

# **MATH233: LINEAR ALGEBRA 2**

## **Homework #1**

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## Problem 1

Let  $x = (2, 1 + i, i)$  and  $y = (2 - i, 2, 1 + 2i)$  be vectors in  $\mathbb{C}^3$ . Compute  $\langle x, y \rangle$ ,  $\|x\|$ ,  $\|y\|$ , and  $\|x + y\|$ . Then verify both the Cauchy-Schwarz inequality and triangle inequality.

## Solution

**Part A:** Compute  $\langle x, y \rangle$ ,  $\|x\|$ ,  $\|y\|$ , and  $\|x + y\|$

$$\begin{aligned}\langle x, y \rangle &= \sum_{i=1}^n x_i \overline{y_i} \\ &= 2 \cdot (2 - i) + (1 + i) \cdot 2 + i \cdot (1 - 2i) \\ &= (4 - 2i) + (2 + 2i) + i + 2 \\ &= \boxed{8 + 5i}\end{aligned}$$

$$\begin{aligned}\|x\| &= \sqrt{\langle x, x \rangle} \\ &= \sqrt{2^2 + (1 + i)(1 - i) + i \cdot (-i)} \\ &= \boxed{\sqrt{7}}\end{aligned}$$

$$\begin{aligned}\|y\| &= \sqrt{\langle y, y \rangle} \\ &= \sqrt{(2 - i)(2 + i) + 2^2 + (1 + 2i)(1 - 2i)} \\ &= \boxed{\sqrt{14}}\end{aligned}$$

$$\begin{aligned}\|x + y\| &= \sqrt{\langle x + y, x + y \rangle} \\ &= \sqrt{(4 - i)(4 + i) + (3 + i)(3 - i) + (1 + 3i)(1 - 3i)} \\ &= \boxed{\sqrt{37}}\end{aligned}$$

**Part B:** Verify Cauchy-Schwarz inequality and triangle inequality

By Cauchy-Schwarz inequality,  $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ . Verify:

$$|\langle x, y \rangle| = \sqrt{8^2 + 5^2} = \sqrt{89} \leq \|x\| \cdot \|y\| = \sqrt{7}\sqrt{14} = \sqrt{98}$$

By triangle inequality,  $\|x + y\| \leq \|x\| + \|y\|$ . Verify:

$$\|x + y\| = \sqrt{37} \leq \|x\| + \|y\| = \sqrt{7} + \sqrt{14}$$

## Problem 2

As is defined in Example 3, let  $\mathbf{V} = \mathcal{C}([0, 1])$ , the vector space of real-valued continuous functions on  $[0, 1]$ . For  $f, g \in \mathbf{V}$ , define:

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

In  $\mathcal{C}([0, 1])$ , let  $f(t) = t$  and  $g(t) = e^t$ . Compute  $\langle f, g \rangle$ ,  $\|f\|$ ,  $\|g\|$ , and  $\|f + g\|$ . Then verify both the Cauchy-Schwarz inequality and the triangle inequality.

## Solution

**Part A:** Compute  $\langle x, y \rangle$ ,  $\|x\|$ ,  $\|y\|$ , and  $\|x + y\|$

$$\begin{aligned} \langle x, y \rangle &= \int_0^1 f(t)g(t)dt & \|f\| &= \sqrt{\langle f, f \rangle} \\ &= te^t \Big|_0^1 + \int_0^1 e^t dt & &= \sqrt{\int_0^1 t^2 dt} = \sqrt{\frac{1}{3}t^3 \Big|_0^1} \\ &= e - e^t \Big|_0^1 & &= \boxed{\frac{\sqrt{3}}{3}} \\ &= \boxed{1} \end{aligned}$$

$$\begin{aligned} \|g\| &= \langle g, g \rangle & \|f + g\| &= \langle t + e^t, t + e^t \rangle \\ &= \sqrt{\int_0^1 e^{2t} dt} = \sqrt{\frac{1}{2}te^{2t} \Big|_0^1} & &= \sqrt{\int_0^1 e^{2t} + t^2 + 2te^t dt} \\ &= \boxed{\sqrt{\frac{1}{2}e^2 - \frac{1}{2}}} & &= \sqrt{\int_0^1 e^{2t} dt + \int_0^1 t^2 dt + 2 \int_0^1 te^t dt} \\ & & &= \boxed{\sqrt{\frac{1}{2}e^2 + \frac{11}{6}}} \end{aligned}$$

