Colored Convex Linear Orders and Logical Limit Laws

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Abstract

We extend previous work on logical limit laws for several classes of ordered structures to the case of structures equipped with a coloring.

1 Introduction

In [1], first-order logical limit laws were proven for convex linear orders by adapting a Markov chain-style proof of Ehrenfeucht. We present a generalization of this argument to the case of convex linear orders equipped with a coloring (henceforth, "colored convex linear orders" or "CCLOs"). These colorings are expressed by expanding the language of convex linear orders to include a countable number of unary predicates, each indicating the color of a point. Every point is assigned a color, and multiple points may have the same color. Many of the proofs here will follow similar arguments to those in of [1]. We present this note as one set of examples demonstrating how Markov chain arguments may be extended to show limit laws for broader classes of ordered structures.

2 Preliminaries

The language of t-colored convex linear orders, for $t \in \mathbb{N}$, is given by $\mathcal{L}_t = \{<, E, C_1(x), \ldots, C_t(x)\}$, where $<$ is a total order on points, E is an equivalence relation whose classes are $<$-intervals, and $C_1(x), \ldots, C_t(x)$ are unary predicates (each corresponding to a "color"). A t-colored convex linear order (t-CCLO) is a finite $\mathcal{L}_t$-structure $\mathfrak{M}$ such that, for each point x in $\mathfrak{M}$, there is exactly one $1 \le i \le t$ where $C_i(x)$ holds. Stated formally, we require that each $C_i(x)$ satisfies:

$$C_i(x) \iff \neg \bigvee_{\substack{1\le \ell \le t \\ \ell \neq i}}^{t} C_\ell(x)$$

We say that x is an *i-colored point* when $C_i(x)$ holds.

Definition 2.1. *Let $\bullet_i$ denote the CCLO with one class, containing one i-colored point.*

Definition 2.2. *For CCLOs $\mathfrak{M}$, $\mathfrak{N}$, define $\mathfrak{M} \oplus \mathfrak{N}$ to be the CCLO such that $\mathfrak{N}$ comes after $\mathfrak{M}$ with respect to $<$.*

Definition 2.3. *Let $\mathfrak{M}$ be a CCLO. Define $\widehat{\mathfrak{M}}^i$ to be the CCLO obtained by adding one i-colored point to the $<$-last class of $\mathfrak{M}$.*

We will denote the empty CCLO by $\varnothing$. As this structure contains no classes, $\widehat{\varnothing}^i$ is not well-defined.

Lemma 2.4. *Any t-CCLO of size n can be constructed uniquely, in n steps, by applying $\widehat{(-)}^i$ and $- \oplus \bullet_i$ to $\varnothing$.*

Proof. We follow an inductive argument in the same spirit as Lemma 2.4 of [1]. Let $\mathfrak{N}$ be a CCLO of size n having t colors. If $n = 1$, $\mathfrak{N}$ contains a single point of some color i; this is equivalent to $\varnothing \oplus \bullet_i$.

Assume now that any CCLO of size $n-1$ can be constructed from the above operations. For some CCLO $\mathfrak{N}$ of size n, let $\mathfrak{M}$ denote $\mathfrak{N}$ minus the $<$-last point. If the last class of $\mathfrak{N}$ contains exactly one i-colored point, $\mathfrak{N} \simeq \mathfrak{M} \oplus \bullet_i$. Otherwise, the last point of $\mathfrak{N}$ is obtained as $\widehat{\mathfrak{M}}^i$. $\square$

Example 2.5. *Suppose we are working in the language of CCLOs with two colors, drawn pointwise as $\circ$ and $\bullet$. Depicting E-classes with square brackets, and reading $<$ as left-to-right, we can visualize all 2-CCLOs of size n as shown in 2.5 for $n = 0, 1, 2$.*

