Generation of harmonics on an electric guitar

Mattia Surricchio
Fundamentals of acoustics

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1 Introduction

Have you ever wondered what are those incredible high pitch sounds that (sometimes) come out of a distorted eletric guitar (often followed by some crazy stuff with a vibrato)?

Those are called ***harmonics***!

But how do they sound like? Here's a brief example by **Joe Satriani** in *"The Summer Song"*

2 How to create harmonics

In order to create a harmonic, guitarists are taught to gently touch the string with their right hand at the ***fifth***, ***seventh*** or ***twelfth*** while picking the string

with the other hand.
It is important to notice two things:

1. The aim of the *gentle touch* is to partially stop the vibratory motion of the string.

2. The guitar player is **not** pressing any fret on the guitar keyboard! Here's the differences in sound between the first string played at the fifth fret pressing the guitar keyboard and the same string played with the *gentle touch*.

   Note | Harmonic

3 Harmonic decomposition

So far we have introduced ***harmonics*** and how to create them on an eletric guitar. But why does this happen? Why do musicians believe (*erroneously*) that natural harmonics are only possible at those frets mentioned above?

First thing first, we have to introduce a fundamental mathematical tool used in almost every field of engineering: the **Fourier Analysis**.

3.1 Fourier Analysis

Intuitively the Fourier Analysis allows us to decompose a general function into the weighted sum of sine or cosine functions. This turns to be extremely useful expecially when dealing with sound waves and audio signals.
But what does "*weighted sum of sine or cosine*" mean? Here's a brief example using a synthesizer.

Example 3.1. Using a synthesizer (is this case we're using Massive by Native Instrument) we create a simple sound with the following spectrum.

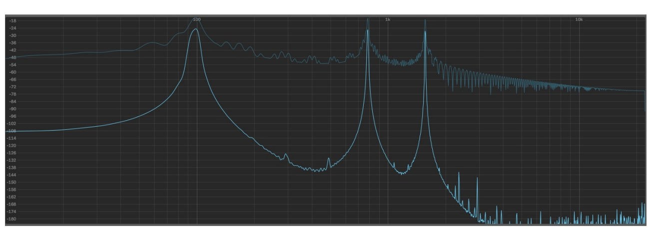


Figure 1: single synthesizer spectrum

As we can see, we have 3 main components at around 100 Hz, 780 Hz and 1.56 KHz and the note played on the keyboard is a G2.

Now we use 3 different synthesizer, each one playing only one of the frequencies mentioned above and then we sum them up. The output spectrum is the following:

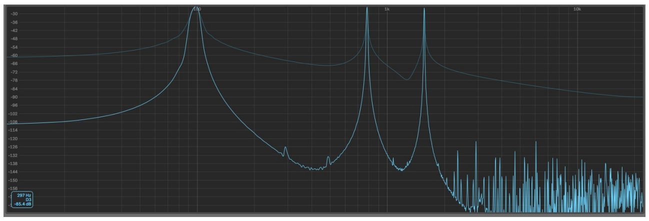


Figure 2: multiple synthesizer spectrum

As you may notice, the spectrum is pretty much the same! (except for some high frequency noise caused by the use of multiple digital synthesizers)

Now we want to mathematically formalize the Fourier Analysis. For simplicity we will assume that the signal/wave we are considering is periodic.

Definition 3.1. Fourier Series. Given a generic periodic function $s(t)$ of period T we can always express it as:

$$s(t) = \sum_{n=-\infty}^{\infty} C_n * e^{j2\pi nt/T} \tag{1}$$

Where C_n is a complex coefficient which indicates the amplitude of the n-th complex wave, its value is given by:

$$C_n = \frac{1}{T} \int_0^T s(t) * e^{-j2\pi nt/T} dt \tag{2}$$

The meaning of **3.1** is still somewhat misterious: why do we have complex coefficient with $s(t)$ which is a real signal?
In order to have $s(t)$ real we need to remove the imaginary part of the sum. This turns into:

$$C_{-n} = C_n^* \tag{3}$$

Example 3.2. Given real signal with C_1 , C_2 , C_3, C_{-1} , C_{-2} , C_{-3} complex coefficients, we can expand them as the sum of a real part plus an imaginary part, obtaining $a_1 + ib_1$, $a_2 + ib_2$, $a_3 + ib_3$ ecc...