**Part B:** Verify Cauchy-Schwarz inequality and triangle inequality

By Cauchy-Schwarz inequality,  $\langle f, g \rangle \leq \|f\| \cdot \|g\|$ . Verify:

$$|\langle f, g \rangle| = 1 \leq \|f\| \cdot \|g\| = \frac{\sqrt{3}}{3} \cdot \sqrt{\frac{1}{2}e^2 - \frac{1}{2}} = \sqrt{\frac{1}{6}e^2 - \frac{1}{6}} \approx 1.065$$

By triangle inequality,  $\|f + g\| \leq \|f\| + \|g\|$ . Verify:

$$\|f + g\| = \sqrt{\frac{1}{2}e^2 + \frac{11}{6}} \approx 2.35 \leq \|f\| + \|g\| = \sqrt{\frac{1}{2}e^2 - \frac{1}{6}} \approx 2.63$$

### Problem 3

Let  $\{v_1, v_2, v_3, \dots, v_k\}$  be an orthogonal set in  $\mathbf{V}$ , and let  $a_1, a_2, \dots, a_k$  be scalars. Prove that

$$\left\| \sum_{i=1}^k a_i v_i \right\|^2 = \sum_{i=1}^k |a_i|^2 \|v_i\|^2.$$

### Solution

*Proof.* In an orthogonal set  $\mathbf{V}$ ,  $\forall i \neq j, \langle v_i, v_j \rangle = 0$ . Thus:

$$\begin{aligned} \left\| \sum_{i=1}^k a_i v_i \right\|^2 &= \left\langle \sum_{i=1}^k a_i v_i, \sum_{j=1}^k a_j v_j \right\rangle \\ &= \sum_{j=1}^k \left( \left\langle \sum_{i=1}^k a_i v_i, a_j v_j \right\rangle \right) \\ &= \sum_{j=1}^k \left( \sum_{i=1}^k \langle a_i v_i, a_j v_j \rangle \right) \\ &= \underbrace{\sum_{i \neq j} \langle a_i v_i, a_j v_j \rangle}_{\text{all 0s}} + \sum_{i=1}^k \langle a_i v_i, a_i v_i \rangle \\ &= \sum_{i=1}^k a_i \overline{a_i} \langle v_i, v_i \rangle \\ &= \sum_{i=1}^k |a_i|^2 \|v_i\|^2 \end{aligned}$$

□

### Problem 4

Given

$$\begin{aligned} \mathbf{V} &= \mathbb{R}^3 \\ S &= \{(1, 0, 1), (0, 1, 1), (1, 3, 3)\}, \\ x &= (1, 1, 2) \end{aligned}$$

apply the Gram-Schmidt process to the subset  $S$  of the inner product space  $\mathbf{V}$  to obtain an orthogonal basis for  $\text{span}(S)$ . Then normalize the vectors in this basis to obtain an orthonormal basis  $\beta$  for  $\text{span}(S)$ , and compute the Fourier coefficients of the given vector  $x$  relative to  $\beta$ . Finally, use the following theorem to verify your results:

