1 $\Delta(\mathbb{R})$ being Empty implies $Y(\mathbb{R})$ is Connected

Lemma 1.1. $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \{p \in \mathbb{P}^2_{[u:v:w]}(\mathbb{R}) \mid Q_1(p) \geqslant 0 \text{ or } \Delta(p) \leqslant 0\} = (Q_1 \geqslant 0) \cup (\Delta \leqslant 0)$

Proposition 1.2. If the topological type of $\Delta(\mathbb{R})$ is empty, then $Y(\mathbb{R})$ is Connected.

Proof. It suffices for us to show that $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ is connected, since every fiber of the image is connected and π_2 is a close map.

Since $\Delta(\mathbb{R})$ is empty, Δ has no real roots, so either $\Delta > 0$ or $\Delta < 0$ for all $p \in \mathbb{P}^2_{[u;v;w]}(\mathbb{R})$.

If $\Delta < 0$, then from Lemma 1.1 $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \mathbb{P}^2_{[u;v;w]}(\mathbb{R})$ is connected.

If $\Delta > 0$, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = (Q_1 \geqslant 0)$.

If Q_1 is positive definite, then $\pi_2(\mathbb{R})$ is again surjective. If Q_1 is negative definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \emptyset$, so $Y(\mathbb{R}) = \emptyset$ is connected. If Q_1 is indefinite, then $(Q_1 \geqslant 0)$ is either an oval or the complement of an oval, which are both connected. $\square$

2 $\Delta(\mathbb{R})$ being One Oval implies $Y(\mathbb{R})$ is Connected

Proposition 2.1. If the topological type of $\Delta(\mathbb{R})$ is one oval, then $Y(\mathbb{R})$ is connected.

Proof. It suffices for us to show that $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ is connected, since every fiber of the image is connected and π_2 is a close map.

If Q_1 is positive definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \mathbb{P}^2_{[u:v:w]}(\mathbb{R})$ is connected.

If Q_1 is negative definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \{p \in \mathbb{P}^2_{[u:v:w]}(\mathbb{R}) \mid \Delta(p) \leqslant 0\}$, which is connected since $\Delta(\mathbb{R})$ is one oval, so both $\Delta(\mathbb{R})$ and its complement is connected.

If Q_1 is indefinite, assume for the sake of contradiction that $Y(\mathbb{R})$ is disconnected, since $(Q_1 \geqslant 0)$ and $(\Delta(p) \leqslant 0)$ are both connected, it has to be the case that $\pi_2(Y(\mathbb{R}))$ has 2 connected components being $(Q_1 \geqslant 0)$ and $(\Delta(p) \leqslant 0)$.

But we note that for all $p \in (Q_1 = 0)$, we have that $\Delta(p) = -Q_2(p) \leqslant 0$, so $(Q_1 = 0)$ is contained in $(\Delta(p) \leqslant 0)$. But this means that $\pi_2(Y(\mathbb{R}))$ is clearly connected. So we have a contradiction.

Thus, in all cases, $Y(\mathbb{R})$ is connected. $\square$

3 $\Delta(\mathbb{R})$ being Four Ovals implies $Y(\mathbb{R})$ is Connected

Lemma 3.1. Let $f : X \to Y$ be a quotient map and the fiber of every point in Y is connected. Then for each connected open or closed subset $U \subset Y$, $f^{-1}(U)$ is connected in X.

Proof. Since $f^{-1}(U)$ is open/closed and saturated, the restriction of f onto $f^{-1}(U)$ gives a quotient map $f' : f^{-1}(U) \to U$. Since f' is a quotient map with connected base and every fiber is connected, $f^{-1}(U)$ is connected. $\square$

Lemma 3.2. Let $f : X \to Y$ be a close map such that the fiber of every point in $f(X)$ is connected. Suppose $f(X)$ has a finite number of connected components, then X and $f(X)$ has the same number of connected components.

Proof. We can without loss take $Y = f(X)$ and just consider f to be surjective. Let m, n be the number of connected components in X and Y respectively. We first note that clearly $m \geqslant n$ since the continuous image of a connected set is connected.

Now for each connected component C_i ($1 \leqslant i \leqslant n$) in $f(X)$, C_i is closed, so $f^{-1}(C_i)$ is also connected in X by Lemma 3.1. The union of all $f^{-1}(C_i)$ is X, so X has at most n connected components, so $m \leqslant n$.

Thus, we have that $m = n$. ☐

Proposition 3.3. If the topological type of $\Delta(\mathbb{R})$ is four ovals, then $Y(\mathbb{R})$ is connected.

Proof. It suffices for us to show that $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ is connected, since every fiber of the image is connected and π_2 is a close map.

If Q_1 is positive definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \mathbb{P}^2_{[u:v:w]}(\mathbb{R})$ is connected.

If Q_1 is negative definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = (\Delta \leqslant 0)$ is either 4 ovals or the complement of four ovals. The latter case is a connected image, so suppose $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ is 4 ovals, then it has 4 connected components.

