1. $(X, x) \Rightarrow K, R, t,$

s.t. $x = PX$

where $P = K(R, t)$

2. $P_1 = (R_1, t_1)$, $P_2 = (R_2, t_2)$

$\Rightarrow P_{1\to2} = P_2 P_1^{-1} = (R_2\ t_2)\begin{pmatrix} R_1 & t_1 \\ & 1 \end{pmatrix}^{-1}$

2.3 let $P_{1\to2} = (R, t)$

$\Rightarrow P_2 = P_{1\to2} P_1 = (R\ t)\begin{pmatrix} R_1 & t_1 \\ & 1 \end{pmatrix}$

$\therefore \{P_1, P_2\}$:

$\Leftrightarrow \{P_1, P_{1\to2}P_1\}$

$\Leftrightarrow \{(I, 0), (R, t)\}$

2.4 $(R, t) = (R_2\ t_2)\begin{pmatrix} R_1 & t_1 \\ & 1 \end{pmatrix}^{-1}$ #

$\therefore R = R_2 R_1^{-1}$, $t = t_2 - R t_1$

2.5 $P_2 = (R\ t)\begin{pmatrix} R_1 & t_1 \\ & 1 \end{pmatrix}$

$= (RR_1,\ Rt_1 + t)$

3. let $R_{1\to2} = R_2 R_1^{-1}$

for rotation-only model.

$H_{1\to2} = K_2 R_{1\to2} K_1^{-1}$

$\Rightarrow R_{1\to2} = K_2^{-1} H_{1\to2} K_1$

$R_2 = R_{1\to2} R_1$ #

here cameras are $K_1(R_1, 0)$, $K_2(R_2, 0)$.

3.2 $\therefore \{K_1(R_1, 0), K_2(R_2, 0)\}$

$\Leftrightarrow \{K_1(R_1, 0), K_2 R(R_1, 0)\}$

$\Leftrightarrow \{K_1(I, 0), K_2(R, 0)\}$

here $R = R_{1\to2}$

4. $\{P_1, P_2\} = \{(R\ 0), (R\ t)\}$

$\Leftrightarrow \{P_1', P_2'\} = \{(I, 0), (I, t)\}$

proof:

$\because R_{1\to2} = R_2 R_1^{-1}$

$t_{1\to2} = t_2 - R_{1\to2} t_1$ (from 2.4)

$\therefore R_{1\to2} = I$, $t_{1\to2} = t$

$\therefore P_1', P_2' = (I, 0), (R_{1\to2}, t_{1\to2})$

$= (I, 0), (I, t)$ (from 2.3)

5. $(I, 0), (R, t) \xrightarrow{\text{rectify}} (I, 0), (I, t_0)$,

where $t_0 = NR^{-\frac{1}{2}} t$

proof: ① $(I, 0), (R, t)$

$\Rightarrow R^{\frac{1}{2}}(I, 0), R^{-\frac{1}{2}}(R, t)$

$\Rightarrow (R^{\frac{1}{2}}, 0), (R^{\frac{1}{2}}, R^{-\frac{1}{2}} t)$

$\Rightarrow (I, 0), (I, s)$, where $s = R^{-\frac{1}{2}} t$

(from 4)

② let $x = \begin{pmatrix}1\\0\\0\end{pmatrix}$ if hori

~~else~~ $x \leftarrow (0,1,0)^T$ else.

$x_2 = s$.

$n \leftarrow x_2 \times x$, $n \leftarrow \frac{n}{\|n\|}$ (from 7)

$\theta = \arccos \frac{x_2 \cdot x}{\|x_2\| \cdot \|x\|}$

$v \leftarrow \theta \cdot n$

$\because$ v is the exponential coordinates

$\therefore$ $N \leftarrow \text{rodrigues}(v)$

7. v is exponential coordinates,

$v = \theta \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\Rightarrow R = \begin{pmatrix} c\theta & -s\theta & \\ s\theta & c\theta & \\ & & 1 \end{pmatrix}$

($\because R = I + s\theta \,\overline{v}_\times + (1-c\theta)\overline{v}_\times^2$)

R rotates point by θ around axis $\begin{pmatrix}0\\0\\1\end{pmatrix}$

here $x_2 \leftarrow s$ if hori

$y_2 \leftarrow s$ else.

③ let $z = (0,0,1)^T$

$\therefore$ if hori:

$x_2 \leftarrow s$,

$y_2 \leftarrow z \times x_2$

else

$y_2 \leftarrow s$

$x_2 \leftarrow y_2 \times z$

$\therefore z_2 \leftarrow x_2 \times y_2$

$x_2 \leftarrow x_2/\|x_2\|$,

$y_2 \leftarrow y_2/\|y_2\|$

$z_2 \leftarrow z_2/\|z_2\|$

$\therefore N = \begin{pmatrix} x_2^T \\ y_2^T \\ z_2^T \end{pmatrix}$

method 2. (use exponential coordinates)

① same as method 1 ①.

②

$\rightarrow NR^{\frac{1}{2}}(I, 0),\ NR^{-\frac{1}{2}}(R, t)$

$\Rightarrow (I, 0), (I, t_0)$

where $t_0 = NR^{-\frac{1}{2}} t$

here t_0 must have form

$\begin{pmatrix}-b\\0\\0\end{pmatrix}$ or $\begin{pmatrix}0\\-b\\0\end{pmatrix}$, where b is baseline.

6. $s \rightarrow N$ in 5.

solve:

method 1: (construct R directly)

① horizontal $= |s(1)| > |s(2)|$

if hori, $ik \leftarrow 1$.

else $ik \leftarrow 2$.

if $s(ik) < 0$, $s \leftarrow -s$.

