Rationality of Real Conic Bundles with Quartic Discriminant Curve

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Outline

Notations

Throughout this talk,

- Unless otherwise specified, we will work over $\mathbb{R}$ as our ground field
- We will denote the projective n-space over the ground field as $\mathbb{P}^n$
- Sometimes, to emphasize the coordinates $[X_0 : \dots : X_n]$ of $\mathbb{P}^n$, we will denote $\mathbb{P}^n$ as $\mathbb{P}^n_{[X_0 : \dots : X_n]}$

Overview of Conics

- A plane conic $C \subset \mathbb{P}^2$ is the curve defined

$$C := (F(X,Y,Z) = 0)$$

  where
  $F(X,Y,Z) = aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ$
  is a homogeneous polynomial of degree 2 in $\mathbb{R}[X,Y,Z]$
- This C has an associated symmetric matrix M_F

$$M_F := \begin{bmatrix} a & \frac{d}{2} & \frac{f}{2} \\ \frac{d}{2} & b & \frac{e}{2} \\ \frac{f}{2} & \frac{e}{2} & c \end{bmatrix}$$

  such that

$$F(X,Y,Z) = \begin{bmatrix} X & Y & Z \end{bmatrix} M_F \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

- C is smooth if and only if M_F has rank 3

Conic Bundles

Definition:

A **conic bundle** over $\mathbb{P}^2$ is a "nice"[1] morphism $\pi : X \to \mathbb{P}^2$ such that

- X is a smooth variety
- The fiber over every point $p \in \mathbb{P}^2$ is a conic
- The generic fiber is a smooth conic

[1] A proper flat $\mathbb{R}$-morphism

Conic Bundles

In our research, we are interested in the conic bundle $\pi_2 : Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^2_{[u:v:w]}$ where:

- $Y_{\tilde{\Delta}/\Delta}$ is a variety defined by the equation[2]:

$$z^2 = Q_1(u,v,w)t_0^2 + 2Q_2(u,v,w)t_0t_1 + Q_3(u,v,w)t_1^2$$

- $Q_1, Q_2, Q_3 \in \mathbb{R}[u,v,w]$ are homogenous polynomials of degree 2
- π_2 is the standard projection that forgets z, t_0, and t_1

We are also interested in the map $\pi_1 : Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^1_{[t_0:t_1]}$, which is the standard projection that forgets z, u, v, and w.

[2] This looks like a very specific choice, but it turns out that every degree 4 conic bundle $X \to \mathbb{P}^2$ is birationally equivalent to some π_2 "up to a class in $\mathbb{Z}/2\mathbb{Z}$" (Theorem 2.6 of [FJS+] based on [Bru08])

Why is π_2 a conic bundle?

Intuitively, every point in $\mathbb{P}^2$ should correspond to some conic in $Y_{\tilde{\Delta}/\Delta}$.

Example of Fibers for π_2:

Concretely, take the point $[1:2:3] \in \mathbb{P}^2_{[u:v:w]}$, then fiber of $[1:2:3]$ is exactly the solutions satisfying:

$$0 = Q_1(1,2,3)\,t_0^2 + 2Q_2(1,2,3)\,t_0t_1 + Q_3(1,2,3)\,t_1^2 - z^2$$

This forms a conic in $\mathbb{P}^2_{[t_0:t_1:z]}$.

Putting π_1 and π_2 together, we have the commutative diagram:

$$\begin{tikzcd}
& Y_{\tilde{\Delta}/\Delta} \arrow[dl, bend right, "\pi_1"'] \arrow[d] \arrow[dr, bend left, "\pi_2"] & \\
& \mathbb{P}^1_{[t_0:t_1]} \times \mathbb{P}^2_{[u:v:w]} \arrow[dl] \arrow[dr] & \\
\mathbb{P}^1_{[t_0:t_1]} & & \mathbb{P}^2_{[u:v:w]}
\end{tikzcd}$$

In this talk, we refer to this as the **double cover model**.

