1. inner product

symmetry: $a\cdot b=b\cdot a$

linearity: $(a+c)\cdot b=a\cdot b+c\cdot b$

homogeneity: $(sa)\cdot b=s(a\cdot b)$

$\|b\|^2=b\cdot b$.

2. $|a-b|^2$

solve: $|a-b|^2$

$=(a-b)\cdot(a-b)$, ∵ is inner product

$=a\cdot a-a\cdot b-b a+b\cdot b$

$=|a|^2-2a\cdot b+|b|^2$

3. R^2: $b\cdot c=|b||c|\cos(\theta)$,

θ is angle from b to c.

prove: $b=(|b|\cos\varphi_b,\ |b|\sin\varphi_b)$

$c=(|c|\cos\varphi_c,\ |c|\sin\varphi_c)$

$\therefore b c$

$=|b||c|\cdot(c\varphi_b c\varphi_c+s\varphi_b s\varphi_c)$

$=|b||c|\cdot\cos(\varphi_b-\varphi_c)$

$=|b||c|\cos\theta$

3.2 R^n: ε.

4. perpendicular

$\Leftrightarrow$ orthogonal

$\Leftrightarrow$ normal.

5. $a=(a_x,a_y)$

$a^\perp=(-a_y,a_x)$,

$\Rightarrow a^\perp$ is counterclockwise perpendicular to a.

5.2 $(a+b)^\perp=a^\perp+b^\perp$

$(ka)^\perp=ka^\perp$

$(a^\perp)^\perp=-a$.

5.3. $\perp$ is "rotate 90° left"

"perp" $\Leftrightarrow$ perpendicular

5.4 $a^\perp\cdot b$

$=a_xb_y-a_yb_x$

5.5 $a^\perp\cdot a=0$

$|a^\perp|^2=|a|^2$

5.6 $a^\perp\cdot b=-b^\perp\cdot a$

(antisymmetric)

proof:

~~$a^\perp\cdot b=a_xb_y-a_yb_x$~~

$a^\perp\cdot b$

$=(a^\perp)^\perp\cdot b^\perp$

$=-ab^\perp$

$=-b^\perp a$. ok.

5.7. $a\cdot b$

$=a^\perp\cdot b^\perp$

proof: $a\cdot b=a_xb_x+a_yb_y$

$a^\perp\cdot b^\perp=a_yb_y+a_xb_x$

5.8. $a^\perp\cdot b=\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}$

proof:

$a^\perp\cdot b$

$=a_xb_y-a_yb_x$

$=\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}$

5.9 $(a^\perp\cdot b)^2+(a\cdot b)^2$

$=|a|^2|b|^2$

proof: $(a^\perp\cdot b)^2+(a\cdot b)^2$

$=(a_xb_y)^2+(a_yb_x)^2$

$+(a_xb_x)^2+(a_yb_y)^2$

$|a|^2=a_x^2+a_y^2$

$|b|^2=b_x^2+b_y^2$

5.10. if $a+b+c=0$

$\Rightarrow a^\perp\cdot b$

$=b^\perp\cdot c$

$=c^\perp\cdot a$

proof:

~~$a^\perp b$~~

$\because a^\perp+b^\perp+c^\perp=0$

$\therefore a^\perp\cdot b+c^\perp\cdot b=0$

$\therefore a^\perp\cdot b=b^\perp\cdot c$

$\because a^\perp\cdot c+b^\perp\cdot c=0$

$\therefore b^\perp\cdot c=a\cdot c^\perp$

$\therefore a^\perp\cdot b=b^\perp\cdot c=c^\perp\cdot a$

~~8~~.9 . cross product

$\Leftrightarrow$ ①,②,③.

① ~~cross product~~ f is linear.

② $f(x,x)=0,\ \forall x\in R^3$

③ $f(i,j)=k,$
$f(j,k)=i,$
$f(k,i)=j.$

proof: $\Rightarrow$ if f is cross product then ①②③ holds

$\Leftarrow$. $\forall a,b\in R^3$.

