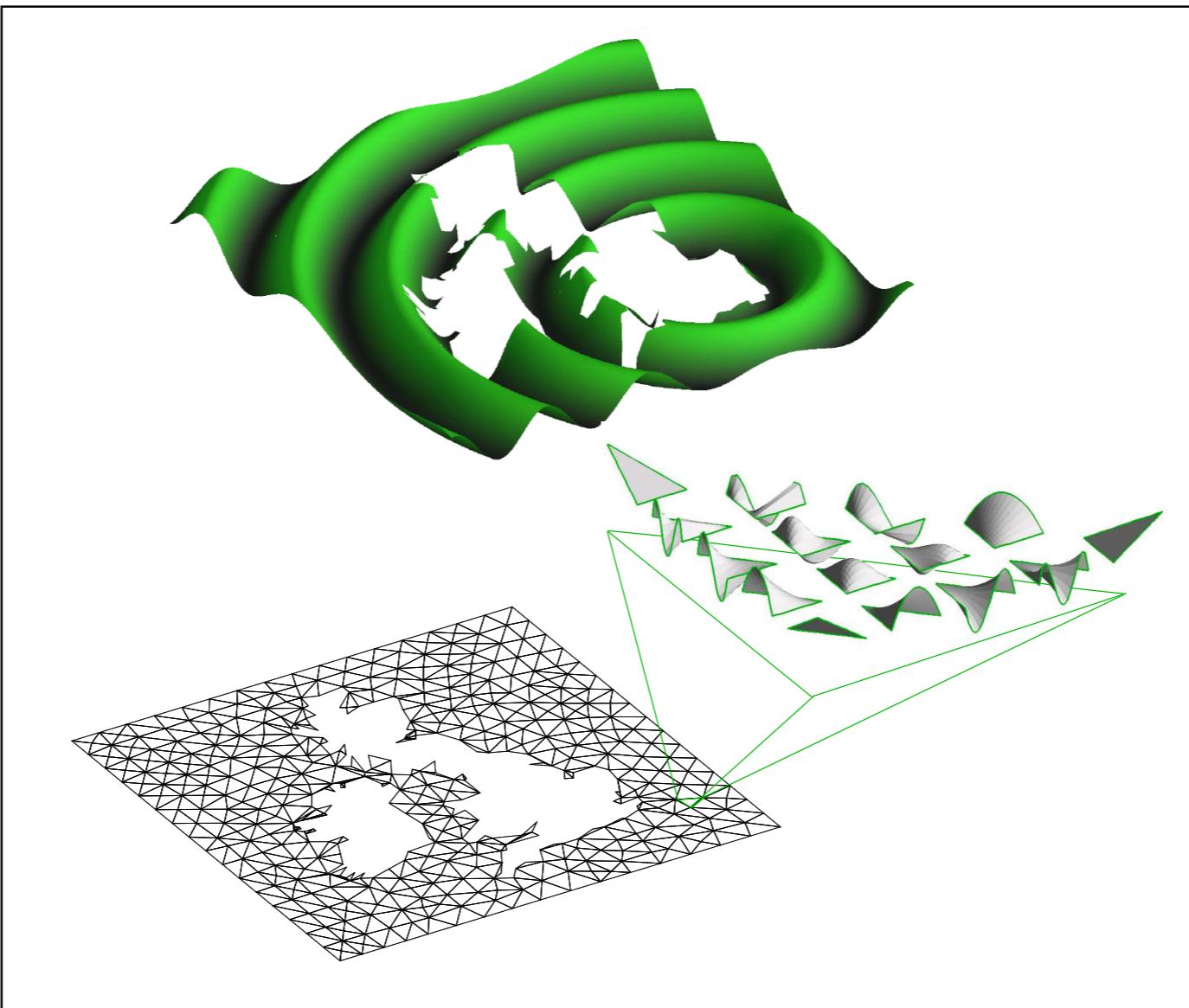
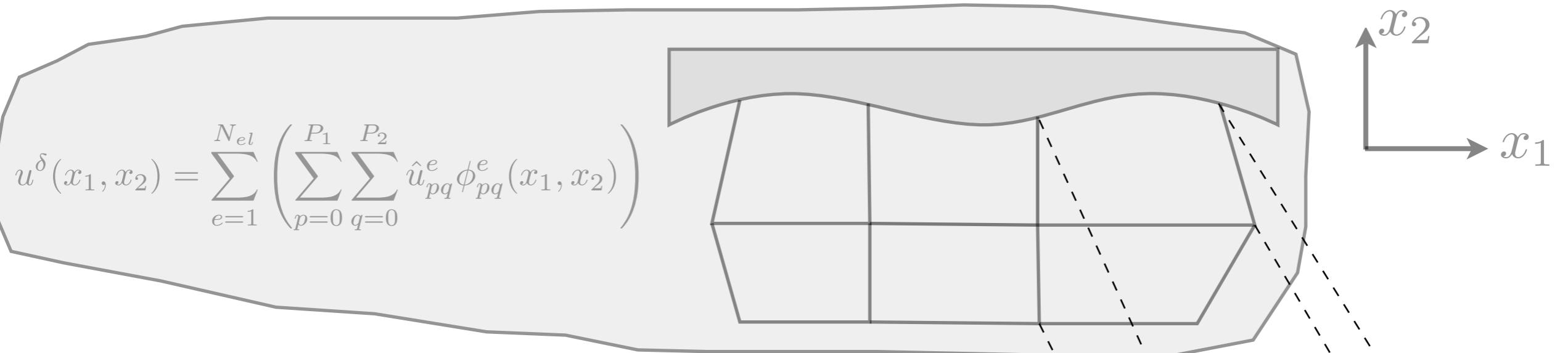


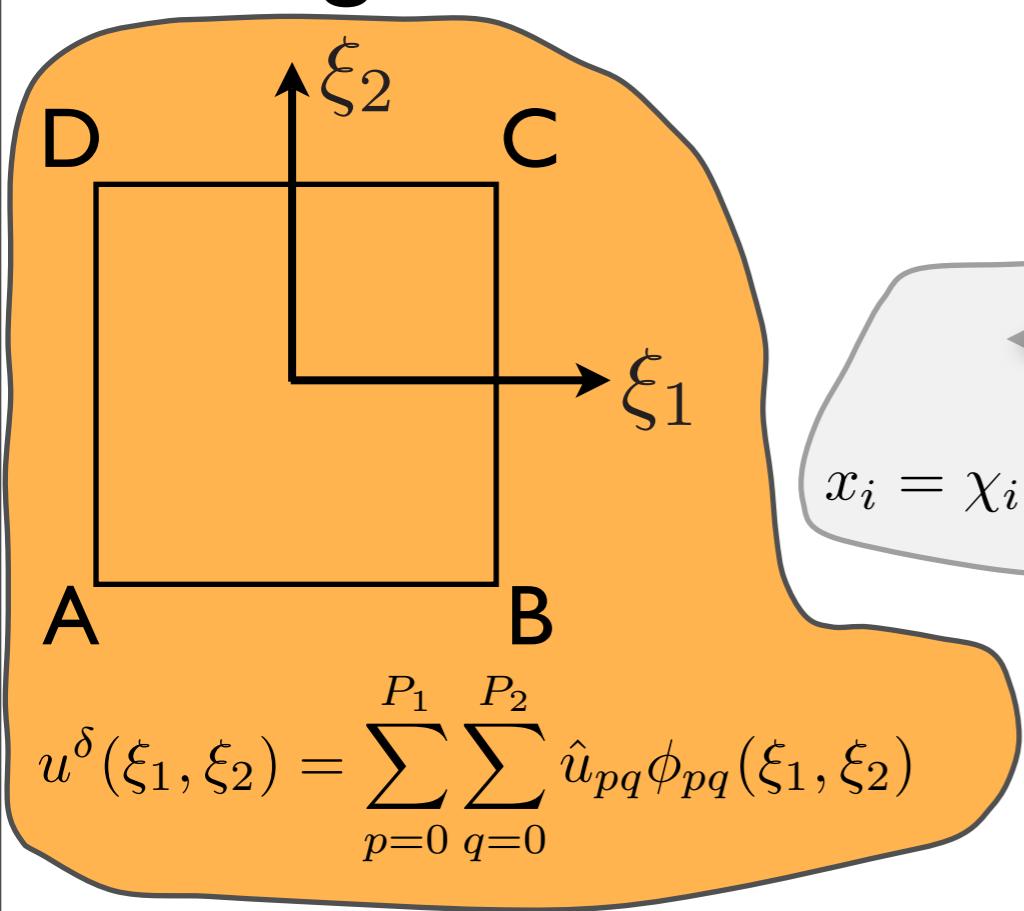
Expansions in Standard Regions



The big picture



StdRegions



$$u^\delta(\xi_1, \xi_2) = \sum_{p=0}^{P_1} \sum_{q=0}^{P_2} \hat{u}_{pq} \phi_{pq}(\xi_1, \xi_2)$$

MultiRegions

SpatialDomains

$$x_i = \chi_i(\xi_1, \xi_2) = \sum_{p=0}^{P_1} \sum_{q=0}^{P_2} \hat{x}_{pq}^i \phi_{pq}(\xi_1, \xi_2)$$

LocalRegions

$$u^\delta(x_1, x_2) = \sum_{p=0}^{P_1} \sum_{q=0}^{P_2} \hat{u}_{pq} \phi_{pq}(x_1, x_2)$$