② if hori, choose z_2, s.t. $z_2 \in \text{plane}(z, 0, x_2)$

else, choose z_2, s.t. $z_2 \in \text{plane}(y_2, 0, z)$

8. $P_1, P_2 = K_1(I, 0),\ K_2(I, t_0)$

with same focus, i.e.,

$f_{1x} = f_{1y} = f_{2x} = f_{2y} = f$

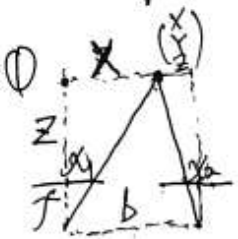
$t_0 = (-b, 0, 0)^T$ or $(0, -b, 0)^T$

$\Rightarrow Q$, s.t., $P = Qp$.

here $p = (x, y, d, 1)^T$

$d = x_1 - x_2$ when hori.

solve: ①



$$\frac{X}{Z} = \frac{x_1 - c_{1x}}{f}, \quad \frac{X-b}{Z} = \frac{x_2 - c_{2x}}{f}$$

② $\therefore$ depth $= \dfrac{bf}{d + c_{21}}$, $c_{21} = c_{2x} - c_{1x}$

③
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} (x - c_{1x}) \cdot \frac{\text{depth}}{f} \\ (y - c_{1y}) \cdot \frac{\text{depth}}{f} \\ \text{depth} \end{pmatrix}$$

$$\therefore \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} X \\ Y \\ Z \\ s \end{pmatrix} = \begin{pmatrix} x - c_{1x} \\ y - c_{1y} \\ f \\ f/\text{depth} \end{pmatrix}$$

$$\therefore \begin{pmatrix} X \\ Y \\ Z \\ s \end{pmatrix} = \begin{pmatrix} x - c_{1x} \\ y - c_{1y} \\ f \\ (d + c_{21})/b \end{pmatrix}$$

$$= Q \begin{pmatrix} x \\ y \\ d \\ 1 \end{pmatrix}$$

where
$$Q = \begin{pmatrix} 1 & & & -c_{1x} \\ & 1 & & -c_{1y} \\ & & 0 & f \\ & & 1/b & c_{21}/b \end{pmatrix}$$

④ hori $\leftarrow |P_2(1,4)| > |P_2(2,4)|$

$i_k \leftarrow$ hori ? 1 : 2.

$b \leftarrow -P_2(i_k, 4) / P_2(i_k, i_k)$

$c_{1x} \leftarrow P_1(1,3)$, $c_{1y} \leftarrow P_1(2,3)$

$f \leftarrow P_1(1,1)$.

$c_{21} \leftarrow P_2(i_k, 3) - P_1(i_k, 3)$

$\therefore \Rightarrow Q$.

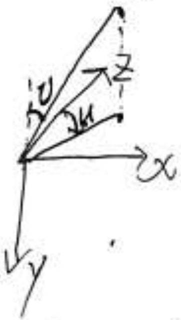
1. spherical projection.

1.2 $(x,y,z) \to (u,v)$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow \left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\|^{-1} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore u \leftarrow r \cdot \tan 2^{-1}\left(\frac{x}{z}\right)$$

$$v \leftarrow r \cdot (\pi - \cos^{-1} y)$$



1.3 $(u,v) \to \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} u \\ v \end{pmatrix} \leftarrow r^{-1} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\therefore v \leftarrow \pi - v$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s(v) \cdot s(u) \\ c(v) \\ s(v) c(u) \end{pmatrix}$$

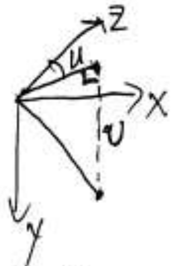
2. cylindrical projection

2.2 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow (x^2+z^2)^{-\frac{1}{2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore \begin{pmatrix} u \\ v \end{pmatrix} \leftarrow \begin{pmatrix} \tan 2^{-1}(x,z) \\ y \end{pmatrix}$$

$$\therefore \begin{pmatrix} u \\ v \end{pmatrix} \leftarrow r \begin{pmatrix} u \\ v \end{pmatrix}$$



2.3 $\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} u \\ v \end{pmatrix} \leftarrow r^{-1} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow \begin{pmatrix} s(u) \\ v \\ c(u) \end{pmatrix}$$

1. trapezoidal distortion of tilted image sensor

given $\theta_x, \theta_y \Rightarrow T$

1.3 let $x = -\theta_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $y = -\theta_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\Rightarrow R_x = e^{x_\times} = \begin{pmatrix} 1 & & \\ & c\theta_x & s\theta_x \\ & -s\theta_x & c\theta_x \end{pmatrix}$$

$$R_y = e^{y_\times} = \begin{pmatrix} c\theta_y & & -s\theta_y \\ & 1 & \\ s\theta_y & & c\theta_y \end{pmatrix}$$

1.4 let $R = R_y R_x$

$$P_z = \begin{pmatrix} R_{33} & & -R_{13} \\ & R_{33} & -R_{23} \\ & & 1 \end{pmatrix}$$

$$\Rightarrow T = P_z R = P_z R_y R_x$$

1.5 Note:

given $v = \theta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow R_v = e^{v_\times} = \begin{pmatrix} c\theta & -s\theta & \\ s\theta & c\theta & \\ & & 1 \end{pmatrix}$$

2. T^{-1}

$\because T = P_z R_y R_x$

$\therefore T^{-1} = R_x^T R_y^T P_z^{-1}$

where $P_z^{-1} = \begin{pmatrix} R_{33}^{-1} & & R_{33}^{-1} R_{13} \\ & R_{33}^{-1} & R_{33}^{-1} R_{23} \\ & & 1 \end{pmatrix}$

2.3 Note:

let $A = \begin{pmatrix} s_1 & & t_1 \\ & s_2 & t_2 \\ & & 1 \end{pmatrix}$

$$\Rightarrow A^{-1} = \begin{pmatrix} s_1^{-1} & & -s_1^{-1} t_1 \\ & s_2^{-1} & -s_2^{-1} t_2 \\ & & 1 \end{pmatrix}$$

3. $T \Rightarrow \frac{\partial T}{\partial \theta_x}, \frac{\partial T}{\partial \theta_y}$

solve:

① $\frac{\partial R}{\partial \theta_x} = R_y \frac{\partial R_x}{\partial \theta_x} = R_y \cdot \begin{pmatrix} 0 & & \\ & -s\theta_x & c\theta_x \\ & -c\theta_x & -s\theta_x \end{pmatrix}$

$\frac{\partial R}{\partial \theta_y} = \frac{\partial R_y}{\partial \theta_y} R_x = \begin{pmatrix} -s\theta_y & & -c\theta_y \\ & 0 & \\ c\theta_y & & -s\theta_y \end{pmatrix} R_x$

