“Modelling Warfare using Lanchester’s equations”

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Contents

Introduction

Brief project description and overview of project aims:

Lanchester's equations are a pair of differential equations which can be used to model attrition rates in a battle. Using military data from the WWII Battle of Ardennes, this project will use Lanchester's equations to model the battle. The equations themselves are relatively straightforward, with the project focussing on how best to fit the model to the data.

Objectives:

1. Fit the Lanchester Equations to the Ardennes battle data
2. Improve the fit by splitting the battle into 2 battles
3. Improve the fit to the data by varying equipment weightings
4. Improve the fit to the data by varying subdivision resources weightings
5. Identify best fitting models as Lanchester linear or Lanchester square or Lanchester logarithmic

Modelling warfare has very practical applications and uses. If the results of a battle could be accurately predicted beforehand, this knowledge could be used in the decision of whether to engage in battle at all, or in changing tactics. The most important battle predictions would be the attrition rates of both sides and therefore the resulting winning side.

In 1916, Frederick Lanchester [6] proposed a set of differential equations as a model of warfare. These differential equations are capable of modelling attrition rates by describing the time dependence of the strength of 2 opposing armies in battle, Red (R) and Blue (B), as a function of time with the function dependent only on A and B.[5] [6]

A generalisation of the model is;

$$\dot{B}(t) = aR(t)^p B(t)^q \qquad (1)$$

$$\dot{R}(t) = bB(t)^p R(t)^q \qquad (2)$$

Where;

$\dot{B}(t)$: Blue forces killed at time t

$\dot{R}(t)$: Red forces killed at time t

B(t): Strength of army force B at time t

R(t): Strength of army force R at time t

a: Attrition parameter for Red

b: Attrition parameter for Blue

p: Exponent parameter of the attacking force

q: Exponent parameter of the defending force

Lanchester identified distinct interpretations of his equations based on different p and q values, known as Lanchester's Laws.

Lanchester's linear law: When p = q = 1, thus force ratios remain equal if aR = bB. This model is theorised as collections of small engagements e.g. ancient warfare or unaimed fire at enemy-occupied areas. The larger force will always win. [4]

Lanchester's square law: When p = 1, q = 0, and the force ratios remain equal when $aR^2 = bB^2$. This model is theorised to fit modern warfare in which both sides are able to concentrate forces or when engaging in aimed fire. [4]

Lanchester's logarithmic law: When p = 0, q = 1. [4]

Hartley's homogeneous, mixed, linear-logarithmic law: When p = 0.4, q = 0.75. He proposed these parameter values based on an analysis of numerous historical battles. [4]

Bracken introduced a tactical parameter to the standard Lanchester equations, to consider when which side is attacking and defending if this switches throughout the battle. [1]

As can be seen, many versions of Lanchester's equations have been proposed attempting to produce a universally applicable model. But it is difficult to validate these as there is a lack of empirical evidence and applications have given inconclusive and conflicting results. There is a lack of battle data, and what little data there is, is often inconsistent with other reported data.

Despite the lack of validation, Lanchester's equations are used extensively in modelling warfare. They can be used to model some battle data, but a universally applicable version does not yet exist. Manipulation of the Lanchester equations may still be able to produce this. This may be possible by introducing more parameters. But with battle data, it must be remembered that not all parameter input values may be available, thus giving the model limited application if there are too many parameters. It is important to maintain the simplicity of the Lanchester equations to an extent.

Lanchester's equations are similar to the more recent salvo combat model, which is a specific mathematical representation of anti-ship missile battles between modern warships. The difference is that Lanchester equations form a continuous time model, whereas the salvo combat model forms a discrete time model. Also, Lanchester equations only consider offensive firepower, whereas the salvo combat model considers offensive and defensive firepower.

Agent models also exist, considering the individual difference of each unit in warfare. These models are extremely accurate but are complex models requiring long computation and would have very long runtimes for large numbers of units at once.

Lanchester equations remain a useful model for its simplicity. I will attempt to implement changes to the Lanchester equations and create my own version of the equations that accurately fits battle data and may have widespread applicability.

Literature Review

On 16th December 1994, the Germans launched their final offensive of World War II, The Ardennes Campaign, which later came to be known as "the Battle of the Bulge." The German offensive consisted of three armies – the fifth Panzer (Armoured), Sixth Panzer, and Seventh Panzer. The campaign is outlined briefly by *DMSI* [2] and in detail by *Hitler's Last Gamble* [3].

The Germans had identified the Ardennes as a thinly defended section of the Allied front line held by the United States forces. This rugged, wooded terrain was near the intersection of the borders of Germany, Belgium and Luxembourg. They over ambitiously planned to break through the Allied front line at the Ardennes and swiftly advance north across Belgium to the port city of Antwerp. This would have split the U.S. and British forces, and given the Germans a tactical position in being able to attack all the Allied armies.

Due to Allied overconfidence that the Germans lacked the material and personnel resources to launch a major attack on the Western Front, they were caught completely off guard. The Germans managed to amass a large force secretly. However, the German force amassed consisted of low quality personnel, transportation networks were inadequate, and the German air force was far inferior to the Allied air force. They launched their attack under cover of poor weather conditions that kept the far superior Allied air forces grounded, also preventing Allied aerial reconnaissance detecting German approach.

The German forces penetrated through the Allied front line, but did not advance as far as planned because of hard fought U.S. defensives on key routes to the north. The U.S. forces were reinforced by units of the U.S. First and Third Armies and the British XXX Corps. Over several days they rallied to slow the German advance. By 23rd December the skies had cleared up, and with the support of the Allied air force the German advance was halted on 24th December. Allied counter offensives began on 25th December.

On 1st January, the German's launched two final air offensive operations but as they had lost the initiative and had a far inferior air force, ultimately suffered greater casualties than the allies over the course of these operations. The Allied forces pushed back the Germans and the front line was restored to how it was on December 16th 1994 by January 16th 1995.

Chapter 1: The Ardennes Battle Data

This chapter introduces the battle data used by Bracken [1] and the data added to this by Fricker [4]. The data included in both studies were taken from the Ardennes Campaign Simulation Data Base (ACSDB) compiled by Data Memory Systems, Inc. (DSMI) [2] under contract to the U.S. Army Concepts Analysis Agency.

1.1 DMSI

DMSI obtained primary source records from the U.S., Great Britain and Germany as well as many other secondary source records. “The mass of records assembled by DMSI researchers was catalogued and then analyzed to compile the detailed data required for the campaign-level data base. Of necessity, much data was interpolated, extrapolated, or otherwise derived from the various sources acquired from the research facilities” [2, p. I-7].

As is common with battle data a lot of the records are contradictory or inconsistent in reporting format making it difficult for comparison, and some are incomplete, particularly the German data. “Due to various reasons, including the poor record keeping practices maintained by the German Army in the last six months of World War II and the destruction of many reports, there is a notable paucity of daily battle casualty statistics for German units at any level of aggregation during the Ardennes period. Therefore, it was determined that the estimation of lacking battle casualty data was necessary” [2, p. II-G-l-1]. Many other German resources required estimation techniques to be included in the DMSI database. German data tended to be recorded periodically for multiple days rather than daily. Of note for the Bracken data, “personnel and equipment inventory data was estimated for the service support elements of German Army Group B” [2, p. II-G-8-1].

1.2 Bracken Data

The Bracken data (Table 1 & 2) consists of daily tallies of manpower (Mp), tanks (T), Armoured Personnel Carriers (APC)s, and artillery (Art) available and killed for both the Allied and German forces over 32 days of battle, 16th December 1944 – 16th January

1945. Data is also included for 15th December 1944, but most of the data for the Germans is incomplete and can't be included in the model. This day is denoted as "day 0". Bracken chose to exclude the equipment resource data for AT/Mortars.

Labelling U.S. forces blue and German forces red, Bracken considered manpower as either combat manpower or total manpower. Combat manpower consists of the infantry, armour, and artillery personnel. Total manpower consists of all personnel; the combat personnel as well as logistics and support personnel.

In terms of manpower, "killed" is defined as personnel killed or wounded, and does not include personnel captured/missing in action (MIA) or those incapacitated by disease and nonbattle injuries. In terms of equipment, "killed" is defined as any equipment destroyed or too damaged to be used, and does not include abandoned equipment.

Fricker [4] explains, Bracken theorised each day of the Ardennes Campaign as an independent battle, an independent observation of Lanchester equations with fixed attrition and exponent parameters for the duration of the Ardennes Campaign. This is a reasonable approach as Lanchester equations reflect fixed capabilities of forces [6]. "The idea would be that casualties occur according to the fixed Lanchester equations using the previous day's force size, but the overall force size for the current day also depends on the transfer of troops in or out of the fighting force" [4, p. 4]. This is an appropriate approach to modelling the data structure in Table 1 & Table 2 as these tables do not consider the transfer of troops in and out of battle between days. By considering each day independently, initial force sizes for each observation included the transfer of troops in and out of battle for the duration of the observation (one day).

1.3 Fricker Air Data

Fricker [4] added a new resource (Table 3) to the Bracken data from the DMSI database: air sorties (As). With the overwhelmingly superior Allied air force being a major factor contributing to the turning point in the battle as the skies cleared, it is reasonable to include this resource in the model. Unfortunately, air sorties weren't recorded in as much detail and the only information included is number of air sorties flown for Allied and German forces. Again, the exactness of German data is questionable. "Data on Allied (US and British) air sorties was taken directly from daily operations reports, while Luftwaffe

Table 1: Ardennes battle manpower data for the Allied (Blue) and German (Red) forces

Day	Manpower							
	Blue				Red			
	Available: Mb(i)		Killed: Mb(i)		Available: Mr(i)		Killed: Mr(i)	
(i)	Combat	Total	Combat	Total	Combat	Total	Combat	Total
0	351005	632105	458	1468	0	0	0	0
1	349247	630557	1589	3062	360716	575838	2191	5590
2	347915	628985	2383	5712	356818	571301	2423	5559
3	358321	640969	2085	5093	353529	568508	2015	4711
4	366495	807140	2175	12101	350750	565173	1993	4332
5	387342	834136	1389	5334	356278	572181	1985	4351
6	403289	859906	1174	3197	354297	570711	2084	4582
7	410817	874600	1905	4815	361684	581177	2046	4531
8	412811	877247	1548	3730	359353	579660	2468	5351
9	426360	895976	1608	3857	362904	584610	2685	5609
10	432094	907490	1527	3635	359750	580731	2538	5563
11	451316	933045	2320	5411	362611	584551	2504	5526
12	451724	948024	1376	3596	361023	583610	2544	5751
13	451291	928230	1277	3435	356892	578737	2121	4511
14	461189	941188	1005	2934	349900	568768	1682	3900
15	465334	946424	1042	2743	346100	564548	1844	4076
16	467620	948226	1159	3022	343134	560993	1550	3635
17	467801	948379	1004	2773	340875	558214	1788	3898
18	474562	956144	832	2631	338278	555741	1724	3821
19	474192	955821	1831	3580	334356	550854	1752	3892
20	481704	965135	2259	4899	328069	544031	2054	4283
21	480952	964928	1639	4093	321195	534885	1709	3767
22	478593	962193	1228	3388	322830	536481	1946	4169
23	475732	959776	1868	4627	324376	540896	1865	4076
24	475685	959011	1276	3928	322337	538328	1676	3756
25	475155	958799	1379	3725	320612	536719	1434	3466
26	472749	956330	1643	4002	319143	534764	1696	3732
27	472535	956090	1281	3502	319259	533256	1536	3967
28	468127	952030	1083	3590	317406	530919	1167	3199
29	467646	952210	1681	4189	316217	528237	1579	4026
30	466072	950879	1597	4277	314858	526387	1504	3866
31	464643	949508	2098	4477	313074	524150	1425	3744
32	455218	937500	1483	3600	310347	521038	1213	3219

Table 2. Ardennes battle equipment data for tanks, APCs, and artillery of the Allied (Blue) and German (Red) forces

	Equipment											
	Blue						Red					
	Available			Killed			Available			Killed		
Day	Tank	APC	Art	Tank	APC	Art	Tank	APC	Art	Tank	APC	Art
(i)	$T_B(i)$	$_B(i)$	$_B(i)$	$T_{\dot{B}}(i)$	$_{\dot{B}}(i)$	$_{\dot{B}}(i)$	$T_R(i)$	$_R(i)$	$_R(i)$	$T_{\dot{R}}(i)$	$_{\dot{R}}(i)$	$_{\dot{R}}(i)$
0	2853	6103	3006	1	0	0	0	0	36	0	0	0
1	2863	6019	2972	12	33	15	747	2046	4789	10	5	6
2	2867	5970	2963	43	46	9	663	2041	4791	7	20	41
3	2840	5908	2950	60	18	6	639	2021	4768	13	12	27
4	2808	6004	3103	64	37	14	669	2009	4727	21	18	19
5	3965	7274	3531	33	11	33	619	1984	4786	11	22	24
6	4082	7295	3609	10	6	10	595	1952	4773	21	19	30
7	4109	7507	3772	15	13	2	615	2065	4858	5	16	16
8	4086	7533	3772	36	6	10	645	2034	4845	24	34	35
9	4062	7486	3847	48	18	5	596	1970	4885	22	20	94
10	4265	8105	3931	24	2	4	544	1875	4750	28	36	59
11	4520	8552	4063	20	2	4	483	1800	4779	14	31	34
12	4511	8629	4093	19	3	3	466	1731	4661	13	31	23
13	4526	8536	4004	18	3	2	450	1659	4638	7	22	31
14	4541	8552	4065	16	2	0	433	1595	4415	7	15	19
15	4516	8565	4087	20	0	0	428	1542	4321	21	7	26
16	4610	8554	4086	10	2	1	403	1532	4314	5	14	17
17	4695	8615	4077	14	0	0	413	1523	4283	9	8	36
18	4701	8593	4087	24	0	0	419	1516	4246	6	10	14
19	4710	8462	4088	26	0	1	431	1451	4242	2	10	35
20	4728	8578	4150	22	0	3	428	1441	4110	12	12	28
21	4686	8564	4153	13	2	0	394	1419	4016	2	6	22
22	4719	8502	4144	13	0	0	396	1409	4014	2	5	23
23	4684	8375	4133	12	0	1	400	1403	3981	8	16	26
24	4703	8418	4131	9	0	0	407	1364	3971	0	2	23
25	4743	8446	4128	7	0	0	398	1360	3944	7	3	21
26	4761	8476	4131	5	0	0	407	1358	3925	2	9	33
27	4745	8348	4090	7	1	0	407	1349	3916	2	8	32
28	4717	8459	4108	2	1	0	393	1341	3895	0	3	13
29	4699	8454	4106	6	0	0	418	1335	3854	13	17	20
30	4678	8374	4081	16	0	2	410	1322	3867	5	6	19
31	4662	8436	4092	11	0	0	434	1318	3856	3	4	14
32	4628	8363	4080	20	0	9	432	1309	3824	2	3	7

Table 3. Sorties flown by Red and Blue in direct support of ground forces

Air Sorties					
Day (i)	Blue $As_B(i)$	Red $As_R(i)$	Day (i)	Blue $As_B(i)$	Red $As_R(i)$
0	15	0	17	1295	8
1	48	108	18	1195	188
2	760	249	19	971	0
3	1341	195	20	24	100
4	477	225	21	1507	47
5	0	0	22	463	0
6	64	19	23	456	0
7	119	32	24	576	45
8	1413	234	25	0	0
9	2143	254	26	257	77
10	1754	129	27	153	0
11	1686	203	28	47	0
12	886	178	29	707	65
13	727	15	30	617	155
14	831	95	31	210	0
15	952	19	32	394	29
16	718	175			

data, for which primary source records are nearly non-existent, was almost completely estimated" [2, p. V-1]. The DMSI database did not include the significant Allied air operations of a strategic nature against industrial targets and railroad transportation networks. However, all estimated daily German air data is included and Fricker noted that this might mean German air operations are overcounted in relation to the Allies air operations.

Fricker included air sortie categories: attack armed reconnaissance, bombing, patrol, immediate support, support. Fricker excluded air sortie categories: aerial resupply, escort, pathfinding, para-drop, photo recon, scramble, and weather recon.

