_graph(flow)
```



4.7.3 Minimum cut

`minimum_cut(G,s,t)`

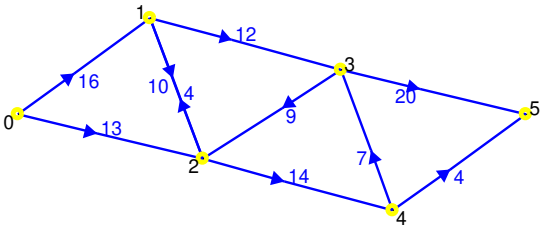
The command `minimum_cut` is used for obtaining minimum cuts in networks. It takes three arguments, a digraph $G(V,E)$ and two vertices $s,t \in V$ such that (G,s,t) is a network with source s and sink t. The returned value is a list of edges in E representing a minimum cut in the network.

The strategy is to apply the command `maxflow`, which finds a maximal flow, and to run depth-first search on the corresponding residual graph to find a S,T partition of V. The minimum cut is then the set of all arcs $vw \in E$ such that $v \in S$ and $w \in T$. The algorithm runs in $O(|V|\,|E|^2)$ time.

```
> G:=digraph(%{[[0,1],16],[[0,2],13],[[1,2],10],[[1,3],12],[[2,1],4],[[2,4],14],
  [[3,2],9],[[3,5],20],[[4,3],7],[[4,5],4]%})
```

a directed weighted graph with 6 vertices and 10 arcs

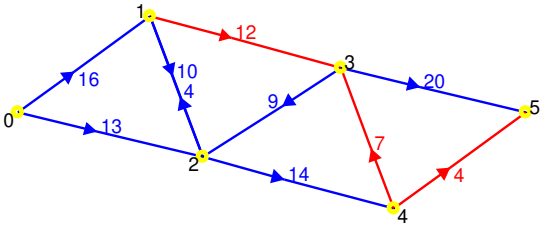
```
> draw_graph(G,spring)
```




```
> cut:=minimum_cut(G,0,5)
```

$$[[1,3],[4,3],[4,5]]$$

```
> draw_graph(highlight_edges(G,cut),spring)
```



By the max-flow min-cut theorem, the sum of edge weights in minimum cut is equal to the value of maximum flow.

```
> w:=0:; for ed in cut do w:=w+get_edge_weight(G,ed); od:; w
```

"Done", "Done", 23

```
> maxflow(G,0,5)
```

23

4.8 Distance in graphs

4.8.1 Vertex distance

`vertex_distance(G,v,w|L)`

The command `vertex_distance` is used for computing the length of the shortest path(s) from the source vertex to some other vertex/vertices of a graph. It takes three arguments, a graph $G(V,E)$, a vertex $v \in V$ called the **source** and a vertex $w \in V$ called the **target** or a list $L \subset V \setminus \{v\}$ of target vertices. The command returns the distance between v and w as the number of edges in a shortest path from v to w, or the list of distances if a list of target vertices is given.

The strategy is to start a breadth-first search [33, p. 35] from the source vertex. Therefore, the algorithm runs in $O(|V|+|E|)$ time.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> vertex_distance(G,1,3)
```

2

```
> vertex_distance(G,1,[3,6,9])
```

$[2, 1, 2]$

4.8.2 All-pairs vertex distance

`allpairs_distance(G)`

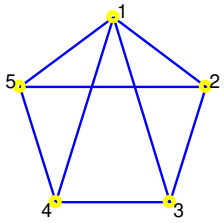
The command `allpairs_distance` is used for computing the matrix of distances between all pairs of vertices in a (weighted) graph. It takes a graph $G(V,E)$ as its only argument and returns a square matrix $D = [d_{ij}]$ with $n = |V|$ rows and columns such that $d_{ij} = \text{distance}(v_i, v_j)$ for all i, $j = 1, 2, ..., n$, where $v_1, v_2, ..., v_n$ are the elements of V. If $v_i v_j \notin E$, then $d_{ij} = +\infty$. The strategy is to apply the algorithm of FLOYD and WARSHALL [28], which runs in $O(|V|^3)$ time.

Note that, if G is weighted, it must not contain negative cycles. A cycle is **negative** if the sum of weights of its edges is negative.

```
> G:=graph([1,2,3,4,5],%{[1,2],[1,3],[1,4],[1,5],[2,3],[3,4],[4,5],[5,2]%})
```

an undirected unweighted graph with 5 vertices and 8 edges

```
> draw_graph(G)
```




```
> allpairs_distance(G)
```

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix}$$

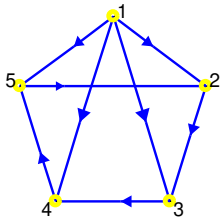
```
> H:=digraph(%{[1,2],[1,3],[1,4],[1,5],[2,3],[3,4],[4,5],[5,2]%})
```

a directed unweighted graph with 5 vertices and 8 arcs

```
> allpairs_distance(H)
```

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ +\infty & 0 & 1 & 2 & 3 \\ +\infty & 3 & 0 & 1 & 2 \\ +\infty & 2 & 3 & 0 & 1 \\ +\infty & 1 & 2 & 3 & 0 \end{pmatrix}$$

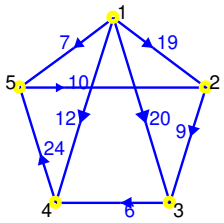
```
> draw_graph(H)
```




```
> H:=assign_edge_weights(H,5,25)
```

a directed weighted graph with 5 vertices and 8 arcs

```
> draw_graph(H)
```




```
> allpairs_distance(H)
```

$$\begin{pmatrix} 0 & 17 & 20 & 12 & 7 \\ +\infty & 0 & 9 & 15 & 39 \\ +\infty & 40 & 0 & 6 & 30 \\ +\infty & 34 & 43 & 0 & 24 \\ +\infty & 10 & 19 & 25 & 0 \end{pmatrix}$$

4.8.3 Diameter

`graph_diameter(G)`

The command `graph_diameter` is used for determining the maximum distance among all pairs of vertices in a graph. It takes a graph $G(V, E)$ as its only argument and returns the maximum distance between two nodes in V, i.e. the cost of a maximum-size shortest path in G if the latter is connected and $+\infty$ otherwise.

If G is weighted, then `graph_diameter` calls `allpairs_distance` and picks the largest element in the output matrix. Thus the complexity of the algorithm is $O(|V|^3)$ in this case. For unweighted graphs it is sufficient to run a breadth-first search from each vertex, which requires $O(|V||E|)$ time.

```
> graph_diameter(graph("petersen"))
```

2

```
> graph_diameter(cycle_graph(19))
```

9

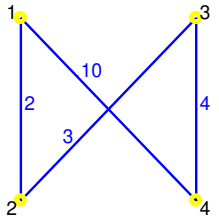
```
> graph_diameter(graph_complement(complete_graph(3,3)))
```

$+\infty$

```
> G:=graph(%{[[1,2],2],[[2,3],3],[[3,4],4],[[4,1],10]%})
```

an undirected weighted graph with 4 vertices and 4 edges

```
> draw_graph(G)
```




```
> graph_diameter(G)
```

9

```
> dijkstra(G,1,4)
```

$[[1, 2, 3, 4], 9]$

4.8.4 Girth

`girth(G)`
`odd_girth(G)`

The commands `girth` and `odd_girth` are used for computing the (odd) girth of an undirected unweighted graph. Both commands take a graph $G(V, E)$ as their only argument and return the girth and odd girth of G, respectively. The (odd) girth of G is defined to be the length of the shortest (odd) cycle in G. If there is no (odd) cycle in G, then the command returns $+\infty$.

The strategy is to apply breadth-first search from each vertex of the input graph. The runtime is therefore $O(|V||E|)$.

```
> girth(graph("petersen"))
```

5

```
> G:=hypercube_graph(3)
```

an undirected unweighted graph with 8 vertices and 12 edges

```
> G:=subdivide_edges(G,["000","001"])
```

an undirected unweighted graph with 9 vertices and 13 edges

```
> girth(G)
```

4

```
> odd_girth(G)
```

5

```
> girth(complete_binary_tree(2))
```

$+\infty$

4.9 Acyclic graphs

4.9.1 Acyclic graphs

`is_acyclic(G)`

The command `is_acyclic` is used for determining whether there are no directed cycles (closed paths) in a digraph. A directed graph with no directed cycle is said to be **acyclic**. `is_acyclic` takes a digraph $G(V, E)$ as its only argument and returns `true` if G is acyclic and `false` otherwise.

The algorithm attempts to find topological order for its vertices. If that succeeds, then the graph is acyclic, otherwise not. The running time is $O(|V| + |E|)$.

```
> is_acyclic(digraph(trail(1,2,3,4,5)))
```

true

```
> is_acyclic(digraph(trail(1,2,3,4,5,2)))
```

false

4.9.2 Topological sorting

`topologic_sort(G)`
`topological_sort(G)` alias

The command `topologic_sort` (alias `topological_sort`) is used for finding a linear ordering of vertices of an acyclic digraph which is consistent with the arcs of the digraph. This procedure is called topological sorting. `topologic_sort` takes a graph $G(V, E)$ as its only argument and returns the list of vertices of G in a particular order: a vertex u precedes a vertex v if $(u, v) \in E$.

Note that topological sorting is possible only if the input graph is acyclic. If this condition is not met, then `topologic_sort` returns an error. Otherwise, it finds the required ordering by applying Kahn's algorithm [47], which runs in $O(|V| + |E|)$ time.

```
> purge(a,b,c,d):; G:=digraph(%{[c,a],[c,b],[c,d],[a,d],[b,d],[a,b]%})
```

"Done", a directed unweighted graph with 4 vertices and 6 arcs

```
> is_acyclic(G)
```

true

```
> topologic_sort(G)
```

$[c, a, b, d]$

4.9.3 st ordering

`st_ordering(G,s,t,⟨p⟩)`
`st_ordering(G,s,t,D,⟨p⟩)`

The command `st_ordering` is used for finding st-orderings in undirected biconnected graphs. It takes three to five arguments. The first three arguments are mandatory: an undirected biconnected[4.3] graph $G(V, E)$, a vertex $s \in V$ called the source, a vertex $t \in V$ called the sink such that $st \in E$. Optionally, one can pass an unassigned identifier `D` and/or a real value $p \in [0, 1]$. The command returns the permutation of the set V which corresponds to a st-numbering of the vertices.

Given a st-numbering, an orientation of each $e = uv \in E$ can be imposed by comparing the ordinals n and m assigned its endpoints u and v, respectively; if $n < m$, then u is the head and v is the tail of the corresponding arc, and vice versa otherwise. If an unassigned identifier `D` is provided, then a copy of G, which is made directed according to these orientations, is assigned to `D`. The oriented variant of G is an acyclic graph (or DAG for short).

If the argument p is not specified, the strategy is to apply TARJAN's algorithm [76] which runs in $O(|V| + |E|)$ time. When $p \in [0, 1]$ is given, a parametrized st-ordering is computed, in which the length of the longest path from s to t in the respective DAG roughly corresponds to $p\,|V|$. Thus by varying p one controls the length of the longest directed path from s to t. The parametrized variant is implemented according to PAPAMANTHOU and TOLLIS [63] and runs in $O(|V|\,|E|)$ time.

```
> purge(a,b,c,d):; G:=graph(%{[a,b],[a,c],[a,d],[b,c],[b,d],[c,d]%})
```

"Done", an undirected unweighted graph with 4 vertices and 6 edges

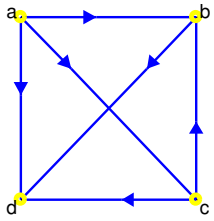
```
> vertices(G)
```

$[a, b, c, d]$

```
> st_ordering(G,a,d,D)
```

$[0, 2, 1, 3]$

```
> draw_graph(D)
```



4.3. For a biconnected input graph, an st-ordering can be computed for any pair $s, t \in V$ such that $st \in E$ [76].

The following program demonstrates the usage of the parametrized st-ordering algorithm for finding a path between vertices u and v in an undirected, biconnected graph $G(V,E)$, the length of which depends on the parameter $p \in [0,1]$.

```
FindPath:=proc(G,u,v,p)
  local tmp,D,W;
  tmp:=!has_edge(G,[u,v]);
  if tmp then G:=add_edge(G,[u,v]); fi;
  purge(D);
  st_ordering(G,u,v,D,p);
  if tmp then D:=delete_arc(D,[u,v]); fi;
  W:=is_weighted(G) ? weight_matrix(G) : adjacency_matrix(G);
  D:=make_weighted(D,-W);
  return bellman_ford(D,u,v)[0];
end:;
```

The procedure `FindPath` uses Bellman-Ford algorithm to find a longest path from the vertex $u \in V$ to the vertex $v \in V$ in the DAG D induced by a parametrized st-ordering of G with parameter p. To trick Bellman-Ford into finding a longest path instead of the shortest one (which it was originally designed for), the edges of D are weighted with negative weights. Since D is acyclic, it contains no negative cycles, so the Bellman-Ford algorithm terminates successfully.

For $p=0$ one obtains a relatively short path, but usually not a minimal one. For $p=1$ one obtains near-Hamiltonian paths. For $p \in (0,1)$, a path of length l which obeys the relation

$$l \approx l_0 + p\,(|V| - l_0),$$

where l_0 is the average path length for $p=0$, is obtained.

After compiling the above program in Xcas (by copying it into a programmation cell which we create by pressing `Alt+P`), we demonstrate it in the following examples.

```
> G:=graph("soccerball")
```

an undirected unweighted graph with 60 vertices and 90 edges

```
> length(FindPath(G,3,33,0))
```

12

```
> length(FindPath(G,3,33,0.5))
```

39

```
> length(FindPath(G,3,33,1))
```

59

4.9.4 Graph condensation

`condensation(G)`

The command `condensation` is used for constructing the condensation graph of a given digraph. It takes a digraph $G(V,E)$ as its only argument and returns the acyclic digraph $G'(V',E')$ obtained by contracting all strongly connected components of G to single vertices. These vertices form the set V'. Two vertices $u, v \in V'$ are connected in G' if and only if there is an arc in G with tail in the component of G corresponding to u and head in the component of G corresponding to v. The order of vertices in V' is the same as the order of components as returned by the command `strongly_connected_components`.

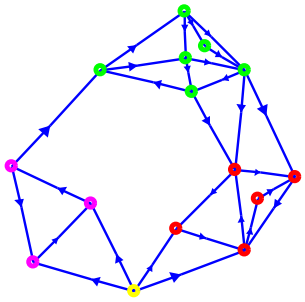
```
> E:=tran([[1,2,3,3,4,4,4,4,5,5,6,6,6,7,8,8,9,9,10,10,10,11,11,12,13,13,14,15],
  [3,1,2,5,1,2,12,13,6,8,7,8,10,10,9,10,5,11,9,11,14,12,14,13,11,15,13,14]]):;
  G:=digraph(set[op(E)])
```

"Done", a directed unweighted graph with 15 vertices and 28 arcs

```
> comp:=strongly_connected_components(G)
```

$$[[11, 12, 13, 14, 15], [5, 6, 7, 8, 9, 10], [1, 2, 3], [4]]$$

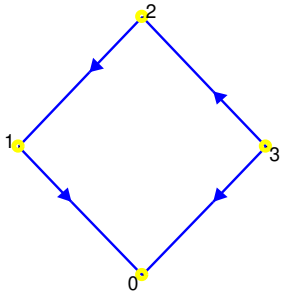
```
> c:=[red,green,magenta]:;
  for k from 0 to 2 do G:=highlight_vertex(G,comp[k],c[k]); od:;
  draw_graph(G,planar,labels=false)
```



```
> C:=condensation(G)
```

a directed unweighted graph with 4 vertices and 4 arcs

```
> draw_graph(C,spring)
```



4.10 Matching in graphs

4.10.1 Maximum matching

`maximum_matching(G)`

The command `maximum_matching` is used for finding maximum matchings [34, p. 43] in undirected unweighted graphs. It takes an undirected graph $G(V, E)$ as its only argument and returns a list of edges $e_1, e_2, ..., e_m \in E$ such that e_i and e_j are not adjacent (i.e. have no common endpoints) for all $1 \leqslant i < j \leqslant m$ and that m is maximal. The return value can be interpreted as the list of matched pairs of vertices in G.

