sub chain is still Markov chain

1. $P(12345) = P_1(1)P_1(2)P_2(3)P_3(4)P_4(5)$ ①

$\Rightarrow P(2345) = P(2)P_2(3)P_3(4)P_4(5)$

$P(1234) = P(1)P_1(2)P_2(3)P_3(4)$

$\Rightarrow P_{23}(4) = \frac{P(234)}{P(23)} = P_3(4)$

$\Rightarrow P(1245) = \sum_3 P(12345)$

$= P(1)P_1(2)\left[\sum_3 P_2(3)P_3(4)\right]P_4(5)$

$= P(1)P_1(2)\left[\sum_3 P_2(3)P_{23}(4)\right]P_4(5)$

$= P(1)P_1(2)P_2(4)P_4(5)$

2. $Z^{(1)}, Z^{(2)}, Z^{(3)}, Z^{(4)}, Z^{(5)}$ Markov chain

$\Leftrightarrow$ ①, $Z^{(1)}, Z^{(2)}, Z^{(4)}, Z^{(5)}$ Markov.