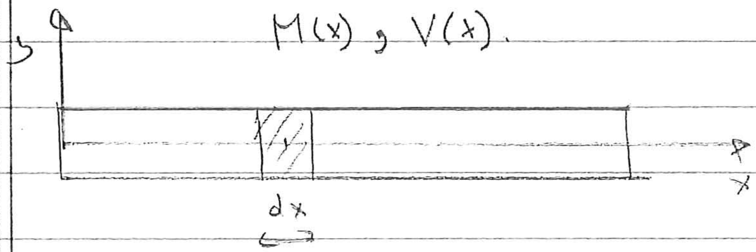
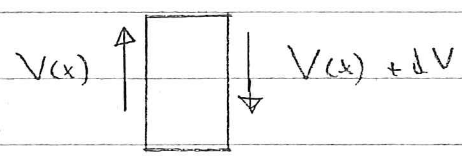
Transverse waves in beams.

$M(x)$, $V(x)$.





$$+V(x) - \left(V(x) + \frac{\partial V}{\partial x} dx\right) = \rho A dx \frac{\partial^2 y}{\partial t^2}$$

$$-\frac{\partial V}{\partial x} = \rho A \frac{\partial^2 y}{\partial t^2}$$

But $\frac{\partial M}{\partial x} = V$

$$-\frac{\partial^2 M}{\partial x^2} = \rho A \frac{\partial^2 y}{\partial t^2}$$

But $\frac{\partial^2 y}{\partial x^2} = \frac{M}{EI}$

$$-EI \frac{\partial^4 y}{\partial x^4} = \rho A \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\frac{EI}{\rho A}\right) \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0}$$

Take $b^2 = EI/\rho A$

B.C.'s: Simply supported: $\begin{cases} y(x_0) = 0 \\ y''(x_0) = 0 \end{cases}$

Clamped: $\begin{cases} y(x_0) = 0 \\ y'(x_0) = 0 \end{cases}$

Free end: $\begin{cases} M = 0 \rightarrow y''(x_0) = 0 \\ V = 0 \rightarrow y'''(x_0) = 0 \end{cases}$

* Solution

Take $y(x,t) = X(x)\, T(t)$

$$X^{(IV)} - \frac{\gamma^2}{b^2} X = 0$$

$$T'' + \gamma^2 T = 0$$

Solution

$$X(x) = C_1 \sin\sqrt{\frac{\gamma}{b}}x + C_2 \cos\sqrt{\frac{\gamma}{b}}x + C_3 \sinh\sqrt{\frac{\gamma}{b}}x + C_4 \cosh\sqrt{\frac{\gamma}{b}}x$$

$$T(t) = C_5 \sin\gamma t + C_6 \cos\gamma t$$

$X(x)$ will give the mode shape, γ corresponds to the frequency

* Example: A simply supported beam of length L

BC's $\quad y(0,t) = y(L,t) = 0$

$\quad y''(0,t) = y''(L,t) = 0$

1) $x = 0$: $\left.\begin{aligned} C_2 + C_4 &= 0 \\ -C_2 + C_4 &= 0 \end{aligned}\right\}$ $\quad C_2 = C_4 = 0$

2) $x = L$

$$\begin{cases} C_1 \sin\sqrt{\frac{\gamma}{b}}L + C_3 \sinh\sqrt{\frac{\gamma}{b}}L = 0 \\ -C_1 \sin\sqrt{\frac{\gamma}{b}}L + C_3 \sinh\sqrt{\frac{\gamma}{b}}L = 0 \end{cases}$$

$\Rightarrow$ only non trivial solution if the determinant vanishes:

$$\sin\sqrt{\frac{\gamma}{b}}L \sinh\sqrt{\frac{\gamma}{b}}L + \sin\sqrt{\frac{\gamma}{b}}L \sinh\sqrt{\frac{\gamma}{b}}L = 0$$

$\Rightarrow$ The only non-trivial solution corresponds to

$$\sin\sqrt{\frac{\gamma}{b}}L = 0 \quad \Rightarrow \quad \sqrt{\frac{\gamma}{b}}L = n\pi \qquad n = 1, 2, 3 \ldots$$

$$\Rightarrow \quad \gamma = \frac{n^2\pi^2}{L^2} b$$

$$\Rightarrow \quad \gamma = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}} \qquad n = 1, 2, \ldots$$

* Only discrete frequencies are allowed in a freestanding beam
  - $\rightarrow$ use this to determine E from γ
  - $\rightarrow$ Discuss dependence on geometry
* It follows from the equations that C_3 must be zero, so that the shape of the beam (i.e. mode form) is given by

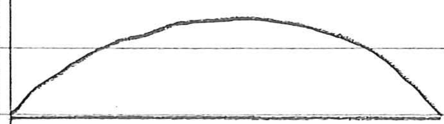
$$X_n(x) = C_1 \sin\frac{n\pi x}{L}$$

* Using superposition we finally find the most general solution for the free vibration of a beam

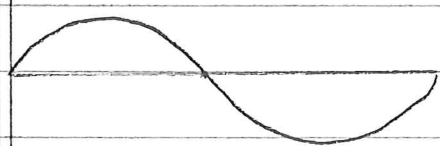
$$y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \sin\gamma_n t + B \cos\gamma_n t\right)$$

with A_n, B_n to be determined from the initial conditions

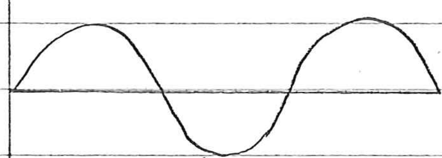
The mode shapes are then



$n=1$



$n=2$



$n=3$