ES120 Spring 2018 – Section 1 Notes

Matt Fernandes

February 1, 2018

Problem 1:

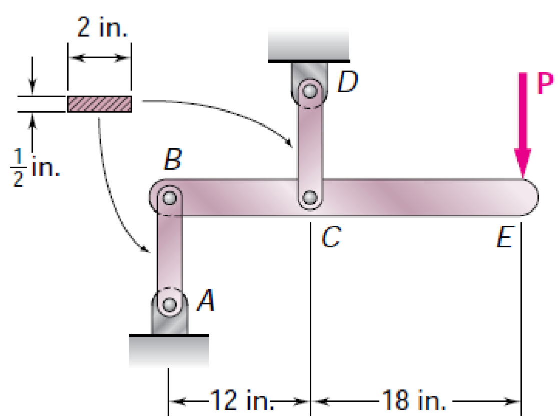


Figure 1

Each of the steel links AB and CD is connected to a support and to member BCE by 1-in.-diameter steel pins acting in single shear. Knowing that the ultimate shearing stress is 30 ksi for the steel used in the pins and that the ultimate normal stress is 70 ksi for the steel used in the links, determine the allowable load P if an overall factor of safety of 3.0 is desired. (Note that the links are not reinforced around the pin holes.)

Solution 1

Use member BCE as free body.

$$\Sigma M_B = 0: \quad 12F_{CD} - 30P = 0 \Rightarrow P = \frac{2}{5}F_{CD} \tag{1}$$

$$\Sigma M_C = 0: \quad 12F_{AB} - 18P = 0 \Rightarrow P = \frac{2}{3}F_{AB} \tag{2}$$

Both links have the same area, pin diameter and material. Therefore, they have the same ultimate load. Failure by pin in single shear

$$A = \frac{\pi}{4}d^2 = 0.7854\text{in}^2 \tag{3}$$

$$F_u = \tau_u A = (30)(0.7854) = 23.562 \text{ kips} \tag{4}$$

Failure by tension in link

$$A = (b-d)t = (2-1)\frac{1}{2} = 0.5 \text{ in}^2 \tag{5}$$

$$F_u = \sigma_u A = (70)(0.5) = 35 \text{ kips} \tag{6}$$

Ultimate load for link and pin is the smaller.

$$F_u = 23.562 \text{ kips} \tag{7}$$

Allowable values of F_{CD} and F_{AB}

$$F_{all} = \frac{F_u}{\text{Factor of Safety}} = \frac{23.562}{3.0} = 7.854 \text{ kips} \tag{8}$$

Allowable load for structure is teh smaller of $\frac{2}{3}F_{all}$ and $\frac{2}{5}F_{all}$.

$$P = \frac{2}{5}(7.854) \tag{9}$$

$$\boxed{P = 3.14 \text{ kips}} \tag{10}$$

Problem 2:

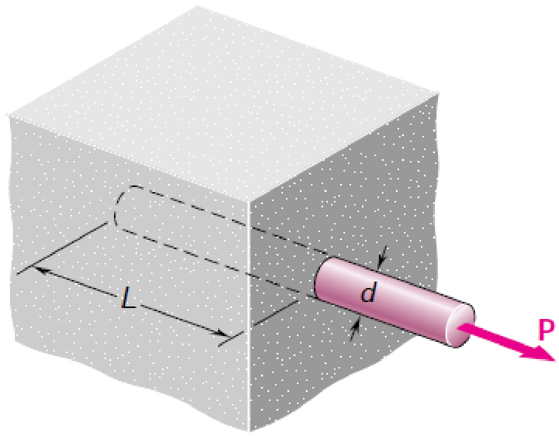


Figure 2

A force P is applied as shown to a steel reinforcing bar that has been embedded in a block of concrete. Determine the smallest length L for which the full allowable normal stress in the bar can be developed. Express the result in terms of the diameter d of the bar, the allowable normal stress σ_{all} in the steel, and the average allowable bond stress τ_{all} between the concrete and the cylindrical surface of the bar. (Neglect the normal stresses between the concrete and the end of the bar.)

Solution 2

For the shear we know that the area in which it is action on is

$$A = \pi d L \tag{11}$$

And thus the reaction force from the shear will be the same as P where

$$P = \tau_{all} A = \tau_{all} \pi d L \tag{12}$$

As for the tensile stress the area will be the cross section of the rod, namely,

$$A = \frac{\pi}{4} d^2 \tag{13}$$

Similarly, the reaction force from the normal stress must balance p such that:

$$P = \sigma_{all} A = \sigma_{all} (\frac{\pi}{4} d^2) \tag{14}$$

Therefore equating both of the above we obtain:

$$\tau_{all} \pi d L = \sigma_{all} \frac{\pi}{4} d^2 \tag{15}$$

Which when solving for L we obtain:

$$\boxed{L = \frac{\sigma_{all} d}{4 \tau_{all}}} \tag{16}$$

ES120 Spring 2018 – Section 2 Notes

Matheus Fernandes

February 8, 2018

Problem 1:

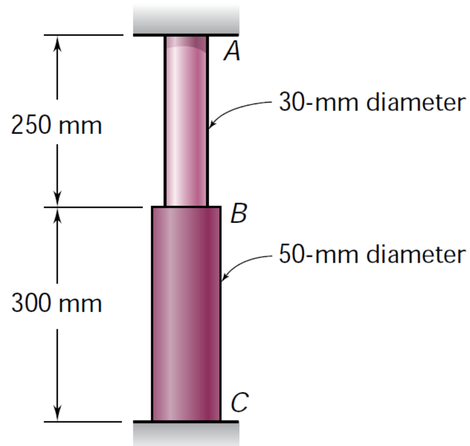


Figure 1

A rod consisting of two cylindrical portions AB and BC is restrained at both ends. Portion AB is made of steel ($E_s = 200$ GPa,$\alpha_s = 11.7 \times 10^6/°C$) and portion BC is made of brass ($E_b = 105$ GPa, $\alpha_b = 20.9 \times 10^6/°C$). Knowing that the rod is initially unstressed, determine the compressive force induced in ABC when there is a temperature rise of 50°C.

Solution 1

Let's first find the cross-sectional area of AB and BC:

$$A_{AB} = \frac{\pi}{4}d_{AB}^2 = \frac{\pi}{4}(30[mm])^2 \Rightarrow 706.86 \times 10^{-6}m^2 \tag{1}$$

$$A_{BC} = \frac{\pi}{4}d_{BC}^2 = \frac{\pi}{4}(50[mm])^2 \Rightarrow 1.9635 \times 10^{-3}m^2 \tag{2}$$

Now, let's assume the device is not constrained and that it is free to move. Let's calculate the free thermal expansion of the rods as if there was nothing acting on them.

$$\delta_t = L_{AB}\alpha_s(\Delta T) + L_{BC}\alpha_b(\Delta T) \tag{3}$$

$$\delta_t = (0.250)(11.7 \times 10^{-6})(50) + (0.300)(20.9 \times 10^{-6})(50) \tag{4}$$

$$\delta_t = 459.75 \times 10^{-6}m \tag{5}$$

Now let's ignore the fact that there is any thermal expansion at all, and just consider an arbitrary shortening due to some load P, namely,

$$\begin{aligned}\delta_P &= \frac{PL}{E_s A_{AB}} + \frac{PL}{E_b A_{BC}} \\ &= \frac{0.250P}{(200 \times 10^9)(706.86 \times 10^{-6})} + \frac{0.300P}{(105 \times 10^9)(1.9635 \times 10^{-3})} \\ &= 3.2235 \times 10^{-9} P\end{aligned} \tag{6}$$

Now we know that the net deflection of both the thermal expansion and this arbitrary force must be zero. Thus the deflection due to the thermal expansion must be transferred as elastic potential energy into the constrained device. In other words...

$$\delta_T = \delta_P \tag{7}$$

Thus equating both of the things we found above we get a closed form solution for P, namely

$$3.2235 \times 10^{-9} P = 459.75 \times 10^{-6} \tag{8}$$

$$\boxed{P = 142.62 \times 10^{-3} N = 142.62 kN} \tag{9}$$

Problem 2:

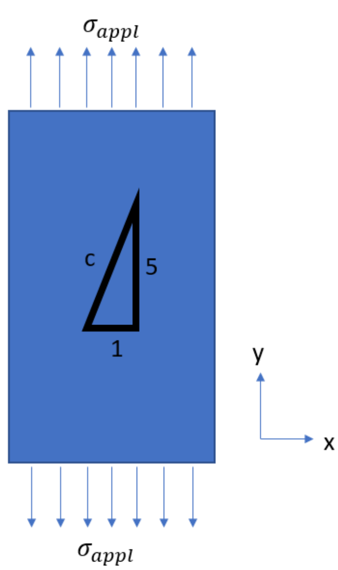


Figure 2

Assuming a triangle is drawn in a block of material which material properties are given by $E = 100$ GPa and $\nu = 100MPa$, determine the length of the hypotenuse of the triangle (c) in the block's deformed state.

Solution 2

For this, we analyze what is known as Hooke's law in the field of linear elasticity. The Hooke's law equations provide you with an insightful relationship between stress and strain and how the couple in different directions. The Hooke's law equations are given as

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z) \tag{10}$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z) \tag{11}$$

$$\epsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_y - \nu\sigma_x) \tag{12}$$

Here based on the applied stress and current state of the geometry, we can obtain the value for stresses as:

$$\sigma_x = \sigma_z = 0 \tag{13}$$

$$\sigma_y = \sigma_{app} \tag{14}$$

We can now plug in what we know above to eq. (10) and eq. (11) to obtain,

$$\epsilon_x = \frac{-\nu\sigma_{app}}{E} \text{ and} \tag{15}$$

$$\epsilon_y = \frac{\sigma_{app}}{E}, \tag{16}$$

respectively. Note, we have ignored eq. (12), simply because the triangle lays on the XY plane, so we do not care about the deformation in the z direction.

We now know that the total length of each triangle in the deformed state will become as illustrated

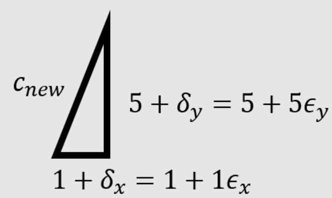


Figure 3: Illustration of deformed triangle. Not to scale!

Therefore, we can solve for each side of the triangle using what we wound in eqs. (15) and (16) to be:

$$L_y = 5 + 5\epsilon_y = 5 + 5\frac{\sigma_{app}}{E} = 5.005 \tag{17}$$

$$L_X = 5 + 5\epsilon_y = 1 - \frac{\nu\sigma_{app}}{E} = 0.9995 \tag{18}$$

So to obtain the new length on the triangle we use these values in conjunction with Pythagorean theorem, such that:

$$\boxed{C_{new} = \sqrt{5.005^2 + 0.9995^2} \approx 5.104} \tag{19}$$

ES120 Spring 2018 – Section 3 Notes

Matheus Fernandes

February 15, 2018

Problem 1:

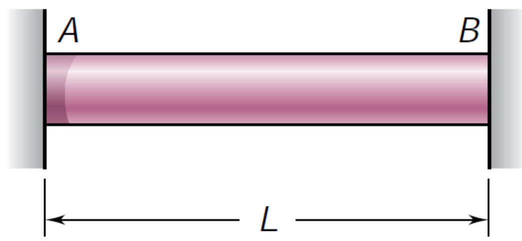


Figure 1

A uniform steel rod of cross-sectional area A is attached to rigid supports and is unstressed at a temperature of 45°F. The steel is assumed to be elastoplastic with $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi. Knowing that $\alpha = 6.5 \times 10^{-6}$/°F, determine the stress in the bar (a) when the temperature is raised to 320 °F, (b) after the temperature has returned to 45°F.

Solution 1

Part A

First lets begin by analyzing the temperature required to cause yielding. It is always a great way to start when trying to break down the problem between it's linear part and the part where plastic deformation will begin to form. To do that, we need to recall the equation for deflection due to a load and balance that with the deflection caused by the thermal strain. Namely,

$$\delta = -\frac{PL}{AE} + L\alpha(\Delta T) \tag{1}$$

We know that because they balance such that

$$\delta = -\frac{PL}{AE} + L\alpha(\Delta T) = -\frac{\sigma_Y L}{E} + L\alpha(\Delta T)_Y = 0 \tag{2}$$

Therefore, we can solve for $(\Delta T)_Y$ for which is the temperature required to begin yielding the material. If we solve for that from the above equation we get:

$$(\Delta T)_Y = \frac{\sigma_Y}{E\alpha} = \frac{36 \times 10^3}{(29 \times 10^6)(6.5 \times 10^{-6})} = 190.98\text{°F} \tag{3}$$

However, since we are raising the temperature to be 320°F and $(320 - 45)$°F $= 275$°F > 190.98°F we know that we are achieving the yield stress and thus the stress in the bar when the temperature is raised is

$$\boxed{\sigma_{max} = -\sigma_Y = -36\text{ksi}} \tag{4}$$

Part B

Now when we cool we the bar back by $\Delta T = 275$ we know that this will create a displacement that will be balanced as it was above in part A. So here we want to assume that there is no yielding so that we can superposed the yielding strain into it. This approach is exactly like the one we do for statically indeterminate problems. So we know that this fictitious displacement is

$$\delta' = \delta'_p + \delta'_T = -\frac{P'L}{AE} + L\alpha(\Delta T)' = 0 \tag{5}$$

Which we can rearrange to be:

$$\begin{aligned} \sigma' =& \frac{P'}{A} \\ =& -E\alpha(\Delta T)' \\ =& -(29\times10^6)(6.5\times10^{-6})(275) = -51.84\times10^3\text{psi} \end{aligned} \tag{6}$$

So the residual stress is the stress from yielding, minus this superposed elastic stress that we computed which becomes

$$\boxed{\sigma_{res} = -\sigma_Y - \sigma' = -36\times10^3 - (-51.8375\times10^3) = 15.84\text{ksi}} \tag{7}$$

Problem 2:

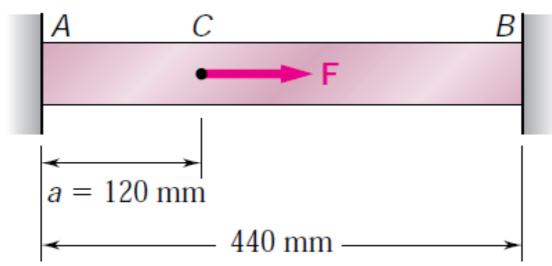


Figure 2

Bar AB has a cross-sectional area of 1200 mm^2 and is made of a steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. Knowing that the force F increases from 0 to 520 kN and then decreases to zero, determine (a) the permanent deflection of point C, (b) the residual stress in the bar.

Solution 2

Part A

Fir this problem we are asked for the permanent deflection after the load has increased and decreased. As you can imagine, if we have stresses which are higher then yielding we have left the elastic regime and now have permanent deformation, meaning, we will not retreat to the original spot we started at. To begin, I like to get all most of my simple calculations out of the way, so let's compute the area we are interested in:

$$A = \pi r^2 = 1200\text{mm}^2 = 1200 \times 10^{-6}\text{m}^2 \tag{8}$$

Now let's compute the force required to yield portion AC:

$$P_{AC} = A\sigma_Y = (1200 \times 10^{-6})(250 \times 10^6) = 300 \times 10^3\text{N} \tag{9}$$

For equilibrium to be true we can do a force balance such that

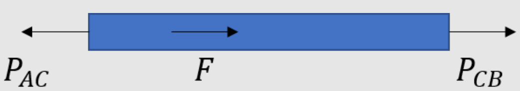


Figure 3

$$F + P_{CB} - P_{AC} = 0 \Rightarrow P_{CB} = P_{AC} - F = 300 \times 10^3 - 520 \times 10^{-3} = -220 \times 10^3 \text{ N} \tag{10}$$

Now using what we have left of the load, we can compute the stress in member BC and the deflection of point C such that

$$\delta_C = -\frac{P_{CB}L_{CB}}{EA} = \frac{(220 \times 10^3)(0.44 - 0.120)}{(200 \times 10^9)(1200 \times 10^{-6})} - 0.293 \times 10^{-3} \text{ m} \tag{11}$$

$$\sigma_{CB} = \frac{P_{CB}}{A} = \frac{220 \times 10^3}{1200 \times 10^{-3}} = 183.33 \times 10^6 \text{ Pa} \tag{12}$$

Now when we unload we can use the compatibility fact that

$$\delta'_C = \frac{P'_{AC}L_{AC}}{EA} = -\frac{P'_{BC}L_{CB}}{EA} = \frac{(F - P'_{AC})L_{BC}}{EA} \tag{13}$$

$$P'_{AC}(\frac{L_{AC}}{EA} + \frac{LBC}{EA}) = \frac{FL_{BC}}{EA} \tag{14}$$

$$P'_{AC} = \frac{FL_{CB}}{L_{AC} + LCB} = \frac{(520 \times 10^3)(0.440 - 0.120)}{0.440} = 378.182 \times 10^3 \text{ N} \tag{15}$$

Now we again know from force balance that

$$P'_{CB} = P'_{AC} - F = 378.182 \times 10^3 - 520 \times 10^3 = -141.818 \times 10^3 \text{ N} \tag{16}$$

$$\sigma'_{AC} = \frac{P'_{AC}}{A} = \frac{378.182 \times 10^3}{1200 \times 10^{-6}} = 315.152 \times 10^6 \text{ Pa} \tag{17}$$

$$\sigma'_{CB} = \frac{P'_{CB}}{A} = \frac{-141.818 \times 10^3}{1200 \times 10^{-6}} = -118.18 \times 10^6 \text{ Pa} \tag{18}$$

$$\delta'_C = \frac{(378.182)(0.120)}{(200 \times 10^9)(1200 \times 10^6)} = 0.189 \times 10^{-3} \text{ m} \tag{19}$$

So therefore, we know that the permanent displacement is the final yield displacement minus the elastic displacement, namely,

$$\boxed{\delta_{CP} = \delta_C - \delta'_C = 0.293^{-3} - 0.189 \times 10^{-3} = 0.1042 \times 10^{-3} \text{ m}} \tag{20}$$

Part B

Now for the stresses, we have already computed much of the work above and we just need to remove the elastic part from the yielding stresses and left over compressive stress from large member, namely,

$$\boxed{\begin{aligned} \sigma_{AC,res} &= \sigma_Y - \sigma'_{AC} = \mathbf{250 \times 10^6 - 315.152 \times 10^6 = -65.2}\,\text{MPa} \\ \sigma_{CB,res} &= \sigma_{CB} - \sigma'_{CB} = \mathbf{-183.3 \times^6 + 118.18 \times 10^6 = -65.2}\,\text{MPa} \end{aligned}} \tag{21}$$

ES120 Spring 2018 – Section 4 Notes

Matheus Fernandes

February 22, 2018

Problem 1:

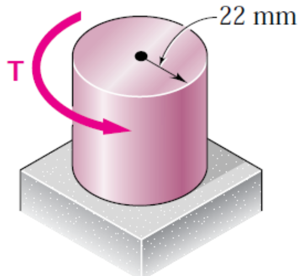


Figure 1

For the cylindrical shaft shown, determine the maximum shearing stress caused by a torque of magnitude T=1.5 kN·m.

Solution 1

In order to determine the maximum shear stress caused by a torque onto the shaft we must convert the physical torque (which is equivalent to a force) to a stress. The relationship between torque and maximum shear stress is given by:

$$\tau_{max} = \frac{Tr}{J}, \tag{1}$$

where J is the polar moment of inertia for a cylinder as described per the document at the end of the notes. We can then obtain the equation for J as:

$$J = \frac{\pi}{2} c^4 \tag{2}$$

such that we can solve for τ_{max} (where r and c for this problem are the same) as

$$\boxed{\tau_{max} = \frac{2T}{\pi c^3} = \frac{(2)(1500)}{\pi (0.022)^3} = 89.7\text{MPa}} \tag{3}$$

Problem 2:

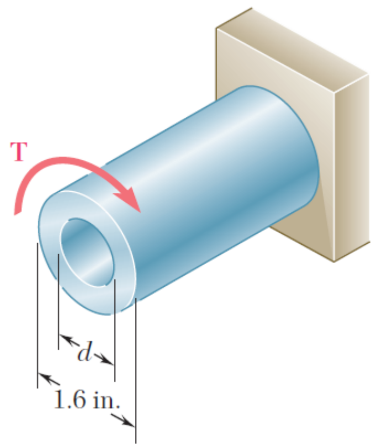


Figure 2

Knowing that the internal diameter of the hollow shaft shown is d=0.9 in., determine the maximum shearing stress caused by a torque of magnitude T=9 kip·in.

