1 $\Delta(\mathbb{R})$ being Four Ovals implies $Y(\mathbb{R})$ is Connected

Lemma 1.1. Let $f : X \to Y$ be a quotient map and the fiber of every point in Y is connected. Then for each connected open or closed subset $U \subset Y$, $f^{-1}(U)$ is connected in X.

Proof. Since $f^{-1}(U)$ is open/closed and saturated, the restriction of f onto $f^{-1}(U)$ gives a quotient map $f' : f^{-1}(U) \to U$. Since f' is a quotient map with connected base and every fiber is connected, $f^{-1}(U)$ is connected. □

Lemma 1.2. Let $f : X \to Y$ be a close map such that the fiber of every point in $f(X)$ is connected. Suppose $f(X)$ has a finite number of connected components, then X and $f(X)$ has the same number of connected components.

Proof. We can without loss take $Y = f(X)$ and just consider f to be surjective. Let m, n be the number of connected components in X and Y respectively. We first note that clearly $m \geqslant n$ since the continuous image of a connected set is connected.

Now for each connected component $C_i, 1 < i < n$ in $f(X)$, C_i is closed, so $f^{-1}(C_i)$ is also connected in X by Lemma 1.1. The union of all $f^{-1}(C_i)$ is X, so X has at most n connected components, so $m \leqslant n$.

Thus, we have that $m = n$. □

Proposition 1.3. If the topological type of $\Delta(\mathbb{R})$ is four ovals, then $Y(\mathbb{R})$ is connected.

Proof. It suffices for us to show that $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ is connected, since every fiber of the image is connected and π_2 is a close map.

If Q_1 is positive definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \mathbb{P}^2_{[u:v:w]}(\mathbb{R})$ is connected.

If Q_1 is negative definite, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = (\Delta \leqslant 0)$ is either 4 ovals or the complement of four ovals. The latter case is a connected image, so suppose $\pi_2(\mathbb{R})(Y(\mathbb{R}))$ is 4 ovals, then it has 4 connected components.

Then Lemma 1.2 tells us that $Y(\mathbb{R})$ has 4 connected components, but this also means that $\pi_1(Y(\mathbb{R}))$ has 4 connected components. But this would imply that $det(M_1t_0^2 + 2M_2t_0t_1 + M_3t_1^2)$ has at least 8 roots in $\mathbb{P}^1_{[t_0:t_1]}(\mathbb{R})$, which is impossible since it can only have at most 6 roots.

If Q_1 is indefinite, then we first note that for all $p \in (Q_1 = 0)$, $\Delta(p) = -Q_2(p)^2 \leqslant 0$, so we have that $(Q_1 = 0) \subset (\Delta \leqslant 0)$.

If $(\Delta \leqslant 0)$ is the complement of 4 ovals, then $(\Delta \leqslant 0)$ itself is already connected, and we know that $(Q_1 \geqslant 0)$ is also connected. Since the union of two connected sets with non-empty intersection is connected, we have that $\pi_2(\mathbb{R})(Y(\mathbb{R})) = (\Delta \leqslant 0) \cup (Q_1 \geqslant 0)$ is connected.

If $(\Delta \leqslant 0)$ is 4 ovals, then since $(Q_1 = 0)$ is connected, $(Q_1 = 0)$ has to be contained in only 1 of the 4 ovals.

Now if $(Q_1 \geqslant 0)$ is an oval, then $(Q_1 \geqslant 0) \subset (\Delta \leqslant 0)$, so $\pi_2(\mathbb{R})(Y(\mathbb{R})) = (\Delta \leqslant 0)$ is 4 ovals. Then using Lemma 1.2 again leads to a contradiction.

If $(Q_1 \geqslant 0)$ is the complement of an oval, then $\pi_2(\mathbb{R})(Y(\mathbb{R})) = \mathbb{P}^2(\mathbb{R})$, so the image is connected.

□