Limit Laws

1 Introduction

It is a classic combinatorial problem to ask,"What does a random structure of size n look like as n grows larger?" Zero-one laws are one method of gaining insight into this question for the case of properties expressible in first-order logic. A class $\mathcal{C}$ of structures in some first-order language is said to admit a zero-one law if, given a sentence φ, the probability that a randomly selected $\mathcal{C}$-structure of size n satisfies φ converges asymptotically to zero or one as n goes to infinity. For example, the class of finite graphs (in the language with a single relation), properties such as containing a specified subgraph or induced subgraph are expressible in first-order logic, whereas properties such as connectedness or Hamiltonicity are not. It is a seminal result that this class admits a zero-one law; the probability that a randomly selected finite graph of size n satisfies a first order property converges asymptotically to zero or one as $n \to \infty$.

The requirement that such a probability for a class of structures converges to either zero or one is rather strict, and in general, not many classes admit a zero-one law. This is particularly evident when dealing with ordered structures. For example, consider the class of finite linear orders with two colors: red and blue. The probability that a randomly selected structure's first point is red (or blue) is 1/2. A class of structures is said to admit a *logical limit law* when the probability that a randomly selected structure of size N satisfies a first order property converges as $n \to \infty$. Using this definition, a zero-one law is a logical limit law which converges to zero or one.

We prove the existance of logical limit laws for various classes of ordered structures. In particular, we show the following result:

Theorem 1.1. *Convex linear orders and layered permutations admit both unlabelled and labelled limit laws. Ordered equivalence relations admit an unlabelled limit law.*

Logical limit law for convex linear orders are proven using a variation of Ehrenfeucht's proof for linear orders. After developing transfer results, we lift this limit law to layered permutations and ordered equivalence relations.

In [3], the limiting probability distributions of several first order properties are computed for random preorders (what Cameron and Stark call "preorders" we refer to as "ordered equivalence relations"). Our result generalizes this as we show that the limiting probability of any first-order property converges (though we do not examine distributions of any particular properties.

It is a fairly well-known result that the class of all permutations does not admit a logical limit law. However, we show that an oft-studied class of permutations, layered permutations, does admit a logical limit law.

2 Convex Linear Orders

We will refer to a linear order with a convex equivalence relation as a *convex linear order*. Let $\mathcal{L}$ denote the language of convex linear orders. $\mathcal{L}$ consists of two relation symbols: $<_1$, (the order on points within equivalence classes) and $<_2$ (the order on equivalence classes themselves). The empty convex linear order with one class (containing one point) will be denoted $\bullet$.

Definition 2.1. *Let $\mathfrak{C}$ be a convex linear order. Denote by $\widehat{\mathfrak{C}}$ the convex linear order obtained by adding one additional point to the last class of $\mathfrak{C}$.*

Definition 2.2. *For convex linear orders $\mathfrak{C}, \mathfrak{D}$, define their concatenation $\mathfrak{C} \oplus \mathfrak{D}$ as the convex linear order where every equivalence class in $\mathfrak{C}$ comes before every equivalence class of $\mathfrak{D}$ (with respect to the order on equivalence classes, $<_2$).*

Lemma 2.3. *Every finite convex linear order containing n points can be uniquely constructed using only $\widehat{(-)}$ and $- \oplus \bullet$ applied to $\bullet$; this construction is done in $n-1$ steps.*

Proof. We proceed by induction. Let $\mathfrak{C}$ be a convex linear order of size n.

If $n = 1$, $\mathfrak{C} \simeq \bullet$. In the $n = 2$ case, there are two possibilities: either $\mathfrak{C} \simeq \widehat{\bullet}$, or $\mathfrak{C} \simeq \bullet \oplus \bullet$.

Now assume that, for arbitrary n, any convex linear order of size $n-1$ can be uniquely constructed from the operations above. Either the last class of $\mathfrak{C}$ contains one point, or it contains more than one point. If the former is true, than

$$\mathfrak{C} \simeq \mathfrak{B} \oplus \bullet$$

for some $\mathfrak{B}$ with $n-1$ points (as we need only add one point to the end of $\mathfrak{B}$ to obtain $\mathfrak{C}$). In this case, $\mathfrak{C}$ cannot be obtained from $\widehat{(-)}$, as the last class of $\mathfrak{C}$ needs to contain exactly one point. If the latter is true, then

$$\mathfrak{C} \simeq \widehat{\mathfrak{B}}$$

as we have added an additional point to the last class of $\mathfrak{B}$ to construct an convex linear order with n points; $\mathfrak{C}$ could not have been constructed from $- \oplus \bullet$ as its last class contains more than one point. $\square$

Starting with $\bullet$ and randomly applying $- \oplus \bullet$ and $\widehat{(-)}$ (each with probability $1/2$) will randomly sample all possible convex linear orders. We write $i\bullet$ to mean $\bigoplus_i \bullet$.

