Example.

Given

- A state space basis of $\left\{ \left| u \right\rangle, \left| d \right\rangle \right\}$ for both Alice and Bob,
- Alice's system is prepared in the direction

$$\hat{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{pmatrix} \sin\frac{\pi}{3}\cos 0 \\ \sin\frac{\pi}{3}\sin 0 \\ \cos\frac{\pi}{3} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix}$$

- Bob's system is prepared in the generic direction

$$\hat{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

- Alice and Bob measure spin along the *z*-axis
- The composite system is a product state

Compute/Identify

- Find Bob's state $\left| \Phi_B \right\rangle$ and wave function, observable σ_B and its eigenvectors and eigenvalues, and use equation (4.13) to compute $\left\langle \sigma_B \right\rangle = \left\langle \Phi_B \middle| \sigma_B \middle| \Phi_B \right\rangle$
- Find Alice's state, wave function, observable, eigenvectors and eigenvalues, and compute $\left\langle \sigma_A \right\rangle$.
- Find Bob's density matrix ρ_B, confirm that it is the projection operator, and compute $\left\langle \sigma_B \right\rangle = \mathrm{Tr}\left(\rho_B \sigma_B \right)$.
- Find the unitary matrix U of Theorem 7.5 such that $U^\dagger \rho_B U$ is a diagonal matrix and use it to help show in 4 different ways that Bob's system is pure
- Find similar information for Alice's density matrix
- Find the product state $\left| \Psi \right\rangle$ and the wave function
- Show that Alice's and Bob's observables are uncorrelated

Solution.

Bob

Spin is prepared in direction $\hat{n}$ and measured along z-axis. Thus, Bob's spin basis is $\left\{ \left|u\right\rangle, \left|d\right\rangle \right\}$ and his observable is $\sigma_B = \sigma_z \overset{(3.20)}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. ✔

The eigenvectors are $\left|u\right\rangle$ and $\left|d\right\rangle$ with corresponding eigenvalues 1 and -1. ✔

Denote Bob's prepared state by $\left|\Phi_B\right\rangle = \phi_u\left|u\right\rangle + \phi_d\left|d\right\rangle$. From the spherical coordinates of $\hat{n}$ we deduce that the angle between $\hat{n}$ and the z -is ϕ and we call θ the phase angle. The conditions for equations 1.02 and 2.2 apply. We use them to solve for ϕ_u and ϕ_d in terms of ϕ and θ :

$$\mathrm{P}_{\sigma_B}\left(\left|u\right\rangle\right) \overset{(1.02)}{=} \cos^2\frac{\phi}{2},$$

$$\mathrm{P}_{\sigma_B}\left(\left|u\right\rangle\right) \overset{(2.2)}{=} \phi_u^*\phi_u.$$

A general solution for $\phi_u^*\phi_u \overset{(1.02,\ 2.2)}{=} \cos^2\frac{\phi}{2}$ is $\phi_u = \cos\frac{\phi}{2}e^{i\theta}$, $\phi_u^* = \cos\frac{\phi}{2}e^{-i\theta}$.

The phase factor $e^{i\theta}$ must be included or else $n_y = 0 \Rightarrow \hat{n} \in xz\text{-plane}$.

The general expressions for ϕ_d and ϕ_d^* are $\phi_d = \sin\frac{\phi}{2}e^{i\theta}$, $\phi_d^* = \sin\frac{\phi}{2}e^{-i\theta}$:

$$\phi_d^*\phi_d \overset{(2.2)}{=} \mathrm{P}_{\sigma_B}\left(\left|d\right\rangle\right) = 1 - \mathrm{P}_{\sigma_B}\left(\left|u\right\rangle\right) = 1 - \cos^2\frac{\phi}{2} = \sin^2\frac{\phi}{2} \quad ✔$$

The conditions for equation (4.13) hold. Hence

$$\left\langle\sigma_B\right\rangle \overset{(4.13)}{=} \left\langle\Phi_B\middle|\sigma_B\middle|\Phi_B\right\rangle = \begin{pmatrix} \phi_u^* & \phi_d^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \phi_u \\ \phi_d \end{pmatrix} = \begin{pmatrix} \phi_u^* & \phi_d^* \end{pmatrix}\begin{pmatrix} \phi_u \\ -\phi_d \end{pmatrix}$$

$$= \phi_u^*\phi_u - \phi_d^*\phi_d = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} \overset{(\text{Dbl Angle Formula})}{=} \cos\phi$$

Check #1: $\left\langle\sigma_B\right\rangle = \mathrm{P}_{\sigma_B}\left(\left|u\right\rangle\right)\bullet 1 + \mathrm{P}_{\sigma_B}\left(\left|d\right\rangle\right)\bullet(-1) = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} = \cos\phi$ ✔

(definition of expected value)

Check #2: $\left\langle\sigma_B\right\rangle \overset{(1.01)}{=} \cos\phi$ (cosine of angle between prep and measurement) ✔

Bob's wave function is $\left\{\phi_u, \phi_d\right\} = \left\{\cos\frac{\phi}{2}e^{i\theta}, \sin\frac{\phi}{2}e^{i\theta}\right\}$. ✔

Alice

$\left|\Psi_A\right\rangle = \psi_u\left|u\right\rangle + \psi_d\left|d\right\rangle$. From Bob's generic results we deduce that

$\psi_u = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = \psi_u^*$. (Alice's phase factor is zero because $m_y = 0$.) Also

$\psi_d = \sin\frac{\pi}{6} = \frac{1}{2} = \psi_d^*$, $\mathrm{P}_{\sigma_A}\left(\left|u\right\rangle\right) = \cos^2\frac{\pi}{6} = \frac{3}{4}$, $\mathrm{P}_{\sigma_A}\left(\left|d\right\rangle\right) = \sin^2\frac{\pi}{6} = \frac{1}{4}$, and

$\sigma_A = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus, $\left\langle\sigma_A\right\rangle = \cos\frac{\pi}{3} = \frac{1}{2}$ and her wave function is

$\left\{\psi_{u,}\ \psi_d\right\} = \left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}$.

Density Matrix

Bob's system is pure since AB is a product space. Thus, his density matrix is

$\rho_B = \left(\rho_{b'b}\right)$ where $\rho_{b'b} \overset{(7.25)}{=} \phi_b^*\phi_{b'}$:

$$\rho_{uu} = \phi_u{}^*\phi_u = \cos^2\frac{\phi}{2},\quad \rho_{ud} = \phi_d{}^*\phi_u = \sin\frac{\phi}{2}\cos\frac{\phi}{2} = \rho_{du},\quad \rho_{dd} = \phi_d{}^*\phi_d = \sin^2\frac{\phi}{2}.$$

Therefore

$$\rho_B = \begin{pmatrix} \cos^2\frac{\phi}{2} & \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & \sin^2\frac{\phi}{2} \end{pmatrix} \overset{(7.22)}{=} \begin{pmatrix} \mathrm{P}_{\sigma_B}\left(\left|u\right\rangle\right) & \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & \mathrm{P}_{\sigma_B}\left(\left|d\right\rangle\right) \end{pmatrix}$$

Check #1: $\mathrm{P}_{\sigma_B}\left(\left|u\right\rangle\right) + \mathrm{P}_{\sigma_B}\left(\left|d\right\rangle\right) = \cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} = 1.$ ✔

Check #2: From (7.12b), ρ_B should be the projection operator $\left|\Phi_B\right\rangle\left\langle\Phi_B\right|$:

$$\left|\Phi_B\right\rangle\left\langle\Phi_B\right| = \begin{pmatrix} \phi_u\phi_u^* & \phi_u\phi_d^* \\ \phi_d\phi_u^* & \phi_d\phi_d^* \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\phi}{2} & \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & \sin^2\frac{\phi}{2} \end{pmatrix} = \rho_B \quad ✔$$

Check #3:

$$\rho_B\sigma_B = \begin{pmatrix} \cos^2\frac{\phi}{2} & \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & \sin^2\frac{\phi}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\phi}{2} & -\sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & -\sin^2\frac{\phi}{2} \end{pmatrix}.$$

$$\Rightarrow \quad \langle\sigma_B\rangle = \text{Tr}\left(\rho_B\sigma_B\right) = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} = \cos\phi \quad ✔$$

Even though the formula above for $\langle\sigma_B\rangle$ in Theorem 7.2 pertains to mixed systems, it is true also for a pure system, like this one, because pure is a special case of mixed where P(*a*) = 1 for one state *a* and P(*a*') = 0 for all other states *a*'.

We next diagonalize ρ_B and confirm (per the Corollary to Theorem 7.5) that its diagonal entries are the eigenvalues of ρ_B. (We would also diagonalize σ_B but it is already diagonalized and we observe that its diagonal entries are indeed its eigenvalues.)

