qgSurf
Determination of intersections between Cartesian planes and raster digital elevation models
Algorithm Theoretical Basis Document

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The intersection between a geological plane and a topographic surface is a subject of interest in geological field mapping and structural geology.
In a GIS, the topographic surface is generally represented as a grid. GDAL, the most diffuse open-source library for handling rasters, describes and manages a grid via parameters such as the cell sizes, the top-right cell corner coordinates and also the grid rotations.
Generally available topographic grids do not present rotations, but this possibility cannot be ruled out in a few cases.
Prior to the 2.0 version, qgSurf assumed no grid rotations for determining the plane-grid intersections. With the 2.0 version, a new algorithm, inspired to the previous one, permit to process also grid with built-in rotations.

The GDAL geotransform

To describe the geographical properties of a raster, GDAL uses the concept of "geotransform".
In essence, the GDAL geotransform is just a matrix that allows to derive the geographic coordinates of a point given its pixel coordinates.
Mathematically, it can be expressed using an augmented matrix:

(1)
$$\begin{bmatrix} \vec{y} \\ 1 \end{bmatrix} = \begin{bmatrix} A & | & \vec{b} \\ 0..0 & | & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$$

so that:

(2)
$$y' = A\vec{x} + \vec{b}$$

In eq. (1) we can substitute the geotransform parameters:

(3)
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} gt_1 & gt_2 & gt_0 \\ gt_4 & gt_5 & gt_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The linear equations for the transformed coordinates are:

(4)
$$\begin{aligned} x' &= gt_1 x + gt_2 y + gt_0 \\ y' &= gt_4 x + gt_5 y + gt_3 \\ 1 &= 0 + 0 + 1 \end{aligned}$$

From the equation in (4), it can be easily observed that the gt0 and gt3 represent grid offsets in the x - and y- directions respectively, gt1 and gt5 are scaling factors for x- and y- directions, while gt2 and gt4 represent rotations/skewing.
In the general case, an orthogonal grid can be transformed by a GDAL geotransform into a grid where x and y axes are no longer perpendicular: an example of a skewed and rotated grid is illustrated in Fig. 1.

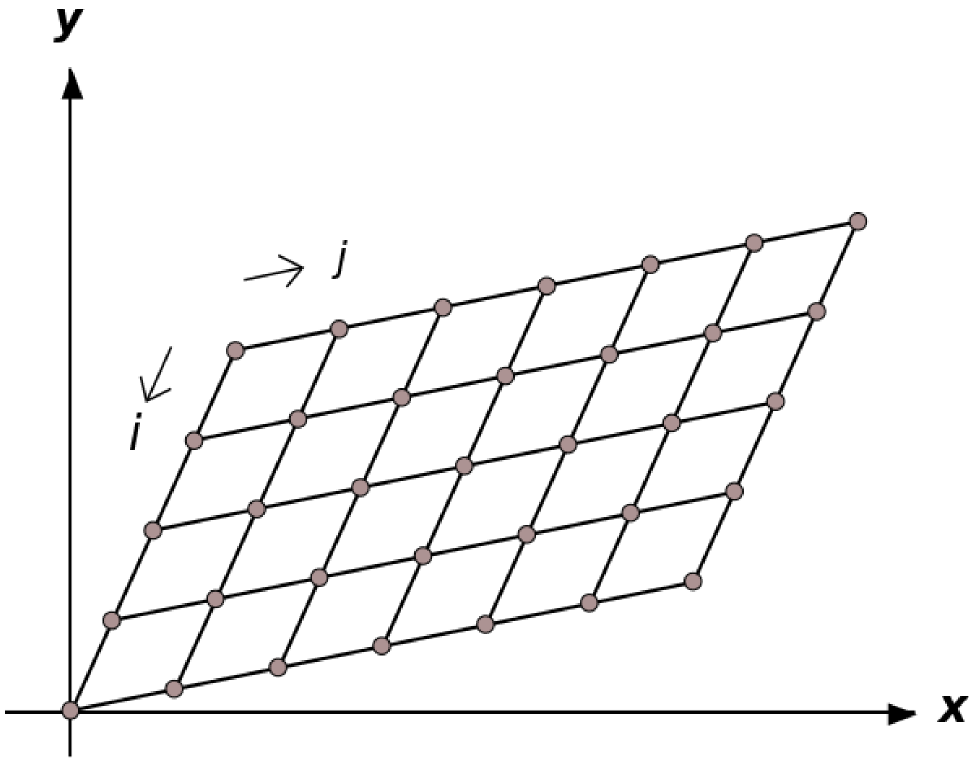

Figure 1: Example of an orthogonal grid transformed by an affine transformation.

