Rationality of Real Conic Bundles with Quartic Discriminant Curve

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Preliminaries

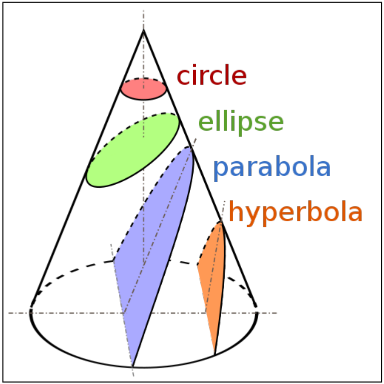
For this talk,

- This talk is intended to be accessible to undergraduates who has taken a first course in Abstract Algebra (ex. MATH1530)
- Unless otherwise specified, we will work over $\mathbb{R}$ as our ground field

Outline

What are Conics?

In Ancient Greece, conics (or conic sections) are defined as the intersection of a cone and a plane, by "slicing" a cone in creative ways.



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[1] Figure taken from
`https://en.wikipedia.org/wiki/File:Conic_Sections.svg`

Affine Space

In Algebraic Geometry, our classical conic sections become part of affine spaces.

Definition

Let $n \geq 0$ an integer, the **affine space** of dimension n is $\mathbb{R}^n$, which we will denote as $\mathbb{A}^n$

Definition

An (affine) **algebraic variety** $V \subset \mathbb{A}^n$ is the set of common $\mathbb{R}$-roots of a collection of polynomials $\{F_i\}_{i \in I}$ where $F_i \in \mathbb{R}[x_1, ..., x_n]$. We write V as

$$V = \mathbb{V}(\{F_i\}_{i \in I})$$

Example of Affine Algebraic Varieties

Example:

In the affine space $\mathbb{A}^n$

- $\mathbb{V}(0) = \mathbb{A}^n$, $\mathbb{V}(1) = \emptyset$

Take $n = 2$, then

- $\mathbb{V}(y - x)$ is the line $y = x$ through $\mathbb{R}^2$
- $\mathbb{V}(y - x^2)$ is the quadratic line $y = x^2$ in $\mathbb{R}^2$
- the classical conic section C is the variety

$$C = \mathbb{V}(ax^2 + by^2 + c + dxy + ey + fx)$$

where $a, b, c, d, e, f \in \mathbb{R}$

Projective Space

The theory of affine varieties is great, but we can generalize conics with what's known as "projective spaces".

Definition

The set of 1-dimensional subspaces of $\mathbb{A}^{n+1}$ is called the **projective space** of dimension n, denoted as $\mathbb{P}^n$. In other words, they are just the set of lines going through the origin in $\mathbb{A}^{n+1}$.

Notations:

- We will denote the line through 0 and $(a_0, ..., a_n)$ as $[a_0 : ... : a_n]$ in $\mathbb{P}^n$.
- Sometimes we will denote $\mathbb{P}^n$ as $\mathbb{P}^n_{[x_0,...,x_n]}$ to emphasize its coordinates.

Why Projective Spaces?

Q: Why do we want to study conics in projective spaces rather than affine spaces?

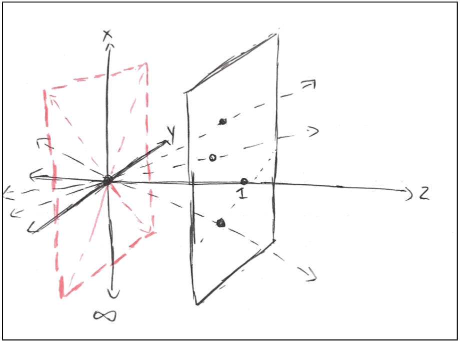
A: There are 2 reasons:

- Geometrically, projective spaces are a natural compactification of affine spaces.
- Algebraically, we can turn conic sections into a class of what's called "homogeneous polynomials", which is generally nicer to work with.

Embedding the Affine Plane

We can embed the affine plane $\mathbb{A}^2$ into $\mathbb{P}^2$ by identifying $\mathbb{A}^2$ with the subset $U_Z = \{[X : Y : Z] \in \mathbb{P}^2 \mid Z \neq 0\}$ via:

$$\varphi_Z : U_Z \to \mathbb{A}^2,\ [X : Y : Z] \mapsto (\frac{X}{Z}, \frac{Y}{Z})$$



This gives a compactification $\mathbb{P}^2 = \mathbb{A}^2 \sqcup \mathbb{P}^1$

Homogeneous Polynomials

Definition

A polynomial $F \in \mathbb{R}[x_0, ..., x_n]$ is called **homogeneous of degree d** if it is a sum of degree d monomials.

