Chapter 2. HZ

~~25~~ ~~27~~ 79

1. plane $\Leftrightarrow \mathbb{R}^2$

prove: point $P \in$ plane $\Leftrightarrow (x,y) \in \mathbb{R}^2$

2. projective space P^2

$:=$ the set of equivalence classes of vectors in $\mathbb{R}^3 - (0,0,0)^T$

$\Leftrightarrow \{$ homogeneous vector $v \mid v \in \mathbb{R}^3 - (0,0,0)^T \}$

2.2 homogeneous vector v is a equivalence class.

2.3. line in $\mathbb{R}^2$

$\Leftrightarrow ax+by+c=0$

$\Leftrightarrow k(a,b,c)^T,\ k \neq 0$

2.4 point $x \in$ line l

$\Leftrightarrow ax+by+c=0$

$\Leftrightarrow k(x,y,1)\cdot l = 0,\ \forall k \neq 0$

2.5

$\therefore$ point $\in P^2$, line $\in P^2$.

3. point $x \in$ line l

$\Leftrightarrow x^T l = 0$

4. dof : degrees of freedom.

dof (point) $= 2$

dof (line) $= 2$.

5. $x \in l,\ x \in l'$

$\Leftrightarrow x = l \times l'$.

here $\times$ is cross product.

6. $(x_1,x_2,x_3)^T \times (y_1,y_2,y_3)^T$

$$= \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i + \left(-\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}\right) j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k$$

7. $x \in l,\ x' \in l$

$\Rightarrow l = x \times x'$

8. line $l = (a,b,c)^T$, $l' = (a,b,c')^T$

$\Rightarrow l \times l' = (c'-c)(b,-a,0)^T$

$\therefore l, l'$ meet at infinity

9. point $(x_1, x_2, 0)^T$ is known as ideal points, or point at infinity.

$l_\infty = (0,0,1)^T$ is line at infinity.

9.3 $l_\infty = \{$ ideal point $\}$

10 ① $\forall$ point $x \in P^2$, $\Rightarrow$ ~~x is ...~~ $x \in \mathbb{R}^2$ or x is point at infinity.

② $\forall$ line $l \in P^2 \Rightarrow l$ is line in $\mathbb{R}^2$ or l is line at infinity.

here $P^2 = \{$ homogeneous vector $v \mid v \in \mathbb{R}^3 - \{(0,0,0)^T\} \}$

11. the study of P^2 is known as projective geometry.

12. P^2 is a set of rays in R^3.

13. duality principle:

to any theorem of 2d projective geometry there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem.

14 the equation of a conic in inhomogeneous coordinates is

$ax^2+bxy+cy^2+dx+ey+f=0$

i.e., a polynomial of degree 2.

14.3 homogenizing this

$\Rightarrow x^T C x = 0$, $C=\begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$

C is a homogeneous representation of a conic.

14.4 dof(conic) $=5$

12.3

such a ray may be thought of as representing a single point ~~[illegible]~~ in P^2.

12.4

In this model, the lines in P^2 are planes passing through the origin.

(use $L = x \times x'$) (or use 15)

[illegible] 15.

15.2 $x \in l$, $y \in l$

$\Rightarrow x + \alpha y \in l$, $\forall \alpha$

15.3.

$\forall z \in l \Rightarrow l^T z = 0$

$\because l^T x = 0$, $l^T y = 0$, $x \neq y$, i.e., x, y linearly independent,

$\therefore \exists \alpha$, s.t. $z = \cancel{k}(x+\alpha y)$

15.4 $\therefore$ ~~line l~~

$\therefore \forall z \in$ line $l \Leftrightarrow z = x + \alpha y$

16. line l tangent to conic C at point x on C

$\Rightarrow l = Cx$.

17. dual conics are also known as conic envelopes.

18. $\begin{cases} \text{point conic.} \\ \text{line conic} \end{cases}$

degenerate conics.

2

19.

a projectivity is an invertible mapping h from P^2 to itself such that points x_1, x_2, x_3 lie on the same line $\Leftrightarrow$ if $h(x_1), h(x_2), h(x_3)$ do.

19.3 A projectivity is also called
a collineation,
a projective transformation.
or a homography.

19.4 projectivities form a group

19.5 P^2 is homogeneous 3d vectors.

20. x_1, x_2, x_3 lie on same line
H non-singular
$\Rightarrow Hx_1, Hx_2, Hx_3$ lie on same line.

prove:

let $l^T x_i = 0, \forall i \in \{1,2,3\}$

$\therefore l^T H^{-1} H x_i = 0$

$\therefore (H^{-T} l)^T (H x_i) = 0$

$\therefore H x_i, i=1,2,3$ lie on the line $H^{-T} l$.

21. $h: P^2 \to P^2$ is a projectivity
$\Leftrightarrow \exists H$ non-singular such that $\forall x \in P^2$
$h(x) = Hx$.

(ε)

22. we call a matrix homogeneous matrix if only the ratio of the matrix elements is significant.

23. dof(projective transformation) $= 8$

23.3
an alternative definition of a projective transformation:

A planar projective transformation is a linear transformation on homogeneous 3d vectors represented by a non-singular 3x3 matrix.

$x' = Hx$.

23.4 H is a homogeneous matrix.

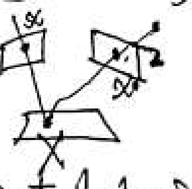
24. Projection along rays through a common point defines a mapping from one plane to another.
The central projection mapping may be expressed by $x' = Hx$ where H is invertible.
If we define Euclidean coordinate systems in the two planes, then it is called a perspectivity rather than a full projectivity.

and the dof of it is 6.

25. removing the projective distortion from a perspective image of a plane

25.3 Note that since the ground and the front are not in the same plane, the projective transformation that must be applied to rectify the front is not the same as the one used for the ground.

26 projective transformations are important mappings representing many more situations than the perspective imaging of a world plane.



27. covariant, contravariant behaviour.

(2

28 point transformation $x' = Hx$

$\Rightarrow$ line transformation $l'^T = l^T H^{-1}$. い

29. the concatenation of two projective transformations ~~between~~ is a projective transformation.

(い) (group)

30. under a point transformation $x' = Hx$, a conic C transforms to $C' = H^{-T} C H^{-1}$. い

31. $GL(n)$: the group of invertible $n \times n$ matrices with real elements is the real general linear group on n dimensions.

31.3 $PL(n)$: projective linear group, is a quotient group of $GL(n)$.

31.4 affine group is the subgroup of $PL(3)$ consisting of matrices for which the last row is $(0,0,1)$.

31.5 the Euclidean group is a subgroup of the affine group for which the upper left 2×2 matrix is orthogonal.

Oriented Euclidean group is a subgroup of the Euclidean group in which the upper left 2×2 matrix has determinant 1.

31.6 perspectivities are not group. (not closed).

32. isometries are transformations of $\mathbb{R}^2$ that preserve Euclidean distance.

~~It is also called Euclidean transformation~~

32.3 $x' = H_E x = \begin{pmatrix} R & t \\ 0^T & 1 \end{pmatrix} x$,

where R is 2×2 rotation matrix. (i.e., R is an orthogonal matrix).

32.4 It's also known as a displacement.

35. an affine trans is a non-singular linear trans followed by a translation.

35.2 $x' = H_A x = \begin{pmatrix} A & t \\ 0^T & 1 \end{pmatrix} x$

35.3 $\because A = UDV^T$ (SVD)

$\therefore A = (UV^T)(VDV^T) = R(\theta)R(-\phi)DR(\phi)$,

($\because U, V$ orthogonal)

($D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$)

$\therefore$ A ~~[illegible]~~ is seen to be the concatenation of a rotation by ϕ, a scaling by λ_1, λ_2 in the rotated x and y directions, a rotation back, and finally another rotation.

35.4 The essence of an affinity is this scaling in orthogonal directions oriented at a particular angle.

33. similarity transformation.

It's an isometry composed with an ~~isometry~~ isotropic scaling.

33.3. In the case of a Euclidean transformation composed with a scaling the similarity has the form:

$x' = \begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix} x$

33.4 A similarity trans is also known as an equi-form trans, because it preserves shape.

34. metric structure implies that the structure is defined up to a similarity. It means Euclidean trans.

32.5 An isometry:

$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ & & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

$\varepsilon = \pm 1$.

If $\varepsilon = 1$ then isometry is orientation-preserving and is a Euclidean transformation.

If $\varepsilon = -1$ then it reverses orientation

32.6 error in 32;

isometries is also called Euclidean transformation. (X)

(because only true when $\varepsilon = 1$)

32.7 H_E is 32.3 is Euclidean trans.

32.8 ~~error~~ in 32.4.

35.5

Under an affinity:

three important invariants are

(i) parallel lines.

(ii) ratio of lengths of parallel line segments.

(iii) ratio of areas.

prove:

(i) l_1, l_2 parallel,

∴ they intersect at $(x_1, x_2, 0)^T$

∵ $H_A\begin{pmatrix}x_1\\x_2\\0\end{pmatrix} = \begin{pmatrix}x_1'\\x_2'\\0\end{pmatrix}$,

∴ l_1', l_2' intersect at infinity

∴ l_1', l_2' parallel.

(ii) $\lambda^2 = (\lambda_1 c\alpha)^2 + (\lambda_2 s\alpha)^2$

∴ $\lambda = \sqrt{(\lambda_1 c\alpha)^2 + (\lambda_2 s\alpha)^2}$

here α is the angle between the line and the x-axis of the orthogonal scaling direction.

∴ we get scaling magnitude λ.

∵ λ is common to all lines with the same direction.

∴ it cancels out in a ratio of parallel segment lengths.

(iii) ∵ from 35.3,

rotations and translations do not affect area, so only the scalings by λ_1 and λ_2 matter.

∴ area is scaled by $\lambda_1\lambda_2$ which is equal to $|A|$.

∴ the area of any shape is scaled by $|A|$.

∴ the scaling cancels out ~~from~~ for a ratio of areas.

35.6 an affinity is orientation-preserving or -reversing according to whether $|A|>0$ or $|A|<0$ respectively.

36. Projective transformations.

36.2 $x' = H_P x = \begin{pmatrix}A & t\\ v^T & v\end{pmatrix} x$

where $v = (v_1, v_2)^T$

37. $H_A\begin{pmatrix}x_1\\x_2\\0\end{pmatrix} = \begin{pmatrix}A & t\\ 0^T & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\0\end{pmatrix} = \begin{pmatrix}A\begin{pmatrix}x_1\\x_2\end{pmatrix}\\ 0\end{pmatrix}$

$H_P\begin{pmatrix}x_1\\x_2\\0\end{pmatrix} = \begin{pmatrix}A & t\\ v^T & v\end{pmatrix}\begin{pmatrix}x_1\\x_2\\0\end{pmatrix} = \begin{pmatrix}A\begin{pmatrix}x_1\\x_2\end{pmatrix}\\ v_1x_1+v_2x_2\end{pmatrix}$

∴ ideal point is mapped to a finite point under H_P.

∴ H_P can model vanishing points.

38.

A projective transformation H

$\Rightarrow H = H_s H_A H_p$

$$= \begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ v^T & v \end{pmatrix}$$

$$= \begin{pmatrix} A & t \\ v^T & v \end{pmatrix}$$

with A a non-singular matrix given by $A = sRK + tv^T$, and K an upper-triangular matrix normalized as $|K| = 1$.

This decomposition is valid provided $v \neq 0$, and is unique if s is chosen positive.

39. $H_E < H_S < H_A < H_P$

$dof(H_E)=3$, $dof(H_S)=4$, $dof(H_A)=6$, $dof(H_P)=8$

40 the cross ratio is the basic projective invariant of P^1.

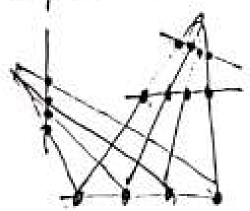
40.2 given 4 points x_i the cross ratio is defined as

$$Cross(x_1, x_2, x_3, x_4) = \frac{|x_1 x_2||x_3 x_4|}{|x_1 x_3||x_2 x_4|}$$

where $|x_i x_j| = det\begin{pmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{pmatrix}$

40.3 Under a projective trans of the plane, a 1D projective trans is induced on any line in the plane.

40.4



41. topology of the projective plane

41.3 the projective plane may be pictured as the unit sphere with opposite points identified.

$\therefore$ a line in P^2 is modelled as a great circle on the unit sphere with opposite points identified.

41.4. the sphere S^2 is a 2-sheeted covering space of P^2.

$\therefore$ P^2 is not simply connected.

41.5 the fundamental group of P^2 is the cyclic group of order 2.

41.6 P^2 is simply a disk with opposite points on its boundary identified, or glued together.

41.7 P^2 is non-orientable.

41.8 P^1 is topologically equivalent to a line segment with the two endpoints identified — namely a circle, S^1.

42. the line at infinity, l_∞ is a fixed line under the projective trans $H \Leftrightarrow H$ is an affinity.

43. Affine rectification.

① H_1, H_2 projective transformations.

② H_1 maps l_∞ from $(0,0,1)^T$ on Euclidean plane π_1 to a finite line l on the plane π_2.

③ H_2 maps l back to $(0,0,1)^T$ on plane π_3

$\Rightarrow H_1H_2$ is affine trans from π_1 to π_3.

44. Line $l=(l_1,l_2,l_3)^T$, $l_3 \neq 0$

H_A is any affine trans.

$$H = H_A\begin{pmatrix}1&&\\&1&\\l_1&l_2&l_3\end{pmatrix}$$

$\Rightarrow H$ maps l to $l_\infty=(0,0,1)^T$

prove:

method 1:

$\cancel{x_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\in l}$

let $x_1 \neq x_2$, $x_1, x_2 \in l$.

$\Rightarrow Hx_1, Hx_2 \in l_\infty$, and $Hx_1 \neq Hx_2$

$\therefore$ line l is mapped to l_∞.

method 2:

$$H^{-T}l = H_A^{-T}\begin{pmatrix}1&&l_1\\&1&l_2\\&&l_3\end{pmatrix}^{-1}\begin{pmatrix}l_1\\l_2\\l_3\end{pmatrix}$$

$$=H_A^{-T}\begin{pmatrix}1&&-l_1/l_3\\&1&-l_2/l_3\\&&1/l_3\end{pmatrix}\begin{pmatrix}l_1\\l_2\\l_3\end{pmatrix}$$

$$=H_A^{-T}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$=\begin{pmatrix}0\\0\\1\end{pmatrix} = l_\infty$$

45. Affine properties may be recovered by simply specifying a line (2 dof). Conversely if affine properties are known, these may be used to determine points and line at infinity.

