Max Grove
HW 11
Q5

a) Use mathematical induction to prove that for any positive integer n, 3 divides $n^3 + 2n$ (leaving no remainder)

   We need to prove that $n^3 + 2n$ can be expressed as the multiple of 3 and another integer, that is, $n^3 + 2n = 3m$. I will prove using induction

   Base Case: n = 1
   $1^3 + 2(1) = 3$; 3 can be expressed as 3m, when m = 1

   Assume that $n^3 + 2n$ can be expressed as 3m.
   We need to prove that $(n+1)^3 + 2(n+1)$ is divisible by 3 with no remainder.

   $(n+1)^3 + 2(n+1) = (n^2 + 2n + 1)(n+1) + 2(n+1) = n^3 + 2n^2 + n + n^2 + 2n + 1 + 2n + 2$
   $= n^3 + 3n^2 + 5n + 3$
   $= n^3 + 2n + 3n^2 + 3n + 3$
   $= 3m + 3n^2 + 3n + 3$, since $n^3 + 2n$ can be expressed as 3m,
   $= 3(m + n^2 + 3n + 3)$.
   Since m and n are integers, $m + n^2 + 3n + 3$ is an integer, so $(n+1)^3 + 2(n+1)$ can be expressed as an integer multiple of 3, which means that 3 divides $n^3 + 2n$.

b) Use strong induction to prove that any positive integer n (n ≥ 2) can be written as a product of primes.

   Base Case: n = 2. 2 can be expressed as the product of 1 prime number: 2

   Assume that every number from 2 to k can be expressed as the product of 2 prime numbers.

   We will prove that (k+1) can be expressed as the product of 2 prime numbers.
   (k+1) can either be prime or composite. If (k+1) is prime, it can be expressed as the product of just that number k+1. If (k+1) is composite, there exists two numbers less than (k+1) that equal (k+1). (k+1) = a*b, where a and b are less than k and are integers. Since a and b are less than (k+1), they are equal to k or less. Given the assumption that every number from 2 to k can be expressed as a product of prime numbers, both a and b can be expressed as the product of 2 prime numbers. Thus (k+1) can be expressed as the product of 2 prime numbers. Thus, any positive integer n (n ≥ 2) can be written as a product of primes.

Question 6

a) Exercise 7.4.1, sections a-g

a) $P(3) = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$

$3(3+1)(3*2+1) / 6 = 3*4*7 / 6 = 14$

b) $P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

c) $p(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

d) In the base case, we prove $P(1) = \sum_{j=1}^{1} j^2 = \frac{1(1+1)(2*1+1)}{6}$

$P(1) = \sum_{j=1}^{1} j^2 = 1^2 = 1$

$\frac{1(1+1)(2*1+1)}{6} = \frac{1 \times 2 \times 3}{1} = 1$

e) In the inductive step, we prove $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

f) The hypothesis is $P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

g) We will prove by induction that for any positive integer n,

$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

We have proven the base case above in step d. We take the hypothesis assumption in part f to be true.

We will prove $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\sum_{j=1}^{k+1} j^2 = (k+1)^2 + \sum_{j=1}^{k} j^2 = (k+1)^2 + \frac{k(k+1)(2k+1)}{6}$$

Let us prove $(k+1)^2 + \frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$6(k+1)^2 + k(k+1)(2k+1) = (k+1)(k+2)(2k+3)$

$6(k+1) + k(2k+1) = (k+2)(2k+3)$

$6k + 6 + 2k^2 + k = 2k2 + 3k + 4k + 6$

$2k^2 + 7k + 6 = 2k^2 + 7k + 6$

Since we have proven $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$, we have proven that for any positive integer n, P(k) = $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

b) Exercise 7.4.3, section c

c) Prove that, for n >= 1, $\sum_{j=1}^{n} \frac{1}{j^2} <= 2 - \frac{1}{n}$

When n = 1, $\sum_{j=1}^{n} \frac{1}{j^2}$ = 1. $2 - \frac{1}{n}$ = 2-1 = 1. 1 <= 1

We assume $\sum_{j=1}^{k} \frac{1}{j^2} <= 2 - \frac{1}{k}$ and will prove $\sum_{j=1}^{k+1} \frac{1}{j^2} <= 2 - \frac{1}{k+1}$

$\sum_{j=1}^{k+1} \frac{1}{j^2} = \frac{1}{(k+1)^2} + \sum_{j=1}^{k} \frac{1}{j^2}$ <= $\frac{1}{(k+1)^2} + 2 - \frac{1}{k}$ (by the hypothesis)

<= $\frac{1}{k(k+1)} + 2 - \frac{1}{k}$

= 2 + $\frac{1}{k(k+1)}$ - $\frac{(k+1)1}{(k+1)k}$

= 2 + $\frac{1-(k+1)}{k(k+1)}$

= 2 + $\frac{(k)}{k(k+1)}$

= 2 + $\frac{1}{(k+1)}$

c) Exercise 7.5.1, section a

a) Prove that for any positive integer n, 4 evenly divides 3^{2n}-1

We are looking to prove 3^{2n}-1 can be expressed as 4m, where m is an integer

Base Case: 3^{2*1} - 1 = 9 - 1 = 8. 8 can be expressed as 4*2

Assume 3^{2n}-1 can be expressed as 4m and prove $3^{2(n+1)}$-1 can be expressed as 4 times an integer

$3^{2(n+1)}$-1 = 3^{2n+2} - 1 = $3^2 3^{2n}$-1 = 9*3^{2n} - 9 + 8 = 9(3^{2n} - 1) + 8

= 9(4m) + 8, by the hypothesis

= 4(9m + 2). Since m is an integer, 4(9m + 2) will be an integer. Thus $3^{2(n+1)}$-1 can be expressed as 4 times an integer. Thus we have proven that for any positive integer n, 4 evenly divides 3^{2n}-1