Graph theory package for Giac/Xcas

Reference manual

October 2018

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Introduction

This document[1] contains an overview of the library of graph theory commands built in Giac computation kernel and supported within the Xcas GUI. The library provides an effective and free replacement for the GraphTheory package in Maple with a high level of syntax compatibility (although there are some minor differences).

For each command, the calling syntax is presented along with the detailed description of its functionality and several examples. The square brackets `[` and `]` in the calling syntax indicate that the respective argument should be a list of particular elements or that its inclusion is optional. The character `|` stands for *or*.

The algorithms in this library are implemented according to relevant scientific publications. Although the development focus was on simplicity, the algorithms are reasonably fast. For some more difficult tasks, such as for solving traveling salesman problem, finding graph colorings and graph isomorphism, freely available third party libraries are used, in particular GNU Linear Programming Kit (GLPK) and nauty. These libraries, included in Giac/Xcas by default, are optional during the compilation; most commands have no dependencies save Giac itself.

This library was written and documented by Luka Marohnić[2]. The author would like to thank Bernard Parisse, the Giac/Xcas project leader, for integrating the package and Jose Capco for suggesting nauty integration.

1. This manual was written in GNU TEXMACS, a scientific document editing platform. All examples were entered as interactive Giac sessions.

2. Email: `luka.marohnic@tvz.hr`

Chapter 1

Constructing graphs

1.1. General graphs

The commands `graph` and `digraph` are used for constructing general graphs.

1.1.1. Undirected graphs

Syntax:		
	`graph(n|V,[opts])`	graph with n vertices or vertex set V and no edges
	`graph(V,E,[opts])`	graph (V, E)
	`graph(E,[opts])`	graph with edge set E, vertices implied
	`graph(V,T,[opts])`	graph (V, E) where E is the edge set of trail T
	`graph(T,[opts])`	graph with edges from trail T, vertices implied
	`graph(V,T1,T2,T3,..,Tk,[opts])`	graph with edge set consisting of edges on the given trails
	`graph(T1,T2,T3,..,Tk,[opts])`	
	`graph(A,[opts])`	graph with adjacency or weight matrix A, vertices implied
	`graph(V,E,A,[opts])`	weighted graph (V, E) with weight matrix A
	`graph(V,Perm,[opts])`	digraph with a single cycle as a permutation of vertices V
	`graph(Str)`	special graph

The command `graph` takes between one and three main arguments, each of them being one of the following structural elements of the resulting graph $G(V, E)$. Throughout this manual, an edge $e \in E$ with endpoints $u, v \in V$ is denoted by $e = uv$. Note that the order of the endpoints does not matter if G is undirected; hence $uv = vu$. If G is directed, uv and vu are treated as separate edges.

- number n or list of vertices V (a vertex may be any atomic object, such as an integer, a symbol or a string); it must be the first argument if used,
- set of edges E (each edge is represented by the list of its endpoints), a permutation, a trail of edges or a sequence of trails; it can be either the first or the second argument if used,
- trail T or sequence of trails $T_1, T_2, ..., T_k$,
- permutation `Perm` of vertices,
- adjacency or weight matrix A,
- string `Str`, representing a special graph.

Optionally, the following options may be appended to the sequence of arguments:

- `directed = true` or `false`,
- `weighted = true` or `false`,
- `color =` an integer or a list of integers representing color(s) of the vertices,
- `coordinates =` a list of vertex 2D or 3D coordinates.

The `graph` command may also be called by passing a string `Str`, representing the name of a special graph, as its only argument. In that case the corresponding graph will be constructed and returned. The supported graphs are listed below.

special graph	name in Giac	special graph	name in Giac
Balaban 10-cage	`balaban10`	Grötzsch graph	`grotzsch`
Balaban 11-cage	`balaban11`	Harries graph	`harries`
Bidiakis cube	`bidiakis`	Harries–Wong graph	`harries-wong`
Biggs-Smith graph	`biggs-smith`	Heawood graph	`heawood`
2nd Blanuša snark	`blanusa`	Herschel graph	`herschel`
Bull graph	`bull`	Hoffman graph	`hoffman`
Butterfly graph	`butterfly`	Icosahedral graph	`icosahedron`
Clebsch graph	`clebsch`	Levi graph (Tutte 8-cage)	`levi`
Chvátal graph	`chvatal`	Ljubljana graph	`ljubljana`
Coxeter graph	`coxeter`	McGee graph	`mcgee`
Desargues graph	`desargues`	Moser spindle	`moser`
Diamond graph	`diamond`	Möbius–Kantor graph	`mobius-kantor`
Dodecahedral graph	`dodecahedron`	Nauru graph	`nauru`
Dürer graph	`durer`	Octahedral graph	`octahedron`
Dyck graph	`dyck`	Pappus graph	`pappus`
Errera graph	`errera`	Petersen graph	`petersen`
F26A graph	`f26a`	Poussin graph	`poussin`
Folkman graph	`folkman`	Robertson graph	`robertson`
Foster graph	`foster`	Truncated icosahedral graph	`soccerball`
Franklin graph	`franklin`	Shrikhande graph	`shrikhande`
Frucht graph	`frucht`	Tetrahedral graph	`tehtrahedron`
Goldner-Harary graph	`goldner-harary`	Tietze graph	`tietze`
Golomb graph	`golomb`	Tutte graph	`tutte`
Gray graph	`gray`	Tutte 12-cage	`tutte12`
Grinberg graph	`grinberg`	Wagner graph	`wagner`

1.1.2. Directed graphs

The `digraph` command is used for creating directed graphs, although it is also possible with the `graph` command by specifying the option `directed=true`. Actually, calling `digraph` is the same as calling `graph` with that option appended to the sequence of arguments. However, creating special graphs is not supported by `digraph` since they are all undirected.

Edges in directed graphs are usually called **arcs**.

1.1.3. Examples

Creating vertices. A graph consisting only of vertices and no edges can be created simply by providing the number of vertices or the list of vertex labels.

```
> graph(5)
```

an undirected unweighted graph with 5 vertices and 0 edges

```
> graph([a,b,c])
```

an undirected unweighted graph with 3 vertices and 0 edges

The commands that return graphs often need to generate vertex labels. In these cases ordinal integers are used, which are 0-based in Xcas mode and 1-based in Maple mode. Examples throughout this manual are made by using the default, Xcas mode.

Creating edges and arcs. Edges/arcs must be specified inside a set so that it can be distinguished from a (adjacency or weight) matrix. If only a set of edges/arcs is specified, the vertices needed to establish these will be created automatically. Note that, when constructing a directed graph, the order of the vertices in an arc matters; in undirected graphs it is not meaningful.

```
> graph(%{[a,b],[b,c],[a,c]%})
```

an undirected unweighted graph with 3 vertices and 3 edges

Edge weights may also be specified.

```
> graph(%{[[a,b],2],[[b,c],2.3],[[c,a],3/2]%})
```

an undirected weighted graph with 3 vertices and 3 edges

If the graph contains isolated vertices (not connected to any other vertex) or a particular order of vertices is desired, the list of vertices has to be specified first.

```
> graph([d,b,c,a],%{[a,b],[b,c],[a,c]%})
```

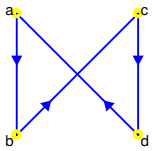
an undirected unweighted graph with 4 vertices and 3 edges

Creating paths and trails. A directed graph can also be created from a list of n vertices and a permutation of order n. The resulting graph consists of a single directed cycle with the vertices ordered according to the permutation.

```
> G:=graph([a,b,c,d],[1,2,3,0])
```

a directed unweighted graph with 4 vertices and 4 arcs

```
> draw_graph(G)
```



Alternatively, one may specify edges as a `trail`.

```
> digraph([a,b,c,d],trail(b,c,d,a))
```

a directed unweighted graph with 4 vertices and 3 arcs

Using trails is also possible when creating undirected graphs. Also, some vertices in a trail may be repeated, which is not allowed in a path.

```
> G:=graph([a,b,c,d],trail(b,c,d,a,c))
```

an undirected unweighted graph with 4 vertices and 4 edges

```
> edges(G)
```

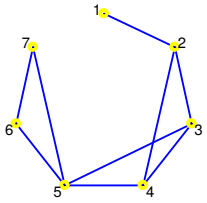
$$\begin{pmatrix} a & c \\ a & d \\ b & c \\ c & d \end{pmatrix}$$

It is possible to specify several trails in a sequence, which is useful when designing more complex graphs.

```
> G:=graph(trail(1,2,3,4,2),trail(3,5,6,7,5,4))
```

an undirected unweighted graph with 7 vertices and 9 edges

```
> draw_graph(G)
```



Specifying adjacency or weight matrix. A graph can be created from a single square matrix $A=[a_{ij}]_n$ of order n. If it contains only ones and zeros and has zeros on its diagonal, it is assumed to be the adjacency matrix for the desired graph. Otherwise, if an element outside the set $\{0,1\}$ is encountered, it is assumed that the matrix of edge weights is passed as input, causing the resulting graph to be weighted accordingly. In each case, exactly n vertices will be created and i-th and j-th vertex will be connected iff $a_{ij}\neq 0$. If the matrix is symmetric, the resulting graph will be undirected, otherwise it will be directed.

```
> G:=graph([[0,1,1,0],[1,0,0,1],[1,0,0,0],[0,1,0,0]])
```

an undirected unweighted graph with 4 vertices and 3 edges

```
> edges(G)
```

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$$

```
> G:=graph([[0,1.0,2.3,0],[4,0,0,3.1],[0,0,0,0],[0,0,0,0]])
```

a directed weighted graph with 4 vertices and 4 arcs

```
> edges(G,weights)
```

$$\{[[0,1],1.0],[[0,2],2.3],[[1,0],4],[[1,3],3.1]\}$$

List of vertex labels can be specified alongside a matrix.

```
> graph([a,b,c,d],[[0,1,1,0],[1,0,0,1],[1,0,0,0],[0,1,0,0]])
```

an undirected unweighted graph with 4 vertices and 3 edges

When creating a weighted graph, one can first specify the list of n vertices and the set of edges, followed by a square matrix A of order n. Then for every edge $\{i,j\}$ or arc (i,j) the element a_{ij} of A is assigned as its weight. Other elements of A are ignored.

```
> G:=digraph([a,b,c],%{[a,b],[b,c],[a,c]%},[[0,1,2],[3,0,4],[5,6,0]])
```

a directed weighted graph with 3 vertices and 3 arcs

```
> edges(G,weights)
```

$$\{[[a,b],1],[[a,c],2],[[b,c],4]\}$$

Creating special graphs. When a special graph is desired, one just needs to pass its name to the `graph` command. An undirected unweighted graph will be returned.

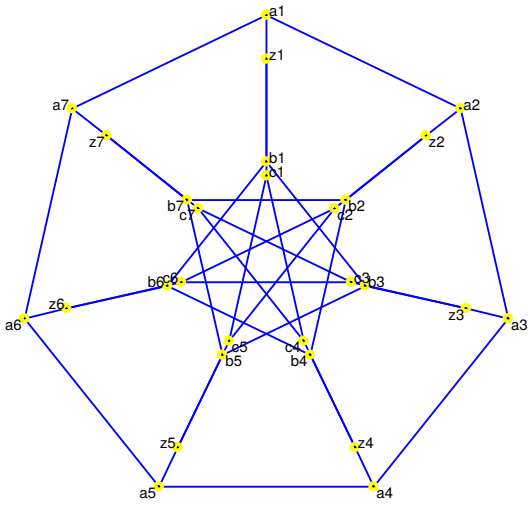
```
> graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> G:=graph("coxeter")
```

an undirected unweighted graph with 28 vertices and 42 edges

```
> draw_graph(G)
```



1.2. Cycle and path graphs

1.2.1. Cycle graphs

The command `cycle_graph` is used for constructing cycle graphs [27, pp. 4].

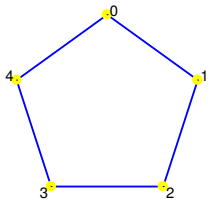
Syntax: `cycle_graph(n)`
`cycle_graph(V)`

`cycle_graph` takes a positive integer n or a list of distinct vertices V as its only argument and returns the graph consisting of a single cycle on the specified vertices in the given order. If n is specified it is assumed to be the desired number of vertices, in which case they will be created and labeled with the first n integers (starting from 0 in Xcas mode and from 1 in Maple mode). The resulting graph will be given the name `Cn`, for example `C4` for $n = 4$.

```
> C5:=cycle_graph(5)
```

C5: an undirected unweighted graph with 5 vertices and 5 edges

```
> draw_graph(C5)
```




```
> cycle_graph(["a","b","c","d","e"])
```

C5: an undirected unweighted graph with 5 vertices and 5 edges

1.2.2. Path graphs

The command `path_graph` is used for constructing path graphs [27, pp. 4].

Syntax: `path_graph(n)`
`path_graph(V)`

`path_graph` takes a positive integer n or a list of distinct vertices V as its only argument and returns a graph consisting of a single path on the specified vertices in the given order. If n is specified it is assumed to be the desired number of vertices, in which case they will be created and labeled with the first n integers (starting from 0 in Xcas mode resp. from 1 in Maple mode).

Note that a path cannot intersect itself. Paths that are allowed to cross themselves are called **trails** (see the command `trail`).

```
> path_graph(5)
```

an undirected unweighted graph with 5 vertices and 4 edges

```
> path_graph(["a","b","c","d","e"])
```

an undirected unweighted graph with 5 vertices and 4 edges

1.2.3. Trails of edges

Syntax: `trail(v1,v2,..,vk)`
`trail2edges(T)`

If the dummy command `trail` is called with a sequence of vertices $v_1, v_2, ..., v_k$ as arguments, it returns the symbolic expression representing the trail which visits the specified vertices in the given order. The resulting symbolic object is recognizable by some commands, for example `graph` and `digraph`. Note that a trail may cross itself (some vertices may be repeated in the sequence).

Any trail T is easily converted to the corresponding list of edges by calling the `trail2edges` command, which takes the trail as its only argument.

```
> T:=trail(1,2,3,4,2):; graph(T)
```

Done, an undirected unweighted graph with 4 vertices and 4 edges

```
> trail2edges(T)
```

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 2 \end{pmatrix}$$

1.3. COMPLETE GRAPHS

1.3.1. Complete (multipartite) graphs

The command `complete_graph` is used for construction of complete (multipartite) graphs.

Syntax: `complete_graph(n)`
`complete_graph(V)` complete graphs
`complete_graph(n1,n2,..,nk)` complete multipartite graphs

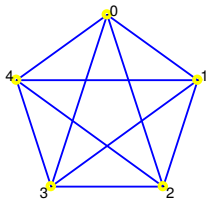
`complete_graph` can be called with a single argument, a positive integer n or a list of distinct vertices V, in which case it returns the complete graph [27, pp. 2] on the specified vertices. If integer n is specified, it is assumed that it is the desired number of vertices and they will be created and labeled with the first n integers (starting from 0 in Xcas mode and from 1 in Maple mode).

If `complete_graph` is given a sequence of positive integers $n_1, n_2, ..., n_k$ as its argument, it returns a complete multipartite graph with partitions of size $n_1, n_2, ..., n_k$.

```
> K5:=complete_graph(5)
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> draw_graph(K5)
```




```
> K3:=complete_graph([a,b,c])
```

an undirected unweighted graph with 3 vertices and 3 edges

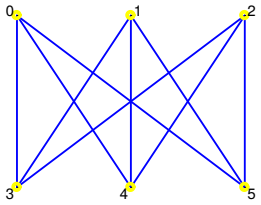
```
> edges(K3)
```

$$\{[a,b],[a,c],[b,c]\}$$

```
> K33:=complete_graph(3,3)
```

an undirected unweighted graph with 6 vertices and 9 edges

```
> draw_graph(K33)
```



1.3.2. Complete trees

The commands `complete_binary_tree` and `complete_kary_tree` are used for construction of complete binary trees and complete k-ary trees, respectively.

Syntax: `complete_binary_tree(n)`
`complete_kary_tree(k,n)`

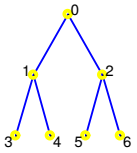
`complete_binary_tree` takes a positive integer n as its only argument and returns a complete binary tree of depth n.

`complete_kary_tree` takes positive integers k and n as its arguments and returns the complete k-ary tree of depth n.

```
> T1:=complete_binary_tree(2)
```

an undirected unweighted graph with 7 vertices and 6 edges

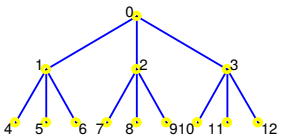
```
> draw_graph(T1)
```




```
> T2:=complete_kary_tree(3,2)
```

an undirected unweighted graph with 13 vertices and 12 edges

```
> draw_graph(T2)
```



1.4. SEQUENCE GRAPHS

1.4.1. Creating graphs from degree sequences

The command `sequence_graph` is used for constructing graphs from degree sequences.

Syntax: `sequence_graph(L)`

`sequence_graph` takes a list L of positive integers as its only argument and, if L represents a graphic sequence, the corresponding graph G with $|L|$ vertices is returned. If the argument is not a graphic sequence, an error is returned.

```
> sequence_graph([3,2,4,2,3,4,5,7])
```

an undirected unweighted graph with 8 vertices and 15 edges

Sequence graphs are constructed in $O(|L|^2 \log |L|)$ time by applying the algorithm of HAVEL and HAKIMI [31].

1.4.2. Validating graphic sequences

The command `is_graphic_sequence` is used to check whether a list of integers represents the degree sequence of some graph.

Syntax: `is_graphic_sequence(L)`

`is_graphic_sequence` takes a list L of positive integers as its only argument and returns `true` if there exists a graph $G(V, E)$ with degree sequence $\{\deg v : v \in V\}$ equal to L and `false` otherwise. The algorithm, which has the complexity $O(|L|^2)$, is based on the theorem of ERDŐS and GALLAI.

```
> is_graphic_sequence([3,2,4,2,3,4,5,7])
```

true

1.5. INTERSECTION GRAPHS

1.5.1. Interval graphs

The command `interval_graph` is used for construction of interval graphs.

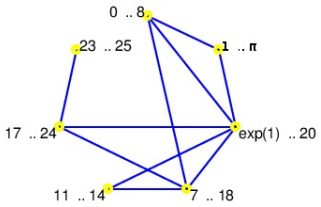
Syntax: `interval_graph(L)`

`interval_graph` takes a sequence or list L of real-line intervals as its argument and returns an undirected unweighted graph with these intervals as vertices (the string representations of the intervals are used as labels), each two of them being connected with an edge if and only if the corresponding intervals intersect.

```
> G:=interval_graph(0..8,1..pi,exp(1)..20,7..18,11..14,17..24,23..25)
```

an undirected unweighted graph with 7 vertices and 10 edges

```
> draw_graph(G)
```



1.5.2. Kneser graphs

The commands `kneser_graph` and `odd_graph` are used for construction of Kneser graphs.

Syntax: `kneser_graph(n,k)`
`odd_graph(d)`

`kneser_graph` takes two positive integers $n \leq 20$ and k as its arguments and returns the Kneser graph $K(n,k)$. The latter is obtained by setting all k-subsets of a set of n elements as vertices and connecting each two of them if and only if the corresponding sets are disjoint. Therefore, each Kneser graph is the complement of the corresponding intersection graph on the same collection of subsets.

Kneser graphs can get exceedingly complex even for relatively small values of n and k. Note that the number of vertices in $K(n,k)$ is equal to $\binom{n}{k}$.

```
> kneser_graph(5,2)
```

an undirected unweighted graph with 10 vertices and 15 edges

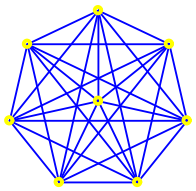
```
> G:=kneser_graph(8,1)
```

an undirected unweighted graph with 8 vertices and 28 edges

```
> is_clique(G)
```

true

```
> draw_graph(G,spring,labels=false)
```



The command `odd_graph` is used for creating **odd graphs**, i.e. Kneser graphs with parameters $n = 2d+1$ and $k = d$ for $d \geqslant 1$.

`odd_graph` takes a positive integer $d \leq 8$ as its only argument and returns d-th odd graph $K(2d+1, d)$. Note that the odd graphs with $d > 8$ will not be constructed as they are too big to handle.

```
> odd_graph(3)
```

an undirected unweighted graph with 10 vertices and 15 edges

1.6. Special graphs

1.6.1. Hypercube graphs

The command `hypercube_graph` is used for construction of hypercube graphs.

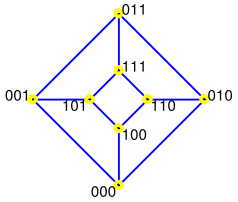
Syntax: `hypercube_graph(n)`

`hypercube_graph` takes a positive integer n as its only argument and returns the hypercube graph of dimension n on 2^n vertices. The vertex labels are strings of binary digits of length n. Two vertices are joined by an edge if and only if their labels differ in exactly one character. The hypercube graph for $n=2$ is a square and for $n=3$ it is a cube.

```
> H:=hypercube_graph(3)
```

an undirected unweighted graph with 8 vertices and 12 edges

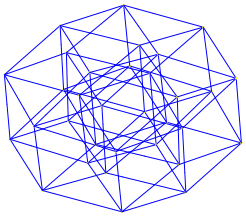
```
> draw_graph(H,planar)
```




```
> H:=hypercube_graph(5)
```

an undirected unweighted graph with 32 vertices and 80 edges

```
> draw_graph(H,plot3d,labels=false)
```



1.6.2. Star graphs

The command `star_graph` is used for construction of star graphs.

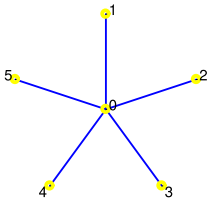
Syntax: `star_graph(n)`

`star_graph` takes a positive integer n as its only argument and returns the star graph with $n+1$ vertices, which is equal to the complete bipartite graph `complete_graph(1,n)` i.e. a n-ary tree with one level.

```
> G:=star_graph(5)
```

an undirected unweighted graph with 6 vertices and 5 edges

```
> draw_graph(G)
```



1.6.3. Wheel graphs

The command `wheel_graph` is used for construction of wheel graphs.

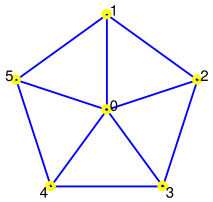
Syntax: `wheel_graph(n)`

`wheel_graph` takes a positive integer n as its only argument and returns the wheel graph with $n+1$ vertices.

```
> G:=wheel_graph(5)
```

an undirected unweighted graph with 6 vertices and 10 edges

```
> draw_graph(G)
```



1.6.4. Web graphs

The command `web_graph` is used for construction of web graphs.

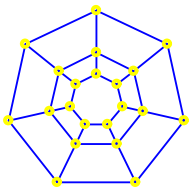
Syntax: `web_graph(a,b)`

`web_graph` takes two positive integers a and b as its arguments and returns the web graph with parameters a and b, namely the Cartesian product of `cycle_graph(a)` and `path_graph(b)`.

```
> G:=web_graph(7,3)
```

an undirected unweighted graph with 21 vertices and 35 edges

```
> draw_graph(G,labels=false)
```



1.6.5. Prism graphs

The command `prism_graph` is used for construction of prism graphs.

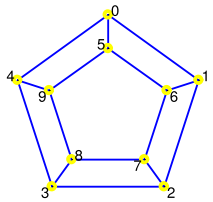
Syntax: `prism_graph(n)`

`prism_graph` takes a positive integer n as its only argument and returns the prism graph with parameter n, namely `petersen_graph(n,1)`.

```
> G:=prism_graph(5)
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> draw_graph(G)
```



1.6.6. Antiprism graphs

The command `antiprism_graph` is used for construction of antiprism graphs.

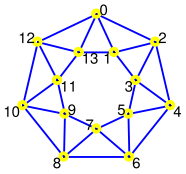
Syntax: `antiprism_graph(n)`

`antiprism_graph` takes a positive integer n as its only argument and returns the antiprism graph with parameter n, which is constructed from two concentric cycles of n vertices by joining each vertex of the inner to two adjacent nodes of the outer cycle.

```
> G:=antiprism_graph(7)
```

an undirected unweighted graph with 14 vertices and 28 edges

```
> draw_graph(G)
```



1.6.7. Grid graphs

The command `grid_graph` resp. `torus_grid_graph` is used for construction of rectangular/triangular resp. torus grid graphs.

Syntax:		
	`grid_graph(m,n)`	rectangular grids
	`grid_graph(m,n,triangle)`	triangular grids
	`torus_grid_graph(m,n)`	toroidal grids

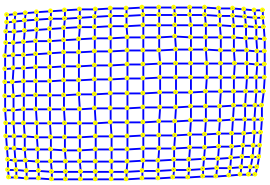
`grid_graph` takes two positive integers m and n as its arguments and returns the m by n grid on $m\,n$ vertices, namely the Cartesian product of `path_graph(m)` and `path_graph(n)`. If the option `triangle` is passed as the third argument, the returned graph is a triangular grid on $m\,n$ vertices defined as the underlying graph of the strong product of two directed path graphs with m and n vertices, respectively [1, Definition 2, pp. 189]. Strong product is defined as the union of Cartesian and tensor products.

`torus_grid_graph` takes two positive integers m and n as its arguments and returns the m by n torus grid on $m\,n$ vertices, namely the Cartesian product of `cycle_graph(m)` and `cycle_graph(n)`.

```
> G:=grid_graph(15,20)
```

an undirected unweighted graph with 300 vertices and 565 edges

```
> draw_graph(G,spring)
```

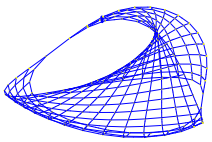


For example, connecting vertices in the opposite corners of the above grid yields a grid-like graph with no corners.

```
> G:=add_edge(G,[["14:0","0:19"],["0:0","14:19"]])
```

an undirected unweighted graph with 300 vertices and 567 edges

```
> draw_graph(G,plot3d)
```



In the next example, the Möbius strip is constructed by connecting the vertices in the opposite sides of a narrow grid graph.

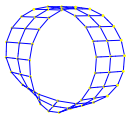
```
> G:=grid_graph(20,3)
```

an undirected unweighted graph with 60 vertices and 97 edges

```
> G:=add_edge(G,[["0:0","19:2"],["0:1","19:1"],["0:2","19:0"]])
```

an undirected unweighted graph with 60 vertices and 100 edges

```
> draw_graph(G,plot3d,labels=false)
```

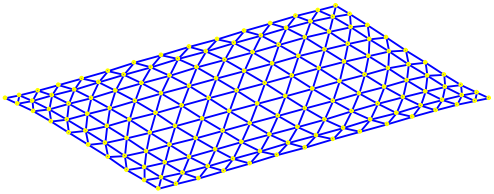


A triangular grid is created by passing the option `triangle`.

```
> G:=grid_graph(10,15,triangle)
```

an undirected unweighted graph with 150 vertices and 401 edges

```
> draw_graph(G,spring)
```

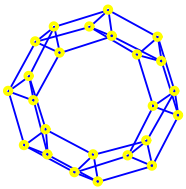


The next example demonstrates creating a torus grid graph with eight triangular levels.

```
> G:=torus_grid_graph(8,3)
```

an undirected unweighted graph with 24 vertices and 48 edges

```
> draw_graph(G,spring,labels=false)
```



1.6.8. Sierpiński graphs

The command `sierpinski_graph` is used for construction of Sierpiński-type graphs S_k^n and ST_k^n [34].

Syntax: `sierpinski_graph(n,k)`
`sierpinski_graph(n,k,triangle)`

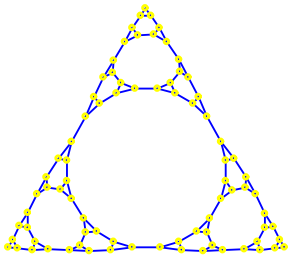
`sierpinski_graph` takes two positive integers n and k as its arguments and optionally the option `triangle` as the third argument. It returns the Sierpiński (triangle) graph with parameters n and k.

The Sierpiński triangle graph ST_k^n is obtained by contracting all non-clique edges in S_k^n. To detect such edges the variant of the algorithm by Bron and Kerbosch, developed by Tomita et al. in [59], is used, which can be time consuming for $n > 6$.

```
> S:=sierpinski_graph(4,3)
```

an undirected unweighted graph with 81 vertices and 120 edges

```
> draw_graph(S,spring)
```



In particular, ST_3^n is the well-known Sierpiński sieve graph of order n.

```
> sierpinski_graph(4,3,triangle)
```

an undirected unweighted graph with 42 vertices and 81 edges

```
> sierpinski_graph(5,3,triangle)
```

an undirected unweighted graph with 123 vertices and 243 edges

A drawing of the graph produced by the last command line is shown in Figure 3.1.

1.6.9. Generalized Petersen graphs

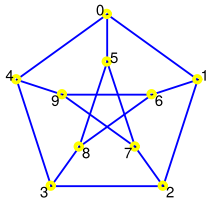
The command `petersen_graph` is used for construction of generalized Petersen graphs $P(n,k)$.

Syntax: `petersen_graph(n)`
`petersen_graph(n,k)`

`petersen_graph` takes two positive integers n and k as its arguments. The second argument may be omitted, in which case $k=2$ is assumed. The graph $P(n,k)$, which is returned, is a connected cubic graph consisting of—in Schläfli notation—an inner star polygon $\{n,k\}$ and an outer regular polygon $\{n\}$ such that the n pairs of corresponding vertices in inner and outer polygons are connected with edges. For $k=1$ the prism graph of order n is obtained.

The well-known Petersen graph is equal to the generalized Petersen graph $P(5,2)$. It can also be constructed by calling `graph("petersen")`.

```
> draw_graph(graph("petersen"))
```



To obtain the dodecahedral graph $P(10,2)$, input:

```
> petersen_graph(10)
```

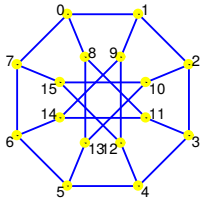
an undirected unweighted graph with 20 vertices and 30 edges

To obtain Möbius–Kantor graph $P(8,3)$, input:

```
> G:=petersen_graph(8,3)
```

an undirected unweighted graph with 16 vertices and 24 edges

```
> draw_graph(G)
```



Note that Desargues, Dürer and Nauru graphs are isomorphic to the generalized Petersen graphs $P(10,3)$, $P(6,2)$ and $P(12,5)$, respectively.

1.6.10. LCF graphs

The command `lcf_graph` is used for construction of cubic Hamiltonian graphs from LCF notation.

Syntax: `lcf_graph(L)`
`lcf_graph(L,n)`

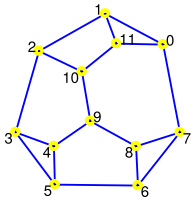
`lcf_graph` takes one or two arguments, a list L of nonzero integers, called *jumps*, and optionally a positive integer n, called the *exponent* (by default, $n=1$). The command returns the graph on $n\,|L|$ vertices obtained by iterating the sequence of jumps n times.

For example, the following command line creates Frucht graph.

```
> F:=lcf_graph([-5,-2,-4,2,5,-2,2,5,-2,-5,4,2])
```

an undirected unweighted graph with 12 vertices and 18 edges

```
> draw_graph(F,planar)
```

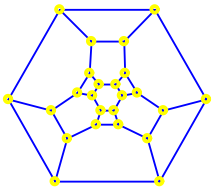


In the next example, the truncated octahedral graph is constructed from LCF notation.

```
> G:=lcf_graph([3,-7,7,-3],6)
```

an undirected unweighted graph with 24 vertices and 36 edges

```
> draw_graph(G,planar,labels=false)
```



1.7. Isomorphic copies of graphs

1.7.1. Creating isomorphic copies from permutations

To create an isomorphic copy of a graph use the `isomorphic_copy` command.

Syntax: `isomorphic_copy(G,sigma)`
`isomorphic_copy(G)`

`isomorphic_copy` takes one or two arguments, a graph $G(V, E)$ and optionally a permutation σ of order $|V|$. It returns a new graph where the adjacency lists are reordered according to σ or a random permutation if the second argument is omitted. The vertex labels are the same as in G. This command discards all vertex and edge attributes present in G.

The complexity of the algorithm is $O(|V| + |E|)$.

```
> G:=path_graph([1,2,3,4,5])
```

an undirected unweighted graph with 5 vertices and 4 edges

```
> vertices(G), neighbors(G)
```

$[1, 2, 3, 4, 5], \{[2], [1, 3], [2, 4], [3, 5], [4]\}$

```
> H:=isomorphic_copy(G)
```

an undirected unweighted graph with 5 vertices and 4 edges

```
> vertices(H), neighbors(H)
```

$[1, 2, 3, 4, 5], \{[2, 3], [1, 5], [1, 4], [3], [2]\}$

```
> H:=isomorphic_copy(G,[2,4,0,1,3])
```

an undirected unweighted graph with 5 vertices and 4 edges

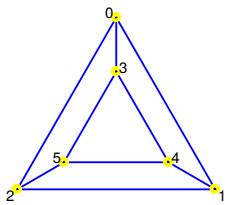
```
> vertices(H), neighbors(H)
```

$[1, 2, 3, 4, 5], \{[4, 5], [5], [4], [1, 3], [1, 2]\}$

```
> P:=prism_graph(3)
```

an undirected unweighted graph with 6 vertices and 9 edges

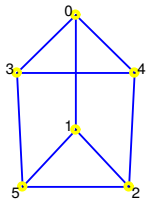
```
> draw_graph(P)
```




```
> H:=isomorphic_copy(P,[3,0,1,5,4,2])
```

an undirected unweighted graph with 6 vertices and 9 edges

```
> draw_graph(H,spring)
```



1.7.2. Permuting vertices

To create an isomorphic copy of a graph by providing the reordered list of vertex labels, use the command `permute_vertices`.

