1. $\forall x, f(x)=\infty \Leftrightarrow epi(f)=\phi$

proof: $f=\infty \Leftrightarrow dom(f)=\phi$

$\because$ dom(f) is projection of epi(f)

$\therefore dom(f)=\phi \Leftrightarrow epi(f)=\phi$

2. $\exists x\in X$, s.t., $f(x)=-\infty \Leftrightarrow$ epi(f) contains a vertical line.

proof: $\exists x$, s.t., $f(x)=-\infty \Leftrightarrow f(x)\leq w, \forall w\in R$

$\Leftrightarrow$ epi(f) contains a vertical line.

3. f is proper if $epi(f)\neq\phi$ and epi(f) doesn't contain a vertical line.

4. forbidden sum $-\infty+\infty$ may happen in $\alpha f(x)+(1-\alpha)f(y)$

$\Leftrightarrow$ f is improper.

5. $C\subset R^n$ convex, $f: C\to[-\infty,\infty]$ convex if epi(f) convex.

definition:

6. $f: C\to[-\infty,\infty]$ convex $\Rightarrow$ dom(f) convex, level set convex.

~~proof~~

7. $f: C\to(-\infty,\infty)$ convex $\Leftrightarrow$ epi(f) convex

proof: ① $\Rightarrow$, let $(x,u),(y,v)\in epi(f)$, $\therefore \underset{\leq u}{f(x)}, f(y)\leq v$

$p=\alpha(x,u)+(1-\alpha)(y,v)\in epi(f)$?

$=(\alpha x+(1-\alpha)y, \alpha u+(1-\alpha)v)$

$\because f(\alpha x+(1-\alpha)y)\leq\alpha f(x)+(1-\alpha)f(y)\leq\alpha u+(1-\alpha)v$, $\therefore p\in epi(f)$

② $\Leftarrow$ let $x,y\in C$, $u=f(x)$, $v=f(y)$, $\therefore (x,u),(y,v)\in epi(f)$

$\because$ epi(f) convex, $\therefore p=\alpha(x,u)+(1-\alpha)(y,v)\in epi(f)$

$\therefore f(\alpha x+(1-\alpha)y)\leq\alpha u+(1-\alpha)v=\alpha f(x)+(1-\alpha)f(y)$

8. $f: C \to [-\infty, \infty)$ convex $\Leftrightarrow$ epi(f) convex

proof: ① $\Rightarrow$ same as the proof 7.①

② Similar to the proof 7.② if $f(x), f(y) \neq -\infty$.

if $f(x) = -\infty$, $\Rightarrow \forall u \in \mathbb{R}$, $(x, u) \in$ epi(f).

$\because p \in$ epi(f), $\therefore f(\alpha x + (1-\alpha)y) \leq \alpha u + (1-\alpha)v$, $\forall u \in \mathbb{R}$, $v = f(y)$

$\therefore f(\alpha x + (1-\alpha)y) = -\infty = \alpha f(x) + (1-\alpha) f(y)$, $\alpha \neq 0$.

9. $f: C \to [-\infty, \infty]$, $\therefore f|_{dom(f)}$ convex $\Leftrightarrow$ epi($f|_{dom(f)}$) convex (from 8)

$\therefore f|_{dom(f)}$ convex $\Leftrightarrow$ epi(f) convex

5. $\because$ definition: $f: C \to [-\infty, \infty]$ convex if $f|_{dom(f)}$ convex.

10. $f: X \to [-\infty, \infty) \Leftrightarrow X = dom(f)$

11. $f: X \to [-\infty, \infty)$ convex $\Rightarrow$ dom(f) convex, level set convex.

proof ① $\because f$ convex $\therefore X$ convex. $\therefore$ dom(f) convex.

② f convex $\Rightarrow$ level set convex. (see page 6)

6. $f: X \to [-\infty, \infty]$ convex $\Rightarrow$ dom(f) convex, level set convex.

12. $f: X \to [-\infty, \infty]$, $\Rightarrow$ level set of f = level set of $f|_{dom(f)}$

proof: ① $x \in$ level set of f, $\therefore f(x) \leq \gamma$, $\therefore x \in dom(f)$.

$\therefore f|_{dom(f)}(x) \leq \gamma$ $\therefore x \in$ level set of $f|_{dom(f)}$

②, $x \in$ level set of $f|_{dom(f)}$, $\therefore f(x) = f|_{dom(f)}(x) \leq \gamma$

$\therefore x \in$ level set of f

6 proof: $\because f$ convex $\therefore f|_{dom(f)}$ convex

$\therefore$ dom($f|_{dom}$) convex, levelset($f|_{dom}$) convex

$\therefore$ dom(f) convex, levelset(f) convex.

