MIT Challenge

Course 1: 6.042J: Mathematics for Computer Science

Meyer, Chlipala *Mateusz Faltyn*

1.1 Proofs

1.1.1 Well Ordering Principle

From Section 1.3 in the course notes.

Definition 1.1 ***(Well Ordering Principle)****: Every nonempty set of nonnegative integers has a smallest element.*

Example 1.1.1 *To prove that $P(n)$ is true for all $n \in \mathbb{N}$ using the Well Ordering Principle:*

- *Define the set, C, of coutnerxamples to P being true. Specifically, define:*

$$C := \{n \in \mathbb{N} | NOT(P(n)) \textit{ is true}\}.$$

- *Assume for proof by contradiction that C is nonempty.*
- *By the Well Ordering Principle, there will be a smallest element, $n \in C$.*
- *Reach a contradiction somehow - often by showing that $P(n)$ is actually true or by showing that there is another member of C that is smaller than n.*
- *Conclude that C must be empty.*

1.1.2 Induction

From Section 1.8 in the course notes.

Definition 1.2 ***(Induction Principle)****: Let P be a predicate on nonnegative integers. If $P(0)$ is true, and$P(n)$ implies $P(n+1)$ for all nonnegative integers n, then $P(m)$ is true for all nonnegative integers m.*

Definition 1.3 ***(Strong Induction Principle)****: Let P be a predicate on nonnegative integers. If $P(0)$ is true, and for all $n \in \mathbb{N}$, $P(0).P(1), \ldots, P(n)$ together imply $P(n+1)$, then $P(m)$ is true for all nonnegative integers m.*

1.1.3 State Machines

Definition 1.4 ***(Preserved Invariant)****: A property that is preserved through a series of operations or steps.*

Definition 1.5 ***(State Machine)****: A binary relation (transition) on a set, except that the elements of the set are called "states. The transition relation is also called the state graph of the machine. A state machine also comes equipped with a designated start state.*

Definition 1.6 ***(Invariant Principle)****: If a preserved invariant of a state machine is true for the start state, then it is true for all reachable states.*

Definition 1.7 ***(Partial Correctness)****: When there is a result from a process, the result is correct.*

Definition 1.8 ***(Termination)****: A process always produces a result.*

1.1.4 Recursive Definition

Definition 1.9 ***(Structural Induction Principle)****: Let P be a predicate on a recursively defined data type R. If $P(b)$ is true for each base case element $b \in R$ and for all two argument constructors c:*

$$[P(r) \wedge P(S)] \rightarrow P(c(r, s)), \forall r, s \in R,$$

and likewise for all constructors taking other numbers of arguments, then $P(r)$ is true for all $r \in R$.

1.2 Structures

1.2.1 Trees

Definition 1.10 ***(Forest and Trees)****: An acyclic graph is called a forest. A connected acyclic graph is called a tree.*

Theorem 1.11 *Every tree has the following properties:*

1. *Every connected subgraph is a tree.*
2. *There is a unique path between every pair of vertices.*
3. *Adding an edge between nonadjacent nodes in a tree creates a graph with a cycle.*
4. *Removing any edge disconnects the graph. That is, every edge is a cut edge.*
5. *If the tree has at least two vertices, then it has at least two leaves.*
6. *The number of vertices in a tree is one larger than the number of edges.*

Definition 1.12 ***(Spanning Tree)****: A subgraph containing all the vertices of a graph G.*

1.3 Counting

1.3.1 Asymptotics

Definition 1.13 ***(Little O)***: *For functions $f, g : \mathbb{R} \to \mathbb{R}$, with g nonnegative, we say f is asymptotically smaller than g:*

$$f(x) = o(g(x)),$$

iff

$$\lim_{x\to\infty} f(x)/g(x) = 0.$$

Definition 1.14 ***(Big O)***: *Given functions $f, g : \mathbb{R} \to \mathbb{R}$, with g nonnegative, we say that:*

$$f(x) = O(g),$$

iff

$$\lim_{x\to\infty} \sup |f(x)|/g(x) < \infty.$$

Corollary 1.15 *If $f = o(g)$ or $f \sim g$, then $f = O(g)$.*

Definition 1.16 ***(Theta)***: *Functions f and g are equal to within a constant factor:*

$$f(x) = \Theta(g),$$

iff

$$f = O(g) \wedge g = O(f).$$

1.4 Probability

1.4.1 Random Walks and PageRank

Definition 1.17 ***(Stationary Distribution)***: *An assignment of probabilities to verticies in a digraph is a stationary distribution if for all verticies x:*

$$Pr[\textit{at } x] = Pr[\textit{go to } x \textit{ at next step}].$$