$\because a=a_x i+a_y j+a_z k$
$b=b_x i+b_y j+b_z k.$

$\because f(a,b)=f(a_x i+a_y j+a_z k,\ b)$
$=a_x f(i,b)+a_y f(j,b)+a_z f(k,b)$

$f(i,b)=b_x f(i,i)+b_y f(i,j)+b_z f(i,k)$
$=-b_y k+b_z f(i,k)$

~~$f(i,k)$~~ $\because f(i+k,i+k)=0$
$\therefore f(i,k)+f(k,i)=0$

$\therefore f(i,b)=b_y k-b_z j$, similarly, $f(j,b), f(k,b)$.

$\therefore f(a,b)=\begin{vmatrix} i & j & k\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$, ok.

8.3 $i\times j=k$
$j\times k=i$
$k\times i=j$

8.4 $a\times b=-b\times a$ (antisymmetry)

8.5 $a\times(b+c)=a\times b+a\times c$

$(sa)\times b=s(a\times b)$

proof:

$a\times(b+c)$
$=\begin{vmatrix} i & j & k\\ a_x & a_y & a_z\\ b_x+c_x & b_y+c_y & b_z+c_z\end{vmatrix}$

$=\begin{vmatrix} i & j & k\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}+\begin{vmatrix} i & j & k\\ a_x & a_y & a_z\\ c_x & c_y & c_z\end{vmatrix}$

$=a\times b+a\times c$

8.6. 8.3–8.5 are true in both left-handed and right-handed coordinate systems.

$a=m+e.$

$r=-m+e$

$\therefore r=a-2m$

$\because m=\frac{a\cdot n}{|n|^2}n$

$=(a\cdot\hat{n})\hat{n}$

($\hat{n}$ is normal of n)

$\therefore r=a-2m$
$=a-2(a\cdot\hat{n})\hat{n}$

8. cross product in R^3.
(vector product)

8.2. definition:

$a\times b$:

$=\begin{vmatrix} i & j & k\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$

here $a,b,i,j,k\in R^3$

6 $c=kv+Mv^{\perp}$

$\Rightarrow k=\frac{c\cdot v}{|v|^2}$

$M=\frac{c\cdot v^{\perp}}{|v|^2}$

6.2 $|Mv^{\perp}|$

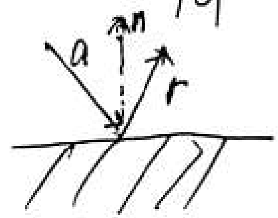
$=\frac{|v^{\perp}\cdot c|}{|v|}$

proof: $\left|\frac{c\cdot v^{\perp}}{|v|^2}v^{\perp}\right|$

$=|M||v|$

$=\frac{|v^{\perp}\cdot c|}{|v|}$

7.



10. $(a\times b)\cdot a = 0$.

proof: $(a\times b)\cdot a$

$= \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \cdot (a_x i + a_y j + a_z k)$

$= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} a_x + \cdots$

$\cancel{=}$

$= \begin{vmatrix} a_x & a_y & a_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

$= 0$.

11. $a\times(b\times c) \neq (a\times b)\times c$

e.x.

$j\times(j\times k) \neq (j\times j)\times k$

12. $|a\times b| = \sqrt{|a|^2|b|^2 - (a\cdot b)^2}$

$= |a||b||\sin\theta|$

proof: $\text{right}^2 = |a|^2|b|^2 - (a\cdot b)^2$

$= (a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2)$

$- (a_x^2 b_x^2 + a_y^2 b_y^2 + a_z^2 b_z^2 + 2a_x b_x a_y b_y + 2a_x b_x a_z b_z + 2a_y b_y a_z b_z)$

$= a_x^2 b_y^2 + a_x^2 b_z^2$

$+ a_y^2 b_x^2 + a_y^2 b_z^2$

$+ a_z^2 b_y^2 + a_z^2 b_x^2$

$- 2a_x b_x a_y b_y$

$- 2a_y b_x a_z b_z$

$- 2a_y b_y a_z b_z$

$= (a_x b_y - a_y b_x)^2$

$+ (a_x b_z - a_z b_x)^2$

$+ (a_y b_z - a_z b_y)^2$

$\text{left}^2 = |a\times b|^2$

$= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}^2 + \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}^2 + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}^2$

ok.

