

# Logical Limit Laws for Layered Permutations and Related Structures

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## Definition

A class  $C$  of structures in some first-order language admits a *zero-one law* if, for any sentence  $\varphi$ , the probability that a randomly selected  $C$ -structure of size  $n$  satisfies  $\varphi$  converges asymptotically to zero or one as  $n \rightarrow \infty$ .

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- Classical example: finite graphs [Glebskii et. al]
- Convergence to zero or one is a rather strict requirement

## Definition

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- “Unlabeled limit law” — class of *unlabeled* structures admits a limit law
- “Labeled limit law” — class of *labeled* structures admits a limit law



## Theorem

Convex linear orders and layered permutations admit both unlabeled and labeled limit laws. Compositions admit an unlabeled limit law.

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## Definition

Let  $\mathcal{L}$  be the language containing two binary relations:  $<$  and  $E$ . A convex linear order is an  $\mathcal{L}$ -structure satisfying:

- $<$  is a total order on points
- $E$  is an equivalence relation
- $x E z, x < y < z \Rightarrow z E x, y$

## Definition

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## Definition

For convex linear orders  $\mathfrak{C}, \mathfrak{D}$ , define  $\mathfrak{C} \oplus \mathfrak{D}$  as the convex linear order placing  $\mathfrak{D}$   $\prec$ -after  $\mathfrak{C}$ .

# Constructing convex linear orders

## Lemma

Every finite convex linear order containing  $n$  points can be uniquely constructed by applying  $\widehat{(-)}$  and/or  $- \oplus \bullet$  to  $\bullet$  repeatedly.

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## Proof

Proceed by induction.

- Base case:  $n = 1$  trivial
- When  $n = 2$ , two possible cases:  $\mathcal{C} \simeq \bullet \oplus \bullet$  or  $\mathcal{C} \simeq \widehat{\bullet}$
- In general: last class of  $\mathcal{C}$  contains one or more points.  
Apply  $- \oplus \bullet$  or  $\widehat{(-)}$  appropriately.



- Ehrenfeucht–Fraïssé game on two structures:  
back-and-forth game between players Spoiler and Duplicator in which corresponding points are marked on each structure
- In game of length  $k$  between  $\mathfrak{A}$  and  $\mathfrak{B}$ , Duplicator has a winning strategy iff  $\mathfrak{A}$  and  $\mathfrak{B}$  agree on all sentences of quantifier depth at most  $k$ .
- Write  $\mathfrak{A} \equiv_k \mathfrak{B}$  in this case



## Lemma

Let  $\mathfrak{M}, \mathfrak{N}, \mathfrak{M}', \mathfrak{N}'$  be convex linear orders such that  $\mathfrak{M} \equiv_k \mathfrak{N}$  and  $\mathfrak{M}' \equiv_k \mathfrak{N}'$ . The following equivalences hold:

- $\mathfrak{M} \oplus \mathfrak{M}' \equiv_k \mathfrak{N} \oplus \mathfrak{N}'$
- $\widehat{\mathfrak{M}} \equiv_k \widehat{\mathfrak{N}}$

## Lemma

For a convex linear order  $\mathfrak{M}$  and  $k \in \mathbb{N}$ , there exists  $\ell \in \mathbb{N}$  such that for all  $s, t > \ell$ ,

$$\bigoplus_s \mathfrak{M} \equiv_k \bigoplus_t \mathfrak{M}$$

# The limit law

- Labeled limit laws: count all possible structures over  $[n] := \{1, \dots, n\}$  as  $n \rightarrow \infty$
- Unlabeled: count all structures *up to isomorphism*

- Labeled limit laws: count all possible structures over  $[n] := \{1, \dots, n\}$  as  $n \rightarrow \infty$
- Unlabeled: count all structures *up to isomorphism*
- Finite linearly ordered structures have no nontrivial automorphisms, hence, no distinction in this case

General idea:

- For first-order sentence  $\varphi$  (with quantifier rank  $k$ ), associate a Markov chain  $M_\varphi$
- States of  $M_\varphi$  are  $\equiv_k$ -classes
- Probability that randomly selected structure of size  $n$  satisfies  $\varphi$  is probability that  $M_\varphi$  is in a state that satisfies  $\varphi$  after  $n$  transitions

# The limit law

For an  $\equiv_k$ -class  $C$ , define

$$C \oplus \bullet := [\mathfrak{M} \oplus \bullet]_{\equiv_k}$$

and

$$\widehat{C} := [\widehat{\mathfrak{M}}]_{\equiv_k}$$

For  $\varphi$  an  $\mathcal{L}$ -sentence (with quantifier depth  $k$ ), construct a Markov chain  $M_\varphi$  as follows:

- Starting state:  $[\bullet]_{\equiv_k}$
- From any  $\equiv_k$ -class  $C$ , there are two possible transitions out: to  $C \oplus \bullet$  or  $\widehat{C}$
- Each transition probability is  $1/2$

## Definition

A Markov chain  $M$  is *fully aperiodic* if there do not exist disjoint sets of  $M$ -states  $P_0, P_1, \dots, P_{d-1}$  for some  $d > 1$  such that for every state in  $P_i$ ,  $M$  transitions to a state in  $P_{i+1}$  with probability 1 (with  $P_{d-1}$  transitioning to  $P_0$ ).

