$\boxed{\text{group}}$

1. Group action.

let X be a set.
let G be a group whose identity is e.

1.2. (left) group action is an operation $\phi: G\times X \to X$ such that $\forall (g,x)\in G\times X$,

$g*x := \phi(g,x) \in X$.

in such a way that the group action axioms are satisfied:

① $g*(h*x) = (g\circ h)*x$ $\forall g,h\in G, x\in X$.

② $e*x = x$, $\forall x\in X$.

1.3 right group action:
$\phi: X\times G \to X$, $x*g := \phi(x,g)$
$(x*g)*h = x*(g\circ h)$
$x*e = x$

1.4. next.

2. permutation.

a bijection $f: S\to S$ from a set S to itself is called a permutation of S.

3 symmetric group defined over any set is the group whose elements are all the bijections from the set to itself, and whose group operation is the composition of functions.

3.2 The symmetric group on set $\{1,2,\cdots,n\}$ is denoted S_n.

3.3 S_n has order $n!$.

3.4 S_n is solvable $\Leftrightarrow n\le 4$

3.5 symmetric groups on infinite sets behave quite differently from symmetric groups on finite sets.

4. Group homomorphism

let $(G,\circ)$, $(H,*)$ be groups.
let $\phi: G\to H$ be a mapping such that $\circ$ has the morphism property under ϕ. That is,

$\phi(a\circ b) = \phi(a)*\phi(b)$

Then ϕ is a group homomorphism.

4.2 $\phi(e_G) = e_H$
$\phi(u^{-1}) = (\phi(u))^{-1}$, $\forall u\in G$.

proof:
① $\phi(e_Ge_G) = \phi(e_G)\phi(e_G)$
$\therefore \phi(e_G) = \phi(e_G)\phi(e_G)$
$\therefore e_H = \phi(e_G)$

② $\phi(u^{-1})\phi(u) = \phi(u^{-1}\cdot u)$
$= \phi(e_G) = e_H$
$\therefore \phi(u^{-1}) = (\phi(u))^{-1}$

4.3 homomorphism means similar form, or similar structure.

5. permutation representation of group.

5.2 let G be a group.
let X be a set.
let $T(X)$ be the symmetric group on X.

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1.8 given a group homomorphism from G into symmetric group $Sym(X)$, we can define a group action of G on X, ~~s.t.~~ from which, we can get the same group homomorphism.

proof: ~~$\phi: G$~~

① $\because \varphi: G \to Sym(X)$.

$\varphi(g) = \phi_g$

let $\phi: G \times X \to X$

$\phi(g, x) := \phi_g(x)$

$= (\varphi(g))(x)$

~~then ϕ is group action~~.

② $\because (gh)x = \phi_{gh}(x)$

$= (\phi_g \phi_h)(x)$ $\quad \because$ homomorphism.

$= \phi_g(\phi_h(x)) = g(\phi_h(x))$

~~$= g(h(x))$~~ $= g(hx)$

$\because ex = \phi_e(x)$

$\because \varphi(e) = \phi_e = e_{Sym(X)}$ $\therefore ex = x$

$\therefore \phi$ is group action.

③ from 1.7. proved.

1.6 given 1.5,

$\Rightarrow \phi_{g_1 g_2} = \phi_{g_1} \phi_{g_2}$.

$\forall g_1, g_2 \in G$.

proof: $\forall x \in X$,

$\phi_{g_1 g_2}(x)$

$= (g_1 g_2) x$

$= g_1(g_2 x)$

$= \phi_{g_1}(g_2 x)$

$= \phi_{g_1}(\phi_{g_2}(x))$

$= (\phi_{g_1} \phi_{g_2})(x)$

$\therefore \phi_{g_1 g_2} = \phi_{g_1} \phi_{g_2}$

1.7. given group action of G on X, we get a group homomorphism from G into symmetric group $Sym(X)$.

proof: define $\varphi: G \to Sym(X)$

$\varphi(g) = \phi_g$.

from 1.6, proved.

1.4 group G is said to act on X (on the left). The set X is called a (left) G-set.

1.5 give group action $\phi: G \times X \to X$,

$\Rightarrow \phi_g: X \to X$, $\phi_g(x) = \phi(g, x)$ is bijective.

proof: ~~first~~ ① ϕ_g is a map.