The strategy is to apply the blossom algorithm of EDMONDS [23], which runs in $O(|V|^2 |E|)$ time, on connected components of G (one at a time). The implementation combines an efficient initial heuristic with implicit blossom-shrinking.

```
> maximum_matching(graph("octahedron"))
```

$$[[0,4],[1,3],[2,5]]$$

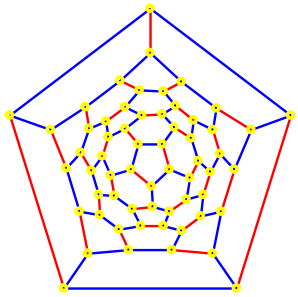
```
> G:=graph("soccerball")
```

an undirected unweighted graph with 60 vertices and 90 edges

```
> M:=maximum_matching(G):; length(M)
```

"Done", 30

```
> draw_graph(highlight_edges(G,M),labels=false)
```



```
> G:=random_graph(10000,0.02)
```

an undirected unweighted graph with 10000 vertices and 998282 edges

```
> size(maximum_matching(G))
```

5000

1.592 sec

4.10.2 Maximum matching in bipartite graphs

`bipartite_matching(G)`
`bipartite_matching(G,minimize|maximize,⟨epsilon=<real>⟩)`

The command `bipartite_matching` is used for finding maximum (weighted) matchings in undirected bipartite graphs. It takes an undirected bipartite graph $G(V, E)$ as its mandatory argument and returns the maximum matching in G as a list of edges. The strategy is to apply the algorithm of Hopcroft and Karp [44] which runs in $O\left(\sqrt{|V|}\,|E|\right)$ time.

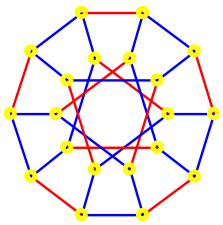
The optional second argument can be either `minimize` or `maximize`. If one is given, then G must be a weighted graph. In that case, `bipartite_matching` finds a minimum/maximum weighted matching (i.e. solves an assignment problem) using the out-of-kilter algorithm implementation in GLPK. If the latter is not available, then the task is converted to a transportation problem and solved by the `tpsolve` command. Note that negative edge weights are allowed. The return value is a sequence containing the weight of the matching followed by the list of edges in the matching.

The out-of-kilter algorithm requires all edge weights in G to be integer-valued. If that is not the case, then `bipartite_matching` divides the weights by the value `epsilon` set by the optional third argument (by default 10^{-5}) and rounds them subsequently.

```
> G:=graph("desargues")
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> draw_graph(highlight_edges(G,bipartite_matching(G)),labels=false)
```



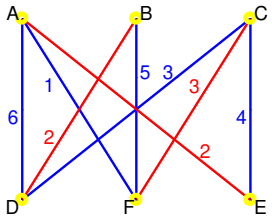
```
> purge(A,B,C,D,E,F):; G:=graph(%{[[A,D],6],[[A,E],2],[[A,F],1],[[B,D],2],[[B,
  F],5],[[C,D],3],[[C,E],4],[[C,F],3]%})
```

"Done", an undirected weighted graph with 6 vertices and 8 edges

```
> w_min,ed_min:=bipartite_matching(G,minimize)
```

$$7, [[A, E], [B, D], [C, F]]$$

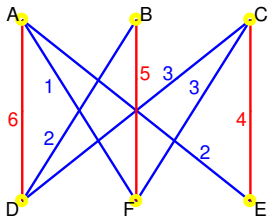
```
> draw_graph(highlight_edges(G,ed_min))
```




```
> w_max,ed_max:=bipartite_matching(G,maximize)
```

$$15, [[A, D], [B, F], [C, E]]$$

```
> draw_graph(highlight_edges(G,ed_max))
```




```
> R:=random_bipartite_graph(5000,20000)
```

an undirected unweighted graph with 5000 vertices and 20000 edges

```
> size(bipartite_matching(R))
```

1417

259 msec

```
> w,ed:=bipartite_matching(assign_edge_weights(R,1..100),maximize):;
```

16.508 sec

```
> size(ed)
```

1417

4.11 Vertex covers

4.11.1 Finding a vertex cover of the specified size

`find_vertex_cover(G,⟨k⟩)`

The command `find_vertex_cover` is used for obtaining a vertex cover (optionally of the given size) of an undirected graph. It takes an undirected graph $G(V,E)$ as its first argument and returns a vertex cover of G, i.e. a set of vertices C such that each edge $e \in E$ has an endpoint in C. If the optional second argument $k \geqslant 0$ is given, then size of the returned cover must be equal to k. If there exists no such cover, then the return value is `false`.

The strategy for $k \geqslant 0$ is to compute an approximation of minimal vertex cover in G using the algorithm described in Section 4.11.2. If the obtained cover is of size $m \leqslant k$, then $k-m$ vertices among the remaining ones are added to the cover at random. Otherwise, if $m > k$, then a cover of size k is obtained by solving the ILP formulation of the problem.

If k is not specified, `find_vertex_cover` approximates minimum vertex cover by using Alon's algorithm [64] which runs in $O(|V|^2)$ time and tries to minimize the number of vertices in the cover. Cover vertices are gathered one at a time, each time choosing a vertex with the maximum incident edges. Ties are broken by randomly choosing a candidate vertex with the smallest number of edges connecting it to other candidates. After a vertex is chosen, it is added to the cover and all edges incident to it are deleted. This procedure is repeated until there are no more edges in the graph.

```
> G:=random_graph(1000,0.1)
```

an undirected unweighted graph with 1000 vertices and 49981 edges

```
> length(find_vertex_cover(G))
```

956

```
> H:=random_graph(100,0.1)
```

an undirected unweighted graph with 100 vertices and 476 edges

```
> size(find_vertex_cover(H))
```

75

```
> size(minimum_vertex_cover(H))
```

68

4.11.2 Minimum vertex cover

`minimum_vertex_cover(G,⟨approx⟩)`
`vertex_cover_number(G)`

The command `minimum_vertex_cover` is used for finding minimum vertex covers of undirected graphs. The command `vertex_cover_number` is convenient when only the cardinality of a minimum cover is sought. Both commands take an undirected graph $G(V,E)$ as their mandatory argument and return a minimum vertex cover (MVC) $C \subseteq V$ of G resp. the cardinality $|C|$. The routine for computing MVC in Giac operates as follows. First, G is splitted into connected components. Then MVC is found for each component separately. The union of the obtained covers is a MVC for G. There are three different kinds of components recognized by the algorithm.

1. If component $C(V_C, E_C)$ is a tree, cyclic, or clique graph, then its MVC is straightforward to construct. This requires $O(|V_C|)$ time.

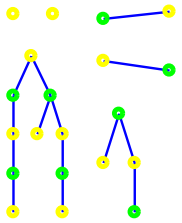
2. If C is bipartite, then its MVC can be obtained by applying Kőnig's theorem which obtains MVC from maximum matching M (`vertex_cover_number` simply computes the size of M). This requires $O\big(\sqrt{|V_C|}\,|E_C|\big)$ time.
3. If C is neither a tree, cyclic, clique, or bipartite graph, then solving the integer linear programming (ILP) formulation of the MVC problem is attempted. The inclusion of a vertex $v \in V$ is represented by a binary variable $x_v \in \{0,1\}$. The objective to be minimized is $\sum_{v \in V} x_v$ and the constraints are $x_v + x_w \geqslant 1$ for each $vw \in E$. The branching and reduction techniques by AKIBA and IWATA [3] are used in the process. Note that MVC problem is NP-hard, which means that exponential time is required for certain choices of G.

If the optional second argument `approx` is provided, then the solution in the case 3 is approximated as follows. First, an approximately maximum independent set M in the component C induced by $V_C \subseteq V$ is computed by applying the greedy algorithm described in Section 4.12.5. The complement $V_C \setminus M$ is then returned as an approximate MVC of C. This strategy usually gives very good (near-optimal) results in a short amount of time. It is also used by the exact solver for obtaining an initial feasible solution.

```
> G:=random_graph(20,0.1)
```

an undirected unweighted graph with 20 vertices and 14 edges

```
> draw_graph(highlight_vertex(G,minimum_vertex_cover(G)),labels=false)
```



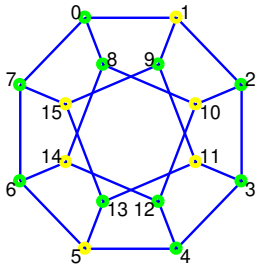
```
> P:=petersen_graph(8,2)
```

an undirected unweighted graph with 16 vertices and 24 edges

```
> C:=minimum_vertex_cover(P)
```

$[0, 2, 3, 4, 6, 7, 8, 9, 12, 13]$

```
> draw_graph(highlight_vertex(P,C))
```




```
> R:=random_graph(100,0.15)
```

an undirected unweighted graph with 100 vertices and 729 edges

```
> size(minimum_vertex_cover(R,approx))
```

```
> vertex_cover_number(R)
```

75

6.437 sec

4.12 Cliques and independent sets

4.12.1 Clique graphs

`is_clique(G)`

Given an undirected graph $G(V, E)$, a subset $S \subset V$ is called a **clique** in G if any two distinct vertices from S are adjacent in G, i.e. if the subgraph of G induced by the set S is complete. To check whether an undirected graph is a clique, one can use the `is_clique` command. It takes an undirected graph $G(V, E)$ as its only argument and returns `true` if every pair of distinct vertices is connected by a unique edge in E. Otherwise, it returns `false`.

```
> K5:=complete_graph(5)
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> is_clique(K5)
```

true

```
> G:=delete_edge(K5,[1,2])
```

an undirected unweighted graph with 5 vertices and 9 edges

```
> is_clique(G)
```

false

4.12.2 Finding maximal cliques

`find_cliques(G,⟨C⟩)`
`find_cliques(G,k,⟨C⟩)`
`find_cliques(G,m..n,⟨C⟩)`

A clique in an undirected graph $G(V, E)$ is **maximal** if it cannot be extended by adding more vertices from V to it. To count all maximal cliques in a graph (and optionally list them) one can use the `find_cliques` command. It takes an undirected graph $G(V, E)$ as the mandatory first argument. If no other arguments are given, the command returns a list of pairs, each pair consisting of two integers: clique cardinality k and the number $n_k > 0$ of k-cliques in G, respectively. (Therefore, the sum of second members of all returned pairs is equal to the total count of all maximal cliques in G.) If two arguments are passed to `find_cliques`, the second argument must be a positive integer k or an interval with integer bounds m .. n. In the first case the number of k-cliques is returned; in the second case, only cliques with cardinality between m and n (both inclusive) are counted.

If `C` is specified as the last argument, it must be an unassigned identifier. Maximal cliques are in that case assigned to `C` as a list of lists of cliques of equal size. This option is therefore used for listing all maximal cliques.

The strategy used to find all maximal cliques is a variant of the algorithm of Bron and Kerbosch developed by Tomita et al. [78]. Its worst-case running time is $O(3^{|V|/3})$. However, the algorithm is usually very fast, typically taking only a moment for graphs with few hundred vertices or less.

```
> G:=sierpinski_graph(3,3)
```

an undirected unweighted graph with 27 vertices and 39 edges

```
> find_cliques(G)
```

$$\begin{pmatrix} 2 & 12 \\ 3 & 9 \end{pmatrix}$$

```
> G:=random_graph(100,0.5)
```

an undirected unweighted graph with 100 vertices and 2507 edges

```
> find_cliques(G,5)
```

3529

```
> G:=random_graph(500,0.25)
```

an undirected unweighted graph with 500 vertices and 31163 edges

```
> find_cliques(G,5..7)
```

$$\begin{pmatrix} 5 & 149442 \\ 6 & 17105 \\ 7 & 349 \end{pmatrix}$$

1.457 sec

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> purge(C):; find_cliques(G,C)
```

"Done", (3 8)

```
> C
```

$[[0, 2, 4], [0, 2, 5], [0, 3, 4], [0, 3, 5], [1, 2, 4], [1, 2, 5], [1, 3, 4], [1, 3, 5]]$

4.12.3 Maximum clique

`maximum_clique(G)`
`clique_number(G)`

Any largest maximal clique in an undirected graph is called **maximum clique**. The command `maximum_clique` can be used to find one in a graph. If only the size of a maximum clique is desired, one can use the command `clique_number`.

`maximum_clique` takes an undirected graph G as its only argument and returns a maximum clique in G as a list of vertices. The clique may subsequently be extracted from G using the command `induced_subgraph`.

The strategy used to find maximum clique is an improved variant of the classical algorithm of CARRAGHAN and PARDALOS developed by ÖSTERGÅRD [60].

`clique_number` has the same calling syntax, but returns only the size of a maximum clique in G.

```
> G:=sierpinski_graph(5,5)
```

an undirected unweighted graph with 3125 vertices and 7810 edges

```
> maximum_clique(G)
```

$[1560, 1561, 1562, 1563, 1564]$

```
> G:=random_graph(300,0.3)
```

an undirected unweighted graph with 300 vertices and 13603 edges

```
> maximum_clique(G)
```

$[53, 93, 103, 179, 183, 224, 277, 289]$

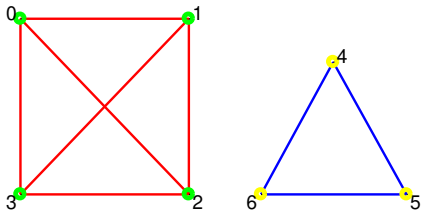
```
> G:=graph_complement(complete_graph(4,3))
```

an undirected unweighted graph with 7 vertices and 9 edges

```
> cliq:=maximum_clique(G)
```

$[0, 1, 2, 3]$

```
> draw_graph(highlight_subgraph(G,induced_subgraph(G,cliq)))
```



`clique_number` takes an undirected graph G as its only argument and returns the number of vertices forming a maximum clique in G.

```
> clique_number(G)
```

4

4.12.4 Maximum independent set

`maximum_independent_set(G)`
`independence_number(G)`

The command `maximum_independent_set` is used for finding maximum independent sets in undirected graphs. For obtaining just the size of a maximum independent set use the command `independence_number`.

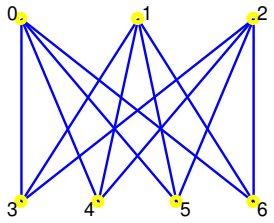
`maximum_independent_set` takes an undirected graph G as its only argument and finds a maximum clique in the complement of G (see Section 4.12.3), which corresponds to a maximum independent set in G.

`independence_number` has the same calling syntax, but returns only the size of a maximum independent set in G.

