ES120 Spring 2018 – Midterm 2 Review w/ Solutions

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Document Disclaimer

The list provided below is by no means comprehensive and if you find anything missing that you would like to add please let me know. This review session has been created without prior knowledge of the problems in the exam and should not be treated in any way as hints to problems that will be asked in the exam. We will do our best to go over the topics of the course in detail however please do your own reading of chapters 4, 5 and 9 as well as other topics not included in the book. If you find any typos please let me know and I will update and push a new version to Github.

You may also find my notes from a previous year helpful: `http://fer.me/es120notes`

Topics Covered Summary

4. **Pure bending**
   - **Geometry** – any cross section perpendicular to the axis of the member remains plane and remains perpendicular to the centroid line.
   - **Normal strain and normal stress** – normal strain $\epsilon_{xx} = -\frac{y}{\rho}$ and normal stress $\sigma_{xx} = -\frac{Ey}{\rho}$, where ρ is the radius of the beam. We also have $\epsilon_{xx} = -\frac{y}{c}\epsilon_m$ and $\sigma_{xx} = -\frac{y}{c}\sigma_m$, where ϵ_m and σ_m are maximum strain and stress in the beam.
   - **Force** & **Position centroid line** – the force in the beam can be calculated by integrating normal stress over the cross section: $F = \int \sigma_{xx}\, \mathrm{d}A$, if the material is within elastic range, $= -\frac{E}{\rho}\int y\, \mathrm{d}A$, where $Q = \int y\, \mathrm{d}A$ is called first moment (see Appendix A.2 in text book). Force in beam should be zero, that means $Q = 0$. Solving $Q = 0$ gives the position of centroid line.
   - **Moment** – moment in the beam is calculated by $M = -\int y\sigma_{xx}\, \mathrm{d}A = \frac{E}{\rho}\int y^2\, \mathrm{d}A$, where $I = \int y^2\, \mathrm{d}A$ is called the second moment (see Appendix A.3 for details). Now we have $M = \frac{EI}{\rho}$, and $\frac{1}{\rho} = \frac{M}{EI}$ (curvature - moment relation).
   - **Connect stress with moment** – $\sigma_{xx} = -\frac{My}{I}$ and $\sigma_m = \frac{Mc}{I}$. Introducing elastic section modulus $S = I/c$, so that we have $\sigma_m = \frac{M}{S}$.
   - **Composite beams** – two materials with young's modulus E_1 and E_2, let $n = E_2/E_1$, the resistance to bending of the bar would remain the same if both portions were made of the first material E_1, provided that the **width** of each element of the lower portion were multiplied by the factor n. To obtain the **stress** σ_2 for material 2, we must multiply by n (see more on textbook page 230).
   - **Reinforced concrete beams** – (1) replace the total cross-sectional area of the steel bars A_s by an equivalent area nA_s; (2) only the portion of the cross section in compression should be used in the transformed section (see textbook 233).
   - **Eccentric axial loading** – $\sigma_{xx} = (\sigma_{xx})_{centric} + (\sigma_{xx})_{bending} = \frac{P}{A} - \frac{My}{I}$

5. **Analysis and Design of Beams for Bending**
   - **Shear and Bending moments diagrams** – Drawing shear forces through a uniform beam. Remember which direction is positive and which direction is negative. Note that the bending moment is the integral of the shear diagram, see Figure 5.7 of textbook.