13. $\because |a\times b| = |a||b||\sin\theta|$

$\therefore |a\times b| = |\text{area}|$.

area of the parallelogram determined by a and b.

14. P_1, P_2, P_3 don't lie in a straight line.

$\Rightarrow n = a\times b$,

where $a = P_2 - P_1$,

$b = P_3 - P_1$.

n is normal vector of the plane through P_1, P_2, P_3.

14.2. $\because$ cross product involves subtraction of various quantities,

$\therefore$ this method for finding is vulnerable to numerical inaccuracies, especially when the angle between a and b is small.

Linear algebra

1. ~~?~~ R^4 cannot be dealt with geometrically, but the algebraic approach works well.

2. generalizations of dot product will become important when in inner product space.

2.3 No useful generalization of cross product in R^3 exists in higher dimensions.

3. linear algebra focuses not on ~~ab~~ arbitrary vector spaces, but on finite-dimensional vector space.

4. $\mathrm{span}(v_1, \cdots, v_m) = \langle v_1, \cdots, v_m \rangle$
$:= \{$ linear combination of $(v_1, \cdots, v_m)\}$

4.2 $\mathrm{span}(v_1, \cdots, v_m)$ is a subspace of V

5. A vector space is called finite dimensional if $\exists$ set $S \subset V$, S finite, $\langle S \rangle = V$.

5.2 A vector space that is not finite dimensional is called infinite dimensional.

5.3 infinite-dimensional vector spaces are the center of attention in functional analysis. Functional analysis uses tools from both analysis and algebra.

6 linearly independent.
linearly dependent.
basis
standard basis

6.3 dimension.

7. As sets, R^2 can be identified with C.
(addition is same on both spaces, as is scalar multiplication by real numbers).

7.2 As vector space, R^2 has dimension 2; ~~the~~ C has dimension 1.

8. $U_1 < V$, $U_2 < V$, V finite-dimensional, $\Rightarrow \dim(U_1+U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

9. sum of subspaces in the theory of vector spaces are analogous to unions of subsets in set theory. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets.

10. The interesting part of linear algebra is not vector space, but linear maps.

11. ~~linear map~~ $T: V \to W$
with the following properties;
① $T(u+v) = Tu + Tv$, $\forall u, v \in V$.
(additivity)
② $T(av) = aTv$, $\forall a \in F$, $v \in V$.
(homogeneity)
Then, T is called linear map, or linear transformation.

12.

$\mathcal{L}(V,W) := \{$ linear maps from V to $W \}$

13. $P(F)$ is vector space.

with $P(F)$ is the set of all polynomials with coefficients in F.

14. $T: P(\mathbb{R}) \to P(\mathbb{R})$

$Tp = p'$

$\Rightarrow T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$

14.3 $T: P(\mathbb{R}) \to \mathbb{R}$

$Tp = \int_0^1 p(x)\,dx$

$\Rightarrow T \in \mathcal{L}(P(\mathbb{R}), \mathbb{R})$

14.4 $I: V \to V$

$Iv = v$

$\Rightarrow I$ linear, i.e. $I \in \mathcal{L}(V, V)$

14.5 $0 \in \mathcal{L}(V,W)$

14.6 $T: P(\mathbb{R}) \to P(\mathbb{R})$

$(Tp)(x) = x^2 p(x)$

$\Rightarrow \cancel{T}\ T$ linear.

14.7 backward shift

$T: F^\infty \to F^\infty$

$T(x_1, x_2, x_3, \cdots)$

$= (x_2, x_3, \cdots)$

$\Rightarrow T$ linear.

14.8

$T: F^n \to F^m$

$T(x_1, \cdots, x_n)$

$= \left(\sum_{i=1}^{n} a_{1i} x_i, \right.$

$\cdots,$

$\left. \sum_{i=1}^{n} a_{mi} x_n \right)$

$\Rightarrow T$ linear.

15. $(v_1, \cdots, v_n)$ basis of V,

$T: V \to W$ linear,

$\forall v = a_1 v_1 + \cdots + a_n v_n$

$\Rightarrow Tv = a_1 T v_1 + \cdots + a_n T v_n$

16. given basis $(v_1, \cdots, v_n)$ ~~of~~ V, $\forall w_1, \cdots, w_n \in W$

$\Rightarrow$ we can construct $T \in \mathcal{L}(V,W)$.

solve: $T: V \to W$,

$T(v_j) = w_j$, $j = 1, \cdots, n$.

