Module 7b: Weighting Estimators

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Gov 2003 (Harvard)

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  - $K_M(i)$ is the number of times i is used as a match.
- Weighting estimators choose the weights directly to reduce imbalance.

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- Key idea: reweight sample to be representative of population.

Horvitz-Thompson for treatment effects

- Applying HT potential outcomes: weight by inverse propensity score.

$$\widehat{\text{ATE}} = \widehat{\tau}_{ipw} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{D_i Y_i}{\widehat{\pi}(\mathbf{X}_i)} - \frac{(1 - D_i)Y_i}{1 - \widehat{\pi}(\mathbf{X}_i)}\right)$$

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    - A kind of “continuous” version of matching with replacement.

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- Practically, weighted least squares gives automatic normalization:

$$(\widehat{\alpha}_{\text{wls}}, \widehat{\tau}_{\text{wls}}) = \underset{\alpha,\tau}{\arg\min} \sum_{i=1}^n \left(\frac{D_i}{\widehat{\pi}(\mathbf{X}_i)} + \frac{1-D_i}{1-\widehat{\pi}(\mathbf{X}_i)} \right) (Y_i - \alpha - \tau D_i)^2$$

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  - Moment conditions for the propensity score model with parameters θ:

$$\mathbb{E}\left[\underbrace{\left(\frac{D_i}{\pi_\theta(\mathbf{X}_i)} - \frac{1-D_i}{1-\pi_\theta(\mathbf{X}_i)}\right)\frac{\partial \pi_\theta(\mathbf{X}_i)}{\partial \theta}}_{\text{score for treatment model}}\right] = 0$$

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- Replace with sample versions and use delta method to get asymptotic variance.

Estimated versus known pscores

```r
ht.est <- function(y, d, w) {
  n <- length(y)
  (1/n) * sum((y * d * w) - (y * (1-d) * w))
}
n <- 200
x <- rbinom(n, size = 1, prob = 0.5)
dprobs <- 0.5*x + 0.4*(1-x)
d <- rbinom(n, size = 1, prob = dprobs)
y <- 5 * d - 10 * x + rnorm(n, sd = 5)

true.w <- ifelse(d == 1, 1/dprobs, 1/(1-dprobs))
pprobs <- predict(glm(d ~ x))
est.w <- ifelse(d == 1, 1/pprobs, 1/(1 - pprobs))
ht.est(y, d, est.w)
```

```
## [1] 5.22
```

```r
ht.est(y, d, true.w)
```

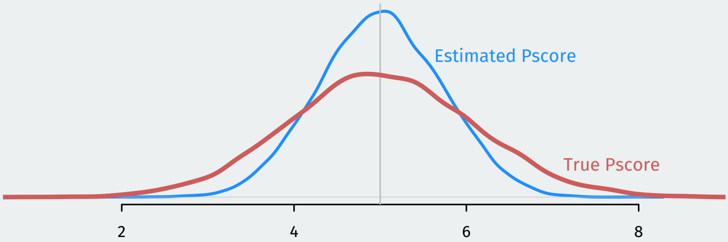
```
## [1] 5.56
```

Sampling distribution of the HT estimators

```
sims <- 10000
true.holder <- rep(NA, sims)
est.holder <- rep(NA, sims)
for (i in 1:sims) {
  x <- rbinom(n, size = 1, prob = 0.5)
  dprobs <- 0.5*x + 0.4*(1-x)
  d <- rbinom(n, size = 1, prob = dprobs)

  y <- 5 * d - 10 * x + rnorm(n, sd = 5)
  true.w <- ifelse(d == 1, 1/dprobs, 1/(1-dprobs))
  pprobs <- predict(glm(d ~ x))
  est.w <- ifelse(d == 1, 1/pprobs, 1/(1 - pprobs))
  est.holder[i] <- ht.est(y, d, est.w)
  true.holder[i] <- ht.est(y, d, true.w)
}
```

Sampling distribution of the HT estimators



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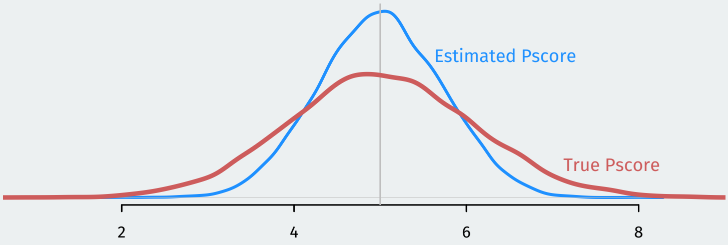


```
var(est.holder)
```

```
## [1] 0.506
```

```
var(true.holder)
```

```
## [1] 1.15
```

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- **Removing chance variations** using $\widehat{\pi}(\mathbf{X}_i)$ adjusts for any small imbalances that arise because of a finite sample.
- True PS only adjusts for the **expected** differences between samples.
- Only true if propensity score model is **correctly specified!!**

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&= \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{\mu}_1(\mathbf{X}_i) - \hat{\mu}_0(\mathbf{X}_i)\right. \\
&\qquad\qquad \left. + \left(\frac{D_i(Y_i - \hat{\mu}_1(\mathbf{X}_i))}{\widehat{\pi}(\mathbf{X}_i)} - \frac{(1-D_i)(Y_i - \hat{\mu}_0(\mathbf{X}_i))}{1-\widehat{\pi}(\mathbf{X}_i)}\right)\right\}
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    - Can allow each model to be more flexible without hurting asymptotics.

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&\qquad\qquad \left. - \left(\frac{D_i-\widehat{\pi}(\mathbf{X}_i)}{\widehat{\pi}(\mathbf{X}_i)}\hat{\mu}_1(\mathbf{X}_i) - \frac{D_i-\widehat{\pi}(\mathbf{X}_i)}{1-\widehat{\pi}(\mathbf{X}_i)}\hat{\mu}_0(\mathbf{X}_i)\right)\right\} \\
&= \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{\mu}_1(\mathbf{X}_i) - \hat{\mu}_0(\mathbf{X}_i)\right. \\
&\qquad\qquad \left. + \left(\frac{D_i(Y_i-\hat{\mu}_1(\mathbf{X}_i))}{\widehat{\pi}(\mathbf{X}_i)} - \frac{(1-D_i)(Y_i-\hat{\mu}_0(\mathbf{X}_i))}{1-\widehat{\pi}(\mathbf{X}_i)}\right)\right\}
\end{aligned}
$$

- **Double robustness**: consistent if either $\widehat{\pi}(\mathbf{X}_i)$ or $\hat{\mu}_d(\mathbf{X}_i)$ are consistent.
    - Can allow each model to be more flexible without hurting asymptotics.
- **Efficient**: lowest asymptotic variance among consistent estimators when PS model is correct.

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  - Find parameters that get closest to satisfying sample moment conditions

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    - Can include X_{ik}^2 etc to balance other parts of the distribution.