Rationality of Real Conic Bundles with Quartic Discriminant Curve

Mattie Ji† (Brown University)

University of Michigan Mathematics REU - advised by Dr. Lena Ji

What is Rationality?

Given an algebraic variety, a natural question to ask is how simple it is:

- One notion of simplicity is its closeness to projective spaces.
- One way to measure this is to ask whether or not a variety is isomorphic to a projective space outside some lower-dimensional subsets.
- Conic bundles have been a very rich source of examples of varieties with varying levels of similarity to projective space.

Definition. *A variety X is rational over a field k if there exists non-empty open subsets $U \subseteq X$ and $V \subseteq \mathbb{P}_k^{dim X}$ such that U and V are isomorphic.*

There are two relevant results about rationality we need here:

Lemma (Lang-Nishimura Lemma)**.** *If X is a projective k-rational variety, then $X(k)$ is non-empty.*

Fact. *If X is a smooth projective $\mathbb{R}$-rational variety, then $X(\mathbb{R})$ is connected.*

What is a Conic Bundle?

Unless specified otherwise, we will always be working over the real numbers.

Definition. *A plane conic $C \subseteq \mathbb{P}^2$ is the zero locus of a homogenous polynomial of degree 2 in $\mathbb{R}[X, Y, Z]$.*

Definition. *A conic bundle over $\mathbb{P}^2$ is a proper flat morphism $\pi : X \to \mathbb{P}^2$ such that*

- *X is a smooth variety*
- *$\pi^{-1}(p)$ is a conic for all $p \in \mathbb{P}^2$*
- *The generic fiber is a smooth conic*

Example. *In our research, we are interested in the conic bundle $\pi_2 : Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^2_{[u:v:w]}$ where:*

- *$Y_{\tilde{\Delta}/\Delta}$ is a variety defined by the equation*
$$z^2 = Q_1(u, v, w)t_0^2 + 2Q_2(u, v, w)t_0t_1 + Q_3(u, v, w)t_1^2$$
- *$Q_1, Q_2, Q_3 \in \mathbb{R}[u, v, w]$ are homogenous polynomials of degree 2*
- *π_2 is the standard projection that forgets z, t_0, and t_1*

$$\begin{tikzcd} Y_{\tilde{\Delta}/\Delta} \arrow[d] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^1_{[t_0:t_1]} \times \mathbb{P}^2_{[u:v:w]} \arrow[r] & \mathbb{P}^2_{[u:v:w]} \end{tikzcd}$$

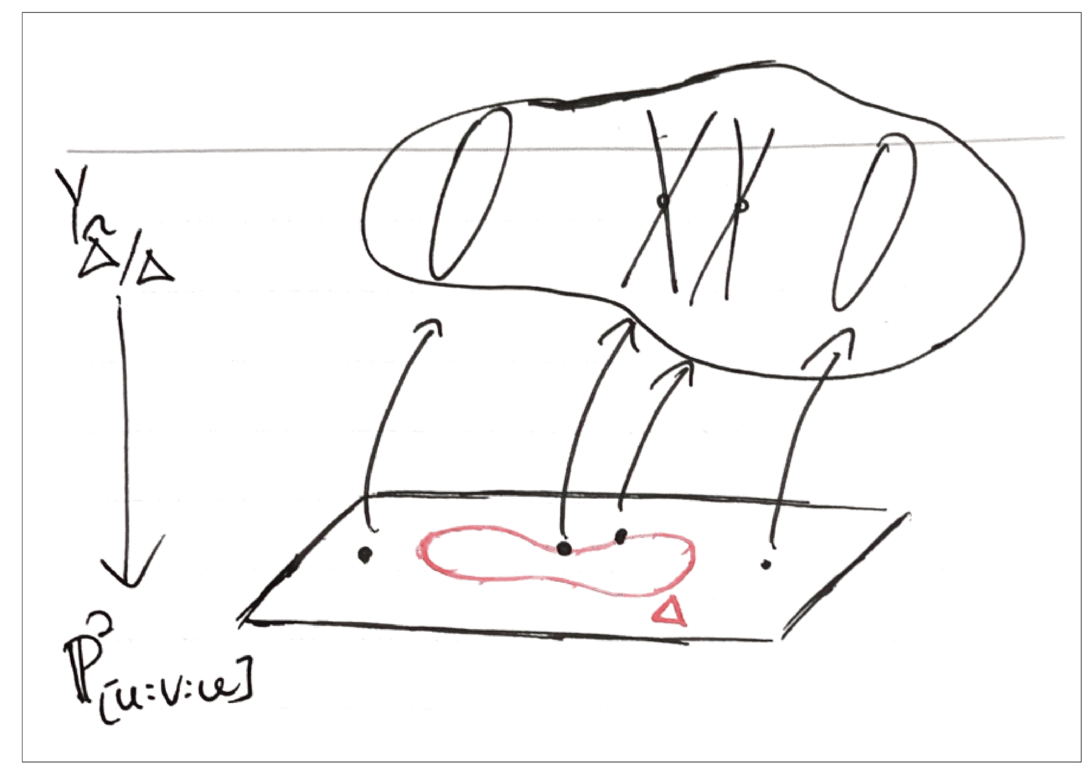
To see why π_2 is a conic bundle, take the point $[1:2:3] \in \mathbb{P}^2_{[u:v:w]}$, the fiber of $[1:2:3]$ is exactly the solutions satisfying:
$$z^2 = Q_1(1, 2, 3)t_0^2 + 2Q_2(1, 2, 3)t_0t_1 + Q_3(1, 2, 3)t_1^2$$
This forms a conic in $\mathbb{P}^2_{\mathbb{R},[t_0:t_1:z]}$.