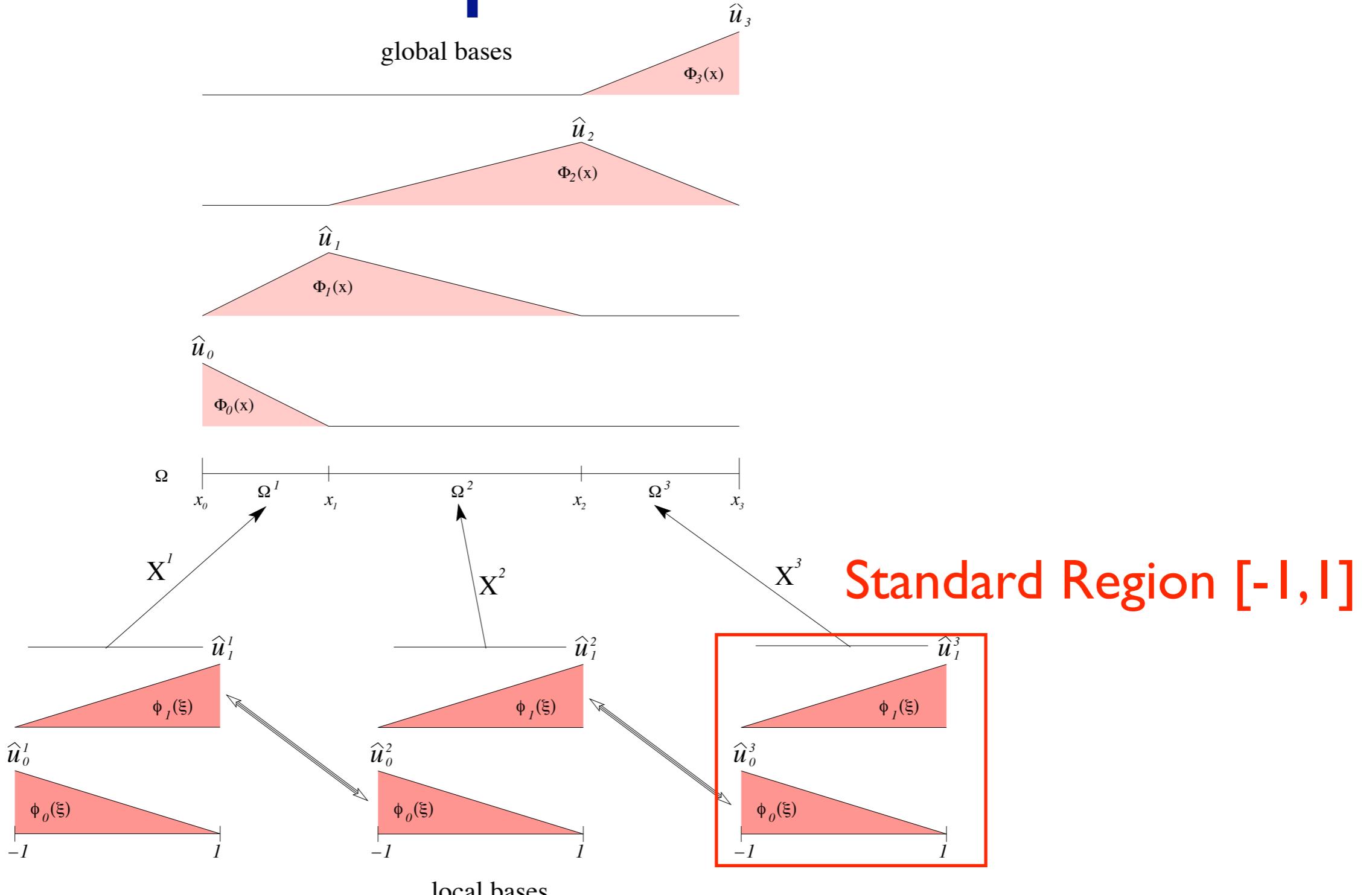
Outline

- Choice of Tensorial Expansions
 - Standard Segments
(StdRegions::StdSegExp)
 - Standard Quadrilaterals
(StdRegions::StdQuadExp)
 - Standard Triangles
(StdRegions::StdTriExp)
- Sum Factorisation of tensorial bases (notes: 3.1.6)

1	Fundamental Concepts in One Dimension	1
1.1	Method of Weighted Residuals	2
1.2	Galerkin Formulation	5
1.2.1	Descriptive Formulation	5
1.2.2	Two-Domain Linear Finite Element Example	8
1.2.3	Mathematical Formulation	11
1.2.4	Mathematical Properties of the Galerkin Approximation	13
1.2.5	Residual Equation for C^0 Test and Trial Functions	15
1.3	One-Dimensional Expansion Bases	16
1.3.1	Elemental Decomposition: The h -Type Extension	16
1.3.2	Polynomial Expansions: The p -Type Extension	22
1.3.3	Modal Polynomial Expansions	28
1.3.4	Nodal Polynomial Expansions	31
1.4	Elemental Operations	34
1.4.1	Numerical Integration	34
1.4.2	Differentiation	39
1.5	Error Estimates	42
1.5.1	h -Convergence of Linear Finite Elements	42
1.5.2	L^2 Error of the p -Type Interpolation in a Single Element	44
1.5.3	General Error Estimates for hp Elements	45
1.6	Implementation of a 1D Spectral/ hp Element Solver:	46
1.6.1	Exercises	46
1.6.2	Convergence Examples	50
2	Multi-dimensional Expansion Bases	53
2.1	Quadrilateral and Hexahedral Tensor Product Expansions	55
2.1.1	Standard Tensor Product Extensions	55
2.1.2	Polynomial Space of Tensor Product Expansions	59
2.2	Generalised Tensor Product Modal Expansions	60
2.2.1	Coordinate Systems	62
2.2.2	Orthogonal Expansions	68
2.2.3	Modified C^0 Expansions	75
2.3	Non-Tensorial Nodal Expansions in a Simplex	83
2.3.1	The Lagrange Polynomial and Lebesgue Constant	85
2.3.2	Generalised Vandemonde Matrix	86
2.3.3	Electrostatic Points	87
2.3.4	Fekete Points	88
2.4	Other Useful Tensor Product Extensions	91
2.4.1	Nodal Elements in a Prismatic Region	92
2.4.2	Expansions in Homogeneous Domains	92
2.4.3	Cylindrical Domains	93
2.5	Exercises: Construction of Multi-Dimensional Elemental Mass Matrices	93

Classic linear finite elements

expansion



Polynomial Expansion

1.3.2.1 Construction of a Polynomial Expansion

In an hp elemental discretisation we can apply a polynomial expansion of any order within each elemental region. It is therefore appropriate to start our discussion of p -type methods by considering what makes an acceptable p -type expansion in a single domain.

The steps involved in designing an elemental p -type expansion, which we will also later adopt in constructing the unstructured basis in section 2.2, are:

- Determine a favourable expansion within a standard region.
- Modify the expansion so that it can easily be numerically implemented.

In the first step, a favourable expansion is typically an orthogonal or near orthogonal set of functions within the standard regions. In the second step, the computational considerations of implementing this basis are taken into account and the basis is modified, if necessary, to facilitate this process. Typically, the basis is decomposed into contributions on the boundary and interior of the standard region since this simplifies the elemental decomposition process.

Choice of an Expansion Bases

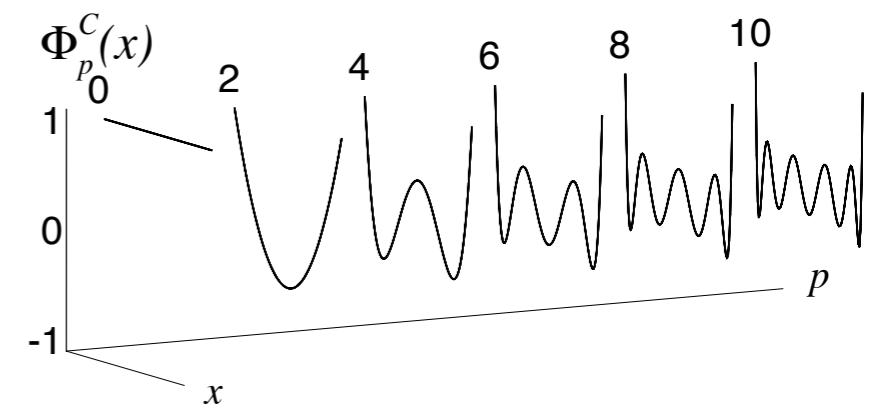
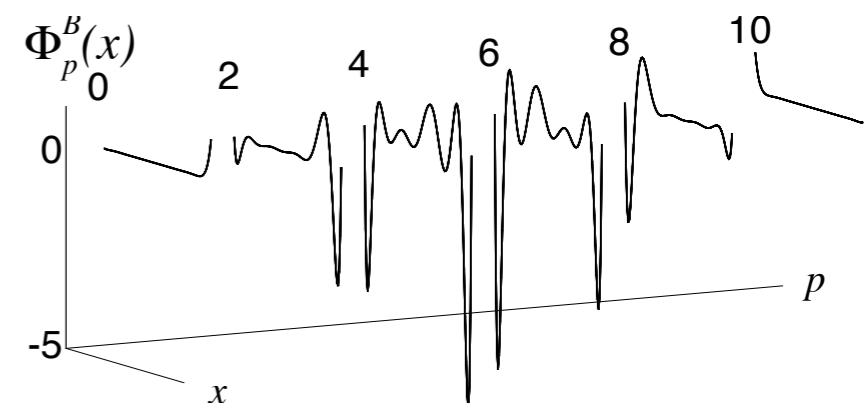
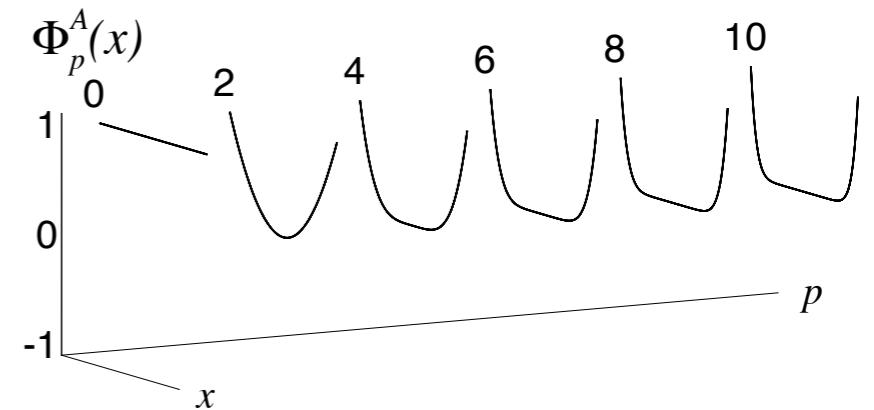
Consider three expansion:

- $\Phi_p^A(x)$, increases the order of x is a *moment* expansion (each order contributing an extra moment to the expansion).
 - Basis is *hierarchical modal*.
- $\Phi_p^B(x)$ is a Lagrange polynomial based on a series of $P + 1$ nodal points x_q .
 - Lagrange polynomial is a non-hierarchical basis
 - The Lagrange basis has the property that $\Phi_p^B(x_q) = \delta_{pq}$

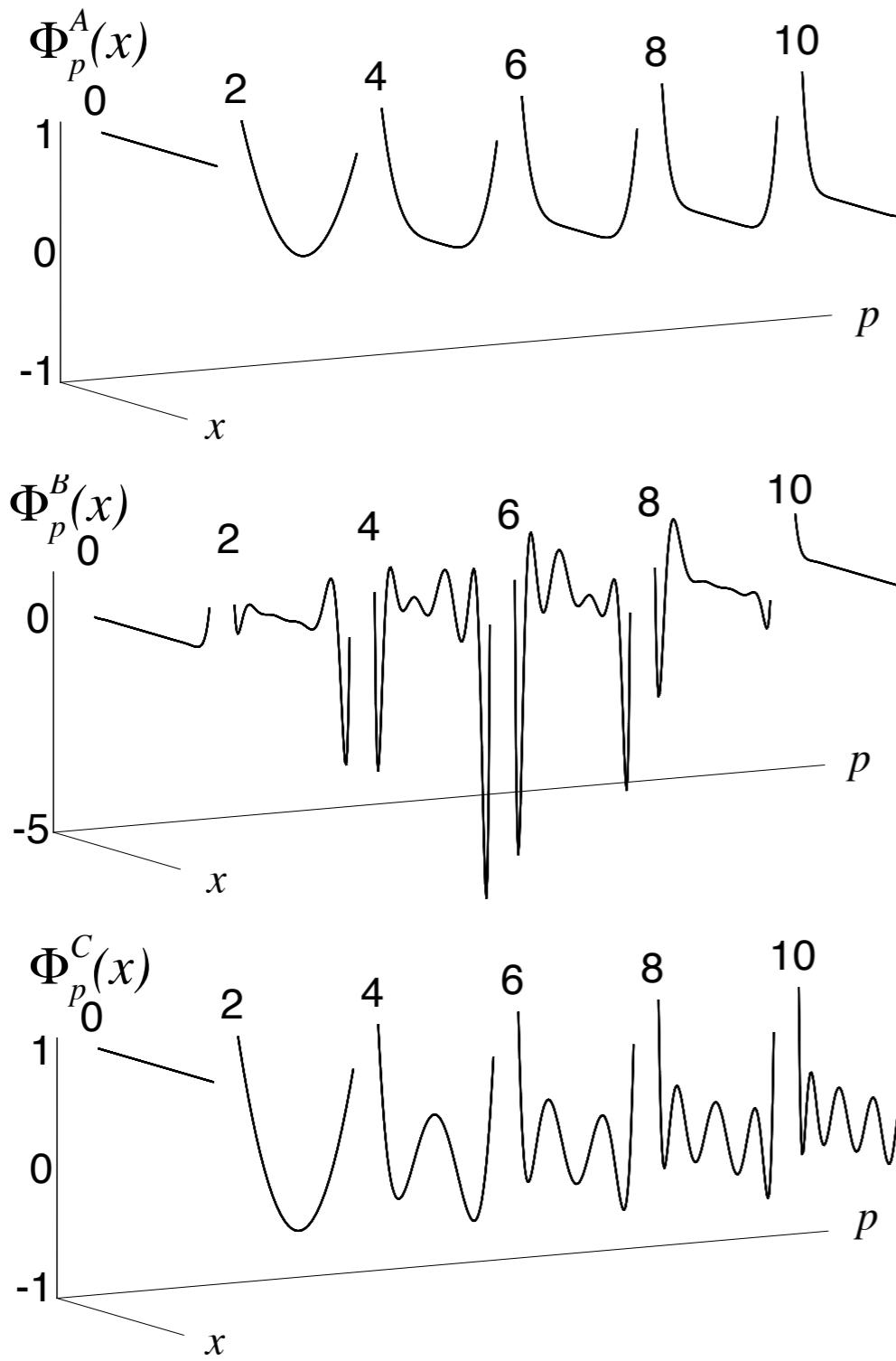
$$u^\delta(x) = \sum_{p=0}^P \hat{u}_p \Phi_p^B(x),$$

