Module 3(b): Inference for Blocked and Matched Pair Designs

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Gov 2003 (Harvard)

1/ Block randomized experiments

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    - Some confusion in the literature: can blocking hurt?
    - Care needed: comparison depends on sample assumptions (Pashley & Miratrix, 2021, JEBS)

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- SATEs within blocks defined by V_i:

$$\tau_{\text{v, fs}} = \frac{1}{n_{\text{v}}} \sum_{i:V_i=1} \{Y_i(1) - Y_i(0)\} \qquad \tau_{\text{nv, fs}} = \frac{1}{n_{\text{nv}}} \sum_{i:V_i=0} \{Y_i(1) - Y_i(0)\}$$

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- Iterated expectations gives us:

$$\tau_{\text{fs}} = \underbrace{\left(\frac{n_{\text{v}}}{n_{\text{v}} + n_{\text{nv}}}\right)}_{\text{fraction voters}} \tau_{\text{v, fs}} + \underbrace{\left(\frac{n_{\text{nv}}}{n_{\text{v}} + n_{\text{nv}}}\right)}_{\text{fraction nonvoters}} \tau_{\text{nv, fs}}$$

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- Blocking ensures balance across blocks:
  - When $p_{\text{v}} = p_{\text{nv}}$, distribution of treatment is exactly the same in each block.
  - With complete randomization, treatment might be very imbalanced across V_i.
  - No possibility of "chance" imbalances skewing the estimates.

Estimators in blocked designs

- Within-strata difference in means:

$$\widehat{\tau}_{\text{v}} = \overline{Y}_{1,\text{v}} - \overline{Y}_{0,\text{v}} = \frac{1}{n_{1,\text{v}}} \sum_{i:V_i=1} D_i Y_i - \frac{1}{n_{0,\text{v}}} \sum_{i:V_i=1} (1 - D_i) Y_i$$

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  - Otherwise, standard $\widehat{\tau}_{\text{diff}}$ under block design will be **biased**.

Sampling variance of blocking estimator

- Each block is a completely randomized experiment so we have:

$$\mathbb{V}(\widehat{\tau}_{\mathrm{v}} \mid \mathbf{O}) = \frac{S^2_{1,\mathrm{v}}}{n_{1,\mathrm{v}}} + \frac{S^2_{0,\mathrm{v}}}{n_{0,\mathrm{v}}} - \frac{S^2_{\tau_i,\mathrm{v}}}{n_{\mathrm{v}}}$$

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$$\widehat{\mathbb{V}}_b = \left(\frac{n_{\mathrm{v}}}{n}\right)^2 \left(\frac{\widehat{\sigma}^2_{1,\mathrm{v}}}{n_{1,\mathrm{v}}} + \frac{\widehat{\sigma}^2_{0,\mathrm{v}}}{n_{0,\mathrm{v}}}\right) + \left(\frac{n_{\mathrm{nv}}}{n}\right)^2 \left(\frac{\widehat{\sigma}^2_{1,\mathrm{nv}}}{n_{1,\mathrm{nv}}} + \frac{\widehat{\sigma}^2_{0,\mathrm{nv}}}{n_{0,\mathrm{nv}}}\right)$$

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  - $\widehat{\sigma}^2_{d,\mathrm{v}}$ are the within-strata **observed outcome variances**

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$$\widehat{\tau}_b = \sum_{j=1}^{J} w_j \widehat{\tau}_j, \qquad \hat{\mathbb{V}}(\widehat{\tau}_b) = \sum_{j=1}^{J} w_j^2 \hat{\mathbb{V}}(\widehat{\tau}_j)$$

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- Difference can be positive or negative (blocking can hurt or help)

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Efficiency of blocking

- Efficiency of block versus CR depends on the sampling scheme.
    - Usually blocking will be more efficient/lower variance, but not always.
- Finite sample difference in sampling variances:

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    - Blocking always more efficient for PATE under stratified sampling

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    - Possible to get optimal blocks with **pair matching** ($J = n/2$).

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- Across-pair variance estimator (conservative for SATE):

$$\widehat{\mathbb{V}}(\widehat{\tau}_p) = \frac{1}{J(J-1)} \sum_{j=1}^{J} \{W_j(Y_{1j} - Y_{2j} - \widehat{\tau}_p)\}^2$$