MATH233: LINEAR ALGEBRA 2
Homework #1

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Problem 1

Let $x = (2, 1+i, i)$ and $y = (2-i, 2, 1+2i)$ be vectors in $\mathbb{C}^3$. Compute $\langle x, y\rangle$, $\|x\|$, $\|y\|$, and $\|x+y\|$. Then verify both the Cauchy-Schwarz inequality and triangle inequality.

Solution

Part A: Compute $\langle x, y\rangle$, $\|x\|$, $\|y\|$, and $\|x+y\|$

$$\begin{aligned}
\langle x, y\rangle &= \sum_{i=1}^{n} x_i \overline{y_i} \\
&= 2 \cdot (2+i) + (1+i) \cdot 2 + i \cdot (1-2i) \\
&= (4+2i) + (2+2i) + i + 2 \\
&= \boxed{8+5i}
\end{aligned}$$

$$\begin{aligned}
\|x\| &= \sqrt{\langle x, x\rangle} \\
&= \sqrt{2^2 + (1+i)(1-i) + i \cdot (-i)} \\
&= \boxed{\sqrt{7}}
\end{aligned}$$

$$\begin{aligned}
\|y\| &= \sqrt{\langle y, y\rangle} \\
&= \sqrt{(2-i)(2+i) + 2^2 + (1+2i)(1-2i)} \\
&= \boxed{\sqrt{14}}
\end{aligned}$$

$$\begin{aligned}
\|x+y\| &= \sqrt{\langle x+y, x+y\rangle} \\
&= \sqrt{(4-i)(4+i) + (3+i)(3-i) + (1+3i)(1-3i)} \\
&= \boxed{\sqrt{37}}
\end{aligned}$$

Part B: Verify Cauchy-Schwarz inequality and triangle inequality
By Cauchy-Schwarz inequality, $\langle x, y\rangle \leq \|x\| \cdot \|y\|$. Verify:

$$|\langle x, y\rangle| = \sqrt{8^2 + 5^2} = \sqrt{89} \leq \|x\| \cdot \|y\| = \sqrt{7}\sqrt{14} = \sqrt{98}$$

By triangle inequality, $\|x+y\| \leq \|x\| + \|y\|$. Verify:

$$\|x+y\| = \sqrt{37} \leq \|x\| + \|y\| = \sqrt{7} + \sqrt{14}$$

Problem 2

As is defined in Example 3, let $\mathbf{V} = \mathscr{C}([0,1])$, the vector space of real-valued continuous functions on $[0,1]$. For $f, g \in \mathbf{V}$, define:

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\mathrm{d}t$$

In $\mathscr{C}([0,1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g\rangle$, $\|f\|$, $\|g\|$, and $\|f+g\|$. Then verify both the Cauchy-Schwarz inequality and the triangle inequality.

Solution

Part A: Compute $\langle x, y\rangle$, $\|x\|$, $\|y\|$, and $\|x+y\|$

$$\begin{aligned}\langle x, y\rangle &= \int_0^1 f(t)g(t)\mathrm{d}t \\ &= te^t\Big|_0^1 + \int_0^1 e^t\mathrm{d}t \\ &= e - e^t\Big|_0^1 \\ &= \boxed{1}\end{aligned}$$

$$\begin{aligned}\|f\| &= \sqrt{\langle f, f\rangle} \\ &= \sqrt{\int_0^1 t^2\mathrm{d}t} = \sqrt{\frac{1}{3}t^3\Big|_0^1} \\ &= \boxed{\frac{\sqrt{3}}{3}}\end{aligned}$$

$$\begin{aligned}\|g\| &= \langle g, g\rangle \\ &= \sqrt{\int_0^1 e^{2t}\mathrm{d}t} = \sqrt{\frac{1}{2}te^{2t}\Big|_0^1} \\ &= \boxed{\sqrt{\frac{1}{2}e^2 - \frac{1}{2}}}\end{aligned}$$

$$\begin{aligned}\|f+g\| &= \left\langle t+e^t, t+e^t\right\rangle \\ &= \sqrt{\int_0^1 e^{2t} + t^2 + 2te^t\mathrm{d}t} \\ &= \sqrt{\int_0^1 e^{2t}\mathrm{d}t + \int_0^1 t^2\mathrm{d}t + 2\int_0^1 te^t\mathrm{d}t} \\ &= \boxed{\sqrt{\frac{1}{2}e^2 + \frac{11}{6}}}\end{aligned}$$