We begin by using the characteristic polynomial $\text{p}_{\rho_B}(\lambda)$ to find the eigenvalues of ρ_B:

$$0 = \text{p}_{\rho_B}(\lambda) \equiv \mathbf{det}\left(\rho_B - \lambda I\right) = \begin{vmatrix} \cos^2\frac{\phi}{2} - \lambda & \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & \sin^2\frac{\phi}{2} - \lambda \end{vmatrix} = \lambda^2 - \lambda = \lambda\left(\lambda - 1\right)$$

$$\Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = 0. \qquad \text{(a)}$$

We find the eigenvectors of ρ_B by solving $\rho_B\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$, resulting in

$$\left|\lambda_1\right\rangle = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \end{pmatrix} \text{ and } \left|\lambda_2\right\rangle = \begin{pmatrix} -\sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{pmatrix}.$$

Let *U* be a unitary matrix whose columns are the eigenvectors of ρ_B:

$$U = \begin{pmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \quad \Rightarrow \quad U^\dagger = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}.$$

Then

$$\rho_B U = \begin{pmatrix} \cos^2\frac{\phi}{2} & \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & \sin^2\frac{\phi}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\phi}{2} & 0 \\ \sin\frac{\phi}{2} & 0 \end{pmatrix}.$$

So

$$\overline{\rho}_B \equiv U^\dagger \rho_B U = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\phi}{2} & 0 \\ \sin\frac{\phi}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \left(\overline{\rho}_{b'b}\right).$$

Thus, $\overline{\rho}_{uu} = 1 \overset{(a)}{=} \lambda_1$ and $\overline{\rho}_{dd} = 0 \overset{(a)}{=} \lambda_2$ ✔

We show below in four different ways that Bob's system is pure:

$$\rho_B^2 = \begin{pmatrix} \cos^2\frac{\phi}{2} & \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & \sin^2\frac{\phi}{2} \end{pmatrix}\begin{pmatrix} \cos^2\frac{\phi}{2} & \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & \sin^2\frac{\phi}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\frac{\phi}{2}\left[\cancel{\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2}}\right] & \sin\frac{\phi}{2}\cos\frac{\phi}{2}\left[\cancel{\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2}}\right] \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2}\left[\cancel{\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2}}\right] & \sin^2\frac{\phi}{2}\left[\cancel{\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2}}\right] \end{pmatrix}$$

$$= \rho_B \quad ✔$$

$$\mathrm{Tr}\left(\rho_B^2\right) = \cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} = 1 \quad ✔$$

$$\overline{\rho}_B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \overline{\rho}_B \quad ✔$$

$$\mathrm{Tr}\left(\overline{\rho}_B^2\right) = 1 + 0 = 1 \quad ✔$$

Plugging Alice's values into Bob's generic formulas yield

$$\rho_A = \frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad \overline{\rho}_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \left|\lambda_1\right\rangle = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \quad \left|\lambda_2\right\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}.$$

So

$$\rho_A^2 = \rho_A, \quad \mathrm{Tr}\left(\rho_A^2\right) = 1, \quad \text{and} \quad \rho_A = \left|\Psi_A\right\rangle\left\langle\Psi_A\right|.$$

Composite System

The product state is the tensor product

$$\left|\Psi\right\rangle=\left|\Psi_A\right\rangle\otimes\left|\Phi_B\right\rangle=\begin{pmatrix}\psi_u\begin{pmatrix}\phi_u\\ \phi_d\end{pmatrix}\\ \psi_d\begin{pmatrix}\phi_u\\ \phi_d\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\psi_u\phi_u\\ \psi_u\phi_d\\ \psi_d\phi_u\\ \psi_d\phi_d\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\sqrt{3}\cos\frac{\phi}{2}e^{i\theta}\\ \sqrt{3}\sin\frac{\phi}{2}e^{i\theta}\\ \cos\frac{\phi}{2}e^{i\theta}\\ \sin\frac{\phi}{2}e^{i\theta}\end{pmatrix}. \checkmark$$

The product state wave function is

$$\{\psi_{ab}\}=\{\ \psi_{uu},\ \psi_{ud},\ \psi_{du},\ \psi_{dd}\ \}$$

$$=\left\{\frac{\sqrt{3}}{2}\cos\frac{\phi}{2}e^{i\theta},\frac{\sqrt{3}}{2}\sin\frac{\phi}{2}e^{i\theta},\cos\frac{\phi}{2}e^{i\theta},\sin\frac{\phi}{2}e^{i\theta}\right\}. \checkmark$$

$\mathrm{Corr}(A,B)=\langle\sigma_A\sigma_B\rangle-\langle\sigma_A\rangle\langle\sigma_B\rangle$:

$$\sigma_A\sigma_B=\sigma_A\otimes\sigma_B=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\left(\begin{array}{cc|cc}1&0&0&0\\0&-1&0&0\\ \hline 0&0&-1&0\\0&0&0&1\end{array}\right)$$

$$\langle\sigma_A\sigma_B\rangle=\langle\Psi|\sigma_A\sigma_B|\Psi\rangle$$

$$=\frac{e^{-i\theta}}{2}\frac{e^{i\theta}}{2}\begin{pmatrix}\sqrt{3}\cos\frac{\phi}{2} & \sqrt{3}\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & \sin\frac{\phi}{2}\end{pmatrix}$$

$$\times\begin{pmatrix}1&&&\\&-1&\mathbf{0}&\\&\mathbf{0}&-1&\\&&&1\end{pmatrix}\begin{pmatrix}\sqrt{3}\cos\frac{\phi}{2}\\ \sqrt{3}\sin\frac{\phi}{2}\\ \cos\frac{\phi}{2}\\ \sin\frac{\phi}{2}\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} \frac{\sqrt{3}}{2}\cos\frac{\phi}{2} & \frac{\sqrt{3}}{2}\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \end{pmatrix}\begin{pmatrix} \sqrt{3}\cos\frac{\phi}{2} \\ -\sqrt{3}\sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \end{pmatrix}$$

$$= \frac{1}{4}\left(3\left[\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\right] - \left[\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\right] \right) = \frac{1}{2}\left(\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\right)$$

$$= \frac{1}{2}\cos\phi$$

Therefore $\text{Corr}(\text{A,B}) = \langle \sigma_A \, \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \; = \frac{1}{2}\cos\phi - \left(\frac{1}{2}\right)(\cos\phi) = 0$ ✔

As expected, σ_A and σ_B are uncorrelated. ■

Note 1. Exercise 7.9 is practically a subset of this exercise except that Alice's as well as Bob's state is generic.

Note 2. Had the system been prepared in the *z*-direction and measured in some direction $\hat{n}$ (instead of vice-versa), the problem would essentially be the same but we would not have been able to use key formulas like 2.2 [$\text{P}(|u\rangle) = \psi_u^* \psi_u$], 4.13 [$\langle \sigma \rangle = \langle \Psi | \sigma | \Psi \rangle$], or 7.13 [$\langle \sigma \rangle = \text{Tr}\;(\rho\,\sigma)$]. That is, care must be taken to measure spin in the *z*-direction to use many of the formulas in the book.

We saw in Bob's generic system that his density matrix ρ_B has one eigenvalue +1 and the other 0. This behavior is required by Theorem 7.8.

Summary – What Alice knows about her system and Bob's system if the composite system is pure

- Alice knows the wave function (or, equivalently, the composite state) of the composite system AB since it is pure
- At one extreme, if the composite state is a product state (i.e., not entangled) then
    - Alice knows everything about her own state, namely the probability of each possible outcome.
    - But she has no knowledge of Bob's system since the two systems are uncorrelated.
- At the other extreme, if the composite state is maximally entangled, then

- Alice knows nothing about her own state because the probability of each outcome is the same as every other
- But after she performs a measurement of her state she knows everything about Bob's state because the two states are fully correlated. For example, if Alice measures up then she knows that Bob will measure down.

Section 7.8 Quantum-fying the spin measurement apparatus $\mathcal{A}$

We assign the **spin apparatus** $\mathcal{A}$ three states.

$|b\rangle$ = Blank = Read-out in apparatus window prior to a measurement

$|+\rangle$ = Read-out of +1

$|-\rangle$ = Read-out of -1

Alice's states are $|u\rangle$ and $|d\rangle$.

The basis for the composite system (consisting of Alice plus the apparatus) is

$$\{\, |ub\rangle,\ |u\ +\rangle,\ |u\ -\rangle,\ |db\rangle,\ |d\ +\rangle,\ |d\ -\rangle \,\}$$

Let L be the apparatus Hermitian measurement operator, defined as

$$\begin{aligned} L|u\,b\rangle &= |u\ +\rangle \\ L|d\,b\rangle &= |d\ -\rangle \end{aligned} \qquad \text{(i)}$$

After a measurement, if $\mathcal{A}$ reads +1 then the spin is up and if it reads -1 then the spin is down.

If the initial spin is in a superposition state $\alpha_u|u\rangle + \alpha_d|d\rangle$ then

$$L\big(\alpha_u|ub\rangle + \alpha_d|db\rangle\big) = \alpha_u|u\ +\rangle + \alpha_d|d\ -\rangle, \qquad \text{(ii)}$$

an entangled state where P(+1) = $\alpha_u^{\,*}\alpha_u$. It is maximally entangled if in the singlet state of $\alpha_u = -\alpha_d$:

$$\mathrm{P}_d = \mathrm{P}(-1) = \alpha_d^*\alpha_d = (-\alpha_u)^*(-\alpha_u) = \alpha_u^*\alpha_u = \mathrm{P}_u$$

Moreover, over time the system evolves via a unitary operator U operating on the initial state $\alpha_u|u\ +\rangle + \alpha_d|d\ -\rangle$.