For example, in $\mathbb{R}[x, y, z]$,

$$6x^5 + 7y^5 + \pi x^4 y + 3x^2 y^2 z + 9z^5$$

is a homogeneous polynomial of degree 5.

Observation:

Let F be a homogeneous polynomial of degree d and $\lambda \in \mathbb{R}$,

$$F(\lambda a_0, ..., \lambda a_n) = \lambda^d F(a_0, ..., a_n)$$

for all $(a_0, ..., a_n) \in \mathbb{R}^{n+1}$. In particular, if $(a_0, ..., a_n)$ is a root of F, then so is $(\lambda a_0, ..., \lambda a_n)$.

Connection to Conics

Classically, conic sections have been considered as real roots of the polynomial

$$f(x, y) = ax^2 + by^2 + c + dxy + ey + fx \in \mathbb{R}[x, y]$$

With our embedding, we can homogenize $f(x, y)$ into:

$$F(X, Y, Z) = aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ$$

Then we note that on $Z = 1$, $F(X, Y, Z)$ becomes $f(x, y)$. This is in fact a bijective correspondence.

Definition:

A **plane conic** $C \subset \mathbb{P}^2_{[X:Y:Z]}$ is the real roots of a homogeneous polynomial of degree 2 in $\mathbb{R}[X, Y, Z]$.

Matrices and Conics

Take any homogenous polynomial of degree 2

$$F(X, Y, Z) := aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ$$

We note that this polynomial has an associated symmetric matrix

$$M_F = \begin{bmatrix} a & \frac{d}{2} & \frac{f}{2} \\ \frac{d}{2} & b & \frac{e}{2} \\ \frac{f}{2} & \frac{e}{2} & c \end{bmatrix}$$

such that

$$F(X, Y, Z) = \begin{bmatrix} X & Y & Z \end{bmatrix} M_F \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

This also makes $F(X, Y, Z)$ into what's called a quadratic form of 3 variables.

Smoothness of Conics

It turns out that the rank of the matrix M_F determines the geometry of the conic C.

Fact:

Let $C_{\mathbb{C}}$ be all the complex roots of F (in particular $C \subseteq C_{\mathbb{C}}$)

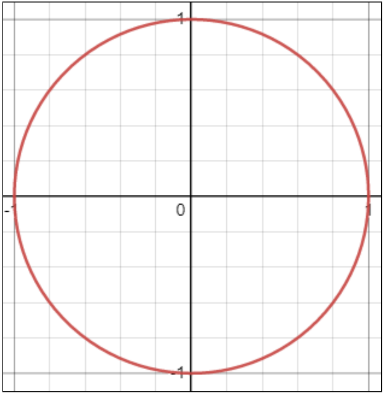
- If M_F has rank 3, then $C_{\mathbb{C}}$ is a smooth conic
- If M_F has rank 2, then $C_{\mathbb{C}}$ is the union of two distinct lines meeting at a point.
- If M_F has rank 1, then $C_{\mathbb{C}}$ is a double line.

Example: $rank(M_F) = 3$

Let $M_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $F(X, Y, Z) = X^2 + Y^2 - Z^2$.

Then C is a smooth conic.

On the chart $(Z \neq 0)$,

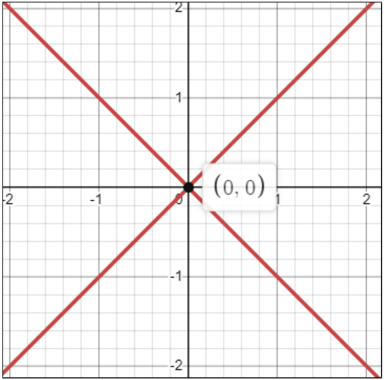


Example: $rank(M_F) = 2$

Let $M_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $F(X, Y, Z) = X^2 - Y^2$.

Then C is the union of two lines meeting at the origin.

