University of Strathclyde

Fantastic sheaves and where to find them

An MSP101 talk

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October 22, 2020 (Day 300 of the COVID Era)

Geometry

Geometry

Idea:

sheaves are 'continuously parametrized' objects on a topological space.

We are going to see

1. Sheaf condition as a continuity requirement
2. 'Local-to-global' behaviour
3. Sheaf cohomology

Basics

Let X be a topological space, $\mathcal{O}(X)$ its frame of open sets.

Definition

A **presheaf** on X is a functor

$$F : \mathcal{O}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

Elements of $F(U)$ are called **sections**. The map $F(V \subseteq U)$ is called **restriction**, and we write $s|_V := F(V \subseteq U)(s)$.

Definition

A **sheaf** on X is a presheaf F such that for every open covering $\{U_i\}_{i \in I}$:

$$\textbf{sheaf condition}: \quad F(\operatorname{colim} U_i) \cong \lim F(U_i).$$

The sheaf condition is a 'continuity' or a 'locality' condition.

Sheaf condition

Let's unpack it:

1. The colimit of a covering is

$$\operatorname{colim} U_i = \bigcup_i U_i =: U.$$

   Elements of $F(U)$ are 'globally defined sections' (wrt to U).

2. Elements of $\lim F(U_i)$ are 'locally defined sections' which satisfy a **compatibility condition**:

$$\lim F(U_i) = \{(s_i)_{i\in I} \mid s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \text{ for all } i,j\in I\}$$

3. There is a universal morphism

$$\begin{aligned} \varphi : F(\operatorname{colim} U_i) &\longrightarrow \lim F(U_i) \\ s &\longmapsto (s|_{U_i})_{i\in I} \end{aligned}$$

Sheaf condition

$$\begin{aligned} \varphi : F(\operatorname{colim} U_i) &\longrightarrow \lim F(U_i) \\ s &\longmapsto (s|_{U_i})_{i \in I} \end{aligned}$$

Then:

1. φ mono means

   separation axiom: $s = t \in F(U)$ iff $s|_{U_i} = t|_{U_i}$ forall $i \in I$.

2. φ epi means

   glueing axiom: every compatible assignment of sections $(s_i)_{i \in I} \in \lim F(U_i)$ 'glues' to a global section $s \in F(U)$.

A presheaf satisfying separation is called **separated**.

A sheaf satisfies both.

Sheaf condition

Example

The canonical example is $C(-;\mathbb{R})$ of continuous real functions.
Also: smooth functions, measurable functions, etc. (in fact many structures can be encoded directly in a *structure sheaf*).

Non-example

Separated presheaves which are not sheaves:

1. Constant functions: if U and V are disjoint open sets, there's no way to glue two constant functions with different values on U and V, *even though they agree on* $U \cap V$.
2. Bounded functions: choose an infinite covering, even though every local section might be bounded there's no guarantee their glueing will be bounded (e.g. $\lambda x.x^2|_{(n,n+1)}$)

Non-separated presheaves are rare to find in practice

Sheaf condition

The sheaf condition can be read also in terms of **systems theory**:

Suppose X is a 'system', and its open sets are 'parts'.

Then the **functor of behaviours** is a presheaf:

$$\begin{array}{ccc} U & & \{\text{behaviours of the part } U\} \\ \subseteq \Big\uparrow & \overset{B}{\longmapsto} & \Big\downarrow \text{behaviour induced by forgetting } U \setminus V \\ V & & \{\text{behaviours of the part } V\} \end{array}$$

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Separation: *behaviours can be distinguished at the level of parts.*

Glueing: *compatible behaviours of the parts form global behaviours.*

Sheaf cohomology

Hence:

(pre)sheaves mediate the passage from local to global.

Most importantly, they enable the right tooling to study obstructions to this passage, namely **sheaf cohomology**.