Conic Bundles

This commutative diagram also induces a diagram between **their real points**:

$$\begin{tikzcd}
 & Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) \arrow[d] \arrow[ddl, bend right, "\pi_1(\mathbb{R})"'] \arrow[ddr, bend left, "\pi_2(\mathbb{R})"] & \\
 & \mathbb{P}^1_{[t_0:t_1]}(\mathbb{R}) \times \mathbb{P}^2_{[u:v:w]}(\mathbb{R}) \arrow[dl] \arrow[dr] & \\
\mathbb{P}^1_{[t_0:t_1]}(\mathbb{R}) & & \mathbb{P}^2_{[u:v:w]}(\mathbb{R})
\end{tikzcd}$$

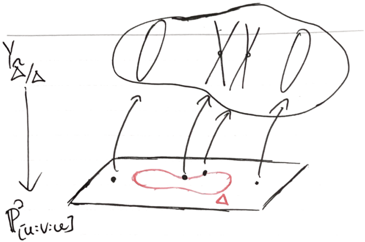
The Discriminant Curve

Smoothness Criterion:

Let M be the symmetric matrix associated to each conic of $Y_{\tilde{\Delta}/\Delta}$, the curve defined by $\det(M) = 0$ is called the **discriminant curve** Δ:

$$\Delta = (Q_1 Q_3 - Q_2^2 = 0) \subset \mathbb{P}^2_{[u:v:w]}$$

The fiber of $s \in \mathbb{P}^2_{[u:v:w]}$ is smooth if and only if $s \notin \Delta$



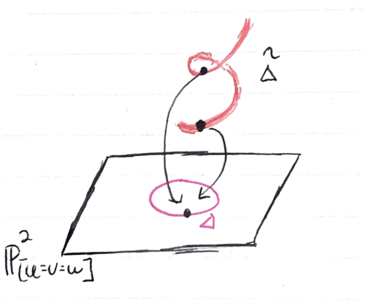
The Double Cover of Δ

Fact:

There exists a curve $\tilde{\Delta} \subset \mathbb{P}^4_{[u:v:w:r:s]}$ defined by

$$\tilde{\Delta} := (Q_1 - r^2 = Q_2 - rs = Q_3 - s^2 = 0)$$

such that the projection $\tilde{\Delta} \to \Delta$ is a double cover.
$\tilde{\Delta}$ is called the **double cover** of Δ.



Quartic Plane Curves

$Q_1Q_3 - Q_2^2$ is a degree 4 homogeneous real polynomial.

Definition

The roots of a degree 4 homogenous polynomial over $\mathbb{P}^2$ is known as a **quartic**.

Theorem (Zeuthen, 1874 [Zeu74])

Let Δ be a smooth quartic over $\mathbb{R}$, then $\Delta(\mathbb{R})$ can be classified into 1 of the 6 following topological types:

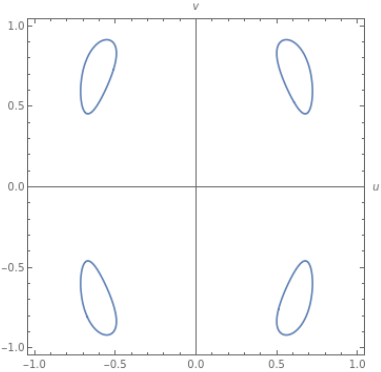
1. No real points
2. One oval
3. Two nested ovals
4. Two non-nested ovals
5. Three ovals
6. Four ovals

Example: Four Ovals

The homogeneous equation defines a smooth quartic whose real component has 4 ovals:

$$0 = -11u^4 - 5u^2v^2 - 2v^4 + 11u^2w^2 + 4v^2w^2 - 3w^4$$

The real components on the chart $(w \neq 0)$

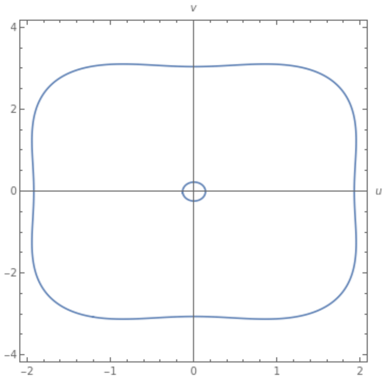


Example: Two Nested Ovals

This homogeneous equation defines a smooth quartic whose real component has 2 nested ovals:

$$0 = -3u^4 - \frac{7}{10}u^2v^2 - \frac{169}{400}v^4 + \frac{67}{6}u^2w^2 + \frac{949}{240}v^2w^2 - \frac{121}{576}w^4$$

The real components on the chart $(w \neq 0)$:



Overview of Rationality

- A variety X is k-rational over a field k if there exists non-empty open sets $U \subset X$ and $V \subset \mathbb{P}_k^{\dim X}$ such that U and V are isomorphic over k.
- If X is not rational over k, we say that X is irrational over k.