```
> G:=complete_graph(3,4)
```

an undirected unweighted graph with 7 vertices and 12 edges

```
> draw_graph(G)
```




```
> maximum_independent_set(G)
```

$[3, 4, 5, 6]$

```
> independence_number(G)
```

4

```
> maximum_clique(graph_complement(G))
```

$[3, 4, 5, 6]$

4.12.5 Greedy clique finding

`greedy_clique(G,⟨n⟩)`
`greedy_independent_set(G,⟨n⟩)`

The commands `greedy_clique` and `greedy_independent_set` are used for finding large cliques and independent sets in undirected graphs.

`greedy_clique` takes an undirected graph $G(V, E)$ as its mandatory first argument and an integer $n \geqslant 2$ as the optional second argument (by default, $n = 5$). It attempts to find a large clique in G, returning the list of corresponding vertex labels.

The strategy is to apply a greedy randomized adaptive search procedure (GRASP) tailored for the local maximum clique problem as described by ABELLO and PARDALOS in [1, procedure `grasp`, pp. 4–5] with `maxitr`$=n$ (the number of iterations).

`greedy_independent_set` attempts to find a large independent set of the graph G, returning the list of corresponding vertex labels. The strategy is to call `greedy_clique` on the complement of G.

```
> G1:=random_graph(1500,0.75)
```

an undirected unweighted graph with 1500 vertices and 843214 edges

```
> size(c1:=greedy_clique(G1))
```

27

1.512 sec

```
> is_clique(induced_subgraph(G1,c1))
```

true

It can be shown that the expected number of cliques of size k in G_1 is equal to $\binom{1000}{k} 0.75^{\binom{k}{2}}$. This number gets below 1 for about $k = 34$, which indicates that a maximum clique should be of size around 34.

In the following example, the same is tried with a large sparse graph, now running 50 iterations instead of the default 5.

```
> G2:=random_graph(3000,0.01)
```

an undirected unweighted graph with 3000 vertices and 44882 edges

```
> c2:=greedy_clique(G2,50)
```

$$[241, 350, 1216, 2116]$$

```
> is_clique(induced_subgraph(G2,c2))
```

true

By the same argument as before, but now with the formula $\binom{3000}{k} 0.01^{\binom{k}{2}}$, it is suggested that the maximum clique in G_2 should be of size about 4.

In the next example a random regular graph is used.

```
> G3:=random_regular_graph(100,30)
```

an undirected unweighted graph with 100 vertices and 1500 edges

```
> greedy_clique(G3)
```

$$[7, 29, 38, 46, 68]$$

Finally, we also find an independent set using 8 iterations.

```
> greedy_independent_set(G3,8)
```

$$[2, 5, 7, 8, 12, 13, 56, 60, 74, 75, 76, 94]$$

4.12.6 Minimum clique cover

`clique_cover(G,⟨b⟩)`

A **minimum clique cover** for an undirected graph G is any minimal set $S = \{C_1, C_2, ..., C_k\}$ of cliques in G such that for every vertex v in G there exists $i \leqslant k$ such that $v \in C_i$. Such a cover can be obtained by calling the `clique_cover` command. It takes an undirected graph $G(V, E)$ as its mandatory argument and returns the smallest possible cover. Optionally, a positive integer b may be passed as the second argument. In that case the requirement $k \leqslant b$ is set. If no such cover is found, then an empty list is returned.

The strategy is to find a minimal vertex coloring in the complement G^c of G. Each set of equally colored vertices in G^c corresponds to a clique in G. Therefore, the color classes of G^c correspond to the elements $C_1, ..., C_k$ of a minimal clique cover in G.

There is a special case in which G is triangle-free (i.e. contains no 3-cliques), which is computed separately by the algorithm. In that case, G contains only 1- and 2-cliques. Therefore, every clique cover in G consists of a set $M \subset E$ of matched edges together with the singleton cliques (i.e. the isolated vertices in V which remain unmatched). The total number of cliques in the cover is equal to $|V| - |M|$, hence to find a minimal cover one just needs to find a maximum matching in G, which can be done in polynomial time.

```
> G:=random_graph(30,0.2)
```

an undirected unweighted graph with 30 vertices and 98 edges

```
> clique_cover(G)
```

[[9, 13, 29], [1, 21, 23], [17, 28], [19, 20, 22, 25], [8, 16, 18], [5, 6, 15], [2, 4], [0, 27], [24, 26], [12, 14], [7, 11], [3, 10]]

```
> clique_cover(graph("octahedron"))
```

$$[[2, 4, 5], [1, 3, 6]]$$

The vertices of Petersen graph can be covered with five, but not with three cliques.

```
> clique_cover(graph("petersen"),3)
```

[]

```
> clique_cover(graph("petersen"),5)
```

$[[7, 9], [6, 8], [0, 5], [3, 4], [1, 2]]$

4.12.7 Clique cover number

`clique_cover_number(G)`

The command `clique_cover_number` is used for computing the clique cover number of a graph. It takes an undirected graph $G(V, E)$ as its only argument and returns the minimum number of cliques in G needed to cover the vertex set V. (More precisely, it calls the `clique_cover` command and returns the length of the output list.) This number, denoted by $\theta(G)$, is equal to the chromatic number $\chi(G^c)$ of the complement graph G^c of G.

```
> clique_cover_number(graph("petersen"))
```

5

```
> clique_cover_number(graph("soccerball"))
```

30

4.12.8 Split graphs

`is_split_graph(G,⟨part⟩)`

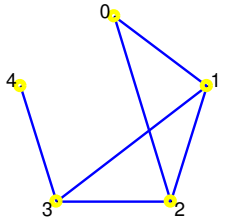
The command `is_split_graph` is used for detecting split graphs and obtaining decompositions of such graphs. It takes single mandatory argument, an undirected graph $G(V, E)$, and returns `true` if and only if G is a split graph, which means that V can be partitioned into a clique $C \subset V$ and an independent set $D = V \setminus C$. If the keyword `part` is given and G is a split graph, then the list containing C and D (in that order) is also returned.

The strategy is to apply the criterion of HAMMER and SIMEONE [39] which uses only the degree sequence of G. The algorithm operates in $O(|V| \log |V|)$ time.

```
> K:=graph(5,%{[0,1],[0,2],[1,2],[1,3],[2,3],[3,4]%})
```

an undirected unweighted graph with 5 vertices and 6 edges

```
> draw_graph(K)
```




```
> is_split_graph(K)
```

true

```
> is_split_graph(K,part)
```

true, [[3, 2, 1], [0, 4]]

4.12.9 Simplicial vertices

`simplicial_vertices(G)`

A vertex in an undirected graph is called **simplicial** if its neighborhood is a clique. The command `simplicial_vertices` is used for finding such vertices in the given graph. It takes an undirected graph $G(V, E)$ as its only argument and returns the list of simplicial vertices in V, i.e. the vertices for which the corresponding neighborhood-induced subgraphs are complete.

The strategy is to use the algorithm of Kloks et al. [49], which in the present implementation has the worst-case running time $O(|E|^{3/2})$.

```
> G:=graph("goldner-harary")
```

an undirected unweighted graph with 11 vertices and 27 edges

```
> simplicial_vertices(G)
```

[0, 1, 2, 3, 4, 5]

```
> H:=sierpinski_graph(3,6)
```

an undirected unweighted graph with 216 vertices and 645 edges

```
> simplicial_vertices(H)
```

[0, 43, 86, 129, 172, 215]

```
> is_clique(induced_subgraph(H,neighbors(H,43)))
```

true

4.13 Network analysis

4.13.1 Counting triangles

`number_of_triangles(G,⟨L⟩)`

The command `number_of_triangles` is used for counting triangles in graphs. It takes a graph G as its first, mandatory argument and returns the number n of 3-cliques in G if G is undirected resp. the number m of directed cycles of length 3 if G is directed. If an unassigned identifier L is given as the second argument, then the list of triangles is assigned to L. Note that triangle listing is supported only for undirected graphs.

For undirected graphs the algorithm of Schank and Wagner [67, Algorithm *forward*], improved by Latapy [52], is used, which runs in $O(|E|^{3/2})$ time. For digraphs, the strategy is to compute the trace of A^3 where A is the adjacency matrix of G encoded in a sparse form. This algorithm requires $O(|V|\,|E|)$ time.

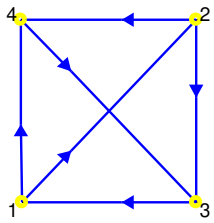
```
> number_of_triangles(graph("tetrahedron"))
```

4

```
> G:=digraph([1,2,3,4],%{[1,2],[1,4],[2,3],[2,4],[3,1],[4,3]%})
```

a directed unweighted graph with 4 vertices and 6 arcs

```
> draw_graph(G,spring)
```




```
> number_of_triangles(G)
```

2

```
> G:=sierpinski_graph(7,3,triangle)
```

an undirected unweighted graph with 1095 vertices and 2187 edges

```
> number_of_triangles(G)
```

972

Petersen graph is triangle-free, i.e. contains no 3-cliques.

```
> number_of_triangles(graph("petersen"))
```

0

Counting triangles in undirected graphs is very fast, as illustrated by the following example.

```
> G:=random_graph(10^5,10^6)
```

an undirected unweighted graph with 100000 vertices and 1000000 edges

```
> number_of_triangles(G)
```

25306

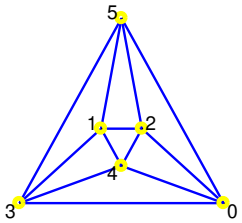
1.155 sec

To list all triangles in a graph, pass an unassigned identifier as the second argument; the list of triangles will be assigned to it.

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(G)
```




```
> purge(L):; number_of_triangles(G,L)
```

"Done", 8

```
> L
```

$$[[1,3,5],[1,3,4],[0,3,5],[0,3,4],[1,2,5],[1,2,4],[0,2,5],[0,2,4]]$$

4.13.2 Clustering coefficient

```
clustering_coefficient(G,⟨opt⟩)
clustering_coefficient(G,v)
clustering_coefficient(G,v1,v2,..,vk)
clustering_coefficient(G,[v1,v2,..,vk])
```

The command `clustering_coefficient` is used for computing the average clustering coefficient (or simply: clustering coefficient) of an undirected graph as well as the local clustering coefficient of a particular vertex in that graph. It takes one or two arguments, an undirected graph $G(V,E)$ and optionally a vertex $v \in V$ or a list/sequence of vertices $v_1, v_2, ..., v_k \in V$. If G is the only argument, then the clustering coefficient $c(G)$ [12, p. 5] is returned. Otherwise, the local clustering coefficient $c_G(v)$ [12, p. 4] of v resp. a list of local clustering coefficients of $v_1, v_2, ..., v_k$ is returned. The second argument may also be one of the following options.

- `exact` — the clustering coefficient $c(G)$ is returned as a rational number (by default it is a floating point number). Note that local clustering coefficient is always returned in exact form
- `approx` — an approximation of the clustering coefficient $c(G)$, lying within 0.5×10^{-2} of the exact value with probability $p = 1 - 10^{-5}$, is returned

In any case, the return value is—by definition—a rational number in the range $[0,1]$.

The clustering coefficient of G is defined as the mean of $c_G(v)$, $v \in V$:

$$c(G) = \frac{1}{|V|} \sum_{v \in V} c_G(v).$$

$c(G)$ can be interpreted as the probability that, for a randomly selected pair of incident edges (u,v) and (v,w) in G, the vertices u and w are connected. The number $c_G(v)$ is interpreted analogously but for a fixed $v \in V$. It represents the probability that two neighbors of v are connected to each other.

For example, assume that G represents a social network in which $uv \in E$ indicates that u and v are friends (which is a symmetric relation). In this context, $c(v)$ represents the probability that two friends of v are also friends of each other.

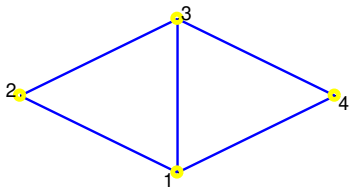
The time complexity of computing $c(G)$ is $O(|E|^{3/2})$, whereas the algorithm of Schank and Wagner [68, Algorithm 1, p. 269] for approximating $c(G)$ runs in $O(\log |V|)$ time.

Note that the command `random_graph` can, by using a preferential attachment rule, generate realistic random networks with adjustable clustering coefficient.

```
> G:=graph("diamond")
```

an undirected unweighted graph with 4 vertices and 5 edges

```
> draw_graph(G,spring)
```



The command lines below compute $c(G)$, $c_G(1)$ and $c_G(2)$.

```
> clustering_coefficient(G,exact)
```

$$\frac{5}{6}$$

```
> clustering_coefficient(G,1)
```

$$\frac{2}{3}$$

```
> clustering_coefficient(G,2)
```

$$1$$

The next example demonstrates the performance of `clustering_coefficient` on a large graph.

```
> G:=random_graph(25000,10,100)
```

an undirected unweighted graph with 25000 vertices and 988571 edges

```
> clustering_coefficient(G)
```

0.637641843214

1.469 sec

```
> clustering_coefficient(G,approx)
```

0.637380772718

491 msec

The probability that two neighbors of a vertex in G are connected is therefore about 64%.

4.13.3 Network transitivity

`network_transitivity(G)`

The command `network_transitivity` is used for computing the transitivity (also called **triangle density** or the **global clustering coefficient**) of a network. It takes a graph G as its only argument and returns the transitivity $T(G)$ of G [12, p. 5] as a rational number in the range $[0, 1]$:

$$T(G) = \frac{3\, N_{\text{triangles}}}{N_{\text{triplets}}}.$$

$T(G)$ is a measure of transitivity of a non-symmetric relation between the vertices of a network. If G is a digraph, then a **triplet** in G is any directed path (v, w, z) where $v, w, z \in V$. For example, in a Twitter-like social network this could mean that v following w and w following z. The triplet (v, w, z) is **closed** if $(v, z) \in E$, i.e. if v also follows z [82, p. 243]. A closed triplet is called a **triangle**. If G is undirected, then $N_{\text{triangles}}$ equals the number of 3-cliques and N_{triplets} equals the number of two-edge paths in V.

The complexity of computing $T(G)$ is $O(\Delta_G\,|E|)$ for digraphs, where Δ_G is the maximum vertex degree in G, resp. $O(|E|^{3/2})$ for undirected graphs.

```
> G:=graph(%{[1,2],[2,3],[2,4],[3,4],[4,1]%})
```

an undirected unweighted graph with 4 vertices and 5 edges

```
> network_transitivity(G)
```

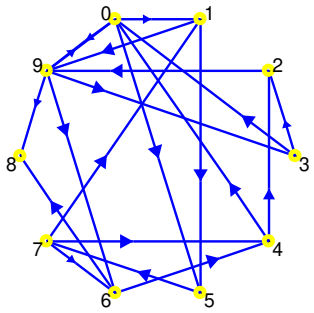
$$\frac{3}{4}$$

Observe that the above result is different than $c(G)$ obtained in Section 4.13.2. Hence $c(G) \neq T(G)$ in general [12, p. 5].

```
> G:=random_digraph(10,20)
```

a directed unweighted graph with 10 vertices and 20 arcs

```
> draw_graph(G)
```



In the above digraph, the triplet $(7, 6, 8)$ is open while the triplet $(7, 6, 4)$ is closed. Triangles $(1, 5, 7)$ and $(3, 2, 9)$ are not closed by definition.

```
> network_transitivity(G)
```

$$\frac{1}{8}$$

The transitivity algorithms are suitable for large networks, as demonstrated in the examples below.

```
> G:=random_digraph(1000,500000)
```

a directed unweighted graph with 1000 vertices and 500000 arcs

```
> nt:=network_transitivity(G):;
```

2.137 sec

```
> evalf(nt)
```

0.500525000328

```
> H:=random_graph(30000,10,50)
```

an undirected unweighted graph with 30000 vertices and 1015074 edges

```
> evalf(network_transitivity(H))
```

0.135416551785

1.418 sec

4.13.4 Centrality measures

```
betweenness_centrality(G,⟨v⟩)
closeness_centrality(G,⟨v⟩)
communicability_betweenness_centrality(G,⟨v⟩)
degree_centrality(G,⟨v⟩)
harmonic_centrality(G,⟨v⟩)
information_centrality(G,⟨v⟩,⟨approx⟩)
katz_centrality(G,alpha,⟨v⟩)
```

Giac provides several commands for measuring vertex centrality in networks. These commands compute the indicated centrality measure for the vertex $v \in V$ in a graph $G(V, E)$ with $|V| > 1$. If v is omitted, then the list of values for every vertex in V is returned, in order as provided by `vertices(G)`.