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Čech

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Čech

Reasons to prefer Čech to plain cohomology:

1. It's computational (it's a simplicial cohomology in disguise)
2. Easily generalized to non-abelian sheaves (e.g. **Set**-sheaves)

For this talk though, let's stick to the *abelian* version, hence we will assume F *is a sheaf valued in* **Ab** (or any abelian cat if you know what that means)

Čech cohomology

Take a 'good' cover $\{U_i\}_{i\in I}$ of X. (e.g. contractible intersections)

Consider the following **chain complex**: (meaning im $d_n \subseteq \ker d_{n+1}$)

$$0 \to \prod_i F(U_i) \xrightarrow{d_0} \prod_{i,j} F(U_i \cap U_j) \xrightarrow{d_1} \prod_{i,j,k} F(U_i \cap U_j \cap U_k) \xrightarrow{d_2} \cdots$$

where

$$d_0((s_i)_i) = (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$$

$$d_1((s_{i,j})_{i,j}) = (s_{i,j}|_{U_i\cap U_j\cap U_k} - s_{j,k}|_{U_i\cap U_j\cap U_k} + s_{k,i}|_{U_i\cap U_j\cap U_k})_{i,j,k}$$

and so on...

Čech cohomology

The cohomology of the complex measures how far is im d_n from coinciding with ker d_{n+1}: ('exactness')

$$
\begin{tikzcd}
0 \arrow[r] & \prod_i F(U_i) \arrow[r, "d_0"] \arrow[d, squiggly] & \prod_{i,j} F(U_i \cap U_j) \arrow[r, "d_1"] \arrow[d, squiggly] & \prod_{i,j,k} F(U_i \cap U_j \cap U_k) \arrow[r, "d_2"] \arrow[d, squiggly] & \cdots \\
& H^0 = \ker d_0 & H^1 = \dfrac{\ker d_1}{\operatorname{im} d_0} & H^2 = \dfrac{\ker d_2}{\operatorname{im} d_1} &
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Čech cohomology

The cohomology of the complex measures how far is im d_n from coinciding with $\ker d_{n+1}$: ('exactness')

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$$H^0 = \ker d_0 \qquad H^1 = \frac{\ker d_1}{\operatorname{im} d_0} \qquad H^2 = \frac{\ker d_2}{\operatorname{im} d_1}$$

Meaning:

$H^0 =$ local sections $(s_i)_i$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j \in I$

$\overset{\text{glueing}}{=}$ global sections.

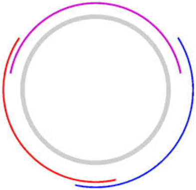
$H^1 =$ compatible local sections on $U_i \cap U_j$

which do not arise from an assignment on the opens

$\approx$ 1-holes

$H^n = \cdots \approx$ n-holes

Čech cohomology: example



Cover S^1 as in the picture,

let F = constant sheaf at K field.

Čech complex: $0 \longrightarrow K^3 \xrightarrow{d_0} K^3 \xrightarrow{d_1} 0$

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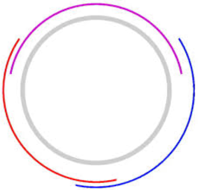

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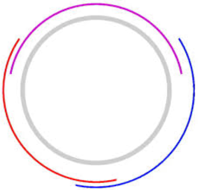

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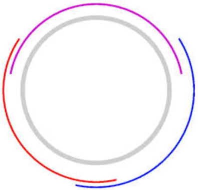

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$d_1 = 0$

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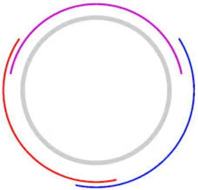

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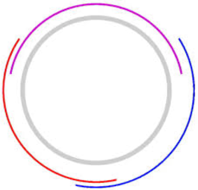

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Hence

$H^0 = K^1$ — 1 0-hole = 1 connected component

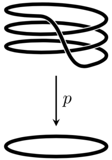
$H^1 \cong K^3/K^2 \cong K^1$ — 1 1-hole

$H^n = 0 \cong K^0$ — 0 n-holes for all $n \geq 2$

Čech cohomology

Cohomology is not completely determined by the topology of the space though, it is really determined by the structure of the sheaf (it's the cohomology of its étalé space) .

Example



Repeat the previous computation with a different *locally* constant sheaf.

Hence **topological obstructions** $\neq$ **local-to-global obstructions**!

Čech cohomology: systems theory

Let B (presheaf of behaviours) be a **pre**sheaf of abelian groups, let $\{U_i\}_{i \in I}$ be an open covering of U.