Solution 2

Similar to the previous problem we know that the maximum shear stress is dictated by the equation

$$\tau_{max} = \frac{Tr}{J}, \tag{4}$$

where now our radii differ in that for an annulus. So our c's become

$$c_1 = 0.5 * 0.9 = 0.45 \text{ in} \tag{5}$$

$$c_2 = 0.5 * 1.6 = 0.8 \text{ in} \tag{6}$$

Our second moment of an annulus can be calculated by subtracting the moment of the inner from the outer such that:

$$J = \frac{\pi}{2}(c_2^4 - c_2^4) = \frac{\pi}{2}(0.8^4 - 0.45^4) = 0.5790 \text{ in}^4 \tag{7}$$

Applying that in to the equation for τ_{max} (remembering that r is the outer radius) we get

$$\boxed{\tau_{max} = \frac{(9)(0.8)}{0.5790} = 12.44 \text{ ksi}} \tag{8}$$

Problem 3:

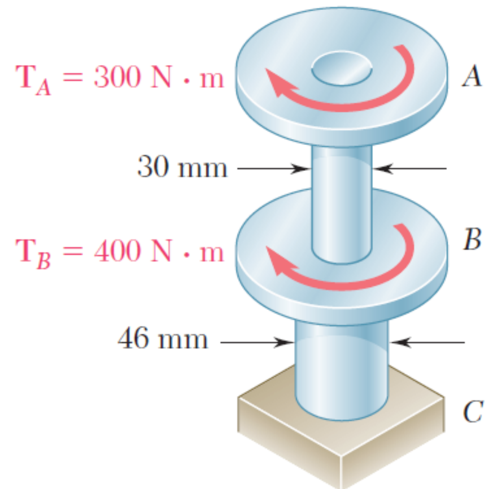


Figure 3

The torques shown are exerted on pulleys A and B. Knowing that each shaft is solid, determine the maximum shearing stress (a) in shaft AB, (b) in shaft BC.

Solution 3

This problem is very similar to the ones we have previously looked at however, in this case we need to make sure we keep in mind how the torques change over the shaft.

Part A

For the AB shaft it is quite easy to figure out the torque. We can once again pull out our handy equations

$$\tau_{max} = \frac{Tr}{J} \tag{9}$$

$$J = \frac{\pi}{2}c^4 \text{ (from second moment of inertia table)} \tag{10}$$

where again $r = c = 0.015$ m for this problem are the same. We can see from the schematic that the only torque acting on this section of the shaft is T_A so that we can obtain the following:

$$\tau_{max} = \frac{Tr}{J} = \frac{(2)(300)}{\pi(0.015)^3} \tag{11}$$

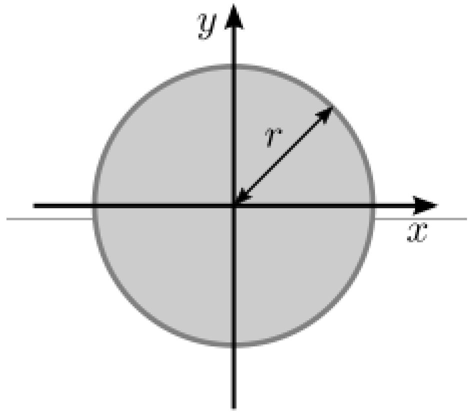
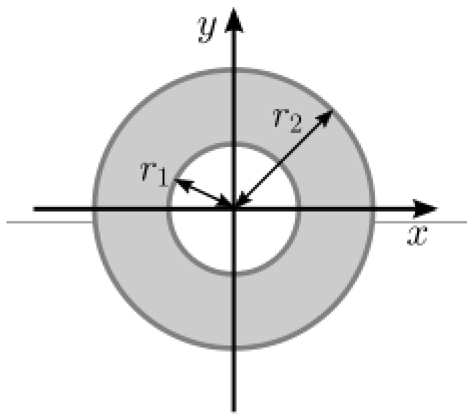
$$\boxed{\tau_{max} = 56.6 \text{ MPa}} \tag{12}$$

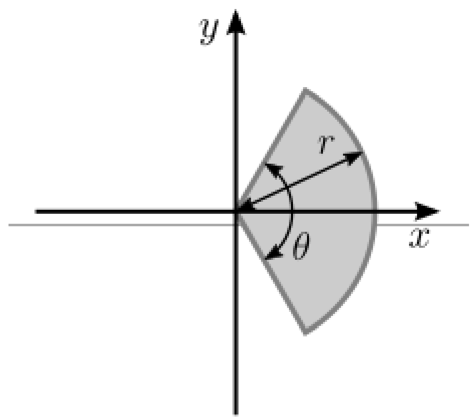
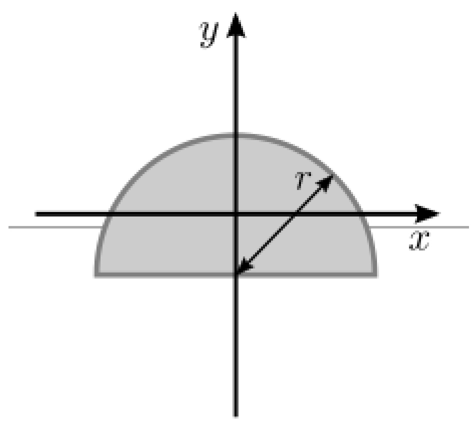
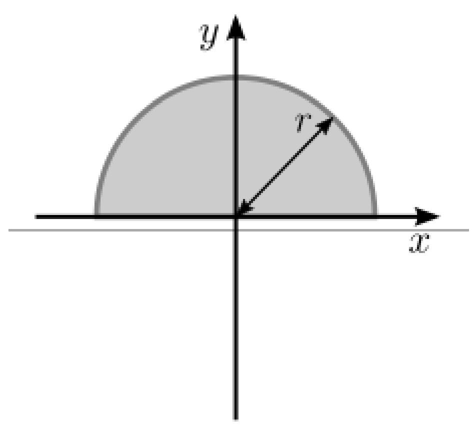
Part B

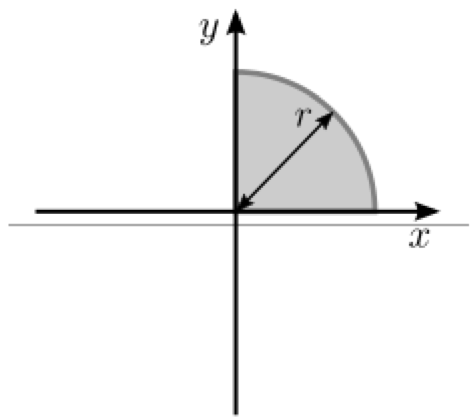
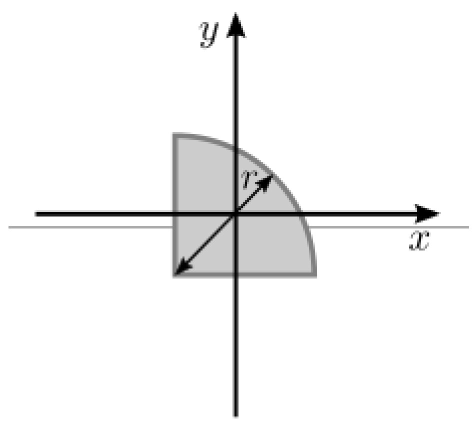
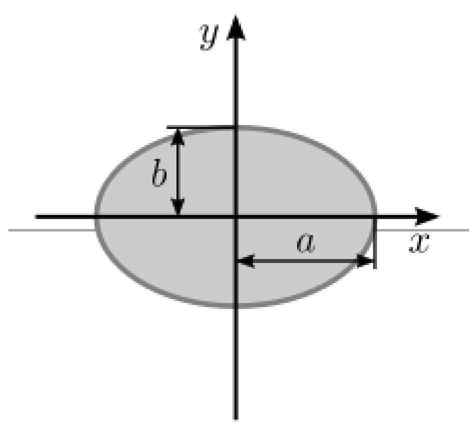
For this section of the shaft, we know that $r = c = 0.023$ m have to be the same again due to the uniform cross-section, however, we know that torque now has to change to $T_{total} = T_A + T_B$. So that the equation above becomes:

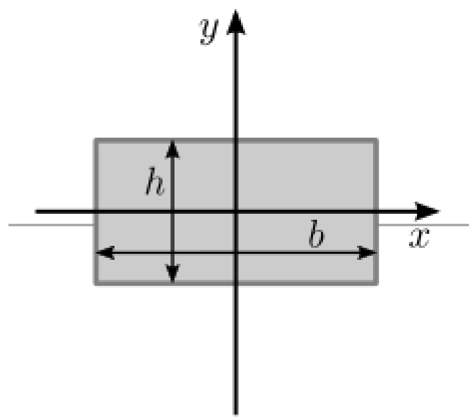
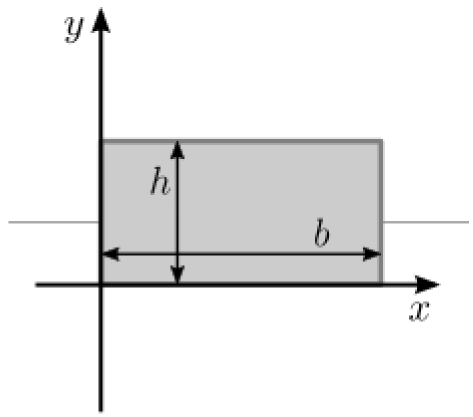
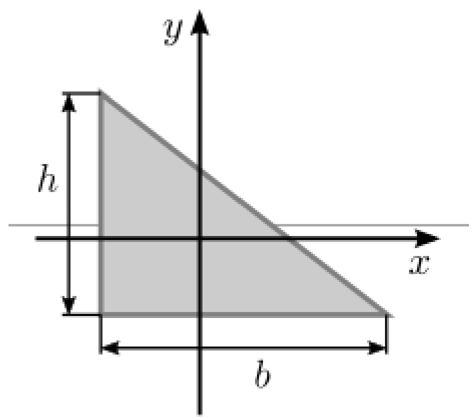
$$\tau_{max} = \frac{Tr}{J} = \frac{(2)(700)}{\pi(0.023)^3} \tag{13}$$

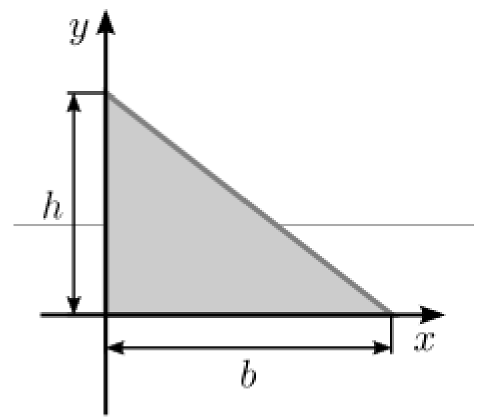
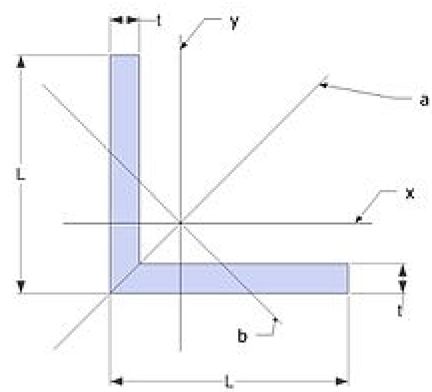
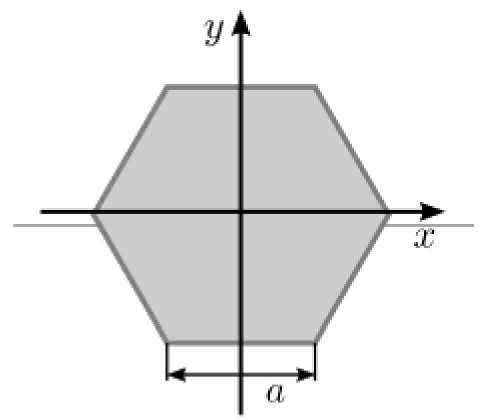
$$\boxed{\tau_{max} = 36.6 \text{ MPa}} \tag{14}$$

Description	Figure	Area moment of inertia	Comment
A filled circular area of radius r		$I_x = \frac{\pi}{4} r^4$ $I_y = \frac{\pi}{4} r^4$ $I_z = \frac{\pi}{2} r^4$ [1]	I_z is the Polar moment of inertia.
An annulus of inner radius r_1 and outer radius r_2		$I_x = \frac{\pi}{4}\left({r_2}^4 - {r_1}^4\right)$ $I_y = \frac{\pi}{4}\left({r_2}^4 - {r_1}^4\right)$ $I_z = \frac{\pi}{2}\left({r_2}^4 - {r_1}^4\right)$	For thin tubes, $r \equiv r_1 \approx r_2$ and $r_2 \equiv r_1 + t$. So, for a thin tube, $I_x = I_y \approx \pi r^3 t$. I_z is the Polar moment of inertia.
A filled circular sector of angle θ in radians and radius r with respect to an axis through the centroid of the sector and the center of the circle		$I_x = (\theta - \sin\theta)\frac{r^4}{8}$	This formula is valid only for 0 ≤ θ ≤ π

			
A filled semicircle with radius *r* with respect to a horizontal line passing through the centroid of the area		$I_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4 \approx 0.1098 r^4$ $I_y = \frac{\pi r^4}{8}$ [2]	
A filled semicircle as above but with respect to an axis collinear with the base		$I_x = \frac{\pi r^4}{8}$ $I_y = \frac{\pi r^4}{8}$ [2]	I_x: This is a consequence of the parallel axis theorem and the fact that the distance between the x axes of the previous one and this one is $\frac{4r}{3\pi}$
A filled quarter circle with radius *r* with the			

axes passing through the bases		$I_x = \frac{\pi r^4}{16}$ $I_y = \frac{\pi r^4}{16}$ [3]	
A filled quarter circle with radius *r* with the axes passing through the centroid		$I_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4 \approx 0.0549 r^4$ $I_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4 \approx 0.0549 r^4$ [3]	This is a consequence of the parallel axis theorem and the fact that the distance between these two axes is $\frac{4r}{3\pi}$
A filled ellipse whose radius along the *x*-axis is *a* and whose radius along the *y*-axis is *b*		$I_x = \frac{\pi}{4} a b^3$ $I_y = \frac{\pi}{4} a^3 b$	
A filled rectangular area with a base width of			

b and height *h*		$I_x = \frac{bh^3}{12}$ $I_y = \frac{b^3h}{12}$ [4]	
A filled rectangular area as above but with respect to an axis collinear with the base		$I_x = \frac{bh^3}{3}$ $I_y = \frac{b^3h}{3}$ [4]	This is a result from the parallel axis theorem
A filled triangular area with a base width of *b* and height *h* with respect to an axis through the centroid		$I_x = \frac{bh^3}{36}$ $I_y = \frac{b^3h}{36}$ [5]	
A filled triangular area as above but with			This is a

respect to an axis collinear with the base		$I_x = \frac{bh^3}{12}$ $I_y = \frac{b^3h}{12}$ [5]	consequence of the parallel axis theorem
An equal legged angle, commonly found in engineering applications		$I_x = I_y = \frac{t(5L^2 - 5Lt + t^2)(L^2 - Lt + t^2)}{12(2L - t)}$ $I_{(xy)} = \frac{L^2t(L-t)^2}{4(t-2L)}$ $I_a = \frac{t(2L-t)(2L^2 - 2Lt + t^2)}{12}$ $I_b = \frac{t(2L^4 - 4L^3t + 8L^2t^2 - 6Lt^3 + t^4)}{12(2L-t)}$	$I_{(xy)}$ is the often unused product of inertia, used to define inertia with a rotated axis
A filled regular hexagon with a side length of *a*		$I_x = \frac{5\sqrt{3}}{16}a^4$ $I_y = \frac{5\sqrt{3}}{16}a^4$	The result is valid for both a horizontal and a vertical axis through the centroid, and therefore is also valid for an axis with arbitrary direction that passes through the origin.

ES120 Spring 2018 – Section 5 Notes

Matheus Fernandes

March 1, 2018

Problem 1:

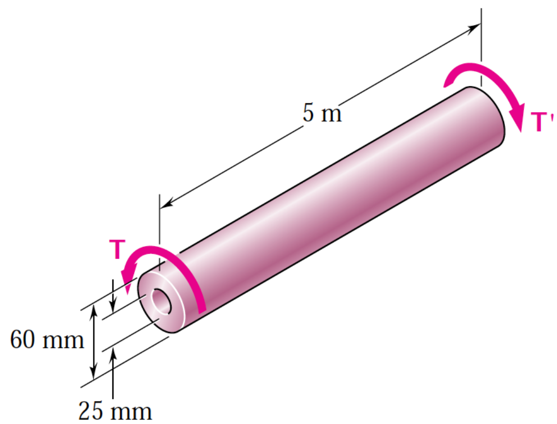


Figure 1

The hollow shaft shown is made of a steel that is assumed to be elastoplastic with $\tau_Y = 145$ MPa and $G = 77.2$ GPa. The magnitude T of the torques is slowly increased until the plastic zone first reaches the inner surface of the shaft; calculate T.

Solution 1

For this problem we know that the inner and outer radii are given by:

$$c_1 = 12.5 \text{ mm} \qquad c_2 = 30 \text{ mm} \,, \tag{1}$$

respectively. We know that when the plastic zone reaches the inner surface the stress must be equal to τ_Y given that the rest of the cross section still has resistance to the shear. Therefore, that resistance will not allow the entire corss-section to plastically deform throughout the cross-section. Using this and the fact that the incremental change in torque is a result of the increment of change of the shear and the area we can compute the corresponding torque by integration, namely,

$$dT = \rho\tau dA = \rho\tau_Y(2\pi\rho d\rho) = 2\pi\tau_Y\rho^2 d\rho \tag{2}$$

where ρ is the radius and the equation relates the incremental change in torque as a function of the incremental change in radius. Integrating this equation in ρ we get:

$$\begin{aligned} T &= 2\pi\tau_Y \int_{c_1}^{c_2} \rho^2 d\rho = \frac{2\pi}{3}\tau_Y (c_2^3 - c_1^3) \\ &= \frac{2\pi}{3}(145 \times 10^6)\left[(30 \times 10^{-3})^3 - (12.5 \times 10^{-3})^3\right] = \boxed{7.61 \times 10^3\ \mathrm{N \cdot m}} \end{aligned} \tag{3}$$

Problem 2:

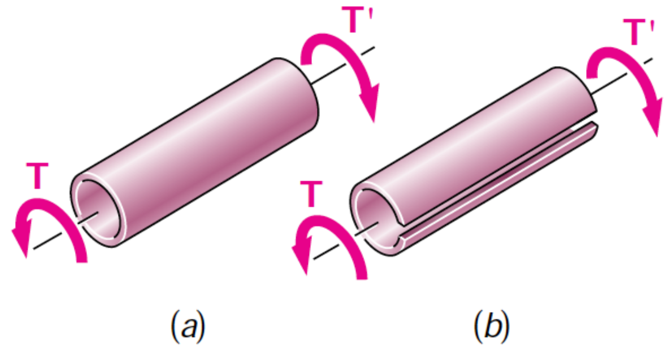


Figure 2

Equal torques are applied to thin-walled tubes of the same length L, same thickness t, and same radius c. One of the tubes has been slit lengthwise as shown. Determine (a) the ratio τ_b/τ_a of the maximum shearing stresses in the tubes, (b) the ratio ϕ_b/ϕ_a of the angles of twist of the shafts.

Solution 2

For this problem lets separate into separate geometries, namely 'a' for the one without the slit and 'b' for the one with the slit. This is not part a and part b of the problem, but simply a step towards the goals of the problem.

Here we are only calculating the different values for the different geometries. So for the geometry without the slit we can compute the following maximum shear stresses using the fact that we know this to be a thin-walled hollow shaft, which can be read in section 3.13 of the book. We know that the torque applied to a hollow member with constant shear flow is

$$\tau_a = \frac{T}{2t\mathcal{A}} \tag{4}$$

where $\mathcal{A}$ is the area bounded by the centerline of the wall cross section. Which given that t is very thin, we can approximate it to be simply c which gives us $\mathcal{A} = \pi c^2$. Therefore,

$$\tau_a = \frac{T}{2\pi c^2 t} \tag{5}$$

$$J \approx 2\pi c^3 t \tag{6}$$

So therefore, we can solve for the angle of twist as

$$\phi_a = \frac{TL}{GJ} = \frac{TL}{2\pi c^3 tG} \tag{7}$$

Now for the case for the geometry with the slit, we can approximate it as a non-circular member as described in section 3.12 of the book. For this approximation we can compute the length of the wider face a to be the circumference of the cylinder

$$a = 2\pi c \tag{8}$$

and the length of the lesser wide face to be

$$b = t. \tag{9}$$

We can see that

$$\frac{a}{b} = \frac{2\pi c}{t} >> 1 \tag{10}$$

which from Table 3.1 of the book indicates that $c_1 = c_2 = \frac{1}{3}$. From the equation in section 3.12 we can therefore compute the τ_b to be

$$\tau_b = \frac{T}{c_1 ab^2} = \frac{3T}{2\pi ct^2} \tag{11}$$

Now for the angle of twist we also know form section 3.12 of the book that it is given by:

$$\phi_b = \frac{TL}{c_2 ab^3 G} = \frac{3TL}{2\pi ct^3 G} \tag{12}$$

Part (a)

Now solving for the question on the stress ratio we can plug in our findings to get:

$$\frac{\tau_b}{\tau_a} = \frac{3T}{2\pi ct^2} \cdot \frac{2\pi c^2 t}{T} = \boxed{\frac{3c}{t}} \tag{13}$$

Part (b)

Now solving for the question on the twist ratio we can plug in our findings to get:

$$\frac{\phi_b}{\phi_a} = \frac{3TL}{2\pi ct^3 G} \cdot \frac{2\pi c^3 tG}{TL} = \boxed{\frac{3c^2}{t^2}} \tag{14}$$

Now let's look at these solutions a little deeper for some intuition. It is cool to see that when we add a slit to the thin shell cylinder we get higher maximum shear stresses than without the slit (assuming $c > t$)). Furthermore, not only is our angle of twist also larger for the geometry with the slit but it scales $\left(\frac{c}{t}\right)^2$ which is much faster than the maximum shear stresses.

