

Fantastic sheaves and where to find them

An MSP101 talk

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Overview

Goal: to show sheaves are useful modelling gadgets.

Two avenues:

1. **Geometry:** local-to-global behaviour, cohomology
2. **Logic:** local modalities, system specifications, models

Basics

Let X be a topological space, $\mathcal{O}(X)$ its frame of open sets.

Definition

A **presheaf** on X is a functor

$$F : \mathcal{O}(X)^{\text{op}} \longrightarrow \mathbf{Set}$$

Elements of $F(U)$ are called **sections**. The map $F(V \subseteq U)$ is called **restriction**, and we write $s|_V := F(V \subseteq U)(s)$.

Definition

A **sheaf** on X is a presheaf F such that for every open covering $\{U_i\}_{i \in I}$:

$$\text{sheaf condition : } F(\text{colim } U_i) \cong \lim F(U_i).$$

The sheaf condition is a ‘continuity’ or a ‘locality’ condition.

Sheaf condition

Let's unpack it:

1. The colimit of a covering is

$$\operatorname{colim} U_i = \bigcup_i U_i =: U.$$

Elements of $F(U)$ are 'globally defined sections' (wrt to U).

2. Elements of $\lim F(U_i)$ are 'locally defined sections' which satisfy a **compatibility condition**:

$$\lim F(U_i) = \{(s_i)_{i \in I} \mid s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for all } i, j \in I\}$$

3. There is a universal morphism

$$\varphi : F(\operatorname{colim} U_i) \longrightarrow \lim F(U_i)$$

$$s \longmapsto (s|_{U_i})_{i \in I}$$

Sheaf condition

$$\begin{aligned}\varphi : F(\operatorname{colim} U_i) &\longrightarrow \lim F(U_i) \\ s &\longmapsto (s|_{U_i})_{i \in I}\end{aligned}$$

Then:

1. φ mono means

separation axiom: $s = t \in F(U)$ iff $s|_{U_i} = t|_{U_i}$ for all $i \in I$.

2. φ epi means

glueing axiom: every compatible assignment of sections $(s_i)_{i \in I} \in \lim F(U_i)$ 'glues' to a global section $s \in F(U)$.

A presheaf satisfying separation is called **separated**.

A sheaf satisfies both.

Sheaf condition

Example

The canonical example is $C(-; \mathbb{R})$ of continuous real functions.

Also: smooth functions, measurable functions, etc. (in fact many structures can be encoded directly in a *structure sheaf*).

Non-example

Separated presheaves which are not sheaves:

1. Constant functions: if U and V are disjoint open sets, there's no way to glue two constant functions with different values on U and V , *even though they agree on $U \cap V$* .
2. Bounded functions: choose an infinite covering, even though every local section might be bounded there's no guarantee their glueing will be bounded (e.g. $\lambda x. x^2|_{(n, n+1)}$)

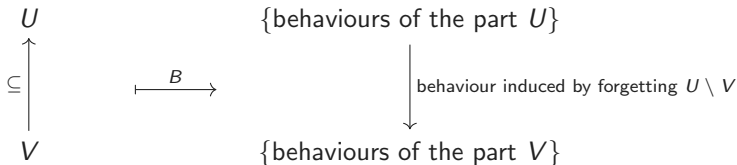
Non-separated presheaves are rare to find in practice (though easy to construct)

Sheaf condition

The sheaf condition can be read also in terms of **systems theory**:

Suppose X is a 'system', and its open sets are 'parts'.

Then the **functor of behaviours** is a presheaf:

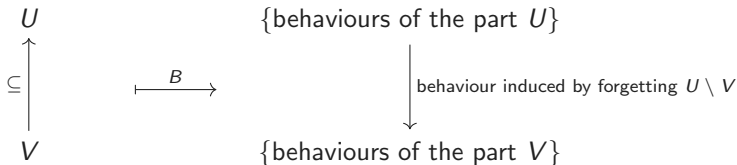


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Separation: *behaviours can be distinguished at the level of parts.*

Glueing: *compatible behaviours of the parts form global behaviours.*

Sheafification

The inclusion of sheaves into presheaves has a left adjoint:

$$\mathrm{Sh}(X) \begin{array}{c} \xleftarrow{a} \\ \xrightarrow{i} \end{array} \mathrm{Psh}(X)$$

Informally, you get it by putting ‘locally’ in front of properties.

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Example

1. $\underline{S}^a(U) =$ **locally** constant maps $U \rightarrow S$ (constant *sheaf* at S)
2. $Bdd_{\mathbb{R}}^a(U) =$ **locally** bounded maps $U \rightarrow \mathbb{R}$

In general,

$$F^a(U) = \operatorname{colim}_{\mathcal{U} \text{ hyper.}} \lim F(\mathcal{U}) = \text{formal glueings over (hyper)covers}$$

Sheaf cohomology

Hence:

(pre)sheaves mediate the passage from local to global.

Most importantly, they enable the right tooling to study obstructions to this passage, namely **sheaf cohomology**.