Its **augmented** **Čech complex** is:

$$0 \longrightarrow B(U) \xrightarrow{d_{-1}} \prod_i B(U_i) \xrightarrow{d_0} \prod_{i,j} B(U_i \cap U_j) \xrightarrow{d_1} \cdots$$

where $d_{-1}(s) = (s|_{U_i})_i$.

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$\rightsquigarrow$ **emergent behaviour** ($= 0$ if no emergence) [Adam, 2017]

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$H^{\geq 1}$ = same as before! $\rightsquigarrow$ **higher-order emergent behaviour...?**

Logic

Logic

Idea:

sheaves are 'locally defined sets' on a 'site'.

We are going to see

1. Sites & topoi
2. The internal language of a topos of sheaves
3. The 'locally' modality
4. Logical-flavoured applications: forcing, internalization

Sites

Definition

A **sieve** on U is a collection S of morphisms $\{U_i \to U\}_{i \in I}$ closed by precomposition on the left:

$$\underbrace{V \to \underbrace{U_i \to U}_{\in S}}_{\implies \in S}$$

You generate a sieve by choosing some morphisms into U and then closing.

Definition

A **site** is a small category $\mathbf{C}$ together with a **Grothendieck topology** J, i.e. a choice of sieves for each object $U : \mathbf{C}$:

$$J(U) = \textbf{covering sieves for } U$$

such that $J(U)$ satisfies some very reasonable conditions.

Sites

Definition

A presheaf $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf iff for any *sieve* S on $U : \mathbf{C}$,

$$F(U) \cong \lim F(S).$$

Warning: it's not always true that $U \cong \operatorname{colim} S$!

Example

For X topological space:

$$\operatorname{Sh} X = \operatorname{Sh}(\mathcal{O}(X), \text{jointly epimorphic families})$$

$$\operatorname{Psh} X = \operatorname{Sh}(\mathcal{O}(X), \text{trivial coverings})$$

Takeaway: locality is not a fixed concept! It depends on the topology, but not the one you expect.

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The topos of presheaves

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$$G^F(U) \cong \mathrm{Nat}(よU, G^F) \cong \mathrm{Nat}(よU \times F, G)$$

3. **Subobject classifier**: $\Omega(U) =$ 'principal sieves on U'.

$$\begin{tikzcd} P \arrow[r, "!"] \arrow[d, tail, "\forall\varphi"'] & 1 \arrow[d, "\mathsf{true}"] & * \arrow[d, mapsto] \\ F \arrow[r, "\exists!\ulcorner\varphi\urcorner"] & \Omega & 1_U \end{tikzcd}$$

$$s \in F(U) \longmapsto \{V \xrightarrow{f} U \mid s|_f \in P(V)\}$$

Intuition: 'ways s enters in P at U'.

$P(s)$ true at U <u>iff</u> s entered through 1_U <u>iff</u> $s \in P(U)$ already.

Internal language of topoi

A category satisfying these properties is called an **elementary topos**.

A topos $\simeq$ to a topos of sheaves on a site is called **Grothendieck**.

The definition is tailored to provide a rich **internal language**:

A type	A object
$t(x) : A\,[x : X]$	$X \xrightarrow{t} A$
$\varphi(x) : \mathrm{Prop}\,[x : X]$	$X \xrightarrow{\varphi} \Omega$ (hence $\{x \mid \varphi\} \rightarrowtail X$)
$\forall/\exists y : Y\ \varphi(x, y)\,[x : X]$	$\Omega^{X \times Y} \underset{\forall y : Y}{\overset{\exists y : Y}{\underset{\xleftarrow{\pi_Y^*}}{\rightrightarrows}}} \Omega^Y$

$+$ type/term constructors coming from exponentials, limits, colimits, ...

Basically the internal language of Set!