ES120 Spring 2018 – Section 6 Notes

Matheus Fernandes

March 22, 2018

Problem 1:

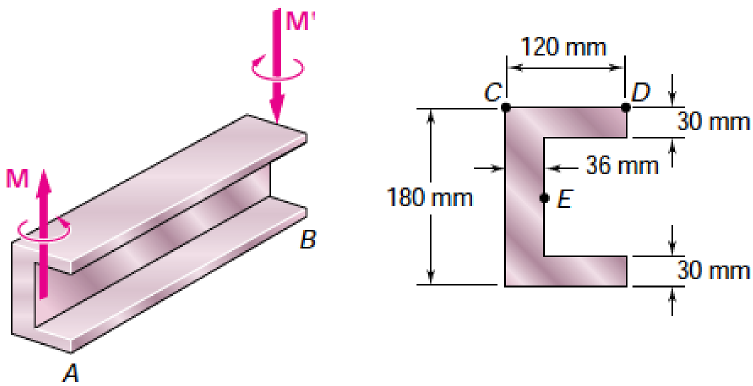


Figure 1

Two equal and opposite couples of magnitude $M = 25$ [kN·m] are applied to the channel-shaped beam AB. Observing that the couples cause the beam to bend in a horizontal plane, determine the stress at (a) point C, (b) point D, (c) point E.

Solution 1

To do this problem, we need to identify what is the axis of the moment M applied to the beam. Since the axis of both moment arms are in the positive and negative y_o direction (according to the right hand rule), we need to first find what is the centroid of the structure within the plane of the cross section. To do so, we need to divide the cross section into three separate rectangles as shown fig. 2. Using the dimensions provided by the problem statement we can obtain measures for the area A of the different rectangles, and the distance $\hat{x}_o$ in the direction x_o of the center of the rectangle. For these distances, we choose our reference axis to be y_o.

Now we can easily repeat this approach for all of the 3 rectangles in the cross section and formulate the first two columns of the table below. The last column is simply computed by multiplying the first two columns together.

	A, [mm^2]	$\bar{x}_o$, [mm]	$A\bar{x}_o$, [mm^3]
#1	3600	60	216×10^3
#2	4320	18	77.76×10^3
#3	3600	60	216×10^3
Σ	$11,520$		509.76×10^3

Using this table we have all of the information we need to compute the centroid distance $\bar{x}$ from the axis $\bar{y}_o$ using the equation:

$$\begin{aligned} \bar{x}\Sigma A &= \Sigma A\bar{x}_o \\ \bar{x}(11520) &= 509.76 \times 10^3 \\ \bar{x} &= 44.25 \text{ [mm]} \end{aligned} \tag{1}$$

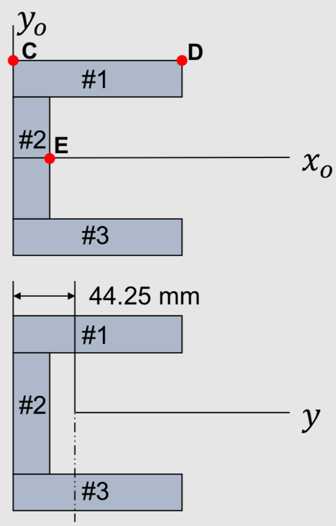


Figure 2: Diagram depicting different sections and centroid of cross-section.

Using the structure's centroid information, we can now compute the distance d from a rectangles center to the centroid of the structure. We will get a different d for each of the rectangles

$$\begin{aligned} d_1 &= 60 - 44.25 = 15.75 \text{ [mm]} \\ d_2 &= 44.25 - 1.8 = 26.26 \text{ [mm]} \\ d_3 &= 60 - 44.25 = 15.75 \text{ [mm]} \end{aligned} \tag{2}$$

We're ready to compute the second moment of inertia for the three different rectangles using the parallel axis theorem. **The parallel axis theorem states** that if the body is made to rotate instead about a new axis which is parallel to the first axis and displaced from it by a distance d, then the moment of inertia I with respect to the new axis is related to I_{cm} (center of mass) by

$$I = I_{cm} + Ad^2 \tag{3}$$

In other words, we must simply compute the moment of inertia of a given rectangle and offset the centroid by the distance between the two axis squared multiplied by the area. Always remember to define b and h according to the axis of rotation, which for this case is y_o. Let's use this information to compute the moment of inertias for this problem

$$I_1 = I_3 = \frac{1}{12}b_1h_1^3 + A_1d_1^2 = \frac{1}{12}(30)(120)^3 + (3600)(15.75)^2 = 5.2130 \times 10^6 \text{ [mm}^4\text{]} \tag{4}$$

$$I_2 = \frac{1}{12}b_2h_2^3 + A_2d_2^2 = \frac{1}{12}(120)(36)^3 + (4320)(26.25)^3 = 3.4433 \times 10^6 \text{ [mm}^4\text{]} \tag{5}$$

All of the work done so far gets us to the final goal of computing the moment of inertia of this slightly complex cross section. So the final step of this journey is to add these offset moment of inertias with respect to the centroid of the cross section together, namely

$$I = I_1 + I_2 + I_3 = 2I_1 + I_2 = 13.8694 \times 10^{-6} \text{ [m}^4\text{]} \tag{6}$$

Now for the different parts of the problem we must compute the distance of the points to our centroid axis y, namely:

$$\begin{aligned} y_1 &= -44.25 \text{ [mm]} \\ y_2 &= 120 - 44.25 = 75.75 \text{ [mm]} \\ y_3 &= 36 - 44.25 = -8.25 \text{ [mm]} \end{aligned} \tag{7}$$

At this point we have all of the information we need to compute the stresses at the different points requested for the different parts of the problem. So let's do this!

Part (a)

For Point C

$$\boxed{\sigma_C = -\frac{My_C}{I} = -\frac{(25 \times 10^3)(-0.04425)}{13.8694 \times 10^{-6}} = 79.8 \times 10^6 \text{ [Pa]}} \tag{8}$$

Part (b)

For Point D

$$\boxed{\sigma_D = -\frac{My_D}{I} = -\frac{(25 \times 10^3)(0.07575)}{13.8694 \times 10^{-6}} = -136.5 \times 10^6 \text{ [Pa]}} \tag{9}$$

Part (c)

For Point E

$$\boxed{\sigma_E = -\frac{My_E}{I} = -\frac{(25 \times 10^3)(-0.00825)}{13.8694 \times 10^{-6}} = -14.87 \times 10^6 \text{ [Pa]}} \tag{10}$$

Problem 2:

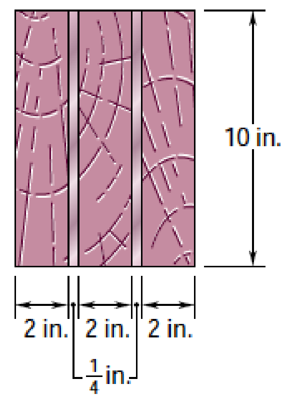


Figure 3

Three wooden beams and two steel plates are securely bolted together to form the composite member shown. Using the data given below, determine the largest permissible bending moment when the member is bent about a horizontal axis.

	Wood	**Steel**
Modulus of elasticity	2×10^6 psi	30×10^6psi
Allowable stress	2000 psi	$22,000$ psi

Solution 2

We can easily identify that this problem is a composite problem. So now, we need to transform one material property into another using geometry. Therefore, let's pick wood as our reference material and scale the steel such that it is made of the same material but is instead thicker to compensate for the it's stiffness compared to the wood. To do that we introduce the ration n with respect to the wood, namely,

$$n_w = 1 \quad \text{for wood} \tag{11}$$

$$n_s = \frac{E_s}{E_w} = \frac{30 \times 10^6}{2 \times 10^6} = 15 \quad \text{for steel} \tag{12}$$

Now we compute the properties of the geometric section, which is simply dependent on the geometry and does not need to be related to ns. This is a much simpler calculation then what we saw in the previous example because our axis is now the horizontal axis and the material is uniform w.r.t. the horizontal axis. Therefore, to compute the moment of inertia we simply compute:

$$I_s = \frac{1}{12}\left(\frac{1}{4} + \frac{1}{4}\right)(10)^3 = 41.6667 \text{ [in}^4\text{]} \tag{13}$$

$$I_w = \frac{1}{12}(2 + 2 + 2)(10)^3 = 500 \text{ [in}^4\text{]} \tag{14}$$

Now for the transformed section, we must only need to transform respective sections we found above to the scale we derived at the beginning of the problem. Therefore, that becomes,

$$I_{\text{trans}} = n_s I_s + n_w I_w = (15)(41.6667) + (1)(500) = 1125 \text{ [in}^4\text{]} \tag{15}$$

To obtain the larges permissible bending moment and relate that to the allowable stress we need to manipulate our equation below such that we get it as a function of the stress. The tricky part is to know that there is an n in the solution below. This makes sense, because we need to weigh the stress according to the ratio of the stiffness. The only reason we can treat this difference in Young's moduli as a fraction of the geometry is because we're dealing with linear elasticity (and it's not non-linear).

$$|\sigma| = \left|\frac{nMy}{I}\right| \Rightarrow M = \left|\frac{\sigma_{\text{allow}} I}{ny}\right| \tag{16}$$

Now we need to evaluate this for both, the wood and the steed and see which one will fail first. The smaller value of the two will dictate the largest permissible bending moment of the member. Therefore,

$$M_{\text{wood}} = \frac{(2000)(1125)}{(1)(5)} = 450 \times 10^3 \text{ [lb} \cdot \text{in]} \tag{17}$$

$$M_{\text{steel}} = \frac{(22000)(1125)}{(15)(5)} = 330 \times 10^3 \text{ [lb} \cdot \text{in]} \tag{18}$$

$$\boxed{\text{Choosing the smaller value of the two, the steel will fail first at } M = 330 \text{ [kip} \cdot \text{in]}} \tag{19}$$

ES120 Spring 2018 – Section 7 Notes

Matheus Fernandes

March 29, 2018

Problem 1:

For the beam and loading shown, (a) determine the equations of the shear and bending-moment curves, (b) draw the shear and bending-moment diagrams.

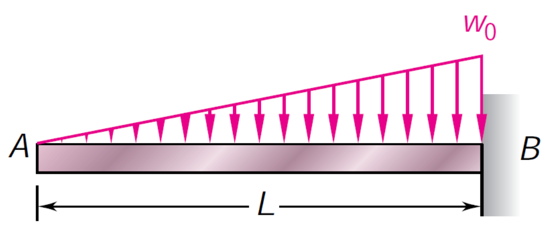


Figure 1

Solution 1

Part (a)

To do this problem we must begin by draying the free body diagram for an arbitrary distance x from edge A to point J.

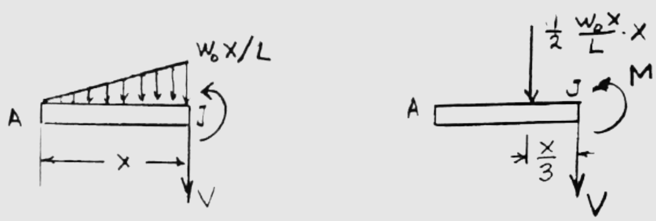


Figure 2

Using this figure, we can compute the force balance in the y direction such that

$$-\frac{1}{2}\frac{w_o x}{L}x - V = 0 \tag{1}$$

$$V = -\frac{w_o x^2}{2L} \tag{2}$$

Now to obtain the equation for the bending moment, we simply integrate V with respect to x, namely:

$$M = -\frac{w_o x^3}{6L} \tag{3}$$

Now, to obtain the maximum value of both, we evaluate the above equations at point B, where $x = L$

$$V(L) = -\frac{w_o L}{2} \tag{4}$$

$$M(L) = -\frac{w_o L^2}{6} \tag{5}$$

Part (b)

Using this information, we can draw the shear and bending moment diagrams as below. Note, that we know the way the function should look like based on the power of x, i.e. $V(x) \sim -x^2$ amd $M(x) \sim -x^3$

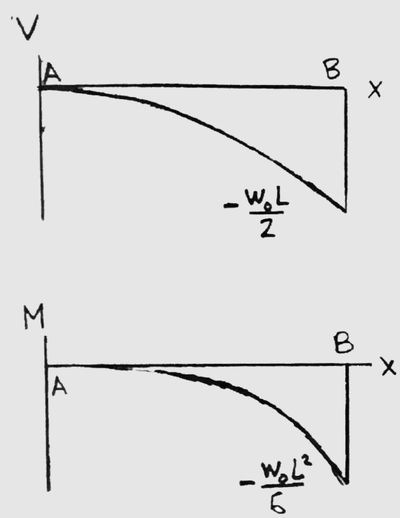


Figure 3

Problem 2:

Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (a) of the shear, (b) of the bending moment.

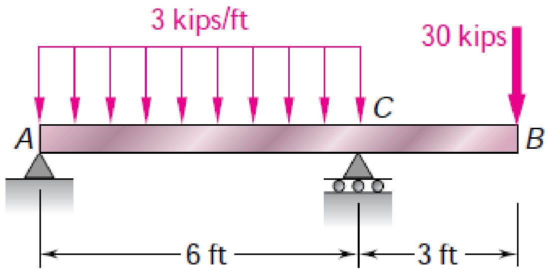


Figure 4

Solution 2

This problem is a bit more complex than the previous one. First, we need to begin finding the unknown reaction forces along the bottom of the beam. To do this, let's use sum of moments about points C and A, respectively (we know that the distributed load can be approximated as a point load at the center between A and C and of the total amount $3 * 6 = 18$ kips):

$$\Sigma M_C = 0 : -6R_A + (3)(18) - (3)(30) = 0 \Rightarrow R_A = -6 \text{ kips (downward)} \tag{6}$$

$$\Sigma M_A = 0 : 6R_C - (3)(18) - (9)(30) = 0 \Rightarrow R_C = 54 \text{ kips (upward)} \tag{7}$$

Using this information, we can separate this problem into two parts, namely A to C and C to B. So let's begin by solely analyzing A to C. This is valid for $- < x < 6$

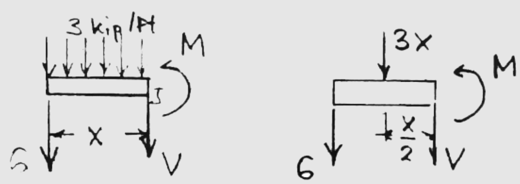


Figure 5

If we do sum of forces in y

$$\Sigma F_y = 0 : -6 - 3x - V = 0 \Rightarrow V = -6 - 3x \text{ kips} \tag{8}$$

So to obtain the bending moment we can take the integral such that we obtain

$$M(x) = -6x - 1.5x^2 \tag{9}$$

Which if we evaluate at pint C which is $x = 6$ we obtain

$$M(6) = -6(6) - 1.5(6)^2 = -90 \tag{10}$$

Now lets analyze C to B

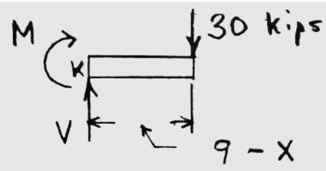


Figure 6

Similar to before, lets do a sum of forces in the y direction

$$\Sigma F_y = 0 : V - 30 = 0 \Rightarrow V = 30 \text{ kips} \tag{11}$$

We can do this integral one of two ways: 1. matching the solution to solve for the integration constant of an indefinite integral or 2. find the bounds we need to use in the integral.
Method 1: Now here we have to be careful when taking the integral as the integration constant does matter, namely we will have

$$M = 30x + k \tag{12}$$

We know that at $x = 6$, M has to match the solution from the previous part (from A to C.) Since we can evaluate to be $M(6) = -90$ then we can solve for the constant k at point C, namely,

$$M = 30(6) + k = -90 \Rightarrow k = -270 \text{ kips} \tag{13}$$

Method 2: We perform the following definite integral

$$M(x) = \int_9^x 30(x)dx \Rightarrow 30x - 30(9) = 30x - 270 \tag{14}$$

So our final expression therefore becomes

$$M(x) = 30x - 270 \tag{15}$$

Using this information we are ready to draw our shear and bending moment diagrams.

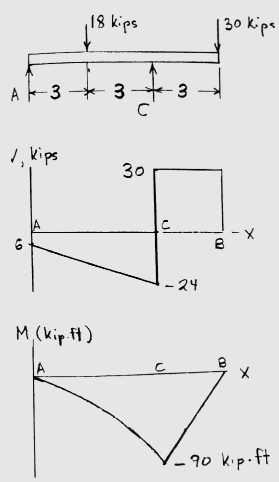


Figure 7

From these graphs, we can decipher that out maximum absolute values for the shear and bending moments are

$$\boxed{|V|_{\text{max}} = 30 \text{ kips}} \tag{16}$$

$$\boxed{|M|_{\text{max}} = 90 \text{ kips} \cdot \text{ft}} \tag{17}$$

Problem 3:

(a) Using singularity functions, write the equations defining the shear and bending moment for the beam and loading shown. (b) Use the equation obtained for M to determine the bending moment at point C and check your answer by drawing the free-body diagram of the entire beam.

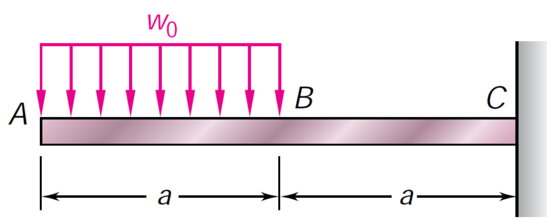


Figure 8

Solution 3

Singularity functions are your friends and are by no means as scary as they sound. They are simply a way to help you write the governing equations in a much simpler way. They simply indicate where a change in the stresses or boundary conditions occur.

Just to formalize the singularity function, let's define what it is:

Singularity Function Definition

$$< x - a >^n = \begin{cases} (x-a)^n & \text{for } x \geq a \\ 0 & \text{for } x < a \end{cases} \tag{18}$$

Let's first begin by drawing the free body diagram of the cantilever beam

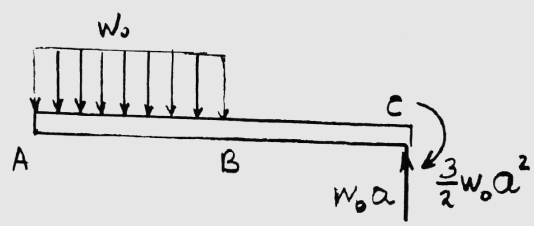


Figure 9

Part (a)

Simply based on the diagram we can write the following for $w(x)$

$$w(x) = w_o - w_o < x - a >^0 \tag{19}$$

Note that $w(x) = -\frac{dV}{dx}$ so we can obtain $V(x)$ as

$$\boxed{V(x) = -w_o x + w_o < x - a >^1} \tag{20}$$

Again note that $V(x) = \frac{dM}{dx}$ so we can obtain $M(x)$ as

$$\boxed{M(x) = -\frac{1}{2}w_o x^2 + \frac{1}{2}w_o < x - a >^2} \tag{21}$$

Part (b)

This asks us to obtain the bending moment at point C. At point C we will have a value of $x = 2a$. So we can evaluate our function $M(x)$ at point $2a$ such that we obtain

$$M(2a) = -\frac{1}{2}w_o(2a)^2 + \frac{1}{2}w_o < (2a) - a >^2 = -\frac{1}{2}w_o(2a)^2 + \frac{1}{2}w_o(a)^2 = \boxed{-\frac{3}{2}w_o a^2} \tag{22}$$

Extra Check

To verify that we did this correctly we can do a sum of moments about point C such that,

$$\Sigma M_c = 0: \quad \left(\frac{3a}{2}\right)(w_o a) + M_c = 0 \Rightarrow \boxed{M_c = -\frac{3}{2}w_o a^2} \quad \checkmark \tag{23}$$

ES120 Spring 2018 – Section 8 Notes

Matheus Fernandes

April 19, 2018

Problem 1:

The vertical shear is 25 kN in a beam having the cross section shown. Knowing that $d = 50$ mm, determine the shearing stress at (a) point a, (b) point b.

