A.1. $a_\times a = 0$

A.2 $a_\times b = -b_\times a$

A.3 $\cancel{a_\times b_\times = ba^T - (a\cdot b)I}$

$ab^T + b_\times a_\times^T = (a\cdot b)I$

A.4 $a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$

A.5 $(a\times b)_\times = ba^T - ab^T$

proof: $\because (a\times b)\times u = b(a\cdot u) - a(b\cdot u)$ $\forall u$

$\therefore$ proved.

A.6 $(a\times b)_\times = a_\times b_\times - b_\times a_\times$

(use A.3)

A.6.2 $G\in\mathbb{R}^{3\times3}$, G invertible, $((Ga)\times(Gb))_\times = G(a\times b)_\times G^T$

(use A.6)

A.7

$$(Ga)\times(Gb) = |G|G^{-T}(a\times b)$$

(from A.8②)

A.8 ① $a_\times G + G^T a_\times = tr(G)a_\times - (Ga)_\times$

② $(Ga)_\times = |G|G^{-T}a_\times G^{-1}$

proof ②: let $G = (h_1, h_2, h_3)$, $a = (a_1, a_2, a_3)^T$

goal: $G^T(Ga)_\times G = |G|a_\times$

$$\Leftrightarrow Z = \begin{pmatrix} h_1^T \\ h_2^T \\ h_3^T \end{pmatrix}(Ga)_\times(h_1\ h_2\ h_3) = |G|\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

$\because z_{ij} = h_i^T(Ga)_\times h_j = h_i\cdot((Ga)\times h_j)$

$= |(h_i,\ a_1h_1+a_2h_2+a_3h_3,\ h_j)|$

$\therefore$ $\cancel{z_{11}=z_{22}=z_{33}=0}$, $z_{12} = -|G|a_3$, $z_{13} = |G|a_2$

$z_{23} = -|G|a_1$ $\because Z$ skew-symmetric. $\therefore$ proved.

A.8 ①

proof:

let $Z = a_\times G + G^T a_\times$

let $Y = tr(G)a_\times - (Ga)_\times$

$b = Ga$

$\because z_{12} = -a_3 g_{22} + a_2 g_{32} - a_3 g_{11} + a_1 g_{31}$

$Y_{12} = tr(G)(-a_3) + b_3$

$= tr(G)(-a_3) + g_{31}a_1 + g_{32}a_2 + g_{33}a_3$

$\therefore z_{12} = Y_{12}$

Similarly, $z_{13} = Y_{13}$, $z_{23} = Y_{23}$

$\therefore Z = Y$.

proved.

1. $A=(a,b,c)\in R^{3\times3}$

$\Rightarrow |A| = a\cdot(b\times c)$

$\because |A| = |(b,c,a)| = |c\,a\,b|$

$= b\cdot(c\times a) = c\cdot(a\times b)$

$= (c\times a)\cdot b = (a\times b)\cdot c$

B.1. $A=(a,b,c)\in R^{3\times3}$

$\Rightarrow |A| = a\cdot(b\times c)$

B.1.2 $\because |A| = |(a,b,c)| = |(c,a,b)|$

$\therefore |A| = \cancel{(a\times b)\cdot c}\ c\cdot(a\times b)$

$\therefore a\cdot(b\times c) = (a\times b)\cdot c$

B.2. $R\in SO(3), \Rightarrow R(v\times w) = (Rv)\times(Rw)$

proof: let $R=\begin{pmatrix} r_1^T \\ r_2^T \\ r_3^T \end{pmatrix}$, $x=Rv$, $y=Rw$

① $R(v\times w)$

$= \begin{pmatrix} r_1\cdot(v\times w) \\ r_2\cdot(v\times w) \\ r_3\cdot(v\times w) \end{pmatrix}$

② $z=x\times y = \begin{pmatrix} x_2y_3-x_3y_2 \\ x_3y_1-x_1y_3 \\ x_1y_2-x_2y_1 \end{pmatrix}$

③ $z_1 = x_2y_3 - x_3y_2$

$= (r_2^Tv)(r_3^Tw) - (r_3^Tv)(r_2^Tw)$

$= v^T(r_2r_3^T - r_3r_2^T)w$

$= v^T(r_3\times r_2)_\times w$ (from A5)

