Conic Bundles over the Real Projective Plane

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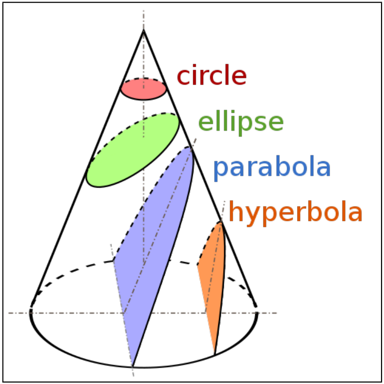
What are Conics?

In Ancient Greece, conics (or conic sections) are defined as the intersection of a cone and a plane, by "slicing" a cone in creative ways.



[1]

[1] Figure taken from
https://en.wikipedia.org/wiki/File:Conic_Sections.svg

Affine Space

In Algebraic Geometry, our classical conic sections become part of affine spaces.

Definition

Let k be a field, $n \geq 0$ an integer, the **affine space** of dimension n is k^n, which we will denote as $\mathbb{A}_k^n$

Definition

An (affine) **algebraic variety** $V \subset \mathbb{A}_k^n$ is the set of common k-roots of a collection of polynomials $\{F_i\}_{i \in I}$ where $F_i \in k[x_1, ..., x_n]$. We write V as

$$V = \mathbb{V}(\{F_i\}_{i \in I})$$

Example of Affine Algebraic Varieties

Example:

In the affine space $\mathbb{A}_k^n$

- $\mathbb{V}(0) = \mathbb{A}_k^n$, $\mathbb{V}(1) = \text{O}$
- $\mathbb{V}(x_1 - a_1, ..., x_n - a_n) = \{(a_1, ..., a_n)\}$
- Take $k = \mathbb{R}, n = 2$, then the classical conic section C is the variety

$$C = \mathbb{V}(ax^2 + by^2 + c + dxy + ey + fx)$$

where $a, b, c, d, e, f \in \mathbb{R}$

Projective Space

The theory of affine varieties is great, but we can generalize conics with what's known as "projective spaces".

Definition

The set of 1-dimensional subspaces of $\mathbb{A}_k^{n+1}$ is called the **projective space** of dimension n, denoted as $\mathbb{P}_k^n$. In other words, they are just the set of lines going through the origin in $\mathbb{A}_k^{n+1}$.

Notations:

- We will denote the line through 0 and $(a_0, ..., a_n)$ as $[a_0 : ... : a_n]$ in $\mathbb{P}_k^n$.
- Sometimes we will denote $\mathbb{P}_k^n$ as $\mathbb{P}_{k,[x_0,...,x_n]}^n$ to emphasize its coordinates.

Why Projective Spaces?

Q: Why do we want to study conics in projective spaces rather than affine spaces?

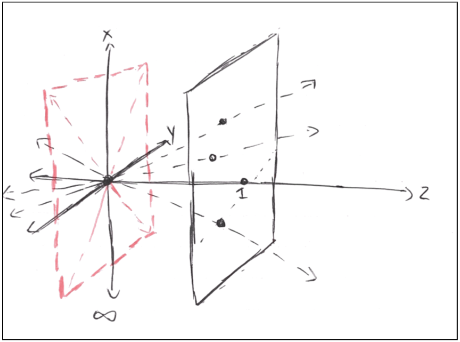
A: There are 2 reasons:

- Geometrically, projective spaces are a natural compactification of affine spaces.
- Algebraically, we can turn conic sections into a class of what's called "homogeneous polynomials", which is generally nicer to work with.

Embedding the Affine Plane

We can embed the affine plane $\mathbb{A}_k^2$ into $\mathbb{P}_k^2$ by identifying $\mathbb{A}^2$ with the subset $U_Z = \{[X : Y : Z] \in \mathbb{P}_k^2 \mid Z \neq 0\}$ via:

$$\varphi_Z : U_Z \to \mathbb{A}_k^2,\ [X : Y : Z] \mapsto (\frac{X}{Z}, \frac{Y}{Z})$$



This gives the compactification $\mathbb{P}_k^2 = \mathbb{A}_k^2 \sqcup \mathbb{P}_k^1$

Homogeneous Polynomials

Definition

A polynomial $F \in k[x_0, ..., x_n]$ is called **homogeneous of degree d** if it is a sum of degree d monomials.