Fricker modelled the Ardennes Campaign from start to finish as one battle by estimating "initial force sizes that reflect all of the troops that eventually fought in the campaign and then subtract[ing] the casualty attrition from this total on a daily basis" [4, p. 4]. Fricker developed an algorithm that estimated each initial battle resource size "by sequentially stepping through each resource from day 0 to day 32, accounting for any local reserves

Table 4. The reformatted Ardennes battle manpower data for the Allied (Blue) and German (Red) forces

Day	Reformatted Manpower							
	Blue				Red			
	Available: $Mp_B(i)$		Killed: $Mp_{\dot{B}}(i)$		Available: $Mp_R(i)$		Killed: $Mp_{\dot{B}}(i)$	
(i)	Combat	Total	Combat	Total	Combat	Total	Combat	Total
0	513514	1075857	458	1468	385955	656278	0	0
1	513056	1074389	1589	3062	385955	656278	2191	5590
2	511467	1071327	2383	5712	383764	650688	2423	5559
3	509084	1065615	2085	5093	381341	645129	2015	4711
4	506999	1060522	2175	12101	379326	640418	1993	4332
5	504824	1048421	1389	5334	377333	636086	1985	4351
6	503435	1043087	1174	3197	375348	631735	2084	4582
7	502261	1039890	1905	4815	373264	627153	2046	4531
8	500356	1035075	1548	3730	371218	622622	2468	5351
9	498808	1031345	1608	3857	368750	617271	2685	5609
10	497200	1027488	1527	3635	366065	611662	2538	5563
11	495673	1023853	2320	5411	363527	606099	2504	5526
12	493353	1018442	1376	3596	361023	600573	2544	5751
13	491977	1014846	1277	3435	358479	594822	2121	4511
14	490700	1011411	1005	2934	356358	590311	1682	3900
15	489695	1008477	1042	2743	354676	586411	1844	4076
16	488653	1005734	1159	3022	352832	582335	1550	3635
17	487494	1002712	1004	2773	351282	578700	1788	3898
18	486490	999939	832	2631	349494	574802	1724	3821
19	485658	997308	1831	3580	347770	570981	1752	3892
20	483827	993728	2259	4899	346018	567089	2054	4283
21	481568	988829	1639	4093	343964	562806	1709	3767
22	479929	984736	1228	3388	342255	559039	1946	4169
23	478701	981348	1868	4627	340309	554870	1865	4076
24	476833	976721	1276	3928	338444	550794	1676	3756
25	475557	972793	1379	3725	336768	547038	1434	3466
26	474178	969068	1643	4002	335334	543572	1696	3732
27	472535	965066	1281	3502	333638	539840	1536	3967
28	471254	961564	1083	3590	332102	535873	1167	3199
29	470171	957974	1681	4189	330935	532674	1579	4026
30	468490	953785	1597	4277	329356	528648	1504	3866
31	466893	949508	2098	4477	327852	524782	1425	3744
32	464795	945031	1483	3600	326427	521038	1213	3219

Table 5. The reformatted Ardennes battle equipment data for tanks, APCs, and artillery of the Allied (Blue) and German (Red) forces

	Reformatted Equipment											
	Blue						Red					
	Available			Killed			Available			Killed		
Day	Tank	APC	Art	Tank	APC	Art	Tank	APC	Art.	Tank	APC	Art
(i)	$T_B(i)$	$_B(i)$	$_B(i)$	$T_{\dot{B}}(i)$	$_{\dot{B}}(i)$	$_{\dot{B}}(i)$	$T_R(i)$	$_R(i)$	$_R(i)$	$T_{\dot{R}}(i)$	$_{\dot{R}}(i)$	$_{\dot{R}}(i)$
0	5350	8821	4275	1	0	0	747	2161	5130	0	0	0
1	5349	8821	4275	12	33	15	747	2161	5130	10	5	6
2	5337	8788	4260	43	46	9	737	2156	5124	7	20	41
3	5294	8742	4251	60	18	6	730	2136	5083	13	12	27
4	5234	8724	4245	64	37	14	717	2124	5056	21	18	19
5	5170	8687	4231	33	11	33	696	2106	5037	11	22	24
6	5137	8676	4198	10	6	10	685	2084	5013	21	19	30
7	5127	8670	4188	15	13	2	664	2065	4983	5	16	16
8	5112	8657	4186	36	6	10	659	2049	4967	24	34	35
9	5076	8651	4176	48	18	5	635	2015	4932	22	20	94
10	5028	8633	4171	24	2	4	613	1995	4838	28	36	59
11	5004	8631	4167	20	2	4	585	1959	4779	14	31	34
12	4984	8629	4163	19	3	3	571	1928	4745	13	31	23
13	4965	8626	4160	18	3	2	558	1897	4722	7	22	31
14	4947	8623	4158	16	2	0	551	1875	4691	7	15	19
15	4931	8621	4158	20	0	0	544	1860	4672	21	7	26
16	4911	8621	4158	10	2	1	523	1853	4646	5	14	17
17	4901	8619	4157	14	0	0	518	1839	4629	9	8	36
18	4887	8619	4157	24	0	0	509	1831	4593	6	10	14
19	4863	8619	4157	26	0	1	503	1821	4579	2	10	35
20	4837	8619	4156	22	0	3	501	1811	4544	12	12	28
21	4815	8619	4153	13	2	0	489	1799	4516	2	6	22
22	4802	8617	4153	13	0	0	487	1793	4494	2	5	23
23	4789	8617	4153	12	0	1	485	1788	4471	8	16	26
24	4777	8617	4152	9	0	0	477	1772	4445	0	2	23
25	4768	8617	4152	7	0	0	477	1770	4422	7	3	21
26	4761	8617	4152	5	0	0	470	1767	4401	2	9	33
27	4756	8617	4152	7	1	0	468	1758	4368	2	8	32
28	4749	8616	4152	2	1	0	466	1750	4336	0	3	13
29	4747	8615	4152	6	0	0	466	1747	4323	13	17	20
30	4741	8615	4152	16	0	2	453	1730	4303	5	6	19
31	4725	8615	4150	11	0	0	448	1724	4284	3	4	14
32	4714	8615	4150	20	0	9	445	1720	4270	2	3	7

(X_{lr}) or the addition of reinforcements (X_r) as they may result, while first using local reserves for any force increase before assuming that reinforcements were added” [4, p. 6]. Fricker reflected this reformatted data in Tables 4 and 5.

This chapter presented Tables 1-5 which include all the data sourced for this analysis.

Chapter 2: Fitting the Lanchester Equations

This chapter will cover how to fit the Ardennes battle data to the Lanchester equations. Considering the Lanchester equations in general form (equations1&2), the resources data must be expressed as single forces and losses of red and blue.

2.1 Aggregated Forces

It must be identified how much each resource influences overall force. For example, one tank will not be equivalent in force to one personnel of manpower. Bracken [1] assigned each resource an effectiveness weighting to estimate their contribution to overall forces. These weights were derived from standard U.S. Army Concepts Analysis Agency practices.

Table 6. Bracken Weightings

Manpower Weighting	1
Tank Weighting	20
APC Weighting	5
Art Weighting	40
Air Sorties Weighting	0

By aggregating resource forces, Bracken formed homogenous equations that represent overall forces for Blue and Red. Overall force losses can be aggregated similarly, aggregating resource force losses instead of resource forces.

Aggregated overall forces are considered both including and excluding air force:

$$\text{Force} = (\text{Mp}*\text{MpW}) + (\text{T}*\text{TW}) + (\text{APC}*\text{APCW}) + (\text{Art}*\text{ArtW})\ [+(\text{As}*\text{AsW})] \qquad (3)$$

Where MpW is manpower weighting, TW is tank weighting, APCW is APC weighting, ArtW is artillery weighting, and AsW is air sorties weighting.

Manpower, tank, APC, artillery and air sortie inputs can either all be "available" or all be "killed" to calculate either force or force loss respectively.

2.2 Bracken approach

Bracken [1] calculated these aggregated forces for combat manpower and total manpower, and applied the general form of lanchester equations as well as his own version including a tactical parameter d which considers the impact of whether a force is attacking or defending. By searching over a grid in the {a,b,p,q,d} space for the minimum sum of square residuals (SSR), the best model parameters could be found. Bracken specified search ranges for each parameter, limiting the model. "However, it does guarantee that the identified parameters are optimal over the options made available" [1, p. 9].

2.3 Linear Regression

Similar to Fricker [4], this analysis more accurately estimates the model parameters by using linear regression on logarithmically transformed Lanchester equations. The Lanchester equations are originally in a nonlinear format. Willard [9] logarithmically transformed equations (1) & (2) by estimating α and β parameters:

$$\dot{B} = a\,R^p\,B^q$$

$$\dot{R} = b\,B^p\,R^q$$

$$\frac{\dot{B}}{\dot{R}} = \left(\frac{a}{b}\right)\frac{R^{p-q}}{B^{p-q}}$$

$$\alpha = \frac{a}{b}$$

$$\beta = p - q$$

$$\frac{\dot{B}}{\dot{R}} = \alpha\left(\frac{R}{B}\right)^{\beta}$$

$$\log\left(\frac{\dot{B}}{\dot{R}}\right) = \beta\ \log\left(\frac{R}{B}\right) + \log(\alpha) \qquad (4)$$

Equation (4) is in a linear form that can be solved by linear regression. Using linear regression to solve these equations can minimise the sum of square residuals better than Bracken's brute force approach as it does not require search restrictions. Also, there are many statistical techniques that can only be used when analysing linear models to assess the significance of parameter results, making linear models preferable.

2.4 Lanchester Equations in general form

α and β are not found in the general form of the Lanchester equations. The parameters that need to be found are p, q, a, and b. To extract these parameter values from α and β, a second equation must be solved by logarithmic transformation as well:

$$\dot{B}\dot{R} = ab(\,BR\,)^{p+q}$$

$$\gamma = ab$$

$$\delta = p + q$$

$$\dot{B}\dot{R} = \gamma(\,BR\,)^{\delta}$$

$$\log(\dot{B}\dot{R}) = \delta \log(BR) + \log(\gamma) \qquad (5)$$

From δ and β;

$$p = \frac{1}{2}(\delta + \beta)$$

$$q = \frac{1}{2}(\delta - \beta)$$

From α and γ;

$$a = \sqrt{\alpha\gamma}$$

$$b = \sqrt{\frac{\gamma}{\alpha}}$$

By solving equations (4) and (5) by linear regression; a, b, p, and q can be calculated and a general form of Lanchester equations can be found. The Lanchester equations can then be used to find estimated $\dot{B}(t)$ and $\dot{R}(t)$ values with known B(t) and R(t) values. These can then be compared to the actual $\dot{B}(t)$ and $\dot{R}(t)$ values from the Ardennes battle data, and by considering the residual data, the degree of error can be critically evaluated.

The linear regression techniques outlined in this chapter will be used in this analysis to fit the Lanchester equations.

Chapter 3: Optimization Techniques

This chapter will discuss some optimisation techniques that could be used to improve the fit of the Lanchester Equations and how they may be implemented.

3.1 Splitting the data

Bracken modelled only the first ten days of the Ardennes Campaign. Fricker reformatted the data and modelled the Campaign from start to finish. This analysis will split the Ardennes campaign in to two battles, split at the turning point of the battle.

The Germans attacked days 0 – 5 and the Allies attacked days 6 – 32, with the Germans launching an attack on day 17 but having lost the initiative by this stage. Day 0 data for the Germans is incomplete and can't be included in the model.

It is identified that on day 8 the skies cleared and the Allies gained the advantage of being able to use their superior air force. On day 9 the German advance was halted. On day 10 Allied counter offensives began.

Ultimately the turning point of the battle was day 10 when the Allies began pushing back the Germans. Therefore, the two splits analysed will be day 1-10 / day 11-32.

3.2 Equipment Weighting

These weightings are considered fixed values by Bracken and Fricker. This analysis will consider searching over a range for the weightings that improve the fit of the Lanchester equations. The influence individual resources will have on force will be battle specific. It can be assumed that the weightings derived from standard U.S. Army Concepts Analysis Agency practices provide a reasonably good estimate for the Ardennes battle data, and that the search range for better weightings can be centred around these weightings. Manpower weighting will remain 1, and other weightings will be optimised around this. An appropriate range must be used to allow significant changes without allowing unrealistic weightings with poor physical interpretation.

3.3 Resource subdivisions

Fricker doesn't use the calculated reinforcements and local reserves for each resource in his analysis other than to calculate initial resource size for the campaign. By implementing Fricker's algorithm, a MATLAB programme was developed to replicate the reinforcements and local reserves for each resource for each day of the Ardennes Campaign (see Appendix 1). With this data, resources in Table 1 & 2 can be divided into reinforcements (REINF), local reserves (LRes) and surviving (Surv) resources for each day, and each resource can be expressed as the sum of these subdivisions (see Appendices 2-11 for subdivisions of each resource). The Bracken data for each resource consisted of the sum of surviving and reinforcement resources, and did not consider local reserves. The Bracken data can be described as the "used" resources each day, whereas the sum of surviving, reinforcements and local reserves can be described as the "available" resources each day.

But each of these subdivisions will not have the same influence on the force of the resource. Local reserves may only participate in battle for part of a day or not at all that day. The surviving resources may be fatigued /injured/damaged from previous days of battle whilst still being able to participate in battle, whereas reinforcements are new resources that will not suffer from any fatigue/injury/damage. This logic identifies ranked weightings as;

$$\text{LRes Weighting} < \text{Surv Weighting} < \text{REINF Weighting}$$

If surviving resources, local reserves and reinforcements weightings could vary freely, some unrealistic weightings might be observed that lack physical interpretation. Therefore, surviving resources is assigned a weighting of 1 and the weightings of local reserves and reinforcements can vary within predefined search ranges around this for each resource to optimise the fit of the Lanchester equations within the search ranges. It is assumed that any killed will first be decreased from surviving resources.

3.4 Reformatted resources subdivisions

A similar process can be implemented for Fricker's reformatted data. For each day, a reformatted resource can be divided into surviving resources, reinforcements, local reserves, and nonlocal reserves (Res) and expressed as its sum of these subdivisions (see Appendices 2-11 for subdivisions of each resource). These external reserves will have an even lower weighting than local reserves as they are yet to join the battle and are further away from the locus of the battle, not readily available to contribute to battle that day. This logic identifies ranked weightings as;

$$\text{Res Weighting} < \text{LRes Weighting} < \text{Surv Weighting} < \text{REINF Weighting}$$

Similarly, assigning surviving resources a weighting of 1 for consistency; reserves, local reserves and reinforcements can vary freely within defined search ranges around this for each resource to optimise the fit of the Lanchester equations within the search ranges.

3.5 Logistics & support data

By considering total manpower and combat manpower, support manpower can be reasonably deducted by subtracting combat manpower from total manpower. It can be seen from this that the Blue manpower have a larger proportion of their manpower consisting of support personnel than Red manpower. As support forces increase the effectiveness of combat forces [8], this larger proportion of support personnel in the Blue army makes their combat effectiveness higher than that of the Red army. With the large number of Blue APCs engaging during the Ardennes Campaign, if APCs are included as support forces despite having combat capabilities, the support to combat ratio of Blue increases even higher above Red forces.

However, when this analysis calculated local reserves and reinforcement resources for non-combat manpower, some negative values were produced (see Appendices 12 & 13). This identifies possible weakness in the algorithm used to estimate local reserves and reinforcements. Assumptions are made that any force increase uses local reserves before adding reinforcements. There can also be contradiction and inconsistency in records and reports about which personnel count as combat manpower, and this can lead to overcounting of combat or total manpower. It is illogical to include these negative support data values in this analysis.

To optimise the fit of the Lanchester equations, this analysis will test splitting the data, varying equipment weightings and varying subdivision weightings as outlined in this chapter.

Chapter 4: Analysis & Results

This chapter will include an outline of forces data and analysis of the results of applying the optimisation techniques outlined in the previous chapter to the modelling process.

Analysis of Data

Figure 1 outlines combat forces for modelling Lanchester equations before considering any optimisation techniques.