$= v^T(-r_1)_\times w$

$= -v\cdot(r_1\times w)$

$= -|(v, r_1, w)|$

$= r_1\cdot(v\times w)$

④ $z_2 = x_3y_1 - x_1y_3$

$= v^T(r_1\times r_3)_\times w$

$= v^T(-r_2)_\times w$

$= r_2\cdot(v\times w)$

⑤ $z_3 = x_1y_2 - x_2y_1$

$= v^T(r_2\times r_1)_\times w$

$= v^T(-r_3)_\times w$

$= r_3\cdot(v\times w)$

⑥ $\therefore R(v\times w) = (Rv)\times(Rw)$

B.3 $R\in SO(3)$, $\forall w\in R^3$

$\Rightarrow R w_\times R^T = (Rw)_\times$

(from B.2)

B.2 method 2:

(from A.7)

1. $s = \|x\|$

$\Rightarrow \frac{\partial s}{\partial x_i} = \frac{x_i}{s}$

$\therefore \frac{\partial s}{\partial x} = \frac{1}{s} x$

2. $s = \|x\|_2$, $\bar{x} = \frac{x}{s}$

$$\frac{\partial \bar{x}_j}{\partial x_i} = \frac{\partial \left(\frac{x_j}{s}\right)}{\partial x_i} = \begin{cases} \frac{1}{s}\left(1 - \left(\frac{x_i}{s}\right)^2\right), & i=j \\ -\frac{1}{s}\left(\frac{x_i}{s}\right)\left(\frac{x_j}{s}\right), & i \neq j \end{cases}$$

2.0. $s = \|x\|_2$, $y = \bar{x} = \frac{x}{s}$

$$\Rightarrow \frac{\partial y_j}{\partial x_i} = \frac{1}{s} 1_{i=j} - \frac{1}{s} y_i y_j$$

3. $s = \|x\|$, $y = \frac{x}{s}$,

$$\Rightarrow \frac{\partial y}{\partial x_i} = \frac{1}{s} e_i - \frac{1}{s} y_i y = \frac{1}{s}(I - yy^T) e_i$$

4. $s = \|x\|$, $y = \frac{x}{s}$

$$\Rightarrow \frac{\partial y_x}{\partial x_i} = \left(\frac{\partial y}{\partial x_i}\right)_x$$

$$= \frac{1}{s}(e_i - y_i y)_x$$

5.

~~$s=\|x\|, y$~~ $s = \|x\|$, $y = \frac{x}{s}$

given $\frac{\partial y}{\partial x_i}$, $\Rightarrow \frac{\partial (yy^T)}{\partial x_i}$

solve: $\frac{\partial yy^T}{\partial x_i}$

$$= \frac{\partial y}{\partial x_i} \cdot y^T + y \cdot \left(\frac{\partial y}{\partial x_i}\right)^T$$

$$= \frac{1}{s}\left(e_i y^T + y e_i^T - 2 y_i (yy^T)\right)$$

6. $x \in R^3$, $\Rightarrow xx^T + x_x x_x^T = \|x\|^2 I$

7. $R = I + s\theta\, y_x + (1 - c\theta) y_x^2$

here $\theta = \|x\|$, $y = \frac{x}{\theta}$, use v, u instead of x, y.

$$\Leftrightarrow R = c\theta I + s\theta\, y_x + (1 - c\theta) yy^T$$

$(\because yy^T + y_x y_x^T = I)$

8. $R = I + s\theta\, y_x + (1 - c\theta) y_x^2 = R(x)$
$= e^{x_x}$

$$\Rightarrow \frac{\partial R}{\partial x_i} = s\theta\, y_i y_x^2 + c\theta\, y_i y_x$$

$$+ \frac{s\theta}{\theta}(e_i - y_i y)_x$$

$$+ \frac{1 - c\theta}{\theta}\left(e_i y^T + y e_i^T - 2 y_i yy^T\right)$$

1. $k \in R,\ u(x), v(x) \in R^m,\ w \in R^m$
$x \in R^n,\ A \in R^{m \times m},\ B(x) \in R^{m}$ ~~$w \in R$~~