Lemma 2.4. *Let $\mathfrak{M}, \mathfrak{N}, \mathfrak{M}', \mathfrak{N}'$ be convex linear orders such that $\mathfrak{M} \equiv_k \mathfrak{N}$ and $\mathfrak{M}' \equiv_k \mathfrak{N}'$. Then, $\mathfrak{M} \oplus \mathfrak{M}' \equiv_k \mathfrak{N} \oplus \mathfrak{N}'$.*

Proof. We will show that in any EF game of length k, Duplicator has a winning strategy. Consider such a game between $\mathfrak{M} \oplus \mathfrak{M}'$ and $\mathfrak{N} \oplus \mathfrak{N}'$. If Spoiler chooses any element in $\mathfrak{M}$ or $\mathfrak{M}'$ (respectively, $\mathfrak{N}$ or $\mathfrak{N}'$), Duplicator responds by playing as if they would in a length-k game between $\mathfrak{N}$ or $\mathfrak{N}'$ ($\mathfrak{M}$ or $\mathfrak{M}'$); a winning strategy exists due to the fact that $\mathfrak{M} \equiv_k \mathfrak{N}$ and $\mathfrak{M}' \equiv_k \mathfrak{N}'$. This gives Duplicator a winning strategy in the EF game between $\mathfrak{M} \oplus \mathfrak{M}'$ and $\mathfrak{M}' \equiv_k \mathfrak{N}'$. By definition of $\oplus$, all elements in $\mathfrak{M}$ come before those in $\mathfrak{M}'$ in $\mathfrak{M} \oplus \mathfrak{M}'$ (and likewise for $\mathfrak{N}$, $\mathfrak{N}'$ in $\mathfrak{N} \oplus \mathfrak{N}'$). The sets of points picked out in $\mathfrak{M}$ and $\mathfrak{N}$ (or $\mathfrak{M}'$ and $\mathfrak{N}'$) are isomorphic, and since $\oplus$ preserves isomorphism, the sets of points picked in $\mathfrak{M} \oplus \mathfrak{M}'$ are isomorphic to $\mathfrak{N} \oplus \mathfrak{N}'$ □

Lemma 2.5. *Suppose $\mathfrak{M} \equiv_k \mathfrak{N}$, then, $\widehat{\mathfrak{M}} \equiv_k \widehat{\mathfrak{N}}$.*

Proof. We again show that in an EF game of length k, Duplicator has a winning strategy. For any move by Spoiler in $\mathfrak{M}$ (or $\mathfrak{N}$), Duplicator responds by playing as they would normally would in an EF game between $\mathfrak{M}$ and $\mathfrak{N}$; because $\mathfrak{M} \equiv_k \mathfrak{N}$, Duplicator always has a winning move in response to Spoiler for any such play. If Duplicator plays the last point in the last class of $\widehat{\mathfrak{M}}$ or $\widehat{\mathfrak{N}}$ (that is, the point added by $\widehat{(-)}$), Duplicator can always respond with the corresponding point at the end of $\widehat{\mathfrak{N}}$ or $\widehat{\mathfrak{M}}$. Hence, Duplicator has a response for any of Spoiler's moves in a length k EF game between $\widehat{\mathfrak{M}}$ and $\widehat{\mathfrak{N}}$, so $\widehat{\mathfrak{M}} \equiv_k \widehat{\mathfrak{N}}$. □

Lemma 2.6. *For two finite linear orders N, M having n and m points respectively, $N \equiv_k M$ iff $n = m$ or $n, m \geq 2^k - 1$.*

