New focused approaches to topics within model selection and approximate Bayesian inversion

Thesis presented for the degree of Philosophiae Doctor (PhD)

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April 1, 2016

Thesis papers

Paper I

JULLUM, M. & HJORT, N. L. (2016). Parametric or nonparametric: The FIC approach. *Minor revision submitted for publication in Statistica Sinica*

Paper II

JULLUM, M. & HJORT, N. L. (2015). What price semiparametric Cox regression? *Submitted for publication in Scandinavian Journal of Statistics*

Paper III

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Paper IV

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Focused inference for the layman

Data: Quantified information gathered from recordings, measurements or surveys

Statistics: Answer scientific questions under uncertainty based on mathematical modelling and analysis of data

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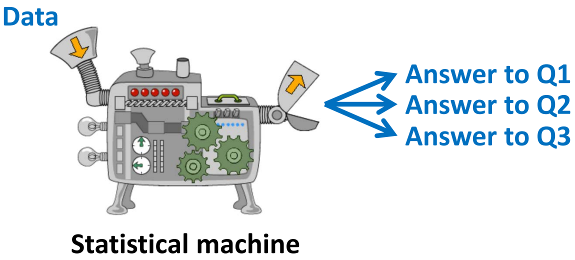

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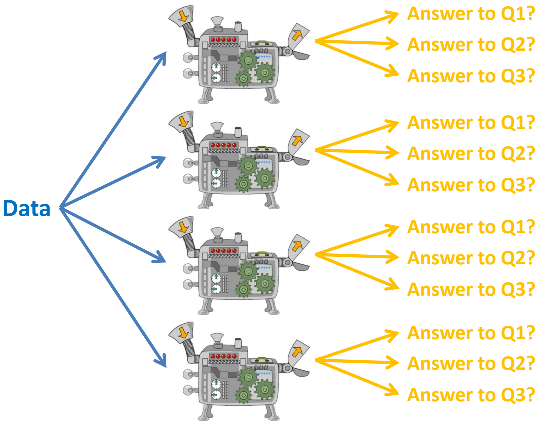
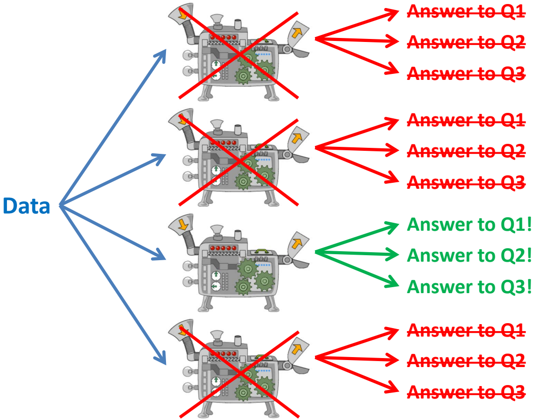


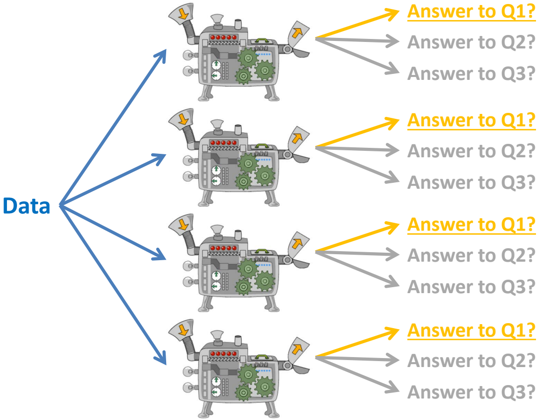
Focused inference for the layman

Traditional (unfocused) approach



Focused inference for the layman

Focused approach (question 1)