Part B: Verify Cauchy-Schwarz inequality and triangle inequality
By Cauchy-Schwarz inequality, $\langle f, g\rangle \leq \|f\| \cdot \|g\|$. Verify:

$$|\langle f, g\rangle| = 1 \leq \|f\| \cdot \|g\| = \frac{\sqrt{3}}{3} \cdot \sqrt{\frac{1}{2}e^2 - \frac{1}{2}} = \sqrt{\frac{1}{6}e^2 - \frac{1}{6}} \approx 1.065$$

By triangle inequality, $\|f+g\| \leq \|f\| + \|g\|$. Verify:

$$\|f+g\| = \sqrt{\frac{1}{2}e^2 + \frac{11}{6}} \approx 2.35 \leq \|f\| + \|g\| = \sqrt{\frac{1}{2}e^2 - \frac{1}{6}} \approx 2.63$$

Problem 3

Let $\{v_1, v_2, v_3, \cdots, v_k\}$ be an orthogonal set in $\mathbf{V}$, and let $a_1, a_2, \cdots, a_k$ be scalars. Prove that

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2 .$$

Solution

Proof. In an orthogonal set $\mathbf{V}$, $\forall i! = j, \langle v_i, v_j \rangle = 0$. Thus:

$$\begin{aligned}
\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 &= \left\langle \sum_{i=1}^{k} a_i v_i, \sum_{j=1}^{k} a_j v_j \right\rangle \\
&= \sum_{j=1}^{k} \left(\left\langle \sum_{i=1}^{k} a_i v_i, a_j v_j \right\rangle \right) \\
&= \sum_{j=1}^{k} \left(\sum_{i=1}^{k} \langle a_i v_i, a_j v_j \rangle \right) \\
&= \underbrace{\sum_{i \neq j} \langle a_i v_i, a_j v_j \rangle}_{\text{all 0s}} + \sum_{i=1}^{k} \langle a_i v_i, a_i v_i \rangle \\
&= \sum_{i=1}^{k} a_i \overline{a_i} \langle v_i, v_i \rangle \\
&= \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2
\end{aligned}$$

□

Problem 4

Given

$$\begin{aligned}
&\mathbf{V} = \mathbb{R}^3 \\
&S = \{(1,0,1), (0,1,1), (1,3,3)\}, \\
&x = (1,1,2)
\end{aligned}$$

apply the Gram-Schmidt process to the subset S of the inner product space $\mathbf{V}$ to obtain an orthogonal basis for span(S). Then normalize the vectors in this basis to obtain an orthonormal basis β for span(S), and compute the Fourier coefficients of the given vector x relative to β. Finally, use the following theorem to verify your results:
Let $\mathbf{V}$ be a nonzero finite-dimensional inner product space. Then $\mathbf{V}$ has an orthonormal basis β. Furthermore, if $\beta = \{v_1, v_2, \cdots, v_n\}$ and $x \in \mathbf{V}$, then

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i.$$

Problem 4 continued on next page. . .

Solution

Part A: Apply the Gram-Schmidt process to obtain the orthonormal basis β
Suppose $S' = \{v_1, v_2, v_3\}$ and $S = \{w_1, w_2, w_3\}$. We have $v_1 = w_1 = (1, 0, 1)$. And

$$\begin{aligned}
v_2 &= w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 \\
&= (0, 1, 1) - \frac{\langle (0,1,1), (1,0,1) \rangle}{\langle (1,0,1), (1,0,1) \rangle} (1, 0, 1) \\
&= (0, 1, 1) - \frac{1}{2}(1, 0, 1) \\
&= \left(-\frac{1}{2}, 1, \frac{1}{2} \right)
\end{aligned}$$

$$\begin{aligned}
v_3 &= w_3 - \frac{\langle v_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle v_3, v_2 \rangle}{\|v_2\|} v_2 \\
&= (1, 3, 3) - \frac{\langle (1,3,3), (1,0,1) \rangle}{\langle (1,0,1), (1,0,1) \rangle} - \frac{\left\langle (1,3,3), \left(-\frac{1}{2}, 1, \frac{1}{2}\right) \right\rangle}{\left\langle \left(-\frac{1}{2}, 1, \frac{1}{2}\right), \left(-\frac{1}{2}, 1, \frac{1}{2}\right) \right\rangle} \\
&= \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3} \right)
\end{aligned}$$

These vectors can be normalized to obtain the orthonormal basis $\beta = \{u_1, u_2, u_3\}$, where

$$\begin{aligned}
u_1 &= \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{2}} (1, 0, 1) \\
u_2 &= \frac{1}{\|v_2\|} v_2 = \frac{\sqrt{6}}{3} \left(-\frac{1}{2}, 1, \frac{1}{2} \right) \\
u_3 &= \frac{1}{\|v_3\|} v_3 = \sqrt{3} \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3} \right)
\end{aligned}$$