Figure 1:

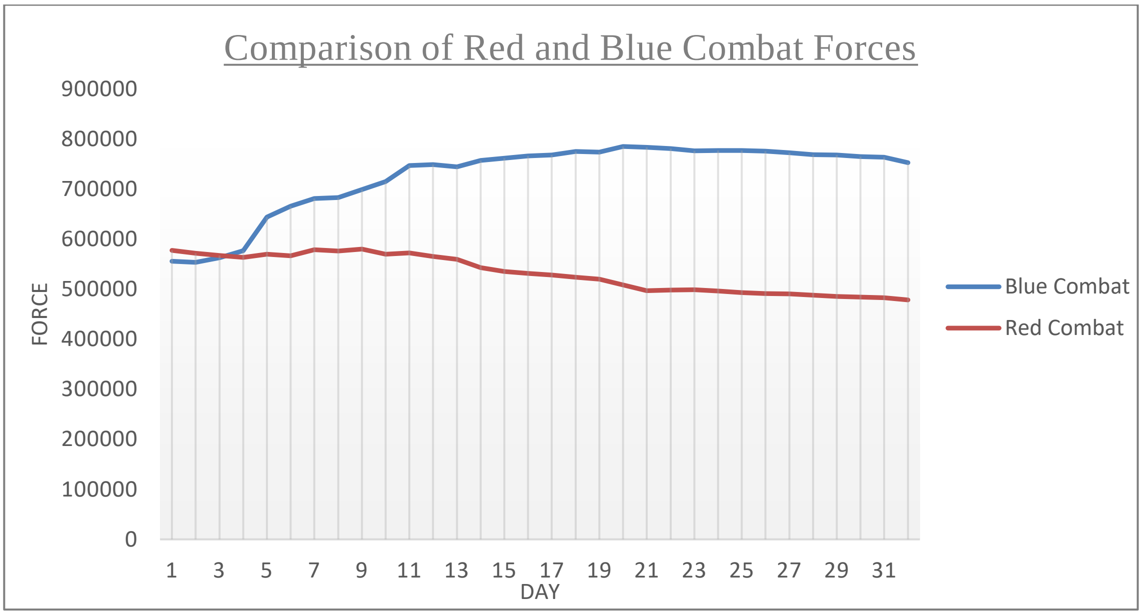
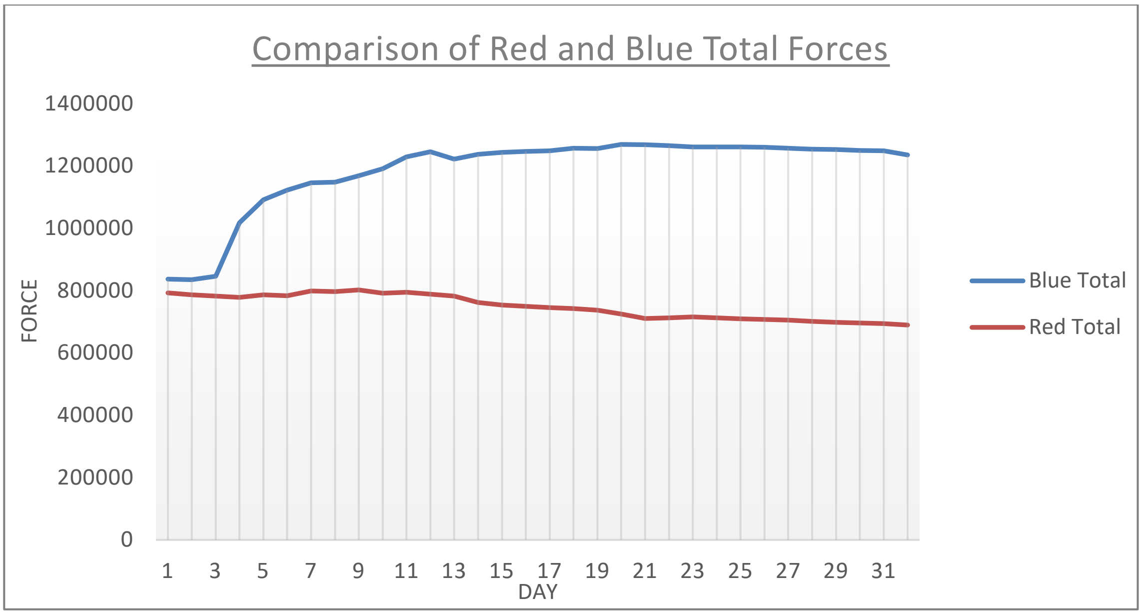


Figure 2 outlines total forces for modelling Lanchester equations before considering any optimisation techniques.

Figure 2:



4.1 Splitting the data

Fricker modelled the Ardennes Campaign from start to finish, day 1-32. Day 0 couldn't be included because of the missing German data. This section will compare the results from modelling day 1-32 with the results from modelling the campaign as two battles split at the turning point of the battle; day 1-10 and day 11-32.

This comparison will be analysed using the Bracken & Fricker data considering combat, total, reformatted combat and reformatted total force, each considered including and excluding air force. This will identify how well the Lanchester equations fits to each of these conditions to analyse variation in fits and identify which fit is best.

Before considering any other optimisation techniques, the aggregation of forces (equation 3) can be carried out with Bracken's fixed weightings and not considering subdivisions of resources using a MATLAB code that was developed (see Appendix 14). Linear regression is used to solve equation (4) and to calculate values for α and β, and find the variance of the model to assess how well it fits the data.

Table 7. Comparison of Lanchester equations fit including and excluding Air Force (6 s.f.)

		Excluding Air Force			Including Air Force		
	Day	alpha	beta	**Variance**	alpha	beta	**Variance**
Combat	1-32	0.450544	0.652411	0.121636	0.470210	0.733204	0.149430
	1-10	0.545500	2.53337	0.485355	0.571645	2.32539	0.522395
	11-32	0.184829	-1.53405	0.149637	0.169175	-1.66186	0.137399
Total	1-32	0.911344	0.246170	0.0142401	0.947410	0.321135	0.0237491
	1-10	1.09638	0.895519	0.0670789	1.15725	1.03180	0.0932888
	11-32	0.177591	-2.76280	0.412511	0.158803	-2.89816	0.372433
Reformatted Combat	1-32	0.641071	1.57346	0.0285523	1.09458	2.91337	0.113690
	1-10	12703.6	33.0712	0.497327	4.16251	6.83623	0.344360
	11-32	0.0374867	-5.97728	0.221984	0.0432757	-5.28583	0.124230
Reformatted Total	1-32	0.621036	-0.532960	0.00367991	1.19447	0.733209	0.00704552
	1-10	216588	26.7907	0.179452	217.850	11.6014	0.261115
	11-32	0.0246498	-6.47466	0.517825	0.0181501	-6.96298	0.423276
Mean				0.224940			0.214368

It can be seen from Table 7 that in all cases for day 1-32, the variances are weak and therefore these models don't fit the data well. The models for day 1-10 and day 11-32 generally have better variances, but still don't fit the data overly well with the best variances being around 0.5 and "total" day 1-10 being particularly weak (0.0670789).

Excluding air force, the best fit to the data was "reformatted total" day 11-32 with a variance of 0.517825. Including air force, the best fit to the data was "combat" day 1-10 with a variance of 0.522395. The mean variance is slightly better when excluding air force, indicating that generally the models perform slightly better excluding air force. However, it is a small difference and more importantly it can be seen that models for day 1-32 and day 1-10 improve when air force is included except for "reformatted combat" day 1-10, and models of day 11-32 consistently weaken when air force is included. Air force is a stronger variable in the early days of the Campaign and a weaker variable towards the end of the Campaign.

Based on the information in table 7, the rest of this analysis will not consider models of day 1-32, and will instead focus on improving the models for day 1-10 and day 11-32.

4.2 Equipment Weightings

This section will analyse the improvement of the models when allowing the equipment weightings to vary.

Better equipment weightings that improve the fit of the model were searched for within an appropriate range with reasonable minimum and maximum values so that any results would have a physical interpretation (see appendix 15). With manpower weighting being fixed as 1, it would be hard to justify a tank having a weighting of 1, or an APC having a weighting of 100 or more. Tank weighting, artillery weighting and air sortie weighting were all allowed to vary between ten either side of Bracken's respective weightings. As it is hard to justify weighting an APC equivalent to one manpower, APC was only allowed to vary as low as 2. APC could vary as high as 20 to give a similar search range as the other resources.

It can be seen from Tables 8 and 9 that all variances have improved. "Reformatted combat" day 1-10 gives the best variance excluding air force (0.678354), and "combat" day 1-10 gives the best variance including air force (0.716455). From the mean, models including air force have improved better than models excluding air force as they have an extra variable varying.

An important thing to note is how almost all optimal equipment weightings within the search ranges are found at the very edges of the ranges. This suggests that the global minima of the equipment weightings are found outside of the search ranges. This is surprising because finding minima much further outside of the search range would lack physical interpretation, suggesting inappropriate equipment weightings. Appendices 16 and 17 investigated the results when allowing the model to vary all the equipment weightings between 0 and 100 in increments of 5. Very surprisingly, most of the optimal weightings were still found at the edges of the search ranges, suggesting global minima further outside of this range. Some models were fitting best assigning all equipment resources a weighting of 100, greatly reducing the influence manpower has on the model. Other models were fitting best setting 1 or even 2 equipment resources to 0, suggesting that those resources had no effect at all on the model. Suggesting for example, it didn't matter whether a force had 1 tank or 1000000 tanks, this would not affect the outcome of the battle in any way. Clearly this model lacks physical interpretation.

The models reflected in tables 8 and 9 identify realistic equipment weighting variables with a physical interpretation and can be considered an improvement on models with fixed Bracken equipment weightings.

Table 8. Lanchester equations fit with equipment weighted

		Equipment Weightings			Results		
	Day	TW (10:1:30)	APCW (2:1:20)	ArtW (30:1:50)	alpha	beta	**Variance**
Combat	1-10	10	20	30	0.736836	3.02674	0.642301
	11-32	10	20	30	0.117405	-2.02678	0.235629
Total	1-10	30	20	50	1.36299	1.41458	0.157499
	11-32	10	20	30	0.113119	-3.19866	0.493732
Reformatted Combat	1-10	10	20	50	120027	37.9632	0.678354
	11-32	10	20	30	0.0146629	-6.69666	0.303929
Reformatted Total	1-10	17	20	50	480877	32.2201	0.292193
	11-32	10	20	30	0.0150425	-6.70054	0.579521
Mean							0.422895

Table 9. Lanchester equations fit with equipment weighted, including air force

		Equipment Weightings				Results		
	Day	TW (10:1:30)	APCW (2:1:20)	ArtW (30:1:50)	AsW (20:1:40)	alpha	beta	**Variance**
Combat	1-10	10	20	50	40	0.658104	3.01481	0.716455
	11-32	10	20	30	20	0.107345	-2.14570	0.223310
Total	1-10	30	20	50	40	1.46483	1.54681	0.200257
	11-32	10	20	30	20	0.101451	-3.33150	0.467535
Reformatted Combat	1-10	10	20	50	20	58.7082	14.4419	0.605424
	11-32	10	20	35	20	0.0110404	-7.33626	0.250235
Reformatted Total	1-10	10	20	50	40	321.085	12.4246	0.381297
	11-32	10	20	30	20	0.00999109	-7.29246	0.536734
Mean								0.422656

4.3 Subdivisions of forces

This section will analyse the improvement of the models when allowing the weightings for subdivisions of resources to vary.

Reinforcements, local reserves and reserves will not have the same contribution to the force of the resource. Appropriate weightings have been identified as;

Res Weighting < LRes Weighting < Surv Weighting < REINF Weighting

Where reserves are included when considering Fricker's reformatted data and excluded when considering Bracken's original data, and surviving resources are assigned a weighting of 1.

Weightings were found under the search operations that reinforcements weighting must be at least 1 and at most 3, and that local reserves and reserves can have a weighting of 0 and at most 0.9. Because of the amount of iterations required to search through subdivisions for each resource, small increments are not used. REINF increments in 0.4s, LReserves increments in 0.3s. This decreases the computation runtime massively compared to searching in increments of 0.1s, while still searching over the same range.

Table 10. Lanchester equations fit with subdivisions weighted

			Weightings				**Results**		
	Day	Resources	Surviving	REINF (1:0.4:3)	Lreserves (0:0.3:0.9)	Reserves	alpha	beta	**variance**
Combat	1-10	Manpower	1	1	0.9	0	107126000	196251000	0.501572
		Tank	1	1	0.9	0			
		APC	1	1	0.9	0			
		Artillery	1	1	0.9	0			
	11-32	Manpower	1	3	0.9	0	0.0983257	-3.42772	0.271406
		Tank	1	3	0.9	0			
		APC	1	3	0.9	0			
		Artillery	1	3	0.9	0			
Total	1-10	Manpower	1	1	0.9	0	1.09876	0.907362	0.0680541
		Tank	1	1	0.9	0			
		APC	1	1	0	0			
		Artillery	1	1	0.9	0			
	11-32	Manpower	1	2.2	0.9	0	0.0972551	-4.01067	0.523632
		Tank	1	3	0.9	0			
		APC	1	3	0.9	0			
		Artillery	1	3	0.9	0			
Mean									0.341166

Tables 10 and 11 do not include reformatted models as this would involve 12 for loops and MATLAB did not have the memory to produce matrices of the required size.

As expected from previous analysis, including air force improved models of day 1-10 best, and excluding air force improved models of day 11-32 best. Other cases did not improve models much.

Like equipment, the optimal weightings are found at the edges of search ranges suggesting that global minima are found outside the search range. Appendices 18 and 19 investigated the results when allowing the model to vary all the subdivision weightings between 0 and 5. Similar unusual results are observed with many optimal weightings still being at the edges of the search range suggesting a global minimum further still outside of this search range. It would lack physical interpretation to consider these weightings, with some models suggesting high local reserves weightings and low reinforcements weightings.

Table 11. Lanchester equations fit with subdivisions weighted, including air force

			Weightings				**Results**		
	Day	Resources	Surviving	REINF (1:0.4:3)	Lreserves (0:0.3:0.9)	Reserves	alpha	beta	**variance**
Combat	1-10	Manpower	1	1	0.9	0	0.572837	2.38535	0.538861
		Tank	1	1	0.9	0			
		APC	1	1	0.9	0			
		Artillery	1	1	0.9	0			
	11-32	Manpower	1	3	0.9	0	0.0760007	-3.90031	0.250868
		Tank	1	3	0.9	0			
		APC	1	3	0.9	0			
		Artillery	1	3	0.9	0			
Total	1-10	Manpower	1	1	0.9	0	1.16026	1.04499	0.0945866
		Tank	1	1	0.9	0			
		APC	1	1	0	0			
		Artillery	1	1	0.9	0			
	11-32	Manpower	1	1.4	0.9	0	0.0840531	-4.19910	0.483459
		Tank	1	3	0.9	0			
		APC	1	3	0.9	0			
		Artillery	1	3	0.9	0			
Mean									0.341944

The models reflected in tables 10 and 11 all showed improvement, but generally not as much as in the previous section with equipment weightings suggesting that varying equipment weightings has a stronger impact on the fit of the Lanchester equations than varying the subdivisions of resources.

4.4 Alternative model

This section considers an alternative way of aggregating forces.

Reserves can be allowed to vary by considering an alternative model. Instead of calculating overall force as an aggregation of each resource (3), an alternative method would be to calculate overall force as an aggregation of overall subdivisions of forces. This can be done by calculating an overall Surviving, REINF, LReserves and reserves from the resources:

Surviving = Mp Surviving + T Surviving + APC Surviving + Art Surviving (6)

REINF = Mp REINF + T REINF + APC REINF + Art REINF (7)

LReserves = Mp LReserves + T LReserves + APC LReserves + Art LReserves (8)

Reserves = Mp Reserves + T Reserves + APC Reserves + Art Reserves (9)

Where each individual subdivision is already weighted as a force using Bracken's weightings.

And the equations for overall forces would be:

Force = (Surv*SurvW) + (REINF*REINFW) + (LRes*LResW) + (Res*ResW) [+ As*AsW] (10)

A MATLAB programme was developed to model this (see appendix 20).

This model generalises resources rather than considering subdivisions for resources individually, and so the variances for models excluding reserves are at most the same or slightly weaker than the previous subdivisions models. However, with this model requiring less computation this enables it to handle varying reserves as well to give reformatted models as well. And by varying reserves from 0 – 0.9, results no longer need to be reformatted or non-reformatted as reserves can be excluded by having a weighting of zero, or included if a higher weighting improves the variance.