Degree centrality of a vertex $v \in V$ is computed using the formula

$$C_{\mathrm{D}}(v) = \frac{\deg(v)}{|V|-1}.$$

Closeness centrality of a vertex $v \in V$ is computed using the formula

$$C(v) = \frac{|V|-1}{\sum_{u \in V} d(u,v)},$$

where $d(u,v)$ is the distance from u to v [7].

Harmonic centrality of a vertex $v \in V$ is computed using the formula [54]

$$H(v) = \sum_{u \in V} \frac{1}{d(u,v)}.$$

The worst case complexity in both cases is $O(|V|^3)$ when closeness/harmonic centrality is computed for all vertices in a weighted graph G. For unweighted graphs, the complexity drops to $O(|V|^2 + |V|\,|E|)$.

Betweenness centrality of a vertex $v \in V$ is computed using the formula

$$C_{\mathrm{B}}(v) = \sum_{s \neq v \neq t \in V} \frac{\sigma_{st}(v)}{\sigma_{st}},$$

where σ_{st} is total number of shortest paths from s to t and $\sigma_{st}(v)$ is the number of those paths which pass through v [30]. The strategy is to use the algorithm by BRANDES [13] which operates in $O(|V|\,|E|)$ time and $O(|V| + |E|)$ space. Edge weights are ignored.

Communicability betweenness centrality of a vertex $v \in V$ is computed using the formula

$$C_{\mathrm{CB}}(v) = \frac{1}{(n-1)(n-2)} \sum_{s \neq v} \sum_{t \neq s,v} \left(1 - \frac{(\mathrm{e}^{A_v})_{st}}{(\mathrm{e}^{A})_{st}}\right),$$

where A is the adjacency matrix of G and A_v is the same matrix with the row corresponding to v filled with zeros [26]. It is always $C_{\mathrm{CB}}(v) \in [0,1]$ and this value is interpreted as the weighted proportion of walks which pass through v, but do not start nor end in v. Matrix exponentials are computed in floating-point arithmetic for efficiency; hence the output is always inexact. Note that G must be (strongly) connected for $C_{\mathrm{CB}}(v)$ to be defined for all $v \in V$. If this condition is not met, then `communicability_betweenness_centrality` returns an error.

Information centrality of a vertex $v \in V$ is computed using the formula [71]

$$C_{\mathrm{I}}(v) = \frac{|V|}{\sum_{u \in V} (B_{uu} + B_{vv} - 2\,B_{uv})},$$

where $B = (L + J)^{-1}$, L is the Laplacian matrix of G and J is the $|V| \times |V|$ matrix in which every entry is 1 [69]. The graph G must be undirected and connected (the matrix B does not exist if G is disconnected). If the optional argument `approx` is given, then the result is computed using the floating-point arithmetic, which is faster. Note that information centrality takes edge weights into account. If G is weighted, then the corresponding weight function must be nonnegative.

Katz centrality of a vertex $v \in V$ is computed using the formula

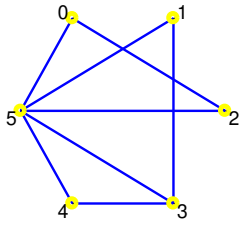
$$C_{\mathrm{K}}(v) = \sum_{w \in V} (I - \alpha\,A^T)^{-1}_{v,w}$$

where A is the adjacency matrix of G, $\alpha < \frac{1}{|\lambda|}$ is attenuation (given as the second argument), and λ is eigenvalue of A with the largest magnitude [48]. Note that if the above condition on α is not met, the result is meaningless. If α is a floating-point value, then the computation uses the floating-point arithmetic, otherwise it is exact.

```
> G:=graph(6,%{[0,2],[0,5],[1,3],[1,5],[2,5],[3,4],[3,5],[4,5]%})
```

an undirected unweighted graph with 6 vertices and 8 edges

```
> draw_graph(G,circle)
```




```
> degree_centrality(G)
```

$$\left[\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{2}{5}, 1\right]$$

```
> closeness_centrality(G)
```

$$\left[\frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{5}{7}, \frac{5}{8}, 1\right]$$

```
> harmonic_centrality(G)
```

$$\left[\frac{7}{2}, \frac{7}{2}, \frac{7}{2}, 4, \frac{7}{2}, 5\right]$$

```
> betweenness_centrality(G)
```

$$\left[0, 0, 0, \frac{1}{2}, 0, \frac{13}{2}\right]$$

```
> information_centrality(G)
```

$$\left[\frac{72}{61}, \frac{36}{29}, \frac{72}{61}, \frac{72}{49}, \frac{36}{29}, \frac{72}{37}\right]$$

```
> katz_centrality(G,0.1)
```

[1.29598461169, 1.30920894446, 1.29598461169, 1.42822793941, 1.30920894446, 1.66386150517]

```
> communicability_betweenness_centrality(G)
```

[0.188215207081, 0.199985710756, 0.188215207081, 0.363437500766, 0.199985710756, 0.8472283\
14216]

The above results show that, according to each of the implemented centrality measures, the vertex with label 5 is more important than other vertices.

4.14 Graph coloring

To color vertices or edges of a graph $G(V, E)$ means to assign to each vertex $v \in V$ or edge $e \in E$ a positive integer. Each integer represents a distinct color. The key property of graph coloring is that the colors of a pair of adjacent vertices, or incident edges, must be mutually different. Two different colorings of G may use different number of colors.

value	1	2	3	4	5	6	7
color	red	green	yellow	blue	magenta	cyan	black

Table 4.1. Vertex/edge colors in XCAS.

4.14.1 Greedy vertex coloring

`greedy_color(G,⟨p⟩)`

The command `greedy_color` is used for coloring vertices of a graph in a greedy fashion. It takes one mandatory argument, a graph $G(V,E)$. Optionally, a permutation p of order $|V|$ may be passed as the second argument. Vertices are colored one by one in the order specified by p (or in the default order if p is not given) such that each vertex gets the smallest available color. The list of vertex colors is returned in the order of `vertices(G)`.

Generally, different choices of permutation p produce different colorings. The total number of different colors may not be the same each time. The complexity of the algorithm is $O(|V|+|E|)$.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> greedy_color(G)
```

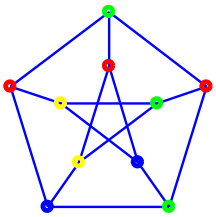
$$[1,2,1,2,3,2,1,3,3,2]$$

```
> L:=greedy_color(G,randperm(10))
```

$$[2,1,2,4,1,1,2,4,3,3]$$

Observe that a different number of colors is obtained by executing the last command line. To display the colored graph, input:

```
> draw_graph(highlight_vertex(G,vertices(G),L),labels=false)
```



The first six positive integers are always mapped to the standard XCAS colors, as indicated in Table 4.1. Note that the color 0 (black) and color 7 (white) are swapped; a vertex with color 0 is white (uncolored) and vertex with color 7 is black. Also note that XCAS maps numbers greater than 7 to colors too, but the number of available colors is limited.

4.14.2 Minimal vertex coloring

`minimal_vertex_coloring(G,⟨sto⟩)`

A vertex coloring of G is **minimal** (or **optimal**) if the total number of used colors is minimal. To obtain such a coloring use the command `minimal_vertex_coloring`. It takes one mandatory argument, an undirected graph $G(V,E)$ where $V=\{v_1,v_2,...,v_n\}$. Optionally, the keyword `sto` may be passed as the second argument. The command returns the vertex colors $c_1,c_2,...,c_n$ in order of `vertices(G)` or, if the second argument is given, stores the colors as vertex attributes and returns the modified copy of G.

The minimal vertex coloring problem (MVCP) is converted to the equivalent integer linear programming problem and solved by using the branch-and-bound method with specific branching and backtracking techniques [20], for which the GLPK library is required. Simplicial vertices are removed prior to the conversion, to be handled afterwards. The lower resp. the upper bound for the number n of colors is obtained by finding a maximal clique (n cannot be smaller than its cardinality) resp. by applying the heuristic proposed by Brélaz in [14] (which will use at least n colors). Note that the algorithm performs some randomization when applying heuristics, hence coloring a graph generally does not take the same amount of computation time in each instance. Since the vertex coloring problem is NP hard, the algorithm may take exponential time on some graphs.

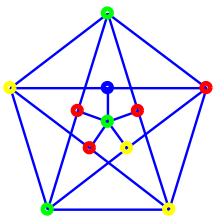
```
> G:=graph("groetzsch")
```

an undirected unweighted graph with 11 vertices and 20 edges

```
> coloring:=minimal_vertex_coloring(G)
```

$$[2, 1, 3, 2, 3, 4, 1, 3, 1, 1, 2]$$

```
> draw_graph(highlight_vertex(G,vertices(G),coloring),labels=false)
```



4.14.3 Chromatic number

`chromatic_number(G,⟨c⟩)`
`chromatic_number(G,approx|interval)`

The command `chromatic_number` is used for exact computation or approximation of the chromatic number of a graph. It takes one mandatory argument, an undirected graph $G(V, E)$, and optionally a second argument. To obtain only upper and lower bound for the chromatic number (which is much faster than computing exactly) the option `approx` or `interval` should be passed as the second argument. Alternatively, an unassigned identifier `c` is passed as the second argument; in that case the corresponding coloring is assigned to it in form of a list of colors of the individual vertices, ordered as in `vertices(G)`.

The command returns the chromatic number χ_G of the graph G in the case of exact computation. If the option `approx` or `interval` is given, then an interval `lb..ub` is returned, where `lb` is the best lower bound and `ub` the best upper bound for χ_G found by the algorithm.

The strategy is call `minimal_vertex_coloring` in the case of exact computation. When approximating the chromatic number, the algorithm will establish the lower bound by finding a maximum clique. The timeout for this operation is set to 5 seconds as it can be time consuming. If no maximum clique is not found in that time, then the largest clique found is used. An upper bound is established by by using the heuristic proposed by Brélaz in [14]. Obtaining the bounds for χ_G is usually very fast; however, their difference grows with $|V|$.

```
> chromatic_number(graph("groetzsch"),cols)
```

4

```
> cols
```

$$[4, 3, 2, 1, 1, 2, 1, 1, 3, 1, 2]$$

```
> G:=random_graph(30,0.75)
```

an undirected unweighted graph with 30 vertices and 325 edges

```
> chromatic_number(G)
```

11

```
> G:=random_graph(300,0.05)
```

an undirected unweighted graph with 300 vertices and 2188 edges

```
> chromatic_number(G,approx)
```

4...7

4.14.4 Mycielski graphs

`mycielski(G)`

The command `mycielski` is used for constructing Mycielski graphs. It takes an undirected graph $G(V, E)$ as its only argument and returns the corresponding Mycielski graph M (also called the **Mycielskian** of G) with $2|V|+1$ vertices and $3|E|+|V|$ edges. If G is triangle-free then M is also triangle-free and $\chi_M = \chi_G + 1$.

```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> M:=mycielski(P)
```

an undirected unweighted graph with 21 vertices and 55 edges

```
> apply(number_of_triangles,[P,M])
```

$[0, 0]$

```
> chromatic_number(P)
```

3

```
> chromatic_number(M)
```

4

`mycielski` can be applied iteratively, producing arbitrarily large graphs from the most simple ones. For example, Grötzsch graph is obtained as the Mycielskian of a cycle graph on 5 vertices, which is the Mycielskian of a path graph on two vertices.

```
> G1:=path_graph(2)
```

an undirected unweighted graph with 2 vertices and 1 edge

```
> G2:=mycielski(G1)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> is_isomorphic(G2,cycle_graph(5))
```

true

```
> G3:=mycielski(G2)
```

an undirected unweighted graph with 11 vertices and 20 edges

```
> is_isomorphic(G3,graph("groetzsch"))
```

true

All three graphs are triangle-free. Since it is obviously $\chi_{G_1} = 2$, it follows $\chi_{G_2} = 3$ and $\chi_{G_3} = 4$.

```
> apply(chromatic_number,[G1,G2,G3])
```

$[2, 3, 4]$

4.14.5 *k*-coloring

`is_vertex_colorable(G,k,⟨c⟩)`

The command `is_vertex_colorable` is used for determining whether the vertices of a graph can be colored with at most k colors. It takes two or three arguments: a graph $G(V, E)$, a positive integer k and optionally an unassigned identifier `c`. The command returns `true` if G can be colored using at most k colors and `false` otherwise. If the third argument is given, then a coloring using at most k colors is assigned to `c` as a list of vertex colors, in the order of `vertices(G)`.

The strategy is to first apply a simple greedy coloring procedure which runs in linear time. If the number of required colors is greater than k, the heuristic proposed by Brélaz in [14] is used, which runs in quadratic time. If the number of required colors is still larger than k, the algorithm attempts to find the chromatic number χ_G using k as the upper bound in the process.

```
> G:=graph("groetzsch")
```

an undirected unweighted graph with 11 vertices and 20 edges

```
> is_vertex_colorable(G,3)
```

false

```
> is_vertex_colorable(G,4)
```

true

```
> G:=graph("brouwer-haemers")
```

an undirected unweighted graph with 81 vertices and 810 edges

```
> chromatic_number(G,approx)
```

3...8

```
> is_vertex_colorable(G,4)
```

false

4.14.6 Minimal edge coloring

`minimal_edge_coloring(G,⟨sto⟩)`
`minimal_edge_coloring(G,sto)`

The command `minimal_edge_coloring` is used for finding a minimal coloring of edges in a graph, satisfying the following two conditions: any two mutually incident edges are colored differently and the total number n of colors is minimal. The theorem of Vizing [21, p. 103] implies that every simple undirected graph falls into one of two categories: *1* if $n = \Delta$ or *2* if $n = \Delta + 1$, where Δ is the maximum degree of the graph.

`minimal_edge_coloring` takes one or two arguments, a graph $G(V, E)$ and optionally the keyword `sto`. If the latter is given, a minimal coloring is stored in the input graph (each edge $e \in E$ gets a color c_e stored as an attribute) and a modified copy of G is returned. Otherwise, the command returns a sequence of two objects: integer `1` or `2`, indicating the category, and the list of edge colors $c_{e_1}, c_{e_2}, ..., c_{e_m}$ according the order of edges $e_1, e_2, ..., e_m \in E$ as returned by `edges`.

The strategy is to find a minimal vertex coloring of the line graph of G by using the algorithm described in Section 4.14.2.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

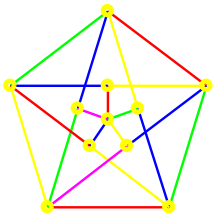
```
> minimal_edge_coloring(G)
```

$$2, [1, 2, 3, 3, 2, 1, 4, 3, 2, 1, 1, 4, 1, 3, 2]$$

```
> H:=minimal_edge_coloring(graph("groetzsch"),sto)
```

an undirected unweighted graph with 11 vertices and 20 edges

```
> draw_graph(H,labels=false)
```



```
> G:=random_graph(50,0.15)
```

an undirected unweighted graph with 50 vertices and 157 edges

```
> minimal_edge_coloring(G):;
```

15.504 sec

4.14.7 Chromatic index

`chromatic_index(G,⟨c⟩)`

The command `chromatic_index` is used for computing the chromatic index of an undirected graph. It takes one or two arguments, an undirected graph $G(E, V)$ and optionally an unassigned identifier `c`. The command returns the minimal number $\chi'(G)$ of colors needed to color each edge in G such that two incident edges never share the same color. If the second argument is given, it specifies the destination for storing the coloring in form of a list of colors according to the order of edges in E as returned by the command `edges`.

The example below demonstrates how to color the edges of a graph with colors obtained by passing unassigned identifier `c` to `chromatic_index` as the second argument.