1.2 $v(x) = k(x) \cdot u(x)$

$\Rightarrow \frac{dv}{dx} = u \cdot k' + k u'$

1.3 $k(x) = u(x)^T \cdot v(x)$

$\Rightarrow \frac{dk}{dx} = v^T \cdot u' + u^T \cdot v'$

1.4 $v(x) = A \cdot u(x)$

$\Rightarrow \frac{dv}{dx} = A \cdot u'$

~~1.5 $v(x) = B(x) w$~~

~~$\Rightarrow \frac{dv}{dx} =$~~

2. $A(t) \in R^{m \times n}$
$w \in R^n$
$v(t) \in R^m,\ t \in R$

$v = Aw$

$\Rightarrow \frac{\partial v}{\partial t} = \frac{\partial}{\partial t}(A(t) w) = \frac{\partial A}{\partial t} w$

proof: let $A = \begin{pmatrix} A_1^T \\ \vdots \\ A_m^T \end{pmatrix}$

$\because A(t) w = \begin{pmatrix} A_1^T w \\ \vdots \\ A_m^T w \end{pmatrix}$

$\therefore \frac{\partial v}{\partial t} = \begin{pmatrix} \frac{\partial (A_1^T w)}{\partial t} \\ \vdots \\ \frac{\partial (A_m^T w)}{\partial t} \end{pmatrix}$

$= \begin{pmatrix} w^T \frac{\partial A_1}{\partial t} \\ \vdots \\ w^T \frac{\partial A_m}{\partial t} \end{pmatrix}$

$= \begin{pmatrix} \left(\frac{\partial A_1}{\partial t}\right)^T w \\ \vdots \\ \left(\frac{\partial A_m}{\partial t}\right)^T w \end{pmatrix}$

$= \left(\frac{\partial A}{\partial t}\right) w$

5.

$(\bar{v}^T u I + \bar{v} u^T) \vec{v}_x^2$

$=$

$(\bar{v}^T u I - u\bar{v}^T + \bar{v}u^T - u\bar{v}^T)\vec{v}_x^2$

$(\because u\bar{v}^T \vec{v}_x^2 = 0)$

$(\because \bar{v}^T u I - u\bar{v}^T = -\bar{v}_x u_x)$

$(\because \bar{v}u^T - u\bar{v}^T = (u\times\bar{v})_x)$

$= -\bar{v}_x u_x \vec{v}_x^2 + (u\times\bar{v})_x \vec{v}_x^2$

$= -\bar{v}_x u_x \vec{v}_x^2 + (u_x\bar{v}_x - \bar{v}_x u_x)\vec{v}_x^2$

$= -2\bar{v}_x u_x \vec{v}_x^2 + u_x \bar{v}_x \vec{v}_x^2$

$(\because u_x \bar{v}_x \vec{v}_x^2 = u_x \bar{v}_x(\bar{v}\bar{v}^T - I) = -u_x\bar{v}_x)$

$= -2\bar{v}_x u_x \vec{v}_x^2 - u_x\bar{v}_x$

4. $\dfrac{\partial \bar{v}_x^2 u}{\partial v}$

$= \dfrac{\partial(\bar{v}\bar{v}^T - I)u}{\partial v}$

$= \dfrac{\partial \bar{v}\bar{v}^T u}{\partial v}$

$= (\bar{v}^T u)\dfrac{\partial\bar{v}}{\partial v} + \bar{v}\cdot\dfrac{\partial(\bar{v}^T u)}{\partial v}$

$= (\bar{v}^T u)\dfrac{\partial\bar{v}}{\partial v} + \bar{v}u^T\dfrac{\partial\bar{v}}{\partial v}$

$= (\bar{v}^T u I + \bar{v}u^T)\dfrac{\partial\bar{v}}{\partial v}$

$= (\bar{v}^T u I + \bar{v}u^T)\left(-\dfrac{\vec{v}_x^2}{\theta}\right)$

$= -\dfrac{1}{\theta}(\bar{v}^T u \cdot I + \bar{v}u^T)(\bar{v}_x)^2$

1. $\dfrac{\partial(\bar{v}_x u)}{\partial v}$

$= \dfrac{\partial(-u_x\bar{v})}{\partial\bar{v}}\cdot\dfrac{\partial\bar{v}}{\partial v}$