Kripke–Joyal semantics

Let X be a topological space, $U \subseteq X$, φ, ψ formulae of $\mathrm{Sh}\,X$:

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$U \models \varphi \wedge \psi$	<u>iff</u>	$U \models \varphi$ and $U \models \psi$
$U \models \varphi \Rightarrow \psi$	<u>iff</u>	for all $V \subseteq U$, $V \models \varphi$ implies $V \models \psi$
$U \models \forall s\!:\!F\ \varphi(s)$	<u>iff</u>	for any $s \in F(U)$ and $V \subseteq U$, $V \models \varphi(s)$

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$U \vDash \varphi \Rightarrow \psi$	<u>iff</u>	for all $V \subseteq U$, $V \vDash \varphi$ implies $V \vDash \psi$
$U \vDash \forall s \colon F\ \varphi(s)$	<u>iff</u>	for any $s \in F(U)$ and $V \subseteq U$, $V \vDash \varphi(s)$
$U \vDash \varphi \vee \psi$	<u>iff</u>	there exists an open covering $\{U_i\}_{i \in I}$ of U such that $U_i \vDash \varphi$ or $U_i \vDash \psi$ for every $i \in I$
$U \vDash \exists s \colon F\ \varphi(s)$	<u>iff</u>	there exists an open covering $\{U_i\}_{i \in I}$ of U such that for all $i \in I$ there exists $s_i \in F(U_i)$ such that $U_i \vDash \varphi(s_i)$.

Kripke–Joyal semantics

Let X be a topological space, $U \subseteq X$, φ, ψ formulae of $\mathrm{Sh}\,X$:

$$
\begin{array}{lll}
U \vDash \varphi \wedge \psi & \underline{\text{iff}} & U \vDash \varphi \text{ and } U \vDash \psi \\
U \vDash \varphi \Rightarrow \psi & \underline{\text{iff}} & \text{for all } V \subseteq U,\ V \vDash \varphi \text{ implies } V \vDash \psi \\
U \vDash \forall s \colon F\ \varphi(s) & \underline{\text{iff}} & \text{for any } s \in F(U) \text{ and } V \subseteq U,\ V \vDash \varphi(s) \\
U \vDash \varphi \vee \psi & \underline{\text{iff}} & \text{there exists an open covering } \{U_i\}_{i \in I} \text{ of } U \\
 & & \text{such that } U_i \vDash \varphi \text{ or } U_i \vDash \psi \text{ for every } i \in I \\
U \vDash \exists s \colon F\ \varphi(s) & \underline{\text{iff}} & \text{there exists an open covering } \{U_i\}_{i \in I} \text{ of } U \\
 & & \text{such that for all } i \in I \text{ there exists } s_i \in F(U_i) \\
 & & \text{such that } U_i \vDash \varphi(s_i).
\end{array}
$$

This interpretation is **local**: if $\{U_i\}_{i \in I}$ is an open covering of U,

$$U \vDash \varphi \quad \underline{\text{iff}} \quad \text{for every } i \in I,\ U_i \vDash \varphi$$

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This interpretation is **local**: if $\{U_i\}_{i \in I}$ is an open covering of U,

$$U \vDash \varphi \quad \underline{\text{iff}} \quad \text{for every } i \in I,\ U_i \vDash \varphi$$

and **intuitionistically sound**: if $\varphi \vdash \psi$ in HIL, then

$$U \vDash \varphi \quad \underline{\text{implies}} \quad U \vDash \psi.$$

Lawvere–Tierney topologies

Definition

A **Lawvere–Tierney topology** on a topos $\mathcal{E}$ is an operator $\Box : \Omega \to \Omega$ such that

1. $\mathcal{E} \vDash \varphi \Rightarrow \Box\varphi$ (global truth entails local truth) ,
2. $\mathcal{E} \vDash \Box\Box\varphi \Rightarrow \Box\varphi$ (stability under refinements)
3. If $\mathcal{E} \vDash \varphi \Rightarrow \psi$ then $\mathcal{E} \vDash \Box\varphi \Rightarrow \Box\psi$ (soundness)

Define $\text{▯} : \mathcal{E} \to \mathcal{E}$:

$$\text{▯}X := \{\text{singletons of } X\}\big/_{\Box(=_X)}$$

Proposition

An LT topology extends to a lex monad $\Box = \text{▯▯}$.

Definition

A **sheaf** in $\mathcal{E}$ is a modal type for $\Box$, meaning $\Box X \cong X$.