Even though we will only be concerned with measurements taken when we turn the apparatus on, that is, when the screen is initially blank, in order to verify that L is a bona-fide operator we proceed to see if we can generate a consistent Hermitian matrix for L.

In the 1st matrix equation below we have labeled the rows and columns. For example, the (u + , u b) element of L is called ℓ_{21}, and the (d b) element of the initial and final state vectors are labeled δ and m_4, respectively. The matrix equation provides the general formula for L acting on an arbitrary state vector.

$$L\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\\ \chi\\ \xi\end{pmatrix}=\begin{array}{c c}
 & \begin{array}{cccccc} u\,b & u+ & u- & d\,b & d+ & d- \end{array}\\
\begin{array}{c} u\,b\\ u+\\ u-\\ d\,b\\ d+\\ d-\end{array} &
\begin{pmatrix}
\ell_{11} & \ell_{12} & \ell_{13} & \ell_{14} & \ell_{15} & \ell_{16}\\
\ell_{21} & \ell_{22} & \ell_{23} & \ell_{24} & \ell_{25} & \ell_{26}\\
\ell_{31} & \ell_{32} & \ell_{33} & \ell_{34} & \ell_{35} & \ell_{36}\\
\ell_{41} & \ell_{42} & \ell_{43} & \ell_{44} & \ell_{45} & \ell_{46}\\
\ell_{51} & \ell_{52} & \ell_{53} & \ell_{54} & \ell_{55} & \ell_{56}\\
\ell_{61} & \ell_{62} & \ell_{63} & \ell_{64} & \ell_{65} & \ell_{66}
\end{pmatrix}
\end{array}\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\\ \chi\\ \xi\end{pmatrix}$$

$$=\begin{pmatrix}
\ell_{11}\alpha+\ell_{12}\beta+\ell_{13}\gamma+\ell_{14}\delta+\ell_{15}\chi+\ell_{16}\xi\\
\ell_{21}\alpha+\ell_{22}\beta+\ell_{23}\gamma+\ell_{24}\delta+\ell_{25}\chi+\ell_{26}\xi\\
\ell_{31}\alpha+\ell_{32}\beta+\ell_{33}\gamma+\ell_{34}\delta+\ell_{35}\chi+\ell_{36}\xi\\
\ell_{41}\alpha+\ell_{42}\beta+\ell_{43}\gamma+\ell_{44}\delta+\ell_{45}\chi+\ell_{46}\xi\\
\ell_{51}\alpha+\ell_{52}\beta+\ell_{53}\gamma+\ell_{54}\delta+\ell_{55}\chi+\ell_{56}\xi\\
\ell_{61}\alpha+\ell_{62}\beta+\ell_{63}\gamma+\ell_{64}\delta+\ell_{65}\chi+\ell_{66}\xi
\end{pmatrix}=\begin{array}{cc}\begin{pmatrix}m_1\\ m_2\\ m_3\\ m_4\\ m_5\\ m_6\end{pmatrix} & \begin{array}{c} u\,b\\ u+\\ u-\\ d\,b\\ d+\\ d-\end{array}\end{array}$$

To solve for the elements of L we begin with the two equations in formula (i).

$$L\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}=L\big|u\,b\big\rangle=\big|u+\big\rangle=\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}m_1\\m_2\\m_3\\m_4\\m_5\\m_6\end{pmatrix}:$$

$$0=m_1=\ell_{11}1+\ell_{12}0+\ell_{13}0+\ell_{14}0+\ell_{15}0+\ell_{16}0=\ell_{11}$$
$$1=m_2=\ell_{21}1+\ell_{22}0+\ell_{23}0+\ell_{24}0+\ell_{25}0+\ell_{26}0=\ell_{21}$$
$$0=m_3=\ell_{31}1+\ell_{32}0+\ell_{33}0+\ell_{34}0+\ell_{35}0+\ell_{36}0=\ell_{31}$$
$$\vdots$$
$$0=m_6=\ell_{61}1+\ell_{62}0+\ell_{63}0+\ell_{64}0+\ell_{65}0+\ell_{66}0=\ell_{61}$$

$\Rightarrow$ 1st column of L is 0, 1, 0, 0, 0, 0 $\overset{\text{(Hermitian)}}{\Rightarrow}$ 1st row of L is 0, 1, 0, 0, 0, 0 :

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \ell_{22} & \ell_{23} & \ell_{24} & \ell_{25} & \ell_{26} \\ 0 & \ell_{32} & \ell_{33} & \ell_{34} & \ell_{35} & \ell_{36} \\ 0 & \ell_{42} & \ell_{43} & \ell_{44} & \ell_{45} & \ell_{46} \\ 0 & \ell_{52} & \ell_{53} & \ell_{54} & \ell_{55} & \ell_{56} \\ 0 & \ell_{62} & \ell_{63} & \ell_{64} & \ell_{65} & \ell_{66} \end{pmatrix}$$

$$L\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = L\left|d\,b\right\rangle = \left|d\,-\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{pmatrix}:$$

$$0 = m_1 = \ell_{11}0 + \ell_{12}0 + \ell_{13}0 + \ell_{14}1 + \ell_{15}0 + \ell_{16}0 = \ell_{14}$$
$$\vdots$$
$$0 = m_5 = \ell_{51}0 + \ell_{52}0 + \ell_{53}0 + \ell_{54}1 + \ell_{55}0 + \ell_{56}0 = \ell_{54}$$
$$1 = m_6 = \ell_{61}0 + \ell_{62}0 + \ell_{63}0 + \ell_{64}1 + \ell_{65}0 + \ell_{66}0 = \ell_{64}$$

$\Rightarrow$ 4th column of L is 0, 0, 0, 0, 0, 1 $\overset{\text{(Hermitian)}}{\Rightarrow}$ 4th row of L is 0, 0, 0, 0, 0, 1 :

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \ell_{22} & \ell_{23} & 0 & \ell_{25} & \ell_{26} \\ 0 & \ell_{32} & \ell_{33} & 0 & \ell_{35} & \ell_{36} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \ell_{52} & \ell_{53} & 0 & \ell_{55} & \ell_{56} \\ 0 & \ell_{62} & \ell_{63} & 1 & \ell_{65} & \ell_{66} \end{pmatrix}$$

To determine additional elements of L we must define additional actions by L. When the on-off button is pushed a 2nd time we define the result to be “off” (i.e., a blank screen) while preserving the system state. That is, we define

$$L\left|u\,+\right\rangle = \left|u\,b\right\rangle$$
$$L\left|d\,-\right\rangle = \left|d\,b\right\rangle \qquad \text{(iii)}$$

$$L\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}=L\left|u\,+\right\rangle=\left|u\;b\right\rangle=\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}:$$

$$\begin{aligned}
1=m_1&=\ell_{11}0+\ell_{12}1+\ell_{13}0+\ell_{14}0+\ell_{15}0+\ell_{16}0=\ell_{12}\\
0=m_2&=\ell_{21}0+\ell_{22}1+\ell_{23}0+\ell_{24}0+\ell_{25}0+\ell_{26}0=\ell_{22}\\
&\;\vdots\\
0=m_6&=\ell_{61}0+\ell_{62}1+\ell_{63}0+\ell_{64}0+\ell_{65}0+\ell_{66}0=\ell_{62}
\end{aligned}$$

$\Rightarrow$ 2nd column of L is 1, 0, 0, 0, 0, 0 $\overset{\text{(Hermitian)}}{\Rightarrow}$ 2nd row of L is 1, 0, 0, 0, 0, 0 :

$$L=\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&\ell_{33}&0&\ell_{35}&\ell_{36}\\0&0&0&0&0&1\\0&0&\ell_{53}&0&\ell_{55}&\ell_{56}\\0&0&\ell_{63}&1&\ell_{65}&\ell_{66}\end{pmatrix}$$

$$L\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}=L\left|d\,-\right\rangle=\left|d\;b\right\rangle=\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}:$$

$$\begin{aligned}
0=m_1&=\ell_{11}0+\ell_{12}0+\ell_{13}0+\ell_{14}0+\ell_{15}0+\ell_{16}1=\ell_{16}\\
0=m_2&=\ell_{21}0+\ell_{22}0+\ell_{23}0+\ell_{24}0+\ell_{25}0+\ell_{26}1=\ell_{26}\\
0=m_3&=\ell_{31}0+\ell_{32}0+\ell_{33}0+\ell_{34}0+\ell_{35}0+\ell_{36}1=\ell_{36}\\
1=m_4&=\ell_{41}0+\ell_{42}0+\ell_{43}0+\ell_{44}0+\ell_{45}0+\ell_{46}1=\ell_{46}\\
0=m_5&=\ell_{51}0+\ell_{52}0+\ell_{53}0+\ell_{54}0+\ell_{55}0+\ell_{56}1=\ell_{56}\\
0=m_6&=\ell_{61}0+\ell_{62}0+\ell_{63}0+\ell_{64}0+\ell_{65}0+\ell_{66}1=\ell_{66}
\end{aligned}$$

$\Rightarrow$ 6th column of L is 0, 0, 0, 1, 0, 0 $\overset{\text{(Hermitian)}}{\Rightarrow}$ 6th row of L is 0, 0, 0, 1, 0, 0 :

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ell_{33} & 0 & \ell_{35} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \ell_{53} & 0 & \ell_{55} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The four elements yet to be defined actually don't matter since the states $\left| \, u \quad - \right\rangle$ and $\left| \, d \quad + \right\rangle$ can never be obtained. The simplest solution is to define elements (3,3) and (5,5) to be 1 and (3,5) and (5,3) to be 0. That is, starting in either of these states means staying in the same state.

