Maxie Dion Schmidt
Tentative Doctoral Thesis Abstract

Abstract

We are initially motivated by a few special cases of invertibly matrix based constructions for expressing the ordinary generating functions (OGFs) of special convolution type sums. These sums arise in classical number theoretic applications where we can express power series that enumerate many multiplicative arithmetic functions. We have a base of prior literature published over 2016-2021 with significant contributions by the author (MDS) to prove generating function based factorization theorems. These so-termed factorization theorems express the coefficients generated by Lambert series (or more generally, Dirichlet divisor sums) and GCD type sums that compute the sequences of partial sums of any arithmetic function f over the integers $1 \leq d \leq n$ such that d is relatively prime to the $n \geq 1$. Typical examples of sums of this type include, respectively, the class of Dirichlet convolutions $g = f * 1$ which are inverted by a standard Möbius inversion technique, and the Euler totient function, $\phi(n)$.

We spend several sections at the middle of the thesis recalling the proofs and relevant constructions behind the articles published in the cited *Acta Arithmetica*, the *Ramanujan Journal*, and the *American Mathematical Monthly*, among others in the past few years. These results suggest and motivate the study of an even more general class of generating function expansions (and matrix-based factorization theorems) for convolution type sums. Consider the following sum types where the bivariate kernel, or convolution weight function denoted by $\mathcal{D} : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{C}$, is lower triangular and invertible when composed of finite dimensional $N \times N$ square matrices:

$$(f \boxdot_{\mathcal{D}} g)(n) := \sum_{k=1}^{n} f(k) g(n+1-k) \mathcal{D}(n,k), n \geq 1.$$

Sums of this type have familiar expressions by matrix-vector products involving Topelitz matrices, which are themselves common and well studied to express the discrete convolutions of sequences.

The new interpretation and analysis of the later stated results for these matrix based constructions of special convolution type sums is two fold. First, and the most obvious rationale for looking at series based enumerations of special sequence types, is to provide a more traditional analog to the strictly combinatorial perspective whereby we express sequences (or arithmetic functions) of interest in applications. As is typical with more traditionally combinatorial generating function methods, we find revealing structures that offer appeals to methods from both purely algebraic formal power series constructions and analytic objects. Such a generalization of the combinatorics behind OGFs of standard classical series to express new identities for special sums in multiplicative number theory and other less usual applications, is a natural extension of the Lambert series generating function for many divisor sums that arise in studying classical elementary number theory (e.g., see Hardy and Wright). The second component in which our perspective on these topics is viewed is in formulating a rigorous exploration for expressing so-called canonical relations between sequences and characteristic series that can appear in seemingly disparate branches of additive and multiplicative number theory and mathematical applications. The idea to this second key component of the dissertation offers a unique new way to generalize the previously non-standard relationships that arise in studying the factorization theorems by Merca and Schmidt over 2016-2018. These initially studied relationships express traditionally multiplicative only based constructions naturally by partition theoretic sequences and product based generating function constructions.

These topics have been studied and classified to a certain extent (while our interpretation and motivating context is certainly new) by Rota and others in their original works defining generalized Möbius functions. This topic, typically related and introduced to study posets and relationships between sums of functions over partially ordered sets, has deeper settings when explicitly defined by incidence algebras. The Dirichlet divisor sums and convolution formulas we have already discovered in recent print are, in this instance, a simple example of the convolution operators that are associated to the intervals $[1, n] \subseteq \mathbb{N}$ of positive integers ordered by divisibility. This straightforward example leads us to the celebrated (classical, ordinary) Möbius function, $\mu(a, b) \equiv \mu(b/a)$. We stress that the new interpretations we build on differentate the new work found in this dissertation from the algebraic perspectives primarily due to Rota, et. al.