Proof. This is Lemma 2.6.3 in [1]. □

Lemma 2.7. *For a convex linear order $\mathfrak{M}$, there exists $\ell \in \mathbb{N}$ such that for all $s, t > \ell$,*

$$\bigoplus_s \mathfrak{M} \equiv_k \bigoplus_t \mathfrak{M}$$

Proof. We reduce this to a case of the previous lemma. Let O_s be a linear order with s points, each corresponding to a copy of $\mathfrak{M}$ in $\bigoplus_s \mathfrak{M}$, and define O_t likewise for t and $\bigoplus_t \mathfrak{M}$. In an EF game of length k between $\bigoplus_s \mathfrak{M}$ and $\bigoplus_t \mathfrak{M}$, we will show that Duplicator has a winning strategy. If Spoiler picks a point in the ith copy of $\mathfrak{M}$ in $\bigoplus_s \mathfrak{M}$, we view this as Spoiler picking the ith point in O_s if it were playing a length-k EF game between O_s and O_t. By Lemma 0.6, Duplicator has a response in O_t; suppose this response is the jth point. To have a winning strategy in the EF game between $\bigoplus_s \mathfrak{M}$ and $\bigoplus_t \mathfrak{M}$, Duplicator can select the same point in $\mathfrak{M}$ which Spoiler selected, but in the jth copy of $\mathfrak{M}$ in $\bigoplus_t \mathfrak{M}$. □

Lemma 2.8. *Let M be a finite, aperiodic Markov chain with initial state S, and let $Pr^n(S, Q)$ denote the probability that M is in state Q after n steps. Then, for any Q, $\lim_{n \to \infty} Pr^n(S, Q)$ converges.*

Proof. This is Theorem 0.3.1 in [2]. □

Given a first-order sentence φ having quantifier rank k, we compute the limiting probability of φ by associating to it a Markov chain M_φ. For a $\equiv_k$-class C, define

$$C \oplus \bullet := [\mathfrak{M} \oplus \bullet]_{\equiv_k}$$

and

$$\widehat{C} := \left[\widehat{\mathfrak{M}}\right]_{\equiv_k}$$

where $\mathfrak{M} \in C$ (any choice of $\mathfrak{M}$ yields an equivalent result by Lemma 2.4).

The states of M_φ are $\equiv_k$ classes of $\mathcal{L}$-structures (of which there are finitely many). For a class C, there are two possible transitions out of C: one to $C \oplus \bullet$, and one to $\widehat{C}$, each having probability 1/2. The starting state of M_φ is $\bullet$ (as a slight abuse of notation, we will write $\bullet$ to also mean $[\bullet]_{\equiv_k}$).

Lemma 2.9. *For any φ, M_φ is aperiodic.*

Proof. Suppose M_φ were periodic, then, there would exist disjoint sets of M-states $P_0, P_1, \cdots, P_{d-1}$ for some $d > 1$ such that for every state in P_i, M_φ must move to a state in P_{i+1} with probability 1 (with P_{d-1} transitioning to P_0). For any $Q \in P_0$, $Q \oplus i\bullet$ is in P_0 iff $d \mid i$. But by Lemma 2.4 and Lemma 2.7, $Q \oplus i\bullet \equiv_k Q \oplus (i+1)\bullet$ for large enough i, contradicting this claim. □

Theorem 2.10. *Convex linear orders admit a logical limit law.*

Proof. Given a first-order sentence φ, we compute the limiting probability of M_φ. For each state S_n in M_φ, either each structure in S_n satisfies φ or no structures in S_n satisfy φ. Let S_φ denote the set of states ($\equiv_k$ classes) in M_φ for which all structures in that class satisfy φ. By Lemma 1.3, the probability that a uniformly, randomly selected structure of size n satisfies φ is precisely the probability that, after n steps, M_φ is in a state S_φ such that $S_\varphi \models \varphi$. This probability is given by

$$\sum_{Q \in S_\varphi} Pr^n(\bullet, Q)$$

where $Pr^n(\bullet, Q)$ denotes the transition probability from $\bullet$ to P after n steps.

We wish to show that $\lim_{n\to\infty} \sum_{Q \in S_\varphi} Pr^n(\bullet, Q)$ converges. Because M_φ has finitely many states,

$$\lim_{n\to\infty} \sum_{Q \in S_\varphi} Pr^n(\bullet, Q) = \sum_{Q \in S_\varphi} \lim_{n\to\infty} Pr^n(\bullet, Q)$$

So, it suffices to show that $\lim_{n\to\infty} Pr^n(\bullet, Q)$ exists for every state Q of M_φ. Because M_φ always begins with the same state ($\bullet$), by the fundamental theorem of Markov chains [2] each of these probabilities exists. □

3 Layered Permutations

In many cases, it is useful to transfer a logical limit law on one class of structures to another class of similar structures. This is possible when some modest interdefinability conditions are defined.