That completes the construction of a Hermitian matrix for L:

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Returning to entanglement and collapse, at first the apparatus knows the spin state and is entangled, but Alice has not yet looked at the display, and so she is not yet entangled. After she looks at the apparatus, she becomes entangled, and from her perspective the spin wave function has collapsed. From Bob's perspective the system has not collapsed; it is just a 3-way entangled system (spin, apparatus, Alice).

Thus, an entangled entity observes a particle while an unentangled entity experiences a wave. As in Special Relativity where there is no contradiction that two observers see a different speed for an object, there is no contradiction here that entangled Alice sees a particle yet unentangled Bob sees a wave.

We next further clarify this in terms of locality.

Section 7.9 Entanglement and Locality

Definition. A system is **local** if no information can be sent faster than light.

Extend Alice's system A to contain her measuring apparatus and Alice herself, and the same for Bob's system B. Let $\{|a\rangle\}$ be a basis for A and $\{|b\rangle\}$ be a basis for B. Then $\{|ab\rangle\}$ is a basis for AB. Let $\{\psi(ab)\}$ be the composite system wave function, possibly entangled:

$$|\Psi\rangle = \left(\ \psi(ab)\ \right) = \begin{pmatrix} \psi(11) \\ \psi(12) \\ \psi(21) \\ \psi(22) \end{pmatrix}.$$

Alice's complete description of her system is contained in her density matrix ρ:

$$\rho_{aa'} = \sum_b \psi^*(a'b)\,\psi(ab) \qquad (7.31)$$

We will show there is nothing Bob can do to instantly change ρ, thus proving that even if AB is maximally entangled, no faster-than-light signal can be sent by Bob to change Alice's system. That is, quantum mechanics does not violate "locality".

Bob's system, including any changes he makes, must be described by a unitary matrix:

$$U_B = \left(\ u_{bb'}\ \right) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

The corresponding unitary matrix for the composite system is

$$U = I_A \otimes U_B = \begin{pmatrix} 1\otimes U_B & 0\otimes U_B \\ 0\otimes U_B & 1\otimes U_B \end{pmatrix} = \left(\begin{array}{cc|cc} u_{11} & u_{12} & & \\ u_{21} & u_{22} & \multicolumn{2}{c}{0} \\ \hline & & u_{11} & u_{12} \\ \multicolumn{2}{c|}{0} & u_{21} & u_{22} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \left(u_{bb'}\right) & 0 \\ \hline 0 & \left(u_{bb'}\right) \end{array}\right).$$

When U acts on the initial wave function it creates a new, or final, wave function

$$\left|\Psi_f\right\rangle = U\left|\Psi\right\rangle = \left(\begin{array}{cc|cc} u_{11} & u_{12} & & \\ u_{21} & u_{22} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & u_{11} & u_{12} \\ & & u_{21} & u_{22} \end{array}\right)\left(\begin{array}{c} \psi(11) \\ \psi(12) \\ \hline \psi(21) \\ \psi(22) \end{array}\right)$$

$$= \left(\begin{array}{c} u_{11}\psi(11)+u_{12}\psi(12) \\ u_{21}\psi(11)+u_{22}\psi(12) \\ \hline u_{11}\psi(21)+u_{12}\psi(22) \\ u_{21}\psi(21)+u_{22}\psi(22) \end{array}\right) = \left(\sum_{b'} u_{bb'}\psi(ab')\right) \equiv \left(\psi_f(ab)\right). \qquad (7.32)$$

Changing $b' \to b''$ and $a \to a'$, the corresponding bra vector is

$$\left\langle\Psi_f\right| = \left(\sum_{b''}\psi^*(a'b'')u^*_{bb''}\right) = \left(\psi^*_f(a'b)\right). \qquad (7.33)$$

Also

$$U^\dagger_{\mathrm{B}} = \left(u^*_{b''b}\right) = \begin{pmatrix} u^*_{11} & u^*_{21} \\ u^*_{12} & u^*_{22} \end{pmatrix}.$$

So

$$\left(\delta_{b'b''}\right) = I = U^\dagger_{\mathrm{B}}U = \begin{pmatrix} u^*_{11} & u^*_{21} \\ u^*_{12} & u^*_{22} \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} u^*_{11}u_{11}+u^*_{21}u_{21} & u^*_{11}u_{12}+u^*_{21}u_{22} \\ u^*_{12}u_{11}+u^*_{22}u_{21} & u^*_{12}u_{12}+u^*_{22}u_{22} \end{pmatrix} = \left(\sum_b u^*_{bb''}u_{bb'}\right). \qquad (7.34)$$

Then Alice's final density function is

$$\rho^f_{aa'} \overset{(7.31)}{=} \sum_b \psi^*_f(a'b)\psi_f(ab) \overset{(7.32,\ 7.33)}{=} \sum_b\sum_{b''}\sum_{b'}\psi^*(a'b'')u^*_{bb''}u_{bb'}\psi(ab')$$

$$= \sum_{b''}\sum_{b'}\psi^*(a'b'')\left[\sum_b u^*_{bb''}u_{bb'}\right]\psi(ab') \overset{(7.34)}{=} \sum_{b''}\sum_{b'}\psi^*(a'b'')\delta_{b'b''}\psi(ab')$$

$$= \sum_{b''} \psi^*(a'b'')\,\psi(ab'') \overset{(b''\to b)}{=} \sum_{b} \psi^*(a'b)\,\psi(ab)$$

$$\overset{(7.31)}{=} \rho_{aa'} \quad ✔$$

So nothing Bob does affects Alice's density function.

Section 7.10 Bell's Theorem

Susskind does not state Bell's Theorem. He argues that it violates a different definition of locality (that he also does not state).

Example 1. (Waves do not have definite properties, not even hidden properties, until they collapse – i.e., Quantum Mechanics example). Entangle 2 particles to have opposite spins. Let Alice and Bob make repeated spin measurements, Alice in the directions 0°, 120°, and -120°, Bob in the opposite directions 180°, 300°, and 60°, each choosing directions randomly. Compare their first measurements, their 2nd measurements, their 3rd measurements, etc. Find the expectation that compared spins have opposite signs.

Solution. First, observe that for any pair of opposite directions, $P_A(+1) = P_B(-1)$.

Suppose Alice measures +1 in one of her directions. When Bob randomly chooses his direction, there is a 1/3 chance he picks the opposite direction of Alice and 2/3 chance he picks a different direction.

If he chooses the opposite direction, then his result is -1 since the particles are oppositely entangled. That is, his conditional probability of picking the opposite spin is

$$\text{P}(\text{Bob measures an opposite spin}) = \text{P}_\text{B}(-1) = 1.$$

If he chooses a different direction, then we use (1.02) to compute Alice's spin probability in her opposite direction:

$$\text{P}_\text{A}(+1) \overset{(1.02)}{=} \mathbf{cos}^2 \frac{120°}{2} = \frac{1}{4}.$$

So, Bob's conditional probability is

$$\text{P}(\text{Bob measures an opposite spin}) = \text{P}_\text{B}(-1) = \text{P}_\text{A}(+1) = \frac{1}{4}.$$

Thus,

$$\text{E}(\text{Opposite spins}) = \frac{1}{3}(1) + \frac{2}{3}\frac{1}{4} = \frac{1}{2}.$$

The calculation is the same if Alice measures -1 in her direction. Thus,

E(Opposite spins) = $\frac{1}{2}$. ■

In the next example, the doors represent directions, the color blue represents +1 spin, and red represents -1 spin.

Example 2. (Waves have definite properties – i.e., Classical Mechanics example). Suppose there are 3 doors and behind each door are 2 colored balls. Two of the doors have a blue ball in front and a red behind, and the other door has a red ball in front and a blue ball behind. Alice and Bob each randomly select a door. Alice always selects the ball in front and Bob always selects the ball in back. Find the expectation that the balls selected are different colors.

Solution.

Event 1	**P(Event 1)**	**Event 2**	**P(Event 2)**	**P(Opp colors)**	**Contribution to E(Outcome)**
Alice picks Blue	2/3	Bob picks same door	1/3	1	$\frac{2}{3}\frac{1}{3}(1)=\frac{2}{9}$
Alice picks Blue	2/3	Bob picks diff door	2/3	1/2	$\frac{2}{3}\frac{2}{3}\frac{1}{2}=\frac{2}{9}$
Alice picks Red	1/3	Bob picks same door	1/3	1	$\frac{1}{3}\frac{1}{3}(1)=\frac{1}{9}$
Alice picks Red	1/3	Bob picks diff door	2/3	0	$\frac{1}{3}\frac{2}{3}(0)=0$

E(Opposite spins) = E(opposite colors) $=\frac{2}{9}+\frac{2}{9}+\frac{1}{9}+0=\frac{5}{9}$ ■

The issue being investigated in examples 1 and 2 above is whether or not entangled particles have a definite spin prior to being measured. Einstein and Bohr famously argued about this but since the calculated result of a single measurement is the same under either assumption, they thought this to be an unresolvable disagreement. However, years later Bell generated an inequality based upon examples similar to the two above. His inequality shows there is indeed a measurable consequence for each assumption.

