# Stationary Stochastic Processes

## Lecture 2 - Gaussian processes

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# Lecture overview

- Spectral density in discrete time
- Periodogram and Whittle likelihood
- Convolutions and filters

# Gaussian processes - Wiener process

**Definition**

A Gaussian process (GP) is a collection of random variables, any finite number of which have a multivariate Gaussian distribution.

- $X(t) \sim \mathrm{GP}\left(m(t), r(t,s)\right)$ means:
  - $X(t) \sim N\left((m(t), r(t,t)\right)$
  - $\mathrm{Cov}(X(t), X(s)) = r(t,s)$
- **Wiener process** (**Brownian motion**). GP with $X(0) = 0$
  - **Independent increments** $X(t_4) - X(t_3)$ indep of $X(t_2) - X(t_1)$ for all $0 < t_1 < t_2 < t_3 < t_4$
  - $X(t+h) - X(t) \sim N(0, \sigma^2 h)$
- Properties:
  - $X(t) \sim N(0, \sigma^2 t)$
  - $r(s,t) = \sigma^2 \min(s,t)$ [not stationary]
  - **Continuous**, but **nowhere differentiable**.
  - Let $W(t)$ be standard $(\sigma = 1)$ Wiener. Then $X(t) = e^{-\alpha t} W(e^{2\alpha t})$ is an Ornstein-Uhlenbeck process.

# Spectral density in discrete time

- Time series: $\{X_t, t \in \mathbb{Z}\}$'
- For every covariance function $r(\tau)$ of a stationary sequence $\{X_t, t \in \mathbb{Z}\}$ there is a positive and integrable $R(f)$ such that

$$r(\tau) = \int_{-1/2+0}^{1/2} e^{i2\pi f\tau} R(f) df$$

- An absolutely summable function $r(\tau)$ ($\sum |r(\tau)| < \infty$) is a covariance function if

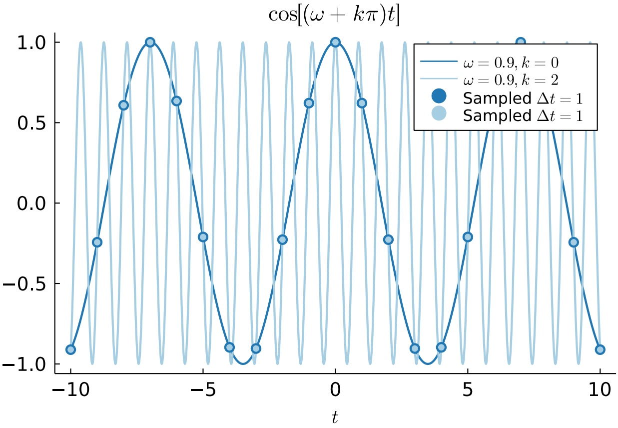
$$R(f) = \sum_{\tau=-\infty}^{\infty} e^{-i2\pi f\tau} r(\tau)$$

  is symmetric, non-negative, and integrable.
- Discrete time ($\Delta_t = 1$): $f \in (-1/2, 1/2]$. Enough since

$$\cos\left((2\pi f + k\pi)t\right) = \begin{cases} \cos(2\pi ft) & k, t \text{ integers with } k \text{ even} \\ \cos\left((\pi - 2\pi f)t\right) & k, t \text{ integers with } k \text{ odd} \end{cases}$$

  so variation at angular frequencies $\omega = 2\pi f$ higher than $\pi$ ($f >1/2$) cannot be distinguished from corresponding $\omega \in (-\pi, \pi]$ ($f \in (-1/2, 1/2]$). **Aliasing**.

# Aliasing




- **Sampling** a continous time process at $\Delta_t$ distance.
- **Nyquist frequency**: $f_n = 1/(2\Delta_t)$. Highest detectable freq.
- **Under-sampling** (Fig 4.8 in LRS).

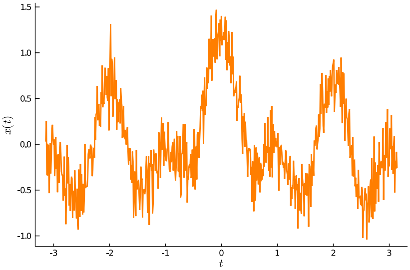
# Discrete Fourier Transform - a statistical view

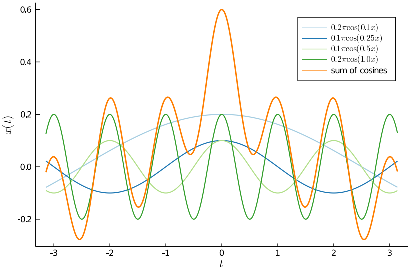
- **Fitting time trends** with **polynomial basis functions**

$$x_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \varepsilon$$

- **Fitting time trends** with **periodic basis functions**

$$x_t = \beta_0 + \beta_1 \cos(0.1t) + \beta_2 \cos(t) + \beta_3 \cos(2t) + \varepsilon$$



# Discrete Fourier Transform - a statistical view

- Regress on trigonometric bases $\cos(2\pi f_k t)$ and $\sin(2\pi f_k t)$ for all Fourier frequencies

$$f_k \in \{k/n \text{ for } k = -\lceil n/2 \rceil + 1, \ldots, \lfloor n/2 \rfloor\}$$