```
> J5:=flower_snark(5)
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> chromatic_index(J5)
```

Chapter 5
Traversing graphs

5.1 Walks and tours

5.1.1 Eulerian graphs

`is_eulerian(G,⟨T⟩)`

The command `is_eulerian` is used for determining whether a graph contains an Eulerian trail, i.e. a trail which passes through each of its edges exactly once [34, p. 395]. A graph is **Eulerian** if it has such a trail. An Eulerian trail may be closed, in which case it is an **Eulerian circuit**.

`is_eulerian` takes one or two arguments, a (di)graph $G(V, E)$ and optionally an unassigned identifier `T`, and returns `true` if G is Eulerian and `false` otherwise. If the second argument is given and G is undirected, then an Eulerian trail is computed and assigned to `T`.

The strategy for finding an Eulerian trail is to apply Hierholzer's algorithm [41]. It works by covering one cycle at a time in the input graph. The required time is $O(|E|)$.

```
> is_eulerian(complete_graph(4))
```

false

```
> is_eulerian(complete_graph([1,2,3,4,5]),T); T
```

true, $[1, 2, 3, 4, 1, 5, 2, 4, 5, 3, 1]$

```
> is_eulerian(graph("tetrahedron"))
```

false

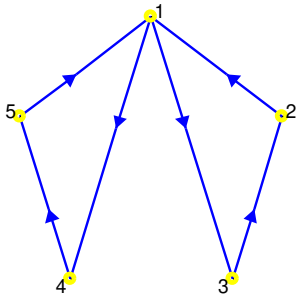
```
> is_eulerian(graph("octahedron"))
```

true

```
> G:=digraph(%{[1,4],[1,3],[2,1],[3,2],[4,5],[5,1]%})
```

a directed unweighted graph with 5 vertices and 6 arcs

```
> draw_graph(G)
```




```
> is_eulerian(G)
```

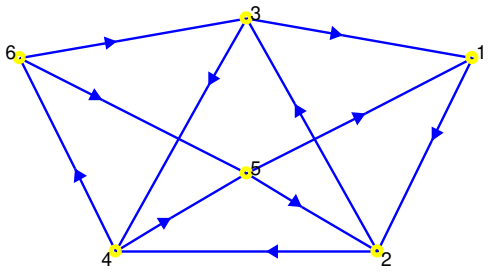
```
Input digraph has an Eulerian circuit
```

true

```
> H:=digraph(%{[1,2],[2,3],[2,4],[3,4],[3,1],[4,5],[4,6],[5,1],[5,2],[6,3],[6,
  5]%})
```

a directed unweighted graph with 6 vertices and 11 arcs

```
> draw_graph(H,spring)
```




```
> is_eulerian(H)
```

```
Input digraph has an Eulerian trail starting at 6 and ending at 1
```

true

5.1.2 Hamiltonian graphs

`is_hamiltonian(G,⟨hc⟩)`

The command `is_hamiltonian` is used for determining whether a graph is Hamiltonian. The command can also construct a Hamiltonian cycle in the input graph if the latter is Hamiltonian. It takes one or two arguments, a (di)graph $G(V, E)$ and optionally an unassigned identifier `hc`. The command returns `true` if G is Hamiltonian and `false` otherwise. If the second argument is given, a Hamiltonian cycle is assigned to `hc`.

The strategy is to apply a simple backtracking algorithm for finding a Hamiltonian cycle. However, some known characterizations of (non)hamiltonicity are applied first, as follows.

- If G is directed then the following criteria are applied. If G is not strongly connected, then it is not Hamiltonian. Otherwise, the criterion of GHOUILA and HOURI [53] is applied: if $\deg(v) \geqslant |V|$ for all $v \in V$, then G is Hamiltonian. Otherwise, the criterion of MEYNIEL [53] is applied: if $\deg(v) + \deg(w) \geqslant 2|V| - 1$ for any pair of non-adjacent vertices $v, w \in V$, then G is Hamiltonian.
- If G is undirected then the criteria listed by DELEON [19] are applied, in the following order.

  Connectivity criterion — If G is not biconnected, then it is not Hamiltonian.

  DIRAC criterion — If $\delta(G) \geqslant \frac{|V|}{2}$, where $\delta(G) = \min\{\deg(v) : v \in V\}$, then G is Hamiltonian.

  Bipartite-graph criterion — If G is bipartite with vertex partition $V = V_1 \cup V_2$ and $|V_1| \neq |V_2|$, then G is not Hamiltonian.

  ORE criterion — If $\deg(u) + \deg(v) \geqslant |V|$ holds for every pair u, v of non-adjacent vertices from V, then G is Hamiltonian.

  BONDY–CHVÁTAL theorem — If the closure $\mathrm{cl}(G)$ of G (obtained by finding a pair u, v of non-adjacent vertices from V such that $\deg(u) + \deg(v) \geqslant |V|$, adding a new edge uv to E and repeating the process until exhaustion) is Hamiltonian, then G is Hamiltonian. Note that in this case the previously tried criteria are applied to $\mathrm{cl}(G)$; vertex degrees in $\mathrm{cl}(G)$ are generally higher than those in G and hence the probability of success is greater.

Nash–Williams criterion (for large edge-density) — If $\delta(G) \geqslant \max\left\{\frac{|V|+2}{3}, \beta\right\}$, where β is the independence number of G, then G is Hamiltonian.

The backtracking algorithm is space-efficient but may take a long time on larger graphs.

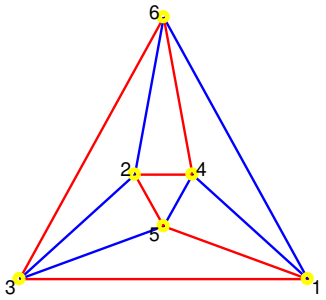
```
> is_hamiltonian(graph("soccerball"))
```

true

```
> is_hamiltonian(graph("octahedron"),hc)
```

true

```
> draw_graph(highlight_trail(graph("octahedron"),hc))
```




```
> is_hamiltonian(graph("herschel"))
```

false

```
> is_hamiltonian(graph("petersen"))
```

false

```
> is_hamiltonian(hypercube_graph(6))
```

true

8.455 sec

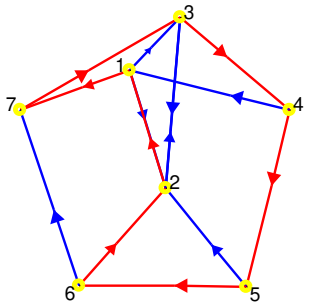
```
> G:=digraph(%{[1,2],[1,3],[1,7],[2,1],[2,3],[3,2],[3,4],[4,1],[4,5],[5,2],[5,
  6],[6,2],[6,7],[7,3]%})
```

a directed unweighted graph with 7 vertices and 14 arcs

```
> purge(hc):; is_hamiltonian(G,hc)
```

"Done", true

```
> draw_graph(highlight_trail(G,hc),spring)
```



5.2 Optimal routing

5.2.1 Shortest unweighted paths

`shortest_path(G,s,t|T)`

The command `shortest_path` is used for finding shortest paths in unweighted graphs. It takes three arguments: an undirected unweighted graph $G(V,E)$, the source vertex $s \in V$ and the target vertex $t \in V$ or a list T of target vertices. The shortest path from source to target is returned. If more targets are specified, then the list of shortest paths from the source to each of these vertices is returned.

The strategy is to run breadth-first traversal on the graph G starting from the source vertex s. The complexity of the algorithm is therefore $O(|V|+|E|)$.

```
> G:=graph("dodecahedron")
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> shortest_path(G,1,16)
```

$$[1, 0, 10, 18, 16]$$

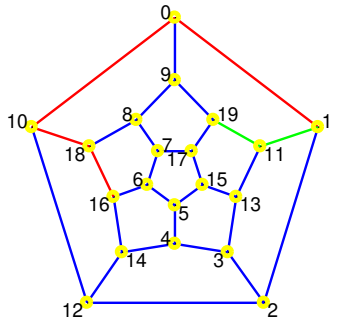
```
> paths:=shortest_path(G,1,[16,19])
```

$$[[1, 0, 10, 18, 16], [1, 11, 19]]$$

```
> H:=highlight_trail(G,paths,[red,green])
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> draw_graph(H)
```



5.2.2 Cheapest weighted paths

`dijkstra(G,s,⟨t|T⟩)`
`bellman_ford(G,s,⟨t|T⟩)`

The commands `dijkstra` and `bellman_ford` are used for finding cheapest paths in weighted (directed) graphs. Both commands take two or three arguments: a weighted (di)graph $G(V,E)$, a vertex $s \in V$ and optionally a vertex $t \in V$ or a list T of vertices in V. It returns the cheapest path from s to t. If more target vertices are given, then the sequence of such paths to each target vertex $t \in T$ is returned. If no target vertex is specified, then all vertices in $V \setminus \{s\}$ are assumed to be targets. If `dijkstra` is used, then the weights of edges in E must all be nonnegative. `bellman_ford` takes negative weights, but does not work if the input graph contains negative cycles (in which the weights of the corresponding edges sum up to a negative value).

A cheapest path from s to t is represented with a list `[[v1,v2,...,vk],c]` where the first element consists of path vertices with $v_1 = s$ and $v_k = t$, while the second element c is the weight (cost) of that path, equal to the sum of weights of its edges.

`dijsktra` computes the cheapest path using Dijkstra's algorithm which runs in $O(|V|^2)$ time [22]. `bellman_ford` applies a similar algorithm of Bellman and Ford (see [9] and [29]), which runs in $O(|V|\,|E|)$ time but imposes fewer requirements upon its input.

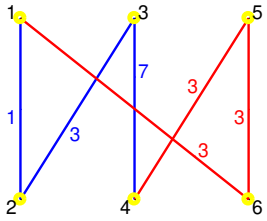
```
> G:=graph(%{[[1,2],1],[[1,6],3],[[2,3],3],[[3,4],7],[[4,5],3],[[5,6],3]%})
```

an undirected weighted graph with 6 vertices and 6 edges

```
> res:=dijkstra(G,1,4)
```

$[[1, 6, 5, 4], 9]$

```
> draw_graph(highlight_trail(G,res[0]))
```




```
> dijkstra(G,1)
```

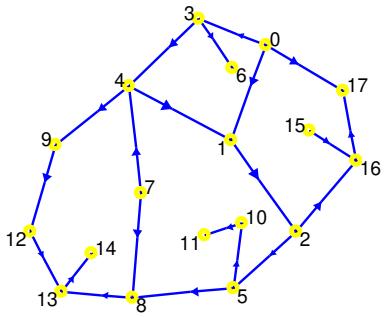
$[[1], 0], [[1, 2], 1], [[1, 6], 3], [[1, 2, 3], 4], [[1, 6, 5, 4], 9], [[1, 6, 5], 6]$

In the following example, a longest path in an unweighted acyclic graph is found using Bellman-Ford algorithm and negative unit weights.

```
> G:=random_network(3,2,acyclic=true)
```

a directed unweighted graph with 18 vertices and 21 arcs

```
> draw_graph(G,planar)
```




```
> W:=make_weighted(G,-adjacency_matrix(G))
```

a directed weighted graph with 18 vertices and 21 arcs

```
> L:=[]:; for v in vertices(W) do L.append(bellman_ford(W,v)); od:;
```

"Done", "Done"

```
> lp:=[]:; for p in L do if length(lp) < length(p[0]) then lp:=p[0]; fi; od:;
```

"Done", "Done"

```
> lp
```

$[0, 3, 4, 1, 2, 5, 8, 13, 14]$

5.2.3 *k*-shortest paths

`kspaths(G,s,t,k)`

The command `kspaths` is used for obtaining *k*-shortest paths from the given source to the given destination in a (weighted) graph. It takes four arguments, a (weighted) (di)graph $G(V,E)$, a source vertex $s \in V$, a sink vertex $t \in V$ and a positive integer k. It returns a list containing a largest number not greater than k of shortest (cheapest) paths from s to t, sorted in ascending order with respect to their costs (weights). The cost of a path is equal to the sum of weights of its edges if G is weighted resp. to the number of edges if G is unweighted.

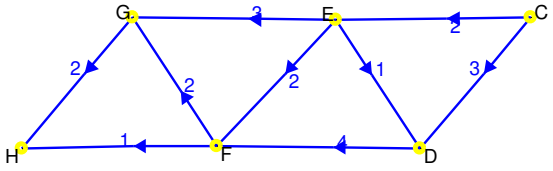
The strategy is to apply the algorithm of YEN [86] which uses DIJKSTRA's algorithm [22] as a subroutine. The algorithm runs in $O(k\,|V|^3)$ time.

```
> purge(C,D,E,F,G,H):;
```

```
> DG:=digraph(%{[[C,D],3],[[C,E],2],[[D,F],4],[[E,D],1],[[E,F],2],[[E,G],3],[[F,
  G],2],[[F,H],1],[[G,H],2]%})
```

a directed weighted graph with 6 vertices and 9 arcs

```
> draw_graph(DG,spring)
```




```
> kspaths(DG,C,H,5)
```

$$[[C,E,F,H],[C,E,G,H],[C,E,D,F,H],[C,E,F,G,H],[C,D,F,H]]$$

5.2.4 Traveling salesman problem

`traveling_salesman(G,⟨opts⟩)`
`traveling_salesman(G,M,⟨opts⟩)`

The command `traveling_salesman` is used for solving traveling salesman problem (TSP)[5.1]. It takes the following arguments: a (di)graph $G(V,E)$, a weight matrix M (optional) and a sequence of options (optional). The supported options are `approx` and `vertex_distance` for undirected graphs and `is_included=arc|[arcs]` for directed graphs.

If the input graph G is unweighted and M is not specified, then a Hamiltonian cycle (tour) is returned (the adjacency matrix of G is used for the edge weights). If G is weighted, two objects are returned: the optimal value for the traveling salesman problem and a Hamiltonian cycle which achieves the optimal value. If M is given and G is unweighted, M is used as the weight matrix for G.

If the option `vertex_distance` is provided but M is not, then for each edge $e \in E$ the Euclidean distance between its endpoints is used as the weight of e. Therefore it is required for each vertex in G to have a predefined position.

If the option `approx` is provided, then a near-optimal tour is returned. In this case it is required that G is a complete undirected weighted graph. For larger graphs, this is significantly faster than finding optimal tour. Results thus obtained are usually only a few percent larger than the corresponding optimal values, despite the fact that the reported guarantee is generally much weaker (around 30%).

If the option `is_included=arc|[arcs]` is provided, then the algorithm finds a shortest Hamiltonian cycle in G which contains the specified arc(s).

If G is a digraph, then an optional argument `k` may be passed, which must be a positive integer. If $k > 1$ then the first k shortest Hamiltonian cycles are returned (the return value is a sequence containing the list of tour costs and the list of the corresponding Hamiltonian cycles).

5.1. For the details on traveling salesman problem and a historical overview see [18].

The strategy for undirected graphs is to formulate TSP as an integer linear programming problem and to solve it by branch-and-cut method, using the GLPK library along with the routine written by its author Andrew Makhorin (see the example program TSPSOL which is a part of GLPK) for generating subtour elimination constraints on the fly. The branching rule is implemented according to Padberg and Rinaldi [62]. In addition, the algorithm combines the method of Christofides [16], the method of farthest insertion and a variant of the powerful tour improvement heuristic developed by Lin and Kernighan [40] to generate near-optimal feasible solutions during the branch-and-cut process.

For Euclidean TSP instances, i.e. in cases when G is a complete undirected graph with vertex distances as edge weights, the algorithm usually finishes in a few seconds for TSP with up to, say, 42 cities. For problems with 100 or more cities, the option `approx` is recommended since finding the optimal value takes a long time. Note that TSP is NP-hard, meaning that no polynomial time algorithm is known. Hence the algorithm may take exponential time to find the optimum in some instances.

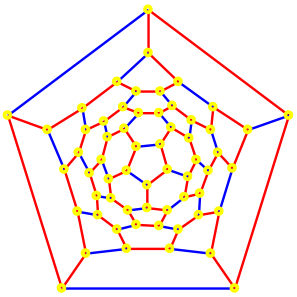
For directed graphs, an integer linear programming formulation of the problem, using enhanced subtour elimination constraints of Miller, Tucker and Zemlin [66], is solved by branch-and-cut method.

The following example demonstrates finding a Hamiltonian cycle in the truncated icosahedral ("soccer ball") graph. The result is visualized by using `highlight_trail`.

