300.206A(003): LINEAR ALGEBRA 2
Homework #3

Due on November 10 2021

Professor Atanas Iliev

Minghang Li

November 10, 2021

Problem 1

Given the following matrix

$$A = \begin{bmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{bmatrix}$$

find a basis for the generalized eigenspace of L_A consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of A.

Solution

We first find the eigenvalues of A. Let

$$\det(A - \lambda I) = 0$$

We have

$$\begin{aligned} &-\lambda^3 + 3\lambda^2 + 30\lambda + 26 = 0 \\ &-(\lambda + 1)(\lambda^2 - 4\lambda + 4) = 0 \end{aligned}$$

We can compute that $\lambda_1 = -1$, $\lambda_2 = 2$ with multiplicity of 2. thus $\dim(K_{\lambda_1}) = 1$ and $\dim(K_{\lambda_2})2$. Also note that $\dim(E_{\lambda_1}) = 1$ and $\dim(E_{\lambda_2}) = 1$, which means that the basis for K_{λ_1} is a single eigenvector and for K_{λ_2} is a single cycle of length 2.
Eigenvector corresponding to λ_1 is

$$\beta_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

We need to find a vector v such that

$$(A - 2I)v \neq 0 \quad (A - 2I)^2 v = 0$$

$$(A - 2I)^2 = \begin{bmatrix} -18 & 9 & 9 \\ -54 & 27 & 27 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of the solution space $(A - 2I)^2 x = 0$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Now from that basis we choose a vector such that $(A - 2I)v \neq 0$:

$$(A - 2I) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

which means that

$$\beta_2 = \left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is a basis for K_{λ_2}.

Problem 1 continued on next page. . .

Then the Jordan canonical basis of A is

$$\beta = \beta_1 \cup \beta_2 = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

And the Jordan canonical form of A is

$$J = [A]_\beta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 2

Given the following matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

find a basis for the generalized eigenspace of L_A consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of A.

Solution

We first find the eigenvalues of A. Let

$$\det(A - \lambda I) = 0$$

We have

$$\lambda^2 - 10\lambda^3 + 37\lambda^2 - 60\lambda + 36 = 0$$
$$(\lambda - 3)^2(\lambda - 2)^2 = 0$$

We can compute that $\lambda_1 = 3$ with the multiplicity of 2, $\lambda_2 = 2$ with multiplicity of 2, thus $\dim(K_{\lambda_1}) = 2$ and $\dim(K_{\lambda_2})2$. Also note that $\dim(E_{\lambda_1}) = 1$ and $\dim(E_{\lambda_2}) = 1$, which means that the basis for K_{λ_1} is a single cycle of length 2 and the basis for K_{λ_2} is a union of two cycles of length 1. We need to find a vector v such that

$$(A - 2I)v \neq 0 \quad (A - 2I)^2 v = 0$$

$$(A - 2I)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

Basis of the solution space $(A - 2I)^2 x = 0$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Now from that basis we choose a vector such that $(A - 2I)v \neq 0$:

$$(A-2I)\begin{bmatrix}0\\-1\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\0\\0\end{bmatrix}$$

which means that

$$\beta_1 = \left\{\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right\}$$

is a basis for K_{λ_1}.
Now choose eigenvectors for A corresponding to eigenvalue 3:

$$A - 3I = \begin{bmatrix}-1 & 1 & 0 & 0\\0 & -1 & 1 & 0\\0 & 0 & 0 & 0\\0 & 1 & -1 & 0\end{bmatrix}$$

basis of the solution space of $(A - 3I)x = 0$ is

$$\beta_2 = \left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

Then the Jordan canonical basis of A is

$$\beta = \beta_1 \cup \beta_2 = \left\{\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

And the Jordan canonical form of A is

$$J = [A]_\beta = \begin{bmatrix}2 & 1 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 3 & 0\\0 & 0 & 0 & 3\end{bmatrix}$$

Problem 3

For the linear operator T on $P_2(\mathbb{R})$ defined by

$$T(f(x)) = 2f(x) - f'(x)$$

Find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of T.