$= -u_x\dfrac{\partial\bar{v}}{\partial v}$

1.3

$\dfrac{\partial\bar{v}}{\partial v} = \dfrac{1}{\theta}(I - \bar{v}\bar{v}^T)$

$= -\dfrac{\bar{v}_x^2}{\theta}$

2. $\dfrac{\partial s\theta}{\partial v} = c\theta\cdot\dfrac{\partial\theta}{\partial v} = c\theta\,\bar{v}^T$

3. $\dfrac{\partial(1-c\theta)}{\partial v} = s\theta\cdot\dfrac{\partial\theta}{\partial v} = s\theta\,\bar{v}^T$

7. $R^T \bar{v}_x$

$$= c\theta \bar{v}_x - s\theta \bar{v}_x^2 + (1-c\theta)\bar{v}\bar{v}^T\bar{v}_x$$

$$\left(\because \bar{v}^T \bar{v}_x u = |(\bar{v}, \bar{v}, u)| = 0, \forall u \in R^3 \quad \therefore \bar{v}^T \bar{v}_x = 0 \right)$$

$$= c\theta \bar{v}_x - s\theta \bar{v}_x^2$$

8. $R^T (c\theta \bar{v}_x + s\theta \bar{v}_x^2) u \bar{v}^T$

$$= (c\theta R^T \bar{v}_x + s\theta R^T \bar{v}_x^2) u \bar{v}^T$$

$$= (c^2\theta \bar{v}_x - c\theta s\theta \bar{v}_x^2 + s\theta c\theta \bar{v}_x^2 - s^2\theta \bar{v}_x^3) u \bar{v}^T$$

$$= (c^2\theta \bar{v}_x - s^2\theta \bar{v}_x^3) u \bar{v}^T$$

$$\left(\because \bar{v}_x^3 = \bar{v}_x(\bar{v}\bar{v}^T - I) = -\bar{v}_x \right)$$

$$= (c^2\theta \bar{v}_x + s^2\theta \bar{v}_x) u \bar{v}^T$$

$$= \bar{v}_x u \bar{v}^T = -u_x \bar{v}\bar{v}^T$$

6. $\cancel{\partial Ru}$

$$Ru = u + s\theta \bar{v}_x u + (1-c\theta)\bar{v}_x^2 u \Rightarrow \frac{\partial (Ru)}{\partial v}$$

Solve: $\dfrac{\partial (Ru)}{\partial v} = s\theta \dfrac{\partial (\bar{v}_x u)}{\partial v} + \bar{v}_x u \dfrac{\partial s\theta}{\partial v} + (1-c\theta)\dfrac{\partial (\bar{v}_x^2 u)}{\partial v} + \bar{v}_x^2 u \dfrac{\partial (1-c\theta)}{\partial v}$

$$= s\theta \cdot u_x \frac{\bar{v}_x^2}{\theta} + \bar{v}_x u \cdot c\theta \bar{v}^T + \bar{v}_x^2 u \cdot s\theta \bar{v}^T + (1-c\theta)\left(-\frac{1}{\theta}\right)(-2\bar{v}_x u_x \bar{v}_x^2 - u_x \bar{v}_x)$$

$$= (c\theta \bar{v}_x + s\theta \bar{v}_x^2) u \bar{v}^T + \frac{s\theta}{\theta} u_x \bar{v}_x^2 + \frac{1-c\theta}{\theta}(2\bar{v}_x u_x \bar{v}_x^2 + u_x \bar{v}_x)$$

$$\overset{(\text{from } 10)}{=} -Ru_x \left(\bar{v}\bar{v}^T + \frac{1}{\theta}(R^T - I)\bar{v}_x \right)$$

here $T = s\theta u_x \bar{v}_x^2 + (1-c\theta) u_x \bar{v}_x + T_{part}$.

10. $\because R^T \frac{\partial (Ru)}{\partial v}$

$= -u_x \bar{v} \bar{v}^T + \frac{1}{\theta}(-1) u_x (R^T - I) \bar{v}_x$

$\therefore \frac{\partial (R\cancel{u})}{\partial v}$

$= -R u_x \left(\bar{v}\bar{v}^T + \frac{1}{\theta}(R^T - I)\bar{v}_x \right)$

$\therefore$ 6 proved.