Then Lemma 3.2 tells us that $Y(\mathbb{R})$ has 4 connected components, but this also means that $\pi_1(Y(\mathbb{R}))$ has 4 connected components. But this would imply that $det(M_1 t_0^2 + 2M_2 t_0 t_1 + M_3 t_1^2)$ has at least 8 roots in $\mathbb{P}^1_{[t_0:t_1]}(\mathbb{R})$, which is impossible since it can only have at most 6 roots.

If Q_1 is indefinite, then we first note that for all $p \in (Q_1 = 0)$, $\Delta(p) = -Q_2(p)^2 \leqslant 0$, so we have that $(Q_1 = 0) \subset (\Delta \leqslant 0)$.

If $(\Delta \leqslant 0)$ is the complement of 4 ovals, then $(\Delta \leqslant 0)$ itself is already connected, and we know that $(Q_1 \geqslant 0)$ is also connected. Since the union of two connected sets with non-empty intersection is connected, we have that $\pi_2(\mathbb{R})(Y(\mathbb{R})) = (\Delta \leqslant 0) \cup (Q_1 \geqslant 0)$ is connected.

If $(\Delta \leqslant 0)$ is 4 ovals, then since $(Q_1 = 0)$ is connected, $(Q_1 = 0)$ has to be contained in only 1 of the 4 ovals.

Now if $(Q_1 \geqslant 0)$ is an oval, then $(Q_1 \geqslant 0) \subset (\Delta \leqslant 0)$, so $\pi_2(\mathbb{R})(Y(\mathbb{R})) = (\Delta \leqslant 0)$ is 4 ovals. Then using Lemma 3.2 again leads to a contradiction.

If $(Q_1 \geqslant 0)$ is the complement of an oval, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \mathbb{P}^2(\mathbb{R})$, so the image is connected. ☐

4 Topological Type of $\Delta(\mathbb{R})$ and Number of Connected Components

Lemma 4.1. If $(\Delta \leqslant 0)$ is connected, then $Y(\mathbb{R})$ is connected.

Proof. We already know that $(Q_1 = 0) \subset (\Delta \leqslant 0)$ and $\pi_2(\mathbb{R})(Y(\mathbb{R})) = (\Delta \leqslant 0) \cup (Q_1 \geqslant 0)$. Since both $(\Delta \leqslant 0)$ and $(Q_1 \geqslant 0)$ are connected and they have non-empty intersection, $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ is connected.

Since π_2 is closed and the fiber of every point in its image is connected, we have that $Y(\mathbb{R})$ is connected. ☐

Proposition 4.2. If $\pi_1(\mathbb{R})(Y(\mathbb{R}))$ has 3 connected components, then the topological type of $\Delta(\mathbb{R})$ is 3 ovals.

Proof. From Proposition 1.2, 2.1, and 3.3, we know that $Y(\mathbb{R})$ can only be disconnected if $\Delta(\mathbb{R})$ is either 2 non-nested ovals, 2 nested ovals, or 3 ovals. We first note by Lemma 3.2 we know that $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ also needs to have 3 connected components.

Now if $\Delta(\mathbb{R})$ is 2 non-nested ovals, then $(\Delta \leqslant 0)$ either has 1 or 2 connected components. Since $(Q_1 = 0) \subset (\Delta \leqslant 0)$ and $(Q_1 \geqslant 0)$, we know that the union of $(\Delta \leqslant 0)$ and $(Q_1 \geqslant 0)$ won't increase the number of connected components, so $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ has at most 2 connected components.

Similarly, if $\Delta(\mathbb{R})$ is 2 nested ovals, then $(\Delta \leqslant 0)$ still only has either 1 or 2 connected components, so the same reasoning shows that $\pi_2(Y(\mathbb{R}))$ has at most 2 connected components. $\square$

Proposition 4.3. If $\pi_1(\mathbb{R})(Y(\mathbb{R}))$ has 2 connected components, then the topological type of $\Delta(\mathbb{R})$ is either 2 nested ovals or 2 non-nested ovals.

Proof. It suffices for us to show that $\Delta(\mathbb{R})$ cannot have the topological type of 3 ovals. Indeed, Lemma 3.2 tells us that since $\pi_1(\mathbb{R})(Y(\mathbb{R}))$ has 2 connected components, $\pi_2(Y(\mathbb{R}))$ also has 2 connected components.

Now assume for the sake of contradiction that $\Delta(\mathbb{R})$ has 3 ovals. Now if $(\Delta \leqslant 0)$ is the complement of 3 ovals, then it's connected, then Lemma 3.1 leads to a contradiction.

Now if $(\Delta \leqslant 0)$ is 3 ovals, then $(Q_1 = 0)$ is contained in exactly one oval. If $(Q_1 \geqslant 0)$ is also an oval, then $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ has 3 connected components, hence a contradiction. If $(Q_1 \geqslant 0)$ is the complement of an oval, then π_2 is surjective, which means $Y(\mathbb{R})$ is connected, hence a contradiction. $\square$