Let  $\mathbf{V}$  be a nonzero finite-dimensional inner product space. Then  $\mathbf{V}$  has an orthonormal basis  $\beta$ . Furthermore, if  $\beta = \{v_1, v_2, \dots, v_n\}$  and  $x \in \mathbf{V}$ , then

$$x = \sum_{i=1}^n \langle x, v_i \rangle v_i.$$

## Solution

**Part A:** Apply the Gram-Schmidt process to obtain the orthonormal basis  $\beta$

Suppose  $S' = \{v_1, v_2, v_3\}$  and  $S = \{w_1, w_2, w_3\}$ . We have  $v_1 = w_1 = (1, 0, 1)$ . And

$$\begin{aligned}
 v_2 &= w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 \\
 &= (0, 1, 1) - \frac{\langle (0, 1, 1), (1, 0, 1) \rangle}{\langle (1, 0, 1), (1, 0, 1) \rangle} (1, 0, 1) \\
 &= (0, 1, 1) - \frac{1}{2} (1, 0, 1) \\
 &= \left(-\frac{1}{2}, 1, \frac{1}{2}\right) \\
 v_3 &= w_3 - \frac{\langle v_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle v_3, v_2 \rangle}{\|v_2\|^2} v_2 \\
 &= (1, 3, 3) - \frac{\langle (1, 3, 3), (1, 0, 1) \rangle}{\langle (1, 0, 1), (1, 0, 1) \rangle} (1, 0, 1) - \frac{\langle (1, 3, 3), (-\frac{1}{2}, 1, \frac{1}{2}) \rangle}{\langle (-\frac{1}{2}, 1, \frac{1}{2}), (-\frac{1}{2}, 1, \frac{1}{2}) \rangle} \left(-\frac{1}{2}, 1, \frac{1}{2}\right) \\
 &= \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)
 \end{aligned}$$

These vectors can be normalized to obtain the orthonormal basis  $\beta = \{u_1, u_2, u_3\}$ , where

$$\begin{aligned}
 u_1 &= \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{2}} (1, 0, 1) \\
 u_2 &= \frac{1}{\|v_2\|} v_2 = \frac{\sqrt{6}}{3} \left(-\frac{1}{2}, 1, \frac{1}{2}\right) \\
 u_3 &= \frac{1}{\|v_3\|} v_3 = \sqrt{3} \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)
 \end{aligned}$$

**Part B:** Compute the Fourier coefficients w.r.t  $x$

$$\begin{aligned}
 c_1 &= \langle x, u_1 \rangle = \frac{3\sqrt{2}}{2} \\
 c_2 &= \langle x, u_2 \rangle = \frac{\sqrt{6}}{2} \\
 c_3 &= \langle x, u_3 \rangle = 0
 \end{aligned}$$

**Part C:** Verify the results

$$\begin{aligned}
 \langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2 + \langle x, u_3 \rangle u_3 &= \frac{3}{2} (1, 0, 1) + \left(-\frac{1}{2}, 0, \frac{1}{2}\right) + (0, 0, 0) \\
 &= (1, 1, 2) = x
 \end{aligned}$$

**Problem 5**

Let  $S = \{(1, 0, i), (1, 2, 1)\}$  in  $\mathbb{C}^3$ . Compute  $S^\perp$ .

**Solution**

Let  $x = (x_1, x_2, x_3) \in S^\perp$ , this means  $x$  is orthogonal to every vector in  $S$ , i.e.

$$\langle (x_1, x_2, x_3), (1, 0, i) \rangle = 0 \quad \text{and} \quad \langle (x_1, x_2, x_3), (1, 2, 1) \rangle = 0$$

Expand the inner product we have

$$\begin{cases} x_1 - ix_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases} \implies \begin{cases} x_1 = ix_3 \\ x_2 = \frac{-1-i}{2}x_3 \end{cases}$$

Then we can conclude that

$$S^\perp = \text{span} \left\{ \left( i, \frac{-i-1}{2}, 1 \right) \right\}$$

## Problem 6

Given inner product space  $\mathbf{V}$ , linear operator  $T$  on  $\mathbf{V}$  and vector  $x$  in  $\mathbf{V}$

$$\mathbf{V} = \mathbb{C}^2$$

$$T(z_1, z_2) = (2z_1 + iz_2, (1-i)z_1)$$

$$x = (3-i, 1+2i)$$

Evaluate  $T^*$  at the vector  $x$ .