$\because T$ must be linear.

$\therefore T(a_1 v_1 + \cdots + a_n v_n)$

$= a_1 w_1 + \cdots + a_n w_n$.

now we ~~proof~~ prove T is linear.

$\forall x \in V$, $x = a_1 v_1 + \cdots + a_n v_n$

~~$\forall x$~~ $\forall y \in V$, $y = b_1 v_1 + \cdots + b_n v_n$

$\forall k \in F$.

$T(kx) = T(ka_1 v_1 + \cdots + ka_n v_n)$

$= ka_1 w_1 + \cdots + ka_n w_n$

$= kTx$

$T(x+y) = T((a_1+b_1)v_1, \cdots, (a_n+b_n)v_n)$

$= (a_1+b_1)w_1 + \cdots + (a_n+b_n)w_n$ ~~$\cdots$~~

$= Tx + Ty$

$\therefore T$ linear.

17. $\forall S, T \in \mathcal{L}(V,W)$, $\forall a \in F$,

$(S+T)v := Sv + Tv$

$(aT)v := a(Tv)$

$\Rightarrow \mathcal{L}(V,W)$ is a vector space.

proof: ① $\forall$ ~~x~~ $x, y \in V$, $k \in F$.

$(S+T)(kx) = S(kx) + T(kx)$

$= k((S+T)x)$

$(S+T)(x+y) = S(x+y) + T(x+y)$

$= Sx + Sy + Tx + Ty$

$\therefore S+T \in \mathcal{L}(V,W)$ $= (S+T)x + (S+T)y$

$\because (ST)(x+y)$

$= S(T(x+y))$

$= S(Tx+Ty)$

$= S(Tx)+S(Ty)$

$= (ST)x+(ST)y$

$\therefore ST \in L(U,W)$

19. $T \in L(V,W)$

~~then~~ $\text{null}\,T := \{v \in V \mid Tv = 0\}$

$\Rightarrow \text{null}\,T < V$

19.2. kernel $T := \text{null}\,T$

19.3 T injective $\Leftrightarrow \text{null}\,T = \{0\}$

19.4 injective $\Leftrightarrow$ one-to-one

20. $T \in L(V,W)$

$\text{range}\,T := \{Tv \mid v \in V\}$

$\Rightarrow \text{range}\,T < W$.

20.3 T surjective $\Leftrightarrow \text{range}\,T = W$

20.4 onto means the same as surjective.

$= (k_1(k_2 S))x$

$\therefore (k_1 k_2)S = k_1(k_2 S)$

8). $(1 \cdot S)x$

$= 1 \cdot Sx = Sx$

$\therefore 1 \cdot S = S$.

$\therefore L(V,W)$ is vectorspace.

18. $T \in L(U,V)$, $S \in L(V,W)$

~~#~~ define $ST : U \to W$.

$(ST)v := S(Tv)$

$\Rightarrow ST \in L(U,W)$

proof: $\because ST$ is composition of two functions,

$\therefore ST$ is function.

$\because (ST)(kx) = S(T(kx))$

$= k(S(Tx))$

$= k((ST)x)$

4) $(S+T)x = Sx+Tx$

$(T+S)x = Sx+Tx$

$\therefore S+T = T+S$.

$\therefore L(V,W)$ is abelian group.