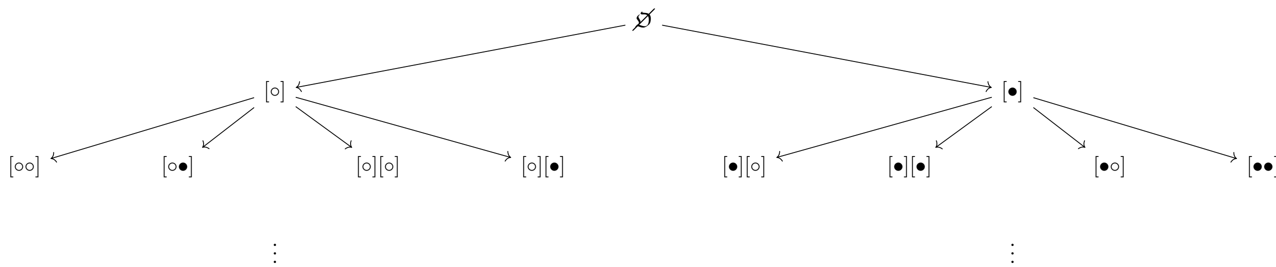

Figure 1: Constructing all 2-CCLOs of size $0, 1, 2$

We write $\mathfrak{M} \equiv_k \mathfrak{N}$ to mean structures $\mathfrak{M}$, $\mathfrak{N}$ agree up to first-order sentences with a maximum quantifier depth of k. This is equivalent to requiring that Duplicator has a winning strategy in a length k Ehrenfeucht–Fraïssé game [2].

Lemma 2.6. *Let $\mathfrak{M}, \mathfrak{N}, \mathfrak{M}', \mathfrak{N}'$ be CCLOs with $\mathfrak{M} \equiv_k \mathfrak{N}$ and $\mathfrak{M}' \equiv_k \mathfrak{N}'$. Then, $\mathfrak{M} \oplus \mathfrak{M}' \equiv_k \mathfrak{N} \oplus \mathfrak{N}'$.*

Lemma 2.7. *Suppose $\mathfrak{M} \equiv_k \mathfrak{N}$, then, $\widehat{\mathfrak{M}}^i \equiv_k \widehat{\mathfrak{N}}^i$.*

Lemma 2.8. *For a CCLO $\mathfrak{M}$ and $k \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ such that for all $s, t > \ell$,*

$$\bigoplus_s \mathfrak{M} \equiv_k \bigoplus_t \mathfrak{M}$$

Proofs of 2.6, 2.7, and 2.8 are identical to those of Lemmas 2.7, 2.8, and 2.10 respectively in [1].

3 Constructing a Markov chain

Fix a first-order sentence φ in $\mathcal{L}_t$ with quantifier rank k. We associate a Markov chain M_φ to φ in a manner similar to the uncolored case.

For a $\equiv_k$-class $\mathcal{C}$, and any $\mathfrak{M} \in \mathcal{C}$, define

$$\mathcal{C} \oplus \bullet_i := [\mathfrak{M} \oplus \bullet_i]_{\equiv_k} \;,\; \widehat{\mathcal{C}}^i := \left[\widehat{\mathfrak{M}}^i\right]_{\equiv_k}$$

By Lemmas 2.7 and 2.6, any choice of representative $\mathfrak{M}$ will yield a $\equiv_k$-equivalent result. We define M_φ recursively. The starting state is $\varnothing$. There are t possible transitions out of $\varnothing$ to $\bullet_1, \ldots, \bullet_t$, each having probability $1/t$. These initial transitions move only to CCLOs obtained from $- \oplus \bullet_i$ due to the fact that $\widehat{(-)}^i$ is not well-defined on $\varnothing$. For every $\mathcal{C} \not\simeq \varnothing$, there are $1/2t$ transitions out: one to $\widehat{\mathcal{C}}^i$ and one to $\mathcal{C} \oplus \bullet_i$ (for each $1 \leq i \leq t$). Because any t-CCLO can be constructed uniquely by applying $- \oplus \bullet_i$ and $\widehat{(-)}^i$ to $\varnothing$ repeated n times, this procedure will uniformly randomly sample all t-CCLO structures of size n.

$$
\begin{tikzcd}[row sep=small]
 & \bullet_1 \\
\varnothing \arrow[ur, "1/t"] \arrow[dr, "1/t"'] & \vdots \\
 & \bullet_t
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[row sep=small]
 & \mathcal{C} \oplus \bullet_1 \\
 & \vdots \\
 & \mathcal{C} \oplus \bullet_t \\
\mathcal{C} \arrow[uuur, "1/2t"] \arrow[ur, "1/2t"'] \arrow[dr, "1/2t"] \arrow[dddr, "1/2t"'] & \\
 & \widehat{\mathcal{C}}^1 \\
 & \vdots \\
 & \widehat{\mathcal{C}}^t
\end{tikzcd}
$$

In order for this Markov chain to converge, we require that it is aperiodic in the sense of Definition 2.11 of [1].

