CENX570: Simulation and Modelling

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Project Phase 1

Simulation of rare events: A report

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Abstract- *Simulation of rare events is a process of developing a digital twin of a system that helps to analyze the actual system and to study the occurrence of rare events based on various inputs. Researchers use techniques in Simulation of rare events to avoid prohibitively large amount of computer time needed. In this project, we will have a detailed look at rare events and review some of the techniques used in research for the simulation of rare events. We will also discuss expected research directions.*

I. INTRODUCTION

Rare events: Rare events are events that are expected to occur infrequently or, more technically, those that have low probabilities (say, order of 10^{-3} or less) of occurring according to a probability model [1]. Rare-event simulation involves estimating extremely small but important probabilities. Any important measures of performance in communications networks are defined in terms of rare event probabilities. Obtaining accurate estimates of such rare event probabilities using computer simulation can require execution times that are prohibitively long, but simulation techniques enable using rare events in predicting performance of a system.

Computer time needed to achieve a specified relative error (standard simulation)

	1 replication per second		1,000 replications per second	
	Desired Relative Error		Desired Relative Error	
γ	100%	10%	100%	10%
10^{-3}	17 min	1 day	1 sec	2 min
10^{-5}	1 day	4 months	2 min	3 hr
10^{-7}	4 months	32 years	3 hr	12 days
10^{-9}	32 years	3,169 years	12 days	3 years

Figure: Time needed to simulate rare events

A. Application of rare-events simulations

Rare-event simulation is used in many application areas, including communication and networking where packets loss and data loss, communication systems where the rare event could be the loss of important information, is crucial for system metrics. Another example where it is widely used is finance and insurance, where the rare event could be a large financial gain or loss, power systems where the rare event could be a major power outage, network reliability where the rare event could mean that certain nodes in the network are disconnected, nuclear physics where the rare event could be that particles cross a given protection shield, and aviation safety where the rare event could be an aircraft collision.

It is also encountered in computer graphics (image synthesis by Monte Carlo methods), computational statistics, computational physics, computational biology, military applications, and so on [3].

II. RELATED RESEARCH

Computation of such rare-event probabilities is challenging. Analytical solutions are usually not available for nontrivial problems, and standard Monte Carlo simulation is computationally inefficient. Therefore, much research effort has focused on developing advanced stochastic simulation methods that are more efficient. In spite of the advances in computational power, using simulation to obtain rare event probabilities such as cell/packet loss or delay in networks still requires prohibitively long execution times [2]. Regenerative techniques must be used to make the rare events simulation feasible and efficient. M. Devetsikiotis et. al [8] suggest the application of regenerative techniques to obtain correct confidence intervals for the estimates involved.

The authors present a methodology that uses Importance Sampling (IS) dynamically, within each regeneration cycle, to drive the system back to the regeneration state, after an accurate estimate has been obtained. We will now look at three broad classifications namely Importance Sampling, subset simulation and Splitting.

A. Importance Sampling (IS)

Importance sampling (IS) refers to a collection of Monte Carlo methods where a mathematical expectation with respect to a target distribution is approximated by a weighted average of random draws from another distribution.

IS-based techniques can be broadly grouped into two categories: techniques where the individual stochastic elements are modified or biased, and those where the global evolution of the system is manipulated. J. Keith et al. [2] in their work, reviewed various techniques to simulate rare events in communication networks. Their research work contributed to the following work, which details different Importance sampling techniques

The IS estimator is now constructed similarly to (6) by utilizing the law of large numbers:

$$p_{\mathcal{E}} \approx p_{\mathcal{E}}^{IS} = \frac{1}{N}\sum_{i=1}^{N} \frac{I_{\mathcal{E}}(U_i)p(U_i)}{q(U_i)} = \frac{1}{N}\sum_{i=1}^{N} I_{\mathcal{E}}(U_i)w(U_i),$$

where $U_1, \ldots, U_N$ are i.i.d. samples from $q(u)$, called the *importance sampling density* (ISD), and $w(U_i) = \frac{p(U_i)}{q(U_i)}$ is the *importance weight* of sample U_i.

Figure: Importance Sampling Estimator

i. Modification Of Individual Stochastic Elements

The process of simulation involves developing models of the various key functions of the system, interconnecting these models to mimic behaviour of the actual system, and finally, performing the actual simulation trials. Examples include models of sources, traffic routing, and packet lengths. An important category of IS techniques involves modifying the underlying probability distributions of one or more of these random number generators in the simulation model [2].