Syntax: `permute_vertices(G,L)`
`permute_vertices(G)`

`permute_vertices` takes one or two arguments, a graph $G(V, E)$ and optionally a list L of length $|V|$ containing all vertices from V, and returns a copy of G with vertices rearranged in order they appear in L or at random if L is not given. All vertex and edge attributes are copied, which includes vertex position information (if present). That means the resulting graph will look the same as G when drawn.

The complexity of the algorithm is $O(|V| + |E|)$.

```
> G:=path_graph([1,2,3,4,5])
```

an undirected unweighted graph with 5 vertices and 4 edges

```
> vertices(G), neighbors(G)
```

$[1, 2, 3, 4, 5], \{[2], [1, 3], [2, 4], [3, 5], [4]\}$

```
> H:=permute_vertices(G,[3,5,1,2,4])
```

an undirected unweighted graph with 5 vertices and 4 edges

```
> vertices(H), neighbors(H)
```

$[3, 5, 1, 2, 4], \{[2, 4], [4], [2], [1, 3], [3, 5]\}$

1.7.3. Relabeling vertices

To relabel the vertices of a graph without changing their order, use the command `relabel_vertices`.

Syntax: `relabel_vertices(G,L)`

`relabel_vertices` takes two arguments, a graph $G(V, E)$ and a list L of vertex labels of length $|V|$. It returns a copy of G with L as the list of vertex labels.

The complexity of the algorithm is $O(|V|)$.

```
> G:=path_graph([1,2,3,4])
```

an undirected unweighted graph with 4 vertices and 3 edges

```
> edges(G)
```

$\{[1, 2], [2, 3], [3, 4]\}$

```
> H:=relabel_vertices(G,[a,b,c,d])
```

an undirected unweighted graph with 4 vertices and 3 edges

```
> edges(H)
```

$\{[a, b], [b, c], [c, d]\}$

1.8. Subgraphs

1.8.1. Extracting subgraphs

To extract the subgraph of a graph formed by a subset of the set of its edges, use the command `subgraph`.

Syntax: `subgraph(G,L)`

`subgraph` takes two arguments, a graph $G(V, E)$ and a list of edges L. It returns the subgraph $G'(V', L)$ of G, where $V' \subset V$ is a subset of vertices of G incident to at least one edge from L.

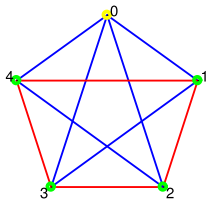
```
> K5:=complete_graph(5)
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> S:=subgraph(K5,[[1,2],[2,3],[3,4],[4,1]])
```

an undirected unweighted graph with 4 vertices and 4 edges

```
> draw_graph(highlight_subgraph(K5,S))
```



1.8.2. Induced subgraphs

To obtain the subgraph of a graph induced by a subset of its vertices, use the command `induced_subgraph`.

Syntax: `induced_subgraph(G,L)`

`induced_subgraph` takes two arguments, a graph $G(V, E)$ and a list of vertices L. It returns the subgraph $G'(L, E')$ of G, where $E' \subset E$ contains all edges which have both endpoints in L [27, pp. 3].

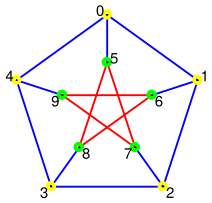
```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> S:=induced_subgraph(G,[5,6,7,8,9])
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> draw_graph(highlight_subgraph(G,S))
```



1.8.3. Underlying graphs

For every graph $G(V, E)$ there is an undirected and unweighted graph $U(V, E')$, called the **underlying graph** of G, where E' is obtained from E by dropping edge directions. To construct U, use the command `underlying_graph`.

Syntax: `underlying_graph(G)`

`underlying_graph` takes a graph $G(V, E)$ as its only argument and returns an undirected unweighted copy of G in which all vertex and edge attributes, together with edge directions, are discarded.

The complexity of the algorithm is $O(|V| + |E|)$.

```
> G:=digraph(%{[[1,2],6],[[2,3],4],[[3,1],5],[[3,2],7]%})
```

a directed weighted graph with 3 vertices and 4 arcs

```
> U:=underlying_graph(G)
```

an undirected unweighted graph with 3 vertices and 3 edges

```
> edges(U)
```

$$\{[1,2],[1,3],[2,3]\}$$

1.8.4. Fundamental cycles

The command `fundamental_cycle` is used for extracting cycles from unicyclic graphs (also called 1-trees). To find a fundamental cycle basis of an undirected graph, use the command `cycle_basis`.

Syntax: `fundamental_cycle(G)`
`cycle_basis(G)`

`fundamental_cycle` takes one argument, an undirected connected graph $G(V, E)$ containing exactly one cycle (i.e. a unicyclic graph), and returns that cycle as a graph. If G is not unicyclic, an error is returned.

`cycle_basis` takes an undirected graph $G(V, E)$ as its only argument and returns a basis B of the cycle space of G as a list of fundamental cycles in G, with each cycle represented as a list of vertices. Furthermore, $|B| = |E| - |V| + \kappa(G)$, where $\kappa(G)$ is the number of connected components of G. Every cycle C in G such that $C \notin B$ can be obtained from cycles in B using only symmetric differences.

The strategy is to construct a spanning tree T of G using depth-first search and look for edges in E which do not belong to the tree. For each non-tree edge e there is a unique fundamental cycle C_e consisting of e together with the path in T connecting the endpoints of e. The vertices of C_e are easily obtained from the search data. The complexity of this algorithm is $O(|V| + |E|)$.

```
> G:=graph(trail(1,2,3,4,5,2,6))
```

an undirected unweighted graph with 6 vertices and 6 edges

```
> C:=fundamental_cycle(G)
```

an undirected unweighted graph with 4 vertices and 4 edges

```
> edges(C)
```

$$\begin{pmatrix} 2 & 5 \\ 2 & 3 \\ 4 & 5 \\ 3 & 4 \end{pmatrix}$$

Given a tree graph G and adding an edge from the complement G^c to G one obtains a 1-tree graph.

```
> G:=random_tree(25)
```

an undirected unweighted graph with 25 vertices and 24 edges

```
> ed:=choice(edges(graph_complement(G)))
```

$$[10, 20]$$

```
> G:=add_edge(G,ed)
```

an undirected unweighted graph with 25 vertices and 25 edges

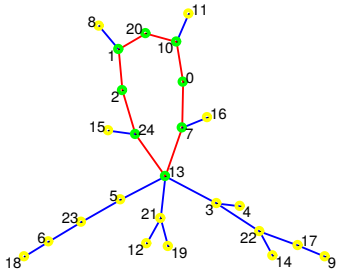
```
> C:=fundamental_cycle(G)
```

an undirected unweighted graph with 8 vertices and 8 edges

```
> edges(C)
```

$$\begin{pmatrix} 10 & 20 \\ 0 & 10 \\ 1 & 20 \\ 1 & 2 \\ 2 & 24 \\ 13 & 24 \\ 7 & 13 \\ 0 & 7 \end{pmatrix}$$

```
> draw_graph(highlight_subgraph(G,C),spring)
```

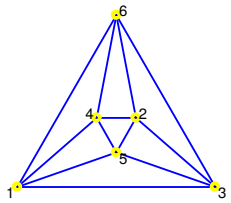


In the next example, a cycle basis of octahedral graph is computed.

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(G)
```




```
> cycle_basis(G)
```

$$\{[6,3,1],[5,4,6,3,1],[4,6,3,1],[5,4,6,3],[2,5,4,6,3],[2,5,4,6],[2,5,4]\}$$

Given a tree graph T, one can create a graph with cycle basis cardinality equal to k by simply adding k randomly selected edges from the complement T^c to T.

```
> tree1:=random_tree(15)
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> G1:=add_edge(tree1,rand(3,edges(graph_complement(tree1))))
```

an undirected unweighted graph with 15 vertices and 17 edges

```
> tree2:=random_tree(12)
```

an undirected unweighted graph with 12 vertices and 11 edges

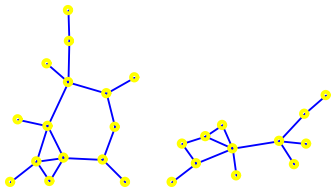
```
> G2:=add_edge(tree2,rand(2,edges(graph_complement(tree2))))
```

an undirected unweighted graph with 12 vertices and 13 edges

```
> G:=disjoint_union(G1,G2)
```

an undirected unweighted graph with 27 vertices and 30 edges

```
> draw_graph(G,spring,labels=false)
```



```
> nops(cycle_basis(G))
```

5

```
> number_of_edges(G)-number_of_vertices(G)+nops(connected_components(G))
```

5

1.9. Operations on graphs

1.9.1. Graph complement

The command `graph_complement` is used for construction of complement graphs.

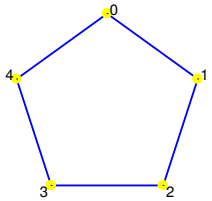
Syntax: `graph_complement(G)`

`graph_complement` takes a graph $G(V, E)$ as its only argument and returns the complement graph $G^c(V, E^c)$ of G, where E^c is the largest set containing only edges/arcs not present in G. The complexity of the algorithm is $O(|V|^2)$.

```
> C5:=cycle_graph(5)
```

C5: an undirected unweighted graph with 5 vertices and 5 edges

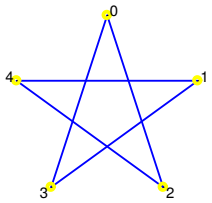
```
> draw_graph(C5)
```




```
> G:=graph_complement(C5)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> draw_graph(G)
```



1.9.2. Seidel switching

The command `seidel_switch` is used for Seidel switching in graphs.

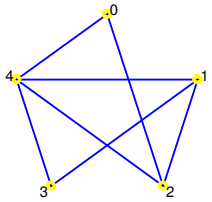
Syntax: `seidel_switch(G,L)`

`seidel_switch` takes two arguments, an undirected and unweighted graph $G(V, E)$ and a list of vertices $L \subset V$. The result is a copy of G in which, for each vertex $v \in L$, its neighbors become its non-neighbors and vice versa.

```
> S:=seidel_switch(cycle_graph(5),[1,2])
```

an undirected unweighted graph with 5 vertices and 7 edges

```
> draw_graph(S)
```



1.9.3. Transposing graphs

The command `reverse_graph` is used for reversing arc directions in digraphs.

Syntax: `reverse_graph(G)`

`reverse_graph` takes a graph $G(V, E)$ as its only argument and returns the reverse graph $G^T(V, E')$ of G where $E' = \{(j, i): (i, j) \in E\}$, i.e. returns the copy of G with the directions of all edges reversed.

Note that `reverse_graph` is defined for both directed and undirected graphs, but gives meaningful results only for directed graphs.

G^T is also called the **transpose graph** of G because adjacency matrices of G and G^T are transposes of each other (hence the notation).

```
> G:=digraph(6, %{[1,2],[2,3],[2,4],[4,5]%})
```

a directed unweighted graph with 6 vertices and 4 arcs

```
> GT:=reverse_graph(G)
```

a directed unweighted graph with 6 vertices and 4 arcs

```
> edges(GT)
```

$\{[2, 1], [3, 2], [4, 2], [5, 4]\}$

1.9.4. Union of graphs

The command `graph_union` is used for constructing unions of graphs.

Syntax: `graph_union(G1,G2,..,Gn)`

`graph_union` takes a sequence of graphs $G_k(V_k, E_k)$ for $k = 1, 2, ..., n$ as its argument and returns the graph $G(V, E)$ where $V = V_1 \cup V_2 \cup \cdots \cup V_k$ and $E = E_1 \cup E_2 \cup \cdots \cup E_k$.

```
> G1:=graph([1,2,3],%{[1,2],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> G2:=graph([1,2,3],%{[3,1],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> G:=graph_union(G1,G2)
```

an undirected unweighted graph with 3 vertices and 3 edges

```
> edges(G)
```

$$\{[1,2],[1,3],[2,3]\}$$

1.9.5. Disjoint union of graphs

To construct disjoint union of graphs use the command `disjoint_union`.

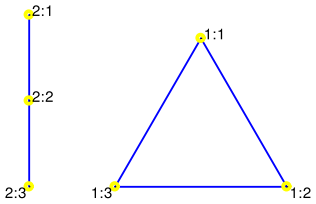
Syntax: `disjoint_union(G1,G2,..,Gn)`

`disjoint_union` takes a sequence of graphs $G_k(V_k, E_k)$ for $k = 1, 2, ..., n$ as its only argument and returns the graph obtained by labeling all vertices with strings `k:v` where $v \in V_k$ and all edges with strings `k:e` where $e \in E_k$ and calling `graph_union` subsequently. As all vertices and edges are labeled differently, it follows $|V| = \sum_{k=1}^{n} |V_k|$ and $|E| = \sum_{k=1}^{n} |E_k|$.6

```
> G:=disjoint_union(cycle_graph([1,2,3]),path_graph([1,2,3]))
```

an undirected unweighted graph with 6 vertices and 5 edges

```
> draw_graph(G)
```



1.9.6. Joining two graphs

The command `graph_join` is used for joining two graphs together.

Syntax: `graph_join(G,H)`

`graph_join` takes two graphs G and H as its arguments and returns the graph $G + H$ which is obtained by connecting all the vertices of G to all vertices of H. The vertex labels in the resulting graph are strings of the form `1:u` and `2:v` where u is a vertex in G and v is a vertex in H.

```
> G:=path_graph(2)
```

an undirected unweighted graph with 2 vertices and 1 edge

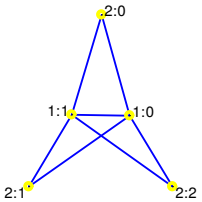
```
> H:=graph(3)
```

an undirected unweighted graph with 3 vertices and 0 edges

```
> GH:=graph_join(G,H)
```

an undirected unweighted graph with 5 vertices and 7 edges

```
> draw_graph(GH,spring)
```



1.9.7. Power graphs

The command `graph_power` is used for computing powers of graphs.

Syntax: `graph_power(G,k)`

`graph_power` takes two arguments, a graph $G(V,E)$ and a positive integer k. It returns the k-th power G^k of G with vertices V such that $v,w \in V$ are connected with an edge if and only if there exists a path of length at most k in G.

The graph G^k is constructed from its adjacency matrix A_k which is obtained by adding powers of the adjacency matrix A of G:

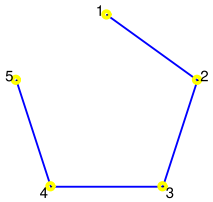
$$A_k = \sum_{i=1}^{k} A^k.$$

The above sum is obtained by assigning $A_k \leftarrow A$ and repeating the instruction $A_k \leftarrow (A_k + I)\, A$ for $k-1$ times, so exactly k matrix multiplications are required.

```
> G:=graph(trail(1,2,3,4,5))
```

an undirected unweighted graph with 5 vertices and 4 edges

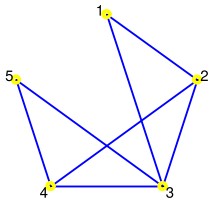
```
> draw_graph(G,circle)
```




```
> P2:=graph_power(G,2)
```

an undirected unweighted graph with 5 vertices and 7 edges

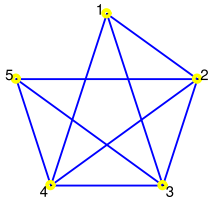
```
> draw_graph(P2,circle)
```




```
> P3:=graph_power(G,3)
```

an undirected unweighted graph with 5 vertices and 9 edges

```
> draw_graph(P3,circle)
```



1.9.8. Graph products

There are two distinct operations for computing a product of two graphs: the Cartesian product and the tensor product. These operations are available in Giac as the commands `cartesian_product` and `tensor_product`, respectively.

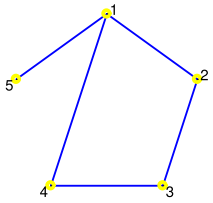
Syntax: `cartesian_product(G1,G2,..,Gn)`
`tensor_product(G1,G2,..,Gn)`

`cartesian_product` takes a sequence of graphs $G_k(V_k, E_k)$ for $k = 1, 2, \ldots, n$ as its argument and returns the Cartesian product $G_1 \times G_2 \times \cdots \times G_n$ of the input graphs. The Cartesian product $G(V, E) = G_1 \times G_2$ is the graph with list of vertices $V = V_1 \times V_2$, labeled with strings `v1:v2` where $v_1 \in V_1$ and $v_2 \in V_2$, such that (`u1:v1`,`u2:v2`)$\in E$ if and only if u_1 is adjacent to u_2 and $v_1 = v_2$ **or** $u_1 = u_2$ and v_1 is adjacent to v_2.

```
> G1:=graph(trail(1,2,3,4,1,5))
```

an undirected unweighted graph with 5 vertices and 5 edges

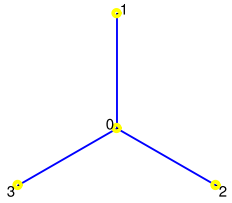
```
> draw_graph(G1,circle)
```




```
> G2:=star_graph(3)
```

an undirected unweighted graph with 4 vertices and 3 edges

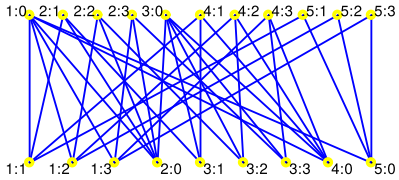
```
> draw_graph(G2,circle=[1,2,3])
```




```
> G:=cartesian_product(G1,G2)
```

an undirected unweighted graph with 20 vertices and 35 edges

```
> draw_graph(G)
```

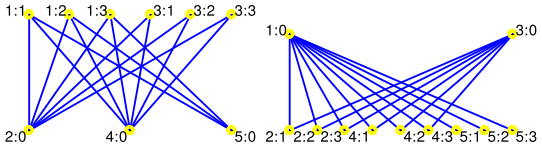


`tensor_product` takes a sequence of graphs $G_k(V_k, E_k)$ for $k = 1, 2, \ldots, n$ as its argument and returns the tensor product $G_1 \times G_2 \times \cdots \times G_n$ of the input graphs. The tensor product $G(V, E) = G_1 \times G_2$ is the graph with list of vertices $V = V_1 \times V_2$, labeled with strings `v1:v2` where $v_1 \in V_1$ and $v_2 \in V_2$, such that (`u1:v1`,`u2:v2`)$\in E$ if and only if u_1 is adjacent to u_2 **and** v_1 is adjacent to v_2.

```
> T:=tensor_product(G1,G2)
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> draw_graph(T)
```



1.9.9. Transitive closure graph

The command `transitive_closure` is used for constructing transitive closure graphs.

Syntax: `transitive_closure(G)`
`transitive_closure(G,weighted[=true or false])`

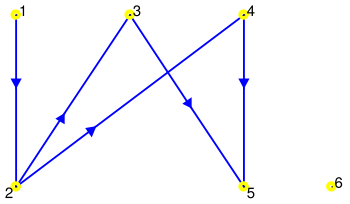
`transitive_closure` takes one or two arguments, a graph $G(V, E)$ and optionally the option `weighted=true` (or simply `weighted`) or the option `weighted=false` (which is the default). The command returns the transitive closure $T(V, E')$ of the input graph G by connecting $u \in V$ to $v \in V$ in T if and only if there is a path from u to v in G. If G is directed, then T is also directed. When `weighted=true` is specified, T is weighted such that the weight of edge $v\ w \in E'$ is equal to the length (or cost, if G is weighted) of the shortest path from v to w in G.

The lengths/weights of the shortest paths are obtained by the command `allpairs_distance` if G is weighted resp. the command `vertex_distance` if G is unweighted. Therefore T is constructed in at most $O(|V|^3)$ time if `weighted[=true]` is given and in $O(|V|\,|E|)$ time otherwise.

```
> G:=digraph([1,2,3,4,5,6],%{[1,2],[2,3],[2,4],[4,5],[3,5]%})
```

a directed unweighted graph with 6 vertices and 5 arcs

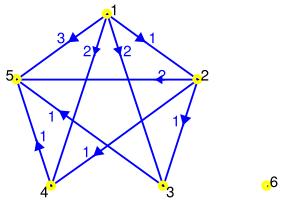
```
> draw_graph(G)
```




```
> T:=transitive_closure(G,weighted)
```

a directed weighted graph with 6 vertices and 9 arcs

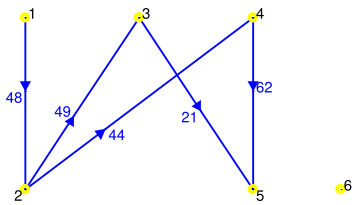
```
> draw_graph(T)
```



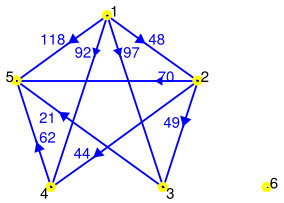

```
> G:=assign_edge_weights(G,1,99)
```

a directed weighted graph with 6 vertices and 5 arcs

```
> draw_graph(G)
```




```
> draw_graph(transitive_closure(G,weighted=true))
```



1.9.10. Line graph

The command `line_graph` is used for construction of line graphs [27, pp. 10].

Syntax: `line_graph(G)`

`line_graph` takes an undirected graph G as its only argument and returns the line graph $L(G)$ with $|E|$ distinct vertices, one vertex for each edge in E. Furthermore, two vertices v_1 and v_2 in $L(G)$ are adjacent if and only if the corresponding edges $e_1, e_2 \in E$ have a common endpoint. The vertices in $L(G)$ are labeled with strings in form `v-w`, where $e = vw \in E$.

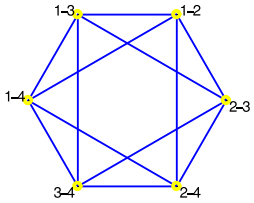
```
> K4:=complete_graph([1,2,3,4])
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> L:=line_graph(K4)
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(L,spring)
```



1.9.11. Plane dual graph

The command `plane_dual` is used for construction of dual graphs from undirected biconnected planar graphs. To determine whether a graph is planar [27, pp. 12] use the command `is_planar`.

Syntax: `plane_dual(G)`
`plane_dual(F)`
`is_planar(G)`
`is_planar(G,F)`

`plane_dual` takes a biconnected planar graph $G(V, E)$ or the list F of faces of a planar embedding of G as its only argument and returns the graph H with faces of G as its vertices. Two vertices in H are adjacent if and only if the corresponding faces share an edge in G. The algorithm runs in $O(|V|^2)$ time.

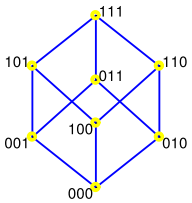
Note that the concept of dual graph is normally defined for multigraphs. By the strict definition, every planar multigraph has the corresponding dual multigraph; moreover, the dual of the latter is equal to the former. Since Giac generally does not support multigraphs, a simplified definition suitable for simple graphs is used; hence the requirement that the input graph is biconnected.

In the example below, the dual graph of the cube graph is obtained.

```
> H:=hypercube_graph(3)
```

an undirected unweighted graph with 8 vertices and 12 edges

```
> draw_graph(H,spring)
```

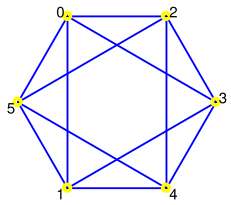


The cube has six faces, hence its plane dual graph D has six vertices. Also, every face obviously shares an edge with exactly four other faces, so the degree of each vertex in D is equal to 4.

```
> D:=plane_dual(H)
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(D,spring)
```



`is_planar` takes one or two arguments, the input graph G and optionally an unassigned identifier `F`. It returns `true` if G is planar and `false` otherwise. If the second argument is given and G is planar and biconnected, the list of faces of G is stored to `F`. Each face is represented as a cycle (a list) of vertices. The strategy is to use the algorithm of DEMOUCRON et al. [26, pp. 88], which runs in $O(|V|^2)$ time.

```
> is_planar(graph("petersen"))
```

false

```
> is_planar(graph("durer"))
```

true

In the next example, a graph isomorphic to D is obtained when passing a list of faces of H to `plane_dual`. The order of vertices is determined by the order of faces.

```
> is_planar(H,F); F
```

$$\text{true}, \begin{pmatrix} 010 & 000 & 001 & 011 \\ 001 & 000 & 100 & 101 \\ 010 & 011 & 111 & 110 \\ 100 & 000 & 010 & 110 \\ 111 & 011 & 001 & 101 \\ 101 & 100 & 110 & 111 \end{pmatrix}$$

```
> is_isomorphic(plane_dual(F),D)
```

true

1.9.12. Truncating planar graphs

The command `truncate_graph` performs truncation of biconnected planar graphs.

Syntax: `truncate_graph(G)`

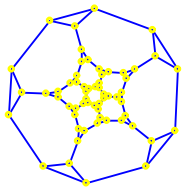
`truncate_graph` takes a biconnected planar graph $G(V, E)$ as its only argument and returns the graph obtained by truncating the respective polyhedron, i.e. by "cutting off" its vertices. The resulting graph has $2|E|$ vertices and $3|E|$ edges. The procedure of truncating a graph by subdividing its edges is described in [3].

The algorithm requires computing a planar embedding of G, which is done by applying DEMOUCRON's algorithm. Hence its complexity is $O(|V|^2)$.

```
> G:=truncate_graph(graph("dodecahedron"))
```

an undirected unweighted graph with 60 vertices and 90 edges

```
> draw_graph(G,planar,labels=false)
```



Truncating the plane dual of G represents the **leapfrog operation** on G, which can be used for constructing fullerene graphs [3]. By performing the leapfrog operation on a fullerene graph one obtains a larger fullerene. For example, the dual of the Errera graph is a fullerene (see here); hence by truncating Errera graph (i.e. the dual of its dual) one obtains a fullerene.

```
> G:=truncate_graph(graph("errera"))
```

an undirected unweighted graph with 90 vertices and 135 edges

```
> is_planar(G,F)
```

true

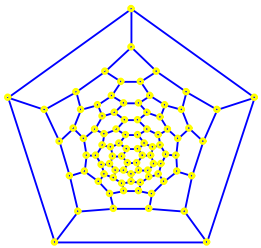
Now F contains a list of faces of the graph G. Since G is a fullerene, every face is a 5- or 6-cycle.

```
> set[op(apply(length,F))]
```

$\mathrm{set}[5,6]$

When drawing fullerenes, it is recommended to use the circular method since it usually produces best results. Any face of the planar embedding of a given fullerene be chosen as the outer face, as in the example below.

```
> draw_graph(G,circle=rand(F))
```



1.10. Random graphs

1.10.1. Random general graphs

The commands `random_graph` and `random_digraph` are used for generating general (di)graphs at random according to various models, including preferential attachment.

Syntax:

`random_graph(n|L,p)`	Erdős–Rényi model
`random_graph(n|L,m)`	
`random_digraph(n|L,p)`	
`random_digraph(n|L,m)`	
`random_graph(n|L,[p0,p1,...])`	custom vertex degree distribution
`random_graph(n|L,f)`	
`random_graph(n|L,d,k)`	preferential attachment

`random_graph` and `random_digraph` can both take two arguments: a positive integer n or a list of labels L of length n. The second argument is a positive real number $p<1$ or a positive integer m. The return value is a (di)graph on n vertices (with elements of L as vertex labels) selected uniformly at random, i.e. a (di)graph in which each edge/arc is present with probability p or which contains exactly m edges/arcs chosen uniformly at random (Erdős–Rényi model).

Erdős–Rényi model is implemented according to BAGATELJ and BRANDES [4, algorithms 1 and 2]. The corresponding algorithms run in linear time and are suitable for generating large graphs.

`random_graph` can also generate graphs with respect to a given probability distribution of vertex degrees if the second argument is a discrete probability density function given as a list of probabilities or weights $(p_0, p_1, ..., p_{n-1})$ or as a weight function $f : \mathbb{N} \cup \{0\} \to [0, +\infty\rangle$ such that $f(i) = p_i$ for $i = 0, 1, ..., n-1$. Trailing zeros in the list of weights, if present, may be omitted. The numbers p_i are automatically scaled by $1/\sum_{i=1}^{n-1} p_i$ to achieve the sum of 1 and a graph with that precise distribution of vertex degrees is generated at random using the algorithm described in [43, pp. 2567] with some modifications. First, a degree sequence d is generated randomly by drawing samples from the given distribution and repeating the process until a graphic sequence is obtained. Then the algorithm for constructing a feasible solution from d [31] is applied. Finally, the edges of that graph are randomized by choosing suitable pairs of nonincident edges and "rewiring" them without changing the degree sequence. Two edges uv and wz can be rewired in at most two ways, becoming either uz and wv or uw and vz (if these edges are not in the graph already). Letting m denote the number of edges, at most

$$N = \left\lceil \left(\log_2 \frac{m}{m-1} \right)^{-1} \right\rceil < m$$

such choices is made, assuring that the probability of rewiring each edge at least once is larger than $\frac{1}{2}$. The total complexity of this algorithm is $O(n^2 \log n)$.

Additionally, to support generation of realistic networks, `random_graph` can be used with integer parameters $d > 0$ and $k \geqslant 0$ as the second and the third argument, respectively, in which case a preferential attachment rule is applied in the following way. For $n \geqslant 2$, the resulting graph $G(V, E)$ initially contains two vertices v_1, v_2 and one edge $v_1 v_2$. For each $i = 3, ..., n$, the vertex v_i is added to V along with edges $v_i v_j$ for $\min\{i-1, d\}$ mutually different values of j, which are chosen at random in the set $\{1, 2, ..., i-1\}$ with probability

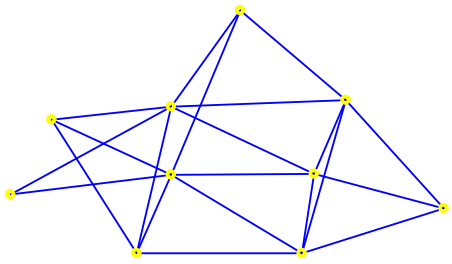
$$p_j = \frac{\deg v_j}{\sum_{r=1}^{i-1} \deg v_r}.$$

Subsequently, additional at most k random edges connecting the neighbors of v_i to each other are added to E, allowing the user to control the clustering coefficient of G. This method is due to SCHANK and WAGNER [52, Algorithm 2, pp. 271]. The time complexity of the implementation is $O(n^2 d + n k)$.

```
> G:=random_graph(10,0.5)
```

an undirected unweighted graph with 10 vertices and 21 edges

```
> draw_graph(G,spring,labels=false)
```



```
> G:=random_graph(1000,0.05)
```

an undirected unweighted graph with 1000 vertices and 24870 edges

```
> is_connected(G)
```

true

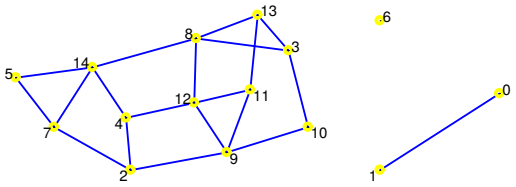
```
> minimum_degree(G),maximum_degree(G)
```

$20, 71$

```
> G:=random_graph(15,20)
```

an undirected unweighted graph with 15 vertices and 20 edges

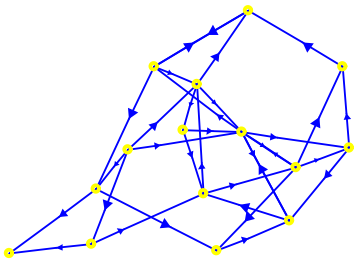
```
> draw_graph(G,spring)
```




```
> DG:=random_digraph(15,0.15)
```

a directed unweighted graph with 15 vertices and 33 arcs

```
> draw_graph(DG,labels=false,spring)
```



In the following example, a random graph is generated such that the degree of each vertex is drawn from $\{0, 1, ..., 10\}$ according to weights specified in the table below.

degree	0	1	2	3	4	5	6	7	8	9	10
weight	0	0	9	7	0	5	4	3	0	1	1

That is, the degrees are generated with probabilities $0, 0, \frac{3}{10}, \frac{7}{30}, 0, \frac{1}{6}, \frac{2}{15}, \frac{1}{10}, 0, \frac{1}{30}, \frac{1}{30}$, respectively.