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Čech

Reasons to prefer Čech to plain cohomology:

1. It's computational (it's a simplicial cohomology in disguise)
2. Easily generalized to non-abelian sheaves (e.g. **Set**-sheaves)

For this talk though, let's stick to the *abelian* version, hence we will assume F is a *sheaf valued in **Ab*** (or any abelian cat if you know what that means)

Čech cohomology

Take a 'good' cover $\{U_i\}_{i \in I}$ of X . (e.g. contractible intersections)

Consider the following **chain complex**: (meaning $\text{im } d_n \subseteq \ker d_{n+1}$)

$$0 \rightarrow \prod_i F(U_i) \xrightarrow{d_0} \prod_{i,j} F(U_i \cap U_j) \xrightarrow{d_1} \prod_{i,j,k} F(U_i \cap U_j \cap U_k) \xrightarrow{d_2} \dots$$

where

$$d_0((s_i)_i) = (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})_{i,j}$$

$$d_1((s_{i,j})_{i,j}) = (s_{i,j}|_{U_i \cap U_j \cap U_k} - s_{j,k}|_{U_i \cap U_j \cap U_k} + s_{k,i}|_{U_i \cap U_j \cap U_k})_{i,j,k}$$

and so on...

Čech cohomology

The cohomology of the complex measures how far is $\text{im } d_n$ from coinciding with $\ker d_{n+1}$: ('exactness')

$$\begin{array}{ccccc} 0 \longrightarrow \prod_i F(U_i) & \xrightarrow{d_0} & \prod_{i,j} F(U_i \cap U_j) & \xrightarrow{d_1} & \prod_{i,j,k} F(U_i \cap U_j \cap U_k) \xrightarrow{d_2} \dots \\ \downarrow \text{~~~~~} & & \downarrow \text{~~~~~} & & \downarrow \text{~~~~~} \\ H^0 = \ker d_0 & & H^1 = \frac{\ker d_1}{\text{im } d_0} & & H^2 = \frac{\ker d_2}{\text{im } d_1} \end{array}$$

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Meaning:

H^0 = local sections $(s_i)_i$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \forall i, j \in I$
 $\stackrel{\text{glueing}}{=} \text{global sections.}$

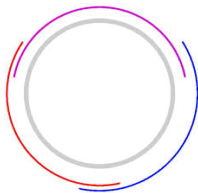
H^1 = compatible local sections on $U_i \cap U_j$

which do not arise from an assignment on the opens

\approx 1-holes

$H^n = \dots \approx n\text{-holes}$

Čech cohomology: example

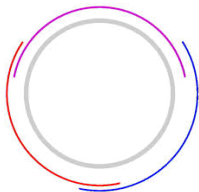


Cover S^1 as in the picture,

let $F =$ constant sheaf at K field.

$$\check{\text{Cech complex:}} \quad 0 \longrightarrow K^3 \xrightarrow{d_0} K^3 \xrightarrow{d_1} 0$$

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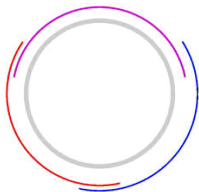
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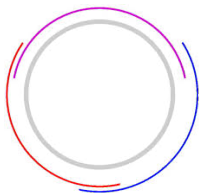
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$$d_0 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \implies \operatorname{rk}(d_0) = 2 \implies \begin{cases} \dim \ker d_0 = 1, \\ \dim \operatorname{im} d_0 = 2 \end{cases}$$

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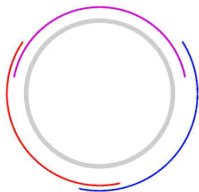
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$$d_1 = 0$$

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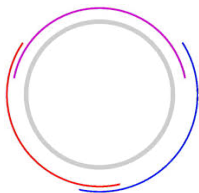
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$$d_1 = 0$$

$$\implies \dim \ker d_1 = 3$$

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$$\implies \dim \ker d_1 = 3$$

Hence

$$H^0 = K^1$$

1 0-hole = 1 connected component

$$H^1 \cong K^3 / K^2 \cong K^1$$

1 1-hole

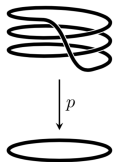
$$H^n = 0 \cong K^0$$

0 n-holes for all $n \geq 2$

Čech cohomology

Cohomology is not completely determined by the topology of the space though, it is really determined by the structure of the sheaf (it's the cohomology of its étalé space) .

Example



Repeat the previous computation with a different *locally* constant sheaf.

Hence **topological obstructions** \neq **local-to-global obstructions**!

Čech cohomology: systems theory

Let B (presheaf of behaviours) be a **presheaf** of abelian groups, let $\{U_i\}_{i \in I}$ open covering of U .

Its **augmented Čech complex** is:

$$0 \longrightarrow B(U) \xrightarrow{d_{-1}} \prod_i B(U_i) \xrightarrow{d_0} \prod_{i,j} B(U_i \cap U_j) \xrightarrow{d_1} \dots$$

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$H^{\geq 1}$ = same as before! \rightsquigarrow **higher-order emergent behaviour...**?

Thanks for your attention!

Questions?