University of Strathclyde

Fantastic sheaves and where to find them

An MSP101 talk

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Overview

Goal: to show sheaves are useful modelling gadgets.

Two avenues:

1. **Geometry**: local-to-global behaviour, cohomology
2. **Logic**: local modalities, system specifications, models

Basics

Let X be a topological space, $\mathcal{O}(X)$ its frame of open sets.

Definition

A **presheaf** on X is a functor

$$F : \mathcal{O}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

Elements of $F(U)$ are called **sections**. The map $F(V \subseteq U)$ is called **restriction**, and we write $s|_V := F(V \subseteq U)(s)$.

Definition

A **sheaf** on X is a presheaf F such that for every open covering $\{U_i\}_{i \in I}$:

$$\textbf{sheaf condition} : \quad F(\operatorname{colim} U_i) \cong \lim F(U_i).$$

The sheaf condition is a ‘continuity’ or a ‘locality’ condition.

Sheaf condition

Let's unpack it:

1. The colimit of a covering is

$$\operatorname{colim} U_i = \bigcup_i U_i =: U.$$

   Elements of $F(U)$ are 'globally defined sections' (wrt to U).

2. Elements of $\lim F(U_i)$ are 'locally defined sections' which satisfy a **compatibility condition**:

$$\lim F(U_i) = \{(s_i)_{i \in I} \mid s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for all } i, j \in I\}$$

3. There is a universal morphism

$$\begin{aligned} \varphi : F(\operatorname{colim} U_i) &\longrightarrow \lim F(U_i) \\ s &\longmapsto (s|_{U_i})_{i \in I} \end{aligned}$$

Sheaf condition

$$\varphi : F(\operatorname{colim} U_i) \longrightarrow \lim F(U_i)$$
$$s \longmapsto (s|_{U_i})_{i\in I}$$

Then:

1. φ mono means

   separation axiom: $s = t \in F(U)$ iff $s|_{U_i} = t|_{U_i}$ forall $i \in I$.

2. φ epi means

   glueing axiom: every compatible assignment of sections $(s_i)_{i\in I} \in \lim F(U_i)$ 'glues' to a global section $s \in F(U)$.

A presheaf satisfying separation is called **separated**.

A sheaf satisfies both.

Sheaf condition

Example

The canonical example is $C(-;\mathbb{R})$ of continuous real functions.
Also: smooth functions, measurable functions, etc. (in fact many structures can be encoded directly in a *structure sheaf*).

Non-example

Separated presheaves which are not sheaves:

1. Constant functions: if U and V are disjoint open sets, there's no way to glue two constant functions with different values on U and V, *even though they agree on* $U \cap V$.
2. Bounded functions: choose an infinite covering, even though every local section might be bounded there's no guarantee their glueing will be bounded (e.g. $\lambda x.x^2|_{(n,n+1)}$)

Non-separated presheaves are rare to find in practice (though easy to construct)

Sheaf condition

The sheaf condition can be read also in terms of **systems theory**:

Suppose X is a 'system', and its open sets are 'parts'.

Then the **functor of behaviours** is a presheaf:

$$\begin{tikzcd} U & & \{\text{behaviours of the part } U\} \arrow[d, "\text{behaviour induced by forgetting } U \setminus V"] \\ V \arrow[u, "\subseteq"] & \arrow[r, mapsto, "B"] & \{\text{behaviours of the part } V\} \end{tikzcd}$$

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Separation: *behaviours can be distinguished at the level of parts*.

Glueing: *compatible behaviours of the parts form global behaviours*.

Sheafification

The inclusion of sheaves into presheaves has a left adjoint:

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[r, hook, shift right, "i"'] & \mathrm{Psh}(X) \arrow[l, shift right, "\cdot^a"'] \end{tikzcd}$$

Informally, you get it by putting 'locally' in front of properties.

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$$\mathrm{Sh}(X) \underset{i}{\overset{\cdot^a}{\rightleftarrows}} \mathrm{Psh}(X)$$

Informally, you get it by putting 'locally' in front of properties.

Example

1. $\underline{S}^a(U) =$ locally constant maps $U \to S$ (constant *sheaf* at S)
2. $Bdd_{\mathbb{R}}^a(U) =$ locally bounded maps $U \to \mathbb{R}$

In general,

$$F^a(U) = \operatorname*{colim}_{\mathcal{U}\text{ hyper.}} \lim F(\mathcal{U}) = \text{formal glueings over (hyper)covers}$$

Sheaf cohomology

Hence:

(pre)sheaves mediate the passage from local to global.