Table 12. Lanchester equations fit with alternative model

		Weightings				Results		
	Day	Surviving	REINF (1:0.4:3)	Lreserves (0:0.3:0.9)	Reserves (0:0.3:0.9)	alpha	beta	**variance**
Combat	1-10	1	1	0.9	0.8	13.6630	12.6438	0.536278
	11-32	1	3	0.9	0.9	0.0150309	-8.16466	0.343748628
Total	1-10	1	3	0.9	0.9	0.482595	-1.29147	0.184928
	11-32	1	2.6	0.9	0.9	0.0148090	-7.42724	0.561583
Mean								0.406634

Table 13. Lanchester equations fit with alternative model, including air force

		Weightings				Results		
	Day	Surviving	REINF (1:0.4:3)	Lreserves (0:0.3:0.9)	Reserves (0:0.3:0.9)	alpha	beta	**variance**
Combat	1-10	1	1	0.9	0.3	0.821777	3.36847	0.544834
	11-32	1	3	0.9	0.2	0.0595030	-4.48690	0.252154
Total	1-10	1	1	0.9	0.9	51.1605	8.85291	0.245431
	11-32	1	2.6	0.9	0.1	0.0657590	-4.62522	0.471296
Mean								0.378429

This alternative model improves on the subdivision models.

4.5 Combined Model

Equipment and subdivision weightings can be combined using the alternative model.

The alternative model considers subdivisions of forces with considerably less computation. Therefore, this model can be used to combine subdivision weightings with equipment weightings. Equipment weightings can be used to calculate force for each of the individual subdivisions before they are aggregated for overall subdivision force (see appendix 20).

Table 14. Lanchester equations fit with combined model

		Weightings						Results		
	Day	TW (10:2:30)	APCW (2:2:20)	ArtW (30:2:50)	REINF (1:0.4:3)	Lreserves (0:0.3:0.9)	Reserves (0:0.3:0.9)	alpha	beta	**variance**
Combat	1-10	10	20	46	1	0.9	0.9	541.065	22.0326	0.695175
	11-32	10	20	30	3	0.9	0.9	0.00474560	-8.89461	0.429911
Total	1-10	10	2	30	3	0.9	0.9	0.388086	-1.79931	0.284505
	11-32	10	20	30	2.6	0.9	0.9	0.00777751	-7.73690	0.626707
Mean										0.509075

Table 15. Lanchester equations fit with combined model, including air force

		Weightings							Results		
	Day	TW (10:2:30)	APCW (2:2:20)	ArtW (30:2:50)	AsW (20:2:40)	REINF (1:0.4:3)	Lreserves (0:0.3:0.9)	Reserves (0:0.3:0.9)	alpha	beta	**variance**
Combat	1-10	10	20	50	24	1	0.9	0.6	2.98118	7.25182	0.748726
	11-32	10	20	30	20	3	0.9	0.3	0.0192984	-5.89993	0.361445
Total	1-10	10	20	50	40	1	0.9	0.9	91.1903	10.1284	0.377727
	11-32	14	20	30	20	2.6	0.9	0.6	0.0127571	-6.73520	0.572618
Mean											0.515129

4.6 General Lanchester form

This section will find the general form of Lanchester equations with the best fitting results.

The best fitting results for modelling the Ardennes Campaign for both day 1-10 or day 11-32 were using the combined model. Varying equipment and subdivision weightings within reasonable ranges, day 1-10 was best modelled with combat manpower and including air force, producing a variance of 0.748726. Day 11-32 was best modelled with total manpower and excluding air force, producing a variance of 0.626707.

From MATLAB programme (Appendix 20); p, q, a and b are calculated for these two models. From (equations 1&2), this gives general form equations of:

Day 1-10:

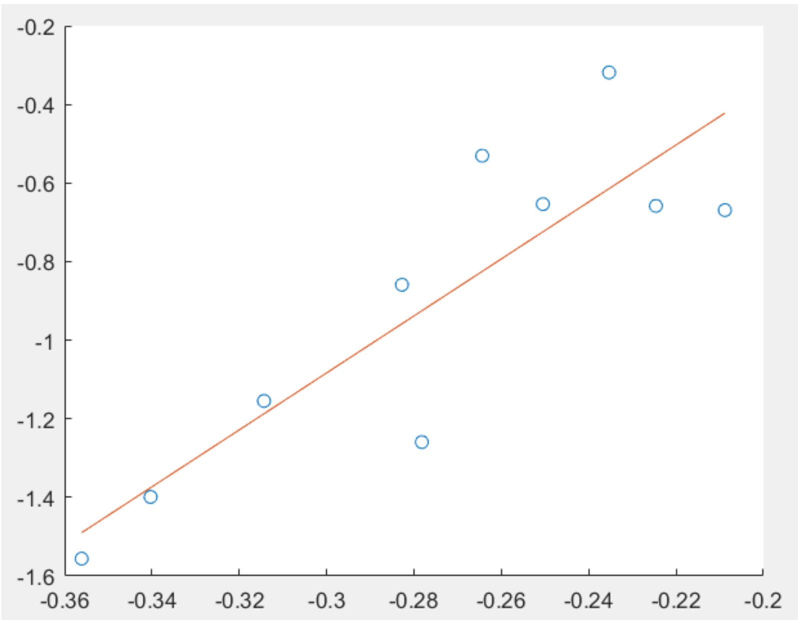
$$\dot{B}(t) = (0.000098)\, R(t)^{(4.30)}\, B(t)^{(-2.95)}$$

$$\dot{R}(t) = (0.000033)\, B(t)^{(4.30)}\, R(t)^{(-2.95)}$$

considering combat manpower and including air force,

with equipment and subdivision weighted as shown in table 15.

Figure 3:



Day 11-32:

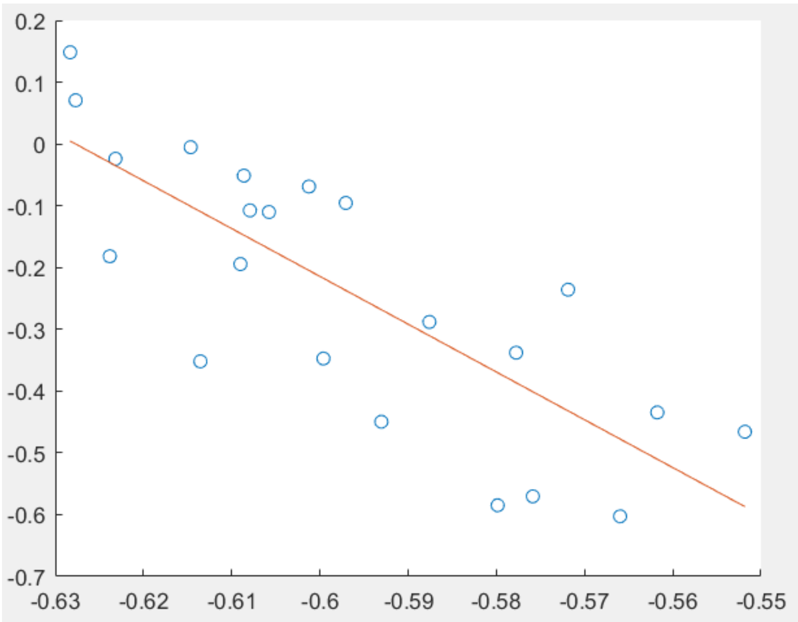
$$\dot{B}(t) = (0.0000000094)\ R(t)^{(-2.98)}\ B(t)^{(4.75)}$$

$$\dot{R}(t) = (0.0000012)\ B(t)^{(-2.98)}\ R(t)^{(4.75)}$$

considering total manpower and excluding air force,

with equipment and subdivision weighted as shown in table 14.

Figure 4:



The different gradients and positive/negative p and q are indicative of Red forces being more effective day 1-10, and Blue forces being more effective day 11-32.

Critical Evaluation

Chapter 1 introduced the source data for this analysis.

Chapter 2 covered how the data can be fit to the Lanchester equations.

Chapter 3 outlined optimisation techniques that could be implemented to improve the fit to the data by applying Lanchester equations.

Chapter 4 detailed the analysis and findings, displaying results and finding the optimal weightings that improved the fit of the Lanchester equations.

This section will review the project objectives and how well they have been completed.

Objective 1: "Fit the Lanchester Equations to the Ardennes battle data"

Chapter 2 covered how to fit the Lanchester Equations to the Ardennes battle data. The Lanchester equations required that Blue and Red resources be represented as single forces. To do this, section 2.1 explains how the resources were assigned weightings that indicated their influence on overall force. The resource forces could then be aggregated to represent overall force. Section 2.3 explains how linear regression is used to estimate α and β parameters, and section 2.4 explains how to derive the Lanchester equations parameters; p, q, a, and b from this.

Section 4.6 finally fits the Lanchester Equations to the Ardennes battle data after implementing optimisation techniques.

Objective 2: "Improve the fit by splitting the battle into 2 battles"

It was identified in section 3.1 that the turning point of the battle was day 10. The skies had cleared allowing the Allies to fully utilise their superior air force, the German's had lost the initiative by this stage and the Allies started pushing the Germans back.

Within Chapter 4 the battle data was split into "day 1-10" and "day 11-32." It was found that splitting the battle data at the turning point of the battle was effective in improving the fit to the model. The 2 different splits produced opposite gradients indicating that Red forces were winning day 1-10, but ultimately Blue forces won day 11-32, which matches the literature review.

Including air force improved the fit of the model for day 1-10 better than day 11-32, possibly indicating the importance of air force on days 8-10 as the skies cleared in turning the battle around.

Objective 3: “Improve the fit to the data by varying equipment weightings”

Section 3.2 briefly mentions how equipment weightings will be varied to improve fit of the Lanchester equations. The weighting ranges were subjectively decided to be as wide a range as possible while maintaining a physical interpretation. This was carried out in section 4.2.

The optimal weightings found within search ranges were almost always at the edges of the search range, suggesting global minima outside of the range even when the range is very large. This was true even when allowing the programme to search over a larger range that lacked physical interpretation. Models with wide search ranges for weightings produce good models, with variances as strong as 0.899901 in the case of “combat” day 1-10. But the weightings found have no physical interpretation with some equipment weightings of 0 and some of 100. This unusual behaviour might be explained by the model missing a variable that has more importance than the variables currently included, but it is difficult to imagine what this might be with the current variables already included. Including air force in the model did nothing to change this unusual behaviour. Another explanation is that the Lanchester equations are simply not an appropriate way of modelling this data. The Lanchester equations are a very simple form of modelling, which is the reason for an interest in testing their capabilities.

Objective 4: “Improve the fit to the data by varying subdivision resources weightings”

Sections 3.3 and 3.4 outline the procedure of splitting resources into subdivisions of reinforcements, local reserves, reserves and surviving resources. Logic was used to infer how resources should be weighted. Assigning surviving resources a weighting of 1, reinforcements would have a weighting higher than 1 with no fatigue / injury / damage. Local reserves and reserves that may or may not even be used in battle that day were weighted lower than surviving resources.

With the amount of resources subdivisions to search over, computation times were longer for finding subdivision weightings. Section 4.3 searched for subdivision weightings that optimised the fit of the model, but search increments had to be kept small. Even still, it was not possible for MATLAB to create the matrices required for models including reserves requiring 12 for loops. MATLAB did not have the memory required to store the size of the matrices required by the programme.

An alternative model was considered in section 4.4 that could include reserves data. This model calculated overall force as an aggregation of overall subdivisions of forces instead of an aggregation of resource forces.

It was found that varying equipment weightings improved the fit of the model better than varying subdivision weightings, but varying both sets of weightings together using the alternative model in section 4.5 gave the best fits to the model.

Objective 5: "Identify best fitting models as Lanchester linear or Lanchester square"

Section 4.6 modelled the best fitting Lanchester equations to the data. Day 1-10 had p and q parameters 4.3 and -2.95 respectively. Day 11-32 had p and q parameters -2.98 and 4.75 respectively.

These parameters do not fit the conditions of Lanchester linear ($q - p = 0$) or Lanchester square ($p - q = 1$) or Lanchester logarithmic ($q - p = 1$). For both equations, as one number is negative, one take away the other results in a number further away from 0 or 1.

In all cases, for models day 1-10 and day 11-32, the result of p and q is plus or minus a number between 6 and 7. Clearly this does not fit the other conditions. This might be because the Ardennes campaign is a mixture of small engagements and concentrated forces.

The models fit the data well enough to recognise a general trend in the data but without much precise predictive power. As the Lanchester equations don't fit any of the lanchester conditions, the results conclude the Ardennes campaign is neither lanchester linear, lanchester square, or lanchester logarithmic.

Conclusions

This analysis found the optimal fit to the data using Lanchester equations over the options available. This study does not suggest that these new-found weightings are an optimal fit for lanchester models of war. It suggests that the weightings of these resources in a lanchester model will vary depending on the campaign. Instead of modelling around fixed weightings, Lanchester models will fit data best when considering an appropriate range of weighting for each resource.

With more time, validation could have been given to the best fitting models. The error could have been analysed in detail through the residual data. This could have revealed the difference between the model's estimated losses and actual losses in terms of force. A mean difference and standard deviation could be found, like Bracken [1] and a significance test carried out on the difference between actual and estimated losses.

Other validation techniques could have been used such as splitting the data into a training set and a validation set by using the holdout method, periodically leaving out daily tallies and analysing how the fit on the training set fits the validation set. The data set of 32 days might be too short for this method to work well. The final models have only been fitted on the Ardennes battle data, but no testing has been done for other battle data. The great tank battle of Kursk has similar battle data available in daily tallies [7]. The Lanchester models found in this analysis could be applied directly to Kursk battle data to see how well it fits other data as means of validation.

Additional parameters could be considered that increase the effectiveness of one forces resources but not the other forces making the equations heterogenous. One side might have an advantage in battle through a resource e.g. it might be found that the Allied air sorties had increased effectiveness over the Germans because of the large allied air force and the German's inability to fully engage them. Or German APCs might influence German forces more than Allied forces as transportation was essential if the Germans were to succeed.

More research could have been done into the ratios of support to combat forces, with [8] identifying an Allied ratio of 0.8 compared to a German ratio of 0.5.

Appendices

Appendix 1

```
clear all;
close all;
clc;

resourceAvailable = xlsread('Table1Manpower.xlsx',
'B8:B17');
resourceKilled = xlsread('Table1Manpower.xlsx',
'D8:D17');

X = resourceAvailable;
x = resourceKilled;
xr = 0;
xlr = 0;

for t = 1: numel(X)-1

    if and(X(t+1) >= (X(t) - x(t)), xlr(t) == 0)
        xr(t+1) = X(t+1) - X(t) + x(t);
        xlr(t+1) = xlr(t);
    else
        if and(X(t+1) >= (X(t)-x(t)), xlr(t) >=
(X(t+1)-X(t)+x(t)))
            xlr(t+1) = xlr(t) - (X(t+1)-X(t)+x(t));
            xr(t+1) = 0;
        else
            if and(X(t+1) >= (X(t)-x(t)), 0 < xlr(t) <
(X(t+1)-X(t)+x(t)))
                xr(t+1) = xr(t) + (X(t+1)-X(t)+x(t)) -
xlr(t);
                xlr(t+1) = 0;
            else
                if X(t+1) < (X(t)-x(t))
                    xlr(t+1) = xlr(t) + (X(t)-x(t)-
X(t+1));
                    xr(t+1) = 0;
                end
            end
        end
    end
end

output = [X,x,xr',xlr']
```

Appendix 2

Table. Blue Combat Manpower divided into surviving, reinforcements, local reserves (& reserves)