Since $s(t)$ has to be real, we want the imaginary part to be null. Using (3) we obtain that $C_1 = a_1 + ib_1$ and $C_{-1} = a_1 - ib_1$. If we sum them together we obtain:

$$a_1 + ib_1 + a_1 - ib_1 = a_1 + a_1 = 2a_1 \tag{4}$$

which is a pure real number.
The iteration over all the coefficients gives us a real signal $s(t)$

NB: the coefficient C_0 turns out to be purely real using (2) as $e^{-j2\pi nt/T}$ is equal to 0 for $n = 0$

3.2 Fourier decomposition of a vibrating string

As we said above, we can use the **Fourier Analysis** to decompose any periodic signals into the weighted sum of sine and cosine function at multiple frequencies of the ***fundamental frequency***.

Using the waves equation for a standing wave we end up with:

$$y(x,t) = \sum [A_n cos(w_n t) + B_n sin(w_n t)] sin(\frac{w_n}{v} x) \tag{5}$$

with

$$f_n = \frac{w_n}{2\pi} = n\frac{v}{2L} \tag{6}$$

and L length of the string.
(5) can be rewritten as:

$$y(x,t) = \sum [C_n sin(w_n t + \phi_n)] sin(\frac{w_n}{v} x) \tag{7}$$

which clearly shows how our vibrating string can be decomposed into the sum of single components at multiple frequencies. These components are called **modes**.

4 Physical study of harmonics

So far we've discovered that every periodic signal can be decompose with the **Fourier Analysis** summing *"signals"* at multiple frequencies of the fundamental.

This leads us to a natural question: *Is there any correlation between the Fourier Analysis and the high pitch of the harmonics generated by a guitarist while playing?*

The answer is **yes**.

4.1 Modes on a guitar string

Picking a string will start a vibratory motion that can be decomposed into **modes** using the **Fourier Analysis**, we expect the string to vibrate as a linear combination of its ***harmonic modes***.

Referring to a guitar neck of length L, the fundamental frequency must "fit" the length L of the neck.

This is the most intuitive behavior, we expect the string to vibrate as a whole after being picked, as shown in the figure.

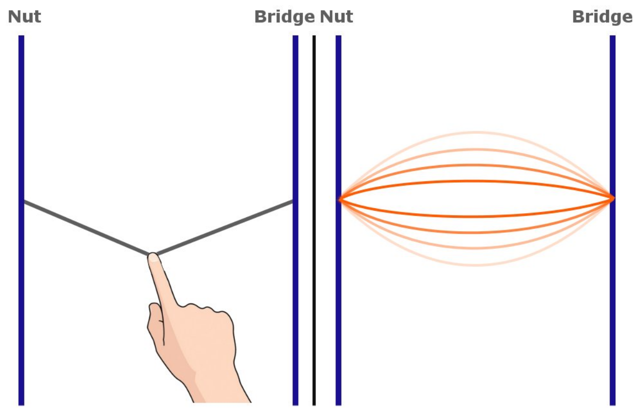


Figure 3: Vibration of a string

What we see in the image (and also in real life) is the ***fundamental frequency*** of vibration, but as mentioned in (1) it's not the only one! We have other (theoretically) infinite harmonics that will contribute to our vibration. They will look something like this:

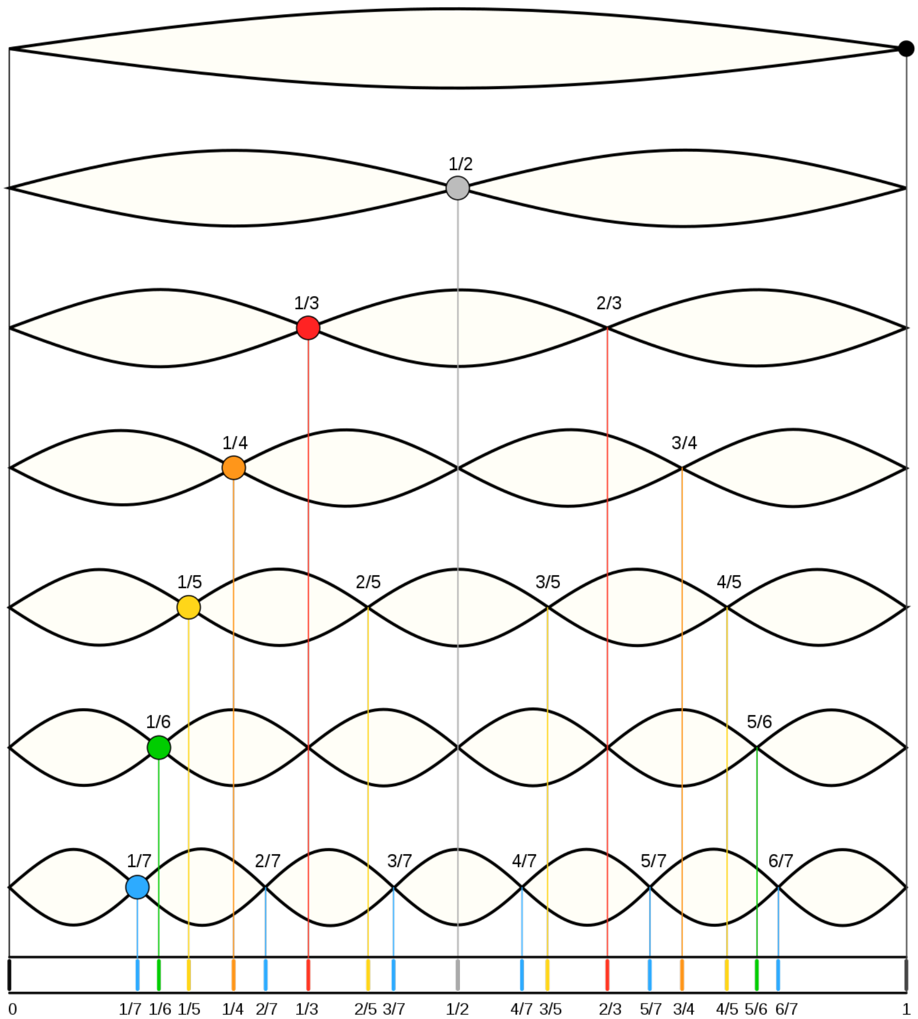


Figure 4: Harmonics at multiple frequencies of the fundamental

4.2 Visualizing modes on a vibrating string

Now we will introduce a simple experiment in order to understand how harmonics look in real life: looking at a vibrating string with the naked eye, will only show the main mode as shown in Figure 3! We need something to *"enhance"* the higher harmonics.

Given a string of length L we attach it to a mechanical wave driver connected to a sin wave generator, the driver will convert the sine wave signal created by the generator into mechanical vibrations.

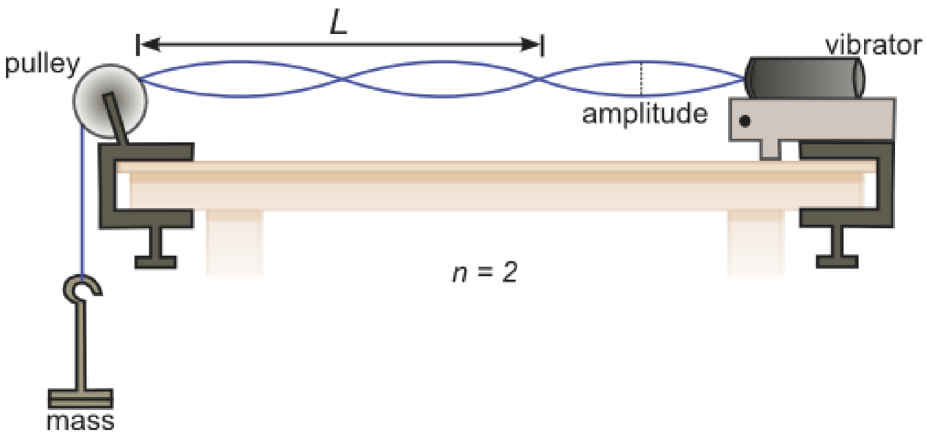


Figure 5: Experiment

Once we have connected our components we can start with the experiment. The experiment will follow as shown in the video