Suppose the particles do have a definite spin prior to measurement. If the front balls are all blue (or all red) then E(opposite color) = 1. Suppose there are balls of both colors. There must be 2 blue balls and 1 red, or vice-versa. Example 2 proves that E(opposite spin) = $\frac{5}{9}$. Thus, no matter what, E(opposite spin) $\geq \frac{5}{9}$.

However, if the particles have no definite spin until the moment of measurement, then Example 1, which uses QM calculations, predicts that E(opposite spin) = $\frac{1}{2}$.

Theorem (Bell's Inequality). Two particles are entangled with opposite spin. They are repeatedly measured along axes chosen randomly and independently from 3 different directions. If they have definite spins (even if they are hidden values that we cannot access), then E(opposite spin) $> \frac{1}{2}$.

Years after Bell developed his famous theorem, the technology became available to make a large number of spin measurements. The measured result was ½. Thus, it was found that particles do not have definite spin until the moment a measurement is made.

Example 3. (Single spin measurement). A classical computer can simulate quantum spin.

Solution. We assume as usual there is an apparatus with a Measure Button and a Display Window that can be placed along any axis in 3-space. We imagine a classical computer that initially stores 2 complex numbers in memory, α_u and α_d that are normalized as usual by $\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$. The computer solves the Schrödinger equation to update the α's. The computer also stores a unit vector representing the apparatus' 3D orientation. Finally, the computer stores +1 or -1 and displays the number in the Display Window.

Alice selects an orientation and pushes the Measure Button on the apparatus. The computer uses a random number generator to output +1 or -1 with probabilities $\alpha_u^*\alpha_u$ and $\alpha_d^*\alpha_d$, respectively. Then the Schrödinger equation takes over until the apparatus is (possibly) re-oriented and the button is pushed again.

There is no known experiment that Alice can perform to distinguish that the computer is not a quantum computer. ■

Example 4. (Multiple measurements). A classical computer can also simulate a pair of separated, entangled particles assuming information from the computer travels instantaneously to each apparatus.

Solution. Denote the composite system by

$$\left|\Psi\right\rangle = \psi_{uu}\left|uu\right\rangle + \psi_{ud}\left|ud\right\rangle + \psi_{du}\left|du\right\rangle + \psi_{dd}\left|dd\right\rangle.$$

Let L_A and L_B be Alice's and Bob's respective measurement operators. Then $\mathrm{P}_{L_A}(a) \overset{(7.22)}{=} \rho_{aa} \overset{(7.17)}{=} \sum_b \psi^*_{ab}\psi_{ab}$ and similarly for Bob $\mathrm{P}_{L_B}(b) = \sum_a \psi^*_{ab}\psi_{ab}$. For example, $\mathrm{P}_{L_A}(u) = \psi^*_{uu}\psi_{uu} + \psi^*_{ud}\psi_{ud}$.

The computer initially stores 4 complex numbers: ψ_{uu}, ψ_{ud}, ψ_{du}, ψ_{dd} and it updates them using the Schrödinger equation for the combined system. Alice's Display Window shows only her spin (+1 or -1) as does Bob's. Each apparatus can be independently oriented and each has its own Measure Button. When Alice pushes her button, the composite computer uses a random number generator to instantly display +1 or -1 on Alice's Display Window according to the probabilities $\mathrm{P}_{L_A}(u)$ and $\mathrm{P}_{L_A}(d)$, and similarly for Bob.

As in Example 3, there is no known experiment that Alice or Bob can perform to distinguish that this is not a quantum computer. ■

The computer in the above example is non-local because it sends signals faster than the speed of light. Yet this system is *local* with respect to our definition of locality because even instantaneous information from Bob about his wave function cannot affect Alice's density function (which encapsulates everything she can know about her system). This is the argument that Susskind makes to support his claim that Bell's Theorem (and also the Einstein-Bohr argument) was about a computer simulation and not about the real world. He does not explicitly give his alternate definition of "simulation locality" nor does he describe how it relates to Bell Theorem (which he doesn't state).

Example 5. (Multiple measurements). A classical computer cannot simulate a pair separated, entangled particles if information from the computer travels to each apparatus at less than or equal to the speed of light. Thus, the combined system is local with respect to definition 2.

Solution. First of all, if the computer is situated, say, with Bob, then the slower-than-light updating of her Display Window is a clear giveaway to Alice that the computer is classical.

But perhaps part of the composite computer resides with Alice and part with Bob. They can now see instantaneous results in their Display Windows.

As usual let Alice's and Bob's system states be $\left|\Psi_A\right\rangle = \psi_u\left|u\right\rangle + \psi_d\left|d\right\rangle$ and $\left|\Phi_B\right\rangle = \phi_u\left|u\right\rangle + \phi_d\left|d\right\rangle$, respectively. Alice's part of the computer can only generate

ψ_u and ψ_d and, similarly, Bob's part can only generate ϕ_u and ϕ_d. The composite computer cannot instantly determine ψ_{uu}, ψ_{ud}, ψ_{du}, and ψ_{dd} as would be necessary if, say, the spins were entangled in the singlet state (opposite spins). The computer can only use ψ_u and ψ_d for Alice and ϕ_u and ϕ_d for Bob.
So, how do Alice and Bob figure out the computer is not a quantum computer? Suppose that Alice and Bob make a large number of measurements, independently and randomly choosing from directions 0°, 120°, and -120°. Since the spins have distinct values, according to Bell's Theorem, E(opp spin) > $\frac{1}{2}$.
But, also according to Bell's theorem, a true quantum computer would yield E(opp spin) = $\frac{1}{2}$. So, Alice and Bob are able to determine that the computer is not a quantum computer. (They could also just compare their spins and observe that more than half of them are not opposites.) ■

Chapter 8. Particles and Waves

TABLE 8.1

Concept	Discrete	Continuous
Observable L	Discrete set of outcomes (eigenvectors) $\{\,\lvert\lambda\rangle\,\}$	Continuous set of outcomes (eigenvectors) $\{\,\lvert x\rangle\,\}$
Vector Space of States	Finite dimensional complex vector space	Vector space of complex-valued functions f
State Vector	$\lvert\Psi\rangle = \sum_{\lambda}\psi(\lambda)\lvert\lambda\rangle$ (8.1)	$\lvert\Psi\rangle = \int_x \psi(x)\,dx$ (A)
Bra Vector	$\langle\Psi\rvert = \sum_{\lambda}\langle\lambda\rvert\psi^*(\lambda)$	$\langle\Psi\rvert = \int_x \psi^*(x)\,dx$
Wave Function	$\{\,\psi(\lambda)\,\}$	$\{\,\psi(x)\,\}$
Probability	$\mathrm{P}(\lambda) = \psi^*(\lambda)\,\psi(\lambda)$ (3.11)	$\mathrm{P}(a,b) = \int_a^b p(x)\,dx = \int_a^b \psi^*(x)\,\psi(x)\,dx$ (B)
Inner Product	$\langle\Psi\vert\Phi\rangle \overset{(1.14)}{=} \sum_{\lambda}\psi^*(\lambda)\,\phi(\lambda)$	$\langle\Psi\vert\Phi\rangle = \int_x \psi^*(x)\,\phi(x)\,dx$ (8.2)
Normalization	$\sum_{\lambda}\psi^*(\lambda)\,\psi(\lambda) = 1$	$\int_{-\infty}^{\infty}\psi^*(x)\,\psi(x)\,dx = 1$ (8.3)
Kronecker Delta & Dirac Delta Functions	$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$	δ is the function that satisfies $\int_{-\infty}^{\infty}\delta(x-x')F(x')\,dx' = F(x)$ (8.4) for every continuous function F having compact support
Integration By Parts	$\int_a^b F\,dG = FG\big\rvert_a^b - \int_a^b G\,dF$	$\int_{-\infty}^{\infty} F\frac{dG}{dx}dx = -\int_{-\infty}^{\infty} G\frac{dF}{dx}dx$ (F) for wave functions F and G
Basis Vector	$\lvert\lambda\rangle = \sum_{\lambda'}\delta_{\lambda\lambda'}\lvert\lambda'\rangle$ $\langle\lambda\rvert = \sum_{\lambda'}\langle\lambda'\rvert\delta_{\lambda\lambda'}$ (G)	$\lvert x\rangle = \int_{x'=-\infty}^{\infty}\delta(x-x')\,dx'$ $\langle x\rvert = \int_{x'=-\infty}^{\infty}\delta(x-x')\,dx'$ (H)
Inner Product with a Basis Vector	$\langle\lambda\vert\Psi\rangle = \psi(\lambda)$ (1.5)	$\langle x\vert\Psi\rangle = \psi(x)$ (8.13)

Quantum Mechanics is not so much about particles and waves as it is about the set of non-classical principles given in Chapter 3 that govern their behavior. We now extend the principles and concepts from the discrete systems we have so far studied to continuous systems (where we will at last develop wave examples.)