Lemma 3.1. *M_φ is aperiodic for all φ.*

Proof. We follow the same argument as Lemma 2.13 of [1]. Suppose M_φ were periodic. Then, there would exist disjoint sets of M_φ-states ($\equiv_k$-classes) $P_0, P_1, \ldots, P_{d-1}$ for some $d > 1$ such that for every state in P_i, M_φ transitions to a state in P_{i+1} with probability 1 (with P_{d-1} transitioning to P_0). Writing $j\bullet_i$ to mean $\bigoplus_j \bullet_i$, we have that for any $\mathcal{C} \in P_0$, $\mathcal{C} \oplus j\bullet_i$ is in P_0 iff $d \mid j$. From Lemmas 2.6 and 2.8, $\mathcal{C} \oplus j\bullet_i \equiv_k \mathcal{C} \oplus (j+1)\bullet_i$ for sufficiently large j, contradicting this. $\square$

Theorem 3.2. *The class of t-CCLOs admits a logical limit law for all $t \in \mathbb{N}$.*

Proof. Consider M_φ for some fixed φ. In any M_φ state (a $\equiv_k$-class) S of M_φ, either every structure in S satisfies φ or no structures in S satisfy φ. By the definitions of $- \oplus \bullet_i$ and $\widehat{(-)}^i$ for $\equiv_k$-classes, moving n steps in M_φ (starting from $\varnothing$) is equivalent to uniformly randomly selecting a CCLO of size n and taking its $\equiv_k$-class. Hence, the probability of M_φ being in a state which satisfies φ after n steps is equal to the probability that a randomly selected CCLO of size n satisfies φ. It is sufficient to show that the probability of M_φ being in a satisfactory state after n steps converges as $n \to \infty$; this follows from the fact that M_φ is finite and aperiodic. $\square$

4 Reduction to the uncolored case

We briefly note that limit laws for uncolored convex linear orders can be obtained as a special case of 3.2. An uncolored structure may be equivalently viewed as a colored structure with exactly one color. Hence, the relation $C_1(x)$ holds for every point x, so that there is no distinction in terms of color on the points.

We have two operations for building such structures: $\widehat{(-)}^1$ and $- \oplus \bullet_1$. These are equivalent to the corresponding operators $\widehat{(-)}$ and $- \oplus \bullet$ in Definition 2.2 and Lemma 2.4 respectively of [1] (the subscripts are dropped hereafter). Following the procedure in 3, we construct M_φ for first-order sentence φ as:

$$
\begin{tikzcd}[row sep=small]
 & & \widehat{\bullet} & \cdots \\
\varnothing \arrow[r, "1"] & \bullet \arrow[ur, "1/2"] \arrow[dr, "1/2"'] & & \\
 & & \bullet \oplus \bullet & \cdots
\end{tikzcd}
$$

The initial transition has probability 1, as there is only one way to construct $\bullet$ from the empty order. From this diagram, it can be seen that moving n steps in M_φ is equivalent to moving $n-1$ steps in the Markov chain defined by [1], due to the fact that the latter is defined starting at $\bullet$ rather than $\varnothing$. The two Markov chains will converge to the same limiting probability as $n \to \infty$.

References

[1] Samuel Braunfeld and Matthew Kukla. Logical Limit Laws for Layered Permutations and Related Structures. *Enumerative Combinatorics and Applications*, 2(S4PP2), 2021.

[2] Joel Spencer. *The Strange Logic of Random Graphs*, volume 22. Springer Science & Business Media, 2001.