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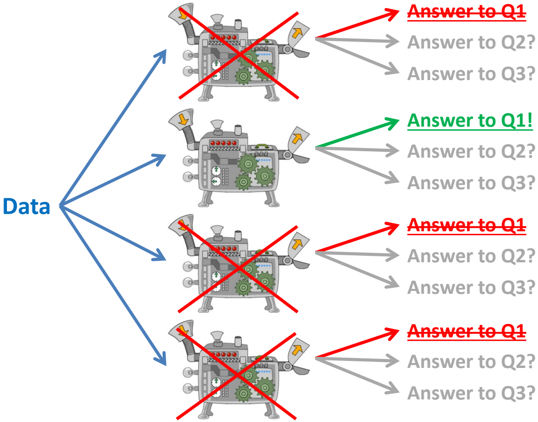

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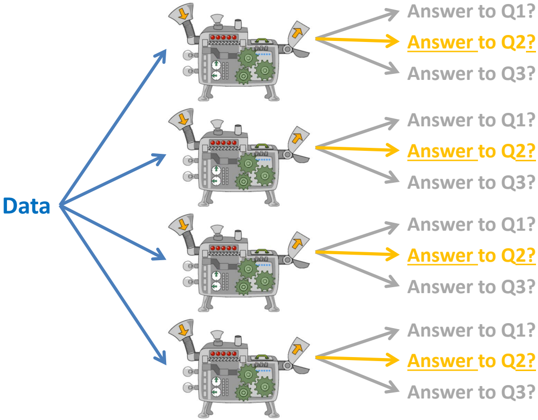

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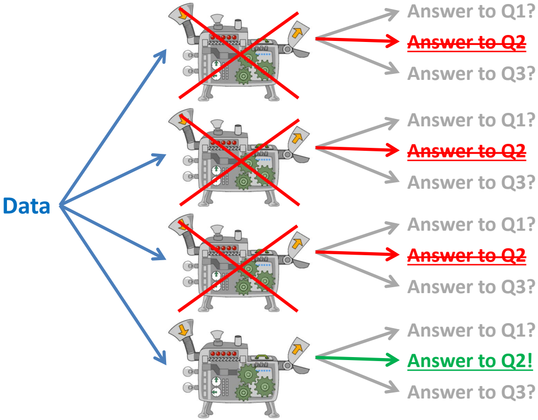

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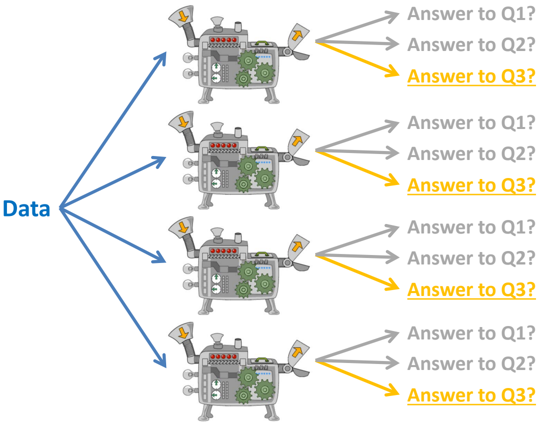
Focused approach (question 3)



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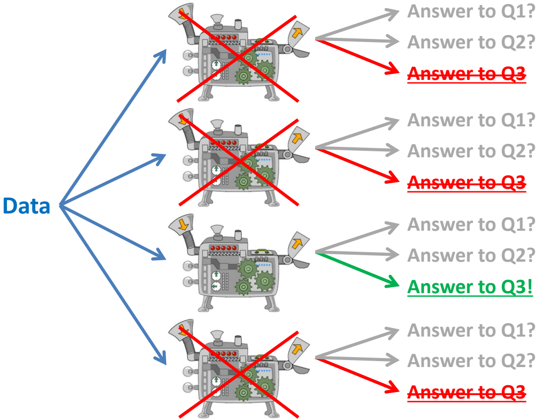
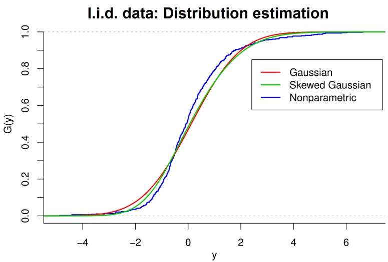
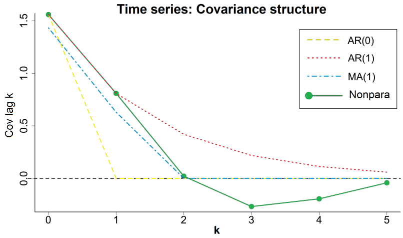
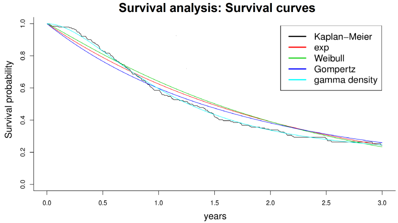


