Module 4: Linear Regression and Randomized Experiments

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Matthew Blackwell

Gov 2003 (Harvard)

Where are we? Where are we going?

- So far: analysis of experiments from Fisher and Neyman approaches.
  - Neyman: Unbiased estimators, (conservative) variances.
  - Fisher: exact tests of the sharp null.
- Today: how does the workhorse estimator, OLS, fit into this story?
- Why might we use regression?
  - **Simplicity**: known tool that is already very common.
  - **Increased precision**: we may want to add covariates for more precise effect estimates.

1/ Regression with no covariates

Analyzing experiments with regression?

- Under complete randomization, can we use OLS to estimate ATEs?
    - Literally just `lm(y ~ d)`?
- Recall that the OLS estimator solves the least squares problem:

$$(\widehat{\tau}_{\text{ols}}, \hat{\alpha}_{\text{ols}}) = \arg\min_{\tau,\alpha} \sum_{i=1}^{n} (Y_i - \alpha - \tau D_i)^2$$

- Remember coefficient on a binary r.v. is mechanically the diff. in means:

$$\widehat{\tau}_{\text{ols}} = \overline{Y}_1 - \overline{Y}_0 = \widehat{\tau}_{\text{diff}}$$

    - $\rightsquigarrow$ standard Neyman analysis for unbiasedness, sampling variance.
- Generalizes to discrete treatments with > 2 levels.

Justifying the linear model

- Mechanically the same, but can we justify the linear model itself?
    - Key assumptions: **linearity** and **mean independence of errors.**
- Some simple manipulations of the consistency assumption:

$$
\begin{aligned}
Y_i &= D_i Y_i(1) + (1 - D_i) Y_i(0) \\
&= Y_i(0) + D_i \{Y_i(1) - Y_i(0)\} \\
&= Y_i(0) + D_i \tau_i \\
&= \mathbb{E}[Y_i(0)] + D_i \tau + \{Y_i(0) - \mathbb{E}[Y_i(0)]\} + D_i (\tau_i - \tau) \\
&= \alpha + D_i \tau + \epsilon_i
\end{aligned}
$$

- "Linear" functional form fully justified by consistency alone with:
    - Intercept $\alpha = \mathbb{E}[Y_i(0)]$ average control outcome.
    - Slope $\tau = \mathbb{E}[Y_i(1) - Y_i(0)]$ the PATE.
    - Error is deviation for control PO + treatment effect heterogeneity.

Mean independent errors

$$\varepsilon_i = (Y_i(0) - \mathbb{E}[Y_i(0)]) + D_i \cdot (\tau_i - \tau)$$

- What about mean independence $\mathbb{E}[\varepsilon_i \mid D_i] = 0$?
- Using randomization, we know D_i is independent of $Y_i(1), Y_i(0)$, so:

$$\begin{aligned}
\mathbb{E}[\varepsilon_i \mid D_i] &= \mathbb{E}\left\{[Y_i(0) - \mathbb{E}[Y_i(0)] + D_i \cdot (\tau_i - \tau) \mid D_i\}\right] \\
&= \mathbb{E}[Y_i(0) \mid D_i] - \mathbb{E}[Y_i(0)] + D_i\left(\mathbb{E}[\tau_i \mid D_i] - \tau\right) \\
&= \mathbb{E}[Y_i(0)] - \mathbb{E}[Y_i(0)] + D_i \underbrace{\left(\mathbb{E}[\tau_i] - \tau\right)}_{\tau = \mathbb{E}[\tau_i]} \\
&= 0
\end{aligned}$$

- Randomization + consistency $\rightsquigarrow$ linear model.
  - Does not imply homoskedasticity or normal errors, though!

Homoskedasticity

- Software default assumption: **Homoskedasticity**

$$\mathbb{V}[\varepsilon_i \mid \mathbf{D}] = \sigma^2, \qquad \forall i$$

- But in general, based on previous error definition:

$$\mathbb{V}[\varepsilon_i \mid \mathbf{D}] = \mathbb{V}[\varepsilon_i \mid D_i] = D_i\sigma_1^2 + (1 - D_i)\sigma_0^2$$

  - $\rightsquigarrow$ homoskedasticity true when $\sigma_1^2 = \mathbb{V}[Y_i(1)] = \mathbb{V}[Y_i(0)] = \sigma_0^2$
  - True under constant treatment effects!