Applying IS in this fashion typically requires considerable prior knowledge about the system. This prior knowledge relates how the modified random number distributions affect the distribution of the target event(s) of interest. A chief issue when using this technique is determining how much the distributions of the random number generators should be modified or biased. This issue of tuning/optimization is covered below.

ii. Global Modification Via Trajectory Splitting

An alternative way to increase the relative number of visits to the important rare event region is to use (trajectory) splitting, which we will see in next section.

Importance sampling is a Monte Carlo simulation variance reduction technique that has achieved dramatic results in estimating performance measures associated with certain rare events. It involves simulating the system under a change of measure that accentuates paths to the rare-event and then un-biasing the resultant output from the generated path by weighing it with the 'likelihood ratio' [4]. [9] utilize likelihood ratio (LR) and importance sampling (IS) methods to optimize complex computer simulation models involving rare events.

P. Shahabuddin and S. Juneja [4] review some of the recent developments for efficient estimation of rare-events, most of which involve application of importance sampling techniques to achieve variance reduction. The zero-variance importance sampling measure is well known and, in many cases, has a simple representation. Though not implementable, it proves useful in selecting good and implementable importance sampling changes of measure that are in some sense close to it and thus provides a unifying framework for such selections.

Using asymptotically optimal Importance sampling (IS), Philips Heidelberger [5] studied fast simulation of rare events and the probabilities of long waiting times or buffer overflow rare events in queueing and reliability models, and system failure events in reliability models of highly dependable computing systems.

B. Subset Simulation

The Subset Simulation (SS) method is an advanced stochastic simulation method for estimating rare events which is based on Markov chain Monte Carlo (MCMC). The basic idea behind SS is to represent a very small probability p $\mathcal{E}$ of the rare event E as a product of larger probabilities of "more-frequent" events and then estimate these larger probabilities separately [1]. Let,

$$\mathbb{R}^D \equiv \mathcal{E}_0 \supset \mathcal{E}_1 \ldots \supset \mathcal{E}_L \equiv \mathcal{E}$$

be a sequence of nested subsets of the uncertain excitation space starting from the entire space $\mathcal{E}_0 = R^D$ and shrinking to the target rare event $\mathcal{E}_L = \mathcal{E}$.

Using the notion of conditional probability and exploiting the nesting of the subsets, the target probability pE can be factorized as follows:

$$p_{\mathcal{E}} = \prod_{i=1}^{L} \mathbb{P}(\mathcal{E}_i | \mathcal{E}_{i-1}).$$

C. Splitting

The idea of splitting is to “split” (or clone) a simulation run into separate runs whenever it gets “near” the rare event of interest. These runs share a common history up to the splitting point, but conditional on that history, they evolve independently of each other. In this way, more computer time is spent on promising runs that are close to the rare event [3]. Splitting can be regarded as an indirect IS technique which, instead of modifying individual stochastic sources, changes the entire transition probability structure of the system.

sample paths of $h(X_t)$. The rare-event set $\mathcal{R}$ is defined as the set of states whose level is at least as large as some constant $l > 0$. That is, $\mathcal{R} \equiv \{x \in \chi : h(x) \geq l\}$. Let T_R be the time the process first enters $\mathcal{R}$, and let T_S be the time the process first returns to level 0. Specifically,

$$T_R \equiv \inf\{t > 0 : h(X_t) \geq l\}, \qquad T_S \equiv \inf\{t > 0 : h(X_t) = 0\},$$

where it is assumed that the process has just left level 0 at time $t = 0$ $(h(X_0) > 0$ and

The fundamental idea of splitting is based on the assumption that there exist some well identifiable intermediate system states that are visited much more often than the target states themselves and behave as gateway states to reach the target states. For example, if the target states represent a full queue in a queuing system, the states that correspond to the case when the queue is at least half full can be regarded as intermediate states. A very important feature of splitting is that the step-by-step evolution of the system follows the original probability measure.