```
> G:=random_graph(10000,[0,0,9,7,0,5,4,3,0,1,1])
```

an undirected unweighted graph with 10000 vertices and 21231 edges

```
> frequencies(degree_sequence(G))
```

$$\begin{pmatrix} 2 & 0.3125 \\ 3 & 0.2256 \\ 5 & 0.163 \\ 6 & 0.1331 \\ 7 & 0.0987 \\ 9 & 0.0311 \\ 10 & 0.036 \end{pmatrix}$$

In the example below, a random graph is generated such that the vertex degrees are distributed according to the following weight function:

$$f(k) = \begin{cases} 0, & k = 0, \\ k^{-3/2}\,\mathrm{e}^{-k/3}, & k \geqslant 1. \end{cases}$$

```
> G:=random_graph(10000,k->when(k<1,0,k^-1.5*exp(-k/3)))
```

an undirected unweighted graph with 10000 vertices and 8017 edges

```
> length(connected_components(G))
```

2266

The command line below computes the average size of a connected component in G.

```
> round(mean(apply(length,connected_components(G))))
```

4

The next example demonstrates how to generate random graphs with adjustable clustering coefficient.

```
> G1:=random_graph(10000,5,10)
```

an undirected unweighted graph with 10000 vertices and 105628 edges

```
> clustering_coefficient(G1)
```

0.469236344448

```
> G2:=random_graph(10000,5,20)
```

an undirected unweighted graph with 10000 vertices and 121957 edges

```
> clustering_coefficient(G2)
```

0.612673551668

```
> G3:=random_graph(10000,10,5)
```

an undirected unweighted graph with 10000 vertices and 143646 edges

```
> clustering_coefficient(G3)
```

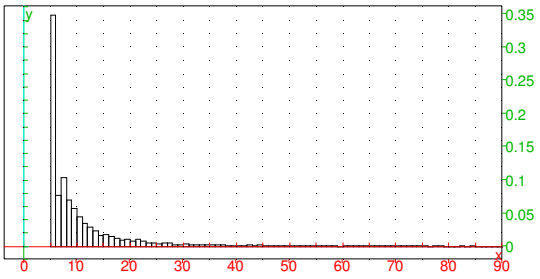
0.113671512462

The distribution of vertex degrees in a graph generated with preferential attachment rule roughly obeys the power law in its tail, as shown in the example below.

```
> G:=random_graph(10000,5,2)
```

an undirected unweighted graph with 10000 vertices and 67875 edges

```
> histogram(degree_sequence(G))
```



1.10.2. Random bipartite graphs

The command `random_bipartite_graph` is used for generating bipartite graphs at random.

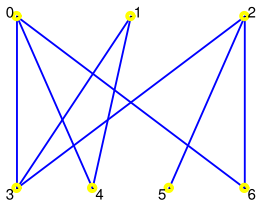
Syntax: `random_bipartite_graph(n,p|m)`
`random_bipartite_graph([a,b],p|m)`

`random_bipartite_graph` takes two arguments. The first argument is either a positive integer n or a list of two positive integers a and b. The second argument is either a positive real number $p<1$ or a positive integer m. The command returns a random bipartite graph on n vertices (or with two partitions of sizes a and b) in which each possible edge is present with probability p (or m edges are inserted at random).

```
> G:=random_bipartite_graph([3,4],0.5)
```

an undirected unweighted graph with 7 vertices and 8 edges

```
> draw_graph(G)
```




```
> G:=random_bipartite_graph(30,60)
```

an undirected unweighted graph with 30 vertices and 60 edges

1.10.3. Random trees

The command `random_tree` is used for generating tree graphs at random.

Syntax:

`random_tree(n|V)`	unrooted unlabeled trees
`random_tree(n|V,d)`	trees with limited maximum degree
`random_tree(n|V,root)`	rooted unlabeled trees
`random_tree(V,root=v)`	

`random_tree` takes one or two arguments: a positive integer n or a list $V=\{v_1, v_2, ..., v_n\}$ and optionally an integer $d \geqslant 2$ or the option `root[=v]`, where $v \in V$. It returns a random tree $T(V, E)$ on n vertices such that

- if the second argument is omitted, then T is uniformly selected among all unrooted unlabeled trees on n vertices,
- if d is given as the second argument, then $\Delta(T) \leqslant d$, where $\Delta(T)$ is the maximum vertex degree in T,
- if `root[=v]` is given as the second argument, then T is uniformly selected among all rooted unlabeled trees on n vertices. If v is specified then the vertex labels in V (required) will be assigned to vertices in T such that v is the first vertex in the list returned by the command `vertices`.

Rooted unlabeled trees are generated uniformly at random using RANRUT algorithm [44, pp. 274]. The root of a tree T generated this way, if not specified as v, is always the first vertex in the list returned by `vertices`. The average time complexity of RANRUT algorithm is $O(n \log n)$ [2].

Unrooted unlabeled trees, also called **free** trees, are generated uniformly at random using Wilf's algorithm[1.1] [65], which is based on RANRUT algorithm and runs in about the same time as RANRUT itself.

Trees with bounded maximum degree are generated using a simple algorithm which starts with an empty tree and adds edges at random one at a time. It is much faster than RANRUT but selects trees in a non-uniform manner. To force the use of this algorithm even without vertex degree limit (for example, when n is very large), one can set $d = +\infty$.

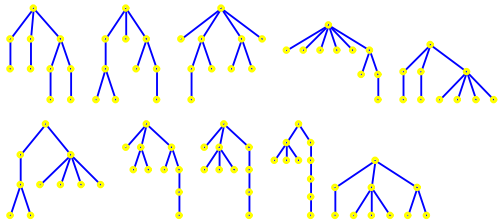
For example, the command line below creates a forest containing 10 randomly selected free trees on 10 vertices.

1.1. The original Wilf's algorithm has a minor flaw in the procedure `Free` [65, pp. 207]. In the formula $p = \binom{1+a_{n/2}}{2} / a_n$ in step (T1) the denominator a_n stands for the number of all rooted unlabeled trees on n vertices. However, one should divide by the number t_n of all *unrooted* unlabeled trees instead, which can be obtained from $a_1, a_2, ..., a_n$ by applying the formula in [46, pp. 589]. This implementation includes the correction.

```
> G:=disjoint_union(apply(random_tree,[10$10]))
```

an undirected unweighted graph with 100 vertices and 90 edges

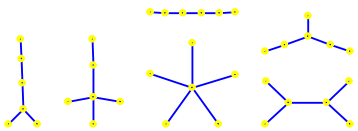
```
> draw_graph(G,tree,labels=false)
```



The following example demonstrates the uniformity of random generation of free trees. Letting $n=6$, there are exactly 6 distinct free trees on 6 vertices, created by the next command line.

```
> trees:=[star_graph(5),path_graph(6),graph(trail(1,2,3,4),trail(5,4,6)),
  graph(%{[1,2],[2,3],[2,4],[4,5],[4,6]%}),graph(trail(1,2,3,4),trail(3,5,6)),
  graph(trail(1,2,3,4),trail(5,3,6))]:;
```

```
> draw_graph(disjoint_union(trees),spring,labels=false)
```



Now, generating a random free tree on 6 nodes always produces one of the above six graphs, which is determined by using the command `is_isomorphic`. 1200 trees are generated in total and the number of occurrences of `trees[k]` is stored in `hits[k]` for every $k=1,2,...,6$ (note that in Xcas mode it is actually $k=0,...,5$).

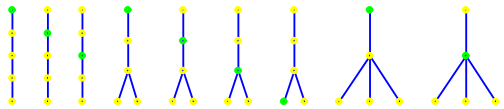
```
> hits:=[0$6]:;
  for k from 1 to 1200 do
      T:=random_tree(6);
      for j from 0 to 5 do
          if is_isomorphic(T,trees[j]) then hits[j]++; fi;
      od;
  od:;
  hits
```

$$[198, 194, 192, 199, 211, 206]$$

To show that the algorithm also selects rooted trees on n vertices with equal probability, one can reproduce the example in [44, pp. 281], in which $n=5$. First, all distinct rooted trees on 5 vertices are created and stored in `trees`; there are exactly nine of them. Their root vertices are highlighted to be distinguishable. Then, 4500 rooted trees on 5 vertices are generated at random, highlighting the root vertex in each of them. As in the previous example, the variable `hits[k]` records how many of them are isomorphic to `trees[k]`.

```
> trees:=[
  highlight_vertex(graph(trail(1,2,3,4,5)),1),
  highlight_vertex(graph(trail(1,2,3,4,5)),2),
  highlight_vertex(graph(trail(1,2,3,4,5)),3),
  highlight_vertex(graph(trail(1,2,3),trail(4,3,5)),1),
  highlight_vertex(graph(trail(1,2,3),trail(4,3,5)),2),
  highlight_vertex(graph(trail(1,2,3),trail(4,3,5)),3),
  highlight_vertex(graph(trail(1,2,3),trail(4,3,5)),4),
  highlight_vertex(graph(trail(1,2,3),trail(4,2,5)),1),
  highlight_vertex(graph(trail(1,2,3),trail(4,2,5)),2)
  ]:;
```

```
> draw_graph(disjoint_union(trees),labels=false)
```

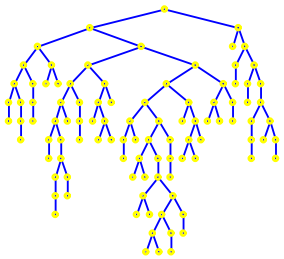


```
> hits:=[0$9]:;
  for k from 1 to 4500 do
      T:=random_tree(5,root);
      HT:=highlight_vertex(T,vertices(T)[0]);
      for j from 0 to 8 do
          if is_isomorphic(HT,trees[j]) then hits[j]++; fi;
      od;
  od:;
  hits
```

$$[534, 483, 486, 485, 496, 521, 498, 489, 508]$$

In the following example, a random tree on 100 vertices with maximum degree at most 3 is drawn.

```
> draw_graph(random_tree(100,3))
```



1.10.4. Random planar graphs

The command `random_planar_graph` is used for generating random planar graphs.

Syntax: `random_planar_graph(n|L,p)`
`random_planar_graph(n|L,p,k)`

`random_planar_graph` takes two or three arguments, a positive integer n (or a list L of length n), a positive real number $p<1$ and optionally an integer $k\in\{0,1,2,3\}$ (by default, $k=1$). The command returns a random k-connected planar graph on n vertices (using the elements of L as vertex labels).

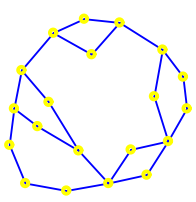
The result is obtained by first generating a random maximal planar graph and then attempting to remove each edge with probability p, maintaining the k-connectivity of the graph (if $k=0$, the graph may be disconnected). The running time is $O(n)$ if $k=0$, $O(n^2)$ if $k\in\{1,2\}$ and $O(n^3)$ if $k=3$.

The following command line generates a biconnected planar graph.

```
> G:=random_planar_graph(20,0.8,2)
```

an undirected unweighted graph with 20 vertices and 25 edges

```
> draw_graph(G,planar,labels=false)
```

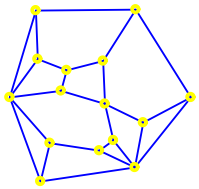


The command line below generates a triconnected planar graph.

```
> G:=random_planar_graph(15,0.9,3)
```

an undirected unweighted graph with 15 vertices and 25 edges

```
> draw_graph(G,planar,labels=false)
```



The next command line generates a disconnected planar graph with high probability.

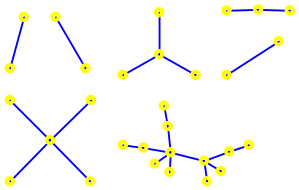
```
> G:=random_planar_graph(30,0.9,0)
```

an undirected unweighted graph with 30 vertices and 23 edges

```
> is_forest(G)
```

true

```
> draw_graph(G,spring,labels=false)
```



By default, a connected planar graph is generated, like in the following example.

```
> G:=random_planar_graph(15,0.618)
```

an undirected unweighted graph with 15 vertices and 19 edges

```
> is_connected(G)
```

true

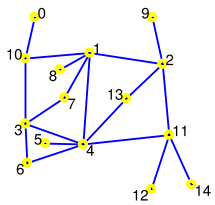
```
> is_biconnected(G)
```

false

```
> articulation_points(G)
```

$[1, 2, 4, 10, 11]$

```
> draw_graph(G,planar)
```



1.10.5. Random graphs from a given degree sequence

The command `random_sequence_graph` is used for generating a random undirected graph with a specified degree sequence.

Syntax: `random_sequence_graph(L)`

`random_sequence_graph` takes the degree sequence L (a list of nonnegative integers) as its only argument. It returns an asimptotically uniform random graph with the degree sequence equal to L using the algorithm developed by Bayati et al. [5].

The algorithm slows down quickly and uses $O(|L|^2)$ of auxiliary space, so it is best used for a few hundreds of vertices or less.

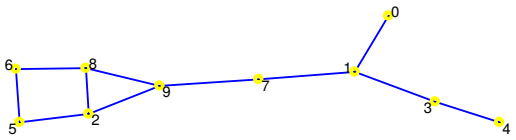
```
> s:=[1,3,3,2,1,2,2,2,3,3]:; is_graphic_sequence(s)
```

Done, true

```
> G:=random_sequence_graph(s)
```

an undirected unweighted graph with 10 vertices and 11 edges

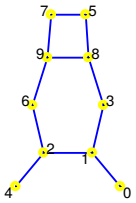
```
> draw_graph(G,spring)
```




```
> H:=random_sequence_graph(s)
```

an undirected unweighted graph with 10 vertices and 11 edges

```
> draw_graph(H,spring)
```



1.10.6. Random regular graphs

The command `random_regular_graph` is used for generating random regular graphs on a specified set of vertices.

Syntax: `random_regular_graph(n or L,d)`
`random_regular_graph(n or L,d,connected)`

`random_regular_graph` takes two mandatory arguments, a positive integer n (or a list L of length n) and a nonnegative integer d. Optionally, the option `connected` may be specified as a third argument, indicating that the generated graph must be connected. The command creates n vertices (using elements of L as vertex labels) and returns a random d-regular (connected) graph on these vertices.

Note that a d-regular graph on n vertices exists if and only if $n > d+1$ and $n\,d$ is even. If these conditions are not met, `random_regular_graph` returns an error.

The strategy is to use the algorithm developed by Steger and Wormald [53, algorithm 2]. The runtime is negligible for $n \leqslant 100$. However, for $n > 200$ the algorithm is considerably slower. Graphs are generated with approximately uniform probability, which means that for $n \to \infty$ and d not growing so quickly with n the probability distribution converges to uniformity.

```
> G:=random_regular_graph(16,3)
```

an undirected unweighted graph with 16 vertices and 24 edges

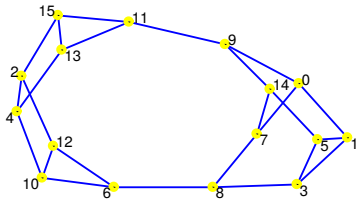
```
> is_regular(G)
```

true

```
> degree_sequence(G)
```

[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]

```
> draw_graph(G,spring)
```



1.10.7. Random tournaments

The command `random_tournament` is used for generating random tournaments.

Syntax: `random_tournament(n)`
`random_tournament(L)`

`random_tournament` takes a positive integer n or a list L of length n as its only argument and returns a random tournament on n vertices. If L is specified, its elements are used to label the vertices.

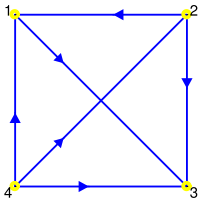
```
> G:=random_tournament([1,2,3,4])
```

a directed unweighted graph with 4 vertices and 6 arcs

```
> is_tournament(G)
```

true

```
> draw_graph(G)
```



1.10.8. Random network graphs

The command `random_network` is used for generation of random networks.

Syntax: `random_network(a,b,[opts])`
`random_network(a,b,p,[opts])`

`random_network` takes two to four arguments: a positive integer a, a positive integer b, an optional real number p such that $0 < p \leqslant 1$ (by default $p = 0.5$) and optionally a sequence of options `opts`. The supported options are `acyclic[=true|false]` and `weights=a..b`.

The command returns a network graph with $a^2 b$ vertices which is composed as follows (the method of generating the network skeleton is due to GOLDFARB and GRIGORIADIS [28]).

Firstly, grid graphs $F_1, F_2, ..., F_b$ (called **frames**), each of them with $a \times a$ vertices, are generated. If the option `acyclic[=true]` is used (by default is `acyclic=false`), then an acyclic orientation is computed for each frame using st-ordering (see Section 4.9.3) with two opposite corners of the frame as source and sink, otherwise all vertices in the frame are connected to their neighbors (forth and back). In addition, for each $k < b$ the vertices of F_k are connected one to one with the vertices of the next frame F_{k+1} using a random permutation of those vertices. The first vertex of the first frame is the source and the last vertex of the last frame is the sink of the network (some arcs may have to be removed to achieve that). Finally, the removal of each arc is attempted with probability $1 - p$ (unless its removal disconnects the network), making each arc present with probability p.

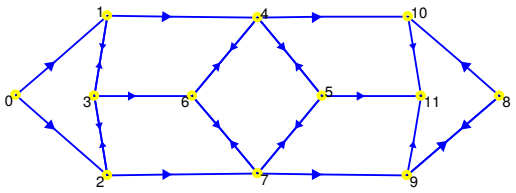
if the option `weights=a..b` is specified, arc weights in the network are randomized in the interval $[a, b] \subset \mathbb{R}$. If a, b are integers, the weights are also integers.

For example, the command line below creates a random network, consisting of 3 frames of size 2×2, in which each arc is present with the probability 0.9.

```
> N1:=random_network(2,3,0.9)
```

a directed unweighted graph with 12 vertices and 25 arcs

```
> draw_graph(N1,spring)
```




```
> is_network(N1)
```

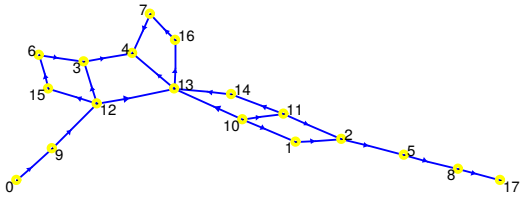
[0], [11]

In the next example, passing the option `acyclic` forces the output graph to be acyclic.

```
> N2:=random_network(3,2,0.618,acyclic)
```

a directed unweighted graph with 18 vertices and 22 arcs

```
> draw_graph(N2,spring)
```




```
> is_network(N2)
```

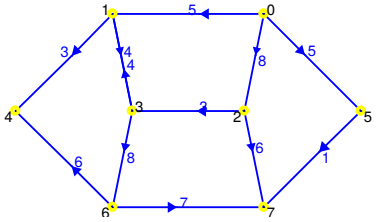
$$\begin{pmatrix} 0 & 10 \\ 4 & 17 \end{pmatrix}$$

Arc weights can be randomized, as demonstrated below.

```
> N3:=random_network(2,2,0.75,weights=1..9)
```

a directed unweighted graph with 8 vertices and 12 arcs

```
> draw_graph(N3,spring)
```



1.10.9. Randomizing edge weights

The command `assign_edge_weights` is used for assigning weights to edges of graphs at random.

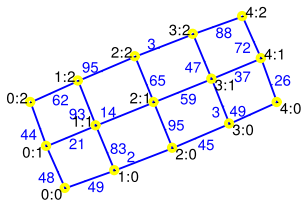
Syntax: `assign_edge_weights(G,a..b)`
`assign_edge_weights(G,m,n)`

`assign_edge_weights` takes two or three arguments: a graph $G(V, E)$ and an interval a .. b of real numbers or a sequence of two positive integers m and n. The command operates such that for, each edge $e \in E$, the weight of e is chosen uniformly from the real interval $[a, b)$ or from the set of integers lying between m and n, including both m and n. After assigning weights to all edges, a modified copy of G is returned.

```
> G:=assign_edge_weights(grid_graph(5,3),1,99)
```

an undirected weighted graph with 15 vertices and 22 edges

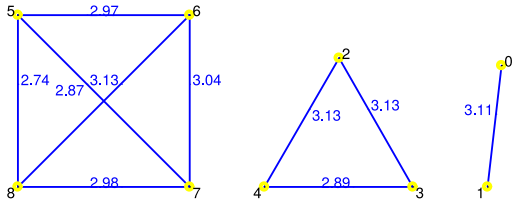
```
> draw_graph(G,spring)
```




```
> G:=assign_edge_weights(graph_complement(complete_graph(2,3,4)),e..pi)
```

an undirected weighted graph with 9 vertices and 10 edges

```
> draw_graph(G)
```



Chapter 2

Modifying graphs

2.1. Promoting to directed and weighted graphs

2.1.1. Converting edges to arcs

To promote an existing undirected graph to a directed one, use the command `make_directed`.

Syntax: `make_directed(G)`
`make_directed(G,A)`

`make_directed` is called with one or two arguments, an undirected graph $G(V, E)$ and optionally a numerical square matrix $A = [a_{ij}]$ of order $|V|$. Every edge $\{i, j\} \in E$ is replaced with the pair of arcs (i, j) and (j, i) and, if matrix A is specified, its elements a_{ij} and a_{ji} are assigned as weights of these arcs, respectively. Thus a directed (weighted) copy of G is constructed and subsequently returned.

```
> make_directed(cycle_graph(4))
```

C4: a directed unweighted graph with 4 vertices and 8 arcs

```
> make_directed(cycle_graph(4),[[0,0,0,1],[2,0,1,3],[0,1,0,4],[5,0,4,0]])
```

C4: a directed weighted graph with 4 vertices and 8 arcs

2.1.2. Assigning weight matrix to unweighted graphs

To promote an existing unweighted graph to a weighted one, use the command `make_weighted`.

Syntax: `make_weighted(G)`
`make_weighted(G,A)`

`make_weighted` takes one or two arguments, an unweighted graph $G(V, E)$ and optionally a square matrix $A = [a_{ij}]$ of order $|V|$. If the matrix specification is omitted, a square matrix of ones is assumed. Then a copy of G is returned in which each edge/arc $(i, j) \in E$ gets the element a_{ij} in A assigned as its weight. If G is undirected, it is assumed that A is a symmetric matrix.

```
> make_weighted(graph(%{[1,2],[2,3],[3,1]%}),[[0,2,3],[2,0,1],[3,1,0]])
```

an undirected weighted graph with 3 vertices and 3 edges

2.2. Modifying vertices of a graph

2.2.1. Adding and removing vertices

For adding and removing vertices to/from graphs use the commands `add_vertex` and `delete_vertex`, respectively.

Syntax: `add_vertex(G,v|L)`
`delete_vertex(G,v|L)`

The command `add_vertex` takes two arguments, a graph $G(V, E)$ and a single label v or a list of labels L, and returns the graph $G'(V \cup \{v\}, E)$ or $G''(V \cup L, E)$ if a list L is given.

```
> K5:=complete_graph([1,2,3,4,5])
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> add_vertex(K5,6)
```

an undirected unweighted graph with 6 vertices and 10 edges

```
> add_vertex(K5,[a,b,c])
```

an undirected unweighted graph with 8 vertices and 10 edges

Note that vertices already present in G will not be added. For example:

```
> add_vertex(K5,[4,5,6])
```

an undirected unweighted graph with 6 vertices and 10 edges

The command `delete_vertex` takes two arguments, a graph $G(V, E)$ and a single label v or a list of labels L, and returns the graph

$$G'(V \setminus \{v\}, \{e \in E\colon e \text{ is not incident to } v\})$$

or, if L is given,

$$G''(V \setminus L, \{e \in E\colon e \text{ is not incident to any } v \in L\}).$$

If any of the specified vertices does not belong to G, an error is returned.

```
> delete_vertex(K5,2)
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> delete_vertex(K5,[2,3])
```

an undirected unweighted graph with 3 vertices and 3 edges

2.3. MODIFYING EDGES OF A GRAPH

2.3.1. Adding and removing edges

For adding and removing edges or arcs to/from graphs use the commands `add_edge` or `add_arc` and `delete_edge` or `delete_arc`, respectively.

Syntax:
```
add_edge(G,e|E|T)
add_arc(G,e|E|T)
delete_edge(G,e|E|T)
delete_arc(G,e|E|T)
```

The command `add_edge` takes two arguments, an undirected graph G and an edge e or a list of edges E or a trail of edges T (entered as a list of vertices), and returns the copy of G with the specified edges inserted. Edge insertion implies that its endpoints will be created if they are not already present in G.

```
> C4:=cycle_graph(4)
```

an undirected unweighted graph with 4 vertices and 4 edges

```
> add_edge(C4,[1,3])
```

an undirected unweighted graph with 4 vertices and 5 edges

```
> add_edge(C4,[1,3,5,7])
```

an undirected unweighted graph with 6 vertices and 7 edges

The command `add_arc` works similarly to `add_edge` but applies only to directed graphs. Note that the order of endpoints in an arc matters.

```
> add_arc(digraph(trail(a,b,c,d,a)),[[a,c],[b,d]])
```

a directed unweighted graph with 4 vertices and 6 arcs

When adding edge/arc to a weighted graph, its weight should be specified alongside its endpoints, or it will be assumed that it equals to 1.

```
> add_edge(graph(%{[[1,2],5],[[3,4],6]%}),[[2,3],7])
```

an undirected weighted graph with 4 vertices and 3 edges

The commands `delete_edge` and `delete_arc` take two arguments, the input graph G and an edge/arc e or a list of edges/arcs E or a trail of edges T. It returns a copy of G in which the specified edges/arcs are removed. Note that this operation does not change the set of vertices of G.

```
> K33:=relabel_vertices(complete_graph(3,3),[A,B,C,D,E,F])
```

an undirected unweighted graph with 6 vertices and 9 edges

```
> has_edge(K33,[A,D])
```

true

```
> delete_edge(K33,[A,D])
```

an undirected unweighted graph with 6 vertices and 8 edges

Note that G itself is not changed.

```
> has_edge(K33,[B,D])
```

true

```
> delete_edge(K33,[[A,D],[B,D]])
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> DG:=digraph(trail(1,2,3,4,5,2,4))
```

a directed unweighted graph with 5 vertices and 6 arcs

```
> delete_arc(DG,[[2,3],[4,5],[5,2]])
```

a directed unweighted graph with 5 vertices and 3 arcs

```
> delete_arc(DG,[3,4,5,2])
```

a directed unweighted graph with 5 vertices and 3 arcs

2.3.2. Accessing and modifying edge weights

The commands `get_edge_weight` and `set_edge_weight` are used to access and modify the weight of an edge in a weighted graph, respectively.

Syntax: `set_edge_weight(G,e,w)`
`set_edge_weight(G,e)`

`set_edge_weight` takes three arguments: a weighted graph $G(V, E)$, edge $e \in E$ and the new weight w, which may be any number. It returns the modified copy of G.

The command `get_edge_weight` takes two arguments, a weighted graph $G(V, E)$ and an edge or arc $e \in E$. It returns the weight of e.

```
> G:=set_edge_weight(graph(%{[[1,2],4],[[2,3],5]%}),[1,2],6)
```

an undirected weighted graph with 3 vertices and 2 edges

```
> get_edge_weight(G,[1,2])
```

6

2.3.3. Contracting edges

The command `contract_edge` is used for contracting edges in undirected graphs.

Syntax: `contract_edge(G,e)`

`contract_edge` takes two arguments, an undirected graph $G(V, E)$ and an edge $e = (v, w) \in E$, and merges v and w to a single vertex, deleting the edge e. The resulting vertex inherits the label of v. The modified copy of G is returned.

```
> K5:=complete_graph(5)
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> contract_edge(K5,[1,2])
```

an undirected unweighted graph with 4 vertices and 6 edges

To contract a set $\{e_1, e_2, ..., e_k\} \subset E$ of edges in G, none two of which are incident (i.e. when the given set is a matching in G), one can use the `foldl` command. In the following example, the complete graph K_5 is obtained from Petersen graph by edge contraction.

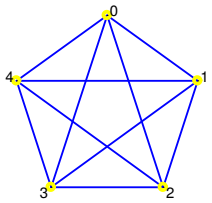
```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> G:=foldl(contract_edge,P,[0,5],[1,6],[2,7],[3,8],[4,9])
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> draw_graph(G)
```



2.3.4. Subdividing edges

The command `subdivide_edges` is used for graph subdivision.

Syntax: `subdivide_edges(G,e|S)`
`subdivide_edges(G,e|S,r)`

`subdivide_edges` takes two or three arguments: a graph $G(V, E)$, a single edge/arc $e \in E$ or a list of edges/arcs $S \subset E$ and optionally a positive integer r (which defaults to 1). Each of the specified edges/arcs will be subdivided with exactly r new vertices, labeled with the smallest available nonnegative integers. The resulting graph, which is homeomorphic to G, is returned.

If the endpoints of the edge being subdivided have valid coordinates, the coordinates of the inserted vertices will be computed accordingly.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> G:=subdivide_edges(G,[2,3])
```

an undirected unweighted graph with 11 vertices and 16 edges

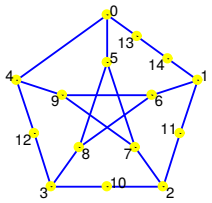
```
> G:=subdivide_edges(G,[[1,2],[3,4]])
```

an undirected unweighted graph with 13 vertices and 18 edges

```
> G:=subdivide_edges(G,[0,1],2)
```

an undirected unweighted graph with 15 vertices and 20 edges

```
> draw_graph(G)
```



2.4. Using attributes

2.4.1. Graph attributes

The graph structure maintains a set of attributes as tag-value pairs which can be accessed and/or modified by using the commands `set_graph_attribute`, `get_graph_attribute`, `list_graph_attributes` and `discard_graph_attribute`.

Syntax:
```
set_graph_attribute(G,tag1=value1,tag2=value2,...)
set_graph_attribute(G,[tag1=value1,tag2=value2,...])
set_graph_attribute(G,[tag1,tag2,...],[value1,value2,...])
get_graph_attribute(G,tag1,tag2,...)
get_graph_attribute(G,[tag1,tag2,...])
list_graph_attributes(G)
discard_graph_attribute(G,tag1,tag2,...)
discard_graph_attribute(G,[tag1,tag2,...])
```

The command `set_graph_attribute` is used for modifying the existing graph attributes or adding new ones. It takes two arguments, a graph G and a sequence or list of graph attributes in form `tag=value` where `tag` is a string. Alternatively, attributes may be specified as a sequence of two lists `[tag1,tag2,...]` and `[value1,value2,...]`. The command sets the specified values to the indicated attribute slots, which are meant to represent some global properties of the graph G, and returns the modified copy of G.

The previously set graph attribute values can be fetched with the command `get_graph_attribute` which takes two arguments: a graph G and a sequence or list of tags. The corresponding values will be returned in a sequence or list, respectively. If an attribute is not set, `undef` is returned as its value.

To list all graph attributes of G for which the values are set, use the command `list_graph_attributes` which takes G as its only argument.

To discard a graph attribute, use the command `discard_graph_attribute`. It takes two arguments: a graph G and a sequence or list of tags to be cleared, and returns the modified copy of G.

Two tags being used by the CAS commands are directed and weighted, so it is not advisable to overwrite their values using this command; use the `make_directed`, `make_weighted` and `underlying_graph` commands instead. Another attribute used internally is name, which holds the name of the respective graph (as a string).

```
> G:=digraph(trail(1,2,3,1))
```

a directed unweighted graph with 3 vertices and 3 arcs

```
> G:=set_graph_attribute(G,"name"="C3","message"="this is some text")
```

C3: a directed unweighted graph with 3 vertices and 3 arcs

```
> get_graph_attribute(G,"message")
```

this is some text

```
> list_graph_attributes(G)
```

[directed = true, weighted = false, name = $C3$, message = this is some text]

```
> G:=discard_graph_attribute(G,"message")
```

C3: a directed unweighted graph with 3 vertices and 3 arcs

```
> list_graph_attributes(G)
```

[directed = true, weighted = false, name = $C3$]

2.4.2. Vertex attributes

For every vertex of a graph, the list of attributes in form of tag-value pairs is maintained, which can be accessed/modified by using the commands `set_vertex_attribute`, `get_vertex_attribute`, `list_vertex_attributes` and `discard_vertex_attribute`.

Syntax:
```
set_vertex_attribute(G,v,tag1=value1,tag2=value2,...)
set_vertex_attribute(G,v,[tag1=value1,tag2=value2,...])
set_vertex_attribute(G,v,[tag1,tag2,...],[value1,value2,...])
get_vertex_attribute(G,v,tag1,tag2,...)
get_vertex_attribute(G,v,[tag1,tag2,...])
list_vertex_attributes(G,v)
discard_vertex_attribute(G,v,tag1,tag2,...)
discard_vertex_attribute(G,v,[tag1,tag2,...])
```

The command `set_vertex_attribute` is used for modifying the existing vertex attributes or adding new ones. It takes three arguments, a graph $G(V, E)$, a vertex $v \in V$ and a sequence or list of attributes in form `tag=value` where `tag` is a string. Alternatively, attributes may be specified as a sequence of two lists `[tag1,tag2,...]` and `[value1,value2,...]`. The command sets the specified values to the indicated attributes of the vertex v and returns the modified copy of G.

The previously set attribute values for v can be fetched with the command `get_vertex_attribute` which takes three arguments: G, v and a sequence or list of tags. The corresponding values will be returned in a sequence or list, respectively. If an attribute is not set, `undef` is returned as its value.