Most importantly, they enable the right tooling to study obstructions to this passage, namely **sheaf cohomology**.

~~Sheaf~~ Čech cohomology

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~~Sheaf~~ cohomology
Čech

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Čech

Reasons to prefer Čech to plain cohomology:

1. It's computational (it's a simplicial cohomology in disguise)
2. Easily generalized to non-abelian sheaves (e.g. **Set**-sheaves)

For this talk though, let's stick to the *abelian* version, hence we will assume F *is a sheaf valued in* **Ab** (or any abelian cat if you know what that means)

Čech cohomology

Take a 'good' cover $\{U_i\}_{i\in I}$ of X. (e.g. contractible intersections)

Consider the following **chain complex**: (meaning $\operatorname{im} d_n \subseteq \ker d_{n+1}$)

$$0 \to \prod_i F(U_i) \xrightarrow{d_0} \prod_{i,j} F(U_i \cap U_j) \xrightarrow{d_1} \prod_{i,j,k} F(U_i \cap U_j \cap U_k) \xrightarrow{d_2} \cdots$$

where

$$d_0((s_i)_i) = (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$$

$$d_1((s_{i,j})_{i,j}) = (s_{i,j}|_{U_i\cap U_j\cap U_k} - s_{j,k}|_{U_i\cap U_j\cap U_k} + s_{k,i}|_{U_i\cap U_j\cap U_k})_{i,j,k}$$

and so on...

Čech cohomology

The cohomology of the complex measures how far is im d_n from coinciding with ker d_{n+1}: ('exactness')

$$\begin{tikzcd}
0 \arrow[r] & \prod_i F(U_i) \arrow[r, "d_0"] \arrow[d, squiggly] & \prod_{i,j} F(U_i \cap U_j) \arrow[r, "d_1"] \arrow[d, squiggly] & \prod_{i,j,k} F(U_i \cap U_j \cap U_k) \arrow[r, "d_2"] \arrow[d, squiggly] & \cdots \\
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Meaning:

$H^0 =$ local sections $(s_i)_i$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}\ \forall i, j \in I$

$\overset{\text{glueing}}{=}$ global sections.

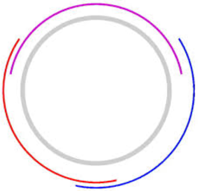
$H^1 =$ compatible local sections on $U_i \cap U_j$

which do not arise from an assignment on the opens

$\approx$ 1-holes

$H^n = \cdots \approx$ n-holes

Čech cohomology: example



Cover S^1 as in the picture,

let $F =$ constant sheaf at K field.

Čech complex: $\quad 0 \longrightarrow K^3 \xrightarrow{d_0} K^3 \xrightarrow{d_1} 0$

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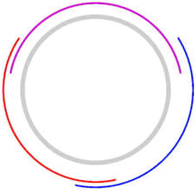

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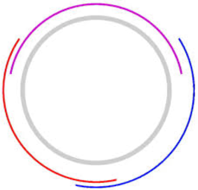

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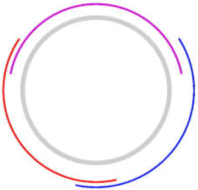

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$d_1 = 0$

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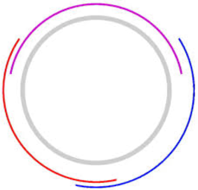

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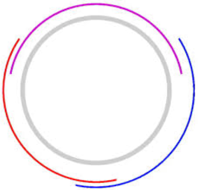

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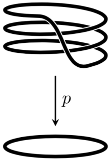
Hence

$H^0 = K^1$	1 0-hole = 1 connected component
$H^1 \cong K^3/K^2 \cong K^1$	1 1-hole
$H^n = 0 \cong K^0$	0 n-holes for all $n \geq 2$

Čech cohomology

Cohomology is not completely determined by the topology of the space though, it is really determined by the structure of the sheaf (it's the cohomology of its étalé space) .

Example



Repeat the previous computation with a different *locally* constant sheaf.

Hence **topological obstructions** $\neq$ **local-to-global obstructions**!

