# MATH233: LINEAR ALGEBRA 2
# Homework #1

Due on September 22 2021

*Professor Tony Xu*

**Mischa "Matchy" Volynskaya**

October 12, 2021

# Problem 1

Let $A$ and $B$ be $n \times n$ matrices. Recall that the trace of $A = (a_{ij})$ is defined by $tr(A) = \sum_{i=1}^{n} a_{ii}$. Prove that $tr(AB) = tr(BA)$.

---

## Solution

*Proof.* Let $A = (a_{ij})$, $B = (b_{ij})$, AB $= (\alpha_{ij})$ and BA $= (\beta_{ij})$.
By the definition of matrix multiplication, we have

$$\alpha_{ij} = (\sum_{k=1}^{n} a_{ik}b_{kj}), \text{ and } \beta_{ij} = (\sum_{k=1}^{n} b_{ik}a_{kj})$$

It's trivial to see that

$$\begin{aligned} tr(AB) &= \sum_{i=1}^{n} \alpha_{ii} \\ &= \sum i = 1^n \left( \sum_{k=1}^{n} a_{ik}b_{kj} \right) \end{aligned}$$

$$\begin{aligned} tr(BA) &= \sum_{i=1}^{n} \beta_{ii} \\ &= \sum_{i=1}^{n} \left( \sum_{k=1}^{n} b_{ik}a_{kj} \right) \end{aligned}$$

Thus, $tr(AB) = tr(BA)$.

$\square$

# Problem 2

Let $A$ and $B$ be $n \times n$ matrices such that $AB$ is invertible. Prove that $A$ and $B$ are invertible.

---

## Solution

*Proof.* Let $C = B(AB)^{-1}$ and $D = (AB)^{-1}A$. Then we have

$$\begin{aligned} AC &= A(B(AB)^{-1}) \\ &= (AB)(AB)^{-1} \\ &= I \end{aligned}$$

It's trivial to see that $CA = I$ because

$$\begin{aligned} I &= CC^{-1} \\ &= CIC^{-1} \\ &= C(AC)C^{-1} \\ &= (CA)(CC^{-1}) \\ &= CA \end{aligned}$$

Since $AC = CA = I$, $C = A^{-1}$, which means $A$ is invertible.

Problem 2 continued on next page. . .

Also we have

$$\begin{aligned} DB &= ((AB)^{-1}A)B \\ &= (AB)^{-1}(AB) \\ &= I \end{aligned}$$

By the proof elaborated above, we know that $D = B^{-1}$, which means $B$ is invertible.
So, we have proven when $AB$ is invertible, $A$ and $B$ are both invertible. $\square$

# Problem 3

## Solution

Suppose we have a matrix

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that

$$Q\vec{\beta} = \vec{\beta'}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}$$

$$\begin{cases} -4a + 3b = 2 \\ 2a - b = -4 \\ -4c + 3d = 1 \\ 2c - d = 1 \end{cases}$$

Solving these aligns gives

$$\begin{cases} a = -5 \\ b = -6 \\ c = 1 \\ d = 1 \end{cases}$$

That is to say,

$$Q = \begin{pmatrix} -5 & -6 \\ 1 & 1 \end{pmatrix}$$

# Problem 4

## Solution

$$Ae_1' = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} Ae_2' = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} Ae_3' = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$$

Suppose $A' = \left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{smallmatrix}\right)$, then we know that

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = a_1\vec{e_1}' + b_1\vec{e_2}' + c_1\vec{e_3}' = \begin{pmatrix} a_1 + b_1 + c_1 \\ a_1 + c_1 \\ a_1 + b_1 + 2c_1 \end{pmatrix}$$

Problem 4 continued on next page... 

$$\begin{pmatrix}0\\3\\1\end{pmatrix} = a_2\vec{e_1}' + b_2\vec{e_2}' + c_2\vec{e_3}' = \begin{pmatrix}a_2+b_2+c_2\\a_2+c_2\\a_2+b_2+2c_2\end{pmatrix}$$

$$\begin{pmatrix}0\\4\\2\end{pmatrix} a_3\vec{e_1}' + b_3\vec{e_2}' + c_3\vec{e_3}' = \begin{pmatrix}a_3+b_3+c_3\\a_3+c_3\\a_3+b_3+2c_3\end{pmatrix}$$

We solve that

$$A' = \begin{pmatrix}4 & 2 & 2\\-2 & -3 & -4\\-1 & 1 & 2\end{pmatrix}$$

# Problem 5

## Solution

*Proof.* We know that $B = Q^{-1}AQ$. Then we have

$$\begin{aligned} tr(B) &= tr((Q^{-1})(AQ)) \\ &= tr((AQ)(Q^{-1})) \quad \text{(By problem (1))} \\ &= tr(A(QQ^{-1})) \\ &= tr(A) \end{aligned}$$

$\square$

# Problem 6

## Solution

$$A(x,y,z)\begin{pmatrix}1 & 2 & 1\\-1 & 1 & 2\\1 & 0 & 1\end{pmatrix}$$

If a matrix is invertible, then its determinant $\det(A) \neq 0$. We compute the determinant of matrix A:

$$\begin{aligned} \det(A) &= \det(\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}) - 2\cdot\det(\begin{pmatrix}-1 & 2\\1 & 1\end{pmatrix}) + \det(\begin{pmatrix}-1 & 1\\1 & 0\end{pmatrix}) \\ &= 1 + 2\cdot 3 - 1 \\ &= 6 \end{aligned}$$

So matrix $A$ is invertible. Write the augmented matrix:

$$\left[\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\-1 & 1 & 2 & 0 & 1 & 0\\1 & 0 & 1 & 0 & 0 & 1\end{array}\right]$$

By performing row reduction we have

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1/6 & -1/3 & 1/2\\0 & 1 & 0 & 1/2 & 0 & -1/2\\0 & 0 & 1 & -1/6 & 1/3 & 1/2\end{array}\right]$$

Problem 6 continued on next page. . . 