Part B: Compute the Fourier coefficients w.r.t x

$$\begin{aligned}
c_1 &= \langle x, u_1 \rangle = \frac{3\sqrt{2}}{2} \\
c_2 &= \langle x, u_2 \rangle = \frac{\sqrt{6}}{2} \\
c_3 &= \langle x, u_3 \rangle = 0
\end{aligned}$$

Part C: Verify the results

$$\begin{aligned}
\langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2 + \langle x, u_3 \rangle u_3 &= \frac{3}{2} (1, 0, 1) + \left(-\frac{1}{2}, 0, \frac{1}{2} \right) + (0, 0, 0) \\
&= (1, 1, 2) = x
\end{aligned}$$

Problem 5

Let $S = \{(1, 0, i), (1, 2, 1)\}$ in $\mathbb{C}^3$. Compute $S^\perp$

Solution

Let $x = (x_1, x_2, x_3) \in S^\perp$, this means x is orthogonal to every vector in S., i.e.

$$\langle (x_1, x_2, x_3), (1, 0, i)\rangle = 0 \quad \text{and} \quad \langle (x_1, x_2, x_3), (1, 2, 1)\rangle = 0$$

Expand the inner product we have

$$\begin{cases} x_1 - ix_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases} \implies \begin{cases} x_1 = ix_3 \\ x_2 = \dfrac{-1-i}{2} x_3 \end{cases}$$

Then we can conclude that

$$S^\perp = \operatorname{span}\left\{\left(i, \frac{-i-1}{2}, 1\right)\right\}$$

Problem 6

Given inner product space $\mathbf{V}$, linear operator T on $\mathbf{V}$ and vector x in $\mathbf{V}$

$$\begin{aligned} &\mathbf{V} = \mathbb{C}^2 \\ &T(z_1, z_2) = (2z_1 + iz_2, (1-i)z_1) \\ &x = (3-i, 1+2i) \end{aligned}$$

Evaluate T^* at the vector x.

Solution

Take the standard ordered basis for $\mathbb{C}^2$ and write

$$[T]_\beta = \begin{pmatrix} 2 & i \\ 1-i & 0 \end{pmatrix} \Longrightarrow [T^*]_\beta = [T]_\beta^* = \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix}$$

Then

$$\begin{aligned} T^*(x) &= [T]_\beta^* x \\ &= \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 3-i \\ 1+2i \end{pmatrix} \\ &= \begin{pmatrix} 5+i \\ -1-3i \end{pmatrix} \end{aligned}$$

Problem 7

For the set of data $\{(-3,9),(-2,6),(0,2),(1,1)\}$, use the least square approximation to find the best fits with both (i) a linear function and (ii) a quadratic function. Compute the error E in both cases.

Solution

(i) To find the least square line $y = cx + d$, we let

$$A = \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad x = \begin{pmatrix} c \\ d \end{pmatrix} \qquad y = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}$$

Find the solution to the system by

$$A^*Ax = A^*y$$

$$\begin{pmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 14 & -4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -38 \\ 18 \end{pmatrix}$$

Expand the computation we can solve

$$\begin{cases} 14c - 4d = -38 \\ -4c + 4d = 18 \end{cases} \Longrightarrow \begin{cases} c = -2 \\ d = 2.5 \end{cases}$$

So the linear function is

$$y = 2x + 2.5$$

and the error E can be computed by

$$\begin{aligned} E &= \|Ax - y\|^2 \\ &= \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2.5 \end{pmatrix} - \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} \\ &= 1 \end{aligned}$$

(ii) To compute the quadratic form $y = ax + bx + c$, suppose

$$A = \begin{pmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad y = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}$$

Find the solution to the system by

$$A^* A x = A^* y$$

$$\begin{pmatrix} 98 & -34 & 14 \\ -34 & 14 & -4 \\ 14 & -4 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 106 \\ -38 \\ 18 \end{pmatrix}$$

Expand the computation we can solve

$$\begin{cases} 98a - 34b + 14c = 106 \\ -34a + 14b - 4c = -38 \\ 14a - 4b + 4c = 18 \end{cases} \Longrightarrow \begin{cases} a = \dfrac{1}{3} \\ b = -\dfrac{4}{3} \quad c = 2 \end{cases}$$

So the linear function is

$$y = \frac{1}{3}x^2 - \frac{4}{3}x + 2$$

and the error E can be computed by

$$\begin{aligned} E &= \|Ax - y\|^2 \\ &= \begin{pmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/3 \\ -4/3 \\ 2 \end{pmatrix} - \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} \\ &= 0 \end{aligned}$$

Problem 8

Let $\mathbf{V}$ be a complex inner product space, and let T be a linear operator on $\mathbf{V}$. Define

$$T_1 = \frac{1}{2}\left(T + T^*\right) \text{ and } T_2 = \frac{1}{2i}\left(T - T^*\right).$$

1. Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.
2. Suppose also that $T = U_1 + iU_2$, where U_1 and U_2 are self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$.
3. Prove that T is normal if and only if $T_1T_2 = T_2T_1$.