46. three collinear points a', b', c' on an image corresponding to collinear world points with interval ratio $a:b$.

$\Rightarrow$ vanishing point on the line $a'b'c'$

47. homogeneous coordinate $(x, y, 0)$ $\overset{\text{relation}}{\Longleftrightarrow}$ limit point, $\lim_{\lambda\to 0}(x, y, \lambda)$

48. The circular points I, J, also called the absolute points, are a pair of complex conjugate ideal points.

The canonical coordinates are:

$$I=\begin{pmatrix}1\\i\\0\end{pmatrix},\quad J=\begin{pmatrix}1\\-i\\0\end{pmatrix}.$$

49. The circular points I, J are fixed points under the projective trans $H \Leftrightarrow H$ is a similarity.

prove: ① $\Leftarrow$ ② $\Rightarrow$ later.

(A reflection swaps I and J.)

54. a conic is determined uniquely (up to scale) by five points in general position. Similarly, five lines in general position define a dual conic.

55. The dual conic C^*_∞ is fixed under the projective trans $H \Leftrightarrow H$ is a similarity.
(see 49)

55.3 $C^*_\infty = IJ^T + JI^T$

C^*_∞ is a ~~degenerated~~ degenerate line conic. (rank 2), which consists of the two circular points.

55.4 $C = lm^T + ml^T$ is composed of two lines l, m.

56 ① degenerate point conics include two lines (rank 2), and a repeated line (rank 1).

② degenerate line conics include two points (rank 2), and a repeated point (rank 1).

53.4
dual (line) conic is a conic which defines an equation on lines.

53.5
point conic C
dual conic C^*

C^* is cofactor matrix of C,
$\because C$ symmetric,
$\therefore C^*$ is adjoint matrix of C.

53.6
for non-singular symmetric matrix $C \Rightarrow C^* = C^{-1}$ (up to scale)

53.7
the line l tangent to conic C satisfies $l^T C^* l = 0$

50. cross product
$a \times b = [a]_\times b = (a^T [b]_\times)^T$

where $[a]_\times := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$ is skew-symmetric matrix.

51. matrix M is a square matrix.
M^* is cofactor matrix of M.
$\mathrm{adj}(M)$ is (classical) adjoint of M.

51.3 If M invertible,
$\Rightarrow M^* = |M| M^{-T}$

51.4 $\forall M$ a square matrix,
$\Rightarrow \mathrm{adj}(M) \cdot M = M \cdot \mathrm{adj}(M) = |M| I$

52. $[a]_\times$ is singular, a is its null-vector (right or left).

53.3 point conic is a conic which defines an equation on points

57. circular points arises because every circle intersects l_∞ at the circular points.

58. Algebraically, the circular points are the orthogonal directions of Euclidean geometry, $(1,0,0)^T$ and $(0,1,0)^T$, packaged into a single complex conjugate entity,

e.g. $I = (1,0,0)^T + i(0,1,0)^T$

$\therefore$ It's not so surprising that once the circular points are identified, orthogonality, and other metric properties, are then determined.

59 ① C^*_∞ has 4 degrees of freedom.

② l_∞ is the null vector of C^*_∞.

60 under a point trans $x' = Hx$, a point conic C transforms to $C' = H^{-T}CH^{-1}$ (see 30)

prove: $\because x = H^{-1}x'$

x^TCx

$= x'^T H^{-T} C H^{-1} x'$

$\therefore C' = H^{-T}CH^{-1}$

61. Under a point trans $x' = Hx$, a dual conic C^* transforms to $C^{*'} = HC^*H^T$

proof:

$\because l' = H^{-T}l$,

$\therefore l = H^T l'$

$\because l^T C^* l$

$= l'^T H C^* H^T l'$

$\therefore C^{*'} = HC^*H^T$

62. Once the conic C^*_∞ is identified on the projective plane then Euclidean angles may be measured by

$$\cos\theta = \frac{l^T C^*_\infty m}{\sqrt{l^T C^*_\infty l}\,\sqrt{m^T C^*_\infty m}} \quad (1)$$

prove: let $l = (l_1, l_2, l_3)^T$, $m = (m_1, m_2, m_3)^T$

① $\because$ $\cos\theta = \dfrac{l_1m_1 + l_2m_2}{\sqrt{l_1^2+l_2^2}\,\sqrt{m_1^2+m_2^2}}$ (2)

$\because C^*_\infty = IJ^T + JI^T = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$

$\therefore$ (1) reduces to (2) in a Euclidean coordinate system.

② $\because l' = H^{-T}l$, $m' = H^{-T}m$, $C^{*'}_\infty = HC^*_\infty H^T$

$\therefore l'^T C^{*'}_\infty m' = l^T H^{-1} \cdot HC^*_\infty H^T \cdot H^{-T} m$

$= l^T C^*_\infty m$

③ similarly for $l'^T C^{*'}_\infty l'$, $m'^T C^{*'}_\infty m'$.

④ the scale of the homogeneous objects cancels between the numerator and denominator in (1).

$\therefore$ (1) is invariant to projective trans.

$\therefore$ proved.

68. singular values are real and non-negative.

prove: A^TA symmetric and positive-semi-definite.

$\therefore$ its eigenvalues are real and non-negative

$\therefore$ proved. (here see 67)

69. A real symmetric,

$\Rightarrow A = UDU^T$, where U orthogonal, D real diagonal.

$\because$ a real symmetric matrix has real eigenvalues, and the eigenvectors are orthogonal.

70. Once C_∞^* is identified on the projective plane then projective distortion may be rectified up to a similarity.

that maps the imaged circular points to their canonical position on l_∞.

$\Rightarrow$ the trans between the world ~~and~~ plane and the rectified image is a similarity.

prove: ① the trans is ~~a~~ projective

② the circular points are fixed.

$\therefore$

66. A square, $\Rightarrow A = UDV^T$,

U, V orthogonal matrices, D diagonal matrix with non-negative entries.

(SVD)

66.3 SVD exists for non-square A.

67. the singular values of A are the square-roots of the eigenvalues of A^TA.

proof: $A = UDV^T$

$\therefore A^TA = VDU^TUDV^T = VD^2V^T$

$\therefore D^2$ are eigenvalue matrix of A^TA.

63. (see 55) C_∞^* fixed under projective trans H $\Leftrightarrow$ H is a similarity.

proof:

(1) $\Leftarrow$

$$H_S C_\infty^* H_S^T$$

$$= \begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} sR^T & 0 \\ t^T & 1 \end{pmatrix}$$

$$= \begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} sR^T & 0 \\ 0^T & 0 \end{pmatrix}$$

$$= s^2 \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = s^2 C_\infty^* = C_\infty^* \text{ (up to scale)}$$

$\therefore C_\infty^*$ is fixed.

$\therefore$

(2) $\Rightarrow$ later.

64. C_∞^* is dual to the circular points.

(3)

65. If the circular points are identified in an image, and the image is then rectified by a projective trans H

71. The polar line $l = Cx$ of point x w.r.t conic C intersects the conic in two points. The two lines tangent to C at these points intersect at x.

proof: $y \in C$,

$x \in$ tangent line Cy

$\Leftrightarrow x^T C y = 0$

$\Leftrightarrow (Cx)^T y = 0$

$\Leftrightarrow y \in$ line l.

72. If point x is on C then the polar is the tangent line to the conic at x.

73. the conic induces a map between points and lines of P^2.

The map is a projective construction since it involves only intersections and tangency, both properties that are preserved under projective transformations.

73.3 ① conic C, point x

$\Rightarrow$ polar $l = Cx$.

② conic C, given l, C non-singular

$\Rightarrow$ ~~x,~~

$x = C^{-1} l$, s.t., l is polar of x w.r.t C.

74. A correlation is an invertible mapping from points of P^2 to lines of P^2. It's represented by a 3x3 non-singular matrix A as $l = Ax$.

75. If point x, y satisfying $y^T C x = 0$ then x, y are conjugate w.r.t. C.

75.3

x on the polar of $y \Leftrightarrow y$ on the polar of x.

76. dual conjugacy relationship for lines:

two lines l and m are conjugate if $l^T C^* m = 0$

77. A symmetric,

$\Rightarrow \exists T$ invertible, s.t. $A = T^T D T$

$D =$ ~~diag~~ $diag(\varepsilon_1, \cdots, \varepsilon_n)$

here $\varepsilon_i \in \{1, -1, 0\}$

~~$\varepsilon_i = 1$ occurs before $\varepsilon_i = -1$~~

~~$\varepsilon_i = 1$, $\varepsilon_j = -1$, if $i < j$, and one is 1 the other -1.~~

~~$\varepsilon_i = -1$, $\varepsilon_j = 0$, if $i < j$, and one is -1, the other 0.~~

~~$\varepsilon_i = 1$, $\varepsilon_j = 0$, if $i < j$, and one is 1 the other 0~~

Values $\varepsilon_i = 1$ occur before values $\varepsilon_i = -1$ which in turn precede $\varepsilon_i = 0$.

proof: $\because A = U^T D_0 U$, U orthogonal, D_0 real diagonal.

$D_0 = T_1^T D T_1$, $\therefore A = T^T D T$, $T = T_1 U$.

78. Any conic is equivalent under projective transformation to one with a diagonal matrix D. ∵

∴ The types of conics may now be enumerated :

$D = diag(1,1,1)$
$D = diag(1,1,-1)$, circle.
$D = diag(1,1,0)$
$D = diag(1,-1,0)$, two lines $x = \pm y$
$D = diag(1,0,0)$. , single line $x=0$ counted twice.

79. intersections and tangency are preserved under projective transformations.

proof: ① $x \in l \cap m$. ∴ ~~x = l × m~~ $l^T x = m^T x = 0$

$x' = Hx$, $l' = H^{-T} l$, $m' = H^{-T} m$

~~∴ x' = l' × m'~~

∵ $l'^T x' = l^T H^{-1} H x = l^T x = 0$

$m'^T x' = 0$, ∴ $x' \in l' \cap m'$

② ~~x~~ $\in C$,

l tangent to C at x

∴ $l = Cx$

∵ $C' = H^{-T} C H^{-1}$

$x' = Hx$, $l' = H^{-T} l$

∵ $C' x' = H^{-T} C x = H^{-T} l = l'$

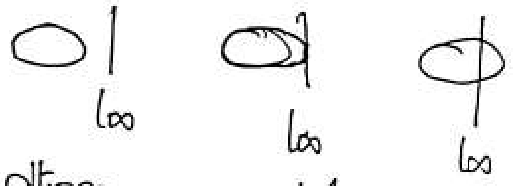
∴ l' tangent to C' at x'.

80. Non-degenerate, proper conics in Euclidean geometry ~~include~~ are ellipse, parabola, ~~and~~ or hyperbola.

80.3 ∵ the Euclidean classification depends only on the relation of l_∞ to the conic,

又 ∵ l_∞ is fixed under affine trans, and intersections are preserved

∴ the classification is unaltered by an affinity.



ellipse　parabola　hyperbola.

80.4.

The ~~three~~ types of conic are projectively equivalent to a circle.

81. fixed points and lines.

81.3. Eigenvector corresponds to a fixed point of the transformation.

$He = \lambda e$.

e and λe represent the same point.

81.4 ∵ $l' = H^{-T} l$, ∴ fixed lines correspond to the eigenvectors of H^T.

81.5 $\begin{cases} \text{line is fixed as a set.} \\ \text{line is fixed pointwise.} \end{cases}$

H2

Chapter 3

Projective geometry and transformations of 3D.

1\. A projective transformation acting on P^3 is a linear trans on homogeneous 4d vectors represented by a non-singular 4x4 matrix:

$X' = HX.$

1.3 H is homogeneous.

$dof(H) = 15.$

1.4 The map is collineation which preserves incidence relations

2\. points and planes are dual in P^3.

lines are self-dual in P^3.

3\. a plane in 3d space is:

$$n \cdot \tilde{X} + d = 0, \text{ where}$$

$n = (\pi_1, \pi_2, \pi_3)^T$, $\tilde{X} = (X, Y, Z)^T$,

$d = \pi_4$

3.3. homogenizing it by replacements

$$X = X_1/X_4, \quad Y = X_2/X_4, \quad Z = X_3/X_4$$

$\Rightarrow \pi^T X = 0.$

where $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)^T$.

$$X = (X_1, X_2, X_3, X_4)$$

3.4 the distance of the plane from the origin is $\frac{|d|}{\|n\|}$.

4\. $|A| = |A^T|$

5\. $a \cdot (b \times c)$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6\. In P^2, l passes x_1, x_2, $\Rightarrow l = x_1 \times x_2$

proof: $\forall x \in l$,

$$M = (x, x_1, x_2)$$

$$= \begin{pmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{pmatrix}$$

$\because |M| = 0$

$\therefore x \cdot (x_1 \times x_2) = 0$

$\therefore$ line $l = x_1 \times x_2$.

7\. In P^3, plane π passes x_1, x_2, x_3,

$$\Rightarrow \pi = \begin{pmatrix} (\tilde{x}_1 - \tilde{x}_3) \times (\tilde{x}_2 - \tilde{x}_3) \\ -\tilde{x}_3^T (\tilde{x}_1 \times \tilde{x}_2) \end{pmatrix}$$

here $x_i^T = w_i(\tilde{x}_i^T, 1)$

proof: $\forall x \in \pi$,

$\because |M| = 0$, $\therefore |M| = |x, x_1, x_2, x_3| = 0$

$$\therefore \begin{vmatrix} X & X_1 & X_2 & X_3 \\ Y & Y_1 & Y_2 & Y_3 \\ Z & Z_1 & Z_2 & Z_3 \\ W & W_1 & W_2 & W_3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} X & \tilde{X}_1 & \tilde{X}_2 & \tilde{X}_3 \\ Y & \tilde{Y}_1 & \tilde{Y}_2 & \tilde{Y}_3 \\ Z & \tilde{Z}_1 & \tilde{Z}_2 & \tilde{Z}_3 \\ W & 1 & 1 & 1 \end{vmatrix} = 0$$

$\therefore x^T \pi = 0$, $\pi = (D_{234}, -D_{134}, D_{124}, -D_{123})^T$

here $D_{234} = \begin{vmatrix} \tilde{Y}_1 & \tilde{Y}_2 & \tilde{Y}_3 \\ \tilde{Z}_1 & \tilde{Z}_2 & \tilde{Z}_3 \\ 1 & 1 & 1 \end{vmatrix} = ((\tilde{Y}_1 - \tilde{Y}_3) \times (\tilde{Y}_2 - \tilde{Y}_3))$,

similarly, other components computed.