And we know that

$$A^{-1} = \begin{pmatrix} 1/6 & -1/3 & 1/2 & \\ 2 & 1/2 & 0 & -1/2 \\ 3 & -1/6 & 1/3 & 1/2 \end{pmatrix}$$

# Problem 7

## Solution

*Proof.* Let

$$L_A : F^n \to F^m$$

be the linear transformation coefficient matrix

$$A = \left[\vec{A}_1, \vec{A}2, ..., \vec{A}n\right]$$

describes and we have

$$A\vec{x} = \vec{b}$$

If we have a solution $\vec{s} \in F^n$ where

$$\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

to the system, then

$$\vec{b} = A\vec{s} \in R(L_A)$$

Vice versa, if $\vec{b} \in R(L_A)$, then for $A$ there exists $\vec{s}$ where

$$\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and

$$b = s_1\vec{A}_1 \vec{+} s_2\vec{A}_2 + \cdots + s_n\vec{A}_n = A\vec{s}$$

That means $\vec{s}$ is a solution of the system $A\vec{x} = \vec{b}$. So $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b} \in R(L_A)$.

Suppose $rank(A) = m$, which means $R(L_A) \subseteq F^m$ and that gives $\vec{b} \in F^m$.
It is known that $R(L_A) = \text{span}(\{\vec{A}1, \vec{A}2, ..., \vec{A}n\})$. Thus $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b} \in \text{span}(\{\vec{A}1, \vec{A}2, ..., \vec{A}n\})$.
But $\vec{b} \in \text{span}(\{\vec{A}1, \vec{A}2, ..., \vec{A}n\})$ if and only if $\text{span}(\{\vec{A}1, \vec{A}2, ..., \vec{A}n\}) = \text{span}(\{\vec{A}1, \vec{A}2, ..., \vec{A}n, \vec{b}\})$. It is equivalent to

$$\dim(\text{span}(\{\vec{A}1, \vec{A}2, ..., \vec{A}n\}) = \dim(\text{span}(\{\vec{A}1, \vec{A}2, ..., \vec{A}n, \vec{b}\})))$$

We know that

$$m = \dim(R(L_A)) = \dim(\text{span}(\{\vec{A}1, \vec{A}2, ..., \vec{A}n\}))$$

Because $\vec{b} \in F^m$, it is true that

$$m = \dim(\text{span}(\{\vec{A}1, \vec{A}2, ..., \vec{A}n, \vec{b}\}))$$

So the system has a solution. ☐

# Problem 8

## Solution

*Proof.* Proving $\det(A) = 0$ is equivalent to proving $A$ is not invertible. Suppose that $A$ is invertible, then we have

$$AA^{-1} = I$$

It's trivial to see that when $A$ is nilpotent, we have

$$\begin{aligned}(AA^{-1})^k &= I \\ (A^k)(A^{-k}) &= I \\ OA^{-k} &= I \\ O &= I\end{aligned}$$

This can never happen. By contradiction we have proven that $A$ is not invertible, thus $\det(A) = 0$ when $A$ is nilpotent. $\square$

# Problem 9

## Solution

*Proof.* It's trivial to see that

$$QQ^T = I = Q^TQ$$

and

$$\det(Q) = \det(Q^T)$$

We also know that fact that

$$\det(AB) = \det(A)\det(B)$$

and

$$\det(I) = 1$$

Putting these facts altogether we know that

$$\begin{aligned}\det(I) &= \det(QQ^T) \\ &= \det(Q)\det(Q^T) \\ &= [\det(Q)]^2\end{aligned}$$

So either $\det(Q) = 1$ or $\det(Q) = -1$. $\square$

# Problem 10

## Solution

*Proof.* If $n = 2$, then obviously

$$\det(M) = \det \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = ac$$

For induction from $n-1 \Rightarrow n$, we take the determinant by expanding along the first column of $M$. Let $\tilde{M}_{ij}$ be the matrix obtained from $M$ by deleting the $i$th and $j$th column. Let $k$ be the largest row of $A$. First, note that $M_{i1} = 0$ for all $i > k$. For $i \leq k$ , $M_{i1} = A_{i1}$ and

$$\det(\tilde{M}_{i1}) = \det \begin{pmatrix} \tilde{A}_{i1} & \tilde{B} \\ O & C \end{pmatrix}$$

by the induction hypothesis as $\tilde{M}_{i1}$ is block upper triangular. Then

$$\begin{aligned}
\det(M) &= \sum_{i=1}^{n} (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1}) \\
&= \sum_{i=1}^{k} (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1}) \\
&= \left( \sum_{i=1}^{k} (-1)^{i+1} A_{i1} \det(\tilde{A}_{i1}) \right) \det(C) \\
&= \det(A) \det(C)
\end{aligned}$$

$\square$