Module 5(b): Sensitivity Analysis and Partial Identification

Fall 2021

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Gov 2003 (Harvard)

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    2. **Partial identification**: abandon point identification and try to find bounds for the ATE under different assumptions.

1/ Sensitivity analysis

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- Standard **omitted variable bias formula**:

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- Partial R^2 is the incremental predictive value of one variable:

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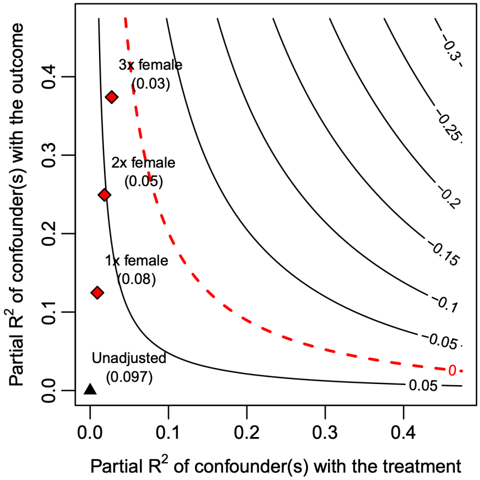
  - From these we can determine the bias and thus the true value of τ

Sensitivity analysis example



2/ Partial identification and bounds

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- If Y is bounded $[y_L, y_U]$, τ logically must be in $[y_L - y_U, y_U - y_L]$.
- Can we improve using data? Rewrite the ATE with $p = \mathbb{P}(D_i = 1)$:

$$\begin{aligned}\tau = &\ \mathbb{E}[Y_i \mid D_i = 1]p + \mathbb{E}[Y_i(1) \mid D_i = 0](1-p) \\ &- \mathbb{E}[Y_i(0) \mid D_i = 1]p - \mathbb{E}[Y_i \mid D_i = 0](1-p)\end{aligned}$$

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$$\tau \leq \mathbb{E}[Y_i \mid D_i = 1]p + y_U(1-p) - y_L p - \mathbb{E}[Y_i \mid D_i = 0](1-p)$$

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  - These bounds have width of $|y_U - y_L|$ which is half of the logical bounds.
  - But always will contain 0. Weak assumptions $\rightsquigarrow$ weak inferences

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    - Interval will still always include zero.

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  2. Covers the true value of the parameter with probability $1 - \alpha$

     $$\mathbb{P}(\tau \in [\hat{\delta}_L, \hat{\delta}_U]) \geq 1 - \alpha$$

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  - If $\delta_L < \tau < \delta_U$, then coverage converges to 1.