- **Relations among load shear and bending moment** – The overall relationship of all of these are integrals. Specifically they are related through these equations: $-w = \frac{dV}{dx}, V = \frac{dM}{dx}$. In other words, they are the area under the curve of each other.
- **Design of Prismatic Beams for Bending** – How to design beam cross-sections such that you achieve the most economical design possible to efficiently withstand specific load conditions. This is a procedure detailed on page 333 of textbook. The main idea of the procedure is to use bending moment diagrams in conjunction with the fact that the maximum stress occurs at the edge of the section to come up with a cross-section height that is appropriate for the loading conditions.
- **Using Singularity Functions for Shear and Bending Moments** – Singularity functions are defined as $< x - a >^n = \begin{cases} (x-a)^n & \text{for } n \geq a \\ 0 & \text{for } n < a \end{cases}$ They are useful to organize the discontinuities in the distribution of shear forces, bending moments and weights through the beam. It simplifies by not having to account for each part of the beam in separate equations, but instead condenses all information into one equation. This is also useful when performing the integrals from weight to shear forces to bending moments
- **Nonprismatic beams** – Unlike primsmatic beams, we now relax the assumption that the beam cross-section is constant. Now the goal of this is to have a beam of constant strength for a particular loading condition and allow the cross section of the beam to be controlled via the loading conditions. This is very similar to prismatic beam design only now $S = \frac{|M|}{\sigma_{\text{all}}}$. This is covered in more detail in an example problem below.

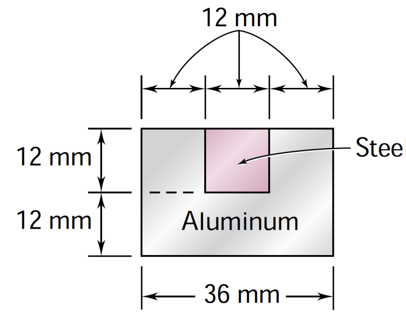
9. <u>**Deflection of Beams**</u>

- **Equation of the elastic curve** – $\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$.
- **Integration** – $y = \int_0^x dx \int_0^x \frac{M(x)}{EI} dx + C_1 x + C_2$, where C_1 and C_2 are determined by boundary conditions.
- **Statically indeterminate beams** – superposition of deflection (see more in textbook on page 560).
- **Beam vibrations** – Know the governing PDE and the general solutions for varying boundary conditions. Read lecture notes on this and also notes posted to Canvas.

Review Problems

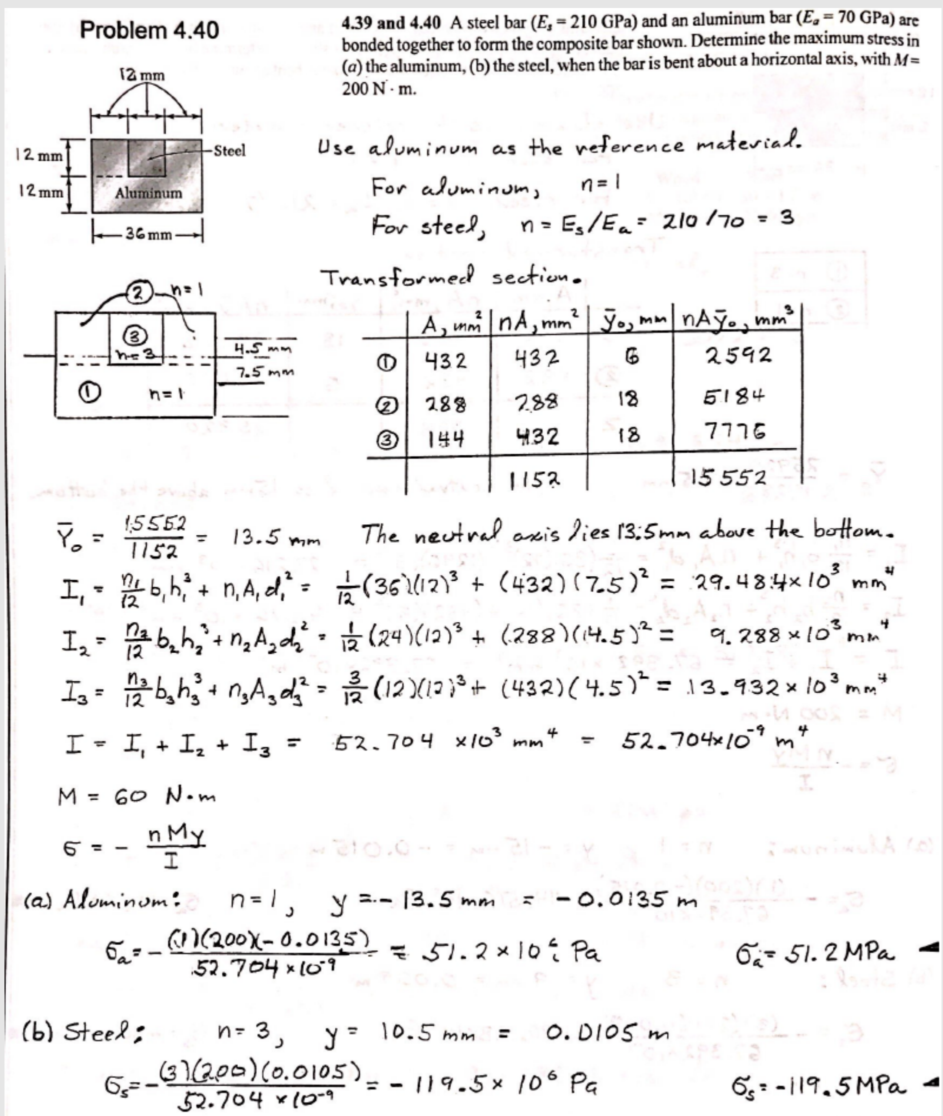
Try to work these out on your own and solutions will be pushed to Github later.