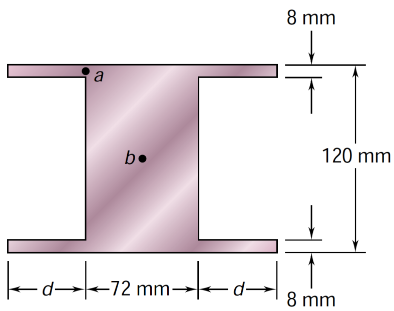


Figure 1

Solution 1

For this problem, let's first begin by computing the moment of inertia of the cross section using the outside parts and the inside part separately and using parallel axis theorem

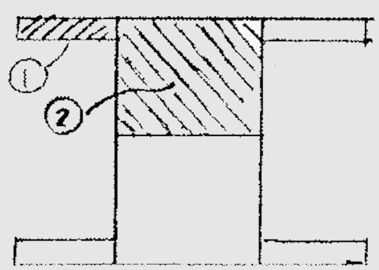


Figure 2

$$I_1 = \frac{1}{12}(50)(8)^3 + (50)(8)(56)^2 = 1.256 \times 10^6 \text{ mm}^4 \quad (1)$$

$$I_2 = \frac{1}{3}(72)(63)^3 = 5.184 \times 10^6 \text{ mm}^4 \quad (2)$$

Such that the total section's second moment of inertia becomes

$$I = 4I_1 + 2I_2 = 15.3933 \times 10^{-6}\ \mathrm{m}^4 \tag{3}$$

Now we can compute the first moment with respect to the neutral axis Q

$$Q_1 = A_1\bar{y}_1 = (50)(8)(56) = 22.4 \times 10^3\ \mathrm{mm}^3 \tag{4}$$

$$Q_2 = A_2\bar{y}_2 = (72)(60)(30) = 129.6 \times 10^3\ \mathrm{mm}^3 \tag{5}$$

Part (a)

Now to determine the shearing stress at point a, we obtain the average shearing stress exerted on the element since it is located at the center of the element

$$\tau_a = \frac{VQ_a}{It_a} = \frac{(25 \times 10^3)(22.4 \times 10^{-6})}{(15.3933 \times 10^{-6})(8 \times 10-3)} \tag{6}$$

$$\boxed{\tau_a = 4.55\ \mathrm{MPa}} \tag{7}$$

Part (b)

Similarly, to obtain the shear at point b, since it is at the center of the element, we only need to evaluate the average shearing stress exerted on the element. Remember, for these types of calculation we need to be working from the outside inward. So here we need to take account both of the first moments with respect to the neutral axis, namely

$$Q_b = 2Q_1 + Q_2 = 174.4 \times 10^{-6}\ \mathrm{m}^3 \tag{8}$$

With this, we are ready to compute the average shear using the same equation as the previous problem, namely,

$$\tau_b = \frac{VQ_b}{It_b} = \frac{(25 \times 10^3)(174.4 \times 10^{-6})}{(15.3933 \times 10^{-6})(72 \times 10^{-3})} \tag{9}$$

$$\boxed{\tau_b = 3.93\ \mathrm{MPa}} \tag{10}$$

Problem 2:

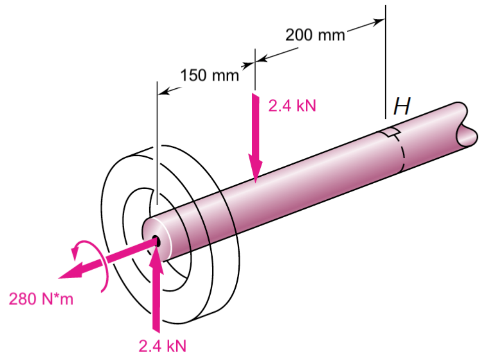


Figure 3

The axle of an automobile is acted upon by the forces and couple shown. Knowing that the diameter of the solid axle is 1.25 in., determine (a) the principal planes and principal stresses at point H located on top of the axle, (b) the maximum shearing stress at the same point.

Solution 2

For this part, what we need to do is evaluate it the same way we did in the torsion problem, noting that H is located on the outer portion of the rod. So that the shear due to torsion becomes

$$\tau = \frac{Tc}{J} = \frac{2T}{\pi c^3} \tag{11}$$

Plugging in the values we get:

$$\tau = \frac{(2)(280)}{\pi(15)^3} = 52.8 \text{ kPa} \tag{12}$$

The bending second moment of inertia is

$$I = \frac{\pi}{4}c^4 = 39761 \text{ mm}^4 \tag{13}$$

The bending moment is

$$M = (150)(2400) = 360000 \text{ N} \cdot \text{m} \tag{14}$$

Thus the bending stress is

$$\sigma = -\frac{My}{I} = -\frac{(36 \times 10^4)(15)}{29761} = -135.8 \text{ MPa} \tag{15}$$

So now if we do a drawing of what is going on in term of the stress state at H we see the following:

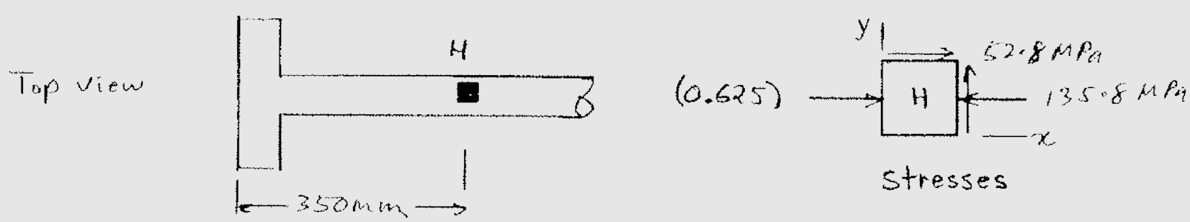


Figure 4

So, if we define a x and y axis we can decompose the stress states into the following

$$\sigma_x = -135.8 \text{ MPa} \tag{16}$$

$$\sigma_y = 0 \text{ MPa} \tag{17}$$

$$\tau_{xy} = 52.8 \text{ MPa} \tag{18}$$

So we van obtain the average stress as

$$\sigma_{\text{ave}} = \frac{1}{2}(\sigma_x + \sigma_y) = -67.9 \text{ MPa} \tag{19}$$

The radius of the circle

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(-67.9)^2 + (52.8)^2} = 86 \text{ MPa} \tag{20}$$

Part (a)

So to calculate the first principal stress we use

$$\sigma_1 = \sigma_{\text{ave}} + R = -67.9 + 86 = 18.1 \text{ MPa} \tag{21}$$

$$\sigma_2 = \sigma_{\text{ave}} - R = -67.9 - 86 = -153.9 \text{ MPa} \tag{22}$$

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{(2)(52.8)}{-67.9} = -1.5552 \tag{23}$$

$$\boxed{\theta_p = -28.6^\circ \text{ and } 61.6^\circ} \tag{24}$$

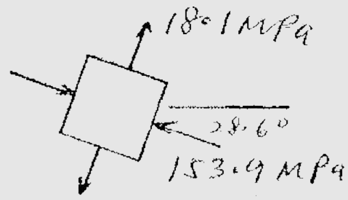


Figure 5

Part (b)

$$\boxed{\tau_{\text{max}} = R = 86 \text{ MPa}} \tag{25}$$

Problem 3:

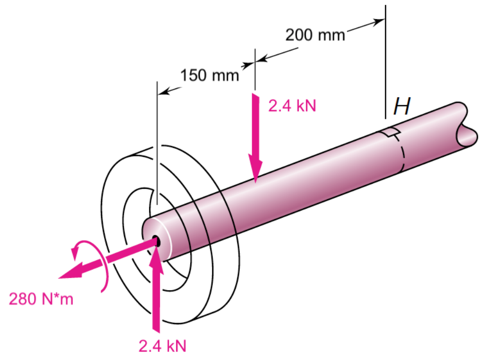


Figure 6

Solve the previous problem using Mohr's circle

Solution 3

Using the solutions from the previous problems, namely,

$$\sigma_x = -135.8 \text{ MPa} \tag{26}$$

$$\sigma_y = 0 \text{ MPa} \tag{27}$$

$$\tau_{xy} = 52.8 \text{ MPa} \tag{28}$$

$$\sigma \text{ave} = \frac{1}{2}(\sigma_x + \sigma_y) = -67.9 \text{ MPa} \tag{29}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(-67.9)^2 + (52.8)^2} = 86 \text{ MPa} \tag{30}$$

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{(2)(52.8)}{-67.9} = -1.5552 \tag{31}$$

We can plot the the points from the previous problem, namely

$$X : (-135.8, -52.8); \quad Y : (0, 52.8); \quad \text{center} : (-67.9, 0) \tag{32}$$

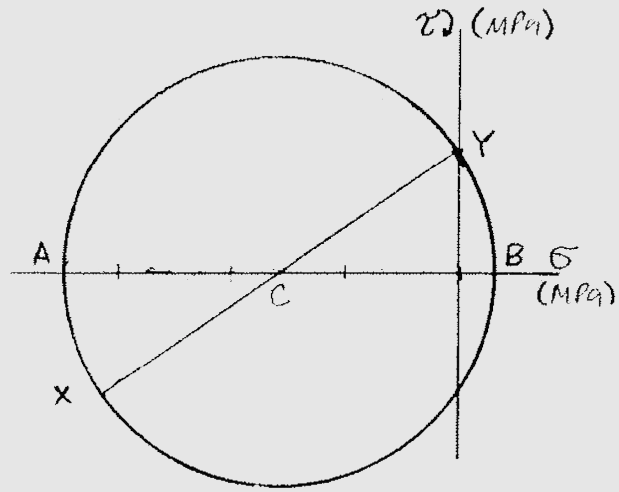


Figure 7

Part (a)

$$\boxed{\theta_p = -28.6^\circ \text{ and } 61.6^\circ} \tag{33}$$

Part (b)

$$\sigma_1 = \sigma_{\text{ave}} + R = -67.9 + 86 = 18.1 \text{ MPa} \tag{34}$$

$$\sigma_2 = \sigma_{\text{ave}} - R = -67.9 - 86 = -153.9 \text{ MPa} \tag{35}$$

$$\boxed{\tau_{\text{max}} = R = 86 \text{ MPa}} \tag{36}$$

ES120 Spring 2018 – Section 9 Notes

Bolei Deng, Matheus Fernandes

April 26, 2018

Problem 1:

Determine the dimension d so that the aluminum and steel struts will have the same weight, and compute the critical load for each strut.

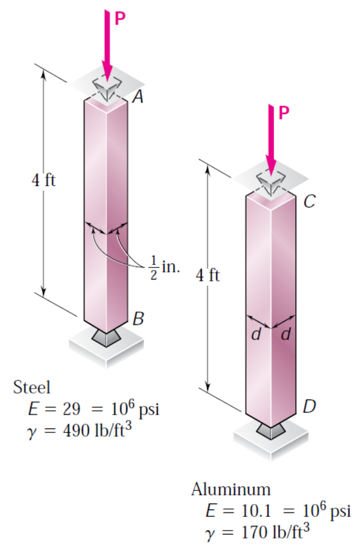


Figure 1

Solution 1

For this problem we must first obtain the total weight of the different beams, namely

$$W = \gamma L d_s^2, \tag{1}$$

for the steel the weight is

$$W_s = (0.2835)(4 * 12)(0.5)^2 = 3.4028 \text{ lb} \tag{2}$$

for the aluminum the weight as a function of the dimension d is

$$W_a = (0.09838)(4 * 12)d^2 = 4.7222d^2 \tag{3}$$

So we want them to be the same weight, thus,

$$W_s = W_a \Rightarrow 3.4028 = 4.7222d^2 \Rightarrow \boxed{d = 0.849 \text{ in}} \tag{4}$$

Now to compute the critical load for each strut we need to solve for the P_{cr} given by

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{5}$$

For the steel strut, we have that

$$I = \frac{1}{12} d_s^4 = \frac{1}{12}\left(\frac{1}{2}\right)^4 = 5.208 \times 10^{-3} \text{ in}^4 \tag{6}$$

$$P_{cr} = \frac{\pi^2 (29 \times 10^6)(5.2083 \times 10^{-3})}{(4 * 12)^2} = \boxed{647 \text{ lb}} \tag{7}$$

For the aluminum strut, we have that

$$I = \frac{1}{12} d_s^4 = \frac{1}{12}(0.849)^4 = 43.271 \times 10^{-3} \text{ in}^4 \tag{8}$$

$$P_{cr} = \frac{\pi^2 (10.1 \times 10^6)(13.271 \times 10^{-3})}{(4 * 12)^2} = \boxed{1872 \text{ lb}} \tag{9}$$

Problem 2:

Knowing that a factor of safety of 2.6 is required, determine the largest load **P** that can be applied to the structure shown. Use $E = 200$ GPa and consider only buckling in the plane of the structure.

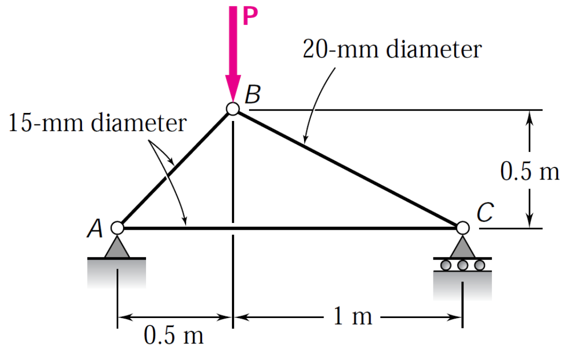


Figure 2

Solution 2

For this problem we must first find the length of the individual members

$$L_{BC} = \sqrt{1^2 + 0.5^2} = 1.1180\text{ m} \tag{10}$$

$$L_{AB} = \sqrt{0.5^2 + 0.5^2} = 0.70711\text{ m} \tag{11}$$

We can also compute the second moment of inertias as:

$$I_{BC} = \frac{\pi}{64}(20)^4 = 7.854 \times 10^{-9}\text{ m}^4 \tag{12}$$

$$I_{AB} = \frac{\pi}{64}(15)^4 = 2.485 \times 10^{-9}\text{ m}^4 \tag{13}$$

We can now compute what the critical load for both members BC and AB using

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{14}$$

$$P_{cr,BC} = \frac{\pi^2(200 \times 10^9)(7.854 \times 10^{-9})}{(1.1180)^2} = 12.403\text{ kN} \tag{15}$$

$$P_{cr,AB} = \frac{\pi^2(200 \times 10^9)(2.485 \times 10^{-9})}{(0.70711)^2} = 9.8106\text{ kN} \tag{16}$$

Given our factor of safety FS we can compute the allowable force recalling that

$$F_{all} = \frac{P_{cr}}{FS} \tag{17}$$

So for the different members that becomes

$$F_{all,BC} = \frac{12.403}{2.6} = 4.770\text{kN} \tag{18}$$

$$F_{all,AB} = \frac{9.8106}{2.6} = 3.773\text{kN} \tag{19}$$

Now we can obtain the free body diagram for point B as

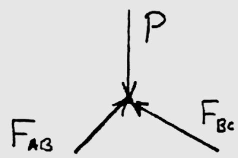


Figure 3

Performing force balance in the horizontal and vertical directions respectively, we obtain

$$\Sigma F_x = 0: \quad \frac{0.5}{0.70711} F_{AB} - \frac{1}{1.1180} F_{BC} = 0 \Rightarrow F_{BC} = 0.7905 F_{AB} \tag{20}$$

$$\Sigma F_y = 0: \quad \frac{0.5}{0.70711} F_{AB} + \frac{0.5}{1.1180} F_{BC} - P = 0 \Rightarrow P = 1.06066 F_{AB} \tag{21}$$

Which combining both we can also obtain a relationship between P and F_{BC}, namely

$$P = (1.06066) \frac{F_{BC}}{0.79057} = 1.3416 F_{BC} \tag{22}$$

Therefore, solving for the allowable value for P yields,

$$P < 1.06066 F_{all,AB} = (1.06066)(3.773) = 4.0 \text{ kN} \tag{23}$$

$$P < 1.3416 F_{all,BC} = (1.3416)(4.770) = 6.4 \text{ kN} \tag{24}$$

The smallest of the the two gives us the largest load namely

$$\boxed{P_{all} = 4.0 \text{ kN}} \tag{25}$$

Problem 3:

Using the method of work and energy, determine the deflection at point D caused by the load **P**.

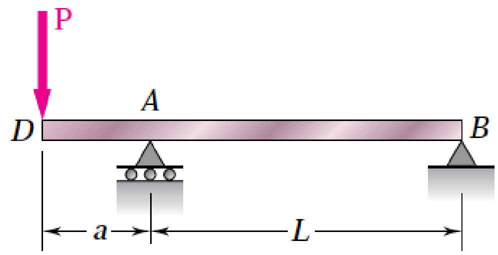
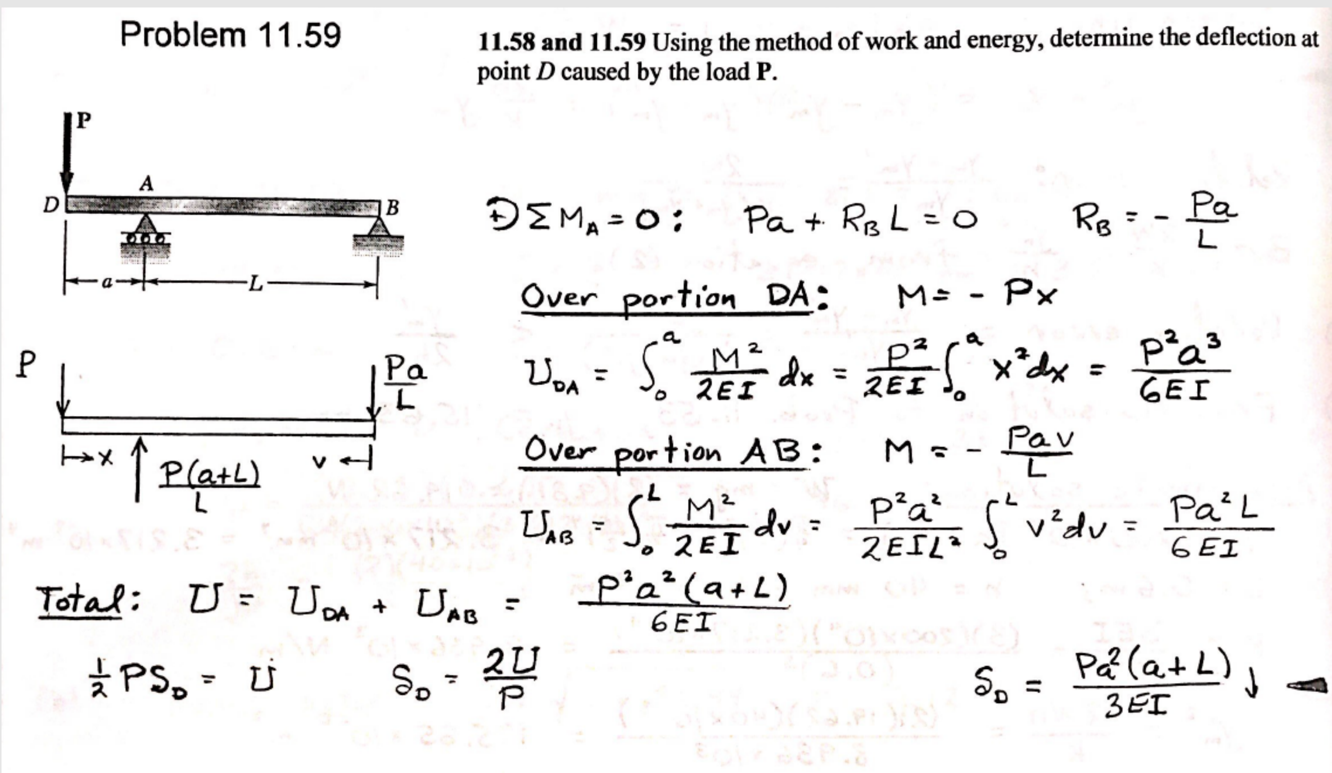


Figure 4

Solution 3



Problem 11.59

11.58 and 11.59 Using the method of work and energy, determine the deflection at point D caused by the load **P**.

$\circlearrowleft \Sigma M_A = 0: \quad Pa + R_B L = 0 \qquad R_B = -\frac{Pa}{L}$

<u>Over portion DA:</u> $\quad M = -Px$

$$U_{DA} = \int_0^a \frac{M^2}{2EI}dx = \frac{P^2}{2EI}\int_0^a x^2 dx = \frac{P^2a^3}{6EI}$$

<u>Over portion AB:</u> $\quad M = -\frac{Pav}{L}$

$$U_{AB} = \int_0^L \frac{M^2}{2EI}dv = \frac{P^2a^2}{2EIL^2}\int_0^L v^2 dv = \frac{Pa^2L}{6EI}$$

<u>Total:</u> $\quad U = U_{DA} + U_{AB} = \frac{P^2a^2(a+L)}{6EI}$

$\frac{1}{2}P\delta_D = U \qquad \delta_D = \frac{2U}{P} \qquad \delta_D = \frac{Pa^2(a+L)}{3EI} \downarrow$

Figure 5

Problem 4:

For the uniform rod and loading shown and using Castigliano's theorem, determine the deflection of point B.