```
> G:=graph("soccerball")
```

an undirected unweighted graph with 60 vertices and 90 edges

```
> draw_graph(highlight_trail(G,traveling_salesman(G)),labels=false)
```



A matrix may be passed alongside an undirected graph to specify the edge weights. The alternative is to pass a weighted graph as the single argument.

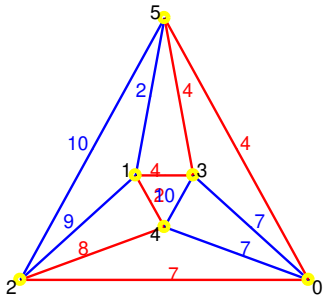
```
> G:=assign_edge_weights(graph("octahedron"),1,10)
```

an undirected weighted graph with 6 vertices and 12 edges

```
> c,t:=traveling_salesman(G)
```

$29.0, [4, 1, 3, 5, 0, 2, 4]$

```
> draw_graph(highlight_trail(G,t))
```

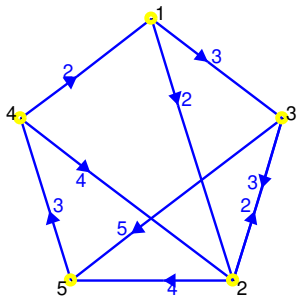


The following examples demonstrate finding shortest Hamiltonian cycle(s) in a weighted directed graph.

```
> D:=digraph(%{[[1,2],2],[[1,3],3],[[2,3],2],[[2,5],4],[[3,2],3],[[3,5],5],[[4,
  1],2],[[4,1],1],[[4,2],4],[[5,4],3]%})
```

a directed weighted graph with 5 vertices and 9 arcs

```
> draw_graph(D,circle=[1,3,2,5,4])
```




```
> traveling_salesman(D)
```

$$14.0, [1, 2, 3, 5, 4, 1]$$

```
> traveling_salesman(D,is_included=[2,5])
```

$$15.0, [1, 3, 2, 5, 4, 1]$$

```
> traveling_salesman(D,2)
```

$$[14.0, 15.0], \begin{pmatrix} 1 & 2 & 3 & 5 & 4 & 1 \\ 1 & 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

In the next example, an instance of Euclidean TSP with 42 cities is solved to optimality. The vertex positions are pairs of integers randomly chosen on the grid $[0, 1000] \times [0, 1000] \in \mathbb{Z}^2$.

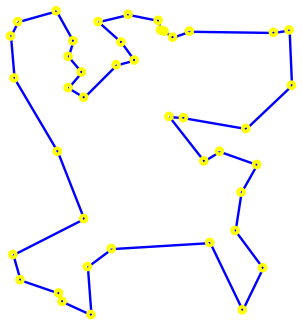
```
> G:=set_vertex_positions(complete_graph(42),[randvector(2,1000)$(k=1..42)])
```

an undirected unweighted graph with 42 vertices and 861 edges

```
> c,t:=traveling_salesman(G,vertex_distance):;
```

10.74 sec

```
> draw_graph(subgraph(G,trail2edges(t)),labels=false)
```



For large instances of Euclidean TSP the `approx` option may be used, as in the following example with 555 cities.

```
> H:=set_vertex_positions(complete_graph(555),[randvector(2,10000)$(k=1..555)])
```

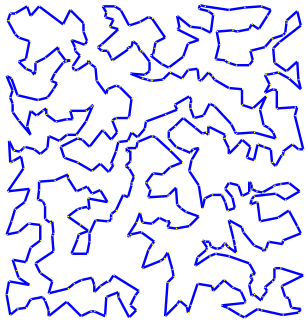
an undirected unweighted graph with 555 vertices and 153735 edges

```
> ac,t:=traveling_salesman(H,vertex_distance,approx):;
```

```
The tour cost is within 32% of the optimal value
```

33.36 sec

```
> draw_graph(subgraph(H,trail2edges(t)))
```



Near-optimal tours produced by using the `approx` option are usually only slightly more expensive than the optimal ones. For example, a sub-optimal tour for the previous instance G with 42 cities is obtained by the following command.

```
> ac,st:=traveling_salesman(G,vertex_distance,approx):;
```

```
The tour cost is within 39% of the optimal value
```

Although it is guaranteed that the near-optimal cost `ac` is for at most 28% larger than `c` (the optimum), the relative difference is actually smaller than 3%, as computed below.

```
> 100*(ac-c)/c
```

2.7119936606

5.3 Spanning trees

5.3.1 Construction of spanning trees

`spanning_tree(G,⟨r⟩)`

The command `spanning_tree` is used for construction of spanning trees in graphs. It takes one or two arguments, an undirected graph $G(V, E)$ and optionally a vertex $r \in V$. It returns the spanning tree T (rooted in r) of G, obtained by depth-first traversal in $O(|V| + |E|)$ time.

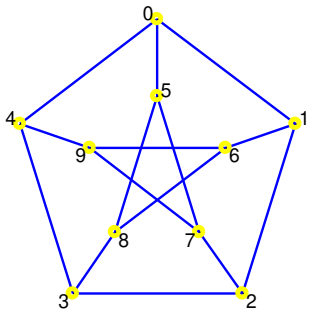
```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> T1:=spanning_tree(P)
```

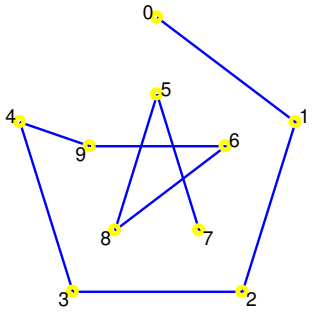
an undirected unweighted graph with 10 vertices and 9 edges

```
> draw_graph(P)
```



By extracting T_1 from P as a subgraph, it inherits vertex positions from P.

```
> draw_graph(subgraph(P,edges(T1)))
```




```
> T2:=spanning_tree(P,4)
```

an undirected unweighted graph with 10 vertices and 9 edges

```
> edges(T1)
```

$$[[0,1],[1,2],[2,3],[3,4],[4,9],[5,7],[5,8],[6,8],[6,9]]$$

```
> edges(T2)
```

$$[[0,1],[0,4],[1,2],[2,3],[3,8],[5,7],[5,8],[6,9],[7,9]]$$

5.3.2 Minimal spanning tree

`minimal_spanning_tree(G)`

The command `minimal_spanning_tree` is used for obtaining minimal spanning trees in undirected graphs. It takes an undirected graph $G(V, E)$ as its only argument and returns its minimal spanning tree as a graph. If G is not weighted, then the unity weights are assumed for all edges.

The strategy is to apply KRUSKAL's algorithm which runs in $O(|E|\log|V|)$ time.

```
> A:=[[0,1,0,4,0,0],[1,0,1,0,4,0],[0,1,0,3,0,1],[4,0,3,0,1,0],[0,4,0,1,0,4],[0,
  0,1,0,4,0]]:;
```

```
> G:=graph(A)
```

an undirected weighted graph with 6 vertices and 8 edges

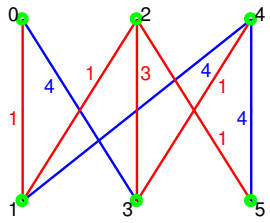
```
> T:=minimal_spanning_tree(G)
```

an undirected weighted graph with 6 vertices and 5 edges

```
> edges(T,weights)
```

[[[0, 1], 1], [[1, 2], 1], [[2, 3], 3], [[2, 5], 1], [[3, 4], 1]]

```
> draw_graph(highlight_subgraph(G,T))
```



5.3.3 Counting the spanning trees and forests in a graph

`number_of_spanning_trees(G)`

The command `number_of_spanning_trees` is used for counting spanning trees and forests in a graph. It takes an undirected graph $G(V, E)$ as its only argument and returns the total number of (labeled) spanning trees in G if G is connected or the total number of spanning forests in G otherwise. The number of spanning forests is equal to the product of numbers obtained by counting spanning trees in each of the connected components of G.

The strategy is to use Kirchhoff's Theorem [84, Theorem 2.2.12, p. 86]. The number of spanning trees is equal to the first principal minor of the Laplacian matrix of G.

```
> number_of_spanning_trees(graph("octahedron"))
```

384

```
> number_of_spanning_trees(graph("dodecahedron"))
```

5184000

```
> number_of_spanning_trees(hypercube_graph(4))
```

42467328

```
> number_of_spanning_trees(graph("soccerball"))
```

375291866372898816000

5.3.4 Vertex reachability

`is_reachable(G,u,v)`
`reachable(G,u)`

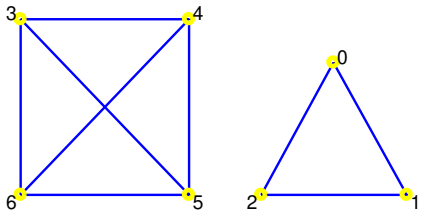
The commands `is_reachable` and `reachable` are used for obtaining information about reachability of vertices in a graph from a given vertex. Both commands takes a (di)graph $G(V, E)$ as their first argument and a vertex $u \in V$. `is_reachable` additionally takes a vertex $v \in V$ as its third argument and returns `true` if there is a path from u to v in G, else returns `false`. `reachable` returns the list of vertices in V which are reachable from u (including u itself).

The strategy is to start a breadth-first search from u. Hence both algorithms run in at most $O(|V| + |E|)$ time.

```
> C34:=graph_complement(complete_graph(3,4))
```

an undirected unweighted graph with 7 vertices and 9 edges

```
> draw_graph(C34)
```




```
> is_reachable(C34,3,5); is_reachable(C34,3,2)
```

true, false

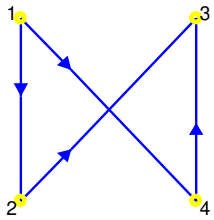
```
> reachable(C34,3)
```

$[4, 5, 6, 3]$

```
> G:=digraph([1,2,3,4],%{[1,2],[1,4],[2,3],[4,3]%})
```

a directed unweighted graph with 4 vertices and 4 arcs

```
> draw_graph(G)
```




```
> is_reachable(G,1,3); is_reachable(G,2,4)
```

true, false

```
> reachable(G,1), reachable(G,2), reachable(G,3), reachable(G,4)
```

$[2, 4, 3], [3], [], [3]$

Chapter 6
Visualizing graphs

6.1 Drawing graphs

6.1.1 Overview

`draw_graph(G,⟨opts⟩)`

The `draw_graph` command is used for visualizing graphs. It takes one or two arguments, the mandatory first one being a graph $G(V, E)$. This command assigns 2D or 3D coordinates to each vertex $v \in V$ and produces a visual representation of G based on these coordinates. The second (optional) argument is a sequence of options. Each optional argument is one of the following.

- `labels=true|false` — controls the visibility of vertex labels and edge weights (by default `true`, i.e. the labels and weights are displayed)
- `spring` — applies the multilevel force-directed algorithm
- `tree⟨=r|[r1,r2,...]⟩` — draws the tree or forest graph G, optionally specifying the root node for each tree (by default the first node is used)
- `bipartite` — draws the bipartite graph G, separating vertex partitions from one another
- `circle⟨=L⟩` or `convexhull⟨=L⟩` — draws the graph G by spreading the *hull vertices* from list $L \subset V$ (assuming $L = V$ by default) across the unit circle and putting all other vertices in origin and applying a force-directed vertex placement algorithm to generate the layout while keeping the hull vertices fixed
- `planar` or `plane` — draws the planar graph G using a force-directed algorithm
- `plot3d` — draws the graph G as if the `spring` option was enabled, but with vertex positions in 3D instead of 2D
- `[x,y,⟨z⟩]` — sets the anchor point of the layout, which corresponds to the bottom-left corner in 2D drawings
- `size=[width,height,⟨depth⟩]` — sets the size of the layout's bounding box
- `scale=<real>` — sets the scaling factor of the layout
- `title=<string>` — prints the title above the drawing (left-aligned)
- an unassigned identifier — sets the destination for storing the list of vertex positions (in order as returned by `vertices`)

The style options `spring`, `tree`, `circle`, `planar` and `plot3d` cannot be mixed, i.e. at most one can be specified. The option `labels` may be combined with any of the style options. Note that edge weights will not be displayed when using `plot3d` option when drawing a weighted graph.

If no style option is specified, the algorithm first checks whether G is a tree or a bipartite graph, in which cases it is drawn accordingly. Otherwise, the graph is drawn as if the option `circle` was specified.

Tree, circle and bipartite drawings are obtained in linear time, while force-directed algorithms require $O(|V|^2)$ time.

6.1.2 Spring method

When the option `spring` is specified, then G is drawn using the force-directed algorithm described in [45]. The idea, originally due to Fruchterman and Reingold [31], is to simulate physical forces in a spring-electrical model where the vertices and edges represent equally charged particles and springs connecting them, respectively.

In a spring-electrical model, each vertex is being repulsed by every other vertex with a force inversely proportional to the distance between them. At the same time, it is attracted to each of its neighbors with a force proportional to the square of the distance. Assuming that x_v is the vector representing the position of the vertex $v \in V$, the total force F_v applied to v is equal to

$$F_v = \sum_{w \in V \setminus \{v\}} -\frac{CK^2}{\|x_v - x_w\|^2}(x_v - x_w) + \sum_{w \in N(v)} \frac{\|x_v - x_w\|}{K}(x_v - x_w),$$

where $N(v)$ is the set of neighbors of v and C, K are certain positive real constants (actually, K may be any positive number, it affects only the scaling of the entire layout). Applying the forces iteratively and updating vertex positions in each iteration (starting from a random layout) leads the system to the state of minimal energy. By applying a certain "cooling" scheme to the model which cuts down the force magnitude in each iteration. the layout "freezes" after a number of iterations large enough to achieve a minimal-energy state.

The above force-directed method is computationally expensive and a pleasing layout usually cannot be obtained for larger graphs since the algorithm, starting with a random initial layout, gets easily "stuck" in a local energy minimum. To avoid this, a multilevel scheme is applied. The input graph is iteratively coarsened, either by removing the vertices from a maximal independent vertex set or by contracting the edges of a maximal matching in each iteration. Each coarsening level is processed by the force-directed algorithm, starting from the deepest (coarsest) one and "lifting" the obtained layout to the first upper level, using it as the initial layout for that level. The lifting done using the prolongation-matrix technique described in [46]. To support drawing large graphs (with, say, 1000 vertices or more), the matrices used in the lifting process are stored in sparse form. The multilevel scheme also speeds up the layout process significantly.

If the structure of the input graph is symmetric, a layout obtained by using a force-directed method typically reveals these symmetries, which is a unique property among graph drawing algorithms. To make the symmetries more prominent, a 2D layout is rotated such that the axis, with respect to which the layout exhibits the largest *symmetry score*, becomes vertical. Since symmetry detection is computationally expensive (up to $O(|V|^7)$ when using the symmetry measure of PURCHASE [83], for instance), the algorithm accounts only the convex hull and the barycenter of the layout, which may not always be enough to produce the optimal result. Nevertheless, this approach is fast and works for highly symmetrical graphs most of the time.

The spring method is also used for creating 3D graph layouts, which are obtained by passing the option `plot3d` to the `draw_graph` command.