Problem 1: Chapter 4 Review



A steel bar ($E_s = 210$ GPa) and an aluminum bar ($E_a = 70$ GPa) are bonded together to form the composite bar shown. Determine the maximum stress in (a) the aluminum, (b) the steel, when the bar is bent about a horizontal axis, with $M = 200$ Nm.

Solution 1



Problem 4.40

4.39 and 4.40 A steel bar ($E_s = 210$ GPa) and an aluminum bar ($E_a = 70$ GPa) are bonded together to form the composite bar shown. Determine the maximum stress in (*a*) the aluminum, (*b*) the steel, when the bar is bent about a horizontal axis, with $M =$ 200 N · m.

Use aluminum as the reference material.

For aluminum, $n = 1$

For steel, $n = E_s/E_a = 210/70 = 3$

Transformed section.

	A, mm²	nA, mm²	$\bar{y}_o$, mm	$nA\bar{y}_o$, mm³
1	432	432	6	2592
2	288	288	18	5184
3	144	432	18	7776
		1152		15552

$\bar{Y}_o = \frac{15552}{1152} = 13.5$ mm The neutral axis lies 13.5 mm above the bottom.

$I_1 = \frac{n_1}{12} b_1 h_1^3 + n_1 A_1 d_1^2 = \frac{1}{12}(36)(12)^3 + (432)(7.5)^2 = 29.484 \times 10^3 \text{ mm}^4$

$I_2 = \frac{n_2}{12} b_2 h_2^3 + n_2 A_2 d_2^2 = \frac{1}{12}(24)(12)^3 + (288)(4.5)^2 = 9.288 \times 10^3 \text{ mm}^4$

$I_3 = \frac{n_3}{12} b_3 h_3^3 + n_3 A_3 d_3^2 = \frac{3}{12}(12)(12)^3 + (432)(4.5)^2 = 13.932 \times 10^3 \text{ mm}^4$

$I = I_1 + I_2 + I_3 = 52.704 \times 10^3 \text{ mm}^4 = 52.704 \times 10^{-9} \text{ m}^4$

$M = 60$ N·m

$\sigma = -\frac{nMy}{I}$

(a) Aluminum: $n = 1$, $y = -13.5$ mm $= -0.0135$ m

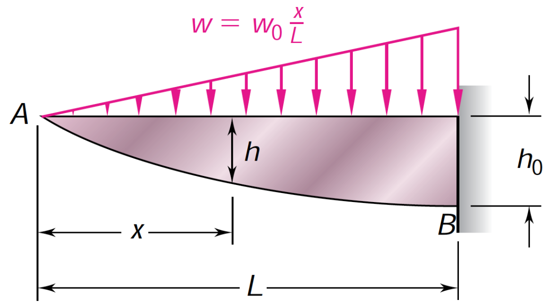
$\sigma_a = -\frac{(1)(200)(-0.0135)}{52.704 \times 10^{-9}} = 51.2 \times 10^6$ Pa $\quad \sigma_a = 51.2$ MPa