For example, in $\mathbb{R}[x, y, z]$,

$$6x^5 + 7y^5 + \pi x^4 y + 3x^2 y^2 z + 9z^5$$

is a homogeneous polynomial of degree 5.

Observation:

Let F be a homogeneous polynomial of degree d and $\lambda \in k$,

$$F(\lambda a_0, ..., \lambda a_n) = \lambda^d F(a_0, ..., a_n)$$

for all $(a_0, ..., a_n) \in k^{n+1}$. In particular, if $(a_0, ..., a_n)$ is a root of F, then so is $(\lambda a_0, ..., \lambda a_n)$.

Connection to Conics

Classically, conic sections have been considered as real roots of the polynomial

$$f(x,y) = ax^2 + by^2 + c + dxy + ey + fx \in \mathbb{R}[x,y]$$

With our embedding, we can homogenize $f(x,y)$ into:

$$F(X,Y,Z) = aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ$$

Then we note that on $Z = 1$, $F(X,Y,Z)$ becomes $f(x,y)$. This is in fact a bijective correspondence.

Definition:

Let k be a field of characteristic $\neq 2$, a **plane conic** $C \subset \mathbb{P}^2_{[X:Y:Z],k}$ is the k-roots of a homogeneous polynomial of degree 2 in $k[X,Y,Z]$.

Matrices and Conics

Take any homogenous polynomial of degree 2

$$F(X,Y,Z) := aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ$$

We note that this polynomial has an associated symmetric matrix

$$M_F = \begin{bmatrix} a & \frac{d}{2} & \frac{f}{2} \\ \frac{d}{2} & b & \frac{e}{2} \\ \frac{f}{2} & \frac{e}{2} & c \end{bmatrix}$$

such that

$$F(X,Y,Z) = \begin{bmatrix} X & Y & Z \end{bmatrix} M_F \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

This also makes $F(X,Y,Z)$ into what's called a **quadratic form** of 3 variables.

Smoothness of Conics

It turns out that the rank of the matrix M_F determines the geometry of the conic C.

Fact:

Let $\overline{k}$ be the algebraic closure of k,

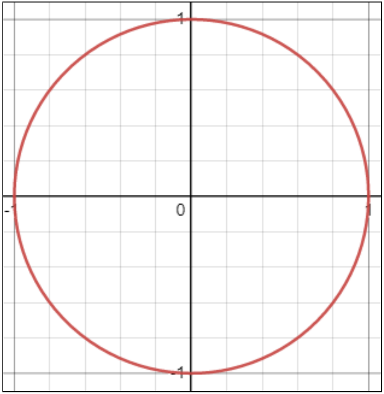
- If M_F has rank 3, then C is a smooth conic
- If M_F has rank 2, then $C_{\overline{k}}$, by considering all $\overline{k}$-roots of F, is the union of two distinct lines meeting at a point.
- If M_F has rank 1, then $C_{\overline{k}}$ is a double line.

Example: $rank(M_F) = 3$

Let $k = \mathbb{R}$, $M_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $F(X, Y, Z) = X^2 + Y^2 - Z^2$.

Then C is a smooth conic.

On the chart $(Z \neq 0)$,

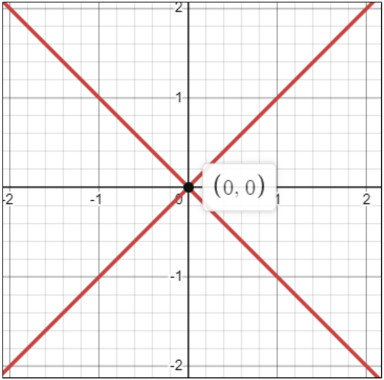


Example: $rank(M_F) = 2$

Let $k = \mathbb{R}$, $M_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $F(X, Y, Z) = X^2 - Y^2$.