Definition 3.1. *(Uniform Interdefinability)*
Let $\mathcal{L}_0$, $\mathcal{L}_1$ be languages, and $\mathcal{C}_0$, C_1 be classes of finite $\mathcal{L}_0$, $\mathcal{L}_1$ structures respectively, with a common domain of $[n] = \{1, \cdots, n\}$. $\mathcal{C}_0$ and $\mathcal{C}_1$ are said to be ***uniformly interdefinable*** *if there exists a map $f_I : \mathcal{C}_0 \to \mathcal{C}_1$ (which is a bijection on structures of size n, for all n), along with formulae $\varphi_{R_0}, \varphi_{R_1}$ for each relation in $\mathcal{L}_0$ and $\mathcal{L}_1$ respectively such that, for each $\mathfrak{M}_0$ in $\mathcal{C}_0$ and $\mathfrak{M}_1$ in $\mathcal{C}_1$:*

- $\mathfrak{M}_0 \models R_0(\bar{x}) \iff f_I(\mathfrak{M}_0) \models \varphi_{R_0}(\bar{x})$
- $\mathfrak{M}_1 \models R_1(\bar{x}) \iff f_I^{-1}(\mathfrak{M}_1) \models \varphi_{R_1}(\bar{x})$

Lemma 3.2. *Let $\mathcal{L}_0$, $\mathcal{L}_1$ be languages, $\mathcal{C}_0$, $\mathcal{C}_1$ classes of $\mathcal{L}_0$, $\mathcal{L}_1$ structures respectively, f a map from the set of $\mathcal{L}_0$-structures to the set of $\mathcal{L}_1$-structures, and g a map from the set of $\mathcal{L}_0$-sentences to the set of $\mathcal{L}_1$-sentences such that, for any $\mathcal{C}_0$-structure $\mathfrak{M}$ and $\mathcal{L}_0$-sentence φ:*

1. $\mathfrak{M} \models \varphi \iff f(\mathfrak{M}) \models g(\varphi)$
2. *f is a bijection between $\mathcal{C}_0$ and $\mathcal{C}_1$ structures of size n*
3. *The class $\mathcal{C}_1$ admits a logical limit law*

Then, $\mathcal{C}_0$ admits a logical limit law as well.

Proof. Let φ be a sentence in $\mathcal{L}_1$ and a_1 the number of size n $\mathcal{C}_1$-structures satisfying φ. Likewise, let a_0 be the number of size n $\mathcal{L}_0$ structures which satisfy $g(\varphi)$. For a randomly selected $\mathcal{C}_1$ structure $\mathfrak{S}$ (of size n), the probability that $\mathfrak{S} \models \varphi$ is $\frac{a_1}{|\mathcal{C}_1|}$, and the probability that $f(\mathfrak{S}) \models g(\varphi)$ in $\mathcal{C}_0$ is $\frac{a_0}{|\mathcal{C}_0|}$. f is a bijection, so $|\mathcal{C}_1| = |\mathcal{C}_0|$ and we have that $a_1 = a_0$ by (1). Hence, the probabilities are equal for any φ; because $\mathcal{C}_1$ admits a limit law, $\mathcal{C}_0$ admits a limit law. □

Lemma 3.3. *If $\mathcal{C}_0$, $\mathcal{C}_1$ are uniformly interdefinable classes of $\mathcal{L}_0$, $\mathcal{L}_1$ structures, there exist f and g as above.*

Proof. We take $f = f_I$ and g as the map which sends an $\mathcal{L}_0$ sentence to the $\mathcal{L}_1$ sentence where each occurrence of R_0 is replaced by φ_{R_0}, and each occurrence of R_1 is replaced by φ_{R_1}. Given a $\mathcal{C}_0$ structure $\mathfrak{M}_0$ and an arbitrary $\mathcal{L}_0$ formula φ, $f_I(\mathfrak{M}_0) \models g(\varphi)$ by the definition of uniform interdefinability when φ is atomic; when φ is nonatomic, the result follows from standard induction on the complexity of φ.