(b) Steel: $n = 3$, $y = 10.5$ mm $= 0.0105$ m

$\sigma_s = -\frac{(3)(200)(0.0105)}{52.704 \times 10^{-9}} = -119.5 \times 10^6$ Pa $\quad \sigma_s = -119.5$ MPa

Problem 2: Chapter 5 Review

The beam AB, consisting of a cast-iron plate of uniform thickness b and length L, is to support the distributed load $w(x)$ shown. (a) Knowing that the beam is to be of constant strength, express h in terms of x, L, and h_0. (b) Determine the smallest value of h_0 if $L = 750$ mm, $b = 30$ mm, $w_0 = 300$ kN/m, and $\sigma_{all} = 200$ MPa.



Solution 2

We know from class the relationship between the weight distribution, shear forces and moments to be:

$$-w = \frac{dV}{dx} \tag{1}$$

$$V = \frac{dM}{dx} \tag{2}$$

therefore, knowing that

$$w(x) = w_0 \frac{x}{L} \tag{3}$$

We can simply integrate the above to obtain:

$$V(x) = -\int_0^x w_0 \frac{x}{L} dx = -\frac{w_0 x^2}{2L} \tag{4}$$

Now to obtain the moment we integrate again

$$M(x) = \int_0^x -\frac{w_0 x^2}{2L} dx = -\frac{w_0 x^3}{6L} \tag{5}$$

Part (a)

Here we have all of the information we need to determine the how the cross-section should look like. This was indicated in chapter 5.1 that the design of the beam is usually controlled by the maximum absolute value of the bending moment that will occur in the beam. The largest normal stress in the beam is found at the surface of the beam in the critical section where $|M|_{\max}$ occurs. The section modulus S is therefore defined in chapter 5.4 as

$$S = \frac{|M|_{\max}}{\sigma_{all}} \tag{6}$$

Which we can use the polar moment of inertia to solve for S of a rectangular cross-section such that

$$S = \frac{1}{6} b h^2 \tag{7}$$

Equating eq. (6) and eq. (7) we can obtain a closed form solution for h, namely,

$$\frac{1}{6}bh^2 = \frac{w_0 x^3}{6L\sigma_{\text{all}}} \tag{8}$$

which solving for h explicitly yields

$$h = \sqrt{\frac{w_0 x^3}{\sigma_{\text{all}} bL}} \tag{9}$$

Now knowing that we value at $x = L$ for h is h_o, we can normalize h by:

$$h(x = L) = h_o = \sqrt{\frac{w_o L^2}{\sigma_{\text{all}} b}} \tag{10}$$

Factoring that out from h, we obtain

$$\boxed{h(x) = h_0 \left(\frac{x}{L}\right)^{3/2}} \tag{11}$$

Part(b)

Here we only need to plug in the values we are given in the problem statement into our equation for h_0, namely,

$$\boxed{h_o = \sqrt{\frac{(300 \times 10^3)(0.75)^2}{(200 \times 10^6)(0.030)}} = 167.7 \times 10^{-3} \text{ mm}} \tag{12}$$

Problem 3: Chapter 9 Review

For the loading shown, determine (a) the equation of the elastic curve for the cantilever beam AB, (b) the deflection at the free end, (c) the slope at the free end.

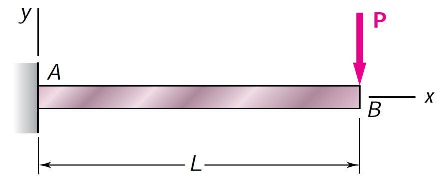
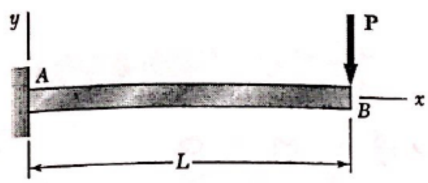


Fig. P9.2

Solution 3

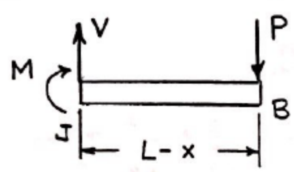
Problem 9.2

9.1 through 9.4 For the loading shown, determine (*a*) the equation of the elastic curve for the cantilever beam *AB*, (*b*) the deflection at the free end, (*c*) the slope at the free end.