② $\frac{\partial P_z}{\partial \theta_x} = \begin{pmatrix} \frac{\partial R}{\partial \theta_x}(3,3) & & -\frac{\partial R}{\partial \theta_x}(1,3) \\ & \frac{\partial R}{\partial \theta_x}(3,3) & -\frac{\partial R}{\partial \theta_x}(2,3) \\ & & 0 \end{pmatrix}$

same to $\frac{\partial P_z}{\partial \theta_y}$

③ $\frac{\partial T}{\partial \theta_x} = P_z \frac{\partial R}{\partial \theta_x} + \frac{\partial P_z}{\partial \theta_x} R$

same to $\frac{\partial T}{\partial \theta_y}$

1. $a, u, v, w \in R^b$, $X \in R^{b\times n}$

$X^T = (X_1, X_i, X_b)$

$v^T = (v_1, v_i, v_b)$

$v_i' = a_i X_i^T \in R^{1\times n}$

$\Rightarrow v' = a \cdot X \in b\times(1\times n) = b\times n$

2. $u, v, w \in R^b$, $X \in R^{b\times n}$

$A \in R^{m\times b}$, $A = (A_1, A_2, A_m)^T$

$X = (X_1, X_i, X_b)^T$

$v_i' = \begin{pmatrix} A_{1i} \\ A_{mi} \end{pmatrix} X_i^T \in R^{m\times n}$

$\Rightarrow v' \in R^{b\times(m\times n)}$

move axis ~~reshape~~ $(v') = \begin{pmatrix} A_1 \\ A_2 \\ A_m \end{pmatrix} \begin{pmatrix} X \\ X \\ X \end{pmatrix} = (m\cdot b)\times((m\cdot b)\times n) = (m\cdot b)\times n$

3. $Y = X \cdot H^T$, $y = Hx$, $x = k_1\binom{u}{1}$, proof:

$y = k_2\binom{v}{1}$, $H_{m,n} = 1$, $x \in R^n$,

$y \in R^m$, $H \in R^{m\times n}$, $Y^* = (\cdots, y, \cdots)^T \in R^{b\times m}$

$X^* = (\cdots, x, \cdots)^T \in R^{b\times n}$

$\Rightarrow dV/dH$,

where $V = (\cdots, v, \cdots)^T \in R^{b\times(m-1)}$

solve:

① $y_i' = \begin{pmatrix} c \\ x^T \\ 0 \end{pmatrix}$, $m\times n$

⑤ $v_i' = \left(\frac{y_i}{y_m}\right)' = -w^2 y_i y_m' + w y_i'$

$= \begin{pmatrix} w \\ -w^2 y_i \end{pmatrix} x^T \in m\times n$

(from 4)

here $w = 1/y_m$

let $A = \begin{pmatrix} A_{11} & A_{12} & A_{1b} \\ A_{21} & A_{22} & A_{2b} \\ A_{m1} & A_{m2} & A_{mb} \end{pmatrix}$

$\therefore v' = b\times(m\times n)$

$= \begin{pmatrix} v_1' \\ v_2' \\ v_b' \end{pmatrix}$

$= \begin{pmatrix} \begin{pmatrix} A_{11}X_1^T \\ A_{m1}X_1^T \end{pmatrix} \\ \begin{pmatrix} A_{12}X_2^T \\ A_{m2}X_2^T \end{pmatrix} \\ \begin{pmatrix} A_{1b}X_b^T \\ A_{mb}X_b^T \end{pmatrix} \end{pmatrix}$

move axis ~~reshape~~

$= \begin{pmatrix} \begin{pmatrix} A_{11}X_1^T \\ A_{12}X_2^T \\ A_{1b}X_b^T \end{pmatrix} \\ \vdots \end{pmatrix} = \begin{pmatrix} A_1 \cdot X \\ \vdots \end{pmatrix}$

$= \begin{pmatrix} A_1 \\ A_2 \\ A_m \end{pmatrix} \begin{pmatrix} X \\ X \\ X \end{pmatrix}$

$$\overset{\text{reshape}}{=} \begin{pmatrix} V_1 \\ V_i \\ V_{m-1} \end{pmatrix}'$$

$$= \begin{pmatrix} V_1' \\ V_i' \\ V_{m-1}' \end{pmatrix} = (m-1) \times b \times (m \times n)$$

4. $t, k \in R$, $u, v \in R^m$,

$x \in R^n$, $t = f(k)$

$\Rightarrow \frac{dt}{dx}$,

solve: $k' = v^T u' + u^T v'$

$t' = f'(k) k'$

③ let

$V = (V_1, V_i, V_{m-1})$

$Y = (Y_1, Y_i, Y_m) = K_2(V, 1)$

④ V_i' : $b \times (m \times n)$

$$\overset{\text{move axis}}{=} \begin{pmatrix} 0 \\ w \\ -(w^2 \cdot Y_i) \end{pmatrix} \cdot \begin{pmatrix} X \\ X \\ X \end{pmatrix}$$

$= (m \cdot b) \times ((m \cdot b) \times n)$

$= (m \cdot b) \times n$ (from 2)

where $w = 1/Y_m$

⑤ $V' = (b \times (m-1)) \times (m \times n)$

$$\overset{\text{transpose}}{=} \begin{pmatrix} V_1^T \\ V_i^T \\ V_{m-1}^T \end{pmatrix}'$$

1. $y=g(x)$, $g=K=\begin{pmatrix} f_1 & & c_1 \\ & f_2 & c_2 \\ & & 1 \end{pmatrix}$

$\Rightarrow \frac{dy}{dK}, \frac{dy}{dx}$.

1.2. $y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} f_1 & \\ & f_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\therefore Y=(f_1, f_2)X+(c_1, c_2)$

here $Y=\begin{pmatrix} \vdots \\ y^T \\ \vdots \end{pmatrix}$, $X=\begin{pmatrix} \vdots \\ x^T \\ \vdots \end{pmatrix}$

1.3 $\frac{dy}{dc}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1.4 fix aspect ratio s:

i.e., $s=f_1/f_2$, $\therefore f_1=sf_2$.

$\therefore y=\begin{pmatrix} sf_2 & \\ & f_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\therefore \frac{dy}{df}=\begin{pmatrix} 0 & sx_1 \\ 0 & x_2 \end{pmatrix}$

1.5 not fix aspect ratio:

$\frac{dy}{df}=\begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix}$

1.6 $\frac{dy}{dx}=\begin{pmatrix} f_1 & \\ & f_2 \end{pmatrix}$

1.7 $\therefore \frac{dy}{d(f_1,f_2,c_1,c_2)}$

$$=\begin{cases} \begin{pmatrix} 0 & sx_1 & 1 & 0 \\ 0 & x_2 & 0 & 1 \end{pmatrix}, & \text{fix } s. \\ \begin{pmatrix} x_1 & & 1 & \\ & x_2 & & 1 \end{pmatrix}, & \text{otherwise} \end{cases}$$

1.8 $\therefore \frac{dY}{dX}=\begin{pmatrix} f_1 & 0 & 0 & f_2 \\ & \vdots & & \end{pmatrix} \in R^{b\times(2\times2)}$

