1. $\forall x, f(x)=\infty \Leftrightarrow epi(f)=\phi$

proof: $f=\infty \Leftrightarrow dom(f)=\phi$

$\because$ dom(f) is projection of epi(f)

$\therefore dom(f)=\phi \Leftrightarrow epi(f)=\phi$

2. $\exists x\in X$, s.t., $f(x)=-\infty \Leftrightarrow$ epi(f) contains a vertical line.

proof: $\exists x$, s.t., $f(x)=-\infty \Leftrightarrow f(x)\leq w, \forall w\in R$

$\Leftrightarrow$ epi(f) contains a vertical line.

3. f is proper if $epi(f)\neq\phi$ and epi(f) doesn't contain a vertical line.

4. forbidden sum $-\infty+\infty$ may happen in $\alpha f(x)+(1-\alpha)f(y)$

$\Leftrightarrow$ f is improper.

5. $C\subset R^n$ convex, $f: C\to[-\infty,\infty]$ convex if epi(f) convex.

definition:

6. $f: C\to[-\infty,\infty]$ convex $\Rightarrow$ dom(f) convex, level set convex.

~~proof~~

7. $f: C\to(-\infty,\infty)$ convex $\Leftrightarrow$ epi(f) convex

proof: ① $\Rightarrow$, let $(x,u),(y,v)\in epi(f)$, $\therefore f(x)\leq u, f(y)\leq v$

$p=\alpha(x,u)+(1-\alpha)(y,v)\in epi(f)$?

$=(\alpha x+(1-\alpha)y, \alpha u+(1-\alpha)v)$

$\because f(\alpha x+(1-\alpha)y)\leq\alpha f(x)+(1-\alpha)f(y)\leq\alpha u+(1-\alpha)v$, $\therefore p\in epi(f)$

② $\Leftarrow$ let $x,y\in C$, $u=f(x)$, $v=f(y)$, $\therefore (x,u),(y,v)\in epi(f)$

$\because$ epi(f) convex, $\therefore p=\alpha(x,u)+(1-\alpha)(y,v)\in epi(f)$

$\therefore f(\alpha x+(1-\alpha)y)\leq\alpha u+(1-\alpha)v=\alpha f(x)+(1-\alpha)f(y)$

8. $f: C \to [-\infty, \infty)$ convex $\Leftrightarrow$ epi(f) convex

proof: ① $\Rightarrow$ same as the proof 7.①

② Similar to the proof 7.② if $f(x), f(y) \neq -\infty$.

if $f(x) = -\infty$, $\Rightarrow \forall u \in R$, $(x, u) \in epi(f)$.

$\because p \in epi(f)$, $\therefore f(\alpha x + (1-\alpha)y) \le \alpha u + (1-\alpha)v$, $\forall u \in R$, $v = f(y)$

$\therefore f(\cancel{p}\ \alpha x + (1-\alpha)y) = -\infty = \alpha f(x) + (1-\alpha) f(y)$, $\alpha \neq 0$.

9. $f: C \to [-\infty, \infty]$, $\therefore f|_{dom(f)}$ convex $\Leftrightarrow epi(f|_{dom(f)})$ convex (from 8)

$\therefore f|_{dom(f)}$ convex $\Leftrightarrow$ epi(f) convex

5. $\because$ definition: $f: C \to [-\infty, \infty]$ convex if $f|_{dom(f)}$ convex.

10. $f: \cancel{X} \to [-\infty, \infty) \Leftrightarrow X = dom(f)$

11. $f: X \to [-\infty, \infty)$ convex $\Rightarrow$ dom(f) convex, level set convex.

proof ① $\because$ f convex $\therefore$ X convex. $\therefore$ dom(f) convex.

② f convex $\Rightarrow$ level set convex. (see page 6)

6 $f: X \to [-\infty, \infty]$ convex $\Rightarrow$ dom(f) convex, level set convex.

12 $f: X \to [-\infty, \infty]$, $\Rightarrow$ levelset of f = level set of $f|_{dom(f)}$

proof: ① $x \in$ level set of f, $\therefore f(x) \le \gamma$, $\therefore x \in dom(f)$

$\therefore f|_{dom(f)}(x) \le \gamma$ $\therefore x \in$ level set of $f|_{dom(f)}$

② $x \in$ level set of $f|_{dom(f)}$, $\therefore f(x) = f|_{dom(f)}(x) \le \gamma$

$\therefore x \in$ level set of f

6 proof: $\because$ f convex $\therefore f|_{dom(f)}$ convex

$\therefore dom(f|_{dom})$ convex, levelset$(f|_{dom})$ convex

$\therefore$ dom(f) convex, levelset(f) convex.