② if $\phi_g(x_1) = \phi_g(x_2)$

$\therefore gx_1 = gx_2$

$\therefore g^{-1}(gx_1) = g^{-1}(gx_2)$

$\therefore (g^{-1}g)x_1 = (g^{-1}g)x_2$

$\therefore x_1 = x_2$

$\therefore \phi_g$ is injective (one-to-one).

③ $\forall y \in X$, $\exists x = g^{-1}y$,

s.t. $\phi_g(x) = gx = y$

$\therefore \phi_g$ is surjective. (onto)

$\therefore \phi_g$ is bijective.

A permutation representation of G is a group homomorphism from G to $T(X)$.

5.3 let $\phi: G \times X \to X$ be group action.

$\phi_g: X \to X$

$\phi_g(x) = \phi(g, x)$

The permutation representation of G associated to group action is the group homomorphism $G \to T(X)$ which sends g to ϕ_g.

proof: $\because \phi_g$ is bijective $\therefore \phi_g \in T(X)$

$\because \phi_{g_1 g_2} = \phi_{g_1} \phi_{g_2}$.

$\therefore \varphi: G \to T(X)$

$\varphi(g) = \phi_g$ is group homomorphism

5.4 next.

1.9. group homomorphism

$\Leftrightarrow$ group action.

(from 1.8).

1.10 next.

5.4 permutation representation of group

$\Leftrightarrow$ permutation representation of ~~group~~ group associated to group action.

(from 1.9).

1.10 transitive:

the action of G on X is called transitive if ① X is non-empty, ② $\forall x, y \in X$, $\exists g \in G$, s.t. $g \cdot x = y$.

1.10.3 the action of the symmetric group of X is transitive.

proof: $\forall x, y \in X$,

$\exists g \in \mathrm{sym}(X)$,

s.t. $g(x) = y$.

$\because$ $g \cdot x = \phi(g, x)$

$= \phi_g(x)$

$= (\varphi(g))(x)$

$= g(x) = y$

(here $\varphi(g) = g$, $\forall g \in \mathrm{sym}(X)$)

$\therefore$ action of sym(X) is transitive.

1.11. faithful (or effective)

the action of G on X is called faithful

$\Leftrightarrow$ $\forall g, h \in G$, $g \neq h$,

$\exists x \in X$, s.t. $g \cdot x \neq h \cdot x$.

$\Leftrightarrow$ $\forall g \in G$, $g \neq e$, $\exists x \in X$,

s.t. $g \cdot x \neq x$.

(ε).

1.12 free

the action of G on X is called free

$\Leftrightarrow$ $\forall g, h \in G$, $\exists x \in X$,

$g \cdot x = h \cdot x \Rightarrow g = h$.

$\Leftrightarrow$ $\forall g \in G$, $\exists x \in X$, $g \cdot x = x$

$\Rightarrow g = e$.

ε.

1.12.2

a free action on a non-empty set is faithful.

$\boxed{\text{affine space}}$

1. associativity axioms.

$\mathcal{E}$ is a set, V a vector space over K.

$+ : \mathcal{E}\times V \to \mathcal{E}$

$- : \mathcal{E}\times\mathcal{E} \to V$

satisfying:

(A1): $\forall p, q \in \mathcal{E}$, $p+(q-p) = q$

(A2): $\forall p\in\mathcal{E}, \forall u, v \in V$,

$$(p+u)+v = p+(u+v)$$

(A3): $\forall p, q \in \mathcal{E}, \forall u \in V$.

$$(p-q)+u = (p+u)-q$$

Then $(\mathcal{E}, +, -)$ is an affine space.

2. group action

$\mathcal{E}$ is a set, V a vector space over K.

$\phi: \mathcal{E}\times V \to \mathcal{E}$ is a free and transitive group action of V on $\mathcal{E}$.

Then $(\mathcal{E}, \phi)$ is an affine space.

3. Weyl's axioms

$\mathcal{E}$ set, V vector space over K.

$- : \mathcal{E}\times\mathcal{E}\to V$

satisfying:

(W1): $\forall p\in\mathcal{E}, \forall v \in V$,

$\exists q \in \mathcal{E}$, q is unique,

s.t. $v = q-p$

(W2): $\forall p, q, r \in \mathcal{E}$,

$(r-q)+(q-p) = r-p$

Then $(\mathcal{E}, -)$ is an affine space.

4. associativity axioms $\Rightarrow$ Weyl's axioms.

proof: ① $\forall p, q \in \mathcal{E}$,

$q-p = (p+(q-p))-p$ (A1)

$= (p-p)+(q-p)$ (A3)

$\therefore$ ~~(A1)~~: $p-p=0$, $\forall p\in\mathcal{E}$.