Model selection

- Unknown data generating mechanism G_0 for data $Y_1, \ldots, Y_n$
- Parametric approaches
- Nonparametric approach
- Several appropriate models – which one should we trust?







Motivation and research objective

- Traditional model selection approaches (AIC, BIC, DIC,...) cannot handle selection among parametrics and nonparametrics
- Different models have strengths and weaknesses at different parts of the data space
- What you want to learn should reflect your choice of model

Main research objective Papers I-III

Construct focused/interest driven model selection criteria for selection among a set of parametric and nonparametric type models

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Construct focused/interest driven model selection criteria for selection among a set of parametric and nonparametric type models

Focus parameter

- A general population quantity of interest – not(!) a model specific parameter
- A quantity μ, written as functional T of the distribution G: $\mu = T(G)$

Examples

- Expectation: $\mu = T(G) = \mathrm{E}_G\left(Y_i\right)$
- $\Pr\left(Y_i > 2\right)$: $\mu = T(G) = 1 - G(2)$
- Interquartile range: $\mu = T(G) = G^{-1}(3/4) - G^{-1}(1/4)$

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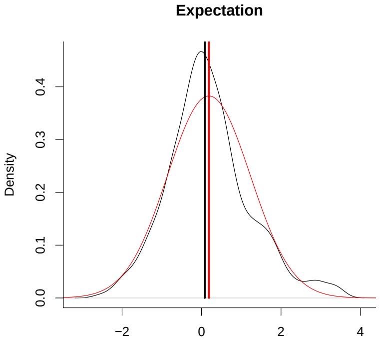

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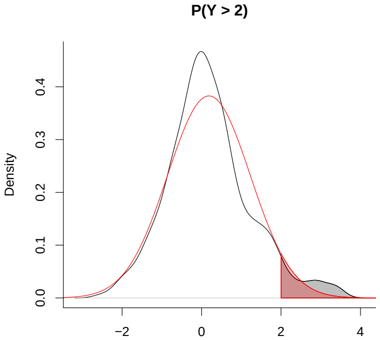

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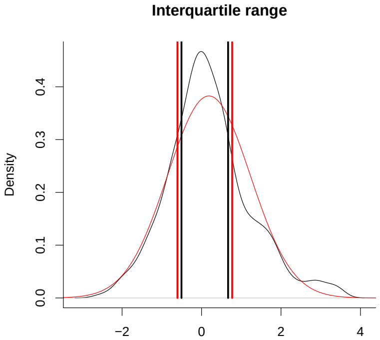

Criterion idea

- Model selection problem $\rightarrow$ Best estimator for μ
- Estimate μ by plug-in estimation for each model M: $\widehat{\mu}_M = T(\widehat{G}_M)$
- Performance measure:

$$\text{risk} = \text{mse}(\widehat{\mu}_M) = \mathrm{E}\left\{(\widehat{\mu}_M - \mu_{\text{true}})^2\right\} = \text{bias}^2(\widehat{\mu}_M) + \text{Var}(\widehat{\mu}_M)$$

Basic idea: Focused information criterion (FIC)

- Estimate the mean squared error (mse) as squared bias + variance:

$$\text{FIC}(M) = \widehat{\text{mse}}(\widehat{\mu}_M) = \widehat{\text{bias}^2}(\widehat{\mu}_M) + \widehat{\text{Var}}(\widehat{\mu}_M)$$