13. $f: C \to (-\infty, \infty]$ convex $\Leftrightarrow$ epi(f) convex.

proof: ① $\Rightarrow$ same as the proof of 7.①

② $\Leftarrow$ if $f(x), f(y) \neq \infty$, similar to the proof of 7.②

if $f(x), f(y) = \infty$, $\Rightarrow f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$

if $f(x) = \infty$, $f(y) \neq \infty$, $\alpha \neq 0$, $\Rightarrow$ convex.

$\therefore f$ convex.

14. $\because f: C \to [-\infty, \infty]$ convex $\Leftrightarrow$ epi(f) convex. (definition)

$f: C \to [-\infty, \infty)$ convex $\Leftrightarrow$ epi(f) convex

$f: C \to (-\infty, \infty]$ convex $\Leftrightarrow$ epi(f) convex

$\therefore$ The definition $f: C \to [-\infty, \infty]$ convex $\Leftrightarrow$ epi(f) convex

is consistent with the definition of convex function.

15. epi(f) convex $\Rightarrow$ dom(f) convex

16. page 7: 3.①: $\exists w = \begin{cases} f(x), & \text{if } f(x) \neq -\infty \\ 0, & \text{if } f(x) = -\infty \end{cases}$, $\therefore w \in \mathbb{R}$.

15. proof: $\forall x, y \in$ dom(f), $\therefore f(x), f(y) < \infty$,

method 1: let $u = (f(x) = -\infty \,?\, 0 : f(x))$, $v = (f(y) = -\infty \,?\, 0 : f(y))$

$\therefore f(x) \leq u$, $f(y) \leq v$, $\therefore (x, u), (y, v) \in$ epi

$\because$ epi(f) convex, $\therefore \alpha(x, u) + (1-\alpha)(y, v) = (\alpha x + (1-\alpha)y, \alpha u + (1-\alpha)v) \in$ epi

$\therefore f(\alpha x + (1-\alpha)y) \leq \alpha u + (1-\alpha)v < \infty$

$\therefore \alpha x + (1-\alpha)y \in$ dom $\therefore$ dom convex.

method 2: $\because$ dom(f) is projection of epi, epi convex, $\therefore$ dom convex.

17. X convex, X_1 is projection of X, $\Rightarrow X_1$ convex

proof: $\forall x_1, y_1 \in X_1$, $\exists x_2, y_2$, s.t., $(x_1, x_2) \in X, (y_1, y_2) \in X$

$\because X$ convex, $\therefore \alpha(x_1, x_2) + (1-\alpha)(y_1, y_2) \in X$.

$\therefore (\alpha x_1 + (1-\alpha) y_1, \alpha x_2 + (1-\alpha) y_2) \in X$

$\therefore \alpha x_1 + (1-\alpha) y_1 \in X_1$ $\therefore X_1$ convex

11.① $\because f$ convex $\therefore$ epi(f) convex $\therefore$ dom(f) convex (from 15)

18. $X = \{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\}$, X not empty.

$\therefore X$ convex $\Leftrightarrow X_1, X_2$ convex

proof: ① $\Rightarrow$ (from 17) (require X not empty)

② $\Leftarrow$ $\forall (x_1, x_2), (y_1, y_2) \in X$ $\because X_1, X_2$ convex

$\therefore \alpha(x_1, x_2) + (1-\alpha)(y_1, y_2)$ (X can be empty)

$= (\alpha x_1 + (1-\alpha) y_1, \alpha x_2 + (1-\alpha) y_2) \in X$

19. indicator function $\delta: R^n \to (-\infty, \infty]$ of $X \subset R^n$, is defined by

$$\delta(x|X) = \begin{cases} 0, & \text{if } x \in X \\ \infty, & \text{if } x \notin X \end{cases}$$

20 X convex $\Leftrightarrow \delta(x|X)$ convex.

proof: ① $\Leftarrow$ $\delta(x|X)$ convex, $\therefore$ epi(δ) convex, $\therefore X = $ dom(δ) convex.

② $\Rightarrow$ epi(δ) $= \{(x, w) \mid x \in X, w \in R, f(x) \le w\}$

$= \{(x, w) \mid x \in X, w \in R, w \ge 0\}$

$\because X$ convex, $\{w \mid w \ge 0\}$ convex, $\therefore$ epi(δ) convex.

21 X nonempty $\Leftrightarrow \delta_X$ proper.

proof: ① X nonempty $\therefore$ epi(δ) nonempty $\because$ epi(δ) doesn't contain vertical line $\therefore \delta_X$ proper. ② δ_X proper, $\therefore$ epi $\neq \emptyset$ $\therefore X \neq \emptyset$