Solution

Let α be a standard ordered basis for $P_2(\mathbb{R})$. Then

$$A = [T]_\alpha = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Note that eigenvalue of A is $\lambda = 2$ with multiplicity of 3, thus $\dim(K_\lambda) = 3$. Also note that $\dim(E_\lambda) = 1$, which means that the basis of K_λ is a single cycle of length 3.
We need to find a vector v such that

$$(A - \lambda I)v \neq 0 \quad (A - \lambda I)^2 v \neq 0 \neq (A - \lambda I)^3 v = 0$$

$$(A - 2I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of the solution space of $(A - 2I)^3 x = 0$ is :

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Now from that basis choose a vector such that $(A - 2I)v \neq 0$ and $(A - 2I)^2 v \; neq 0$:

$$(A - 2I) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$(A - 2I)^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Which means that the basis β of K_λ is

$$\beta = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

And the Jordan canonical form is

$$J = [T]_\beta = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 4

For the linear operator T defined by

$$T(A) = 2A + A^t \forall A \in M_{2\times 2}(\mathbb{R})$$

Find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of T.

Solution

Let α be $\{1, t, t^2, e^t, te^t\}$ Then

$$A = [T]_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that this matrix is almost in Jordan canonical form except for A_{23} which should be 1 instead of 2. Thus, almost all vectors of $\alpha = \{1, t, t^2, e^t, te^t\}$ can be left unchanged. So instead of t^2 we can put $\frac{1}{2}y^2$ in the basis such that $T\left(\frac{1}{2}t^2\right) = t$. Thus, for $\beta = \left\{1, t, \frac{1}{2}t^2, e^t te^t\right\}$, the Jordan canonical form is

$$J = [T]_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5

Let $T : V \mapsto W$ be a linear transformation. Prove the following results:

1. $N(T) = N(-T)$.
2. $N(T^k) = N((-T)^k)$.
3. If $V = W$ (so that T is a linear operator on V) and λ is an eigenvalue of T, then for any positive integer k,
$$N((T - \lambda I_V)^k) = N((-T + \lambda I_V)^k).$$

Solution

1. $T(x) = 0 \iff -T(x) = 0 \iff (-T)(x) = 0$, which means that $N(T) = N(-T)$.
2. $T^k(x) = 0 \iff (-1)^k T^k(x) = 0 \iff (-T)^k(x) = 0$, which means that $N(T^k) = N((-T)^k)$
3. Since (b) holds, $N((T - \lambda I_V)^k) = N((-(\lambda I_V - T))^k) = N((\lambda I_V - T)^k)$

Problem 6

Let U be a linear operator on finite-dimensional vector space V. Prove the following results:

1. If $\text{rank}(U^m) = \text{rank}(U^{m+1})$ for some positive integer m, then $N(U^m) = N(U^k)$ for any positive integer $k \geq m$.

2. Let T be a linear operator on V, and let λ be an eigenvalue of T. Prove that if $\text{rank}((T - \lambda I)^m) = \text{rank}((T - \lambda I)^{m+1})$ for some positive integer m, then $K^\lambda = N((T - \lambda I)^m)$

Solution

1. Note that
$$U^{m+1}(V) = U^m(U(V)) \subseteq U^m(V)$$
which means that
$$\text{rank}(U^{m+1}) \leq \text{rank}(U^m)$$
Since we know that their ranks are equal, then
$$U^{m+1}(V) = U^m(V)$$
Now
$$U^{m+2}(V) = U((U^{m+1}(V))) = U((U^m(V))) = U^{m+1}(V) = U^m(V)$$
Inductively,
$$U^k(V) = U^{m+l}(V) = U^m(V)$$
i.e.
$$\text{rank}(U^k(V)) = \text{rank}(U^m(V))$$
Since V is finite-dimensional with dimension n, we know that
$$\dim(R(U)) + \dim(N(U)) = n$$
for every linear operator U on V. Then
$$\dim(N(U^m)) = n - \text{rank}(U^m) = n - \text{rank}(U^k) = \dim(N(U^k)) \quad \forall k \geq m$$
Then, (from the conclusion in the book) we know that
$$N(U^m) \subseteq N(U^k)$$
Since dimensions are equal, then $N(U^m) = N(U^k)$.