```
> G1:=graph(trail(1,2,3,4,5,2))
```

an undirected unweighted graph with 5 vertices and 5 edges

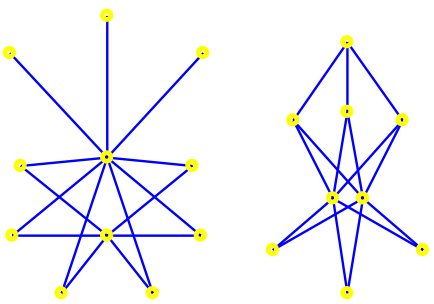
```
> G2:=star_graph(3)
```

an undirected unweighted graph with 4 vertices and 3 edges

```
> G:=tensor_product(G1,G2)
```

an undirected unweighted graph with 20 vertices and 30 edges

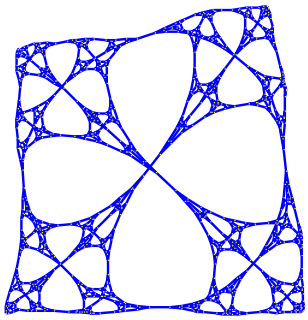
```
> draw_graph(G,spring,labels=false)
```



```
> S:=sierpinski_graph(5,4)
```

an undirected unweighted graph with 1024 vertices and 2046 edges

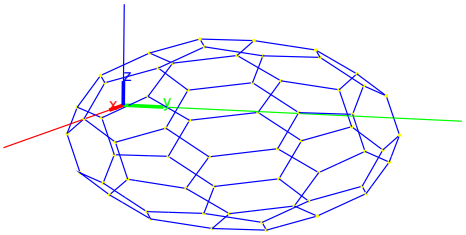
```
> draw_graph(S,spring)
```



8.568 sec

Note that vertex labels are automatically suppressed because of the large number of vertices.

```
> draw_graph(graph("soccerball"),plot3d,labels=false)
```



click k: kill 3d view

```
> G1:=graph("icosahedron"):; G2:=graph("dodecahedron"):;
```

"Done", "Done"

```
> G1:=highlight_edges(G1,edges(G1),red)
```

an undirected unweighted graph with 12 vertices and 30 edges

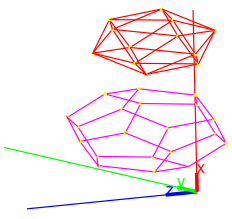
```
> G2:=highlight_edges(G2,edges(G2),magenta)
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> G:=disjoint_union(G1,G2)
```

an undirected unweighted graph with 32 vertices and 60 edges

```
> draw_graph(G,plot3d,labels=false)
```



click k: kill 3d view

6.1.3 Drawing trees

When the `tree[=r]` option is specified and the input graph G is a tree (and $r \in V$), it is drawn using a fast but simple node positioning algorithm inspired by the algorithm of WALKER [81], using the vertex r as the root node. If r is not specified, then the first node which admits the minimum height of the tree is used instead.

When drawing a rooted tree, one usually requires the following aesthetic properties [15].

A1. The layout displays the hierarchical structure of the tree, i.e. the y-coordinate of a node is given by its level.

A2. The edges do not cross each other.

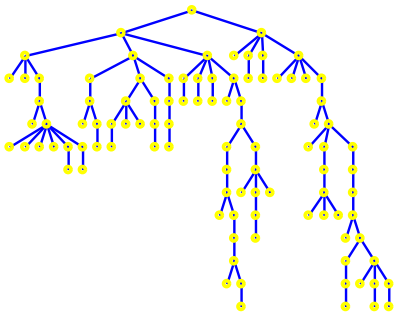
A3. The drawing of a sub-tree does not depend on its position in the tree, i.e. isomorphic sub-trees are drawn identically up to translation.

A4. The order of the children of a node is displayed in the drawing.

A5. The algorithm works symmetrically, i.e. the drawing of the reflection of a tree is the reflected drawing of the original tree.

The algorithm implemented in GIAC generally satisfies all the above properties except **A3**. Instead, it tries to spread the inner sub-trees evenly across the available horizontal space. It works by organizing the structure of the input tree into levels by using depth-first search and laying out each level subsequently, starting from the deepest one and climbing up to the root node. In the end, another depth-first traversal is made, shifting the sub-trees horizontally to avoid intersections between their edges. The algorithm runs in $O(|V|)$ time and uses the minimum of horizontal space to draw the tree with respect to the specified root node r.

```
> draw_graph(random_tree(100))
```



6.1.4 Drawing planar graphs

The algorithm for drawing planar graphs implemented in GIAC applies augmentation techniques to extend the input graph G to a graph G', which is homeomorphic to a triconnected graph, by adding temporary edges. The augmented graph G' is drawn using TUTTE's barycentric method (see [79] and [34, p. 293]) which puts each vertex in the barycenter of its neighbors. It is guaranteed that a (non-strict) convex drawing will be produced, without edge crossings. Finally, the duplicate of the outer face and temporary edges inserted during the augmentation stage are removed from layout.

TUTTE's algorithm requires that vertices of the outer face are initially placed on the boundary of a convex polygon. In order to produce a more flexible layout, the present algorithm duplicates the outer face such that the subgraph induced by the vertices on both the outer face and its duplicate is a prism graph. The duplicates of the outer-face vertices form a fixed regular polygon, allowing the original beneath it to take a more natural shape, and are deleted once the layout is frozen.

The augmentation process consists of two stages. The first stage consists of decomposing G into biconnected components (blocks) by using a depth-first search [33, p. 25] and decomposing each block into faces by using DEMOUCRON's algorithm (see [33, p. 88] and [57]). Embeddings obtained for each blocks are combined by adding one temporary edge for each articulation point, thus joining the two corresponding blocks. Figure 6.1a shows the outer faces of two blocks B_1 and B_2, connected by an articulation point (cut vertex). The temporary edge (shown in green) is added to join B_1 and B_2 into a single block. After "folding up" the tree of blocks, the algorithm picks the largest face in the resulting biconnected graph to be the outer face of the planar embedding.

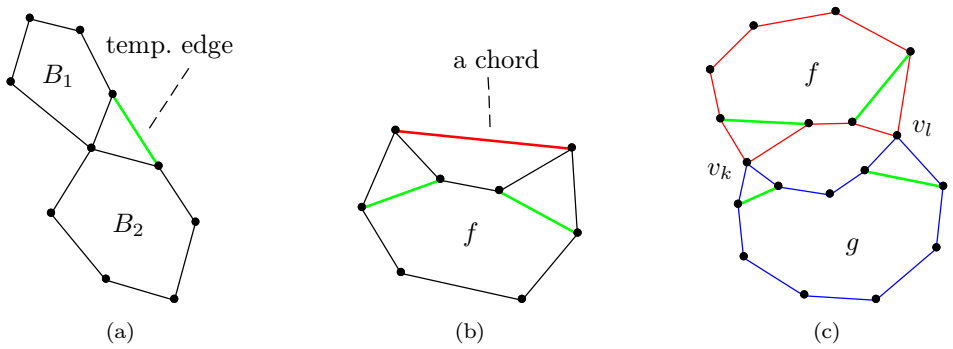


Fig. 6.1. (a) Joining blocks by adding a temporary edge; (b) a chorded face f; (c) faces f and g having two vertices but no edges in common

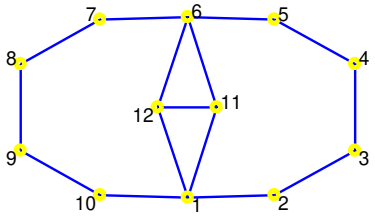
The second stage of the augmentation process consists of recursively decomposing each non-convex inner face into several convex polygons by adding temporary edges. An inner face $f = (v_1, ..., v_n)$ is non-convex if there exist k and l such that $1 \leqslant k < l - 1 < n$ and either $v_k v_l \in E$, in which case the edge $v_k v_l$ is a *chord* (see Figure 6.1b for an example) or there exists a face $g = (w_1, w_2, ..., v_k, ..., v_l, ..., w_{m-1}, w_m)$ such that the vertices $v_{k+1}, ..., v_{l-1}$ are not contained in g (see Figure 6.1c for an example). In Figure 6.1, temporary edges added by the algorithm are drawn in green.

This method of drawing planar graphs operates in $O(|V|^2)$ time, which makes it usable for graphs with up to several thousands vertices. A drawback of this method is that it sometimes creates clusters of vertices which are very close to each other, resulting in a very high ratio of the area of the largest inner face to the area of the smallest inner face (not considering the outer, infinite-area face). In order to cope with this issue, `draw_graph` redraws the layout several times (unless the graph is triconnected), each time randomizing the graph augmentation, an picks the layout which minimizes the above ratio. The number of redraws is proportional to $\frac{1}{|V|}$, hence `draw_graph` does not hang on large graphs. For $|V| < 500$, at least two drawings are computed. Of course, one can always call `draw_graph` repeatedly in attempt to obtain a better result. This strategy usually works well for biconnected graphs; however, 1-connected sparse graphs are often drawn poorly. In such cases consider using the `spring` option.

```
> G:=graph(trail(1,2,3,4,5,6,7,8,9,10,1),trail(11,12,6,11,1,12))
```

an undirected unweighted graph with 12 vertices and 15 edges

```
> draw_graph(G,planar)
```



Note that the inner diamond-like shape in the above drawing would end up flattened—making the two triangular faces invisible—if the input graph was not augmented; since the vertices with labels 11 and 12 are "attracted" to each other (namely, the two large faces are "inflating" themselves to become convex), they would end up in the same position.

In the second example the input graph is connected but not biconnected (it has two articulation points). It is obtained by removing a vertex from the Sierpiński triangle graph ST_3^3. Note that the syntax mode is set to Xcas in this example, so the first vertex label is zero.

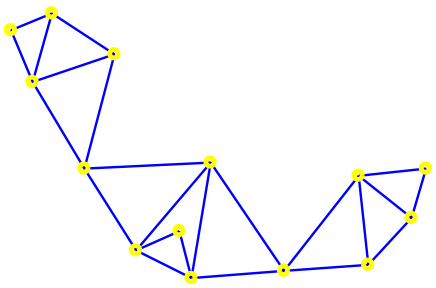
```
> G:=sierpinski_graph(3,3,triangle)
```

an undirected unweighted graph with 15 vertices and 27 edges

```
> G:=delete_vertex(G,3)
```

an undirected unweighted graph with 14 vertices and 23 edges

```
> draw_graph(G,planar,labels=false)
```



6.1.5 Circular graph drawings

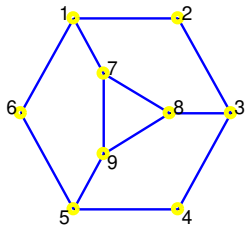
The drawing method selected by specifying the option `circle=L` or `convexhull=L` when calling `draw_graph` on a triconnected graph $G(V, E)$, where $L \subset V$ is a set of vertices in G, uses the following strategy. First, positions of the vertices from L are fixed so that they form a regular polygon on the unit circle. Other vertices, i.e. all vertices from $V \setminus L$, are placed in origin. Then an iterative force-directed algorithm [65], similar to TUTTE's barycentric method, is applied to obtain the final layout.

This approach gives best results for symmetrical graphs such as generalized Petersen graphs. In addition, if the input graph is planar and triconnected and the outer hull represents a face in a planar embedding, then the drawing will contain no edge crossings. Some very short edges may, however, end up crossing each other since the number of force-update iterations is limited.

```
> G:=graph([1,2,3,4,5,6,7,8,9],%{[1,2],[1,6],[1,7],[2,3],[3,4],[3,8],[4,5],[5,
  6],[5,9],[7,8],[7,9],[8,9]%})
```

an undirected unweighted graph with 9 vertices and 12 edges

```
> draw_graph(G,circle=[1,2,3,4,5,6])
```



To draw a planar triconnected graph, one should pass one of its faces as the outer hull. In the following example we draw a circular planar layout of the Frucht graph.

```
> G:=graph("frucht")
```

an undirected unweighted graph with 12 vertices and 18 edges

Get the list of faces:

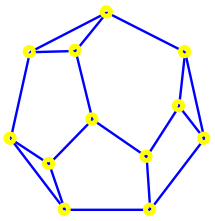
```
> purge(F):; is_planar(G,F)
```

"Done", true

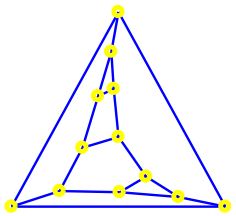
```
> apply(length,F)
```

[7, 4, 3, 5, 3, 5, 3, 6]

```
> draw_graph(G,circle=F[0],labels=false)
```



```
> draw_graph(G,circle=F[6],labels=false)
```



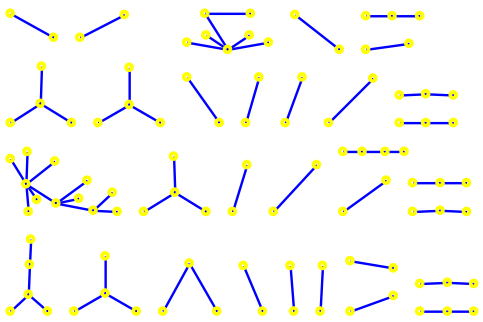
6.1.6 Drawing disconnected graphs

When the input graph has two or more connected components, each component is drawn separately and the drawings are subsequently arranged such that the bounding box of the whole drawing has the smallest perimeter under condition that as little space as possible is wasted inside the box.

```
> G:=random_planar_graph(100,0.9,0)
```

an undirected unweighted graph with 100 vertices and 68 edges

```
> draw_graph(G,planar)
```



6.1.7 Setting layout position, size, and title

Graph drawings may be moved around, scaled, and resized using optional arguments. This is useful for combining several drawings in one figure or adding graph drawings to larger graphical scenes. Individual drawings may also be given a title. Note that the graph attribute `name`, when one is provided, is used as the title in case the latter is not set.

Below we draw the first two discovered snarks in one figure.

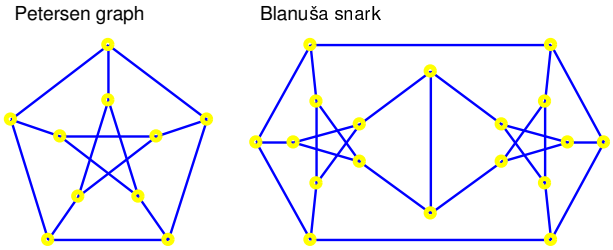
```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> B:=graph("blanusa")
```

an undirected unweighted graph with 18 vertices and 27 edges

```
> draw_graph(P,[0,0],size=[0,1],title="Petersen graph",labels=false);
  draw_graph(B,[1.32,0],size=[0,1],title="Blanuša snark",labels=false)
```

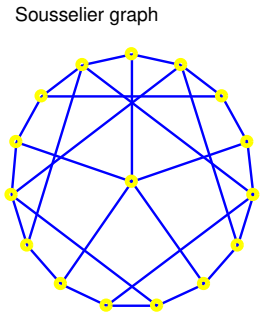


If graph has a name, it is printed automatically. To override this behavior, set the `title` option to `""` (an empty string).