8. under the point trans $X' = HX$,
a plane transforms as $\pi' = H^{-T}\pi$.

9. $\forall X \in \pi$.

$\Rightarrow X = Mx$

where $\pi^T M = 0$, i.e., the columns of the 4×3 matrix M generate the rank 3 null-space of π^T.

9.3 M is not unique, of course.

9.4 This representation is simply the analogue of a line l in P^2 defined as a linear combination of its 2D null-space as $x = \mu a + \lambda b$, where $l^T a = l^T b = 0$.

10. a line is defined by the join of two points or the intersection of two planes.

10.3 $dof(\text{line}) = 4$, $dof(\text{point}) = 3$, $dof(\text{plane}) = 3$.

10.4. Lines are very awkward to represent in 3d space.

A number of line representations have been proposed.

11. The line joining A, B (non-coincident) is represented by the span of the row space of the 2×4 matrix W.

$W = \begin{pmatrix} A^T \\ B^T \end{pmatrix}$

$\Rightarrow$ (i) the span of $W^T = (A, B)$ is the pencil of points $\lambda A + \mu B$ on the line.

(ii) the 2d right null-space of W is the pencil of planes with the line as axis.

proof: (ii) suppose that $\langle P, Q\rangle$ = null-space

$\therefore WP = 0$ $\therefore A^T P = B^T P = 0$.

$\therefore P$ is a plane containing $\langle W^T\rangle$

Similarly, Q containing $\langle W^T\rangle$.

$\therefore$ the line i.e. $\langle W^T\rangle$ is the ? plane intersection.

$\because \forall \pi \in \langle P, Q\rangle$,

$A^T\pi = B^T\pi = 0$

$\therefore \pi$ containing $\langle W^T\rangle$.

$\therefore$ proved.

2. the dual representation of a line:

The line is represented as the span of the row space of the 2×4 matrix W^*

$W^* = \begin{pmatrix} P^T \\ Q^T \end{pmatrix}$,

P, Q plane.

$\Rightarrow$ (i) $\langle W^{*T}\rangle = \langle P, Q\rangle$

= the pencil of planes $\lambda P + \mu Q$ with the line as axis.

(ii) the 2d null-space of W^* is the pencil of points on the line.

13. $W^* W^T = W W^{*T} = 0_{2\times2}$

14. example:

X-axis:

$W = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $W^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

where A = origin,

B = ideal point in the X-direction

P = XY-plane

Q = XZ plane.

15.3 $\dim(\text{range}(A)) = \text{rank}(A)$

where $A \in R^{m \times n}$

15.4 row space $\perp$ null space

15.5 rank-nullity theorem

$A \in R^{m \times n} \Rightarrow \text{rank}(A) + \text{nullity}(A) = n$

proof: $\because$ $\dim(\text{row space}) = \text{rank}(A)$

$\dim(\text{row space}) + \dim(\text{null space}) = n$ (15.4)

$\therefore$ proved.

16(1) π = join of point X and line W.

$\Rightarrow \{\pi\}$ = null space of M.

$M = \begin{pmatrix} W \\ X^T \end{pmatrix}$ $\because$

(2) point X = intersection of line W with plane π

$\Rightarrow \{X\}$ = ~~AA~~ null space of M.

$M = \begin{pmatrix} W^* \\ \pi^T \end{pmatrix}$

17. Plücker representation

a line joining points A, B is represented by a 4×4 skew-symmetric homogeneous matrix L.

$L = AB^T - BA^T$.

~~here the line joining points A, B is represented by L.~~

18.3 $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$ $\because$

18.4. $C = AB$, B square, invertible.

$\Rightarrow \text{rank}(C) = \text{rank}(A)$.

18.5 U, V vector space $\Rightarrow U+V$ is vs.

18.6 $\dim(U+V) \leq \dim(U) + \dim V$, here U, V vs.

proof: $\forall x \in U+V, \Rightarrow x = x_u + x_v$, $x_u \in U$, $x_v \in V$

$\because x_u + x_v \in \langle U \cup V \rangle$ $\therefore x \in \langle U \cup V \rangle$

$\therefore U+V < \langle U \cup V \rangle$

$\therefore \dim(U+V) \leq \dim\langle U \cup V \rangle$

$= \dim\langle \text{basis } U \cup \text{basis } V \rangle$

$\leq \dim U + \dim V$

18.7 matrix $A, B \Rightarrow \text{rank}(A+B) \leq r(A) + r(B)$

proof: $\because \langle A+B \rangle < \langle A \rangle + \langle B \rangle$ $\because$

$\therefore \dim\langle A+B \rangle \leq \dim(\langle A \rangle + \langle B \rangle)$

$\leq \dim\langle A \rangle + \dim\langle B \rangle$

$\therefore$ proved.

25. Under the point transformation $X' = HX$, L^* transforms as $L^{*'} = H^{-T} L^* H^{-1}$.

26. L^* can be obtained directly from L by the simple rule:

$l_{12} : l_{13} : l_{14} : l_{23} : l_{42} : l_{34} = l^*_{34} : l^*_{42} : l^*_{23} : l^*_{14} : l^*_{13} : l^*_{12}$ ③

27. A skew-symmetric matrix $\Rightarrow \mathrm{rank}(A) \neq 1$.

proof: 若 $r(A)=1$, $\Rightarrow \exists\, u, v \neq 0$, s.t. $A = uv^T$

$\because A^T = -A$, $\therefore vu^T = -uv^T$

$\because A = uv^T \Rightarrow$ ~~…~~ $u \in \langle A \rangle$

$\because \dim\langle A \rangle = 1$, $u \neq 0$, $\therefore \langle u \rangle = \langle A \rangle$

$\because A = -A^T = -vu^T$, $\Rightarrow \langle v \rangle = \langle A \rangle$

$\therefore \exists\, k \neq 0$, s.t. $u = kv$.

$\because vu^T = -uv^T \Rightarrow kvv^T = -kvv^T$

$\therefore kvv^T = 0$ $\therefore vv^T = 0$, error. $\therefore$ proved.

21. dual Plucker representation L^*. $L^* = PQ^T - QP^T$

here the line is formed by the intersection of two planes P, Q.

22. correction:

$\pi' = H^{-1}\pi$ (X)

$\pi' = H^{-T}\pi$ (V)

23. In P^2,

~~$l' = H^{-T} l$ if $x' = Hx$~~

$H^{-T}(x_1 \times x_2) = (Hx_1) \times (Hx_2)$ (up to scale)

proof:

$\because l' = H^{-T} l$, $l' = x_1' \times x_2'$

$l = x_1 \times x_2$

$\therefore (Hx_1) \times (Hx_2) = H^{-T}(x_1 \times x_2)$

24. M is any 3x3 matrix, x, y vectors,

$\Rightarrow M^*(x \times y) = (Mx) \times (My)$

where M^* is cofactor matrix of M.

19. properties of Plücker representation L.

19.3 ① $\mathrm{rank}(L) = 2$.

② $LW^{*T} = 0 \in R^{4 \times 2}$

here $W^{*T} = (P, Q)$, P, Q planes.

19.4 $\therefore$ null space$(L) = \langle W^{*T} \rangle$

see 12.(ii), vs 19.3 ②

19.5 $\mathrm{dof}(L) = 4$.

19.6 matrix L is independent of the points A, B used to define it.

19.7. Under point trans $X' = HX$,

$\Rightarrow L' = HLH^T$.

proof: $L' = A'B'^T - B'A'^T$

$= HAB^TH^T - HBA^TH^T$

$= HLH^T$

20. example:

X-axis is represented as: $L = AB^T - BA^T = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

where A origin, B ideal point in the X-direction.

proof:

① let $\pi = L^* X$, $L^* X \neq 0$

$\because L^* = PQ^T - QP^T \therefore L \in \pi$

② $\because X^T \pi = X^T L^* X$

$= X^T PQ^T X - X^T QP^T X$

$= (X^T Q)(X^T P) - (X^T Q)(X^T P)$

$= 0$

$\therefore X \in \pi$

③ $\therefore L^* X$ is the plane π when $L^* X \neq 0$

④ $L^* X = 0 \Leftrightarrow PQ^T X - QP^T X = 0$

$\Leftrightarrow Q^T X = P^T X = 0$

$\Leftrightarrow X \in Q,\ X \in P$

$\Leftrightarrow X \in L$

proof 1:

let $C = A + \mu B$

$\therefore \hat{L} = AC^T - CA^T$

$= AA^T + \mu AB^T$

$- AA^T - \mu BA^T$

$= \mu (AB^T - BA^T)$

$= \mu L$

$\therefore \hat{L} = L$ (up to scale)

proof 2:

(later)

30. plane π is defined by the join of point X and line L.

$\Rightarrow \pi = L^* X$, when $L^* X \neq 0$, 29.

and $L^* X = 0 \Leftrightarrow X \in L$

28.3

$L = AB^T - BA^T$, A, B different points.

$\Rightarrow L \neq 0$

proof: If $L = 0$

$\Rightarrow AB^T = BA^T$

$\Rightarrow A, B$ linear dependent

$\Rightarrow A, B$ same point.

$\therefore L \neq 0$

28.4 $\operatorname{rank}(L) = 2$

proof:

L skew-symmetric, $L \neq 0$

$\Rightarrow r(L) \geq 2$

$\Rightarrow r(L) = 2$ ($\because L = AB^T - BA^T$ $\therefore r(L) \leq 2$)

L is independent of A, B. (see 19.6)

31. point X is defined by the intersection of line L with plane π.

$\Rightarrow X = L\pi$, when $L\pi \neq 0$

and $L\pi = 0 \Leftrightarrow L \in \pi$

32. e.x.,

the intersection of the X-axis with plane $X=1$ is given by $X = L\pi$ as

$$X = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & & & \\ 0 & & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

33. Plücker line coordinates are the six non-zero elements of Plücker matrix L: $L = \{l_{12}, l_{13}, l_{14}, l_{23}, l_{42}, l_{34}\}$

33.3 $\because |L| = 0 \Rightarrow l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0$

34. let $M = (A, B)$

$\therefore L_{ij}$ is the determinant of rows i and j of M.

proof:

$\because L = AB^T - BA^T$

$\therefore L_{ij} = b_j a_i - a_j b_i$

$$= \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$$

here $A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix}$

35. $\because L_{ij} = -L_{ji}$.

$\therefore$ Plücker line coordinates ~~has~~ have $C_4^2 = 6$ elements.

36. proof2 of 29. (see 19.6)

$\because M' = (A', B')$, $M = (A, B)$

$\therefore M' = M\Lambda$.

$\therefore L' = L|\Lambda|$.

$\therefore L' = L$ (up to a scale)

37. $\because$ rank$(M) = 2$

$\therefore L \neq 0$

3

38. Leibniz formula :

$A : n \times n$

$\Rightarrow |A| = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)}$

where S_n is the set of all permutation of the integers $\{1, 2, \cdots, n\}$

$\sigma(i)$ is the element in position i after the reordering σ.

$\text{sign}(\sigma) = \begin{cases} 1, & \sigma \text{ is even} \\ -1, & \sigma \text{ is odd.} \end{cases}$

38.3 σ is even (odd) if the new sequence can be obtained by an even (odd) number of switches of numbers.

39. prove 38.3.

proof: $\because |L| = a^2+b^2+c^2-2ab-2bc+2ac$

$= (a-b+c)^2$

here $a = l_{12}l_{34}$, $b = l_{13}l_{24}$, $c = l_{14}l_{23}$

$\therefore |L| = 0 \Leftrightarrow a-b+c=0$

$\Leftrightarrow l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0$

40. $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

$= k_{12}l_{34} + k_{13}l_{42} + k_{14}l_{23}$
$+ k_{23}l_{14} + k_{42}l_{13} + k_{34}l_{12}$

where $k_{ij} = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$

$l_{ij} = \begin{vmatrix} c_i & d_i \\ c_j & d_j \end{vmatrix}$

41. line L : join of A, B.

line $\hat{L}$: join of $\hat{A}, \hat{B}$.

L and $\hat{L}$ intersect (i.e., coplanar)

$\Leftrightarrow A, B, \hat{A}, \hat{B}$ coplanar.

$\Leftrightarrow |A, B, \hat{A}, \hat{B}| = 0$

$\Leftrightarrow (L|\hat{L}) = 0$

proof:

$\because$ the bilinear product

$(L|\hat{L}) := l_{12}\hat{l}_{34} + l_{13}\hat{l}_{42}$
$+ l_{14}\hat{l}_{23} + l_{23}\hat{l}_{14}$
$+ l_{42}\hat{l}_{13} + l_{34}\hat{l}_{12}$

$\therefore (L|\hat{L}) = |A, B, \hat{A}, \hat{B}|$ (from 40)

$\therefore$ proved.

42. $L = AB^T - BA^T$

$\Rightarrow l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0$

(see 33)

proof: $L = AB^T - BA^T$

$\Rightarrow L_{ij} = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$

$\Rightarrow (L|L) = |A, B, A, B|$ (from 40)

$\Rightarrow (L|L) = 0$

$\Rightarrow l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0$

43. $L \in P^5$ is a line in P^3

$\Leftrightarrow (L|L) = 0$

proof: ① $\Rightarrow$: (from 42)

② $\Leftarrow$: let $Q = (q_{12}, q_{13}, q_{14}, q_{23}, q_{42}, q_{34})^T \in P^5$

$(Q|Q) = 0$, suppose $q_{12} \neq 0$

③. in ②, we supposed $q_{12} \neq 0$,

If $q_{ij} \neq 0$, $i<j$, then let $m, n \in \{1,2,3,4\}/\{i,j\}$

with $m<n$.

then M is defined by:

row i: q_{ij} 0

row j: 0 q_{ij}

row m: $-q_{jm}$ q_{im}

row n: $-q_{jn}$ q_{in}

$\therefore l_{st} = q_{ij}q_{st}$, when $(s,t) \neq (m,n)$

$l_{mn} = -q_{jm}q_{in} + q_{jn}q_{im} = q_{ij}q_{mn}$

($\because q_{ij}q_{mn} + q_{jm}q_{in} - q_{jn}q_{im} = 0$ by $(Q|Q)=0$)

$\therefore Q$ is line L.

then define $M = (A, B)$

$= \begin{pmatrix} q_{12} & 0 \\ 0 & q_{12} \\ -q_{23} & q_{13} \\ q_{42} & q_{14} \end{pmatrix}$

$\therefore A, B \in P^3$, $A \neq B$

$\therefore$ line $L = (l_{12}, l_{13}, l_{14}, l_{23}, l_{42}, l_{34})$

with

$l_{12} = q_{12}^2$, $l_{13} = q_{12}q_{13}$, $l_{14} = q_{12}q_{14}$

$l_{23} = q_{12}q_{23}$, $l_{42} = q_{12}q_{42}$,

$l_{34} = -q_{14}q_{23} - q_{13}q_{42} = q_{12}q_{34}$

($\because (Q|Q) = 0$)

$\therefore L = (q_{12}, q_{13}, q_{14}, q_{23}, q_{42}, q_{34})$

up to scale.