Furthermore, by Definition 3.1, f_I is a bijection on structures of size n, therefore, f_I, g satisfy 3.2.1 and 3.2.2. □

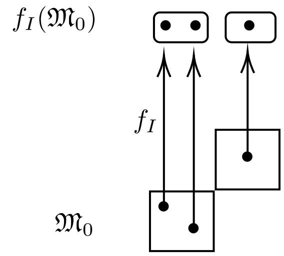

Figure 1: An illustration of the map f_I. Blocks of the layered permutation $\mathfrak{M}_0$ are mapped to equivalence classes of $f_I(\mathfrak{M}_0)$, and points are mapped in an order-preserving manner.

Let $\mathcal{L}_0 = \{<_1, <_2\}$ and $\mathcal{L}_1 = \{E, <\}$ denote the languages of layered permutations and convex linear orders respectively, and let $\mathcal{C}_0, \mathcal{C}_1$ classes of isomorphism types of $\mathcal{L}_0, \mathcal{L}_1$ structures respectively. Define a map from layered permutations to convex linear orders which sends blocks of a layered permutation to convex equivalence classes, and points in each block of a layered permutation to points in the same equivalence class such that $<_1$ agrees with $<$. Formally, this is a map $f : \mathcal{C}_0 \to \mathcal{C}_1$ is defined such that for $\mathfrak{M}_0$ in $\mathcal{C}_0$ and $\mathfrak{M}_1$ in $\mathcal{C}_1$:

- $f(\mathfrak{M}_0) \models a < b \iff \mathfrak{M}_0 \models a <_1 b$
- $f(\mathfrak{M}_0) \models a \; E \; b \iff \mathfrak{M}_0 \models (a <_1 b \wedge a >_2 b) \vee (b <_1 a \wedge b >_2 a)$

The relations $<_1$, $<_2$ in the language of layered permutations are rewritten inductively in the language of convex equivalence relations using the following rules on atomic formulas:

- $\varphi_{<_1} : a <_1 b \rightsquigarrow a < b$
- $\varphi_{<_2} : a <_2 b \rightsquigarrow (a \; E \; b \wedge b < a) \vee (\neg(a \; E \; b) \wedge a < b)$

We show that this map satisfies the requirements of uniform interdefinability.

Lemma 3.4. *Layered permutations and convex linear orders are uniformly interdefinable.*

Proof. A finite layered permutation is determined, up to isomorphism, by the number of points in each of its blocks; likewise, a finite convex linear order is determined in the same manner with the number of points in each equivalence class. Because the map f sends blocks of $\mathfrak{M}_0$ to equivalence classes of $f(\mathfrak{M}_0)$ of the same size, f is injective, hence, a bijection.
Let $\mathfrak{M}_0$, $\mathfrak{M}_0'$ be two $\mathcal{C}_0$ structures (that is, layered permutations) with $f(\mathfrak{M}_0) = f(\mathfrak{M}_0')$. From the definition of $<_1$, it is clear that

$$\mathfrak{M}_0 \models a <_1 b \iff f(\mathfrak{M}_0) \models a < b$$

We verify

$$\mathfrak{M}_0 \models a <_2 b \iff f(\mathfrak{M}_0) \models (a\ E\ b \wedge b < a) \vee (\neg a\ E\ b \wedge a < b)$$

($\Rightarrow$) Suppose a and b are in the same block, then, because $\mathfrak{M}_0 \models a <_2 b$, we have that $\mathfrak{M}_0 \models b <_1 a$, so $f(\mathfrak{M}_0) \models b < a$ (because the orders $<_1$ and $<$ agree). When a and b are in different blocks, $f(\mathfrak{M}_0) \models \neg a\ E\ b$; furthermore, because $\mathfrak{M}_0 \models a <_2 b$, and a, b are in different blocks, $\mathfrak{M}_0 \models a <_1 b$, so $f(\mathfrak{M}_0) \models a < b$.

($\Leftarrow$) Now suppose a and b are in the same equivalence class. Then, $f(\mathfrak{M}_0) \models b < a$ and $\mathfrak{M}_0 \models b <_1 a$ Because a, b are in same block, they are in the same class of $\mathfrak{M}_0$, furthermore, $\mathfrak{M}_0 \models b <_1 a$ and a, b are in the same class, so $\mathfrak{M}_0 \models a <_2 b$. When a and b are in different equivalence classes, $a < b$, so $a <_1 b$ in $\mathfrak{M}_0$. Since a and b are in different blocks, the orders $<_1$ and $<_2$ agree, giving $a <_2 b$.