- Under homoskedasticity, variance of the OLS estimator is:

$$\mathbb{V}[\widehat{\tau}_{\text{ols}} \mid \mathbf{D}] = \frac{\sigma^2}{\sum_{i=1}^{n}\left(D_i - \overline{D}\right)^2}$$

Variance estimation

- "Standard" variance estimator under homoskedasticity:

$$\widehat{\mathbb{V}}_{const} = \frac{\frac{1}{n-2}\sum_{i=1}^{n} \hat{\varepsilon}_i}{\sum_{i=1}^{n}(D_i - \overline{D})^2} = \frac{\frac{1}{n-2}\sum_{i=1}^{n}(Y_i - \hat{\alpha}_{\text{ols}} - \widehat{\tau}_{\text{ols}} D_i)^2}{\sum_{i=1}^{n}(D_i - \overline{D})^2}$$

- We can rewrite this as a function of the **pooled** variance $\widehat{\sigma}^2_{Y|D}$:

$$\widehat{\mathbb{V}}_{const} = \widehat{\sigma}^2_{Y|D}\left(\frac{1}{n_0} + \frac{1}{n_1}\right)$$

$$\widehat{\sigma}^2_{Y|D} = \frac{1}{n-2}\left(\sum_{i:D_i=0}(Y_i - \overline{Y}_0)^2 + \sum_{i:D_i=1}(Y_i - \overline{Y}_1)^2\right)$$

- **Inconsistent**: $\widehat{\mathbb{V}}_{const} - \mathbb{V}[\widehat{\tau}] \overset{p}{\to} c \neq 0$ unless
  - Homoskedasticity holds: $\sigma_1^2 = \sigma_0^2$
  - Design is balanced: $n_1 = n_0$

Robust SEs

- Eicker-Huber-White (EHW) robust/sandwich variance estimator:

$$\widehat{\mathbb{V}}_{\text{EHW}} = \left(\sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i'\right)^{-1} \left(\sum_{i=1}^{n} \hat{\varepsilon}_i^2 \mathbf{X}_i \mathbf{X}_i'\right) \left(\sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i'\right)^{-1}$$

$$= (\mathbb{X}'\mathbb{X})^{-1} \left(\sum_{i=1}^{n} \hat{\varepsilon}_i^2 \mathbf{X}_i \mathbf{X}_i'\right) (\mathbb{X}'\mathbb{X})^{-1} \quad \text{where} \quad \mathbb{X} = [1 \;\; \mathbf{D}]$$

- Recall the PATE-targeted variance of the difference-in-means:

$$\mathbb{V}(\widehat{\tau}_{\text{diff}}) = \frac{\sigma_0^2}{n_0} + \frac{\sigma_1^2}{n_1} = \frac{\mathbb{V}[Y_i(0)]}{n_0} + \frac{\mathbb{V}[Y_i(1)]}{n_1}$$

- To see this, we can derive $\widehat{\mathbb{V}}_{\text{EHW}}$ under our case:

$$\widehat{\mathbb{V}}_{\text{EHW}} = \frac{\tilde{\sigma}_1^2}{n_1} + \frac{\tilde{\sigma}_0^2}{n_0}, \quad \text{where} \quad \tilde{\sigma}_d^2 = \frac{1}{n_d} \sum_{i: D_i = d} \left(Y_i - \overline{Y}_d\right)^2$$

  - $\tilde{\sigma}_0^2, \tilde{\sigma}_1^2$ consistent for $\sigma_0^2, \sigma_1^2 \rightsquigarrow \widehat{\mathbb{V}}_{\text{EHW}}$ consistent for $\mathbb{V}(\widehat{\tau}_{\text{diff}})$

