UNDERSTADING LINEAR MAPS WITH GAUSS-JORDAN ELIMINATION

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Abstract. We study some basic properties of linear maps by using column-wise Gauss-Jordan elimination.

1. Gauss-Jordan elimination

1.1. Given an $m \times n$ matrix A, we denote $f_A : \mathbb{R}^n \to \mathbb{R}^m$ the linear map defined by A in coordinates of the canonical basis. We denote $\mathbb{R}^{m\times n}$ the space of matrices of shape $m \times n$.

1.2. We say that a column-wise Gauss-Jordan pair is a tuple of matrices (A, B) where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times n}$. Gauss-Jordan pairs are meant to represent the steps in the execution of Gauss-Jordan elimination.

1.3. Column-wise elementary operations can be of either of three kinds:

(1) Permutation of columns.
(2) Replacing a column c_i with λc_i where λ is a non-zero scalar.
(3) Replacing a column c_i with $c_i + \mu c_j$ where μ is any scalar.

We denote $e(A)$ the transform of a matrix A by an elementary operation e.

1.4. Hereinafter our discussion will be based entirely on column-wise operations, so we will drop "column-wise" from our statements.

1.5. Elementary operations can be regarded as linear maps $\mathbb{R}^n \to \mathbb{R}^n$. We say that E is the matrix representing the elementary operation e whenever $e(A) = AE$ for all matrices $A \in \mathbb{R}^{m\times n}$.

1.6. Elementary operations are injective as linear maps $\mathbb{R}^n \to \mathbb{R}^n$, hence $\operatorname{rank}(A) = \operatorname{rank}(e(A)) = \operatorname{rank}(AE)$, i.e., they preserve the rank.

1.7. We can extend elementary operations to Gauss-Jordan pairs as follows

$$e : (A, B) \mapsto (e(A), e(B)) = (AE, BE)$$

1.8. We denote $(A, B) \sim (A', B')$ the fact that (A, B) can be transformed into (A', B') by applying a sequence of elementary operations.

1.9. Gauss-Jordan elimination is the method whereby the pair (A, I_n) is transformed, by means of applying a sequence of elementary operations, into a pair (L, B) with L in reduced lower column echelon form, which means the following:

(1) L is a lower column echelon matrix.
(2) The leading entries of L are 1.
(3) The leading entries of L are the only non-zero entries in their row.

Following our notation: $(A, I_n) \sim (L, B)$.

1.10. Given an input A, Gauss-Jordan elimination leads to a unique matrix L in reduced lower echelon form. We can stress this fact by defining a Gauss-Jordan transform $L : A \mapsto L(A)$ that takes A to the unique reduced lower column echelon form reachable by Gauss-Jordan elimination.

1.11. If $(A, I_n) \sim (L, B)$ then $L = AB$.

Proof. Let $E = E_1 \cdots E_k$ the product of the matrices encoding the elementary operations that have been applied. By definition $L = AE$ and $B = I_n E = E$, so it follows that $L = AB$. □

1.12. Observe that $\operatorname{rank}(B) = \operatorname{rank}(I_n) = n$.

1.13. Observe that if A is a square, full-rank matrix, $L(A) = I_n$.

1.14. If A is a square, full-rank matrix, then $(A, I_n) \sim (I_n, A^{-1})$.

Proof. We know that $(A, I_n) \sim (I_n, B)$ for some matrix B. Then $I_n = AB$, so $B = A^{-1}$. □

1.15. For the next discussion, let $(A, I_n) \sim (L, B)$, with L in reduced lower echelon form. Let $B = [b_1 \dots b_n]$ be the matrix B specified by its columns b_i.

1.16. The rank of A is precisely the number of non-zero columns in L. Moreover, these columns are the generators of $\operatorname{Im}(f_A)$. Since any subset of non-zero columns of L is a linearly independent set, it follows that $\dim \operatorname{Im}(f_A) = \operatorname{rank}(A)$.

1.17. Observe that the last $n - \operatorname{rank}(A)$ columns of L are zero.

1.18. The last $n - \operatorname{rank}(A)$ columns of B form a basis of the null-space $\ker(f_A)$.

Proof. The following are known facts:

(1) $b_1, \dots, b_n$ is a basis of $\mathbb{R}^n$.
(2) $\{Ab_i \mid 1 \leq i \leq \operatorname{rank}(A)\}$ is a linearly independent set.
(3) $Ab_i = 0$ for $\operatorname{rank}(A) + 1 \leq i \leq n$. Consequently,

$$S = \operatorname{span}\{b_i \mid \operatorname{rank}(A) + 1 \leq i \leq n\}\} \subset \ker(f_A).$$

Let's check that $\ker(f_A) \subset S$. By (1) we can write any $v \in \ker(f_A)$ as $v = \sum \lambda_i b_i$. Using $v \in \ker(f_A)$ and (3), we have $0 = f_A(v) = \sum_{i \leq \operatorname{rank}(A)} \lambda_i A b_i$. By (2) this cannot hold unless $\lambda_i = 0$ for $i = 1, \dots, \operatorname{rank}(A)$. Then $v \in S$. □

1.19. It follows that $\dim \ker(f_A) = n - \operatorname{rank}(A)$.

1.20. The following important result follows from all the discussion above. For any linear map $f_A : \mathbb{R}^n \to \mathbb{R}^m$ the following identity holds:

$$\dim \ker(f_A) + \dim \operatorname{Im}(f_A) = n.$$