5) $(k(S+T))x$

$= k((S+T)x)$

$= k(Sx+Tx)$

$= kSx+kTx$

$= (kS)x+(kT)x$

$= (kS+kT)x$

$\therefore k(S+T) = kS+kT$

6) $((k_1+k_2)S)x$

$= (k_1+k_2)(Sx)$

$= k_1 Sx + k_2 Sx$

$= (k_1 S)x + (k_2 S)x$

$= (k_1 S + k_2 S)x$

$\therefore (k_1+k_2)S = k_1 S + k_2 S$

7) $((k_1 k_2)S)x$

$= (k_1 k_2)(Sx)$

$= k_1(k_2 Sx) = k_1((k_2 S)x)$

② $(aT)(kx) = a(T(kx))$

$= k(aTx)$

$= k((aT)x)$

$(aT)(x+y) = a(T(x+y))$

$= aTx + aTy$

$= (aT)x + (aT)y$

$\therefore aT$ linear, i.e. $aT \in L$

1) $(0+S)x = Sx$

$\therefore 0+S = S$; $0 \in L(V,W)$

2) $((S_1+S_2)+S_3)x$

$= (S_1+S_2)x + S_3 x$

$= S_1 x + S_2 x + S_3 x$

$\because (S_1+(S_2+S_3))x$

$= S_1 x + S_2 x + S_3 x$

$\therefore (S_1+S_2)+S_3 = S_1+(S_2+S_3)$

3) $(-S+S)x = (-S)x + Sx$

$= -Sx + Sx = 0$

$\therefore -S+S = 0$

21. V finite dimensional,

$T \in \mathcal{L}(V, W)$,

$\Rightarrow$ ① range T finite dimensional

② $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$

22. V, W, finite dimensional,

$\dim V > \dim W$

$T \in \mathcal{L}(V, W)$, i.e., T is linear

$\Rightarrow$ T is not injective.

23. V, W, finite dimensional,

$\dim V < \dim W$

$T \in \mathcal{L}(V, W)$

$\Rightarrow$ T is not surjective.

24. V, W, finite dimensional,

$T \in \mathcal{L}(V, W)$,

T bijective

$\Rightarrow \dim V = \dim W$.

25. a homogeneous system of linear equations in which there are more variables than equations must have nonzero solutions.

(use 22).

$(Tx = 0 \ni x)$

26. an inhomogeneous system of linear equations in which there are more equations than variables has no solution for almost all of c.

proof:

$Tx = c$.

$c \neq 0$.

$T \in \mathcal{L}(V, W)$

$\dim V < \dim W$

$\Rightarrow T$ is not surjective.

(23)

$\therefore \dim \operatorname{range} T < \dim W$.

$\therefore$

$P\{Tx = c \text{ has solution}\} = 0$.

27. V finite,

$T \in \mathcal{L}(V, W)$

$(v_1, \cdots, v_n)$ basis of V.

$\Rightarrow T$ is determined by

Tv_i, $i = 1, \cdots, n$.

(see 16)

28. V finite,

$T: V \to W$,

$\{v_1, \cdots, v_n\} \subset V$ basis,

then

① T linear $\Rightarrow$ T is determined by

Tv_i, $i = 1, \cdots, n$. (see 27)

② give Tv_i, $i = 1, \cdots, n$.

we can construct T, s.t. T linear.

T is unique. (see 16)

③ T linear $\Leftrightarrow$ Tv_i, $i = 1, \cdots, n$.

29. matrix of T with respect to bases $(v_1, \cdots, v_n)$ and $(w_1, \cdots, w_m)$

$$\mathcal{M}(T, (v_1, \cdots, v_n), (w_1, \cdots, w_m)) := \begin{array}{cc} \begin{array}{ccccc} v_1 & \cdots & v_k & \cdots & v_n \end{array} & \\ \begin{pmatrix} & & a_{1k} & & \\ & & \vdots & & \\ & & a_{mk} & & \end{pmatrix} & \begin{array}{c} w_1 \\ \vdots \\ w_m \end{array} \end{array}$$

, here top vectors and right vectors are aid for remembering, not exist actually.

$Tv_k = \sum_{i=1}^{m} a_{ik} w_i$

30. $T \in \mathcal{L}(F^2, F^3)$

$T(x,y) = (x+3y, 2x+5y, 7x+9y)$

$\Rightarrow M(T)$ (with respect to standard basis) is $\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$

proof: $\because T(1,0) = (1,2,7)$

$T(0,1) = (3,5,9)$.

31. given bases $(v_1, \dots, v_n) \subset V$, $(w_1, \dots, w_m) \subset W$.