To list all attributes of v for which the values are set, use the command `list_vertex_attributes` which takes two arguments, G and v.

The command `discard_vertex_attribute` is used for discarding attribute(s) assigned to some vertex $v \in V$. It takes three arguments: G, v and a sequence or list of tags to be cleared, and returns the modified copy of G.

The attributes label, color, shape and pos are also used internally. These hold the vertex label, color, shape and coordinates in a drawing, respectively. If the color is not set for a vertex, the latter is drawn in yellow. The shape attribute may have one of the following values: square, triangle, diamond, star or plus. If the shape attribute is not set or has a different value, the circled shape is applied when drawing the vertex.

The following example shows how to change individual labels and colors.

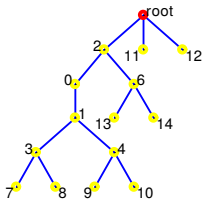
```
> T:=complete_binary_tree(3)
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> T:=set_vertex_attribute(T,5,"label"="root","color"=red)
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> draw_graph(T,tree="root")
```



A vertex may also hold custom attributes.

```
> T:=set_vertex_attribute(T,"root","depth"=3,"shape"="square")
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> list_vertex_attributes(T,"root")
```

[label = root, color = $r\,e\,d$, shape = square, depth = 3]

```
> T:=discard_vertex_attribute(T,"root","color")
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> list_vertex_attributes(T,"root")
```

[label = root, shape = square, depth = 3]

2.4.3. Edge attributes

For every edge of a graph, the list of attributes in form of key-value pairs is maintained, which can be accessed and/or modified by using the commands `set_edge_attribute`, `get_edge_attribute`, `list_edge_attributes` and `discard_edge_attribute`.

Syntax:
```
set_edge_attribute(G,e,tag1=value1,tag2=value2,...)
set_edge_attribute(G,e,[tag1=value1,tag2=value2,...])
set_edge_attribute(G,e,[tag1,tag2,...],[value1,value2,...])
get_edge_attribute(G,e,tag1,tag2,...)
get_edge_attribute(G,e,[tag1,tag2,...])
list_edge_attributes(G,e)
discard_edge_attribute(G,e,tag1,tag2,...)
discard_edge_attribute(G,e,[tag1,tag2,...])
```

The command `set_edge_attribute` is used for modifying the existing edge attributes or adding new ones. It takes three arguments, a graph $G(V, E)$, an edge/arc $e \in E$ and a sequence or list of attributes in form `tag=value` where `tag` is a string. Alternatively, attributes may be specified as a sequence of two lists `[tag1,tag2,...]` and `[value1,value2,...]`. The command sets the specified values to the indicated attributes of the edge/arc e and returns the modified copy of G.

The previously set attribute values for e can be fetched with the command `get_edge_attribute` which takes three arguments: G, e and a sequence or list of tags. The corresponding values will be returned in a sequence or list, respectively. If some attribute is not set, `undef` is returned as its value.

To list all attributes of e for which the values are set, use the command `list_edge_attributes` which takes two arguments, G and e.

To discard attribute(s) assigned to e call the command `discard_edge_attribute`, which takes three arguments: G, e and a sequence or list of tags to be cleared, and returns the modified copy of G.

The attributes weight, color, style, pos and temp are also used internally. They hold the edge weight, color, line style, the coordinates of the weight label anchor (and also the coordinates of the arrow) and `true` if the edge is temporary. If the color attribute is not set for an edge, the latter is drawn in blue, unless it is a temporary edge, in which case it is drawn in light gray. The style attribute may have one of the following values: dashed, dotted or bold. If the style attribute is not set or has a different value, the solid line style is applied when drawing the edge.

The following example illustrates the possibilities of using edge attributes.

```
> T:=complete_binary_tree(3)
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> T:=set_edge_attribute(T,[1,4],"cost"=12.8,"message"="this is some text")
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> list_edge_attributes(T,[1,4])
```

$[\text{cost} = 12.8, \text{message} = \text{this is some text}]$

```
> T:=discard_edge_attribute(T,[1,4],"message")
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> T:=set_edge_attribute(T,[1,4],"style"="dotted","color"=magenta)
```

an undirected unweighted graph with 15 vertices and 14 edges

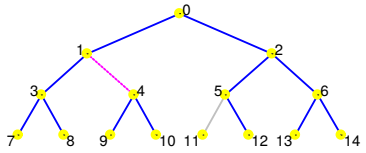
```
> list_edge_attributes(T,[1,4])
```

$[\text{color} = m\,a\,g\,e\,n\,t\,a, \text{style} = \text{dotted}, \text{cost} = 12.8]$

```
> T:=set_edge_attribute(T,[5,11],"temp"=true)
```

an undirected unweighted graph with 15 vertices and 14 edges

```
> draw_graph(T)
```



Chapter 3

Import and export

3.1. Importing graphs

3.1.1. Loading graphs from dot files

The command `import_graph` is used for importing a graph from text file in dot format.

Syntax: `import_graph(filename)`

`import_graph` takes a string `filename` as its only argument and returns the graph constructed from instructions written in the file `filename` or `undef` on failure. The passed string should contain the path to a file in dot format. The file extension `.dot` may be omitted in the `filename` since dot is the only supported format. The alternative extension is `.gv`[3.1], which must be explicitly specified.

If a relative path to the file is specified, i.e. if it does not contain a leading forward slash, the current working directory (which can be obtained by calling the `pwd` command) will be used as the reference. The working directory can be changed by using the command `cd`.

For example, assume that the file `example.dot` is saved in the directory `path/to/dot/` with the following contents:

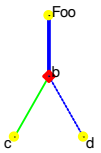
```
graph "Example graph" {
  a [label="Foo"];
  b [shape=diamond,color=red];
  a -- b [style=bold];
  b -- c [color=green];
  b -- d [style=dotted];
}
```

To import the graph, input:

```
> G:=import_graph("path/to/dot/example.dot")
```

Example graph: an undirected unweighted graph with 4 vertices and 3 edges

```
> draw_graph(G)
```



3.1.2. The dot file format overview

Giac has some basic support for dot language. Each dot file is used to hold exactly one graph and should consist of a single instance of the following environment:

3.1. Although it is recommended to use `.gv` as the extension for dot files to avoid a certain confusion between different file types, Giac uses the `.dot` extension because it coincides with the format name. This may change in the future.

```
strict? (graph | digraph) name? {
    ...
}
```

The keyword `strict` may be omitted, as well as the `name` of the graph, as indicated by the question marks. The former is used to differentiate between simple graphs (strict) and multigraphs (non-strict). Since Giac supports only simple graphs, `strict` is redundant.

For specifying undirected graphs the keyword `graph` is used, while the `digraph` keyword is used for undirected graphs.

The `graph`/`digraph` environment contains a series of instructions describing how the graph should be built. Each instruction ends with the semicolon (`;`) and has one of the following forms.

syntax	*creates*
`vertex_name [attributes]?`	isolated vertices
`V1 <edgeop> V2 <edgeop> ... <edgeop> Vk [attributes]?`	edges and trails
`graph [attributes]`	graph attributes

Here, `attributes` is a comma-separated list of tag-value pairs in form `tag=value`, `<edgeop>` is `--` for undirected and `->` for directed graphs. Each of `V1`, `V2` etc. is either a vertex name or a set of vertex names in form `{vertex_name1 vertex_name2 ...}`. In the case a set is specified, each vertex from that set is connected to the neighbor operands. Every specified vertex will be created if it does not exist yet.

Lines beginning with `#` are ignored. C-like comments are recognized and skipped as well.

Using the dot syntax it is easy to specify a graph with adjacency lists. For example, the following is the contents of a file which defines the octahedral graph with 6 vertices and 12 edges.

```
# octahedral graph
graph "octahedron" {
  1 -- {3 6 5 4};
  2 -- {3 4 5 6};
  3 -- {5 6};
  4 -- {5 6};
}
```

3.2. EXPORTING GRAPHS

The command `export_graph` is used for saving graphs to disk in dot or LaTeX format.

Syntax: `export_graph(G,filename)`
`export_graph(G,filename,latex[=<params>])`

The argument `filename` should be a string containing a path to the desired destination file (which is created if it does not exist). The remark on relative paths in Section 3.1.1 applies here as well.

3.2.1. Saving graphs in dot format

`export_graph` takes two mandatory arguments, a graph G and a string `filename`, and writes G to the file specified by `filename`. If only two arguments are given the graph is saved in dot format. The file name may be entered with or without `.dot` extension. The command returns 1 on success and 0 on failure.

```
> export_graph(G,"path/to/dot/copy_of_example")
```

3.2.2. Saving graph drawings in LaTeX format

When calling `export_graph`, an optional third argument in form `latex[=<params>]` may be given. In that case the drawing of G (obtained by calling the `draw_graph` command) will be saved to the LaTeX file indicated by `filename` (the extension `.tex` may be omitted). Optionally, one can specify a parameter or list of parameters `params` which will be passed to `draw_graph`.

For example, let us create a picture of the Sierpiński sieve graph of order $n=5$, i.e. the graph ST_3^5.

```
> G:=sierpinski_graph(5,3,triangle)
```

an undirected unweighted graph with 123 vertices and 243 edges

```
> export_graph(G,"some/directory/st53.tex",latex=[spring,labels=false])
```

1

The LaTeX file obtained by exporting a graph is easily converted into an EPS file, which can subsequently be inserted[3.2] in a paper, report or some other document. A Linux user simply needs to launch a terminal emulator, navigate to the directory in which the exported file, in this case `st53.tex`, is stored and enter the following command:

```
latex st53.tex && dvips st53.dvi && ps2eps st53.ps
```

This will produce the (properly cropped) `st53.eps` file in the same directory. Afterwards, it is recommended to enter

```
rm st53.tex st53.aux st53.log st53.dvi st53.ps
```

to delete the intermediate files. The above two commands can be combined in a simple shell script which takes the name of the exported file (without the extension) as its input argument:

```
#!/bin/bash
# convert LaTeX to EPS
latex $1.tex
dvips $1.dvi
ps2eps $1.ps
rm $1.tex $1.aux $1.log $1.dvi $1.ps
```

Assuming that the script is stored under the name `latex2eps` in the same directory as `st53.tex`, to do the conversion it is enough to input:

```
bash latex2eps st53
```

The drawing produced in our example is shown in Figure 3.1.

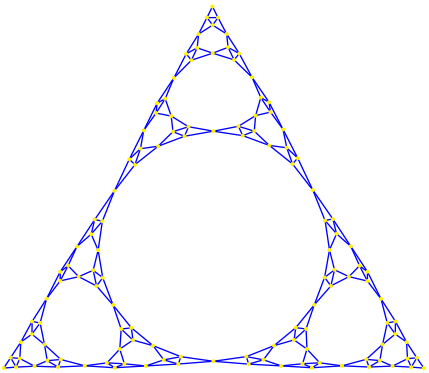


Fig. 3.1. drawing of the Sierpiński graph ST_3^5 using LaTeX and PSTricks

3.2. Alternatively, a PSTricks picture from the body of the `.tex` file can be copied to some other LaTeX document.

Chapter 4

Graph properties

4.1. Basic properties

4.1.1. Determining the type of a graph

The commands `is_directed` and `is_weighted` are used for determining the type of a graph: whether is it directed or not resp. weighted or not.

Syntax: `is_directed(G)`
`is_weighted(G)`

Both commands take a graph G as their only argument. `is_directed` resp. `is_weighted` returns `true` if G is directed resp. weighted, else it returns `false`.

```
> G:=graph(trail(1,2,3,4,5,1,3))
```

an undirected unweighted graph with 5 vertices and 6 edges

```
> is_directed(G)
```

false

```
> is_directed(make_directed(G))
```

true

```
> is_weighted(G)
```

false

```
> is_weighted(make_weighted(G,randmatrix(5,5,99)))
```

true

4.1.2. Listing vertices and edges

The command `vertices` or `graph_vertices` resp. `edges` is used for extracting the set of vertices resp. the set of edges from a graph. To obtain the number of vertices resp. the number of edges, use the `number_of_vertices` resp. the `number_of_edges` command.

Syntax: `vertices(G)`
`graph_vertices(G)`
`edges(G)`
`edges(G,weights)`
`number_of_vertices(G)`
`number_of_edges(G)`

`vertices` or `graph_vertices` takes a graph $G(V, E)$ as its only argument and returns the set of vertices V in the same order in which they were created.

`edges` takes one or two arguments, a graph $G(V, E)$ and optionally the identifier `weights`. The command returns the set of edges E (in a non-meaningful order). If `weights` is specified, each edge is paired with the corresponding weight (in this case G must be a weighted graph).

`number_of_vertices` resp. `number_of_edges` takes the input graph $G(V, E)$ as its only argument and returns $|V|$ resp. $|E|$.

```
> G:=hypercube_graph(2)
```

an undirected unweighted graph with 4 vertices and 4 edges

```
> vertices(G)
```

$[00, 01, 10, 11]$

```
> C:=graph("coxeter")
```

an undirected unweighted graph with 28 vertices and 42 edges

```
> vertices(C)
```

$[a1, a2, a7, z1, a3, z2, a4, z3, a5, z4, a6, z5, z6, z7, b1, b3, b6, b2, b4, b7, b5, c1, c4, c5, c2, c6, c3, c7]$

```
> number_of_vertices(C), number_of_edges(C)
```

$28, 42$

```
> H:=digraph([[0,2.32,0,0.25],[0,0,0,1.32],[0,0.50,0,0],[0.75,0,3.34,0]])
```

a directed weighted graph with 4 vertices and 6 arcs

```
> edges(H)
```

$$\begin{pmatrix} 0 & 1 \\ 0 & 3 \\ 1 & 3 \\ 2 & 1 \\ 3 & 0 \\ 3 & 2 \end{pmatrix}$$

```
> edges(H,weights)
```

$\{[[0, 1], 2.32], [[0, 3], 0.25], [[1, 3], 1.32], [[2, 1], 0.5], [[3, 0], 0.75], [[3, 2], 3.34]\}$

4.1.3. Equality of graphs

Two graphs are considered **equal** if they are both (un)weighted and (un)directed and if the commands `vertices` and `edges` give the same results for both graphs. To determine whether two graphs are equal use the command `graph_equal`.

Syntax: `graph_equal(G1,G2)`

`graph_equal` takes two arguments, graphs G_1 and G_2, and returns `true` if G_1 is equal to G_2 with respect to the above definition. Else, it returns `false`.

```
> G1:=graph([1,2,3],%{[1,2],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> G2:=graph([1,3,2],%{[1,2],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> graph_equal(G1,G2)
```

false

```
> G3:=graph(trail(1,2,3))
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> graph_equal(G1,G3)
```

true

```
> G4:=digraph(trail(1,2,3))
```

a directed unweighted graph with 3 vertices and 2 arcs

```
> graph_equal(G1,G4)
```

false

4.1.4. Vertex degrees

The command `vertex_degree` is used for computing the degree of a vertex, i.e. counting the vertices adjacent to it. The related specialized commands are `vertex_out_degree`, `vertex_in_degree`, `degree_sequence`, `minimum_degree` and `maximum_degree`.

Syntax:
```
vertex_degree(G,v)
vertex_in_degree(G,v)
vertex_out_degree(G,v)
degree_sequence(G)
minimum_degree(G,v)
maximum_degree(G,v)
```

`vertex_degree` takes two arguments, a graph $G(V,E)$ and a vertex $v \in V$, and returns the cardinality of the set $\{w \in V : (v,w) \in E\}$, i.e. the number of vertices in V which are adjacent to v. Note that the edge directions are ignored in case G is a digraph.

When dealing with directed graphs, one can also use the specialized command `vertex_out_degree` resp. `vertex_in_degree` which takes the same arguments as `vertex_degree` but returns the number of arcs $(v,w) \in E$ resp. the number of arcs $(w,v) \in E$, where $w \in V$.

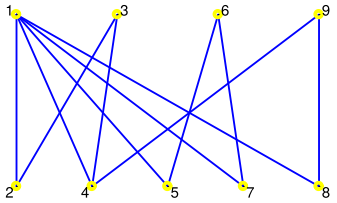
To obtain the list of degrees of all vertices $v \in V$, use the command `degree_sequence` which takes a graph $G(V,E)$ as its only argument and returns the list of degrees of vertices from V in the same order as returned by the command `vertices`. If G is a digraph, arc directions are ignored.

To compute the minimum vertex degree $\delta(G)$ and the maximum vertex degree $\Delta(G)$ in an undirected graph G, use the commands `minimum_degree` and `maximum_degree`, respectively. Both commands take G as the only argument and return $\delta(G)$ resp. $\Delta(G)$.

```
> G:=graph(trail(1,2,3,4,1,5,6,7,1,8,9,4))
```

an undirected unweighted graph with 9 vertices and 11 edges

```
> draw_graph(G)
```




```
> vertex_degree(G,1)
```

5

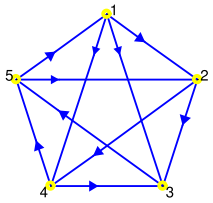
```
> degree_sequence(G)
```

$[5,2,2,3,2,2,2,2,2]$

```
> T:=random_tournament([1,2,3,4,5])
```

a directed unweighted graph with 5 vertices and 10 arcs

```
> draw_graph(T)
```




```
> vertex_out_degree(T,1)
```

3

```
> vertex_in_degree(T,5)
```

2

The command line below shows that Petersen graph is cubic (3-regular).

```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> minimum_degree(P), maximum_degree(P)
```

3, 3

```
> is_regular(P,3)
```

true

4.1.5. Regular graphs

The command `is_regular` is used for determining whether a graph is regular.

Syntax: `is_regular(G)`
`is_regular(G,d)`

`is_regular` takes one or two arguments, a graph $G(V, E)$ and optionally a nonnegative integer or an unassigned identifier d. If G is undirected, the return value is `true` if $\delta_G = \Delta_G$, i.e. if the minimal vertex degree is equal to the maximal vertex degree in G, otherwise `false` is returned. If G is a digraph, it is also required for each vertex $v \in V$ to have the same in- and out-degree. If the second argument is given, G is tested for d-regularity in case d is an integer, otherwise Δ_G is written to d in case the latter is an identifier and G is regular.

The complexity of the algorithm is $O(|V|)$.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> is_regular(G,d)
```

true

```
> d
```

3

```
> is_regular(G,2)
```

false

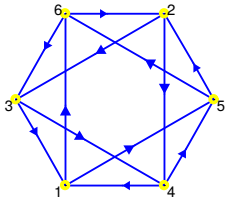
```
> is_regular(graph("grotzsch"))
```

false

```
> G:=digraph(%{[1,5],[1,6],[2,3],[2,4],[3,1],[3,4],[4,1],[4,5],[5,2],[5,6],[6,
  2],[6,3]%})
```

a directed unweighted graph with 6 vertices and 12 arcs

```
> draw_graph(G,spring)
```




```
> is_regular(G,4)
```

true

```
> H:=add_arc(delete_arc(G,[5,6]),[6,5])
```

a directed unweighted graph with 6 vertices and 12 arcs

```
> is_regular(H,4)
```

false

```
> is_regular(underlying_graph(H))
```

true

4.1.6. Strongly regular graphs

The command `is_strongly_regular` is used for determining whether a graph is strongly regular.

Syntax: `is_strongly_regular(G)`
`is_strongly_regular(G,srg)`

`is_strongly_regular` takes one or two arguments, a graph $G(V, E)$ and optionally an unassigned identifier `srg`. It returns `true` if G is regular and there are integers λ and μ such that every two adjacent vertices resp. non-adjacent vertices in V have exactly λ resp. μ common neighbors. Else, it returns `false`. If the second argument is given, the list $[k, \lambda, \mu]$, where k is the degree of G, is stored to `srg`.

The complexity of the algorithm is $O(k\,|V|^2)$.

```
> G:=graph("clebsch")
```

an undirected unweighted graph with 16 vertices and 40 edges

```
> is_regular(G)
```

true

```
> is_strongly_regular(G)
```

true

```
> H:=graph("shrikhande")
```

an undirected unweighted graph with 16 vertices and 48 edges

```
> is_strongly_regular(H,s)
```

true

```
> s
```

$[6, 2, 2]$

```
> is_strongly_regular(cycle_graph(5))
```

true

```
> is_strongly_regular(cycle_graph(6))
```

false

4.1.7. Vertex adjacency

The command `has_edge` is used for checking whether two vertices in an undirected graph are adjacent. For digraphs, there is an analogous command `has_arc`.

The command `neighbors` is used for obtaining the list of vertices in a graph that are adjacent to the particular vertex or the complete adjacency structure of the graph, in sparse form.

The command `departures` resp. `arrivals` is used for obtaining all neighbors of a vertex v in a digraph which are the heads resp. the tails of the corresponding arcs.

Syntax: `has_edge(G,[u,v])`
`has_arc(G,[u,v])`
`neighbors(G)`
`neighbors(G,v)`
`departures(G)`
`departures(G,v)`
`arrivals(G)`
`arrivals(G,v)`

`has_edge` takes two arguments, an undirected graph $G(V, E)$ and a list `[u,v]` where $u, v \in V$. The command returns `true` if $uv \in E$ and `false` otherwise. The syntax for `has_arc` is the same, except now G is required to be directed. Note, however, that the order of vertices u and v matters in digraphs. The complexity of both algorithms is $O(\log |V|)$.

`neighbors` takes one or two arguments, a graph $G(V, E)$ and optionally a vertex $v \in V$. The command returns the list of neighbors of v in G if v is given. Otherwise, it returns the list of lists of neighbors for all vertices in V, in order of `vertices(G)`. Note that edge directions are ignored in case G is a digraph.

`departures` resp. `arrivals` takes one or two arguments, a digraph $G(V, E)$ and optionally a vertex $v \in V$, and returns the list L_v containing all vertices $w \in V$ for which $vw \in E$ resp. $wv \in E$. If v is omitted, the list of lists L_v for every $v \in V$ is returned.

```
> G:=graph(trail(1,2,3,4,5,2))
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> has_edge(G,[1,2])
```

true

```
> has_edge(G,[2,1])
```

true

```
> has_edge(G,[1,3])
```

false

```
> D:=digraph(trail(1,2,3,4,5,2,1))
```

a directed unweighted graph with 5 vertices and 6 arcs

```
> has_arc(D,[1,2])
```

true

```
> has_arc(D,[2,1])
```

true

```
> has_arc(D,%{1,2%})
```

true

```
> has_arc(D,[4,5])
```

true

```
> has_arc(D,[5,4])
```

false

```
> has_arc(D,%{4,5%})
```

false

```
> neighbors(G,3)
```

$[2, 4]$

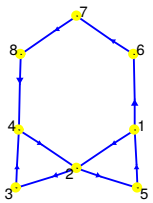
```
> neighbors(G)
```

$\{[2], [1, 3, 5], [2, 4], [3, 5], [2, 4]\}$

```
> G:=digraph(trail(1,2,3,4,2,5,1,6,7,8,4))
```

a directed unweighted graph with 8 vertices and 10 arcs

```
> draw_graph(G,spring)
```




```
> departures(G,2); arrivals(G,2); departures(G,1); arrivals(G,1)
```

$[3, 5], [1, 4], [2, 6], [5]$

4.1.8. Tournament graphs

The command `is_tournament` is used for determining whether a graph is a tournament.

Syntax: `is_tournament(G)`

`is_tournament` takes a graph $G(V, E)$ as its only argument and returns `true` if G is directed and for each pair of vertices $u, v \in V$ it is either $uv \in E$ or $vu \in E$, i.e. there is exactly one arc between u and v. Else, it returns `false`.

```
> T1:=digraph(%{[1,2],[2,3],[3,1]%})
```

a directed unweighted graph with 3 vertices and 3 arcs

```
> is_tournament(T1)
```

true

```
> T2:=digraph(%{[1,2],[2,3],[3,1],[1,3]%})
```

a directed unweighted graph with 3 vertices and 4 arcs

```
> is_tournament(T2)
```

false

4.1.9. Bipartite graphs

The command `is_bipartite` is used for determining if a graph is bipartite.

Syntax: `is_bipartite(G)`
`is_bipartite(G,P)`

`is_bipartite` takes one or two arguments, a graph $G(V, E)$ and optionally an unassigned identifier `P`. It returns `true` if there is a partition of V into two sets S and T such that every edge from E connects a vertex in S to one in T. Else, it returns `false`. If the second argument is given and G is bipartite, the partition of V is stored to `P` as a list of two lists of vertices, the first one containing the vertices from S and the second one containing vertices from T.

```
> K32:=complete_graph(3,2)
```

an undirected unweighted graph with 5 vertices and 6 edges

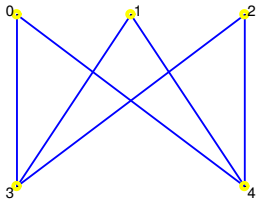
```
> is_bipartite(K32,bp)
```

true

```
> bp
```

$[[0, 1, 2], [3, 4]]$

```
> draw_graph(K32,bipartite)
```




```
> adjacency_matrix(K32)
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

```
> G:=cycle_graph(5)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> is_bipartite(G)
```

false

4.1.10. Edge incidence

The command `incident_edges` is used for obtaining edges incident to a given vertex in a graph.

Syntax: `indcident_edges(G,v)`
`indcident_edges(G,L)`

`incident_edges` takes two argument, a graph $G(V, E)$ and a vertex $v \in V$ or a list of vertices $L \subset V$. The command returns the list of edges $e_1, e_2, ..., e_k \in E$ which have v as one of its endpoints.

Note that edge directions are ignored when G is a digraph. To obtain only outgoing or incoming edges, use the commands `departures` and `arrivals`, respectively.

```
> G:=cycle_graph([1,2,3,4,5])
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> incident_edges(G,1)
```

$$\begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}$$

```
> incident_edges(G,[2,4,5])
```

$$\begin{pmatrix} 1 & 2 \\ 1 & 5 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{pmatrix}$$

```
> G:=random_tournament([1,2,3,4,5])
```

a directed unweighted graph with 5 vertices and 10 arcs

```
> incident_edges(G,2)
```

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 5 \\ 4 & 2 \end{pmatrix}$$

4.2. Algebraic properties

4.2.1. Adjacency matrix

The command `adjacency_matrix` is used for obtaining the adjacency matrix of a graph.

Syntax: `adjacency_matrix(G)`

`adjacency_matrix` takes a graph $G(V, E)$, where $V = \{v_1, v_2, ..., v_n\}$, as its only argument and returns the square matrix $A = [a_{ij}]$ of order n such that, for $i, j = 1, 2, ..., n$,

$$a_{ij} = \begin{cases} 1, & \text{if the set } E \text{ contains edge/arc } v_i v_j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mathrm{tr}\,(A) = 0$. Also, the adjacency matrix of an undirected graph is always symmetrical.

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> A:=adjacency_matrix(G)
```

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

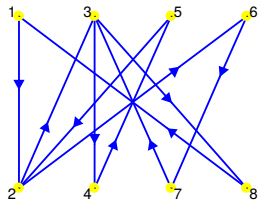
```
> transpose(A)==A
```

true

```
> D:=digraph(trail(1,2,3,4,5,2,6,7,3,8,1))
```

a directed unweighted graph with 8 vertices and 10 arcs

```
> draw_graph(D)
```




```
> A:=adjacency_matrix(D)
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
> transpose(A)==A
```

false

4.2.2. Laplacian matrix

The command `laplacian_matrix` is used for computing the Laplacian matrix of a graph.

Syntax: `laplacian_matrix(G)`
`laplacian_matrix(G,normal)`

`laplacian_matrix` takes an undirected graph $G(V, E)$, where $V = \{v_1, v_2, ..., v_n\}$, and returns the symmetric matrix $L = D - A$, where A is the adjacency matrix of G and

$$D = \begin{pmatrix} \deg(v_1) & 0 & 0 & \cdots & 0 \\ 0 & \deg(v_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \deg(v_n) \end{pmatrix}.$$

The option `normal` may be passed as the second argument. In that case, the normalized Laplacian $L^{\text{sym}} := I - D^{-1/2} A D^{-1/2}$ of G is returned.

```
> G:=path_graph(4)
```

an undirected unweighted graph with 4 vertices and 3 edges

```
> A:=adjacency_matrix(G)
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
> L:=laplacian_matrix(G)
```

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

```
> diag(degree_sequence(G))-A==L
```

true

```
> laplacian_matrix(G,normal)
```

$$\begin{pmatrix} 1 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 1 & \frac{-1}{2} & 0 \\ 0 & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 1 \end{pmatrix}$$

The smallest eigenvalue of a Laplacian matrix of an undirected graph is always zero. Moreover, its multiplicity is equal to the number of connected components in the corresponding graph [27, pp. 280].

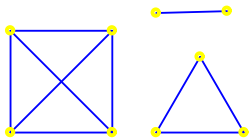
```
> sort(eigenvals(L))
```

$$0, -\sqrt{2}+2, 2, \sqrt{2}+2$$

```
> H:=disjoint_union(complete_graph(4),cycle_graph(3),path_graph(2))
```

an undirected unweighted graph with 9 vertices and 10 edges

```
> draw_graph(H,labels=false)
```



```
> eigenvals(laplacian_matrix(H))
```

$$0, 0, 0, 4, 4, 4, 3, 3, 2$$

```
> nops(connected_components(H))
```

3

4.2.3. Incidence matrix

The command `incidence_matrix` is used for obtaining the incidence matrix of a graph.

Syntax: `incidence_matrix(G)`

`incidence_matrix` takes a graph $G(V, E)$, where $V = \{v_1, v_2, ..., v_n\}$ and $E = \{e_1, e_2, ..., e_m\}$, as its only argument and returns the $n \times m$ matrix $B = [b_{ij}]$ such that, for all $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$,

$$b_{ij} = \begin{cases} 1, & \text{if the vertex } v_i \text{ is incident to the edge } e_j, \\ 0, & \text{otherwise} \end{cases}$$

if G is undirected resp.

$$b_{ij} = \begin{cases} 1, & \text{if the vertex } v_i \text{ is the head of the arc } e_j, \\ -1, & \text{if the vertex } v_i \text{ is the tail of the arc } e_j, \\ 0, & \text{otherwise} \end{cases}$$

if G is directed.

```
> K4:=complete_graph([1,2,3,4])
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> edges(K4)
```

$$\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 2 & 3 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}$$

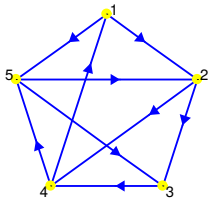
```
> incidence_matrix(K4)
```

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

```
> DG:=digraph(trail(1,2,3,4,5,3),trail(1,5,2,4,1))
```

a directed unweighted graph with 5 vertices and 9 arcs

```
> draw_graph(DG)
```




```
> edges(DG)
```

$$\begin{pmatrix} 1 & 2 \\ 1 & 5 \\ 2 & 3 \\ 2 & 4 \\ 3 & 4 \\ 4 & 1 \\ 4 & 5 \\ 5 & 2 \\ 5 & 3 \end{pmatrix}$$

```
> incidence_matrix(DG)
```

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

4.2.4. Weight matrix

The command `weight_matrix` is used for obtaining the weight matrix of a weighted graph.

Syntax: `weight_matrix(G)`

`weight_matrix` takes a graph $G(V,E)$, where $V=\{v_1,v_2,...,v_n\}$, as its only argument and returns the square matrix $M=[m_{ij}]$ of order n such that m_{ij} equals zero if v_i and v_j are not adjacent and the weight of the edge/arc $v_i\,v_j$ otherwise, for all $i,j=1,2,...,n$ (note that the weight of an edge/arc may be any real number).

Note that $\mathrm{tr}\,(M)=0$. Also, the weight matrix of an undirected graph is always symmetrical.

```
> G:=graph(%{[[1,2],1],[[2,3],2],[[4,5],3],[[5,2],4]%})
```

an undirected weighted graph with 5 vertices and 4 edges

```
> weight_matrix(G)
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 4 & 0 & 3 & 0 \end{pmatrix}$$

4.2.5. Characteristic polynomial

The command `graph_charpoly` or `charpoly` is used for obtaining the characteristic polynomial of an undirected graph.

Syntax: `graph_charpoly(G)`
`graph_charpoly(G,x)`
`charpoly(G)`
`charpoly(G,x)`

`graph_charpoly` or `charpoly` takes one or two arguments, an undirected graph $G(V,E)$ and optionally a value or symbol x. The command returns $p(x)$, where p is the characteristic polynomial of the adjacency matrix of G.

```
> G:=graph(%{[1,2],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> charpoly(G,x)
```

$$x^3-2\,x$$

```
> charpoly(G,3)
```

21

```
> G:=graph("shrikhande")
```

an undirected unweighted graph with 16 vertices and 48 edges

```
> charpoly(G,x)
```

$$x^{16}-48\,x^{14}-64\,x^{13}+768\,x^{12}+1536\,x^{11}-5888\,x^{10}-15360\,x^9+23040\,x^8+81920\,x^7-36864\,x^6-245760\,x^5-32768\,x^4+393216\,x^3+196608\,x^2-262144\,x-196608$$

4.2.6. Graph spectrum

The command `graph_spectrum` is used for computing graph spectra.