On the chart $(Z \neq 0)$,

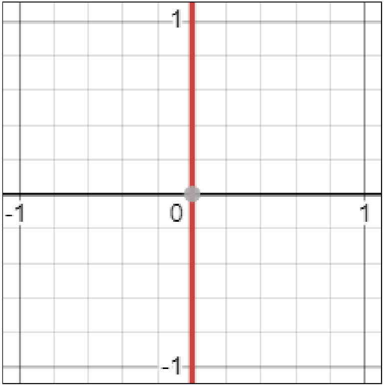


Example: $rank(M_F) = 1$

Let $M_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $F(X,Y,Z) = X^2$.

Then C is a line, we say it's "double" because of the square.

On the chart $(Z \neq 0)$,



Outline

Conic Bundles

Definition:

A **conic bundle** over $\mathbb{P}^2$ is a "nice"[2] morphism $\pi : X \to \mathbb{P}^2$ such that

- X is a smooth variety
- The fiber over every point $p \in \mathbb{P}^2$ is a conic
- The generic fiber is a smooth conic

[2] A proper flat $\mathbb{R}$-morphism

In our research, we are interested in the conic bundle $\pi_2 : Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^2_{[u:v:w]}$ where:

- $Y_{\tilde{\Delta}/\Delta}$ is a variety defined by the equation[3]:

$$z^2 = Q_1(u,v,w)t_0^2 + 2Q_2(u,v,w)t_0t_1 + Q_3(u,v,w)t_1^2$$

- $Q_1, Q_2, Q_3 \in \mathbb{R}[u,v,w]$ are homogenous polynomials of degree 2
- π_2 is the standard projection that forgets z, t_0, and t_1

[3]This looks like a very specific choice, but it turns out that every degree 4 conic bundle $X \to \mathbb{P}^2$ is birationally equivalent to some π_2 "up to a class in $\mathbb{Z}/2\mathbb{Z}$" (Theorem 2.6 of [FJS+] based on [Bru08])

Why is π_2 a conic bundle?

Intuitively, every point in $\mathbb{P}^2$ should correspond to some conic in $Y_{\tilde{\Delta}/\Delta}$.

Example of Fibers for π_2:

Concretely, take the point $[1:2:3] \in \mathbb{P}^2_{[u:v:w]}$, then fiber of $[1:2:3]$ is exactly the solutions satisfying:

$$0 = Q_1(1,2,3)\, t_0^2 + 2Q_2(1,2,3)\, t_0 t_1 + Q_3(1,2,3)\, t_1^2 - z^2$$

This forms a conic in $\mathbb{P}^2_{[t_0:t_1:z]}$.

Quadric Surface Bundle

We are also interested in the map $\pi_1 : Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^1_{[t_0:t_1]}$, which is the standard projection that forgets z, u, v, and w.

Example of Fibers for π_1:

Similarly, take the point $[1:3] \in \mathbb{P}^1_{[t_0:t_1]}(\mathbb{R})$, then fiber of $[1:3]$ is exactly the solutions satisfying:

$$\boxed{z^2} = Q_1(\boxed{u,v,w})(1)^2 + 2Q_2(\boxed{u,v,w})(1)(3) + Q_3(\boxed{u,v,w})(3)^2$$

$$= (1)Q_1(\boxed{u,v,w}) + 2(3)Q_2(\boxed{u,v,w}) + (9)Q_3(\boxed{u,v,w})$$

This forms a degree 2 surface (known as a **quadric**) in $\mathbb{P}^3_{[u:v:w:z]}(\mathbb{R})$.

$\pi_1 : Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^1_{[t_0:t_1]}$ is an example of a **quadric surface bundle**.

Conic Bundles

Putting π_1 and π_2 together, we have the commutative diagram:

$$\begin{tikzcd}
& Y_{\tilde{\Delta}/\Delta} \arrow[d] \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] & \\
& \mathbb{P}^1_{[t_0:t_1]} \times \mathbb{P}^2_{[u:v:w]} \arrow[dl] \arrow[dr] & \\
\mathbb{P}^1_{[t_0:t_1]} & & \mathbb{P}^2_{[u:v:w]}
\end{tikzcd}$$

In this talk, we refer to this as the **double cover model**.