- $\Phi_p^c(x)$, is a hierarchical modal expansion.
 - Based on the Legendre polynomial $L_p(x)$. which is orthogonal in the Legendre inner product

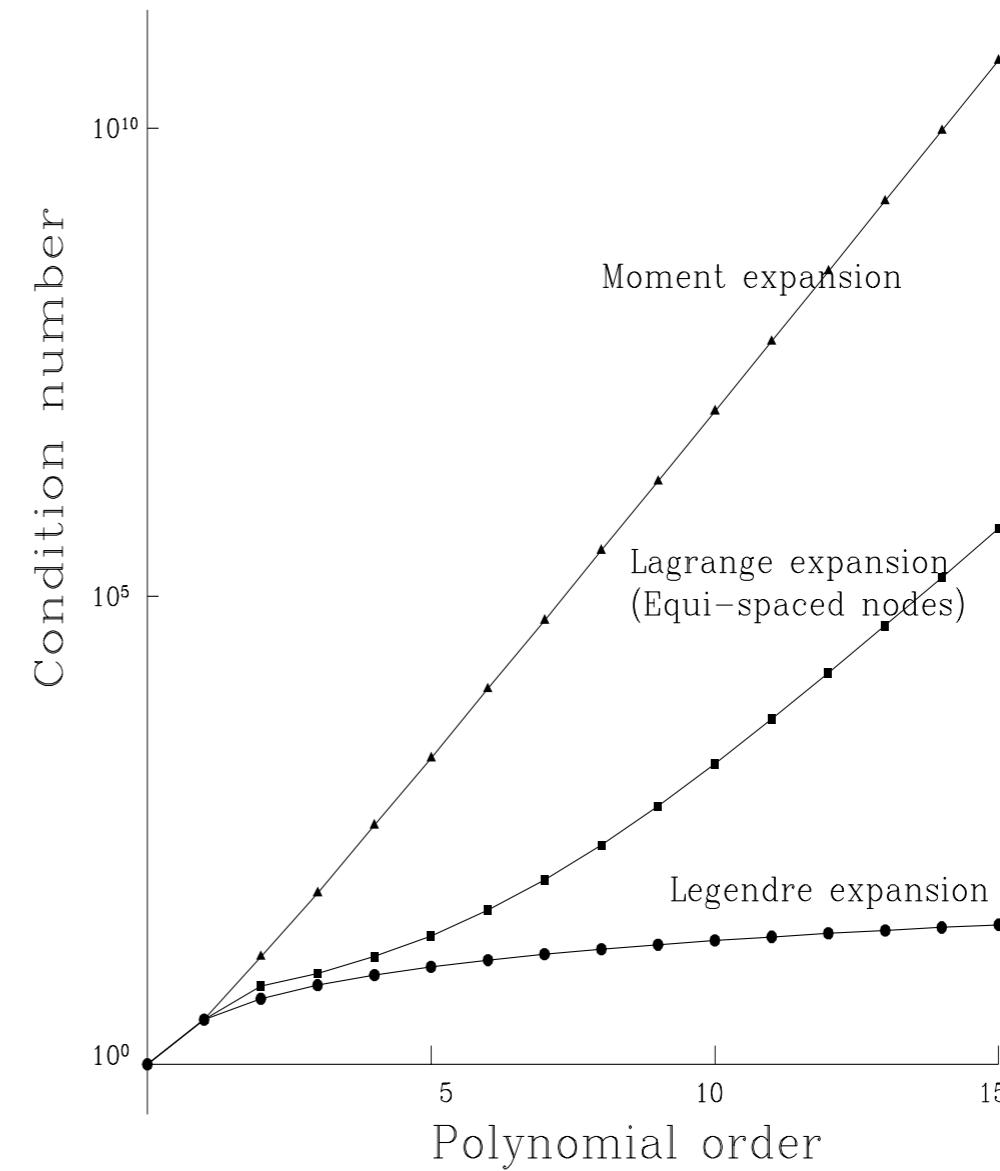
$$(L_p(x), L_q(x)) = \int_{-1}^1 L_p(x) L_q(x) dx = \left(\frac{2}{2p+1} \right) \delta_{pq}.$$



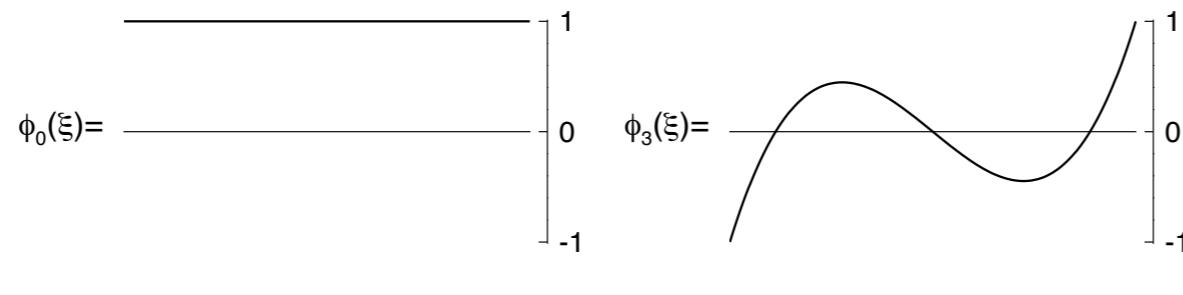
Choice of an Expansion Bases



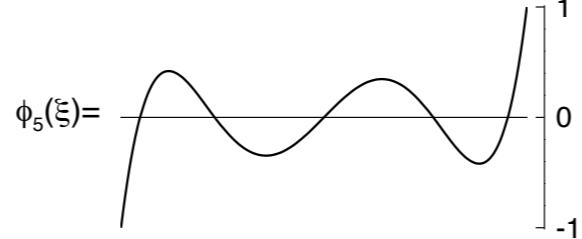
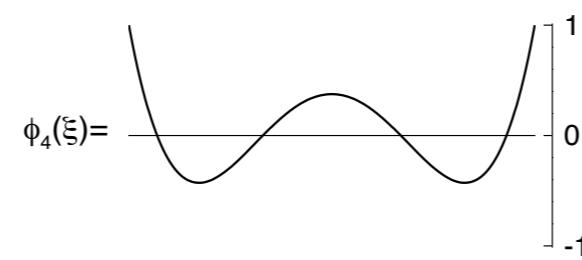
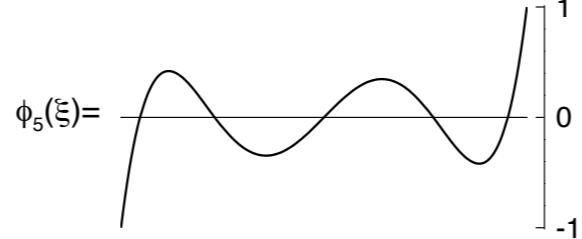
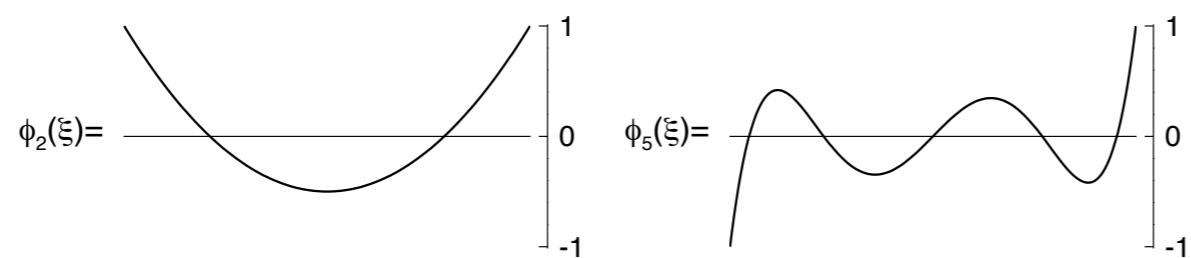
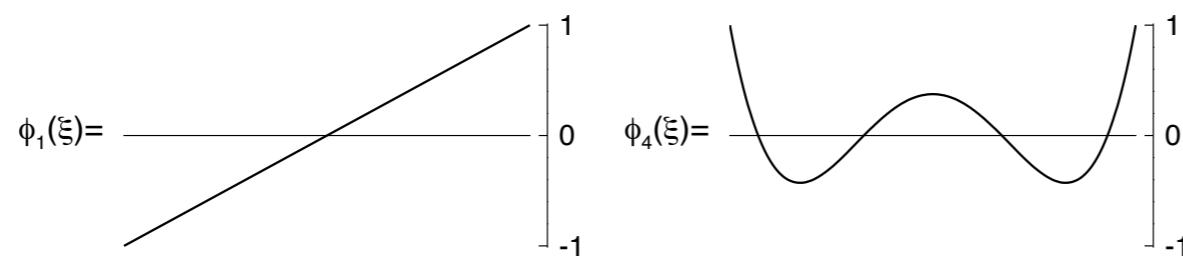
$$\kappa_2 = \|\mathbf{M}\|_2 \|\mathbf{M}^{-1}\|_2$$



Legendre expansion



$$\Omega_{st} = \{\xi \mid -1 \leq \xi \leq 1\} ,$$



$$\phi_p(\xi) \mapsto L_p(\xi) \equiv P_p^{(0,0)}(\xi) , \quad 0 \leq p \leq P .$$