Given that the fundamental frequency is **13.7 Hz** and the mass connected to the pulley is about **150-200g**, we can easily calculate the density of the string. Supposing that we're using a string of length **64.77cm** (which is the length nut-bridge of a standard Fender Stratocaster), we can derive easily the linear density u:

$$u = Tv^2 \tag{8}$$

We can also calculate v using:

$$v = \lambda f \tag{9}$$

λ_0 (wavelength of the fundamental mode) is given by:

$$\lambda_0 = 2L \tag{10}$$

Using these formulas we can easily calculate everything we need in order to describe the physics of our vibrating string. Using (6) we get:

$$f_n = \frac{w_n}{2\pi} = n\frac{v}{2L} = n\frac{v}{\lambda_0} = nf_0 \tag{11}$$

which is coherent with what has been shown in the experiment, we'll have harmonics at $13.7 * 2 = 27.4Hz$, $13.7 * 3 = 41,1Hz$ ecc...

4.3 The strobe experiment

So far we have shown each harmonic independently, but how can we see them on a real string after it has been picked?
In order to achieve our goal we can exploit the frequency sampling of a digital camera. Knowing that a video-camera usually works with 30 frames/second (30Hz), we're able to *artificially* see the complex patterns designed by the vibrating string.

When a guitar is played, many of the mentioned modes of vibration are happening all at once, which creates the rich sound of the guitar.The different harmonics combine to determine the exact shape of the total vibration of the string, and could create something like a sawtooth wave. The shape of the vibration also depends on how and where the string is plucked.

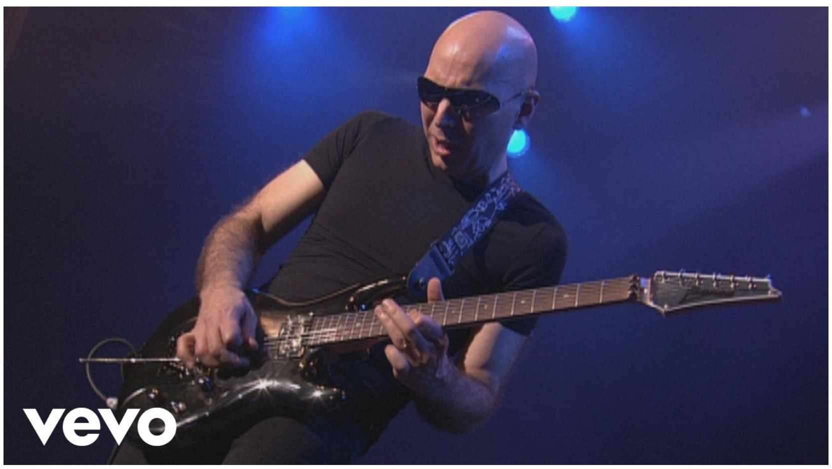
The reason these vibrations aren't obvious to the naked eye is because they happen too fast to see. For example, the fundamental frequency of the A string is at 110 Hz, so the string is vibrating back and forth more than one hundred times a second. The higher harmonics vibrate even faster.

A cell phone camera (or any video camera) can make these rapid oscillations visible because it records images at a certain frame rate. As mentioned above, 30 frames per second is a typical frame rate for a phone camera.
If you're recording the motion of a guitar string, the camera gets a snapshot of the string's position 30 times a second. If the string is vibrating at a frequency

which is a multiple of 30 Hz, it will always come back to the same position at the moment the camera captures a frame, so in the video recording it looks frozen in place.