Table 8.1 provides a side-by-side overview of the discrete and continuous cases. It also includes key equation numbers, both mine and the book's. For easier comparison, for the discrete case I have changed the notation from ψ_λ to $\psi(\lambda)$. The sections below provide clarification and detail.

In Table 8.1, the values *x* could be points in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \cdots$ or even other types of more abstract spaces. For this book, however, it is sufficient to consider just the case where $x \in \mathbb{R}$.

Vector Space

It is easy to show that all the conditions of a vector space hold if we define addition of two functions to be $(f+g)(x) = f(x) + g(x)$ and scalar multiplication by a complex number to be $(\alpha f)(x) = \alpha\left[\, f(x)\,\right]$.

Probability

For the discrete case, if $\{\,|\lambda\rangle\,\}$ is a basis for the state $|\Psi\rangle$, then

$$|\Psi\rangle = \sum_\lambda \psi(\lambda)|\lambda\rangle \tag{8.1}$$

and probability is determined by

$$\mathrm{P}(\lambda) = \psi^*(\lambda)\psi(\lambda). \tag{3.11}$$

For the continuous case there are both a probability density function and a cumulative distribution function. If $\{\,|x\rangle\,\}$ is a basis for the state $|\Psi\rangle$, then

$$|\Psi\rangle = \int_x \psi(x)dx. \tag{A}$$

The density function is

$$p(x) = \psi^*(x)\psi(x). \tag{B1}$$

The cumulative distribution function can act on a finite or infinite interval:

$$\mathrm{P}(a,b) = \int_a^b p(x)\,dx = \int_a^b \psi^*(x)\,\psi(x)\,dx. \tag{B2}$$

Inner Product

Let discrete $\left|\Phi\right\rangle = \sum_{\lambda} \phi(\lambda)\left|\lambda\right\rangle$ and continuous $\left|\Phi\right\rangle = \int_{x} \phi(x)dx$.

The discrete form of the inner product is

$$\left\langle \Psi \middle| \Phi \right\rangle = \sum_{\lambda} \psi^*(\lambda)\phi(\lambda). \tag{1.14}$$

The continuous form is

$$\left\langle \Psi \middle| \Phi \right\rangle = \int_{x} \psi^*(x)\phi(x)dx. \tag{8.2}$$

Dirac Delta

Definition. A function *F* has **compact support** if there is some closed interval [a,b] outside of which *F*(x) = 0.

In this book it is assumed that all wave functions are continuous and have compact support.

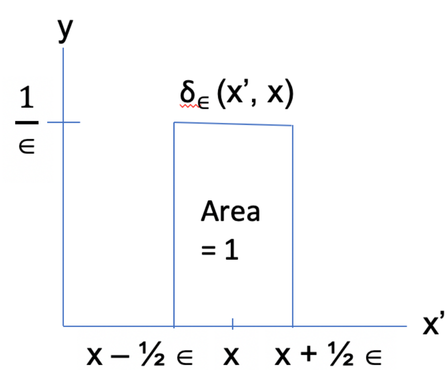
Intuitively the Dirac delta function $\delta(x - x')$ is a density function at some value *x*. In physics it is often defined informally as the function δ that satisfies

$$\int_{-\infty}^{\infty} \delta(x - x')F(x')dx' = F(x) \tag{8.4}$$

for every continuous function *F* having compact support.

The reason this definition is "informal" is that in fact there is no such function. We give a more rigorous definition shortly.

The Dirac delta function can be visualized as the limit of a set of shrinking step functions of increasing height as follows.



For $\in > 0$ define step function

$$\delta_{\in}(x', x) = \begin{cases} \dfrac{1}{\in} & \text{if } x' \in \left(x - \dfrac{1}{2}\in, x + \dfrac{1}{2}\in\right) \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\int_{-\infty}^{\infty} \delta(x - x')dx' = \lim_{\in \to 0} \int_{-\infty}^{\infty} \delta_{\in}(x', x)dx' \tag{C}$$

It is understood that the limit and integral sign cannot be freely interchanged. This, too, doesn't make sound mathematical sense since it violates Fubini's theorem. Even worse, we do make the interchange when necessary or convenient.

It follows immediately from (C) that

$$\int_{-\infty}^{\infty} \delta\left(x - x'\right) dx' = 1: \qquad \text{(D)}$$

$$\int_{-\infty}^{\infty} \delta\left(x - x'\right) dx' \overset{\text{(C)}}{=} \lim_{\in \to 0} \int_{-\infty}^{\infty} \delta_{\in}\left(x', x\right) dx' = \lim_{\in \to 0} 1 = 1 \quad ✔$$

and

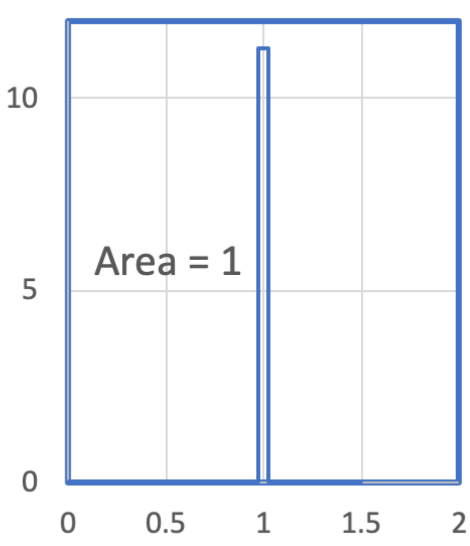
$$\delta\left(x - x'\right) = \begin{cases} \infty & \text{if } x' = x \\ 0 & \text{otherwise} \end{cases} : \qquad \text{(E)}$$

$$\delta\left(x - x'\right) \overset{\text{(C)}}{=} \lim_{\in \to 0} \delta_{\in}\left(x', x\right) = \begin{cases} \infty & \text{if } x' = x \\ 0 & \text{otherwise} \end{cases} . \quad ✔$$

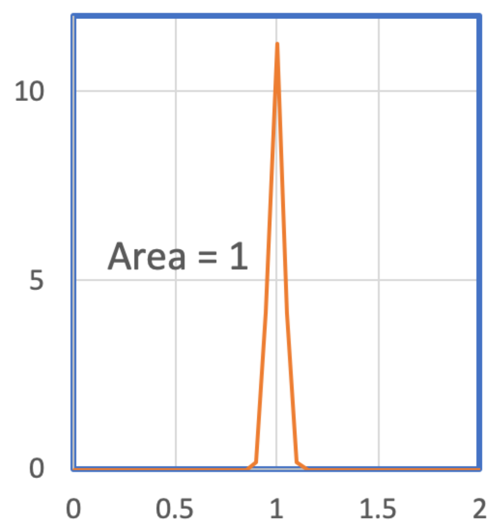
When referring to equation (E) in light of (D), we informally say that δ **approaches infinity at a rate that keeps the area under the curve $\delta\left(x - x'\right)$ at unity.**

Since, by (E), $\delta\left(x - x'\right)$ can equal infinity, δ is not a (real-valued) function. This is part of why the definition (C) does not make precise mathematical sense.

Thie informal definition (C) could also have been made in terms of the normal distribution density function $\frac{n}{\sqrt{\pi}} e^{-\left(n\left[x' - x\right]\right)^2}$ instead of $\delta_{\in}\left(x', x\right)$. To see this, let $\in = \frac{\sqrt{\pi}}{n}$ and compare $\delta_{\in}\left(x', x\right) = \delta_{\frac{\sqrt{\pi}}{n}}\left(x', x\right)$ to $\frac{n}{\sqrt{\pi}} e^{-\left(n\left[x' - x\right]\right)^2}$. The graphs below show the comparison for x = 1 and n = 20.



$$\delta_{\frac{\sqrt{\pi}}{20}}\left(x',1\right)$$



$$\frac{20}{\sqrt{\pi}} e^{-\left(20\left[x'-1\right]\right)^2}$$

Unfortunately, the fact that the above definitions are informal and mathematically incorrect makes proving some of its properties difficult. See, for example, the argument below to normalize ψ_p in equation (8.17).

A mathematically rigorous definition of δ involves the concept of measure, specifically Lebesgue measure. The Dirac delta "function" is not a function at all but a measure. The argument of a measure, unlike a function, is a set. The Dirac delta function $\delta(x)$ is defined for any measurable set containing *zero* to be 1, and to be 0 for any other measurable set. The Lebesgue integral with respect to δ satisfies $\int_{x=-\infty}^{\infty} F(x)\delta\{dx\} = F(0)$ or, by translation, $\int_{x'=-\infty}^{\infty} F(x-x')\delta\{dx'\} = F(x)$ for any continuous function *F* with compact support. Note that the argument of δ is a set.

There is a cumulative distribution function associated with the density function δ:

$$\Delta(x) \equiv \int_{-\infty}^{\infty} 1_{(-\infty,x]}\,\delta(dx) \quad \text{where} \quad 1_{(-\infty,x]} = \begin{cases} 1 & \text{if } x' \le x \\ 0 & \text{otherwise} \end{cases}.$$

So, δ is the derivative of Δ but only in the sense that $\int_{-\infty}^{\infty} \delta\{dx\} = \int_{-\infty}^{\infty} d\Delta(x)$ where these integrals are understood to be Riemann-Stieltjes integrals.

