Theorem 0.1. The following are equivalent:

- $\pi_1(\mathbb{R}) : Y(\mathbb{R}) \to \mathbb{P}^1_{[t_0,t_1]}(\mathbb{R})$ is surjective
- For all $[t_0, t_1] \in \mathbb{P}^1_{[t_0,t_1]}(\mathbb{R})$, the correspondent quadratic form:

$$z^2 = Q_1(u,v,w)t_0^2 + 2Q_2(u,v,w)t_0t_1 + Q_3(u,v,w)t_1^2$$

  has a real solution

- Let M_1, M_2, M_3 be the symmetric matrix associated to Q_1, Q_2, Q_3 the matrix

$$M_{[t_0,t_1]} = \left(\begin{array}{c|c} M_1t_0^2 + 2M_2t_0t_1 + M_3t_1^2 & \mathbf{0} \\ \hline \mathbf{0} & -1 \end{array}\right)$$

  is indefinite

- $M_{[t_0,t_1]}$ is not negative-definite (since the matrix cannot be positive definite with the -1 term)

Theorem 0.2 (Sylvester's Criterion)**.** Let $M \in M_{n\times n}(\mathbb{R})$ be an $n \times n$ real matrix, and let $M_1, ..., M_n$ be real matrices such that M_k is the $k \times k$ upper left corner matrix of M.

Then M is negative-definite if and only if for all odd k, $det(M_k) < 0$, and for all even k, $det(M_k) > 0$.

Proposition 0.3. Suppose either M_1 or M_3 is negative definite, then $\pi_1(\mathbb{R}) : Y(\mathbb{R}) \to \mathbb{P}^1_{[t_0,t_1]}(\mathbb{R})$ is not surjective.

Proof. Suppose M_1 is negative definite, then on $[1, 0] \in \mathbb{P}^1_{[t_0,t_1]}(\mathbb{R})$, the matrix $M_{[t_0,t_1]}$ becomes

$$M_{[1,0]} = \left(\begin{array}{c|c} M_1 & \mathbf{0} \\ \hline \mathbf{0} & -1 \end{array}\right)$$

Let M_1, M_2, M_3, M_4 be the upper left corner matrix as described in Theorem 0.2. Since M_1 is negative definite, we have that $det(M_1) < 0, det(M_2) > 0, det(M_3) < 0$. We also have that

$$det(M_4) = det(M) = (-1)det(M_3) > 0$$

So Theorem 0.2 shows that $M_{[1,0]}$ is a negative definite matrix, then Theorem 0.1 shows that $\pi_1(\mathbb{R})$ is not surjective.

Suppose M_3 is negative definite, then a nearly identical argument follows by considering the point $[0, 1] \in \mathbb{P}^1_{[t_0,t_1]}(\mathbb{R})$ □