Lemma 0.1. Let $Y_{\tilde{\Delta}/\Delta}$ be determined by

$$z^2 = Q_1(u,v,w)t_0^2 + 2Q_2(u,v,w)t_0t_1 + Q_3(u,v,w)t_1^2$$

Let $X_{\tilde{\Delta}/\Delta}$ be determined by

$$z^2 = Q_3(u,v,w)t_0^2 + 2Q_2(u,v,w)t_0t_1 + Q_1(u,v,w)t_1^2$$

Then $Y(\mathbb{R})$ is connected if and only if $X(\mathbb{R})$ is connected.

Proof. The map $f : Y(\mathbb{R}) \to X(\mathbb{R})$ that swaps t_0 and t_1 is a homeomorphism. □

Lemma 0.2. $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \{p \in \mathbb{P}^2_{[u:v:w]}(\mathbb{R}) \mid Q_1(p) \geqslant 0 \text{ or } \Delta(p) \leqslant 0\}$

Proposition 0.3. If the topological type of $\Delta(\mathbb{R})$ is one oval, then $Y(\mathbb{R})$ is connected.

Proof. It suffices for us to show that $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ is connected, since every fiber of the image is connected and π_2 is a close map.

If Q_1 is positive definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \mathbb{P}^2_{[u:v:w]}(\mathbb{R})$ is connected.

If Q_1 is negative definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \{p \in \mathbb{P}^2_{[u:v:w]}(\mathbb{R}) \mid \Delta(p) \leqslant 0\}$, which is connected since $\Delta(\mathbb{R})$ is one oval, so both $\Delta(\mathbb{R})$ and its complement is connected.

If Q_1 is indefinite, assume for the sake of contradiction that $Y(\mathbb{R})$ is disconnected, since $(Q_1 \geqslant 0)$ and $(\Delta(p) \leqslant 0)$ are both connected, it has to be the case that $\pi_2(Y(\mathbb{R}))$ has 2 connected components being $(Q_1 \geqslant 0)$ and $(\Delta(p) \leqslant 0)$.

Now consider the set $(Q_3 \geqslant 0)$. If Q_3 is negative-definite or positive definite, then we are done by just substituting Q_1 with Q_3 and Q_3 with Q_1 using Lemma 0.1. Now if Q_3 is indefinite, then $(Q_3 \geqslant 0)$ is a connected non-empty set.

Let $S_1 = (Q_1 \geqslant 0)$ and $S_2 = (\Delta(p) \leqslant 0)$, we note that for all $p \in \mathbb{P}^2(\mathbb{R})\backslash(S_1 \cup S_2)$, $\Delta(p) > 0$ and $Q_1(p) \leqslant 0$. So in particular $Q_1(p)Q_3(p) = \Delta(p) + Q_2(p)^2 > 0$ implies that $Q_3(p) < 0$.

Now for points in S_1 and S_2, we note that Q_3 cannot have both $p \in S_1$ and $q \in S_2$ such that $Q_3(p) \geqslant 0$ and $Q_3(q) \geqslant 0$, because this would imply that $(Q_3 \geqslant 0)$ has two connected components. Thus, $(Q_3 \geqslant 0)$ can only be contained in one of S_1 and S_2.

If $(Q_3 \geqslant 0)$ is contained in S_2, then we can again make the substitution of Q_1 with Q_3 and Q_3 and Q_1 using Lemma 0.1.

If $(Q_3 \geqslant 0)$ is contained in S_1, we note that for all point p in $(Q_1 > 0)$, $Q_1(p) > 0$ and $\Delta(p) = Q_1(p)Q_3(p) - Q_2^2(p) > 0$, so $Q_1(p)Q_3(p) > Q_2^2(p) \geqslant 0$ implies that $Q_3(p) > 0$. Therefore, we know that for all $p \in (Q_1 > 0)$, $Q_3(p) > 0$, and for all $p \in (Q_1 < 0)$, $Q_3(p) < 0$.

But since Q_3 is smooth, there has to be some point $q \in (Q_1 = 0)$ such that $Q_3(q) = 0$, but this means that at that point. $Q_1(q) = Q_3(q) = 0$, so $\Delta(q) = 0 - Q_2(q)^2 \leqslant 0$, so $q \in (Q_1 \geqslant 0) \cap (\Delta \leqslant 0)$, which is a contradiction.

Thus, we conclude that $Y(\mathbb{R})$ is connected. □