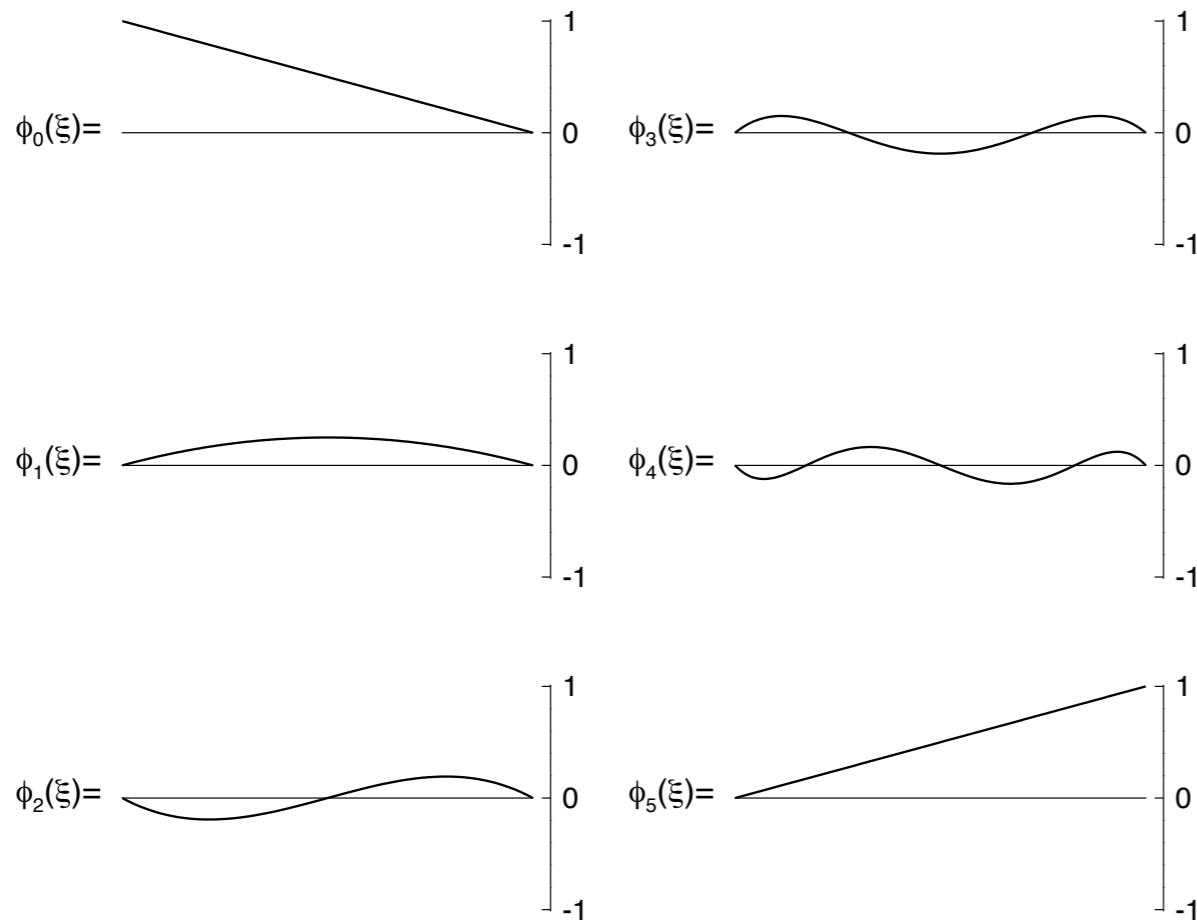
Jacobi
Polynomials

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_n^{\alpha,\beta}(x) P_i^{\alpha,\beta}(x) d\xi = C \delta_{ni} ,$$

Boundary-Interior Decomposition

- “Best” choice for our expansion appears to be the Legendre polynomial.
- But also want to combine the expansion with the h -type elemental decomposition.
- Difficulty arises when we try to ensure a degree of continuity in the global expansion at elemental boundaries.
- Numerically efficient way of achieving C^0 continuity is to design an expansion where only some modes have a magnitude at elemental boundary
- This type of decomposition is known as *boundary* and *interior* decomposition.

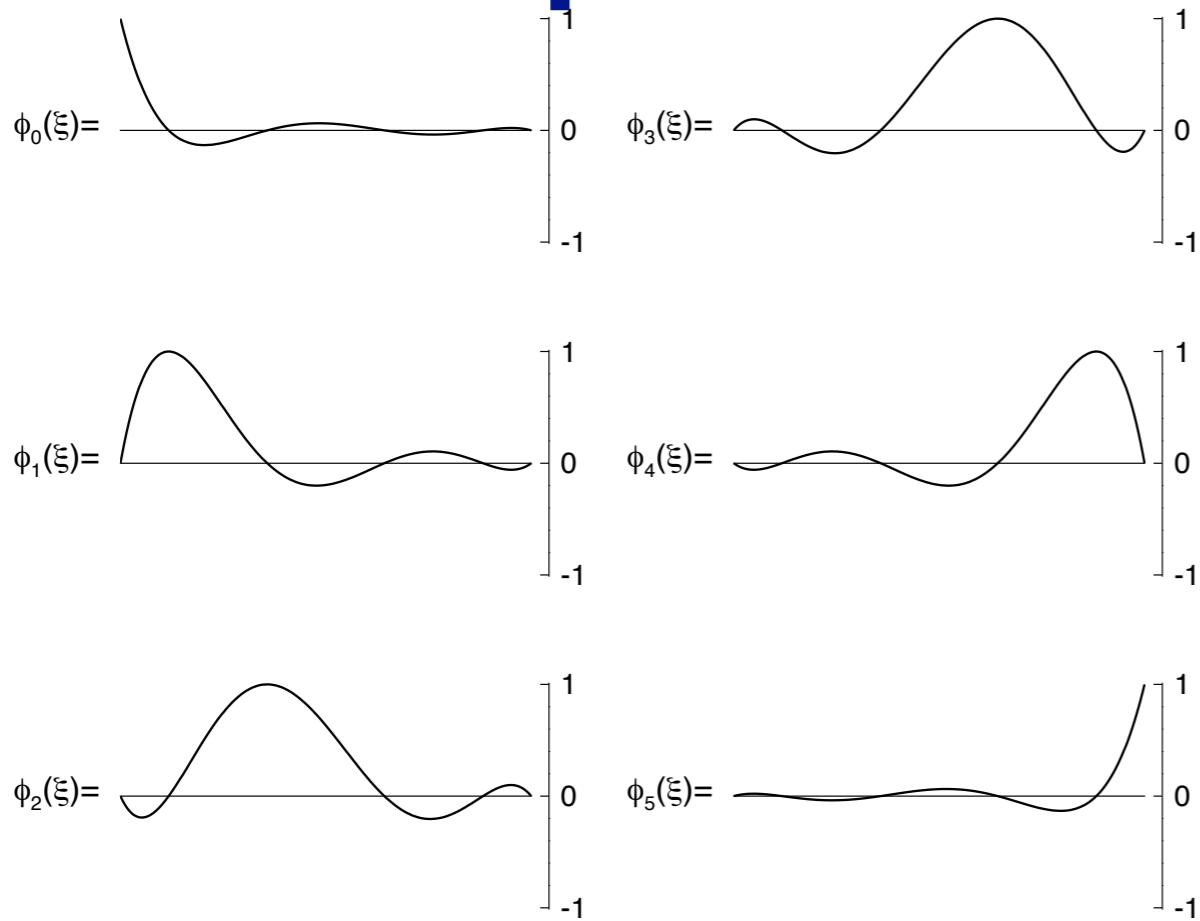
P-type finite elements



$$\Omega_{st} = \{\xi \mid -1 \leq \xi \leq 1\} ,$$

$$\phi_p(\xi) \mapsto \psi_p(\xi) = \begin{cases} \left(\frac{1-\xi}{2}\right) & p = 0 \\ \left(\frac{1-\xi}{2}\right) \left(\frac{1+\xi}{2}\right) P_{p-1}^{1,1}(\xi) & 0 < p < P \\ \left(\frac{1+\xi}{2}\right) & p = P \end{cases}$$

Spectral Elements



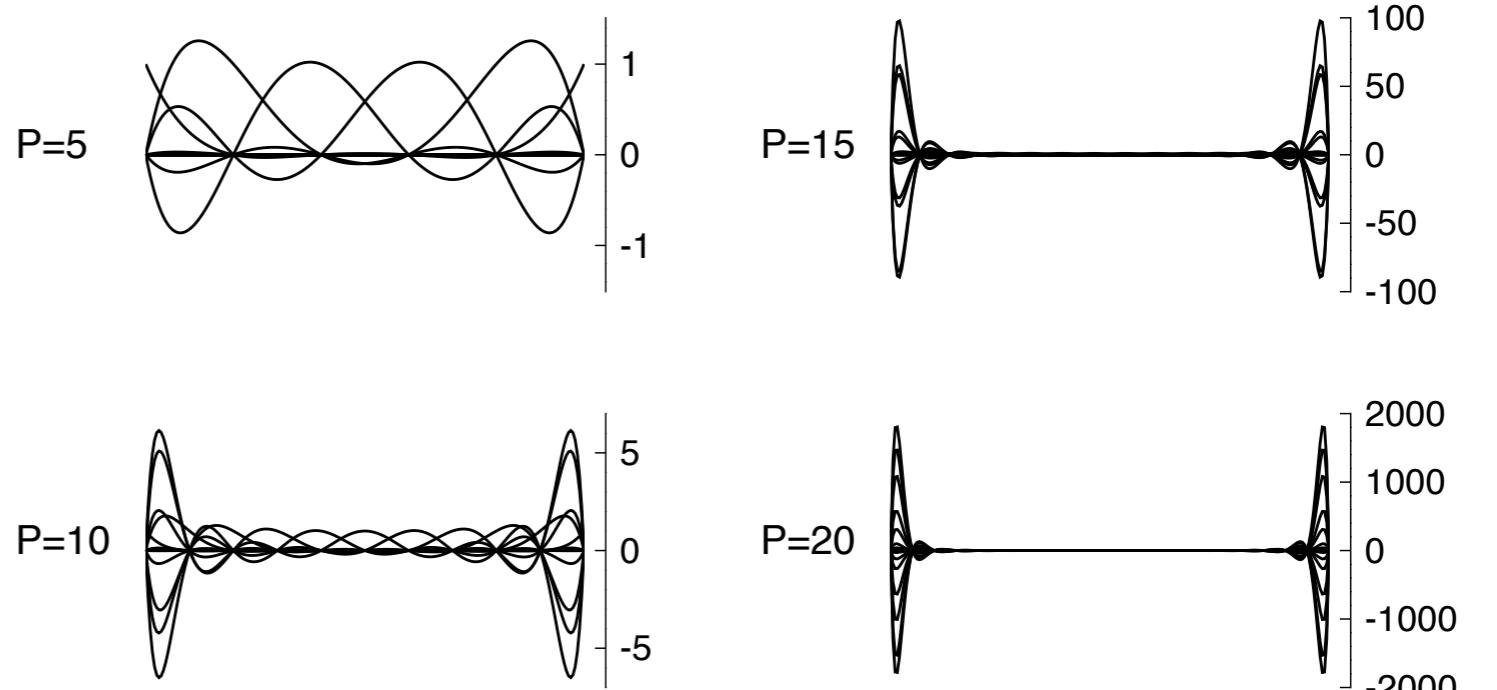
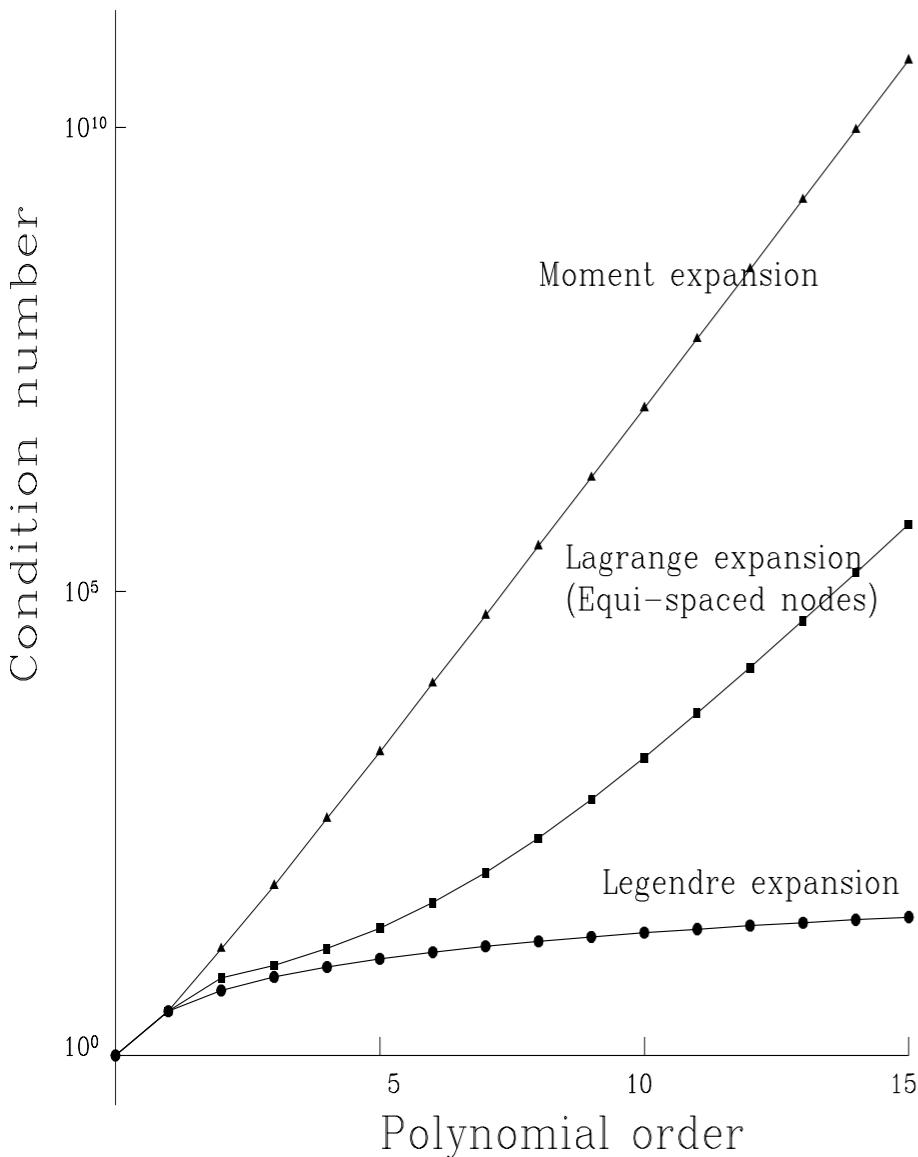
$$\Omega_{st} = \{\xi \mid -1 \leq \xi \leq 1\} ,$$

