Recall that on Thursday you showed me that the following are equivalent:

- $\pi_1(\mathbb{R}) : y(\mathbb{R}) \to \mathbb{P}^1_{[t_0,t_1]}(\mathbb{R})$ is surjective
- For all $[t_0, t_1] \in \mathbb{P}^1_{[t_0,t_1]}(\mathbb{R})$, the correspondent quadratic form:

$$z^2 = Q_1(u,v,w)t_0^2 + 2Q_2(u,v,w)t_0t_1 + Q_3(u,v,w)t_1^2$$

  has a real solution
- Let M_1, M_2, M_3 be the symmetric matrix associated to Q_1, Q_2, Q_3 the matrix

$$M_{[t_0,t_1]} = \left(\begin{array}{c|c} M_1t_0^2 + 2M_2t_0t_1 + M_3t_1^2 & \mathbf{0} \\ \hline \mathbf{0} & -1 \end{array}\right)$$

  is indefinite
- $M_{[t_0,t_1]}$ is not negative-definite (since the matrix cannot be positive definite with the -1 term)

Question:
Choose M_1, M_2, M_3 to be diagonal matrices $[a_1, b_1, c_1], [a_2, b_2, c_2], [a_3, b_3, c_3]$ respectively, then the matrix M becomes

$$M_{[t_0,t_1]} = \begin{bmatrix} a_1t_0^2 + 2a_2t_0t_1 + a_3t_1^2 & 0 & 0 & 0 \\ 0 & b_1t_0^2 + 2b_2t_0t_1 + b_3t_1^2 & 0 & 0 \\ 0 & 0 & c_1t_0^2 + 2c_2t_0t_1 + c_3t_1^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Now choose M_1, M_2, M_3 to all be negative-definite, since they are diagonal this just means that every non-zero entry in these matrices are negative. Now consider when $t_0 = 0$, then

$$M_{[0,t_1]} = \begin{bmatrix} a_3t_1^2 & 0 & 0 & 0 \\ 0 & b_3t_1^2 & 0 & 0 \\ 0 & 0 & c_3t_1^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Then isn't this matrix always negative definite? I am pretty sure $[0, t_1]$ is still a valid point in $\mathbb{P}^1(\mathbb{R})$ as long as t_1 is non-zero. So then $\pi_1(\mathbb{R})$ fails to be surjective, and $\tilde{\Delta}(\mathbb{R})$ is empty since M_1 and M_3 are negative definite.

Conclusion:
So what I found is that assuming that either M_1 or M_3 is negative definite, then $\pi_1(\mathbb{R})$ can never be surjective. If M_1 is negative definite, choose $t_0 = 1, t_1 = 0$, if M_3 is negative definite, choose $t_0 = 0$, $t_1 = 1$.

For the case of M_1, this amounts to asking if this is solvable:

$$z^2 = Q_1(u,v,w)$$

, but $Q_1(u,v,w) \leqslant 0$ and $z^2 \geqslant 0$, so the only solution occurs when they are all 0, but that point does not exist.

Constraint: If $a_1 = 2a_2 = a_3 > 0$, then $M_{[t_0,t_1]}$ can never be negative definite, as

$$a_1 t_0^2 + 2a_2 t_0 t_1 + a_3 t_1^2 = a_1(t_0^2 + t_0 t_1 + t_1^2) > 0$$

so $\pi_1(\mathbb{R})$ is always surjective.

Note: I think we can also set a_2 to be negative as long as $|a_2| = a_1/2$.
u, v, w, r, s can't all be 0.
Know:

- $\tilde{\Delta}(\mathbb{R}) \neq \emptyset$ imply $\pi(\mathbb{R})$ is surjective
- Current Approach: Find examples where $\pi(\mathbb{R})$ is surjective by setting constraints, then use computer to see find $\tilde{\Delta}$ has any real solutions.