Then C is the union of two lines meeting at the origin.

On the chart $(Z \neq 0)$,

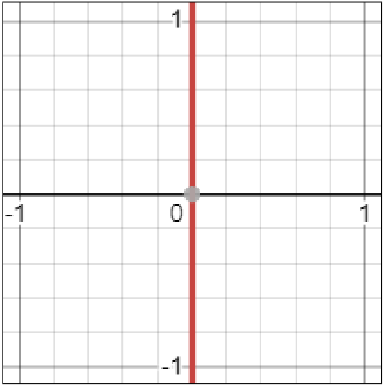


Example: $rank(M_F) = 1$

Let $k = \mathbb{R}$, $M_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $F(X, Y, Z) = X^2$.

Then C is a line, we say it's "double" because of the square.

On the chart $(Z \neq 0)$,



Conic Bundles

Definition[2]:

A **conic bundle** is a morphism $\pi : X \to S$ between (smooth) varieties X, S such that the fiber of every point $p \in S$, defined as $\pi^{-1}(\{p\})$, is a conic, and the generic fiber is a smooth conic.

[2] We need a few more ideas in algebraic geometry to properly define conic bundles, but for the purpose of this talk we will adopt this more convenient definition.

Conic Bundles

Example:

In our research, we are interested in the conic bundle $\pi : Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^2_{\mathbb{R},[u:v:w]}$ where:

- $Y_{\tilde{\Delta}/\Delta}$ is a variety defined by the equation:

$$z^2 = Q_1(u,v,w)t_0^2 + 2Q_2(u,v,w)t_0t_1 + Q_3(u,v,w)t_1^2$$

- $Q_1, Q_2, Q_3 \in \mathbb{R}[u,v,w]$ are homogenous polynomials of degree 2
- π is the standard projection

$$\begin{tikzcd}
Y_{\tilde{\Delta}/\Delta} \arrow[d] \arrow[dr, "\pi"] & \\
\mathbb{P}^1_{\mathbb{R},[t_0:t_1]} \times \mathbb{P}^2_{\mathbb{R},[u:v:w]} \arrow[r] & \mathbb{P}^2_{\mathbb{R},[u:v:w]}
\end{tikzcd}$$

Why is π a conic bundle?

Intuitively, for a conic bundle $\pi : X \to S$, every point in S should correspond to some conic in X.

Example of Fibers for π:

Concretely, take the point $[1:2:3] \in \mathbb{P}^2_{\mathbb{R},[u:v:w]}$, then fiber of $[1:2:3]$ is exactly the solutions satisfying:

$$z^2 = Q_1(1,2,3)t_0^2 + 2Q_2(1,2,3)t_0t_1 + Q_3(1,2,3)t_1^2$$

This forms a conic in $\mathbb{P}^2_{\mathbb{R},[t_0:t_1:z]}$.

Thus, $\pi : Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^2_{\mathbb{R},[u:v:w]}$ is an example of a **conic bundle**.

The Discriminant Curve

We would like to identify if a given fiber of π is smooth:

Smoothness Criterion

Given fixed $[u:v:w] \in \mathbb{P}^2_{\mathbb{R},[u:v:w]}$, we can rewrite its assoicated conic as:

$$0 = Q_1(u,v,w)t_0^2 + 2Q_2(u,v,w)t_0t_1 + Q_3(u,v,w)t_1^2 + (-1)z^2 \ (*)$$

This gives the symmetric matrix:

$$M = \begin{bmatrix} Q_1(u,v,w) & Q_2(u,v,w) & 0 \\ Q_2(u,v,w) & Q_3(u,v,w) & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

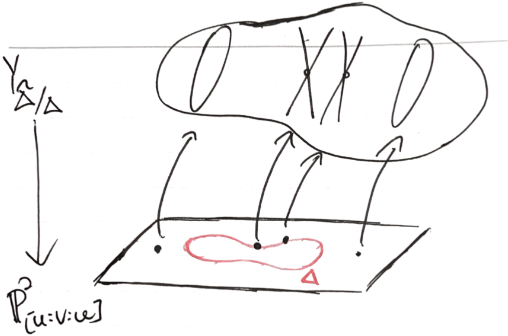
The conic $(*)$ is smooth if and only if $\det(M) \neq 0$.