```
> S:=set_graph_attribute(graph("sousselier"),"name"="Sousselier graph")
```

Sousselier graph: an undirected unweighted graph with 16 vertices and 27 edges

```
> draw_graph(S,labels=false)
```



6.2 Vertex positions

6.2.1 Setting vertex positions

`set_vertex_positions(G,L)`

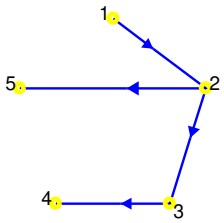
The command `set_vertex_positions` is used for assigning custom coordinates to vertices of a graph to be used when drawing the graph. It takes two arguments, a graph $G(V, E)$ and the list L of positions to be assigned to vertices in order of `vertices(G)`. The positions may be complex numbers, lists of coordinates or geometrical objects created by using the command `point`. `set_vertex_positions` returns a modified copy G' of G containing the specified layout.

Any subsequent call to `draw_graph` with G' as an argument and without specifying the drawing method will result in displaying vertices at the stored coordinates. However, if a drawing style is specified, the stored layout is ignored, although it remains stored in G'.

```
> G:=digraph([1,2,3,4,5],%{[1,2],[2,3],[3,4],[2,5]%})
```

a directed unweighted graph with 5 vertices and 4 arcs

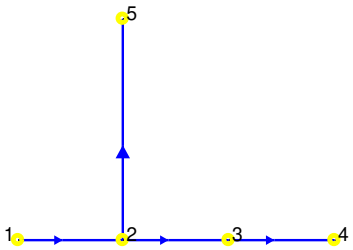
```
> draw_graph(G,circle)
```




```
> H:=set_vertex_positions(G,[[0,0],[0.5,0],[1.0,0],[1.5,0],[0.5,1]])
```

a directed unweighted graph with 5 vertices and 4 arcs

```
> draw_graph(H)
```



6.2.2 Generating vertex positions

Vertex positions can be generated for a particular graph G by using the `draw_graph` command with the additional argument P which should be an unassigned identifier. After the layout is obtained, it is assigned to P as a list of positions (complex numbers for 2D drawings or points for 3D drawings) for each vertex in order of `vertices(G)`.

By using the `set_vertex_positions` command, a desired layout obtained by calling `draw_graph` can be easily stored to the graph for future reference. In particular, each subsequent call of `draw_graph` with G as an argument will display the stored layout. The example below illustrates this property by setting a custom layout to the octahedral graph.

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> purge(P):; draw_graph(G,P,spring):;
```

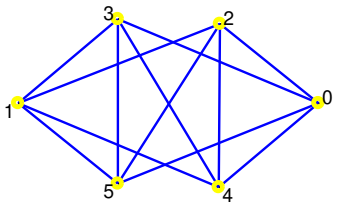
"Done","Done"

Now P contains vertex coordinates, which can be permanently stored to G:

```
> G:=set_vertex_positions(G,P)
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(G)
```



Note that, after a particular layout is fixed, it stays valid after a removal of vertices or edges, contracting an edge, or adding a new edge. The stored layout is invalidated after a new vertex is added to the graph (unless its position is specified by `set_vertex_attribute` upon the creation) or after discarding the `position` attribute of an existing vertex.

6.2.3 Custom layout example: spectral graph drawing

In this section it is demonstrated how `set_vertex_positions` can be used to obtain spectral drawings of graphs.

Let $G(V,E)$ be an undirected, connected graph with $|V|=m$. Furthermore, let $\rho: V \to \mathbb{R}^n$ be a **graph drawing** of G in $\mathbb{R}^n$. The **matrix of a graph drawing** ρ is a $m \times n$ matrix R whose i-th row corresponds to the row vector $\rho(v_i)$ containing coordinates of the point representing v_i in $\mathbb{R}^n$. Typically, it is desired that n is (much) smaller than m. The drawing is **balanced** if sum of entries in each column of R equals to 1.

The **energy** of a drawing R is given by

$$\varepsilon(R) = \sum_{v_i v_j \in E} w_{ij} \, \|\rho(v_i) - \rho(v_j)\|^2,$$

where w_{ij} is the weight of $v_i v_j$. (If G is unweighted, then $w_{ij}=1$ for all $1 \leqslant i, j \leqslant m$.) A drawing is considered to be "good" if it minimizes the energy function ε under certain constraints which prevent vertices from occupying the same position. The function ε can be expressed in terms of the (non-normalized) Laplacian matrix L:

$$\varepsilon(R) = \operatorname{tr}(R^T L R).$$

In order to avoid trivial minimum-energy layouts, it is reasonable to assume that the columns of R are pairwise orthogonal and that they have unit length. Therefore,

$$R^T R = I,$$

where I is identity matrix. A drawing which satisfies the above condition is called **orthogonal drawing**.

The following theorem provides a way to find minimum graph drawings under this condition. Note that, since G is undirected and connected, its Laplacian matrix has exactly one zero eigenvalue and $m-1$ strictly positive—but not necessarily distinct—eigenvalues (see Section 4.2.2). We assume that $n < m$. For details on spectral graph drawing, see [51].

Theorem 6.1. *Let the eigenvalues of L be $0=\lambda_1<\lambda_2\leqslant\lambda_3\leqslant\cdots\leqslant\lambda_m$. Then the minimal energy of any balanced orthogonal drawing of G in $\mathbb{R}^n$ is equal to $\lambda_2+\lambda_3+\cdots+\lambda_{n+1}$. The $m\times n$ matrix R whose columns are unit eigenvectors $u_2,\ldots,u_{n+1}$ associated with $\lambda_2,\ldots,\lambda_{n+1}$, respectively, represents a balanced orthogonal drawing of minimal energy.*

In particular, for $n=2$ and $n=3$ one obtains graph drawings in the two- resp. three-dimensional Euclidean space. For example, we can write a function `spectral_layout` which takes G as its argument and returns a copy of G with a spectral 2D layout stored to it. The code is given below.

```
spectral_layout:=proc(G)
  local L,R,evals,ev,S,m,k;
  m:=number_of_vertices(G);
  L:=evalf(laplacian_matrix(G));      // Laplacian matrix (approx)
  evals:=[eigenvals(L)];              // list of eigenvalues
```

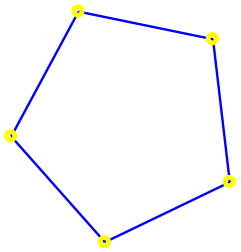
```
  ev:=tran(eigenvectors(L));          // list of corresponding eigenvectors
  S:=sort(zip(evals,[k$(k=0..m-1)])); // sort eigenvalues
  R:=tran([ev[S[1][1]],ev[S[2][1]]]); // get the second and third eigenvector
  return set_vertex_positions(G,R);
end:;
```

After compiling the above program in Xcas (by copying it into a programmation cell which we create by pressing ⟨key|Alt+P⟩), we demonstrate it in the following examples.

```
> C:=cycle_graph(5)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> draw_graph(spectral_layout(C),labels=false)
```



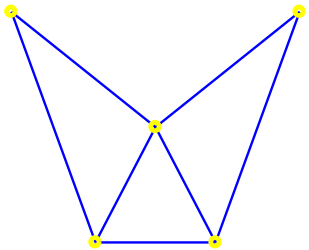
```
> A:=[[0,1,1,0,0],[1,0,1,1,1],[1,1,0,1,0],[0,1,1,0,1],[0,1,0,1,0]]
```

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

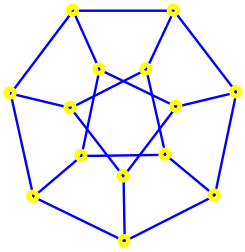
```
> G:=graph(A)
```

an undirected unweighted graph with 5 vertices and 7 edges

```
> draw_graph(spectral_layout(G),labels=false)
```



```
> draw_graph(spectral_layout(petersen_graph(7,2)),labels=false)
```



Unlike the methods used by `draw_graph` command, spectral graph layout takes edge weights into account. Assuming that all weights are positive, endpoints of edges with larger weights tend to get closer to each other, as in the next example.

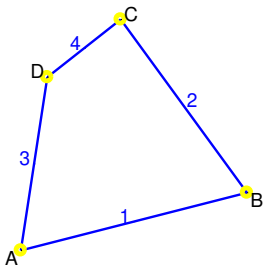
```
> W:=[[0,1,0,3],[1,0,2,0],[0,2,0,4],[3,0,4,0]]
```

$$\begin{pmatrix} 0 & 1 & 0 & 3 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \\ 3 & 0 & 4 & 0 \end{pmatrix}$$

```
> purge(A,B,C,D):; H:=graph([A,B,C,D],W)
```

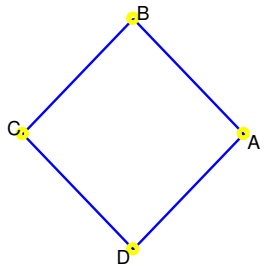
"Done", an undirected weighted graph with 4 vertices and 4 edges

```
> draw_graph(spectral_layout(H))
```



On the other hand, the underlying graph of H is unweighted and hence drawn as a regular polygon.

```
> draw_graph(spectral_layout(underlying_graph(H)))
```



6.3 Highlighting parts of graphs

6.3.1 Highlighting vertices

`highlight_vertex(G,v,⟨c⟩)`
`highlight_vertex(G,[v1,v2,..,vk],⟨c⟩)`
`highlight_vertex(G,[v1,v2,..,vk],[c1,c2,..,ck])`

The command `highlight_vertex` changes color of one or more vertices in a graph. It takes two or three arguments: a graph $G(V, E)$, a vertex $v \in V$ or a list of vertices $v_1, v_2, ..., v_k \in V$ and optionally the new color c or a list of colors $c_1, c_2, ..., c_k$ for the selected vertices (the default is green). It returns a modified copy of G in which the specified vertices are colored with the specified color(s).

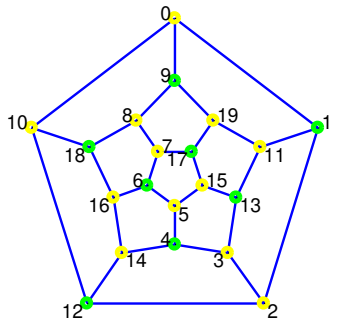
```
> G:=graph("dodecahedron")
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> L:=maximum_independent_set(G)
```

$$[1, 4, 6, 9, 12, 13, 17, 18]$$

```
> draw_graph(highlight_vertex(G,L))
```



6.3.2 Highlighting edges and trails

```
highlight_edges(G,e,⟨c⟩)
highlight_edges(G,[e1,e2,..,ek],⟨c⟩)
highlight_edges(G,[e1,e2,..,ek],[c1,c2,..,ck])
highlight_trail(G,T,⟨c⟩)
highlight_trail(G,[T1,T2,..,Tk],⟨c⟩)
highlight_trail(G,[T1,T2,..,Tk],[c1,c2,..,ck])
```

To highlight an edge or a set of edges in a graph, use the `highlight_edges` command. If the edges form a trail, then it is usually more convenient to use the `highlight_trail` command.

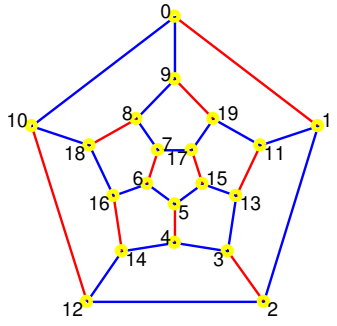
`highlight_edges` takes two or three arguments: a graph $G(V, E)$, an edge $e \in E$ or a list of edges $e_1, e_2, ..., e_k \in E$ and optionally the new color c or a list of colors $c_1, c_2, ..., c_k$ for the selected edges (the default color is red). It returns a modified copy of G in which the specified edges are colored with the specified color.

`highlight_trail` takes two or three arguments: a graph $G(V, E)$, a trail T or a list of trails $T_1, T_2, ..., T_k$ and optionally the new color c or a list of colors $c_1, c_2, ..., c_k$. The command returns the copy of G in which edges between consecutive vertices in each of the given trails are highlighted with color c (by default red) or the trail T_i is highlighted with color c_i for $i = 1, 2, ..., k$.

```
> M:=maximum_matching(G)
```

$$[[0, 1], [2, 3], [4, 5], [6, 7], [8, 18], [9, 19], [10, 12], [11, 13], [14, 16], [15, 17]]$$

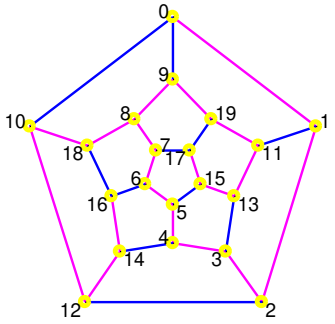
```
> draw_graph(highlight_edges(G,M))
```



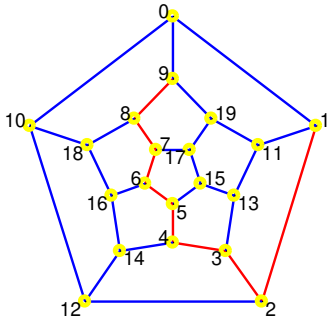
```
> S:=spanning_tree(G)
```

an undirected unweighted graph with 20 vertices and 19 edges

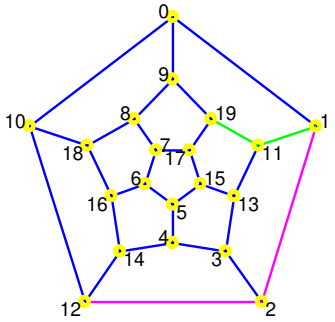
```
> draw_graph(highlight_edges(G,edges(S),magenta))
```




```
> draw_graph(highlight_trail(G,[1,2,3,4,5,6,7,8,9]))
```




```
> draw_graph(highlight_trail(G,shortest_path(G,1,[19,12]),[green,magenta]))
```



6.3.3 Highlighting subgraphs

`highlight_subgraph(G,S,⟨weights⟩)`
`highlight_subgraph(G,S,c1,c2,⟨weights⟩)`
`highlight_subgraph(G,[S1,S2,..,Sk],⟨c1,c2⟩)`

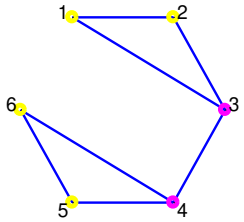
The command `highlight_subgraph` is used for highlighting subgraph(s) of a graph. It takes two or four input arguments: a graph $G(V, E)$, a subgraph $S(V', E')$ of G or a list of subgraphs $S_1, S_2, ..., S_k$ in G and optionally the new colors c_1, c_2 for the edges and vertices of the selected subgraph(s), respectively. It returns a modified copy of G with the selected subgraph(s) colored as specified. If colors are not given, then red and green are used, respectively.

The option `weights` may be passed as an additional argument if G and S are weighted graphs. In that case, the weights of edges in $E' \subset E$ in G are overwritten with those defined in S for the same edges, which is useful for e.g. visualizing maximum flows in networks.

```
> G:=graph(%{[1,2],[2,3],[3,1],[3,4],[4,5],[5,6],[6,4]%})
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> draw_graph(highlight_vertex(G,articulation_points(G),magenta))
```



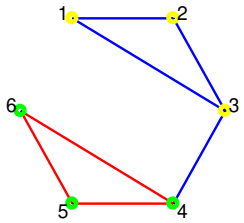

```
> B:=biconnected_components(G)
```

$$[[4,5,6],[3,4],[1,2,3]]$$

```
> S:=induced_subgraph(G,B[0])
```

an undirected unweighted graph with 3 vertices and 3 edges

```
> draw_graph(highlight_subgraph(G,S))
```



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