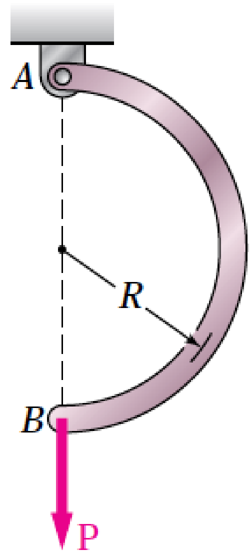
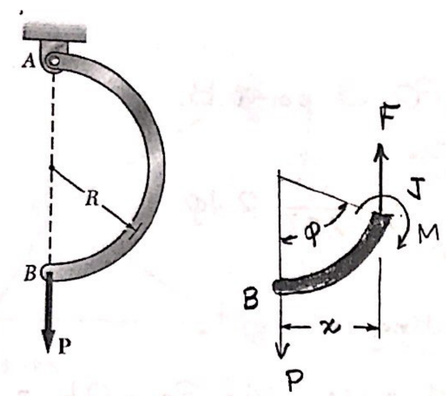


Figure 6

Solution 4



Use polar coordinate φ.

Calculate the bending moment $M(\varphi)$ using free body BJ.

$$+\circlearrowleft \Sigma M_J = 0: \quad Px - M = 0$$

$$M = Px = PR\sin\varphi$$

Strain energy: $U = \int_0^L \frac{M^2}{2EI}\,ds$

$$U = \int_0^{\pi} \frac{(PR\sin\varphi)^2}{2EI}(R\,d\varphi) = \frac{P^2R^3}{2EI}\int_0^{\pi} \sin^2\varphi\,d\varphi$$

$$= \frac{P^2R^3}{2EI}\int_0^{\pi} \frac{1-\cos 2\varphi}{2}\,d\varphi$$

$$= \frac{P^2R^2}{2EI}\left(\frac{1}{2}\varphi\Big|_0^{\pi} - \frac{1}{4}\sin 2\varphi\Big|_0^{\pi}\right) = \frac{\pi P^2 R}{4EI}$$

By Castigliano's theorem, $\delta = \frac{\partial U}{\partial P}$

$$\delta = \frac{\pi P R^3}{2EI} \downarrow$$

Figure 7

ES120 Spring 2018 – Midterm 1 Review w/ Solutions

Matheus Fernandes

March 8, 2018

Document Disclaimer

The list provided below is by no means comprehensive and if you find anything missing that you would like to add please let me know. This review session has been created without prior knowledge of the problems in the exam and should not be treated in any way as hints to problems that will be asked in the exam. We will do our best to go over the topics of the course in detail however please do your own reading of chapters 1, 2 and 3 as well as other topics not included in the book. If you find any typos please let me know and I will update and push a new version to Github ASAP.

You may also find my notes from a previous year helpful: `http://fer.me/es120notes`

Topics Covered Summary

1. **Introduction – Concepts of Stress**
   - **Normal Stress** – $\sigma = \frac{P}{A}$, where A is perpendicular to direction of force
   - **Shearing Stress** – $\tau_{ave} = \frac{P}{A}$, where A is parallel to the direction of force
   - **Stresses under general loading conditions** - Determining the different components of stress from FBD such as σ_{xx}, σ_{yy} and τ_{xy}.
   - **Ultimate stress** – $\sigma_U = \frac{P_u}{A}$
   - **Factor of Safety** – F.S.$= \frac{\text{ultimate load}}{\text{allowable load}} = \frac{\text{ultimate stress}}{\text{allowable stress}}$
   - **Truss Systems** – How to efficiently solve a truss system using method of sections
2. **Stress and Strain – Axial Loading**
   - **Strain** – $\epsilon = \frac{\delta}{L}$
   - **Elastic Stress-Strain Diagram** – Linear Relationship
   - **Plastic Stress-Strain Diagram** – Ideal plasticity with yield stress σ_Y
   - **True Stress and True Strain** – Difference between True and Engineering is the cross-sectional area. True stress uses A of deformed specimen.
   - **Hooke's Law** – $\sigma = E\epsilon$
   - **General Stress State** – Symmetric positive definite matrix of σ_{ij}
   - **Modulus of Elasticity** – E
   - **Elastic vs. Plastic Behavior of Material** – Necking, yield stress, rupture etc.
   - **Fatigue** – In cases of cyclic loading, rupture will occur at a stress much lower than the static breaking strength; this phenomenon is called fatigue.
   - **Deformations of Members Under Axial Loading** – $\delta = \frac{PL}{AE}$
   - **Statically Indeterminate Problems** – Problems that cannot be determined using statics, but where we need to formulate a compatibility constraint. This occurs when we have more reaction forces to solve for than we have equations.
   - **Problems involving temperature changes** – Thermal strain $\epsilon_T = \alpha\Delta T$. This does not create a stress until it is statically constrained and as per superposition the thermal strain becomes mechanical strain.
   - **Superposition Method** – In statically indeterminate problems we remove redundant loads and apply superposition to solve for the different unknown loads.

- **Thermal Stress in a Film on a Substrate** – This is the problem he worked out in class. My notes can be found here `http://fer.me/l/pbFeFA`
- **Poisson's Ratio** – Relates lateral and axial strains through $\nu = -\frac{\text{lateral strain}}{\text{axial strain}}$
- **Multiaxial loading** – Loading through multiple axis and the relationship to strain
- **Generalized Hooke's Law** – Generalized relationship between all stresses, strains and material parameters through equations of form $\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$, $\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z)$, $\epsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_y - \nu\sigma_x)$
- **Plane Strain** – $\epsilon_z = 0$
- **Plane Stress** – $\sigma_z = 0$
- **Bulk Modulus** – Change of volume per unit volume described by $k = \frac{E}{3(1-2\nu)}$
- **Shearing Strain** – Nondimensional deformation due to shearing stress γ_{xy}
- **Hooke's law for shearing stress and strain** – $\tau_{xy} = G\gamma_{xy}$
- **Modulus of Rigidity** – Empirical value to relates shear stress to shear strain G. Analogous to modulus of elasticity E.
- **Relation among E, ν, G** – $\frac{E}{2G} = 1 + \nu$
- **Stress Concentrations** – $K = \frac{\sigma_{max}}{\sigma_{avg}}$
- **Plastic Deformation** – Elastoplastic material stress strain curve. Gain intuition from this curve.
- **Residual Stresses** – Stresses left in a part post plastic deformation

3. **Torsion**

- **Deformation in a Circular shaft** – $\gamma = \frac{\rho\phi}{L}$
- **Average shearing strain** – $\gamma = \frac{\rho}{c}\gamma_{max}$
- **Torsion Stresses** – Shear stresses due to torsion $\tau = \frac{T\rho}{J}$ where J is the polar moment of inertia: `http://fer.me/git/es120notes/blob/master/Section4/J-list.pdf`
- **Torsion Stresses in the Elastic Range** – As long as the yield strength is not exceeded in any part of circualr shaft, the shearing stress in that shaft varies linearly with distance ρ from the axis of the shaft such that $\tau = \frac{\rho}{c}\tau_{max}$
- **Angle of Twist in the Elastic Range** – $\phi = \frac{TL}{JG}$
- **Statically Indeterminate Shafts** – This is analogous to non-torsional statically indeterminate problems, where we need to find a compatibility equation to constrain the different reaction forces we cannot solve for using statics.
- **Design of Transmission Shafts** – This simply builds the relationship of what we have learned to power, frequency and torque, namely, $T = \frac{P}{2\pi f}$
- **Stress concentration in circular shafts** – $\tau_{max} = K\frac{Tc}{J}$, where $\frac{Tc}{J}$ is the stress computed for the smaller-diameter shaft and K is a tabulated stress-concentration factor obtained from an empirical curve.
- **Plastic Deformation in Circular Shafts** – $R_t = \frac{T_u c}{J}$, where R_t is the modulus of rupture, and T_u is the ultimate torque of the shaft
- **Circular shafts made of Elastoplastic Material** – $T = \frac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{\phi_Y^3}{\phi^3}\right)$ is the relationship between the torque and the angle of twist of a elastoplastic shaft. There are also relationships derived in section 3.10 that discuss it in the form of the radius ρ. It is worth reviewing the derivation of this section in detail.
- **Torsion of non-circular members** – Using tabulated values we can find an expression for maximum shear $\tau_{max} = \frac{T}{c_1 ab^2}$ and angle of twist $\phi = \frac{TL}{c_2 ab^3 G}$ where a and b are the lengths of the sides and c_1 and c_2 are coefficients determined empirically. Note that for large values of a/b the coefficients become $c_1 = c_2 \approx \frac{1}{3}$
- **Thin-walled hollow shafts** – Integrate through an idea of shear flow to obtain a relationship between an equivalent area and the torque such that we can concentrate the shear only on the thin amount of material to obtain $\tau = \frac{T}{2t\mathcal{A}}$, where $\mathcal{A}$ is the area of the region from the centerline of the thin-wall to the center of the shaft
- **Dynamics Torsion Problems** – What is the frequency of a rods vibration if twisted and allowed to freely vibrate torsionally. This can be solved to be a wave equation of form $\frac{\partial^2\rho}{\partial t^2} = c^2\frac{\partial^2\rho}{\partial x^2}$. My notes on this matter can be found here: `http://fer.me/l/Nh1grt`

Review Problems

Problem 1:

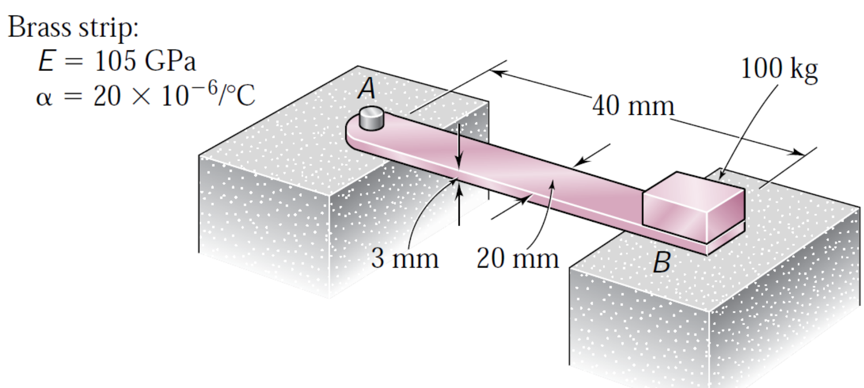


Figure 1

The brass strip AB has been attached to a fixed support at A and rests on a rough support at B. Knowing that the coefficient of friction is 0.60 between the strip and the support at B, determine the decrease in temperature for which slipping will impend.

Solution 1

As always, lets begin by drawing a FBD of the problem to better understand the forces acting on the body.

Figure 2: FBD of stated problem

The weight acting on the device will create a friction force between the bottom of the brass strip and the rough surface. We can compute that force as a result of the normal force.

Now lets do force balance on both directions.

$$\Sigma F_y = 0 : N - W = 0 \Rightarrow N = W \tag{1}$$

$$\Sigma F_x = 0 : P - \mu N = 0 \Rightarrow P = \mu W = \mu mg \tag{2}$$

We know that for slipping to impend we need to find when the the thermal strain will cause a horizontal force that is grater than what the friction will be able to overcome. So assuming there there is no slipping we know based on superposition that the displacement of the thermal strain has to balance the mechanical displacement giving us the following compatibility constraint:

$$\delta = -\frac{PL}{EA} + L\alpha(\Delta T) = 0 \tag{3}$$

Therefore solving this equation for ΔT we obtain:

$$\Delta T = \frac{P}{EA\alpha} = \frac{\mu m g}{EA\alpha} \tag{4}$$

And if we plug in the values from the problem statement we obtain

$$\boxed{\Delta T = \frac{(0.6)(100)(9.81)}{(105 \times 10^9)(60 \times 10^{-6})(20 \times 10^{-6})} = 4.67°\text{C}} \tag{5}$$

Problem 2:

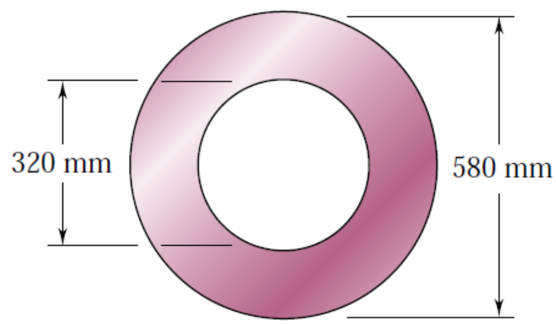


Figure 3

One of the two hollow steel drive shafts of an ocean liner is 75 m long and has the cross section shown. Knowing that $G = 77.2$ GPa and that the shaft transmits 44 MW to its propeller when rotating at 144 rpm, determine (a) the maximum shearing stress in the shaft, (b) the angle of twist of the shaft.

Solution 2

For this problem we need to recall how to associate power to frequency and torque. But before that we need to convert rpm into Hz for the frequency, namely

$$f = 144\,\text{rmp} = \frac{144}{60} = 2.4\,\text{Hz} \tag{6}$$

Now from the equation relating power and torque we can solve for torque, namely

$$P = 2\pi f T \Rightarrow T = \frac{P}{2\pi f} = \frac{44 \times 10^6}{2\pi(2.4)} = 2.917 \times 10^6\,\text{N}\cdot\text{m} \tag{7}$$

So that we can compute the maximum shear stress and the angle of twist, we need to first determine the polar second moment of inertia of a shaft with this cross-section (`http://fer.me/git/es120notes/blob/master/Section4/J-list.pdf`), namely,

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}\left(0.290^4 - 0.160^4\right) = 10.08 \times 10^{-3}\,\text{m} \tag{8}$$

Part (a)

To find the maximum shear we know that we need to evaluate our shear-torque relation at $\rho =$ outer edge so that we obtain

$$\boxed{\tau_{max} = \frac{Tc_2}{J} = \frac{(2.917 \times 10^6)(0.290)}{(10.08 \times 10^{-3})} = 83.9 \text{ MPa}} \quad (9)$$

Part (b)

To obtain the angle of twist ,we now depend on the length of the rod and must enter that information such that

$$\boxed{\phi = \frac{TL}{GJ} = \frac{(2.9178 \times 10^6)(75)}{(77 \times 10^9)(10.08 \times 10^{-3})} = 2.81.9 \times 10^{-3} \text{ rad} = 16.15^\circ} \quad (10)$$

Problem 3:

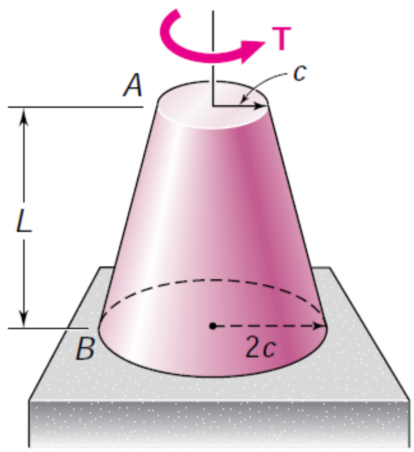


Figure 4

A torque **T** is applied as shown to a solid tapered shaft AB. Show by integration that the angle of twist at A is

$$\phi = \frac{7TL}{12\pi G c^4}$$

Solution 3

A rule of thumb is that if something varies in cross-section, modulus or weight as a function of space, it is easy to spot an integral coming somewhere within the problem. So, since this problem varies in cross-section as a function of space, we know that we likely will have to integrate in space. Let's begin by defining some coordinate system to make our lives a bit easier.

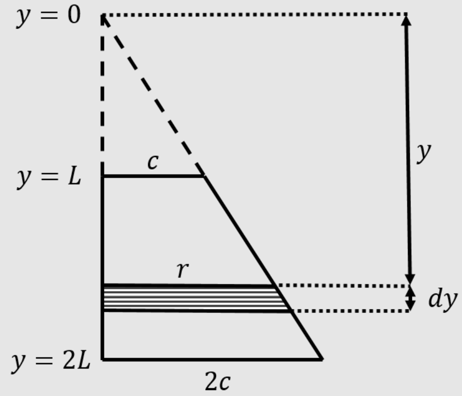


Figure 5: Coordinate system definition for problem.

Here,

$$r = \frac{cy}{L} \tag{11}$$

So if we consider a infinitesimal length dy and assume there will be an infinitesimal angle of twist $d\phi$ though that infinitesimal space, we can obtain the following relationship

$$d\phi = \frac{Tdy}{GJ} \tag{12}$$

We know from `http://fer.me/git/es120notes/blob/master/Section4/J-list.pdf` what J is

$$J = \frac{\pi}{2} r^4 \tag{13}$$

We also know from eq. (11) what r is. So plugging this information into eq. (12) we obtain

$$d\phi = \frac{2TL^4 dy}{\pi G c^4 y^4} \tag{14}$$

Now that we have this differential equation, we must INTEGRATE (not pull teeth) 🙂 both sides

$$\phi = \int_L^{2L} \frac{2TL^4}{\pi G c^4} \frac{dy}{y^4} = \frac{2TL^4}{\pi G c^4} \int_L^{2L} \frac{dy}{y^4} \tag{15}$$

where we separate what is a function of y and what isn't. We must only integrate was is a function of y since the differential equation is a derivative of space. So this integral becomes,

$$\begin{aligned} \phi &= \frac{2TL^4}{\pi G c^4} \left\{ -\frac{1}{3y^3} \right\}_L^{2L} \\ &= \frac{2TL^4}{\pi G c^4} \left\{ -\frac{1}{24L^3} + \frac{1}{3L^3} \right\} \\ &= \frac{2TL^4}{\pi G c^4} \left\{ \frac{7}{24L^3} \right\} \\ &\boxed{= \frac{7TL}{12\pi G c^4}} \end{aligned} \tag{16}$$

See, the integral wasn't that bad...

ES120 Spring 2018 – Midterm 1 Solutions

Matheus Fernandes, Bolei Deng

March 9, 2018

Length: 53 minutes

You are allowed to use a calculator when solving the problems, as well as the equation sheet posted on the web site. Please make sure your answers are clear and legible. No credit will be given if we cannot read an answer or figure out how you derived it! All questions are weighted equally.

Problem 1:

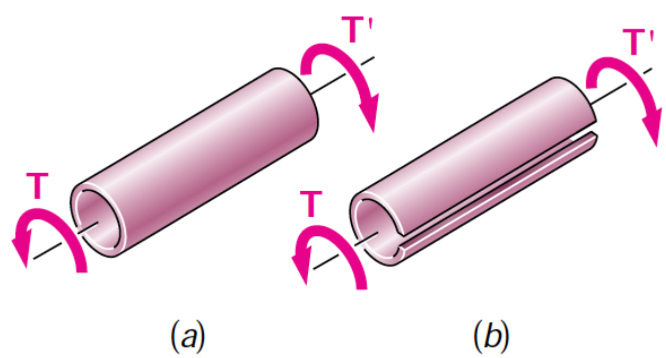


Figure 1

Equal torques are applied to thin-walled tubes of the same length L, same thickness t, and same radius c. One of the tubes has been slit lengthwise as shown. Determine (a) the ratio τ_b/τ_a of the maximum shearing stresses in the tubes, (b) the ratio ϕ_b/ϕ_a of the angles of twist of the shafts, (c) the radii of solid cylindrical shafts of the same material with the same stiffness as the two thin-walled tubes.

Solution 1

For this problem lets separate into separate geometries, namely 'a' for the one without the slit and 'b' for the one with the slit. This is not part a and part b of the problem, but simply a step towards the goals of the problem.