$$h_p(x) = \frac{\prod_{q=0, q \neq p}^{Q-1} (x - x_q)}{\prod_{q=0, q \neq p}^{Q-1} (x_p - x_q)} .$$

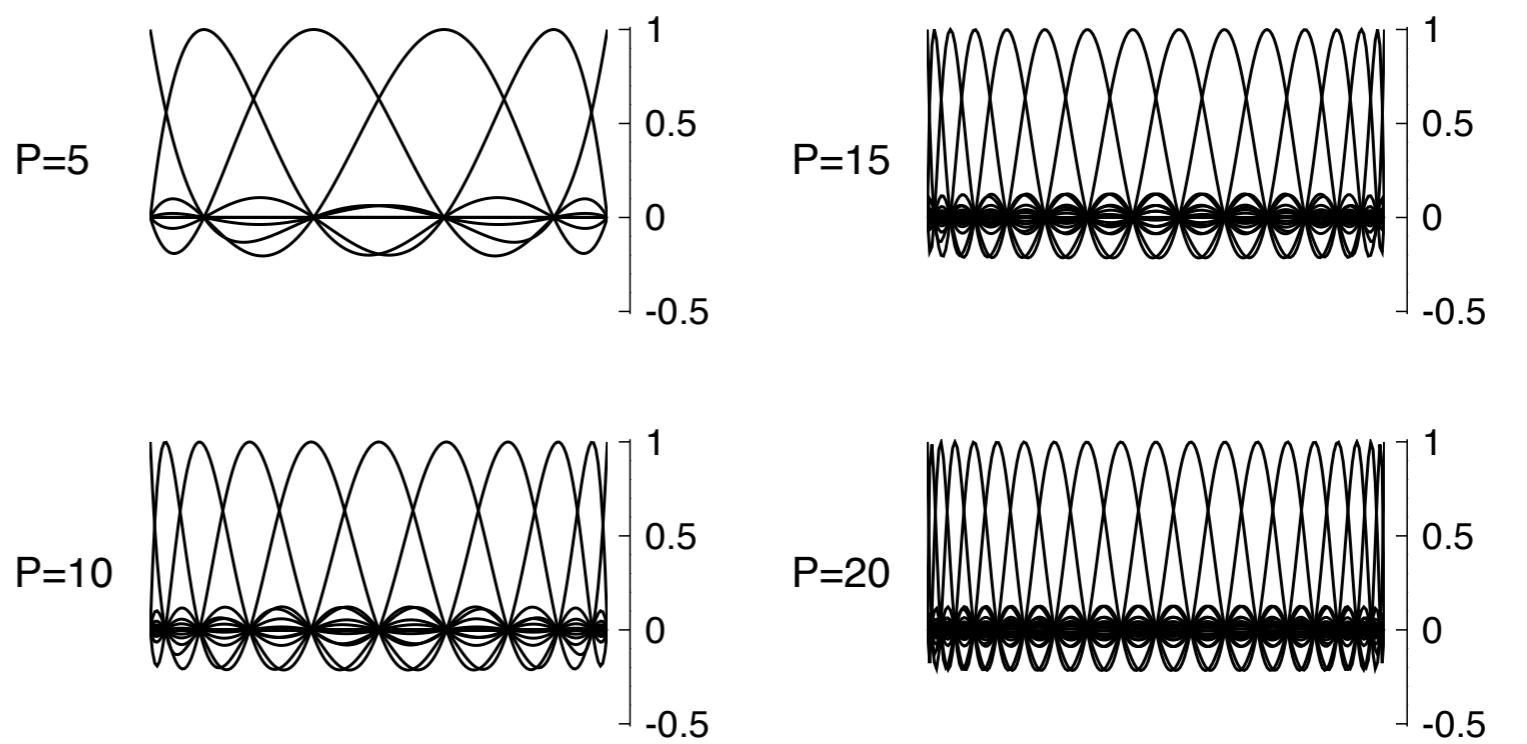
$$\phi_p(\xi) \mapsto h_p^{gl}(\xi) = \begin{cases} 1, & \xi = \xi_p, \\ (\xi - 1)(\xi + 1) \frac{\partial L_P(\xi)}{\partial \xi} & 0 \leq p \leq P. \\ \frac{P(P+1)L_P(\xi_p)(\xi_p - \xi)}{P(P+1)L_P(\xi_p)(\xi_p - \xi)}, & \text{otherwise}, \end{cases} \quad (20)$$

Collocation property: $u_\delta(\xi_q) = \sum_{p=0}^P \tilde{u}_p h_p(\xi_q) = \sum_{p=0}^P \tilde{u}_p \delta_{pq} = \tilde{u}_q$.

Spectral element Conditioning:



Lagrange nodal expansions through the equi-spaced points for polynomial orders of $P = 5, 10, 15$ and 20 .



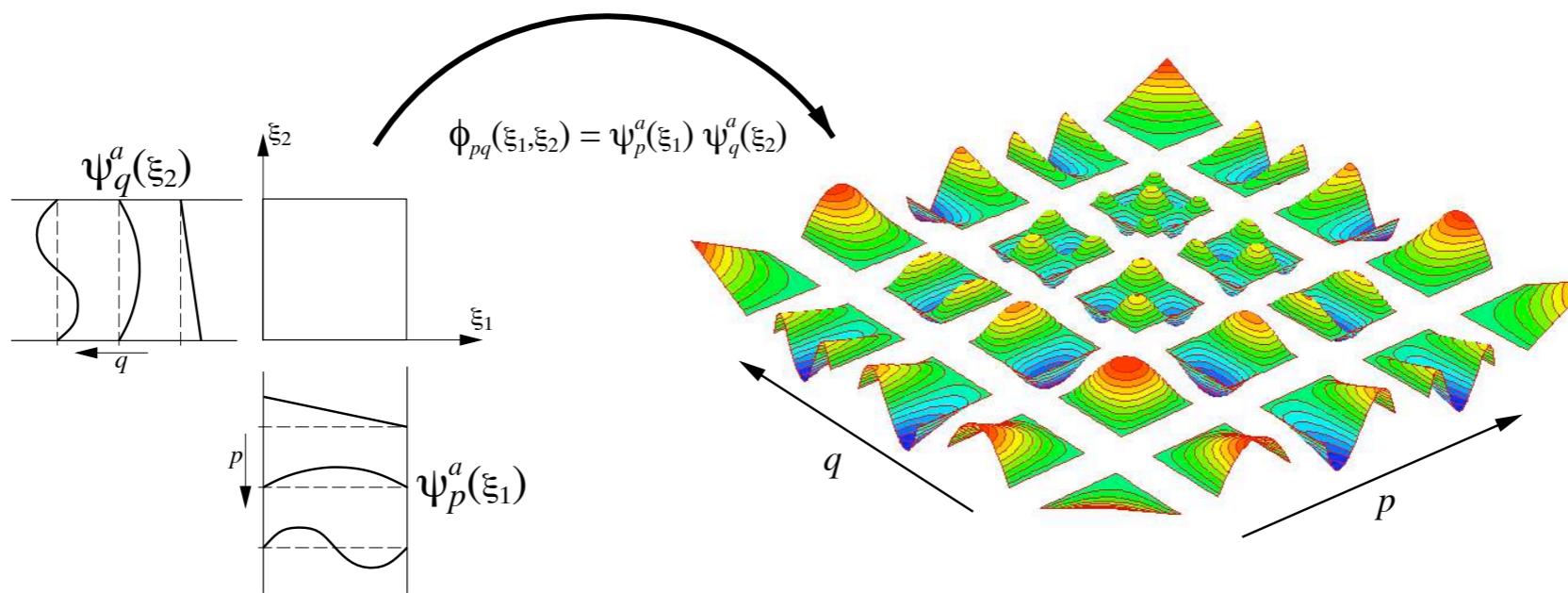
Lagrange nodal expansions through the Gauss-Lobatto points for polynomial orders of $P = 5, 10, 15$ and 20 .