## Lemma

Let  $M$  be a finite, fully aperiodic Markov chain with initial state  $S$ , and let  $Pr^{n-1}(S, Q)$  denote the probability that  $M$  is in state  $Q$  after  $n - 1$  steps. For any  $Q$ ,  $\lim_{n \rightarrow \infty} Pr^{n-1}(S, Q)$  converges.



## Theorem

$M_\varphi$  is fully aperiodic for any first-order sentence  $\varphi$ .

## Proof

Suppose  $M_\varphi$  were not fully aperiodic.

- There would exist disjoint sets of  $M_\varphi$ -states ( $\equiv_k$ -classes)  $P_0, P_1, \dots, P_{d-1}$  for  $d > 1$  where every state in  $P_i$ ,  $M_\varphi$  transitions to a state in  $P_{i+1}$  with probability 1 ( $P_{d-1}$  transitioning to  $P_0$ ).
- Thus, for any  $Q \in P_0$ ,  $Q \oplus i\bullet$  is in  $P_0$  iff  $d \mid i$ .
- By equivalence lemmas, this is not possible □

## Theorem

Convex linear orders admit a logical limit law.

## Proof

Fix a first-order sentence  $\varphi$ , and consider  $M_\varphi$ .

- For each state  $S$  in  $M_\varphi$ , either each structure in  $S$  satisfies  $\varphi$  or no structures in  $S$  satisfy  $\varphi$ .
- Let  $S_\varphi$  denote the set of states in  $M_\varphi$  for which all structures in that state satisfy  $\varphi$ .
- $\widehat{(-)}$  and  $- \oplus \bullet$  are well-defined on  $\equiv_k$ -classes, hence, moving  $n - 1$  steps in  $M_\varphi$  is equivalent to starting with any structure in the current state, applying  $\widehat{(-)}$  or  $- \oplus \bullet$   $n - 1$  times, and taking the  $\equiv_k$ -class.

## Proof (continued)

- The probability that after  $n$  steps,  $M_\varphi$  is in a state of  $S_\varphi$  equals probability that uniformly randomly selected structure of size  $n$  satisfies  $\varphi$
- Suffices to show that  $\lim_{n \rightarrow \infty} \sum_{Q \in S_\varphi} Pr^{n-1}(\bullet, Q)$  converges, which follows from Markov chain lemma □

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# Transfer lemmas

Fix languages  $\mathcal{L}_0, \mathcal{L}_1$  and classes  $C_0, C_1$  of  $\mathcal{L}_0, \mathcal{L}_1$  structures respectively.

## Lemma

Let  $f$  be a map from the set of  $\mathcal{L}_0$ -structures to the set of  $\mathcal{L}_1$ -structures, and  $g$  a map from the set of  $\mathcal{L}_0$ -sentences to the set of  $\mathcal{L}_1$ -sentences such that, for any  $C_0$ -structure  $\mathfrak{M}$  and  $\mathcal{L}_0$ -sentence  $\varphi$ :

- 1  $\mathfrak{M} \models \varphi \iff f(\mathfrak{M}) \models g(\varphi)$
- 2  $f$  is a bijection between  $C_0$  and  $C_1$  structures of size  $n$
- 3 The class  $C_1$  admits a logical limit law

Then,  $C_0$  admits a logical limit law as well.

## Definition

Classes  $C_0$  and  $C_1$  of structures (over a with a common domain of  $[n]$ ) are said to be *uniformly interdefinable* if there exists a map  $f_I : C_0 \rightarrow C_1$  (bijective on structures), along with formulae  $\varphi_{R_{0,i}}, \varphi_{R_{1,i}}$  for each relation  $R_{0,i}$  in  $\mathcal{L}_0$  and  $R_{1,i}$  in  $\mathcal{L}_1$  such that, for each  $\mathfrak{M}_0$  in  $C_0$  and  $\mathfrak{M}_1$  in  $C_1$ :

- $\mathfrak{M}_0 \models R_{0,i}(\bar{x}) \iff f_I(\mathfrak{M}_0) \models \varphi_{R_{0,i}}(\bar{x})$
- $\mathfrak{M}_1 \models R_{1,i}(\bar{x}) \iff f_I^{-1}(\mathfrak{M}_1) \models \varphi_{R_{1,i}}(\bar{x})$

## Theorem

Let  $C_0, C_1$  be uniformly interdefinable classes of  $\mathcal{L}_0, \mathcal{L}_1$  structures. If  $C_1$  admits a logical limit law,  $C_0$  admits one as well.