- Choose the model and estimator with the smallest FIC score

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Average/weighted FIC

- Generalisation of FIC for a (weighted) set of focus parameters
- Performance measure:
  risk $= \int \mathrm{mse}(\widehat{\mu}_M(t))\,\mathsf{d}W(t) = \int \mathsf{E}\left[\{\widehat{\mu}_M(t) - \mu_{\mathsf{true}}(t)\}^2\right]\mathsf{d}W(t)$,
- $\mathsf{AFIC}(M) = \int \widehat{\mathrm{mse}}(\widehat{\mu}_M(t))\,\mathsf{d}W(t) = \int \mathrm{FIC}(\widehat{\mu}_M(t))\,\mathsf{d}W(t)$

FIC/AFIC asymptotics (pm vs. np)

- Parametric model biased: Pr (FIC/AFIC selects pm) $\to 0$
- Parametric model correct: Pr (FIC selects pm) $\to \chi_1^2(2) \approx 0.843$

Original FIC

- Idea based on the original FIC by Claeskens & Hjort (2003)
- Our approach does not require a local misspecification framework and works for nonparametrics and with non-nested parametric models

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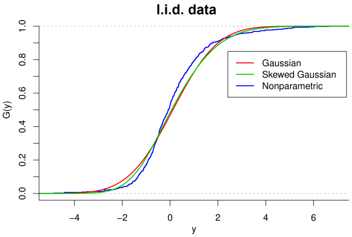
Paper I

JULLUM, M. & HJORT, N. L. (2016). Parametric or nonparametric: The FIC approach. *Minor revision submitted for publication in Statistica Sinica*

- Main contribution: Develop and study the FIC construction routine for i.i.d. data
- G: The cumulative distribution function
- Nonparametric estimation: Empirical distribution function $\widehat{G}_n$
- Parametric estimation: Ordinary maximum likelihood estimation for parametric families F_θ

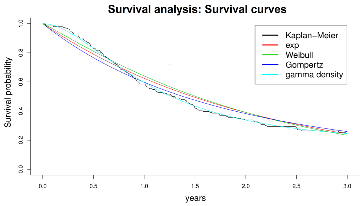
- Typical focus parameters: Smooth functions of means and quantiles
- Also discuss corresponding FIC schemes for density estimation and regression



Paper II

JULLUM, M. & HJORT, N. L. (2015). What price semiparametric Cox regression? *Submitted for publication in Scandinavian Journal of Statistics*

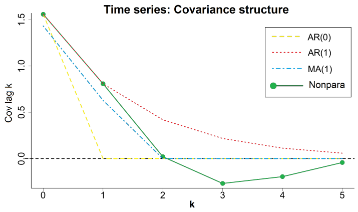
- Main contribution: Lifting the FIC framework to censored survival time data with covariates
- $G = \{A(\cdot), \beta\}$ corresponding to the hazard rate function: $\alpha(\cdot)\exp(x^{\mathrm{t}}\beta)$
- 'Nonparametric' estimation: Semiparametric Cox regression
- Parametric estimation: Joint ML estimation of θ, β, with a parametric hazard rate function $\alpha_\theta(\cdot)$
- Typical focus parameters: Survival probabilities, quantiles and cumulative hazards, conditional on covariate values
- Also investigate the asymptotic relative efficiency (ARE) for various focus parameters



Paper III

HERMANSEN, G. H., HJORT, N. L. & JULLUM, M. (2015). Parametric or nonparametric: The FIC approach for stationary time series. *Technical report, Department of Mathematics, University of Oslo*

- Main contribution: Lifts the FIC framework to stationary Gaussian time series
- G: Spectral measure/distribution
- Nonparametric estimation: Periodogram $\widehat{G}_n$
- Typical parametric alternatives: Autoregressive and moving average models, estimated by ML or using the Whittle approximation
- Typical focus parameters: Differences in spectral distribution, covariance lags and correlation lags



Forward and inverse problems

- Consider

$$y = H(x) + \varepsilon$$

  - y: observable data
  - x: latent cause/source
  - H: (causal) mechanism operator
  - ε: noise term

- Forward problem: 'Finding' y based on x
- Inverse problem: 'Finding' x based on y