$\therefore Q$ is line L.

proved.

44. $\therefore$ the lines in P^3 are in bijective correspondence with Plücker quadric or Klein quadric in P^5.

(from 43)

45. A quadric is a surface in P^3 defined by $X^TQX=0$, where Q is a symmetric 4x4 matrix.

45.3 Q : matrix Q, or the quadric surface it defines.

45.4 $dof(Q)=9$, (14.4 in C2)

45.5 nine points in general position define a quadric.

45.6 A quadric defines a polarity between a point and a plane, in a similar manner to the polarity defined by a conic ~~defined~~ between a point and a line. (73 in C2)

45.7 the plane $\pi=QX$ is the polar plane of X with respect to Q. (71 in C2)

45.8 the intersection of a plane π with a quadric Q is a conic C.

A coordinate system for the plane can be defined by the complement space to π as $X=Mx$. (see 9)

45.8.3 $X\in\pi, X\in Q$

$\Leftrightarrow X=Mx,\ X^TQX=0$

$\Leftrightarrow X=Mx,\ x^TCx=0,\ C=M^TQM$

$\Leftrightarrow X=Mx,\ x\in C,\ C=M^TQM.$

45.9 Under the point trans $X'=HX$, a (point) quadric transforms as

$Q'=H^{-T}QH^{-1}$ (30 in C2)

46 dual quadrics are equations on planes: the tangent planes π to the point quadric Q satisfy $\pi^TQ^*\pi=0$, where $Q^*=$ adjoint Q, or Q^{-1} if Q is invertible.

46.3 Under the point trans $X'=HX$, a dual quadric transforms as $Q^{*\prime}=HQ^*H^T$

47. classification of quadrics (see 78 in C2)

47.3 the projective type of a quadric is uniquely determined by its rank and signature.

47.4 the signature of a diagonal matrix D, $d_{ii}\in\{1,-1,0\}$, denoted $\sigma(D)$, is defined to be the number of $+1$ entries minus the number of -1 entries.

47.5 Q real symmetric, $\sigma(Q):=\sigma(D)$, s.t. $Q=H^TDH$, where H is real, D in 47.4 σ is well defined, being independent of H.

47.6 quadrics fall into two classes: ruled and unruled quadrics.

47.7 a ruled quadric is one that contains a straight line.

48. the twisted cubic may be considered to be a 3d analogue of a 2d conic.

48.3 a conic in 2d projective plane may be described as a parametrized curve given by the equation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A \begin{pmatrix} 1 \\ \theta \\ \theta^2 \end{pmatrix}, \quad A \in 3\times 3, \text{ non-singular.}$$

48.4 In an analogous manner, a twisted cubic is defined to be a curve in P^3 given by:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = A \begin{pmatrix} 1 \\ \theta \\ \theta^2 \\ \theta^3 \end{pmatrix}, \quad A \in 4\times 4, \text{ non-singular.}$$

48.5 c non-singular twisted cubic

$\Rightarrow$ c is not contained within any plane of P^3.

It intersects a general plane at three distinct points.

48.6 $dof(c) = 12$.

48.7 there is a unique c through 6 points in general position.

48.8 all non-degenerate twisted cubics are projectively equivalent.

49. The 15 degrees of freedom of a projective trans are accounted for as seven for a similarity (three for rotation, three for translation, one for isotropic scaling), five for affine scalings, and three for the projective part of the trans.

49.3

Group	Matrix
dof(Projective) = 15	$\begin{pmatrix} A & t \\ v^T & v \end{pmatrix}$
dof(affine) = 12	$\begin{pmatrix} A & t \\ 0^T & 1 \end{pmatrix}$
dof(similarity) = 7	$\begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix}$
dof(Euclidean) = 6	$\begin{pmatrix} R & t \\ 0^T & 1 \end{pmatrix}$

50. Any particular translation and rotation is equivalent to a rotation about a screw axis together with a translation along the screw axis. The screw axis is parallel to the rotation axis.

50.4 a is unit direction of the rotation axis $\Leftrightarrow Ra = a$

51. In P^2, identifying the line at infinity l_∞ allows affine properties of the plane to be measured. Identifying the circular points on l_∞ then allows the ~~measurements~~ measurement of metric properties.

In P^3 the corresponding geometric entities are the plane at infinity π_∞ and the absolute conic Ω_∞.

52. the plane at infinity has the canonical position $\pi_\infty = (0,0,0,1)^T$ in affine 3-space.

53 π_∞ is fixed under the projective trans $H \Leftrightarrow H$ is an affinity. (42 in C2)

53.3 π_∞ is, in general, fixed as a set, It is not fixed pointwise.

54. The absolute conic Ω_∞ is a (point) conic on π_∞.

54.3 In a metric frame $\pi_\infty = (0,0,0,1)^T$, and points on Ω_∞ satisfy

$$\begin{cases} X_1^2 + X_2^2 + X_3^2 = 0 \\ X_4 = 0 \end{cases}$$

54.4 For points with $X_4 = 0$,

$\Rightarrow (X_1, X_2, X_3) I \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = 0$

$\Rightarrow \Omega_\infty$ corresponds to a conic C with matrix $C = I$.

55. Ω_∞ is fixed under the projective trans $H \Leftrightarrow H$ is a similarity trans.

proof: $\because \Omega_\infty \in$ plane at infinity, H fix it

$\Rightarrow H$ must fix the plane at infinity

$\Rightarrow H$ affine

$\therefore H = H_A = \begin{pmatrix} A & t \\ 0^T & 1 \end{pmatrix}$

Restricting Ω_∞ to the plane at infinity, it is $I_{3\times3}$.

$\because \Omega_\infty$ is fixed by H_A.

$\therefore A^{-T} I A^{-1} = I$ (up to scale)

$\therefore A^T A = I$, (up to scale)

~~∴ A is~~ $\therefore A = sR$, R orthogonal.

$\therefore H_A$ similarity.

56. Ω_∞ is only fixed as a set by a general similarity; it is not fixed pointwise.

56.4 ① a line intersects a conic in two points.

② All circles intersect Ω_∞ in two points.

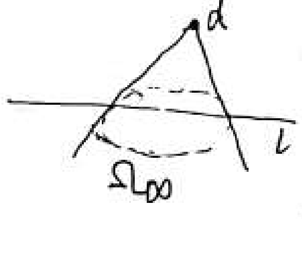
③ All spheres intersect π_∞ in Ω_∞.

57. In P^1, a conic C and two points, x_1 and x_2, in general position have the invariant $I = \frac{(x_1^T C x_2)^2}{(x_1^T C x_1)(x_2^T C x_2)}$

ε (iv, b on P_8)

58. two directions d_1 and d_2 are orthogonal if $d_1^T \Omega_\infty d_2 = 0$

58.4 A plane normal direction d and the intersection line l of the plane with π_∞ are in pole-polar relation w.r.t. Ω_∞



59. Ω_∞ is defined by two equations. It's a conic on the plane at infinity.

60. The dual of Ω_∞ is a degenerate dual quadric in 3d space called the absolute dual quadric, and denoted Q_∞^*.

60.3 Geometrically Q_∞^* consists of the planes tangent to Ω_∞, so that Ω is the rim of Q_∞^*. This is called a rim quadric. Algebraically Q_∞^* has the canonical form $Q_\infty^* = \begin{pmatrix} I_{3\times3} & 0 \\ 0^T & 0 \end{pmatrix}$

61. Q_∞^* is made up of just ~~those~~ planes tangent to the absolute conic.

proof: $\forall \pi = (v^T, k)^T \in Q_\infty^*$

$\Leftrightarrow \pi^T Q_\infty^* \pi = 0$

$\Leftrightarrow v^T v = 0 \quad (\because Q_\infty^* = \begin{pmatrix} I & 0 \\ 0^T & 0 \end{pmatrix})$

$\Leftrightarrow v^T I v = 0$

$\Leftrightarrow$ the line represented by v is tangent to the absolute conic

$\Leftrightarrow$ plane π is tangent to the absolute conic.

($\because$ v represents the line in which plane π meets the plane at infinity)

ε

62. absolute conic as the limit of a series of squashed ellipsoids.

63. $dof(Q_\infty^*) = 8$

64. Q_∞^* - fixed under projective trans $H \Leftrightarrow H$ similarity.

proof: Q_∞^* fixed under $H = \begin{pmatrix} A & t \\ v^T & k \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} I & 0 \\ 0^T & 0 \end{pmatrix} = H \begin{pmatrix} I & 0 \\ 0^T & 0 \end{pmatrix} H^T \doteq \begin{pmatrix} AA^T & Av \\ v^T A^T & v^T v \end{pmatrix}$ (up to scale)

$\Leftrightarrow v = 0$, A is scaled orthogonal.

$\Leftrightarrow H$ similarity.

65. the plane at infinity π_∞ is the null-vector of Q_∞^*.

66. the angle between plane π_1 and plane π_2 is given by

$\cos\theta = (\pi_1^T Q_\infty^* \pi_2) / \sqrt{(\pi_1^T Q_\infty^* \pi_1)(\pi_2^T Q_\infty^* \pi_2)}$

Chapter 6 Hz

Camera Models

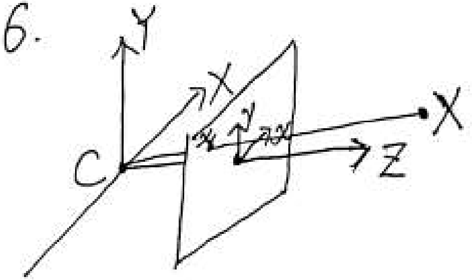
1. A camera is a mapping between the 3D world and 2D image.
2. Camera models are matrices with particular properties that represent the camera mapping.
3. The principal camera of interest is central projection.
4. All cameras modelling central projection are specializations of the general projective camera.

The anatomy of this most general camera model is examined using the tools of projective geometry.

5. The specialized models fall into two major classes — those that model cameras with a finite center, and those that model cameras with center "at infinity".

5.4 affine camera is the natural generalization of parallel projection.

6.



C: camera center, or optical center.

image plane,

principal axis, principal point,

principal plane.

$(X, Y, Z)^T \rightarrow (fX/Z, fY/Z)^T$

6.4. $\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{pmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$

6.5 $x = PX$,

$P = diag(f, f, 1)(I \; 0)$,

X world point, x image point.

7. principal point offset:

$(X, Y, Z)^T \rightarrow (fX/Z + p_x, fY/Z + p_y)^T$

where $(p_x, p_y)^T$ are principal point.

$\therefore \; x = K(I \; 0)X_{cam}$

K is called camera calibration matrix.

$K = \begin{pmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{pmatrix}$

X_{cam} is expressed in camera coordinate frame.

8. camera rotation and translation.

$\tilde{X}_{cam} = R(\tilde{X} - \tilde{C})$,

where $\tilde{C}$ represents the coordinates of the camera center in the world coordinate frame,

$\therefore$ α_x, α_y represent the focal length of the camera in terms of pixel dimensions in the x and y direction respectively.

$\tilde{x}_0 = (x_0, y_0)$ is the principal point in terms of pixel dimensions.

9.4 A CCD camera has 10 degrees of freedom.

10. finite projective camera

$$K = \begin{pmatrix} \alpha_x & s & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{pmatrix}$$

s is referred to as the skew parameter.

10.2 A camera $P = KR(I \; -\tilde{C})$ for which K is of the form above is called a finite projective camera.

10.3 $dof(P) = 11$

proof:

$\because$ $\tilde{X}_{cam} = R\tilde{X}$,

$= A_{W\to C}\tilde{X}$

$\because$ $A_{W\to C} = A^{-1}_{C\to W}$ (change of basis)

$\because$ $A_{C\to W} = (r_1 \; r_2 \; r_3)$

$A_{C\to W}$ orthogonal

$\therefore$ $A_{W\to C} = A^{T}_{C\to W} = \begin{pmatrix} r_1^T \\ r_2^T \\ r_3^T \end{pmatrix}$

$\therefore$ $R = \begin{pmatrix} r_1^T \\ r_2^T \\ r_3^T \end{pmatrix}$

9. CCD cameras.

$$K = \begin{pmatrix} \alpha_x & & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{pmatrix} = \begin{pmatrix} m_x & & \\ & m_y & \\ & & 1 \end{pmatrix} \begin{pmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{pmatrix}$$

where m_x is the number of pixels per unit distance in image coordinates in x direction. Similarly for m_y

R is 3x3 rotation matrix representing the orientation of the camera coordinate frame. (see 8.6)

8.3 $X_{cam} = \begin{pmatrix} R & -R\tilde{C} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$

$= \begin{pmatrix} R & -R\tilde{C} \\ 0 & 1 \end{pmatrix} X$

$\therefore$ $x = KR(I \; -\tilde{C})X$

X is in a world coordinate frame.

8.4 $P = KR(I \; -\tilde{C})$

$dof(P) = 9$, ($\because dof(K)=3, dof(R)=3, dof(\tilde{C})=3$)

8.5 $P = K(R \; t)$, where $t = -R\tilde{C}$

8.6 $R = \begin{pmatrix} r_1^T \\ r_2^T \\ r_3^T \end{pmatrix}$, $\Leftarrow$

r_i is the coordinates of i-th camera coordinate axis in the world coordinate frame ~~where~~ (here no translation).

$\therefore$ ~~A~~ $\in$ the line $X(\lambda)$

camera center

~~$\therefore$ A arbitrary~~

$\therefore$ camera center $\in X_A(\lambda), \forall A$

$\therefore C \in X_A(\lambda), \forall A$

$\therefore C =$ camera center.

proof 2:

$\because PC = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not defined,

又 $\because$ the camera center is the unique point in space for which the image is undefined.

$\therefore$ C is camera center.

12.4. finite camera

$\Rightarrow C = \begin{pmatrix} -M^{-1}p_4 \\ 1 \end{pmatrix}$, where $P = (M, p_4)$

11. general projective cameras

11.2 A general projective camera is one represented by an arbitrary homogeneous 3×4 matrix of rank 3.

11.3 It has 11 degrees of freedom.

12. $PC = 0$, P is a general projective camera $\Rightarrow$ C is camera center.

proof 1:

(Note C must exist).