1.9 $\frac{dY}{d(f_1,f_2,c_1,c_2)}$

$$=\begin{cases} \begin{pmatrix} 0 & sx_1 & 1 & 0 & 0 & x_2 & 0 & 1 \\ & & \vdots & & & & & \end{pmatrix}, & \text{fix } s. \\ \begin{pmatrix} x_1 & 0 & 1 & 0 & 0 & x_2 & 0 & 1 \\ & & \vdots & & & & & \end{pmatrix}, & \text{not fix } s. \end{cases}$$

2. $y=f(x)$,

$f=d_1$, $d_1=(k_1,\cdots,k_6,p_1,p_2,s_1,\cdots,s_4)$

$\Rightarrow \frac{dy}{dd_1}, \frac{dy}{dx}$

2.2. $x=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

let $r^2=x_1^2+x_2^2$,

$a_0=2x_1x_2$, $a_1=r^2+2x_1^2$

$a_2=r^2+2x_2^2$

$k_t=1+k_1r^2+k_2r^4+k_3r^6$

$k_b=1+k_4r^2+k_5r^4+k_6r^6$

$k=k_t/k_b$

$b_1=p_1a_0+p_2a_1+s_1r^2+s_2r^4$

$b_2=p_1a_2+p_2a_0+s_3r^2+s_4r^4$

$y=kx+b$

② $\frac{dk_t}{dx} = \frac{dk_t}{d(r^2)} \frac{d(r^2)}{dx}$

$= (k_1 + 2k_2 r^2 + 3k_3 r^4) \frac{d(r^2)}{dx}$

$\frac{dk_b}{dx} = (k_4 + 2k_5 r^2 + 3k_6 r^4) \frac{d(r^2)}{dx}$

$\therefore \frac{d(k_b^{-1})}{dx} = \frac{-1}{k_b^2} \frac{dk_b}{dx}$

③ $\frac{da_0}{dx} = (2x_2, 2x_1)$

$\because a_1 = r^2 + 2x_1^2$

$\therefore \frac{da_1}{dx} = \frac{da_1}{d(r^2)} \frac{dr^2}{dx} + \left(\frac{da_1}{dx}\right)_{\text{direct path}}$ ① $\frac{d(r^2)}{dx} = (2x_1, 2x_2)$

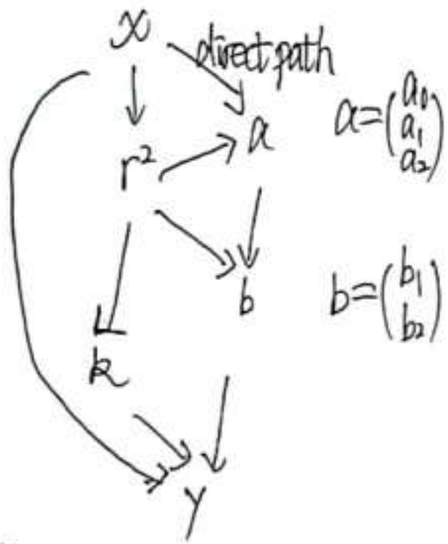
$= \frac{dr^2}{dx} + (4x_1, 0)$

2.4

$\frac{dy}{d(p_1, p_2, s_1, s_2, s_3, s_4)}$

$= \begin{pmatrix} a_0 & a_1 & r^2 & r^4 & & \\ a_2 & a_1 & & & r^2 & r^4 \end{pmatrix}$

2.5. $\frac{dy}{dx}$ (forward)



2.3 $\frac{dy}{dk} = x$

$\frac{dk}{dk_t} = \frac{1}{k_b}$, $\frac{dk}{dk_b} = -\frac{k_t}{k_b^2}$

$\therefore \frac{dk}{d(k_1, k_2, k_3)} = \frac{1}{k_b}(r^2, r^4, r^6)$

$\frac{dk}{d(k_4, k_5, k_6)} = -\frac{k_t}{k_b^2}(r^2, r^4, r^6)$

$= -k\left(\frac{dk}{d(k_1, k_2, k_3)}\right)$

$\therefore \frac{dy}{d(k_1, k_2, k_3)} = \frac{1}{k_b} x (r^2, r^4, r^6)$

$\frac{dy}{d(k_4, k_5, k_6)} = -k \frac{dy}{d(k_1, k_2, k_3)}$

③ $\frac{d(k\cancel{X})}{dX}$: $b\times(2\cdot2)$

for $k_b^{-1} x k_t'$,

$$=(k_b^{-1})_{b\times1}(x)_{b\times2\times1}(k_t')_{b\times1\times2}\Big)_{b\times4}$$

$$=b\times4$$

for $k_t k_b^{-1} I$

$$=(k_t)_{b\times1}(k_b^{-1})_{b\times1}(I)_{1\times4}$$

④ $\frac{d\mathbf{Y}}{d\mathbf{X}}=\frac{d(kX)}{dX}+\frac{db}{dX}$

2.10.

The derivatives ~~here~~ in the batch mode are elementwise.

$$\frac{dY}{d(k_4,k_5,k_6)}$$

$$=-k_{b\times1}\cdot\frac{dY}{d(k_1k_2k_3)}$$

2.7. $Y_{b\times2}=k_{b\times1}X_{b\times2}+b_{b\times2}$

2.8 $\frac{dY}{d(p_1,p_2,s_1,\cdots,s_4)}$

$$=(a_0\ a_1\ r^2\ r^4\ 0\ 0\ a_2\ a_0\ 0\ 0\ r^2\ r^4)_{b\times\cancel{12}}$$

2.9 ① $\frac{dr^2}{dX}=2X$

② $\frac{db_1}{dX}=\left(\frac{db_1}{dx}\right)_{b\times2}$, $\frac{db_2}{dX}=b\times2$

$$\left(\frac{db}{dX}\right)_{b\times4}=\left(\frac{db_1}{dX},\frac{db_2}{dX}\right)$$, concatenate axis 2.

$$\frac{da_2}{dx}=\frac{dr^2}{dx}+(0,4x_2)$$

④ $\frac{db}{dx}=\begin{pmatrix}db_1/dx\\ db_2/dx\end{pmatrix}$

$$=\begin{pmatrix}p_1a_0'+p_2a_1'+s_1(r^2)'+2s_2r^2(r^2)'\\ p_1a_2'+p_2a_0'+s_3(r^2)'+2s_4r^2(r^2)'\end{pmatrix}$$