Bayesian solution to the inverse problem

- Apply Bayes' formula $p(x|y) \propto p(y|x)p(x)$
- Consult posterior distribution $p(x|y)$

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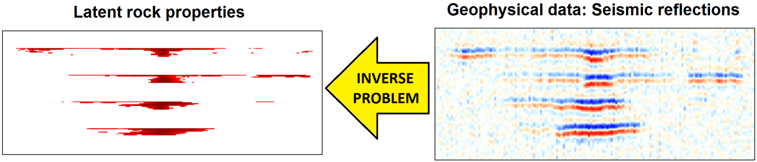
Inverse problem within the geosciences



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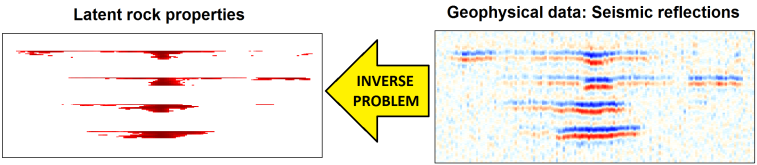

Forward model

$$\mathbf{r} \longrightarrow \mathbf{m} \longrightarrow \mathbf{d}$$

- $\mathbf{r}$: Latent rock properties
- $\mathbf{m}$: Latent geophysical properties
- $\mathbf{d}$: Geophysical data

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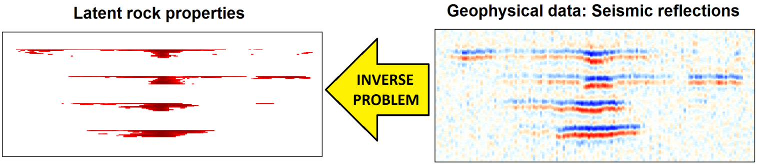

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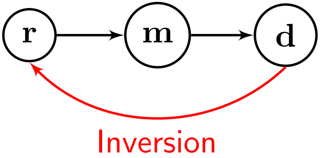

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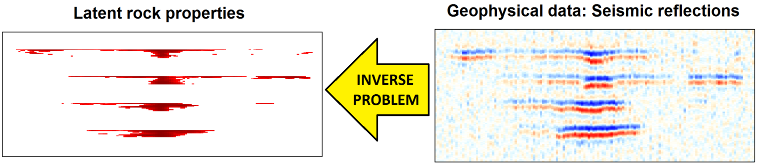

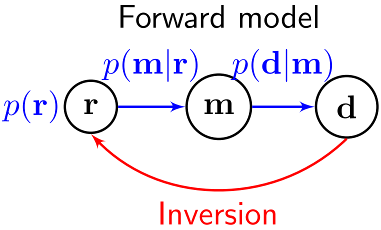

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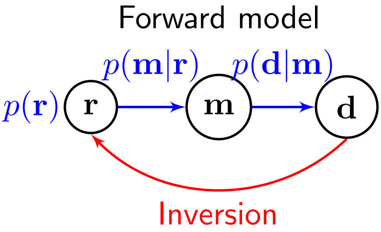

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Posterior distribution: $p(\mathbf{r}|\mathbf{d}) \propto \int p(\mathbf{d}|\mathbf{m})p(\mathbf{m}|\mathbf{r})p(\mathbf{r})\,d\mathbf{m}$

- High dimensional problem
- Enormous amount of highly correlated data $\mathbf{d}$
- Complex dependency structures
- Analytical expression for posterior seldom available
- MCMC can be very time consuming

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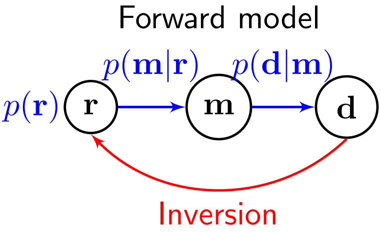

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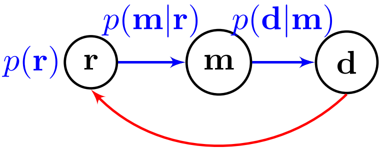