	Blue Combat Manpower							
Day (i)	Reformatted $_B(i)$	Available $_B(i)$	Used $_B(i)$	**Surviving** $_B(i)$	**REINF** $_B(i)$	**LReserves** $_B(i)$	**Reserves** $_B(i)$	**Killed** $_b(i)$
0	513514	351005	351005	351005	0	0	162509	458
1	513056	350547	349247	349247	0	1300	162509	1589
2	511467	348958	347915	347915	0	1043	162509	2383
3	509084	358321	358321	346575	11746	0	150763	2085
4	506999	366495	366495	356236	10259	0	140504	2175
5	504824	387342	387342	364320	23022	0	117482	1389
6	503435	403289	403289	385953	17336	0	100146	1174
7	502261	410817	410817	402115	8702	0	91444	1905
8	500356	412811	412811	408912	3899	0	87545	1548
9	498808	426360	426360	411263	15097	0	72448	1608
10	497200	432094	432094	424752	7342	0	65106	1527
11	495673	451316	451316	430567	20749	0	44357	2320
12	493353	451724	451724	448996	2728	0	41629	1376
13	491977	451291	451291	450348	943	0	40686	1277
14	490700	461189	461189	450014	11175	0	29511	1005
15	489695	465334	465334	460184	5150	0	24361	1042
16	488653	467620	467620	464292	3328	0	21033	1159
17	487494	467801	467801	466461	1340	0	19693	1004
18	486490	474562	474562	466797	7765	0	11928	832
19	485658	474192	474192	473730	462	0	11466	1831
20	483827	481704	481704	472361	9343	0	2123	2259
21	481568	480952	480952	479445	1507	0	616	1639
22	479929	479313	478593	478593	0	720	616	1228
23	478701	478085	475732	475732	0	2353	616	1868
24	476833	476217	475685	475685	0	532	616	1276
25	475557	475155	475155	474941	214	0	402	1379
26	474178	473776	472749	472749	0	1027	402	1643
27	472535	472535	472535	472133	402	0	0	1281
28	471254	471254	468127	468127	0	3127	0	1083
29	470171	470171	467646	467646	0	2525	0	1681
30	468490	468490	466072	466072	0	2418	0	1597
31	466893	466893	464643	464643	0	2250	0	2098
32	464795	464795	455218	455218	0	9577	0	1483

Appendix 3

Table. Red Combat Manpower divided into surviving, reinforcements, local reserves (& reserves)

	Red Combat Manpower							
Day (i)	Reformatted $_R$(i)	Available $_R$(i)	Used $_R$(i)	**Surviving** $_R$(i)	**REINF** $_R$(i)	**LReserves** $_R$(i)	**Reserves** $_R$(i)	**Killed** $_r$(i)
0	385955	0	0	0	0	0	385955	0
1	385955	360716	360716	360716	0	0	25239	2191
2	383764	358525	356818	356818	0	1707	25239	2423
3	381341	356102	353529	353529	0	2573	25239	2015
4	379326	354087	350750	350750	0	3337	25239	1993
5	377333	356278	356278	352094	4184	0	21055	1985
6	375348	354297	354297	354293	4	0	21051	2084
7	373264	361684	361684	352213	9471	0	11580	2046
8	371218	359638	359353	359353	0	285	11580	2468
9	368750	362904	362904	357170	5734	0	5846	2685
10	366065	360219	359750	359750	0	469	5846	2538
11	363527	362611	362611	357681	4930	0	916	2504
12	361023	361023	361023	360107	916	0	0	2544
13	358479	358479	356892	356892	0	1587	0	2121
14	356358	356358	349900	349900	0	6458	0	1682
15	354676	354676	346100	346100	0	8576	0	1844
16	352832	352832	343134	343134	0	9698	0	1550
17	351282	351282	340875	340875	0	10407	0	1788
18	349494	349494	338278	338278	0	11216	0	1724
19	347770	347770	334356	334356	0	13414	0	1752
20	346018	346018	328069	328069	0	17949	0	2054
21	343964	343964	321195	321195	0	22769	0	1709
22	342255	342255	322830	322830	0	19425	0	1946
23	340309	340309	324376	324376	0	15933	0	1865
24	338444	338444	322337	322337	0	16107	0	1676
25	336768	336768	320612	320612	0	16156	0	1434
26	335334	335334	319143	319143	0	16191	0	1696
27	333638	333638	319259	319259	0	14379	0	1536
28	332102	332102	317406	317406	0	14696	0	1167
29	330935	330935	316217	316217	0	14718	0	1579
30	329356	329356	314858	314858	0	14498	0	1504
31	327852	327852	313074	313074	0	14778	0	1425
32	326427	326427	310347	310347	0	16080	0	1213

Appendix 4

Table. Blue Total Manpower divided into surviving, reinforcements, local reserves (& reserves)

	Blue Total Manpower							
Day (i)	Reformatted B(i)	Available B(i)	Used B(i)	**Surviving** B(i)	**REINF** B(i)	**LReserves** B(i)	**Reserves** B(i)	**Killed** b(i)
0	1075857	632105	632105	632105	0	0	443752	1468
1	1074389	630637	630557	630557	0	80	443752	3062
2	1071327	628985	628985	627575	1410	0	442342	5712
3	1065615	640969	640969	623273	17696	0	424646	5093
4	1060522	807140	807140	635876	171264	0	253382	12101
5	1048421	834136	834136	795039	39097	0	214285	5334
6	1043087	859906	859906	828802	31104	0	183181	3197
7	1039890	874600	874600	856709	17891	0	165290	4815
8	1035075	877247	877247	869785	7462	0	157828	3730
9	1031345	895976	895976	873517	22459	0	135369	3857
10	1027488	907490	907490	892119	15371	0	119998	3635
11	1023853	933045	933045	903855	29190	0	90808	5411
12	1018442	948024	948024	927634	20390	0	70418	3596
13	1014846	944428	928230	928230	0	16198	70418	3435
14	1011411	941188	941188	940993	195	0	70223	2934
15	1008477	946424	946424	938254	8170	0	62053	2743
16	1005734	948226	948226	943681	4545	0	57508	3022
17	1002712	948379	948379	945204	3175	0	54333	2773
18	999939	956144	956144	945606	10538	0	43795	2631
19	997308	955821	955821	953513	2308	0	41487	3580
20	993728	965135	965135	952241	12894	0	28593	4899
21	988829	964928	964928	960236	4692	0	23901	4093
22	984736	962193	962193	960835	1358	0	22543	3388
23	981348	959776	959776	958805	971	0	21572	4627
24	976721	959011	959011	955149	3862	0	17710	3928
25	972793	958799	958799	955083	3716	0	13994	3725
26	969068	956330	956330	955074	1256	0	12738	4002
27	965066	956090	956090	952328	3762	0	8976	3502
28	961564	952588	952030	952030	0	558	8976	3590
29	957974	952210	952210	948998	3212	0	5764	4189
30	953785	950879	950879	948021	2858	0	2906	4277
31	949508	949508	949508	946602	2906	0	0	4477
32	945031	945031	937500	937500	0	7531	0	3600

Appendix 5

Table. Red Combat Manpower divided into surviving, reinforcements, local reserves (& reserves)

	Red Total Manpower							
Day (i)	Reformatted $_R$(i)	Available $_R$(i)	Used $_R$(i)	**Surviving** $_R$(i)	**REINF** $_R$(i)	**LReserves** $_R$(i)	**Reserves** $_R$(i)	**Killed** $_r$(i)
0	656278	0	0	0	0	0	656278	0
1	656278	575838	575838	575838	0	0	80440	5590
2	650688	571301	571301	570248	1053	0	79387	5559
3	645129	568508	568508	565742	2766	0	76621	4711
4	640418	565173	565173	563797	1376	0	75245	4332
5	636086	572181	572181	560841	11340	0	63905	4351
6	631735	570711	570711	567830	2881	0	61024	4582
7	627153	581177	581177	566129	15048	0	45976	4531
8	622622	579660	579660	576646	3014	0	42962	5351
9	617271	584610	584610	574309	10301	0	32661	5609
10	611662	580731	580731	579001	1730	0	30931	5563
11	606099	584551	584551	575168	9383	0	21548	5526
12	600573	583610	583610	579025	4585	0	16963	5751
13	594822	578737	578737	577859	878	0	16085	4511
14	590311	574226	568768	568768	0	5458	16085	3900
15	586411	570326	564548	564548	0	5778	16085	4076
16	582335	566250	560993	560993	0	5257	16085	3635
17	578700	562615	558214	558214	0	4401	16085	3898
18	574802	558717	555741	555741	0	2976	16085	3821
19	570981	554896	550854	550854	0	4042	16085	3892
20	567089	551004	544031	544031	0	6973	16085	4283
21	562806	546721	534885	534885	0	11836	16085	3767
22	559039	542954	536481	536481	0	6473	16085	4169
23	554870	540896	540896	538785	2111	0	13974	4076
24	550794	538328	538328	536820	1508	0	12466	3756
25	547038	536719	536719	534572	2147	0	10319	3466
26	543572	534764	534764	533253	1511	0	8808	3732
27	539840	533256	533256	531032	2224	0	6584	3967
28	535873	530919	530919	529289	1630	0	4954	3199
29	532674	528237	528237	527720	517	0	4437	4026
30	528648	526387	526387	524211	2176	0	2261	3866
31	524782	524150	524150	522521	1629	0	632	3744
32	521038	521038	521038	520406	632	0	0	3219

Appendix 6

Table. Blue Tanks divided into surviving, reinforcements, local reserves (& reserves)

Day	Blue Tanks							
	Reformatted	Available	Used	**Surviving**	**REINF**	**LReserves**	**Reserves**	**Killed**
(i)	$_B(i)$	$_B(i)$	$_B(i)$	$_B(i)$	$_B(i)$	$_B(i)$	$_B(i)$	$_b(i)$
0	5350	2853	2853	2853	0	0	2497	1
1	5349	2863	2863	2852	11	0	2486	12
2	5337	2867	2867	2851	16	0	2470	43
3	5294	2840	2840	2824	16	0	2454	60
4	5234	2808	2808	2780	28	0	2426	64
5	5170	3965	3965	2744	1221	0	1205	33
6	5137	4082	4082	3932	150	0	1055	10
7	5127	4109	4109	4072	37	0	1018	15
8	5112	4094	4086	4086	0	8	1018	36
9	5076	4062	4062	4058	4	0	1014	48
10	5028	4265	4265	4014	251	0	763	24
11	5004	4520	4520	4241	279	0	484	20
12	4984	4511	4511	4500	11	0	473	19
13	4965	4526	4526	4492	34	0	439	18
14	4947	4541	4541	4508	33	0	406	16
15	4931	4525	4516	4516	0	9	406	20
16	4911	4610	4610	4505	105	0	301	10
17	4901	4695	4695	4600	95	0	206	14
18	4887	4701	4701	4681	20	0	186	24
19	4863	4710	4710	4677	33	0	153	26
20	4837	4728	4728	4684	44	0	109	22
21	4815	4706	4686	4686	0	20	109	13
22	4802	4719	4719	4693	26	0	83	13
23	4789	4706	4684	4684	0	22	83	12
24	4777	4703	4703	4694	9	0	74	9
25	4768	4743	4743	4694	49	0	25	7
26	4761	4761	4761	4736	25	0	0	5
27	4756	4756	4745	4745	0	11	0	7
28	4749	4749	4717	4717	0	32	0	2
29	4747	4747	4699	4699	0	48	0	6
30	4741	4741	4678	4678	0	63	0	16
31	4725	4725	4662	4662	0	63	0	11
32	4714	4714	4628	4628	0	86	0	20

Appendix 7

Table. Red Tanks divided into surviving, reinforcements, local reserves (& reserves)

Day	Red Tanks							
	Reformatted	Available	Used	**Surviving**	**REINF**	**LReserves**	**Reserves**	**Killed**
(i)	$_R$(i)	$_R$(i)	$_R$(i)	$_R$(i)	$_R$(i)	$_R$(i)	$_R$(i)	$_r$(i)
0	747	0	0	0	0	0	747	0
1	747	747	747	0	747	0	0	10
2	737	737	663	663	0	74	0	7
3	730	730	639	639	0	91	0	13
4	717	717	669	669	0	48	0	21
5	696	696	619	619	0	77	0	11
6	685	685	595	595	0	90	0	21
7	664	664	615	615	0	49	0	5
8	659	659	645	645	0	14	0	24
9	635	635	596	596	0	39	0	22
10	613	613	544	544	0	69	0	28
11	585	585	483	483	0	102	0	14
12	571	571	466	466	0	105	0	13
13	558	558	450	450	0	108	0	7
14	551	551	433	433	0	118	0	7
15	544	544	428	428	0	116	0	21
16	523	523	403	403	0	120	0	5
17	518	518	413	413	0	105	0	9
18	509	509	419	419	0	90	0	6
19	503	503	431	431	0	72	0	2
20	501	501	428	428	0	73	0	12
21	489	489	394	394	0	95	0	2
22	487	487	396	396	0	91	0	2
23	485	485	400	400	0	85	0	8
24	477	477	407	407	0	70	0	0
25	477	477	398	398	0	79	0	7
26	470	470	407	407	0	63	0	2
27	468	468	407	407	0	61	0	2
28	466	466	393	393	0	73	0	0
29	466	466	418	418	0	48	0	13
30	453	453	410	410	0	43	0	5
31	448	448	434	434	0	14	0	3
32	445	445	432	432	0	13	0	2

Appendix 8

Table. Blue APCs divided into surviving, reinforcements, local reserves (& reserves)

	Blue APCs							
Day	Reformatted	Available	Used	**Surviving**	**REINF**	**LReserves**	**Reserves**	**Killed**
(i)	B(i)	B(i)	B(i)	B(i)	B(i)	B(i)	B(i)	b(i)
0	8821	6103	6103	6103	0	0	2718	0
1	8821	6103	6019	6019	0	84	2718	33
2	8788	6070	5970	5970	0	100	2718	46
3	8742	6024	5908	5908	0	116	2718	18
4	8724	6006	6004	6004	0	2	2718	37
5	8687	7274	7274	5969	1305	0	1413	11
6	8676	7295	7295	7263	32	0	1381	6
7	8670	7507	7507	7289	218	0	1163	13
8	8657	7533	7533	7494	39	0	1124	6
9	8651	7527	7486	7486	0	41	1124	18
10	8633	8105	8105	7509	596	0	528	2
11	8631	8552	8552	8103	449	0	79	2
12	8629	8629	8629	8550	79	0	0	3
13	8626	8626	8536	8536	0	90	0	3
14	8623	8623	8552	8552	0	71	0	2
15	8621	8621	8565	8565	0	56	0	0
16	8621	8621	8554	8554	0	67	0	2
17	8619	8619	8615	8615	0	4	0	0
18	8619	8619	8593	8593	0	26	0	0
19	8619	8619	8462	8462	0	157	0	0
20	8619	8619	8578	8578	0	41	0	0
21	8619	8619	8564	8564	0	55	0	2
22	8617	8617	8502	8502	0	115	0	0
23	8617	8617	8375	8375	0	242	0	0
24	8617	8617	8418	8418	0	199	0	0
25	8617	8617	8446	8446	0	171	0	0
26	8617	8617	8476	8476	0	141	0	0
27	8617	8617	8348	8348	0	269	0	1
28	8616	8616	8459	8459	0	157	0	1
29	8615	8615	8454	8454	0	161	0	0
30	8615	8615	8374	8374	0	241	0	0
31	8615	8615	8436	8436	0	179	0	0
32	8615	8615	8363	8363	0	252	0	0

Appendix 9

Table. Red APCs divided into surviving, reinforcements, local reserves (& reserves)

Day	Red APCs							
	Reformatted	Available	Used	**Surviving**	**REINF**	**LReserves**	**Reserves**	**Killed**
(i)	R(i)	R(i)	R(i)	R(i)	R(i)	R(i)	R(i)	r(i)
0	2161	0	0	0	0	0	2161	0
1	2161	2046	2046	0	2046	0	115	5
2	2156	2041	2041	2041	0	0	115	20
3	2136	2021	2021	2021	0	0	115	12
4	2124	2009	2009	2009	0	0	115	18
5	2106	1991	1984	1984	0	7	115	22
6	2084	1969	1952	1952	0	17	115	19
7	2065	2065	2065	1950	115	0	0	16
8	2049	2049	2034	2034	0	15	0	34
9	2015	2015	1970	1970	0	45	0	20
10	1995	1995	1875	1875	0	120	0	36
11	1959	1959	1800	1800	0	159	0	31
12	1928	1928	1731	1731	0	197	0	31
13	1897	1897	1659	1659	0	238	0	22
14	1875	1875	1595	1595	0	280	0	15
15	1860	1860	1542	1542	0	318	0	7
16	1853	1853	1532	1532	0	321	0	14
17	1839	1839	1523	1523	0	316	0	8
18	1831	1831	1516	1516	0	315	0	10
19	1821	1821	1451	1451	0	370	0	10
20	1811	1811	1441	1441	0	370	0	12
21	1799	1799	1419	1419	0	380	0	6
22	1793	1793	1409	1409	0	384	0	5
23	1788	1788	1403	1403	0	385	0	16
24	1772	1772	1364	1364	0	408	0	2
25	1770	1770	1360	1360	0	410	0	3
26	1767	1767	1358	1358	0	409	0	9
27	1758	1758	1349	1349	0	409	0	8
28	1750	1750	1341	1341	0	409	0	3
29	1747	1747	1335	1335	0	412	0	17
30	1730	1730	1322	1322	0	408	0	6
31	1724	1724	1318	1318	0	406	0	4
32	1720	1720	1309	1309	0	411	0	3

