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UNDERSTADING LINEAR MAPS WITH GAUSS-JORDAN ELIMINATION

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Abstract. We study some basic properties of linear maps by using column-wise Gauss-Jordan elimination.

1. Gauss-Jordan elimination

1.1. Given an $m \times n$ matrix A, we denote $f_A : \mathbb{R}^n \to \mathbb{R}^m$ the linear map defined by A in coordinates of the canonical basis. We denote $\mathbb{R}^{m\times n}$ the space of matrices of shape $m \times n$.

1.2. We say that a pair of matrices (A, B) where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times n}$ is a (column-wise) Gauss-Jordan pair.

1.3. Our discussion is based entirely on column-wise operations, so we will implicitly assume elimination as being column-wise in this text. Likewise, matrix pairs can be defined column-wise and row-wise.

1.4. Gauss-Jordan pairs are meant to represent the steps in the execution of Gauss-Jordan elimination.

1.5. Column-wise elementary operations can be either of three kinds:

(1) Permutating columns.
(2) Replacing a column c_i with λc_i where $\lambda \neq 0$.
(3) Replacing a column c_i with $c_i + \mu c_j$ where μ can be any scalar.

We will generally denote the transform of a matrix A by an elementary operation e as $e(A)$.

1.6. Elementary operations are linear maps $\mathbb{R}^n \to \mathbb{R}^n$. We say that E is the matrix representing the elementary operation e if the transform matrix of A can be written $e(A) = AE$ for all $A \in \mathbb{R}^{m\times n}$.

1.7. Elementary operations are injective to injective linear maps $\mathbb{R}^n \to \mathbb{R}^n$, hence the transformation $A \mapsto AE$ preserves the rank.

1.8. We can extend operations to Gauss-Jordan pairs as follows $(A, B) \mapsto (AE, BE)$.

1.9. Whenever (A, B) can be transformed into (A', B') by applying a sequence of elementary operations, we denote it as $(A, B) \sim (A', B')$.

1.10. Column-wise Gauss-Jordan elimination is the method whereby the Gauss-Jordan pair (A, I_n) is transformed into another pair (L, B) with L in lower (column) echelon form by means of applying a sequence of elementary operations. Following our notation: $(A, I_n) \sim (L, B)$.

1.11. If $(A, I_n) \sim (L, B)$ then $L = AB$.

Proof. Let $E = E_1 \cdots E_k$ the product of the matrices encoding the elementary operations that have been applied. By definition $L = AE$ and $B = I_n E = E$, so it follows that $L = AB$. □

1.12. Observe that $\operatorname{rank}(B) = \operatorname{rank}(I_n) = n$.

1.13. If A is a square, full-rank matrix, then $(A, I_n) \sim (I_n, A^{-1})$.

Proof. If $(A, I_n) \sim (I_n, B)$ for some matrix B, then we know that $I_n = AB$, so B is the inverse of A. □

1.14. For the next discussion, let $(A, I_n) \sim (L, B)$, with L in lower echelon form. Let $B = [b_1 \dots b_n]$ be the matrix B specified by its columns B_i.

1.15. The rank of A is precisely the number of non-zero columns in L: these are the generators of $\operatorname{Im}(f_A)$. Since any subset of non-zero columns of L is a linearly independent set, it follows that then $\dim \operatorname{Im}(f_A) = \operatorname{rank}(A)$.

1.16. Likewise, the last $n - \operatorname{rank}(A)$ columns of L are zero.

1.17. The last $n - \operatorname{rank}(A)$ columns of B form a basis of the null-space of f_A.

Proof. We already know the following:

(1) $b_1, \dots, b_n$ is a basis of $\mathbb{R}^n$.
(2) $\{Ab_i \mid 1 \leq i \leq \operatorname{rank}(A)\}$ is a linearly independent set.
(3) $Ab_i = 0$ for $\operatorname{rank}(A) + 1 \leq i \leq n$. Consequently,

$$S = \operatorname{span}\{b_i \mid \operatorname{rank}(A) + 1 \leq i \leq n\}\} \subset \ker(f_A).$$

But there is no vector v outside of S with $v \in \ker(f_A)$. Let's reason by contradiction assuming there is one. Since $\{b_1, \dots, b_n\}$ is a basis, we can write $v = \sum \lambda_i b_i$. Then $f_A(v) = 0 = \sum_{i \leq \operatorname{rank}(A)} \lambda_i Ab_i$, which is in contradiction with point (1). We conclude that $S = \ker(f_A)$. □

1.18. It follows that $\dim \ker(f_A) = n - \operatorname{rank}(A)$.

1.19. The following important result follows from all the discussion above. For any linear map $f_A : \mathbb{R}^n \to \mathbb{R}^m$ the following identity holds:

$$\dim \ker(f_A) + \dim \operatorname{Im}(f_A) = n.$$