Here we are only calculating the different values for the different geometries. So for the geometry without the slit we can compute the following maximum shear stresses using the fact that we know this to be a thin-walled hollow shaft, which can be read in section 3.13 of the book. We know that the torque applied to a hollow member with constant shear flow is

$$\tau_a = \frac{T}{2t\mathcal{A}} \tag{1}$$

where $\mathcal{A}$ is the area bounded by the centerline of the wall cross section. Which given that t is very thin, we can approximate it to be simply c which gives us $\mathcal{A} = \pi c^2$. Therefore,

$$\tau_a = \frac{T}{2\pi c^2 t} \tag{2}$$

$$J \approx 2\pi c^3 t \tag{3}$$

So therefore, we can solve for the angle of twist as

$$\phi_a = \frac{TL}{GJ} = \frac{TL}{2\pi c^3 tG} \tag{4}$$

Now for the case for the geometry with the slit, we can approximate it as a non-circular member as described in section 3.12 of the book. For this approximation we can compute the length of the wider face a to be the circumference of the cylinder

$$a = 2\pi c \tag{5}$$

and the length of the lesser wide face to be

$$b = t. \tag{6}$$

We can see that

$$\frac{a}{b} = \frac{2\pi c}{t} >> 1 \tag{7}$$

which from Table 3.1 of the book indicates that $c_1 = c_2 = \frac{1}{3}$. From the equation in section 3.12 we can therefore compute the τ_b to be

$$\tau_b = \frac{T}{c_1 ab^2} = \frac{3T}{2\pi ct^2} \tag{8}$$

Now for the angle of twist we also know form section 3.12 of the book that it is given by:

$$\phi_b = \frac{TL}{c_2 ab^3 G} = \frac{3TL}{2\pi ct^3 G} \tag{9}$$

Part (a)

Now solving for the question on the stress ratio we can plug in our findings to get:

$$\frac{\tau_b}{\tau_a} = \frac{3T}{2\pi ct^2} \cdot \frac{2\pi c^2 t}{T} = \boxed{\frac{3c}{t}} \tag{10}$$

Part (b)

Now solving for the question on the twist ratio we can plug in our findings to get:

$$\frac{\phi_b}{\phi_a} = \frac{3TL}{2\pi ct^3 G} \cdot \frac{2\pi c^3 tG}{TL} = \boxed{\frac{3c^2}{t^2}} \tag{11}$$

Now let's look at these solutions a little deeper for some intuition. It is cool to see that when we add a slit to the thin shell cylinder we get higher maximum shear stresses than without the slit (assuming $c > t$)). Furthermore, not only is our angle of twist also larger for the geometry with the slit but it scales $\left(\frac{c}{t}\right)^2$ which is much faster than the maximum shear stresses.

Part (c)

For this part we expect two solutions for the radii given that there are two different tubes of different stiffnesses. To obtain a similar stiffness between two members, we must equate the relationship between the torque required per unit twist, namely ϕ.

Therefore, the torsional stiffness of a solid cylindrical shaft is

$$\phi_{\text{cyl}} = \frac{TL}{G\frac{\pi}{2}r_{\text{cyl}}^4} \tag{12}$$

For shaft (a) we must equate the two equations for torsional stiffness, namely

$$\phi_a = \frac{TL}{2\pi c^3 tG} = \frac{TL}{G\frac{\pi}{2}r_{\text{cyl}}^4} = \phi_{\text{cyl}} \tag{13}$$

$$\boxed{r_{\text{cyl}}^{(a)} = \sqrt[4]{4c^3t}} \tag{14}$$

For shaft (b) we must equate the two equations for torsional stiffness, namely

$$\phi_b = \frac{3TL}{2\pi ct^3G} = \frac{TL}{G\frac{\pi}{2}r_{\text{cyl}}^4} = \phi_{\text{cyl}} \tag{15}$$

$$\boxed{r_{\text{cyl}}^{(b)} = \sqrt[4]{\frac{4}{3}ct^3}} \tag{16}$$

Note, for this part we were very lenient if you attempted and got something remotely close. However, you had to identify that there are two radii that you are solving for in this problem.

Problem 2:

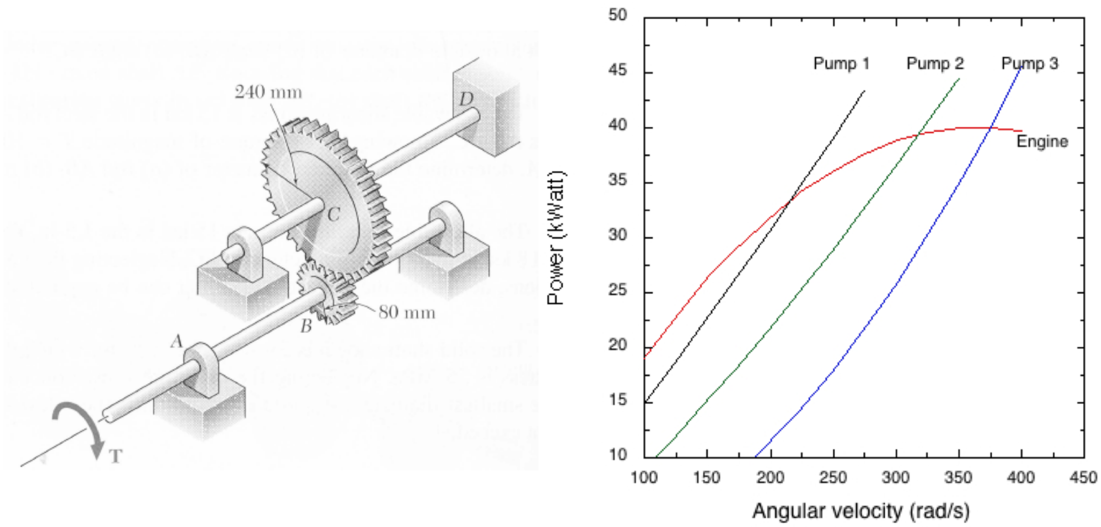


Figure 2

Consider the figure below and imagine that A represents a diesel engine. Gears B and C represent a gearbox, and D represents the load, in this case a pump. The maximum output of the engine is 40 kW. The characteristics of the engine (as measured at point D for a fixed fuel supply) and of several pumps are shown in the graph below.

(a) Given that you want to maximize power to the pump, which pump would you select?

(b) Having selected the pump, determine the corresponding angular velocities at points A and D. What are the internal torques in sections AB and CD of the shaft. What is the total elastic twist angle ϕ in the system as a result of these torques?

(c) Knowing that the maximum allowable shear stress of the shaft material (mild steel) is 105 MPa, determine the required diameter of shaft AB and shaft CD. Use metric units in your answer.

Solution 2

Part (a)

If we want to maximize power we can see by the intersection of the graph for the different pumps and the engine that the maximum power occurs when pump 3 intersects with the engine. Therefore, we would want to select **Pump 3**.

Part (b)

From the graph provided in the problem, we can see that the intersection occurs at a power $P = 40,000$ kW and $\omega = 375$ rad/s (we accept anywhere from 350-400 rad/s). Because the engine is acting at point A, we know that $\omega_{AB} = 375$ rad/s.

Using the gear ratio we can develop the following relationship between the angular momentum of the two shafts

$$\omega_{CD} = \frac{80}{240}\omega_{AB} \Rightarrow \omega_{CD} = 125 \text{ [rad/s]} \tag{17}$$

To obtain the torque, we can use the relationship between power and angular velocity,

$$T = \frac{p}{\omega} \tag{18}$$

Note, that even if you didn't remember this equation or could find it in the equation sheet, it would have been easy to come up with using dimensional analysis.

Now we can compute the different torques, namely,

$$\boxed{T_{AB} = \frac{40000}{375} = 106.66 \text{ N}\cdot\text{m}} \tag{19}$$

$$\boxed{T_{CD} = \frac{40000}{125} = 320 \text{ N}\cdot\text{m}} \tag{20}$$

Similarly, to compute the angle of twist we have the equation relating torque and angle of twist:

$$\phi = \frac{TL}{GI_p} \tag{21}$$

Where the polar moment of inertia for this cross section is

$$I_p = \frac{\pi}{2} r^4 \tag{22}$$

Plugging this it, it becomes

$$\phi = \frac{TL}{G\frac{\pi}{2}r^4} \tag{23}$$

Now we can compute the different twist angles for the different members, namely,

$$\phi_{AB} = \frac{T_{AB}L_{AB}}{G\frac{\pi}{2}r_{AB}^4} \tag{24}$$

$$\phi_{CD} = \frac{T_{CD}L_{CD}}{G\frac{\pi}{2}r_{CD}^4} \tag{25}$$

Using the information we were provided this far in the problem, we can come up with the total angle of twist to be

$$\boxed{\phi_{\text{total}} = \phi_{AB} + \phi_{CD} = \frac{106.66L_{AB}}{G\frac{\pi}{2}r_{AB}^4} + \frac{320L_{CD}}{G\frac{\pi}{2}r_{CD}^4}} \tag{26}$$

Note, we did not take points off for incorrect numerical values. Since the original statement did not include any numerical values, whether you used any or not, we did not grade on value correctness but on the correct approach. We were very lenient on this problem as long as the approach is correct.

Part (c)

For this part we just need to remember the equation relating the shear stress to the radius:

$$\tau = \frac{T\rho}{I_p} \quad \text{where} \quad I_p = \frac{\pi}{2} r^4 \tag{27}$$

Thus, for member AB

$$\tau_{AB} = \frac{T_{AB}r_{AB}}{\frac{\pi}{2}r_{AB}^4} = \tau_{\max} = 105 \times 10^6 \text{ Pa} \tag{28}$$

$$r_{AB} = 0.0084 \text{ m} \tag{29}$$

$$\boxed{d_{AB} = 0.001693 \text{ m}} \tag{30}$$

For member DC

$$\tau_{DC} = \frac{T_{DC} r_{DC}}{\frac{\pi}{2} r_{DC}^4} = \tau_{\max} = 105 \times 10^6 \text{ Pa} \tag{31}$$

$$r_{DC} = 0.00546 \text{ m} \tag{32}$$

$$\boxed{d_{DC} = 0.011737 \text{ m}} \tag{33}$$

Note, we did not take many points off for incorrect numerical values. There are various final results depending on what value of angular velocity you used. As mentioned, you could have used any angular velocity ranging from 350-400 for full credit. We have focused more on the approach for this problem.

Problem 3:

α_{Cu}	16 x 10^{-6}/K
E_{Cu}	120 GPa
ν_{Cu}	0.35
σ_y	200 MPa
α_{Si}	3 x 10^{-6}/K
Room temperature	20°C
T_o	100°C

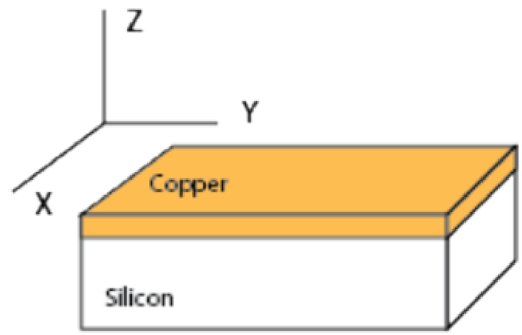


Figure 3

At the heart of your iPhone or laptop, there is a small microprocessor. This microprocessor consists essentially of a rectangular piece of silicon coated with many layers of other materials such as copper (to carry the signals) and silicon dioxide (serves as dielectric). Residual stresses in these coatings are a major reliability concern of manufacturers such as Intel or AMD. Let's try to estimate the residual stresses that develop in a thin layer of copper on a silicon substrate.

a. Assume that we have a thick silicon substrate and a very thin layer of copper as indicated in the figure. Both silicon and copper are stress-free at room temperature. When you turn on your iPhone or laptop, the temperature of the microprocessor increases from room temperature to T_o. If the thermal expansion coefficient of silicon is α_{Si} and that of Cu is α_{Cu}, where $\alpha_{Cu} >> \alpha_{Si}$ do you develop compressive or tensile normal stresses in the copper? Why?

b. Estimate the stress in the copper coating at T_o using the data in the table below. Assume that the copper is isotropic and elastic, and that the same thermal strain develops in all directions in the plane of the coating. Further assume that the stress perpendicular to the coating is zero and that the elastic deformation of the substrate is negligible.

c. What happens if the temperature, T_o , is so large that the stress in the copper in absolute value exceeds the yield stress of the copper? In that case, what is the stress in the copper at T_o ? What happens when you turn off your iPhone or laptop and the chip cools down to room temperature? Make a sketch of stress in the copper as a function of temperature.

Solution 3

Before we start, let's define our notations clearly here: ϵ_{xx}^{Cu} indicates the total strain of the copper layer, $\left(\epsilon_{xx}^{Cu}\right)_T$ is the stress of copper layer induced by thermal expansion and $\left(\epsilon_{xx}^{Cu}\right)_E$ is the strain of copper layer by elastic deformation. The similar definition applies for the silicon layer.

Part (a)

$$\begin{aligned} \left(\epsilon_{xx}^{Cu}\right)_T &= \alpha_{Cu}\Delta T = \alpha_{Cu}(T_0 - T_R) \\ \left(\epsilon_{xx}^{Si}\right)_T &= \alpha_{Si}\Delta T = \alpha_{Si}(T_0 - T_R) \end{aligned} \tag{34}$$

since $\alpha_{Cu} >> \alpha_{Si}$, as we increase the temperature, we would image that the copper layer tends to expand much more than the silicon substrate. While because the copper layer is coated on the silicon substrate which expands much less, the silicon substrate is actually "dragging" the copper layer back. Thus we conclude that there is $\boxed{\textbf{compressive normal stress}}$ within copper layer.

Note: stating compressive stress with proper explanation gets 5pts.

Part (b)

We know that the total strain is the summation of thermal strain and elastic strain, thus,

$$\begin{aligned}\epsilon_{xx}^{Cu} &= \left(\epsilon_{xx}^{Cu}\right)_T + \left(\epsilon_{xx}^{Cu}\right)_E \\ \epsilon_{xx}^{Si} &= \left(\epsilon_{xx}^{Si}\right)_T + \left(\epsilon_{xx}^{Si}\right)_E\end{aligned} \tag{35}$$

According to the problem description *the elastic deformation of the substrate is negligible*, we know that

$$\left(\epsilon_{xx}^{Si}\right)_E = 0 \tag{36}$$

therefore,

$$\epsilon_{xx}^{Si} = \left(\epsilon_{xx}^{Si}\right)_T \tag{37}$$

Furthermore, as the copper layer and the silicon substrate are safely bonded, their total strains should be the same, i.e.,

$$\epsilon_{xx}^{Si} = \epsilon_{xx}^{Cu} \tag{38}$$

Combining Eq. 35, Eq. 37, and Eq. 38, we know that

$$\left(\epsilon_{xx}^{Cu}\right)_E = \left(\epsilon_{xx}^{Cu}\right)_T - \left(\epsilon_{xx}^{Si}\right)_T \tag{39}$$

From the description *the stress perpendicular to the coating is zero* we know that we are looking at a plane stress problem. Since both the copper layer and the silicon substrate are isotropic, we now generalize our result to y direction and considering Eq.(34),

$$\begin{aligned}\left(\epsilon_{xx}^{Cu}\right)_E &= \left(\epsilon_{xx}^{Cu}\right)_T - \left(\epsilon_{xx}^{Si}\right)_T = \left(\alpha_{Cu} - \alpha_{Si}\right)\left(T_0 - T_R\right) \\ \left(\epsilon_{yy}^{Cu}\right)_E &= \left(\epsilon_{yy}^{Cu}\right)_T - \left(\epsilon_{yy}^{Si}\right)_T = \left(\alpha_{Cu} - \alpha_{Si}\right)\left(T_0 - T_R\right)\end{aligned} \tag{40}$$

Now, for plane stress problem, the Generalized Hooke's Law can be deducted to,

$$\begin{aligned}\left(\epsilon_{xx}^{Cu}\right)_E &= \frac{1}{E}\left(\sigma_{xx}^{Cu} - \nu_{Cu}\,\sigma_{yy}^{Cu}\right) \\ \left(\epsilon_{yy}^{Cu}\right)_E &= \frac{1}{E}\left(\sigma_{yy}^{Cu} - \nu_{Cu}\,\sigma_{xx}^{Cu}\right)\end{aligned} \tag{41}$$

Our goal is to solve for the stress in the copper layer, i.e., solve for σ_{xx}^{Cu} and σ_{yy}^{Cu}, substitute eq. (40) into the plane stress equations yields the equation for stresses:

$$\begin{aligned}\left(\alpha_{Cu} - \alpha_{Si}\right)\left(T_0 - T_R\right) &= \frac{1}{E}\left(\sigma_{xx}^{Cu} - \nu_{Cu}\,\sigma_{yy}^{Cu}\right) \\ \left(\alpha_{Cu} - \alpha_{Si}\right)\left(T_0 - T_R\right) &= \frac{1}{E}\left(\sigma_{yy}^{Cu} - \nu_{Cu}\,\sigma_{xx}^{Cu}\right)\end{aligned} \tag{42}$$

solve for stresses,

$$\boxed{\sigma_{xx}^{Cu} = \sigma_{yy}^{Cu} = \frac{E\left(\alpha_{Cu} - \alpha_{Si}\right)\left(T_0 - T_R\right)}{1 - \nu_{Cu}} = 192\ \text{MPa}} \tag{43}$$

the stress calculated is less than yield stress.

Note: writing proper thermal expansion equations, strain-stress relations gets 2pts; solving the 1D (should be 2D) problem correctly gets 3pts; mentioning 2D plane stress problem with proper derivation about thermal expansion gets 4pts; obtaining the final stress expression gets 5pts.

Part (c)

If T_0 is so large that the stress in the copper in absolute value exceeds the yield stress of the copper, the copper begins to $\boxed{\text{plastically deform}}$ (1pt). The stress in copper will be $\boxed{\text{equal to the yield strength } -\sigma_Y}$ (1pt) beyond that temperature.
As your turn off your iPhone, the chips cool down to room temperature. Since the silicon substrate will go back to original shape, the copper layer will also go back to zero strain, with certain $\boxed{\text{residue stress}}$ (1pts).

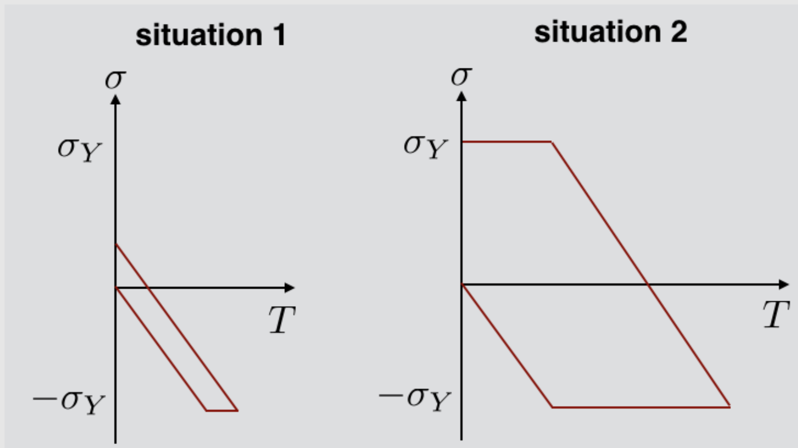


Figure 4

Note: plot either 1 or 2 is considered as full score.

ES120 Spring 2018 – Midterm 2 Review w/ Solutions

Bolei Deng, Matheus Fernandes

April 5, 2018

Document Disclaimer

The list provided below is by no means comprehensive and if you find anything missing that you would like to add please let me know. This review session has been created without prior knowledge of the problems in the exam and should not be treated in any way as hints to problems that will be asked in the exam. We will do our best to go over the topics of the course in detail however please do your own reading of chapters 4, 5 and 9 as well as other topics not included in the book. If you find any typos please let me know and I will update and push a new version to Github.

You may also find my notes from a previous year helpful: `http://fer.me/es120notes`

Topics Covered Summary

4. **Pure bending**
   - **Geometry** – any cross section perpendicular to the axis of the member remains plane and remains perpendicular to the centroid line.
   - **Normal strain and normal stress** – normal strain $\epsilon_{xx} = -\frac{y}{\rho}$ and normal stress $\sigma_{xx} = -\frac{Ey}{\rho}$, where ρ is the radius of the beam. We also have $\epsilon_{xx} = -\frac{y}{c}\epsilon_m$ and $\sigma_{xx} = -\frac{y}{c}\sigma_m$, where ϵ_m and σ_m are maximum strain and stress in the beam.
   - **Force & Position centroid line** – the force in the beam can be calculated by integrating normal stress over the cross section: $F = \int \sigma_{xx}\, \mathrm{d}A$, if the material is within elastic range, $= -\frac{E}{\rho}\int y\, \mathrm{d}A$, where $Q = \int y\, \mathrm{d}A$ is called first moment (see Appendix A.2 in text book). Force in beam should be zero, that means $Q = 0$. Solving $Q = 0$ gives the position of centroid line.
   - **Moment** – moment in the beam is calculated by $M = -\int y\sigma_{xx}\, \mathrm{d}A = \frac{E}{\rho}\int y^2\, \mathrm{d}A$, where $I = \int y^2\, \mathrm{d}A$ is called the second moment (see Appendix A.3 for details). Now we have $M = \frac{EI}{\rho}$, and $\frac{1}{\rho} = \frac{M}{EI}$ (curvature - moment relation).
   - **Connect stress with moment** – $\sigma_{xx} = -\frac{My}{I}$ and $\sigma_m = \frac{Mc}{I}$. Introducing elastic section modulus $S = I/c$, so that we have $\sigma_m = \frac{M}{S}$.
   - **Composite beams** – two materials with young's modulus E_1 and E_2, let $n = E_2/E_1$, the resistance to bending of the bar would remain the same if both portions were made of the first material E_1, provided that the **width** of each element of the lower portion were multiplied by the factor n. To obtain the **stress** σ_2 for material 2, we must multiply by n (see more on textbook page 230).
   - **Reinforced concrete beams** – (1) replace the total cross-sectional area of the steel bars A_s by an equivalent area nA_s; (2) only the portion of the cross section in compression should be used in the transformed section (see textbook 233).
   - **Eccentric axial loading** – $\sigma_{xx} = (\sigma_{xx})_{centric} + (\sigma_{xx})_{bending} = \frac{P}{A} - \frac{My}{I}$
5. **Analysis and Design of Beams for Bending**
   - **Shear and Bending moments diagrams** – Drawing shear forces through a uniform beam. Remember which direction is positive and which direction is negative. Note that the bending moment is the integral of the shear diagram, see Figure 5.7 of textbook.