There are two relevant facts about rationality:

- **Lang–Nishimura Lemma:** If X is a projective k-rational variety, then $X(k)$ is non-empty.
- **General Topological Fact:** If X is a smooth projective $\mathbb{R}$-rational variety, then $X(\mathbb{R})$ is connected.

In particular,

$$Y_{\tilde{\Delta}/\Delta} \text{ is } \mathbb{R}\text{-rational} \implies Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) \text{ is non-empty and connected}$$

Criterion for Rationality

The $\mathbb{C}$-rationality of $Y_{\tilde{\Delta}/\Delta}$ is quite clear:

Theorem (Iskovskikh, 1987 [Isk87])

$Y_{\tilde{\Delta}/\Delta}$ is always $\mathbb{C}$-rational

The $\mathbb{R}$-rationality of $Y_{\tilde{\Delta}/\Delta}$ is more complicated, but the work of S. Frei, L. Ji, S. Sankar, B. Viray, and I. Vogt previously gave a very useful criterion:

Proposition (Proposition 6.1, [FJS$^+$])

Let $Y_{\tilde{\Delta}/\Delta}$ be defined as before. If $\tilde{\Delta}(\mathbb{R}) \neq \emptyset$, then $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational

In our REU, we instead investigate what happens when $\tilde{\Delta}(\mathbb{R}) = \emptyset$.

Outline

Our Main Result

Theorem:

Let $\mathcal{CB}_{\emptyset/*}$ denote the set of real conic bundles $X \to \mathbb{P}^2$ of the form $Y_{\tilde{\Delta}/\Delta}$ with smooth Δ of topological type $*$ and smooth $\tilde{\Delta}$ that has no real points,

1. $\mathcal{CB}_{\emptyset/\emptyset}$ contains both rational and irrational members
2. $\mathcal{CB}_{\emptyset/\text{1-oval}}$ contains both rational and irrational members[3]
3. $\mathcal{CB}_{\emptyset/\text{2 non-nested ovals}}$, $\mathcal{CB}_{\emptyset/\text{2 nested ovals}}$, and $\mathcal{CB}_{\emptyset/\text{3-ovals}}$ contains both rational and irrational members[4]
4. $\mathcal{CB}_{\emptyset/\text{4-ovals}}$ contains only rational members[5].

[3] The case of irrational 1 oval was done in [FJS$^+$]

[4] The case of irrational 2 non-nested ovals was done in [FJS$^+$]

[5] We proved this based on ideas joint with S. Frei, S. Sankar, B. Viray, and I. Vogt

Rational Examples

It turns out that there's another rationality construction for rational examples when $\tilde{\Delta}(\mathbb{R})$ is empty:

Theorem (Witt, 1937 [Wit37])

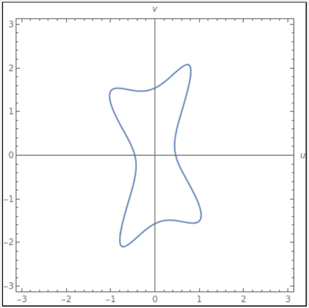
If $\pi_1 : Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^1_{[t_0:t_1]}$ is surjective on real points, then $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational.

Using this criterion, we can find and check that rational members exist for all topological types of $\Delta(\mathbb{R})$.

Rational Examples

Example of Rational One Oval with $\tilde{\Delta}(\mathbb{R}) = \emptyset$

Take $Q_1 := -u^2 + uv - w^2, Q_2 := 3u^2 + uv - v^2 + w^2$, and $Q_3 = -u^2 - 2uv - 2w^2$, then one can verify that $\tilde{\Delta}(\mathbb{R}) = \emptyset$, π_1 is surjective on real points, and $\Delta(\mathbb{R})$ is one oval, as seen on the chart $(w \neq 0)$:



Outline

Obstruction by Disconnected Real Loci

- Recall that if $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational, then $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected and non-empty.
- What if we take $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ to be disconnected?

The key insight is looking at $\pi_1(\mathbb{R}) : Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) \to \mathbb{P}^1_{[t_0:t_1]}(\mathbb{R})$:

$$Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) \text{ is disconnected } \iff \pi_1(Y_{\tilde{\Delta}/\Delta}(\mathbb{R})) \text{ is disconnected}$$

Key Observation

Observation:

Take any point $[a:b] \in \mathbb{P}^1_{[t_0:t_1]}(\mathbb{R})$, then fiber of $[a:b]$ is exactly the solutions satisfying:

$$z^2 = Q_1(u,v,w)(a)^2 + 2Q_2(u,v,w)(ab) + Q_3(u,v,w)(b)^2$$

This equation has a solution if and only if the matrix

$$M_{[a:b]} := \left[\begin{array}{c|c} M_1a^2 + 2M_2ab + M_3b^2 & 0 \\ \hline 0 & -1 \end{array}\right]$$

is **NOT** negative definite, where M_1, M_2, M_3 are the symmetric matrices associated to Q_1, Q_2, Q_3 respectively.