Syntax: `graph_spectrum(G)`

`graph_spectrum` takes a graph G as its only argument and returns the list in which every element is an eigenvalue of the adjacency matrix of G paired with its multiplicity.

```
> C5:=cycle_graph(5)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> gs:=graph_spectrum(C5)
```

$$\begin{pmatrix} 2 & 1 \\ \frac{\sqrt{5}-1}{2} & 2 \\ \frac{-\sqrt{5}-1}{2} & 2 \end{pmatrix}$$

```
> p:=charpoly(C5,x)
```

$$x^5 - 5\,x^3 + 5\,x - 2$$

```
> expand(roots(p))==expand(gs)
```

true

The above result indicates that `gs` and `roots(p)` are equal.

4.2.7. Seidel spectrum

The command `seidel_spectrum` is used for computing Seidel spectra.

Syntax: `seidel_spectrum(G)`

`seidel_spectrum` takes a graph $G(V, E)$ as its only argument and returns the list in which every element is an eigenvalue of the Seidel adjacency matrix S paired with its multiplicity. The matrix S, which can be interpreted as the difference of the adjacency matrices of G and its complement G^c, is computed as $J - I - 2\,A$, where J is all-one $n \times n$ matrix, I is the identity matrix of order n, A is the adjacency matrix of G and $n = |V|$.

```
> seidel_spectrum(graph("clebsch"))
```

$$\begin{pmatrix} -3 & 10 \\ 5 & 6 \end{pmatrix}$$

```
> seidel_spectrum(graph("levi"))
```

$$\begin{pmatrix} -5 & 9 \\ -1 & 10 \\ 3 & 9 \\ 5 & 1 \\ 23 & 1 \end{pmatrix}$$

4.2.8. Integer graphs

The command `is_integer_graph` is used for determining whether a graph is an integral graph.

Syntax: `is_integer_graph(G)`

`is_integer_graph` takes a graph G as its only argument and returns `true` if the spectrum of G consists only of integers. Else it returns `false`.

```
> G:=graph("levi")
```

an undirected unweighted graph with 30 vertices and 45 edges

```
> is_integer_graph(G)
```

true

```
> factor(charpoly(G,x))
```

$$x^{10}\,(x-3)\,(x-2)^9\,(x+2)^9\,(x+3)$$

```
> graph_spectrum(G)
```

$$\begin{pmatrix} -3 & 1 \\ -2 & 9 \\ 0 & 10 \\ 2 & 9 \\ 3 & 1 \end{pmatrix}$$

4.3. Graph isomorphism

4.3.1. Isomorphic graphs

The command `is_isomorphic` is used for determining whether two graphs are isomorphic.

Syntax: `is_isomorphic(G1,G2)`
`is_isomorphic(G1,G2,m)`
`canonical_labeling(G)`

`is_isomorphic` takes two or three arguments: a graph $G_1(V_1, E_1)$, a graph $G_2(V_2, E_2)$ and optionally an unassigned identifier `m`. The command returns `true` if G_1 and G_2 are isomorphic and `false` otherwise. If the third argument is given and G_1 and G_2 are isomorphic, the list of pairwise matching of vertices in G_1 and G_2, representing the isomorphism between the two graphs, is stored to `m`.

Note that the algorithm takes vertex colors into account. Namely, only vertices sharing the same color can be mapped to each other. Vertex colors can be set by calling the `highlight_vertex` command.

This command, as well as the commands `canonical_labeling` and `graph_automorphisms` described later in this section, is using `nauty` library developed by Brendan McKay [40], which is one of the fastest implementations for graph isomorphism.

For example, entering the command line below one shows that Petersen graph is isomorphic to Kneser graph $K(5, 2)$.

```
> is_isomorphic(graph("petersen"),kneser_graph(5,2))
```

true

In the following example, G_1 and G_3 are isomorphic while G_1 and G_2 are not isomorphic.

```
> G1:=graph(trail(1,2,3,4,5,6,1,3))
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> G2:=graph(trail(1,2,3,4,5,6,1,4))
```

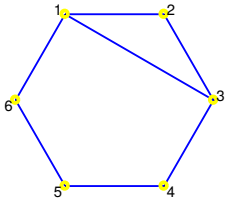
an undirected unweighted graph with 6 vertices and 7 edges

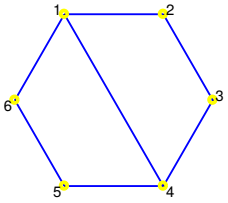
```
> G3:=graph(trail(1,2,3,4,5,6,1,5))
```

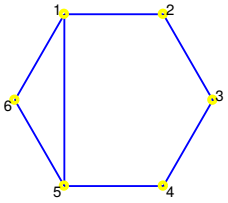
an undirected unweighted graph with 6 vertices and 7 edges

```
> draw_graph(G1,circle)
> draw_graph(G2,circle)
> draw_graph(G3,circle)
```

The drawings are ordered from left to right.








```
> is_isomorphic(G1,G2)
```

false

```
> is_isomorphic(G1,G3)
```

true

```
> is_isomorphic(G1,G3,mapping):; mapping
```

Done, $[1=5, 2=6, 3=1, 4=2, 5=3, 6=4]$

```
> H1:=highlight_vertex(G1,5):; H3:=highlight_vertex(G3,5):;
```

Done, Done

```
> is_isomorphic(H1,H3)
```

false

```
> H1:=highlight_vertex(H1,1):; H3:=highlight_vertex(H3,3):;
```

Done, Done

```
> is_isomorphic(H1,H3)
```

true

In the next example, D_1 and D_3 are isomorphic while D_1 and D_2 are not isomorphic.

```
> D1:=digraph(trail(1,2,3,1,4,5))
```

a directed unweighted graph with 5 vertices and 5 arcs

```
> D2:=digraph(trail(1,2,3,4,5,3))
```

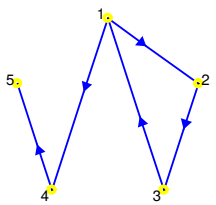
a directed unweighted graph with 5 vertices and 5 arcs

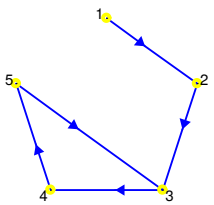
```
> D3:=digraph([1,2,3,4,5],trail(3,4,5,3,1,2))
```

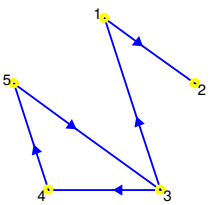
a directed unweighted graph with 5 vertices and 5 arcs

```
> draw_graph(D1,circle)
> draw_graph(D2,circle)
> draw_graph(D3,circle)
```

The drawings are ordered from left to right.








```
> is_isomorphic(D1,D2)
```

false

```
> is_isomorphic(D1,D3)
```

true

Isomorphism testing with nauty is very fast and can be used for large graphs, as in the example below.

```
> G:=random_graph(10000,0.01)
```

an undirected unweighted graph with 10000 vertices and 499867 edges

```
> H:=isomorphic_copy(G,randperm(10000))
```

an undirected unweighted graph with 10000 vertices and 499867 edges

```
> is_isomorphic(G,H)
```

true

1.7 sec

To make the edge structures of G and H slightly different, a random edge from H is "misplaced".

```
> ed:=edges(H)[rand(number_of_edges(H))]
```

$[813, 3021]$

```
> has_edge(H,[813,3022])
```

false

```
> H:=add_edge(delete_edge(H,ed),[813,3022])
```

an undirected unweighted graph with 10000 vertices and 499867 edges

```
> is_isomorphic(G,H)
```

false

4.3.2. Canonical labeling

Graph isomorphism testing in nauty is based on computing the canonical labelings for the input graphs. The **canonical labeling** of G is a particular ordering of the vertices of G. Rearranging the vertices with respect to that ordering produces the **canonical representation** of G. Two graphs are isomorphic if and only if their canonical representations share the same edge structure.

The command `canonical_labeling` is used for computing the canonical labeling as a permutation. One can reorder the vertices by using this permutation with the `isomorphic_copy` command.

`canonical_labeling` takes a graph $G(V, E)$ as its only argument and returns the permutation representing the canonical labeling of G. Note that the colors of the vertices are taken into account.

In the next example it is demonstrated how to prove that G_1 and G_3 are isomorphic by comparing their canonical representations C_1 and C_3 with the `graph_equal` command. Before testing C_1 and C_3 for equality, their vertices have to be relabeled so that the command `vertices` gives the same result for both graphs.

```
> L1:=canonical_labeling(G1)
```

$[4, 3, 5, 1, 2, 0]$

```
> L3:=canonical_labeling(G3)
```

$[2, 1, 3, 5, 0, 4]$

```
> C1:=relabel_vertices(isomorphic_copy(G1,L1),[1,2,3,4,5,6])
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> C3:=relabel_vertices(isomorphic_copy(G3,L3),[1,2,3,4,5,6])
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> graph_equal(C1,C3)
```

true

4.3.3. Graph automorphisms

The command `graph_automorphisms` is used for finding generators of the automorphism group of a graph.

Syntax: `graph_automorphisms(G)`

`graph_automorphisms` takes a graph G as its only argument and returns a list containing the generators of $\mathrm{Aut}(G)$, the automorphism group of G (see [27, pp. 4] and [7, pp. 115]). Each generator is given as a list of cycles, which can be turned to a permutation by calling the command `cycles2permu`.

Note that vertex colors are taken into account. Only vertices sharing the same color can be mapped to each other. The color of a vertex can be set by calling the command `highlight_vertex`.

```
> g:=graph_automorphisms(graph("petersen"))
```

$$\{\begin{pmatrix} 3 & 7 \\ 4 & 5 \\ 8 & 9 \end{pmatrix}, \begin{pmatrix} 2 & 6 \\ 3 & 8 \\ 4 & 5 \\ 7 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 6 & 9 \\ 7 & 8 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 4 \\ 5 & 6 \\ 7 & 9 \end{pmatrix}\}$$

```
> cycles2permu(g[2])
```

$$[0, 4, 3, 2, 1, 5, 9, 8, 7, 6]$$

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> G:=highlight_vertex(G,4)
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> graph_automorphisms(G)
```

$$\{\begin{pmatrix} 2 & 6 \\ 3 & 9 \\ 7 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 9 \\ 6 & 8 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 1 & 2 \\ 5 & 8 \\ 6 & 7 \end{pmatrix}\}$$

In the above result, all permutations map the vertex 4 to itself, because it is the single green-colored vertex in G (it cannot be mapped to any other vertex because colors do not match).

Frucht graph (see the page 23) is an example of a graph with automorphism group containing only the identity, so the set of its generators is empty:

```
> graph_automorphisms(graph("frucht"))
```

$$\{\}$$

4.4. GRAPH POLYNOMIALS

4.4.1. Tutte polynomial

The command `tutte_polynomial` is used for computing Tutte polynomials.

Syntax: `tutte_polynomial(G)`
`tutte_polynomial(G,x,y)`

`tutte_polynomial` takes one or three arguments, an undirected graph $G(V, E)$ and optionally two variables or values x and y. It returns the the bivariate Tutte polynomial[4.1] T_G of G or the value $T_G(x, y)$ if the optional arguments are given. If G is weighted, it is treated as a multigraph: the weight w of an edge e, which must be a positive integer, is interpreted as the multiplicity of e, for each $e \in E$. Note, however, that loops are not supported.

The strategy is to apply the recursive definition of Tutte polynomial [29] together with the vorder heuristic proposed by Haggard et al. [30] and improved by Monagan [41]. The subgraphs appearing in the computation tree are cached and reused when possible, pruning the tree significantly. Subgraphs are stored (and compared) in their canonical form, for which the nauty library is used.

Note that finding Tutte polynomials is NP-hard in general, hence the problem becomes intractable for larger and/or denser graphs.

```
> K4:=complete_graph(4)
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> tutte_polynomial(K4,x,y)
```

$$x^3 + 3\,x^2 + 4\,x\,y + 2\,x + y^3 + 3\,y^2 + 2\,y$$

```
> tutte_polynomial(K4,x,1)
```

$$x^3 + 3\,x^2 + 6\,x + 6$$

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> f:=tutte_polynomial(G)
```

$$x^9 + 6\,x^8 + 21\,x^7 + 56\,x^6 + 12\,x^5\,y + 114\,x^5 + 70\,x^4\,y + 170\,x^4 + 30\,x^3\,y^2 + 170\,x^3\,y + 180\,x^3 + 15\,x^2\,y^3 + 105\,x^2\,y^2 + 240\,x^2\,y + 120\,x^2 + 10\,x\,y^4 + 65\,x\,y^3 + 171\,x\,y^2 + 168\,x\,y + 36\,x + y^6 + 9\,y^5 + 35\,y^4 + 75\,y^3 + 84\,y^2 + 36\,y$$

This result coincides with that in [7, pp. 103], which is supposed to be correct. Alternatively, it can be verified by applying the recursive definition with an arbitrary edge $e \in E$, as below.

```
> ed:=edges(G)[0]
```

$$[0, 1]$$

```
> Gdelete:=delete_edge(G,ed)
```

an undirected unweighted graph with 10 vertices and 14 edges

```
> Gcontract:=contract_edge(G,ed)
```

an undirected unweighted graph with 9 vertices and 14 edges

```
> expand(f-tutte_polynomial(Gdelete)-tutte_polynomial(Gcontract))
```

$$0$$

The value $T_G(1, 1)$ is equal to the number of spanning forests in G [9, pp. 345]—in this case, the number of spanning trees in Petersen graph. For verification, the same number is computed by using the specialized command `number_of_spanning_trees`, which uses a different (much faster) algorithm.

4.1. See [29], [7, pp. 97] and [9, pp. 335].

```
> tutte_polynomial(G,1,1)
```

2000

```
> number_of_spanning_trees(G)
```

2000

For a graph G and its dual G^* the following relation holds: $T_G(x, y) = T_{G^*}(y, x)$. Therefore, if $T_G(x, y) = T_G(y, x)$ then G and G^* are isomorphic (since Tutte polynomial is a graph invariant). A simple example of such graph is tetrahedral graph. Since it is planar and biconnected, its dual can be determined by using the command `plane_dual`.

```
> G:=graph("tetrahedron")
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> is_biconnected(G) and is_planar(G)
```

true

```
> H:=plane_dual(G)
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> T:=tutte_polynomial(G)
```

$$x^3 + 3\,x^2 + 4\,x\,y + 2\,x + y^3 + 3\,y^2 + 2\,y$$

```
> expand(T-subs(T,[x,y],[y,x]))
```

0

```
> is_isomorphic(G,H)
```

true

Multiple edges can be specified as edge weights.

```
> M:=make_weighted(G)
```

an undirected weighted graph with 4 vertices and 6 edges

```
> M:=set_edge_weight(set_edge_weight(M,[1,2],2),[3,4],3)
```

an undirected weighted graph with 4 vertices and 6 edges

```
> edges(M,weights)
```

$$\{[[1,2],2],[[1,3],1],[[1,4],1],[[2,3],1],[[2,4],1],[[3,4],3]\}$$

```
> tutte_polynomial(M,x,y)
```

$$x^3 + x^2\,y^2 + 2\,x^2\,y + 3\,x^2 + 3\,x\,y^3 + 6\,x\,y^2 + 6\,x\,y + 2\,x + y^6 + 3\,y^5 + 6\,y^4 + 7\,y^3 + 5\,y^2 + 2\,y$$

4.4.2. Chromatic polynomial

The command `chromatic_polynomial`, is used for computing chromatic polynomials.

Syntax: `chromatic_polynomial(G)`
`chromatic_polynomial(G,t)`

`chromatic_polynomial` takes one or two arguments, an undirected unweighted graph $G(V, E)$ and optionally a variable or value t. It returns the chromatic polynomial P_G of G or the value $P_G(t)$ if the second argument is given.

P_G and the Tutte polynomial T_G satisfy the following relation (see [29] and [9, pp. 346]):

$$P_G(t) = (-1)^{|V|-\kappa(G)}\, t^{\kappa(G)}\, T_G(1-t, 0), \tag{4.1}$$

where $\kappa(G)$ is the number of connected components of G. `chromatic_polynomial` uses (4.1) to compute P_G.

The value $P_G(k)$, where $k>0$ is an integer, is equal to the number of all distinct k-colorings of vertices in G. As shown in the example below, Petersen graph cannot be colored by using only two colors, but is 3-colorable with 120 distinct colorings (all using the same three colors).

```
> P:=chromatic_polynomial(graph("petersen"),x)
```

$$x\,(x-2)\,(x-1)\,(x^7-12\,x^6+67\,x^5-230\,x^4+529\,x^3-814\,x^2+775\,x-352)$$

```
> subs(P,x=2)
```

0

```
> subs(P,x=3)
```

120

4.4.3. Flow polynomial

The command `flow_polynomial` is used for computing flow polynomials.

Syntax: `flow_polynomial(G)`
`flow_polynomial(G,x)`

`flow_polynomial` takes one or two arguments, an undirected unweighted graph $G(V,E)$ and optionally a variable or value x. It returns the flow polynomial Q_G of G or the value $Q_G(x)$ if the second argument is given.

Q_G and the Tutte polynomial T_G satisfy the following relation (see [29] and [7, pp. 110]):

$$Q_G(x)=(-1)^{|E|-|V|+\kappa(G)}\,T_G(0,1-x), \tag{4.2}$$

where $\kappa(G)$ is the number of connected components of G. `flow_polynomial` uses (4.2) to compute Q_G.

The value $Q_G(k)$, where $k>0$ is an integer, is equal to the number of all nowhere-zero k-flows in G. In such flows, the total flow f_v entering and leaving vertex v is congruent modulo k, hence $f_v\in\{1,2,...,k-1\}$ for all $v\in V$ [9, pp. 347]. As shown in the example below, Petersen graph has zero 4-flows and 240 5-flows.

```
> Q:=flow_polynomial(graph("petersen"))
```

$$x^6-15\,x^5+95\,x^4-325\,x^3+624\,x^2-620\,x+240$$

```
> Q | x=4
```

0

```
> Q | x=5
```

240

4.4.4. Reliability polynomial

The command `reliability_polynomial` is used for computing reliability polynomials.

Syntax: `reliability_polynomial(G)`
`reliability_polynomial(G,p)`

`reliability_polynomial` takes one or two arguments, an undirected graph $G(V,E)$ and optionally a variable or value p. It returns the all-terminal reliability polynomial R_G of G or the value $R_G(p)$ if the second argument is given. If G is weighted, it is treated as a multigraph: the weight w of an edge e, which must be a positive integer, is interpreted as the multiplicity of e, for each $e\in E$.

R_G and the Tutte polynomial T_G satisfy the following relation [41]:

$$R_G(p) = (1-p)^{|V|-\kappa(G)}\, p^{|E|-|V|+\kappa(G)}\, T_G(1, p^{-1}), \tag{4.3}$$

where $\kappa(G)$ is the number of connected components of G. `reliability_polynomial` uses (4.3) to compute R_G.

If G is a connected network, then the value $R_G(p)$, where $p \in [0,1]$, is equal to the probability that G does not fail (i.e. stays connected) after removing each edge with probability p [27, pp. 354–355].

In the following example, it is clear that the graph G will stay connected with probability $(1-p)^2$ if each of its two edges is removed with probability p.

```
> G:=graph(%{[1,2],[2,3]%})
```

an undirected unweighted graph with 3 vertices and 2 edges

```
> R:=reliability_polynomial(G,p)
```

$$p^2 - 2\,p + 1$$

```
> factor(R)
```

$$(p-1)^2$$

Adding a new edge should increase the reliability of G, since the latter is connected. Indeed, the difference $S - R$ below is positive for $0 < p < 1$.

```
> S:=reliability_polynomial(add_edge(G,[1,3]),p)
```

$$2\,p^3 - 3\,p^2 + 1$$

```
> factor(S-R)
```

$$2\,p\,(p-1)^2$$

Multiple edges can be specified as edge weights.

```
> M:=graph(%{[[1,2],2],[[2,3],1],[[3,1],1]%})
```

an undirected weighted graph with 3 vertices and 3 edges

```
> factor(reliability_polynomial(M))
```

$$(x-1)^2\,(2\,x^2 + 2\,x + 1)$$

The following graph represents the Arpanet (early internet) in December 1970.

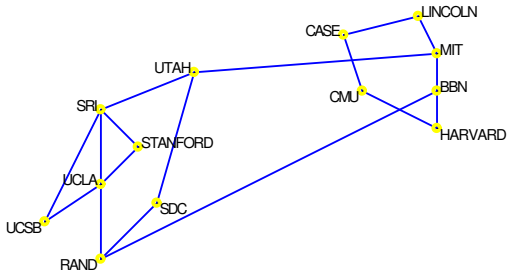
```
> V:=["MIT","LINCOLN","CASE","CMU","HARVARD","BBN","UCSB","UCLA","STANFORD",
  "SRI","RAND","UTAH","SDC"]:;
> A:=graph(V,trail("BBN","HARVARD","CMU","CASE","LINCOLN","MIT","UTAH","SRI",
  "STANFORD","UCLA","UCSB","SRI","UCLA","RAND","BBN","MIT"),trail("RAND","SDC",
  "UTAH"))
```

an undirected unweighted graph with 13 vertices and 17 edges

```
> Arpanet:=set_vertex_positions(A,[[1.0,1.0],[0.9,1.2],[0.5,1.1],[0.6,0.8],[1.0,
  0.6],[1.0,0.8],[-1.1,0.1],[-0.8,0.3],[-0.6,0.5],[-0.8,0.7],[-0.8,-0.1],[-0.3,
  0.9],[-0.5,0.2]])
```

an undirected unweighted graph with 13 vertices and 17 edges

```
> draw_graph(Arpanet)
```



Which edge should be added to the Arpanet to improve the reliability the most? Below is an analysis for the edge from Stanford to CMU.

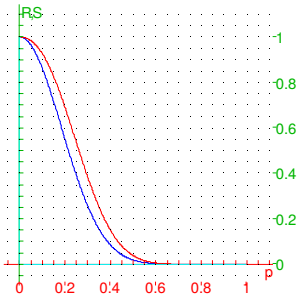
```
> R:=reliability_polynomial(Arpanet,p)
```

$$(p-1)^{12}\,(280\,p^5+310\,p^4+186\,p^3+63\,p^2+12\,p+1)$$

```
> S:=reliability_polynomial(add_edge(Arpanet,["STANFORD","CMU"]),p)
```

$$(p-1)^{12}\,(976\,p^6+1118\,p^5+703\,p^4+276\,p^3+72\,p^2+12\,p+1)$$

```
> labels=["p","R,S"]; plot([R,S],p=0..1,color=[blue,red])
```



The improvement is defined as the area enclosed by the above two curves, which can be computed as an integral.

```
> improvement:=integrate(S-R,p=0..1)
```

$$\frac{443879}{10581480}$$

```
> evalf(improvement)
```

$$0.0419486688063$$

4.5. Connectivity

4.5.1. Connected, biconnected and triconnected graphs

The commands `is_connected`, `is_biconnected` and `is_triconnected` are used for determining if a graph is connected, biconnected or triconnected (3-connected), respectively.

Syntax: `is_connected(G)`
`is_biconnected(G)`
`is_triconnected(G)`

Each of the above commands takes a graph $G(V, E)$ as its only argument and returns `true` if G possesses the required level of connectivity. Else, it returns `false`.

If G is directed, the edge directions are simply ignored (the commands operate on the underlying graph of G).

The strategy for checking 1- and 2-connectivity is to use depth-first search (see [26, pp. 20] and [54]). Both algorithms run in $O(|V|+|E|)$ time. The algorithm for checking 3-connectivity is quite simple but less efficient: it works by choosing a vertex $v \in V$ and checking if the subgraph induced by $V \setminus \{v\}$ is biconnected, moving on to the next vertex if so, and repeating the process until all vertices are visited exactly once or a non-biconnected subgraph is found for some v. In the latter case the input graph is not triconnected. The complexity of this algorithm is $O(|V|\,|E|)$.

```
> G:=graph_complement(complete_graph(2,3,4))
```

an undirected unweighted graph with 9 vertices and 10 edges

```
> is_connected(G)
```

false

```
> C:=connected_components(G)
```

$\{[0,1],[2,3,4],[5,6,7,8]\}$

```
> H:=induced_subgraph(G,C[2])
```

an undirected unweighted graph with 4 vertices and 6 edges

```
> is_connected(H)
```

true

```
> is_biconnected(path_graph(5))
```

false

```
> is_biconnected(cycle_graph(5))
```

true

```
> is_triconnected(graph("petersen"))
```

true

```
> is_triconnected(cycle_graph(5))
```

false

4.5.2. Connected and biconnected components

The command `connected_components` resp. `biconnected_components` is used for decomposing a graph into connected resp. biconnected components.

Syntax: `connected_components(G)`
`biconnected_components(G)`

`connected_components` resp. `biconnected_components` takes a graph $G(V,E)$ as its only argument and returns the minimal partition $\{V_1, V_2, ..., V_k\}$ of V such that the subgraph $G_i \subset G$ induced by V_i is connected resp. biconnected for each $i = 1, 2, ..., k$. The partition is returned as a list of lists $V_1, V_2, ..., V_k$.

If G is directed, the edge directions are simply ignored (the commands operate on the underlying graph of G).

The connected components of G are readily obtained by depth-first search in $O(|V|+|E|)$ time. To find the biconnected components of G, TARJAN's algorithm is used [54], which also runs in linear time.

```
> G:=graph_complement(complete_graph(3,5,7))
```

an undirected unweighted graph with 15 vertices and 34 edges

```
> is_connected(G)
```

false

```
> C:=connected_components(G)
```

$\{[0,1,2],[3,4,5,6,7],[8,9,10,11,12,13,14]\}$

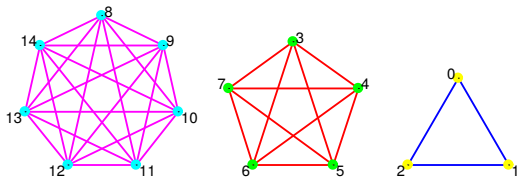
```
> G:=highlight_subgraph(G,induced_subgraph(G,C[1]))
```

an undirected unweighted graph with 15 vertices and 34 edges

```
> G:=highlight_subgraph(G,induced_subgraph(G,C[2]),magenta,cyan)
```

an undirected unweighted graph with 15 vertices and 34 edges

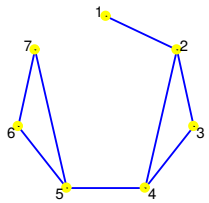
```
> draw_graph(G)
```




```
> H:=graph(trail(1,2,3,4,2),trail(4,5,6,7,5))
```

an undirected unweighted graph with 7 vertices and 8 edges

```
> draw_graph(H)
```




```
> is_biconnected(H)
```

false

```
> biconnected_components(H)
```

$\{[1,2],[2,3,4],[4,5],[5,6,7]\}$

4.5.3. Vertex connectivity

The command `vertex_connectivity` is used for computing vertex connectivity in undirected graphs.

Syntax: `vertex_connectivity(G)`

`vertex_connectivity` takes an undirected connected graph $G(V,E)$ as its only argument and returns the largest integer k for which G is k-vertex-connected, meaning that G remains connected after removing fewer than k vertices from V.

The strategy is to use the algorithm by Esfahanian and Hakimi [21], which is based on the maximum-flow computing approach by Even [22, Section 6.2]. The algorithm makes $|V|-\delta-1+\frac{\delta(\delta-1)}{2}$ calls to `maxflow` command, where δ is the minimum vertex degree in G.

```
> vertex_connectivity(graph("petersen"))
```

3

```
> vertex_connectivity(graph("clebsch"))
```

5

```
> G:=random_planar_graph(1000,0.5,2)
```

an undirected unweighted graph with 1000 vertices and 1876 edges

```
> is_biconnected(G)
```

true

```
> vertex_connectivity(G)
```

2

3.28 sec

4.5.4. Graph rank

The command `graph_rank` is used for computing graph rank.

Syntax: `graph_rank(G)`
`graph_rank(G,S)`

`graph_rank` takes one or two arguments, a graph $G(V,E)$ and optionally a set of edges $S \subset E$ (by default $S = E$), and returns $|V| - k$ where k is the number of connected components of the spanning subgraph of G with edge set S.

```
> G:=graph(%{[1,2],[3,4],[4,5]%})
```

an undirected unweighted graph with 5 vertices and 3 edges

```
> graph_rank(G)
```

3

```
> graph_rank(G,[[1,2],[3,4]])
```

2

4.5.5. Articulation points

The command `articulation_points` is used for obtaining the set of articulation points (cut-vertices) of a graph.

Syntax: `articulation_points(G)`

`articulation_points` takes a graph $G(V,E)$ as its only argument and returns the list of articulation points of G. A vertex $v \in V$ is an **articulation point** of G if the removal of v increases the number of connected components of G.

The articulation points of G are found by depth-first search in $O(|V| + |E|)$ time [26].

```
> articulation_points(path_graph([1,2,3,4]))
```

$[2, 3]$

```
> length(articulation_points(cycle_graph(1,2,3,4)))
```

0

4.5.6. Strongly connected components

The command `strongly_connected_components` is used for decomposing digraphs into strongly connected components. A digraph H is **strongly connected** if for each pair (v, w) of distinct vertices in H there is a directed path from v to w in H. The command `is_strongly_connected` can be used to determine whether a graph is strongly connected.

Syntax: `strongly_connected_components(G)`
`is_strongly_connected(G)`

`strongly_connected_components` takes a digraph $G(V, E)$ as its only argument and returns the minimal partition $\{V_1, V_2, ..., V_k\}$ of V such that the subgraph $G_i \subset G$ induced by V_i is strongly connected for each $i = 1, 2, ..., k$. The result is returned as a list of lists $V_1, V_2, ..., V_k$.

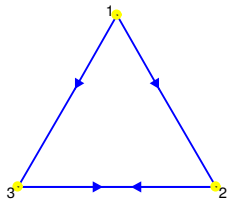
`is_strongly_connected` takes a digraph G as its only argument and returns `true` if G has exactly one strongly connected component and `false` otherwise.

The strategy is to use Tarjan's algorithm for strongly connected components [54], which runs in $O(|V| + |E|)$ time.

```
> G:=digraph([1,2,3],%{[1,2],[1,3],[2,3],[3,2]%})
```

a directed unweighted graph with 3 vertices and 4 arcs

```
> draw_graph(G)
```




```
> is_connected(G)
```

true

```
> is_strongly_connected(G)
```

false

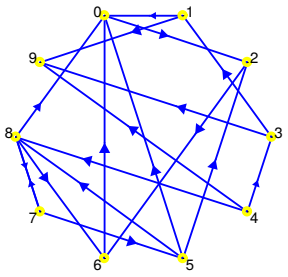
```
> strongly_connected_components(G)
```

$\{[1], [2, 3]\}$

```
> G:=random_digraph(10,18)
```

a directed unweighted graph with 10 vertices and 18 arcs

```
> draw_graph(G)
```




```
> strongly_connected_components(G)
```

$\{[0, 2, 6], [1], [3], [4], [5, 7, 8], [9]\}$

4.5.7. Edge connectivity

The command `edge_connectivity` is used for computing the edge connectivity of an undirected graph.

Syntax: `edge_connectivity(G)`

`edge_connectivity` takes an undirected connected graph $G(V, E)$ as its only argument and returns the largest integer k for which G is k-edge connected, meaning that G remains connected after fewer than k edges are removed from E.

The strategy is to apply MATULA's algorithm [58, Section 13.3.1], which constructs a dominating set $D \subset V$ and calls `maxflow` command $|D| - 1$ times.

```
> G:=cycle_graph([1,2,3,4,5])
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> edge_connectivity(G)
```

2

```
> K5:=complete_graph(5)
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> edge_connectivity(K5)
```

4

```
> edge_connectivity(graph("petersen"))
```

3

```
> edge_connectivity(graph("clebsch"))
```

5

4.5.8. Edge cuts

The command `is_cut_set` is used for determining whether a particular subset of edges of a graph is an edge cut.

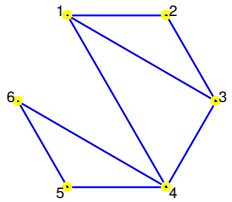
Syntax: `is_cut_set(G,L)`

`is_cut_set` takes two arguments, a graph $G(V, E)$ and a subset $L \subset E$ of edges, and returns `true` if the graph $G'(V, E \setminus L)$ has more connected components than G. Else it returns `false`.

```
> G:=graph(trail(1,2,3,4,5,6,4,1,3))
```

an undirected unweighted graph with 6 vertices and 8 edges

```
> draw_graph(G)
```




```
> E:=[[1,4],[3,4]]
```

$$\begin{pmatrix} 1 & 4 \\ 3 & 4 \end{pmatrix}$$

```
> is_cut_set(G,E)
```

true

```
> is_connected(delete_edge(G,E))
```

false

4.5.9. Two-edge-connected graphs

The command `is_two_edge_connected` is used for determining whether an undirected graph is two-edge-connected. The command `two_edge_connected_components` is used for splitting a graph into components having this property.