Outline

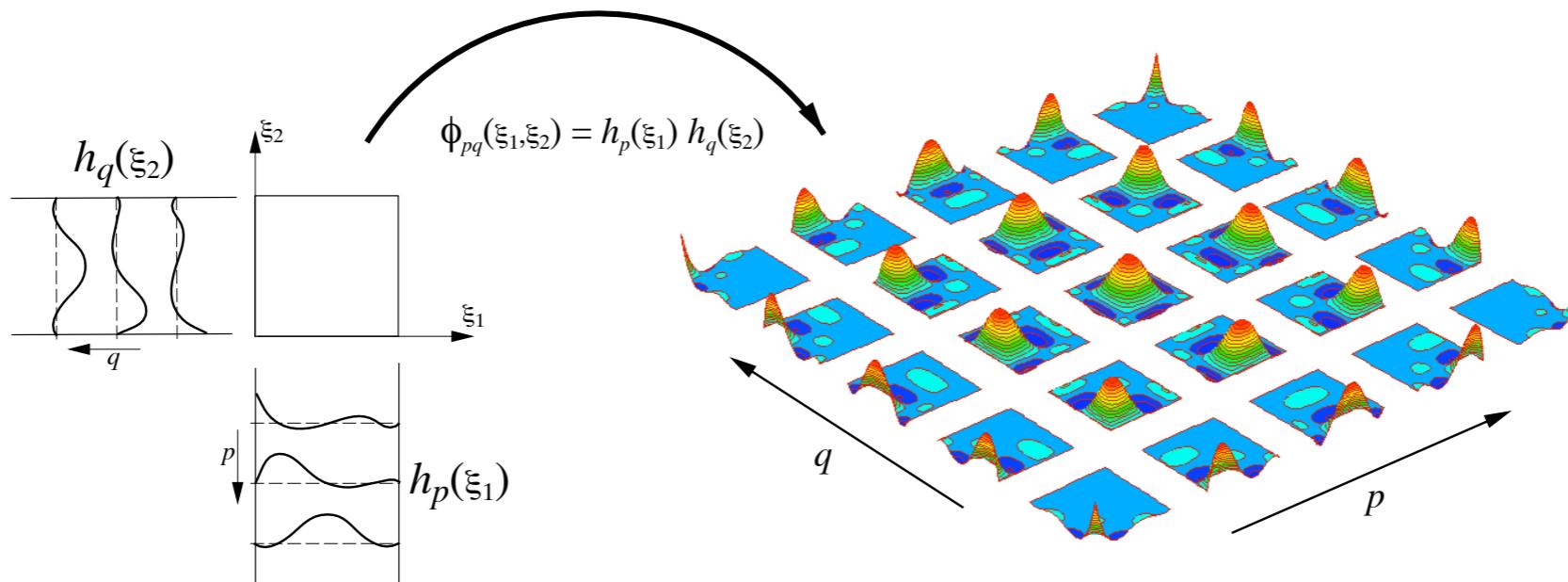
- Choice of Tensorial Expansions
 - Standard Segments
(StdRegions::StdSegExp)
 - Standard Quadrilaterals
(StdRegions::StdQuadExp)
 - Standard Triangles
(StdRegions::StdTriExp)
- Sum Factorisation of tensorial bases (notes: 3.1.6)

1	Fundamental Concepts in One Dimension	1
1.1	Method of Weighted Residuals	2
1.2	Galerkin Formulation	5
1.2.1	Descriptive Formulation	5
1.2.2	Two-Domain Linear Finite Element Example	8
1.2.3	Mathematical Formulation	11
1.2.4	Mathematical Properties of the Galerkin Approximation	13
1.2.5	Residual Equation for C^0 Test and Trial Functions	15
1.3	One-Dimensional Expansion Bases	16
1.3.1	Elemental Decomposition: The h -Type Extension	16
1.3.2	Polynomial Expansions: The p -Type Extension	22
1.3.3	Modal Polynomial Expansions	28
1.3.4	Nodal Polynomial Expansions	31
1.4	Elemental Operations	34
1.4.1	Numerical Integration	34
1.4.2	Differentiation	39
1.5	Error Estimates	42
1.5.1	h -Convergence of Linear Finite Elements	42
1.5.2	L^2 Error of the p -Type Interpolation in a Single Element	44
1.5.3	General Error Estimates for hp Elements	45
1.6	Implementation of a 1D Spectral/ hp Element Solver:	46
1.6.1	Exercises	46
1.6.2	Convergence Examples	50
2	Multi-dimensional Expansion Bases	53
2.1	Quadrilateral and Hexahedral Tensor Product Expansions	55
2.1.1	Standard Tensor Product Extensions	55
2.1.2	Polynomial Space of Tensor Product Expansions	59
2.2	Generalised Tensor Product Modal Expansions	60
2.2.1	Coordinate Systems	62
2.2.2	Orthogonal Expansions	68
2.2.3	Modified C^0 Expansions	75
2.3	Non-Tensorial Nodal Expansions in a Simplex	83
2.3.1	The Lagrange Polynomial and Lebesgue Constant	85
2.3.2	Generalised Vandemonde Matrix	86
2.3.3	Electrostatic Points	87
2.3.4	Fekete Points	88
2.4	Other Useful Tensor Product Extensions	91
2.4.1	Nodal Elements in a Prismatic Region	92
2.4.2	Expansions in Homogeneous Domains	92
2.4.3	Cylindrical Domains	93
2.5	Exercises: Construction of Multi-Dimensional Elemental Mass Matrices	93

Spectral element/P-type finite element



p-type finite element- hierarchical basis



Spectral element - collocation basis

Outline

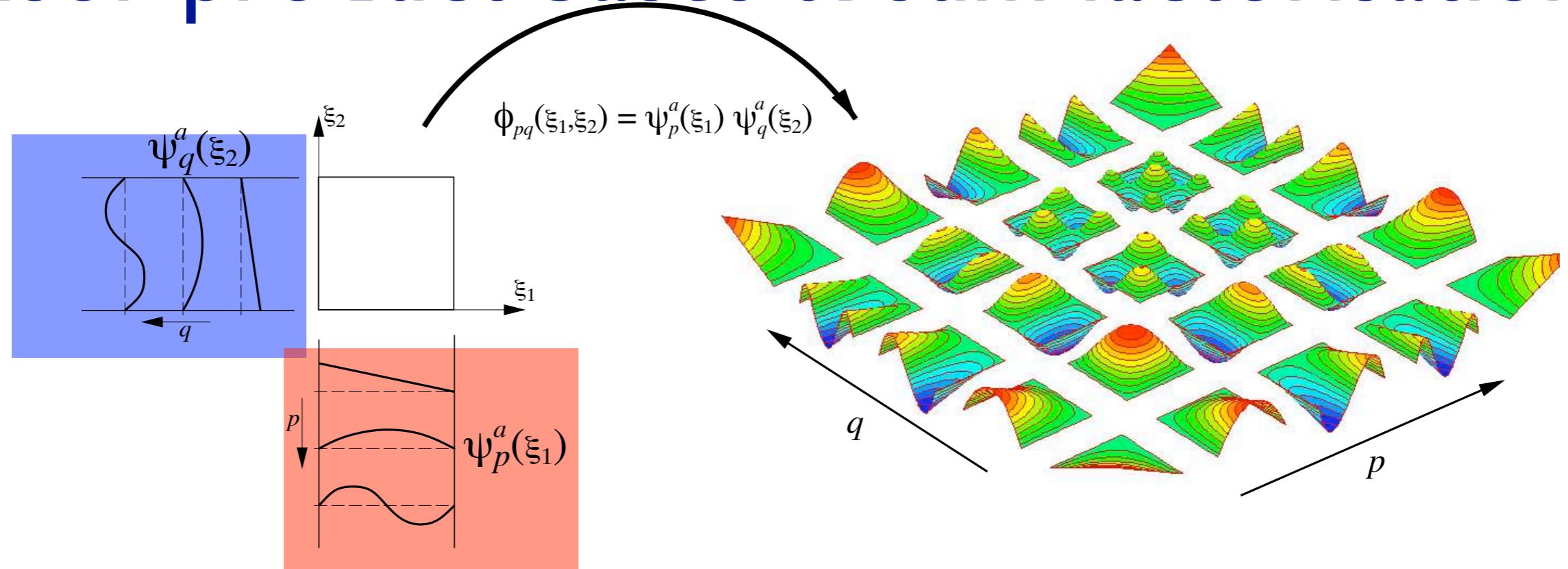
- Choice of Tensorial Expansions

- Standard Segments
(`StdRegions::StdSegExp`)
- Standard Quadrilaterals
(`StdRegions::StdQuadExp`)
- Standard Triangles
(`StdRegions::StdTriExp`)

- Sum Factorisation of tensorial bases (notes: 3.1.6)

1	Fundamental Concepts in One Dimension	1
1.1	Method of Weighted Residuals	2
1.2	Galerkin Formulation	5
1.2.1	Descriptive Formulation	5
1.2.2	Two-Domain Linear Finite Element Example	8
1.2.3	Mathematical Formulation	11
1.2.4	Mathematical Properties of the Galerkin Approximation	13
1.2.5	Residual Equation for C^0 Test and Trial Functions	15
1.3	One-Dimensional Expansion Bases	16
1.3.1	Elemental Decomposition: The h -Type Extension	16
1.3.2	Polynomial Expansions: The p -Type Extension	22
1.3.3	Modal Polynomial Expansions	28
1.3.4	Nodal Polynomial Expansions	31
1.4	Elemental Operations	34
1.4.1	Numerical Integration	34
1.4.2	Differentiation	39
1.5	Error Estimates	42
1.5.1	h -Convergence of Linear Finite Elements	42
1.5.2	L^2 Error of the p -Type Interpolation in a Single Element	44
1.5.3	General Error Estimates for hp Elements	45
1.6	Implementation of a 1D Spectral/ hp Element Solver:	46
1.6.1	Exercises	46
1.6.2	Convergence Examples	50
2	Multi-dimensional Expansion Bases	53
2.1	Quadrilateral and Hexahedral Tensor Product Expansions	55
2.1.1	Standard Tensor Product Extensions	55
2.1.2	Polynomial Space of Tensor Product Expansions	59
2.2	Generalised Tensor Product Modal Expansions	60
2.2.1	Coordinate Systems	62
2.2.2	Orthogonal Expansions	68
2.2.3	Modified C^0 Expansions	75
2.3	Non-Tensorial Nodal Expansions in a Simplex	83
2.3.1	The Lagrange Polynomial and Lebesgue Constant	85
2.3.2	Generalised Vandemonde Matrix	86
2.3.3	Electrostatic Points	87
2.3.4	Fekete Points	88
2.4	Other Useful Tensor Product Extensions	91
2.4.1	Nodal Elements in a Prismatic Region	92
2.4.2	Expansions in Homogeneous Domains	92
2.4.3	Cylindrical Domains	93
2.5	Exercises: Construction of Multi-Dimensional Elemental Mass Matrices	93