Appendix 10

Table. Blue Artillery divided into surviving, reinforcements, local reserves (& reserves)

	Blue Artillery							
Day (i)	Reformatted B(i)	Available B(i)	Used B(i)	**Surviving** B(i)	**REINF** B(i)	**LReserves** B(i)	**Reserves** B(i)	**Killed** b(i)
0	4275	3006	3006	3006	0	0	1269	0
1	4275	3006	2972	2972	0	34	1269	15
2	4260	2991	2963	2963	0	28	1269	9
3	4251	2982	2950	2950	0	32	1269	6
4	4245	3103	3103	2976	127	0	1142	14
5	4231	3531	3531	3089	442	0	700	33
6	4198	3609	3609	3498	111	0	589	10
7	4188	3772	3772	3599	173	0	416	2
8	4186	3772	3772	3770	2	0	414	10
9	4176	3847	3847	3762	85	0	329	5
10	4171	3931	3931	3842	89	0	240	4
11	4167	4063	4063	3927	136	0	104	4
12	4163	4093	4093	4059	34	0	70	3
13	4160	4090	4004	4004	0	86	70	2
14	4158	4088	4065	4065	0	23	70	0
15	4158	4088	4087	4087	0	1	70	0
16	4158	4088	4086	4086	0	2	70	1
17	4157	4087	4077	4077	0	10	70	0
18	4157	4087	4087	4087	0	0	70	0
19	4157	4088	4088	4087	1	0	69	1
20	4156	4150	4150	4087	63	0	6	3
21	4153	4153	4153	4147	6	0	0	0
22	4153	4153	4144	4144	0	9	0	0
23	4153	4153	4133	4133	0	20	0	1
24	4152	4152	4131	4131	0	21	0	0
25	4152	4152	4128	4128	0	24	0	0
26	4152	4152	4131	4131	0	21	0	0
27	4152	4152	4090	4090	0	62	0	0
28	4152	4152	4108	4108	0	44	0	0
29	4152	4152	4106	4106	0	46	0	0
30	4152	4152	4081	4081	0	71	0	2
31	4150	4150	4092	4092	0	58	0	0
32	4150	4150	4080	4080	0	70	0	9

Appendix 11

Table. Red Artillery divided into surviving, reinforcements, local reserves (& reserves)

	Red Artillery							
Day (i)	Reformatted R(i)	Available R(i)	Used R(i)	**Surviving** R(i)	**REINF** R(i)	**LReserves** R(i)	**Reserves** R(i)	**Killed** r(i)
0	5130	36	36	36	0	0	5094	0
1	5130	4789	4789	36	4753	0	341	6
2	5124	4791	4791	4783	8	0	333	41
3	5083	4768	4768	4750	18	0	315	27
4	5056	4741	4727	4727	0	14	315	19
5	5037	4786	4786	4722	64	0	251	24
6	5013	4773	4773	4762	11	0	240	30
7	4983	4858	4858	4743	115	0	125	16
8	4967	4845	4845	4842	3	0	122	35
9	4932	4885	4885	4810	75	0	47	94
10	4838	4791	4750	4750	0	41	47	59
11	4779	4779	4779	4732	47	0	0	34
12	4745	4745	4661	4661	0	84	0	23
13	4722	4722	4638	4638	0	84	0	31
14	4691	4691	4415	4415	0	276	0	19
15	4672	4672	4321	4321	0	351	0	26
16	4646	4646	4314	4314	0	332	0	17
17	4629	4629	4283	4283	0	346	0	36
18	4593	4593	4246	4246	0	347	0	14
19	4579	4579	4242	4242	0	337	0	35
20	4544	4544	4110	4110	0	434	0	28
21	4516	4516	4016	4016	0	500	0	22
22	4494	4494	4014	4014	0	480	0	23
23	4471	4471	3981	3981	0	490	0	26
24	4445	4445	3971	3971	0	474	0	23
25	4422	4422	3944	3944	0	478	0	21
26	4401	4401	3925	3925	0	476	0	33
27	4368	4368	3916	3916	0	452	0	32
28	4336	4336	3895	3895	0	441	0	13
29	4323	4323	3854	3854	0	469	0	20
30	4303	4303	3867	3867	0	436	0	19
31	4284	4284	3856	3856	0	428	0	14
32	4270	4270	3824	3824	0	446	0	7

Appendix 12

Table. Blue Support Manpower divided into surviving, reinforcements, local reserves (& reserves)

	Blue Support Manpower							
Day (i)	Reformatted B(i)	Available B(i)	Used B(i)	Surviving B(i)	REINF B(i)	LReserves B(i)	Reserves B(i)	Killed b(i)
0	562343	281100	281100	281100	0	0	281243	1010
1	561333	280090	281310	281310	0	-1220	281243	1473
2	559860	278617	279660	278250	1410	-1043	281243	3329
3	556531	276698	276698	270748	5950	0	279833	3008
4	553523	279640	279640	118635	161005	0	273883	9926
5	543597	430719	430719	414644	16075	0	112878	3945
6	539652	442849	442849	429081	13768	0	96803	2023
7	537629	454594	454594	445405	9189	0	83035	2910
8	534719	460873	460873	457310	3563	0	73846	2182
9	532537	462254	462254	454892	7362	0	70283	2249
10	530288	467367	467367	459338	8029	0	62921	2108
11	528180	473288	473288	464847	8441	0	54892	3091
12	525089	478638	478638	460976	17662	0	46451	2220
13	522869	494080	477882	478825	-943	16198	28789	2158
14	520711	490979	490979	501959	-10980	0	29732	1929
15	518782	478070	478070	475050	3020	0	40712	1701
16	517081	479389	479389	478172	1217	0	37692	1863
17	515218	478743	478743	476908	1835	0	36475	1769
18	513449	478809	478809	476036	2773	0	34640	1799
19	511650	479783	479783	477937	1846	0	31867	1749
20	509901	479880	479880	476329	3551	0	30021	2640
21	507261	480791	480791	477606	3185	0	26470	2454
22	504807	481522	482242	480884	1358	-720	23285	2160
23	502647	480720	483073	482102	971	-2353	21927	2759
24	499888	478932	479464	475602	3862	-532	20956	2652
25	497236	480142	480142	476640	3502	0	17094	2346
26	494890	481298	482325	481069	1256	-1027	13592	2359
27	492531	480195	480195	476835	3360	0	12336	2221
28	490310	481334	483903	483903	0	-2569	8976	2507
29	487803	478827	481352	478140	3212	-2525	8976	2508
30	485295	479531	481949	479091	2858	-2418	5764	2680
31	482615	479709	481959	479053	2906	-2250	2906	2379
32	480236	480236	482282	482282	0	-2046	0	2117

Appendix 13

Table . Red Support Manpower divided into surviving, reinforcements, local reserves (& reserves)

	Red Support Manpower							
Day (i)	Reformatted R(i)	Available R(i)	Used R(i)	Surviving R(i)	REINF R(i)	LReserves R(i)	Reserves R(i)	Killed r(i)
0	270323	0	0	0	0	0	270323	0
1	270323	215122	215122	215122	0	0	55201	3399
2	266924	212776	214483	213430	1053	-1707	54148	3136
3	263788	212406	214979	212213	2766	-2573	51382	2696
4	261092	211086	214423	213047	1376	-3337	50006	2339
5	258753	220087	220087	212931	7156	0	38666	2366
6	256387	216418	216418	213541	2877	0	39969	2498
7	253889	228964	228964	223387	5577	0	24925	2485
8	251404	220022	220307	217293	3014	-285	31382	2883
9	248521	227440	227440	222873	4567	0	21081	2924
10	245597	220512	220981	219251	1730	-469	25085	3025
11	242572	226870	226870	222417	4453	0	15702	3022
12	239550	223503	223503	219834	3669	0	16047	3207
13	236343	220258	221845	220967	878	-1587	16085	2390
14	233953	217868	218868	218868	0	-1000	16085	2218
15	231735	215650	218448	218448	0	-2798	16085	2232
16	229503	213418	217859	217859	0	-4441	16085	2085
17	227418	211333	217339	217339	0	-6006	16085	2110
18	225308	209223	217463	217463	0	-8240	16085	2097
19	223211	207126	216498	216498	0	-9372	16085	2140
20	221071	204986	215962	215962	0	-10976	16085	2229
21	218842	202757	213690	213690	0	-10933	16085	2058
22	216784	200699	213651	213651	0	-12952	16085	2223
23	214561	200587	216520	214409	2111	-15933	13974	2211
24	212350	199884	215991	214483	1508	-16107	12466	2080
25	210270	199951	216107	213960	2147	-16156	10319	2032
26	208238	199430	215621	214110	1511	-16191	8808	2036
27	206202	199618	213997	211773	2224	-14379	6584	2431
28	203771	198817	213513	211883	1630	-14696	4954	2032
29	201739	197302	212020	211503	517	-14718	4437	2447
30	199292	197031	211529	209353	2176	-14498	2261	2362
31	196930	196298	211076	209447	1629	-14778	632	2319
32	194611	194611	210691	210059	632	-16080	0	2006

Appendix 14

```
clear all;
close all;
clc;

B = xlsread('Table6Combat&TotalForces.xlsx', 'H6:H15');
b = xlsread('Table6Combat&TotalForces.xlsx',
'H90:H99');
R = xlsread('Table6Combat&TotalForces.xlsx', 'Q6:Q15');
r = xlsread('Table6Combat&TotalForces.xlsx',
'Q90:Q99');

for t = 1: numel(B);

    x(t) = (R(t)/B(t));
    y(t) = (b(t)/r(t));
    logx(t) = log(x(t));
    logy(t) = log(y(t));
end

scatter(logx,logy);
polycoefficients = polyfit(logx,logy,1);
linearbestfit =
polycoefficients(1)*logx+polycoefficients(2);
hold on;
plot(logx,linearbestfit);

beta = polycoefficients(1);
alpha = exp(polycoefficients(2));
residuals = (logy - linearbestfit)';
variance = 1-(sum(residuals.^2)/sum((logy-
mean(logy)).^2));

output = [alpha, beta, variance]
```