Conic Bundles

This commutative diagram also induces a diagram between **their real points**:

$$\begin{tikzcd}
 & & Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) \arrow[d] \arrow[dll, bend right, "\pi_1(\mathbb{R})"'] \arrow[drr, bend left, "\pi_2(\mathbb{R})"] & & \\
 & & \mathbb{P}^1_{[t_0:t_1]}(\mathbb{R}) \times \mathbb{P}^2_{[u:v:w]}(\mathbb{R}) \arrow[dll] \arrow[drr] & & \\
\mathbb{P}^1_{[t_0:t_1]}(\mathbb{R}) & & & & \mathbb{P}^2_{[u:v:w]}(\mathbb{R})
\end{tikzcd}$$

The Discriminant Curve

We would like to identify if a given fiber of π_2 is smooth:

Smoothness Criterion

Given fixed $[u : v : w] \in \mathbb{P}^2_{[u:v:w]}(\mathbb{R})$, we can rewrite its assoicated conic as:

$$0 = Q_1(u,v,w)t_0^2 + 2Q_2(u,v,w)t_0t_1 + Q_3(u,v,w)t_1^2 + (-1)z^2 \; (*)$$

This gives the symmetric matrix:

$$M = \begin{bmatrix} Q_1(u,v,w) & Q_2(u,v,w) & 0 \\ Q_2(u,v,w) & Q_3(u,v,w) & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The conic $(*)$ is smooth if and only if $\det(M) \neq 0$.

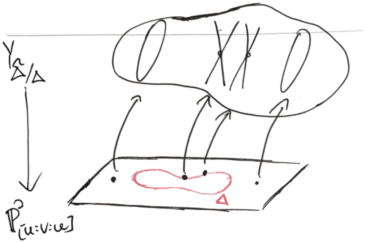
The Discriminant Curve

Smoothness Criterion:

Let M be the symmetric matrix associated to each conic of $Y_{\tilde{\Delta}/\Delta}$, the curve defined by $\det(M) = 0$ is called the **discriminant curve** Δ:

$$\Delta = (Q_1 Q_3 - Q_2^2 = 0) \subset \mathbb{P}^2_{[u:v:w]}$$

The fiber of $s \in \mathbb{P}^2_{[u:v:w]}$ is smooth if and only if $s \notin \Delta$



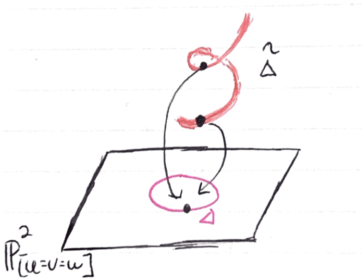
The Double Cover of Δ

Fact:

There exists a curve $\tilde{\Delta} \subset \mathbb{P}^4_{[u:v:w:r:s]}$ defined by

$$\tilde{\Delta} \coloneqq (Q_1 - r^2 = Q_2 - rs = Q_3 - s^2 = 0)$$

such that the projection $\tilde{\Delta} \to \Delta$ is a double cover.
$\tilde{\Delta}$ is called the **double cover** of Δ.



Quartic Plane Curves

$Q_1Q_3 - Q_2^2$ is a degree 4 homogeneous real polynomial.

Definition

The roots of a degree 4 homogenous polynomial over $\mathbb{P}^2$ is known as a **quartic**.

Theorem (Zeuthen, 1874 [Zeu74])

Let Δ be a smooth quartic over $\mathbb{R}$, then $\Delta(\mathbb{R})$ can be classified into 1 of the 6 following topological types:

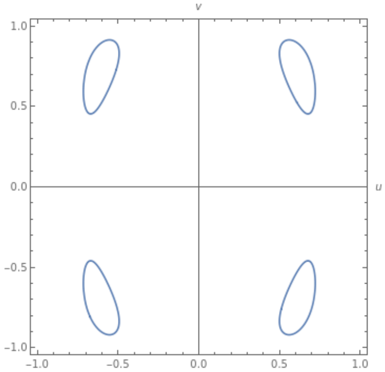
1. No real points
2. One oval
3. Two nested ovals
4. Two non-nested ovals
5. Three ovals
6. Four ovals