Integration By Parts

We are able to drop the term $FG|_a^b$ because wave functions are zero at $\pm\infty$:

$$\int_{-\infty}^{\infty} F \frac{dG}{dx} dx = -\int_{-\infty}^{\infty} G \frac{dF}{dx} dx \qquad \text{(F)}$$

Inner Product with a Basis Vector

Consider $\langle x|\Psi\rangle$. When $|x\rangle$ is one of the basis vectors used in the definition of $|\Psi\rangle$, the inner product takes a simpler form:

$$\langle x|\Psi\rangle = \psi(x) \qquad (8.13)$$

This equation is needed in the derivation of equation (8.18), which Susskind leaves to the reader.

While equation (8.13) is perhaps an obvious generalization of the discrete equation (1.5) shown in Table 8.1, Susskind mentions that (8.13) holds because $|x\rangle$ is represented by the Dirac delta function. This is worth showing and adds a little more insight into (8.13).

In order to have a discrete process to imitate, we write the discrete basis vector in terms of the Kronecker delta:

$$\langle\lambda| = \sum_{\lambda'} \langle\lambda'|\delta_{\lambda\lambda'} . \qquad \text{(G)}$$

Then,

$$\begin{aligned}\langle\lambda|\Psi\rangle &\overset{\text{(G, 8.1)}}{=} \sum_{\lambda'} \langle\lambda'|\delta_{\lambda\lambda'} \sum_{\lambda''} \psi(\lambda'')|\lambda''\rangle = \sum_{\lambda'} \delta_{\lambda\lambda'} \sum_{\lambda''} \psi(\lambda'')\langle\lambda'|\lambda''\rangle \\ &= \sum_{\lambda'} \delta_{\lambda\lambda'}\psi(\lambda') = \psi(\lambda) \ .\end{aligned} \qquad (1.5)$$

In the continuous version of (G), the Kronecker delta is replaced by the Dirac delta and takes the form

$$\langle x| = \int_{x'} \delta(x - x')dx' . \qquad \text{(H)}$$

The continuous case of Equation (1.5) is Equation (8.2), which can be rewritten

$$\langle\Phi|\Psi\rangle = \int_{x'} \phi^*(x')\psi(x')dx' . \qquad \text{(I)}$$

Replacing $\langle\Phi| = \int \phi^*(x')dx'$ by (H) on both sides of (I) yields (8.13):

$$\langle x|\Psi\rangle \overset{\text{(I)}}{=} \int_{x'} \delta^*(x - x')\psi(x')dx' \overset{(8.4)}{=} \psi(x). \quad ✔$$

In other words, the wave function, $\psi(x)$, of a particle at position x is the projection of a state vector $|\Psi\rangle$ onto the eigenvector of position.

Linear Operators

Recall that observables are represented by Hermitian linear operators. By definition, L is a **linear operator** if $L(\alpha x + \beta y) = \alpha Lx + \beta Ly$.

Recall that L is **Hermitian** if $L^\dagger = L \overset{(3.06)}{\Leftrightarrow} \langle \Psi | L | \Phi \rangle = \langle \Phi | L | \Psi \rangle^* \quad \forall\, |\Psi\rangle, |\Phi\rangle$.

In the two examples below, ψ is a complex-valued function of a real variable x.

Example 1. **Multiplication Operator**: $X\,\psi(x) \equiv x\psi(x)$ (8.5)

$$\langle \Psi | X | \Phi \rangle = \int \psi^*(x)\, x\, \phi(x)\, dx$$

$$\langle \Phi | X | \Psi \rangle = \int \phi^*(x)\, x\, \psi(x)\, dx$$

$$\langle \Phi | X | \Psi \rangle^* = \int \phi(x)\, x^*\, \psi^*(x)\, dx = \int \phi(x)\, x\, \psi^*(x)\, dx = \langle \Psi | X | \Phi \rangle$$

(because $x^* = x$ since $x \in \mathbb{R}$).

Thus, by Theorem (3.06), X is Hermitian.

Example 2. **Differentiation Operator**: $D\,\psi(x) \equiv \dfrac{d\,\psi(x)}{dx}$ (8.6)

$$\langle \Psi | D | \Phi \rangle = \int \psi^*(x) \frac{d\,\phi(x)}{dx} dx \tag{8.7}$$

$$\langle \Phi | D | \Psi \rangle = \int \phi^*(x) \frac{d\,\psi(x)}{dx} dx \overset{(F)}{=} -\int \psi(x) \frac{d\phi^*(x)}{dx} dx = -\langle \Psi | D | \Phi \rangle^*$$

Thus, D is anti-Hermitian. But, we next show that this means that both iD and $-iD$ are Hermitian. In particular, $i\hbar D$ is Hermitian.

Theorem. If M is anti-Hermitian, then both iM and $-iM$ are Hermitian.

Proof. Let $z, w \in \mathbb{C}$. Then $(zw)^* = z^* w^*$. In particular,

$$(iz)^* = i^* z^* = -iz^* \tag{i}$$

$$(-iz)^* = (-i)^* z^* = iz^* \tag{ii}$$

Let $M=\left(m_{jk}\right)$ be an $\left(nxn\right)$-complex matrix. Then

$$\left[iM\right]^{\dagger}=\left(im_{jk}\right)^{T^{*}}=\left(im_{kj}\right)^{*}=\left(\left[im_{kj}\right]^{*}\right)\overset{(i)}{=}\left(-im_{kj}^{*}\right)=-iM^{\dagger} \qquad \text{(iii)}$$

$$\left[-iM\right]^{\dagger}=\left(-im_{jk}\right)^{T^{*}}=\left(-im_{kj}\right)^{*}=\left(\left[-im_{kj}\right]^{*}\right)\overset{(ii)}{=}\left(im_{kj}^{*}\right)=iM^{\dagger} \qquad \text{(iv)}$$

Suppose M is anti-Hermitian:

$$M^{\dagger}=-M \qquad \text{(v)}$$

Then

$\left[iM\right]^{\dagger}\overset{(iii)}{=}-iM^{\dagger}\overset{(v)}{=}iM$. That is, iM is Hermitian. ✔

$\left[-iM\right]^{\dagger}\overset{(iv)}{=}iM^{\dagger}\overset{(v)}{=}-iM$. That is, -$iM$ is Hermitian. ✔ ■

Eigenvectors and Eigenvalues of the Multiplication Operator: Position

The multiplication operator X is the Hermitian operator that represents the position observable. That is, a position is simply a number on the x-axis, and multiplication by any real number is possible. So, an outcome of X is a real number, a position.

Let $\left|\Psi\right\rangle$ be a state vector for the observable X, and x_0 an outcome. By Principle 2, an outcome is an eigenvalue of X. That is,

$$X\left|\Psi\right\rangle=x_0\left|\Psi\right\rangle.$$

By the definition (8.5) of X, the LHS is

$$X\left|\Psi\right\rangle\overset{(A)}{=}\int_{-\infty}^{\infty}X\,\psi\left(x\right)dx\overset{(8.5)}{=}\int_{-\infty}^{\infty}x\,\psi\left(x\right)dx.$$

The RHS is

$$x_0\left|\Psi\right\rangle\overset{(A)}{=}\int_{-\infty}^{\infty}x_0\,\psi\left(x\right)dx.$$

Thus,

$x\,\psi\left(x\right)=x_0\,\psi\left(x\right)$ a.e. (almost everywhere; i.e., except on a set of measure zero)

$$\Leftrightarrow \quad \left(x-x_0\right)\psi\left(x\right)=0 \text{ a.e.} \qquad (8.11)$$

This means that except on some set of measure zero, if $x\neq x_0$ then $\psi\left(x\right)=0$.

We seek a function ψ that has the property (8.11) and that also satisfies the normalization requirement:

$$\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx = 1. \tag{8.3}$$

We claim that the Dirac delta function meets these conditions. Let

$$\psi(x) = \delta(x_0 - x) = \begin{cases} \infty & \text{if } x = x_0 \\ 0 & \text{Otherwise} \end{cases} \tag{J}$$

If $x \neq x_0$, $\psi(x) = 0$. Thus, ψ satisfies (8.11). Also

$$\delta^2(x_0 - x) \overset{(J)}{=} \left\{ \begin{array}{ll} \infty^2 & \text{if } x = x_0 \\ 0 & \text{Otherwise} \end{array} \right\} = \delta(x_0 - x).$$

So, intuitively ψ satisfies (8.3):

$$\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx \overset{(J)}{=} \int_{-\infty}^{\infty} \delta^2(x_0 - x)\,dx = \int_{-\infty}^{\infty} \delta(x_0 - x)\,dx \overset{(D)}{=} 1,$$

This argument lacks rigor because we haven't shown that $\delta^2(x_0 - x)$ is just infinite enough that the area under the curve is unity.

Definition. We will refer to $\boldsymbol{\psi}(\boldsymbol{x})$ as the **wave function in the position representation**.