Appendix 15

```
clear all;
close all;
clc;

ManpowerBsurviving =
xlsread('Table7ManpowerWeightings.xlsx', 'D6:D15');
ManpowerBREINF =
xlsread('Table7ManpowerWeightings.xlsx', 'E6:E15');
ManpowerBLreserves =
xlsread('Table7ManpowerWeightings.xlsx', 'F6:F15');
ManpowerBreserves =
xlsread('Table7ManpowerWeightings.xlsx', 'G6:G15');

TankBsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'D6:D15');
TankBREINF = xlsread('Table9EquipmentWeightings.xlsx',
'E6:E15');
TankBLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'F6:F15');
TankBreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'G6:G15');

APCBsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'D48:D57');
APCBREINF = xlsread('Table9EquipmentWeightings.xlsx',
'E48:E57');
APCBLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'F48:F57');
APCBreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'G48:G57');

ArtBsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'D90:D99');
ArtBREINF = xlsread('Table9EquipmentWeightings.xlsx',
'E90:E99');
ArtBLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'F90:F99');
ArtBreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'G90:G99');

SortiesB = xlsread('Table6Combat&TotalForces.xlsx',
'G6:G15');
```

```
Manpowerb = xlsread('Table6Combat&TotalForces.xlsx',
'B90:B99');
Tankb = xlsread('Table6Combat&TotalForces.xlsx',
'D90:D99');
APCb = xlsread('Table6Combat&TotalForces.xlsx',
'E90:E99');
Artb = xlsread('Table6Combat&TotalForces.xlsx',
'F90:F99');


ManpowerRsurviving =
xlsread('Table7ManpowerWeightings.xlsx', 'M6:M15');
ManpowerRREINF =
xlsread('Table7ManpowerWeightings.xlsx', 'N6:N15');
ManpowerRLreserves =
xlsread('Table7ManpowerWeightings.xlsx', 'O6:O15');
ManpowerRreserves =
xlsread('Table7ManpowerWeightings.xlsx', 'P6:P15');

TankRsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'M6:M15');
TankRREINF = xlsread('Table9EquipmentWeightings.xlsx',
'N6:N15');
TankRLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'O6:O15');
TankRreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'P6:P15');

APCRsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'M48:M57');
APCRREINF = xlsread('Table9EquipmentWeightings.xlsx',
'N48:N57');
APCRLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'O48:O57');
APCRreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'P48:P57');

ArtRsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'M90:M99');
ArtRREINF = xlsread('Table9EquipmentWeightings.xlsx',
'N90:N99');
ArtRLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'O90:O99');
ArtRreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'P90:P99');
```

```
SortiesR = xlsread('Table6Combat&TotalForces.xlsx',
'P6:P15');

Manpowerr = xlsread('Table6Combat&TotalForces.xlsx',
'K90:K99');
Tankr = xlsread('Table6Combat&TotalForces.xlsx',
'M90:M99');
APCr = xlsread('Table6Combat&TotalForces.xlsx',
'N90:N99');
Artr = xlsread('Table6Combat&TotalForces.xlsx',
'O90:O99');


Manpowersurvivingweighting = 1;
ManpowerREINFweighting = [1:0.4:3];
ManpowerLreservesweighting = [0:0.3:0.9];
Manpowerreservesweighting = 0;
ManpowerWeighting = 1;

Tanksurvivingweighting = 1;
TankREINFweighting = [1:0.4:3];
TankLreservesweighting = [0:0.3:0.9];;
Tankreservesweighting = 0;
TankWeighting = 20;

APCsurvivingweighting = 1;
APCREINFweighting = [1:0.4:3];
APCLreservesweighting = [0:0.3:0.9];
APCreservesweighting = 0;
APCWeighting = 5;

Artsurvivingweighting = 1;
ArtREINFweighting = [1:0.4:3];
ArtLreservesweighting = [0:0.3:0.9];
Artreservesweighting = 0;
ArtWeighting = 40;

SortiesWeighting = 30;

for i = 1: numel(ManpowerBsurviving)
    for mprw = 1: numel(ManpowerREINFweighting)
        for mplrw = 1:
numel(ManpowerLreservesweighting)
            for mpresw =
1:numel(Manpowerreservesweighting)
                for trw = 1:numel(TankREINFweighting)
                    for tlrw =
1:numel(TankLreservesweighting)
```

```
                                                        for tresw =
1:numel(Tankreservesweighting)
                                                                for tw =
1:numel(TankWeighting)
                                                                        for apcrw =
1:numel(APCREINFweighting)
                                                                                for apclrw =
1:numel(APCLreservesweighting)
                                                                                        for apcresw =
1:numel(APCreservesweighting)
                                                                                                for apcw =
1:numel(APCWeighting)
                                                                                                        for
artrw = 1:numel(ArtREINFweighting)
                                                                                                                for
artlrw = 1:numel(ArtLreservesweighting)

for artresw = 1:numel(Artreservesweighting)

for artw = 1:numel(ArtWeighting)

for asw = 1:numel(SortiesWeighting)

Bmatrix(mprw,mplrw,mpresw,trw,tlrw,tresw,tw,apcrw,apclr
w,apcresw,apcw,artrw,artlrw,artresw,artw,asw,i) =
(((ManpowerBsurviving(i)*Manpowersurvivingweighting)+(M
anpowerBREINF(i)*ManpowerREINFweighting(mprw))+(Manpowe
rBLreserves(i)*ManpowerLreservesweighting(mplrw))+(Manp
owerBreserves(i)*Manpowerreservesweighting(mpresw)))*Ma
npowerWeighting)+(((TankBsurviving(i)*Tanksurvivingweig
hting)+(TankBREINF(i)*TankREINFweighting(trw))+(TankBLr
eserves(i)*TankLreservesweighting(tlrw))+(TankBreserves
(i)*Tankreservesweighting(tresw)))*TankWeighting(tw))+(
((APCBsurviving(i)*APCsurvivingweighting)+(APCBREINF(i)
*APCREINFweighting(apcrw))+(APCBLreserves(i)*APCLreserv
esweighting(apclrw))+(APCBreserves(i)*APCreservesweight
ing(apcresw)))*APCWeighting(apcw))+(((ArtBsurviving(i)*
Artsurvivingweighting)+(ArtBREINF(i)*ArtREINFweighting(
artrw))+(ArtBLreserves(i)*ArtLreservesweighting(artlrw)
)+(ArtBreserves(i)*Artreservesweighting(artresw)))*ArtW
eighting(artw))+(SortiesB(i)*SortiesWeighting(asw));

Rmatrix(mprw,mplrw,mpresw,trw,tlrw,tresw,tw,apcrw,apclr
w,apcresw,apcw,artrw,artlrw,artresw,artw,asw,i) =
(((ManpowerRsurviving(i)*Manpowersurvivingweighting)+(M
anpowerRREINF(i)*ManpowerREINFweighting(mprw))+(Manpowe
rRLreserves(i)*ManpowerLreservesweighting(mplrw))+(Manp
```

```
owerRreserves(i)*Manpowerreservesweighting(mpresw)))*Ma
npowerWeighting)+(((TankRsurviving(i)*Tanksurvivingweig
hting)+(TankRREINF(i)*TankREINFweighting(trw))+(TankRLr
eserves(i)*TankLreservesweighting(tlrw))+(TankRreserves
(i)*Tankreservesweighting(tresw)))*TankWeighting(tw))+(
((APCRsurviving(i)*APCsurvivingweighting)+(APCRREINF(i)
*APCREINFweighting(apcrw))+(APCRLreserves(i)*APCLreserv
esweighting(apclrw))+(APCRreserves(i)*APCreservesweight
ing(apcresw)))*APCWeighting(apcw))+(((ArtRsurviving(i)*
Artsurvivingweighting)+(ArtRREINF(i)*ArtREINFweighting(
artrw))+(ArtRLreserves(i)*ArtLreservesweighting(artlrw)
)+(ArtRreserves(i)*Artreservesweighting(artresw)))*ArtW
eighting(artw))+(SortiesR(i)*SortiesWeighting(asw));

bmatrix(mprw,mplrw,mpresw,trw,tlrw,tresw,tw,apcrw,apclr
w,apcresw,apcw,artrw,artlrw,artresw,artw,asw,i) =
((Manpowerb(i)*Manpowersurvivingweighting)*ManpowerWeig
hting)+((Tankb(i)*Tanksurvivingweighting)*TankWeighting
(tw))+((APCb(i)*APCsurvivingweighting)*APCWeighting(apc
w))+((Artb(i)*Artsurvivingweighting)*ArtWeighting(artw)
);

rmatrix(mprw,mplrw,mpresw,trw,tlrw,tresw,tw,apcrw,apclr
w,apcresw,apcw,artrw,artlrw,artresw,artw,asw,i) =
((Manpowerr(i)*Manpowersurvivingweighting)*ManpowerWeig
hting)+((Tankr(i)*Tanksurvivingweighting)*TankWeighting
(tw))+((APCr(i)*APCsurvivingweighting)*APCWeighting(apc
w))+((Artr(i)*Artsurvivingweighting)*ArtWeighting(artw)
);

end

end

end
                                                    end
                                                end
                                            end
                                        end
                                    end
                                end
                            end
                        end
                    end
                end
            end
        end
```

```
    end
end

for MPRW = 1:mprw
    for MPLRW = 1:mplrw
        for MPRESW = 1:mpresw
            for TRW = 1:trw
                for TLRW = 1:tlrw
                    for TRESW = 1:tresw
                        for TW = 1:tw
                            for APCRW = 1:apcrw
                                for APCLRW = 1:apclrw
                                    for APCRESW =
1:apcresw
                                        for APCW =
1:apcw
                                            for ARTRW =
1:artrw
                                                for
ARTLRW = 1:artlrw
                                                    for
ARTRESW = 1:artresw

for ARTW = 1:artw

for ASW = 1:asw

for I = 1:i

LinearIndexing(I,ASW,ARTW,ARTRESW,ARTLRW,ARTRW,APCW,APC
RESW,APCLRW,APCRW,TW,TRESW,TLRW,TRW,MPRESW,MPLRW,MPRW)
=
sub2ind(size(Bmatrix),MPRW,MPLRW,MPRESW,TRW,TLRW,TRESW,
TW,APCRW,APCLRW,APCRESW,APCW,ARTRW,ARTLRW,ARTRESW,ARTW,
ASW,I);

end

end

end
                                                    end
                                                end
                                            end
                                        end
                                    end
                                end
```

```
                                    end
                                end
                            end
                        end
                    end
                end
            end
        end

Bmatrixsorted = Bmatrix(LinearIndexing);
Rmatrixsorted = Rmatrix(LinearIndexing);
bmatrixsorted = bmatrix(LinearIndexing);
rmatrixsorted = rmatrix(LinearIndexing);

Bmatrix2D =
reshape(Bmatrixsorted,i,mprw*mplrw*mpresw*trw*tlrw*tres
w*tw*apcrw*apclrw*apcresw*apcw*artrw*artlrw*artresw*art
w*asw);
Rmatrix2D =
reshape(Rmatrixsorted,i,mprw*mplrw*mpresw*trw*tlrw*tres
w*tw*apcrw*apclrw*apcresw*apcw*artrw*artlrw*artresw*art
w*asw);
bmatrix2D =
reshape(bmatrixsorted,i,mprw*mplrw*mpresw*trw*tlrw*tres
w*tw*apcrw*apclrw*apcresw*apcw*artrw*artlrw*artresw*art
w*asw);
rmatrix2D =
reshape(rmatrixsorted,i,mprw*mplrw*mpresw*trw*tlrw*tres
w*tw*apcrw*apclrw*apcresw*apcw*artrw*artlrw*artresw*art
w*asw);

v = 0;

for m =
1:(mprw*mplrw*mpresw*trw*tlrw*tresw*tw*apcrw*apclrw*apc
resw*apcw*artrw*artlrw*artresw*artw*asw)
    B = Bmatrix2D(:,m);
    R = Rmatrix2D(:,m);
    b = bmatrix2D(:,m);
    r = rmatrix2D(:,m);

        for t = 1:i
            x(t) = (R(t)/B(t));
            y(t) = (b(t)/r(t));
            logx(t) = log(x(t));
            logy(t) = log(y(t));
        end
```

```
        polycoefficients = polyfit(logx,logy,1);
        linearbestfit =
polycoefficients(1)*logx+polycoefficients(2);
        residuals = (logy - linearbestfit)';
        variance(m) = 1-(sum(residuals.^2)/sum((logy-
mean(logy)).^2));

        if variance(m) > v
        v = variance(m);
        M = m;
        end
end

myB = Bmatrix2D(:,M);
myR = Rmatrix2D(:,M);
myb = bmatrix2D(:,M);
myr = rmatrix2D(:,M);

for t = 1:i
    x(t) = (myR(t)/myB(t));
    y(t) = (myb(t)/myr(t));
    logx(t) = log(x(t));
    logy(t) = log(y(t));

    X(t) = (myB(t)*myR(t));
    Y(t) = (myb(t)*myr(t));
    logX(t) = log(X(t));
    logY(t) = log(Y(t));
end

scatter(logx,logy);
polycoefficients = polyfit(logx,logy,1);
linearbestfit =
polycoefficients(1)*logx+polycoefficients(2);
residuals = (logy - linearbestfit)';
hold on;
plot(logx,linearbestfit);

beta = polycoefficients(1);
alpha = exp(polycoefficients(2));
myVariance = v;

polycoefficients2 = polyfit(logX,logY,1);
delta = polycoefficients2(1);
gamma = exp(polycoefficients2(2));

p = (delta+beta)/2;
```

```
q = (delta-beta)/2;
a = sqrt(alpha*gamma);
bee = sqrt(gamma/alpha);


ManpowerREINFweighting =
ManpowerREINFweighting(ceil(M/(asw*artw*artresw*artlrw*
artrw*apcw*apcresw*apclrw*apcrw*tw*tresw*tlrw*trw*mpres
w*mplrw)));

if mplrw == 0
    ManpowerLreservesweighting = 0;
else
    if
mod((ceil((M/(asw*artw*artresw*artlrw*artrw*apcw*apcres
w*apclrw*apcrw*tw*tresw*tlrw*trw*mpresw)))),mplrw) == 0
        ManpowerLreservesweighting =
ManpowerLreservesweighting(mplrw);
    else
        ManpowerLreservesweighting =
ManpowerLreservesweighting(mod(ceil(M/(asw*artw*artresw
*artlrw*artrw*apcw*apcresw*apclrw*apcrw*tw*tresw*tlrw*t
rw*mpresw)),mplrw));
    end
end

if mpresw == 0
    Manpowerreservesweighting = 0;
else
    if
mod((ceil((M/(asw*artw*artresw*artlrw*artrw*apcw*apcres
w*apclrw*apcrw*tw*tresw*tlrw*trw)))),mpresw) == 0
        Manpowerreservesweighting =
Manpowerreservesweighting(mpresw);
    else
        Manpowerreservesweighting =
Manpowerreservesweighting(mod(ceil(M/(asw*artw*artresw*
artlrw*artrw*apcw*apcresw*apclrw*apcrw*tw*tresw*tlrw*tr
w)),mpresw));
    end
end

if trw == 0
    TankREINFweighting = 0;
else
```

```
    if
mod((ceil((M/(asw*artw*artresw*artlrw*artrw*apcw*apcres
w*apclrw*apcrw*tw*tresw*tlrw)))),trw) == 0
        TankREINFweighting = TankREINFweighting(trw);
    else
        TankREINFweighting =
TankREINFweighting(mod(ceil(M/(asw*artw*artresw*artlrw*
artrw*apcw*apcresw*apclrw*apcrw*tw*tresw*tlrw)),trw));
    end
end

if tlrw == 0
    TankLreservesweighting = 0;
else
    if
mod((ceil((M/(asw*artw*artresw*artlrw*artrw*apcw*apcres
w*apclrw*apcrw*tw*tresw)))),tlrw) == 0
        TankLreservesweighting =
TankLreservesweighting(tlrw);
    else
        TankLreservesweighting =
TankLreservesweighting(mod(ceil(M/(asw*artw*artresw*art
lrw*artrw*apcw*apcresw*apclrw*apcrw*tw*tresw)),tlrw));
    end
end

if tresw == 0
    Tankreservesweighting = 0;
else
    if
mod((ceil((M/(asw*artw*artresw*artlrw*artrw*apcw*apcres
w*apclrw*apcrw*tw)))),tresw) == 0
        Tankreservesweighting =
Tankreservesweighting(tresw);
    else
        Tankreservesweighting =
Tankreservesweighting(mod(ceil(M/(asw*artw*artresw*artl
rw*artrw*apcw*apcresw*apclrw*apcrw*tw)),tresw));
    end
end

if tw == 0
    TankWeighting = 0;
else
    if
mod((ceil((M/(asw*artw*artresw*artlrw*artrw*apcw*apcres
w*apclrw*apcrw)))),tw) == 0
```

```
        TankWeighting = TankWeighting(tw);
    else
        TankWeighting =
TankWeighting(mod(ceil(M/(asw*artw*artresw*artlrw*artrw
*apcw*apcresw*apclrw*apcrw)),tw));
    end
end

if apcrw == 0
    APCREINFweighting = 0;
else
    if
mod((ceil((M/(asw*artw*artresw*artlrw*artrw*apcw*apcres
w*apclrw)))),apcrw) == 0
        APCREINFweighting = APCREINFweighting(apcrw);
    else
        APCREINFweighting =
APCREINFweighting(mod(ceil(M/(asw*artw*artresw*artlrw*a
rtrw*apcw*apcresw*apclrw)),apcrw));
    end
end

if apclrw == 0
    APCLreservesweighting = 0;
else
    if
mod((ceil((M/(asw*artw*artresw*artlrw*artrw*apcw*apcres
w)))),apclrw) == 0
        APCLreservesweighting =
APCLreservesweighting(apclrw);
    else
        APCLreservesweighting =
APCLreservesweighting(mod(ceil(M/(asw*artw*artresw*artl
rw*artrw*apcw*apcresw)),apclrw));
    end
end

if apcresw == 0
    APCreservesweighting = 0;
else
    if
mod((ceil((M/(asw*artw*artresw*artlrw*artrw*apcw)))),ap
cresw) == 0
        APCreservesweighting =
APCreservesweighting(apcresw);
    else
```

```
        APCreservesweighting =
APCreservesweighting(mod(ceil(M/(asw*artw*artresw*artlr
w*artrw*apcw)),apcresw));
    end
end

if apcw == 0
    APCWeighting = 0;
else
    if
mod((ceil((M/(asw*artw*artresw*artlrw*artrw)))),apcw)
== 0
        APCWeighting = APCWeighting(apcw);
    else
        APCWeighting =
APCWeighting(mod(ceil(M/(asw*artw*artresw*artlrw*artrw)
),apcw));
    end
end

if artrw == 0
    ArtREINFweighting = 0;
else
    if mod((ceil((M/(asw*artw*artresw*artlrw)))),artrw)
== 0
        ArtREINFweighting = ArtREINFweighting(artrw);
    else
        ArtREINFweighting =
ArtREINFweighting(mod(ceil(M/(asw*artw*artresw*artlrw))
,artrw));
    end
end

if artlrw == 0
    ArtLreservesweighting = 0;
else
    if mod((ceil((M/(asw*artw*artresw)))),artlrw) == 0
        ArtLreservesweighting =
ArtLreservesweighting(artlrw);
    else
        ArtLreservesweighting =
ArtLreservesweighting(mod(ceil(M/(asw*artw*artresw)),ar
tlrw));
    end
end

if artresw == 0
```

```
    Artreservesweighting = 0;
else
    if mod((ceil((M/(asw*artw)))),artresw) == 0
        Artreservesweighting =
Artreservesweighting(artresw);
    else
        Artreservesweighting =
Artreservesweighting(mod(ceil(M/(asw*artw)),artresw));
    end
end

if artw == 0
    ArtWeighting = 0;
else
    if mod(ceil(M/asw),artw) == 0
        ArtWeighting = ArtWeighting(artw);
    else
        ArtWeighting =
ArtWeighting(mod(ceil(M/asw),artw));
    end
end

if asw == 0
    SortiesWeighting = 0;
else
    if mod(M,asw) == 0
        SortiesWeighting = SortiesWeighting(asw);
    else
        SortiesWeighting =
SortiesWeighting(mod(M,asw));
    end
end


parameters =  ["alpha", "beta", "delta", "gamma", "p",
"q", "a", "b"]';
parametervalues = [alpha, beta, delta, gamma, p, q, a,
bee]';
parametersoutput = [cellstr(parameters),
num2cell(parametervalues)]

weightings = ["surviving manpower weighting", "manpower
reinforcements weighting", "manpower local reserves
weighting", "manpower reserves weighting", "manpower
weighting", "surviving tank weighting", "tank
reinforcements weighting", "tank local reserves
weighting", "tank reserves weighting", "tank
```

```
weighting", "surviving APC weighting", "APC
reinforcements weighting", "APC local reserves
weighting", "APC reserves weighting", "APC weighting",
"surviving artillery weighting", "arterilly
reinforcements weighting", "artillery local reserves
weighting", "artillery reserves weighting", "artillery
weighting", "air sorties weighting", "Variance"]';
weightingvalues = [Manpowersurvivingweighting,
ManpowerREINFweighting, ManpowerLreservesweighting,
Manpowerreservesweighting, ManpowerWeighting,
Tanksurvivingweighting, TankREINFweighting,
TankLreservesweighting, Tankreservesweighting,
TankWeighting, APCsurvivingweighting,
APCREINFweighting, APCLreservesweighting,
APCreservesweighting, APCWeighting,
Artsurvivingweighting, ArtREINFweighting,
ArtLreservesweighting, Artreservesweighting,
ArtWeighting, SortiesWeighting, myVariance]';
weightingsoutput = [cellstr(weightings),
num2cell(weightingvalues)]
```

Appendix 16

Table. Lanchester equations fit with equipment weighted wide ranges

		Equipment Weightings			Results		
	Day	TW (0:5:100)	APCW (0:5:100)	ArtW (0:5:100)	alpha	beta	**Variance**
Combat	1-10	10	100	25	6.27915	4.56031	0.833126
	11-32	0	100	0	0.0134935	-3.04806	0.434261
Total	1-10	100	100	100	3.85499	2.89850	0.522034
	11-32	5	100	0	0.0107884	-4.41438	0.670216
Reformatted Combat	1-10	0	100	90	51382200000	48.4286	0.813990
	11-32	0	100	0	0.000409481	-7.01563	0.437989
Reformatted Total	1-10	0	100	100	51770600000	45.0072	0.625270
	11-32	0	100	10	0.00144870	-7.09735	0.676760
Mean							0.626706

Appendix 17

Table. Lanchester equations fit with equipment weighted wide ranges, including air force

	Day	Equipment Weightings				Results		
		TW (0:5:100)	APCW (0:5:100)	ArtW (0:5:100)	AsW (0:5:100)	alpha	beta	**Variance**
Combat	1-10	0	100	70	100	2.90693	4.42112	0.899901
	11-32	0	100	0	0	0.0134935	-3.04806	0.434261
Total	1-10	65	100	100	100	3.67627	3.02196	0.588407
	11-32	5	100	0	0	0.0107884	-4.41438	0.670216
Reformatted Combat	1-10	0	90	100	10	47434600	38.9278	0.832976
	11-32	0	100	0	0	0.000409481	-7.01563	0.437989
Reformatted Total	1-10	0	100	100	25	12581200	29.6984	0.674585
	11-32	0	100	0	0	0.00144870	-7.09735	0.676760
Mean								0.651887

Appendix 18

Table. Lanchester equations fit with subdivisions weighted wide ranges

	Day	Resources	Weightings				Results		
			Surviving	REINF (0:1:5)	Lreserves (0:1:5)	Reserves	alpha	beta	**variance**
Combat	1-10	Manpower	1	0	5	0	0.557229	3.36473	0.707277
		Tank	1	0	0	0			
		APC	1	0	5	0			
		Artillery	1	1	5	0			
	11-32	Manpower	1	3	0	0	0.0528765	-6.50956	0.498049
		Tank	1	3	5	0			
		APC	1	2	5	0			
		Artillery	1	5	4	0			
Total	1-10	Manpower	1	0	0	0	1.73596	2.62510	0.527388
		Tank	1	0	0	0			
		APC	1	0	5	0			
		Artillery	1	0	5	0			
	11-32	Manpower	1	1	0	0	0.0347945	-6.83630	0.662520
		Tank	1	5	5	0			
		APC	1	5	2	0			
		Artillery	1	5	4	0			
Mean									0.598809

Appendix 19

Table 27.