② let $p\in\mathcal{E}$, $v\in V$ as in (W1)

let $q = p+v$

then $q-p$

$= (p+v)-p$

$= v$

let $r\in\mathcal{E}$ be any other element s.t. $v = r-p$.

$\because q = p+v$

$= p+(r-p)$

$= r$ (A1)

$\therefore q$ is unique.

③ let $p, q, r\in\mathcal{E}$ as in (W2)

then

$r-p = (q+(r-q))-p$ (A1)

$= (q-p)+(r-q)$ (A3)

$\therefore$ (W2) established.

5. Weyl's axioms $\Rightarrow$ group action.

proof:

① define $+ : \mathcal{E}\times V \to \mathcal{E}$

$p+v = q$

where $v = q-p$.

from (W1), we know $+$ is a map.

② $\phi: \mathcal{E}\times V\to\mathcal{E}$,

$\phi(p, v) := p+v$

we will prove ϕ is group action.

$\therefore$ we must verify:

(RGA1): $(p+u)+v = p+(u+v)$ (W1) established.

(RGA2): $p+0 = p$

③ $\forall p\in\mathcal{E}, u, v \in V$.

by (W1),

$\therefore \exists q\in\mathcal{E}$, s.t. $q-p = u$

$\exists r \in\mathcal{E}$, s.t. $r-q = v$

$\exists s\in\mathcal{E}$, s.t. $s-p = u+v$

$\therefore s-q = (s-q)+u-u = (s-q)+(q-p)-u$ #

$= (s-p)-u$ (W2).

④ Now that the mappings + and − are defined, we verify (A1)(A2)(A3)

⑤ $p+(q-p)=q$

$\Leftrightarrow p+v=q$

where $v=q-p$

$\Leftrightarrow$ ~~p+v=q~~

if $v=q-p$

then $p+v=q$

(A1) established.

⑥ $(p+u)+v=p+(u+v)$

$\Leftrightarrow$ (RGA1)

$\therefore$ (A2) established.

⑦. ~~(p-q)+u~~ (A3):

$= \cancel{(q+((p-q)+u)-q)}$

$(p+u)-q=((q+(p-q))+u)-q$ (A1)

$=(q+((p-q)+u))-q$ (RGA1)

let $v=(p-q)+u$.

$\therefore (p+u)-q=(q+v)-q$

$=v$ $(q+v=q+v)$ (− definition)

OK.

① $+: \mathcal{E}\times V\Rightarrow\mathcal{E}$

$p+v:=\phi(p,v)$

② $\because \phi$ transitive.

$\forall p,q\in\mathcal{E},\ \exists v\in V,$

s.t. $p+v=q$.

if $p+v_1=p+v_2$

$\therefore p+(v_1-v_2)$

$=(p+v_1)+(-v_2)$ (RGA1)

$=(p+v_2)+(-v_2)$

$=p+(v_2-v_2)$ (RGA1)

$=p+0$

$=p$ (RGA2).

$\therefore v_1-v_2=0$ (action is free)

$\therefore v_1=v_2$

③ $\therefore$ we can define

$-: \mathcal{E}*\mathcal{E}\to V$

$q-p=v$

where $p+v=q$

⑤ $\therefore \phi$ is action.

⑥ we now show that the action is free, that is;

$\forall v\in V, \exists p\in\mathcal{E}, p+v=p$

$\Rightarrow v=0$.

$\because v=p-p$

$\because p-p=0$

$\therefore v=0$

⑦ now we show that the action is transitive, that is,

$\forall p,q\in\mathcal{E},\ \exists v\in V,$

s.t. $p+v=q$.

let $v=q-p$

$\therefore p+v=q$. proved.

6. group action $\Rightarrow$ associativity axioms.

proof: let $\phi: \mathcal{E}\times V\to\mathcal{E}$ be a free and transitive group action.

$=u+v-u$

$=v$

$=r-q$

$\therefore s-q=r-q$

$\therefore s=r$ (W1)

$\therefore (p+u)+v$

$=q+v$

$=r$

$=s$

$=p+(u+v)$

$\therefore$ (RGA1) established.

④ $p+0=p$

$\Leftrightarrow p-p=0$

$\because \forall q\in\mathcal{E},$

$q-p=(q-p)+(p-p)$ (W2)

$\therefore p-p=0$

$\therefore$ (RGA2) established.