$T: V \to W$.

then ① T linear $\Rightarrow M(T, \{v_j\}, \{w_j\})$

② $M \Rightarrow T$ exists, unique, T linear

③ T linear $\Leftrightarrow M$

32. given matrix A, B, of same size,

$A = M(T)$, $B = M(S)$,

$T \in \mathcal{L}(V,W)$, $S \in \mathcal{L}(V,M)$,

then $A + B := M(T+S)$

32.2 $A+B$ is independent of choices of bases.

proof: (see 32.3).

32.3 $A+B$ is adding corresponding entries in them.

proof: $\because Tv_k = \sum_{i=1}^{m} a_{ik} w_i$

$Sv_k = \sum_{i=1}^{m} b_{ik} w_i$

$\because (T+S)v_k = Tv_k + Sv_k$

$= \sum_{i=1}^{m} (a_{ik} + b_{ik}) w_i$

$\therefore A+B = M(T+S) = \begin{pmatrix} a_{11}+b_{11}, \dots, a_{1k}+b_{1k}, \dots, a_{1n}+b_{1n} \\ \vdots \quad \vdots \quad \vdots \\ a_{mk}+b_{mk}, \end{pmatrix}$

proved.

33. ~~give A, B of same size~~, give A matrix, $c \in F$.

$cA := M(cT)$

where $A = M(T)$.

then cA is obtained by multiplying each entry in A by c.

proof: $\because Tv_k = \sum_{i=1}^{m} a_{ik} w_i$

$(cT)v_k = c(Tv_k)$

$= \sum_{i=1}^{m} (c a_{ik}) w_i$

$\therefore cA = M(cT)$

$= \begin{pmatrix} \vdots & c a_{1k} & \vdots \\ \vdots & \vdots & \vdots \\ & c a_{mk} & \end{pmatrix}$

34. $(v_1, \dots, v_n)$ basis of V,

$(w_1, \dots, w_m)$ basis of W.

$(u_1, \dots, u_p)$ basis of U.

$S: U \to V$, $T: V \to W$.

S linear, T linear.

we want $M(TS) = M(T)M(S)$

$\therefore \forall A, B$ matrices,

we ~~chose~~ get S, T,

s.t. $A = M(T)$, $B = M(S)$

then $AB := M(TS)$.

34.3. $(AB)(j,k) = \sum_{r=1}^{n} a_{j,r} b_{r,k}$

proof:

$\because M(T) = \begin{pmatrix} \vdots & a_{1k} & \vdots \\ \vdots & \vdots & \vdots \\ & a_{mk} & \end{pmatrix}$, $m \times n$

$M(S) = \begin{pmatrix} \vdots & b_{1k} & \vdots \\ \vdots & \vdots & \vdots \\ & b_{nk} & \end{pmatrix}$, $n \times p$

$(TS)u_k$

$= T\left(\sum_{r=1}^{n} b_{rk} v_r\right)$

$= \sum_{r=1}^{n} b_{rk} T v_r$

$= \sum_{r=1}^{n} b_{rk} \sum_{j=1}^{m} a_{jr} w_j$

$= \sum_{j=1}^{m} \left(\sum_{r=1}^{n} a_{jr} b_{rk}\right) w_j$

$\therefore M(TS) = \begin{pmatrix} \vdots & c_{1k} & \vdots \\ \vdots & \vdots & \vdots \\ & c_{mk} & \end{pmatrix}$, $m \times p$

where $c_{jk} = \sum_{r=1}^{n} a_{jr} b_{rk}$

35. V finite dimensional,

$\Rightarrow$ ① $\forall v \in V$,

$\exists$; unique, $S: U \to V$, S linear.

here $\dim U = 1$. $Su = v$.

② $\forall S \in L(U,V)$, $\dim U = 1$,

$\Rightarrow S$ is determined only by Su.

$\{u\}$ is basis of U.

③ let $S = f(v)$., then f is bijective.
(see 28)

36. $v \in V$, $T \in L(V,W)$,

~~let $S = f(v)$~~ given f_V, f_W.

f is defined by (35).

$\Rightarrow f_W(Tv) = T f_V(v)$

proof: ~~T~~ let $S = f_V(v)$

$\therefore Tv = T(Su) = (TS)u$

$\therefore f_W(Tv) = TS = T \cdot f_V(v)$

37. define $M(v) := M(v, \{v_1, \cdots, v_n\})$
$:= M(f_V(v))$

$\{v_1, \cdots, v_n\} \subset V$ basis,

$T \in L(V,W)$.