- **Relations among load shear and bending moment** – The overall relationship of all of these are integrals. Specifically they are related through these equations: $-w = \frac{dV}{dx}, V = \frac{dM}{dx}$. In other words, they are the area under the curve of each other.
- **Design of Prismatic Beams for Bending** – How to design beam cross-sections such that you achieve the most economical design possible to efficiently withstand specific load conditions. This is a procedure detailed on page 333 of textbook. The main idea of the procedure is to use bending moment diagrams in conjunction with the fact that the maximum stress occurs at the edge of the section to come up with a cross-section height that is appropriate for the loading conditions.
- **Using Singularity Functions for Shear and Bending Moments** – Singularity functions are defined as $< x - a >^n = \begin{cases} (x-a)^n & \text{for } n \geq a \\ 0 & \text{for } n < a \end{cases}$ They are useful to organize the discontinuities in the distribution of shear forces, bending moments and weights through the beam. It simplifies by not having to account for each part of the beam in separate equations, but instead condenses all information into one equation. This is also useful when performing the integrals from weight to shear forces to bending moments
- **Nonprismatic beams** – Unlike primsmatic beams, we now relax the assumption that the beam cross-section is constant. Now the goal of this is to have a beam of constant strength for a particular loading condition and allow the cross section of the beam to be controlled via the loading conditions. This is very similar to prismatic beam design only now $S = \frac{|M|}{\sigma_{\text{all}}}$. This is covered in more detail in an example problem below.

9. <u>**Deflection of Beams**</u>

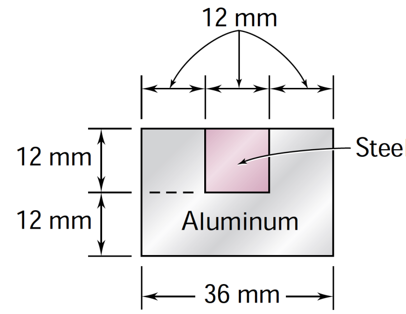
- **Equation of the elastic curve** – $\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$.
- **Integration** – $y = \int_0^x dx \int_0^x \frac{M(x)}{EI} dx + C_1 x + C_2$, where C_1 and C_2 are determined by boundary conditions.
- **Statically indeterminate beams** – superposition of deflection (see more in textbook on page 560).
- **Beam vibrations** – Know the governing PDE and the general solutions for varying boundary conditions. Read lecture notes on this and also notes posted to Canvas.

Review Problems

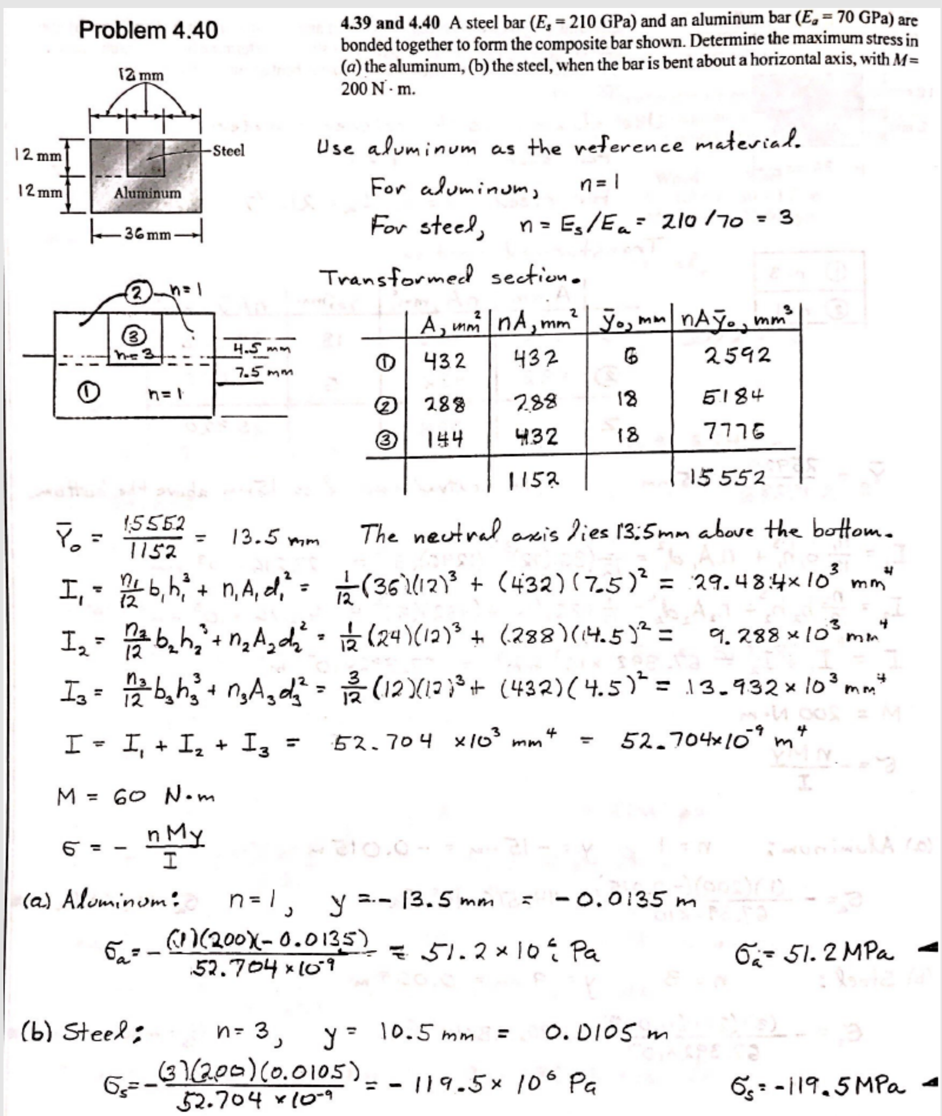
Try to work these out on your own and solutions will be pushed to Github later.

Problem 1: Chapter 4 Review

A steel bar ($E_s = 210$ GPa) and an aluminum bar ($E_a = 70$ GPa) are bonded together to form the composite bar shown. Determine the maximum stress in (a) the aluminum, (b) the steel, when the bar is bent about a horizontal axis, with $M = 200$ Nm.



Solution 1



Problem 4.40

4.39 and 4.40 A steel bar ($E_s = 210$ GPa) and an aluminum bar ($E_a = 70$ GPa) are bonded together to form the composite bar shown. Determine the maximum stress in (*a*) the aluminum, (*b*) the steel, when the bar is bent about a horizontal axis, with $M = 200$ N · m.

Use aluminum as the reference material.

For aluminum, $n = 1$

For steel, $n = E_s/E_a = 210/70 = 3$

Transformed section.

	A, mm²	nA, mm²	$\bar{y}_o$, mm	$nA\bar{y}_o$, mm³
①	432	432	6	2592
②	288	288	18	5184
③	144	432	18	7776
		1152		15552

$\bar{Y}_o = \frac{15552}{1152} = 13.5$ mm The neutral axis lies 13.5 mm above the bottom.

$I_1 = \frac{n_1}{12} b_1 h_1^3 + n_1 A_1 d_1^2 = \frac{1}{12}(36)(12)^3 + (432)(7.5)^2 = 29.484 \times 10^3 \text{ mm}^4$

$I_2 = \frac{n_2}{12} b_2 h_2^3 + n_2 A_2 d_2^2 = \frac{1}{12}(24)(12)^3 + (288)(4.5)^2 = 9.288 \times 10^3 \text{ mm}^4$

$I_3 = \frac{n_3}{12} b_3 h_3^3 + n_3 A_3 d_3^2 = \frac{3}{12}(12)(12)^3 + (432)(4.5)^2 = 13.932 \times 10^3 \text{ mm}^4$

$I = I_1 + I_2 + I_3 = 52.704 \times 10^3 \text{ mm}^4 = 52.704 \times 10^{-9} \text{ m}^4$

$M = 60$ N·m

$\sigma = -\frac{nMy}{I}$

(a) Aluminum: $n = 1$, $y = -13.5$ mm $= -0.0135$ m

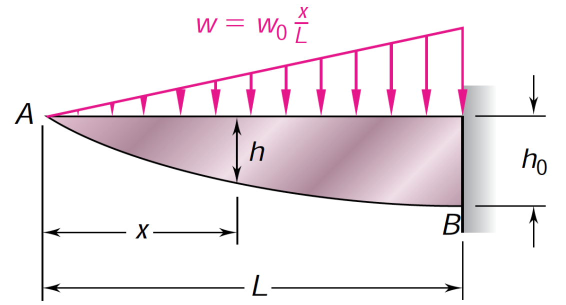
$\sigma_a = -\frac{(1)(200)(-0.0135)}{52.704 \times 10^{-9}} = 51.2 \times 10^6$ Pa $\sigma_a = 51.2$ MPa

(b) Steel: $n = 3$, $y = 10.5$ mm $= 0.0105$ m

$\sigma_s = -\frac{(3)(200)(0.0105)}{52.704 \times 10^{-9}} = -119.5 \times 10^6$ Pa $\sigma_s = -119.5$ MPa

Problem 2: Chapter 5 Review

The beam AB, consisting of a cast-iron plate of uniform thickness b and length L, is to support the distributed load $w(x)$ shown. (a) Knowing that the beam is to be of constant strength, express h in terms of x, L, and h_0. (b) Determine the smallest value of h_0 if $L = 750$ mm, $b = 30$ mm, $w_0 = 300$ kN/m, and $\sigma_{all} = 200$ MPa.



Solution 2

We know from class the relationship between the weight distribution, shear forces and moments to be:

$$-w = \frac{dV}{dx} \tag{1}$$

$$V = \frac{dM}{dx} \tag{2}$$

therefore, knowing that

$$w(x) = w_0 \frac{x}{L} \tag{3}$$

We can simply integrate the above to obtain:

$$V(x) = -\int_0^x w_0 \frac{x}{L} dx = -\frac{w_0 x^2}{2L} \tag{4}$$

Now to obtain the moment we integrate again

$$M(x) = \int_0^x -\frac{w_0 x^2}{2L} dx = -\frac{w_0 x^3}{6L} \tag{5}$$

Part (a)

Here we have all of the information we need to determine the how the cross-section should look like. This was indicated in chapter 5.1 that the design of the beam is usually controlled by the maximum absolute value of the bending moment that will occur in the beam. The largest normal stress in the beam is found at the surface of the beam in the critical section where $|M|_{\text{max}}$ occurs. The section modulus S is therefore defined in chapter 5.4 as

$$S = \frac{|M|_{\text{max}}}{\sigma_{all}} \tag{6}$$

Which we can use the polar moment of inertia to solve for S of a rectangular cross-section such that

$$S = \frac{1}{6} bh^2 \tag{7}$$

Equating eq. (6) and eq. (7) we can obtain a closed form solution for h, namely,

$$\frac{1}{6}bh^2 = \frac{w_0 x^3}{6L\sigma_{\text{all}}} \tag{8}$$

which solving for h explicitly yields

$$h = \sqrt{\frac{w_0 x^3}{\sigma_{\text{all}} bL}} \tag{9}$$

Now knowing that we value at $x = L$ for h is h_o, we can normalize h by:

$$h(x = L) = h_o = \sqrt{\frac{w_o L^2}{\sigma_{\text{all}} b}} \tag{10}$$

Factoring that out from h, we obtain

$$\boxed{h(x) = h_0 \left(\frac{x}{L}\right)^{3/2}} \tag{11}$$

Part(b)

Here we only need to plug in the values we are given in the problem statement into our equation for h_0, namely,

$$\boxed{h_o = \sqrt{\frac{(300 \times 10^3)(0.75)^2}{(200 \times 10^6)(0.030)}} = 167.7 \times 10^{-3}\ \text{mm}} \tag{12}$$

Problem 3: Chapter 9 Review

For the loading shown, determine (a) the equation of the elastic curve for the cantilever beam AB, (b) the deflection at the free end, (c) the slope at the free end.

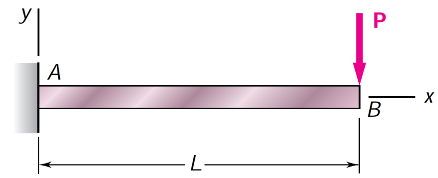
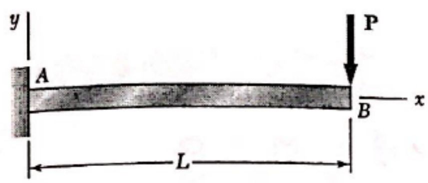


Fig. P9.2

Solution 3

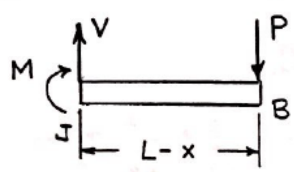
Problem 9.2

9.1 through 9.4 For the loading shown, determine (*a*) the equation of the elastic curve for the cantilever beam *AB*, (*b*) the deflection at the free end, (*c*) the slope at the free end.



$[x=0,\ y=0]$

$[x=0,\ \frac{dy}{dx}=0]$



$\circlearrowleft \Sigma M_J = 0: \quad -M - P(L-x) = 0$

$M = -P(L-x)$

$EI\frac{d^2y}{dx^2} = -P(L-x) = -PL + Px$

$EI\frac{dy}{dx} = -PLx + \frac{1}{2}Px^2 + C_1$

$[x=0,\ \frac{dy}{dx}=0] \qquad 0 = -0 + 0 + C_1$

$C_1 = 0$

$EIy = -\frac{1}{2}PLx^2 + \frac{1}{6}Px^3 + C_1x + C_2$

$[x=0,\ y=0] \qquad 0 = -0 + 0 + 0 + C_2 \qquad C_2 = 0$

(a) Elastic curve. $\quad y = -\frac{Px^2}{6EI}(3L-x)$

$\frac{dy}{dx} = -\frac{Px}{2EI}(2L-x)$

(b) y @ $x = L$: $\quad y_B = -\frac{PL^2}{6EI}(3L-L) = -\frac{PL^3}{3EI}$

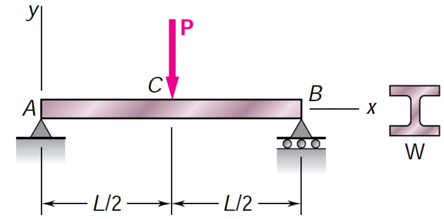
$y_B = \frac{PL^3}{3EI} \downarrow$

(c) $\frac{dy}{dx}$ @ $x = L$: $\quad \left.\frac{dy}{dx}\right|_B = -\frac{PL}{2EI}(2L-L) = -\frac{PL^2}{2EI}$

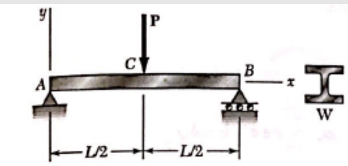
$\theta_B = \frac{PL^2}{2EI}$ ↻

Problem 4: Chapter 9 Review

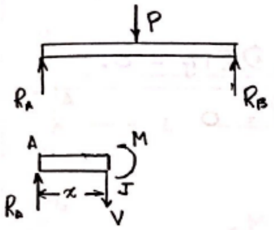
Knowing that beam AB is a W130 × 23.8 rolled shape and that $P = 50$ kN, $L = 1.25$ m, and $E = 200$ GPa, determine (a) the slope at A, (b) the deflection at C.



Solution 4



$[x=0, y=0]$ $[x=\frac{L}{2}, \frac{dy}{dx}=0]$ $[x=L, y=0]$



Use symmetry boundary condition at C.

By symmetry, $R_A = R_B = \frac{1}{2}P$

Using free body AJ, $0 \le x \le \frac{L}{2}$

$+\circlearrowleft \Sigma M_J = 0: \quad M - R_A x = 0$

$M = R_A x = \frac{1}{2}Px$

$EI\frac{d^2y}{dx^2} = \frac{1}{2}Px$

$EI\frac{dy}{dx} = \frac{1}{4}Px^2 + C_1$

$EIy = \frac{1}{12}Px^3 + C_1 x + C_2$

$[x=0, y=0] \qquad 0 = 0 + 0 + C_2 \qquad C_2 = 0$

$[x=\frac{L}{2}, \frac{dy}{dx}=0] \qquad 0 = \frac{1}{4}P\left(\frac{L}{2}\right)^2 + C_1 \qquad C_1 = -\frac{1}{16}PL^2$

Elastic curve. $\quad y = \frac{PL}{48EI}(4x^3 - 3L^2x)$

$\frac{dy}{dx} = \frac{PL}{16EI}(4x^2 - L^2)$

Slope at $x = 0$. $\quad \left.\frac{dy}{dx}\right|_A = -\frac{PL^3}{16EI} \qquad \theta_A = \frac{PL^2}{16EI}$

Deflection at $x = \frac{L}{2}$. $\quad y_C = -\frac{PL^3}{48EI} \qquad y_C = \frac{PL^3}{48EI} \downarrow$

Data: $P = 50 \times 10^3$ N, $I = 8.80 \times 10^6\ \text{mm}^4 = 8.80 \times 10^{-6}\ \text{m}^4$

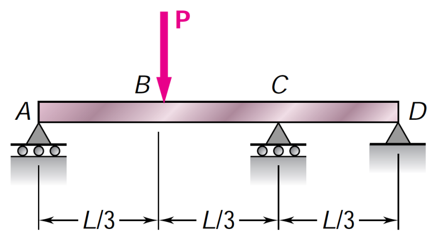
$E = 200 \times 10^9$ Pa $\qquad EI = 1.76 \times 10^6\ \text{N}\cdot\text{m}^2 \qquad L = 1.25$ m

(a) $\theta_A = \frac{(50\times10^3)(1.25)^2}{(16)(1.76\times10^6)} \qquad \theta_A = 2.77\times10^{-3}$ rad

(b) $y_C = \frac{(50\times10^3)(1.25)^3}{(48)(1.76\times10^6)} = 1.156\times10^{-3}$ m $\qquad y_C = 1.156$ mm $\downarrow$

Problem 5: Chapter 9 Review

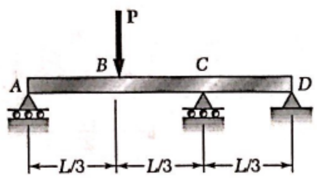
For the uniform beam shown, determine the reaction at each of the three supports.



Solution 5

Problem 9.81

9.81 and 9.82 For the uniform beam shown, determine the reaction at each of the three supports.



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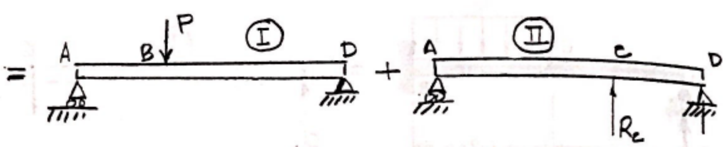


Consider R_C as redundant and replace loading system by I and II.

Loading I. (Case 5 of Appendix D) $a = \frac{2L}{3}$, $b = \frac{L}{3}$, $x = \frac{L}{3}$ at C.

$$(y_C)_1 = \frac{Pb}{6EIL}\left[x^3 - (L^2 - b^2)x\right] = \frac{P(L/3)}{6EIL}\left[\left(\frac{L}{3}\right)^3 - \left\{L^2 - \left(\frac{L}{3}\right)^2\right\}\frac{L}{3}\right]$$

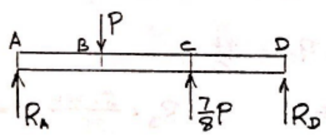
$$= -\frac{7PL^3}{486EI}$$

Loading II. (Case 5 of Appendix D) $a = \frac{2L}{3}$, $b = \frac{L}{3}$

$$(y_C)_2 = \frac{R_C a^2 b^2}{3EIL} = \frac{R_C (L/3)^2 (2L/3)^2}{3EIL} = \frac{4R_C L^3}{243EI}$$

Superposition and constraint. $y_C = (y_C)_1 + (y_C)_2 = 0$

$$-\frac{7PL^3}{486EI} + \frac{4R_C}{243EI} = 0 \qquad R_C = \frac{7}{8}P \uparrow$$



$+\circlearrowleft \Sigma M_D = 0:$

$$-R_A L + P\left(\frac{2L}{3}\right) - \left(\frac{7}{8}P\right)\left(\frac{L}{3}\right) = 0$$

$$R_A = \frac{3}{8}P \uparrow$$

$+\uparrow \Sigma F_y = 0:$ $R_A + R_D - P + \frac{7}{8}P = 0$

$$R_D = P - \frac{7}{8}P - \frac{3}{8}P = -\frac{1}{4}P$$

$$R_D = \frac{1}{4}P \downarrow$$

ES120 Spring 2018 – Midterm 2 Solutions

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April 11, 2018

Length: 53 minutes

You are allowed to use a calculator when solving the problems, as well as the equation sheet posted on the web site. Please make sure your answers are clear and legible. No credit will be given if we cannot read an answer or figure out how you derived it! All questions are weighted equally.