Example: Four Ovals

The homogeneous equation defines a smooth quartic whose real component has 4 ovals:

$$0 = -11u^4 - 5u^2v^2 - 2v^4 + 11u^2w^2 + 4v^2w^2 - 3w^4$$

The real components on the chart $(w \neq 0)$

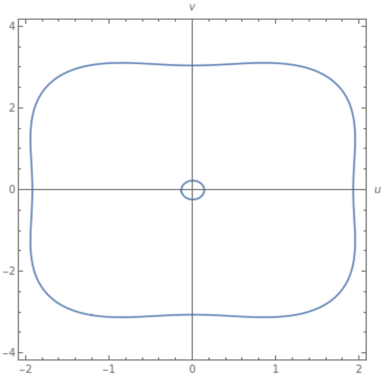


Example: Two Nested Ovals

This homogeneous equation defines a smooth quartic whose real component has 2 nested ovals:

$$0 = -3u^4 - \frac{7}{10}u^2v^2 - \frac{169}{400}v^4 + \frac{67}{6}u^2w^2 + \frac{949}{240}v^2w^2 - \frac{121}{576}w^4$$

The real components on the chart $(w \neq 0)$:



Outline

Overview of Rationality

- A variety X is k-rational over a field k if there exists non-empty open sets $U \subset X$ and $V \subset \mathbb{P}_k^{\dim X}$ such that U and V are isomorphic over k.
- If X is not rational over k, we say that X is irrational over k.

There are two relevant facts about rationality:

- **Lang–Nishimura Lemma:** If X is a projective k-rational variety, then $X(k)$ is non-empty.
- **General Topological Fact:** If X is a smooth projective $\mathbb{R}$-rational variety, then $X(\mathbb{R})$ is connected.

In particular,

$$Y_{\tilde{\Delta}/\Delta} \text{ is } \mathbb{R}\text{-rational} \implies Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) \text{ is non-empty and connected}$$

Why Rationality

Q: Why do we care about rationality?

A: Given an algebraic variety, a natural question to ask is how simple it is:

- One notion of simplicity is its closeness to projective spaces.
- It turns out that rational algebraic varieties are "birationally equivalent" to projective spaces.
- Conic bundles have been a very rich source of examples of varieties with varying levels of similarity to projective space.

Criterion for Rationality

The $\mathbb{C}$-rationality of $Y_{\tilde{\Delta}/\Delta}$ is quite clear:

Theorem (Iskovskikh, 1987 [Isk87])

$Y_{\tilde{\Delta}/\Delta}$ is always $\mathbb{C}$-rational

The $\mathbb{R}$-rationality of $Y_{\tilde{\Delta}/\Delta}$ is more complicated, but the work of S. Frei, L. Ji, S. Sankar, B. Viray, and I. Vogt previously gave a very useful criterion:

Proposition (Proposition 6.1, Frei et al, 2022 [FJS+])

Let $Y_{\tilde{\Delta}/\Delta}$ be defined as before. If $\tilde{\Delta}(\mathbb{R}) \neq \emptyset$, then $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational

In our REU, we instead investigate what happens when $\tilde{\Delta}(\mathbb{R}) = \emptyset$.

Outline

Main Result 1

Theorem (Ji and Ji, 2022 [JJ22]):

Let $\mathcal{CB}_{\emptyset/*}$ denote the set of real conic bundles $X \to \mathbb{P}^2$ of the form $Y_{\tilde{\Delta}/\Delta}$ with smooth Δ of topological type $*$ and smooth $\tilde{\Delta}$ that has no real points,

1. $\mathcal{CB}_{\emptyset/\emptyset}$ contains both rational and irrational members
2. $\mathcal{CB}_{\emptyset/\text{1-oval}}$ contains both rational and irrational members[4]
3. $\mathcal{CB}_{\emptyset/\text{2 non-nested ovals}}$, $\mathcal{CB}_{\emptyset/\text{2 nested ovals}}$, and $\mathcal{CB}_{\emptyset/\text{3-ovals}}$ contains both rational and irrational members[5]
4. $\mathcal{CB}_{\emptyset/\text{4-ovals}}$ contains only rational members[6].

[4] The case of irrational 1 oval was done in [FJS$^+$]

[5] The case of irrational 2 non-nested ovals was done in [FJS$^+$]

[6] We proved this based on ideas joint with S. Frei, S. Sankar, B. Viray, and I. Vogt

Rational Examples

It turns out that there's another rationality construction for rational examples when $\tilde{\Delta}(\mathbb{R})$ is empty:

Theorem (Witt, 1937 [Wit37])

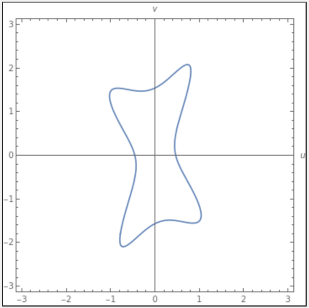
If $\pi_1 : Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^1_{[t_0:t_1]}$ is surjective on real points, then $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational.

