**Proposition 0.1.** If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has two connected components and the hyper-elliptic curve $\Gamma$ has six real Weierstrass points, then $\Delta(\mathbb{R})$ is two nested ovals.

*Proof.* It suffices for us to show that $\Delta(\mathbb{R})$ is not two non-nested ovals, so we will assume for the sake of contradiction that it is. Since $\Gamma$ has six real Weierstrass points and $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has two connected components, the signatures of $\pi_1(\mathbb{R})$ must follow $(0,4),(1,3),(0,4),(1,3),(2,2),(1,3)$. In particular, $\pi_1(\mathbb{R})$ contains some point $t$ whose fiber has signature $(2,2)$.

Theorem 2.6(1) of FSJVV then tells us that, we can without loss choose $Q_1$ such that its associated matrix $M_1$ has 1 negative eigenvalue and 2 positive eigenvalues. Geometrically, $(Q_1 \geqslant 0)$ is the closure of the compliment of a disk.

Since $Q_1$ is indefinite and $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is disconnected, the image of $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ under $\pi_2(\mathbb{R})$ has to be $(Q_1Q_3 - Q_2^2 \leqslant 0)$, meaning that $(Q_1 > 0)$ is a subset of $(Q_1Q_3 - Q_2^2 \leqslant 0)$.

However, when $\Delta(\mathbb{R})$ is two non-nested ovals, since $(Q_1Q_3 - Q_2^2 \leqslant 0)$ is the union of two disjoint disks say $D_1, D_2$, but $(Q_1 > 0) = D_3^c$ for some disk $D_3$ that's contained in either $D_1$ or $D_2$. But this means that $Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) = D_1 \cup D_2 \cup D_3^c = \mathbb{P}^2_{\mathbb{R}}$, so we have a contradiction. □

**Proposition 0.2.** If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has two connected components and the hyper-elliptic curve $\Gamma$ has four real Weierstrass points, then $\Delta(\mathbb{R})$ is two non-nested ovals.

**educated guess:**

*Proof.* It suffices for us to show that $\Delta(\mathbb{R})$ is not two nested ovals, so we will assume for the sake of contradiction that it is. Since $\Gamma$ has four real Weierstrass points and $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has two connected components, the signatures of $\pi_1(\mathbb{R})$ must follow $(0,4),(1,3),(0,4),(1,3)$.

We want to show to show that disconnected two nested ovals implies the existence of a fiber with signature $(2,2)$. I guess maybe the idea is that we can modify the $Q_1, Q_2, Q_3$ up to $PGL_2$-action such that $(Q_1 \geqslant 0)$ becomes the compliment of a disk (or equivalently $Q_1$'s associated matrix $M_1$ has signature $(2,1)$). The intuition for this guess is that as long as $(Q_1 \geqslant 0) \subset (Q_1Q_3 - Q_2^2 \leqslant 0)$, we could try to modify it to whatever we want? □

**Question 0.3.** Let $Q_1, Q_2, Q_3$ be the associated quadratic forms of $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$, let $Q_4$ be a quadratic form such that $(Q_4 \geqslant 0) \cup (Q_1Q_3 - Q_2^2 \leqslant 0) = (Q_1 \geqslant 0) \cup (Q_1Q_3 - Q_2^2 \leqslant 0)$, does there exist a $PGL_2$-action on $Q_1, Q_2, Q_3$ such that the new $Q_1$ has the same signature as $Q_4$? (This might be too general)