	Day	Resources	Surviving	REINF (0:1:5)	Lreserves (0:1:5)	Reserves	alpha	beta	**variance**
			Weightings				**Results**		
Combat	1-10	Manpower	1	0	5	0	0.590047	2.98613	0.729115
		Tank	1	0	0	0			
		APC	1	0	5	0			
		Artillery	1	1	5	0			
	11-32	Manpower	1	3	0	0	0.0398933	-7.02511	0.435380
		Tank	1	0	5	0			
		APC	1	0	5	0			
		Artillery	1	5	4	0			
Total	1-10	Manpower	1	0	0	0	1.91050	2.79981	0.602853
		Tank	1	0	0	0			
		APC	1	0	5	0			
		Artillery	1	0	5	0			
	11-32	Manpower	1	2	2	0	0.0518756	-5.41909	0.597830
		Tank	1	5	5	0			
		APC	1	5	3	0			
		Artillery	1	5	2	0			
Mean									0.591294

Appendix 20

```
clear all;
close all;
clc;

ManpowerBsurviving =
xlsread('Table7ManpowerWeightings.xlsx', 'D6:D15');
ManpowerBREINF =
xlsread('Table7ManpowerWeightings.xlsx', 'E6:E15');
ManpowerBLreserves =
xlsread('Table7ManpowerWeightings.xlsx', 'F6:F15');
ManpowerBreserves =
xlsread('Table7ManpowerWeightings.xlsx', 'G6:G15');

TankBsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'D6:D15');
TankBREINF = xlsread('Table9EquipmentWeightings.xlsx',
'E6:E15');
TankBLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'F6:F15');
TankBreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'G6:G15');

APCBsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'D48:D57');
APCBREINF = xlsread('Table9EquipmentWeightings.xlsx',
'E48:E57');
APCBLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'F48:F57');
APCBreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'G48:G57');

ArtBsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'D90:D99');
ArtBREINF = xlsread('Table9EquipmentWeightings.xlsx',
'E90:E99');
ArtBLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'F90:F99');
ArtBreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'G90:G99');

SortiesB = xlsread('Table6Combat&TotalForces.xlsx',
'G6:G15');

Manpowerb = xlsread('Table6Combat&TotalForces.xlsx',
'B90:B99');
```

```
Tankb = xlsread('Table6Combat&TotalForces.xlsx',
'D90:D99');
APCb = xlsread('Table6Combat&TotalForces.xlsx',
'E90:E99');
Artb = xlsread('Table6Combat&TotalForces.xlsx',
'F90:F99');


ManpowerRsurviving =
xlsread('Table7ManpowerWeightings.xlsx', 'M6:M15');
ManpowerRREINF =
xlsread('Table7ManpowerWeightings.xlsx', 'N6:N15');
ManpowerRLreserves =
xlsread('Table7ManpowerWeightings.xlsx', 'O6:O15');
ManpowerRreserves =
xlsread('Table7ManpowerWeightings.xlsx', 'P6:P15');

TankRsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'M6:M15');
TankRREINF = xlsread('Table9EquipmentWeightings.xlsx',
'N6:N15');
TankRLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'O6:O15');
TankRreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'P6:P15');

APCRsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'M48:M57');
APCRREINF = xlsread('Table9EquipmentWeightings.xlsx',
'N48:N57');
APCRLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'O48:O57');
APCRreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'P48:P57');

ArtRsurviving =
xlsread('Table9EquipmentWeightings.xlsx', 'M90:M99');
ArtRREINF = xlsread('Table9EquipmentWeightings.xlsx',
'N90:N99');
ArtRLreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'O90:O99');
ArtRreserves =
xlsread('Table9EquipmentWeightings.xlsx', 'P90:P99');

SortiesR = xlsread('Table6Combat&TotalForces.xlsx',
'P6:P15');
```

```
Manpowerr = xlsread('Table6Combat&TotalForces.xlsx',
'K90:K99');
Tankr = xlsread('Table6Combat&TotalForces.xlsx',
'M90:M99');
APCr = xlsread('Table6Combat&TotalForces.xlsx',
'N90:N99');
Artr = xlsread('Table6Combat&TotalForces.xlsx',
'O90:O99');


%Resource weightings
ManpowerWeighting = 1;
TankWeighting = [10:2:30];
APCWeighting = [2:2:20];
ArtWeighting = [30:2:50];
SortieWeighting = [20:2:40];

%subdivision weightings
SurvivingWeighting = 1;
REINFWeighting = [1:0.4:3];
LReservesWeighting = [0:0.3:0.9];
ReservesWeighting = [0:0.3:0.9];

for i = 1:numel(ManpowerBsurviving)
    for tw = 1:numel(TankWeighting)
        for apcw = 1:numel(APCWeighting)
            for artw = 1:numel(ArtWeighting)
                for asw = 1:numel(SortieWeighting)
                    for reinf = 1:numel(REINFWeighting)
                        for lres =
1:numel(LReservesWeighting)
                            for res =
1:numel(ReservesWeighting)

Bmatrix(tw,apcw,artw,asw,reinf,lres,res,i) =
(((ManpowerBsurviving(i)*ManpowerWeighting) +
(TankBsurviving(i)*TankWeighting(tw)) +
(APCBsurviving(i)*APCWeighting(apcw)) +
(ArtBsurviving(i)*ArtWeighting(artw)))*SurvivingWeighti
ng)+ (((ManpowerBREINF(i)*ManpowerWeighting) +
(TankBREINF(i)*TankWeighting(tw)) +
(APCBREINF(i)*APCWeighting(apcw)) +
(ArtBREINF(i)*ArtWeighting(artw)))*REINFWeighting(reinf
)) + (((ManpowerBLreserves(i)*ManpowerWeighting) +
(TankBLreserves(i)*TankWeighting(tw)) +
(APCBLreserves(i)*APCWeighting(apcw)) +
(ArtBLreserves(i)*ArtWeighting(artw)))*LReservesWeighti
```

```
ng(lres)) + (((ManpowerBreserves(i)*ManpowerWeighting)
+ (TankBreserves(i)*TankWeighting(tw)) +
(APCBreserves(i)*APCWeighting(apcw)) +
(ArtBreserves(i)*ArtWeighting(artw)))*ReservesWeighting
(res)) + (SortiesB(i)*SortieWeighting(asw));

Rmatrix(tw,apcw,artw,asw,reinf,lres,res,i) =
(((ManpowerRsurviving(i)*ManpowerWeighting) +
(TankRsurviving(i)*TankWeighting(tw)) +
(APCRsurviving(i)*APCWeighting(apcw)) +
(ArtRsurviving(i)*ArtWeighting(artw)))*SurvivingWeighti
ng)+ (((ManpowerRREINF(i)*ManpowerWeighting) +
(TankRREINF(i)*TankWeighting(tw)) +
(APCRREINF(i)*APCWeighting(apcw)) +
(ArtRREINF(i)*ArtWeighting(artw)))*REINFWeighting(reinf
)) + (((ManpowerRLreserves(i)*ManpowerWeighting) +
(TankRLreserves(i)*TankWeighting(tw)) +
(APCRLreserves(i)*APCWeighting(apcw)) +
(ArtRLreserves(i)*ArtWeighting(artw)))*LReservesWeighti
ng(lres)) + (((ManpowerRreserves(i)*ManpowerWeighting)
+ (TankRreserves(i)*TankWeighting(tw)) +
(APCRreserves(i)*APCWeighting(apcw)) +
(ArtRreserves(i)*ArtWeighting(artw)))*ReservesWeighting
(res)) + (SortiesR(i)*SortieWeighting(asw));

bmatrix(tw,apcw,artw,asw,reinf,lres,res,i) =
((Manpowerb(i)*SurvivingWeighting)*ManpowerWeighting)+(
(Tankb(i)*SurvivingWeighting)*TankWeighting(tw))+((APCb
(i)*SurvivingWeighting)*APCWeighting(apcw))+((Artb(i)*S
urvivingWeighting)*ArtWeighting(artw));

rmatrix(tw,apcw,artw,asw,reinf,lres,res,i) =
((Manpowerr(i)*SurvivingWeighting)*ManpowerWeighting)+(
(Tankr(i)*SurvivingWeighting)*TankWeighting(tw))+((APCr
(i)*SurvivingWeighting)*APCWeighting(apcw))+((Artr(i)*S
urvivingWeighting)*ArtWeighting(artw));
                            end
                        end
                    end
                end
            end
        end
    end
end

for TW = 1:tw
    for APCW = 1:apcw
```

```
        for ARTW = 1:artw
            for ASW = 1:asw
                for REINF = 1:reinf
                    for LRES = 1:lres
                        for RES = 1:res
                            for I = 1:i

LinearIndexing(I,RES,LRES,REINF,ASW,ARTW,APCW,TW) =
sub2ind(size(Bmatrix),TW,APCW,ARTW,ASW,REINF,LRES,RES,I
);
                            end
                        end
                    end
                end
            end
        end
    end
end

Bmatrixsorted = Bmatrix(LinearIndexing);
Rmatrixsorted = Rmatrix(LinearIndexing);
bmatrixsorted = bmatrix(LinearIndexing);
rmatrixsorted = rmatrix(LinearIndexing);

Bmatrix2D =
reshape(Bmatrixsorted,i,tw*apcw*artw*asw*reinf*lres*res
);
Rmatrix2D =
reshape(Rmatrixsorted,i,tw*apcw*artw*asw*reinf*lres*res
);
bmatrix2D =
reshape(bmatrixsorted,i,tw*apcw*artw*asw*reinf*lres*res
);
rmatrix2D =
reshape(rmatrixsorted,i,tw*apcw*artw*asw*reinf*lres*res
);

v = 0;

for m = 1:(tw*apcw*artw*asw*reinf*lres*res)
    B = Bmatrix2D(:,m);
    R = Rmatrix2D(:,m);
    b = bmatrix2D(:,m);
    r = rmatrix2D(:,m);

        for t = 1:i
            x(t) = (R(t)/B(t));
```

```
            y(t) = (b(t)/r(t));
            logx(t) = log(x(t));
            logy(t) = log(y(t));
        end

        polycoefficients = polyfit(logx,logy,1);
        linearbestfit =
polycoefficients(1)*logx+polycoefficients(2);
        residuals = (logy - linearbestfit)';
        variance(m) = 1-(sum(residuals.^2)/sum((logy-
mean(logy)).^2));

        if variance(m) > v
        v = variance(m);
        M = m;
        end
end

myB = Bmatrix2D(:,M);
myR = Rmatrix2D(:,M);
myb = bmatrix2D(:,M);
myr = rmatrix2D(:,M);

for t = 1:i
    x(t) = (myR(t)/myB(t));
    y(t) = (myb(t)/myr(t));
    logx(t) = log(x(t));
    logy(t) = log(y(t));

    X(t) = (myB(t)*myR(t));
    Y(t) = (myb(t)*myr(t));
    logX(t) = log(X(t));
    logY(t) = log(Y(t));
end

scatter(logx,logy);
polycoefficients = polyfit(logx,logy,1);
linearbestfit =
polycoefficients(1)*logx+polycoefficients(2);
residuals = (logy - linearbestfit)';
hold on;
plot(logx,linearbestfit);

beta = polycoefficients(1);
alpha = exp(polycoefficients(2));
myVariance = v;

polycoefficients2 = polyfit(logX,logY,1);
```

```
delta = polycoefficients2(1);
gamma = exp(polycoefficients2(2));

p = (delta+beta)/2;
q = (delta-beta)/2;
a = sqrt(alpha*gamma);
bee = sqrt(gamma/alpha);

TankWeighting =
TankWeighting(ceil(M/(res*lres*reinf*asw*artw*apcw)));

if apcw == 0
    APCWeighting = 0;
else
    if mod((ceil((M/(res*lres*reinf*asw*artw)))),apcw)
== 0
        APCWeighting = APCWeighting(apcw);
    else
        APCWeighting =
APCWeighting(mod(ceil(M/(res*lres*reinf*asw*artw)),apcw
));
    end
end

if artw == 0
    ArtWeighting = 0;
else
    if mod((ceil((M/(res*lres*reinf*asw)))),artw) == 0
        ArtWeighting = ArtWeighting(artw);
    else
        ArtWeighting =
ArtWeighting(mod(ceil(M/(res*lres*reinf*asw)),artw));
    end
end

if asw == 0
    SortieWeighting = 0;
else
    if mod((ceil((M/(res*lres*reinf)))),asw) == 0
        SortieWeighting = SortieWeighting(asw);
    else
        SortieWeighting =
SortieWeighting(mod(ceil(M/(res*lres*reinf)),asw));
    end
end

if reinf == 0
```

```
    REINFWeighting = 0;
else
    if mod((ceil((M/(res*lres)))),reinf) == 0
        REINFWeighting = REINFWeighting(reinf);
    else
        REINFWeighting =
REINFWeighting(mod(ceil(M/(res*lres)),reinf));
    end
end

if lres == 0
    LReservesWeighting = 0;
else
    if mod(ceil(M/res),lres) == 0
        LReservesWeighting = LReservesWeighting(lres);
    else
        LReservesWeighting =
LReservesWeighting(mod(ceil(M/res),lres));
    end
end

if res == 0
    ReservesWeighting = 0;
else
    if mod(M,res) == 0
        ReservesWeighting = ReservesWeighting(res);
    else
        ReservesWeighting =
ReservesWeighting(mod(M,res));
    end
end


parameters =  ["alpha", "beta", "delta", "gamma", "p",
"q", "a", "b"]';
parametervalues = [alpha, beta, delta, gamma, p, q, a,
bee]';
parametersoutput = [cellstr(parameters),
num2cell(parametervalues)]

weightings = ["Tank Weighting", "APC Weighting",
"Artillery Weighting", "Air Sortie Weighting",
"surviving weighting", "reinforcements weighting",
"local reserves weighting", "reserves weighting",
"variance"]';
weightingvalues = [TankWeighting, APCWeighting,
ArtWeighting, SortieWeighting, SurvivingWeighting,
```

```
REINFWeighting, LReservesWeighting, ReservesWeighting,
myVariance]';
weightingsoutput = [cellstr(weightings),
num2cell(weightingvalues)]
```

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