Using this criterion, we can find and check that rational members exist for all topological types of $\Delta(\mathbb{R})$.

Rational Examples

Example of Rational One Oval with $\tilde{\Delta}(\mathbb{R}) = \emptyset$

Take $Q_1 := -u^2 + uv - w^2, Q_2 := 3u^2 + uv - v^2 + w^2$, and $Q_3 = -u^2 - 2uv - 2w^2$, then one can verify that $\tilde{\Delta}(\mathbb{R}) = \emptyset$, π_1 is surjective on real points, and $\Delta(\mathbb{R})$ is one oval, as seen on the chart $(w \neq 0)$:



Now we will move on to irrational conic bundles:

- Recall that if $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational, then $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected and non-empty.
- What if we take $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ to be disconnected?

The key insight is looking at $\pi_1(\mathbb{R}) : Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) \to \mathbb{P}^1_{[t_0:t_1]}(\mathbb{R})$:

$$Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) \text{ is disconnected } \iff \pi_1(Y_{\tilde{\Delta}/\Delta}(\mathbb{R})) \text{ is disconnected}$$

Key Observation

Observation:

Take any point $[a : b] \in \mathbb{P}^1_{[t_0:t_1]}(\mathbb{R})$, then fiber of $[a : b]$ is exactly the solutions satisfying:

$$z^2 = Q_1(u, v, w)(a)^2 + 2Q_2(u, v, w)(ab) + Q_3(u, v, w)(b)^2$$

This equation has a solution if and only if the matrix

$$M_{[a:b]} := \left[\begin{array}{c|c} M_1 a^2 + 2M_2 ab + M_3 b^2 & 0 \\ \hline 0 & -1 \end{array}\right]$$

is **NOT** negative definite, where M_1, M_2, M_3 are the symmetric matrices associated to Q_1, Q_2, Q_3 respectively.

Key Observation (Continued)

We will denote the signature of a real symmetric matrix M as (p, n), where p is the number of positive eigenvalues and n is the number of negative eigenvalues.

Lemma (Adapted from Degtjarev Et al., [DIK00])

Given the setup with $Y_{\tilde{\Delta}/\Delta}$, let $M_{[t_0:t_1]}$ be the matrix defined the same as previously, then the signature of $M_{[t_0:t_1]}$ can only change by ± 1 at real solutions of a degree 6 polynomial[7]

$$\Gamma(t) := -det(M_1 t^2 + 2M_2 t + M_3)$$

Notably, on each interval defined by real points of $\Gamma(t)$, the signature of $M_{[t_0:t_1]}$ stays the same.

[7] It is possible that one of the roots is the point at infinity, but we can without loss choose an appropriate basis to avoid this.

Disconnected Examples

Now, as long as we can find examples where at least two non-adjacent intervals with signature $(0, 4)$, this will produce a disconnected $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$.

Irrational Example with $\Delta(\mathbb{R})$ 3 ovals

Define $Q_1 := -u^2 - v^2 - w^2$, $Q_2 := u^2 + 5v^2 + 9w^2$, and $Q_3 := -24v^2 - 80w^2$. Then we have that

$$\Gamma(t) = t^6 - 30t^5 + 340t^4 - 1800t^3 + 4384t^2 - 3840t$$

with 6 real roots

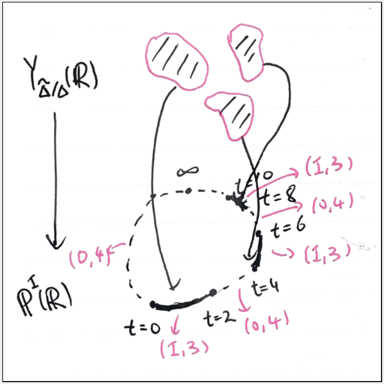
$$t = 0, 2, 4, 6, 8, 10$$

Disconnected Examples

Example (Continued)

The signatures of $M_{[t_0:t_1]}$ follow the pattern:

$$(0,4),(1,3),(0,4),(1,3),(0,4),(1,3)$$



The Explicit Construction

The choice of Q_1, Q_2, Q_3 in the previous example may seem arbitrary, but you can "reverse engineer" the appropriate Q_1, Q_2, Q_3 by considering

$$\begin{bmatrix} a_0t^2 + b_0t + c_0 & 0 & 0 \\ 0 & a_1t^2 + b_1t + c_1 & 0 \\ 0 & 0 & a_2t^2 + b_2t + c_2 \end{bmatrix}$$

$$= t^2(M_1) + t(2M_2) + M_3$$

and compare the coefficients.