Problem 1:

Consider a prismatic beam of length L, moment of inertia I, and Young's modulus E, simply supported on either end. The beam supports several discrete loads P (all equal) as shown in the picture. Use singularity functions to derive an expression for the deflection of the beam.

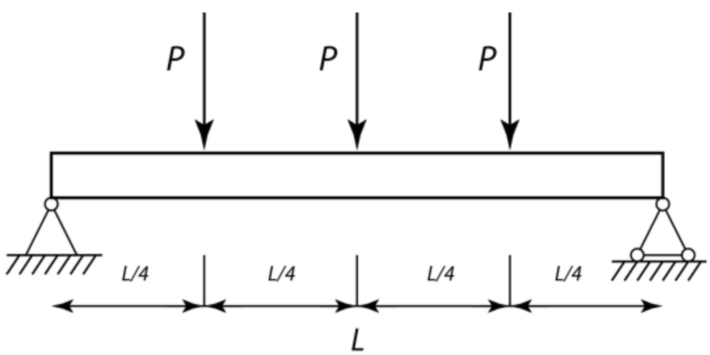


Figure 1: Problem 1 Schematic

Solution 1

To solve this problem, you can do force balance in y direction to yield:

$$\Sigma F_y = 0 \Rightarrow R_a - 3P + R_b = 0 \tag{1}$$

We can also note that the weight is distributed symmetrically so the $3P$ has to be allocated half to R_a and half to R_b such that:

$$R_a = \frac{3}{2}P \tag{2}$$

$$R_b = \frac{3}{2}P \tag{3}$$

Using this information we can come up with an equation for the shear force namely

$$V(x) = P\frac{3}{2} - P\left\langle x - \frac{L}{4}\right\rangle^0 - P\left\langle x - \frac{L}{2}\right\rangle^0 - P\left\langle x - \frac{3L}{4}\right\rangle^0 \tag{4}$$

By integrating we obtain $M(x)$ as

$$M(x) = Px\frac{3}{2} - P\left\langle x - \frac{L}{4}\right\rangle^1 - P\left\langle x - \frac{L}{2}\right\rangle^1 - P\left\langle x - \frac{3L}{4}\right\rangle^1 \tag{5}$$

Using the equation:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{6}$$

we obtain

$$EI\frac{d^2y}{dx^2} = Px\frac{3}{2} - P\left\langle x - \frac{L}{4}\right\rangle^1 - P\left\langle x - \frac{L}{2}\right\rangle^1 - P\left\langle x - \frac{3L}{4}\right\rangle^1 \tag{7}$$

which if we integrate we obtain

$$EI\frac{dy}{dx} = Px^2\frac{3}{4} - \frac{P}{2}\left\langle x - \frac{L}{4}\right\rangle^2 - \frac{P}{2}\left\langle x - \frac{L}{2}\right\rangle^2 - \frac{P}{2}\left\langle x - \frac{3L}{4}\right\rangle^2 + c_1 \tag{8}$$

Finally, if we integrate one last time, we obtain

$$EIy(x) = Px^3\frac{1}{4} - \frac{P}{6}\left\langle x - \frac{L}{4}\right\rangle^3 - \frac{P}{6}\left\langle x - \frac{L}{2}\right\rangle^3 - \frac{P}{6}\left\langle x - \frac{3L}{4}\right\rangle^3 + c_1x + c_2 \tag{9}$$

Here we note that we need 3 different boundary conditions to be able to solve for these constants. For this, the boundary conditions are at the left edge are

$$y(0) = 0 \text{ and } y''(0) = 0 \tag{10}$$

and at the right edge it is

$$y(L) = 0 \text{ and } y''(L) = 0. \tag{11}$$

Using the the following boundary conditions we obtain

$$y(0) = 0 \Rightarrow C_2 = 0 \tag{12}$$

$$y(L) = 0 \Rightarrow C_1 = -\frac{PL^2}{6}\left(\frac{3}{2} - \left(\frac{3}{4}\right)^3 - \left(\frac{1}{2}\right)^3 - \left(\frac{1}{4}\right)^3\right) = -\frac{5PL^2}{32} \tag{13}$$

So the final equation for the deflection becomes

$$\boxed{EIy(x) = Px^3\frac{1}{4} - x\frac{5PL^2}{32} - \frac{P}{6}\left\langle x - \frac{L}{4}\right\rangle^3 - \frac{P}{6}\left\langle x - \frac{L}{2}\right\rangle^3 - \frac{P}{6}\left\langle x - \frac{3L}{4}\right\rangle^3} \tag{14}$$

Note, that we have one final boundary condition, which we we need to satisfy, or at least verify that it is met. Namely that there is also no moment on the right side $y''(L) = 0$ If we plug L into the equation for $M(x)$ we can check that

$$M(L) = EI\frac{d^2y(x=L)}{dx^2} = PL\frac{3}{2} - P\left\langle L - \frac{L}{4}\right\rangle^1 - P\left\langle L - \frac{L}{2}\right\rangle^1 - P\left\langle L - \frac{3L}{4}\right\rangle^1 = 0 \tag{15}$$

and we obtain

$$M(L) = EI\frac{d^2y(x=L)}{dx^2} = PL\frac{3}{2} - PL\frac{3}{2} = 0 \checkmark \tag{16}$$

We can do the same thing for the boundary condition $y(L) = 0$ and verify that it is correct.

The final solution can be plotted using Matlab and normalizing $X^* = \frac{X}{L}$ and $P^* = \frac{P}{EI}$ to obtain the following plot

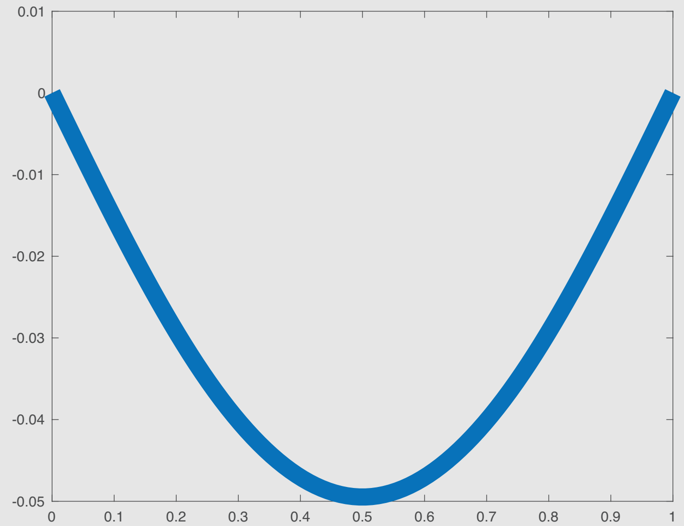


Figure 2: Deformation plot for beam in problem 1.

Matlab Code for Plotting Beam Deformation

```
function Plots()

X=linspace(0,1);

plot(X,P1(X,1,1),'Linewidth',12)
set(gcf, 'Color', 'None')

end


function val = P1(X,L,P)
ct=0;
for x = X
    ct=ct+1;
    val(ct) = P*x^3/4-5*P*L^2/32*x;

    if x>=L/4
        val(ct)=val(ct)-P/6*(x-L/4)^3;
    end

    if x>=L/2
        val(ct)=val(ct)-P/6*(x-L/2)^3;
    end

    if x>=3*L/4
        val(ct)=val(ct)-P/6*(x-3*L/4)^3;
    end
end


end
```

Note: The plot is not necessary for full credit and is just to illustrate the deformation visually.

Problem 2:

Consider a prismatic beam of length L, moment of inertia I, and Young's modulus E, clamped on the left. The beam is also supposed to rest on a support on the right side, but because of faulty construction there is a small gap δ. The beam is loaded by a uniform distributed load w_o (pointing down). (1) Determine the load wo at which the tip of the beam touches the support. (2) Calculate the force exerted on the support once the beam touches.

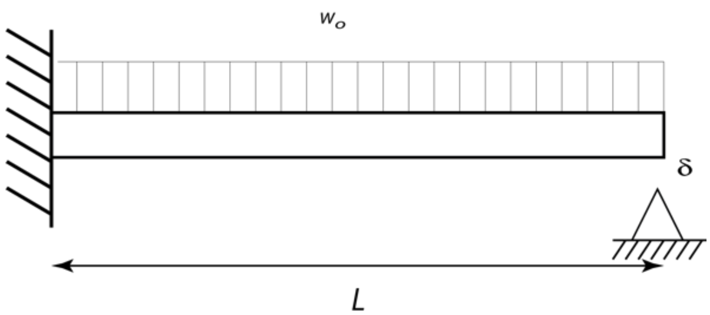


Figure 3: Problem 2 Schematic

Solution 2

For this problem we can simply consult Appendix D. Using appendix D case 2, we can get the following equation for the deflection

$$y(x) = -\frac{w_o}{24EI}\left(x^4 - 4Lx^3 + 6L^2x^2\right) \tag{17}$$

Part (1)

For this part we know that we need to match the deflection at the end of the beam to the gap δ, namely

$$y(x) = -\frac{w_o}{24EI}\left(L^4 - 4L^4 + 6L^4\right) = -\delta \tag{18}$$

Solving for w_o we obtain:

$$\boxed{w_o = \frac{8\delta EI}{L^4}} \tag{19}$$

Part (2)

For this part we now need to assume that we have case 1 of Appendix D. We will call the deflection resulting from the reaction force y^*. So the deflection equation here is

$$y^*(x) = -\frac{P}{6EI}\left(x^3 - 3Lx^2\right) \tag{20}$$

We know that on the right side of the beam we have the compatibility equation:

$$-y(L) + y^*(L) = \delta \Rightarrow y^*(L) = \delta + y(L) \tag{21}$$

Such that

$$y^*(L) = \delta - \frac{w_oL^4}{8EI} \tag{22}$$

Plugging in the left side of the equation

$$y^*(L) = -\frac{P}{6EI}\left(L^3 - 3L^3\right) = \delta - \frac{w_o L^4}{8EI} \tag{23}$$

Which we can solve for P as

$$\boxed{P = \frac{3\delta EI}{L^3} - \frac{3w_o L}{8} \quad \downarrow \quad \text{Pointing down}} \tag{24}$$

Problem 3:

In a sailboat with a fractional rig, the tension in the backstay (i.e., the steel cable that runs from the masthead to the stern) is often used to bend the mast and thus shape the sails for optimum sailing performance in a race (bending the mast tends to depower the sails). As the wind increases, sailors increase the tension in the backstay using a system of pulleys or sometimes hydraulics. The figure below shows the geometry. Assume the mast can be modeled as a vertical beam that is simply supported at the deck level and at the point where the forestay (i.e., cable running from bow to mast) is connected to the mast. Assume further that the mast is initially straight. The bending stiffness of the mast is EI. The height of the mast is H; the height of the point where the forestay is connected is h. The distance from the bow to the mast is a, and from the mast to the stern is b. Determine the shape of the mast as a function of the tension S in the backstay. No need to consider buckling here. You do not need to account for any effects of wind in this problem.

Note that we assume the cables not to deform so that $Y(h) = 0$.

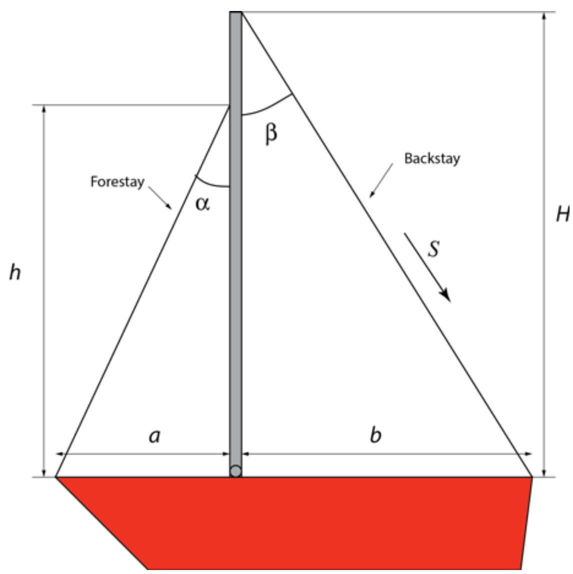


Figure 4: Problem 3 Schematic

Solution 3

First let's draw a free body diagram of the mast with the different tension components of the forestay and the backstay. We note that the attachment of the mast to the hull is a pin joint, based on the picture (no moments).

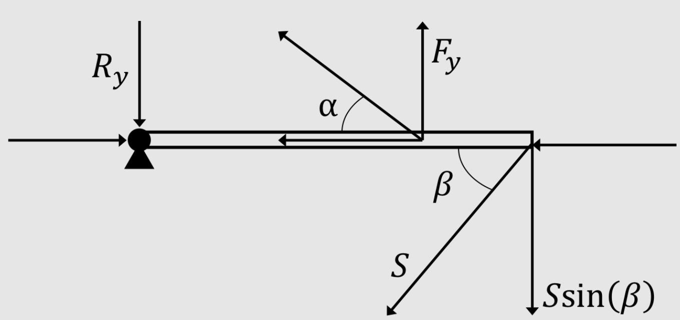


Figure 5: Free body diagram for depicted problem.

For this we need to begin by finding the reaction forces of the Forestay F_y and R_y. To do this, do sum of forces in the Y direction and sum of moments at the joint between the mast and the hull, respectively:

$$\Sigma F_y = 0: \quad -S\cos(\beta) + R_f - F_y \tag{25}$$

$$\Sigma M = 0: \quad -F_y(h) + S\sin(\beta)(H) = 0 \Rightarrow F_y = S\sin(\beta)\left(\frac{H}{h}\right) \tag{26}$$

Thus, using eq. (25),

$$R_y = S\sin(\beta)\left(\frac{H}{h} - 1\right) \quad \text{which is positive given that } \frac{H}{h} > 1 \tag{27}$$

Using these results we are ready to obtain the equations for the shear force, namely,

$$V(x) = -R_y + F_y < x - h >^0 \tag{28}$$

We can now integrate this to obtain the moment

$$M(x) = -R_y x + F_y < x - h >^1 \tag{29}$$

Again, knowing that

$$EI\frac{d^2y}{dx^2} = M(x) = -R_y x + F_y < x - h >^1 \tag{30}$$

We can integrate to obtain

$$EI\frac{dy}{dx} = -\frac{R_y}{2}x^2 + \frac{F_y}{2} < x - h >^2 + c_1 \tag{31}$$

and integrate again to obtain

$$EIy(x) = -\frac{R_y}{6}x^3 + \frac{F_y}{6} < x - h >^3 + c_1 x + c_2 \tag{32}$$

Here we use two boundary conditions, namely,

$$y(0) = 0 \quad \text{and} \quad y(h) = 0. \tag{33}$$

Using the first BC, we can solve

$$y(0) = 0 \Rightarrow c_2 = 0 \tag{34}$$

Using the second BC, we obtain

$$y(h) = -\frac{R_y}{6}h^3 + \frac{f_y}{6}(0) + c_1 h = 0 \Rightarrow c_1 = \frac{R_y}{6}h^2 \tag{35}$$

Putting all of this together, we obtain the final expression for the shape of the mast

$$EIy(x) = -\frac{R_y}{6}x^3 + \frac{F_y}{6} < x - h >^3 + \frac{R_y h^2}{6}x \tag{36}$$

$$y(x) = \frac{R_y}{EI6}(xh^2 - x^3) + \frac{F_y}{EI6} < x - h >^3 \tag{37}$$

Which if we plug in what we know for R_y and F_y, we can obtain an expression as a function of S

$$\boxed{y(x) = \frac{S\sin(\beta)\left(\frac{H}{h} - 1\right)}{EI6}(xh^2 - x^3) + \frac{S\sin(\beta)\left(\frac{H}{h}\right)}{EI6} < x - h >^3} \tag{38}$$

Which can also be written in terms of b instead of β (from simple geometry, $\sin(\beta) = \frac{b}{\sqrt{H^2+b^2}}$) is

$$\boxed{y(x) = \frac{Sb\left(\frac{H}{h}-1\right)}{\sqrt{H^2+b^2}EI6}(xh^2-x^3) + \frac{Sb\left(\frac{H}{h}\right)}{\sqrt{H^2+b^2}EI6} < x-h >^3} \tag{39}$$

Note that this solution is only valid for $h < H$.

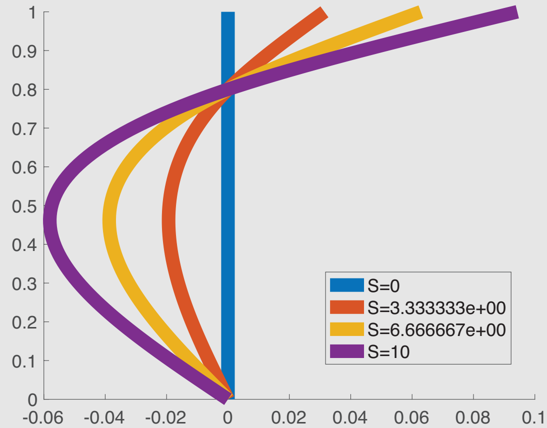


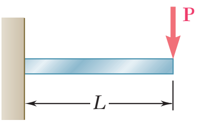
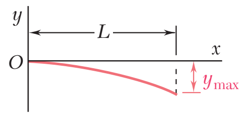
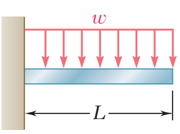
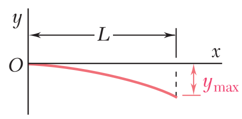
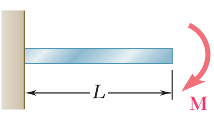
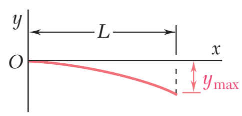
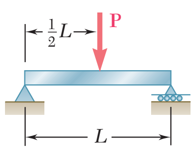
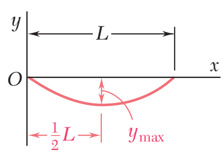
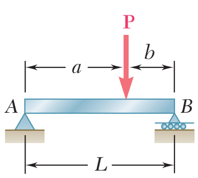
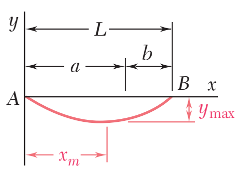
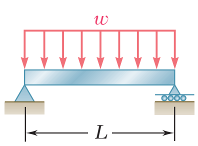
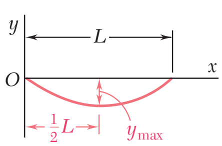
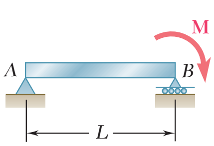
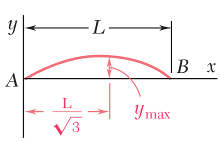
Figure 6: Deformation plot for beam in problem .

Matlab Code for Plotting Beam Deformation

```
function Plots2()

X=linspace(0,1);

S=linspace(0,10,4);
hold on
for s =S
plot(-P1(X,1,0.8,1,s,1,1),X,'Linewidth',12)
end
legend(cellstr(num2str(S', 'S=%-d')),'fontsize',16)
set(gca, 'fontsize', 16)

end


function val = P1(X,H,h,b,s,E,I)
ct=0;
for x = X
    ct=ct+1;
    val(ct) = (H/h-1)*(x*h^2-x^3);

    if x>=h
        val(ct)=val(ct)+(H/h)*(x-h)^3;
    end
    val(ct)=val(ct)*s*b/(sqrt(H^2+b^2)*E*I*6);

end


end
```

ES 120 -- Introduction to the Mechanics of Solids

APPENDIX D Beam Deflections and Slopes

Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
1 		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = \frac{P}{6EI}(x^3 - 3Lx^2)$
2 		$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
3 		$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y = -\frac{M}{2EI}x^2$
4 		$-\frac{PL^3}{48EI}$	$\pm\frac{PL^2}{16EI}$	For $x \leq \frac{1}{2}L$: $y = \frac{P}{48EI}(4x^3 - 3L^2x)$
5 		For $a > b$: $-\frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\frac{L^2 - b^2}{3}}$	$\theta_A = -\frac{Pb(L^2 - b^2)}{6EIL}$ $\theta_B = +\frac{Pa(L^2 - a^2)}{6EIL}$	For $x < a$: $y = \frac{Pb}{6EIL}[x^3 - (L^2 - b^2)x]$ For $x = a$: $y = -\frac{Pa^2b^2}{3EIL}$
6 		$-\frac{5wL^4}{384EI}$	$\pm\frac{wL^3}{24EI}$	$y = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
7 		$\frac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\frac{ML}{6EI}$ $\theta_B = -\frac{ML}{3EI}$	$y = -\frac{M}{6EIL}(x^3 - L^2x)$