⑤ $\frac{d(kx)}{dx}=\frac{d(k_t\cdot k_b^{-1}x)}{dx}$

$$=k_b^{-1}xk_t'+k_tx(k_b^{-1})'+k_tk_b^{-1}I$$ ㉡

(from 5)

⑥ $\therefore\frac{dy}{dx}=\frac{d(kx+b)}{dx}=\frac{d(kx)}{dx}+\frac{db}{dx}$

2.6 $\frac{d\mathbf{Y}}{d(k_1k_2k_3)}$, here $Y=\begin{pmatrix}\vdots\\ y^T\\ \vdots\end{pmatrix}$

$$=\frac{1}{k_b}\left(\mathbf{X}_{b\times2\times1}\cdot(r^2\ r^4\ r^6)_{b\times1\times3}\right)_{b\times6}$$

3. $x \in R^n$,

$\binom{u}{1} = wx$, $w = \frac{1}{x_n}$

$\Rightarrow \frac{du}{dx}$

3.2. $u_i = wx_i$

$\therefore \frac{du}{dx} = \begin{pmatrix} w & & -w^2 x_1 \\ & w & -w^2 x_{n-1} \end{pmatrix}$

$= w(I \ -u)$

4. $u \in R^m, v \in R^n, A \in R^{m \times n}$

$\Rightarrow u^T A v = (u \cdot v^T)$, vector $\cdot$ A, vector

4.2. right $= u_1 v_1 a_{11} + u_1 v_2 a_{12} + \cdots$

left $= a_{11} u_1 v_1 + a_{12} u_1 v_2 + \cdots$

$\therefore$ left = right.

5. $j, k \in R$,

$u \in R^m, x \in R^n$

$y = jku = j(x) k(x) u(x)$

$\Rightarrow \frac{dy}{dx}$

solve: $\frac{dy}{dx} = u(jk)' + jku'$

$= u \cdot (kj' + jk') + jku'$

$= kuj' + juk' + jku'$

6. $y = \cancel{A}x + b$, $y \in R^m, b \in R^m$,

$x \in R^n, A \in R^{m \times n}$

$\Rightarrow \frac{dy}{dA}, \frac{dy}{dx}, \frac{dy}{db}$

6.2 $\frac{dy}{dx} = A$, $\frac{dy}{db} = I$

6.3 $\frac{dy_1}{dA} = \begin{pmatrix} x^T \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x^T$

$\frac{dy_i}{dA} = \begin{pmatrix} 0 \\ x^T \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} x^T$

$\frac{dy_m}{dA} = \begin{pmatrix} 0 \\ 0 \\ x^T \end{pmatrix}$

6.4 $\frac{dy}{dA} = \left(I_{(m \cdot m)} \otimes x^T_{1 \times n} \right)_{m \times (m \cdot n)}$

$= (I, \mathrm{vec}\, x^T)_{m \times (m \cdot n)}$

7: $x = \binom{u}{1}$, $x \in R^n$

$\Rightarrow \frac{dx}{du}$

7.2 $\frac{dx}{du} = \binom{I}{0}_{n \times (n-1)}$, I is $(n-1) \times (n-1)$

8. $y = Ax$, $x = \cancel{\tfrac{1}{x_n}} x_n \binom{u}{1}$

$y = y_m \binom{v}{1} \Rightarrow \frac{dv}{du}$

$y \in R^m, x \in R^n, A \in R^{m \times n}$

$= I_{m\times m\times 1} \cdot x_{1\times 1\times n}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{m\times m\times 1} x_{1\times 1\times n}$

$= \begin{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} x^T \\ \begin{pmatrix}0\\1\\0\end{pmatrix} x^T \\ \begin{pmatrix}0\\0\\1\end{pmatrix} x^T \end{pmatrix}_{m\times(m\times n)}$

② $\because$ $\cancel{y}^T (I_{(m\cdot m)\times 1}\, x_{1\times n})$

$= \begin{pmatrix} y_1\\0\\0\end{pmatrix} x^T + \begin{pmatrix}0\\y_i\\0\end{pmatrix} x^T + \begin{pmatrix}0\\0\\y_m\end{pmatrix} x^T$

$= (yx^T)_{1\times(m\cdot n)}$

10. $Y_{b\times m} (I_{(m\cdot m)\times 1}\, x_{1\times n})_{m\times(m\cdot n)}$

$= (Y_{(b\cdot m)\times 1}\, x_{1\times n})_{b\times(m\cdot n)}$

i.e., $Y_{b\times m}(I.\mathrm{vec}\, x^T)_{m\times(m\cdot n)}$

$= (Y_{1}\mathrm{vec}\, x^T)_{b\times(m\cdot n)}$

$= w\begin{pmatrix} a_{11} - v_1 a_{31}, & a_{12} - v_1 a_{32} \\ a_{21} - v_2 a_{31}, & a_{22} - v_2 a_{32} \end{pmatrix}$

8.4 $\frac{dV}{dU}$,

solve: we use u, v instead of U, V.

$\frac{dv}{du} = w_{b\times 1}(B_{1\times(m-1)\cdot(n-1))} - \{v_{b\times(m-1)\times 1} \cdot b_{1\times 1\times(n-1)}\}_{b\times(m-1)\cdot(n-1)})$

9. $x \in R^n, y \in R^m$

$y^T (I_{(m\cdot m)\times 1}\, x_{1\times n})_{m\times(m\cdot n)}$

$= (yx^T)_{1\times(m\cdot n)}$

proof:

① $I_{(m\cdot m)\times 1}\, x_{1\times n}$

8.2

$\frac{dv}{du} = \frac{dv}{dy}\frac{dy}{dx}\frac{dx}{du}$

$= w(I \; -v)\cdot A \cdot \begin{pmatrix} I \\ 0 \end{pmatrix}$, $w = 1/y_m$

$= w(I \; -v)\begin{pmatrix} B \\ b^T \end{pmatrix}$ $\quad A = \begin{pmatrix} B & c \\ b^T & d \end{pmatrix}$

$= w(B - vb^T)$

8.3 if $m=3, n=3$,

$\Rightarrow \frac{dv}{du}$

solve: $\because$ $B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$b^T = (a_{31}, a_{32})$, $w = 1/y_3$