Syntax: `is_two_edge_connected(G)`
`two_edge_connected_components(G)`

`is_two_edge_connected` takes an undirected graph $G(V, E)$ as its only argument and returns `true` if G has no bridges, i.e. edges which removal increases the number of connected components of G.

`two_edge_connected_components` takes an undirected graph $G(V, E)$ and returns the list of two-edge-connected components of G, each of them represented by the list of its vertices. To obtain a component as a graph, use the `induced_subgraph` command.

The strategy for finding bridges [55] is similar to finding articulation points. Once the bridges of G are found, it is easy to split G into two-edge-connected components by removing the bridges and returning the list of connected components of the resulting graph. Both algorithms run in $O(|V| + |E|)$ time.

```
> is_two_edge_connected(cycle_graph(4))
```

true

```
> is_two_edge_connected(path_graph(4))
```

false

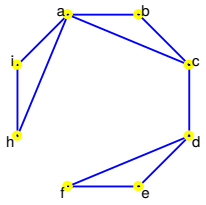
```
> G:=graph(%{["a","b"],["b","c"],["a","c"],["d","e"],["e","f"],["d","f"],["c",
  "d"],["a","h"],["a","i"],["h","i"]%})
```

an undirected unweighted graph with 8 vertices and 10 edges

```
> is_two_edge_connected(G)
```

false

```
> draw_graph(G)
```



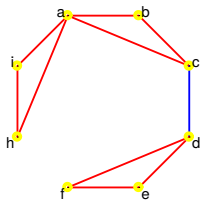

```
> C:=two_edge_connected_components(G)
```

$$\{[a, b, c, h, i], [d, e, f]\}$$

To visualize the bridges of G, one can highlight the edges of each component. The remaining (unhighlighted) edges are the bridges.

```
> for c in C do G:=highlight_edges(G,edges(induced_subgraph(G,c))); od:;
```

```
> draw_graph(G)
```



4.6. TREES

4.6.1. Tree graphs

The command `is_tree` is used for determining whether a graph is a tree.

Syntax: `is_tree(G)`

`is_tree` takes a graph $G(V,E)$ as its only argument and returns `true` if G is undirected, connected and $|V|=|E|+1$. Else it returns `false`.

The only expensive step in the algorithm is determining whether G is connected. The condition $|V|=|E|+1$ is checked first, hence the algorithm runs in $O(|V|)$ time.

```
> is_tree(complete_binary_tree(3))
```

true

```
> is_tree(cycle_graph(5))
```

false

4.6.2. Forest graphs

The command `is_forest` is used for determining whether a graph is a forest.

Syntax: `is_forest(G)`

`is_forest` takes the a $G(V,E)$ as its only argument and returns `true` if every connected component of G is a tree and `false` otherwise.

The algorithm runs in $O(|V|+|E|)$ time.

```
> F:=disjoint_union(apply(random_tree,[k$(k=10..30)]))
```

an undirected unweighted graph with 420 vertices and 399 edges

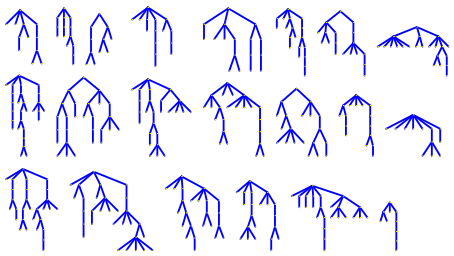
```
> is_connected(F)
```

false

```
> is_forest(F)
```

true

```
> draw_graph(F)
```



4.6.3. Height of a tree

The command `tree_height` is used for determining the height of a tree with respect to the specified root node. The **height** of a tree T is the length of the longest path in T that has the root node of T as one of its endpoints.

Syntax: `tree_height(G,r)`

`tree_height` takes two arguments, a tree graph $G(V,E)$ and a vertex $r \in V$, which is used as the root node. The command returns the height of G with respect to r.

The strategy is to start a depth-first search from the root node and look for the deepest node. Therefore the algorithm runs in $O(|V|)$ time.

```
> G:=random_tree(1000)
```

an undirected unweighted graph with 1000 vertices and 999 edges

```
> r:=rand(1000)
```

296

```
> tree_height(G,r)
```

20

4.6.4. Lowest common ancestor of a pair of nodes

The command `lowest_common_ancestor` is used for computing the lowest common ancestor (LCA) of a pair of nodes in a tree or for each element of a list of such pairs.

Syntax: `lowest_common_ancestor(G,r,u,v)`
`lowest_common_ancestor(G,r,[[u1,v1],[u2,v2],..,[uk,vk]])`

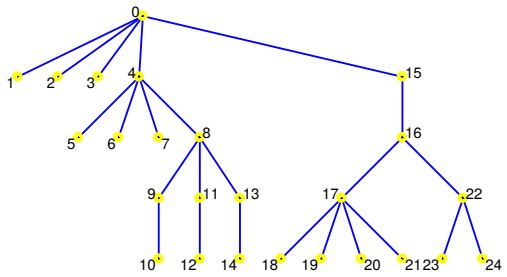
`lowest_common_ancestor` takes two mandatory arguments, a tree graph $G(V,E)$ and the root node $r \in V$. There are two possibilities for specifying the nodes to operate on: either the nodes u, $v \in V$ are given as the third and the fourth argument, or a list of pairs of nodes $(u_1,v_1),(u_2,v_2),\ldots,(u_k,v_k)$, where $u_i,v_i \in V$ and $u_i \neq v_i$ for $i=1,2,\ldots,k$, is given as the third argument. The command returns the LCA of u and v or the list containing LCA of every pair of nodes u_i,v_i for $i=1,2,\ldots,k$. Note that this is much faster than calling `lowest_common_ancestor` k times with a single pair of vertices each time.

The strategy is to use Tarjan's offline LCA algorithm [56], which runs in nearly linear time.

```
> G:=random_tree(25)
```

an undirected unweighted graph with 25 vertices and 24 edges

```
> draw_graph(G)
```




```
> lowest_common_ancestor(G,0,19,22)
```

16

```
> lowest_common_ancestor(G,0,[[5,13],[17,24],[9,16]])
```

$[4,16,0]$

4.6.5. Arborescence graphs

The command `is_arborescence` is used for determining whether a directed unweighted graph is an arborescence (which is the digraph form of a rotted tree).

Syntax: `is_arborescence(G)`

`is_arborescence` takes a digraph $G(V, E)$ as its only argument and returns `true` if there is a vertex $u \in V$ such that for any other $v \in V$ there is exactly one directed path from u to v. Else it returns `false`.

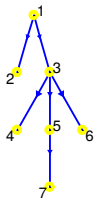
```
> T:=digraph(%{[1,2],[1,3],[3,4],[3,5],[3,6],[5,7]%})
```

a directed unweighted graph with 7 vertices and 6 arcs

```
> is_arborescence(T)
```

true

```
> draw_graph(T)
```



4.7. NETWORKS

4.7.1. Network graphs

The command `is_network` is used for determining whether a graph is a flow network. In this context, a flow network is directed, connected graph with at least one vertex with in-degree 0 (the **source**) and at least one vertex with out-degree 0 (the **sink**).

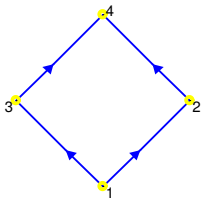
Syntax: `is_network(G)`
`is_network(G,s,t)`

`is_network` takes one or three arguments, a digraph $G(V, E)$ and optionally the source vertex s and the sink vertex t. If these vertices are given, the command returns `true` if G is a network with respect to s, t and `false` otherwise. If the graph G is given as the only argument, the command returns a sequence of two objects, the list of all sources in G and the list of all sinks in G, respectively. If one of these lists is empty, then G is implicitly not a network (both lists are empty if G is not connected).

```
> N:=digraph(%{[1,2],[1,3],[2,4],[3,4]%})
```

a directed unweighted graph with 4 vertices and 4 arcs

```
> draw_graph(N,spring)
```




```
> is_network(N,1,4)
```

true

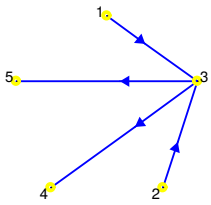
```
> is_network(N,2,3)
```

false

```
> G:=digraph(%{[1,3],[2,3],[3,4],[3,5]%})
```

a directed unweighted graph with 5 vertices and 4 arcs

```
> draw_graph(G,circle)
```




```
> is_network(G)
```

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$$

4.7.2. Maximum flow

The command `maxflow` is used for computing the maximum flow in a network.

Syntax: `maxflow(G,s,t)`
`maxflow(G,s,t,M)`

`maxflow` takes three or four arguments: a network graph $G(V, E)$, the source $s \in V$, the sink $t \in V$ and optionally an unassigned identifier `M`. It returns the optimal value for the maximum flow problem for the network (G, s, t). If the fourth argument is given, an optimal flow is written to `M` in form of a matrix.

The strategy is to use the algorithm of EDMONDS and KARP [20], which solves the maximum flow problem in $O(|V|\,|E|^2)$ time.

```
> A:=[[0,1,0,4,0,0],[0,0,1,0,3,0],[0,1,0$3,1],[0,0,3,0,1,0],[0$3,1,0,4],[0$6]]
```

$$\begin{pmatrix} 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

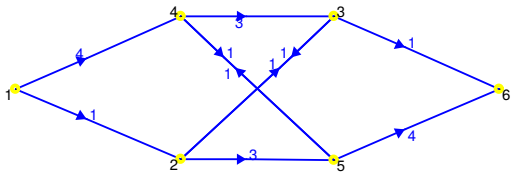
```
> N:=digraph([1,2,3,4,5,6],A)
```

a directed weighted graph with 6 vertices and 10 arcs

```
> is_network(N)
```

$$\begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

```
> draw_graph(N,spring)
```




```
> maxflow(N,1,6,M)
```

4

```
> M
```

$$\begin{pmatrix} 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

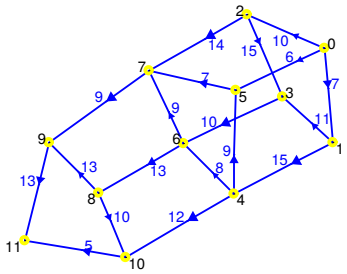
```
> N:=random_network(2,3,0.9,acyclic,weights=5..15)
```

a directed weighted graph with 12 vertices and 19 arcs

```
> is_network(N)
```

$$\begin{pmatrix} 0 \\ 11 \end{pmatrix}$$

```
> draw_graph(N,spring)
```

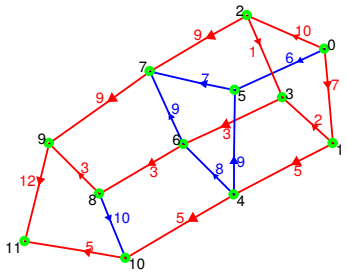


```
> maxflow(N,0,11,F)
```

17

To visualize the optimal flow F, one can use the `highlight_subgraph` command with the option `weights` to display the actual flow in the highlighted edges. Non-highlighted edges have zero flow.

```
> draw_graph(highlight_subgraph(N,digraph(vertices(N),F),weights),spring)
```



4.7.3. Minimum cut

The command `minimum_cut` is used for obtaining minimum cuts in networks.

Syntax: `minimum_cut(G,s,t)`

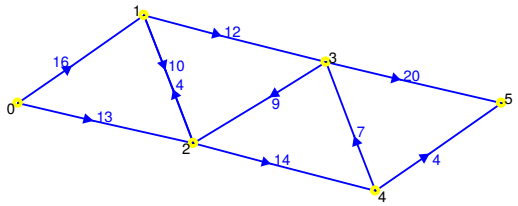
`minimum_cut` takes three arguments, a digraph $G(V,E)$ and two vertices $s,t \in V$ such that (G,s,t) is a network with source s and sink t. The returned value is a list of edges in E representing a minimum cut in the network.

The strategy is to apply the command `maxflow`, which finds a maximal flow, and to run depth-first search on the corresponding residual graph to find a S,T partition of V. The minimum cut is then the set of all arcs $vw \in E$ such that $v \in S$ and $w \in T$. The algorithm runs in $O(|V|\,|E|^2)$ time.

```
> G:=digraph(%{[[0,1],16],[[0,2],13],[[1,2],10],[[1,3],12],[[2,1],4],[[2,4],14],
  [[3,2],9],[[3,5],20],[[4,3],7],[[4,5],4]%})
```

a directed weighted graph with 6 vertices and 10 arcs

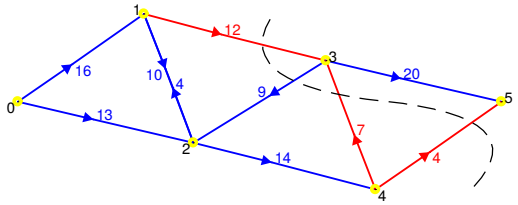
```
> draw_graph(G,spring)
```




```
> cut:=minimum_cut(G,0,5)
```

$$\begin{pmatrix} 1 & 3 \\ 4 & 3 \\ 4 & 5 \end{pmatrix}$$

```
> draw_graph(highlight_edges(G,cut),spring)
```



By the max-flow min-cut theorem, the sum of edge weights in minimum cut is equal to the value of maximum flow.

```
> w:=0:; for ed in cut do w:=w+get_edge_weight(G,ed); od:; w
```

Done, Done, 23

```
> maxflow(G,0,5)
```

23

4.8. Distance in graphs

4.8.1. Vertex distance

The command `vertex_distance` is used for computing the length of the shortest path(s) from the source vertex to some other vertex/vertices of a graph.

Syntax: `vertex_distance(G,v,w)`
`vertex_distance(G,v,L)`

`vertex_distance` takes three arguments, a graph $G(V, E)$, a vertex $v \in V$ called the **source** and a vertex $w \in V$ called the **target** or a list $L \subset V \setminus \{v\}$ of target vertices. The command returns the distance between v and w as the number of edges in a shortest path from v to w, or the list of distances if a list of target vertices is given.

The strategy is to use breadth-first search [26, pp. 35] starting from the source vertex. Therefore, the algorithm runs in $O(|V|+|E|)$ time.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> vertex_distance(G,1,3)
```

2

```
> vertex_distance(G,1,[3,6,9])
```

$$[2, 1, 2]$$

4.8.2. All-pairs vertex distance

The command `allpairs_distance` is used for computing the matrix of distances between all pairs of vertices in a (weighted) graph.

Syntax: `allpairs_distance(G)`

`allpairs_distance` takes a graph $G(V, E)$ as its only argument and returns a square matrix $D = [d_{ij}]$ with $n = |V|$ rows and columns such that $d_{ij} = \text{distance}(v_i, v_j)$ for all $i, j = 1, 2, ..., n$, where $v_1, v_2, ..., v_n$ are the elements of V. If $v_i v_j \notin E$, then $d_{ij} = +\infty$. The strategy is to apply the algorithm of FLOYD and WARSHALL [23], which runs in $O(|V|^3)$ time.

Note that, if G is weighted, it must not contain negative cycles. A cycle is **negative** if the sum of weights of its edges is negative.

```
> G:=graph([1,2,3,4,5],%{[1,2],[1,3],[1,4],[1,5],[2,3],[3,4],[4,5],[5,2]%})
```

an undirected unweighted graph with 5 vertices and 8 edges

```
> allpairs_distance(G)
```

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix}$$

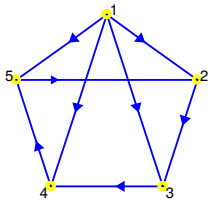
```
> H:=digraph(%{[1,2],[1,3],[1,4],[1,5],[2,3],[3,4],[4,5],[5,2]%})
```

a directed unweighted graph with 5 vertices and 8 arcs

```
> allpairs_distance(H)
```

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ +\infty & 0 & 1 & 2 & 3 \\ +\infty & 3 & 0 & 1 & 2 \\ +\infty & 2 & 3 & 0 & 1 \\ +\infty & 1 & 2 & 3 & 0 \end{pmatrix}$$

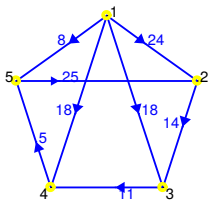
```
> draw_graph(H)
```




```
> H:=assign_edge_weights(H,5,25)
```

a directed weighted graph with 5 vertices and 8 arcs

```
> draw_graph(H)
```



```
> allpairs_distance(H)
```

$$\begin{pmatrix} 0 & 24 & 18 & 18 & 8 \\ +\infty & 0 & 14 & 25 & 30 \\ +\infty & 41 & 0 & 11 & 16 \\ +\infty & 30 & 44 & 0 & 5 \\ +\infty & 25 & 39 & 50 & 0 \end{pmatrix}$$

4.8.3. Diameter

The command `graph_diameter` is used for determining the maximum distance among all pairs of vertices in a graph.

Syntax: `graph_diameter(G)`

`graph_diameter` takes a graph $G(V, E)$ as its only argument and returns the number $\max\{\text{distance}(u, v) : u, v \in V\}$. If G is disconnected, $+\infty$ is returned.

This command calls `allpairs_distance` and picks the largest element in the output matrix. Hence the complexity of the algorithm is $O(|V|^3)$.

```
> graph_diameter(graph("petersen"))
```

2

```
> graph_diameter(cycle_graph(19))
```

9

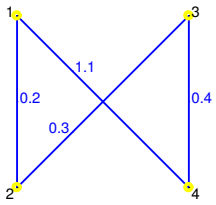
```
> graph_diameter(disjoint_union(graph("petersen"),cycle_graph(19)))
```

$+\infty$

```
> G:=graph(%{[[1,2],0.2],[[2,3],0.3],[[3,4],0.4],[[4,1],1.1]%})
```

an undirected weighted graph with 4 vertices and 4 edges

```
> draw_graph(G)
```




```
> graph_diameter(G)
```

0.9

```
> dijkstra(G,1,4)
```

$[[1, 2, 3, 4], 0.9]$

4.8.4. Girth

The commands `girth` and `odd_girth` are used for computing the (odd) girth of an undirected unweighted graph.

Syntax: `girth(G)`

`girth` resp. `odd_girth` takes a graph $G(V, E)$ as its only argument and returns the girth resp. odd girth of G. The (odd) girth of G is defined to be the length of the shortest (odd) cycle in G. If there is no (odd) cycle in G, the command returns $+\infty$.

The strategy is to apply breadth-first search from each vertex of the input graph. The runtime is therefore $O(|V||E|)$.

```
> girth(graph("petersen"))
```

5

```
> G:=hypercube_graph(3)
```

an undirected unweighted graph with 8 vertices and 12 edges

```
> G:=subdivide_edges(G,["000","001"])
```

an undirected unweighted graph with 9 vertices and 13 edges

```
> girth(G)
```

4

```
> odd_girth(G)
```

5

```
> girth(complete_binary_tree(2))
```

$+\infty$

4.9. ACYCLIC GRAPHS

4.9.1. Acyclic graphs

The command `is_acyclic` is used for checking for absence of directed cycles in digraphs. A directed graph with no directed cycle is said to be **acyclic**.

Syntax: `is_acyclic(G)`

`is_acyclic` takes a digraph $G(V, E)$ as its only argument and returns `true` if G is acyclic and `false` otherwise.

The algorithm attempts to find topological order for its vertices. If that succeeds, the graph is acyclic, otherwise not. The algorithm runs in $O(|V| + |E|)$ time.

```
> is_acyclic(digraph(trail(1,2,3,4,5)))
```

true

```
> is_acyclic(digraph(trail(1,2,3,4,5,2)))
```

false

4.9.2. Topological sorting

The command `topologic_sort` or `topological_sort` is used for finding a linear ordering of vertices of an acyclic digraph which is consistent with the arcs of the digraph. This procedure is called topological sorting.

Syntax: `topologic_sort(G)`
`topological_sort(G)` (alias)

`topologic_sort` takes a graph $G(V, E)$ as its only argument and returns the list of vertices of G in a particular order: a vertex u precedes a vertex v if $u\,v \in E$, i.e. if there is an arc from u to v.

Note that topological sorting is possible only if the input graph is acyclic. If this condition is not met, `topologic_sort` returns an error. Otherwise, it finds the required ordering by applying KAHN's algorithm [38], which runs in $O(|V| + |E|)$ time.

```
> G:=digraph(%{[c,a],[c,b],[c,d],[a,d],[b,d],[a,b]%})
```

a directed unweighted graph with 4 vertices and 6 arcs

```
> is_acyclic(G)
```

true

```
> topologic_sort(G)
```

$[c, a, b, d]$

4.9.3. st ordering

The command `st_ordering` is used for finding st-orderings in undirected biconnected graphs.

Syntax: `st_ordering(G,s,t,[p])`
`st_ordering(G,s,t,D,[p])`

`st_ordering` takes three to five arguments. The first three arguments are mandatory: an undirected biconnected graph $G(V, E)$, a vertex $s \in V$ called the source, a vertex $t \in V$ called the sink such that $st \in E$. Optionally, one can pass an unassigned identifier `D` and/or a real value $p \in [0, 1]$. The command returns the permutation σ of the vertex set V which corresponds to st-numbering of the vertices. Now, an orientation of each $e = uv \in E$ can be determined by the ordinals n and m of its endpoints u and v, respectively, which are assigned by the permutation σ: if $n < m$, then u is the head and v is the tail of the corresponding arc, and vice versa otherwise. If an identifier `D` is given, a copy of G, which is made directed according to these orientations, is stored to `D`. The oriented variant of G is an acyclic graph (or DAG for short).

The requirement that the input graph is biconnected implies that st-ordering can be computed for any pair $s, t \in V$ such that $st \in E$.

If p is not specified, the strategy is to apply Tarjan's algorithm [57] which runs in $O(|V| + |E|)$ time. When $p \in [0, 1]$ is given, a parametrized st-ordering is computed, in which the length of the longest path from s to t in the respective DAG roughly corresponds to $p\,|V|$. Thus by varying p one controls the length of the longest directed path from s to t. The parametrized branch of the algorithm is implemented according to Papamanthou [48] and runs in $O(|V|\,|E|)$ time.

```
> G:=graph(%{[a,b],[a,c],[a,d],[b,c],[b,d],[c,d]%})
```

an undirected unweighted graph with 4 vertices and 6 edges

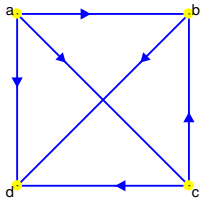
```
> vertices(G)
```

$[a, b, c, d]$

```
> st_ordering(G,a,d,D)
```

$[0, 2, 1, 3]$

```
> draw_graph(D)
```



The following program demonstrates using the parametrized st-ordering algorithm in order to find a path between vertices u and v in an undirected, biconnected graph $G(V, E)$. The path length is controllable by varying the parameter $p \in [0, 1]$.

```
FindPath:=proc(G,u,v,p)
  local tmp,D,W;
  tmp:=!has_edge(G,[u,v]);
  if tmp then G:=add_edge(G,[u,v]); fi;
  purge(D);
  st_ordering(G,u,v,D,p);
  if tmp then D:=delete_arc(D,[u,v]); fi;
  W:=is_weighted(G)?weight_matrix(G):adjacency_matrix(G);
  D:=make_weighted(D,-W);
  return bellman_ford(D,u,v)[0];
end:;
```

The procedure `FindPath` uses the Bellman-Ford algorithm to find a longest path from the vertex $u \in V$ to the vertex $v \in V$ in the DAG D induced by a parametrized st-ordering of G with parameter p. To trick Bellman-Ford into finding a longest path instead of the shortest one (which it was designed for), the edges of D are weighted with negative weights. Since D is acyclic, it contains no negative cycles, so the Bellman-Ford algorithm terminates successfully.

For $p=0$ one obtains a relatively short path, but usually not a minimal one. For $p=1$ one obtains near-Hamiltonian paths. For $p \in (0,1)$ one obtains a path of length l which obeys the relation

$$l \approx l_0 + p\,(|V| - l_0),$$

where l_0 is the average path length for $p=0$.

```
> G:=graph("soccerball")
```

an undirected unweighted graph with 60 vertices and 90 edges

```
> P1:=FindPath(G,3,33,0):; length(P1)
```

Done, 12

```
> P2:=FindPath(G,3,33,0.5):; length(P2)
```

Done, 39

```
> P3:=FindPath(G,3,33,1):; length(P3)
```

Done, 59

4.10. MATCHING IN GRAPHS

4.10.1. Maximum matching

The command `maximum_matching` is used for finding maximum matchings [27, pp. 43] in undirected unweighted graphs.

Syntax: `maximum_matching(G)`

`maximum_matching` takes an undirected graph $G(V,E)$ as its only argument and returns a list of edges $e_1, e_2, ..., e_m \in E$ such that e_i and e_j are not adjacent (i.e. have no common endpoints) for all $1 \leqslant i < j \leqslant m$ and that m is maximal. The return value can be interpreted as the list of matched pairs of vertices in G.

The strategy is to apply the blossom algorithm of EDMONDS [19], which runs in $O(|V|^2\,|E|)$ time.

```
> maximum_matching(graph("octahedron"))
```

$$\begin{pmatrix} 1 & 6 \\ 3 & 2 \\ 5 & 4 \end{pmatrix}$$

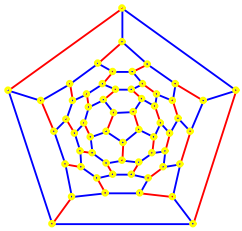
```
> G:=graph("soccerball")
```

an undirected unweighted graph with 60 vertices and 90 edges

```
> M:=maximum_matching(G):; length(M)
```

Done, 30

```
> draw_graph(highlight_edges(G,M),labels=false)
```



```
> G:=random_graph(1000,10,5)
```

an undirected unweighted graph with 1000 vertices and 13993 edges

```
> length(maximum_matching(G))
```

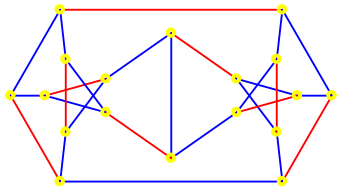
500

181 sec

```
> G:=graph("blanusa")
```

an undirected unweighted graph with 18 vertices and 27 edges

```
> draw_graph(highlight_edges(G,maximum_matching(G)),labels=false)
```



4.10.2. Maximum matching in bipartite graphs

The command `bipartite_matching` is used for finding maximum matchings in undirected, unweighted bipartite graphs. It applies the algorithm of Hopcroft and Karp [35], which is more efficient than the general algorithm used by the command `maximum_matching`.

Syntax: `bipartite_matching(G)`

`bipartite_matching` takes an undirected bipartite graph $G(V, E)$ as its only argument and returns a sequence containing two elements: the size of the matching and the list of edges connecting matched pairs of vertices. The algorithm runs in $O\big(\sqrt{|V|}\,|E|\big)$ time.

```
> G:=graph("desargues")
```

an undirected unweighted graph with 20 vertices and 30 edges

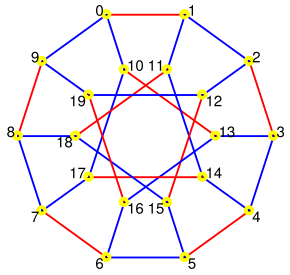
```
> is_bipartite(G)
```

true

```
> M:=bipartite_matching(G)
```

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 8 & 9 \\ 10 & 13 \\ 11 & 18 \\ 12 & 15 \\ 14 & 17 \\ 16 & 19 \end{pmatrix}$$

```
> draw_graph(highlight_edges(G,M))
```



4.11. CLIQUES

4.11.1. Clique graphs

To check whether an undirected graph is complete, one can use the `is_clique` command.

Syntax: `is_clique(G)`

`is_clique` takes an undirected graph $G(V, E)$ as its only argument and returns `true` if every pair of distinct vertices is connected by a unique edge in E. Else, it returns `false`.

```
> K5:=complete_graph(5)
```

an undirected unweighted graph with 5 vertices and 10 edges

```
> is_clique(K5)
```

true

```
> G:=delete_edge(K5,[1,2])
```

an undirected unweighted graph with 5 vertices and 9 edges

```
> is_clique(G)
```

false

4.11.2. Maximal cliques

Given an undirected graph $G(V, E)$, a subset $S \subset V$ is called a **clique** in G if any two distinct vertices from S are adjacent in G, i.e. if the subgraph of G induced by the set S is complete. A clique is **maximal** if it cannot be extended by adding more vertices from V to it. To count all maximal cliques in a graph (and optionally list them) one can use the `clique_stats` command.

Syntax: `clique_stats(G,[C])`
`clique_stats(G,k,[C])`
`clique_stats(G,m..n,[C])`

`clique_stats` takes an undirected graph $G(V, E)$ as the mandatory first argument. If no other arguments are given, the command returns a list of pairs, each pair consisting of two integers: clique cardinality k and the number $n_k > 0$ of k-cliques in G, respectively. (Therefore, the sum of second members of all returned pairs is equal to the total count of all maximal cliques in G.) If two arguments are passed to `clique_stats`, the second argument must be a positive integer k or an interval with integer bounds m .. n. In the first case the number of k-cliques is returned; in the second case, only cliques with cardinality between m and n (inclusive) are counted.

If `C` is specified as the last argument, it must be an unassigned identifier. Maximal cliques are in that case stored to `C` as a list of lists of cliques of equal size. This option is therefore used for listing all maximal cliques.

The strategy used to find all maximal cliques is a variant of the algorithm of Bron and Kerbosch developed by Tomita et al. [59]. Its worst-case running time is $O(3^{|V|/3})$. However, the algorithm is usually very fast, typically taking only a moment for graphs with few hundred vertices or less.

```
> G:=random_graph(50,0.5)
```

an undirected unweighted graph with 50 vertices and 633 edges

```
> clique_stats(G)
```

$$\begin{pmatrix} 3 & 2 \\ 4 & 123 \\ 5 & 465 \\ 6 & 388 \\ 7 & 73 \\ 8 & 6 \end{pmatrix}$$

```
> G:=random_graph(100,0.5)
```

an undirected unweighted graph with 100 vertices and 2448 edges

```
> clique_stats(G,5)
```

4080

```
> G:=random_graph(500,0.25)
```

an undirected unweighted graph with 500 vertices and 31400 edges

```
> clique_stats(G,5..7)
```

$$\begin{pmatrix} 5 & 158436 \\ 6 & 19507 \\ 7 & 383 \end{pmatrix}$$

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> clique_stats(G,C)
```

(3 8)

```
> C
```

$$\begin{pmatrix} 1 & 3 & 6 \\ 1 & 3 & 5 \\ 1 & 6 & 4 \\ 1 & 5 & 4 \\ 3 & 6 & 2 \\ 3 & 5 & 2 \\ 6 & 4 & 2 \\ 5 & 4 & 2 \end{pmatrix}$$

4.11.3. Maximum clique

Any largest maximal clique in an undirected graph is called **maximum clique**. The command `maximum_clique` can be used to find one in a graph. If only the size of a maximum clique is desired, one can use the command `clique_number`.

Syntax: `maximum_clique(G)`
`clique_number(G)`

`maximum_clique` takes an undirected graph G as its only argument and returns a maximum clique in G as a list of vertices. The clique may subsequently be extracted from G using the command `induced_subgraph`.

The strategy used to find maximum clique is an improved variant of the classical algorithm by CARRAGHAN and PARDALOS developed by ÖSTERGÅRD [45].

In the following examples, the results were obtained almost instantly.

```
> G:=sierpinski_graph(5,5)
```

an undirected unweighted graph with 3125 vertices and 7810 edges

```
> maximum_clique(G)
```

$[1560, 1561, 1562, 1563, 1564]$

```
> G:=random_graph(300,0.3)
```

an undirected unweighted graph with 300 vertices and 13352 edges

```
> maximum_clique(G)
```

$[60, 80, 111, 201, 248, 252, 288]$

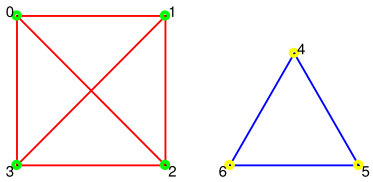
```
> G:=graph_complement(complete_graph(4,3))
```

an undirected unweighted graph with 7 vertices and 9 edges

```
> cliq:=maximum_clique(G)
```

$[0, 1, 2, 3]$

```
> draw_graph(highlight_subgraph(G,induced_subgraph(G,cliq)))
```



`clique_number` takes an undirected graph G as its only argument and returns the number of vertices forming a maximum clique in G.

```
> clique_number(G)
```

4

4.11.4. Minimum clique cover

A **minimum clique cover** for an undirected graph G is any minimal set $S = \{C_1, C_2, ..., C_k\}$ of cliques in G such that for every vertex v in G there exists $i \leqslant k$ such that $v \in C_i$. Such a cover can be obtained by calling the `clique_cover` command.

Syntax: `clique_cover(G)`
`clique_cover(G,k)`

`clique_cover` takes an undirected graph $G(V, E)$ as its mandatory argument and returns the smallest possible cover. Optionally, a positive integer k may be passed as the second argument. In that case the requirement that k is less or equal to the given integer is set. If no such cover is found, `clique_cover` returns empty list.