9. $R^T\left(\frac{s\theta}{\theta} u_x \bar{v}_x^2 + \frac{1-c\theta}{\theta}(2\bar{v}_x u_x \bar{v}_x^2 + u_x \bar{v}_x)\right)$

$= \frac{1}{\theta}(I - s\theta\bar{v}_x + (1-c\theta)\bar{v}_x^2)$

$\cdot (s\theta u_x \bar{v}_x^2 + (1-c\theta)(2\bar{v}_x u_x \bar{v}_x^2 + u_x \bar{v}_x))$

$= \cancel{\theta} \ \frac{1}{\theta} \cdot T$

$T = s\theta u_x \bar{v}_x^2 + (1-c\theta)(2\bar{v}_x u_x \bar{v}_x \bar{v}_x + u_x \bar{v}_x)$

$- s^2\theta \bar{v}_x u_x \bar{v}_x^2 - s\theta(1-c\theta)(2\bar{v}_x^2 u_x \bar{v}_x^2 + \bar{v}_x u_x \bar{v}_x)$

$+ (1-c\theta)s\theta \bar{v}_x^2 u_x \bar{v}_x^2 + (1-c\theta)^2(2\bar{v}_x^2 \bar{v}_x u_x \bar{v}_x^2 + \bar{v}_x^2 u_x \bar{v}_x)$

$= s\theta u_x \bar{v}_x^2 + (1-c\theta) u_x \bar{v}_x$

$+ (u^T\bar{v})\left(-2(1-c\theta) + s^2\theta + (1-c\theta)^2\right)\bar{v}_x^2$

$= u_x \left(s\theta \bar{v}_x^2 + (1-c\theta)\bar{v}_x^3 \right) = -u_x (R^T - I)\bar{v}_x$

$(\because \bar{v}_x^3 = -\bar{v}_x)$ $(\because s^2\theta + (1-c\theta)^2 - 2(1-c\theta) = 0)$

11.

$$\frac{\partial R}{\partial v_i} = \frac{v_i \bar{v}_x + (v \times (I-R)e_i)_x}{\|v\|^2} R$$

proof: $\forall u$, independent of v,

$$\frac{\partial R}{\partial v_i} u = \frac{\partial (Ru)}{\partial v_i} = \frac{\partial (Ru)}{\partial v} e_i$$

$$= -Ru_x \left(\bar{v}\bar{v}^T + \frac{1}{\theta}(R^T - I)\bar{v}_x \right) e_i$$

$(\because R^T \bar{v}_x = \bar{v}_x R^T$ (from the form of R))

$$= -Ru_x \left(\bar{v}\bar{v}^T + \frac{1}{\theta}\bar{v}_x (R^T - I) \right) e_i$$

$$= R\left[\left(\bar{v}\bar{v}^T + \frac{1}{\theta}\bar{v}_x (R^T - I) \right) e_i \right]_x u$$

$$= \left[\left(\bar{v}\bar{v}^T + \frac{1}{\theta}\bar{v}_x (I - R) \right) e_i \right]_x Ru$$

$\left(\because R\left[\left(\bar{v}\bar{v}^T + \frac{1}{\theta}\bar{v}_x (R^T - I) \right) e_i \right]_x = \left[\left(\bar{v}\bar{v}^T + \frac{1}{\theta}\bar{v}_x (I - R) \right) e_i \right]_x R \right.$ (from 13) $\left. \right)$

$\therefore$ proved.

9.3.

$$\bar{v}_x u_x \bar{v}_x = -(u^T \bar{v}) \bar{v}_x$$

$$T_{part} = (1-c\theta)\left(-2(u^T\bar{v})\bar{v}_x^2\right)$$
$$+ s^2\theta (u^T \bar{v}) \bar{v}_x^2$$
$$-s\theta(1-c\theta)\left(-2\bar{v}_x (u^T\bar{v})\bar{v}_x^2 - (u^T\bar{v})\bar{v}_x\right)$$
$$+(1-c\theta)s\theta\, \bar{v}_x (-u^T\bar{v})\bar{v}_x^2$$
$$+(1-c\theta)^2\left(2\bar{v}_x^2(-u^T\bar{v})\bar{v}_x^2 + \bar{v}_x(-u^T\bar{v})\bar{v}_x\right)$$