$\therefore \frac{dv}{du} = w(B - vb^T)$

solve:

$$\frac{dv}{dH}$$

$$= w\begin{pmatrix} 1 & & -v_1 \\ & 1 & -v_2 \end{pmatrix}_{(2\cdot 3)\times 1} \overset{x^T}{\cancel{(v_1, v_2, 1)}}$$

$$= w\left(x^T \; 0 \; -v_1 x^T \; 0 \; x^T \; -v_2 x^T\right)$$

$$= b\times(2\times(3\cdot 3))$$

here $w = \frac{1}{y_3}$

12. $Y \in R^{b\times m}$, $x \in R^n$, $A \in R^{m\times n}$

$$\Rightarrow \left(Y_{(b\cdot m)\times 1}\, x^T\right)_{b\times(m\cdot n)} A.\mathrm{vec}$$

$$= \begin{pmatrix} \cdots \\ yx^T \\ \cdots \end{pmatrix}_{b\times(m\cdot n)} A.\mathrm{vec}$$

$$= \begin{pmatrix} \cdots \\ (yx^T).\mathrm{vec} \cdot A.\mathrm{vec} \\ \cdots \end{pmatrix}$$

$$\overset{\text{from 4}}{=} \begin{pmatrix} \cdots \\ y^T A x \\ \cdots \end{pmatrix} = YAx = b\times 1 = b$$

$$\Rightarrow \frac{dv}{dH}$$

11.2. solve: ① method 2 : slice batch

(see previous method : slice feature).

② $\frac{dv}{dH} = \frac{dv}{dy}\frac{dy}{dH}$

$$= w\left(I \; -v\right)_{(m-1)\times m} \left(I_{(m\cdot m)\times 1}\, x_{1\times n}\right)_{m\times(m\cdot n)}$$

$$= w\left(\left(I \; -v\right)_{((m-1)\cdot m)\times 1}\, x^T\right)_{(m-1)\times(m\cdot n)} \text{ from (1)}$$

③ $\frac{dV}{dH} = \begin{pmatrix} \cdots \\ dv/dH \\ \cdots \end{pmatrix}_{b\times((m-1)\cdot(m\cdot n))}$

~~11~~. 11.3

let $k_1 = 1$, $m, n = 3$

$$\Rightarrow \frac{dV}{dH}.$$

proof: $Y_{b\times m}\left(I_{(m\cdot m)\times 1}\, x_{1\times n}\right)_{m\times(m\cdot n)}$

$$= \begin{pmatrix} \cdots \\ y^T \\ \cdots \end{pmatrix}_{b\times m} \left(I_{(m\cdot m)\times 1}\, x_{1\times n}\right)_{m\times(m\cdot n)}$$

$$= \begin{pmatrix} \cdots \\ y^T\left(I_{(m\cdot m)\times 1}\, x_{1\times n}\right)_{m\times(m\cdot n)} \\ \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cdots \\ (yx^T)_{1\times(m\cdot n)} \\ \cdots \end{pmatrix}_{b\times(m\cdot n)}$$

$$= Y_{b\times m\times 1}\, x_{1\times 1\times n}$$

$$= Y_{(b\cdot m)\times 1}\, x_{1\times n}$$

$$= \left(Y_{(b\cdot m)\times 1}\, x^T\right)_{b\times(m\cdot n)}$$

11. $y = Hx$, $x = k_1\binom{u}{1}$, $y = k_2\binom{v}{1}$,

$x \in R^n$, $y \in R^m$, $H \in R^{m\times n}$

$= (I.\mathrm{vec}\, x^T)_{m\times(m\cdot n)} \left(\frac{dA}{da}\right)_{(m\cdot n)\times l}$

$= (A_1' x, A_2' x, A_l' x)$

(from 16)

here $A' = \frac{dA}{da} = (A'_{1}.\mathrm{vec}, \cdots, A'_{l}.\mathrm{vec})$

$A_i' \in R^{m\times n}$

③ $A' \Rightarrow (A_1' x, A_i' x, A_l' x)$

(from 14.3)

16. $T = (A_1.\mathrm{vec}, A_2.\mathrm{vec}, A_l.\mathrm{vec}) \in R^{(m\cdot n)\times l}$

$A_i \in R^{m\times n}$, $Y \in R^{b\times m}$, $x \in R^n$

$\Rightarrow (Y.\mathrm{vec}\, x^T)_{b\times(m\cdot n)} T_{(m\cdot n)\times l}$

$= Y(A_1 x, A_2 x, A_l x) = b\times l$

proof: let $Y = \begin{pmatrix} \vdots \\ y_i^T \\ \vdots \end{pmatrix} = \begin{pmatrix} Y_1^T \\ \vdots \\ Y_b^T \end{pmatrix}$

$\therefore$ left $= \begin{pmatrix} \vdots \\ (Y_i x^T).\mathrm{vec} \\ \vdots \end{pmatrix}_{b\times(m\cdot n)} T$

(from 4) $= (Y_i^T A_j x)_{b\times l} = Y(A_1 x, \cdots, A_l x)$

14.3 given $T = (A.\mathrm{vec}, B.\mathrm{vec})$

$\Rightarrow (Ax, Bx)$

solve:

① $T \xrightarrow{\text{transpose}} \begin{pmatrix} A.\mathrm{vec} \\ B.\mathrm{vec} \end{pmatrix}_{2\times(m\cdot n)}$

$\xrightarrow{\text{reshape}} \begin{pmatrix} A \\ B \end{pmatrix}_{(2\cdot m)\times n}$

② $\Rightarrow (Ax, Bx)$, (from 14.2)

15. $y = Ax + b$, $y \in R^m$, $b \in R^m$,

$x \in R^n$, $A \in R^{m\times n}$,

$A = A(a)$, a is vector, $a \in R^{l}$

$\Rightarrow \frac{dy}{da}$: $m\times l$

solve: ① $\frac{dy}{da} = \frac{dy}{dA} \frac{dA}{da}$

13. $y = Hx$, $x = \begin{pmatrix} u \\ 1 \end{pmatrix}$, $y = k_2 \begin{pmatrix} v \\ 1 \end{pmatrix}$,

$x \in R^n$, $y \in R^m$, $H \in R^{m\times n}$,

$H = H(h)$, $h \in R$,

$\Rightarrow \frac{dv}{dh}$.

solve: $\frac{dv}{dh} = \frac{dv}{dH} \cdot \frac{dH}{dh}$

$= w\left((I\ -v)_{((m-1)-m)\times 1}\, x^T\right)_{(m-1)\times(m\cdot n)} \frac{dH}{dh}$

$= w(I\ -v)\left(\frac{dH}{dh}\right).\mathrm{mat}\, x$ (from 12)

14. $A, B \in R^{m\times n}$, $x \in R^n$

$\Rightarrow (Ax, Bx)$

14.2 solve: $\begin{pmatrix} A \\ B \end{pmatrix} x = (2m) \stackrel{\text{reshape}}{=} 2\times m = \begin{pmatrix} (Ax)^T \\ (Bx)^T \end{pmatrix}$