Tensor product bases & sum factorisation



Inner product:

$$I_{pq} = \int_{\Omega^e} \phi_{pq}(\xi_1, \xi_2) u(\xi_1, \xi_2) = \sum_i \sum_j \phi_{pq}(\xi_{1,i}, \xi_{2,j}) u(\xi_{1,i}, \xi_{2,j})$$

$$I_{pq} = \sum_i \sum_j \psi_p^a(\xi_{1,i}) \psi_q^a(\xi_{2,j}) u(\xi_{1,i}, \xi_{2,j}) \sim O(P^4)$$

$$I_{pq} = \sum_i \psi_p^a(\xi_{1,i}) f_q(\xi_{2,i}) \sim O(P^3)$$

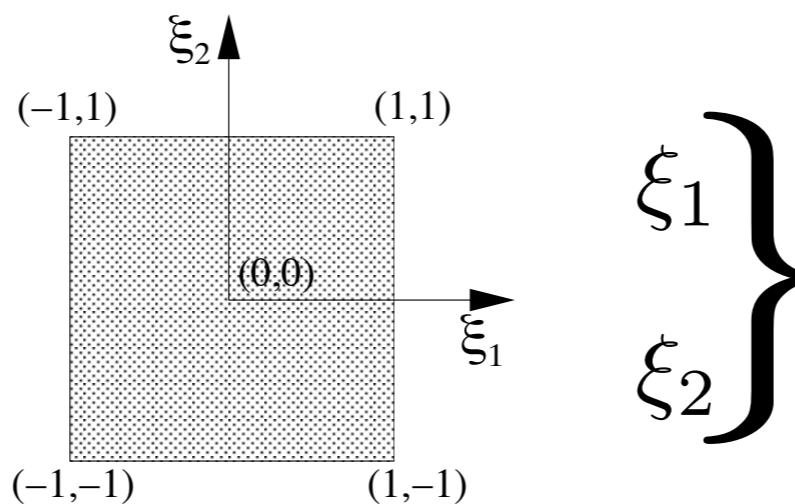
Outline

- Choice of Tensorial Expansions
 - Standard Segments
(StdRegions::StdSegExp)
 - Standard Quadrilaterals
(StdRegions::StdQuadExp)
 - Standard Triangles
(StdRegions::StdTriExp)
- Sum Factorisation of tensorial bases (notes: 3.1.6)

1	Fundamental Concepts in One Dimension	1
1.1	Method of Weighted Residuals	2
1.2	Galerkin Formulation	5
1.2.1	Descriptive Formulation	5
1.2.2	Two-Domain Linear Finite Element Example	8
1.2.3	Mathematical Formulation	11
1.2.4	Mathematical Properties of the Galerkin Approximation	13
1.2.5	Residual Equation for C^0 Test and Trial Functions	15
1.3	One-Dimensional Expansion Bases	16
1.3.1	Elemental Decomposition: The h -Type Extension	16
1.3.2	Polynomial Expansions: The p -Type Extension	22
1.3.3	Modal Polynomial Expansions	28
1.3.4	Nodal Polynomial Expansions	31
1.4	Elemental Operations	34
1.4.1	Numerical Integration	34
1.4.2	Differentiation	39
1.5	Error Estimates	42
1.5.1	h -Convergence of Linear Finite Elements	42
1.5.2	L^2 Error of the p -Type Interpolation in a Single Element	44
1.5.3	General Error Estimates for hp Elements	45
1.6	Implementation of a 1D Spectral/ hp Element Solver:	46
1.6.1	Exercises	46
1.6.2	Convergence Examples	50
2	Multi-dimensional Expansion Bases	53
2.1	Quadrilateral and Hexahedral Tensor Product Expansions	55
2.1.1	Standard Tensor Product Extensions	55
2.1.2	Polynomial Space of Tensor Product Expansions	59
2.2	Generalised Tensor Product Modal Expansions	60
2.2.1	Coordinate Systems	62
2.2.2	Orthogonal Expansions	68
2.2.3	Modified C^0 Expansions	75
2.3	Non-Tensorial Nodal Expansions in a Simplex	83
2.3.1	The Lagrange Polynomial and Lebesgue Constant	85
2.3.2	Generalised Vandemonde Matrix	86
2.3.3	Electrostatic Points	87
2.3.4	Fekete Points	88
2.4	Other Useful Tensor Product Extensions	91
2.4.1	Nodal Elements in a Prismatic Region	92
2.4.2	Expansions in Homogeneous Domains	92
2.4.3	Cylindrical Domains	93
2.5	Exercises: Construction of Multi-Dimensional Elemental Mass Matrices	93

Unstructured coordinate system

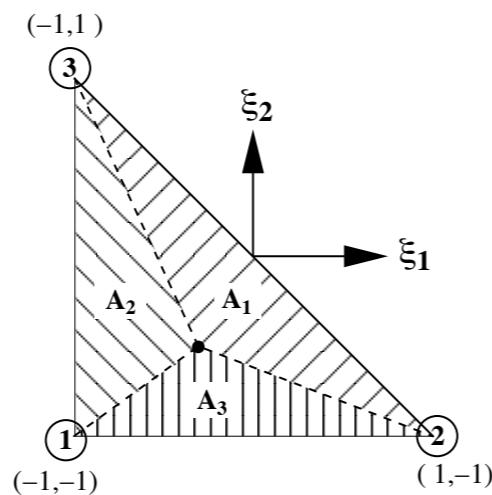
Quadrilateral
system



ξ_1
 ξ_2

2 coordinates in
2 dimensions
(rotationally symmetric)

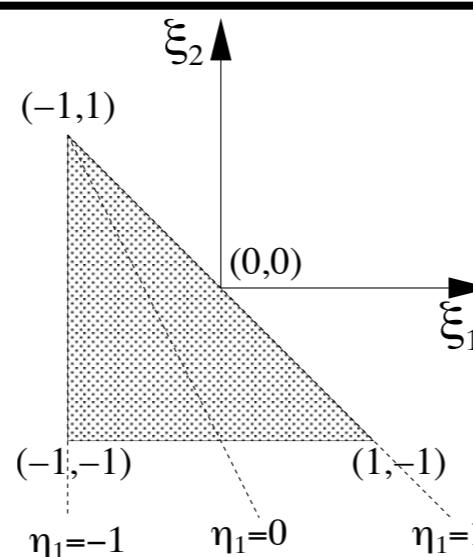
Barycentric/Area
system



$$l_1 = \frac{A_1}{A}$$
$$l_2 = \frac{A_2}{A}$$
$$l_3 = \frac{A_3}{A}$$

3 coordinates in
2 dimensions
(rotationally symmetric)

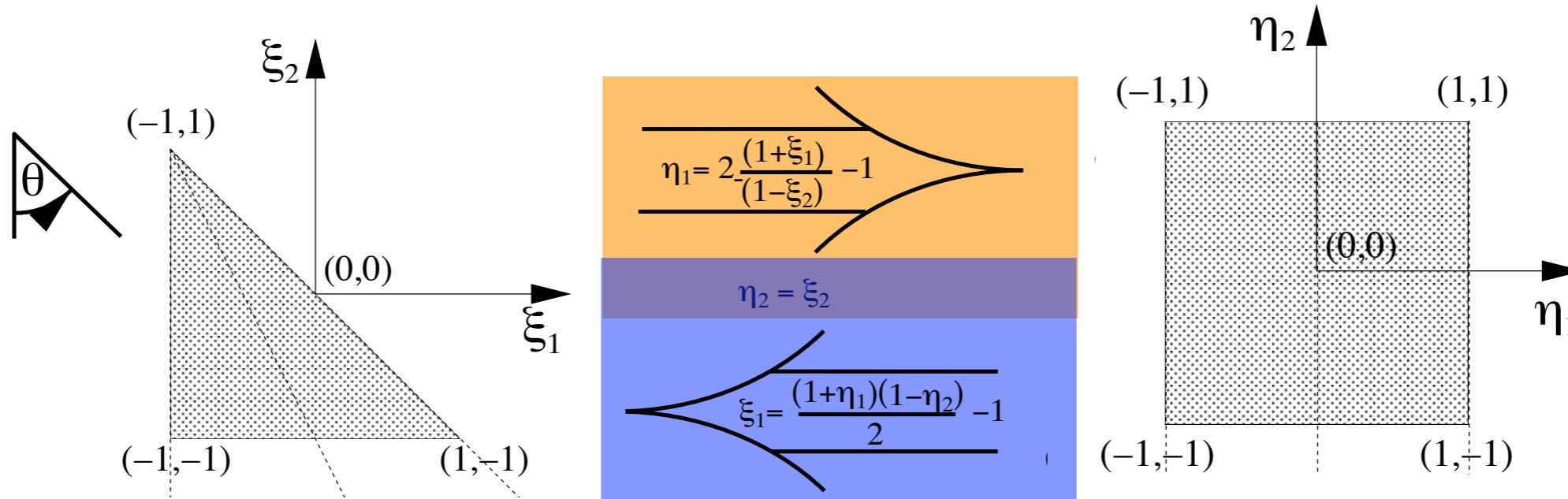
Collapsed
coordinates



η_1
 η_2

2 coordinates in
2 dimensions
(not rotationally symmetric)

Collapsed coordinate system



$$\mathcal{T}_{st} = \{(\xi_1, \xi_2) | -1 \leq \xi_1, \xi_2; \xi_1 + \xi_2 \leq 0\}$$

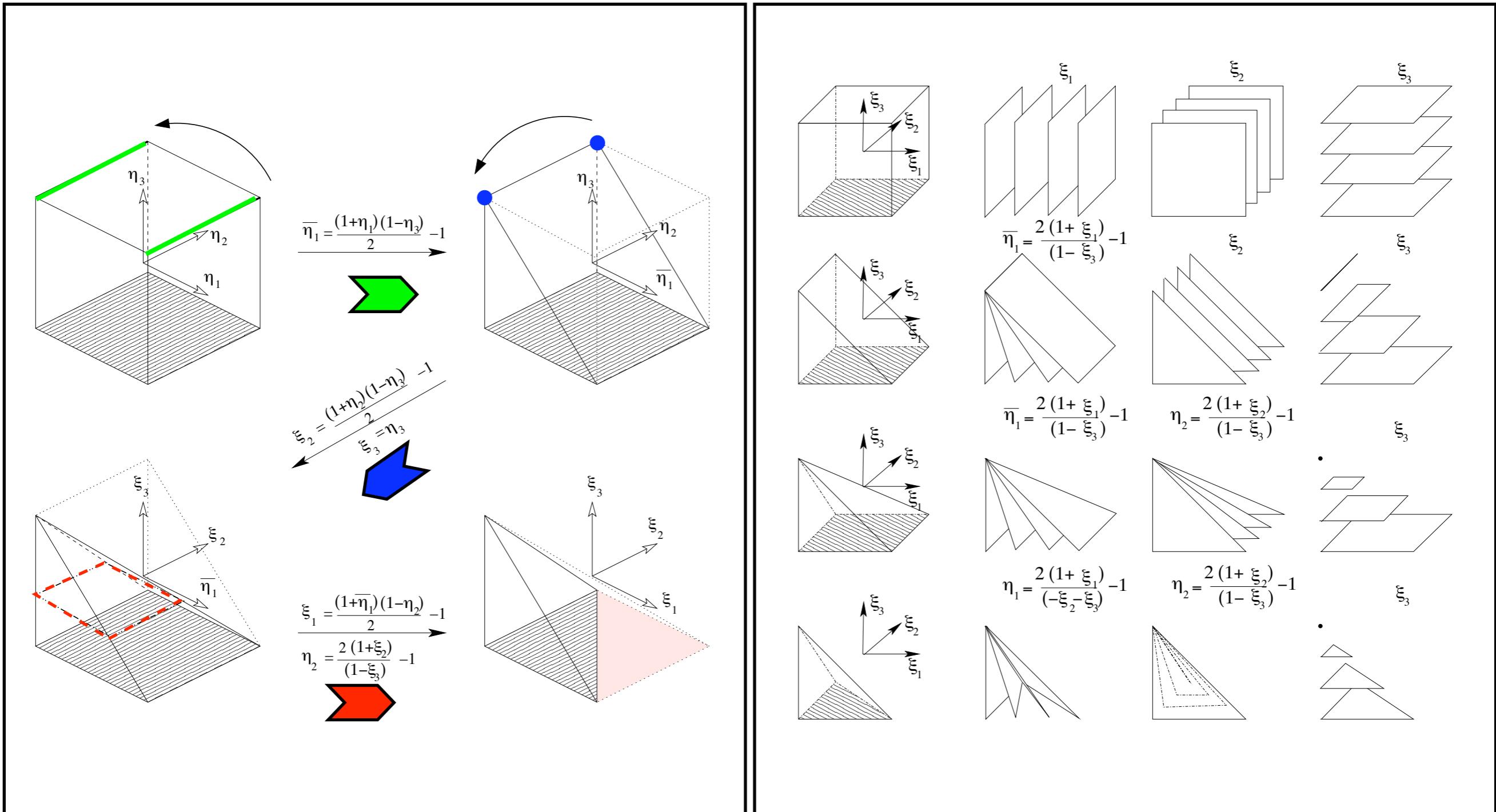
$$\mathcal{T}_{st} = \{(\eta_1, \eta_2) | -1 \leq \eta_1, \eta_2 \leq 1\},$$