$\forall A \in$ 3d space P^3.

The line containing C and A is represented by the join $X(\lambda) = \lambda A + (1-\lambda) C$.

$\because PX(\lambda) = \lambda PA$.

$\therefore$ all points on the line are mapped to the same point PA

10.4 The set of camera matrices of finite projective cameras is identical with the set of homogeneous 3×4 matrices for which ~~the~~ the left hand 3×3 submatrix is non-singular.

proof: $\Rightarrow \because P = KR(I \; -\tilde{C})$.

$= (KR \; -KR\tilde{C})$.

$\therefore P = (M, p_4)$, M non-singular.

$\Leftarrow$ let $P = (M, p_4)$, M non-singular.

$\because M = KR$, (RQ matrix decomposition)

$(\exists K, R$, s.t. K has form 10, R rotation)

$\therefore P = M(I \; M^{-1}p_4) = KR(I \; -\tilde{C})$

with $\tilde{C} = -M^{-1}p_4$

12.5 camera at infinity (M is singular)

$\Rightarrow C = (d^T, 0)^T$, where $Md = 0$.

C is a point at infinity.

13. P, a general projective camera

$P = (p_1, p_2, p_3, p_4)$

$\Rightarrow$ p_i are the vanishing points of the world coordinate X, Y and Z axes respectively. ($i = 1, 2, 3$)

($\because p_1 = P\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$)

p_4 is the image of the world origin.

14. P, a general projective camera,

$P = \begin{pmatrix} P_1^T \\ P_2^T \\ P_3^T \end{pmatrix}$

$\Rightarrow$ P_3 is the principal plane.

P_1, P_2 represent planes in space through the camera center, corresponding to points that map to the image lines $x=0$ and $y=0$ respectively.

proof: ① a point $X \in$ principle plane

$\Leftrightarrow PX = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$\Leftrightarrow P_3^T X = 0$

$\therefore P_3$ is the vector representing the principal plane.

②. $P_1^T X = 0 \Rightarrow PX = \begin{pmatrix} 0 \\ y \\ w \end{pmatrix}$

$\because PC = 0 \therefore P_1^T C = 0 \therefore C \in P_1$

$\therefore P_1$ is defined by the camera center and the line $x=0$ in the image. ③ Similarly for P_2.

15. the image point $x_0 = Mm_3$ is the principal point of the camera, where $M = \begin{pmatrix} m_1^T \\ m_2^T \\ m_3^T \end{pmatrix}$

proof: $\because$ the principal plane is $P_3 = (p_{31}, p_{32}, p_{33}, p_{34})^T$

$\therefore$ the normal to it is $(p_{31}, p_{32}, p_{33})^T$

$\therefore$ The normal is represented by a point on the plane at infinity, $(p_{31}, p_{32}, p_{33}, 0)^T = \hat{P}_3$

$\therefore$ principal point $x_0 = P\hat{P}_3 = (M \; p_4)\hat{P}_3 = Mm_3$.

16. the principal axis is the ray passing through the camera center C with direction vector m_3^T.

The principal axis vector $v = \det(M)\, m_3$ is directed towards the front of the camera.

(see 8.6)

17. $x \to$ ray, s.t. the ray maps to x.

The form of the ray may be specified in several ways, (Plücker representation, join of two points).

17.3. $X(\lambda) = P^+ x + \lambda C$, where P^+ is pseudo-inverse of P. ($P^+ = P^T(PP^T)^{-1}$)

17.4 finite camera: $X(\mu) = \mu\begin{pmatrix} M^{-1}x \\ 0 \end{pmatrix} + \begin{pmatrix} -M^{-1}p_4 \\ 1 \end{pmatrix}$

18. depth of points

$X=(X,Y,Z,T)^T$ 3D point,

$P=(M \; p_4)$ is camera matrix for a finite camera

$x=PX$, $x=w(x,y,1)^T$

$\Rightarrow$ the depth of the point X in front of the principal plane is

$$depth(X;P)=\frac{sign(\det M)\,w}{T\|m_3\|}$$

proof:

$$w=P_3^T X = T P_3^T\begin{pmatrix}\tilde{X}\\1\end{pmatrix} = T P_3^T\left(\begin{pmatrix}\tilde{X}\\1\end{pmatrix}-\begin{pmatrix}\tilde{C}\\1\end{pmatrix}\right)$$

$(\because PC=0)$

$\therefore w = T m_3^T(\tilde{X}-\tilde{C})$

$\therefore$

$$\frac{w}{T\|m_3\|}=\frac{m_3^T(\tilde{X}-\tilde{C})}{\|m_3\|}$$

$\therefore depth(X;P)$

$$=\frac{sign(\det M)\, m_3^T(\tilde{X}-\tilde{C})}{\|m_3\|} \quad (\text{see } 16)$$

$$=\frac{sign(\det M)\,w}{T\|m_3\|}$$

where $M=\begin{pmatrix}m_1^T\\m_2^T\\m_3^T\end{pmatrix}$, proved.

18.3

$depth(kX;P)$

$$=\frac{sign(\det M)(kw)}{(kT)\|m_3\|}$$

$= depth(X;P)$

$depth(X;kP)$

$$=\frac{sign(\det M)\cdot sign(k)\,(kw)}{T\|m_3\|\cdot|k|}=depth(X;P)$$

$\because$ the $depth(X;P)$ is independent of the particular homogeneous representation of X and P.

Chapter 8 HZ

More Single View Geometry

1. Chapter 6 introduced the projection matrix as the model for the action of a camera on points.
This chapter describes the link between other 3D entities and their images under perspective projection. These entities include planes, lines, conics and quadrics.

2. The most general trans that can occur between a scene plane and an image plane under perspective imaging is a plane projective transformation.

proof: (assume that the camera center does not lie on the scene plane).
Choose the world coordinate frame such that the XY-plane corresponds to a plane π in the scene, so that points on the scene plane have zero Z-coordinate.

$$\because x = PX$$
$$= (p_1, p_2, p_3, p_4)\begin{pmatrix} X \\ Y \\ 0 \\ 1 \end{pmatrix}$$
$$= (p_1, p_2, p_4)\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}$$
$$= H x_\pi.$$

where $x_\pi = \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}$

2.4. If the camera is affine, (i.e, $P_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$).
$\Rightarrow$ the scene and image planes are related by an affine trans.

2.5. $P = K(R\ t) \Rightarrow H = K(r_1, r_2, t)$.
where $R = (r_1, r_2, r_3)$.

3. A line in 3-space projects to a line in the image.

proof: $\because X(\mu) = A + \mu B$

$\therefore x(\mu) = P(X(\mu)) = PA + \mu PB$
$= a + \mu b$

4. The set of points in space mapping to a line l via the camera matrix P is the plane $P^T l$.

proof: $x = PX,\ x \in l$
$\Leftrightarrow x^T l = 0,\ x = PX$
$\Leftrightarrow X^T P^T l = 0,\ x = PX$
$\Leftrightarrow X^T \pi = 0,\ \pi = P^T l,\ x = PX.$
$\Leftrightarrow X \in \pi,\ \pi = P^T l,\ x = PX.$

5. camera P, line L represented as a ~~Pl~~ Plücker matrix, l is its image.
$\Rightarrow l_\times = PLP^T$

proof: $a = PA,\ b = PB,\quad a, b \in l,\ A, B \in L.$

let $M = PLP^T$, $\therefore M = P(AB^T - BA^T)P^T$
$= ab^T - ba^T$
$= (a \times b)_\times$ ($\because \langle a \times b \rangle$ nullspace)
$= l_\times$ ($\because l = a \times b$)

5.4 ① The relation between l_i and L_{jk} is linear.
② This relation is quadratic in the elements of P.

9.4 extended cross-product.

10. line L, $P_\perp L=0$

$\Rightarrow$ camera center $\in L$.

proof: $\because$ $P_\perp L=0$

$\therefore (P_2\wedge P_3 | L)=0$ $\therefore$ $P_2\wedge P_3$, L intersect.

Similarly, $P_3\wedge P_1$, L intersect, $P_1\wedge P_2$, L intersect.

$\because$ camera center $= P_1\cap P_2\cap P_3$

$\therefore$ center $\in L$. (ε).

11. A cone is a degenerate quadric.

12. Under the camera P the conic C back-projects to the cone $Q_{co}=P^TCP$

proof: $x\in C \Leftrightarrow x^TCx=0$

$\Leftrightarrow X^TP^TCPX=0$

$\Leftrightarrow X^TQ_{co}X=0$, $Q_{co}=P^TCP$

$\Leftrightarrow X\in Q_{co}$, $Q_{co}=P^TCP$

12.4 $Q_{co}C=P^TCPC=0 \Rightarrow C$ is the vertex of the cone. ε

here C is camera center.

12.5 $P=K(I\ 0) \Rightarrow Q_{co}=\begin{pmatrix} K^TCK & 0 \\ 0^T & 0\end{pmatrix}$

proof:

$l=a\times b=(PA)\times(PB)$

$\therefore l_1=\begin{vmatrix} P_2^TA & P_2^TB \\ P_3^TA & P_3^TB\end{vmatrix}$

$=(P_2\wedge P_3 | L)$ (ε) (3.14 on p73)

The other components follow in a similar manner.

9. ① $a\times b := \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3\end{vmatrix} = a_\times b$

$i=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $j=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $k=\begin{pmatrix}0\\0\\1\end{pmatrix}$.

② $a\times b\times c := \begin{vmatrix} i & j & k & l \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4\end{vmatrix}$

$=(-1)^{n+1}(a,b)_\times c$, $n=4$.

③ $a\times b\times c\times d := \begin{vmatrix} i & j & k & l & m \\ a_1 & \cdots & & & a_5 \\ \vdots & & & & \vdots \\ d_1 & \cdots & & & d_5\end{vmatrix}$

$=(-1)^{n+1}(a,b,c)_\times d$, $n=5$

6. the map between the Plücker line coordinates L and the image line coordinates is represented by a matrix.

7. line projection matrix $\mathcal{P}$ is defined:

$\mathcal{P}:=\begin{pmatrix} P_2\wedge P_3 \\ P_3\wedge P_1 \\ P_1\wedge P_2\end{pmatrix}$, (we use $P_\perp$)

where $P=\begin{pmatrix} P_1^T \\ P_2^T \\ P_3^T\end{pmatrix}$,

$P_i\wedge P_j$ the Plücker line coordinates of the intersection of planes P_i and P_j.

7.4 $\operatorname{rank}(P_\perp)=3$. ε

8. $P_\perp$ line projection matrix,

L line represented by Plücker line coor.

$\Rightarrow$ the image line $l=P_\perp L$ (not right strictly)

$=\begin{pmatrix} (P_2\wedge P_3|L) \\ (P_3\wedge P_1|L) \\ (P_1\wedge P_2|L)\end{pmatrix}$

where $(L|\hat{L})$ is defined in in C3.

13. The camera calibration matrix K is the (affine) trans between x and the ray's direction $d = K^{-1}x$ measured in the camera's Euclidean coordinate frame.

proof: ∵ ray : $\tilde{X} = \lambda d$

∴ $x = K(I\ 0)\begin{pmatrix}\lambda d\\ 1\end{pmatrix} = Kd$ (up to scale)

∴ $d = K^{-1}x$

14. θ, the angle between two rays, with directions d_1, d_2 corresponding to image points x_1, x_2 respectively,

$$\Rightarrow \cos\theta = \frac{d_1^T d_2}{\|d_1\|\|d_2\|} = \frac{x_1^T M x_2}{\sqrt{x_1^T M x_1}\sqrt{x_2^T M x_2}}$$

$$M = K^{-T}K^{-1}$$

14.3 ∴ if K is known, then the angle between rays can be measured from their corresponding image points.

15. An image line l defines a plane through the camera center with normal direction $n = K^T l$ measured in the camera's Euclidean coordinate frame. (Note: $\|n\| \neq 1$ in general).

proof: ∵ $d = K^{-1}x$, $d^T n = 0$

∴ $x^T K^{-T} n = 0$.

∴ $l = K^{-T} n$

∴ $n = K^T l$.

16. the mapping between π_∞ and an image is given by the planar homography $x = Hd$ with $H = KR$.

proof: ∵ $X_\infty = \begin{pmatrix} d\\ 0\end{pmatrix} \in \pi_\infty$

∴ $x = PX_\infty = KR(I\ -\tilde{C})\begin{pmatrix} d\\ 0\end{pmatrix}$

$= KRd$

$= Hd$

17. the image of absolute conic (IAC) is the conic $\omega = (KK^T)^{-1} = K^{-T}K^{-1}$

proof: ∵ $x \to Hx$

∴ $C \to H^{-T}CH^{-1}$, C is conic

∵ $C = \Omega_\infty = I$ on π_∞, 且 ∵ $H = KR$

∴ IAC $\omega = (KR)^{-T} I (KR)^{-1}$

$= (KK^T)^{-1}$

17.4 Like Ω_∞ the conic ω is an imaginary point conic with no real points.

17.5 conic ω may be thought of as a convenient algebraic device.

18. the angle θ between two rays:

$$\cos\theta = \frac{x_1^T \omega x_2}{\sqrt{x_1^T \omega x_1}\sqrt{x_2^T \omega x_2}}$$

18.3. It is unchanged under projective trans of the image.

19. dual image of absolute conic (DIAC)

$\omega^* = \omega^{-1} = KK^T$,

19.3 $\omega^* = PQ_\infty^* P^T$

19.4 $\omega \Rightarrow K$ (Cholesky factorization)

20 A point x and line l back-projecting to a ray and plane respectively that are orthogonal $\Rightarrow l = \omega x$

Chapter 9 HZ

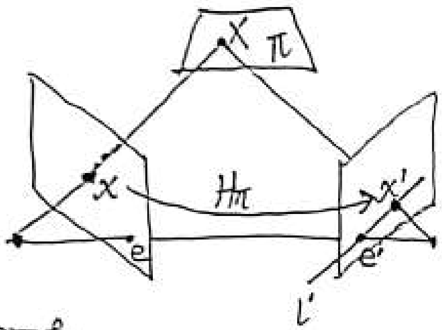
Epipolar geometry and fundamental matrix

1. epipole, epipolar line, epipolar plane, epipolar pencil, baseline, converging cameras.

2. the fundamental matrix is the algebraic representation of epipolar geometry.