Better robust SEs

- Many different "improved" versions of robust variances proposed.
    - Almost all are "finite-sample corrections" (no asymptotic effects)
- HC2 estimator normalizes residuals by the leverage, h_{ii}:

$$\widehat{\mathbb{V}}_{\text{HC2}} = (\mathbb{X}'\mathbb{X})^{-1} \left(\sum_{i=1}^{n} \frac{\widehat{\varepsilon}_i^2}{1 - h_{ii}} \mathbf{X}_i \mathbf{X}_i' \right) (\mathbb{X}'\mathbb{X})^{-1}$$

    - Leverage: $h_{ii} = \mathbf{X}_i (\mathbb{X}'\mathbb{X})^{-1} \mathbf{X}_i'$
    - In this setting, $h_{ii} = n_1^{-1}$ if $D_i = 1$ and n_0^{-1} if $D_i = 0$
- Samii & Aronow (2012): HC2 is exactly the Neyman variance estimator:

$$\widehat{\mathbb{V}}_{\text{HC2}} = \frac{\widehat{\sigma}_0^2}{n_0} + \frac{\widehat{\sigma}_1^2}{n_1}$$

- $\rightsquigarrow$ simple OLS + HC2 = unbiased point and variance estimator.

2/ Linear regression with covariates

Adding covariates

- What if we add covariates to our regression model?

$$(\widehat{\tau}_{\text{adj}}, \hat{\alpha}_{\text{adj}}, \hat{\beta}_{\text{adj}}) = \arg\min_{\tau,\alpha,\beta} \sum_{i=1}^{n} \left(Y_i - \alpha - \tau D_i - \widetilde{\mathbf{X}}_i'\beta \right)^2$$

  - $(\widetilde{\mathbf{X}}_i - \overline{\mathbf{X}})$ are **centered** covariates for notational ease.
- Why might we do this? To increase **precision** of our estimates.
  - We hope $\mathbb{V}[\widehat{\tau}_{\text{adj}}] < \mathbb{V}[\widehat{\tau}_{\text{diff}}]$ so we have smaller CIs, more powerful tests, etc
  - Intuition: less residual variation in Y_i after accounting for $\mathbf{X}_i$
- Questions:
  - Is $\widehat{\tau}$ still unbiased? Consistent?
  - Should we expect an increase in precision?
  - Controversial! Freedman (2008): "Randomization does not justify the regression model"

OLS is biased, but consistent

- Agnostic approach: don't assume correctness of the linear model.
  - $\mathbf{X}_i'\beta$ could be badly misspecified.
  - No assumptions about homoskedasticity.
- Under minimal assumptions, OLS is consistent for the linear projection

$$(\widehat{\tau}_{\text{adj}}, \hat{\alpha}_{\text{adj}}, \hat{\beta}_{\text{adj}}) \xrightarrow{p} (\tau_0, \alpha_0, \beta_0) = \underset{(\tau,\alpha,\beta)}{\arg\min}\, \mathbb{E}\left[\left(Y_i - \alpha - \tau D_i - \widetilde{\mathbf{X}}_i'\beta\right)^2\right]$$

- $\widehat{\tau}_{\text{adj}}$ now **biased** for τ though bias should be small.
- But $\widehat{\tau}_{\text{adj}}$ is **consistent** for τ.
  - Intuition: omitted variable bias. Since $D_i \perp\!\!\!\perp \mathbf{X}_i$, including $\widetilde{\mathbf{X}}_i$ won't (asymptotically) affect coefficient on D_i.
- Freedman (2008) shows the same thing for finite-sample inference.

Variance of adjustment estimator

- Complete randomization + single, mean-zero covariate X_i
  - Generalizes easily to more covariates.
  - Let $\sigma_{0x} = \text{cov}(Y_i(0), X_i)$ and $\sigma_{1x} = \text{cov}(Y_i(1), X_i)$.
  - Probability of treatment $p = n_1/n$
- Freedman (2008) derived gains from adjusting for X_i using OLS:

$$\mathbb{V}[\widehat{\tau}_{\text{diff}}] - \mathbb{V}[\widehat{\tau}_{\text{adj}}] = \frac{\sigma_{0x}\left\{\sigma_{0x} + 2(1-2p)\sigma_{1x}\right\}}{np(1-p)}$$