Remark. *$Y_{\tilde{\Delta}/\Delta}$ looks like a very specific choice, but it turns out that every degree 4 conic bundle $X \to \mathbb{P}^2$ is birationally equivalent to some π_2 "up to a class in $\mathbb{Z}/2\mathbb{Z}$" (Theorem 2.6 of [1])*

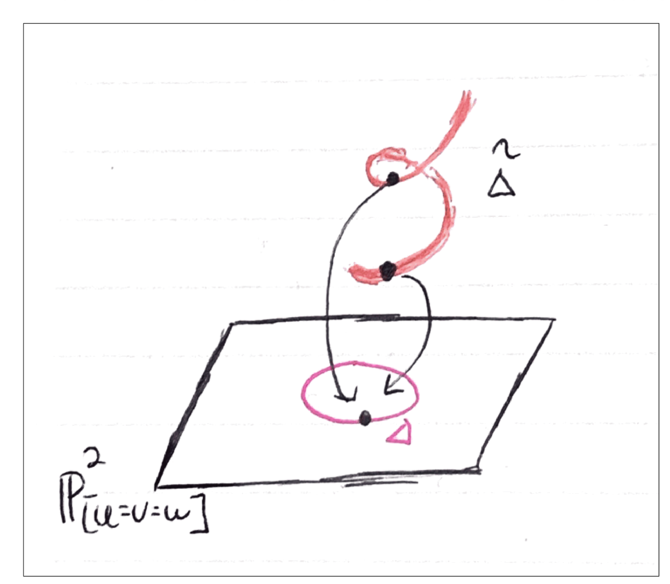
The Discriminant Curve

Definition. *The discriminant curve Δ of $Y_{\tilde{\Delta}/\Delta}$ is the zero locus given by*
$$\Delta = (Q_1Q_3 - Q_2^2 = 0) \subset \mathbb{P}^2_{[u:v:w]}$$

Remark. *The fiber of $s \in \mathbb{P}^2_{[u:v:w]}$ is smooth if and only if $s \notin \Delta$:*



Fact. *There exists a curve $\tilde{\Delta} \subset \mathbb{P}^4_{[u:v:w:r:s]}$ defined by*
$$\tilde{\Delta} = (Q_1 - r^2 = Q_2 - rs = Q_3 - s^2 = 0)$$
such that the projection $\tilde{\Delta} \to \Delta$ is a double cover. $\tilde{\Delta}$ is called the double cover of Δ:



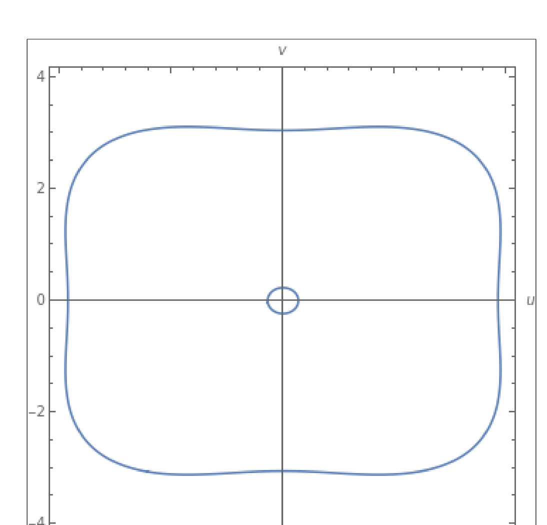
Quartic Curves

Note that $Q_1Q_3 - Q_2^2$ is a homogeneous polynomial of degree 4:

Definition. *A **quartic** is the roots of a degree 4 homogenous polynomial over $\mathbb{P}^2$.*

Theorem (Zeuthen [3])**.** *Let Δ be a smooth quartic over $\mathbb{R}$, then $\Delta(\mathbb{R})$ can be classified into 1 of the 6 following topological types: 1. No real points 2. One oval 3. Two nested ovals 4. Two non-nested ovals 5. Three ovals 6. Four ovals*