$[x = 0,\ y = 0]$

$[x = 0,\ \frac{dy}{dx} = 0]$



$\circlearrowleft \Sigma M_J = 0: \quad -M - P(L-x) = 0$

$M = -P(L-x)$

$EI\frac{d^2y}{dx^2} = -P(L-x) = -PL + Px$

$EI\frac{dy}{dx} = -PLx + \frac{1}{2}Px^2 + C_1$

$[x = 0,\ \frac{dy}{dx} = 0] \qquad 0 = -0 + 0 + C_1$

$C_1 = 0$

$EIy = -\frac{1}{2}PLx^2 + \frac{1}{6}Px^3 + C_1x + C_2$

$[x = 0,\ y = 0] \qquad 0 = -0 + 0 + 0 + C_2 \qquad C_2 = 0$

(a) Elastic curve. $\quad y = -\frac{Px^2}{6EI}(3L - x)$

$\frac{dy}{dx} = -\frac{Px}{2EI}(2L - x)$

(b) y @ $x = L$: $\quad y_B = -\frac{PL^2}{6EI}(3L - L) = -\frac{PL^3}{3EI}$

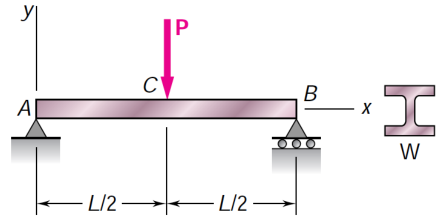
$y_B = \frac{PL^3}{3EI} \downarrow$

(c) $\frac{dy}{dx}$ @ $x = L$: $\quad \left.\frac{dy}{dx}\right|_B = -\frac{PL}{2EI}(2L - L) = -\frac{PL^2}{2EI}$

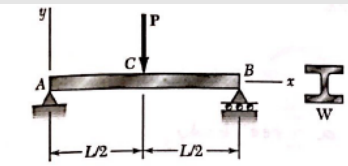
$\theta_B = \frac{PL^2}{2EI} \measuredangle$

Problem 4: Chapter 9 Review

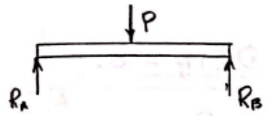
Knowing that beam AB is a W130 $\times$ 23.8 rolled shape and that $P = 50$ kN, $L = 1.25$ m, and $E = 200$ GPa, determine (a) the slope at A, (b) the deflection at C.

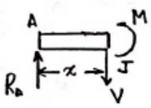


Solution 4



$[x=0, y=0]$ $[x=\frac{L}{2}, \frac{dy}{dx}=0]$ $[x=L, y=0]$





Use symmetry boundary condition at C.

By symmetry, $R_A = R_B = \frac{1}{2}P$

Using free body AJ, $0 \le x \le \frac{L}{2}$

$+\circlearrowleft \Sigma M_J = 0: \quad M - R_A x = 0$

$M = R_A x = \frac{1}{2}Px$

$EI\frac{d^2y}{dx^2} = \frac{1}{2}Px$

$EI\frac{dy}{dx} = \frac{1}{4}Px^2 + C_1$

$EIy = \frac{1}{12}Px^3 + C_1 x + C_2$

$[x=0, y=0] \quad 0 = 0 + 0 + C_2 \qquad C_2 = 0$

$[x=\frac{L}{2}, \frac{dy}{dx}=0] \quad 0 = \frac{1}{4}P(\frac{L}{2})^2 + C_1 \qquad C_1 = -\frac{1}{16}PL^2$

Elastic curve.

$$y = \frac{PL}{48EI}(4x^3 - 3L^2x)$$

$$\frac{dy}{dx} = \frac{PL}{16EI}(4x^2 - L^2)$$

Slope at $x = 0$. $\quad \frac{dy}{dx}\Big|_A = -\frac{PL^3}{16EI} \qquad \theta_A = \frac{PL^2}{16EI}$