$$= (-2)(1-c\theta)(u^T\bar{v})\bar{v}_x^2$$
$$+ s^2\theta\, (u^T\bar{v})\bar{v}_x^2$$
$$- s\theta(1-c\theta)\left(\cancel{2(u^T\bar{v})\bar{v}_x - (u^T\bar{v})\bar{v}_x}\right)$$
$$(u^T\bar{v})\bar{v}_x$$
$$+ (1-c\theta)s\theta\,(u^T\bar{v})\bar{v}_x$$
$$+ (1-c\theta)^2\left(\cancel{2(u^T\bar{v})\bar{v}_x^2 - (u^T\bar{v})\bar{v}_x^2}\right)$$
$$(u^T\bar{v})\bar{v}_x^2$$

$$= \left(-2(1-c\theta) + s^2\theta + (1-c\theta)^2 \right)(u^T\bar{v})\bar{v}_x^2$$

$$= e_i\bar{v}^T + \bar{v}e_i^T$$

$$- c\theta e_i\bar{v}^T - s\theta\, \bar{v}_\times e_i\bar{v}^T - (1-c\theta)\bar{v}\bar{v}^T e_i\bar{v}^T$$

$$- c\theta \bar{v}e_i^T - s\theta\, \bar{v}e_i^T\bar{v}_\times - (1-c\theta)\bar{v}e_i^T\bar{v}\bar{v}^T$$

$$= (1-c\theta)(e_i\bar{v}^T + \bar{v}e_i^T - 2\bar{v}_i\bar{v}\bar{v}^T)$$

$$- s\theta(\bar{v}_\times e_i\bar{v}^T + \bar{v}e_i^T\bar{v}_\times)$$

$$\Big(\because\ \bar{v}_\times(e_i\bar{v}^T) + (e_i\bar{v}^T)^T\bar{v}_\times$$

$$= tr(e_i\bar{v}^T)\bar{v}_\times - (e_i\bar{v}^T\bar{v})_\times$$

$$= \bar{v}_i\bar{v}_\times - \cancel{\bar{v}}e_{i\times} \Big)$$

③ $\therefore \dfrac{\partial R}{\partial v_i} = \bar{v}_i(c\theta\bar{v}_\times + s\theta\bar{v}_\times^2)$

$$- \frac{s\theta}{\theta}(\bar{v}_i\bar{v}_\times - e_{i\times})$$

$$+ \frac{(1-c\theta)}{\theta}(e_i\bar{v}^T + \bar{v}e_i^T - 2\bar{v}_i\bar{v}\bar{v}^T)$$

proved.

12. let $\dfrac{\partial R}{\partial v_i} = \bar{v}_i\bar{v}_\times R + \dfrac{1}{\theta}[\bar{v}_\times(I-R)e_i]_\times R$

$$\Rightarrow \frac{\partial R}{\partial v_i} = \bar{v}_i(c\theta\bar{v}_\times + s\theta\bar{v}_\times^2)$$

$$+ \frac{s\theta}{\theta}(e_i - \bar{v}_i\bar{v})_\times$$

$$+ \frac{1-c\theta}{\theta}(e_i\bar{v}^T + \bar{v}e_i^T - 2\bar{v}_i\bar{v}\bar{v}^T)$$

proof: ① $\bar{v}_\times R = c\theta\bar{v}_\times + s\theta\bar{v}_\times^2$

$$(\because R = c\theta I + s\theta\bar{v}_\times + (1-c\theta)\bar{v}\bar{v}^T)$$

② $(\bar{v}\times(I-R)e_i)_\times R$

$$= (I-R)e_i\bar{v}^T\cancel{R} - \bar{v}e_i^T(I-R)^T R$$

$$= (I-R)e_i\bar{v}^T - \bar{v}e_i^T(R-I) \quad (\because R^T\bar{v} = \bar{v})$$

$$= e_i\bar{v}^T + \bar{v}e_i^T - Re_i\bar{v}^T - \bar{v}e_i^T R$$

14. $\frac{\partial e^{v_\times}}{\partial v_i}\Big|_{v=0} = e_{i\times}$

proof: $\because \frac{\partial R}{\partial v_i} = \bar{v}_i \bar{v}_\times R + \frac{1}{\theta}(\bar{v}_\times (I-R)e_i)_\times R$