2. K_λ is defined as
$$K_\lambda = \{x \in V : (T - \lambda I)^p(x) \text{ for some positive integer } p\}$$
which means that
$$K_\lambda = \{N(T - \lambda I)^p \text{ for some positive integer } p\}$$
Since
$$N(T - \lambda I) \subseteq N(T - \lambda I)^2 \subseteq \cdots \subseteq N(T - \lambda I)^m = N(T - \lambda I)^{m+1} = \cdots$$
clearly,
$$K_\lambda = N(T - \lambda I)^m$$

Problem 7

Let T be a linear operator on a finite-dimensional vector space V with Jordan canonical form

$$\left(\begin{array}{ccc|cc|cc} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right).$$

1. Find the characteristic polynomial of T.
2. Find the dot diagram corresponding to each eigenvalue of T.
3. For which eigenvalue λ_i, does $E_{\lambda_i} = K_{\lambda_i}$?

Solution

1. Note that since matrix is upper triangular, the characteristic polynomial is trivially

$$f(t) = -(t-2)^5(t-3)^2$$

2. The dot diagrams:

$$\begin{array}{ccc} \lambda_1 = 2 & \qquad & \lambda_2 = 3 \\ \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array} & & \begin{array}{cc} \bullet & \bullet \end{array} \end{array}$$

3. This can be read from the Jordan canonical form, we need to find the eigenvalues for which J_{λ_i} is a diagonal matrix. Clearly,

$$J_{\lambda_2} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

which means that

$$E_{\lambda_2} = K_{\lambda_2}$$

Problem 8

Let T be a linear operator on a finite-dimensional vector space V with Jordan canonical form

$$\left(\begin{array}{ccc|cc|cc} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right).$$

1. For each eigenvalue λ_i, find the smallest positive integer p_i for which $K_{\lambda_i} = N((T - \lambda_i I)^{p_i})$.
2. Compute the following numbers for each i where U_i denotes the restriction of $T - \lambda_i I$ to K_{λ_i}.
   (a) $\text{rank}(U_i)$
   (b) $\text{rank}(U_i^2)$
   (c) $\text{nullity}(U_i)$
   (d) $\text{nullity}(U_i^2)$

Solution

1. The integer p_i is the length of the longest cycle in K_{λ_i}. We can easily see that

$$p_1 = 3, \quad p_2 = 1$$

2. Matrix representation of U_1 and U_2 are:

$$U_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad U_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   And we have

$$U_1^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   So we can compute:
   (a) $\text{rank}(U_1) = 3, \quad \text{rank}(U_2) = 0$
   (b) $\text{rank}(U_1^2) = 1, \quad \text{rank}(U_2^2) = 0$
   (c) $\text{nullity}(U_1) = 2, \quad \text{nullity}(U_2) = 2$
   (d) $\text{nullity}(U_1^2) = 4, \quad \text{nullity}(U_2^2) = 2$

Problem 9

Find the minimal polynomial of

$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Solution

First we find the characteristic polynomial of the matrix A by computing $\det(A - \lambda I) = 0$. Then we have:

$$\begin{aligned} -\lambda^3 + 6\lambda^2 - 12\lambda + 8 &= 0 \\ -(\lambda - 2)^3 &= 0 \end{aligned}$$

So the characteristic polynomial $f(\lambda)$ is

$$f(\lambda) = (\lambda - 2)^3$$

Candidated for the minimal polynomial are $\lambda - 2$, $(\lambda - 2)^2$, and $(\lambda - 2)^3$. Obviously, $A - 2I$ is not a zero matrix.

$$\begin{aligned} (A - 2I)^2 &= \begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{bmatrix} \\ &= O \end{aligned}$$

Then the minimal polynomial is $m(\lambda) = (\lambda - 2)^2$

Problem 10

Let T be a linear operator on a finite-dimensional vector space, and let $p(t)$ be the minimal polynomial of T. Prove the following results.