Example. *This homogeneous equation defines a smooth quartic whose real component has 2 nested ovals:*
$$0 = -3u^4 - \frac{7}{10}u^2v^2 - \frac{169}{400}v^4 + \frac{67}{6}u^2w^2 + \frac{949}{240}v^2w^2 - \frac{121}{576}w^4$$
The real components on the chart $(w \neq 0)$:



Our Results:

The $\mathbb{R}$-rationality of $Y_{\tilde{\Delta}/\Delta}$ is complicated, but the work of S. Frei, L. Ji, S. Sankar, B. Viray, and I. Vogt previously gave a very useful criterion:

Theorem (Prop. 6.1 of [1])**.** *Let $Y_{\tilde{\Delta}/\Delta}$ be defined as before. If $\tilde{\Delta}(\mathbb{R}) \neq \emptyset$, then $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational*

In our REU, we instead investigate what happens when $\tilde{\Delta}(\mathbb{R}) = \emptyset$. Based on ideas and results joint with Sarah Frei, Soumya Sankar, Bianca Viray, and Isabel Vogt, we proved the following two theorems:

Theorem (1.1 of [2])**.** *Let $\mathcal{CB}_{\emptyset/*}$ denote the set of real conic bundles $X \to \mathbb{P}^2$ of the form $Y_{\tilde{\Delta}/\Delta}$ with smooth Δ of topological type $*$ and smooth $\tilde{\Delta}$ that has no real points, then:*

1. *$\mathcal{CB}_{\emptyset/\emptyset}$ contains both rational and irrational members*
2. *$\mathcal{CB}_{\emptyset/1\text{-oval}}$ contains both rational and irrational members*
3. *$\mathcal{CB}_{\emptyset/2 \text{ non-nested ovals}}$, $\mathcal{CB}_{\emptyset/2 \text{ nested ovals}}$, and $\mathcal{CB}_{\emptyset/3\text{-ovals}}$ contains both rational and irrational members*
4. *$\mathcal{CB}_{\emptyset/4\text{-ovals}}$ contains only rational members.*

There's also a class of obstructions to rationality that we are interested in - known as the Intermediate Jacobina Torsor (IJT) obstruction. Together, we have our second theorem:

Theorem (1.2 of [2])**.** *Let $Y_{\tilde{\Delta}/\Delta}$ be as before,*

1. *If $\Delta(\mathbb{R})$ is 4 ovals, then $Y_{\tilde{\Delta}/\Delta}$ is rational*
2. *If $\Delta(\mathbb{R})$ is 3 ovals, then $Y_{\tilde{\Delta}/\Delta}$ is rational $\iff$ $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected*
3. *If $\Delta(\mathbb{R})$ is 2 non-nested ovals, then $Y_{\tilde{\Delta}/\Delta}$ is rational $\iff$ $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected and the IJT obstruction vanishes*
4. *If $\Delta(\mathbb{R})$ is empty, then $Y_{\tilde{\Delta}/\Delta}$ is rational $\iff$ $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is non-empty and the IJT obstruction vanishes*

Remark. *A geometric interpretation of this was given in [1]. It was shown that if $\tilde{\Delta}$ has no real points, then the IJT obstruction vanishes if there exists points $P_1, P_2, P_3, P_4 \in \tilde{\Delta}(\mathbb{C})$ and*

- *The set $\{P_1, P_2, P_3, P_4\}$ is closed under complex conjugation and does not span a 2-plane in $\mathbb{P}^4$*
- *The projection of $\{P_1, P_2, P_3, P_4\}$ to $\mathbb{P}^2$ is the set $\Delta \cap \ell$ for a line ℓ in $\mathbb{P}^2$ that does not meet any element of $\Delta(\mathbb{R})$ transversely.*

Acknowledgements

We would like to thank

- Lena Ji for advising this project
- Sarah Frei, Brendan Hassett, Soumya Sankar, Bianca Viray, and Isabel Vogt for helpful discussions and conversations
- János Kollár for the question that motivated this project
- The National Science Foundation (Karen Smith's NSF grant DMS-2101075) for funding this project
- My letter of recommendation writers Nicole Looper and Jungang Li

References

[1] Sarah Frei et al. "Curve classes on conic bundle threefolds and applications to rationality". In: *arXiv e-prints*, arXiv:2207.07093 (), arXiv:2207.07093. arXiv: 2207.07093 [math.AG].

[2] Lena Ji and Mattie Ji. *Rationality of real conic bundles with quartic discriminant curve.* 2022. DOI: 10.48550/ARXIV.2208.08916. URL: https://arxiv.org/abs/2208.08916.

[3] Hieronymus Georg Zeuthen. "Sur les différentes formes des courbes planes du quatrième ordre". fre. In: *Mathematische Annalen* 7 (1874), pp. 410–432.