Topologies on topoi

Facts

1. Any Lawvere–Tierney topology on $\mathrm{Psh}(\mathbf{C})$ gives rise to a Grothendieck topology on $\mathbf{C}$:

$$J(U) = \{S \text{ sieve on } U \mid \Box S = S\}.$$

   and viceversa
2. Moreover, any subtopos $\mathcal{E} \overset{i}{\hookrightarrow} \mathrm{Psh}(\mathbf{C})$ is the topos of sheaves for the LT topology induced by the monad $i_* \dashv i$.
3. Hence, for Grothendieck topoi:

$$\textbf{Grothendieck topologies} \equiv \textbf{LT topologies} \equiv \textbf{subtopoi}$$

   and they all agree on which are the sheaves.

Notice: LT topologies/subtopoi work in non-Grothendieck topoi too!

Forcing

Remark: sheafification is a *localization*, i.e. we introduce new isomorphisms, hence we also enlarge the class of monos $P \overset{\varphi}{\rightarrowtail} X$ such that $\varphi \cong \mathrm{true}_X$:

$$\Box\mathcal{E} \vDash \varphi \quad \underline{\text{iff}} \quad \mathcal{E} \vDash \varphi^{\Box}$$

where $\varphi^{\Box}$ is the '$\Box$-translation', i.e. a recursive application of $\Box$ to subformulae φ.

In particular:

Existence is local: to define a global element $1 \overset{s}{\rightarrowtail} X$ we only need to define (compatible) local elements $U_i \overset{s_i}{\rightarrowtail} X$ for every U_i of a cover of X.

Forcing

Idea: I can use this phenomenon to cook 'impossible objects':

1. Construct a poset P of 'finite' approximations of your object, ordered by extension
2. Take the 'tautological sheaf' on it: $A(p) = p$
3. A will be the desired object in $\mathrm{Sh}\, P$.

Example (Cohen topos)

We want to construct a model of ZFC where $\exists A$ such that

$$\mathbb{N} \rightarrowtail A \rightarrowtail P\mathbb{N}$$

Take $B = PP\mathbb{N}$. Then we *force* the above inclusions:

1. $P =$ 'partially defined monos $B \rightarrowtail 2^{\mathbb{N}}$',
2. $A(p) = \{(k, n) \mid n \in p(k)\}$,
3. By definition: $A \rightarrowtail \Delta K \times \Delta\mathbb{N}$, hence it's a mono $\Delta K \rightarrowtail \Omega^{\Delta\mathbb{N}}$.

Disclaimer: several details under the rug!

Internalization

Idea: $\mathrm{Sh}\,X$ is naturally suited to describe mathematics 'over X'.

Example

Commutative algebra done in $\mathrm{Sh}\,X$ is equivalent to algebraic geometry done over X.

Example (Bohr topos)

Let A be a non-commutative C^*-algebra. Classical physics done in

$$\mathrm{Bohr}(A) = \mathrm{Sh}(\text{commutative subalgebras of } A)$$

is equivalent to quantum physics done over A.

Example (Scott topos)

Let $(X, \Sigma, \mathbb{P})$ be a probability space. Classical calculus done in

$$\mathrm{Scott}(X) = \mathrm{Sh}(\Sigma / \ker \mathbb{P})$$

is equivalent to stochastic calculus done over X.

Systems theory

Idea: the internal language of Psh($\mathbf{S}$) can express constraints on the behaviour of a system (modelled by a site $\mathbf{S}$).

Example

Let $B : \mathrm{Psh}(\mathbf{S})$, with input/outputs maps $I \xleftarrow{f} B \xrightarrow{g} O$. Consider:

$$\mathtt{Tot}(b) :\equiv \forall i : I \ \exists o : O \ (i = f(b) \wedge g(b) = o).$$

Then 'total behaviours in B' $= \{b : B \mid \mathtt{Tot}(b)\} \rightarrowtail B$.

Example

Let $B : \mathrm{Psh}(\mathbf{S})$, let $\square$ be the 'locally' modality better suited for $\mathbf{S}$. Then

$$\mathtt{Extensional} :\equiv \text{`}S \to \square S \text{ is injective`}$$

$$\mathtt{Non\text{-}generative} :\equiv \text{`}S \to \square S \text{ is surjective`}$$

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Thanks for your attention!

Questions?