The Discriminant Curve

Smoothness Criterion (Continued):

The curve defined by $\det(M) = 0$ is called the **discriminant curve** Δ:

$$\Delta = (Q_1 Q_3 - Q_2^2 = 0) \subset \mathbb{P}^2_{[u:v:w]}$$



Quartic Plane Curves

$Q_1 Q_3 - Q_2^2$ is a degree 4 homogeneous real polynomial.

Definition

The roots of a degree 4 homogenous polynomial over $\mathbb{P}^2_{\mathbb{R}}$ is known as a **quartic**.

Theorem (Zeuthen, 1874)

Let Δ be a smooth quartic over $\mathbb{R}$, then $\Delta(\mathbb{R})$ can be classified into 1 of the 6 following topological types:

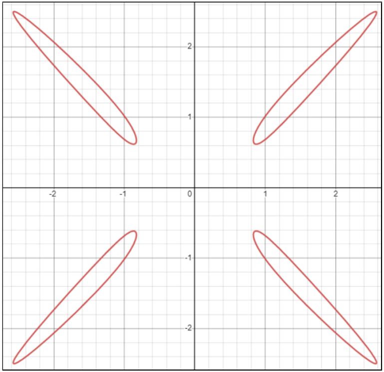
1. No real points
2. One oval
3. Two nested ovals
4. Two non-nested ovals
5. Three ovals
6. Four ovals

Example: Four Ovals

The homogeneous equation defines a smooth quartic whose real component has 4 ovals:

$$0 = \frac{509}{18}x^4 - \frac{6397}{114}x^2y^2 + \frac{2219}{76}y^4 - \frac{2203}{102}x^2z^2 + \frac{4011}{323}y^2z^2 + \frac{2123}{289}z^4$$

The real components on the chart $(z \neq 0)$

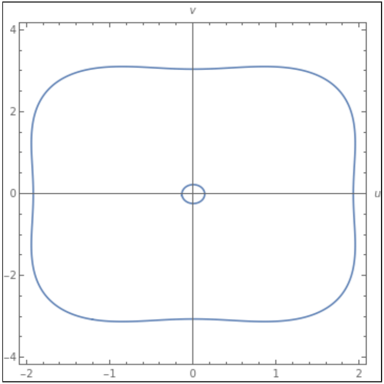


Example: Two Nested Ovals

This homogeneous equation defines a smooth quartic whose real component has 2 nested ovals:

$$0 = -3x^4 - \frac{7}{10}x^2y^2 - \frac{169}{400}y^4 + \frac{67}{6}x^2z^2 + \frac{949}{240}y^2z^2 - \frac{121}{576}z^4$$

The real components on the chart $(z \neq 0)$:



Connectedness

Remark:

Our work at this REU so far has been investigating the relationship between the topological type of $\Delta(\mathbb{R})$ and various properties of $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$

One interesting property we have been investigating so far is the question of connectedness.

Some Results So Far

Theorem

With the previous setup of $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$

- $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has at most 3 connected components
- If the topological type of $\Delta(\mathbb{R})$ is empty, 1 oval, or 4 ovals, then $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is connected
- If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has 2 connected components, then $\Delta(\mathbb{R})$ is either 2 nested ovals or 2 non-nested ovals
- If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has 3 connected components, then $\Delta(\mathbb{R})$ is 3 ovals

Proof.

Exercise :) □