1. T is invertible if and only if $p(0) = 0$.
2. If T is invertible, and $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, Then

$$T^{-1} = -\frac{1}{a_0}(T^{n-1} + a_{n-1}T^{n-1} + \cdots + a_2 T + a_1 I)$$

Solution

1. (a) (Prove "if") Suppose the characteristic function of T is $f(t)$, and assume T is an invertible operator. Then it is one-to-one, which is equivalent to $N(T) = \{0\}$.

   In other words, the image of no vector other than the zero vector by T equals to the zero vector. This further implies that $\lambda = 0$ cannot be an eigenvalue of T, since if so the dimension of its eigenspace would be greater or equal to 1 ($\dim[N(T)] \geq 1$), which contradicts the fact that $N(T) = \{0\}$.

   Therefore, T is not invertible, and $f(0) \neq 0$. Since the characteristic polynomial and the minimal polynomial have the same zeros, we get $p(0) \neq 0$. To say it in another way, if T is invertible, $p(0) = 0$.

(b) (Prove "only if") Let $p(0) \neq 0$, then characteristic polynomial $f(0) \neq 0$ since they have the same zeros. Hence 0 is not an eigenvalue of T. Since T is a linear operator, $T(0) = 0$.

If we assume that for any other vector x and some constant $c \in \mathbb{R}$, we have $T(x) = 0$, from the linearity of T it follows that $T(cx) = cT(x) = 0$, which implies $\lambda = 0$ is an eigenvalue of T. Correspondingly, the eigenvalue dimension is at least 1. This contradicts the previous conclusion.

As a conclusion, $N(T) = \{0\}$, which tells us that T is a one-to-one operator. According to Rank-Nullity theorem, $\dim[\text{rank}(T)] = \dim(V)$, which is equivalent to T being onto.

As a one-to-one and onto transformation, T is invertible.

2. According to Cayley-Hamilton theorem, $p(T) = 0$. Also, T has linearity. Then we can compute:

$$
\begin{aligned}
TT^{-1} &= T\left[-\frac{1}{a_0}(T^{n-1} + a_{n-1}T^{n-1} + \cdots + a_2T + a_1I)\right] \\
&= \left[-\frac{1}{a_0}(TT^{n-1} + a_{n-1}TT^{n-1} + \cdots + a_2TT + a_1TI)\right] \\
&= -\frac{1}{a_0}(T^n + a_{n-1}T^n + \cdots + a_2T^2 + a_1T) \\
&= -\frac{1}{a_0}(p(T) - a_0I) \\
&= \frac{1}{a_0}(0 - a_0I) \\
&= I
\end{aligned}
$$

$$
\begin{aligned}
T^{-1}t &= \left[-\frac{1}{a_0}(T^{n-1} + a_{n-1}T^{n-1} + \cdots + a_2T + a_1I)\right]t \\
&= \left[-\frac{1}{a_0}(T^{n-1}T + a_{n-1}T^{n-1}T + \cdots + a_2TT + a_1IT)\right] \\
&= -\frac{1}{a_0}(T^n + a_{n-1}T^n + \cdots + a_2T^2 + a_1T) \\
&= -\frac{1}{a_0}(p(T) - a_0I) \\
&= \frac{1}{a_0}(0 - a_0I) \\
&= I
\end{aligned}
$$

Since we show that $TT^{-1} = T^{-1}T$, T^{-1} is really the inverse of T.