- Will adjustment decrease the sampling variance?
  - If design in balanced, $p = 1/2$, then adjustment always helps.
  - Design imbalance and correlation with "smaller" potential outcome could lead to adjustment hurting.
- Estimation: EHW robust variance estimators are consistent or asymptotically conservative for $\mathbb{V}[\widehat{\tau}_{\text{adj}}]$

Regression with full interactions

- OLS estimator from fully interacted model, $\widehat{\tau}_{\text{inter}}$:

$$Y_i = \alpha + \tau D_i + \widetilde{\mathbf{X}}_i'\beta + D_i\widetilde{\mathbf{X}}_i'\gamma + \varepsilon_i$$

- Equivalent to running separate Y_i on $\widetilde{\mathbf{X}}_i$ in each D_i group
- As with non-interacted model, $\widehat{\tau}_{\text{inter}}$ is consistent for τ and asymptotically normal.
- Lin (2013): **fully interacted model will never hurt precision asymptotically**.
    - Freedman critique was right, but Lin shows an easy way to resolve.
- EHW robust variance estimator is consistent or asymptotically conservative.

Summarizing regression

- Regression with no covariates: standard Neyman analysis.
- Regression with (uninteracted) covariates:
  - Consistent for SATE/PATE.
  - Usually will help precision, but can hurt.
- Regression with interacted covariates:
  - Consistent for SATE/PATE
  - Asymptotically will never hurt precision.
- Always use robust/HC2 variance estimators unless you have good reasons.

Regression for stratified experiments

- Setup: block randomized experiment with block indicators W_{ij}.
    - Block "fixed effects" $W_{ij} = 1$ if i is in block j, 0 otherwise.
    - Blocks $j \in \{1, \dots, J\}$ with sizes $w_j = n_j/n$ and propensity scores $p_j = n_{1,j}/n_j$
- Can we just include the block FEs in OLS?

$$\left(\widehat{\tau}_{\text{b,fe}}, \hat{\alpha}_1, \dots, \hat{\alpha}_J\right) = \underset{(\tau, \alpha_1, \dots, \alpha_J)}{\arg\min} \sum_{i=1}^{n} \left(Y_i - \tau D_i - \sum_{j=1}^{J} \alpha_j W_{ij} \right)$$

- Converges to a weighted average of block-specific effects, τ_j:

$$\widehat{\tau}_{\text{b,fe}} \xrightarrow{p} \frac{\sum_{j=1}^{J} \omega_j \tau_j}{\sum_{j=1}^{J} \omega_j} \quad \text{where} \quad \omega_j = w_j p_j (1 - p_j)$$

- $\widehat{\tau}_{\text{b,fe}}$ not consistent for the PATE unless:
    - Propensity scores are equal across blocks: $p_j = p$ for all j.
    - ATEs are equal across strata $\tau_j = \tau$ for all j.

Correct analysis of block randomized trials

1. Just use original Neyman analysis aggregating within-strata analyses.
2. Weight OLS by inverse of the propensity score: $1/p_j$.
3. Fully interact block FEs with treatment.

- Latter two allow for additional covariates to be added.

3/ Cluster randomized experiments

Clustering treatments

- Treatment often allocated at a higher level than the data.
    - Counties are treated, but we have individual-level data.
    - Classrooms are treated, but we have student data.
- Has considerable benefits:
    - Often cheaper/easier to implement than individual assignment.
    - Allows for interference within clusters without bias.
- But lots of confusion about how to analyze.
    - More valuable to add more individuals or clusters?
    - What to do with individual-level covariates?