## Proof

Take the transfer maps  $f, g$  to be:

- $f = f_I$
- $g$  is the map sends an  $\mathcal{L}_0$ -sentence to the  $\mathcal{L}_1$ -sentence with each occurrence of  $R_{0,i}$  replaced with  $\varphi_{R_{0,i}}$



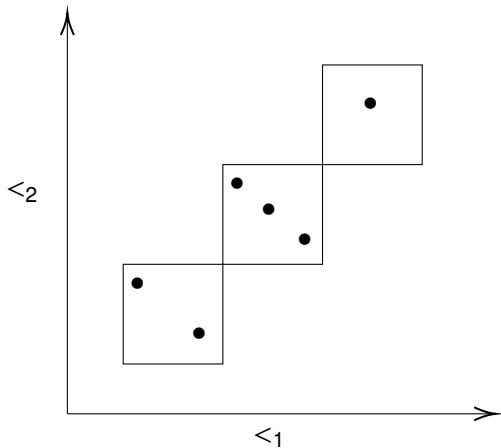
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# Layered permutations

- Permutations can be viewed as structures in the language  $\mathcal{L} = \{<_1, <_2\}$  with two linear orders. The order  $<_1$  gives the unpermuted order of the points (before applying the permutation) and  $<_2$  describes the points in permuted order.
- *Blocks* are maximal subsets which are monotone  $<_1/<_2$ -intervals
- A *layered permutation* is composed of increasing blocks, each of which contains a decreasing permutation

# Layered permutations



# Layered permutations

## Lemma

Layered permutations and ordered equivalence relations are uniformly interdefinable.

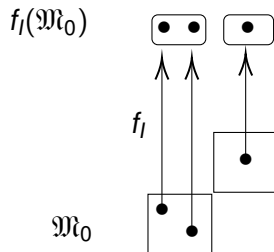
## Proof

Define  $f_l$  to be the map taking blocks of a layered permutation to classes of a convex linear order, and points in an order-preserving manner. The relations  $<_1$  and  $<_2$  are rewritten as:

- $\varphi_{<_1} : a <_1 b \rightsquigarrow a < b$
- $\varphi_{<_2} : a <_2 b \rightsquigarrow (a E b \wedge b < a) \vee (\neg(a E b) \wedge a < b)$



# Layered permutations



# Layered permutations

## Theorem

Layered permutations admit a logical limit law.

## Proof

Layered permutations are uniformly interdefinable with convex linear orders. Because convex linear orders admit a logical limit law, layered permutations admit one as well. □

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- Let  $\mathcal{L}_0 = \{E, <\}$  be the language of convex linear orders
- Define  $\mathcal{L}_1 = \{E, <_1, <_2\}$
- Fractured orders take a convex linear order  $<$  and break it into two parts:  $<_1$  *between*  $E$ -classes, and  $<_2$  *within*  $E$ -classes.

## Definition

A *fractured order* is an  $\mathcal{L}_1$ -structure satisfying:

- 1  $<_1, <_2$  are partial orders
- 2  $E$  is an equivalence relation
- 3 Distinct points  $a, b$  are  $<_1$ -comparable iff they **are not**  $E$ -related
- 4 Distinct points  $a, b$  are  $<_2$ -comparable iff they **are**  $E$ -related
- 5  $a E a', a <_1 b \Rightarrow a' <_1 b$  (convexity)

We denote the class of all finite fractured orders by  $\mathcal{F}$ .



## Theorem

Fractured orders and convex linear orders are uniformly interdefinable.

## Proof

Define  $f_I : \mathcal{F} \rightarrow C_0$  such that:

- $\mathfrak{M}_1 \models a E b \iff f_I(\mathfrak{M}_1) \models a E b$
- $\mathfrak{M}_1 \models a <_1 b \iff f_I(\mathfrak{M}_1) \models \neg a E b \wedge a < b$
- $\mathfrak{M}_1 \models a <_2 b \iff f_I(\mathfrak{M}_1) \models a E b \wedge a < b$

This map satisfies the requirements for uniform interdefinability. □

## Lemma

Let  $\mathcal{L}$  be a language and  $\mathcal{L}' \subset \mathcal{L}$ . Given a class  $\mathcal{C}$  of  $\mathcal{L}$ -structures which admits a logical limit law, any class  $\mathcal{C}'$  of  $\mathcal{L}'$ -structures which expand uniquely to  $\mathcal{C}$ -structures also admits a logical limit law.

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## Proof

Construct the transfer maps  $f$  and  $g$  from earlier:

- $f$  is taken to be the map sending a structure in  $\mathcal{C}'$  to its unique expansion in  $\mathcal{C}$
- This expansion is unique, hence  $f$  is bijective on structures of size  $n$  for all  $n$
- $g$  is given by the identity map on formulas



- *Compositions* are structures in the reduct  $\mathcal{L}_2 \subset \mathcal{L}_1$  given by  $\mathcal{L}_2 = \{E, <_1\}$
- Order defined on equivalence classes, but not on points within each class

## Lemma

Every composition expands uniquely to a fractured order, up to isomorphism.

## Proof

There is a unique way to linearly order each  $E$ -class individually. Because ordering these classes determines  $<_2$ , there is a unique way to define  $<_2$  on any composition, expanding it to a fractured order. □

## Theorem

The class of compositions admit an unlabeled logical limit law.

## Proof

The language of compositions is a reduct of the language of fractured orders, and every composition expands uniquely to a fractured order. The class of fractured orders admits a logical limit law, therefore, by the previous lemma, compositions admit a limit law as well. □



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