$$\eta_1 = 2 \frac{(1+\xi_1)}{(1-\xi_2)} - 1, \quad \eta_2 = \xi_2,$$

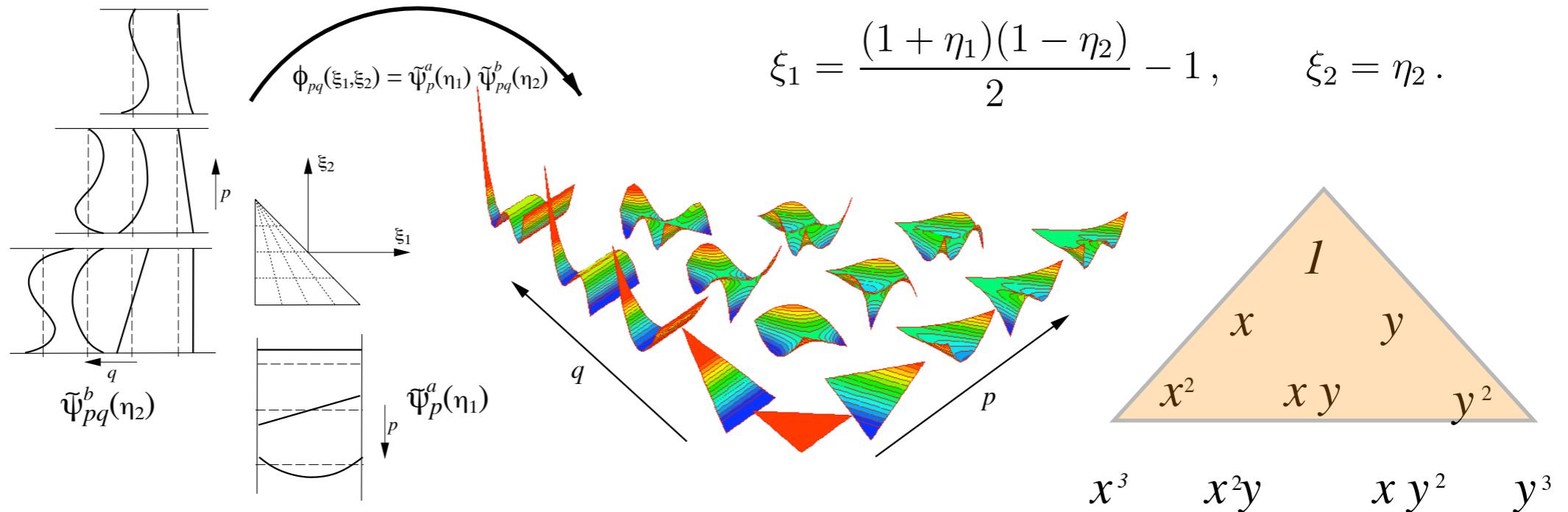
$$\xi_1 = \frac{(1+\eta_1)(1-\eta_2)}{2} - 1, \quad \xi_2 = \eta_2.$$

Analogous system used in cylindrical coordinate system

Hybrid element coordinate systems



Orthogonal Expansion



Principal functions:

$$\tilde{\psi}_p^a(z) = P_p^{0,0}(z), \quad \tilde{\psi}_{pq}^b(z) = \left(\frac{1-z}{2}\right)^p P_q^{2p+1,0}(z)$$

Generalised tensor products:

- Quadrilateral expansion: $\phi_{pq}(\xi_1, \xi_2) = \tilde{\psi}_p^a(\xi_1) \tilde{\psi}_q^a(\xi_2), \quad 0 \leq p, q \leq P$
- Triangular expansion: $\phi_{pq}(\xi_1, \xi_2) = \tilde{\psi}_p^a(\eta_1) \tilde{\psi}_{pq}^b(\eta_2), \quad 0 \leq p, p+q \leq P,$

PKD Orthogonal

Principal functions:

$$\tilde{\psi}_p^a(z) = P_p^{0,0}(z), \quad \tilde{\psi}_{pq}^b(z) = \left(\frac{1-z}{2}\right)^p P_q^{2p+1,0}(z)$$

- Triangular expansion: $\phi_{pq}(\xi_1, \xi_2) = \tilde{\psi}_p^a(\eta_1) \tilde{\psi}_{pq}^b(\eta_2)$

$$\int_{\mathcal{T}_2} \phi_{ij}(\xi_1, \xi_2) \phi_{pq}(\xi_1, \xi_2) d\xi_1 d\xi_2 = \int_{-1}^1 \int_{-1}^1 \underbrace{\tilde{\psi}_i^a(\eta_1)}_{\leftarrow} \underbrace{\tilde{\psi}_{ij}^b(\eta_2)}_{\downarrow} \underbrace{\tilde{\psi}_p^a(\eta_1)}_{\leftarrow} \underbrace{\tilde{\psi}_{pq}^b(\eta_2)}_{\downarrow} \frac{1-\eta_2}{2} d\eta_1 d\eta_2$$

$$\int_{-1}^1 P_i^{0,0}(\eta_1) P_p^{0,0}(\eta_1) d\eta_1 \int_{-1}^1 \left(\frac{1-\eta_2}{2}\right)^i \left(\frac{1-\eta_2}{2}\right)^p P_j^{2i+1,0}(\eta_2) P_q^{2p+1,0}(\eta_2) \left(\frac{1-\eta_2}{2}\right) d\eta_2$$

$$C_{ip} \delta_{ip} \qquad \qquad \qquad C_{jq} \delta_{jq} \quad \text{if } i = p$$

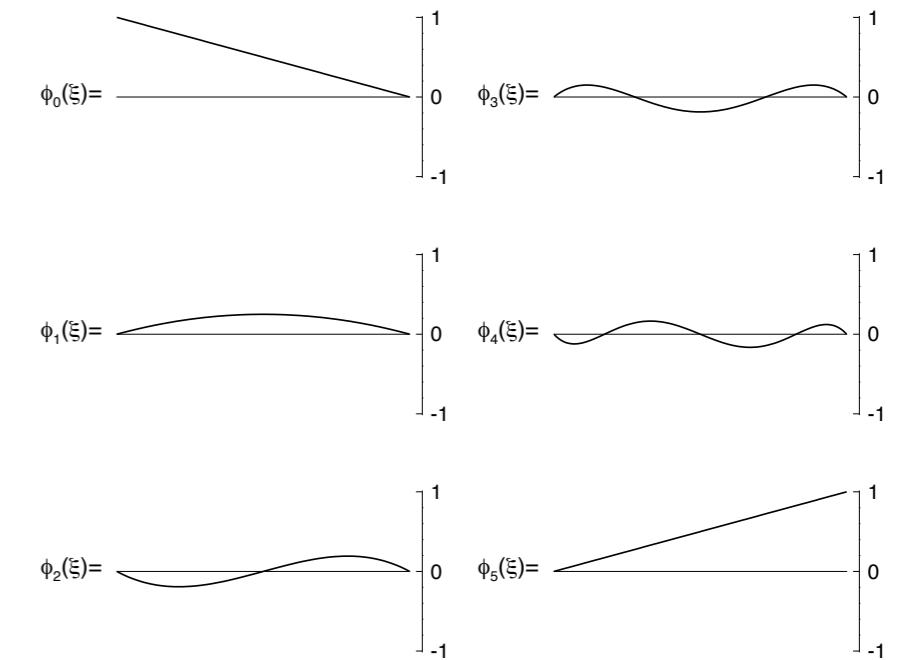
$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_n^{\alpha,\beta}(x) P_i^{\alpha,\beta}(x) d\xi = C \delta_{ni},$$

Modified Dubiner Expansion

Modified Principal Functions:

$$\psi_i^a(z) = \begin{cases} \left(\frac{1-z}{2}\right) & i = 0 \\ \left(\frac{1-z}{2}\right) \left(\frac{1+z}{2}\right) P_{i-1}^{1,1}(z) & 1 \leq i < I \\ \left(\frac{1+z}{2}\right) & i = I \end{cases},$$

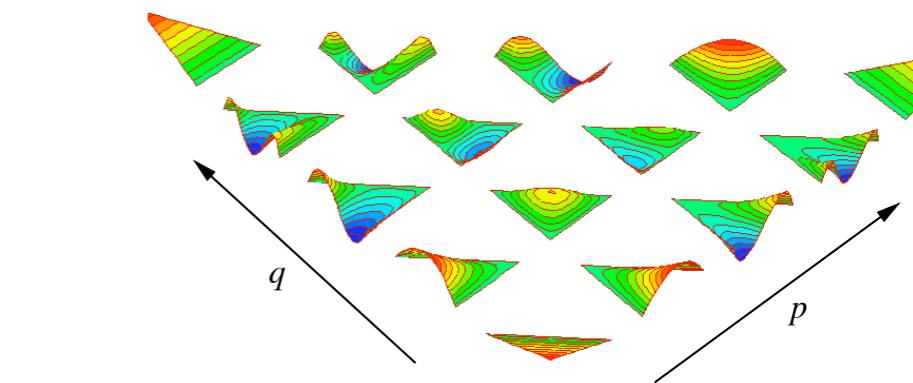
$$\psi_{ij}^b(z) = \begin{cases} \psi_j^a(z) & i = 0, \quad 0 \leq j \leq J \\ \left(\frac{1-z}{2}\right)^{i+1} & 1 \leq i < I, \quad j = 0 \\ \left(\frac{1-z}{2}\right)^{i+1} \left(\frac{1+z}{2}\right) P_{j-1}^{2i+1,1}(z) & 1 \leq i < I, \quad 1 \leq j < J \\ \psi_j^a(z) & i = I, \quad 0 \leq j \leq J \end{cases},$$



Generalised tensor products:

- Quadrilateral expansion: $\phi_{pq}(\xi_1, \xi_2) = \psi_p^a(\xi_1) \psi_q^a(\xi_2)$,

- Triangular expansion: $\phi_{pq}(\xi_1, \xi_2) = \psi_p^a(\eta_1) \psi_{pq}^b(\eta_2)$,



Nektar++ code