3. There is a map $x \to l'$ from a point in one image to its corresponding epipolar line in the other image. ✓

3.4. This mapping is a singular correlation, which is represented by a matrix F, the fundamental matrix.



proof: ✓

$\because$ 1st image plane is projectively equivalent to π.

2nd image plane $\overset{\text{projectively}}{\Longleftrightarrow}$ π (19 in C2)

$\therefore$ 1st image plane $\overset{\text{projectively}}{\Longleftrightarrow}$ 2nd image plane.

$\therefore \exists$ ~~?~~ 2D homography H_π mapping each $x \in$ 1st image plane to $x' \in$ 2nd

$\therefore l' = e' \times x' = e'_x x' = e'_x H_\pi x$

(l' epipolar line)

$\therefore F = e'_x H_\pi$.

$\because r(e'_x) = 2$, $r(H_\pi) = 3$ $\therefore r(F) = 2$

5.4 Note: a plane π is not required in order for F to exist. The plane π is simply used as a means of defining a point map from one image to another. ✓

4. ~~Proof~~. $\exists$ some plane π, s.t. $\{P^+x \mid x \in \text{image plane}\} = \pi$

proof: $\forall\ x_1, x_2, x_3$ different, $\in$ image plane

$\Rightarrow X_1 = P^+x_1,\ X_2 = P^+x_2,\ X_3 = P^+x_3$

X_1, X_2, X_3 different.

denote $\pi = \langle X_1, X_2, X_3 \rangle$

$\forall\ x \in$ image plane,

$\exists\ \lambda_1, \lambda_2, \lambda_3$, s.t. $x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$

$\therefore X = P^+x = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$

$\therefore X \in \pi$

5. $F = [e']_x H_\pi$, where H_π is the transfer mapping from one image to another via any plane π.

rank(F) = 2.

(here π not passing through either of the two camera centers)

11. $P=K(I\ 0)$, $P'=K'(R\ t)$

(here world origin at the first camera)

$\Rightarrow P^+=P^T(PP^T)^{-1}=\begin{pmatrix}K^{-1}\\0^T\end{pmatrix}$

$C=\begin{pmatrix}0\\1\end{pmatrix}$

$e=P\begin{pmatrix}-R^Tt\\1\end{pmatrix}=KR^Tt$ (up to scale)

$e'=P'\begin{pmatrix}0\\1\end{pmatrix}=K't$ ∴

$F=(P'C)_\times P'P^+$

$=(K't)_\times K'RK^{-1}=K'^{-T}t_\times RK^{-1}$

$=K'^{-T}R(R^Tt)_\times K^{-1}=K'^{-T}RK^T(KR^Tt)_\times$

$F=e'_\times K'RK^{-1}=K'^{-T}RK^Te_\times$

12. $\forall$ pair of corresponding points $x\leftrightarrow x'$ in two images $\Rightarrow x'^TFx=0$ ∴

12.4 If image points satisfy $x'^TFx=0$ then the rays defined by these points are coplanar. ∴

7. $\forall$ t vector, M invertible

$\Rightarrow t_\times M=M^*(M^{-1}t)_\times=M^{-T}(M^{-1}t)_\times$ (up to scale)

proof: let $x=M^{-1}t$, ∴ $t=Mx$

$t_\times M=(Mx)_\times M$

$\| $ (from 6.3)

$M^*(M^{-1}t)_\times=M^*x_\times$

$\|$

$M^{-T}(M^{-1}t)_\times=M^{-T}x_\times$

(∵ M invertible $\Rightarrow M^*=\det(M)M^{-T}$)

8 $[a]_\times=a_\times a_\times a_\times$ (up to scale)

or $a_\times^3=-\|a\|^2a_\times$

9 $F=e'_\times M$ is fundamental matrix

$\Rightarrow e'_\times e'_\times F=F$ (up to scale)

∴ $F=e'_\times M$, where $M=e'_\times F$

10. $P=(I\ 0)$, $P'=(M\ t)$

$\Rightarrow F=t_\times M$ ∴

(see 11)

5.5 given two cameras P,P', camera centers C,C', $C\neq C'$.

$\Rightarrow F=[e']_\times P'P^+$

where $e'=P'C$, $PC=0$, $P'C'=0$

proof: ∵ $\pi=\{P^+x\}$, $x\in$ image plane ∴ is a plane

∴ $H_\pi=P'P^+$

∴ $l'=e'_\times(H_\pi x)=e'_\times P'P^+x$

∴ $F=e'_\times P'P^+$

here $e'=P'C$, C is solved from $PC=0$.

P^+ is pseudo-inverse of P. (17.3 in C6)

(∵ $C\neq C'$, ∴ $e'\neq0$)

6. $\forall M\in3\times3$, x,y vectors

$\Rightarrow (Mx)\times(My)=M^*(x\times y)$

M^* is cofactor matrix of M.

6.3 ∴ $(Mx)_\times M=M^*x_\times$

15. ~~若~~ F is fundamental matrix of the pair of cameras (P, P')

$\Rightarrow F^T$ is the fundamental matrix of the pair of cameras (P', P)

proof:

method 1: $\because F = e'_\times H_\pi$

$= (H_\pi e)_\times H_\pi$

$= H_\pi^* e_\times$

$= H_\pi^{-T} e_\times$

$= (e_\times^T H_\pi^{-1})^T$

$= (-e_\times^* H_\pi^{-1})^T$

$= (e_\times H_\pi^{-1})^T$

method 2:

$\because x'^T F x = 0$

$\Leftrightarrow x^T F^T x' = 0$

from 13, proved.

14. ~~two imag~~

given two cameras P, P' with non-coincident centers

$\Rightarrow$ fundamental matrix exists and is unique.

proof: ① $F\ \exists$.

② 若 $\exists F_2 \neq F$.

$\forall x_1, x_2, x_3,\ x_1 \neq x_2 \neq x_3$

$\therefore l'_1 = Fx_1,\ l'_2 = Fx_2,\ l'_3 = Fx_3$

$\lambda_1 l'_1 = F_2 x_1,\ \lambda_2 l'_2 = F_2 x_2,\ \lambda_3 l'_3 = F_2 x_3$

③ (若 $l'_i \neq l'_j$, $i \neq j$

$\Rightarrow l'_i + l'_j = F(x_1 + x_2)$

$k(l'_i + l'_j) = F_2(x_1 + x_2) = \lambda_i l'_i + \lambda_j l'_j$

$\therefore \lambda_i = \lambda_j = k$)

④ 互 $\because \text{rank}(l'_1\ l'_2\ l'_3) = 2$

$\therefore \lambda_1 = \lambda_2 = \lambda_3$ (use ③)

⑤ let $X = (x_1, x_2, x_3)$, $L = (l_1, l_2, l_3)$ $\therefore FX = L,\ F_2 X = \lambda L$

$\therefore F = LX^{-1},\ F_2 = \lambda L X^{-1},\ \therefore F = F_2$ proved. **3**

13. two images acquired by cameras with non-coincident centers,

$\Rightarrow$ fundamental matrix F is the unique 3×3 rank 2 homogeneous matrix which satisfies $x'^T F x = 0$ for all corresponding points $x \leftrightarrow x'$.

proof: ① $F\ \exists$.

②. 若 $\exists F_2 \neq F$,

$\because x'^T F_2 x = 0,\ \forall x \leftrightarrow x'$

$\therefore Fx = F_2 x,\ \forall x \in$ 1st image plane

($\because$ 若 ~~若~~ $\exists x$, s.t. $l' = Fx,\ l'_2 = F_2 x$, $l' \neq l'_2$

$\because \forall x' \in l',\ \Rightarrow 0 = x'^T l' = x'^T F x = x'^T F_2 x$

$= x'^T l'_2$

$\therefore l' = l'_2$ error)

$\therefore F = F_2$ (from 14)

$\therefore$ error. proved.

16. $\cancel{d}$of(F) = 7

17 ① $l' = Fx$ is the epipolar line corresponding to x.

② $l = F^T x'$ is the epipolar line corresponding to x'.

③ $Fe = 0$, $F^T e' = 0$.

18. canonical cameras : (see 10)

$P = (I\ 0)$, $P' = (M\ t)$

$\Rightarrow F = e'_x M = M^{-T} e_x$

where $e' = t$, $e = M^{-1} t$

19. central projection :

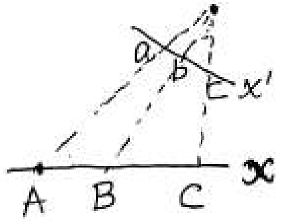
a ray from a point in space is drawn from a 3D world point through a fixed point in space, the center of projection.

20. One special case of a projectivity is a perspectivity, the distinctive ~~of~~ property of which is that lines joining corresponding points are concurrent. (24 in C2)

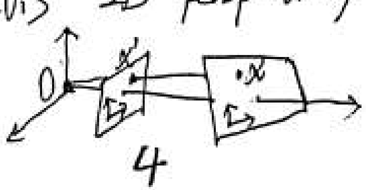
20.3 The composition of two perspectivities is a projectivity, but not, in general, a perspectivity.

20.4

1D perspectivity on the plane



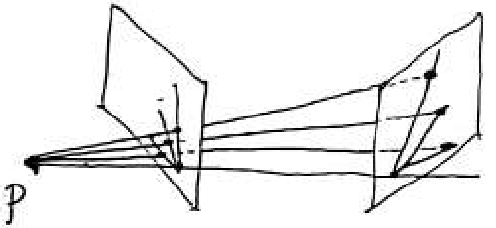
20.5 2D perspectivity



21. The correspondence between epipolar lines, $l_i \leftrightarrow l'_i$, is defined by the pencil of planes with axis the baseline.

21.3 The corresponding lines are related by a perspectivity with center any point p on the baseline.

21.4 The correspondence between epipolar lines in the pencils is a 1D homography.



22. The dof of the fundamental matrix can be counted as follows : 2 for e, 2 for e', and 3 for the epipolar line homography which maps a line through e to a line through e'.

23. l, l' are corresponding epipolar lines, k is any line not passing through the epipole e, $\Rightarrow l' = F k_x l$, $l = F^T k'_x l'$

23.3 let $k = e$, $\Rightarrow l' = F e_x l$, $l = F^T e'_x l'$.

24. pure translation, pure planar motion

The "pure" indicates that there is no change in the internal parameters.

25. In considering pure translations of the camera, one may consider the equivalent situation in which the camera is stationary, and the world undergoes a translation $-t$.

26. $P=K(I\ 0)$, $P'=K(I\ t)$ (pure translation)

$\Rightarrow F=[e']_\times$ (see 11)

$e=e'$, $dof(F)=2$

$\Rightarrow$ $x, x', e=e'$ are collinear if both images are overlaid on top of each other.

$(\because x^T l' = x'^T F x = x^T [e']_\times x = 0$

$\therefore x \in l')$

The collinearity property is termed auto-epipolar.

26.4 If the world undergoes a translation $-t$, the camera is stationary (25),

$\Rightarrow$ points in 3d space move on straight lines parallel to t, and the imaged intersection of these parallel lines is the vanishing point v.

26.5 $v=e=e'$

proof: $v=P\begin{pmatrix} t \\ 0 \end{pmatrix}$

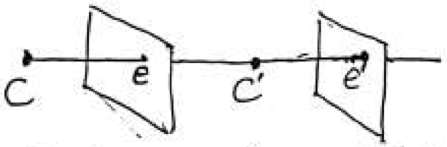
$=Kt$

$\therefore v=e=e'$

26.6 $\because$ imaged parallel lines in 26.4 are the epipolar lines.

26.7 under the motion the epipole is a fixed point, i.e., has the same coordinates in both images, and points appear to move along lines radiating from the epipole

The epipole in this case is termed the Focus of Expansion (FOE).



26.8 In the case of pure translation, ~~If the camera translation is parallel to the axis, then~~ points closer to the camera appear to move faster than those further away. — a common experience when looking out of a train window.

proof: ① $x=\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$, $X=\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$, $x=PX$

$\Rightarrow ZK^{-1}x=\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}=\tilde{X}$

② $\because Z=depth(X;P)$ (18 in C6)

③ $x'=P'X \Rightarrow x'=x+Kt/Z$ (inverse depth Z)

proved.

27. General camera motion.

The first camera may be rotated and corrected to simulate a pure translational motion.

$\Rightarrow F = e'_x H$,

where F is the fundamental matrix for the original pair; e'_x is the fundamental matrix of the translation, and H is the projective transformation corresponding to the correction of the first camera.

proof: $\because \hat{F} = e'_x$, $x'^T \hat{F} \hat{x} = 0$, where $\hat{x} = Hx$ is the corrected point in the first image.

$\therefore x'^T \hat{F} H x = 0$, $\therefore x'^T e'_x H x = 0$

$\therefore F = e'_x H$.

28. $P = K(I\ 0)$, $P' = K'(R\ \cancel{0}\ t)$

$\Rightarrow F = e'_x H$, $H = K'RK^{-1}$

proof: ① $\because x = PX$

$\hat{x}' = \hat{P}'X$, $\hat{P}' = K'(R\ 0)$

$\therefore \hat{x}' = K'(R\ 0)X$

$= K'R(I\ 0)X$

$= K'RK^{-1}K(I\ 0)X$

$= K'RK^{-1}PX$

$= K'RK^{-1}x$

$\therefore H = K'RK^{-1}$

② $\because F = e'_x H$ (27)

29. $P = K(I\ 0)$, $P' = K'(R\ t)$

$\Rightarrow x' = K'RK^{-1}x + K't/Z$

where $x = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$, Z from $X = \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$

proof: $\because X = \begin{pmatrix} ZK^{-1}x \\ 1 \end{pmatrix}$,

$\therefore x' = P'X = ZK'RK^{-1}x + K't$

$= K'RK^{-1}x + \frac{K't}{Z}$ (vs 26.8)

29.3 the first term takes account of the camera rotation and change of internal parameters; the second term takes account of camera translation.

30. geometric representation of the fundamental matrix.

31. The relationships $l' = Fx$ and $x'^T F x = 0$ are projective relationships. In other words they are projectively invariant.

31.2 $\hat{x} = Hx$, $\hat{x}' = H'x' \Rightarrow \hat{l}' = \hat{F}\hat{x}$, with $\hat{F} = H'^{-T} F H^{-1}$

~~⇒~~ the corresponding fundamental matrix.

here H, H' projectivity.

32. H projective $\Rightarrow$ the fundamental matrices corresponding to the pairs of camera matrices (P, P') and $(PH, P'H)$ are same.

proof: $\because x = PX$, $x' = P'X$,

$\therefore x = (PH)(H^{-1}X)$, $x' = (P'H)(H^{-1}X)$

x, x' still matched.

from B, proved.