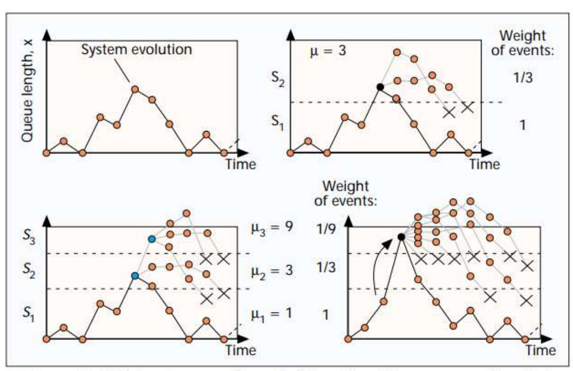

Figure 2. *Splitting increases the probability of reaching rare states by splitting the system trajectory upon entering well-defined subsets of the state-space.*

Figure: Splitting to reach rare states

There are many variations of level splitting and we describe some of the key ideas here.

i. Fixed splitting

In this approach, any simulation run that reaches a new level is split into a fixed number of independent runs. An advantage of fixed splitting is that it can be implemented recursively in a depth-first manner. This means that the computer only needs to store, at most, a single system state per level, corresponding to the simulation history of the current run. To implement fixed splitting, the simulation starts at an initial state and proceeds until it reaches either level 1 or level 0. This process continues so that all offspring of a clone are simulated to conclusion before proceeding to the next clone.

One drawback with fixed splitting is that it is sensitive to the choice of the splitting factor R. If R is too high, the total number of runs and the amount of work explodes. If R is too low, then few simulations reach the rare event.

ii. Fixed-effort splitting

In this approach, a predetermined total number of runs are made at each level. This is different from fixed splitting where the total number of runs at each level is random based on the outcomes of runs at lower levels. The main advantage is that there is no need to know a good splitting factor R in advance, and because the total number of runs at each level is fixed, the overall variance is often lower. The drawback is that the computer must store all of the entrance states to a level to serve as the candidate starting states for simulation of the next level. This can lead to memory problems for models with large state spaces.

iii. Fixed-success splitting and fixed probability of success

In fixed-success splitting, at each level, the total number of trajectories that must reach the next level is fixed; independent replications at the current level continue until the fixed number of successes is achieved. Apparently, this approach is often superior to fixed-splitting and fixed-effort. In fixed probability of success, the levels are learned adaptively so that the probability of reaching the next level is approximately the same at all levels [3].

Sergey Bravyi and Alexander Vargo [7] used the splitting method from Monte Carlo simulation and Bennett's acceptance ratio method for estimating the free energy difference between two canonical ensembles for quantum error correction M. Villen-Altamirano et al. [6] presented a custom method for fast simulation of rare events in teletraffic and high-speed data networks, called RESTART (REpetitive Simulation Trials After Reaching Thresholds) which obtained dramatic computer time savings for an equal confidence of the results.

III. CONCLUSION

Although application of any IS technique requires a problem-specific analytical phase, these techniques have been applied to a number of different networking problems that required obtaining rare event probabilities. Also, there is a trade-off in simulation techniques, pushing the system too hard or altering its parameters, will result in the IS estimator to be much worse than standard simulation and the results may not reflect the reality.

We hope that as computer technology continues to advance, simulation will be used to evaluate more complicated network mechanisms. At the same time, more reliable networks will be characterized by even rarer events. Both trends will increase the importance of IS-based techniques for rare event simulation.

IV. FURTHER RESEARCH

One approach to reduce the computational effort when estimating very low rare-event probabilities is to utilize additional information about the nature of the problem for specific classes of reliability problems. Another more general approach is to construct surrogate models (meta-models) based on using a relatively small number of complex-model simulations as training data. The idea is to use a trained surrogate model to rapidly calculate an approximation of the response of the complex computational model as a substitute when drawing new samples.

Various methods for constructing surrogate models have been applied in reliability engineering, including response surfaces, support vector machines, neural networks, and Gaussian process modelling. The latter method is a particularly powerful one because it also provides a probabilistic assessment of the approximation error. It deserves further exploration, especially with regard to the optimal balance between the accuracy of the surrogate model as a function of the number of training samples from the complex model, and the accuracy of the estimate of the rare-event probability as a function of the total number of samples from both the complex model and the surrogate model.

Another approach could be useful for large datasets with multiple rare event types requiring various levels of modifications. Such data can be split first into multiple data streams and then we can perform importance sampling individually on the data streams. By doing so, large and attribute rich data can be processed to yield case-specific rare events depending on the splitting and importance sampling priorities..

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