Determination of plane-DEM intersections

To determine the plane-DEM intersections, we transform the geometrical problem from 3D to 2d, in order to simply it. We consider the geometric traces of the plane and of the topographic surface with. two sets of parallel vertical planes, the first set oriented parallel to the final j- grid axes orientations, the second set parallel to the final i- grid axis orientation (see Fig. 1). Each plane of the set contains a grid point row (for the j-parallel planes) or column (for the i-parallel planes). The points correspond to the final grid cell centers.
Now we turn to a single vertical plane, for example in the j-direction (see Fig. 2, that correspond to a vertical section).
We have equi-spaced final cell centers along the plane, where the spacing do not necessarily corresponds to the original geotransform cell sizes, given the a general case geotransform can distort (skew) a grid. This spacing is however constant in each set of vertical planes and can be easily calculated as the distance between the first and second geotransformed cell centers. When there is no grid skewing and just a rigid-body rotation (o no rotation at all), the spacing will be equal to the original grid cell sizes.
Within a single vertical plane we consider its intersections with the DEM and the plane surfaces, at each geo-transformed cell centers.
The point intersection with the DEM corresponds trivially to the DEM height for that cell, while the plane intersections can be easily calculated given the plane equation and the considered cell center point coordinates.
Knowing for each cell center along a vertical plane the DEM and plane height, we can determine the plane-DEM intersection between two consecutive cell centers, as described below.

As can be seen in Fig. 2, we have a valid plane-DEM intersection point between two consecutive cell centers (for instance j=0 and j=1 in Fig. 2), when the relative z positions of the plane and the DEM traces switch between the two considered cell centers. On the other hand, when the plane line is always higher (or lower) than a DEM line, there is no intersection (e.g., j=1 and j=2 in Fig. 2). Obviously, when at a cell center line and DEM points coincide, it correspond to an intersection point.

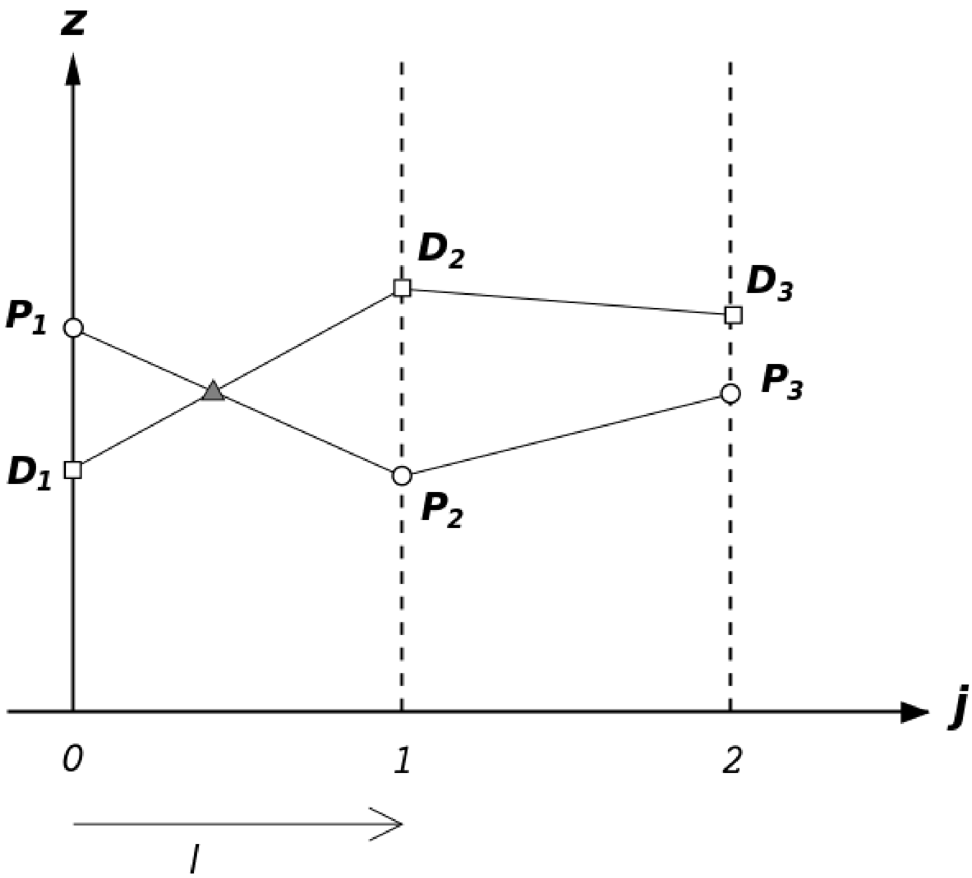

Figure 2: Determination of intersecting plane (P points) and DEM (D points) along the j direction.

The height equations for the DEM and the plane are:

$$z_d = m_d x' + q_d$$
$$z_p = m_p x' + q_p$$

where q_p is the plane elevation at point (x, y), q_d is the grid z value, m_d is the angular coefficient of the DEM trace (for the given direction) and m_p is the angular coefficient of the plane trace in the considered direction.

At an intersection point we have:

$$m_d x' + q_d = m_p x' + q_p \Rightarrow$$
$$(m_d - m_p)x' = q_p - q_d \Rightarrow$$
$$x' = \frac{q_p - q_d}{m_d - m_p}$$

where *x'* is the distance of the intersection point from the left cell center.

The array coordinate of the intersection point is therefore equal to:

$$\frac{x'}{cellsize'} = i'[0 \rightarrow 1]$$

where *cellsize'* is equal to the geotransformed cell spacing in the considered direction and *i'* is the array coordinate in the considered direction, that can then be transformed into geographic coordinates by applying the geotransform, thus solving the investigated problem.