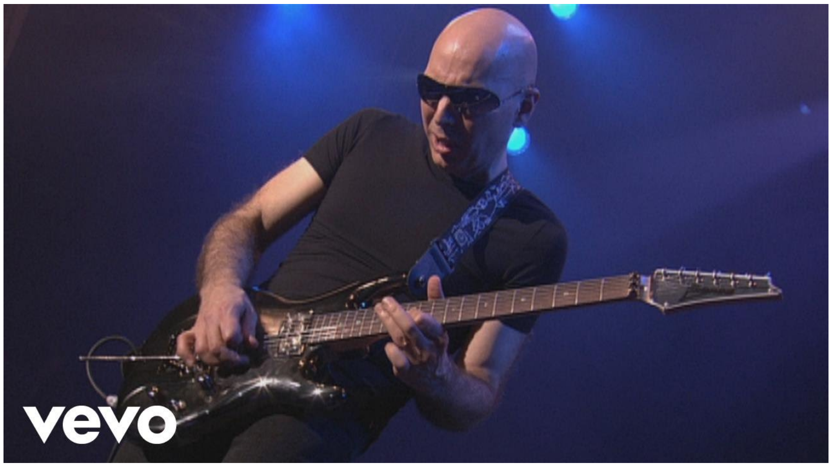
If the frequency is close to a multiple of 30 Hz but not quite, the string will appear to oscillate slowly as it gets caught in a slightly different position in each frame. You can also see this effect with your eyes if you use a strobe light to illuminate the strings.
The flashing light gives you a snapshot of the string position at regular intervals, similar to the camera's frame rate.

Using a low quality CMOS camera, we'll end up with something like this:



We can simulate this behavior even with an high quality camera using the Strobe effect.
Basically what we're doing is turning on/off a light with a certain frequency, when we turn off the light, the camera won't be able to capture the image (because obviously it's all dark in the room).
This is a *"fancy"* way of applying **resampling** (actually it is downsampling,we cannot add more frames than what the camera is able to process) to a signal: for example, if we have 30 frames per seconds, we can turn on the light every 5 frames, so we are "removing" 4 frames in between.

We can also choose the **phase shift**: if we want to accentuate the visual perception of the second harmonic, and we know that it will have its maximum amplitude after t seconds, we can synchronize the light in order to capture the exact moment when the 2 harmonic has its maximum amplitude!



5 Harmonics on guitar

After all the mathematical and physical tools discussed in the previous sections we're finally able to answer the question: *why are we able to create harmonics with the technique explained in Section 2?*

The guitarist is *damping* part of the vibration of the string: at the 12 fret, we hear the first harmonic (the guitarist is damping the fundamental frequency), at the 7 fret we hear the second harmonic (the guitarist is damping both the fundamental and the first) and so on.
It is important to notice (as shown in Figure 4) how the position on the neck, in order to produce each harmonic, corresponds to an almost "stationary" point for the *n-th* mode, while it's a position of "full" vibration for all the other harmonics!

As we said in the **Harmonic decomposition** chapter, *"Why do musicians believe that natural harmonics are only possible at those frets mentioned above (12,7 and 5 fret)*?

As we have shown in figure 4, it's not true that only 3 harmonics are possible. The cause of this *misconception* is the fact that, as shown in the video with all the different harmonics on single string, the more we go up with harmonics the lower is the amplitude of the oscillation, this leads to a lower volume produced by the *n-th* harmonic.
Basically we could also create the $4th$ harmonic at the $3rd$ fret on the guitar, but we're not able to hear it, the vibration is too small.

6 Matlab code

The following Matlab code implements simple FFTs in different cases of playing, in order to show and demonstrate what has been discussed so far.
The full code + material can be found here: GitHub Link

The code will produce two windows containing:

1) The first window contains three FFTs, the first for the open E string (the first one on a guitar), the second for the harmonic at the fifth fret of the first string, and the third FFT for the actual note at the fifth fret

2) The second window shows how the attack transient has a randomic behavior, removing the transient will improve the quality and the detection of the spectrum generated by the harmonic.

7 Conclusions

We have shown how what guitarists have been doing for decades has a real, physical explanation and an almost predictable behavior.

Everything discussed so far will allow us to further extend DSP simulation based on physical modeling of real musical instruments (such as Waveguide modeling and Wave digital filters).