$\xrightarrow{\text{transpose}} (Ax, Bx)$

$$\boxed{J, f \Rightarrow J^TJ,\ J^Tf}$$

1. $$\begin{pmatrix} (A_1B)_{,\text{rvec}} \\ \vdots \\ (A_bB)_{,\text{rvec}} \end{pmatrix} = \left(\begin{pmatrix} A_1 \\ \vdots \\ A_b \end{pmatrix} B\right)_{b\times(\,)}$$

2. $r_1, t_1, r, t \Rightarrow r_2, t_2$

2.3 $R = R_2R_1^{-1}$, $t = t_2 - Rt_1$

$\therefore R_2 = RR_1$, $t_2 = t + Rt_1$

here $R = e^{(r)_\times}$, $R_1 = e^{(r_1)_\times}$, $R_2 = e^{(r_2)_\times}$

2.4
$$\begin{tikzcd}
r_1 \arrow[dr] & r \arrow[d] \arrow[drr] & t_1 \arrow[dr] & t \arrow[d] \\
 & r_2 \arrow[dr] & & t_2 \arrow[dl] \\
 & & x &
\end{tikzcd}$$

given $\frac{dx}{dr_2}$, $\frac{dx}{dt_2}$,

and $\frac{dr_2}{dr_1}$, $\frac{dr_2}{dr}$, $\frac{dt_2}{dr}$, $\frac{dt_2}{dt_1}$, $\frac{dt_2}{dt}$ (see other pages)

$\Rightarrow \frac{dx}{dr_1}$, $\frac{dx}{dt_1}$, $\frac{dx}{dr}$, $\frac{dx}{dt}$

solve: $\frac{dx}{dr_1} = \frac{dx}{dr_2}\frac{dr_2}{dr_1}$,

$$\frac{dx}{dt_1} = \frac{dx}{dt_2}\frac{dt_2}{dt_1}$$

$$\frac{dx}{dr} = \frac{dx}{dr_2}\frac{dr_2}{dr} + \frac{dx}{dt_2}\frac{dt_2}{dr}$$

$$\frac{dx}{dt} = \frac{dx}{dt_2}\frac{dt_2}{dt}$$

2.5 batch: $\frac{dX}{dr_2} = \begin{pmatrix} \vdots \\ (\frac{dx}{dr_2})_{,\text{rvec}} \\ \vdots \end{pmatrix}_{b\times(\,)}$

we use x, not X.

$\therefore \frac{dx}{dr_1}$ $\quad x \in R^{b\times n}$

$$= \left(\left(\frac{dx}{dr_2}\right)_{(b\cdot n)\times 3}\frac{dr_2}{dr_1}\right)_{b\times(n\cdot 3)}$$

(from 1)

similarly for others.

3. $J = (A\ B\ C)$,

$A = \begin{pmatrix} A_1 \\ \vdots \\ A_L \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ \vdots \\ B_L \end{pmatrix}$, $C = \begin{pmatrix} C_1 \\ \vdots \\ C_L \end{pmatrix}$

$f = \begin{pmatrix} f_1 \\ \vdots \\ f_L \end{pmatrix}$, $\quad A_i, B_i, C_i$ is matrix, f_i is vector

$\Rightarrow J^TJ$, J^Tf.

$f_i = \begin{pmatrix} f_{ia} \\ f_{ib} \end{pmatrix}$

here $A_{ia}, \cdots, D_{ib}$ matrix,

f_{ia}, f_{ib} vector.

let $A_{ia}=0$, $C_{ib}=0$, $D_{ia}=0$ 4.

$\Rightarrow J^TJ, \; J^Tf.$

4) solve: ① $J^TJ = \begin{pmatrix} A^T \\ B^T \\ C^T \\ D^T \end{pmatrix} (ABCD)$

$$= \begin{pmatrix} A^TA & A^TB & A^TC & A^TD \\ & B^TB & B^TC & B^TD \\ & & C^TC & C^TD \\ & & & D^TD \end{pmatrix}$$

→ to symmetric

$J^Tf = \begin{pmatrix} A^Tf \\ B^Tf \\ C^Tf \\ D^Tf \end{pmatrix}$

$A^Tf = \sum A_i^T f_i$

$B^Tf = \begin{pmatrix} B_1^T f_1 \\ \vdots \\ B_L^T f_L \end{pmatrix}$

$C^Tf = \sum C_i^T f_i$

$J = (A\,B\,C\,D)$

$f = \begin{pmatrix} f_1 \\ \vdots \\ f_L \end{pmatrix}$

$A = \begin{pmatrix} A_1 \\ \vdots \\ A_L \end{pmatrix}$, $B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_L \end{pmatrix}$

$C = \begin{pmatrix} C_1 \\ \vdots \\ C_L \end{pmatrix}$ $D = \begin{pmatrix} D_1 \\ \vdots \\ D_L \end{pmatrix}$

$A_i = \begin{pmatrix} A_{ia} \\ A_{ib} \end{pmatrix}$, $B_i = \begin{pmatrix} B_{ia} \\ B_{ib} \end{pmatrix}$

$C_i = \begin{pmatrix} C_{ia} \\ C_{ib} \end{pmatrix}$, $D_i = \begin{pmatrix} D_{ia} \\ D_{ib} \end{pmatrix}$,

3.2.

$J^TJ = \begin{pmatrix} A^T \\ B^T \\ C^T \end{pmatrix} (ABC)$

$$= \begin{pmatrix} A^TA & A^TB & A^TC \\ B^TA & B^TB & B^TC \\ C^TA & C^TB & C^TC \end{pmatrix}$$

$J^Tf = \begin{pmatrix} A^T \\ B^T \\ C^T \end{pmatrix} f = \begin{pmatrix} A^Tf \\ B^Tf \\ C^Tf \end{pmatrix}$

$A^TA = \sum A_i^T A_i$

$A^TB = (A_1^TB_1, \cdots, A_L^TB_L)$

$A^TC = \sum A_i^T C_i$

$B^TB = \begin{pmatrix} \ddots & & \\ & B_i^TB_i & \\ & & \ddots \end{pmatrix}$, $B^TC = \begin{pmatrix} B_1^TC_1 \\ \vdots \\ B_L^TC_L \end{pmatrix}$

$C^TC = \sum C_i^T C_i$

5.3 solve J^TJ, J^Tf.