The strategy is to find a minimal vertex coloring in the complement G^c of G (note that these two graphs share the same set of vertices). Each set of equally colored vertices in G^c corresponds to a clique in G. Therefore, the color classes of G^c map to the elements $C_1, ..., C_k$ of a minimal clique cover in G.

There is a special case in which G is triangle-free (i.e. contains no 3-cliques), which is computed separately in the algorithm. In that case, G contains only 1- and 2-cliques. Therefore, every clique cover in G consists of a set $M \subset E$ of matched edges together with the singleton cliques (i.e. the isolated vertices in V which remain unmatched). The total number of cliques in the cover is equal to $|V| - |M|$, hence to find a minimal cover one just needs to find a maximum matching in G, which can be done in polynomial time.

```
> G:=random_graph(30,0.2)
```

an undirected unweighted graph with 30 vertices and 83 edges

```
> clique_cover(G)
```

{[0, 22, 24], [1, 14, 17, 20], [2, 25], [3, 10, 16], [4, 28], [5, 19, 29], [6, 27], [7, 8, 11], [9, 12], [13, 23], [15, 26], [18, 21]}

```
> clique_cover(graph("octahedron"))
```

$$\begin{pmatrix} 1 & 3 & 6 \\ 2 & 4 & 5 \end{pmatrix}$$

The vertices of Petersen graph can be covered with five, but not with three cliques.

```
> clique_cover(graph("petersen"),3)
```

[]

```
> clique_cover(graph("petersen"),5)
```

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 9 \\ 5 & 7 \\ 6 & 8 \end{pmatrix}$$

4.11.5. Clique cover number

The command `clique_cover_number` is used for computing the clique cover number of a graph.

Syntax: `clique_cover_number(G)`

`clique_cover_number` takes an undirected graph $G(V, E)$ as its only argument and returns the minimum number of cliques in G needed to cover the vertex set V. (More precisely, it calls the `clique_cover` command and returns the length of the output list.) This number, denoted by $\theta(G)$, is equal to the chromatic number $\chi(G^c)$ of the complement graph G^c of G.

```
> clique_cover_number(graph("petersen"))
```

5

```
> clique_cover_number(graph("soccerball"))
```

30

```
> clique_cover_number(random_graph(40,0.618))
```

7

4.12. TRIANGLES IN GRAPHS

4.12.1. Counting triangles

The command `number_of_triangles` is used for counting triangles in graphs.

Syntax: `number_of_triangles(G)`
`number_of_triangles(G,L)`

`number_of_triangles` takes a graph G as its first, mandatory argument and returns the number n of 3-cliques in G if G is undirected resp. the number m of directed cycles of length 3 if G is directed. If an unassigned identifier L is given as the second argument, the triangles are also listed and stored to L. Note that triangle listing is supported only for undirected graphs.

For undirected graphs the algorithm of SCHANK and WAGNER [51, Algorithm *forward*], improved by LATAPY [39], is used, which runs in $O(|E|^{3/2})$ time. For digraphs, the strategy is to compute the trace of A^3 where A is the adjacency matrix of G encoded in a sparse form. This algorithm requires $O(|V|\,|E|)$ time.

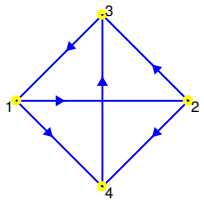
```
> number_of_triangles(graph("tetrahedron"))
```

4

```
> G:=digraph([1,2,3,4],%{[1,2],[1,4],[2,3],[2,4],[3,1],[4,3]%})
```

a directed unweighted graph with 4 vertices and 6 arcs

```
> draw_graph(G,spring)
```




```
> number_of_triangles(G)
```

2

```
> G:=sierpinski_graph(7,3,triangle)
```

an undirected unweighted graph with 1095 vertices and 2187 edges

```
> number_of_triangles(G)
```

972

Petersen graph is triangle-free, i.e. contains no 3-cliques.

```
> number_of_triangles(graph("petersen"))
```

0

Counting triangles in undirected graphs is very fast, as illustrated by the following example.

```
> G:=random_graph(10^5,10^6)
```

an undirected unweighted graph with 100000 vertices and 1000000 edges

147 sec

```
> number_of_triangles(G)
```

25315

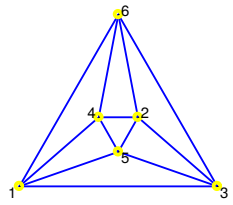
1.62 sec

To list all triangles in a graph, pass an unassigned identifier as the second argument. The triangles will be stored to it as a list of triples of vertices.

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(G)
```




```
> number_of_triangles(G,L)
```

8

```
> L
```

$$\begin{pmatrix} 2 & 4 & 5 \\ 2 & 4 & 6 \\ 2 & 3 & 5 \\ 1 & 4 & 5 \\ 2 & 3 & 6 \\ 1 & 4 & 6 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix}$$

4.12.2. Clustering coefficient

The command `clustering_coefficient` is used for computing the average clustering coefficient (or simply: clustering coefficient) of an undirected graph as well as the local clustering coefficient of a particular vertex in that graph.

Syntax:
```
clustering_coefficient(G,[opt])
clustering_coefficient(G,v)
clustering_coefficient(G,v1,v2,..,vk)
clustering_coefficient(G,[v1,v2,..,vk])
```

`clustering_coefficient` takes one or two arguments, an undirected graph $G(V,E)$ and optionally a vertex $v \in V$ or a list/sequence of vertices $v_1, v_2, ..., v_k \in V$. If G is the only argument, the clustering coefficient $c(G)$ [10, pp. 5] is returned. Otherwise, the local clustering coefficient $c_G(v)$ [10, pp. 4] of v resp. a list of local clustering coefficients of $v_1, v_2, ..., v_k$ is returned. The second argument may also be one of the following options:

`exact` — The clustering coefficient $c(G)$ is returned as a rational number (by default it is a floating point number). Note that local clustering coefficient is always returned in exact form.

`approx` — An approximation of the clustering coefficient $c(G)$, lying within 0.5×10^{-2} of the exact value with probability $p = 1 - 10^{-5}$, is returned.

In any case, the return value is—by definition—a rational number in the range $[0, 1]$.

The clustering coefficient of G is defined as the mean of $c_G(v)$, $v \in V$:

$$c(G) = \frac{1}{|V|} \sum_{v \in V} c_G(v).$$

$c(G)$ can be interpreted as the probability that, for a randomly selected pair of incident edges uv and vw in G, the vertices u and w are connected. The number $c_G(v)$ is interpreted analogously but for a fixed $v \in V$. It represents the probability that two neighbors of v are connected to each other.

For example, assume that G represents a social network in which $uv \in E$ means that u and v are friends (which is a symmetric relation). In this context, $c(v)$ represents the probability that two friends of v are also friends of each other.

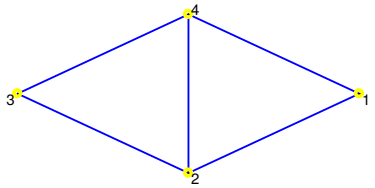
The time complexity of computing $c(G)$ is $O(|E|^{3/2})$, whereas the algorithm of SCHANK and WAGNER [52, Algorithm 1, pp. 269] for approximating $c(G)$ runs in $O(\log |V|)$ time.

In addition, note that the command `random_graph` is able to generate—using a preferential attachment rule—realistic random networks with adjustable clustering coefficient, which are suitable for testing purposes.

```
> G:=graph(%{[1,2],[2,3],[2,4],[3,4],[4,1]%})
```

an undirected unweighted graph with 4 vertices and 5 edges

```
> draw_graph(G,spring)
```



The command lines below compute $c(G)$, $c_G(1)$ and $c_G(2)$.

```
> clustering_coefficient(G,exact)
```

$$\frac{5}{6}$$

```
> clustering_coefficient(G,1)
```

$$1$$

```
> clustering_coefficient(G,2)
```

$$\frac{2}{3}$$

The next example demonstrates the performance of `clustering_coefficient` on a large graph.

```
> G:=random_graph(25000,10,100)
```

an undirected unweighted graph with 25000 vertices and 991473 edges

```
> clustering_coefficient(G)
```

0.635654820498

2.48 sec

```
> clustering_coefficient(G,approx)
```

0.635182159201

0.77 sec

The probability that two neighbors of a vertex in G are connected is therefore about 64%.

4.12.3. Network transitivity

The command `network_transitivity` is used for computing the transitivity (also called **triangle density** or the **global clustering coefficient**) of a network.

Syntax: `network_transitivity(G)`

`network_transitivity` takes a graph G as its only argument and returns the transitivity $T(G)$ of G [10, pp. 5]. By definition, it is a rational number in the range $[0,1]$:

$$T(G) = \frac{3\,N_{\text{triangles}}}{N_{\text{triplets}}}.$$

$T(G)$ is a measure of transitivity of a non-symmetric relation between the vertices of a network. If G is a digraph, a **triplet** in G is any directed path (v,w,z) where $v,w,z \in V$. For example, in a Twitter-like social network this could mean that v following w and w following z. The triplet (v,w,z) is **closed** if $vz \in E$, i.e. if v also follows z [62, pp. 243]. A closed triplet is called a **triangle**. If G is undirected, $N_{\text{triangles}}$ is the number of 3-cliques and N_{triplets} is the number of two-edge paths in V.

The complexity of computing $T(G)$ is $O(\Delta_G\,|E|)$ for digraphs, where Δ_G is the maximum vertex degree in G, resp. $O(|E|^{3/2})$ for undirected graphs.

```
> G:=graph(%{[1,2],[2,3],[2,4],[3,4],[4,1]%})
```

an undirected unweighted graph with 4 vertices and 5 edges

```
> network_transitivity(G)
```

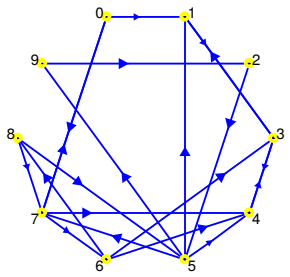
$$\frac{3}{4}$$

Observe that the above result is different than $c(G)$ obtained in Section 4.12.2. Hence $c(G) \neq T(G)$ in general [10, pp. 5].

```
> G:=random_digraph(10,20)
```

a directed unweighted graph with 10 vertices and 20 arcs

```
> draw_graph(G)
```



In the above digraph, the triplet $(5,7,6)$ is open while the triplet $(7,6,4)$ is closed. Triangles $(2,5,9)$ and $(6,8,7)$ are not closed by definition.

```
> network_transitivity(G)
```

$$\frac{5}{33}$$

The transitivity algorithms are suitable for large networks, as demonstrated in the examples below.

```
> G:=random_digraph(1000,500000)
```

a directed unweighted graph with 1000 vertices and 500000 arcs

```
> nt:=network_transitivity(G):;
```

2.91 sec

```
> evalf(nt)
```

0.500523736169

```
> H:=random_graph(30000,10,50)
```

an undirected unweighted graph with 30000 vertices and 1011266 edges

```
> evalf(network_transitivity(H))
```

0.137017372323

2.52 sec

4.13. VERTEX COLORING

To color vertices of a graph $G(V, E)$ means to assign to each vertex $v \in V$ a positive integer. Each integer represents a distinct color. The key property of graph coloring is that the colors of a pair of adjacent vertices must be mutually different. Two different colorings of G may use different number of colors.

4.13.1. Greedy vertex coloring

The command `greedy_color` is used for coloring vertices of a graph in a greedy fashion.

Syntax: `greedy_color(G)`
`greedy_color(G,p)`

`greedy_color` takes one mandatory argument, a graph $G(V, E)$. Optionally, a permutation p of order $|V|$ may be passed as the second argument. Vertices are colored one by one in the order specified by p (or in the default order if p is not given) such that each vertex gets the smallest available color. The list of vertex colors is returned in the order of `vertices(G)`.

Generally, different choices of permutation p produce different colorings. The total number of different colors may not be the same each time. The complexity of the algorithm is $O(|V| + |E|)$.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> greedy_color(G)
```

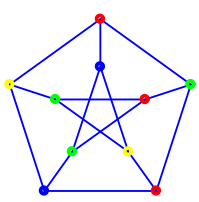
$[1, 2, 1, 2, 3, 2, 1, 3, 3, 2]$

```
> L:=greedy_color(G,randperm(10))
```

$[1, 2, 1, 4, 3, 4, 1, 3, 2, 2]$

Observe that a different number of colors is obtained by executing the last command line. To display the colored graph, input:

```
> draw_graph(highlight_vertex(G,vertices(G),L),labels=false)
```



The first six positive integers are always mapped to the standard Xcas colors, as indicated in Table 4.1. Note that the color 0 (black) and color 7 (white) are swapped; a vertex with color 0 is white (uncolored) and vertex with color 7 is black. Also note that Xcas maps numbers greater than 7 to colors too, but the number of available colors is limited.

4.13.2. Minimal vertex coloring

A vertex coloring of G is **minimal** (or **optimal**) if the total number of used colors is minimal. To obtain such a coloring use the command `minimal_vertex_coloring`.

Syntax: `minimal_coloring(G)`
`minimal_coloring(G,sto)`

`minimal_vertex_coloring` takes one mandatory argument, a graph $G(V, E)$ where $V = \{v_1, v_2, ..., v_n\}$. Optionally, a symbol `sto` may be passed as the second argument. The command returns the vertex colors $c_1, c_2, ..., c_n$ in order of `vertices(G)` or, if the second argument is given, stores the colors as vertex attributes and returns the modified copy of G.

Giac requires the GLPK library to solve the minimal vertex coloring problem (MVCP), which is converted to the equivalent integer linear programming problem and solved by using the branch-and-bound method with specific branch/backtrack techniques [16]. The lower resp. the upper bound for the number n of colors is obtained by finding a maximal clique (n cannot be lower than its cardinality) resp. by applying the heuristic proposed by Brélaz in [11] (which will use at least n colors). Note that the algorithm performs some randomization when applying heuristics, so coloring a graph several times will not take the same amount of computation time in each instance, generally.

In the following example, the Grötzsch graph is colored with the minimal number of colors by first finding the coloring and then assigning it to the graph by using the `highlight_vertex` command.

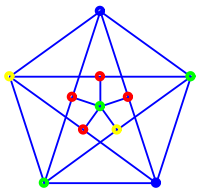
```
> G:=graph("grotzsch")
```

an undirected unweighted graph with 11 vertices and 20 edges

```
> coloring:=minimal_vertex_coloring(G)
```

$[4, 2, 3, 1, 1, 4, 1, 3, 2, 1, 2]$

```
> draw_graph(highlight_vertex(G,vertices(G),coloring),labels=false)
```



Solving MVCP for different graphs of exactly the same size (but which do not share the same edge structure) may take quite different time in each instance. Also note that, since the vertex coloring problem is NP hard, the algorithm may take exponential time on some graphs.

4.13.3. Chromatic number

The command `chromatic_number` is used for exact computation or approximation of the chromatic number of a graph.

value	1	2	3	4	5	6	7
color	red	green	yellow	blue	magenta	cyan	black

Table 4.1. interpretation of abstract vertex/edge colors in Xcas

Syntax: `chromatic_number(G)`
`chromatic_number(G,c)`
`chromatic_number(G,approx or interval)`

`chromatic_number` takes one mandatory argument, a graph $G(V,E)$, and optionally a second argument. To obtain only upper and lower bound for the chromatic number (which is much faster than computing exactly) the option `approx` or `interval` should be passed as the second argument. Alternatively, an unassigned identifier `c` is passed as the second argument; in that case the corresponding coloring will be stored to it in form of a list of colors of the individual vertices, ordered as in `vertices(G)`.

The command returns the chromatic number χ_G of the graph G in the case of exact computation. If the option `approx` or `interval` is given, an interval `lb..ub` is returned, where `lb` is the best lower bound and `ub` the best upper bound for χ_G found by the algorithm.

The strategy is call `minimal_vertex_coloring` in the case of exact computation. When approximating the chromatic number, the algorithm will establish the lower bound by finding a maximum clique. The timeout for this operation is set to 5 seconds as it can be time consuming. If no maximum clique is not found after that time, the largest clique found is used. Then, an upper bound is established by by using the heuristic proposed by BRÉLAZ in [11]. Obtaining the bounds for χ_G is usually very fast; however, their difference grows with $|V|$.

Unless the input graph is sparse enough, the algorithm slows down considerably for, say, $|V|>40$.

```
> chromatic_number(graph("grotzsch"),cols)
```

4

```
> cols
```

$[4,2,3,1,1,4,1,3,2,1,2]$

```
> G:=random_graph(30,0.75)
```

an undirected unweighted graph with 30 vertices and 313 edges

```
> chromatic_number(G)
```

10

```
> G:=random_graph(300,0.05)
```

an undirected unweighted graph with 300 vertices and 2196 edges

```
> chromatic_number(G,approx)
```

4..7

4.13.4. Mycielski graphs

The command `mycielski` is used for constructing Mycielski graphs.

Syntax: `mycielski(G)`

`mycielski` takes an undirected graph $G(V,E)$ as its only argument and returns the corresponding Mycielski graph M (also called the **Mycielskian** of G) with $2|V|+1$ vertices and $3|E|+|V|$ edges. If G is triangle-free then M is also triangle-free and $\chi_M=\chi_G+1$.

```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> M:=mycielski(P)
```

an undirected unweighted graph with 21 vertices and 55 edges

```
> apply(number_of_triangles,[P,M])
```

$[0, 0]$

```
> chromatic_number(P)
```

3

```
> chromatic_number(M)
```

4

`mycielski` can be applied iteratively, producing arbitrarily large graphs from the most simple ones. For example, Grötzsch graph is obtained as the Mycielskian of a cycle graph on 5 vertices, which is the Mycielskian of a path graph on two vertices.

```
> G1:=path_graph(2)
```

an undirected unweighted graph with 2 vertices and 1 edge

```
> G2:=mycielski(G1)
```

an undirected unweighted graph with 5 vertices and 5 edges

```
> is_isomorphic(G2,cycle_graph(5))
```

true

```
> G3:=mycielski(G2)
```

an undirected unweighted graph with 11 vertices and 20 edges

```
> is_isomorphic(G3,graph("grotzsch"))
```

true

All three graphs are triangle-free. Since it is obviously $\chi_{G_1} = 2$, it follows $\chi_{G_2} = 3$ and $\chi_{G_3} = 4$.

```
> apply(chromatic_number,[G1,G2,G3])
```

$[2, 3, 4]$

4.13.5. *k*-coloring

The command `is_vertex_colorable` is used for determining whether the vertices of a graph can be colored with at most k colors.

Syntax: `is_vertex_colorable(G,k)`
`is_vertex_colorable(G,k,c)`

`is_vertex_colorable` takes two or three arguments: a graph $G(V, E)$, a positive integer k and optionally an unassigned identifier `c`. The command returns `true` if G can be colored using at most k colors and `false` otherwise. If the third argument is given, a coloring using at most k colors is stored to `c` as a list of vertex colors, in the order of `vertices(G)`.

The strategy is to first apply a simple greedy coloring procedure which runs in linear time. If the number of required colors is greater than k, the heuristic proposed by Brélaz in [11] is used, which runs in quadratic time. If the number of required colors is still larger than k, the algorithm attempts to find the chromatic number χ_G using k as the upper bound in the process.

```
> G:=graph("grotzsch")
```

an undirected unweighted graph with 11 vertices and 20 edges

```
> is_vertex_colorable(G,3)
```

false

```
> is_vertex_colorable(G,4)
```

true

```
> G:=random_graph(70,0.2)
```

an undirected unweighted graph with 70 vertices and 469 edges

```
> chromatic_number(G,approx)
```

5..6

```
> is_vertex_colorable(G,5)
```

false

818 msec

From the results of the last two command lines it follows $\chi_G = 6$. Finding χ_G by utilizing the next command line is simpler, but requires much more time.

```
> chromatic_number(G)
```

6

92.7 sec

4.14. EDGE COLORING

4.14.1. Minimal edge coloring

The command `minimal_edge_coloring` is used for finding a minimal coloring of edges in a graph, satisfying the following two conditions: any two mutually incident edges are colored differently and the total number n of colors is minimal. The theorem of VIZING [17, pp. 103] implies that every simple undirected graph falls into one of two categories: *1* if $n = \Delta$ or *2* if $n = \Delta + 1$, where Δ is the maximum degree of the graph.

Syntax: `minimal_edge_coloring(G)`
`minimal_edge_coloring(G,sto)`

`minimal_edge_coloring` takes one or two arguments, a graph $G(V, E)$ and optionally the keyword `sto`. If the latter is given, a minimal coloring is stored to the input graph (each edge $e \in E$ gets a color c_e stored as an attribute) and a modified copy of G is returned. Else, the command returns a sequence of two objects: integer `1` or `2`, indicating the category, and the list of edge colors c_{e_1}, $c_{e_2}, ..., c_{e_m}$ according the order of edges $e_1, e_2, ..., e_m \in E$ as returned by the command `edges`.

The strategy is to find a minimal vertex coloring of the line graph of G by using the algorithm described in Section 4.13.2.

```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

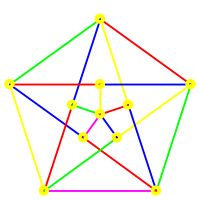
```
> minimal_edge_coloring(G)
```

$2, [1, 2, 3, 2, 3, 3, 4, 1, 2, 3, 1, 4, 1, 4, 2]$

```
> H:=minimal_edge_coloring(graph("grotzsch"),sto)
```

an undirected unweighted graph with 11 vertices and 20 edges

```
> draw_graph(H,labels=false)
```



```
> G:=random_graph(100,0.1)
```

an undirected unweighted graph with 100 vertices and 499 edges

```
> minimal_edge_coloring(G):;
```

20.24 sec

4.14.2. Chromatic index

The command `chromatic_index` is used for computing the chromatic index of an undirected graph.

Syntax: `chromatic_index(G)`
`chromatic_index(G,c)`

`chromatic_index` takes one or two arguments, an undirected graph $G(E, V)$ and optionally an unassigned identifier `c`. The command returns the minimal number $\chi'(G)$ of colors needed to color each edge in G such that two incident edges never share the same color. If the second argument is given, it specifies the destination for storing the coloring in form of a list of colors according to the order of edges in E as returned by the command `edges`.

The example below demonstrates how to color the edges of a graph with colors obtained by passing unassigned identifier `c` to `chromatic_index` as the second argument.

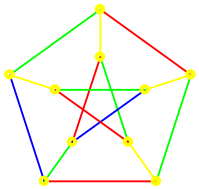
```
> G:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> chromatic_index(G,c)
```

4

```
> draw_graph(highlight_edges(G,edges(G),c))
```

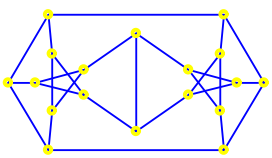


Blanuša snarks, the two graphs with 18 vertices found in 1946 by Danilo Blanuša, were the second and third snarks discovered [8]. For almost fifty years, Petersen graph was the only known snark. The second Blanuša snark is available in Giac by passing the string `"blanusa"` to the `graph` command.

```
> G:=graph("blanusa")
```

an undirected unweighted graph with 18 vertices and 27 edges

```
> draw_graph(G,labels=false)
```



```
> minimum_degree(G),maximum_degree(G)
```

3, 3

To prove that Blanuša snark is bridgeless, it is enough to show that it is biconnected, since each endpoint of a bridge is an articulation point (unless being of degree 1).

```
> is_biconnected(G)
```

true

```
> girth(G)
```

5

```
> chromatic_index(G)
```

4

Chapter 5

Traversing graphs

5.1. Walks and tours

5.1.1. Eulerian graphs

The command `is_eulerian` is used for determining whether an undirected graph contains an Eulerian trail.

Syntax: `is_eulerian(G)`
`is_eulerian(G,T)`

`is_eulerian` takes one or two arguments, an undirected graph $G(V, E)$ and optionally an unassigned identifier `T`, and returns `true` if G is Eulerian and `false` otherwise. If the second argument is given, the corresponding Eulerian trail is stored to `T`.

A graph G is **Eulerian** if it has a trail covering all its edges. Such a trail is called **Eulerian trail**. An Eulerian trail may be closed, in which case it is called **Eulerian cycle**. Note that every edge $e \in E$ must be visited, i.e. "strolled through", exactly once [27, pp. 395]. The edge endpoints (i.e. the vertices in G) may, however, be visited more than once.

The strategy is to apply Hierholzer's algorithm for finding an Eulerian path [33]. It works by covering one cycle at a time in the input graph. The required time is $O(|E|)$.

```
> is_eulerian(complete_graph(4))
```

false

```
> is_eulerian(complete_graph([1,2,3,4,5]),T); T
```

true, $[1, 2, 3, 4, 1, 5, 2, 4, 5, 3, 1]$

```
> is_eulerian(graph("tetrahedron"))
```

false

```
> is_eulerian(graph("octahedron"))
```

true

5.1.2. Hamiltonian graphs

The command `is_hamiltonian` is used for checking hamiltonicity of an undirected graph. The command can also construct a Hamiltonian cycle in the input graph if the latter is Hamiltonian.

Syntax: `is_hamiltonian(G)`
`is_hamiltonian(G,hc)`

`is_hamiltonian` takes one or two arguments, an undirected graph $G(V, E)$ and optionally an unassigned identifier `hc`. The command returns `true` if G is Hamiltonian and `false` otherwise. When failing to determine whether G is Hamiltonian or not, `is_hamiltonian` returns `undef`. If the second argument is given, a Hamiltonian cycle is stored to `hc`.

The strategy is to apply some hamiltonicity criteria presented by DELEON [15] before resorting to the definitive but NP-hard algorithm. If G is not biconnected, it is not Hamiltonian. Else, the criterion of DIRAC is applied: if $\delta(G) \geqslant \frac{|V|}{2}$, where $\delta(G) = \min\{\deg(v) : v \in V\}$, then G is Hamiltonian. Else, if G is bipartite with vertex partition $V = V_1 \cup V_2$ and $|V_1| \neq |V_2|$, then G is not Hamiltonian. Else, the criterion of ORE is applied: if $\deg(u) + \deg(v) \geqslant n$ holds for every pair u, v of non-adjacent vertices from V, then G is Hamiltonian. Else, the theorem of BONDY and CHVÁTAL is applied: if the closure $\mathrm{cl}(G)$ of G (obtained by finding a pair u, v of non-adjacent vertices from V such that $\deg(u) + \deg(v) \geqslant n$, adding a new edge uv to E and repeating the process until exhaustion) is Hamiltonian, then G is Hamiltonian. (Note that in this case the previously tried criteria are applied to $\mathrm{cl}(G)$; since the vertex degrees in $\mathrm{cl}(G)$ are generally higher than those in G, the probability of success also rises.) Else, if the edge density of G is large enough, the criterion of NASH and WILLIAMS is applied: if $\delta(G) \geqslant \max\left\{\frac{n+2}{3}, \beta\right\}$, where β is the independence number of G, then G is Hamiltonian. If all of the above criteria fail, the command `traveling_salesman` is called, either to find a Hamiltonian cycle in G or to determine that none exist.

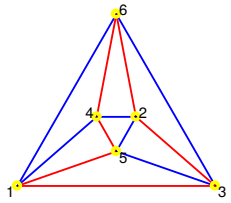
```
> is_hamiltonian(graph("soccerball"))
```

true

```
> is_hamiltonian(graph("octahedron"),hc)
```

true

```
> draw_graph(highlight_trail(graph("octahedron"),hc))
```




```
> is_hamiltonian(graph("herschel"))
```

false

```
> is_hamiltonian(graph("petersen"))
```

false

```
> is_hamiltonian(hypercube_graph(9))
```

true

6.04 sec

5.2. OPTIMAL ROUTING

5.2.1. Shortest unweighted paths

The command `shortest_path` is used for finding shortest paths in unweighted graphs.

Syntax: `shortest_path(G,s,t)`
`shortest_path(G,s,T)`

`shortest_path` takes three arguments: an undirected unweighted graph $G(V, E)$, the source vertex $s \in V$ and the target vertex $t \in V$ or a list T of target vertices. The shortest path from source to target is returned. If more targets are specified, the list of shortest paths from the source to each of these vertices is returned.

The strategy is to run breadth-first traversal on the graph G starting from the source vertex s. The complexity of the algorithm is therefore $O(|V|+|E|)$.

```
> G:=graph("dodecahedron")
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> shortest_path(G,1,16)
```

$[1,6,11,16]$

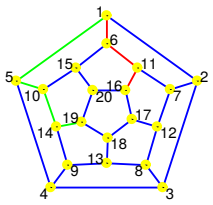
```
> paths:=shortest_path(G,1,[16,19])
```

$\{[1,6,11,16],[1,5,10,14,19]\}$

```
> H:=highlight_trail(G,paths,[red,green])
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> draw_graph(H)
```



5.2.2. Cheapest weighted paths

The commands `dijkstra` and `bellman_ford` are used for finding cheapest paths in weighted (directed) graphs.

Syntax: `dijkstra(G,s,t)`
`dijkstra(G,s,T)`
`bellman_ford(G,s,t)`
`bellman_ford(G,s,T)`

`dijkstra` and `bellman_ford` both take two or three arguments: a weighted (di)graph $G(V,E)$, a vertex $s \in V$ and optionally a vertex $t \in V$ or list T of vertices in V. It returns the cheapest path from s to t or, if more target vertices are given, the list of such paths to each target vertex $t \in T$. If no target vertex is specified, all vertices in $V \setminus \{s\}$ are assumed to be targets. If `dijkstra` is used, the weights of edges in E must all be nonnegative. `bellman_ford` accepts negative weights, but does not work if the input graph contains negative cycles (in which the weights of the corresponding edges sum up to a negative value).

A cheapest path from s to t is represented with a list `[[v1,v2,...,vk],c]` where the first element consists of path vertices with $v_1 = s$ and $v_k = t$, while the second element c is the weight (cost) of that path, equal to the sum of weights of its edges.

`dijsktra` computes the cheapest path using Dijkstra's algorithm which runs in $O(|V|^2)$ time [18]. `bellman_ford` uses somewhat slower algorithm by Bellman and Ford (see [6] and [24]) which runs in $O(|V||E|)$ time but in turn imposes less requirements upon its input.

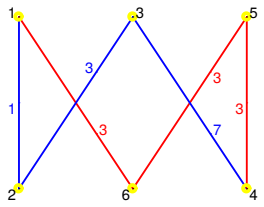
```
> G:=graph(%{[[1,2],1],[[1,6],3],[[2,3],3],[[3,4],7],[[4,5],3],[[5,6],3]%})
```

an undirected weighted graph with 6 vertices and 6 edges

```
> res:=dijkstra(G,1,4)
```

$[[1,6,5,4],9]$

```
> draw_graph(highlight_trail(G,res[0]))
```




```
> dijkstra(G,1)
```

$$[[1],0],[[1,2],1],[[1,6],3],[[1,2,3],4],[[1,6,5,4],9],[[1,6,5],6]$$

5.2.3. Traveling salesman problem

The command `traveling_salesman` is used for solving traveling salesman problem (TSP)[5.1].

Syntax: `traveling_salesman(G,[opts])`
`traveling_salesman(G,M,[opts])`

`traveling_salesman` takes the following arguments: an undirected graph $G(V, E)$, a weight matrix M (optional) and a sequence of options (optional). The supported options are `approx` and `vertex_distance`.

If the input graph G is unweighted and M is not specified, a Hamiltonian cycle (tour) is returned (the adjacency matrix of G is used for the edge weights). If G is weighted, two objects are returned: the optimal value for the traveling salesman problem and a Hamiltonian cycle which achieves the optimal value. If M is given and G is unweighted, M is used as the weight matrix for G.

If the option `vertex_distance` is passed and M is not specified, then for each edge $e \in E$ the Euclidean distance between its endpoints is used as the weight of e. Therefore it is required for each vertex in G to have a predefined position.

If the option `approx` is passed, a near-optimal tour is returned. In this case it is required that G is a complete weighted graph. For larger graphs, this is significantly faster than finding optimal tour. Results thus obtained are usually only a few percent larger than the corresponding optimal values, despite the fact that the reported guarantee is generally much weaker (around 30%).

The strategy is to formulate TSP as a linear programming problem and to solve it by branch-and-cut method, applying the hierarchical clustering method of PFERSCHY and STANĚK [49] to generate subtour elimination constraints. The branching rule is implemented according to PADBERG and RINALDI [47]. In addition, the algorithm combines the method of CHRISTOFIDES [13], the method of farthest insertion and a variant of the powerful tour improvement heuristic developed by LIN and KERNIGHAN [32] to generate near-optimal feasible solutions during the branch-and-cut process.

For Euclidean TSP instances, i.e. in cases when G is a complete graph with vertex distances as the edge weights, the algorithm usually finishes in a few seconds for TSP with up to, say, 42 cities. For problems with 100 or more cities, the option `approx` is recommended as finding the optimal value takes a long time. Note that TSP is NP-hard, meaning that no polynomial time algorithm is known. Hence the algorithm may take exponential time to find the optimum in some instances.

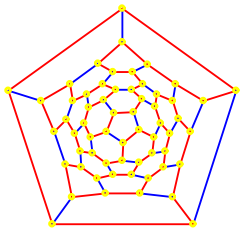
The following example demonstrates finding a Hamiltonian cycle in the truncated icosahedral ("soccer ball") graph. The result is visualized by using `highlight_trail`.

