Lemma 0.1. $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \{p \in \mathbb{P}^2_{[u:v:w]}(\mathbb{R}) \mid Q_1(p) \geqslant 0 \text{ or } \Delta(p) \leqslant 0\}$

Proposition 0.2. If the topological type of $\Delta(\mathbb{R})$ is one oval, then $Y(\mathbb{R})$ is connected.

Proof. It suffices for us to show that $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ is connected, since every fiber of the image is connected and π_2 is a close map.

If Q_1 is positive definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \mathbb{P}^2_{[u:v:w]}(\mathbb{R})$ is connected.

If Q_1 is negative definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \{p \in \mathbb{P}^2_{[u:v:w]}(\mathbb{R}) \mid \Delta(p) \leqslant 0\}$, which is connected since $\Delta(\mathbb{R})$ is one oval, so both $\Delta(\mathbb{R})$ and its complement is connected.

If Q_1 is indefinite, assume for the sake of contradiction that $Y(\mathbb{R})$ is disconnected, since $(Q_1 \geqslant 0)$ and $(\Delta(p) \leqslant 0)$ are both connected, it has to be the case that $\pi_2(Y(\mathbb{R}))$ has 2 connected components being $(Q_1 \geqslant 0)$ and $(\Delta(p) \leqslant 0)$.

But we note that for all $p \in (Q_1 = 0)$, we have that $\Delta(p) = -Q_2(p) \leqslant 0$, so $(Q_1 = 0)$ is contained in $(\Delta(p) \leqslant 0)$. But this means that $\pi_2(Y(\mathbb{R}))$ is clearly connected. So we have a contradiction.

Thus, in all cases, $Y(\mathbb{R})$ is connected. $\square$