Limitations of the Topological Obstruction

During the REU, we also showed that

Proposition (Ji and Ji, 2022 [JJ22])

If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is disconnected, then $\Delta(\mathbb{R})$ is either 2 non-nested ovals, 2 nested ovals, or 3 ovals. More precisely,

- $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has at most 3 connected components
- If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has 2 connected components, then $\Delta(\mathbb{R})$ is either 2 non-nested ovals or 2 nested ovals
- If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has 3 connected components, then $\Delta(\mathbb{R})$ is 3 ovals.

Each case described above does occur.

Remark:

When $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has 2 connected components, the signatures of $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is one of

1. $(0,4),(1,3),(0,4),(1,3),(2,2),(1,3)$
2. $(0,4),(1,3),(0,4),(1,3)$

If pattern (1) occurs, then $\Delta(\mathbb{R})$ is 2 nested ovals.

If pattern (2) occurs, $\Delta(\mathbb{R})$ has always been 2 non-nested ovals experimentally. It's unknown if the pattern (2) implies that $\Delta(\mathbb{R})$ is 2 non-nested ovals.

The Case where $Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) = \emptyset$

- Recall that if $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational, then $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected and non-empty.
- What if we take $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ to be empty instead?

Irrational Example where $\Delta(\mathbb{R}) = \emptyset$

Define $Q_1 := -2u^2 - 3v^2 - 5w^2$, $Q_2 := u^2 + 2v^2 + 3w^2$, and $Q_3 := -2u^2 - 4v^2 - 2w^2$. Then one can verify that the associated $\Delta(\mathbb{R})$ and $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ are both empty, hence $Y_{\tilde{\Delta}/\Delta}$ is irrational over $\mathbb{R}$.

This doesn't get us really far, as $Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) = \emptyset$ implies that $\Delta(\mathbb{R}) = \emptyset$.

Irrational One Oval

S. Frei, L. Ji, S. Sankar, B. Viray, and I. Vogt showed that

Irrational Example (Theorem 1.3(2), [FJS+])

Define $Q_1 := -u^2 - v^2 - 3w^2$, $Q_2 := 3u^2 + 5v^2$, and $Q_3 := -7u^2 - 23v^2 - 12w^2$, then $\Delta(\mathbb{R})$ is one oval, and the associated $Y_{\tilde{\Delta}/\Delta}$ is irrational over any subfield of $\mathbb{R}$.

The main idea behind showing this is irrational relied on what's known as an Intermediate Jacobian Torsor (IJT) obstruction:

- Classically, this has been done over $\mathbb{C}$ by Clemens–Grifiths (1972, [CG72]).
- Over non-closed fields, a refinement of this technique was introduced by Hassett–Tschinkel (2021, [HT21]) and Benoist–Wittenberg (2019, [BW]).

Outline

Main Result 2

A natural question from our discovery would be to characterize the rationality of $Y_{\tilde{\Delta}/\Delta}$ in each topological type of $\Delta(\mathbb{R})$. So far, we have showed that[8]:

Theorem (Ji and Ji, 2022 [JJ22]):

Let $Y_{\tilde{\Delta}/\Delta}$ be as before,

1. If $\Delta(\mathbb{R})$ is 4 ovals, then $Y_{\tilde{\Delta}/\Delta}$ is rational
2. If $\Delta(\mathbb{R})$ is 3 ovals, then $Y_{\tilde{\Delta}/\Delta}$ is rational $\iff$ $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected
3. If $\Delta(\mathbb{R})$ is 2 non-nested ovals, then $Y_{\tilde{\Delta}/\Delta}$ is rational $\iff$ $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected and the IJT obstruction vanishes
4. If $\Delta(\mathbb{R})$ is empty, then $Y_{\tilde{\Delta}/\Delta}$ is rational $\iff$ $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is non-empty and the IJT obstruction vanishes