Solution

1. *Proof.*

$$\begin{aligned} T_1^* &= \left(\frac{1}{2}(T+T^*)\right)^* & T_2^* &= \left(\frac{1}{2i}(T-T^*)\right)^* \\ &= \frac{1}{2}(T^* + (T^*)^*) & &= \frac{1}{2i}(T^* - (T^*)^*) \\ &= \frac{1}{2}(T^* + T) & &= \frac{1}{2i}(-T^* + T) \\ &= T_1 & &= T_2 \end{aligned}$$

$$T_1 + iT_2 = \frac{1}{2}(T+T^*) + \frac{1}{2}(T - T^*) = T$$

□

2. *Proof.* If $T = U_1 + iU_2$ and U_1, U_2 are self-adjoint, then

$$T^* = (U_1 + iU_2)^* = U_1^* - iU_2^* = U_1 - iU_2$$

$$T_1^* = \left(\frac{1}{2}(T+T^*)\right) = \frac{1}{2}((U_1 + iU_2) + (U_1 - iU_2)) = U_1$$

$$T_1^* = \left(\frac{1}{2i}(T-T^*)\right) = \frac{1}{2i}((U_1 + iU_2) - (U_1 - iU_2)) = U_2$$

□

3. *Proof.*

$$T_1T_2 = \frac{1}{2}(T+T^*)\frac{1}{2i}(T-T^*) = \frac{1}{4i}(T^2 - TT^* + T^*T - (T^*)^2)$$

$$T_2T_1 = \frac{1}{2i}(T-T^*)\frac{1}{2}(T+T^*) = \frac{1}{4i}(T^2 + TT^* - T^*T - (T^*)^2)$$

If $T_1T_2 = T_2T_1$, then

$$\frac{1}{4i}(T^2 - TT^* + T^*T - (T^*)^2) = \frac{1}{4i}(T^2 + TT^* - T^*T - (T^*)^2)$$

which is equivalent to

$$2T^*T = TT^* \implies T \text{ is normal}$$

If T is normal, then $T_1T_2 = T_2T_1$ □

Problem 9

Let T be a self-adjoint linear operator on a finite-dimensional inner product space. Prove that $(T+iI)(T-iI)^{-1}$ is unitary using the following conclusion: $T-iI$ is invertible and $[(T-iI)^{-1}]^* = (T+iI)^{-1}$ if T is self-adjoint.

Solution

Proof. Linear operator T is unitary if $TT^* = T^*T = I$.

$$\begin{aligned}
&[(T+iI)(T-iI)^{-1}]^*[(T+iI)(T-iI)^{-1}] \\
&= [(T-iI)-1]^*(T+iI)^*(T+iI)(T-iI)^{-1} \\
&= (T+iI)^{-1}(T+iI)(T-iI)(T-iI)^{-1} \\
&= I
\end{aligned}$$

□

Problem 10

Solution

Proof. If $n = 2$, then obviously

$$\det(M) = \det\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = ac$$

For induction from $n-1 \Rightarrow n$, we take the determinant by expanding along the first column of M. Let $\tilde{M}_{ij}$ be the matrix obtained from M by deleting the ith and jth column. Let k be the largest row of A. First, note that $M_{i1} = 0$ for all $i > k$. For $i \leq k$, $M_{i1} = A_{i1}$ and

$$\det(\tilde{M}_{i1}) = \det\begin{pmatrix} \tilde{A}_{i1} & \tilde{B} \\ O & C \end{pmatrix}$$

by the induction hypothesis as $\tilde{M}_{i1}$ is block upper triangular. Then

$$\begin{aligned}
\det(M) &= \sum_{i=1}^{n}(-1)^{i+1}M_{i1}\det(\tilde{M}_{i1}) \\
&= \sum_{i=1}^{k}(-1)^{i+1}M_{i1}\det(\tilde{M}_{i1}) \\
&= \left(\sum_{i=1}^{k}(-1)^{i+1}A_{i1}\det(\tilde{A}_{i1})\right)\det(C) \\
&= \det(A)\det(C)
\end{aligned}$$

□