```
> G:=graph("soccerball")
```

an undirected unweighted graph with 60 vertices and 90 edges

```
> draw_graph(highlight_trail(G,traveling_salesman(G)),labels=false)
```

5.1. For the details on traveling salesman problem and a historical overview see [14].



A matrix may be passed alongside an undirected graph to specify the edge weights. The alternative is to pass a weighted graph as the single argument.

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

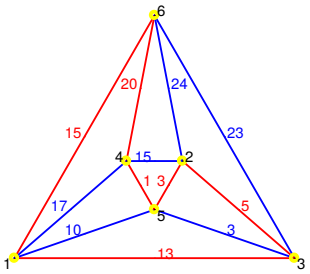
```
> M:=randmatrix(6,6,25)
```

$$\begin{pmatrix} 7 & 13 & 15 & 10 & 17 & 6 \\ 17 & 7 & 23 & 3 & 17 & 5 \\ 15 & 24 & 19 & 15 & 20 & 24 \\ 3 & 16 & 10 & 18 & 1 & 3 \\ 9 & 20 & 9 & 2 & 19 & 15 \\ 17 & 8 & 19 & 20 & 15 & 15 \end{pmatrix}$$

```
> c,t:=traveling_salesman(G,M)
```

$57.0, [4, 5, 2, 3, 1, 6, 4]$

```
> draw_graph(highlight_trail(make_weighted(G,M),t))
```



In the next example, an instance of Euclidean TSP with 42 cities is solved to optimality. The vertex positions are pairs of integers randomly chosen on the grid $[0, 1000] \times [0, 1000] \in \mathbb{Z}^2$.

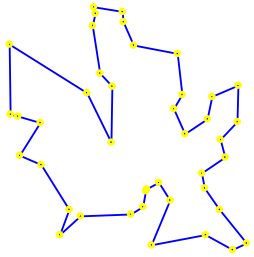
```
> G:=set_vertex_positions(complete_graph(42),[randvector(2,1000)$(k=1..42)])
```

an undirected unweighted graph with 42 vertices and 861 edges

```
> c,t:=traveling_salesman(G,vertex_distance):;
```

10.01 sec

```
> draw_graph(subgraph(G,trail2edges(t)),labels=false)
```



For large instances of Euclidean TSP the `approx` option may be used, as in the following example with 555 cities.

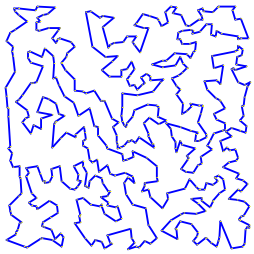
```
> H:=set_vertex_positions(complete_graph(555),[randvector(2,10000)$(k=1..555)])
```

an undirected unweighted graph with 555 vertices and 153735 edges

```
> ac,t:=traveling_salesman(H,vertex_distance,approx):;
```

49.34 sec

```
> draw_graph(subgraph(H,trail2edges(t)))
```



Near-optimal tours produced by the `approx` option are usually only slightly more expensive than the optimal ones. For example, a sub-optimal tour for the previous instance G with 42 cities is obtained by the following command.

```
> ac,st:=traveling_salesman(G,vertex_distance,approx):;
```

```
The tour cost is within 28% of the optimal value
```

Although it is guaranteed that the near-optimal cost `ac` is for at most 28% larger than `c` (the optimum), the actual relative difference is smaller than 3%, as computed below.

```
> 100*(ac-c)/c
```

2.7105821877

5.3. SPANNING TREES

5.3.1. Construction of spanning trees

The command `spanning_tree` is used for construction of spanning trees in graphs.

Syntax: `spanning_tree(G)`
`spanning_tree(G,r)`

`spanning_tree` takes one or two arguments, an undirected graph $G(V, E)$ and optionally a vertex $r \in V$. It returns the spanning tree T (rooted in r) of G, obtained by depth-first traversal in $O(|V| + |E|)$ time.

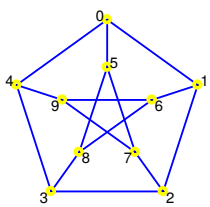
```
> P:=graph("petersen")
```

an undirected unweighted graph with 10 vertices and 15 edges

```
> T1:=spanning_tree(P)
```

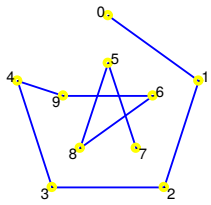
an undirected unweighted graph with 10 vertices and 9 edges

```
> draw_graph(P)
```



By extracting T_1 from P as a subgraph, it inherits vertex positions from P.

```
> draw_graph(subgraph(P,edges(T1)))
```




```
> T2:=spanning_tree(P,4)
```

an undirected unweighted graph with 10 vertices and 9 edges

```
> edges(T1), edges(T2)
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 9 \\ 5 & 7 \\ 5 & 8 \\ 6 & 8 \\ 6 & 9 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 4 \\ 1 & 2 \\ 2 & 3 \\ 3 & 8 \\ 5 & 7 \\ 5 & 8 \\ 6 & 9 \\ 7 & 9 \end{pmatrix}$$

5.3.2. Minimal spanning tree

The command `minimal_spanning_tree` is used for obtaining minimal spanning trees in undirected graphs.

Syntax: `minimal_spanning_tree(G)`

`minimal_spanning_tree` takes an undirected graph $G(V, E)$ as its only argument and returns its minimal spanning tree as a graph. If G is not weighted, it is assumed that the weight of each edge in E is equal to 1.

The strategy is to apply Kruskal's algorithm which runs in $O(|E| \log |V|)$ time.

```
> A:=[[0,1,0,4,0,0],[1,0,1,0,4,0],[0,1,0,3,0,1],[4,0,3,0,1,0],[0,4,0,1,0,4],[0,
  0,1,0,4,0]]:;
```

```
> G:=graph(A)
```

an undirected weighted graph with 6 vertices and 8 edges

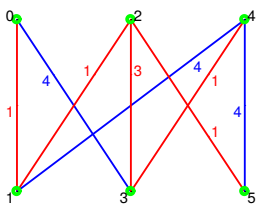
```
> T:=minimal_spanning_tree(G)
```

an undirected weighted graph with 6 vertices and 5 edges

```
> edges(T,weights)
```

$$\{[[0,1],1],[[1,2],1],[[2,5],1],[[2,3],3],[[3,4],1]\}$$

```
> draw_graph(highlight_subgraph(G,T))
```



5.3.3. Counting the spanning trees in a graph

The command `number_of_spanning_trees` is used for counting spanning trees in a graph.

Syntax: `number_of_spanning_trees(G)`

`number_of_spanning_trees` takes an undirected graph $G(V, E)$ as its only argument and returns the total number n of (labeled) spanning trees in G.

The strategy is to use KIRCHHOFF's Theorem [64, Theorem 2.2.12, pp. 86]. The number of spanning trees is equal to the first principal minor of the Laplacian matrix of G.

```
> number_of_spanning_trees(graph("octahedron"))
```

384

```
> number_of_spanning_trees(graph("dodecahedron"))
```

5184000

```
> number_of_spanning_trees(hypercube_graph(4))
```

42467328

```
> number_of_spanning_trees(graph("soccerball"))
```

375291866372898816000

Chapter 6

Visualizing graphs

6.1. Drawing graphs

The `draw_graph` command is used for visualizing graphs. It is capable to produce a drawing of a graph using one of the several built-in methods.

Syntax: `draw_graph(G)`
`draw_graph(G,opts)`

6.1.1. Overview

`draw_graph` takes one or two arguments, the mandatory first one being a graph $G(V, E)$. This command assigns 2D or 3D coordinates to each vertex $v \in V$ and produces a visual representation of G based on these coordinates. The second (optional) argument is a sequence of options. Each option is one of the following.

> `labels=true` or `false` — Control the visibility of vertex labels and edge weights (by default `true`, i.e. the labels and weights are displayed).
>
> `spring` — Apply a multilevel force-directed algorithm.
>
> `tree[=r` or `[r1,r2,...]]` — Draw a tree or forest G, optionally specifying the root node for each tree (by default the first node is used).
>
> `bipartite` — Draw a bipartite graph G, separating the vertex partitions from one another.
>
> `circle[=L]` or `convexhull[=L]` — Draw a graph G by spreading the *hull vertices* from list $L \subset V$ (assuming $L = V$ by default) across the unit circle and putting all other vertices in origin, subsequently applying a force-directed vertex placement algorithm to generate the layout while keeping the hull vertices fixed.
>
> `planar` or `plane` — Draw a planar graph G using a force-directed algorithm.
>
> `plot3d` — Draw a connected graph G as if the `spring` option was enabled, but with vertex positions in 3D instead of 2D.

If an unassigned identifier is passed as an argument, it is used as the destination for storing the computed vertex positions as a list.

The style options `spring`, `tree`, `circle`, `planar` and `plot3d` cannot be mixed, i.e. at most one can be specified. The option `labels` may be combined with any of the style options. Note that edge weights will not be displayed when using `plot3d` option when drawing a weighted graph.

When no style option is specified, the algorithm first checks if the graph G is a tree or if it is bipartite, in which cases it is drawn accordingly. Otherwise, the graph is drawn as if the option `circle` was specified.

Tree, circle and bipartite drawings can be obtained in linear time with a very small overhead, allowing graphs to be drawn quickly no matter the size. The force-directed algorithms are more expensive and operating in the time which is quadratic in the number of vertices. Their performance is, nevertheless, practically instantaneous for graphs with several hundreds of vertices (or less).

6.1.2. Drawing disconnected graphs

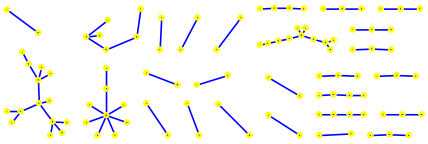
When the input graph has two or more connected components, each component is drawn separately and the drawings are subsequently arranged such that the bounding box of the whole drawing has the smallest perimeter under condition that as little space as possible is wasted inside the box.

For example, the command lines below draw a sparse random planar graph.

```
> G:=random_planar_graph(100,0.9,0)
```

an undirected unweighted graph with 100 vertices and 74 edges

```
> draw_graph(G,planar)
```



6.1.3. Spring method

When the option `spring` is specified, the input graph is drawn using the force-directed algorithm described in [36] (for an example of such drawing see Figure 3.1). The idea, originally due to FRUCHTERMAN and REINGOLD [25], is to simulate physical forces in a spring-electrical model where the vertices and edges represent equally charged particles and springs connecting them, respectively.

In a spring-electrical model, each vertex is being repulsed by every other vertex with a force inversely proportional to the distance between them. At the same time, it is attracted to each of its neighbors with a force proportional to the square of the distance. Assuming that x_v is the vector representing the position of the vertex $v \in V$, the total force F_v applied to v is equal to

$$F_v = \sum_{w \in V \setminus \{v\}} -\frac{CK^2}{\|x_v - x_w\|^2}(x_v - x_w) + \sum_{w \in N(v)} \frac{\|x_v - x_w\|}{K}(x_v - x_w),$$

where $N(v)$ is the set of neighbors of v and C, K are certain positive real constants (actually, K may be any positive number, it affects only the scaling of the entire layout). Applying the forces iteratively and updating vertex positions in each iteration (starting from a random layout) leads the system to the state of minimal energy. By applying a certain "cooling" scheme to the model which cuts down the force magnitude in each iteration. the layout "freezes" after a number of iterations large enough to achieve the minimal energy state.

The force-directed method is computationally expensive and for larger graphs the pleasing layout cannot be obtained most of the time since the algorithm, starting with a random initial layout, gets easily "stuck" in a local energy minimum. To avoid this a multilevel scheme is applied. The input graph is iteratively coarsened, either by removing the vertices from a maximal independent vertex set or by contracting the edges of a maximal matching in each iteration. Each coarsening level is processed by the force-directed algorithm, starting from the deepest (coarsest) one and "lifting" the obtained layout to the first upper level, using it as the initial layout for that level. The lifting done using a prolongation matrix technique described in [37]. To support drawing large graphs (with, say, 1000 vertices or more), the matrices used in the lifting process are stored in sparse form. The multilevel scheme also speeds up the layout process significantly.

If the structure of the input graph is symmetric, a layout obtained by using a force-directed method typically reveals these symmetries, which is a unique property among graph drawing algorithms. To make the symmetries more prominent, the layout is rotated such that the axis, with respect to which the layout exhibits the largest *symmetry score*, becomes vertical. Because symmetry detection is computationally quite expensive (up to $O(|V|^7)$ when using the symmetry measure of PURCHASE [63], for example), the algorithm accounts only the convex hull and the barycenter of the layout, which may not always be enough to produce the optimal result. Nevertheless, this approach is fast and works (most of the time) for highly symmetrical graphs.

For example, the following command lines produce a drawing of the tensor product of two graphs using the force-directed algorithm.

```
> G1:=graph(trail(1,2,3,4,5,2))
```

an undirected unweighted graph with 5 vertices and 5 edges

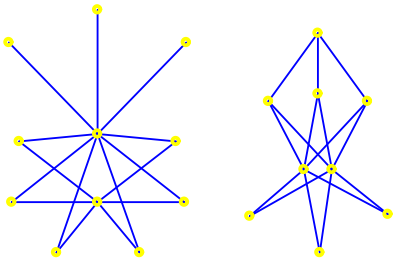
```
> G2:=star_graph(3)
```

an undirected unweighted graph with 4 vertices and 3 edges

```
> G:=tensor_product(G1,G2)
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> draw_graph(G,spring,labels=false)
```

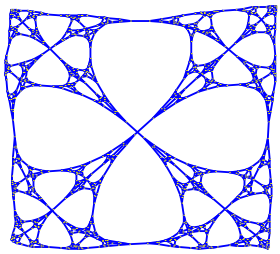


The following example demonstrates drawing a much larger graph.

```
> S:=sierpinski_graph(5,4)
```

an undirected unweighted graph with 1024 vertices and 2046 edges

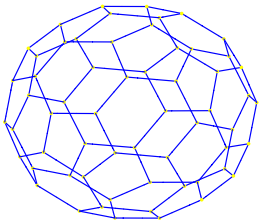
```
> draw_graph(S,spring)
```



Note that vertex labels are automatically suppressed because of the large number of vertices. On our system, the algorithm took less than two seconds to produce the layout.

The spring method is also used for creating 3D graph layouts, which are obtained by passing the option `plot3d` to the `draw_graph` command.

```
> draw_graph(graph("soccerball"),plot3d,labels=false)
```



```
> G1:=graph("icosahedron"):; G2:=graph("dodecahedron"):;
```

Done, Done

```
> G1:=highlight_edges(G1,edges(G1),red)
```

an undirected unweighted graph with 12 vertices and 30 edges

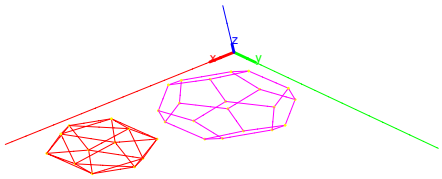
```
> G2:=highlight_edges(G2,edges(G2),magenta)
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> G:=disjoint_union(G1,G2)
```

an undirected unweighted graph with 32 vertices and 60 edges

```
> draw_graph(G,plot3d,labels=false)
```



6.1.4. Drawing trees

When the `tree[=r]` option is specified and the input graph G is a tree (and $r \in V$), it is drawn using a fast but simple node positioning algorithm inspired by the well-known algorithm of WALKER [61], using the first vertex (or the vertex r) as the root node. When drawing a rooted tree, one usually requires the following aesthetic properties [12].

A1. The layout displays the hierarchical structure of the tree, i.e. the y-coordinate of a node is given by its level.

A2. The edges do not cross each other.

A3. The drawing of a sub-tree does not depend on its position in the tree, i.e. isomorphic sub-trees are drawn identically up to translation.

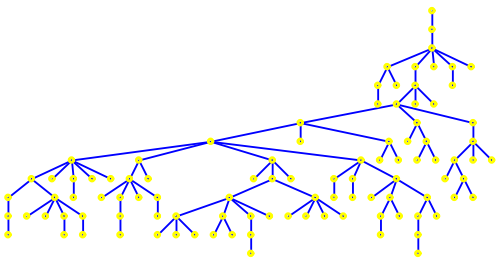
A4. The order of the children of a node is displayed in the drawing.

A5. The algorithm works symmetrically, i.e. the drawing of the reflection of a tree is the reflected drawing of the original tree.

The algorithm implemented in Giac generally satisfies all the above properties but A3. Instead, it tries to spread the inner sub-trees evenly across the available horizontal space. It works by organizing the structure of the input tree into levels by using depth-first search and laying out each level subsequently, starting from the deepest one and climbing up to the root node. In the end, another depth-first traversal is made, shifting the sub-trees horizontally to avoid intersections between their edges. The algorithm runs in $O(|V|)$ time and uses the minimum of horizontal space to draw the tree with respect to the specified root node r.

For example, the following command line draws a random free unlabeled tree on 100 nodes.

```
> draw_graph(random_tree(100))
```



6.1.5. Drawing planar graphs

The algorithm implemented in Giac which draws planar graphs uses augmentation techniques to extend the input graph G to a graph G', which is homeomorphic to some triconnected graph, by adding temporary edges. The augmented graph G' is then drawn using Tutte's barycentric method (see [60] and [27, pp. 293]) which puts each vertex in the barycenter of its neighbors. It is guaranteed that a (non-strict) convex drawing will be produced, without edge crossings. In the end, the duplicate of the outer face and the temporary edges inserted during the augmentation stage are removed.

Tutte's algorithm requires that the vertices of the chosen outer face are initially fixed somewhere the boundary of a convex polygon. In addition, to produce a more flexible layout, the outer face is duplicated such that the subgraph induced by the vertices on both the outer face and its duplicate is a prism graph. Then only the duplicates of the outer face vertices are fixed, allowing the outer face itself to take a more natural shape. The duplicate of the outer face is removed after a layout is produced.

The augmentation process consists of two parts. Firstly, the input graph G is decomposed into biconnected components (blocks) using the depth-first search [26, pp. 25]. Each block is then decomposed into faces (represented by cycles of vertices) using Demoucron's algorithm (see [26, pp. 88] and [42]). Embeddings obtained for each blocks are then combined by adding one temporary edge for each articulation point, joining the two corresponding blocks. Figure 6.1 shows the outer faces of two blocks B_1 and B_2, connected by an articulation point (cut vertex). The temporary edge (shown in green) is added to join B_1 and B_2 into a single block. After "folding up" the tree of blocks, the algorithm picks the largest face in the resulting biconnected graph to be the outer face of the planar embedding.

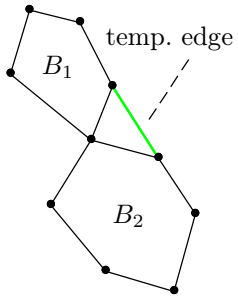


Fig. 6.1. Joining two block by adding a temporary edge.

The second part of the augmentation process consists of recursively decomposing each non-convex inner face into several convex polygons by adding temporary edges. An inner face $f = (v_1, ..., v_n)$ is non-convex if there exist k and l such that $1 \leqslant k < l - 1 < n$ and either $v_k v_l \in E$, in which case the edge $v_k v_l$ is a *chord* (see Figure 6.2 for an example) or there exists a face $g = (w_1, w_2, ..., v_k, ..., v_l, ..., w_{m-1}, w_m)$ such that the vertices $v_{k+1}, ..., v_{l-1}$ are not contained in g (see Figure 6.3 for an example). In Figures 6.1, 6.2 and 6.3, the temporary edges added by the algorithm are drawn in green.

This method of drawing planar graphs operates in $O(|V|^2)$ time. Nevertheless, it is quite fast for graphs up to 1000 vertices, usually producing results in less than a second. A drawback of this method is that it sometimes creates clusters of vertices which are very close to each other, resulting in a very high ratio of the area of the largest inner face to the area of the smallest inner face. However, if the result is not satisfactory, one can simply redraw the graph and repeat the process until a better layout is obtained. The planar embedding will in general be different each time if the graph is not triconnected.

Another drawback of this method is that sparse planar graphs are sometimes drawn poorly.

The following example shows that the above described improvement of the barycentric method handles non-triconnected graphs well.

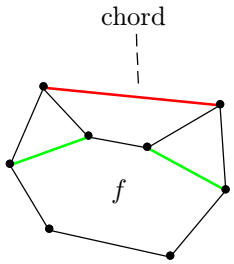


Fig. 6.2. A chorded face f.

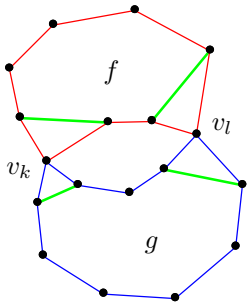
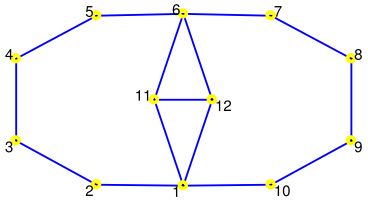


Fig. 6.3. Faces f and g having two vertices but no edges in common.

```
> G:=graph(trail(1,2,3,4,5,6,7,8,9,10,1),trail(11,12,6,11,1,12))
```

an undirected unweighted graph with 12 vertices and 15 edges

```
> draw_graph(G,planar)
```



Note that the inner diamond-like shape in the above drawing would end up flattened—making the two triangular faces invisible—if the input graph was not augmented. It is so because the vertices with labels 11 and 12 are "attracted" to each other (namely, the two large faces are "inflating" themselves to become convex), causing them to merge eventually.

In the following example the input graph G is connected but not biconnected (it has two articulation points). It is obtained by removing a vertex from the Sierpiński triangle graph ST_3^3. Note that the syntax mode is set to Xcas in this example, so the first vertex label is zero.

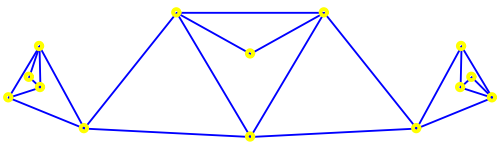
```
> G:=sierpinski_graph(3,3,triangle)
```

an undirected unweighted graph with 15 vertices and 27 edges

```
> G:=delete_vertex(G,3)
```

an undirected unweighted graph with 14 vertices and 23 edges

```
> draw_graph(G,planar,labels=false)
```



In the above example, several redraws were required to obtain a good planar embedding.

6.1.6. Circular graph drawings

The drawing method selected by specifying the option `circle=L` or `convexhull=L` when calling `draw_graph` on a triconnected graph $G(V, E)$, where $L \subset V$ is a set of vertices in G, uses the following strategy. First, positions of the vertices from L are fixed so that they form a regular polygon on the unit circle. Other vertices, i.e. all vertices from $V \setminus L$, are placed in origin. Then an iterative force-directed algorithm [50], similar to Tutte's barycentric method, is applied to obtain the final layout.

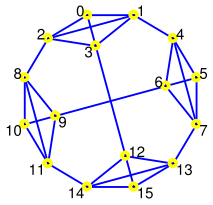
This approach gives best results for symmetrical graphs such as generalized Petersen graphs. In addition, if the input graph is planar and triconnected, and the outer hull represents a face in the respective planar embedding, then the drawing will contain no edge crossings. There is a possibility, however, that some very short edges may cross each other as the number of force update iterations is limited.

In the following example the Sierpiński graph S_4^2 is drawn using the above method. Note that the command lines below are executed in Xcas mode.

```
> G:=sierpinski_graph(2,4)
```

an undirected unweighted graph with 16 vertices and 30 edges

```
> draw_graph(G,circle=[0,1,4,5,7,13,15,14,11,10,8,2])
```



To draw a planar triconnected graph, one should pass one of its faces as the outer hull.

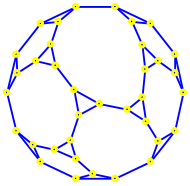
```
> G:=truncate_graph(graph("frucht"))
```

an undirected unweighted graph with 60 vertices and 90 edges

```
> purge(F):; is_planar(G,F)
```

Done, true

```
> draw_graph(G,circle=rand(F),labels=false)
```



6.2. Vertex positions

6.2.1. Setting vertex positions

The command `set_vertex_positions` is used to assign custom coordinates to vertices of a graph to be used when drawing the graph.

Syntax: `set_vertex_positions(G,L)`

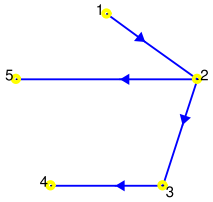
`set_vertex_positions` takes two arguments, a graph $G(V, E)$ and the list L of positions to be assigned to vertices in order of `vertices(G)`. The positions may be complex numbers, lists of coordinates or points (geometrical objects created with the command `point`). `set_vertex_positions` returns the copy G' of G with the given layout stored in it.

Any subsequent call to `draw_graph` with G' as an argument and without specifying the drawing style will result in displaying vertices at the stored coordinates. However, if a drawing style is specified, the stored layout is ignored (although it stays stored in G').

```
> G:=digraph([1,2,3,4,5],%{[1,2],[2,3],[3,4],[2,5]%})
```

a directed unweighted graph with 5 vertices and 4 arcs

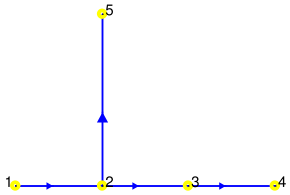
```
> draw_graph(G,circle)
```




```
> H:=set_vertex_positions(G,[[0,0],[0.5,0],[1.0,0],[1.5,0],[0.5,1]])
```

a directed unweighted graph with 5 vertices and 4 arcs

```
> draw_graph(H)
```



6.2.2. Generating vertex positions

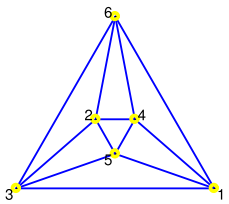
Vertex positions can be generated for a particular graph G by using the `draw_graph` command with the additional argument P which should be an unassigned identifier. After the layout is obtained, it will be stored to P as a list of positions (complex numbers for 2D drawings or points for 3D drawings) for each vertex in order of `vertices(G)`.

This feature combines well with the `set_vertex_positions` command, as when one obtains the desired drawing of the graph G by calling `draw_graph`, the layout coordinates can be easily stored to the graph for future reference. In particular, each subsequent call of `draw_graph` with G as an argument will display the stored layout. The example below illustrates this property by setting a custom layout to the octahedral graph.

```
> G:=graph("octahedron")
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(G)
```



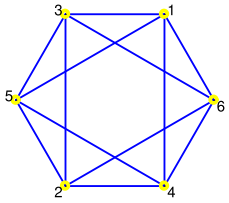

```
> draw_graph(G,P,spring):;
```

Now P contains vertex coordinates, which can be permanently stored to G:

```
> G:=set_vertex_positions(G,P)
```

an undirected unweighted graph with 6 vertices and 12 edges

```
> draw_graph(G)
```



It should be noted that, after a particular layout is fixed, it stays valid when some edges or vertices are removed or when an edge is contracted. The stored layout becomes invalid only if a new vertex is added to the graph (unless its position is specified by `set_vertex_attribute` upon the creation) or if the `position` attribute of an existing vertex is discarded.

6.3. Highlighting parts of graphs

6.3.1. Highlighting vertices

The command `highlight_vertex` is used for changing color of one or more vertices in a graph.

Syntax:
```
highlight_vertex(G,v)
highlight_vertex(G,v,c)
highlight_vertex(G,[v1,v2,..,vk])
highlight_vertex(G,[v1,v2,..,vk],c)
highlight_vertex(G,[v1,v2,..,vk],[c1,c2,..,ck])
```

`highlight_vertex` takes two or three arguments: a graph $G(V, E)$, a vertex $v \in V$ or a list of vertices $v_1, v_2, ..., v_k \in V$ and optionally the new color c or a list of colors $c_1, c_2, ..., c_k$ for the selected vertices (the default color is green). It returns a modified copy of G in which the specified vertices are colored with the specified color.

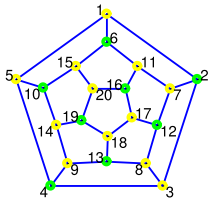
```
> G:=graph("dodecahedron")
```

an undirected unweighted graph with 20 vertices and 30 edges

```
> L:=maximum_independent_set(G)
```

$[2, 4, 6, 12, 13, 10, 16, 19]$

```
> draw_graph(highlight_vertex(G,L))
```



6.3.2. Highlighting edges and trails

To highlight an edge or a set of edges in a graph, use the `highlight_edges` command. If the edges form a trail, it is usually more convenient to use the `highlight_trail` command (see below).

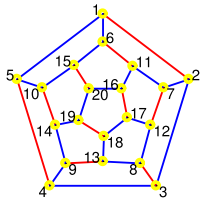
Syntax:
```
highlight_edges(G,e)
highlight_edges(G,e,c)
highlight_edges(G,[e1,e2,..,ek])
highlight_edges(G,[e1,e2,..,ek],c)
highlight_edges(G,[e1,e2,..,ek],[c1,c2,..,ck])
highlight_trail(G,T)
highlight_trail(G,T,c)
highlight_trail(G,[T1,T2,..,Tk])
highlight_trail(G,[T1,T2,..,Tk],c)
highlight_trail(G,[T1,T2,..,Tk],[c1,c2,..,ck])
```

`highlight_edges` takes two or three arguments: a graph $G(V, E)$, an edge $e \in E$ or a list of edges $e_1, e_2, ..., e_k \in E$ and optionally the new color c or a list of colors $c_1, c_2, ..., c_k$ for the selected edges (the default color is red). It returns a modified copy of G in which the specified edges are colored with the specified color.

```
> M:=maximum_matching(G)
```

$$\{[1,2],[5,4],[6,11],[3,8],[7,12],[9,13],[10,15],[14,19],[16,17]\}$$

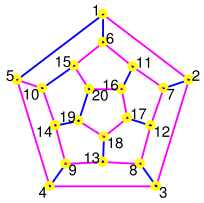
```
> draw_graph(highlight_edges(G,M))
```




```
> S:=spanning_tree(G)
```

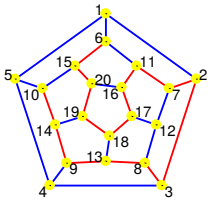
an undirected unweighted graph with 20 vertices and 19 edges

```
> draw_graph(highlight_edges(G,edges(S),magenta))
```

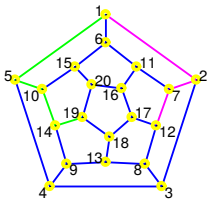


`highlight_trail` takes two or three arguments: a graph $G(V, E)$, a trail T or a list of trails T_1, $T_2, ..., T_k$ and optionally the new color c or a list of colors $c_1, c_2, ..., c_k$. The command returns the copy of G in which edges between consecutive vertices in each of the given trails are highlighted with color c (by default red) or the trail T_i is highlighted with color c_i for $i = 1, 2, ..., k$.

```
> draw_graph(highlight_trail(G,[6,15,20,19,18,17,16,11,7,2,3,8,13,9,14,10]))
```




```
> draw_graph(highlight_trail(G,shortest_path(G,1,[19,12]),[green,magenta]))
```



6.3.3. Highlighting subgraphs

The command `highlight_subgraph` is used for highlighting subgraph(s) of a graph.

Syntax: `highlight_subgraph(G,S,[weights])`
`highlight_subgraph(G,S,c1,c2,[weights])`
`highlight_subgraph(G,[S1,S2,..,Sk])`
`highlight_subgraph(G,[S1,S2,..,Sk],c1,c2)`

`highlight_subgraph` takes two or four mandatory arguments: a graph $G(V, E)$, a subgraph $S(V', E')$ of G or a list of subgraphs $S_1, S_2, ..., S_k$ in G and optionally the new colors c_1, c_2 for the edges and vertices of the selected subgraph(s), respectively. It returns a modified copy of G with the selected subgraph(s) colored as specified. If colors are not given, red and green are used, respectively.

The option `weights` may be passed as an additional argument if G and S are weighted graphs. In that case, the weights of edges in $E' \subset E$ in G are overwritten with those defined in S for the same edges.

```
> G:=graph(%{[1,2],[2,3],[3,1],[3,4],[4,5],[5,6],[6,4]%})
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> A:=articulation_points(G)
```

$[3, 4]$

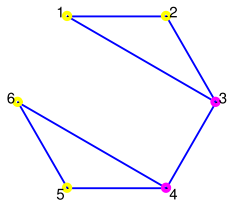
```
> B:=biconnected_components(G)
```

$[[4, 6, 5], [3, 4], [1, 3, 2]]$

```
> H:=highlight_vertex(G,A,magenta)
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> draw_graph(H)
```



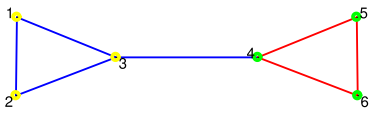

```
> S:=induced_subgraph(G,B[0])
```

an undirected unweighted graph with 3 vertices and 3 edges

```
> H:=highlight_subgraph(G,S)
```

an undirected unweighted graph with 6 vertices and 7 edges

```
> draw_graph(H,spring)
```



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