[8] (1) and (2) were based on or inspired by ideas joint with S. Frei, S. Sankar, B. Viray, and I. Vogt

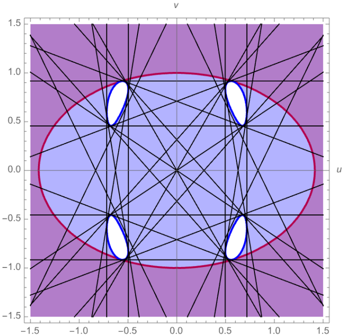
Proof Sketch

For (1) and (2),

Fact:

Every smooth real quartic $\Delta(\mathbb{R})$ has 28 bitangents.

- When $\Delta(\mathbb{R})$ is 4 ovals, it has 28 real bitangents.
- When $\Delta(\mathbb{R})$ is 3 ovals, it has 16 real bitangents.



Proof Sketch

Define W to be the sub-variety of $Y_{\tilde{\Delta}/\Delta}$ given by the equation

$$0 = Q_1(u,v,w)t_0^2 + 2Q_2(u,v,w)t_0t_1 + Q_3(u,v,w)t_1^2$$

There's an induced projection map $\pi_1' : W \to \mathbb{P}^1$ defined similar to π_1.

Viewing W as $W_{\mathbb{C}}$, $W_{\mathbb{C}}$ has 56 geometric lines:

- 32 of those lines each give a geometric section of π_1', call this collection E
- Note that if π_1' has a section over $\mathbb{R}$, then $\pi_1(\mathbb{R})$ is surjective, hence $Y_{\tilde{\Delta}/\Delta}$ is rational

In particular, if $(Q_1Q_3 - Q_2^2 < 0)_{\mathbb{R}}$ contains some non-orientable part of $\mathbb{P}^2$, then each bitangent of $\Delta(\mathbb{R})$ would correspond to two lines in $W_{\mathbb{C}}$.

Proof Sketch

Fact:

If $\Delta(\mathbb{R})$ is 4 ovals or 3 ovals with $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected, then $(Q_1Q_3 - Q_2^2 < 0)_{\mathbb{R}}$ is not orientable.

Then the rest is really a combinatorics argument, as there'd be some real bitangent of $\Delta(\mathbb{R})$ that will end up giving a line in E, which corresponds to a section over $\mathbb{R}$.

Proof Sketch

For (3) and (4), the forward directions are already given. For the converse, we rely on the following proposition

Proposition (Ji and Ji, 2022 [JJ22]):

If $(Q_1Q_3 - Q_2^2 < 0)_{\mathbb{R}}$ contains some non-orientable part of $\mathbb{P}^2$,

$\pi_1(\mathbb{R})$ is surjective $\iff$ the IJT obstruction vanishes for Y

- For both (3) and (4), one can show that the converse conditions forces $\pi_1(\mathbb{R})$ to be surjective, hence $Y_{\tilde{\Delta}/\Delta}$ is rational.
- We leave this as an exercise to the audience.

Outline

Next Steps

The next steps would be to characterize the rationality when $\Delta(\mathbb{R})$ is two nested ovals or one oval.

Experimentally, if $\Delta(\mathbb{R})$ is 2 nested ovals, we are confident that the following statement is most likely true:

- $Y_{\tilde{\Delta}/\Delta}$ is rational $\iff$ the IJT obstruction vanishes $\iff$ $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected

We actually already know that:

- $Y_{\tilde{\Delta}/\Delta}$ is rational $\implies$ the IJT obstruction vanishes $\implies$ $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected

The case when $\Delta(\mathbb{R})$ is one oval is a lot more complicated.

- Experimentally, when π_1 has signautre $(0, 4), (1, 3)$, $\Delta(\mathbb{R})$ is one oval, $\tilde{\Delta}(\mathbb{R}) = \emptyset$, $(Q_1 Q_3 - Q_2^2 < 0)$ is inside the oval
- This situation is tricky because all of our current tools of rationality and irrationality constructions do not satisfy this scenario.

More Information

If you are interested to learn more, please check out our paper:

- Lena Ji and Mattie Ji. Rationality of real conic bundles with quartic discriminant curve, 2022 (https://arxiv.org/abs/2208.08916)

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