33. $(P,P') \Rightarrow \exists H$ invertible, ~~projective~~, s.t. $(PH, P'H)$ canonical form, i.e., $PH = (I\ 0)$

proof:

let $P = \begin{pmatrix} P_1^T \\ P_2^T \\ P_3^T \end{pmatrix}$, $\because r(P) = 3$,

$\therefore \exists P_4$, s.t. $\hat{P} = \begin{pmatrix} P_1^T \\ P_2^T \\ P_3^T \\ P_4^T \end{pmatrix}$, $r(\hat{P}) = 4$.

$\because \hat{P}\hat{P}^{-1} = \begin{pmatrix} I & \\ & 1 \end{pmatrix} \in 4\times 4$

$\therefore P\hat{P}^{-1} = (I\ 0) \in 3\times 4$

$\therefore H = \hat{P}^{-1}$

34. matrix F, $r(F) = 2$, $F = a_\times A$, $F = \tilde{a}_\times \tilde{A}$

$\Rightarrow \exists k \neq 0, v$, s.t. $\tilde{a} = ka$, $\tilde{A} = k^{-1}(A + av^T)$

proof: ① $\because a^T F = a^T a_\times A = 0$,

$\tilde{a}^T F = \tilde{a}^T \tilde{a}_\times \tilde{A} = 0$

$r(F) = 2$

$\therefore \exists k \neq 0$, s.t. $\tilde{a} = ka$

② $\because a_\times A = \tilde{a}_\times \tilde{A} = k a_\times \tilde{A}$

$\therefore a_\times (k\tilde{A} - A) = 0$ $\therefore k\tilde{A} - A = av^T$ for some v.

$\therefore \tilde{A} = k^{-1}(A + av^T)$

35. F fundamental matrix corresponding to each of the pairs (P, P'), $(\tilde{P}, \tilde{P}')$

$\Rightarrow \exists H$ invertible, s.t. $\tilde{P} = PH$, $\tilde{P}' = P'H$

proof: ① suppose each of the two pairs is in canonical form.

$\therefore P = \tilde{P} = (I\ 0)$

$P' = (A\ a)$, $\tilde{P}' = (\tilde{A}\ \tilde{a})$

$\therefore F = a_\times A = \tilde{a}_\times \tilde{A}$ (see 10)

$\therefore P' = (A\ a)$,

$\tilde{P}' = (k^{-1}(A + av^T).\ ka)$

(from 34)

let $H = \begin{pmatrix} k^{-1} I & 0 \\ k^{-1} v^T & k \end{pmatrix}$

$\Rightarrow PH = k^{-1}\tilde{P}$, $P'H = \tilde{P}'$

$\therefore (P, P')$, $(\tilde{P}, \tilde{P}')$ projectively related.

② if $(PH_1, P'H_1)$, $(\tilde{P}H_2, \tilde{P}'H_2)$

H_1, H_2 invertible, 且

and (P, P'), $(\tilde{P}, \tilde{P}')$ canonical,

$\therefore \exists \hat{H}$ invertible, s.t. $\tilde{P} = P\hat{H}$, $\tilde{P}' = P'\hat{H}$.

$\therefore \exists H = H_1^{-1}\hat{H}H_2$ invertible,

$(\tilde{P}H_2) = P\hat{H}H_2 = (PH_1)(H_1^{-1}\hat{H}H_2) = (PH_1)H$

$(\tilde{P}'H_2) = P'\hat{H}H_2 = (P'H_1)(H_1^{-1}\hat{H}H_2) = (P'H_1)H$

proved.

35.4. $\because$ the fundamental matrix captures the projective relationship of two cameras.

35.5 ① A pair of camera matrices determines a unique fundamental matrix.

② A given fundamental matrix determines the pair of camera matrices up to right multiplication by a projective transformation.

35.6 $dof(P, P') = 22$, dof(projective world frame) $= 15$

dof(fundamental matrix) $= 7$. $22 = 15 + 7$

36. A real symmetric, $x^TAx=0, \forall x$,

$\Rightarrow A=0$

proof: $\because A=UDU^T$, U orthogonal,

D real diagonal,

$\therefore y^TDy=0, \forall y$.

$\therefore D=0$ $\therefore A=0$

37. A skew-symmetric $\Leftrightarrow x^TAx=0, \forall x$.

proof: ① $\Rightarrow$ let $y=x^TAx$

$\therefore 2y=y+y^T=x^TAx+x^TA^Tx$

$\therefore 2y=x^T(A+A^T)x=0$ $\therefore y=0$

② $\Leftarrow$ $x^TAx=0, \forall x$.

$\therefore x^TBx=0, \forall x$, $B=A+A^T$

$\therefore B=0$ (36)

$\therefore A=-A^T$

38. A non-zero matrix F is the fundamental matrix corresponding to a pair of camera matrices P, P'

$\Leftrightarrow P'^TFP$ skew-symmetric

proof: P'^TFP skew-symmetric

$\Leftrightarrow X^TP'^TFPX=0, \forall X$

$\Leftrightarrow x'^TFx=0, \forall x\leftrightarrow x'$,

here $x=PX$, $x'=P'X$.

$\Leftrightarrow F$ fundamental matrix

39. in R^3, e_1, e_2, e_3, $\dim\langle e_1,e_2,e_3\rangle=3$

$\Rightarrow \dim\langle e_3\times e_1, e_3\times e_2, e_3\rangle=3$

proof: If $e_3\times e_1=\lambda e_3\times e_2$, $\lambda\neq0$

$\Rightarrow [e_3]_\times(e_1-\lambda e_2)=0$

$\Rightarrow \exists k\neq0$, s.t. $e_1-\lambda e_2=ke_3$

($\because r([e_3]_\times)=2$)

$\Rightarrow \dim\langle e_1,e_2,e_3\rangle=2$, error

$\therefore$ proved.

40 F matrix, $r(F)=2$, S skew-symmetric, $P=(I\ 0)$, $P'=(SF\ e')$, $e'^TF=0$, if $r(P')=3$, i.e., it is a valid camera,

$\Rightarrow F$ is the fundamental matrix corresponding to the pair (P, P').

proof: $(SF\ e')^TF(I\ 0)=\begin{pmatrix}F^TS^TF & 0\\ e'^TF & 0\end{pmatrix}$

which is skew-symmetric.

$\therefore$ proved. (from 38)

40.3 e' is epipole of P'.

proof: $P'C=P'\binom{0}{1}=e'$.

40.4 $r(SF)=2$.

proof: $\because r(S)=2$ (27 in C3)

$r(F)=2$

$\therefore r(SF)\leq2$.

$\because r(SF\ e')=3$, $\therefore r(SF)\geq2$. $\therefore r(SF)=2$

40.5 if $s^Te'\neq0$, then $r(S_\times F\ e')=3$

proof: $\because e'^TF=0$, $s^Te'\neq0 \Rightarrow s\notin\langle F\rangle$

$\Rightarrow \langle S_\times F\rangle=2$ (from 39) $\Rightarrow e'\notin\langle S_\times F\rangle$ ($\because s^Te'\neq0$)

41. The general formular for a pair of canonic camera matrices corresponding to a fundamental matrix F is given by $P=(I\ 0)$,

$P'=([e']_\times F+e'v^T\ \ \lambda e')$, where v is any 3-vector, $\lambda\neq0$.

$\because$ (from 35, 40)

42. The essential matrix is the specialization of the fundamental matrix to the case of normalized image coordinate. The fundamental matrix may be thought of as the generalization of the essential matrix in which the (inessential) assumption of calibrated cameras is removed.

42.3 (calibration matrix K)

42.4 $P = K(R\ t)$, $x = PX$,

① $\hat{x} = K^{-1}x$ is the image point expressed in normalized coordinates.

② $K^{-1}P = (R\ t)$ is called a normalized camera matrix.

43. $P = (I\ 0)$, $P' = (R\ t)$, the fundamental matrix corresponding to the pair of normalized cameras is called essential matrix.

43.3 $\because$ $E = t_\times R = R(R^T t)_\times$

44. the relationship between the fundamental and essential matrices is $E = K'^T F K$

proof: $\because$ $\hat{x}'^T E \hat{x} = 0$

$\forall$ normalized image coordinates correspondence $\hat{x} \leftrightarrow \hat{x}'$

$\Leftrightarrow$ $x'^T K'^{-T} E K^{-1} x = 0$

$\forall$ $x \leftrightarrow x'$

$\Leftrightarrow$ $F = K'^{-T} E K^{-1}$

$\Leftrightarrow$ $E = K'^T F K$

45. $dof(E = t_\times R) = 5$

46. SVD: $A = UDV^T$, $A: m\times n$, $U: m\times n$, $D: n\times n$, $V: n\times n$. $m \geq n$. U is a matrix with orthogonal columns, D diagonal, V orthogonal.

46.3. the singular values of A are the square-roots of the eigenvalues of $A^T A$.

proof: $A^T A = VD^2V^T = VD^2V^{-1}$

46.4 singular values are real and non-negative.

47. matrix E is essential matrix

$\Leftrightarrow$ two of its singular values are equal, and the third is zero.

proof: E essential matrix $\Leftrightarrow E = t_\times R$

$\Leftrightarrow E = SR$, S skew-symmetric

$\Leftrightarrow E = SR$, $S = kUZU^T$, U orthogonal, $Z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Leftrightarrow E = SR$, $S = U\mathrm{diag}(1,1,0)WU^T$, $W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Leftrightarrow E = U\mathrm{diag}(1,1,0)V^T$, $V^T = WU^T R$

47.3 $E = (U\mathrm{diag}(R_{2\times2}, 1))\,\mathrm{diag}(1,1,0)\,(\mathrm{diag}(R_{2\times2}^T, 1))V^T$

R is any 2x2 rotation matrix.

48. the SVD of E is $U\mathrm{diag}(1,1,0)V^T$, $E = SR$,

$\Rightarrow S = UZU^T$, $R = UWV^T$ or $R = UW^TV^T$

48.3 $\because St = 0$. $\therefore t = U\begin{pmatrix}0\\0\\1\end{pmatrix}$ $\therefore t = u_3$, $U = (u_1, u_2, u_3)$

49. $E = U\mathrm{diag}(1,1,0)V^T$, $P = (I\ 0)$,

$\Rightarrow P'$ has 4 possible choices.

$P' = (UWV^T\ u_3)$ or $P' = (UWV^T\ -u_3)$

or $P' = (UW^TV^T\ u_3)$ or $P' = (UW^TV^T\ -u_3)$

Chapter 10.
3D reconstruction of cameras and structure.

The reconstruction task is to find the camera matrices P and P', as well as the 3D points X_i such that $x_i = PX_i$, $x_i' = P'X_i$, $\forall i$.

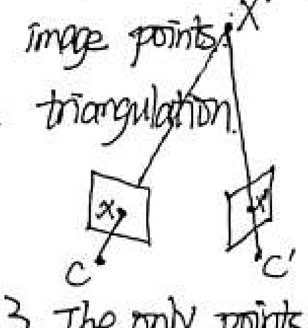
2. outline of reconstruction method

(i) compute the fundamental matrix from point correspondences.

(ii) $F \Rightarrow P, P'$

(iii) for each point correspondence $x_i \leftrightarrow x_i'$, compute the point X_i in space that projects to these two image points.

3. triangulation.



3.3 The only points in 3-space that cannot be determined from their images are points on the baseline between two cameras.

4. ~~Without~~ reconstruction ambiguity.

4.3 Without some knowledge of a scene's placement with respect to a 3D coordinate frame, the scene is determined at best up to a Euclidean transformation (rotation and translation) with respect to the world frame.

4.4 the points X_i and the uncalibrated cameras can be determined at best only up to a projective trans.

proof: $PX_i = (PH^{-1})(HX_i)$

$P'X_i = (P'H^{-1})(HX_i)$

where H invertible, representing a projective trans of P^3.

4.5 It is shown that reconstruction from uncalibrated cameras is possible up to a projective trans.

4.6. for calibrated cameras, the scene is determined at best up to a similarity trans.

proof: let $H_S = \begin{pmatrix} R & t \\ 0^T & \lambda \end{pmatrix}$ be any similarity trans.

$P = K(R_P \; t_P)$

$\because PH_S^{-1} = K(R_P \; t_P)\begin{pmatrix} R^{-1} & -\lambda^{-1}R^{-1}t \\ 0^T & \lambda^{-1} \end{pmatrix}$

$= K(R_P R^{-1} \; t')$,

with $t' = -\lambda^{-1}R_P R^{-1} t + \lambda^{-1} t_P$

$\therefore$ multiplying H_S^{-1} does not change the calibration matrix ~~of~~ P.

$\because$ using 4.4, proved.

4.7 It is shown that for calibrated cameras, reconstruction is possible up to a similarity trans.

4.8 The reconstructed point set X_i and cameras differ from the true reconstruction by a trans belonging to a group (for instance a similarity). One speaks of projective reconstruction, affine recon, similarity recon to indicate the type of ~~[illegible]~~ trans involved.

5. $x_i \leftrightarrow x_i'$ correspondences between points in two images,

F: fundamental matrix uniquely determined by $x_i'^T F x_i = 0$, $\forall i$.

Let $(P_1, P_1', \{X_{1i}\})$ and $(P_2, P_2', \{X_{2i}\})$ be two reconstructions of the $x_i \leftrightarrow x_i'$

$\Rightarrow \exists H$ invertible such that

$P_2 = P_1 H^{-1}$, $P_2' = P_1' H^{-1}$, $X_{2i} = HX_{1i}$

$\forall i$, except for those i s.t. $Fx_i = F^T x_i' = 0$

proof: ① $\because$ F is the fundamental matrix corresponding to (P_1, P_1') and (P_2, P_2')

$\therefore \exists H$ invertible, s.t. $P_2 = P_1 H^{-1}$, $P_2' = P_1' H^{-1}$.

② $\because P_2(HX_{1i}) = P_1 H^{-1} H X_{1i} = P_1 X_{1i} = x_i$

$P_2 X_{2i} = x_i$

$\therefore P_2(HX_{1i}) = P_2 X_{2i}$

$\therefore HX_{1i}, X_{2i} \in$ same ray through the camera center of P_2.

③ Similarly, $HX_{1i}, X_{2i} \in$ same ray through the camera center of P_2'.

④ $\therefore X_{2i} = HX_{1i}$

or $X_{2i}, HX_{1i} \in$ the line joining the two camera centers.

$\Leftrightarrow Fx_i = 0$, $F^T x_i' = 0$

~~$X_{2i} = HX_{1i}$~~ or

6. stratified reconstruction.