Deflection at $x = \frac{L}{2}$. $\quad y_C = -\frac{PL^3}{48EI} \qquad y_C = \frac{PL^3}{48EI} \downarrow$

Data: $P = 50\times10^3$ N, $I = 8.80\times10^6\ mm^4 = 8.80\times10^{-6}\ m^4$

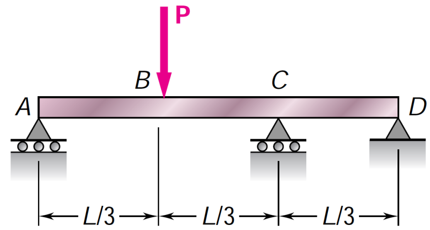
$E = 200\times10^9$ Pa $\qquad EI = 1.76\times10^6\ N\cdot m^2 \qquad L = 1.25$ m

(a) $\theta_A = \frac{(50\times10^3)(1.25)^2}{(16)(1.76\times10^6)} \qquad \theta_A = 2.77\times10^{-3}$ rad

(b) $y_C = \frac{(50\times10^3)(1.25)^3}{(48)(1.76\times10^6)} = 1.156\times10^{-3}$ m $\qquad y_C = 1.156$ mm $\downarrow$

Problem 5: Chapter 9 Review

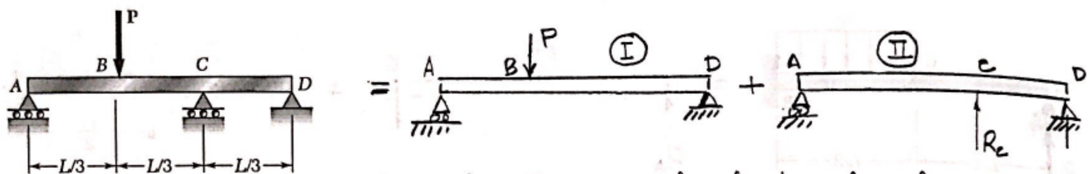
For the uniform beam shown, determine the reaction at each of the three supports.



Solution 5

Problem 9.81

9.81 and 9.82 For the uniform beam shown, determine the reaction at each of the three supports.



Consider R_C as redundant and replace loading system by I and II.

Loading I. (Case 5 of Appendix D) $a = \frac{2L}{3}$, $b = \frac{L}{3}$, $x = \frac{L}{3}$ at C.

$$(y_C)_1 = \frac{Pb}{6EIL}\left[x^3 - (L^2 - b^2)x\right] = \frac{P(L/3)}{6EIL}\left[\left(\frac{L}{3}\right)^3 - \left\{L^2 - \left(\frac{L}{3}\right)^2\right\}\frac{L}{3}\right]$$

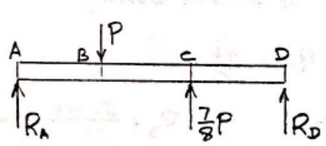
$$= -\frac{7PL^3}{486EI}$$

Loading II. (Case 5 of Appendix D) $a = \frac{2L}{3}$, $b = \frac{L}{3}$

$$(y_C)_2 = \frac{R_C a^2 b^2}{3EIL} = \frac{R_C (L/3)^2 (2L/3)^2}{3EIL} = \frac{4R_C L^3}{243EI}$$

Superposition and constraint. $y_C = (y_C)_1 + (y_C)_2 = 0$

$$-\frac{7PL^3}{486EI} + \frac{4R_C}{243EI} = 0 \qquad R_C = \frac{7}{8}P \uparrow$$



$+\circlearrowleft \Sigma M_D = 0:$

$$-R_A L + P\left(\frac{2L}{3}\right) - \left(\frac{7}{8}P\right)\left(\frac{L}{3}\right) = 0$$

$$R_A = \frac{3}{8}P \uparrow$$

$+\uparrow \Sigma F_y = 0: \quad R_A + R_D - P + \frac{7}{8}P = 0$

$$R_D = P - \frac{7}{8}P - \frac{3}{8}P = -\frac{1}{4}P$$

$$R_D = \frac{1}{4}P \downarrow$$