Čech cohomology: systems theory

Let B (presheaf of behaviours) be a **pre**sheaf of abelian groups, let $\{U_i\}_{i \in I}$ open covering of U.

Its **augmented** **Čech complex** is:

$$0 \longrightarrow B(U) \xrightarrow{d_{-1}} \prod_i B(U_i) \xrightarrow{d_0} \prod_{i,j} B(U_i \cap U_j) \xrightarrow{d_1} \cdots$$

where $d_{-1}(s) = (s|_{U_i})_i$.

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$\rightsquigarrow$ **emergent behaviour** (Adam, 2017)

$H^{\geq 1}$ = same as before! $\rightsquigarrow$ **higher-order emergent behaviour...?**

And now for something completely different...

We would like to define **sheaves on arbitrary categories**: in fact, there are presheaves on any category, why shouldn't we have sheaves?

The topos of presheaves

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3. Subobject classifier: $\Omega(U) =$ subfunctors of $よU = \mathrm{Sub}(よU)$.

$$\begin{tikzcd} P \arrow[r, "!"] \arrow[d, tail, "\forall\varphi"'] & 1 \arrow[d, "\mathsf{true}"] & * \arrow[d, mapsto] \\ F \arrow[r, "\exists!\ulcorner\varphi\urcorner"] & \Omega & 1_{よU} \end{tikzcd}$$

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A category satisfying these properties is called a **Grothendieck topos**. If it's only *finitely* bicomplete, it's an **elementary topos**.

Internal language of topoi

An elementary topos has a rich internal language

description Ω object of 'truth values' or 'propositions'

Internal language of topoi

The internal language can simplify dealing with presheaves by transforming complex sheaf-theoretical theorems/proofs/constructions into easy set-theoretical theorems/proofs/constructions.

classictwistedarrowdiagram

Lawvere–Tierney topologies

The internal language can also be used to describe sheaves!

Definition

Lawvere–Tierney topology

The modality expressed by an LT topology is the ‘locally’ modality.

Definition

A sheaf is a modal type for (the monad induced by) a Lawvere–Tierney topology

Indeed: **a sheaf is ‘something defined locally’**.

Lawvere–Tierney topologies

How does this fit into the ‘sheaf as continuous functors’ perspective?

Sites

Definition

A **sieve** on U is a collection S of morphisms $\{U_i \to U\}_{i \in I}$ closed by precomposition on the left:

$$\underbrace{V \to \underbrace{U_i \to U}_{\in S}}_{\implies \in S}$$

Definition

A **site** is a small category $\mathbf{C}$ together with a **Grothendieck topology** J, i.e. a choice of sieves for each object $U : \mathbf{C}$:

$$J(U) = \textbf{covering sieves for } U$$

such that $J(U)$ satisfies some very reasonable closure conditions.

Then the sheaf condition becomes: for any *sieve* S on $U : \mathbf{C}$,

$$F(U) \cong \lim F(S).$$

Warning: it's not always true that $U \cong \operatorname{colim} S$!

Facts:

1. The subobject classifier of $\mathrm{Psh}(\mathbf{C})$ is given by

$$\Omega(U) = \text{covering sieves for } U$$

2. Any Lawvere–Tierney topology on $\mathrm{Psh}(\mathbf{C})$ gives rise to a Grothendieck topology on $\mathbf{C}$:

$$J(U) = \text{`closed' covering sieves} = \{S \text{ sieve on } U \mid \Box S = S\}.$$

   and viceversa

3. Any Grothendieck topology gives rise to a sheafification functor $a : \mathrm{Psh}(\mathbf{C}) \to \mathrm{Sh}(\mathbf{C})$, so that $a \dashv i$ form a **geometric morphism** $\mathrm{Sh}(\mathbf{C}) \hookrightarrow \mathrm{Psh}(\mathbf{C})$.

4. The monad of this adjunction induces a LT topology on $\mathrm{Psh}(\mathbf{C})$.

5. Hence, for Grothendieck topoi:

$$\text{Grothendieck topologies} \equiv \text{Lawvere–Tierney topologies} \equiv \text{subtopoi}$$

Sites

Example

1. $\mathsf{Sh}(\mathcal{O}(X), \text{open coverings}) = \mathsf{Sh}\, X$.

4. Bohr topos
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Thanks for your attention!

Questions?