Inversion

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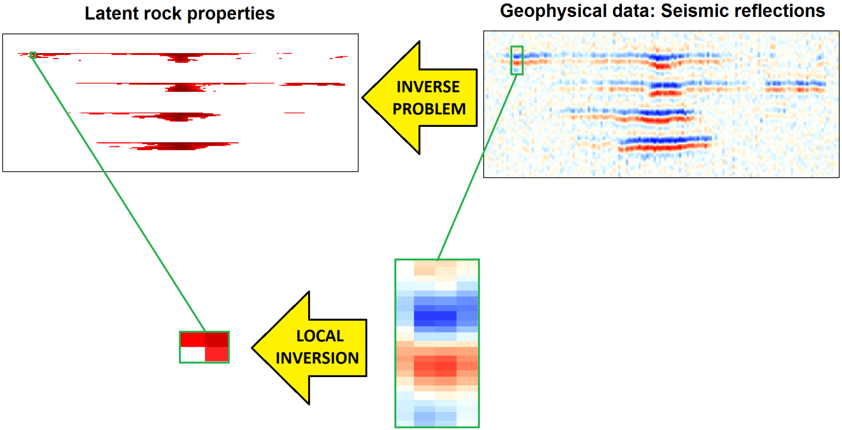
JULLUM, M. & KOLBJØRNSEN, O. (2016). A Gaussian-based framework for local Bayesian inversion of geophysical data to rock properties. *Geophysics* **81**(3), R1–R13.



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- Divide the global inversion problem into several local inversions
  - Approximate marginal posterior $p(\mathbf{r}_i|\mathbf{d})$ for each cell i in the gridded region, rather than the global $p(\mathbf{r}|\mathbf{d})$ for the full region
- Dimension reduction by only using variables spatially close to the cell in focus

- 3 neighborhoods of cells B, C, D, with local variables $\mathbf{r}_B, \mathbf{m}_C, \mathbf{d}_D$
- Approximate marginal posterior:

$$p(\mathbf{r}_i|\mathbf{d}_D) \approx p^*(\mathbf{r}_i|\mathbf{d}_D) \propto \int p^*(\mathbf{d}_D|\mathbf{r}_B)p(\mathbf{r}_B)\,\mathrm{d}\mathbf{r}_{B_{-i}},$$

  with Gaussian approximation

  $p^*(\mathbf{d}_D|\mathbf{r}_B) = \int p^*(\mathbf{d}_D|\mathbf{m}_C)p^*(\mathbf{m}_C|\mathbf{r}_B)\,\mathrm{d}\mathbf{m}_C$
- Weighted Monte Carlo routine for sample based evaluation of posterior

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$$p(\mathbf{r}_i|\mathbf{d}_D) \approx p^*(\mathbf{r}_i|\mathbf{d}_D) \propto \int p^*(\mathbf{d}_D|\mathbf{r}_B)p(\mathbf{r}_B)\,\mathrm{d}\mathbf{r}_{B_{-i}},$$

  with Gaussian approximation
  $p^*(\mathbf{d}_D|\mathbf{r}_B) = \int p^*(\mathbf{d}_D|\mathbf{m}_C)p^*(\mathbf{m}_C|\mathbf{r}_B)\,\mathrm{d}\mathbf{m}_C$
- Weighted Monte Carlo routine for sample based evaluation of posterior

Paper IV

JULLUM, M. & KOLBJØRNSEN, O. (2016). A Gaussian-based framework for local Bayesian inversion of geophysical data to rock properties. *Geophysics* **81**(3), R1–R13.

Properties

- Computationally cheap under stationarity conditions due to reuse of Gaussian approximations
- Offers a range of procedures with a trade-off between accuracy and computationally speed

Main thesis contributions

- Development of a principally new focused model selection strategy for selection among parametric and nonparametric type models
  - Few alternatives available
  - A new paradigm for the FIC
  - Beneficial theoretical behaviour
- Development of a new, locally focused procedure for Bayesian inversion within the geosciences
  - Combination of accuracy and computational speed seems to be out of reach for competing methodology

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