□

Theorem 3.5. *Layered permutations admit a logical limit law.*

Proof. By 3.4, layered permutations are uniformly interdefinable with convex linear orders, hence, the maps f and g in 3.2 exist. Because convex linear orders admit a logical limit law, layered permutations admit one as well. □

4 Ordered Equivalence Relations

Lemma 4.1. *Let $\mathcal{L}$ be a language and $\mathcal{L}'$ a reduct of $\mathcal{L}$. Given a class $\mathcal{C}$ of $\mathcal{L}$-structures which admits a logical limit law, any class $\mathcal{C}'$ of $\mathcal{L}'$-structures which expand uniquely to $\mathcal{C}$-structures also admits a logical limit law.*

Proof. The map f is taken to be the map sending a structure in $\mathcal{C}'$ to its unique expansion in $\mathcal{C}$. Because this expansion is unique, f is bijective on structures of size n for all n. We take g to be the identity map on formulas (as $\mathcal{L}'$ is a reduct of $\mathcal{L}$). It is clear that these maps satisfy the requirements of 3.2. □

This is a situation where two classes of structures are not uniformly interdefinable, but the maps f and g exist.

Let $\mathcal{L}_0 = \{E, <\}$ be the language of convex linear orders as before. Define a language $\mathcal{L}_1 = \{E, \prec_1, \prec_2\}$ consisting of three relation symbols (an equivalence relation and two partial orders), along with a reduct $\mathcal{L}_2 \subset \mathcal{L}_1$ given by $\mathcal{L}_2 = \{E, \prec_1\}$. In a fractured order, we start with a convex linear order $<$ and break it into two parts: $\prec_1$ between E-classes, and $\prec_2$ within E-classes. More formally, define the class of finite fractured orders, $\mathcal{F}$ to be the class of $\mathcal{L}_1$-structures satisfying the following:

1. $\prec_1$, $\prec_2$ are partial orders
2. E is an equivalence relation

3. $a \prec_1 b \iff \neg a \; E \; b$
4. $a \prec_2 b \iff a \; E \; b$
5. $x \; E \; x'$, $x \prec_1 y \Rightarrow x' \prec_1 y$ (convexity)

An ordered equivalence relation consists of an equivalence relation E along with a linear order $\prec_1$ on E-classes (but not on points of the classes themselves). More formally, this is the $\mathcal{L}_2$-reduct of an $\mathcal{F}$-structure. To show that ordered equivalence relations admit a logical limit law, we show that $\mathcal{F}$-structures are uniformly interdefinable with convex linear orders, and that any ordered equivalence relation admits a unique expansion to a fractured order.

Lemma 4.2. *Convex linear orders and finite fractured orders are uniformly interdefinable.*

Proof. Define a map $f_I : \mathcal{F} \to \mathcal{C}_0$ such that:

- $\mathfrak{M}_1 \models a \; E \; b \iff f_I(\mathfrak{M}_1) \models a \; E \; b$
- $\mathfrak{M}_1 \models a \prec_1 b \iff f_I(\mathfrak{M}_1) \models \neg a \; E \; b \wedge a < b$
- $\mathfrak{M}_1 \models a \prec_2 b \iff f_I(\mathfrak{M}_1) \models a \; E \; b \wedge a < b$

The order $<$ is total by axioms 3 and 4 in the above, and convex by axiom 5. Both convex linear orders and ordered equivalence relations are determined (up to isomorphism) by the number of points in each class, therefore the number of size n $\mathcal{F}$-structures is equal to the number of convex linear orders of size n. Because f_I preserves E-classes, it is injective, and therefore a bijection. □

Lemma 4.3. *Every ordered equivalence relation expands uniquely to a fractured order, up to isomorphism.*

Proof. There is a unique way (up to isomorphism) to linearly order each E-class individually. Because ordering these classes determines $\prec_2$, there is a unique (up to isomorphism) way to define $\prec_2$ on any ordered equivalence relation, expanding it to an $\mathcal{F}$-structure. □

Theorem 4.4. *Unlabelled ordered equivalence relations admit a logical limit law.*

Proof. By 4.2, $\mathcal{F}$-structures are uniformly interdefinable with convex linear orders (which admit a logical limit law). Because every unlabelled ordered equivalence relation expands uniquely to a $\mathcal{F}$ structure, by 4.1, we have a limit law for ordered equivalence relations. □

References

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