Theorem 0.1. The following are equivalent:

- $\pi_1(\mathbb{R}) : y(\mathbb{R}) \to \mathbb{P}^1_{[t_0,t_1]}(\mathbb{R})$ is surjective
- For all $[t_0, t_1] \in \mathbb{P}^1_{[t_0,t_1]}(\mathbb{R})$, the correspondent quadratic form:

$$z^2 = Q_1(u, v, w)t_0^2 + 2Q_2(u, v, w)t_0t_1 + Q_3(u, v, w)t_1^2$$

  has a real solution

- Let M_1, M_2, M_3 be the symmetric matrix associated to Q_1, Q_2, Q_3 the matrix

$$M_{[t_0,t_1]} = \left(\begin{array}{c|c} M_1t_0^2 + 2M_2t_0t_1 + M_3t_1^2 & \mathbf{0} \\ \hline \mathbf{0} & -1 \end{array}\right)$$

  is indefinite

- $M_{[t_0,t_1]}$ is not negative-definite (since the matrix cannot be positive definite with the -1 term)

Lemma 0.2. If $\Delta(\mathbb{R}) = \emptyset$, then $\tilde{\Delta}(\mathbb{R}) = \emptyset$

Proof. Suppose for contradiction that $\tilde{\Delta}(\mathbb{R}) \neq$. Then there exist some non-zero (u, v, w, r, s) such that

$$Q_1(u, v, w) = r^2, Q_2(u, v, w) = r * s, Q_3(u, v, w) = s^2$$

but this means that

$$Q_2(u, v, w)^2 - Q_1(u, v, w)Q_3(u, v, w) = 0$$

has solution (u, v, w).

Clearly $(u, v, w) \neq (0, 0, 0)$ as if they do, then this would imply that $0 = r^2, 0 = s^2$, which means that $r, s = 0$. Thus, $\Delta(\mathbb{R})$ then does have some non-zero real solution, so $\Delta(\mathbb{R}) \neq \emptyset$, which is a contradiction.

□

Proposition 0.3. There exist $Y_{\tilde{\Delta}/\Delta}$ such that $\pi_1(\mathbb{R})$ is surjective and $\tilde{\Delta}(\mathbb{R}) = \emptyset$.

Proof. Let $M_1 = [8/7, 3, 2], M_2 = [7/5, 3/5, 10/7], M_3 = [2, 1, 8/5]$, then the matrix becomes

$$M_{[t_0,t_1]} = \begin{bmatrix} \frac{8}{7}t_0^2 + \frac{14}{5}t_0t_1 + 2t_1^2 & 0 & 0 & 0 \\ 0 & 3t_0^2 + \frac{6}{5}t_0t_1 + t_1^2 & 0 & 0 \\ 0 & 0 & 2t_0^2 + \frac{20}{7}t_0t_1 + \frac{8}{5}t_1^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The code in PVS showed that the $\Delta(\mathbb{R})$ for this setup is empty, so Lemma 0.2 gives us that $\tilde{\Delta}(\mathbb{R}) = \emptyset$. It remains for us to show that, for all $[t_0, t_1]$, $M_{[t_0,t_1]}$ is not negative definite. Indeed, Sylvester's Criterion tells us that it suffices for us to show that $\frac{8}{7}t_0^2 + \frac{14}{5}t_0t_1 + 2t_1^2 > 0$ for all non-zero (t_0, t_1), but

$$\begin{aligned} \frac{8}{7}t_0^2 + \frac{14}{5}t_0t_1 + 2t_1^2 &= \frac{8}{7}(t_0^2 + \frac{14 \cdot 7}{40}t_0t_1 + \frac{7}{4}t_1^2) \\ &= \frac{8}{7}(t_0^2 + \frac{49}{20}t_0t_1 + \frac{7}{4}t_1^2) \\ &= \frac{8}{7}[(t_0 + \frac{49}{40}t_1)^2 + (\frac{7}{4} - (\frac{49}{40})^2)t_0^2] && \text{Completing the Square} \\ &\geqslant \frac{8}{7}[\frac{7}{4} - (\frac{49}{40})^2](t_0^2) \end{aligned}$$

Note that $\frac{8}{7}(\frac{7}{4} - (\frac{49}{40})^2 > 0$, so the expression is strictly greater than 0 when $t_0 \neq 0$, so Theorem 0.1 tells us that $\pi_1(\mathbb{R})$ is surjective. □