Cluster randomized trials

- Setup:
  - Clusters: $k \in \{1, \dots, K\}$
  - Randomly choose K_1 treatment clusters, K_0 control.
  - Each cluster has units $i \in \{1, \dots, m_k\}$ with $\sum_{k=1}^{K} m_k = n$
  - Treatment assignment at cluster level: $D_{ik} = D_k$
  - Potential outcomes $Y_{ik}(d)$
- Random assignment at the cluster level: $\{Y_{ik}(1), Y_{ik}(0)\} \perp\!\!\!\perp D_j$.
- Quantity of interest still at individual level:

$$\text{SATE} = \frac{1}{n} \sum_{k=1}^{K} \sum_{i=1}^{m_k} \{Y_{ik}(1) - Y_{ik}(0)\}$$

Analysis of clustered experiments

- Simple setting: all clusters have the same size $m_k = m$ for all k.
- Simple difference in means is unbiased:

$$
\begin{aligned}
\widehat{\tau}_{\text{cl}} &= \frac{1}{mK_1} \sum_{k=1}^{K} \sum_{i=1}^{m_k} D_k Y_{ik} - \frac{1}{mK_0} \sum_{k=1}^{K} \sum_{i=1}^{m_k} (1 - D_k) Y_{ik} \\
&= \frac{1}{K_1} \sum_{k=1}^{K} D_k \overline{Y}_k - \frac{1}{K_0} \sum_{k=1} (1 - D_k) \overline{Y}_k
\end{aligned}
$$

  - $\overline{Y}_k$ is the cluster average: $\frac{1}{m} \sum_{i=1}^{m} Y_{ik}$
  - Unbiasedness follows from Neyman-style analysis at cluster level.
  - Estimator is biased, but consistent (in K) if cluster size varies.
- Neyman-style conservative variance:

$$
\mathbb{V}[\widehat{\tau}_{\text{cl}} \mid \mathbf{O}] \leq \frac{\mathbb{V}[\overline{Y}_k(1)]}{J_1} + \frac{\mathbb{V}[\overline{Y}_k(0)]}{J_0} \quad \text{where for } d = 0, 1 \quad \overline{Y}_k(d) = \frac{1}{m} \sum_{m=1}^{m} Y_{ik}(d)
$$

Cost of clustering

- Standard variance under **individual assignment**:

$$\mathbb{V}[\widehat{\tau}_{\text{diff}}] = \frac{\mathbb{V}[Y_{ik}(1)]}{mK_1} + \frac{\mathbb{V}[Y_{ik}(0)]}{mK_0}$$

- How different is variance under clustering compare to no clustering?

$$\frac{\mathbb{V}[\overline{Y}_k(1)]}{K_1} = \frac{\mathbb{V}[Y_{ik}(1)]}{mK_1}(1 + (m-1)\rho_1)$$

$$\rho_1 = \frac{1}{m(m-1)} \sum_{i=1}^{m} \sum_{j \neq i} \mathbb{E}\left\{(Y_{ik}(1) - \overline{Y}(1))(Y_{jk}(1) - \overline{Y}(1))\right\}$$

- ρ_1 is the intracluster correlation coefficient (ICC)
  - Measures how similar units are within clusters.
  - Usually cluster is less efficient because $\rho_1 > 0$
  - More similarity $\rightsquigarrow$ each unit provides redundant information $\rightsquigarrow$ less efficiency under clustering

Cluster robust standard errors

- What if we want to use OLS at individual level?
  - Adding individual and unit controls.
- **Cluster-robust variance estimator** for OLS:

$$\widehat{\mathbb{V}}_{\text{cl}}[\widehat{\alpha}, \widehat{\tau}] = (\mathbb{X}'\mathbb{X})^{-1} \left(\sum_{k=1}^{K} \mathbb{X}_k' \widehat{\boldsymbol{\varepsilon}}_k \widehat{\boldsymbol{\varepsilon}}_k' \mathbb{X}_k \right) (\mathbb{X}'\mathbb{X})^{-1}$$

  - Here $\mathbb{X}' = [1 \;\; \mathbf{D}]'$, $\mathbb{X}_k' = [1 \;\; \mathbf{D}_k]$, and $\widehat{\boldsymbol{\varepsilon}}_k = (\widehat{\varepsilon}_{1k}, \ldots, \widehat{\varepsilon}_{mk})$
  - Consistent as the number of clusters grows.
- **Cluster at the treatment assignment level** (no higher or lower)
- Vanilla CRVE is biased, Bell & McCaffrey proposed CR2 adjustment similar to HC2 (usually preferable)