- $\cos(2\pi f_k t)$ and $\sin(2\pi f_k t)$ are orthogonal functions/vectors.
- Regress on each basis separately. Each regression costs $O(n)$:

$$\hat{\beta}_k = \sum_{t=1}^{n} \cos(2\pi f_k t) x_t$$

- Total cost is $O(n^2)$. ☹
- Fast Fourier Transform: divide-and-conquer: $O(n \log n)$. ☺

# Periodogram

- **DFT** of time series $x_t$ at Fourier frequencies $f_k$

$$Z_n(f_k) = \sum_{t=0}^{n-1} x_t e^{-i2\pi f_k t}$$

- Time series with $n$ observations in time domain $\Rightarrow$ DFT at $n$ frequencies.
- $Z_n(f_k)$ is complex valued (regression coef for cos and sin).
- How much variation is captured at frequency $f_k$? **Periodogram**

$$I(f_k) = \frac{1}{n} \left| Z_n(f_k) \right|^2$$

# Properties of periodogram and Whittle likelihood

- ´Periodogram

$$I(f_k) = \frac{1}{n} \left| Z_n(f_k) \right|^2$$

- $I(f_k)$ is an asymptotically unbiased but inconsistent estimate of $R(f_k)$. Smoothing etc.
- Asympotically as $n \to \infty$

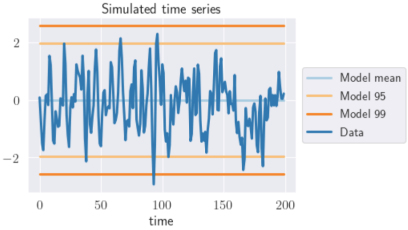
$$I(f_k) \overset{indep}{\sim} \mathrm{Exponential}\left(R(f_k)\right)$$

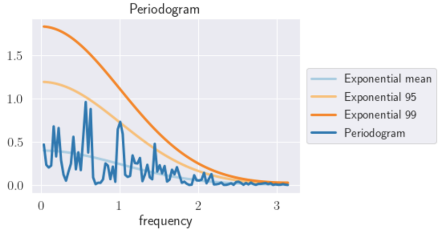
- The Whittle (log-)likelihood uses this asymptotic result

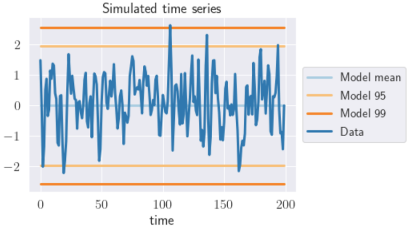
$$\ell(\boldsymbol{\theta}) = \sum_{\mathrm{all} f_k} \left( \log R_{\boldsymbol{\theta}}(f_k) + \frac{I(f_k)}{R_{\boldsymbol{\theta}}(f_k)} \right)$$

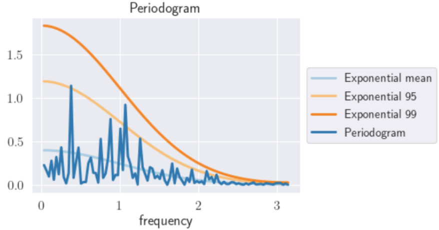
where $R_{\boldsymbol{\theta}}(f_k)$ is spectral density parametrized by $\boldsymbol{\theta}$.

# Gaussian process - squared exponential covariance



# Gaussian processes

**Definition**

A **Gaussian process** (**GP**) is a collection of random variables, any finite number of which have a multivariate Gaussian distribution.

- Linear combinations of Gaussians are Gaussian.
- Integration of GPs gives Gaussian variables.
- Differentiation of GPs gives new GPs.
  - Finite differences $(X(t+h) - X(t))/h$ are Gaussian.
  - Limits of linear combinations are Gaussian

$$X'(t) = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h}$$

  - Integrals are also Gaussian

$$\int_0^1 X(t)dt = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X\left(\frac{k}{n}\right)$$

# Cramér's representation of stationary processes

- **Convergence in quadratic mean** defines the infinite sum

$$X(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi f_k t + \phi_k)$$

- Continuum of frequencies

$$X(t) = \int_{-\infty}^{\infty} e^{i2\pi ft} dZ(f)$$

  where $Z(t)$ is an random complex-valued non-decreasing spectal distribution function with orthogonal increments.
- **Stochastic integral** (enough to define it via q.m. convergence.).
- **Cramérs representation** of any stationary process.

# Convolutions

- Convolution (filtering) using impulse reponse function $h(u)$

$$Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du$$

- Discrete time linear filter

$$Y_t = \sum_{u=-\infty}^{\infty} h(u) X_{t-u}$$

- Impulse reponse: response to a 'unit blip at time $t$', $X(t) = \delta_0(t)$,

$$Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du = h(t)$$

- Frequency function of the filter $h(t)$

$$H(f) = \int_{-\infty}^{\infty} e^{-i2\pi fu} h(u) du$$

$$H(f) = \sum_{u=-\infty}^{\infty} e^{-i2\pi fu} h(u) du$$

# Convolutions

- **Frequency function** of the filter $h(t)$

$$H(f) = \int_{-\infty}^{\infty} e^{-i2\pi fu} h(u) du$$

$$H(f) = \sum_{u=-\infty}^{\infty} e^{-i2\pi fu} h(u) du$$

- **Spectral density of a linear filter**
  $Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du$

$$R_Y(f) = |H(f)|^2 R_X(f)$$

- MA models

$$Y_t = \sum_{u=0}^{\infty} \theta_u \epsilon_{t-u}$$

  so $h(u) = \theta_u$ and $X_t = \epsilon_t$ is white noise with $R_\epsilon(f) = c$ for $-1/2 > f \leq 1/2$.