Key Observation (Continued)

We will denote the signature of a real symmetric matrix M as (p, n), where p is the number of positive eigenvalues and n is the number of negative eigenvalues.

Lemma (Adapted from Degtjarev Et al., [DIK00])

Given the setup with $Y_{\tilde{\Delta}/\Delta}$, let $M_{[t_0:t_1]}$ be the matrix defined the same as previously, then the signature of $M_{[t_0:t_1]}$ can only change by ± 1 at real solutions of a degree 6 polynomial[6]

$$\Gamma(t) := -det(M_1 t^2 + 2M_2 t + M_3)$$

Notably, on each interval defined by real points of $\Gamma(t)$, the signature of $M_{[t_0:t_1]}$ stays the same.

[6] It is possible that one of the roots is the point at infinity, but we can without loss choose an appropriate basis to avoid this.

Disconnected Examples

Now, as long as we can find examples where at least two non-adjacent intervals with signature $(0,4)$, this will produce a disconnected $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$.

Irrational Example with $\Delta(\mathbb{R})$ 3 ovals

Define $Q_1 := -u^2 - v^2 - w^2$, $Q_2 := u^2 + 5v^2 + 9w^2$, and $Q_3 := -24v^2 - 80w^2$. Then we have that

$$\Gamma(t) = t^6 - 30t^5 + 340t^4 - 1800t^3 + 4384t^2 - 3840t$$

with 6 real roots

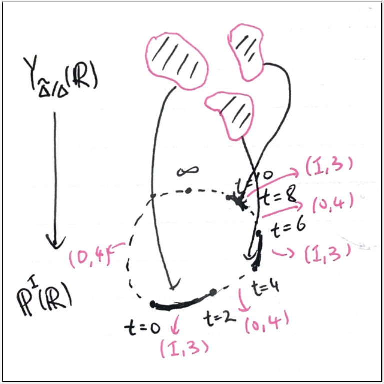
$$t = 0, 2, 4, 6, 8, 10$$

Disconnected Examples

Example (Continued)

The signatures of $M_{[t_0:t_1]}$ follow the pattern:

$$(0,4),(1,3),(0,4),(1,3),(0,4),(1,3)$$



Limitations of the Topological Obstruction

During the REU, we also showed that

Proposition:

If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is disconnected, then $\Delta(\mathbb{R})$ is either 2 non-nested ovals, 2 nested ovals, or 3 ovals. More precisely,

- $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has at most 3 connected components
- If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has 2 connected components, then $\Delta(\mathbb{R})$ is either 2 non-nested ovals or 2 nested ovals
- If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has 3 connected components, then $\Delta(\mathbb{R})$ is 3 ovals.

Each case described above does occur.

The Case where $Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) = \emptyset$

- Recall that if $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational, then $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected and non-empty.
- What if we take $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ to be empty instead?

Irrational Example where $\Delta(\mathbb{R}) = \emptyset$

Define $Q_1 := -2u^2 - 3v^2 - 5w^2$, $Q_2 := u^2 + 2v^2 + 3w^2$, and $Q_3 := -2u^2 - 4v^2 - 2w^2$. Then one can verify that the associated $\Delta(\mathbb{R})$ and $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ are both empty, hence $Y_{\tilde{\Delta}/\Delta}$ is irrational over $\mathbb{R}$.

This doesn't get us really far, as $Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) = \emptyset$ implies that $\Delta(\mathbb{R}) = \emptyset$.

Irrational One Oval

Question:

What about the case for one oval?

S. Frei, L. Ji, S. Sankar, B. Viray, and I. Vogt showed that

Irrational Example (Theorem 1.3(2), [FJS$^+$])

Define $Q_1 := -u^2 - v^2 - 3w^2$, $Q_2 := 3u^2 + 5v^2$, and $Q_3 := -7u^2 - 23v^2 - 12w^2$, then $\Delta(\mathbb{R})$ is one oval, and the associated $Y_{\tilde{\Delta}/\Delta}$ is irrational over any subfield of $\mathbb{R}$.

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