## Solution

Take the standard ordered basis for  $\mathbb{C}^2$  and write

$$[T]_{\beta} = \begin{pmatrix} 2 & i \\ 1-i & 0 \end{pmatrix} \implies [T^*]_{\beta} = [T]_{\beta}^* = \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix}$$

Then

$$\begin{aligned} T^*(x) &= [T]_{\beta}^* x \\ &= \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 3-i \\ 1+2i \end{pmatrix} \\ &= \begin{pmatrix} 5+i \\ -1-3i \end{pmatrix} \end{aligned}$$

## Problem 7

For the set of data  $\{(-3, 9), (-2, 6), (0, 2), (1, 1)\}$ , use the least square approximation to find the best fits with both (i) a linear function and (ii) a quadratic function. Compute the error  $E$  in both cases.

## Solution

(i) To find the least square line  $y = cx + d$ , we let

$$A = \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad x = \begin{pmatrix} c \\ d \end{pmatrix} \quad y = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}$$

Find the solution to the system by

$$\begin{aligned} A^*Ax &= A^*y \\ \begin{pmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} &= \begin{pmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 14 & -4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} &= \begin{pmatrix} -38 \\ 18 \end{pmatrix} \end{aligned}$$

Expand the computation we can solve

$$\begin{cases} 14c - 4d = -38 \\ -4c + 4d = 18 \end{cases} \implies \begin{cases} c = -2 \\ d = 2.5 \end{cases}$$

So the linear function is

$$y = 2x + 2.5$$

and the error  $E$  can be computed by

$$\begin{aligned} E &= \|Ax - y\|^2 \\ &= \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2.5 \end{pmatrix} - \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} \\ &= 1 \end{aligned}$$



(ii) To compute the quadratic form  $y = ax + bx + c$ , suppose

$$A = \begin{pmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad y = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}$$

Find the solution to the system by

$$A^*Ax = A^*y$$

$$\begin{pmatrix} 98 & -34 & 14 \\ -34 & 14 & -4 \\ 14 & -4 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 106 \\ -38 \\ 18 \end{pmatrix}$$

Expand the computation we can solve

$$\begin{cases} 98a - 34b + 14c = 106 \\ -34a + 14b - 4c = -38 \\ 14a - 4b + 4c = 18 \end{cases} \implies \begin{cases} a = \frac{1}{3} \\ b = -\frac{4}{3} \end{cases} \quad c = 2$$

So the linear function is

$$y = \frac{1}{3}x^2 - \frac{4}{3}x + 2$$

and the error  $E$  can be computed by

$$\begin{aligned} E &= \|Ax - y\|^2 \\ &= \begin{pmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/3 \\ -4/3 \\ 2 \end{pmatrix} - \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} \\ &= 0 \end{aligned}$$

## Problem 8

Let  $\mathbf{V}$  be a complex inner product space, and let  $T$  be a linear operator on  $\mathbf{V}$ . Define

$$T_1 = \frac{1}{2}(T + T^*) \quad \text{and} \quad T_2 = \frac{1}{2i}(T - T^*).$$

1. Prove that  $T_1$  and  $T_2$  are self-adjoint and that  $T = T_1 + iT_2$ .
2. Suppose also that  $T = U_1 + iU_2$ , where  $U_1$  and  $U_2$  are self-adjoint. Prove that  $U_1 = T_1$  and  $U_2 = T_2$ .
3. Prove that  $T$  is normal if and only if  $T_1T_2 = T_2T_1$ .

