SYSTEMS OF LINEAR EQUATIONS CAN BE SEEN AS PROJECTIONS

Abstract. Solving a system of linear equations is equivalent to conducting an orthogonal projection in a sense that is made precise below.

1. Problem statement

1.1. **First Version of the Problem.** Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$, find a vector $v \in \mathbb{R}^n$ such that $Av = b$.

1.2. This problem has an exact solution if and only if rank A = rank $A \,|\, b$, or equivalently, if $b \in \text{span}\{A_1, \dots, A_n\}$.

1.3. If A is a square, full-rank matrix, then the exact solution to the problem $\hat{v}$ can be given as $\hat{v} = A^{-1}b$.

1.4. The entries of a solution vector $\hat{v} = (v_1, \dots, v_n)$ can be thought of as a recipe to carry out a linear combination of the column vectors of A, so that the result is exactly b.

1.5. When is the problem not well-defined? Consider the case when rank $A <$ rank $[A \,|\, b]$. In this case $b \notin \text{span}\{A_1, \dots, A_n\}$, i.e., b is not a linear combination of the column vectors of A.

1.6. Second Version of the Problem. One possible way of relaxing the problem statement would go as follows. Find $\hat{v}$ such that $A\hat{v} = \pi(b)$, where $\pi(b)$ is some convenient transform of b satisfying:

i $\pi(b)$ is not too far apart from b
ii $\pi(b) \in \text{span}\{A_1, \dots, A_n\}$

1.7. Whenever they exist, solutions of the first version of the problem would also be solutions for the second version, although we cannot expect the converse to be true.

2. Orthogonal Projection

2.1. A linear map $\pi_v : V \to V$ is deemed an orthogonal projection onto a vector v if it maps v onto itself and any vector orthogonal to v to zero.

2.2. What is the matrix of π_v? It depends on the basis of V we employ to represent vector coordinates. Fixing an orthonormal basis of V, the matrix of π_v is simply the rank one square matrix given by uu^t, where $u = v/\|v\|$.

2.3. Let S be a vector subspace of V. We can define the orthogonal projection onto S as the linear map $\pi_S : V \to V$ that maps any vector of S to itself and any vector orthogonal to S to zero.

2.4. Propostion. Suppose that $S = \operatorname{span}\{u_1, \ldots, u_n\}$ and let's further assume that the vectors $u_1, \ldots, u_n$ are an orthogonal set. Then the orthogonal projection onto S is given by the following linear map:

$$\pi_S = \sum_{i=1}^{n} \pi_{u_i} = \sum_{i=1}^{n} \frac{1}{\|u_i\|^2} u_i u_i^t \tag{2.1}$$

Proof. First, we check that any vector v of S is mapped to itself by π_S. Since v belongs to S we can write $v = \lambda_1 u_1 + \ldots + \lambda_n u_n$. Then

$$\begin{aligned}
\pi_S(v) &= \sum_{i=1}^{n} \frac{1}{\|u_i\|^2} u_i u_i^t v \\
&= \sum_{i=1}^{n} \frac{\lambda_i}{\|u_i\|^2} u_i u_i^t u_i = \sum_{i=1}^{n} \frac{\lambda_i \|u_i\|^2}{\|u_i\|^2} u_i \\
&= \sum_{i=1}^{n} \lambda_i u_i = v.
\end{aligned}$$

Second, any vector v orthogonal to S is mapped to zero by π_S. To justify this, observe note that if v is orthogonal to S, it is orthogonal to all its generators. □

2.5. The projection π_S defined in equation 2.1 can be written in matrix form as $A(A^tA)^{-1}A^t$, where $A = [\, u_1 \,|\, \ldots \,|\, u_n \,]$ denotes the matrix with $u_1, \ldots, u_n$ as column vectors.

2.6. Note that if the vectors $u_1, \ldots, u_n$ were an orthonormal set, then the matrix expression for π_S would be just AA^t, since the middle factor A^tA would be the identity matrix.

2.7. Remark that if a vector u is in the null space of π_S, then it must be orthogonal to S. Can you tell why?

2.8. Now assume that the column vectors of A are linearly independent, but not necessarily orthonormal. We know there is a full-rank, square matrix E such that the matrix $Q = AE$ has orthonormal vectors as columns. Then the projection π_S can be simply written in matrix form as QQ^t, which we can transform a bit using basic matrix multiplication rules:

$$\begin{aligned}
QQ^t &= Q(Q^tQ)^{-1}Q^t = (AE)((AE)^t(AE))^{-1}(AE)^t \\
&= AE(E^tA^tAE)^{-1}E^tA^t = AEE^{-1}(A^tA)^{-1}(E^t)^{-1}E^tA^t \\
&= A(A^tA)^{-1}A^t
\end{aligned}$$

Which proves the following...

2.9. Theorem. For any matrix A with a basis of S as columns, the matrix of the orthogonal projection π_S is given by $A(A^tA)^{-1}A^t$. In other words, the matrix of π_S does not depend on a particular choice of a basis for S.

3. Back to the problem

3.1. Can the reader suggest one possible transform that serves the purpose of solving the linear system problem as described in section 1.6? We might try the orthogonal projection onto the column space of A. Let's denote this linear map simply as π.

3.2. Assuming that the columns of A are linearly independent, the orthogonal projection of b onto the column space of A can be given as $\pi(b) = A(A^tA)^{-1}A^tb$. In view of this expression, the vector $\hat{v} = (A^tA)^{-1}A^tb$ would be a good candidate solution, as it satisfies $A\hat{v} = \pi_A(b)$.

3.3. But it remains to be discussed how the "not too far" part from the requirements in section 1.6 is achieved with the π projection.

3.4. Propostion. The orthogonal projection of a vector v onto a subspace S, denoted $\pi_S(v)$, is the closest possible vector in S to v, in the following sense: for any $w \in S$, $\|w - v\| \geq \|\pi_S(v) - v\|$.

Proof. Suppose the non-trivial case where $v \notin S$. If we set $u = \pi_S(v) - v$ then $u \in \ker(\pi_S)$, which in turn implies that u is orthogonal to S. On the other hand, for any $w \in S$ we have $w = \pi_S(v) + r$, for some other $r \in S$. Then $w - v = \pi_S(v) + r - \pi_S(v) + u = r + u$ with u orthogonal to r, whereas $\pi_S(v) - v = u$. Consequently,

$$\|w - v\|^2 = (r + u)^t(r + u) = \|r\|^2 + \|u\|^2 \geq \|u\|^2 = \|\pi_S(v) - v\|^2$$

$\square$