$\Rightarrow M(Tv) = M(T)M(v)$.

proof: $M(Tv)$

$= M(f_W(Tv))$

$= M(TS)$, $S = f_V(v)$

$= M(T)M(S)$

$= M(T)M(f_V(v))$

$= M(T)M(v)$.

38. T linear, $T: V \to W$.

T invertible $\Leftrightarrow$ T injective,
and T surjective.

39. V and W isomorphic

$\Leftrightarrow \exists T \in L(V,W)$

T invertible.

(definition).

39.3 isomorphic:
iso means equal.
morph means shape.

39.4 homomorphic:

homo means similar.

40. V, W, finite-dimensional
vector spaces,

V, W isomorphic

$\Leftrightarrow \dim V = \dim W$.

40.2 Because every finite-dimensional vector space is isomorphic to some F^n, why bother with abstract vector spaces?

Answer: an investigation of F^n would soon lead to vector spaces that do not equal F^n. So using these abstract vector spaces keeps things simple and brings some new insight.

41. V vector space,

W set,

$T: V \to W$,

T ~~linear~~, invertible,

where invertible is defined in sets.

$(W, +, \cdot)$ is defined by T, s.t. T linear

$Tx + Ty := T(x+y)$

$k \cdot Tx := T(kx)$.

$\Rightarrow W$ is vector space.

proof: $(W, +, \cdot)$ closed for $+$, $\cdot$.

① $(Tx+Ty)+Tz = T((x+y)+z)$

$= T(x+(y+z))$

$= Tx + (Ty+Tz)$

② $T0 + Tx = Tx$

③ $T(-x) + Tx = T(0)$

④ $Tx+Ty = T(x+y) = Ty + Tx$

⑤ $k(Tx+Ty) = kT(x+y)$

$= T(kx+ky)$

$= kTx + kTy$

⑥ $(k_1+k_2)Tx = k_1Tx + k_2Tx$

⑦ $(k_1k_2)Tx = T(k_1k_2x) = k_1(k_2Tx)$

⑧ $1 \cdot Tx = Tx$

$\therefore W$ is vector space.

42. define $Mat(m,n,F)$

$= \{$ m-by-n matrices with entries in $F \}$

$M : L(V,W) \to Mat(m,n,F)$

~~M is defined by~~

$M(T) := M$ in (29)

$\Rightarrow$ ① M is ~~inverteble~~ invertible as map between sets.

(see (31))

⑤ $Mat(m,n,F)$ is vector space.

③ $L(V,W)$ and $Mat(m,n,F)$ isomorphic.

proof: ② ~~∵ M is~~

∵ in $Mat(m,n,F)$, $+$, $\cdot$ are defined s.t. M is linear.

∴ $Mat(m,n,F)$ is vector space. (by 41)

43. $\dim(Mat(m,n,F)) = mn$.

44. V, W, finite dimensional

$\Rightarrow$ ① $L(V,W)$ finite dimensional.

② $\dim(L(V,W)) = \dim V \cdot \dim W$.

45. ~~a linear~~

$T \in L(V,W)$,

$V = W$

$\Leftrightarrow T$ is operator.

$\Leftrightarrow T \in L(V)$

45.3. The deepest and most important parts of linear algebra deal with operators.

46. V finite dimensional,

$T \in L(V)$

then ① $\Leftrightarrow$ ② $\Leftrightarrow$ ③.

①: T invertible

② T injective

③ T surjective.

proof: ① $\Rightarrow$ ②. obvious.

② $\Rightarrow$ ③,

∵ $\dim \operatorname{range} T + \dim \operatorname{null} T = \dim V$

∵ T injective

$\Leftrightarrow \dim \operatorname{null} T = 0$

$\Leftrightarrow \dim \operatorname{range} T = \dim V$

$\Leftrightarrow T$ surjective

∴ ② $\Rightarrow$ ③, ③ $\Rightarrow$ ②

∵ $\left.\begin{matrix} ③ \Rightarrow ② \\ ③ \end{matrix}\right\} \Rightarrow ①$

∴ ③ $\Rightarrow$ ①

proved.