## Solution

1. *Proof.*

$$\begin{aligned} T_1^* &= \left( \frac{1}{2}(T + T^*) \right)^* & T_2^* &= \left( \frac{1}{2i}(T - T^*) \right)^* \\ &= \frac{1}{2}(T^* + (T^*)^*) & &= \frac{1}{2i}(T^* - (T^*)^*) \\ &= \frac{1}{2}(T^* + T) & &= \frac{1}{2i}(-T^* + T) \\ &= T_1 & &= T_2 \\ T_1 + iT_2 &= \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*) = T \end{aligned}$$

□

2. *Proof.* If  $T = U_1 + iU_2$  and  $U_1, U_2$  are self-adjoint, then

$$\begin{aligned} T^* &= (U_1 + iU_2)^* = U_1^* - iU_2^* = U_1 - iU_2 \\ T_1^* &= \left( \frac{1}{2}(T + T^*) \right) = \frac{1}{2}((U_1 + iU_2) + (U_1 - iU_2)) = U_1 \\ T_2^* &= \left( \frac{1}{2i}(T - T^*) \right) = \frac{1}{2i}((U_1 + iU_2) - (U_1 - iU_2)) = U_2 \end{aligned}$$

□

3. *Proof.*

$$\begin{aligned} T_1T_2 &= \frac{1}{2}(T + T^*)\frac{1}{2i}(T - T^*) = \frac{1}{4i}(T^2 - TT^* + T^*T - (T^*)^2) \\ T_2T_1 &= \frac{1}{2i}(T - T^*)\frac{1}{2}(T + T^*) = \frac{1}{4i}(T^2 + TT^* - T^*T - (T^*)^2) \end{aligned}$$

If  $T_1T_2 = T_2T_1$ , then

$$\frac{1}{4i}(T^2 - TT^* + T^*T - (T^*)^2) = \frac{1}{4i}(T^2 + TT^* - T^*T - (T^*)^2)$$

which is equivalent to

$$2T^*T = TT^* \implies T \text{ is normal}$$

If  $T$  is normal, then  $T_1T_2 = T_2T_1$

□

## Problem 9

Let  $T$  be a self-adjoint linear operator on a finite-dimensional inner product space. Prove that  $(T + iI)(T - iI)^{-1}$  is unitary using the following conclusion:  $T - iI$  is invertible and  $[(T - iI)^{-1}]^* = (T + iI)^{-1}$  if  $T$  is self-adjoint.

## Solution

*Proof.* Linear operator  $T$  is unitary if  $TT^* = T^*T = I$ .

$$\begin{aligned} & [(T + iI)(T - iI)^{-1}]^* [(T + iI)(T - iI)^{-1}] \\ &= [(T - iI)^{-1}]^* (T + iI)^* (T + iI)(T - iI)^{-1} \\ &= (T + iI)^{-1} (T + iI)(T - iI)(T - iI)^{-1} \\ &= I \end{aligned}$$

□

## Problem 10

## Solution

*Proof.* If  $n = 2$ , then obviously

$$\det(M) = \det \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = ac$$

For induction from  $n - 1 \Rightarrow n$ , we take the determinant by expanding along the first column of  $M$ . Let  $\tilde{M}_{ij}$  be the matrix obtained from  $M$  by deleting the  $i$ th and  $j$ th column. Let  $k$  be the largest row of  $A$ . First, note that  $M_{i1} = 0$  for all  $i > k$ . For  $i \leq k$ ,  $M_{i1} = A_{i1}$  and

$$\det(\tilde{M}_{i1}) = \det \begin{pmatrix} \tilde{A}_{i1} & \tilde{B} \\ O & C \end{pmatrix}$$

by the induction hypothesis as  $\tilde{M}_{i1}$  is block upper triangular. Then

$$\begin{aligned} \det(M) &= \sum_{i=1}^n (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1}) \\ &= \sum_{i=1}^k (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1}) \\ &= \left( \sum_{i=1}^k (-1)^{i+1} A_{i1} \det(\tilde{A}_{i1}) \right) \det(C) \\ &= \det(A) \det(C) \end{aligned}$$

□