Eigenvectors and Eigenvalues of the Multiplication Operator: Momentum

We move on to the concept of momentum. The **momentum operator** is defined in terms of the differentiation operator:

$$\boldsymbol{P} = -i\hbar\boldsymbol{D}. \tag{8.14}$$

Plugging in the definition of D gives

$$P \overset{(8.6)}{=} -i\hbar\frac{d}{dx} \tag{8.15}$$

Note 1. The $\hbar$ factor is needed to provide units of momentum (mass times velocity).

Note 2. The symbols "P" and "p" are used to denote probability (CDF and density function). We use italics to denote momentum, "*P*" and "*p*" (momentum operator and momentum eigenvalue).

Experiment has shown that we cannot simultaneously measure position and momentum. A state

$$\left|\Psi\right\rangle = \int_x \psi(x)\, dx$$

can be expressed in term of either position or momentum. Of course, the momentum wave function will be different than the position wave function. If we label the momentum wave function $\tilde{\psi}(p)$, then the state $\left|\Psi\right\rangle$ written in terms of momentum is:

$$\left|\Psi\right\rangle = \int_p \tilde{\psi}(p)\, dp\,. \qquad \text{(K)}$$

Since $\left|p\right\rangle$ is one of the basis vectors in equation (K), formula 8.13 applies:

$$\left\langle p\middle|\Psi\right\rangle = \tilde{\psi}(p). \qquad (8.20)$$

We are now set to develop the equations that transform back and forth between $\psi(x)$ and $\tilde{\psi}(p)$.

Step 1. Since a momentum eigenvector $\left|p\right\rangle$ can be considered to be a state of *X*, it can be written in terms of the $\left|x\right\rangle$ basis. It will have its own wave function that we will call $\psi_p(x)$. That is,

$$\left|p\right\rangle = \int_x \psi_p(x)\, dx\,. \qquad \text{(L)}$$

In (8.2), we replace $\left\langle\Psi\right| = \int_x \psi(x)\, dx$ by $\left\langle x\right| \overset{\text{(H)}}{=} \int_{x'} \delta(x - x')\, dx'$ and $\left|\Phi\right\rangle$ by $\left|p\right\rangle \overset{\text{(L)}}{=} \int_{x'} \psi_p(x')\, dx'$ to get

$$\left\langle x\middle|p\right\rangle \overset{(8.2)}{=} \int_{-\infty}^{\infty} \delta(x - x')\psi_p(x')\, dx' \overset{(8.4)}{=} \psi_p(x). \qquad \text{(M)}$$

Step 2: Since $\left|p\right\rangle$ is an eigenvector of *P*,

$$P\left|p\right\rangle = p\left|p\right\rangle.$$

LHS: $P\left|p\right\rangle \overset{\text{(L)}}{=} \int_{-\infty}^{\infty} P\,\psi_p(x)\, dx \overset{(8.15)}{=} \int_{-\infty}^{\infty} (-i\hbar)\frac{d\psi_p(x)}{dx}\, dx\,.$

RHS: $p\left|p\right\rangle \overset{\text{(L)}}{=} \int_{-\infty}^{\infty} p\,\psi_p(x)\, dx\,.$

$$\therefore\; -i\hbar \frac{d\,\psi_p(x)}{dx} = p\,\psi_p(x) \text{ a.e.} \quad \Rightarrow \quad \frac{d\,\psi_p(x)}{dx} = \frac{i\,p}{\hbar}\,\psi_p(x) \text{ a.e.}$$

Solving the differential equation yields

$$\psi_p(x) = Ae^{\frac{i\,p\,x}{\hbar}} \quad \text{a.e.}$$

To solve for *A* we must apply normalization, but (8.3) doesn't work:

$$1 \overset{(8.3)}{=} \int_{-\infty}^{\infty} \psi_p^*(x)\,\psi_p(x)\,dx = \int_{-\infty}^{\infty} Ae^{-\frac{i\,p\,x}{\hbar}}\; Ae^{\frac{i\,p\,x}{\hbar}}\,dx = A^2 \int_{-\infty}^{\infty} dx = A^2\,(\infty) = \infty\,.$$

I believe the problem arises from the impreciseness of the informal Dirac delta definition. An approach that apparently works is to first assume that *x* is periodic with period $2\pi R$. This leads to discrete values p_n for *p* and use of the discrete normalization equation in Table 8.1, above. Then taking $R \to \infty$, discrete $p_n \to$ continuous *p*, and Kronecker $\delta \to$ Dirac δ, this finally leads to

$$A = \frac{1}{\sqrt{2\pi}}\,.$$

Thus,

$$\psi_p(x) = \frac{1}{\sqrt{2\pi}}\,e^{\frac{i\,p\,x}{\hbar}} \quad \text{a.e.} \tag{8.17}$$

Therefore

$$\begin{cases} \langle x|p\rangle \overset{(M)}{=} \psi_p(x) \overset{(8.17)}{=} \dfrac{1}{\sqrt{2\pi}}\,e^{\frac{i\,p\,x}{\hbar}} & \text{a.e.} \\ \langle p|x\rangle = \langle x|p\rangle^* = \dfrac{1}{\sqrt{2\pi}}\,e^{-\frac{i\,p\,x}{\hbar}} & \text{a.e.} \end{cases} \tag{8.18}$$

Step 3. For a discrete basis,

$$I \overset{(7.11)}{=} \sum_\lambda |\lambda\rangle\langle\lambda|\,. \tag{8.21}$$

The continuous versions of this are

$$I = \int_x |x\rangle\langle x|\,dx \tag{8.21}$$

and

$$I = \int_p |p\rangle\langle p|\,dp\,. \tag{8.22}$$

Thus,

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{ip}{\hbar}x} \psi(x)\, dx : \tag{8.24}$$

$$\tilde{\psi}(p) \overset{(8.20)}{=} \langle p|\Psi\rangle = \langle p|I|\Psi\rangle \overset{(8.21)}{=} \langle p|\int_{-\infty}^{\infty} |x\rangle\langle x|\,dx|\Psi\rangle$$

$$= \int_{-\infty}^{\infty} \langle p|x\rangle\langle x|\Psi\rangle\, dx \overset{(8.18,\ 8.13)}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{ip}{\hbar}x} \psi(x)\, dx \quad \blacksquare$$

Example. Alice measures the position wave form $\psi(x)$ of some particle and she wants to know the probability that she would have measured momentum p.

Solution. First, she calculates $\tilde{\psi}(p)$. Then $\mathrm{P}(p) \overset{(B1)}{=} \tilde{\psi}^*(p)\tilde{\psi}(p)$. $\blacksquare$

Solving for $\psi(x)$ in terms of $\tilde{\psi}(p)$ is very similar.

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_p e^{\frac{ip}{\hbar}x} \tilde{\psi}(p)\, dp : \tag{8.25}$$

$$\psi(x) \overset{(8.13)}{=} \langle x|\Psi\rangle = \langle x|I|\Psi\rangle \overset{(8.22)}{=} \langle x|\int_p |p\rangle\langle p|\,dp|\Psi\rangle$$

$$= \int_p \langle x|p\rangle\langle p|\Psi\rangle\, dp \overset{(8.18,\ 8.20)}{=} \frac{1}{\sqrt{2\pi}} \int_p e^{\frac{ip}{\hbar}x} \tilde{\psi}(p)\, dp$$

Equations (8.24) and (8.25) are **reciprocal Fourier transforms**, and are the central equations of Fourier analysis.

Heisenberg Uncertainty Principle

Let $\psi(x)$ be either a position or momentum wave function. By definition,

$$X\psi(x) \overset{(8.5)}{=} x\psi(x)$$

and

$$P\psi(x) \overset{(8.15)}{=} -i\hbar \frac{d\psi(x)}{dx}.$$

So,

$$[X,P]\psi(x) = XP\psi(x) - PX\psi(x) = -i\hbar x \frac{d\psi(x)}{dx} + i\hbar \frac{d}{dx}[x\psi(x)]$$

$$= -i\hbar \left[x\frac{d\psi(x)}{dx} - x\frac{d\psi(x)}{dx} - \psi(x) \right] = i\hbar\psi(x),$$

or

$$\left[X, P\right] = i\hbar \, . \tag{8.29}$$

In classical physics, $X\,P - P\,X = 0$, but not in quantum mechanics where we see from (8.29) that the commutator is not zero. By the Simultaneity Principle developed in Chapter 5, this means that *X* and *P* cannot be simultaneously measured. In fact, the Heisenberg Uncertainty Principle provides a floor for the uncertainty.

Recall that we denote the uncertainty of an operator by its standard deviation.

Δ*X* is the **standard deviation of *X*,**

Δ*P* is the **standard deviation of P.**

When we apply the Heisenberg Uncertainty Principle developed in Chapter 5 to *X* and *P* we get

$$\Delta X \; \Delta P \geq \frac{\hbar}{2}:$$

$$\Delta X \; \Delta P \overset{(5.13)}{\geq} \frac{1}{2}\left| \left\langle \Psi \;\middle|\; \left[X, P\right] \;\middle|\; \Psi \right\rangle \right| \overset{(8.29)}{=} \frac{1}{2}\left| \, i\hbar \left\langle \Psi \middle| \Psi \right\rangle \right| = \frac{1}{2}\left| \, i\hbar \, \right| = \frac{\hbar}{2} \, .$$

No matter how cleverly we measure position and inertia, we can never drive the uncertainty below this small quantity.