6.3 The stratified approach to reconstruction is to begin with a projective reconstruction and then to refine it progressively to an affine and finally a metric reconstruction, if possible.

6.4 affine and metric reconstruction are not possible without further info either about the scene, the motion or the camera calibration.

7. the step to affine reconstruction

7.3 The essence of affine reconstruction is to locate the plane at infinity by some means, since this knowledge is equivalent to an affine reconstruction.

7.4. The plane at infinity cannot be identified unless some extra info is given.

7.5 projective reconstruction $(P, P', \{X_i\})$, the plane at infinity π_∞ in projective recon,

$$H = \begin{cases} \begin{pmatrix} I & 0 \\ \multicolumn{2}{c}{\pi_\infty^T} \end{pmatrix}, & \text{if } \pi_\infty(4) \neq 0, \\ H_v^{-T}, & H_v \text{ is Householder matrix s.t. } H_v \pi_\infty = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ if } \pi_\infty(4) = 0 \end{cases}$$

$\Rightarrow (PH^{-1}, P'H^{-1}, \{HX_i\})$ is an affine reconstruction.

~~proof: ① use 5.3 in ③.~~

~~② $H^{-T}\pi_\infty = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$~~

11. condition of 9

$\Rightarrow$ ~~affine reconstruction~~

$F = t_\times$

$\Rightarrow$ affine reconstruction cameras (P, P'), with $P=(I\ 0)$, $P'=(I\ t)$

$\therefore$ $e=e'=t$

12. $(P_1, P_1', \{X_{1i}\})$ and $(P_2, P_2', \{X_{2i}\})$ two reconstructions of $x_i \leftrightarrow x_i'$,

$\exists H$ invertible, s.t. $P_2 = P_1 H^{-1}$, $P_2' = P_1' H^{-1}$

$\Rightarrow$ the two reconstructions are up to a transformation H.

proof: use proof of 5.

13. ① If (P, P') affine recon with $P=(I\ 0)$, $P'=(M', t)$

$\Rightarrow$ the infinite homography $H_\infty = M'$.

(H_∞ is unchanged by an affine trans of the cameras.)

② Conversely, if H_∞ has been obtained, then the cameras of an affine recon may be chosen as $P=(I\ 0)$, $P'=(H_\infty\ t)$.

9. the motion of the cameras is a pure translation with no rotation and no change in the internal parameters.

$\Rightarrow$ F is skew-symmetric.

proof: $\because$ $P=(I\ 0)$, $P'=(I\ t)$

$\Rightarrow F = t_\times$

$\therefore$ $F = t_\times$ corresponding to $((M, m), (M, m+t))$ (from 8).

10. the motion of the cameras is a pure translation $\underbrace{(\text{condition 9})}_{\text{cameras}}$

$\Rightarrow$ for an affine reconstruction one may choose the two cameras as $P=(I\ 0)$, $P'=(I\ t)$.

proof: $\because$ true reconstruction (M, m), $(M, m+t)$ $\xrightarrow{\text{affine trans}}$ (P, P') (from 8) (use 12)

$\therefore$ (P, P') is the cameras of affine recon.

proof: $\because$ the present reconstruction $\{HX_i\}$ differs from the true reconstruction by a projective trans that fixes the plane at infinity $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, (from 5)

$\because$ a projective trans fixing the plane at infinity is an affine trans. (53 in C3)

$\therefore$ the reconstruction differs by an affine transformation from true recon.

$\therefore$ It is an affine reconstruction.

8. $P=(M\ m)$ 且 $(M, m+t)$, M invertible,

$\Rightarrow \exists H = \begin{pmatrix} M & m \\ 0^T & 1 \end{pmatrix}$, s.t. $PH^{-1} = (I\ 0)$, $P'H^{-1} = (I\ t)$

proof: ok.

18. Just as the key to affine recon is the identification of the plane at infinity, the key to metric recon is the identification of the absolute conic.

19. the image of the absolute conic is a property of the image itself.

19.3. It is unchanged by 3D transformations of ~~the~~ any particular reconstruction.

20. ω, the image of the absolute conic, $P=(M\ m)$ in an affine reconstruction.

$\Rightarrow$ the affine recon may be transformed to a metric recon by applying H:

$H=\begin{pmatrix} A^{-1} & \\ & 1 \end{pmatrix}$, where A is obtained by Cholesky factorization from $AA^T=(M^T\omega M)^{-1}$

~~proof~~

21. direct metric reconstruction using ω.

22. direct method.

Compute homography H such that $HX_i = X_{Ei}$ from five or more ground control points X_{Ei} with known Euclidean positions. Then the metric reconstruction is

$P_M = PH^{-1}$, $P'_M = P'H^{-1}$, $X_{Mi} = HX_i$

16. proof of 13②.

proof: ~~the camera~~

let (P, P') be cameras of an affine recon, $P=(I\ 0)$, $P'=(M'\ t)$.

(we can assume this because of 15)

$\because H_\infty = M'$

$\therefore$ affine recon may be chosen as the cameras of

$P=(I\ 0)$, $P'=(H_\infty\ t)$

i.e., we can use the form and solve it, then get an affine recon.

17. $(P, P', \{X_i\})$ projective recon for which $P=(I\ 0)$. In truth P is known to be an affine camera. $\Rightarrow$ an affine recon is obtained by swapping the last two columns of P and P' and the last two coordinates of each X_i. $\left(\because H=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}\right)$

14. reconstruction:

① i) $F \to (P, P')$ using 4 in C9.

ii) upgrade the projective reconstruction to affine or metric reconstruction.

② i) suppose the form (P, P')

ii) $\Rightarrow F$ form

iii) $\Rightarrow F \Rightarrow (P, P')$ with form ~~specified~~ specified

15. ① We can assume projective recon cameras of are $P=(I\ 0)$, $P'=(M'\ m')$.

③ The cameras of an affine recon may be chosen as $P=(I\ 0)$, $P'=(M', m')$ (finite camera required for 1st camera)

proof:

① $\because (I\ 0)\begin{pmatrix} P \\ P_4 \end{pmatrix} = (M\ m) = P$

$\hat{P}=\begin{pmatrix} P \\ P_4 \end{pmatrix}$, P_4 can be chosen such that $r(\hat{P})=4$.

② $\because$ $\hat{P}$ can be $\begin{pmatrix} M & m \\ 0 & 1 \end{pmatrix}$, $r(\hat{P})=4$

Chapter 11. compute F.

1. $G=\begin{pmatrix}1&0&0\\0&1&0\\-1/f&0&1\end{pmatrix}$, ~~epipole~~ $e=(f,0,1)^T$,

$\Rightarrow$ ① $Ge=\begin{pmatrix}f\\0\\0\end{pmatrix}$,

② G is approximated (to first-order) at the origin by the identity mapping

proof: ① $Ge=\begin{pmatrix}1&0&0\\0&1&0\\-\frac{1}{f}&0&1\end{pmatrix}\begin{pmatrix}f\\0\\1\end{pmatrix}$

$=\begin{pmatrix}f\\0\\0\end{pmatrix}$

② 设 $\begin{pmatrix}\hat{x}\\\hat{y}\\1\end{pmatrix}=Gx=G\begin{pmatrix}x\\y\\1\end{pmatrix}$

$\because \begin{pmatrix}\hat{x}\\\hat{y}\\1\end{pmatrix}=G\begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}x\\y\\1-\frac{x}{f}\end{pmatrix}$

$\because \frac{1}{1-t}=1+t+t^2+\cdots$, $|t|<1$

$\therefore \begin{pmatrix}\hat{x}\\\hat{y}\\1\end{pmatrix}=\begin{pmatrix}x\\y\\1-\frac{x}{f}\end{pmatrix}=\begin{pmatrix}\frac{1}{1-\frac{x}{f}}x\\\frac{1}{1-\frac{x}{f}}y\\1\end{pmatrix}$

$=\begin{pmatrix}x(1+\frac{x}{f}+(\frac{x}{f})^2+\cdots\\y(1+\frac{x}{f}+(\frac{x}{f})^2+\cdots\\1\end{pmatrix}$

$\therefore$ Jacobian $=\frac{\partial(\hat{x},\hat{y})}{\partial(x,y)}$

$=\begin{pmatrix}1+\frac{2x}{f}&0\\\frac{y}{f}&1+\frac{x}{f}\end{pmatrix}+Z$

Z is higher order terms in x and y.

1

$\therefore$ Jacobian $=I$

if $x=y=0$.

2. $H=GRT$, where

T is a translation, $Tx_0=\begin{pmatrix}0\\0\\1\end{pmatrix}$,

R is a rotation about the origin, $R(Te)=\begin{pmatrix}f\\0\\1\end{pmatrix}$,

G is the mapping in 1.

$\Rightarrow$ $He=\begin{pmatrix}f\\0\\0\end{pmatrix}$, H is to first-order a rigid transformation in the neighbourhood of x_0

3. (H,H') are called a matched pair of transformations if $H^{-T}l=H'^{-T}l'$, $\forall$ pair of corresponding epipolar lines l,l'.

4. J, J' images, $F = e'_\times M$,

H' a projective trans of J',

H a projective trans of J

$\Rightarrow$ H matches H'

$\Leftrightarrow H = (I + H'e'a^T)H'M$ for some a.

5. J, J' images, $F = e'_\times M$,

H' a projective trans of J',

H a projective trans of J,

$H'e' = \begin{pmatrix}1\\0\\0\end{pmatrix}$,

$\Rightarrow$ H matches H'

$\Leftrightarrow H = H_A H_0$, where $H_0 = H'M$,

$H_A = \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6. ~~H, H' matched~~,

$$\min_H \sum_i d(Hx_i, H'x'_i)^2$$

H, H' matched.

7.1 If $H'e' = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $H = H_A H_0$ is 5, then ① becomes

$$\min_{H_A} \sum_i d(H_A \hat{x}_i, \hat{x}'_i)^2 \quad ①$$

where $\hat{x}_i = H_0 x_i$, $\hat{x}'_i = H' x'_i$

7.3 let $\hat{x}_i = (\hat{x}_i, \hat{y}_i, 1)^T$

$\hat{x}'_i = (\hat{x}'_i, \hat{y}'_i, 1)^T$

$\therefore$ ① becomes

$$\min_{a,b,c} \sum_i (a\hat{x}_i + b\hat{y}_i + c - \hat{x}'_i)^2 + (\hat{y}_i - \hat{y}'_i)^2$$

7.4 $\therefore$ $\min_{a,b,c} \sum_i (a\hat{x}_i + b\hat{y}_i + c - \hat{x}'_i)^2$

8. outline of resampling algorithm:

input: a pair of images.

output: a pair of images resampled so that the epipolar lines in the two images are horizontal, and such that corresponding points are as close to each other as possible. Any remaining disparity between matching points will be along the horizontal epipolar lines.

(i) identify a seed set of matching points $x_i \leftrightarrow x'_i$. $(n \geq 7)$

(ii) compute F, find e, e'

(iii) select H', s.t. $H'e' = \begin{pmatrix}1\\0\\0\end{pmatrix}$ using 2.

(iv) find H, s.t. $\sum_i d(Hx_i, H'x'_i)$ is minimized. using 7.4

(v) resample images according H and H'.

2.

Chapter 12 HZ

Structure Computation

1. compute the position of a point in 3-space given its images in two views and the camera matrices of those views.

1.3 It is assumed that there are errors only in the measured image coordinates, not in the P, P'.

2. There is no meaningful metric info about the object space in affine and projective reconstruction.

3. ① F is given a priori and then X is determined.

② F and $\{X_i\}$ are estimated simultaneously.

4. two rays corresponding to a matching pair of points $x \leftrightarrow x'$ will meet in space

$\Leftrightarrow$ the two points satisfy the epipolar constraint: $x'^T F x = 0$.

5. A triangulation is said to be invariant under a trans H

if $\tau(x, x', P, P') = H^{-1}\tau(x, x', PH^{-1}, P'H^{-1})$

where τ is a triangulation method used to compute X.

6. The estimated point in linear triangulation methods does not exactly satisfy the geometric relations, and is not an optimal estimate.

7. $\because$ $x = PX \Leftrightarrow x \times (PX) = 0$

$\therefore$ We get three equations for each image point, of which two are linearly independent.

7.3. $\therefore$ $AX = 0$,

with $A = \begin{pmatrix} xP_3^T - P_1^T \\ yP_3^T - P_2^T \\ x'P_3'^T - P_1'^T \\ y'P_3'^T - P_2'^T \end{pmatrix}$

where $x = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$, $x' = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$, $P = \begin{pmatrix} P_1^T \\ P_2^T \\ P_3^T \end{pmatrix}$

$P' = \begin{pmatrix} P_1'^T \\ P_2'^T \\ P_3'^T \end{pmatrix}$

8. Homogeneous method (DLT) and inhomogeneous method.

8.3 They are not projective-invariant.

12. $\min_{t} C = d(x, l(t))^2 + d(x', l'(t))^2$

12.3 $12 \Leftrightarrow 9$.

12.4 (l, l') ~~are~~ ~~is~~ are a pair of matching epipolar lines parametrized by t.

12.5 the optimum point $\hat{x}$ is the closest point on the line l to the point x and $\hat{x}'$ is similarly defined.

12.6 strategy for minimizing 12.

(i) write l in the first image as $l(t)$.

(ii) Using F, compute $l'(t)$ in the second image.

(iii) express C as a function of t.

(iv) Find t.

9.4 Once $\hat{x}'$ and $\hat{x}$ are found, the point $\hat{X}$ may be found by any triangulation method, since the corresponding rays will meet precisely in space.

10. real inner space $\mathbb{R}^n$

< Euclidean space

< affine space.

ε

11. Sampson approximation (first-order geometric correction)

The approximation is accurate if the correction in each image is small (less than a pixel), and is cheap to compute. However, the corrected points will not satisfy the epipolar relation $\hat{x}'^T F \hat{x} = 0$ exactly.

8.4 the inhomogeneous method is affine-invariant, whereas the homogeneous method is not.

8.5 The homogeneous linear method ~~does~~ provides acceptable results. It has the virtue that it generalizes easily to triangulation when more than two views of the point are available.

9. Geometric error cost function.

$$\min_{\hat{x}, \hat{x}'} C(x, x') = \|x - \hat{x}\|^2 + \|x' - \hat{x}'\|^2$$

$$\text{s.t. } \hat{x}'^T F \hat{x} = 0$$

9.3 assuming a Gaussian error distribution,

$\Rightarrow \hat{x}', \hat{x}$ are MLE for the true image point correspondences.