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UNDERSTADING LINEAR MAPS WITH GAUSS-JORDAN ELIMINATION

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Abstract. We study some basic properties of linear maps by using column-wise Gauss-Jordan elimination.

1. Gauss-Jordan elimination

1.1. Given an $m \times n$ matrix A, we denote $f_A : \mathbb{R}^n \to \mathbb{R}^m$ the linear map defined by A in coordinates of the canonical basis. We denote the space of matrices of shape $m \times n$ by $\mathbb{R}^m \times n$.

1.2. For convenience we say that a pair of matrices (A, B) where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times n}$ is a (column-wise) Gauss-Jordan pair.

1.3. Gauss-Jordan (column-wise) pairs intend to represent the steps in the Gauss-Jordan elimination procedure. The motivation is that each pair will represent the state of a Gauss-Jordan elimination A is the matrix under study and B keeps track of the column-wise elementary operations.

1.4. Column-wise elementary operations can be of either of three kinds:

(1) Permutation of columns.
(2) Replacing a column c_i with λc_i with $\lambda \neq 0$.
(3) Replacing a column c_i with $c_i + \mu c_j$ with μ any scalar.

1.5. Column-wise elementary operation can be encoded as linear maps $\mathbb{R}^n \to \mathbb{R}^n$ (endomorphisms of $\mathbb{R}^n$). Hence we can encode such operations via matrix multiplication. In particular, we can encode in this way any composition of elementary operations by simply multiplying their respective matrices.

1.6. Such operations are equivalent to injective linear maps, hence they preserve the rank.

1.7. We can apply column-wise elementary operations on pairs as follows $(A, B) \mapsto (g(A), g(B))$.

1.8. Whenever (A, B) is taken to (A', B') by column-wise elementary operations, we denote it as $(A, B) \sim (A', B')$.

1.9. Column-wise Gauss-Jordan elimination is the method whereby the (column-wise) Gauss-Jordan pair (A, I_n) is transformed into another pair (T, B) with T in upper-column-echelon form by means of iteratively applying a sequence of column-wise elementary operations. Hence $(A, I_n) \sim (T, B)$. Without loss of generality

1.10. If $(A, I_n) \sim (T, B)$ then $T = AB$.

Proof. Let E be the matrix encoding a given column-wise elementary operation on columns. Then if (T, B) has been obtained from (A, I_n) by applying such operation, then $T = AE$ and $B = I_n E$ whence the claim follows. The general case follows by induction. □

1.11. Observe that $\mathrm{rank}(B) = \mathrm{rank}(I_n) = n$.

1.12. If A is a square, full-rank matrix, then $(A, I_n) \sim (I_n, A^{-1})$.

1.13. Let $(A, I_n) \sim (T, B)$, with T in upper-column-echelon form. Let $B = [B_1 \dots B_n]$. Then the last $n - \mathrm{rank}(A)$ columns of T are zero, and $\ker(f_A) = \mathrm{span}\,(B_j \mid j \in \{\mathrm{rank}(A) + 1, \dots, n\})$. Since the columns of B are linearly independent, then $\dim \ker(f_A) = n - \mathrm{rank}(A)$.

1.14. The rank(A) non-zero columns of T form a basis of $\mathrm{Im}(f_A)$, then $\dim \mathrm{Im}(f_A) = \mathrm{rank}(()A)$.

1.15. The following important result follows from the discussion above. For any linear map $f_A : \mathbb{R}^n \to \mathbb{R}^m$ the following identity holds:

$$\dim \ker(f_A) + \dim \mathrm{Im}(f_A) = (n - \mathrm{rank}(A)) + \mathrm{rank}(A) = n.$$