$(\because \lim_{v\to 0} R = I,\ \lim_{v\to 0}(I-R)/\theta = -\bar{v}_\times$ (from 15) $)$

$\therefore \lim_{v\to 0}\frac{\partial R}{\partial v_i} = \bar{v}_i \bar{v}_\times - (\bar{v}_\times^2 e_i)_\times$

$= (\bar{v}_i \bar{v} - \bar{v}_\times^2 e_i)_\times$

$= (\bar{v}_i \bar{v} - \bar{v}\bar{v}^T e_i + e_i)_\times$

$= (e_i)_\times$

15. $\frac{I-R}{\theta} = -\frac{s\theta}{\theta}\bar{v}_\times - \frac{1-c\theta}{\theta}\bar{v}_\times^2$

$\therefore \lim_{\theta\to 0}\frac{I-R}{\theta} = -\bar{v}_\times \quad \therefore \lim_{v\to 0}\frac{I-R}{\theta} = -\bar{v}_\times$ ②

13. $R(\bar{v}\bar{v}^T e_i + \frac{1}{\theta}\bar{v}_\times (R^T - I)e_i)_\times$

$= (\bar{v}\bar{v}^T e_i + \frac{1}{\theta}\bar{v}_\times (I-R)e_i)_\times R$

proof: ① $\Leftrightarrow \bar{v}_i R \bar{v}_\times + \frac{1}{\theta}R(\bar{v}\times(R^T-I)e_i)_\times$

$= \bar{v}_i \bar{v}_\times R + \frac{1}{\theta}(\bar{v}\times(I-R)e_i)_\times R$

$(\because R\bar{v}_\times = \bar{v}_\times R)$

$\Leftrightarrow R(\bar{v}\times(R^T-I)e_i)_\times$

$= (\bar{v}\times(I-R)e_i)_\times R$

② $\Leftrightarrow R((R^T-I)e_i\bar{v}^T - \bar{v}e_i^T(R-I))$

$= ((I-R)e_i\bar{v}^T - \bar{v}e_i^T(I-R^T))R$

$\Leftrightarrow (I-R)e_i\bar{v}^T - R\bar{v}e_i^T(R-I)$

$= (I-R)e_i\bar{v}^T R - \bar{v}e_i^T(R-I)$

$(\because R^T\bar{v} = \bar{v},\ R\bar{v} = \bar{v})$ $\quad\therefore$ proved.

1. $R = e^{v_\times} \Rightarrow \frac{\partial R}{\partial v}$

solve: let $v = \|v\|\bar{v} = \|v\| r = \theta$

① $\because R = c\theta I + s\theta r_\times + (1-c\theta) r r^T$

$\therefore \frac{\partial R}{\partial v_i} = -s\theta r_i I + c\theta r_i r_\times + s\theta \frac{d r_\times}{d v_i} + s\theta r_i r r^T + (1-c\theta)\frac{d r r^T}{d v_i}$

② $\because \frac{d r_\times}{d v_i} = \frac{1}{\theta}(e_i - r_i r)_\times$

$\frac{d r r^T}{d v_i} = \frac{1}{\theta}(e_i r^T + r e_i^T - 2 r_i r r^T)$

③ $\therefore \frac{\partial R}{\partial v_i} = a_0 I + a_1 r r^T + a_2 (e_i r^T + r e_i^T) + a_3 r_\times + a_4 e_{i\times}$

here $a_0 = -s\theta r_i$, $a_1 = \left(s\theta - \frac{2}{\theta}(1-c\theta)\right) r_i$

$a_2 = \frac{1-c\theta}{\theta}$

$a_3 = \left(c\theta - \frac{s\theta}{\theta}\right) r_i$

$a_4 = \frac{s\theta}{\theta}$

④ $\because \lim_{\theta\to 0} a_4 = 1$, $\lim_{\theta \to 0} a_i = 0, i \neq 4$

$\because$ r_i has bound, for $i = 1, 2, 3$

($\because \|r\| = \|\frac{v}{\theta}\| = 1$)

$\therefore \lim_{v\to 0} \frac{\partial R}{\partial v_i} = e_{i\times}$