here $J = (A\,B\,C\,D)$ (see 4)

$$A_{ib} = \frac{dx_2}{d(r,t)} \in R^{(b\cdot 2)\times 6}$$

$$B_{ia} = \frac{dx_1}{d(r_i,t_i)} \quad B_{ib} = \frac{dx_2}{d(r_i,t_i)}$$

$$C_{ia} = \frac{dx_1}{d(K_1,d_1)}, \quad D_{ib} = \frac{dx_2}{d(K_2,d_2)}$$

$f_{ia} = x_1 - \text{true}$, $f_{ib} = x_2 - \text{true}$

i.e., $$J = \frac{d\{(x_1,x_2)\}_{i=1,\cdots,l}}{d(r,t,\{r_i,t_i\}_{i=1,\cdots,l},(K_1,d_1),(K_2,d_2))}$$

$(K_1,d_1), (K_2,d_2), (r,t)$, $r_i := r_{1\to 2}$, $t_i := t_{1\to 2}$, $\{(r_i,t_i)\}_{i=1,\cdots,l}$, $\{X\}_{i=1,\cdots,l}$

$x_1 = \text{real project}(K_1,d_1,r_i,t_i,X)$

$x_2 = \text{real project}(K_2,d_2,r_2,t_2,X)$

$x_1 \in R^{b\times 2}$, $x_2 \in R^{b\times 2}$

$\Rightarrow \{(x_1,x_2)\}_{i=1,\cdots,l}$

$$C^TC = \sum C_i^TC_i = \sum C_{ia}^TC_{ia}$$

$$C^TD = 0$$

$$D^TD = \sum D_{ib}^TD_{ib}$$

$$A^Tf = \sum A_{ib}^Tf_{ib}$$

$$B^Tf = \begin{pmatrix} \cdots \\ B_i^Tf_i \\ \cdots \end{pmatrix}$$

$$B_i^Tf_i = B_{ia}^Tf_{ia} + B_{ib}^Tf_{ib}$$

$$C^Tf = \sum C_{ia}^Tf_{ia}$$

$$D^Tf = \sum D_{ib}^Tf_{ib}$$

② $$A^TA = \sum A_i^TA_i = \sum A_{ib}^TA_{ib}$$

$$A^TB = (\cdots, A_i^TB_i, \cdots)$$

$$= (\cdots, A_{ib}^TB_{ib}, \cdots)$$

$$A^TC = \sum A_i^TC_i = \sum(A_{ia}^TC_{ia} + A_{ib}^TC_{ib})$$

$$= 0$$

$$A^TD = \sum A_{ib}^TD_{ib}$$

$$B^TB = \begin{pmatrix} \cdots & & \\ & B_i^TB_i & \\ & & \cdots \end{pmatrix}$$

here $B_i^TB_i = B_{ia}^TB_{ia} + B_{ib}^TB_{ib}$

$$B^TC = \begin{pmatrix} \cdots \\ B_i^TC_i \\ \cdots \end{pmatrix} = \begin{pmatrix} \cdots \\ B_{ia}^TC_{ia} \\ \cdots \end{pmatrix}$$

$$B^TD = \begin{pmatrix} \cdots \\ B_i^TD_i \\ \cdots \end{pmatrix} = \begin{pmatrix} \cdots \\ B_{ib}^TD_{ib} \\ \cdots \end{pmatrix}$$

6. $(K, d), (r, t)_{1:l}, X_{1:l}$

$\Rightarrow x_{1:l}, x_i \in R^{b\times 2}, X_i \in R^{b\times 3}$

6.3 let $J = \dfrac{dx_{1:l}}{d(K, d, (r,t)_{1:l}, X)}$

$\Rightarrow J^T J, J^T f.$

note $X := X_1 = X_2 = \cdots = X_l$

if solving $\dfrac{dx_{1:l}}{dX}$

$f_i = x_i - \text{true}$

6.4 solve: let $J = (A\ B\ C)$

$f = \begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix}$ $A = \begin{pmatrix} A_1 \\ A_i \\ A_l \end{pmatrix}$, $B = \begin{pmatrix} B_1 & & \\ & B_i & \\ & & B_l \end{pmatrix}$, $C = \begin{pmatrix} C_1 \\ C_i \\ C_l \end{pmatrix}$

$\because A_i = \dfrac{dx_i}{d(K, d)}, B_i = \dfrac{dx_i}{d(r,t)_i}$

$C_i = \dfrac{dx_i}{dX} \in R^{(b\times 2)\times(b\cdot 3)}$

$f_i = x_i - \text{true}.$

$\Rightarrow J^T J, J^T f$ (from 3)

if $j=2$:

$D^T D \mathrel{+}= J_i^T J_i$

$B^T D = \begin{pmatrix} \cdots \\ J_e^T J_i \\ \cdots \end{pmatrix}$

$D^T f \mathrel{+}= J_i^T f_x$

if $j=2$:

$A^T A \mathrel{+}= J_m^T J_m$

$A^T B = (\cdots, J_m^T J_e, \cdots)$

$A^T f \mathrel{+}= J_m^T f_x$

$A^T D \mathrel{+}= J_m^T J_i$

$\forall j$: $B_i^T B_i \mathrel{+}= J_e^T J_e$, $B_i^T f_i \mathrel{+}= J_e^T f_x$

$\therefore B^T B = \begin{pmatrix} \cdots & & \\ & B_i^T B_i & \\ & & \cdots \end{pmatrix}$, $B^T f = \begin{pmatrix} \cdots \\ B_i^T f_i \\ \cdots \end{pmatrix}$

5.4 solve:

① $\dfrac{dx_2}{d(r_2 t_2)} \Rightarrow \dfrac{dx_2}{d(r,t)}\ \dfrac{dx_2}{d(r_1 t_1)}$ (from 2,5)

② $\forall i \in \{1, \cdots, l\}, \forall j \in \{1, 2\}$

$J_e := \begin{cases} B_{ia}, & j=1 \\ B_{ib}, & j=2 \end{cases}$

$J_i := \begin{cases} C_{ia}, & j=1 \\ D_{ib}, & j=2 \end{cases}$

$f_x := \begin{cases} f_{ia}, & j=1 \\ f_{ib}, & j=2 \end{cases}$

$J_m := A_{ib}$

③ $\forall i \in \{1, \cdots, l\}, j \in \{1, 2\}$

if $j=1$: $C^T C \mathrel{+}= J_i^T J_i$, $B^T C = \begin{pmatrix} \cdots \\ J_e^T J_i \\ \cdots \end{pmatrix}$

$C^T f \mathrel{+}= J_i^T f_x$